BENDING OF THIN ELASTIC PLATES CONTAINING LINE DISCONTINUITIES

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ABSTRACT

The purpose of this work is to examine the stress distribution caused by the bending of a thin elastic plate containing a line discontinuity. Specifically, the plate under consideration is of constant thickness and occupies a whole plane exterior to the line discontinuity. The line discontinuity is either a crack or a rigid inclusion.

The loading is applied to the plate at infinity by certain combinations of tractions which leave the plate in equilibirum.

The analysis of the problems considered here is based on an approximate theory which is more refined than the classical theory ordinarily applied to problems of bending of plates. This is because results based on the classical theory may be incorrect, even in first approximation for thin plates, near a boundary, and it is precisely the region near a boundary (in this case, the line discontinuity) which is of primary interest in these problems. In fact one of the principal objectives in this work is to compare the stress distributions near the line discontinuity as predicted by the two theories.

The principal techniques used in this work are based on integral equations and the calculus of variations.

Results based on the two theories are found to agree for thin plates away from the line discontinuity, but differ significantly in the vicinity of the discontinuity, even for very thin plates.

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I. INTRODUCTION

The problem considered in this work is the investigation of the stress distribution caused by the bending of a thin elastic plate containing a line discontinuity of finite length 2c. Specifically we consider a plate of constant thickness h whose midplane occupies the region consisting of all points in the XY-plane except for the segment $Y = 0$, $|X| \leq c$ where the line discontinuity is located (see Figure 1a). The line discontinuity under consideration is either a crack or a rigid inclusion. From the three dimensional point of view (see Figure 1b), a crack is a plane surface perpendicular to the midplane of the plate which is to be free of stress. Thus we shall require that the traction across the crack surface vanish at every point along $Y = 0\pm$, $|X| < c$. A rigid inclusion is such a plane surface which is assumed to be fixed in space. In this case we shall require that the displacement vanish everywhere along $Y = 0\pm$, $|X| < c$. The plate is to be deformed by certain external tractions applied along the cylindrical surface at infinity $(\sqrt{X^2+Y^2} = \infty)$.

The classical theory of bending for thin plates which was first established by Kirchhoff and Gehring (see Love [1]) and clarified later by Kelvin and Tait [2] is known to lead to inaccuracies in stresses, even for thin plates, near the edges of a plate. This is due to the fact that the theory requires, and indeed can accommodate, only two boundary conditions along an edge in connection with the biharmonic differential equation, which is the governing equation of the theory. For example, the physically natural boundary conditions for a free edge of a plate are the vanishing of three components of the traction across that edge. However in the classical theory these conditions are reduced in an approximate way in accordance with Saint-Venant's principle to two conditions, namely the vanishing of the normal stress couple and the vanishing of the so-called "effective Kirchhoff force".

Stoker [3] pointed out that this difficulty may be resolved if one formulates the plate problem as a "boundary layer" problem. He suggested that the starting point would be the three-dimensional theory of elasticity. He would then study the limit problem obtained upon allowing the thickness of the plate to approach zero in the differential equations and he predicted that the differential equation would degenerate and some boundary condition would be lost at the edge. Such an approach has been discussed recently by Friedrichs and Dressler [4].

By taking into account the transverse shear deformation interior to an elastic plate which is omitted in the classical theory, Reissner [5] developed an approximate theory for bending of thin plates which is governed by a sixth-order differential equation and hence requires three boundary conditions along an edge. Quite a few problems [5], [6] have been solved based on Reissner's theory and results so obtained in general provide a qualitatively better approximation to exact values in comparison with the classical theory, particularly in the vicinity of an edge of the plate.

On account of the reasons mentioned in the previous paragraph, the Reissner theory of bending is employed here since it is the stress and displacement fields near the discontinuity that are of primary interest, and in this region we expect that the classical theory may be incorrect even in first approximation.

The problem for bending of an infinite plate containing a crack has been investigated on the basis of classical theory by Williams [7] and by Ang and Williams [8]. In [7] eigenfunction expansions are used in the flexure problem of an isotropic plate containing a semi-infinite crack in order to study qualitatively the character of the stress distributions near the vertex of the crack. In [8] both stretching and bending problems are studied for an orthotropic plate containing a finite crack and solutions are obtained by means of dual integral equations.

There are in the literature many crack problems in elasticity that have been solved [9]. For example we may refer to Sneddon and Elliot [10] for the problem of finding the stress distribution in the neighbourhood of a Griffith crack in a stretched plate and to Sneddon [11] for the similar problem for the case of a penny-shaped crack in a three-dimensional elastic solid.

The problems of an isotropic infinite plate containing an elliptical hole or an elliptical rigid inclusion are considered using Reissner's theory in [12] and in [13] respectively. However, the approximation made in these references is not valid for a slender ellipse and hence the results of [12] and [13] could not be used to examine the limiting case as the ellipse tends to a line.

In Part II, the Reissner theory for bending of thin plates is derived in a somewhat different way than in [5] or in [6]. The present derivation is closer to that given by Green [14].

In Part III the problem of an infinite elastic plate containing a line discontinuity is formulated in terms of Reissner's equations and is reduced to integral equations for the case of a crack and the case of a rigid inclusion. Each of these two cases is separated into two parts, symmetric and antisymmetric with respect to the line discontinuity.

In Part IV the existence of solutions for the symmetric and antisymmetric parts of the crack case and for the antisymmetric part of the rigid inclusion case is established through using Fredholm theorems for integral equations. In the same part an approximate solution for each case is obtained for plates whose thickness is small in comparison with the length 2c of the line discontinuity. The corresponding results based on classical theory are also computed for purposes of comparison. The symmetric part of the rigid inclusion case is purposely omitted since it presents no interesting features beyond those extracted from the other cases. It is shown that for thin plates the results based on classical theory give good approximation in general except in a boundary layer near the edge where the results of two theories are different even for very thin plates. It is found that the angular distribution of stresses around a crack point is different in Reissner's theory than in the classical theory. A discussion of the differences in the results based on the two theories is given in section 4.10.

A variational method is derived in Part V in order to investigate moderately thick plates for which the approximate solution obtained in Part IV is no longer expected to be accurate. It is shown in some special cases that the variational solution tends to the approximate solution obtained in Part IV as the plate thickness tends to zero.

II. DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS FOR BENDING OF PLATES

2.1 Equations of Elasticity Interior to a Plate and Boundary Conditions

Let us attach a set of rectangular coordinate system (XYZ) to a plate of constant thickness such that the XY-plane coincides with the middle plane of the plate (see Figure 2). As shown in Figure 2, the middle plane of the plate occupies a region $\mathcal{D}$ in the XY-plane and its boundary $\bigcup_i C_i$ where $\bigcup_i C_i$ denotes the union of cylindrical contours of the plate's boundary. The plate is assumed to have thickness h and hence every point P(X, Y, Z) interior or on the boundary of the plate belongs to one of the following sets:

(i) Interior set $\mathcal{S} := \{P(X,Y,Z) : (X,Y) \text{ in } \mathcal{D}, Z \text{ in } (-\frac{h}{2}, \frac{h}{2})\}$

(ii) Surface sets $S_{\pm} := \{P(X,Y,Z) : (X,Y) \text{ in } \mathcal{D}, \ Z = \pm \frac{h}{2}\}$ where S_+ and S_- refer to upper surface and lower surface respectively,

(iii) Cylindrical boundary set $B \equiv \bigcup_i B_i$ where $B_i := \{P(X,Y,Z) : (X,Y) \text{ in } C_i, \ Z \text{ in } [-\frac{h}{2}, \frac{h}{2}]\}$.

Now we shall define the stress and strain components for every point in $\mathcal{S}$. Referring to the above coordinate system we denote by σ_x, σ_{xy} and σ_{xz} respectively the vector components in X, Y and Z directions of the traction at a point P in $\mathcal{S}$ across a plane X = const. and similarly by σ_{yx}, σ_y, σ_{yz} and σ_{zx}, σ_{zy}, σ_z respectively for the components of tractions at the same point across planes Y = const. and Z = const. We call these quantities the stress components at the point P in $\mathcal{S}$.

Upon consideration of mechanics, it can be easily shown that $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$ and $\sigma_{zx} = \sigma_{xz}$, and

$$\frac{\partial \sigma_x}{\partial X} + \frac{\partial \sigma_{xy}}{\partial Y} + \frac{\partial \sigma_{xz}}{\partial Z} + F_x = 0 \, , \tag{2.1,1}$$

$$\frac{\partial \sigma_{xy}}{\partial X} + \frac{\partial \sigma_y}{\partial Y} + \frac{\partial \sigma_{yz}}{\partial Z} + F_y = 0 \, , \tag{2.1,2}$$

$$\frac{\partial \sigma_{xz}}{\partial X} + \frac{\partial \sigma_{yz}}{\partial Y} + \frac{\partial \sigma_z}{\partial Z} + F_z = 0 \tag{2.1,3}$$

hold at every point in $\mathcal{S}$. Equations (2.1,1), (2.1,2) and (2.1,3) are known as equations of equilibrium in which F_x, F_y and F_z are the components of the external body force in the X, Y and Z directions, respectively.

The material of the plate is assumed to be isotropic and homogeneous with Young's modulus E, shear modulus G and Poisson's ratio ν. It is also assumed that the plate is subject to small deformations and strains so that the stress-strain relations may be established through Hooke's law. If we denote by U, V and W respectively the displacement components in the X, Y and Z directions at every point interior to the plate, then we have

$$e_x \equiv \frac{\partial U}{\partial X} = \frac{1}{E} (\sigma_x - \nu \sigma_y) - \frac{\sigma_z \nu}{E} \, , \tag{2.1,4}$$

$$e_y \equiv \frac{\partial V}{\partial Y} = \frac{1}{E} (\sigma_y - \nu \sigma_x) - \frac{\sigma_z \nu}{E} \, , \tag{2.1,5}$$

$$e_{xy} \equiv \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} = \frac{1}{G} \sigma_{xy} \, , \tag{2.1,6}$$

$$e_z \equiv \frac{\partial W}{\partial Z} = \frac{\sigma_z}{E} - \frac{\nu}{E} (\sigma_x + \sigma_y) \, , \tag{2.1,7}$$

$$e_{xz} \equiv \frac{\partial W}{\partial X} + \frac{\partial U}{\partial Z} = \frac{1}{G}\sigma_{xz}, \qquad (2.1,8)$$

$$e_{yz} \equiv \frac{\partial W}{\partial Y} + \frac{\partial V}{\partial Z} = \frac{1}{G}\sigma_{yz} \qquad (2.1,9)$$

for P in $\mathcal{S}$.

We assume that the upper surface $Z = \frac{h}{2}$ is subjected to normal traction p(X, Y) per unit area and the lower surface $Z = -\frac{h}{2}$ is free from external forces. Hence, the conditions on both surfaces are respectively

$$\sigma_z = p(X,Y), \quad \sigma_{xz} = \sigma_{yz} = 0 \quad \text{at P in } S_+ , \qquad (2.1,10)$$

$$\sigma_z = \sigma_{xz} = \sigma_{yz} = 0 \qquad \text{at P in } S_- . \qquad (2.1,11)$$

From any point along a cylindrical surface $C(s) \times [-\frac{h}{2}, \frac{h}{2}]$ in $\mathcal{S}$ where C(s) is the projection of the surface on the XY plane and s is its parameter, we may draw a normal to C(s) directed to the right with respect to the positive sense of C(s). If we denote by σ_n, σ_{ns} and σ_{nz} respectively the components in the normal, tangential and Z directions of the traction at the point across a plane perpendicular to the normal, then we must have the following relations:

$$\sigma_n = \sigma_x \cos^2(n,X) + \sigma_y \sin^2(n,X) + 2\sigma_{xy} \sin(n,X)\cos(n,X)$$

$$\sigma_{ns} = (\sigma_y - \sigma_x)\sin(n,X)\cos(n,X) + \sigma_{xy}(\cos^2(n,X) - \sin^2(n,X)) \qquad (2.1,12)$$

$$\sigma_{nz} = \sigma_{xz}\cos(n,X) + \sigma_{yz}\sin(n,X)$$

where cos(n, X) denotes the cosine of the angle between the normal and X-axis. The displacement components along C(s) can be related to U, V and W by an orthogonal transformation

$$\begin{pmatrix} U_n \\ U_s \\ W \end{pmatrix} = \begin{pmatrix} \cos(n,X) & \sin(n,X) & 0 \\ -\sin(n,X) & \cos(n,X) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix} \qquad (2.1,13)$$

where U_n and U_s denote the displacement components in the normal and tangential directions respectively.

Now, we shall investigate the boundary conditions along the cylindrical surface B. For every B_i we may represent as $C_i(s) \times [-\frac{h}{2}, \frac{h}{2}]$ such that $\mathcal{D}$ always appears to be in the left side of $C_i(s)$ as s increases. We shall require the satisfaction of boundary conditions along all of these B_i's by either

$$\begin{cases} \sigma_n(s,Z) = \hat{\sigma}_n(s,Z) \\ \sigma_{ns}(s,Z) = \hat{\sigma}_{ns}(s,Z) \\ \sigma_{nz}(s,Z) = \hat{\sigma}_{nz}(s,Z) \end{cases} \qquad (2.1,14a)$$

corresponding to the case of prescribed surface tractions, or

$$\begin{cases} U_n(s,Z) = \hat{U}_n(s,Z) \\ U_s(s,Z) = \hat{U}_s(s,Z) \\ W(s,Z) = \hat{W}(s,Z) \end{cases} \qquad (2.1,14b)$$

corresponding to the case of prescribed displacements. In (2.1,14a,b) quantities with hats denote given boundary values.

2.2 Approximate Two Dimensional Equations for the Bending of Plates

From now on we shall confine ourselves to problems of transverse bending only. This can be achieved by assuming that the loading along the cylindrical boundary of the plate produces no net resultant force to stretch or compress the middle surface and, in addition, the transverse deflection is small in comparison with the thickness of the plate. Further assumptions can be made since the plate under consideration is assumed to be thin, i.e. the thickness of the plate is small in comparison with its other dimensions. In connection with thin plates the well known approximate theory derived by Kirchhoff-Gehring (see Love [1]) assumes that the stress components σ_z, σ_{xz}, σ_{yz} are small in comparison with the flexural stresses throughout the plate and the normals of the middle surface before bending deform into the normals of the middle surface after bending. We shall refer to this theory as the classical theory of bending for thin plates. By taking into account the transverse shear deformation which was omitted in the classical theory, Reissner [5] developed another approximate theory for thin plates which we shall make use of in the present work, and hence we shall give here a brief derivation.

Reissner in [5] defines the bending and twisting couples M_x, M_y, M_{xy} and the transverse shear resultants Q_x, Q_y as follows:

$$\begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} = \int_{-h/2}^{h/2} Z \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} dZ \qquad (2.2,1)$$

,

$$\begin{pmatrix} Q_x \\ Q_y \end{pmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} dZ \quad . \qquad (2.2,2)$$

Upon omitting body forces, it follows by integrating (2.1,1) to (2.1,3) that these quantities must satisfy the following equations of equilibrium:

$$\frac{\partial M_x}{\partial X} + \frac{\partial M_{xy}}{\partial Y} - Q_x = 0 \ , \qquad (2.2,3)$$

$$\frac{\partial M_{xy}}{\partial X} + \frac{\partial M_y}{\partial Y} - Q_y = 0 \ , \qquad (2.2,4)$$

$$\frac{\partial Q_x}{\partial X} + \frac{\partial Q_y}{\partial Y} + p = 0 \ . \qquad (2.2,5)$$

In addition to these equations stress strain relations were obtained in [5] by using Castigliano's theorem of minimum complementary energy. However, essentially the same results as in [5] can be obtained by assuming certain approximate forms for the stress in the plate and integrating the three-dimensional stress-strain relations over the plate thickness. We shall use this approach in what follows.

From the homogeneity of equations (2.1,1), (2.1,2) and (2.1,3) we may assume

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} M_x(X,Y) \\ M_y(X,Y) \\ M_{xy}(X,Y) \end{pmatrix} h_1(Z) \qquad (2.2,6)$$

$$\begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} = \begin{pmatrix} Q_x(X,Y) \\ Q_y(X,Y) \end{pmatrix} h_2(Z) \ , \qquad (2.2,7)$$

$$\sigma_z = p(X, Y)\, h_3(Z) \tag{2.2,8}$$

where h_1, h_2, h_3 are as yet arbitrary functions satisfying the following conditions:

$$\int_{-h/2}^{h/2} Z\, h_1(Z)\, dZ = 1, \qquad \int_{-h/2}^{h/2} h_2(Z)\, dZ = 1\,, \tag{2.2,9}$$

$$h_2(\pm h/2) = 0, \quad h_3(h/2) = 1,\ h_3(-h/2) = 0\ . \tag{2.2,10}$$

Condition (2.2,9) is required in order to satisfy equations (2.2,1) and (2.2,2). Condition (2.2,10) is required in order to satisfy the surface conditions (2.1,10) and (2.1,11).

Equations (2.2,3) and (2.2,4) can be obtained by multiplying by Z in (2.1,1) and (2.1,2) (after omitting the components of body force) and integrating over the plate thickness. Equation (2.2,5) can also be obtained by integrating (2.1,3) over the plate thickness.

Substituting (2.2,6), (2.2,7), (2.2,8) back in (2.1,1), (2.1,2), (2.1,3) and using (2.2,3), (2.2,4), (2.2,5), we obtain further relations among h_1, h_2 and h_3:

$$\begin{aligned} h_1(Z) &= -\frac{d}{dZ} h_2(Z)\,, \\ h_2(Z) &= \frac{d}{dZ} h_3(Z)\,. \end{aligned} \tag{2.2,11}$$

As an example, let us choose

$$h_1 = \frac{1}{h^2/6}\left(\frac{Z}{h/2}\right)\,. \tag{2.2,12}$$

Then

$$h_2 = \frac{1}{2h/3}[1 - (\frac{Z}{h/2})^2] \quad ,$$

and

$$h_3 = \frac{3}{4}[\frac{2}{3} + \frac{Z}{h/2} - \frac{1}{3}(\frac{Z}{h/2})^3] \quad .$$

This corresponds to the distribution of stress across the thickness used in [5].

Corresponding to the global description of stresses by means of couples and stress resultants, we try to obtain a proper description for displacements. The expression for the work done by the surface traction along any cylindrical surface $C(s) \times [-h/2, h/2]$ in $\mathcal{S}$ is

$$\int_{-h/2}^{h/2} \int_C [\sigma_n U_n + \sigma_{ns} U_s + \sigma_{nz} W] \, ds dZ. \qquad (2.2,13)$$

In this formula σ_n, σ_{ns}, σ_{nz} and U_n, U_s, W are defined as in (2.1,12) and (2.1,13). If the approximate stress distributions (2.2,6) and (2.2,7) are employed, then the work (2.2,13) can be written as

$$\int_C [M_n \int_{-h/2}^{h/2} h_1 U_n dZ + M_{ns} \int_{-h/2}^{h/2} h_1 U_s dZ + Q_n \int_{-h/2}^{h/2} h_2 W dZ] \, ds \qquad (2.2,14)$$

where

$$M_n = M_x \cos^2(n,X) + M_y \sin^2(n,X) + 2M_{xy} \sin(n,X) \cos(n,X)$$
$$M_{ns} = (M_y - M_x) \sin(n,X) \cos(n,X) + M_{xy}(\cos^2(n,X) - \sin^2(n,X)) \qquad (2.2,15)$$
$$Q_n = Q_x \cos(n,X) + Q_y \sin(n,X) \, .$$

The expression (2.2,14) suggests that we may define the generalized displacements as follows:

$$\beta_x(x,y) = \int_{-h/2}^{h/2} U h_1 \, dZ , \tag{2.2,16}$$

$$\beta_y(x,y) = \int_{-h/2}^{h/2} V h_1 \, dZ , \tag{2.2,17}$$

$$w_t(x,y) = \int_{-h/2}^{h/2} W h_2 \, dZ . \tag{2.2,18}$$

The generalized displacements being so defined, we are now able to determine the appropriate two-dimensional stress-strain relations. For the sake of convenience, we write

$$c_{11} = \int_{-h/2}^{h/2} [h_1(Z)]^2 \, dZ ,$$

$$c_{22} = \int_{-h/2}^{h/2} [h_2(Z)]^2 \, dZ ,$$

$$c_{13} = \int_{-h/2}^{h/2} h_1(Z) \, h_3(Z) \, dZ.$$

With the aid of (2.2,11) and (2.2,10), we can easily show that $c_{13} = c_{22}$. Again if $h(Z) = \frac{1}{h^2/6} \left(\frac{Z}{h/2}\right)$ as in the example (2.2,12), then

$$c_{11} = 12/h^3 ,$$

$$c_{22} = 6/5h .$$

Now, multiplying by $h_1(Z)$ in (2.1,4), (2.1,5), (2.1,6) and integrating over the plate thickness, we obtain

$$\frac{\partial \beta_x}{\partial X} = \frac{c_{11}}{E} (M_x - \nu M_y) - \frac{\nu c_{22}}{E} p , \qquad (2.2,19)$$

$$\frac{\partial \beta_y}{\partial Y} = \frac{c_{11}}{E} (M_y - \nu M_x) - \frac{\nu c_{22}}{E} p , \qquad (2.2,20)$$

$$\frac{\partial \beta_x}{\partial Y} + \frac{\partial \beta_y}{\partial X} = \frac{c_{11}}{G} M_{xy} . \qquad (2.2,21)$$

Multiplying by $h_2(Z)$ in (2.1,8) and (2.1,9) and integrating over the plate thickness, we obtain

$$\frac{\partial w_t}{\partial X} + \beta_x = \frac{c_{22}}{G} Q_x , \qquad (2.2,22)$$

$$\frac{\partial w_t}{\partial Y} + \beta_y = \frac{c_{22}}{G} Q_y . \qquad (2.2,23)$$

In analogy with classical plate theory, we define the fluxural rigidity of the plate

$$D = \frac{E}{(1-\nu^2)\, c_{11}}$$

which becomes $D = \frac{Eh^3}{12(1-\nu^2)}$ when $h_1(Z)$ is chosen to be $\frac{1}{h^2/6} (\frac{Z}{h/2})$, as in (2.2,12).

Rearranging equations (2.2,19) to (2.2,23) we have

$$M_x = D \left(\frac{\partial \beta_x}{\partial X} + \nu \frac{\partial \beta_y}{\partial Y} + \frac{\nu c_{22}}{2G} p\right) , \qquad (2.2,24)$$

$$M_y = D \left(\frac{\partial \beta_y}{\partial Y} + \nu \frac{\partial \beta_x}{\partial X} + \frac{\nu c_{22}}{2G} p\right) , \qquad (2.2,25)$$

$$M_{xy} = \frac{1-\nu}{2} D \left(\frac{\partial \beta_x}{\partial Y} + \frac{\partial \beta_y}{\partial X}\right) , \qquad (2.2,26)$$

$$\beta_x = -\frac{\partial w_t}{\partial X} + \frac{c_{22}}{G} Q_x \qquad (2.2,27)$$

$$\beta_y = -\frac{\partial w_t}{\partial Y} + \frac{c_{22}}{G} Q_y \qquad (2.2,28)$$

for every point in $\mathcal{D}$.

The boundary conditions (2.1,14a) and (2.1,14b) are converted to prescribe

$$\text{either} \begin{cases} M_n = \hat{M}_n \\ M_{ns} = \hat{M}_{ns} \\ Q_n = \hat{Q}_n \end{cases} \quad (2.2,29a) \qquad \text{or} \begin{cases} \beta_n = \hat{\beta}_n \\ \beta_s = \hat{\beta}_s \\ w_t = \hat{w}_t \end{cases} \quad (2.2,29b)$$

along any contour $C_i(s)$. Here M_n, M_{ns} and Q_n are given by (2.2,15) and β_n, β_s and w_t are related to β_x, β_y and w_t as follows:

$$\begin{pmatrix} \beta_n \\ \beta_s \\ w_t \end{pmatrix} = \begin{pmatrix} \cos(n,X) & \sin(n,X) & 0 \\ -\sin(n,X) & \cos(n,X) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_x \\ \beta_y \\ w_t \end{pmatrix}. \qquad (2.2,30)$$

Equations (2.2,3) to (2.2,5) and equations (2.2,24) to (2.2,28) form a set of differential equation system for eight unknowns, β_x, β_y, w_t, M_x, M_y, M_{xy}, Q_x and Q_y. To these differential equations, we append either the boundary condition (2.2,29a) or (2.2,29b).

We may remark here that the equations based on the classical theory of bending can be easily deduced by setting $G = \infty$ in the above equation system. The equations of equilibrium remain unchanged while the stress-strain relations take the form:

$$M_x = - D \left(\frac{\partial^2 w_t}{\partial X^2} + \nu \frac{\partial^2 w_t}{\partial Y^2}\right) , \tag{2.2,31}$$

$$M_y = - D \left(\frac{\partial^2 w_t}{\partial Y^2} + \nu \frac{\partial^2 w_t}{\partial X^2}\right) , \tag{2.2,32}$$

$$M_{xy} = - (1-\nu) D \frac{\partial^2 w_t}{\partial X \partial Y} . \tag{2.2,33}$$

Also, there is a noteworthy difference in prescribing boundary conditions. Instead of three conditions as in (2.2,29a,b), the classical plate theory specifies only

either $\left\{ \begin{array}{l} M_n = \hat{M}_n \\ V_n \equiv Q_n + \dfrac{\partial M_{ns}}{\partial s} = \hat{V}_n \end{array} \right.$ (2.2,34a) or $\left\{ \begin{array}{l} \dfrac{\partial w_t}{\partial n} = \dfrac{\partial \hat{w}_t}{\partial n} \\ w_t = \hat{w}_t , \end{array} \right.$ (2.2,34b)

where V_n is known as Kirchhoff force.

2.3 Reduction of the Plate Equations

From now on we shall study the homogeneous differential system, i.e. $p = 0$ at every point in $\mathcal{D}$. Equation (2.2,8) then suggests there exists a stress function χ such that

$$Q_x = \frac{\partial \chi}{\partial Y} , \tag{2.3,1}$$

$$Q_y = - \frac{\partial \chi}{\partial X} . \tag{2.3,2}$$

for every point in $\mathcal{D}$. By using these relations and setting

$$k^2 = \frac{(1-\nu)\,c_{22}D}{2G} = \frac{c_{22}}{c_{11}} \; ,$$

we can put equations (2.2,24) to (2.2,28) into the following form:

$$M_x = -D\left(\frac{\partial^2 w_t}{\partial X^2} + \nu \frac{\partial^2 w_t}{\partial Y^2}\right) + 2k^2 \frac{\partial^2 \chi}{\partial X \partial Y} \; , \qquad (2.3,3)$$

$$M_y = -D\left(\frac{\partial^2 w_t}{\partial Y^2} + \nu \frac{\partial^2 w_t}{\partial X^2}\right) - 2k^2 \frac{\partial^2 \chi}{\partial X \partial Y} \; , \qquad (2.3,4)$$

$$M_{xy} = -(1-\nu)\, D \frac{\partial^2 w_t}{\partial X \partial Y} + k^2 \left(\frac{\partial^2 \chi}{\partial Y^2} - \frac{\partial^2 \chi}{\partial X^2}\right) , \qquad (2.3,5)$$

$$\beta_x = -\frac{\partial w_t}{\partial X} + \frac{2k^2}{(1-\nu)D} \frac{\partial \chi}{\partial Y} \; , \qquad (2.3,6)$$

$$\beta_y = -\frac{\partial w_t}{\partial Y} - \frac{2k^2}{(1-\nu)D} \frac{\partial \chi}{\partial X} \qquad (2.3,7)$$

for every point in $\mathcal{D}$. Substituting equations (2.3,3) to (2.3,5) into equations (2.2,3) and (2.2,4) we obtain a pair of relations:

$$-\frac{\partial}{\partial X}(D \Delta w_t) = \frac{\partial}{\partial Y}(\chi - k^2 \Delta \chi) \; , \qquad (2.3,8)$$

$$\frac{\partial}{\partial Y}(D \Delta w_t) = \frac{\partial}{\partial X}(\chi - k^2 \Delta \chi) \qquad (2.3,9)$$

for every point in $\mathcal{D}$. The symbol Δ in (2.3,8) and (2.3,9) stands for $\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$.

The boundary condition (2.2,29a) can be expressed in terms of w_t and χ through equations (2.3,3) to (2.3,5) and equation (2.2,15). Similarly, the boundary condition (2.2,29b) can be expressed in terms of w_t and χ through (2.3,6), (2.3,7) and (2.2,30).

III. AN INFINITE PLATE CONTAINING A CRACK OR A RIGID LINE INCLUSION

3.1 Formulation of the Problems

Let us consider an infinite plate with constant thickness h containing a crack or a rigid line inclusion of length 2c. We consider only the case in which the crack or the line inclusion is so oriented that it can be represented as one of the cylindrical boundary sets of the plate defined in section 2.1. As shown in Figure 1a, the plate occupies a region $\mathscr{D}$ which consists of all points in the XY-plane except the line segment $Y = 0$, $|X| \leq c$ which corresponds to the crack or the line inclusion.

A crack is to be free of stress; thus it will be required that certain relevant stresses vanish as $Y \to 0^+$ and as $Y \to 0^-$, whenever $|X| < c$ for the case of a crack. A rigid line inclusion is assumed to be fixed in space; thus it will be required that all displacement components vanish as $Y \to 0^+$ and as $Y \to 0^-$, whenever $|X| < c$ for the case of a rigid line inclusion.

It is convenient to introduce dimensionless coordinates x, y and a dimensionless plate thickness ϵ as follows:§

$$X = cx, \quad Y = cy, \qquad \epsilon = \frac{k}{c} = \frac{1}{c}\left(\frac{c_{22}}{c_{11}}\right)^{\frac{1}{2}}$$

In addition, we introduce a new deflection w (which has the units of moment) through the relation

§ If $f_1(Z)$ is chosen to be $\frac{1}{h^2/6}\left(\frac{Z}{h/2}\right)$, then $\epsilon = \frac{h}{\sqrt{10}\,c}$ which is apparently dimensionless. It follows that ϵ is small when h/c is small compared to unity.

$$w_t = \frac{c^2 w}{D} \ . \qquad (3.1,1)$$

In terms of these new variables, equations (2.3,8) and (2.3,9) may be written as follows:

$$-\frac{\partial}{\partial x}(\Delta w) = \frac{\partial}{\partial y}(\chi - \epsilon^2 \Delta\chi) \qquad (3.1,2)$$

and

$$\frac{\partial}{\partial y}(\Delta w) = \frac{\partial}{\partial x}(\chi - \epsilon^2 \Delta\chi) \qquad (3.1,3)$$

respectively, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Similarly, equations (2.3,1) to (2.3,7) can be written as:

$$cQ_x = \frac{\partial \chi}{\partial y} \ , \qquad (3.1,4)$$

$$cQ_y = -\frac{\partial \chi}{\partial x} \ , \qquad (3.1,5)$$

$$M_x = -\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) + 2\epsilon^2 \frac{\partial^2 \chi}{\partial x \partial y} \ , \qquad (3.1,6)$$

$$M_y = -\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) - 2\epsilon^2 \frac{\partial^2 \chi}{\partial x \partial y} \ , \qquad (3.1,7)$$

$$M_{xy} = -(1-\nu)\frac{\partial^2 w}{\partial x \partial y} + \epsilon^2 \left(\frac{\partial^2 \chi}{\partial y^2} - \frac{\partial^2 \chi}{\partial x^2}\right) \ , \qquad (3.1,8)$$

$$\frac{D}{c}\beta_x = -\frac{\partial w}{\partial x} + \frac{2\epsilon^2}{(1-\nu)}\frac{\partial \chi}{\partial y} \ , \qquad (3.1,9)$$

$$\frac{D}{c}\beta_y = -\frac{\partial w}{\partial y} - \frac{2\epsilon^2}{(1-\nu)}\frac{\partial \chi}{\partial x} \qquad (3.1,10)$$

valid at every point in $\mathfrak{D}$.

Now we shall turn our attention to the boundary conditions.

The plate under consideration has two boundaries: the line segment $y = 0$, $|x| \leq 1$ and the periphery of a circle centered at origin with radius $c\rho$ as $\rho \to \infty$ (see Figure 1a).

The boundary condition along the line segment is completely dependent upon the nature of the segment; we shall have the "free edge" conditions

$$\lim_{\substack{|y| \to 0 \\ |x| < 1}} M_y = \lim_{\substack{|y| \to 0 \\ |x| < 1}} M_{xy} = \lim_{\substack{|y| \to 0 \\ |x| < 1}} Q_y = 0 \tag{3.1,11}$$

for the case of a crack. On the other hand we shall have the "fixed edge" condition

$$\lim_{\substack{|y| \to 0 \\ |x| < 1}} \frac{c^2}{D} w = \lim_{\substack{|y| \to 0 \\ |x| < 1}} \beta_x = \lim_{\substack{|y| \to 0 \\ |x| < 1}} \beta_y = 0 \tag{3.1,12}$$

for the case in which the line segment corresponds to a rigid line inclusion.

The plate is to be loaded at infinity. This loading is described by a set of three independent conditions in terms of either moments and shear forces or generalized displacements. For simplicity, we shall take the case in which only moments and shear forces are involved. If ρ and ϕ are polar coordinates in the xy-plane, then the loading at infinity may be described as follows:

$$\begin{aligned} \rho = \infty: \quad & \hat{M}_\rho \equiv \hat{M}_x \cos^2\phi + \hat{M}_y \sin^2\phi + 2\hat{M}_{xy} \sin\phi \cos\phi = g_1(\phi) \\ & \hat{M}_{\rho\phi} \equiv (\hat{M}_y - \hat{M}_x)\sin\phi\cos\phi + \hat{M}_{xy}(\cos^2\phi - \sin^2\phi) = g_2(\phi) \\ & \hat{Q}_\rho \equiv \hat{Q}_x \cos\phi + \hat{Q}_y \sin\phi = g_3(\phi) \end{aligned} \tag{3.1,13}$$

where the hat sign denotes values at the boundary. Here $g_1(\phi)$, $g_2(\phi)$ and $g_3(\phi)$ are given arbitrary functions of the polar angle ϕ subject to the condition that they leave the plate in static equilibrium. Hence, the boundary condition at infinity will be

$$\lim_{\rho \to \infty} \begin{pmatrix} M_\rho(\rho,\phi) \\ M_{\rho\phi}(\rho,\phi) \\ Q_\rho(\rho,\phi) \end{pmatrix} = \begin{pmatrix} g_1(\phi) \\ g_2(\phi) \\ g_3(\phi) \end{pmatrix} \tag{3.1,14}$$

where M_ρ, $M_{\rho\phi}$, Q_ρ are defined as in (3.1,13).

Let us define the boundary operators $B_0^{(c)}$ and $B_0^{(r)}$ by

$$B_0^{(c)}(w,\chi) \equiv \lim_{\substack{|y| \to 0 \\ |x| < 1}} \begin{pmatrix} -\left(\frac{\partial^2}{\partial y^2} + \nu \frac{\partial^2}{\partial x^2}\right) & -2\epsilon^2 \frac{\partial^2}{\partial x \partial y} \\ -(1-\nu)\frac{\partial^2}{\partial x \partial y} & \epsilon^2\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right) \\ 0 & -\frac{1}{c}\frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} w \\ \chi \end{pmatrix}, \tag{3.1,15}$$

$$B_0^{(r)}(w,\chi) \equiv \lim_{\substack{|y| \to 0 \\ |x| < 1}} \begin{pmatrix} -\frac{c}{D}\frac{\partial}{\partial x} & \frac{2\epsilon^2 c}{(1-\nu) D}\frac{\partial}{\partial y} \\ -\frac{c}{D}\frac{\partial}{\partial y} & -\frac{2\epsilon^2 c}{(1-\nu)D}\frac{\partial}{\partial x} \\ \frac{c^2}{D} & 0 \end{pmatrix} \begin{pmatrix} w \\ \chi \end{pmatrix} \tag{3.1,16}$$

and the operator B_∞ by

$$B_{\infty}(w,\chi) = \lim_{\rho\to\infty}\begin{pmatrix} -\cos^2\phi\left(\frac{\partial^2}{\partial x^2}+\nu\frac{\partial^2}{\partial y^2}\right)-\sin^2\phi\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right)-2(1-\nu)\sin\phi\cos\phi\frac{\partial^2}{\partial x\partial y} & 2\epsilon^2(\cos^2\phi-\sin^2\phi)\frac{\partial^2}{\partial x\partial y}+2\epsilon^2\sin\phi\cos\phi\left(\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right) \\ (1-\nu)\sin\phi\cos\phi\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)-(1-\nu)(\cos^2\phi-\sin^2\phi)\frac{\partial^2}{\partial x\partial y} & -4\epsilon^2\sin\phi\cos\phi\frac{\partial^2}{\partial x\partial y}+\epsilon^2(\cos^2\phi-\sin^2\phi)\left(\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right) \\ 0 & \frac{1}{c}\cos\phi\frac{\partial}{\partial x}-\frac{1}{c}\sin\phi\frac{\partial}{\partial y} \end{pmatrix}\begin{pmatrix} w \\ \chi \end{pmatrix} .$$

(3.1,17)

It is clear that $B_0^{(c)}$ and $B_0^{(r)}$ are the boundary operators along the line segment for the case of a crack and for the case of a rigid line inclusion respectively. The operator B_{∞} presents the boundary condition (3.1,14).

Our problem then reduces to the determination of a pair of functions $\{w,\chi\}$ which satisfies the differential equations (3.1,2) and (3.1,3) in $\mathscr{D}$ and which satisfies the boundary condition

$$B_{\infty}(w,\chi) = \vec{g} \qquad \text{at } \rho = \infty . \tag{3.1,18}$$

Here B_{∞} is the operator defined by (3.1,17) and $\vec{g}$ a vector whose components are $g_1(\phi)$, $g_2(\phi)$ and $g_3(\phi)$ given by (3.1,13). We also have the boundary condition

$$B_0\,(w,\chi) = \vec{0} \qquad \text{at } y = 0^{\pm},\ |x| < 1 \tag{3.1,19}$$

where B_0 is either $B_0^{(c)}$ (defined by 3.1,15) for the case of a crack or

$B_0^{(r)}$ (defined by 3.1,16) for the case of a rigid inclusion; $\vec{0}$ denotes the vector with all components zero.

Owing to the fact that it is usually much easier to solve the problem by omitting the condition (3.1,13) (corresponding to the case of a plate without a crack or a rigid inclusion), we proceed as follows. Let

$$w = \tilde{w} + w^* , \tag{3.1,20}$$

$$\chi = \tilde{\chi} + \chi^* \tag{3.1,21}$$

where $\{\tilde{w}, \tilde{\chi}\}$ and $\{w^*, \chi^*\}$ are two pairs of functions with the following properties. Both pairs satisfy the equations (3.1,2) and (3.1,3). We require further that $\{\tilde{w}, \tilde{\chi}\}$ satisfies but

$$B_\infty (\tilde{w}, \tilde{\chi}) = \vec{g} \quad \text{at } \rho = \infty. \tag{3.1,22}$$

Hence, $\{\tilde{w}, \tilde{\chi}\}$ is the solution pair for the plate loaded at infinity as in (3.1,18) but without a crack or a rigid inclusion. By applying the boundary condition operator B_0 to this pair of solutions $\{\tilde{w}, \tilde{\chi}\}$ we define functions f_{1c}, f_{2c}, f_{3c} by

$$\begin{pmatrix} f_{1c}(x) \\ \\ f_{2c}(x) \\ \\ f_{3c}(x) \end{pmatrix} \equiv B_o^{(c)}(\tilde{w}, \tilde{\chi}) = \begin{pmatrix} -(\frac{\partial^2}{\partial y^2} + \nu \frac{\partial^2}{\partial x^2}) & -2\epsilon^2 \frac{\partial^2}{\partial x \partial y} \\ \\ -(1-\nu)\frac{\partial^2}{\partial x \partial y} & \epsilon^2 (\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}) \\ \\ 0 & -\frac{1}{c}\frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \tilde{w} \\ \\ \tilde{\chi} \end{pmatrix}_{\substack{y=0 \\ |x|<1}} \tag{3.1,23}$$

for the case of a crack, and functions f_{1r}, f_{2r}, f_{3r} by

$$\begin{pmatrix} f_{1r}(x) \\ \\ f_{2r}(x) \\ \\ f_{3r}(x) \end{pmatrix} \equiv B_0^{(r)}(\tilde{w},\tilde{\chi}) = \begin{pmatrix} -\frac{c}{D}\frac{\partial}{\partial x} & \frac{2\epsilon^2 c}{(1-\nu)D}\frac{\partial}{\partial y} \\ \\ -\frac{c}{D}\frac{\partial}{\partial y} & -\frac{2\epsilon^2 c}{(1-\nu)D}\frac{\partial}{\partial x} \\ \\ \frac{c^2}{D} & 0 \end{pmatrix} \begin{pmatrix} \tilde{w} \\ \\ \tilde{\chi} \end{pmatrix}_{\substack{y=0 \\ |x|<1}} \qquad (3.1,24)$$

for the case of a rigid inclusion.

The boundary conditions (3.1,18) and (3.1,19) now lead to boundary conditions for the second pair of functions $\{w^*, \chi^*\}$ as follows.

$$B_\infty (w^*, \chi^*) = \vec{0} \qquad \text{at } \rho = \infty \qquad (3.1,25)$$

and

$$B_0 (w^*, \chi^*) = -\vec{f} \text{ at } y = 0^{+}_{-}, \quad |x| < 1 \qquad (3.1,26)$$

where $\vec{f}$ denotes a vector with components defined as in (3.1,23) for the case of a crack or as in (3.1,24) for the case of a rigid inclusion.

Through the linearity of the differential equations and the boundary conditions, it can be easily shown that the pair $\{w, \chi\}$ defined in (3.1,20) and (3.1,21) solves the original problem. Since the pair $\{\tilde{w}, \tilde{\chi}\}$, representing the case of the continuous plate, can be determined without much difficulty, it may be assumed that the vector $\vec{f}$ with components defined either by (3.1,23) or by (3.1,24) is known.

We summarize the results of our analysis up to this point:

Our problem is reduced to finding a pair of functions $\{w^*, \chi^*\}$ of x and y which satisfies the differential equations (3.1,2) and (3.1,3):

$$-\frac{\partial}{\partial x}(\Delta w^*) = \frac{\partial}{\partial y}(\chi^* - \epsilon^2 \Delta \chi^*) ,$$

$$\frac{\partial}{\partial y}(\Delta w^*) = \frac{\partial}{\partial x}(\chi^* - \epsilon^2 \Delta \chi^*)$$

in $\mathcal{D}$. The functions w^* and χ^* must also satisfy the boundary conditions (3.1,25)

$$B_{\infty}(w^*, \chi^*) = \vec{0} \qquad \text{at } \rho = \infty$$

and (3.1,26)

$$B_0(w^*, \chi^*) = -\vec{f} \qquad \text{at } y = 0^{\pm}, \ |x| < 1.$$

The functions w^* and χ^* and their partial derivatives of all orders are required to be continuous in $\mathcal{D}$.

It is known that crack problems lead to infinities in the stress distribution at the crack points. Similar phenomena will be expected to occur in this problem. In order that the total energy contained in the plate be finite, we shall require that these singularities in the moments and stress resultants (computed from w^* and χ^*) be not worse than $O(r^{-1+\delta})$ where r is the distance between the point in $\mathcal{D}$ under consideration and either of the ends $x = 1$, $y = 0$ or $x = -1$, $y = 0$ of the crack or the rigid inclusion (see Fig. 1a), and $\delta > 0$.

3.2 Fourier Transforms and Reduction to Dual Integral Equations

In order to simplify the notation, we use $\{w, \chi\}$ instead of $\{w^*, \chi^*\}$, and we understand that this pair is required to satisfy the differential equations (3.1,2) and (3.1,3) and the boundary conditions (3.1,25) and (3.1,26).

The boundary condition (3.1,25) suggests that we may use a Fourier transform technique to determine w and χ. According to the Fourier integral theorem, any function $\phi(x)$ which is absolutely integrable over the range $(-\infty, \infty)$, possesses Fourier transform and, further, the inversion of its Fourier transform converges to $\phi(x)$ for all x (except on a set of measure zero) provided $\phi(x)$ is of bounded variation. In the present problem, we shall require that the functions[§]

$$\frac{\partial^j w}{\partial x^j}(x,y) \quad , \quad j = 0,\ 1,\ 2,\ 3\ ; \quad \frac{\partial^k \chi}{\partial x^k}(x,y)\ , \quad k = 0,\ 1,$$

meet these conditions for each $y \neq 0$.

The most general solutions to equations (3.1,2) and (3.1,3) which satisfy the boundary condition (3.1,25) are

$$w = w^{(1)} + (\operatorname{sgn} y)\, w^{(2)} \tag{3.2,1}$$

and

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§ In fact, in certain cases w and χ may not possess Fourier transforms. However if their partial derivatives with respect to x and y do, then the problem can be still solved in some cases, as we shall illustrate later.

$$\chi = (\text{sgn } y)\, \chi^{(1)} + \chi^{(2)} \tag{3.2,2}$$

with $$w^{(j)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\, Q_j(\alpha) e^{-|\alpha y|} + R_j(\alpha)\, |y|\, e^{-|\alpha y|} \,] \cdot e^{i\alpha x}\, d\alpha \tag{3.2,3}$$

and $$\chi^{(j)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\, P_j(\alpha) e^{-\sqrt{\alpha^2 + 1/\epsilon^2}\,|y|} - 2i\alpha R_j(\alpha) e^{-|\alpha y|} \,] \cdot e^{i\alpha x}\, d\alpha \,, \tag{3.2,4}$$

for $y \neq 0$ and all x, where Q_j, R_j, P_j are as yet arbitrary functions of α. It is clear that in (3.2,1) and (3.2,2) both w and χ have been separated into two parts. Since $w^{(1)}$ is even in y, we shall refer to $\{w^{(1)}, \chi^{(1)}\}$ as the symmetric solution. Since $(\text{sgn } y)\, w^{(2)}$ is odd in y, the pair $\{w^{(2)}, \chi^{(2)}\}$ will be called the anti-symmetric solution.

Corresponding to (3.2,1) and (3.2,2) we can compute the moments, the shear stress resultants and the generalized displacements. From (3.1,4) to (3.1,10), we have:

$$Q_x = Q_x^{(1)} + (\text{sgn } y)\, Q_x^{(2)} \,, \tag{3.2,5}$$

$$Q_y = (\text{sgn } y)\, Q_y^{(1)} + Q_y^{(2)} \,, \tag{3.2,6}$$

$$M_x = M_x^{(1)} + (\text{sgn } y)\, M_x^{(1)} \,, \tag{3.2,7}$$

$$M_y = M_y^{(1)} + (\text{sgn } y)\, M_y^{(2)} \,, \tag{3.2,8}$$

$$M_{xy} = (\text{sgn})\, M_{xy}^{(1)} + M_{xy}^{(2)} \,, \qquad (3.2,9)$$

$$\beta_x = \beta_x^{(1)} + (\text{sgn } y)\, \beta_x^{(2)} \,, \qquad (3.2,10)$$

$$\beta_y = (\text{sgn } y)\, \beta_y^{(1)} + \beta_y^{(2)} \qquad (3.2,11)$$

where

$$Q_x^{(j)} = \frac{1}{2\pi c}\int_{-\infty}^{\infty} \left[2i\alpha|\alpha|\, R_j(\alpha) e^{-|\alpha y|} - \sqrt{\alpha^2+1/\epsilon^2}\, P_j(\alpha) e^{-\sqrt{\alpha^2+1/\epsilon^2}\,|y|} \right] \cdot e^{i\alpha x}\, d\alpha \,, \qquad (3.2,12)$$

$$Q_y^{(j)} = \frac{-1}{2\pi c}\int_{-\infty}^{\infty} \left[2\alpha^2 R_j(\alpha) e^{-|\alpha y|} + i\alpha P_j(\alpha) e^{-\sqrt{\alpha^2+1/\epsilon^2}\,|y|} \right] \cdot e^{i\alpha x}\, d\alpha \,, \qquad (3.2,13)$$

$$M_x^{(j)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big\{ \left[(1-\nu)\alpha^2 Q_j(\alpha) + ((1-\nu)\alpha^2|y| + 2\nu|\alpha| - 4\epsilon^2\alpha^2|\alpha|) R_j(\alpha) \right] e^{-|\alpha y|} - 2\epsilon^2 i\alpha \sqrt{\alpha^2 + 1/\epsilon^2}^{\,2}\, P_j(\alpha) e^{-\sqrt{\alpha^2+1/\epsilon^2}\,|y|} \Big\} e^{i\alpha x} d\alpha \,, \qquad (3.2,14)$$

$$M_y^{(j)} = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \Big\{ \left[(1-\nu)\alpha^2 Q_j(\alpha) + ((1-\nu)\alpha^2|y| - 2|\alpha| - 4\epsilon^2\alpha^2|\alpha|) R_j(\alpha) \right] e^{-|\alpha y|} - 2\epsilon^2 i\alpha \sqrt{\alpha^2+1/\epsilon^2}\, P_j(\alpha) e^{\sqrt{\alpha^2+1/\epsilon^2}\,|y|} \Big\} e^{i\alpha x}\, d\alpha \,, \qquad (3.2,15)$$

$$M_{xy}^{(j)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big\{ \left[(1-\nu) i\alpha|\alpha| Q_j(\alpha) + (1-\nu)\left(i\alpha|\alpha y| - i\alpha - \frac{4\epsilon^2 i\alpha^3}{1-\nu}\right) R_j(\alpha) \right] e^{-|\alpha y|} + \epsilon^2 (2\alpha^2 + 1/\epsilon^2) P_j(\alpha) e^{-\sqrt{\alpha^2+1/\epsilon^2}\,|y|} \Big\} e^{i\alpha x} d\alpha \,, \qquad (3.2,16)$$

$$\beta_x = -\frac{c}{2\pi D}\int_{-\infty}^{\infty} \{[\, i\alpha Q_j(\alpha) + (i\alpha\,|y| - \frac{4\epsilon^2 i\alpha\,|\alpha|}{(1-\nu)})\, R_j(\alpha)]\, e^{-|\alpha y|} + $$

$$+ \frac{2\epsilon^2}{1-\nu}\sqrt{\alpha^2+1/\epsilon^2}\; P_j(\alpha)\, e^{-\sqrt{\alpha^2+1/\epsilon^2}\,|y|}\} e^{i\alpha x}\, d\alpha\ , \qquad (3.2,17)$$

$$\beta_y = \frac{c}{2\pi D}\int_{-\infty}^{\infty} \{[\, |\alpha|\, Q_j(\alpha) + (\,|\alpha y| -1- \frac{4\epsilon^2\alpha^2}{1-\nu})\, R_j(\alpha)]\, e^{-|\alpha y|} -$$

$$- \frac{2\epsilon^2 i\alpha}{(1-\nu)}\, P_j(\alpha)\, e^{-\sqrt{\alpha^2+1/\epsilon^2}\,|y|}\}\, e^{i\alpha x}\, d\alpha\ . \qquad (3.2,18)$$

The above expressions for the stress resultants, the stress couples and the generalized displacements are valid for $|y| > 0$ and for all x. It is clear that all these quantities vanish as $|y| \to \infty$ from the exponential dependence in their integrands. They also vanish as $|x| \to \infty$ since they are assumed to possess Fourier transforms for all $|y| > 0$. Hence, the boundary condition (3.1,25) is indeed satisfied by the pair $\{w, \chi\}$ as represented in (3.2,1) and (3.2,2).

The pair $\{w, \chi\}$ and their partial derivatives with respect to x and y of all orders are required to be continuous in $\mathcal{D}$. Hence, in particular, we require that all physical quantities be continuous across $y = 0$ for all $|x| > 1$.

In order to determine the arbitrary functions Q_j, R_j, P_j, $j = 1, 2$, use will be made of the continuity properties associated with the pair $\{w, \chi\}$ across $y = 0$ for all $|x| > 1$ and the boundary condition (3.1,26).

Let us first study the continuity properties of w and χ for $y = 0$. From the odd and even behaviors with respect to y, we must have the following for $y = 0\pm$, $|x| > 1$:

$$w^{(2)} = 0 \,, \tag{3.2,19}$$

$$\beta_x^{(2)} = 0 \,, \tag{3.2,20}$$

$$\beta_y^{(1)} = 0 \tag{3.2,21}$$

and

$$Q_y^{(1)} = 0 \,, \tag{3.2,22}$$

$$M_{xy}^{(1)} = 0 \,, \tag{3.2,23}$$

$$M_y^{(2)} = 0 \,. \tag{3.2,24}$$

Next, with regard to the boundary condition (3.1,26), we group our problems into two cases: the case of a crack and the case of a rigid inclusion. In each case, the appropriate boundary condition, combined with the continuity conditions (3.2,19) to (3.2,24) yields two systems of dual integral equations, one for the symmetric part $\{w^{(1)}, \chi^{(1)}\}$ and another for the antisymmetric part $\{w^{(2)}, \chi^{(2)}\}$.

3.2a <u>The Case of a Crack</u>. Using the definition (3.1,15), condition (3.1,26) in the case of a crack now reads

$$\lim_{\substack{y \to 0\pm \\ |x|<1}} \begin{pmatrix} M_y \\ M_{xy} \\ Q_y \end{pmatrix} = - \begin{pmatrix} f_{1c}(x) \\ f_{2c}(x) \\ f_{3c}(x) \end{pmatrix} \tag{3.2,25}$$

where f_{1c}, f_{2c} and f_{3c} are defined as in (3.1,23). Using (3.2,6), (3.2,8) and (3.2,9) equation (3.2,25) yields six equations as follows:

$$\lim_{\substack{|y|\to 0\\ |x|<1}} M_y^{(1)} = -f_{1c}(x) , \qquad (3.2,26)$$

$$\lim_{\substack{|y|\to 0\\ |x|<1}} M_y^{(2)} = 0 , \qquad (3.2,27)$$

$$\lim_{\substack{|y|\to 0\\ |x|<1}} M_{xy}^{(1)} = 0 , \qquad (3.2,28)$$

$$\lim_{\substack{|y|\to 0\\ |x|<1}} M_{xy}^{(2)} = - f_{2c}(x) , \qquad (3.2,29)$$

$$\lim_{\substack{|y|\to 0\\ |x|<1}} Q_y^{(1)} = 0 , \qquad (3.2,30)$$

and $$\lim_{\substack{|y|\to 0\\ |x|<1}} Q_y^{(2)} = -f_{3c}(x) . \qquad (3.2,31)$$

(i) Symmetric Part. Among the equations of continuity (3.2,19) to (3.2,24), and the boundary conditions (3.2,26) to (3.2,31), those which are labeled with the superscript (1) form a system of integral equations for unknowns Q_1, R_1, and P_1; from (3.2,12) to (3.2,18), these integral equations are

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\{(1-\nu)\,\alpha^2 Q_1(\alpha)-2\,|\alpha|\,(1+2\epsilon^2\alpha^2)R_1(\alpha)-2\epsilon^2 i\alpha\sqrt{\alpha^2+1/\epsilon^2}\;P_1(\alpha)\}e^{i\alpha x}d\alpha =$$

$$= - f_{1c}(x) , \qquad |x| < 1 , \qquad (3.2,32)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{(1-\nu)i\alpha\,|\alpha|\,Q_1(\alpha)-(1-\nu)i\alpha\,[1+\frac{4\epsilon^2\alpha^2}{(1-\nu)}]\,R_1(\alpha)+$$

$$+\epsilon^2(2\alpha^2+\frac{1}{\epsilon^2})\,P_1(\alpha)\}\,e^{i\alpha x}\,d\alpha = 0,$$

all x , (3.2,33)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{2\alpha^2\,R_1(\alpha)+i\alpha\,\,P_1(\alpha)\}\,e^{i\alpha x}\,d\alpha = 0,$$

all x , (3.2,34)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\,|\alpha|\,Q_1(\alpha)-[1+\frac{4\epsilon^2\alpha^2}{(1-\nu)}]\,R_1(\alpha)-\frac{2\epsilon^2 i\alpha}{(1-\nu)}\,P_1(\alpha)\}\,\cdot$$

$$\cdot\,e^{i\alpha x}\,d\alpha = 0,$$

$|x|>1$. (3.2,35)

This system can be reduced further to a pair of dual integral equations involving only one unknown. Let us define

$$A(\alpha) = |\alpha|\,Q_1(\alpha)-[1+\frac{4\epsilon^2\alpha^2}{(1-\nu)}]\,R_1(\alpha)-\frac{2\epsilon^2 i\alpha}{(1-\nu)}\,P_1(\alpha)\,. \qquad (3.2,36)$$

Also, the integrands of equations (3.2,33) and (3.2,34) are identically zero from the Fourier integral theorem. Hence equations (3.2,36), (3.2,33) and (3.2,34) form a system of simultaneous algebraic equations as follows:

$$\begin{bmatrix} 1 & -(1+\frac{4\epsilon^2\alpha^2}{(1-\nu)}) & -\frac{2\epsilon^2\alpha}{1-\nu} \\ \alpha & -\alpha(1+\frac{4\epsilon^2\alpha^2}{1-\nu}) & -\frac{\epsilon^2(2\alpha^2+1/\epsilon^2)}{1-\nu} \\ 0 & 2\alpha^2 & \alpha \end{bmatrix}\begin{bmatrix} |\alpha|\,Q_1 \\ R_1 \\ iP_1 \end{bmatrix}=\begin{bmatrix} A(\alpha) \\ 0 \\ 0 \end{bmatrix}. \qquad (3.2,37)$$

Solving (3.2,37), we obtain Q_1, R_1, P_1 in terms of $A(\alpha)$:

$$Q_1(\alpha) = \frac{1+\nu}{2} \frac{A(\alpha)}{|\alpha|} ,$$

$$R_1(\alpha) = -\frac{1-\nu}{2} A(\alpha) , \tag{3.2,38}$$

$$P_1(\alpha) = -i (1-\nu) \alpha A(\alpha) .$$

Substituting (3.2,38) into (3.2,32) and (3.2,35), we obtain a pair of dual integral equations:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\alpha| A(\alpha)[3+\nu - 4\epsilon^2 |\alpha| (\sqrt{\alpha^2+1/\epsilon^2} - |\alpha|)] e^{i\alpha x} d\alpha =$$

$$= \frac{2}{(1-\nu)} f_{1c}(x) , \qquad |x| < 1 , \tag{3.2,39}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(\alpha) e^{i\alpha x} d\alpha = 0, \qquad |x| > 1 . \tag{3.2,40}$$

(ii) Antisymmetric Part. Among the equations (3.2,19) to (3.2,24) and the equations (3.2,26) to (3.2,31), those which are labeled with superscript (2) form a system of integral equations for Q_2, R_2 and P_2; in detail they are, using (3.2,12) to (3.2,18)

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} \{ (1-\nu)i\alpha \cdot |\alpha| Q_2(\alpha) - (1-\nu)i\alpha (1 + \frac{4\epsilon^2\alpha^2}{1-\nu}) R_2(\alpha) +$$

$$+ \epsilon^2(2\alpha^2 + 1/\epsilon^2) P_2(\alpha)\} e^{i\alpha x} d\alpha = - f_{2c}(x) ,$$

$$|x| < 1, \tag{3.2,41}$$

$$- \frac{1}{2\pi c}\int_{-\infty}^{\infty} [2\alpha^2 R_2(\alpha) + i\alpha P_2(\alpha)] e^{i\alpha x} d\alpha = - f_{3c}(x),$$

$$|x| < 1, \tag{3.2,42}$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\{(1-\nu)\alpha^2 Q_2(\alpha)-2\,|\alpha|\,(1+2\epsilon^2\alpha^2)\,R_2(\alpha)-$$

$$-2\epsilon^2 i\alpha\sqrt{\alpha^2+1/\epsilon^2}\,P_2(\alpha)\}\,e^{i\alpha x}d\alpha = 0\,,$$

$$\text{all } x\,, \qquad (3.2,43)$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\{i\alpha Q_2(\alpha)-\frac{4\epsilon^2 i\alpha\,|\alpha|}{1-\nu}\,R_2(\alpha)+\frac{2\epsilon^2}{(1-\nu)}\sqrt{\alpha^2+1/\epsilon^2}\,P_2(\alpha)\}\,\cdot$$

$$\cdot\, e^{i\alpha x}d\alpha = 0,$$

$$|x| > 1\,, \qquad (3.2,44)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Q_2(\alpha)\,e^{i\alpha x}d\alpha = 0,$$

$$|x| > 1\,. \qquad (3.2,45)$$

Equations (3.2,41) to (3.2,45) can be reduced further to a system of coupled dual integral equations. Let us define

$$\Omega(\alpha) = -\,i\alpha Q_2(\alpha)+\frac{4\epsilon^2 i\alpha\,|\alpha|}{(1-\nu)}\,R_2(\alpha)-\frac{2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}}{1-\nu}\,P_2(\alpha)\,,$$

$$(3.2,46)$$

$$\omega(\alpha) = Q_2(\alpha)\,. \qquad (3.2,41)$$

Combining with (3.2,43), we have a system of simultaneous algebraic equations:

$$\begin{pmatrix} \alpha & -\frac{4\epsilon^2\alpha|\alpha|}{(1-\nu)} & \frac{1}{(1-\nu)} \\ 1 & 0 & 0 \\ (1-\nu)\alpha^2 & -2|\alpha|(1+2\epsilon^2\alpha^2) & \alpha \end{pmatrix} \begin{pmatrix} iQ_2 \\ iR_2 \\ 2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}\,P_2 \end{pmatrix} = \begin{pmatrix} -\Omega(\alpha) \\ i\omega(\alpha) \\ 0 \end{pmatrix}. \quad (3.2,48)$$

Solving (3.2,48) we obtain

$$Q_2(\alpha) = \omega(\alpha) ,$$

$$R_2(\alpha) = \frac{1-\nu}{2}\, i\, \frac{\alpha\Omega(\alpha)}{|\alpha|} , \quad (3.2,49)$$

$$P_2(\alpha) = -(1-\nu)\frac{1}{2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}}\left[(1+2\epsilon^2\alpha^2)\Omega(\alpha)+i\alpha\,\omega(\alpha)\right] .$$

Substituting (3.2,49) into (3.2,41), (3.2,42) and (3.2,44), (3.2,45) we obtain a system of coupled dual integral equations for $\Omega(\alpha)$ and $\omega(\alpha)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{\Big[\frac{1/2\epsilon^2}{\sqrt{\alpha^2+1/\epsilon^2}} - 2\epsilon^2\alpha^2\Big(|\alpha| - \frac{\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}}\Big)-\Big(\frac{1-\nu}{2}|\alpha| - \frac{2\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}}\Big)\Big]\Omega(\alpha) +$$

$$+\Big[\frac{1/2\epsilon^2}{\sqrt{\alpha^2+1/\epsilon^2}} - \Big(|\alpha| - \frac{\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}}\Big)\Big]i\alpha\,\omega(\alpha)\Big\}e^{i\alpha x}d\alpha = \quad (3.2,50)$$

$$= \frac{f_{2c}(x)}{(1-\nu)} , \qquad |x| < 1 ,$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{-\Big[\frac{1/2\epsilon^2}{\sqrt{\alpha^2+1/\epsilon^2}}-\Big(|\alpha| - \frac{\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}}\Big)\Big]\, i\alpha\Omega(\alpha)+ \frac{1/2\epsilon^2}{\sqrt{\alpha^2+1/\epsilon^2}}\,\alpha^2\omega(\alpha)\;e^{i\alpha x}d\alpha =$$

$$= \frac{c}{(1-\nu)}\, f_{3c}(x) , \qquad |x| < 1 , \quad (3.2,51)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Omega(\alpha)\, e^{i\alpha x} d\alpha = 0 \,, \qquad |x| > 1 \,, \qquad (3.2,52)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega(\alpha)\, e^{i\alpha x} d\alpha = 0 \,, \qquad |x| > 1 \,. \qquad (3.2,53)$$

3.2b <u>The Case of a Rigid Inclusion</u>. Using the definition (3.1,16), condition (3.1,26) in the case of a rigid inclusion becomes

$$\lim_{\substack{|y|\to 0\\ |x|<1}} \begin{pmatrix} \beta_x \\ \beta_y \\ \frac{c^2}{D} w \end{pmatrix} = - \begin{pmatrix} f_{1r}(x) \\ f_{2r}(x) \\ f_{3r}(x) \end{pmatrix} \qquad (3.2,54)$$

where f_{1r}, f_{2r} and f_{3r} are defined as in (3.1,24). As in the case of a crack, from (3.2,19) to (3.2,24) and (3.2,54) we may deduce two systems of integral equations, one for the symmetric part $\{w^{(1)},\chi^{(1)}\}$ and the other for the antisymmetric part $\{w^{(2)},\chi^{(2)}\}$.

(i) <u>Symmetric Part</u>. The system with superscript (1) gives the following equations:

$$-\frac{c}{2\pi D}\int_{-\infty}^{\infty} \{ i\alpha Q_1(\alpha) - \frac{4\epsilon^2 i\alpha|\alpha|}{(1-\nu)} R_1(\alpha) + \frac{2\epsilon^2}{1-\nu}\sqrt{\alpha^2+1/\epsilon^2}\, P_1(\alpha)\} e^{i\alpha x} d\alpha =$$

$$= - f_{1r}(x) \,, \qquad |x| < 1 \,, \qquad (3.2,55)$$

$$\frac{c^2}{2\pi D}\int_{-\infty}^{\infty} Q_1(\alpha)\, e^{i\alpha x} d\alpha = - f_{3r}(x), \qquad |x| < 1 \,, \qquad (3.2,56)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{|\alpha|\,Q_1(\alpha)-(1+\frac{4\epsilon^2\alpha^2}{1-\nu})\,R_1(\alpha)\;-\;\frac{2\epsilon^2}{1-\nu}\,i\alpha P_1(\alpha)\}\,e^{i\alpha x}d\alpha =$$

$$= 0\ , \qquad \text{all } x\ , \qquad (3.2,57)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\,(1-\nu)\;i\alpha\;|\alpha|\;Q_1(\alpha)\;-\;(1-\nu)i\alpha\;(1+\frac{4\epsilon^2\alpha^2}{1-\nu})R_1(\alpha)\;+$$

$$+\;\epsilon^2(2\alpha^2+1/\epsilon^2)\;P_1(\alpha)\;\}\;e^{i\alpha x}d\alpha = 0\ ,$$

$$|x| > 1\ , \qquad (3.2,58)$$

$$-\frac{1}{2\pi c}\int_{-\infty}^{\infty}[2\alpha^2R_1(\alpha)+i\alpha\;P_1(\alpha)]\;e^{i\alpha x}\;d\alpha = 0\ ,$$

$$|x| > 1\ . \qquad (3.2,59)$$

Similarly, this system of integral equations can be simplified in the following manner. We define

$$G(\alpha) = (1-\nu)i\alpha\;|\alpha|\;Q_1(\alpha)-(1-\nu)i\alpha\;(1+\frac{4\epsilon^2\alpha^2}{1-\nu})\;R_1(\alpha)\;+$$

$$+\;\epsilon^2(2\alpha^2+1/\epsilon^2)\;P_1(\alpha) \qquad (3.2,60)$$

and $$H(\alpha) = -\;2\alpha^2\;R_1(\alpha)\;-\;i\alpha\;P_1(\alpha)\ . \qquad (3.2,61)$$

Solving (3.2,60), (3.2,61) and (3.2,51) we have

$$Q_1(\alpha) = -\frac{1}{2\alpha\;|\alpha|}\;i\;G(\alpha)\;-\;(\frac{1}{2\alpha^2}+\frac{2\epsilon^2}{1-\nu})\;\frac{H(\alpha)}{|\alpha|}$$

$$R_1(\alpha) = -\frac{1}{2\alpha}\;i\;G(\alpha)\;-\;\frac{1}{2\alpha^2}\,H(\alpha) \qquad (3.2,62)$$

$$P_1(\alpha) = G(\alpha)\ .$$

Substituting (3.2,62) into (3.2,55), (3.2,56) and (3.2,58), (3.2,59) we obtain a system of coupled dual integral equations for $G(\alpha)$ and $H(\alpha)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\left[\frac{1}{2\,|\alpha|}+\frac{2\epsilon^2}{(1-\nu)}\left(\sqrt{\alpha^2+1/\epsilon^2}-|\alpha|\right)\right]G(\alpha)-\frac{1}{2\,\alpha\,|\alpha|}\,H(\alpha)\right\}e^{i\alpha x}d\alpha =$$

$$= \frac{D}{c}\,f_{1r}(x)\,, \qquad |x|<1\,, \qquad (3.2,63)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{i}{2\alpha\,|\alpha|}\,G(\alpha)+\left[\frac{1}{2\alpha^2|\alpha|}+\frac{2\epsilon^2}{(1-\nu)\,|\alpha|}\right]H(\alpha)\right\}e^{i\alpha x}d\alpha =$$

$$= \frac{D}{c^2}\,f_{3r}(x)\,, \qquad |x|<1\,, \qquad (3.2,64)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha)\,e^{i\alpha x}d\alpha = 0\,, \qquad |x|>1\,, \qquad (3.2,65)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\alpha)\,e^{i\alpha x}d\alpha = 0\,, \qquad |x|>1\,. \qquad (3.2,66)$$

(ii) Antisymmetric Part. The system with superscript (2) gives the following equations:

$$\frac{c}{2\pi D}\int_{-\infty}^{\infty}\left\{|\alpha|\,Q_2(\alpha)-\left(1+\frac{4\epsilon^2\alpha^2}{1-\nu}\right)R_2(\alpha)-\frac{2\epsilon^2 i\alpha}{1-\nu}\,P_2(\alpha)\right\}e^{i\alpha x}d\alpha =$$

$$= -\,f_{2r}(x), \qquad |x|<1\,, \qquad (3.2,67)$$

$$- \frac{1}{2\pi}\int_{-\infty}^{\infty} \{i\alpha Q_2(\alpha) - \frac{4\epsilon^2 i\alpha|\alpha|}{(1-\nu)} R_2(\alpha) + \frac{2\epsilon^2}{1-\nu}\sqrt{\alpha^2+1/\epsilon^2}\, P_2(\alpha)\}e^{i\alpha x}d\alpha =$$

$$= 0 , \qquad \text{all } x , \qquad (3.2,68)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Q_2(\alpha)e^{i\alpha x}d\alpha = 0 , \qquad \text{all } x , \qquad (3.2,69)$$

$$- \frac{1}{2\pi}\int_{-\infty}^{\infty} \{(1-\nu)\alpha^2 Q_2(\alpha) - 2|\alpha|(1+2\epsilon^2\alpha^2)R_2(\alpha) - 2\epsilon^2 i\alpha\sqrt{\alpha^2+1/\epsilon^2}\, P_2(\alpha)\}\cdot$$

$$\cdot\, e^{i\alpha x}d\alpha = 0 , \qquad |x| > 1 . \qquad (3.2,70)$$

By defining

$$F(\alpha) = -(1-\nu)\alpha^2 Q_2(\alpha) + 2|\alpha|(1+2\epsilon^2\alpha^2)R_2(\alpha) + 2\epsilon^2 i\alpha\sqrt{\alpha^2+1/\epsilon^2}\, P_2(\alpha) \qquad (3.2,71)$$

and solving this equation with the help of (3.2,68) and (3.2,69), we obtain:

$$Q_2(\alpha) = 0$$

$$R_2(\alpha) = \frac{1}{2|\alpha|} F(\alpha) , \qquad (3.2,\ 2)$$

$$P_2(\alpha) = \frac{i\alpha}{\sqrt{\alpha^2+1/\epsilon^2}} F(\alpha) .$$

Substituting (3.2,72) into (3.2,67) and (3.2,70), we obtain a pair of dual integral equations for $F(\alpha)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \{\frac{1}{2|\alpha|} + \frac{2\epsilon^2\alpha^2}{1-\nu}[\frac{1}{|\alpha|} - \frac{1}{\sqrt{\alpha^2+1/\epsilon^2}}]\}F(\alpha)\, e^{i\alpha x}d\alpha =$$

$$= \frac{D}{c} f_{2r}(x) , \qquad |x| < 1, \qquad (3.2,73)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha = 0, \qquad |x| > 1 . \qquad (3.2,74)$$

Dual integral equations have been used previously in certain crack problems in elasticity [8] , [10] , [11] . The contents of those references have been mentioned in the Introduction.

3.3 Reductions to Systems of Singular Integral Equations

In the previous section, we obtained four systems of dual integral equations. For the case of a crack, equations (3.2,39) and (3.2,40) correspond to the case of symmetric deflection and equations (3.2,50) to (3.2,53), to the case of antisymmetric deflection. For the case of a rigid inclusion, equations (3.2,63) to (3.2,66) correspond to the symmetric part and equations (3.2,72) and (3.2,74), to the antisymmetric part.

Instead of reducing the problem to dual integral equations as in the previous section, it is possible to proceed in an alternative way. We shall describe this in the following subsections.

3.3a <u>Symmetric Part - Case of a Crack.</u> Let us define

$$u(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\alpha)\, e^{i\alpha\xi}\, d\alpha \qquad (3.3,1)$$

where $A(\alpha)$ is defined by (3.2,36). The physical meaning of $u(x)$ is clearly described by the equation

$$u(x) = \lim_{y \to 0 \pm} \frac{D}{c}\, \beta_y^{(1)}(x,y) ,$$

from (3.2,18). From (3.2,40) it follows that $u(x) = 0$ for $|x| > 1$, hence, by the Fourier inversion theorem

$$A(\alpha) = \int_{-1}^{1} u(\xi)\, e^{-i\alpha\xi}\, d\xi \ , \qquad \text{all } \alpha \ . \tag{3.3,2}$$

Substituting $A(\alpha)$ into (3.2,39), we obtain an integral equation as follows:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\alpha| \left[(3+\nu) - 4\epsilon^2 |\alpha| \left(\sqrt{\alpha^2 + 1/\epsilon^2} - |\alpha| \right) \right] e^{i\alpha x} d\alpha \ \cdot$$

$$\cdot \int_{-1}^{1} u(\xi)\, e^{-i\alpha\xi}\, d\xi = \frac{2}{(1-\nu)}\, f_{1c}(x), \qquad |x| < 1 \ . \tag{3.3,3}$$

It is quite clear that the order of integrations in (3.3,3) can not be interchanged, hence no explicit use will be made of (3.3,3).

However, the left hand side of equation (3.2,39) is the limiting value of $M_y^{(1)}$ as $|y|$ approaches zero, so we may first express $M_y^{(1)}$ in terms of $u(\xi)$ for $|y| > 0$ from (3.2,15) and (3.2,38) and then require its limiting value to satisfy the boundary condition (3.2,26). Substituting relations (3.2,38) and (3.3,2) into (3.2,15) and interchanging the order of integrations which is justified whenever $|y| > 0$, we have

$$M_y^{(1)}(x,y) = \frac{(1-\nu)}{\pi} \int_{-1}^{1} u(\xi)\, m_y^{(1)}(x-\xi,\ y)\, d\xi \tag{3.3,4}$$

where

$$m_y^{(1)}(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} \Big\{ \Big[-\frac{(3+\nu)}{2} |\alpha| + \frac{1-\nu}{2}\alpha^2 |y| \Big] e^{-|\alpha y|} +$$

$$+ 2\epsilon^2 \Big[\alpha^2 \sqrt{\alpha^2+1/\epsilon^2}\; e^{-\sqrt{\alpha^2+1/\epsilon^2}\,|y|} - |\alpha|^3 e^{-|\alpha y|} \Big] \Big\} e^{i\alpha x} d\alpha . \tag{3.3,5}$$

The integration in (3.3,5) may be carried out explicitly in terms of rational functions and modified Bessel functions of the third kind of integer order. There follows

$$m_y^{(1)}(x,y) = -\frac{(3+\nu)}{2}\frac{(y^2-x^2)}{\rho^4} + (1-\nu)\frac{y^2(y^2-3x^2)}{\rho^6} +$$

$$+2\epsilon^2\Big[-\frac{1}{\epsilon^2\rho^2}K_2\left(\frac{\rho}{\epsilon}\right) + \frac{1}{\epsilon^3\rho}K_3\left(\frac{\rho}{\epsilon}\right) - \frac{x^2y^2}{\epsilon^4\rho^4}K_4\left(\frac{\rho}{\epsilon}\right) -$$

$$-6\,\frac{(x^4-6x^2y^2+y^4)}{\rho^8}\Big] \qquad (3.3,6)$$

where $\rho^2 = x^2 + y^2$ and K_n denotes the modified Bessel function of the third kind of n-th order.

All physical quantities can likewise be expressed in terms of $u(\xi)$; they are

$$\begin{pmatrix} M_x^{(1)} \\ M_y^{(1)} \\ M_{xy}^{(1)} \end{pmatrix} = \frac{1-\nu}{\pi}\int_{-1}^{1} u(\xi)\begin{pmatrix} m_x^{(1)}(x-\xi,y) \\ m_y^{(1)}(x-\xi,y) \\ m_{xy}^{(1)}(x-\xi,y) \end{pmatrix} d\xi \qquad (3.3,7)$$

and

$$\begin{pmatrix} Q_x^{(1)} \\ Q_y^{(1)} \end{pmatrix} = \frac{1-\nu}{\pi c}\int_{-1}^{1} u(\xi)\begin{pmatrix} q_x^{(1)}(x-\xi,y) \\ q_y^{(1)}(x-\xi,y) \end{pmatrix} d\xi \qquad (3.3,8)$$

where m_y is given by (3.3,6) and the remaining kernels are given by the following formulas .

$$m_x^{(1)}(x,y) = -\frac{(1-\nu)}{2}\frac{(x^4+y^4-6x^2y^2)}{\rho^6} -$$

$$-2\epsilon^2\Big[-\frac{1}{\epsilon^2\rho^2}K_2(\frac{\rho}{\epsilon}) + \frac{1}{\epsilon^3\rho}K_3(\frac{\rho}{\epsilon}) - \frac{x^2y^2}{\epsilon^4\rho^4}K_4(\frac{\rho}{\epsilon}) -$$

$$-\frac{6(x^4+y^4-6x^2y^2)}{\rho^8}\Big] \quad , \tag{3.3,9}$$

$$m_{xy}^{(1)}(x,y) = -\frac{2xy}{\rho^4} + \frac{xy}{\epsilon^2\rho^2}K_2(\frac{\rho}{\epsilon}) + \frac{(1-\nu)y(3xy^2-x^3)}{\rho^6} -$$

$$-\epsilon^2\Big[48\left(\frac{xy}{\rho^6} - \frac{2x^3y}{\rho^8}\right) - \frac{6xy}{\epsilon^3\rho^3}K_3(\frac{\rho}{\epsilon}) +$$

$$+2\frac{x^3y}{\epsilon^3\rho^4}K_4(\frac{\rho}{\epsilon})\Big] \quad , \tag{3.3,10}$$

$$q_x^{(1)}(x,y) = \frac{2(3xy^2-x^3)}{\rho^6} - \frac{xy^2}{\epsilon^3\rho^3}K_3(\frac{\rho}{\epsilon}) + \frac{x}{\epsilon^2\rho^2}K_2(\frac{\rho}{\epsilon}) \, , \tag{3.3,11}$$

$$q_y^{(1)}(x,y) = \frac{2(y^3-3x^2y)}{\rho^6} - \frac{y}{\epsilon^2\rho^2}K_2(\frac{\rho}{\epsilon}) + \frac{x^2y}{\epsilon^3\rho^3}K_3(\frac{\rho}{\epsilon}) \, . \tag{3.3,12}$$

To satisfy the boundary condition (3.2,26), we require

$$\lim_{\substack{|y|\to 0\\ |x|<1}} M_y^{(1)} \equiv \lim_{\substack{|y|\to 0\\ |x|<1}} \frac{1-\nu}{\pi}\int_{-1}^{1} u(\xi)\, m_y^{(1)}(x-\xi,y)\, d\xi =$$

$$= -f_{1c}(x) \tag{3.3,13}$$

where the kernel $m_y^{(1)}$ is given in (3.3,6). Apparently, the limiting process in (3.3,13) can not be passed under the integral sign since the kernel $m_y^{(1)}(x-\xi,0)$ is non-integrable. However, if u(x) vanishes at $x = \pm 1$ and is Hölder continuous with some Hölder index μ, $0<\mu<1$ for all x in the closed interval $[-1,1]$, then the stress singularity will not be worse than $O(r^{-1+\delta})$ for some $\delta>0$ (see Appendix A). We shall assume that this condition is fulfilled. Furthermore, if we assume that $\frac{du(x)}{dx}$ exists and is Hölder continuous with Hölder index μ for all x in the open interval $(-1,1)$, then we can write the left hand side of (3.3,13) as

$$\lim_{\substack{|y|\to 0\\ |x|<1}} M_y^{(1)} = \frac{1-\nu}{\pi}\frac{d}{dx}\int_{-1}^{1} u(\xi)\left\{-\frac{(3+\nu)}{2(x-\xi)} + 2\epsilon^2\left[\frac{2}{(x-\xi)^3} - \frac{1}{\epsilon^2(x-\xi)}K_2\left(\frac{|x-\xi|}{\epsilon}\right)\right]\right\} d\xi = -f_{1c}(x) \qquad (3.3,14)$$

(see Appendix B). The integral in (3.3,14) is a Cauchy principal value. Combining (3.3,14) with (3.3,13) and integrating once with respect to x, we obtain a singular integral equation with kernel of Cauchy's type§:

$$\int_{-1}^{1} u(\xi)\left\{\frac{3+\nu}{x-\xi} - 4\epsilon^2\left[\frac{2}{(x-\xi)^3} - \frac{1}{\epsilon^2(x-\xi)}K_2\left(\frac{|x-\xi|}{\epsilon}\right)\right]\right\} d\xi =$$

$$= \frac{2\pi}{(1-\nu)}\int^{x} f_{1c}(\eta)\, d\eta + \text{const.}, \qquad |x|<1 . \qquad (3.3,15)$$

3.3b <u>Antisymmetric Part - Case of a Crack.</u> The reduction for the antisymmetric part of the crack problem can be carried out in a similar way. We omit details and simply list the results here. We define

$$v(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Omega(\alpha)\, e^{i\alpha\xi}\, d\alpha \;, \qquad (3.3,16)$$

$$w(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega(\alpha)\, e^{i\alpha\xi}\, d\alpha \;, \qquad (3.3,17)$$

§ The same integral equation was obtained in [15] in which $f_{1c}(x) = M_0 = \text{const.}$

where $\Omega(\alpha)$ and $\omega(\alpha)$ are defined by (3.2,46) and (3.2,41) respectively. It is clear that from (3.2,17)

$$v(x) = \lim_{y \to 0\pm} \frac{D}{c} \beta_x^{(2)}(x,y)$$

and that from (3.2,3)

$$w(x) = \lim_{y \to 0\pm} w^{(2)}(x,y) .$$

Moreover with the aid of (3.2,44) and (3.2,45), $v(x) = w(x) = 0$ for $|x| > 1$. Hence, by the Fourier inversion theorem

$$\Omega(\alpha) = \int_{-1}^{1} v(\xi)\, e^{-i\alpha\xi}\, d\xi , \qquad \text{all } \alpha \qquad (3.3,18)$$

$$\omega(\alpha) = \int_{-1}^{1} w(\xi)\, e^{-i\alpha\xi}\, d\xi , \qquad \text{all } \alpha . \qquad (3.3,19)$$

All physical quantities can be expressed in terms of $v(\xi)$ and $w(\xi)$ as follows.

$$\begin{Bmatrix} M_x^{(2)} \\ M_y^{(2)} \\ M_{xy}^{(2)} \end{Bmatrix} = \frac{1-\nu}{\pi} \int_{-1}^{1} \{ v(\xi) \begin{Bmatrix} m_x^{(21)}(x-\xi,y) \\ m_y^{(21)}(x-\xi,y) \\ m_{xy}^{(21)}(x-\xi,y) \end{Bmatrix} + w(\xi) \begin{Bmatrix} m_x^{(22)}(x-\xi,y) \\ m_y^{(22)}(x-\xi,y) \\ m_{xy}^{(22)}(x-\xi,y) \end{Bmatrix} \} d\xi , \qquad (3.3,20)$$

$$\begin{Bmatrix} Q_x^{(2)} \\ Q_y^{(2)} \end{Bmatrix} = \frac{1-\nu}{\pi c} \int_{-1}^{1} \{ v(\xi) \begin{Bmatrix} q_x^{(21)}(x-\xi,y) \\ q_y^{(21)}(x-\xi,y) \end{Bmatrix} + w(\xi) \begin{Bmatrix} q_x^{(22)}(x-\xi,y) \\ q_y^{(22)}(x-\xi,y) \end{Bmatrix} \} d\xi \qquad (3.3,21)$$

where

$$m_x^{(21)}(x,y) = -(1-\nu)\frac{(3xy^3-x^3y)}{\rho^6} + 2\epsilon^2[\,24\frac{xy}{\rho^6} - 48\frac{x^3y}{\rho^8} -$$

$$-\frac{3xy}{\epsilon^3\rho^3}K_3(\frac{\rho}{\epsilon}) + \frac{x^3y}{\epsilon^4\rho^4}K_4(\frac{\rho}{\epsilon})] - \qquad (3.3,22)$$

$$-[\frac{xy}{\epsilon^2\rho^2}K_2(\frac{\rho}{\epsilon}) + \nu\frac{2xy}{\rho^4}]$$

$$m_x^{(22)}(x,y) = \frac{2y(y^2-3x^2)}{\rho^6} - \frac{y}{\epsilon^2\rho^2}K_2(\frac{\rho}{\epsilon}) + \frac{x^2y}{\epsilon^3\rho^3}K_3(\frac{\rho}{\epsilon}) \quad ,$$

$$m_y^{(21)}(x,y) = +(1-\nu)\frac{(3xy^3-x^3y)}{\rho^6} - 2\epsilon^2[\,24\frac{xy}{\rho^6} - 48\frac{x^3y}{\rho^8} -$$

$$-\frac{3xy}{\epsilon^3\rho^3}K_3(\frac{\rho}{\epsilon}) + \frac{x^3y}{\epsilon^4\rho^4}K_4(\frac{\rho}{\epsilon})] + \qquad (3.3,23)$$

$$+[\frac{xy}{\epsilon^2\rho^2}K_2(\frac{\rho}{\epsilon}) - \frac{2xy}{\rho^4}]$$

$$m_y^{(22)}(x,y) = -\,m_x^{(22)}(x,y)$$

$$m_{xy}^{(21)}(x,y) = 2\epsilon^2[\,6\frac{(x^4+y^4-6x^2y^2)}{\rho^8} - \frac{3}{\epsilon^2\rho^2}K_2(\frac{\rho}{\epsilon}) + \frac{6x^2}{\epsilon^3\rho^3}K_3(\frac{\rho}{\epsilon}) -$$

$$-\frac{x^4}{\epsilon^4\rho^4}K_4(\frac{\rho}{\epsilon})] - \frac{1}{2\epsilon^2}K_0(\frac{\rho}{\epsilon}) -$$

$$-\frac{2}{\epsilon\rho}K_1(\frac{\rho}{\epsilon}) + \frac{2x^2}{\epsilon^2\rho^2}K_2(\frac{\rho}{\epsilon}) + \frac{(1-\nu)(y^2-x^2)}{2\rho^4} - \qquad (3.3,24)$$

$$-(1-\nu)\frac{(y^4-3x^2y^2)}{\rho^6} \quad ,$$

$$m_{xy}^{(22)}(x,y) = \frac{x}{2\epsilon^3\rho}K_1(\frac{\rho}{\epsilon}) + \frac{3x}{\epsilon^2\rho^2}K_2(\frac{\rho}{\epsilon}) - \frac{x^3}{\epsilon^3\rho^3}K_3(\frac{\rho}{\epsilon}) - \frac{2x(3y^2-x^2)}{\rho^6} \; ,$$

$$q_x^{(21)}(x,y) = \frac{y}{2\epsilon^3\rho} K_1(\frac{\rho}{\epsilon}) + \frac{y}{\epsilon^2\rho^2} K_2(\frac{\rho}{\epsilon}) - \frac{x^2y}{\epsilon^3\rho^3} K_3(\frac{\rho}{\epsilon}) - \frac{2y}{\rho^4} + \frac{8x^2y}{\rho^6}$$

$$q_x^{(22)}(x,y) = -\frac{xy}{2\epsilon^4\rho^2} K_2(\frac{\rho}{\epsilon}) \ , \qquad (3.3,25)$$

$$q_y^{(21)}(x,y) = -\frac{x}{2\epsilon^2\rho} K_1(\frac{\rho}{\epsilon}) - \frac{3x}{\epsilon^2\rho^2} K_2(\frac{\rho}{\epsilon}) + \frac{x^3}{\epsilon^3\rho^3} K_3(\frac{\rho}{\epsilon}) + \frac{2x(3y^2-x^2)}{\rho^6}$$

$$q_y^{(22)}(x,y) = -\frac{1}{2\epsilon^3\rho} K_1(\frac{\rho}{\epsilon}) + \frac{x^2}{2\epsilon^4\rho^2} K_2(\frac{\rho}{\epsilon}) \ . \qquad (3.3,26)$$

Again we shall assume that both $v(x)$ and $w(x)$ vanish at $x = \pm 1$ and are Hölder continuous with some Hölder index μ, $0<\mu<1$ for all x in $[-1,1]$. Further, we assume that $\frac{dv(x)}{dx}$ and $\frac{dw(x)}{dx}$ exist and are Hölder continuous for all x in the open interval $(-1,1)$. Under these assumptions, (3.2,29) and (3.2,31) can be written as

$$\lim_{\substack{|y| \to 0 \\ |x| < 1}} M_{xy}^{(2)} = -\frac{(1-\nu)}{\pi} \Big\{ \int_{-1}^{1} v(\xi)[\frac{1}{2\epsilon^2} K_0(\frac{|x-\xi|}{\epsilon})] \, d\xi + \frac{d}{dx}\int_{-1}^{1} v(\xi)[-\frac{(1-\nu)}{2(x-\xi)} - \frac{2}{(x-\xi)} K_2(\frac{|x-\xi|}{\epsilon}) + \frac{4\epsilon^2}{(x-\xi)^3}] \, d\xi + \int_{-1}^{1} w(\xi)[\frac{1}{\epsilon^2(x-\xi)} K_2(\frac{|x-\xi|}{\epsilon}) + \frac{(x-\xi)}{2\epsilon^3|x-\xi|} K_1(\frac{|x-\xi|}{\epsilon}) - \frac{2}{(x-\xi)^3}] \, d\xi \Big\} = -f_{2c}(x) \qquad (3.3,27)$$

and

$$\lim_{\substack{|y|\to 0\\ |x|<1}} Q_y^{(2)} = \frac{(1-\nu)}{\pi c}\left\{ \int_{-1}^{1} v(\xi)\left[\frac{(x-\xi)}{2\epsilon^3 |x-\xi|} K_3\left(\frac{|x-\xi|}{\epsilon}\right) - \frac{1}{\epsilon^2 (x-\xi)} K_2\left(\frac{|x-\xi|}{\epsilon}\right) - \frac{2}{(x-\xi)^3}\right] d\xi + \right.$$

$$+ \frac{d}{dx}\int_{-1}^{1} w(\xi)\left[-\frac{(x-\xi)}{2\epsilon^3 |x-\xi|} K_1\left(\frac{|x-\xi|}{\epsilon}\right)\right] d\xi =$$

$$= - f_{3c}(x) . \qquad (3.3,28)$$

Equations (3.3,27) and (3.3,28) form a system of coupled singular integral equations.

3.3c The Case of a Rigid Inclusion. The reduction to singular integral equations for the problem of the rigid line inclusion is essentially the same. We list only the results corresponding to the case of antisymmetric deflection. We define

$$t(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha) e^{i\alpha\xi}\, d\alpha , \qquad (3.3,29)$$

where $F(\alpha)$ is defined by (3.2,71). Through (3.2,15), it can be shown that

$$t(x) = \lim_{y\to 0\pm} M_y^{(2)}(x,y) .$$

From (3.2,74), we have $t(x) = 0$ for $|x| > 1$, hence by the Fourier inversion theorem

$$F(\alpha) = \int_{-1}^{1} t(\xi)\, e^{-i\alpha\xi}\, d\xi , \qquad \text{all } \alpha . \qquad (3.3,30)$$

All physical quantities can be expressed in terms of $t(\xi)$ as follows:

$$\begin{pmatrix} M_x^{(2)} \\ M_y^{(2)} \\ M_{xy}^{(2)} \end{pmatrix} = \frac{1}{\pi} \int_{-1}^{1} t(\xi) \begin{pmatrix} m_x^{(2)}(x-\xi, y) \\ m_y^{(2)}(x-\xi, y) \\ m_{xy}^{(2)}(x-\xi, y) \end{pmatrix} d\xi \qquad (3.3,31)$$

,

$$\begin{pmatrix} Q_x^{(2)} \\ Q_y^{(2)} \end{pmatrix} = \frac{1}{\pi c} \int_{-1}^{1} t(\xi) \begin{pmatrix} q_x^{(2)}(x-\xi, y) \\ q_y^{(2)}(x-\xi, y) \end{pmatrix} d\xi \qquad (3.3,32)$$

where

$$m_x^{(2)}(x,y) = \nu \frac{y}{\rho^2} + \frac{(1-\nu)(y^2-x^2)}{2\rho^4} + 2\epsilon^2 [\frac{y}{\epsilon^2 \rho^2} K_2(\frac{\rho}{\epsilon}) - \frac{x^2 y}{\epsilon^3 \rho^3} K_3(\frac{\rho}{\epsilon}) - \frac{2y(y^2-3x^2)}{\rho^6}] , \qquad (3.3,33)$$

$$m_y^{(2)}(x,y) = \frac{y}{\rho^2} - \frac{(1-\nu)y(y^2-x^2)}{2\rho^4} - 2\epsilon^2 [\frac{y}{\epsilon^2 \rho^2} K_2(\frac{\rho}{\epsilon}) - \frac{x^2 y}{\epsilon^3 \rho^3} K_3(\frac{\rho}{\epsilon}) - \frac{2y(y^2-3x^2)}{\rho^6}], \qquad (3.3,34)$$

$$m_{xy}^{(2)}(x,y) = -(1-\nu)\frac{xy^2}{\rho^4} + \frac{(1-\nu)x}{2\rho^2} - \frac{x}{\epsilon\rho} K_1(\frac{\rho}{\epsilon}) - 2\epsilon^2 [\frac{3x}{\epsilon^2 \rho^2} K_2(\frac{\rho}{\epsilon}) - \frac{x^3}{\epsilon^3 \rho^3} K_3(\frac{\rho}{\epsilon}) - 2\frac{(3xy^2-x^3)}{\rho^6}] , \qquad (3.3,35)$$

$$q_x^{(2)}(x,y) = \frac{xy}{\epsilon^2 \rho^2} K_2(\frac{\rho}{\epsilon}) - 2\frac{xy}{\rho^4} , \qquad (3.3,36)$$

$$q_y^{(2)}(x,y) = \frac{1}{\epsilon\rho} K_1(\frac{\rho}{\epsilon}) - \frac{x^2}{\epsilon^2 \rho^2} K_2(\frac{\rho}{\epsilon}) + \frac{x^2-y^2}{\rho^4} . \qquad (3.3,37)$$

Using (3.2,11) and relations (3.2,72), we find that the generalized slope with respect to y is

$$\beta_y^{(2)}(x,y) = \frac{c}{2\pi D}\int_{-\infty}^{\infty}\{(\frac{|y|}{2} - \frac{1}{2|\alpha|})e^{-|\alpha y|} - \frac{2\epsilon^2}{1-\nu}[|\alpha| e^{-|\alpha y|} - \frac{\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}} e^{-\sqrt{\alpha^2+1/\epsilon^2}|y|}]\} \cdot$$

$$\cdot\ e^{i\alpha x}\int_{-1}^{1} t(\xi)e^{-i\alpha\xi}\, d\xi d\alpha\ . \qquad (3.3,38)$$

In order to assure that the integral in (3.3,38) exists for all x and for every $|y| \geqslant 0$, we should require

$$F(0) = 0. \qquad (3.3,39)$$

From (3.3,30), (3.3,39) can be also written as

$$\int_{-1}^{1} t(x)\, dx = 0\ . \qquad (3.3,40)$$

Now, let us define a new function

$$\tau(x) = \int_{-\infty}^{x} t(\xi)\, d\xi \qquad (3.3,41)$$

which exists since t(x) is absolutely integrable and vanishes for all $|x| > 1$ on account of (3.3,40). Integrating the right hand side of (3.3,30) by parts, we obtain the Fourier transform of $\tau(\xi)$ as follows:

$$\frac{F(\alpha)}{i\alpha} = \int_{-1}^{1} \tau(\xi)\, e^{-i\alpha\xi}\, d\xi\ . \qquad (3.3,42)$$

If this relation is used in (3.3,38) we find that

$$\beta_y^{(2)}(x,y) = \frac{c}{\pi D}\int_{-1}^{1} \tau(\xi)\, b_y(x-\xi,y)\, d\xi \qquad (3.3,43)$$

where

$$b_y(x,y) = \frac{2\epsilon^2}{(1-\nu)}\left[\frac{2(3xy^2-x^3)}{\rho^6} - \frac{3x}{\epsilon^2\rho^2} K_2\left(\frac{\rho}{\epsilon}\right) + \frac{x^3}{\epsilon^3\rho^3} K_3\left(\frac{\rho}{\epsilon}\right)\right] +$$

$$+ \frac{x}{2\rho^2} - \frac{xy}{\rho^4}. \qquad (3.3,44)$$

As $|y| \to 0$ we require that (3.3,44) satisfies the boundary condition (3.2,54); i.e.

$$\frac{c}{\pi D}\int_{-1}^{1} \tau(\xi)\left\{\frac{1}{2(x-\xi)} - \frac{2\epsilon^2}{1-\nu}\left[\frac{2}{(x-\xi)^3} + \frac{3}{\epsilon^2(x-\xi)} K_2\left(\frac{|x-\xi|}{\epsilon}\right) - \right.\right.$$

$$\left.\left. - \frac{(x-\xi)}{\epsilon^3|x-\xi|} K_3\left(\frac{|x-\xi|}{\epsilon}\right)\right]\right\} d\xi =$$

$$= - f_{2r}(x), \qquad |x| < 1. \qquad (3.3,45)$$

Equation (3.3,45) is a singular integral equation with kernel of Cauchy's type and its solution determines t(x) uniquely through (3.3,41).

In the above subsections, we have reduced our problems to problems of solving systems of singular integral equations. We have omitted the reductions for the symmetric part of the rigid inclusion case since it presents no interesting features more than those which shall be extracted from the other cases.

It can be shown that under certain conditions the systems of dual integral equations are case by case equivalent to the systems singular integral equations.

Before we go on to study the solutions of the systems of singular integral equations, we shall state here that the solutions to the systems of dual integral equations are unique under certain conditions (see Appendix C).

IV. SOLUTION OF THE INTEGRAL EQUATIONS IN DIFFERENT CASES

4.1 Case I - Symmetric Solution for an Infinite Plate Containing a Crack

In section 3.2a, the crack problem has been separated into two parts, symmetric and antisymmetric. Naturally, the solution pair $\{w, \chi\}$ will be the sum of the pair $\{w^{(1)}, \chi^{(1)}\}$ (symmetric part) and the pair $\{w^{(2)}, \chi^{(2)}\}$ (antisymmetric part) according to (3.2,1) and (3.2,2). Hence, both $\{w^{(1)}, \chi^{(1)}\}$ and $\{w^{(2)}, \chi^{(2)}\}$ are of fundamental importance to the present problem. Besides, from the boundary conditions (3.2,26) to (3.2,31) it is clear that $\{w^{(1)}, \chi^{(1)}\}$ depends only on $f_{1c}(x)$ while $\{w^{(2)}, \chi^{(2)}\}$ depends only on $f_{2c}(x)$ and $f_{3c}(x)$. Since it is possible to load the plate at infinity in such a way that the corresponding solution $\{\tilde{w}, \tilde{\chi}\}$ for the plate without a crack generates either $f_{2c}(x) = f_{3c}(x) = 0$ or $f_{1c}(x) = 0$ along the crack where f_{1c}, f_{2c}, f_{3c} are given by (3.1,23), either of the pairs $\{ w^{(j)}, \chi^{(j)}\}$, j = 1,2 has physical significance in itself.

In this section we shall consider the symmetric solution of the crack problem which is represented by the singular integral equation (3.3,15). For simplicity, we shall replace $f_{1c}(x)$ by $f_1(x)$ hereafter. (3.3,15) can also be written in the form:

$$\frac{1}{\pi}\int_{-1}^{1} \frac{u(\xi)}{\xi - x}\, d\xi + \frac{1}{\pi}\int_{-1}^{1} k(x,\xi)\, u(\xi)\, d\xi = \ell_1(x) + C_1 \,, \qquad |x| < 1 \qquad (4.1,1)$$

where

$$k(x,\xi) = \frac{2 - 4\epsilon^2\left[\frac{2}{(\xi-x)^2} - \frac{K_2\left(\frac{|\xi-x|}{\epsilon}\right)}{\epsilon^2}\right]}{(1+\nu)(\xi - x)} \tag{4.1,2}$$

$$\ell_1(x) = -\frac{2}{(1-\nu^2)}\int^x f_1(t)\,dt \tag{4.1,3}$$

and C_1 is an arbitrary constant.

The existence of solutions to (4.1,1) is discussed in the work of Muskheleshvili [16]. Out of the various classes of functions listed in that reference, we seek our solution in the class of functions which are Hölder continuous for all x in the closed interval [-1,1]. This concept is required in order to fulfill one of the assumptions we made in section 4.3 during the derivation of (3.3,15).

Under the above restriction, (4.1,1) can be transformed to the following Fredholm type integral equation for u(x) as shown in Chapter 14 of [16].

$$u(x) - \frac{1}{\pi}\int_{-1}^{1} M(x,\xi)\,u(\xi)\,d\xi = G_1(x), \qquad |x| < 1 \tag{4.1,4}$$

where

$$M(x,\xi) = \frac{(1-x^2)^{1/2}}{\pi}\int_{-1}^{1} \frac{k(t,\xi)}{(1-t^2)^{1/2}(t-x)}\,dt \tag{4.1,5}$$

and

$$G_1(x) = -\frac{(1-x^2)^{1/2}}{\pi}\int_{-1}^{1} \frac{\ell_1(t)dt}{(1-t^2)^{1/2}(t-x)} \tag{4.1,6}$$

provided that the additional condition

$$C_1 = \frac{1}{\pi^2}\int_{-1}^{1} u(\xi)d\xi \int_{-1}^{1} \frac{k(t,\xi)dt}{(1-t^2)^{1/2}} - \frac{1}{\pi}\int_{-1}^{1} \frac{\ell_1(t)\,dt}{(1-t^2)^{1/2}} \qquad (4.1,7)$$

is fulfilled.

It can be easily shown that the solution u(x) to (4.1,4) vanishes at $x = \pm 1$. Hence, the only assumption made in section 4.3 which remains to be verified is that $\frac{du(x)}{dx}$ exists and is Hölder continuous for all x in the open interval (-1,1). Accordingly, we find that this requirement will be satisfied if $\frac{d\ell_1(x)}{dx}$ (that is $\frac{-2}{(1-\nu^2)} f_1(x)$) is Hölder continuous for all x in the open interval (-1,1), and near the ends not worse than $O((1-x^2)^{-1/2+\delta})$ with $\delta > 0$.

4.2 Thin Plate Solution to Case I

From the definition $\epsilon = 1/c\,(c_{22}/c_{11})^{1/2}$ and the definitions of c_{11}, c_{22}, we find that ϵ depends linearly on the ratio of the plate thickness h to the length 2c of the crack or of the rigid inclusion. In this section, we consider a plate with $\epsilon \ll 1$, i.e. a plate whose thickness is small in comparison with the length of the crack.

Upon observing equation (4.1,1), we shall assume[§]

$$u(x,\epsilon) = u_0(x) + o(1) \qquad \text{as } \epsilon \to 0 \qquad (4.2,1)$$

[§] This assumption ought to be verified. However, it has not yet been possible to carry out this verification because of the complexity of the integral equation (4.1,1).

uniformly for all $|x| \leq 1$ provided that

$$f_1(x,\epsilon) = f_{10}(x) + o(1) \qquad \text{as } \epsilon \to 0 \tag{4.2,2}$$

uniformly for all $|x| \leq 1$. Then, the integral equation for $u_0(x)$ reads as:

$$\frac{1}{\pi}\int_{-1}^{1} \frac{u_0(\xi)}{\xi - x}\, d\xi = \frac{1+\nu}{3+\nu}\, \ell_{10}(x) + C_1, \quad |x| < 1 \tag{4.2,3}$$

where

$$\ell_{10}(x) = -\frac{2}{(1-\nu^2)}\int^{x} f_{10}(t)dt\ . \tag{4.2,4}$$

We seek the solution to (4.2,3) in the same class of functions admitted in the previous section. Following the procedure of §113 in [16], we obtain

$$u_0(x) = -\frac{(1+\nu)}{(3+\nu)} \cdot \frac{(1-x^2)^{1/2}}{\pi}\int_{-1}^{1} \frac{\ell_{10}(t)dt}{(1-t^2)^{1/2}(t-x)} \tag{4.2,5}$$

while the additional condition (4.1,7) determines constant C_1 as follows:

$$C_1 = -\frac{1}{\pi}\int_{-1}^{1} \frac{\ell_{10}(t)dt}{(1-t^2)^{1/2}}\ . \tag{4.2,6}$$

It will be convenient in later analysis if we put this solution in the following form:

$$u_0(x) = \frac{2}{(1-\nu)(3+\nu)}\, h_0(x)\, (1-x^2)^{1/2} \tag{4.2,7}$$

where

$$h_0(x) = -\frac{(1-\nu^2)}{2\pi} \int_{-1}^{1} \frac{\ell_{10}(t)dt}{(1-t^2)^{1/2}(t-x)} . \tag{4.2,8}$$

The function $h_0(x)$ is not defined at $x = \pm 1$, however it is Hölder continuous and bounded for all x in the open interval (-1,1); moreover, it possesses finite limits as $x \to \pm 1$ from interior.

If the assumption (4.2,1) is correct, then $u(x,\epsilon)$ will be well approximated by $u_0(x)$ for thin plates. Hence, we shall replace $u(x,\epsilon)$ by $u_0(x)$ in (3.3,7) and (3.3,8) in order to compute the approximate forms for ϵ small of all the physical quantities.

To examine the stresses interior to the plate for ϵ small, we split the plate into three regions: the region away from the crack, the regions near the vertices $x = \pm 1$, $y = 0$ and the region near the crack but away from the vertices as in the following subsections.

a. Stresses away from the crack. For all points which lie outside an arbitrary fixed ellipse with foci at $x = \pm 1$ and a semi-minor axis $b > 0$, when we let $\epsilon \to 0$, the stress couples and shear force resultants can be computed from (3.3,7) and (3.3,8). We shall consider $M_x^{(1)}$ for example. From (3.3,7) we have

$$M_x^{(1)}(x,y) = \frac{1-\nu}{\pi} \int_{-1}^{1} u(\xi)\, m_x^{(1)}(x-\xi, y, \epsilon)\, d\xi \tag{4.2,9}$$

where $m_x^{(1)}(x,y,\epsilon)$ is defined by (3.3,9). For all points (x,y) outside the ellipse, $R^2 = (x-\xi)^2 + y^2$ is bounded away from zero whenever ξ in $[-1,1]$. Hence if we let $\epsilon \to 0$ and replace $u(\xi)$ by $u_0(\xi)$ in (4.2,9), we have for all (x,y) outside the ellipse

$$\lim_{\epsilon \to 0} M_x^{(1)}(x,y) = -\frac{(1-\nu)^2}{2\pi}\int_{-1}^{1} u_0(\xi)\,\frac{[y^4+(x-\xi)^4-6y^2(x-\xi)^2]}{R^6}\,d\xi \qquad (4.2,10)$$

for any fixed $b > 0$.

Let us define a function

$$\phi_1(x,y) = \frac{1}{\pi}\int_{-1}^{1} \frac{h_0(\xi)(1-\xi^2)^{1/2}}{(x-\xi)^2 + y^2}\,d\xi \ . \qquad (4.2,11)$$

With the aid of (4.2,7), (4.2,10) can be expressed in terms of $\phi_1(x,y)$ as

$$\lim_{\epsilon \to 0} M_x^{(1)} = -\frac{1-\nu}{3+\nu}\left[\phi_1 + 3y\frac{\partial\phi_1}{\partial y} + y^2\frac{\partial^2\phi_1}{\partial y^2}\right] \ . \qquad (4.2,12)$$

Similarly the limiting value as $\epsilon \to 0$ of the other stress couples and resultants can be expressed in terms of ϕ_1:

$$\lim_{\epsilon \to 0} M_y^{(1)} = \frac{1}{3+\nu}\left[(3+\nu)\phi_1 + (5-\nu)\,y\frac{\partial\phi_1}{\partial y} + (1-\nu)\,y^2\frac{\partial^2\phi_1}{\partial y^2}\right] , \qquad (4.2,13)$$

$$\lim_{\epsilon \to 0} M_{xy}^{(1)} = \frac{1}{(3+\nu)}\left[(3-\nu)\,y\frac{\partial\phi_1}{\partial x} + (1-\nu)\,y^2\frac{\partial^2\phi_1}{\partial x\partial y}\right] , \qquad (4.2,14)$$

$$\lim_{\epsilon \to 0} Q_x^{(1)} = \frac{1}{c(3+\nu)}\left[2\frac{\partial\phi_1}{\partial x} + 2y\frac{\partial^2\phi_1}{\partial x\partial y}\right], \qquad (4.2,15)$$

$$\lim_{\epsilon \to 0} Q_y^{(1)} = \frac{1}{c(3+\nu)}\left[4\frac{\partial\phi_1}{\partial y} + 2y\frac{\partial^2\phi_1}{\partial y^2}\right] \qquad (4.2,16)$$

for all (x,y) outside the ellipse.

b. Stresses near the vertex. For computing the stress distribution near the vertex we employ the following local coordinate system[§]:

$$x = 1 + r\cos\theta$$
$$y = r\sin\theta \qquad |\theta| \leqslant \pi$$

to specify the points near the end ($x = 1$, $y = 0$). As $r \to 0$ for any fixed θ, $|\theta| < \pi$, the following asymptotic relations can be established.

$$\left| \frac{r^{1/2}}{\pi} \int_{-1}^{1} \frac{|1-\xi|^{\mu}(1-\xi^2)^{1/2}}{(x-\xi)^2 + y^2} \, d\xi \right| \to 0 \quad \text{for any} \quad \mu > 0, \qquad (4.2,17)$$

$$\left| \frac{r^{1/2}}{\pi} \int_{-1}^{1} (1-\xi^2)^{1/2} h_0(\xi) \log\left[(x-\xi)^2 + y^2\right] d\xi \right| \to 0 \, . \qquad (4.2,18)$$

Now, we consider for example the behavior of M_x for small r and fixed θ, $|\theta| < \pi$. The kernel $m_x^{(1)}$ in (3.3,9), may be written after some algebra in the form:

$$m_x^{(1)}(x,y) = \frac{(1+\nu)}{2} \left(\frac{1}{\rho^2} - \frac{8y^2}{\rho^4} + \frac{8y^4}{\rho^6}\right) + p_x^{(1)}(x,y) \qquad (4.2,19)$$

where

$$p_x^{(1)}(x,y) = \frac{2}{\rho^2} K_2\left(\frac{\rho}{\epsilon}\right) - \frac{2}{\epsilon\rho} K_3\left(\frac{\rho}{\epsilon}\right) + \frac{2x^2y^2}{\epsilon^2\rho^4} K_4\left(\frac{\rho}{\epsilon}\right) -$$

$$- \frac{(x^4 + y^4 - 6x^2y^2)}{\rho^6} \left(1 - 12\frac{\epsilon^2}{\rho^2}\right) . \qquad (4.2,20)$$

[§] An analogous investigation could be carried out for the vertex $x = -1$, $y = 0$.

Then the first equation in (3.3,7) becomes

$$\frac{\pi}{(1-\nu)} M_x^{(1)} = \frac{(1+\nu)}{2} I_1 + I_2 \tag{4.2,21}$$

where

$$I_1 = \int_{-1}^{1} u(\xi) \left[\frac{1}{R^2} - \frac{8y^2}{R^4} + \frac{8y^4}{R^6} \right] d\xi \tag{4.2,22}$$

and

$$I_2 = \int_{-1}^{1} u(\xi)\, p_x^{(1)}(x-\xi, y)\, d\xi \tag{4.2,23}$$

with $R^2 = (x-\xi)^2 + y^2$ and $p_x^{(1)}$ defined as in (4.2,20).

If the thin plate solution $u_0(x)$ given by (4.2,7) is used in (4.2,22) and (4.2,23), we have

$$I_1 = I_{11} + I_{12} \tag{4.2,24}$$

where $I_{11} = \frac{2}{(1-\nu)(3+\nu)} \int_{-1}^{1} h_0(1-)\, (1-\xi^2)^{1/2} \left[\frac{1}{R^2} - \frac{8y^2}{R^4} + \frac{8y^2}{R^6} \right] d\xi$, (4.2,25)

$$I_{12} = \frac{2}{(1-\nu)(3+\nu)} \int_{-1}^{1} [h_0(\xi) - h_0(1-)]\, (1-\xi^2)^{1/2} \cdot \left[\frac{1}{R^2} - \frac{8y^2}{R^4} + \frac{8y^4}{R^6} \right] d\xi \tag{4.2,26}$$

with $h_0(1-) = \lim_{x \to 1-} h_0(x)$

and

$$I_2 = \frac{2}{(1-\nu)(3+\nu)} \int_{-1}^{1} h_0(\xi)\, (1-\xi^2)^{1/2}\, p_x^{(1)}(x-\xi, y)\, d\xi \,. \tag{4.2,27}$$

These integrals may be estimated asymptotically for small r. The integral I_{11} can be evaluated explicitly and its asymptotic form as $r \rightarrow 0$ is found to be

$$I_{11} \sim \frac{2\pi}{(1-\nu)(3+\nu)} \quad h_0 \, (1-) \, (2r)^{-1/2} [\tfrac{3}{4} \cos \tfrac{\theta}{2} + \tfrac{1}{4} \cos \tfrac{5\theta}{2}] \; . \quad (4.2,28)$$

For I_{12}, we have the following estimate:

$$\left| r^{1/2} I_{12} \right| \leq M \frac{r^{1/2}}{\pi} \int_{-1}^{1} |1-\xi|^{\mu} \, (1-\xi^2)^{1/2} [\frac{1}{R^2} + \frac{8y^2}{R^4} + \frac{8y^4}{R^6}] \, d\xi \qquad (4.2,29)$$

where M is a positive constant. The right hand side of (4.2,29) tends to zero as $r \rightarrow 0$ because of (4.2,17). Hence

$$r^{1/2} \, I_{12} \rightarrow 0 \qquad \text{as } r \rightarrow 0. \qquad (4.2,30)$$

The remainder term $p_x^{(1)}(x,y)$ in (4.2,19) can be easily shown to have the property

$$p_x^{(1)} \, (x,y) = 0(\log \rho) \text{ as } \rho \rightarrow 0 \, , \qquad (4.2,31)$$

hence

$$r^{1/2} \, I_2 \rightarrow 0 \qquad \text{as } r \rightarrow 0 \qquad (4.2,32)$$

through using (4.2,18). Substituting these results into (4.2,21), we find that as $r \rightarrow 0$

$$\boxed{M_x^{(1)} \sim \frac{(1+\nu)}{(3+\nu)} \, h_0 \, (1-) \, (2r)^{-1/2} \, (\tfrac{3}{4} \cos \tfrac{\theta}{2} + \tfrac{1}{4} \cos \tfrac{5\theta}{2}) \; .} \qquad (4.2,33)$$

In a similar way it can be shown that, for small ϵ , as $r \to 0$

$$M_y^{(1)} \sim \frac{(1+\nu)}{(3+\nu)} h_0(1-) (2r)^{-1/2} (\frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{5\theta}{2}) , \tag{4.2,34}$$

$$M_{xy}^{(1)} \sim \frac{(1+\nu)}{(3+\nu)} h_0(1-) (2r)^{-1/2} (- \frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{5\theta}{2}) . \tag{4.2,35}$$

The shear stress resultants are found to remain finite as $r \to 0$.

c. Stresses near the crack but away from the vertex. In order to examine the stresses near the crack, we make the change of scale

$$y = \epsilon \eta \tag{4.2,36}$$

and examine the limits of (3.3,7) and (3.3,8) as $\epsilon \to 0$ for fixed $\eta > 0$ and fixed x in (-1, 1).

We illustrate for $M_{xy}^{(1)}$. According to (3.3,7)

$$M_{xy}^{(1)} (x, \epsilon \eta, \epsilon) = \frac{(1-\nu)}{\pi} \int_{-1}^{1} u(\xi, \epsilon) m_{xy}^{(1)}(x-\xi, \eta \epsilon, \epsilon) d\xi \tag{4.2,37}$$

where

$$\begin{aligned} m_{xy}^{(1)} &(x-\xi, \eta \epsilon, \epsilon) = \\ &= - \frac{2\eta \epsilon (x-\xi)}{R^4} + \frac{\eta (x-\xi)}{\epsilon R^2} K_2 (\frac{R}{\epsilon}) + (1-\nu) \frac{\eta \epsilon [3(x-\xi)\eta^2 \epsilon^2 - (x-\xi)^3]}{R^6} \\ &- \epsilon^2 \{ 48 [\frac{\eta \epsilon (x-\xi)}{R^6} - \frac{2\eta \epsilon (x-\xi)^3}{R^8}] - \\ &- 2 [\frac{3\eta (x-\xi)}{\epsilon^2 R^3} K_3 (\frac{R}{\epsilon}) - \frac{\eta (x-\xi)^3}{\epsilon^2 R^4} K_4 (\frac{R}{\epsilon})] \} \end{aligned} \tag{4.2,38}$$

in which $R^2 = (x-\xi)^2 + \eta^2\epsilon^2$. (4.2,39)

The integral (4.2,37) may be written in the form:

$$M_{xy}^{(1)} = \frac{1-\nu}{\pi}\left(\int_{-1}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{1}\right) u(\xi,\epsilon)\, m_{xy}^{(1)}\, d\xi \qquad (4.2,40)$$

where δ is small and positive. In the first and third of these integrals, R is bounded away from zero so that

$$\lim_{\epsilon\to 0}\left(\int_{-1}^{x-\delta}\right) = \lim_{\epsilon\to 0}\left(\int_{x+\delta}^{1}\right) = 0. \qquad (4.2,41)$$

In the second integral in (4.2,40), we use the thin plate solution $u_0(x)$ given by (4.2,7), and expand it into a two term Taylor's series plus a remainder. Also, we change the variable of integration from ξ to ζ where $\zeta = (\xi - x)/\epsilon\eta$. It is then found

$$\lim_{\epsilon\to 0}\int_{x-\delta}^{x+\delta} u(\xi,\epsilon)\, m_{xy}^{(1)}(x-\xi,\epsilon\eta,\epsilon)\, d\xi =$$

$$= u_0'(x)\int_{-\infty}^{\infty}\Big\{\frac{2\zeta^2}{(1-\zeta^2)^2} - \frac{\eta^2\zeta^2}{(1+\zeta^2)} K_2\left(\eta\sqrt{1+\zeta^2}\right) - (1-\nu)\frac{(3\zeta^2-\zeta^4)}{(1+\zeta^2)^3} +$$

$$+ \Big\{48\Big[\frac{\zeta^2}{\eta^2(1+\zeta^2)^3} - \frac{2\zeta^4}{\eta^2(1+\zeta^2)^4}\Big] -$$

$$-2\Big[\frac{3\zeta^2\eta}{(1+\zeta^2)^{3/2}} K_3\left(\eta\sqrt{1+\zeta^2}\right) - \frac{\eta^2\zeta^4}{(1+\zeta^2)^2} K_4\left(\eta\sqrt{1+\zeta^2}\right)\Big]\Big\}\Big\}\, d\zeta =$$

$$= \pi u_0'(x)\left(1 - e^{-y/\epsilon}\right). \qquad (4.2,42)$$

If this result and (4.2,41) are combined in (4.2,40), we find that

$$\lim_{\epsilon \to 0} M_{xy}^{(1)}(x, \epsilon\eta, \epsilon) = (1-\nu)\, u'_0(x)\, (1-e^{-y/\epsilon}) \tag{4.2,43}$$

with $u'_0(x) = \frac{du_0(x)}{dx}$, this result may be written in the form

$$\boxed{M_{xy}^{(1)}(x, y, \epsilon) \sim \frac{2}{(3+\nu)} (1-e^{-y/\epsilon}) \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2} f_{10}(t)dt}{t - x}} \tag{4.2,44}$$

as $\epsilon \to 0$ for fixed $y/\epsilon > 0$ and fixed x in (-1, 1).

Similarly, we find that the other stress couples and resultants are:

$$M_x^{(1)} \sim \frac{(1-\nu)}{(3+\nu)} f_{10}(x) \;, \qquad M_y^{(1)} \sim -f_{10}(x)$$

$$Q_x^{(1)} \sim \frac{2}{(3+\nu)\epsilon c} e^{-y/\epsilon} \frac{1}{\pi\sqrt{1-x^2}} \int_{-1}^{1} \frac{\sqrt{1-t^2}\, f_{10}(x)}{t - x} dt \;,$$

$$Q_y^{(1)} \sim - \frac{2}{(3+\nu)c} (1-e^{-y/\epsilon}) \frac{d}{dx} \left[\frac{1}{\pi\sqrt{1-x^2}} \int_{-1}^{1} \frac{\sqrt{1-t^2} f_{10}(t)dt}{t - x} \right]$$

(4.2,45)

as $\epsilon \to 0$ for fixed $y/\epsilon > 0$ and fixed x in (-1, 1).

It was remarked in section 3.2 that although w may not possess a Fourier transform, the problem still can be solved in some cases if $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ possess Fourier transforms. We may illustrate this point by letting $f_{10}(x) = M_0 = \text{const.}$§ in (4.1,3) and thus

§ This problem has been considered in detail in [15].

$$u_0(x) = \frac{2M_0}{(1-\nu)(3+\nu)} (1-x^2)^{1/2}, \quad |x| \leq 1 . \tag{4.2,46}$$

According to (3.3,2), we obtain

$$A_0(\alpha) = \frac{2\pi M_0}{(1-\nu)(3+\nu)} \cdot \frac{J_1(\alpha)}{\alpha} . \tag{4.2,47}$$

If (4.2,44) is used in (3.2,38) and we find that $w^{(1)}$ in (3.2,1) can be written as:

$$w^{(1)}(x,y) = \frac{2M_0}{(1-\nu)(3+\nu)} \int_0^\infty \frac{J_1(\alpha)}{\alpha} [|y| - \frac{(1+\nu)}{(1-\nu)\alpha}] e^{-\alpha|y|} \cos\alpha x d\alpha \tag{4.2,48}$$

which diverges (because of the behavior of the integrand at $\alpha = 0$) for all (x,y) in $\mathscr{D}$. However, $\frac{\partial w^{(1)}}{\partial x}$ and $\frac{\partial w^{(1)}}{\partial y}$ do possess Fourier transforms, so we could find $w^{(1)}(x,y)$ from its partial derivatives.

4.3 Results Based on Classical Theory of Bending of Plates for Case I

Let us denote by w_c the deflection of the middle surface of the plate under consideration. It is well known that according to the classical theory of bending for plates the stress couples and resultants may be expressed in terms of w_c as:

$$M_{xc} = - D [\frac{\partial^2 w_c}{\partial X^2} + \nu \frac{\partial^2 w_c}{\partial Y^2}] , \tag{4.3,1}$$

$$M_{yc} = -D [\frac{\partial^2 w_c}{\partial Y^2} + \nu \frac{\partial^2 w_c}{\partial X^2}] , \tag{4.3,2}$$

$$M_{xyc} = - (1-\nu) D \frac{\partial^2 w_c}{\partial X \partial Y} , \tag{4.3,3}$$

$$Q_{xc} = - D \frac{\partial}{\partial X} (\Delta w_c) , \tag{4.3,4}$$

$$Q_{yc} = -D \frac{\partial}{\partial Y}(\Delta w_c) \tag{4.3,5}$$

where $D = \frac{Eh^3}{12(1-\nu)^2}$, $\Delta = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}$ and the subscript c refers to the classical theory. As noted in section 2.2, this system can be deduced from equations (2.2,24) to (2.2,26) by taking $G = \infty$ and $h_1(Z) = \frac{1}{h^2/6}(\frac{Z}{h/2})$, however w_c here denotes the deflection of the middle surface of the plate.

It is also well known that for a plate whose upper and lower surfaces are free from external tractions w_c satisfies the biharmonic differential equation

$$\Delta\Delta w_c = 0 \tag{4.3,6}$$

in $\mathscr{D}$ according to the classical theory.

Let us again make a dimensionless coordinate transformation $X = cx$, $Y = cy$ in the above equations and we shall use (x, y) hereafter.

The boundary conditions along the crack are

$$\lim_{\substack{|y| \to 0 \\ |x| < 1}} M_{yc} = -f_{10}(x) \tag{4.3,7}$$

$$\lim_{\substack{|y| \to 0 \\ |x| < 1}} Q_{yc} + \frac{1}{c}\frac{\partial M_{xyc}}{\partial x} = -f_{30}(x) - \frac{1}{c}\frac{d}{dx}f_{20}(x) \tag{4.3,8}$$

where f_{10} is defined by (4.2,2) and

$$f_{20} = \lim_{\epsilon \to 0} f_{2c}(x, \epsilon)$$

$$f_{30} = \lim_{\epsilon \to 0} f_{3c}(x, \epsilon). \tag{4.3,9}$$

Equation (4.3,8) is the Kirchhoff edge condition usually associated with the classical theory and replaces the last two equations in (3.2,25) of the Reissner theory. It is this effect of a reduction in the number of boundary conditions which we wish to study. At infinity, we shall require w_c and all its derivatives vanish.

The Fourier transform technique shall be applied again to (4.3,6) and it is found that the most general solution of (4.3,6) satisfying the condition at infinity is

$$w_c = w_c^{(1)} + (\text{sgn } y)\, w_c^{(2)} \tag{4.3,10}$$

where

$$\frac{D}{c^2} w_c^{(j)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\, Q_{cj}(\alpha) e^{-|\alpha y|} + |y| R_{cj}(\alpha) e^{-|\alpha y|}\,]\, e^{i\alpha x} d\alpha \tag{4.3,11}$$

$$j = 1,\ 2$$

in which Q_{cj}, R_{cj}, $j = 1,2$ are as yet arbitrary functions. Here $w_c^{(1)}$ denotes the symmetric deflection and $w_c^{(2)}$, the antisymmetric part. These superscripts will also be attached to the other physical quantities. Upon satisfaction of the boundary condition (4.3,7) and (4.3,8), we have

$$\lim_{\substack{|y| \to 0 \\ |x| < 1}} M_{yc}^{(1)} = -f_{10}(x)\,, \tag{4.3,12a}$$

$$\lim_{\substack{|y| \to 0 \\ |x| < 1}} [\, Q_{yc}^{(1)} + \frac{1}{c} \frac{\partial M_{xyc}^{(1)}}{\partial x}\,] = 0 \tag{4.3,12b}$$

and

$$\lim_{\substack{|y|\to 0\\ |x|<1}} \left[Q_{yc}^{(2)} + \frac{1}{c}\frac{\partial M_{xyc}^{(2)}}{\partial x} \right] = - f_{30}(x) - \frac{1}{c}\frac{d}{dx} f_{20}(x), \qquad (4.3,13a)$$

$$\lim_{\substack{|y|\to 0\\ |x|<1}} M_{yc}^{(2)} = 0 \; . \qquad (4.3,13b)$$

It is clear that the symmetric part depends on $f_{10}(x)$ only while the antisymmetric part depends on $f_{20}(x)$ and $f_{30}(x)$.

We shall postpone the discussion on the antisymmetric part until section 4.6. For the symmetric part, we define

$$u_c(x) = - \lim_{y \to 0\pm} \frac{D}{c^2} \frac{\partial w_c^{(1)}}{\partial y} \qquad \text{all } x \qquad (4.3,14)$$

which vanishes for all $|x| > 1$ on account of the fact that $w_c^{(1)}$ is even in y and $\frac{\partial w_c^{(1)}}{\partial y}$ is continuous in $\mathscr{D}$. All other physical quantities can be expressed in terms of $u_c(x)$. We omit the detail and list the results as follows.

$$\begin{pmatrix} M_{xc}^{(1)} \\ M_{yc}^{(1)} \\ M_{xyc}^{(1)} \end{pmatrix} = \frac{(1-\nu)}{\pi} \int_{-1}^{1} u_c(\xi) \begin{pmatrix} m_{xc}^{(1)}(x-\xi, y) \\ m_{yc}^{(1)}(x-\xi, y) \\ m_{xyc}^{(1)}(x-\xi, y) \end{pmatrix} d\xi \quad , \qquad (4.3,15)$$

$$c \begin{pmatrix} Q_{xc}^{(1)} \\ Q_{yc}^{(1)} \end{pmatrix} = \frac{(1-\nu)}{\pi} \int_{-1}^{1} u_c(\xi) \begin{pmatrix} q_{xc}^{(1)}(x-\xi, y) \\ q_{yc}^{(1)}(x-\xi, y) \end{pmatrix} d\xi \qquad (4.3,16)$$

where

$$m_{xc}^{(1)}(x,y) = -\frac{(1-\nu)(y^4+x^4-6x^2y^2)}{2\rho^6} , \qquad (4.3,17)$$

$$m_{yc}^{(1)}(x,y) = \frac{(3+\nu)(x^2-y^2)}{2\rho^4} + (1-\nu)\frac{y^2(y^2-3x^2)}{\rho^6} , \qquad (4.3,18)$$

$$m_{xyc}^{(1)}(x,y) = -\frac{2xy}{\rho^4} + (1-\nu)\frac{y(3xy^2-x^3)}{\rho^6} , \qquad (4.3,19)$$

$$q_{xc}^{(1)}(x,y) = 2\frac{(3xy^2-x^3)}{\rho^6} , \qquad (4.3,20)$$

$$q_{yc}^{(1)}(x,y) = 2\frac{(y^3-3x^2y)}{\rho^6} . \qquad (4.3,21)$$

The integral equation for $u_c(x)$ follows directly from the boundary condition (4.3,12a). We have

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u_c(\xi)}{\xi-x}\,d\xi = \frac{(1+\nu)}{(3+\nu)}\ell_{10}(x) + C_1 \qquad |x| < 1 \qquad (4.3,22)$$

where $\ell_{10}(x)$ is given by (4.2,4). Equation (4.3,22) is exactly the same integral equation for $u_0(x)$ which was obtained in section 4.2. Hence $u_c(x)$ is identical to our approximate thin plate solution (4.2,7) of Reissner's theory.

With (4.2,7) and formulas (4.3,15), (4.3,16) we are again able to compute the stresses in different regions.

a. <u>Stress away from the crack - classical theory</u>. When the stress resultant and couples are computed according to classical theory from (4.2,7) and formulas (4.3,15), (4.3,16), we find that

$$M_{xc}^{(1)} = -\frac{(1-\nu)}{(3+\nu)}\left[\phi_1 + 3y\frac{\partial\phi_1}{\partial y} + y^2\frac{\partial\phi_1}{\partial y^2}\right] ,$$

$$M_{yc}^{(1)} = \frac{1}{(3+\nu)}\left[(3+\nu)\phi_1 + (5-\nu)\, y\frac{\partial\phi_1}{\partial y} + (1-\nu)\, y^2\frac{\partial^2\phi_1}{\partial y^2}\right] ,$$

$$M_{xyc}^{(1)} = \frac{1}{(3+\nu)}\left[(3-\nu)\, y\frac{\partial\phi_1}{\partial x} + (1-\nu)\, y^2\frac{\partial^2\phi_1}{\partial x\partial y}\right] ,$$

$$Q_{xc}^{(1)} = \frac{1}{c(3+\nu)}\left[2\frac{\partial\phi_1}{\partial x} + 2y\frac{\partial^2\phi_1}{\partial x\partial y}\right] ,$$

$$Q_{yc}^{(1)} = \frac{1}{c(3+\nu)}\left[4\frac{\partial\phi_1}{\partial y} + 2y\frac{\partial^2\phi_1}{\partial y^2}\right]$$

where $\phi_1(x,y)$ is defined by (4.2,11). These are precisely the same as the limiting values as $\epsilon \to 0$ (4.2,12) to (4.2,16) of the couples and resultants computed according to the Reissner theory, provided we stay away from the crack. Thus for sufficiently thin plates ($\epsilon << 1$), the M's and Q's from classical theory and those for Reissner's theory of bending agree in any region which excludes the crack, as would be expected from our discussion in the Introduction.

b. Stresses near the vertex - classical theory. Using the same method as in the previous section we find for points near the end $x = 1$, $y = 0$ that, as $r \to 0$, the asymptotic expressions for the stress couples are

$$M_{xc}^{(1)} \sim \frac{-1+\nu}{3+\nu} h_0(1-) (2r)^{-1/2} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{5\theta}{2}\right)$$

$$M_{yc}^{(1)} \sim \frac{1}{3+\nu} h_0 (1-) (2r)^{-1/2} \left(\frac{11+5\nu}{4} \cos \frac{\theta}{2} + \frac{1-\nu}{4} \cos \frac{5\theta}{2}\right) \qquad (4.3,23)$$

$$M_{xyc}^{(1)} \sim \frac{-1}{(3+\nu)} h_0 (1-) (2r)^{-1/2} \left(\frac{7+\nu}{4} \sin \frac{\theta}{2} + \frac{1-\nu}{4} \sin \frac{5\theta}{2}\right)$$

where $h_0(x)$ is given by (4.2,8). Similar results hold at $x = -1$, $y = 0$.

Moreover, the shear stress resultants $Q_{xc}^{(1)}$ and $Q_{yc}^{(1)}$ become infinite like $r^{-3/2}$ as $r \to 0$.

c. <u>Stresses near the crack - classical theory.</u> The stresses near the crack can be easily obtained through formulas (4.3,15) and (4.3,16) for small y and fixed x in (-1, 1). We find

$$M_{xc}^{(1)} \sim \frac{1-\nu}{3+\nu} f_{10}(x) \quad , \qquad M_{yc}^{(1)} \sim -f_{10}(x)$$

$$M_{xyc}^{(1)} \sim \frac{2}{(3+\nu)} \frac{1}{\pi \sqrt{1-x^2}} \int_{-1}^{1} \frac{\sqrt{1-t^2}\, f_{10}(t)dt}{t - x} \; ,$$

$$Q_{xc}^{(1)} \sim -\frac{2y}{(3+\nu)c} \frac{d^2}{dx^2} \left[\frac{1}{\pi \sqrt{1-x^2}} \int_{-1}^{1} \frac{\sqrt{1-t^2}\, f_{10}(t)dt}{t - x}\right]$$

$$Q_{yc}^{(1)} \sim -\frac{2}{(3+\nu)c} \frac{d}{dx} \left[\frac{1}{\pi \sqrt{1-x^2}} \int_{-1}^{1} \frac{\sqrt{1-t^2}\, f_{10}(t)dt}{t - x}\right] \qquad (4.3,24)$$

as $y \to 0$ for fixed x in (-1, 1).

We may remark that $M_{xyc}^{(1)}$ and $Q_{yc}^{(1)}$ in (4.3,24) satisfy the Kirchhoff condition (4.3,12b) along the crack, and that $M_{yc} = -f_{10}(x)$ at $y = 0$, $|x| < 1$, thus verifying that the boundary conditions appropriate for the classical bending theory are indeed satisfied at the crack.

4.4 Case II - Antisymmetric Solution for an Infinite Plate Containing a Crack

In this section we shall study the antisymmetric solution of the crack problem which is represented by the system of coupled singular integral equations (3.3,27) and (3.3,28).

Integrating (3.3,27) with respect to x, and after some algebra, we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\frac{v(\xi)d\xi}{\xi-x}+\frac{1}{\pi}\int_{-1}^{1}k_{11}(x,\xi)\,v(\xi)\,d\xi+\frac{1}{\pi}\int_{-1}^{1}k_{12}(x,\xi)w(\xi)d\xi =$$

$$= \ell_2(x) + C_2, \qquad |x| < 1 \qquad (4.4,1)$$

where

$$k_{11}(x,\xi) = \frac{2}{(1+\nu)}\left\{\frac{1+2\epsilon^2\left[\frac{2}{(\xi-x)^2}-\frac{K_2\left(\frac{|\xi-x|}{\epsilon}\right)}{\epsilon^2}\right]+}{\xi - x}\right.$$

$$\left.+\frac{1}{2\epsilon^2}\int^{x}K_0\left(\frac{|\xi-\eta|}{\epsilon}\right)d\eta\right\}, \qquad (4.4,2)$$

$$k_{12}(x,\xi) = \frac{1}{(1+\nu)}\left[\frac{K_2\left(\frac{|\xi-x|}{\epsilon}\right)}{\epsilon^2}-\frac{2}{(\xi-x)^2}\right] \qquad (4.4,3$$

and

$$\ell_2(x) = -\frac{2}{(1-\nu^2)}\int^{x}f_2(\eta)\,d\eta \qquad (4.4,4)$$

in which $f_2(x)$ stands for $f_{2c}(x)$. Integrating (3.3,28) with respect to x, we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\frac{w(\xi)d\xi}{\xi-x} + \frac{1}{\pi}\int_{-1}^{1}k_{21}(x,\xi)v(\xi)d\xi + \frac{1}{\pi}\int_{-1}^{1}k_{22}(x,\xi)w(\xi)d\xi =$$

$$= \ell_3(x) + C_3 \quad , \qquad |x| < 1 \qquad (4.4,5)$$

where

$$k_{21}(x,\xi) = -\epsilon^2\left[\frac{K_2(\frac{|\xi-x|}{\epsilon})}{\epsilon^2} - \frac{2}{(\xi-x)^2}\right] \quad , \qquad (4.4,6)$$

$$k_{22}(x,\xi) = \frac{(\xi-x)}{\epsilon\ |\xi-x|} K_1\left(\frac{|\xi-x|}{\epsilon}\right) - \frac{1}{(\xi-x)} \qquad (4.4,7)$$

and

$$\ell_3(x) = -\frac{2\epsilon^2 c}{(1-\nu)} \int^{x} f_3(\eta)\, d\eta \ . \qquad (4.4,8)$$

in which $f_3(x)$ stands for $f_{3c}(x)$.

We require solutions of (4.4,1) and (4.4,5) to be Hölder continuous with some positive Hölder index μ for all x in the closed interval [-1, 1]. Under the above considerations, (4.4,1) and (4.4,5) can be transformed into a system of Fredholm type integral equations, by procedures discussed in [16], Chapter 19. These Fredholm equations are as follows.

$$v(x) - \frac{1}{\pi}\int_{-1}^{1}M_{11}(x,\xi)\, v(\xi)\, d\xi - \frac{1}{\pi}\int_{-1}^{1}M_{12}(x,\xi)\, w(\xi)\, d\xi =$$

$$= G_2(x) \quad , \qquad |x| < 1 \qquad (4.4,9)$$

and

$$w(x) - \frac{1}{\pi}\int_{-1}^{1} M_{21}(x,\xi)\, v(\xi)\, d\xi - \frac{1}{\pi}\int_{-1}^{1} M_{22}(x,\xi)\, w(\xi)\, d\xi =$$

$$= G_3(x) \;, \qquad |x| < 1 \qquad (4.4,10)$$

where

$$M_{j\ell}(x,\xi) = \frac{(1-x^2)^{1/2}}{\pi}\int_{-1}^{1} \frac{k_{j\ell}(t,\xi)}{(1-t^2)^{1/2}(t-x)}\, dt, \quad j,\ell = 1,2 \qquad (4.4,11)$$

and

$$G_j(x) = -\frac{(1-x^2)^{1/2}}{\pi}\int_{-1}^{1} \frac{\ell_j(t)\, dt}{(1-t^2)^{1/2}(t-x)} \;, \qquad j = 2,3 \qquad (4.4,12)$$

provided that constants C_2, C_3 are chosen as follows.

$$C_2 = \frac{1}{\pi^2}\int_{-1}^{1} \frac{dt}{(1-t^2)^{1/2}} \left\{ \int_{-1}^{1} [k_{11}(t,\xi)v(\xi) + k_{12}(t,\xi)w(\xi)]\, d\xi \right\} -$$

$$- \int_{-1}^{1} \frac{\ell_2(t)dt}{(1-t^2)^{1/2}} \qquad (4.4,13)$$

and

$$C_3 = \frac{1}{\pi^2}\int_{-1}^{1} \frac{dt}{(1-t^2)^{1/2}} \left\{ \int_{-1}^{1} [k_{21}(t,\xi)\, v(\xi) + k_{22}(t,\xi)\, w(\xi)]\, d\xi \right\} -$$

$$- \int_{-1}^{1} \frac{\ell_3(t)dt}{(1-t^2)^{1/2}} . \qquad (4.4,14)$$

Moreover, the requirement made in section 3.3 that $\frac{dv(x)}{dx}$ and

$\frac{dw(x)}{dx}$ exist and are Hölder continuous for all x in the open interval (-1,1) has to be verified. We find that this requirement will be fulfilled if both $\frac{d\ell_2(x)}{dx}$ (that is, $-\frac{2}{(1-\nu^2)} f_2(x)$) and $\frac{d\ell_3(x)}{dx}$ (that is, $-\frac{2\epsilon^2 c}{(1-\nu)} f_3(x)$) are Hölder continuous for x in the open interval (-1,1) and near the ends are not worse than $O(1-x^2)^{-1/2+\delta}$) with some $\delta > 0$.

4.5 Thin Plate Solution to Case II

In order to obtain appropriate approximate solutions for thin plates, it will be convenient to write equation (4.4,1) in the form:

$$-\{\int_{-1}^{1} v(\xi)[\int^{x} \frac{1}{2\epsilon^2} K_0(\frac{|\eta-\xi|}{\epsilon})d\eta \;\; \frac{(1-\nu)}{2(x-\xi)} - \frac{2}{(x-\xi)} K_2(\frac{|x-\xi|}{\epsilon}) + \frac{4\epsilon^2}{(x-\xi)^3}]\, d\xi +$$

$$+\int_{-1}^{1} \frac{dw}{d\xi}[\int^{x} \frac{1}{2\epsilon^2} K_0(\frac{|\eta-\xi|}{\epsilon})d\eta - \frac{1}{(x-\xi)} + \frac{(x-\xi)}{\epsilon|x-\xi|} K_1(\frac{|x-\xi|}{\epsilon})]\; d\xi\} =$$

$$= -\frac{1}{(1-\nu)} \int^{x} f_2(\eta)\, d\eta + C_2, \qquad |x| < 1. \qquad (4.5,1)$$

For a similar reason, we integrate (4.4,5) with respect to x once and then write it as:

$$\{\int_{-1}^{1} v(\xi)[\int^{x} \frac{1}{2\epsilon^2} K_0(\frac{|\eta-\xi|}{\epsilon})d\eta - \frac{1}{(x-\xi)} + \frac{(x-\xi)}{\epsilon\,|x-\xi|} K_1(\frac{|x-\xi|}{\epsilon})]\; d\xi +$$

$$+\int_{-1}^{1} \frac{dw}{d\xi} \int^{x} \frac{1}{2\epsilon^2} K_0(\frac{|\eta-\xi|}{\epsilon})d\eta\; d\xi\} = -\frac{c}{(1-\nu)} \int^{x}\int^{\eta} f_3(\zeta)d\zeta d\eta + C_3 x + C_4,$$

$$|x| < 1. \qquad (4.5,2)$$

As $\epsilon \to 0$, the left sides of both (4.5,1) and (4.5,2) grow without bound. However, this difficulty may be removed if the limits

$$\lim_{\epsilon \to 0} f_2(x,\epsilon) = f_{20}(x)$$
$$\lim_{\epsilon \to 0} f_3(x,\epsilon) = f_{30}(x) \qquad (4.5,3)$$

exist in $|x| < 1$. We shall assume that (4.5,3) holds in the sequel.

Let us assume§

$$v(x,\epsilon) = v_0(x) + \tilde{v}(x,\epsilon)\,, \qquad |x| \leq 1 \qquad (4.5,4)$$

and

$$\tilde{v}(x,\epsilon) = o(1) \text{ as } \epsilon \to 0 \qquad (4.5,5)$$

uniformly in $|x| \leq 1$. Then, if the system of equations (4.5,1) and (4.5,2) does have a limiting solution as $\epsilon \to 0$, we must have

$$\frac{dw}{dx}(x,\epsilon) = -\,v_0(x) + \tilde{\psi}(x,\epsilon)\,, \qquad |x| < 1 \qquad (4.5,6)$$

where

$$\tilde{\psi}(x,\epsilon) = o(1) \qquad \text{as } \epsilon \to 0 \qquad (4.5,7a)$$

for all x, $|x| < 1$; the end points are not included since $\frac{dw}{dx}(x,\epsilon)$ may

§ This assumption again ought to be verified. However a proof has not yet been carried out.

not be defined there, and

$$\int_{-1}^{x} \psi(\xi,\epsilon)d\xi = o(1) \quad \text{as } \epsilon \to 0 \quad \text{for } |x| \leq 1. \qquad (4.5,7b)$$

Substituting these assumed forms into (4.5,1) and (4.5,2) and letting $\epsilon \to 0$, we have

$$-\int_{-1}^{1} v_o(\xi)\frac{(1+\nu)}{2(x-\xi)}d\xi - \lim_{\epsilon\to 0}\int_{-1}^{1}[\tilde{v}(\xi,\epsilon)+\tilde{\psi}(\xi,\epsilon)]\int^{x}\frac{1}{2\epsilon^2}K_0\left(\frac{|\eta-\xi|}{\epsilon}\right)d\eta\, d\xi =$$

$$= -\frac{1}{(1-\nu)}\int^{x} f_{20}(\eta)\, d\eta + C_2, \qquad |x| < 1 \qquad (4.5,8)$$

and

$$-\int_{-1}^{1} v_o(\xi)\frac{1}{(x-\xi)}d\xi + \lim_{\epsilon\to 0}\int_{-1}^{1}[\tilde{v}(\xi,\epsilon)+\tilde{\psi}(\xi,\epsilon)]\int^{x}\frac{1}{2\epsilon^2}K_0\left(\frac{|\eta-\xi|}{\epsilon}\right)d\eta d\xi =$$

$$= -\frac{c}{(1-\nu)}\int^{x}\int^{\eta} f_{30}(\zeta)\, d\zeta\, d\eta + C_3 x + C_4, \qquad |x| < 1. \qquad (4.5,9)$$

Adding (4.5,8) to (4.5,9), we obtain

$$\frac{(3+\nu)}{2\pi}\int_{-1}^{1}\frac{v_0(\xi)}{\xi-x}d\xi = -\frac{1}{(1-\nu)}\left[\int^{x} f_{20}(\eta)d\eta + c\int^{x}\int^{\eta} f_{30}(\zeta)d\zeta\, d\eta \right. +$$

$$\left. + C_3 x + C_5, \qquad |x| < 1. \qquad (4.5,10)\right.$$

The solution of (4.5,10) can be obtained by procedures discussed in §113 of [16]. The solution is

$$v_0(x) = \frac{2}{(1-\nu)(3+\nu)}(1-x^2)^{1/2}\, j_0(x), \qquad |x| \leq 1 \qquad (4.5,11)$$

where

$$j_0(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\int^{t} f_{20}(\eta)d\eta + c\int^{t}\int^{\eta} f_{30}(\zeta)d\zeta\, d\eta}{(1-t^2)^{1/2}\,(t-x)}\, dt - (1-\nu)C_3,$$

$$|x| < 1 \qquad (4.5,12)$$

provided the constant C_5 is chosen to be

$$C_5 = \frac{1}{(1-\nu)}\int_{-1}^{1} \frac{\int^{t} f_{20}(\eta)d\eta + c\int^{t}\int^{\eta} f_{30}(\zeta)d\zeta d\eta}{(1-t^2)^{1/2}}\, dt\ . \qquad (4.5,13)$$

Substituting $v_0(x)$ back into either (4.5,8) or (4.5,9), we obtain another integral equation for $[\tilde{v}(x,\epsilon) + \tilde{\psi}(x,\epsilon)]$ as $\epsilon \to 0$ which can not be solved explicitly. However, a simple estimate shows that both $\tilde{\psi}(x,\epsilon)$ and $\tilde{v}(x,\epsilon)$ for small ϵ give only higher order effects in computing the stress field away from the crack and the stress couples around the vertices. Hence, we shall ignore it as long as the stress field near the crack but away from the vertex, and the shear stress resultants near the vertex, are not considered.

Integrating (4.5,6) with respect to x from x = -1, we obtain

$$w(x,\epsilon) = -\int_{-1}^{x} v_0(\xi)d\xi + \int_{-1}^{x} \tilde{\psi}(\xi,\epsilon)d\xi, \quad |x| \leq 1. \qquad (4.5,14)$$

If the asymptotic property (4.5,7b) is used, then as $\epsilon \to 0$, (4.5,14) gives

$$w_0(x) \equiv \lim_{\epsilon\to 0} w(x,\epsilon) = -\int_{-1}^{x} v_0(\xi)d\xi, \quad |x| \leq 1. \qquad (4.5,15)$$

On account of the fact that $w_0(\pm 1, 0) = 0$, the constant C_3 can be determined through the relation

$$\int_{-1}^{1} v_0(\xi)\, d\xi = 0 \quad . \tag{4.5,15}$$

If the approximate solutions $v_0(x)$ and $w_0(x)$ as given by (4.5,11) and (4.5,15) are used in formulas (3.3,20) and (3.3,21), we can compute the approximate stress field away from the crack and the stress couples near the vertex. Using the same geometrical description as we did in section 4.2, we find the following results.

a. <u>Stress field away from the crack</u>

$$\lim_{\epsilon \to 0} M_x^{(2)} = \frac{1}{(3+\nu)} \left[(1+3\nu)\, y \frac{\partial \phi_2}{\partial x} - (1-\nu)\, y^2 \frac{\partial^2 \phi_2}{\partial x \partial y} \right] , \tag{4.5,16}$$

$$\lim_{\epsilon \to 0} M_y^{(2)} = \frac{1}{(3+\nu)} \left[(1-\nu)\, y \frac{\partial \phi_2}{\partial x} + (1-\nu)\, y^2 \frac{\partial^2 \phi_2}{\partial x \partial y} \right] , \tag{4.5,17}$$

$$\lim_{\epsilon \to 0} M_{xy}^{(2)} = \frac{1}{(3+\nu)} \left[(1+\nu)\, \phi_2 - (1-3\nu)\, y \frac{\partial \phi_2}{\partial y} - (1-\nu) y^2 \frac{\partial^2 \phi_2}{\partial y^2} \right] , \tag{4.5,18}$$

$$\lim_{\epsilon \to 0} Q_x^{(2)} = - \frac{2}{(3+\nu)c} \left[2 \frac{\partial \phi_2}{\partial y} + y \frac{\partial^2 \phi_2}{\partial y^2} \right] , \tag{4.5,19}$$

$$\lim_{\epsilon \to 0} Q_y^{(2)} = \frac{2}{(3+\nu)c} \left[\frac{\partial \phi_2}{\partial x} + y \frac{\partial^2 \phi_2}{\partial x \partial y} \right] \tag{4.5,20}$$

where

$$\phi_2(x,y) = \frac{1}{\pi} \int_{-1}^{1} \frac{(1-\xi^2)^{1/2} j_0(\xi)}{(x-\xi)^2 + y^2}\, d\xi \quad . \tag{4.5,21}$$

b. Stress couples near the vertex $(x = 1,\ y = 0)$

$$M_x \sim -\frac{(1+\nu)}{(3+\nu)} j_0\ (1-)\ (2r)^{-1/2} \left(\frac{7}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{5\theta}{2}\right)$$

$$M_y \sim -\frac{(1+\nu)}{(3+\nu)} j_0\ (1-)\ (2r)^{-1/2} \left(\frac{1}{4}\sin\frac{\theta}{2} - \frac{1}{4}\sin\frac{5\theta}{2}\right)$$

$$M_{xy} \sim \frac{(1+\nu)}{(3+\nu)} j_0\ (1-)\ (2r)^{-1/2} \left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2}\right) \qquad (4.5,22)$$

as $r \to 0$. Here r, θ are local polar coordinates centered at $x = 1$, $y = 0$ as in section 4.2.

4.6 Results Based on Classical Theory for Case II

In section 4.3, we have already discussed the classical theory for bending of plates. For the present case, we shall find a solution $w_c^{(2)}(x,y)$ according to classical theory in the form (4.3,11) such that it satisfies the boundary conditions (4.3,13ab)

$$\lim_{\substack{|y| \to 0 \\ |x| < 1}} M_{yc}^{(2)} = 0\ , \qquad (4.6,1a)$$

$$\lim_{\substack{|y| \to 0 \\ |x| < 1}} \left[Q_{yc}^{(2)} + \frac{1}{c}\frac{\partial M_{xyc}^{(2)}}{\partial x} \right] = -f_{30}(x) - \frac{1}{c}\frac{d}{dx} f_{20}(x) \qquad (4.6,1b)$$

where $M_{yc}^{(2)}$, $Q_{yc}^{(2)}$, $M_{xyc}^{(2)}$ are defined among (4.3,1) to (4.3,5) and $f_{20}(x)$, $f_{30}(x)$ are defined in (4.3,9).

Let us define a function $\Phi_c(x)$ by

$$\Phi_c(x) = \lim_{y \to 0 \pm} \frac{D}{c^2} \frac{\partial w_c^{(2)}(x,y)}{\partial x} \qquad (4.6,2)$$

which vanishes for all $|x| > 1$ since $w_c^{(2)}(x,0) \equiv 0$ for $|x| > 1$. All physical quantities can be expressed in terms of $\Phi_c(x)$ as follows:

$$\begin{pmatrix} M_{xc}^{(2)} \\ M_{yc}^{(2)} \\ M_{xyc}^{(2)} \end{pmatrix} = \frac{1-\nu}{\pi} \int_{-1}^{1} \Phi_c(\xi) \begin{pmatrix} m_{xc}^{(2)}(x-\xi, y) \\ m_{yc}^{(2)}(x-\xi, y) \\ m_{xyc}^{(2)}(x-\xi, y) \end{pmatrix} d\xi \quad (4.6,3)$$

and

$$\begin{pmatrix} Q_{cx}^{(2)} \\ Q_{yc}^{(2)} \end{pmatrix} = \frac{1-\nu}{\pi c} \int_{-1}^{1} \Phi_c(\xi) \begin{pmatrix} q_{xc}^{(2)}(x-\xi, y) \\ q_{yc}^{(2)}(x-\xi, y) \end{pmatrix} d\xi \quad (4.6,4)$$

where

$$m_{xc}^{(2)}(x,y) = (1-\nu)\frac{(3xy^3 - x^3y)}{\rho^6} + (1+\nu)\frac{2xy}{\rho^4}, \quad (4.6,5)$$

$$m_{yc}^{(2)}(x,y) = -(1-\nu)\frac{(3xy^3 - x^3y)}{\rho^6}, \quad (4.6,6)$$

$$m_{xyc}^{(2)}(x,y) = (1+\nu)\frac{(y^2 - x^2)}{2\rho^4} + (1-\nu)\frac{(y^4 - 3x^2y^2)}{\rho^6}, \quad (4.6,7)$$

$$q_{xc}^{(2)}(x,y) = \frac{2y}{\rho^4} - \frac{8x^2y}{\rho^6}, \quad (4.6,8)$$

$$q_{yc}^{(2)}(x,y) = -\frac{2x(3y^2 - x^2)}{\rho^6}. \quad (4.6,9)$$

To satisfy the boundary condition (4.6,16) we obtain an integral equation for $\Phi_c(x)$:

$$\frac{(3+\nu)}{2\pi}\int_{-1}^{1}\frac{\Phi_c(\xi)\,d\xi}{\xi - x} = \frac{1}{(1-\nu)}\left[\int^{x} f_2(\eta)\,d\eta + c\int^{x}\int^{\eta} f_3(\zeta)\,d\zeta\,d\eta\right] -$$

$$-C_3x - C_4, \qquad |x| < 1 \quad (4.6,10)$$

which is exactly of the form (4.5,10). Thus

$$\Phi_c(x) = - v_0(x) \quad , \qquad |x| \leq 1 \qquad (4.6,11)$$

where $v_0(x)$ is given by (4.5,11).

Using (4.6,11), we find from (4.6,3) and (4.6,4) that

(i) for points away from the crack, the stresses based on the classical theory are precisely the limiting values given in (4.5,16) to (4.5,20);

(ii) for points near the vertex (y = 0, x = +1), as $r \to 0$

$$M_{xc}^{(2)} \sim \frac{1}{(3+\nu)} j_0(1-)(2r)^{-1/2} \left(- \frac{9+7\nu}{4} \sin\frac{\theta}{2} + \frac{1-\nu}{4} \sin\frac{5\theta}{2}\right)$$

$$M_{yc}^{(2)} \sim \frac{1}{(3+\nu)} j_0(1-)(2r)^{-1/2} \left(\frac{1-\nu}{4} \sin\frac{\theta}{2} - \frac{1-\nu}{4} \sin\frac{5\theta}{2}\right) \qquad (4.6,12)$$

$$M_{xyc}^{(2)} \sim \frac{1}{(3+\nu)} j_0(1-)(2r)^{-1/2} \left(\frac{5+3\nu}{4} \cos\frac{\theta}{2} - \frac{1-\nu}{4} \cos\frac{5\theta}{2}\right)$$

and $(Q_{xc}^{(2)}, Q_{yc}^{(2)}) \sim 0(r^{-3/2})$.

4.7 Case III - Antisymmetric Solution to an Infinite Plate Containing a Rigid Line Inclusion

In this section, we shall study the antisymmetric solution of the inclusion problem which is represented by the singular integral equation (3.3,45).

We may rewrite (3.3,45) as follows:

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\tau(\xi)d\xi}{\xi-x} - \frac{1}{\pi}\int_{-1}^{1} L(x,\xi)\tau(\xi)d\xi = \ell_{2r}(x), \quad |x| < 1 \qquad (4.7,1)$$

where

$$L(x,\xi) = \frac{1}{(3-\nu)} \quad \frac{2+4\epsilon^2\left[\frac{2}{(\xi-x)^2} + \frac{3}{\epsilon^2} K_2\left(\frac{|\xi-x|}{\epsilon}\right) - \frac{|\xi-x|}{\epsilon^3} K_3\left(\frac{|\xi-x|}{\epsilon}\right)\right.}{\xi - x} \tag{4.7,2}$$

and

$$\ell_{2r}(x) = \frac{2(1-\nu)}{(3-\nu)} \cdot \frac{D}{c} f_{2r}(x) \; . \tag{4.7,3}$$

From (3.3,41), the solution of (4.7,1) is necessarily sought in the class of functions which are bounded and Hölder continuous for all x in the closed interval [-1, 1]. Thus, (4.7,1) can be transformed to a Fredholm type integral equation as follows:

$$\tau(x) + \int_{-1}^{1} N(x,\xi)\, \tau(\xi) d\xi = G_{2r}(x) \quad , \qquad |x| < 1 \tag{4.7,4}$$

where

$$N(x,\xi) = \frac{(1-x^2)^{1/2}}{\pi} \int_{-1}^{1} \frac{L(t,\xi)}{(1-t^2)^{1/2}(t-x)} \, dt \tag{4.7,5}$$

and

$$G_{2r}(x) = - \frac{(1-x^2)^{1/2}}{\pi} \int_{-1}^{1} \frac{\ell_{2r}(t)}{(1-t^2)^{1/2}(t-x)} \tag{4.7,6}$$

provided that the additional condition

$$\int_{-1}^{1} \frac{\ell_{2r}(t) dt}{(1-t^2)^{1/2}} = - \frac{1}{\pi} \int_{-1}^{1} \tau(\xi) d\xi \int_{-1}^{1} \frac{L(t,\xi) dt}{(1-t^2)^{1/2}} \tag{4.7,7}$$

is fulfilled.

Further, in order that $\frac{d\tau(x)}{dx} = t(x)$ exist for all x in (-1,1), we require that

$$\frac{d\ell_{2r}(x)}{dx} \quad \left(\text{that is, } \frac{2(1-\nu)D}{(3-\nu)c} \cdot \frac{df_{2r}(x)}{dx}\right)$$

exist and be Hölder continuous for all x in (-1,1) and near the ends be not worse than $O\left(\frac{1}{(1-x^2)^{1/2-\delta}}\right)$ for some $\delta > 0$.

4.8 Thin Plate Solution for Case III

Equation (4.7,1) suggests that we assume

$$\tau(x,\epsilon) = \tau_0(x) + o(1) \qquad \text{as } \epsilon \to 0 \qquad (4.8,1)$$

uniformly in $|x| \leqslant 1$ if

$$f_{2r}(x,\epsilon) = f_0(x) + o(1) \qquad \text{as } \epsilon \to 0 \qquad (4.8,2)$$

uniformly in $|x| \leqslant 1$. Then, the integral equation for $\tau_0(x)$ reads as

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\tau_0(\xi)d\xi}{\xi - x} = \frac{3-\nu}{1-\nu}\,\ell_0(x) \qquad |x| < 1 \qquad (4.8,3)$$

where

$$\ell_0(x) = \frac{2(1-\nu)}{(3-\nu)}\,\frac{D}{c}\,f_0(x)\,. \qquad (4.8,4)$$

The solution of (4.8,3) can be easily found from (4.7,4) as follows:

$$\tau_0(x) = \frac{(3-\nu)}{(1-\nu)}\,G_0(x)\,, \qquad |x| \leqslant 1 \qquad (4.8,5)$$

where

$$G_0(x) = -\frac{(1-x^2)^{1/2}}{\pi}\int_{-1}^{1}\frac{\ell_0(t)dt}{(1-t^2)^{1/2}(t-x)} . \tag{4.8,6}$$

Differentiating (4.8,5), we obtain

$$t_0(x) \equiv \frac{d\tau_0(x)}{dx} = -2\frac{D}{c}(1-x^2)^{-1/2}k_0(x) \tag{4.8,7}$$

where

$$k_0(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{(1-t^2)^{1/2}f_0(t)dt}{t-x} \tag{4.8,8}$$

is apparently a Hölder continuous function for all x in the closed interval [-1, 1] .

Now, we examine the state of stresses in different regions of the plate by using the approximate solution (4.8,7) and formulas (3.3,31) and (3.3,32). For points away from the inclusion, near the vertex and near the inclusion, we use the same geometrical descriptions as we did in Case I and list the results as follows.

a. <u>Stresses away from the inclusion:</u> By defining

$$\phi_3(x,y) = \frac{1}{\pi}\int_{-1}^{1}\frac{k_0(\xi)\,d\xi}{(1-\xi^2)^{1/2}[(x-\xi)^2+y^2]}$$

$$\phi_4(x,y) = \frac{1}{\pi}\int_{-1}^{1}\frac{k_0(\xi)\cdot(x-\xi)d\xi}{(1-\xi^2)^{1/2}[(x-\xi)^2+y^2]} \tag{4.8,9}$$

(4.8,10)

where $k_0(x)$ is given by (4.8,8), we find that at points away from the inclusion:

$$\lim_{\epsilon\to 0} M_x^{(2)} = \frac{D}{c}[(1-3\nu)y\phi_3 + (1-\nu)y^2\frac{\partial\phi_3}{\partial y}] , \tag{4.8,11}$$

$$\lim_{\epsilon \to 0} M_y^{(2)} = -\frac{D}{c}\left[(3-\nu)\, y\phi_3 + (1-\nu)\, y^2 \frac{\partial \phi_3}{\partial y}\right] \qquad (4.8,12)$$

$$\lim_{\epsilon \to 0} M_{xy}^{(2)} = -(1-\nu)\frac{D}{c}\left[\phi_4 + y^2 \frac{\partial \phi_3}{\partial x}\right] \qquad (4.8,13)$$

$$\lim_{\epsilon \to 0} Q_x^{(2)} = -\frac{2D}{c^2}\frac{\partial \phi_4}{\partial y} \qquad (4.8,14)$$

$$\lim_{\epsilon \to 0} Q_y^{(2)} = -\frac{D}{c^2}\left[\phi_3 + y\frac{\partial \phi_3}{\partial y}\right] \ . \qquad (4.8,15)$$

b. Stresses near the vertex: We find that as $r \to 0$,

$$M_x^{(2)} \sim \frac{D}{c} k_0 (1-)(2r)^{-1/2}\left(\frac{1-7\nu}{4}\sin\frac{\theta}{2} - \frac{1+\nu}{4}\sin\frac{5\theta}{2}\right)$$

$$M_y^{(2)} \sim -\frac{D}{c} k_0 (1-)(2r)^{-1/2}\left(\frac{9+\nu}{4}\sin\frac{\theta}{2} - \frac{1+\nu}{4}\sin\frac{5\theta}{2}\right) \qquad (4.8,16)$$

$$M_{xy}^{(0)} \sim -\frac{D}{c} k_0 (1-)(2r)^{-1/2}\left(\frac{5-3\nu}{4}\cos\frac{\theta}{2} - \frac{1+\nu}{4}\cos\frac{5\theta}{2}\right)$$

where $x = 1 + r\cos\theta$, $y = r\sin\theta$.

Furthermore, the shear stress resultants $Q_x^{(2)}$, $Q_y^{(2)}$ remain finite as $r \to 0$.

c. Stresses near the inclusion but away from the vertices: As $\epsilon \to 0$, for fixed $y/\epsilon > 0$ and for fixed x in $(-1, 1)$, we find

$$M_x^{(2)} \sim \frac{2D}{c}\, \nu\, k_0(x)\,(1-x^2)^{-1/2} \ ,$$
$$M_y^{(2)} \sim -\frac{2D}{c} k_0(x)\,(1-x^2)^{-1/2} \ ,$$
$$M_{xy}^{(2)} \sim -(1-\nu)\frac{D}{c} f_0'(x) \ , \qquad (4.8,17)$$
$$Q_x^{(2)} \sim -\frac{2D}{c^2}(1-e^{-y/\epsilon})\frac{d}{dx}\left[k_0(x)(1-x^2)^{-1/2}\right] \ ,$$
$$Q_y^{(2)} \sim \frac{2D}{c^2} f_0''(x) \ .$$

4.9 Results Based on Classical Theory for Case III

According to the classical theory of bending of plates, the stress couples and shear resultants for the case of a rigid inclusion can still be expressed in terms of w_c as in (4.3,1) to (4.3,5). w_c satisfies the same biharmonic equation (4.3,6) in $\mathcal{D}$ and hence the most general solution for w_c would have the form (4.3,10) and (4.3,11) such that the condition of vanishing of w_c and all its derivatives at infinity is satisfied.

Along the boundary $|x| < 1$, $y = 0$, we require

$$\frac{dw_c}{dy} = - \lim_{\epsilon \to 0} f_{2r}(x) , \tag{4.9,1}$$

$$w_c = - \lim_{\epsilon \to 0} f_{3r}(x) \tag{4.9,2}$$

where f_{2r}, f_{3r} are given in (3.1,24).

If the antisymmetric part $w_c^{(2)}$ alone is considered, then from (4.9,1) and (4.9,2) we find

$$\lim_{\substack{|y| \to 0 \\ |x| < 1}} \frac{dw_c^{(2)}}{dy} = - f_0(x) , \tag{4.9,3a}$$

$$\lim_{\substack{|y| \to 0 \\ |x| < 1}} w_c^{(2)} = 0 \tag{4.9,3b}$$

where

$$f_0(x) = \lim_{\epsilon \to 0} f_{2r}(x) . \tag{4.9,4}$$

We may note here that (4.9,4) is in fact the same as (4.8,2).

Let us define

$$t_c(x) = \lim_{\substack{|y| \to 0 \\ |x| < 1}} M_{yc}^{(2)}(x,y) \tag{4.9,5}$$

which vanishes for $|x| > 1$, $y = 0$ on account of continuity properties. All physical quantities can be expressed in terms of $t_c(x)$ as follows:

$$\begin{pmatrix} M_{xc}^{(2)} \\ M_{yc}^{(2)} \\ M_{xyc}^{(2)} \end{pmatrix} = \frac{1}{\pi}\int_{-1}^{1} t_c(\xi) \begin{pmatrix} m_{xc}^{(2)}(x-\xi,y) \\ m_{yc}^{(2)}(x-\xi,y) \\ m_{xyc}^{(2)}(x-\xi,y) \end{pmatrix} d\xi \tag{4.9,6}$$

$$\begin{pmatrix} Q_{xc}^{(2)} \\ Q_{yc}^{(2)} \end{pmatrix} = \frac{1}{\pi c}\int_{-1}^{1} t_c(\xi) \begin{pmatrix} q_{xc}^{(2)}(x-\xi,y) \\ q_{yc}^{(2)}(x-\xi,y) \end{pmatrix} d\xi \tag{4.9,7}$$

where

$$\begin{aligned} m_{xc}^{(2)}(x,y) &= \nu \frac{y}{\rho^2} + \frac{(1-\nu)y(y^2-x^2)}{2\rho^4} , \\ m_{yc}^{(2)}(x,y) &= \frac{y}{\rho^2} - \frac{(1-\nu)y(y^2-x^2)}{2\rho^4} , \\ m_{xyc}^{(2)}(x,y) &= -(1-\nu)\frac{xy^2}{\rho^4} + \frac{(1-\nu)x}{2\rho^2} , \end{aligned} \tag{4.9,8}$$

$$\begin{aligned} q_{xc}^{(2)}(x,y) &= -\frac{2xy}{\rho^4} , \\ q_{yc}^{(2)}(x,y) &= \frac{x^2-y^2}{\rho^4} . \end{aligned} \tag{4.9,9}$$

Again the subscript c refers to the classical theory of bending.

Upon satisfying (4.9,3a), it follows that the integral equation for

$$\tau_c(x) = \int_{-1}^{x} t_c(\xi)d\xi \qquad (4.9,10)$$

is identical to (4.8,3). Hence, $t_c(x)$ will have the form (4.8,7).

Stress couples and resultants in the plate can be then computed according to (4.9,6) and (4.9,7). For points away from the inclusion we find that these values are exactly the limiting values as $\epsilon \to 0$ (away from the inclusion) based on Reissner's theory listed in (4.8,11) to (4.8,15). For points near the end x=+1, y = 0, we find that

$$M_{xc}^{(2)} \sim -\frac{D}{c}k_0(1-)\,(2r)^{-1/2}\left(\frac{1+7\nu}{4}\sin\frac{\theta}{2} - \frac{1-\nu}{4}\sin\frac{5\theta}{2}\right),$$

$$M_{yc}^{(2)} \sim -\frac{D}{c}k_0\,(1-)\,(2r)^{-1/2}\left(\frac{7+\nu}{4}\sin\frac{\theta}{2} + \frac{1-\nu}{4}\sin\frac{5\theta}{2}\right), \qquad (4.9,11)$$

$$M_{xyc}^{(2)} \sim -\frac{D}{c}k_0(1-)\,(2r)^{-1/2}\left(\frac{5-5\nu}{4}\cos\frac{\theta}{2} + \frac{1-\nu}{4}\cos\frac{5\theta}{2}\right),$$

$$(Q_{xc}^{(2)}, Q_{yc}^{(2)}) \sim O(r^{-3/2})$$

as $r \to 0$. For points near the inclusion but away from the vertices, we find that

$$M_{xc}^{(2)} \sim \frac{2D}{c}\nu k_0(x)(1-x^2)^{-1/2}, \qquad M_{yc}^{(2)} \sim -\frac{2D}{c}k_0(x)(1-x^2)^{-1/2},$$

$$M_{xyc}^{(2)} \sim -(1-\nu)\frac{D}{c}f_0'(x),$$

$$Q_{xc}^{(2)} \sim -\frac{2D}{c^2}\frac{d}{dx}\left[k_0(x)\,(1-x^2)^{-1/2}\right], \quad Q_{yc}^{(2)} \sim \frac{2D}{c^2}f_0''(x) \qquad (4.9,12)$$

as $y \to 0$ for fixed x in (-1, 1).

4.10 Summary of the Results and Discussion

In the three cases we have treated so far the stress fields away from the crack or from the rigid line inclusion are the same for Reissner's theory and the classical theory for thin plates. However, significant differences occur near the vertices of the line segment $y = 0$, $|x| \leq 1$ and near the line segment but away from the vertices. We shall write down again some of the results obtained in previous sections in order to give a discussion.

Let us examine first the stress distribution near the vertex $y = 0$, $x = +1$. For Case I, the case corresponding to symmetric deflection of an infinite plate containing a crack, we have obtained the following.

(i) Results based on the Reissner theory for small ϵ

$$\begin{pmatrix} M_x^{(1)} \\ M_y^{(1)} \\ M_{xy}^{(1)} \end{pmatrix} \sim \frac{(1+\nu)}{(3+\nu)} h_0(1-) (2r)^{-1/2} \begin{pmatrix} \frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2} \\ \frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{5\theta}{2} \\ -\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{5\theta}{2} \end{pmatrix} \quad (4.10,1)$$

$$(Q_x^{(1)}, Q_y^{(1)}) \sim O(1)$$

as $r \to 0$ for $|\theta| < \pi$ with $r = (x-1)^2+y^2$, $\theta = \arctan\frac{y}{x-1}$.

(ii) Results based on the classical theory

$$\begin{pmatrix} M_{xc}^{(1)} \\ M_{yc}^{(1)} \\ M_{xyc}^{(1)} \end{pmatrix} \sim \frac{1}{3+\nu} h_0(1-) (2r)^{-1/2} \begin{pmatrix} -\frac{3(1-\nu)}{4}\cos\frac{\theta}{2} - \frac{1-\nu}{4}\cos\frac{5\theta}{2} \\ \frac{11+5\nu}{4}\cos\frac{\theta}{2} + \frac{1-\nu}{4}\cos\frac{5\theta}{2} \\ -\frac{7+\nu}{4}\sin\frac{\theta}{2} - \frac{1-\nu}{4}\sin\frac{5\theta}{2} \end{pmatrix} . \quad (4.10,2)$$

$$(Q_{xc}^{(1)}, Q_{yc}^{(1)}) \sim O(r^{-3/2})$$

as $r \to 0$ for $|\theta| < \pi$.

Comparing (4.10,1) with (4.10,2) shows that the angular distribution of the stresses based on the two theories is different. Particularly, when $\theta = 0$, we have from (4.10,1) $M_x^{(1)}/M_y^{(1)} \sim 1$ as $r \to 0$, while, according to the classical theory, we have $M_{xc}^{(1)}/M_{yc}^{(1)} \sim -(1-\nu)/(3+\nu)$. Hence, the Reissner theory predicts that along the prolongation of the crack near the vertex, the state of stress is one of uniform hydrostatic tension or compression, while in the classical theory $M_{xc}^{(1)}$ and $M_{yc}^{(1)}$ have opposite sign and different magnitude. Moreover, the angular distribution as shown in (4.10,1) is identical to the corresponding stretching problem reported in [8] according to the classical theory. Thus, if the Reissner theory is expected to be more accurate near edges, we would state that the angular distribution near a vertex of a crack is the same no matter the plate is under the action of stretching or of bending.

The shear force resultants in (4.10,2) become infinite like $r^{-3/2}$ as $r \to 0$. Thus, in order to maintain a finite amount of energy in the neighbourhood of the vertex, the classical theory would have to have the transverse shear modulus $G = \infty$. This is certainly not true for an isotropic elastic solid. In (4.10,1) Q_x and Q_y remain finite as $r \to 0$; thus the above defect will not occur in the refined theory.

For Case II, the case corresponding to antisymmetric deflection of an infinite plate containing a crack, we have near $y = 0$, $x = +1$:

(i) Results based on the Reissner theory for small ϵ

$$\begin{pmatrix} M_x^{(2)} \\ M_y^{(2)} \\ M_{xy}^{(2)} \end{pmatrix} \sim \frac{1+\nu}{3+\nu} j_0(1-)(2r)^{-1/2} \begin{pmatrix} -\frac{7}{4}\sin\frac{\theta}{2} - \frac{1}{4}\sin\frac{5\theta}{2} \\ -\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{5\theta}{2} \\ \frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2} \end{pmatrix} \qquad (4.10,3)$$

as $r \to 0$, for fixed θ, $|\theta| < \pi$.

(ii) Results based on the classical theory

$$\begin{pmatrix} M_{xc}^{(2)} \\ M_{yc}^{(2)} \\ M_{xyc}^{(2)} \end{pmatrix} \sim \frac{1}{3+\nu} j_0(l-)(2r)^{-1/2} \begin{pmatrix} -\frac{9+7\nu}{4} \sin\frac{\theta}{2} + \frac{1-\nu}{4} \sin\frac{5\theta}{2} \\ \frac{1-\nu}{4} \sin\frac{\theta}{2} - \frac{1-\nu}{4} \sin\frac{5\theta}{2} \\ \frac{5+3\nu}{4} \cos\frac{\theta}{2} - \frac{1-\nu}{4} \cos\frac{5\theta}{2} \end{pmatrix} \tag{4.10,4}$$

$$(Q_{xc}^{(2)}, Q_{yc}^{(2)}) \sim O(r^{-3/2})$$

as $r \to 0$, $|\theta| < \pi$.

Formulas (4.10,3) and (4.10,4) show again that the angular distributions based on the two theories are different. Moreover, there is a significant difference in the behaviors of the maximum shear stress computed according to the two theories. It is found that the maximum shear stress according to the classical theory possesses a relative minumum at $\theta = 0$ near the vertex, while, according to the Reissner theory, the maximum shear stress possesses a relative maximum there. Hence, the failure due to tearing of a plate containing a crack would be expected to propagate along the prolongation of the crack, as far as this factor is concerned.

The transverse shear force resultants in this case according to the classical theory again behaves like $O(r^{-3/2})$ as $r \to 0$. In the Reissner theory, we can show that the shear force resultants $Q_x^{(2)}$, $Q_y^{(2)}$ become infinite like $r^{-1/2}$ as $r \to 0$ if the function $\tilde{\psi}(x, \epsilon)$ in (4.5,6) is considered. However, $\tilde{\psi}(x, \epsilon)$ is assumed to be of small order effect when $\epsilon \ll 1$ and the contribution to strain energy due to a stress singularity of $O(r^{-1/2})$ is finite in general.

From (4.8,16) and (4.9,11), it can be easily seen that the stress distribution near the vertex $y = 0$, $x = +1$ is also different based on the different theories for Case III.

Next, we shall examine the stress field near the segment $y = 0$, $|x| \leq 1$ but away from the vertices.

For Case I, we have as $\epsilon \to 0$ for fixed $y/\epsilon > 0$ and fixed x in $(-1,1)$

$$M_x^{(1)} \sim \frac{1-\nu}{3+\nu} f_{10}(x) \;, \qquad M_y^{(1)} \sim - f_{10}(x) \;,$$

$$M_{xy}^{(1)} \sim \frac{2}{3+\nu} (1-e^{-y/\epsilon}) \frac{1}{(1-x^2)^{1/2}} \cdot \frac{1}{\pi} \int_{-1}^{1} \frac{(1-t^2)^{1/2} f_{10}(t) dt}{t - x} \;,$$

$$Q_x^{(1)} \sim \frac{2}{(3+\nu)\epsilon c} e^{-y/\epsilon} \frac{1}{\pi (1-x^2)^{1/2}} \int_{-1}^{1} \frac{(1-t^2)^{1/2} f_{10}(t) dt}{t - x} \;,$$

$$Q_y^{(1)} \sim \frac{2}{(3+\nu) c} (1-e^{-y/\epsilon}) \frac{d}{dx} \left[\frac{1}{\pi (1-x^2)^{1/2}} \int_{-1}^{1} \frac{(1-t^2)^{1/2} f_{10}(t) dt}{t - x} \right] \qquad (4.10,5)$$

based on the Reissner theory. And, we have for $y \to 0$, x in $(-1,1)$:

$$M_{xc}^{(1)} \sim \frac{1-\nu}{3+\nu} f_{10}(x) \;, \qquad M_{yc}^{(1)} \sim - f_{10}(x) \;,$$

$$M_{xyc}^{(1)} \sim \frac{2}{(3+\nu)\pi (1-x^2)^{1/2}} \int_{-1}^{1} \frac{(1-t^2)^{1/2} f_{10}(t) dt}{t-x} \;,$$

$$Q_{xc}^{(1)} \sim -\frac{2y}{(3+\nu)c} \frac{d^2}{dx^2} \left[\frac{1}{\pi (1-x^2)^{1/2}} \int_{-1}^{1} \frac{(1-t^2)^{1/2} f_{10}(t) dt}{t - x} \right] ,$$

$$Q_{yc}^{(1)} \sim -\frac{2}{(3+\nu)c} \frac{d}{dx} \left[\frac{1}{\pi (1-x^2)^{1/2}} \int_{-1}^{1} \frac{(1-t^2)^{1/2} f_{10}(t) dt}{t - x} \right]$$

(4.10,6)

based on the classical theory.

As shown in (4.10,5), the boundary conditions (3.2,26), (3.2,28) and (3.2,30) are indeed satisfied, while (4.10,6) shows that these boundary conditions are satisfied only in an approximate way (see section 4.6). A comparison of (4.10,5) and (4.10,6) demonstrates the presence of a "boundary layer" effect (neglected by the classical theory) in the values of $Q_x^{(1)}$, $Q_y^{(1)}$ and $M_{xy}^{(1)}$ near the crack for thin plates. This effect is not present in the values of $M_x^{(1)}$ and $M_y^{(1)}$. Also it may be observed that while according to the classical theory $Q_{xc}^{(1)} = 0$ along the crack, this is not the case in the Reissner theory. Moreover, the stresses associated $Q_x^{(1)}$ and M's are of about the same magnitude along the crack while according to classical theory the transverse shear stress in thin plates is assumed to be of small order in comparison with the flexural stresses.

Finally, from (4.8,17) and (4.9,12) it can be observed that the boundary layer affects only the value of $Q_x^{(2)}$. We have in the Reissner theory

$$Q_x^{(2)} \sim -\frac{2D}{c^2}(1-e^{-y/\epsilon})\frac{d}{dx}\left[k_0(x)(1-x^2)^{1/2}\right]$$

as $\epsilon \to 0$ for fixed $y/\epsilon > 0$ and for fixed x in (-1,1), while in the classical theory we have

$$Q_{xc}^{(2)} \sim -\frac{2D}{c^2}\frac{d}{dx}\left[k_0(x)\,(1-x^2)^{1/2}\right]$$

as $y \to 0$ for fixed x in (-1,1).

V. APPROXIMATE SOLUTIONS BY A VARIATIONAL METHOD

5.1 A Variational Theorem

In Part IV we have reduced our problems to either a single integral equation or a system of integral equations. In the same part we obtained some approximate results for thin plates through a perturbation method. However, for ϵ of moderately small values the perturbation scheme breaks down since the dependence on ϵ of the higher order terms is not clear. We shall establish certain variational principles in order to obtain furtner approximate solutions to our problems.

In the sequel, the theorem of minimum potential energy from classical linear elasticity shall be used. However, instead of using as admissible displacement and stress states those which satisfy certain boundary conditions, we shall use those which satisfy the equations of equilibrium (2.2,3) to (2.2,5) as well as the stress strain relations (2.2,24) to (2.2,28) in the interior of the plate. Thus the Euler equations obtained according to the variational procedure are the boundary conditions of the problem.

Before we derive the variational method which will be applicable to our problem, we shall compute the strain energy contained in the plate in terms of the moments, the shear force resultants and the generalized displacements which were defined in section 2.2.

Since the plate is assumed to be isotropic and the stress strain relations obey Hooke's law, the strain energy[§] contained

§ See Love [1] .

in the plate is

$$\Pi_s = \frac{1}{2}\iiint_{\mathcal{S}} [\frac{1}{E}(\sigma_x{}^2+\sigma_y{}^2+\sigma_z{}^2) - \frac{2\nu}{E}(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x) + + \frac{1}{G}(\sigma_{xy}{}^2+\sigma_{yz}{}^2+\sigma_{zx}{}^2)]\, dXdYdZ \qquad (5.1,1)$$

where σ_x etc. are defined in section 2.1 and $\mathcal{S}$ is the set consisting of all points interior of the plate.

Let us now consider the plate whose geometry was described as in section 3.1 and whose upper surface and lower surface are free from external tractions. We shall assume that the stress distributions across the plate thickness are approximately (2.2,6), (2.2,7) and (2.2,8). Also we assume that the two-dimensional stress strain relations (2.2,24) to (2.2,28) hold everywhere in $\mathcal{D}$. Under the above assumptions the integration with respect to Z in (5.1,1) can be carried out and (5.1,1) may be expressed in terms of β_x, β_y and w_t as

$$\Pi_s = \frac{D}{4}\iint_{\mathcal{D}} \{(1+\nu)(\frac{\partial\beta_x}{\partial x}+\frac{\partial\beta_y}{\partial y})^2 + (1-\nu)[(\frac{\partial\beta_x}{\partial x}-\frac{\partial\beta_y}{\partial y})^2 + + (\frac{\partial\beta_x}{\partial y}+\frac{\partial\beta_y}{\partial x})^2] + \qquad (5.1,2)$$

$$+ \frac{(1-\nu)}{\epsilon^2}[(\beta_x+\frac{\partial(w_t/c)}{\partial x})^2 + (\beta_y+\frac{\partial(w_t/c)}{\partial y})^2]\}\, dxdy$$

where $D = \dfrac{E}{(1-\nu^2)c_{11}}$, $\epsilon^2 = \frac{1}{c}(\frac{c_{22}}{c_{11}})^{1/2}$ and x,y are dimensionless coordinates as before.

Integrating (5.1, 2) by parts, we obtain

$$\Pi_s = \frac{c}{2} \oint_{C_0 \cup C_\infty} (M_n \beta_n + M_{ns} \beta_s + Q_n w_t)\, ds -$$

$$- \frac{c}{2} \iint \Big[\Big(\frac{\partial M_x}{\partial_x} + \frac{\partial M_{xy}}{\partial y} - cQ_x\Big) \beta_x + \Big(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - cQ_y\Big)\beta_y$$

$$+ \Big(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\Big) w_t \Big]\, dxdy \qquad (5.1, 3)$$

where C_∞ denotes the boundary at infinity and C_0 is the line segment $y = 0\pm$, $|x| \leqslant 1$. The second term in the right side of (5.1, 3) vanishes if the M's and Q's (computed from the set $\{\beta_x, \beta_y, w_t\}$ according to (2.2, 24) to (2.2, 28)) satisfy the equations of equilibrium (2.2, 6) to (2.2, 8).

Hereafter, we shall only consider the reduced problems, that is, the problems associated with the boundary conditions (3.1, 23) and (3.1, 24), where the load at infinity has been transferred to the segment $y = 0$, $|x| < 1$, since they are of principal interest. Again, we shall assume that the total energy contained in the plate is finite. This assumption can be verified if (i) we require that all physical quantities possess Fourier transforms so that the line integral along C_∞ in (5.1, 3) vanishes, and (ii) we require that that components of the vector $\vec{f}$ defined as in (3.1, 23) (case of a crack) or as in (3.1, 24) (case of a rigid inclusion) satisfy the conditions made in Part IV so that the existence of solutions is assured and thus the energy contained in the neighbourhood of the vertices of the crack or rigid inclusion is finite.

Under the above restrictions, (5.1,3) becomes

$$\Pi_s = -\frac{c}{2}\int_{-1}^{1}[\hat{M}_y\hat{\beta}_y + \hat{M}_{xy}\hat{\beta}_x + \hat{Q}_y\hat{w}_t]_{y=0+}\,dx + $$
$$+\frac{c}{2}\int_{-1}^{1}[\hat{M}_y\hat{\beta}_y + \hat{M}_{xy}\hat{\beta}_x + \hat{Q}_y\hat{w}_t]_{y=0-}\,dx . \qquad (5.1,4)$$

where the hat sign denotes boundary values.

We digress for a moment to remark that the uniqueness of the solutions can be easily established with the aid of (5.1,4). We shall illustrate for the case of a crack. For the case of a crack, the boundary condition (3.2,25) shall be used and (5.1,4) can then be written as

$$\Pi_s = +\frac{c}{2}\int_{-1}^{1}[f_{1c}\hat{\beta}_y + f_{2c}\hat{\beta}_x + f_{3c}\hat{w}_t]_{y=0+}\,dx -$$
$$-\frac{c}{2}\int_{-1}^{1}[f_{1c}\hat{\beta}_y + f_{2c}\hat{\beta}_x + f_{3c}\hat{w}_t]_{y=0-}\,dx \qquad (5.1,5)$$

where f_{1c}, f_{2c}, f_{3c} are given by (3.1,23).

If both $\{\beta_x, \beta_y, w_t\}$ and $\{\tilde{\beta}_x, \tilde{\beta}_y, \tilde{w}_t\}$ satisfy the stress strain relations (2.2,24) to (2.2,28), the equations of equilibrium (2.2,6) to (2.2,8) in $\mathfrak{D}$, satisfy the same boundary conditions (3.1,25) and (3.2,25), and possess finite total energy, then from (5.1,5) we have

$$\Pi_s((\beta_x - \tilde{\beta}_x), (\beta_y - \tilde{\beta}_y), (w_t - \tilde{w}_t)) = 0 . \qquad (5.1,6)$$

From the positive definite character of Π_s as shown in (5.1,2), (5.1,6) shows that the difference of these two sets is at most a rigid body displacement. However, the solution corresponding to a rigid body displacement is excluded since it does not possess a Fourier transform and thus the solution to the crack problem is unique.

Similarly, we can prove that the solution to the inclusion problem is also unique.

Now, we shall return to the variational principle. For the case of crack, we shall use the theorem of minimum potential energy as our guide to derive a variational method which is applicable to our problems. Let us define the potential energy as follows

$$\Pi(\beta_x, \beta_y, w_t) =$$

$$= \frac{D}{4}\iint_{\mathcal{D}} \Big\{ (1+\nu)\left(\frac{\partial\beta_x}{\partial x} + \frac{\partial\beta_y}{\partial y}\right)\overline{\left(\frac{\partial\beta_x}{\partial x} + \frac{\partial\beta_y}{\partial y}\right)} +$$

$$+ (1-\nu)\Big[\left(\frac{\partial\beta_x}{\partial x} - \frac{\partial\beta_y}{\partial y}\right)\overline{\left(\frac{\partial\beta_x}{\partial x} - \frac{\partial\beta_y}{\partial y}\right)} + \left(\frac{\partial\beta_x}{\partial y} + \frac{\partial\beta_y}{\partial x}\right)\overline{\left(\frac{\partial\beta_x}{\partial y} + \frac{\partial\beta_y}{\partial x}\right)}\Big] +$$

$$+ \frac{(1-\nu)}{\epsilon^2}\Big[\left(\beta_x + \frac{\partial(w_t/c)}{\partial x}\right)\overline{\left(\beta_x + \frac{\partial(w_t/c)}{\partial x}\right)} + \left(\beta_y + \frac{\partial(w_t/c)}{\partial y}\right)\overline{\left(\beta_y + \frac{\partial(w_t/c)}{\partial y}\right)}\Big]\Big\}$$

$$dx\,dy\; -$$

$$-c\int_{-1}^{1} \mathrm{Re}\,\{f_{1c}\,\overline{\beta}_y + f_{2c}\,\overline{\beta}_x + f_{3c}\,\overline{w}_t\}_{y=0+}\,dx\; +$$

$$+c\int_{-1}^{1} \mathrm{Re}\,\{f_{1c}\overline{\beta}_y + f_{2c}\,\overline{\beta}_x + f_{3c}\,\overline{w}_t\}_{y=0-}\,dx \qquad (5.1,7)$$

where the bar over a symbol denotes its complex conjugate and the symbol Re denotes the real part of a complex function. It is clear that the double integral in (5.1,7) is the strain energy contained in the plate and the line integrals are the work done by external forces along the segment $y = 0, |x| < 1$.

Apart from the solution state $\{\beta_x, \beta_y, w_t\}$, we consider a class of arbitrary displacement sets $\{\beta_x + \delta\beta_x, \beta_y + \delta\beta_y, w_t + \delta w_t\}$ subject to the conditions that their derivatives of all orders are continuous in $\mathfrak{D}$ and vanishing at infinity, and that they possess finite potential energy.

Using the above admissible displacement sets, we compute the first variation of (5.1,7).

$$\delta \Pi(\beta_x, \beta_y, w_t) =$$

$$= -\frac{c}{2}\iint_{\mathfrak{D}} \{(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - cQ_x)\delta\bar{\beta}_x + \overline{(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - cQ_x)}\, \delta\beta_y +$$

$$+ (\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - cQ_y)\, \delta\bar{\beta}_y + \overline{(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - cQ_y)}\, \delta\beta_y +$$

$$+ (\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y})\, \overline{\delta w_t} + \overline{(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y})}\, \delta w_t\}\, dxdy -$$

$$- c\int_{-1}^{1} \text{Re}\,\{(M_y + f_{1c})\,\delta\bar{\beta}_y + (M_{xy} + f_{2c})\,\delta\bar{\beta}_x + (Q_y + f_{3c})\,\delta\bar{w}_t\}_{y=0+}\, dx +$$

$$+ c\int_{-1}^{1} \text{Re}\{(M_y + f_{1c})\delta\bar{\beta}_y + (M_{xy} + f_{2c})\delta\bar{\beta}_x + (Q_y + f_{3c})\,\delta\bar{w}_t\}_{y=0-}\, dx$$

(5.1,8)

in which the M's and Q's computed according to the stress strain relations (2.2,24) to (2.2,28). We also find that the second variation of (5.1,7) is given by

$$\delta^2 \Pi(\beta_x, \beta_y, w_t) = \Pi_s(\delta\beta_x, \delta\beta_y, \delta w_t) \geq 0. \qquad (5.1,9)$$

Equation (5.1,8) shows that out of all admissible displacement sets $\{\beta_x, \beta_y, w_t\}$ the set which satisfies equations (2.2,6) to (2.2,8) and the boundary condition (3.2,25) makes Π an extremum and equation (5.1,9) shows that this extremum is in fact an absolute minimum. On the other hand, if we set $\delta\Pi = 0$ in the first place, then equations (2.2,6) to (2.2,8) and the boundary condition (3.2,25) must be satisfied since $\delta\beta_x$, $\delta\beta_y$ and δw_t are arbitrary in $\mathfrak{D}$ and along its boundaries. Hence, we conclude that equations (2.2,6) to (2.2,8) and the boundary condition (3.2,25) are necessary and sufficient conditions to minimize Π.

If we select the displacement sets $\{\beta_x, \beta_y, w_t\}$ from among the above admissible sets in such a way that they satisfy equations (2.2,6) to (2.2,8) through the stress-strain relations (2.2,24) to (2.2,28), then the potential energy (5.1,7) can be reduced to the following form:

$$\Pi(\beta_x, \beta_y, w_t) =$$

$$= -\frac{c}{2}\int_{-1}^{1} \mathrm{Re}\{(M_y + 2f_{1c})\,\overline{\beta}_y + (M_{xy} + 2f_{2c})\overline{\beta}_x + (Q_y + 2f_{3c})\,\overline{w}_t\}_{y=0+}\,dx\ +$$

$$+\frac{c}{2}\int_{-1}^{1} \mathrm{Re}\{(M_y + 2f_{1c})\,\overline{\beta}_y + (M_{xy} + 2f_{2c})\,\overline{\beta}_x + (Q_y + 2f_{3c})\,\overline{w}_t\}_{y=0-}\,dx. \qquad (5.1,10)$$

It has been assumed that all the physical quantities possess

Fourier transforms. Hence the general solutions of the differential equations as shown in (3.2,1) to (3.2,18) obtained in section 3.2 become the most suitable admissible sets since they satisfy the equations of equilibrium (2.2,6) to (2.2,8), the stress strain relations (2.2,24) to (2.2,28) and the vanishing condition at infinity. They do not, however, necessarily satisfy the boundary conditions along the segment $y = 0, |x| < 1$.

Using the symmetric and antisymmetric representations for stresses and displacements as shown in formulas (3.2,1) to (3.2,18), we may rewrite (5.1,10) as follows:

$$\Pi = \Pi^{(1)} + \Pi^{(2)} \tag{5.1,11}$$

where

$$\Pi^{(1)} = -c\int_{-1}^{1} \mathrm{Re}\{M_y^{(1)}\overline{\beta_y^{(1)}} + M_{xy}^{(1)}\overline{\beta_x^{(1)}} + Q_y^{(1)}\overline{w_t^{(1)}} + 2f_{1c}\overline{\beta_y^{(1)}}\}_{y=0}\,dx \tag{5.1,12}$$

and

$$\Pi^{(2)} = -c\int_{-1}^{1} \mathrm{Re}\{M_y^{(2)}\overline{\beta_y^{(2)}} + M_{xy}^{(2)}\overline{\beta_x^{(2)}} + Q_y^{(2)}\overline{w_t^{(2)}} + 2f_{2c}\overline{\beta_x^{(2)}} + $$

$$+ 2f_{3c}\overline{w_t^{(2)}}\}_{y=0}\,dx \quad . \tag{5.1,13}$$

Equation (5.1,12) and (5.1,13) are independent of each other. Hence, to minimize Π it is sufficient to minimize $\Pi^{(1)}$ and $\Pi^{(2)}$.

It is obvious that the energy expression (5.1,12) corresponds to the symmetric solution for the case of a crack. Owing to the fact that $M_{xy}^{(1)}$, $Q_y^{(1)}$ appear in (5.1,12), we may point out that the moments and shear force resultants based on the admissible sets $\{\beta_x, \beta_y, w_t\}$ are not required to satisfy the boundary conditions along the crack. It is this point which is different from the well known theorem of minimum potential energy.

Similarly, the energy expression (5.1,13) corresponds to the antisymmetric part for the case of a crack.

In our discussion so far, we have treated β_x β_y, w_t as the quantities to be varied in the variational principle. Since, in order to be admissible, a state of displacement and stress characterized by $\{\beta_x, \beta_y, w_t, M_x, M_y, M_{xy}, Q_x, Q_y\}$ must satisfy all of the field equations in the plate, it is possible to select any independent set of three of these quantities, and not necessarily just β_x, β_y, w_t, to be varied in the variational principle.

By considerations similar to the above, we can easily deduce an appropriate energy expression similar to (5.1,10) for the case of a rigid inclusion.

The negative of the work done through the prescribed displacements f_{1r}, f_{2r}, f_{3r} given as in (3.1,24) along C_0 is

$$-\frac{c}{2}\int_{-1}^{1} \mathrm{Re}\,\{2f_{1r}\,\overline{M_{xy}} + 2\,f_{2r}\,\overline{M_y} + 2f_{3r}\,\overline{Q_y}\}_{y=0+}\,dx +$$

$$+\frac{c}{2}\int_{-1}^{1} \mathrm{Re}\,\{2f_{1r}\,\overline{M_{xy}} + 2\,f_{2r}\,\overline{M_y} + 2f_{3r}\,\overline{Q_y}\}_{y=0-}\,dx\,. \qquad (5.1,14)$$

Therefore, the appropriate energy expression for the case of a rigid inclusion becomes

$$\mathfrak{E} =$$

$$= -\frac{c}{2}\int_{-1}^{1} \mathrm{Re}\{(\beta_x + 2f_{1r})\overline{M_{xy}} + (\beta_y + 2f_{2r})\overline{M_y} + (w_t + 2f_{3r})\overline{Q_y}\}_{y=0+} dx +$$

$$+ \frac{c}{2}\int_{-1}^{1} \mathrm{Re}\{(\beta_x + 2f_{1r})\overline{M_{xy}} + (\beta_y + 2f_{2r})\overline{M_y} + (w_t + 2f_{3r})\overline{Q_y}\}_{y=0-} dx \,. \quad (5.1,15)$$

Again, using the symmetric and antisymmetric representation for stresses and displacements as shown in formulas (3.2,1) to (3.2,18), we may write (5.1,15) as follows:

$$\mathfrak{E} = \mathfrak{E}^{(1)} + \mathfrak{E}^{(2)} \quad (5.1,16)$$

where

$$\mathfrak{E}^{(1)} = -c\int_{-1}^{1} \mathrm{Re}\{M_{xy}^{(1)}\overline{\beta_x^{(1)}} + M_y^{(1)}\overline{+\beta_y^{(1)}} + Q_y^{(1)}\overline{w_t^{(1)}} +$$

$$+ 2f_{1r}\overline{M_{xy}^{(1)}} + 2f_{3r}\overline{Q_y^{(1)}}\}_{y=0} dx \quad (5.1,17)$$

and

$$\mathfrak{E}^{(2)} = -c\int_{-1}^{1} \mathrm{Re}\{M_{xy}^{(2)}\overline{\beta_x^{(2)}} + M_y^{(2)}\overline{\beta_y^{(2)}} + Q_y^{(2)}\overline{w_t^{(2)}} + 2f_{2r}M_y^{(2)}\}_{y=0} dx \,.$$

$$(5.1,18)$$

Our variational principle may now be summarized as follows. Among the physical quantities $\beta_x, \beta_y, w_t, M_x, M_y, M_{xy}, Q_x$ and Q_y which are given by the formulas (3.2,1) to (3.2,18), we select three independent ones to form the admissible sets. Among all these sets the set for which

$$\delta \Pi = \delta \Pi^{(1)} + \delta \Pi^{(2)} = 0 \tag{5.1,19}$$

also satisfies the boundary condition (3.2,25) along the line $y = 0$, $|x| < 1$ solves the crack problem.

A similar statement may be made for the case of a rigid inclusion.

5.2 Approximate Solution for Case I

In order to seek an approximate solution for Case I (the case of a crack with symmetric deflection $w^{(1)}$), we use the symmetric parts (with index (1)) in formulas (3.2,1) to (3.2,18) as the set which is appropriate for varying the energy function $\Pi^{(1)}$, (5.1,12). It is found that the most suitable quantities to use for the admissible sets are $\frac{D}{c}\beta_y^{(1)}$, $M_{xy}^{(1)}$ and $cQ_y^{(1)}$, since they vanish at $y = 0, |x| > 1$, and since they appear naturally in $\Pi^{(1)}$. In what follows, we shall transform (5.1,12) into a Hermitian form in terms of these quantities and their Fourier transforms. Then, using a technique similar to that discussed by Noble [17], we can show that the dual integral equations (3.2,39) and (3.2,40) corresponding to Case I arise again from the variational principle.

Using matrix notation, we define

$$\begin{pmatrix} A(\alpha) \\ B(\alpha) \\ C(\alpha) \end{pmatrix} = \begin{pmatrix} 1 & -(1+\frac{4\epsilon^2\alpha^2}{1-\nu}) & -\frac{2\epsilon^2\alpha i}{(1-\nu)} \\ (1-\nu)\alpha i & -\alpha i(1-\nu+4\epsilon^2\alpha^2) & (1+2\alpha^2\epsilon^2) \\ 0 & -2\alpha^2 & -\alpha i \end{pmatrix} \begin{pmatrix} \alpha \quad Q_1(\alpha) \\ R_1(\alpha) \\ P_1(\alpha) \end{pmatrix} \tag{5.2,1}$$

where from (3.2,18), (3.2,16) and (3.2,13), $A(\alpha)$, $B(\alpha)$ and $C(\alpha)$ are seen to be the Fourier transforms of $\frac{D}{c} B_y^{(1)}$, $M_{xy}^{(1)}$ and $cQ_y^{(1)}$ at y = 0 respectively. From the Fourier inversion theorem, we have

$$\begin{pmatrix} \frac{D}{c}\beta_y^{(1)} \\ M_{xy}^{(1)} \\ cQ_y^{(1)} \end{pmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{pmatrix} A(\alpha) \\ B(\alpha) \\ C(\alpha) \end{pmatrix} e^{i\alpha x}\, d\alpha \; . \tag{5.2,2}$$

The left sides of (5.2,2) vanish for all $|x| > 1$ on account of the continuity properties of these functions in $\mathfrak{D}$. Hence, the left sides of (5.2,1) are in fact

$$\begin{pmatrix} A(\alpha) \\ B(\alpha) \\ C(\alpha) \end{pmatrix} = \int_{-1}^{1} \begin{pmatrix} \frac{D}{c}\beta_y^{(1)} \\ M_{xy}^{(1)} \\ cQ_y^{(1)} \end{pmatrix} e^{-i\alpha x}\, dx \; . \tag{5.2,3}$$

Solving (5.2,1) for $|\alpha|\, Q_1(\alpha)$, $R_1(\alpha)$ and $P_1(\alpha)$, we obtain

$$\begin{aligned} |\alpha|\, Q_1(\alpha) &= \frac{1}{2\alpha^2}\left[(1+\nu)\alpha^2 A(\alpha) - i\alpha B(\alpha) - \left(1 + \frac{4\epsilon^2\alpha^2}{1-\nu}\right) C(\alpha)\right] , \\ R_1(\alpha) &= \frac{1}{2\alpha^2}\left[-(1-\nu)\alpha^2 A(\alpha) - i\alpha B(\alpha) - C(\alpha)\right] , \\ P_1(\alpha) &= -(1-\nu) i\alpha A(\alpha) + B(\alpha) \; . \end{aligned} \tag{5.2,4}$$

Substituting (5.2,4) into $M_y^{(1)}$ of (3.2,8), $\frac{D}{c}\beta_x^{(1)}$ of (3.2,10) and $\frac{D}{c^2} w_t^{(1)}$ of (3.2,1), we obtain at y = 0

$$M_y^{(1)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{A(\alpha)(1-\nu) [\frac{3+\nu}{2} |\alpha| + 2\epsilon^2 \alpha^2 (|\alpha| - \sqrt{\alpha^2 + \frac{1}{\epsilon^2}})] +$$

$$+ iB(\alpha) [\frac{1+\nu}{2} \frac{|\alpha|}{\alpha} + 2\epsilon^2 \alpha (|\alpha| - \sqrt{\alpha^2 + 1/\epsilon^2})] +$$

$$+ C(\alpha) \frac{(1+\nu)}{2|\alpha|} \} \quad e^{i\alpha x} \, d\alpha , \qquad (5.2,5)$$

$$\frac{D}{c} \beta_x^{(1)} = - \frac{i}{2\pi} \int_{-\infty}^{\infty} \{A(\alpha)[\frac{1+\nu}{2} \frac{|\alpha|}{\alpha} + 2\epsilon^2 \alpha (|\alpha| - \sqrt{\alpha^2 + 1/\epsilon^2})] +$$

$$+ iB(\alpha) [- \frac{1}{2|\alpha|} + \frac{2\epsilon^2}{(1-\nu)} (|\alpha| - \sqrt{\alpha^2 + 1/\epsilon^2})] +$$

$$+ C(\alpha) [- \frac{1}{2\alpha|\alpha|}] \} \quad e^{i\alpha x} \, d\alpha \qquad (5.2,6)$$

and

$$\frac{D}{c^2} w_t^{(1)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{A(\alpha) \frac{(1+\nu)}{2|\alpha|} + iB(\alpha) (\frac{-1}{2\alpha|\alpha|}) -$$

$$- C(\alpha) \frac{1}{2|\alpha| \alpha^2} (1 + \frac{4\epsilon^2 \alpha^2}{1-\nu}) \} \; e^{i\alpha x} d\alpha . \qquad (5.2,7)$$

We substitute (5.2,5), (5.2,6) (5.2,7) into (5.1,12). Using (5.2,3) after interchanging the order of integrations, we obtain:

$$\Pi^{(1)} =$$

$$= \frac{c^2}{D 2\pi} \int_{-\infty}^{\infty} \{A(\alpha) \overline{A(\alpha)} (1-\nu) [\frac{3+\nu}{2} |\alpha| - 2\epsilon^2 \alpha^2 (\sqrt{\alpha^2 + 1/\epsilon^2}) - |\alpha|] +$$

$$+ B(\alpha) \overline{B(\alpha)} [\frac{1}{2|\alpha|} + \frac{2\epsilon^2}{(1-\nu)} (\sqrt{\alpha^2 + 1/\epsilon^2} - |\alpha|)] +$$

$$+ C(\alpha) \overline{C(\alpha)} \frac{1}{2\alpha^2 |\alpha|} (1 + 4\epsilon^2 \alpha^2 / 1-\nu) +$$

$$+ iB(\alpha)\,\overline{C(\alpha)}\,\frac{1}{2\alpha\,|\alpha|} - i\,\overline{B(\alpha)}\,C(\alpha)\,\frac{1}{2\alpha\,|\alpha|}\,\}\,d\alpha\, -$$

$$- c\int_{-1}^{1}[\,f_{1c}(\alpha)\,\overline{\beta_y^{(1)}(x,0)} + \overline{f_{1c}(x)}\,\beta_y^{(1)}(x,0)]\,dx\,. \qquad (5.2,8)$$

The first integral in (5.2,8) is apparently a Hermitian form in $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ and hence (5.2,8) is an appropriate form to which we shall apply the variational principle again.

Apart from the solution state $\{\frac{D}{c}\beta_y^{(1)}(x,0),\ M_{xy}^{(1)}(x,0),\ cQ_y^{(1)}(x,0)\}$ we consider a class of arbitrary functions

$$\{\frac{D}{c}\,\beta_y^{(1)}+\delta\,\frac{D}{c}\beta_y^{(1)},\ M_{xy}^{(1)}+\delta M_{xy}^{(1)},\ cQ_y^{(1)}+\delta cQ_y^{(1)}\}$$

and their Fourier transforms $\{A(\alpha)+\delta A(\alpha),\ B(\alpha)+\delta B(\alpha),\ C(\alpha)+\delta C(\alpha)\}$ computed according to (5.2,3). The first variation of (5.2,8) is

$$\delta\Pi^{(1)} =$$

$$= \frac{c^2}{D}\,\{\int_{-1}^{1}\frac{D}{c}\,\delta\overline{\beta_y^{(1)}}dx\{\frac{1}{2\pi}\int_{-\infty}^{\infty}A(\alpha)(1-\nu)[\frac{3+\nu}{2}|\alpha| - 2\epsilon^2\alpha^2(\sqrt{\alpha^2+1/\epsilon^2} - |\alpha|)]\,\cdot$$

$$\cdot\ e^{i\alpha x}\,d\alpha - f_{1c}(x)\}\ +$$

$$+\int_{-1}^{1}\frac{D}{c}\,\delta\beta_y^{(1)}dx\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{A(\alpha)}(1-\nu)[\frac{3+\nu}{2}|\alpha| - 2\epsilon^2\alpha^2(\sqrt{\alpha^2+1/\epsilon^2} - |\alpha|)]\,\cdot$$

$$\cdot\ e^{-i\alpha x}\,d\alpha - \overline{f_{1c}(x)}\}\ +$$

$$+\int_{-1}^{1}\delta\overline{M_{xy}^{(1)}}\,dx\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\{B(\alpha)\,[\frac{1}{2|\alpha|}+\frac{2\epsilon^2}{1-\nu}(\sqrt{\alpha^2+1/\epsilon^2} - |\alpha|)] - i\frac{C(\alpha)}{2\alpha|\alpha|}\}\,e^{i\alpha x}d\alpha +$$

$$+\int_{-1}^{1}\delta M_{xy}^{(1)}\,dx\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{B(\alpha)}\,[\frac{1}{2|\alpha|}+\frac{2\epsilon^2}{1-\nu}(\sqrt{\alpha^2+1/\epsilon^2}-|\alpha|)]+i\frac{\overline{C(\alpha)}}{2\,\alpha\,\alpha}e^{-i\alpha x}d\alpha\,+$$

$$+\int_{-1}^{1}c\,\delta\overline{Q_y^{(1)}}\,dx\,\frac{1}{2\pi}\int_{-\infty}^{\infty}[\,i\frac{B(\alpha)}{2\alpha|\alpha|}+C(\alpha)\frac{1}{2|\alpha|\alpha^2}(1+\frac{4\epsilon^2\alpha^2}{1-\nu})]\,e^{i\alpha x}\,d\alpha\,+$$

$$+\int_{-1}^{1}c\,\delta Q_y^{(1)}\,dx\,\frac{1}{2\pi}\int_{-\infty}^{\infty}[\,-i\frac{\overline{B(\alpha)}}{2\alpha|\alpha|}+\overline{C(\alpha)}\frac{1}{2|\alpha|\alpha^2}(1+\frac{4\epsilon^2\alpha^2}{1-\nu})]\;e^{-i\alpha x}d\alpha\quad.$$

(5.2,9)

Also, we find that the second variation

$$\delta^2\Pi^{(1)}\geqslant 0\;. \qquad (5.2,10)$$

When we set $\delta\Pi^{(1)}=0$, (5.2,9) yields three equations since $\delta B_y^{(1)}(x,0)$, $\delta M_{xy}^{(1)}(x,0)$ and $\delta Q_y^{(1)}(x,0)$ are arbitrary and independent of one another; they are

$$\frac{1-\nu}{2\pi}\int_{-\infty}^{\infty}A(\alpha)\,[\frac{3+\nu}{2}|\alpha|-2\epsilon^2\alpha^2(\sqrt{\alpha^2+1/\epsilon^2}-|\alpha|)]\,e^{i\alpha x}d\alpha=f_{1c}(x)\;,$$

$$|x|<1\;, \qquad (5.2,11)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{B(\alpha)[\frac{1}{2|\alpha|}+\frac{2\epsilon^2}{1-\nu}(\sqrt{\alpha^2+1/\epsilon^2}-|\alpha|)]-i\frac{C(\alpha)}{2\alpha|\alpha|}\}\,e^{i\alpha x}d\alpha=0,$$

$$|x|<1. \qquad (5.2,12)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{B(\alpha)\frac{i}{2\alpha|\alpha|}+C(\alpha)\frac{1}{2|\alpha|\alpha^2}(1+\frac{4\epsilon^2\alpha^2}{1-\nu})\}\,e^{i\alpha x}d\alpha=0,$$

$$|x|<1\;. \qquad (5.2,13)$$

Equation (5.2,11) combined with the first equation in (5.2,2) gives exactly the same system of dual integral equations (3.2,39) and (3.2,40) which were obtained directly in Part III.

Equations (5.2,12) and (5.2,13) combined with the last two equations in (5.2,2) form a system of equations which corresponds to the symmetric deflection problem for the case of a rigid inclusion with homogeneous boundary conditions (see equations (3.2,63) to (3.2,66)). This suggests that $B(\alpha) \equiv C(\alpha) \equiv 0$. In fact an argument essentially the same as that given in Appendix C can be used to prove this. Then, from (5.2,2) it follows that

$$M_{xy}^{(1)}(x,0) = Q_y^{(1)}(x,0) = 0$$

for all x.

Making use of these results, the formula (5.2,8) for $\Pi^{(1)}$ now becomes

$$\Pi^{(1)} = \frac{c^2(1-\nu)}{D2\pi} \int_{-\infty}^{\infty} A(\alpha)\overline{A(\alpha)} \left[\frac{3+\nu}{2}|\alpha| - 2\epsilon^2\alpha^2\left(\sqrt{\alpha^2+1/\epsilon^2} - |\alpha|\right)\right] d\alpha -$$

$$-c \int_{-1}^{1} [f_{1c}(x)\,\overline{\beta_y^{(1)}(x,0)} + \overline{f_{1c}(x)}\,\beta_y^{(1)}(x,0)]\, dx\,. \qquad (5.2,14)$$

It is this form which we shall employ to obtain an approximate solution for $\beta_y^{(1)}(x,0)$.

In the usual way (see [17]), we shall assume that our solution may be approximated by a finite linear combination of suitably chosen functions. The coefficients appearing in this linear combination shall then be determined by minimizing (5.2,14).

The selection of a minimizing sequence of functions in general depends on the concept of the solution class. In particular, we wish to select a sequence of functions which is complete with respect to the solution class. For the present problem, we require further that the Fourier transform of each member of the sequence can be evaluated explicitly so that our later computation will be greatly simplified.

In section 3.3, the function u(x) defined by (3.3,1) has been identified as $\frac{D}{c}\beta_y^{(1)}(x,0)$. Hence we shall require that our approximating functions satisfy the same requirements as were imposed on u(x) in the integral equation (4.1,4).

Without loss of generality, we may assume that

$$f_{1c}(-x) = f_{1c}(x) = \overline{f_{1c}(x)} \; . \tag{5.2,15}$$

It follows from the integral equation (4.1,1) that

$$\beta_y^{(1)}(-x,0) = \beta_y^{(1)}(x,0) = \overline{\beta_y^{(1)}(x,0)} \tag{5.2,16}$$

if (5.2,15) holds.

Let us put

$$\frac{D}{c}\beta_y^{(1)}(x,0) = (1-x^2)^{1/2}\, h(x), \quad |x| \leq 1. \tag{5.2,17}$$

Then from (4.14) we shall find that h(x) is Hölder continuous for all x in [-1,1] .

With the above considerations, we find that the sequence of functions $\{\sum_{k=1}^{n} c_k^{(n)}(1-x^2)^{k-1/2}\}$, $n \geq 1$ will be appropriate for our purpose. The reason is clear. They are real and even in x which is in accordance with (5.2,16). They vanish at $x = \pm 1$. If $(1-x^2)^{1/2}$ is factored out, the remaining parts of this sequence is a sequence of polynomials, and it is well known that the sequence of polynomials is complete for continuous functions in a closed interval. Furthermore, the Fourier transform, computed according to (5.2,3), of each member of this sequence can be evaluated explicitly.

Let the approximate solution up to the n-th term be

$$\frac{D}{c}\,\beta^{(1)}_{y\,(n)}\,(x,0+) = \sum_{k=1}^{n} c_k^{(n)}\,\frac{(\frac{1}{2})^k}{\Gamma(\frac{1}{2})\Gamma(k+\frac{1}{2})}\,(1-x^2)^{k-\frac{1}{2}}, \quad |x| \le 1 \tag{5.2,18}$$

where the coefficients are arbitrarily arranged for the sake of convenience. We intend to use the minimum principle to find the optimum $c_k^{(n)}$. From (5.2,3) we obtain the Fourier transform of $\frac{D}{c}\beta^{(1)}_{y(n)}$ as follows (see [18]):

$$A^{(n)}(\alpha) = \sum_{k=1}^{n} c_k^{(n)}\,\frac{J_k(\alpha)}{\alpha^k}\;. \tag{5.2,19}$$

Substituting (5.2,18) and (5.2,19) into (5.2,14) and minimizing $\Pi^{(1)}$ by varying $c_k^{(n)}$, we obtain an nxn system of simultaneous equations:

$$\sum_{\ell=1}^{n} a_{k\ell}\,c_\ell^{(n)} = b_k, \qquad k = 1, \ldots\, n \tag{5.2,20}$$

where

$$a_{k\ell} = \int_0^\infty \frac{J_k(\alpha)J_\ell(\alpha)}{\alpha^{k+\ell}}\,[\,(3+\nu)\,\alpha - 2\epsilon^2\alpha^2(\sqrt{\alpha^2+1/\epsilon^2} - \alpha)]\;d\alpha \tag{5.2,21}$$

and

$$b_k = \frac{2\pi\,(\frac{1}{2})^k}{(1-\nu)\Gamma(\frac{1}{2})\;\Gamma(k+\frac{1}{2})} \int_{-1}^{1} f_{1c}(x)(1-x^2)^{k-\frac{1}{2}}\,dx\;. \qquad (5.2,22)$$

It can be easily shown that the matrix $(a_{k\ell})$ is positive definite and thus non-singular for every $\epsilon \geq 0$ and for every $n \geq 1$. Hence, $c_\ell^{(n)}$ as in (5.2,20) can be uniquely determined if the b_k's are all finite. Since we have required at the beginning that f_{1c} satisfies the conditions given in section 4.1, the b_k's are easily seen to be finite from (5.2,22). Substituting these solutions back into (5.2,14) and denoting by $\Pi_n^{(1)}$ the complementary energy corresponding to an n-term approximation, we obtain

$$\frac{2\pi D}{(1-\nu)c^2}\Pi_n^{(1)} = -\sum_{k=1}^{n} b_k c_k^{(n)}\;. \qquad (5.2,23)$$

It can be shown that the following relation is valid

$$\frac{2\pi D}{(1-\nu)\,c^2}\left(\Pi_{n+1}^{(1)} - \Pi_n^{(1)}\right) = -\frac{\Delta_{n+1}}{\Delta_n}\left[C_{n+1}^{(n+1)}\right]^2 \leq 0, \quad n \geq 1 \qquad (5.2,24)$$

where Δ_n denotes the determinant of the n x n matrix $(a_{k\ell})$. Hence $\{\Pi_n^{(1)}\}$ is a monotone decreasing sequence and is bounded below since it can not be less than $\Pi^{(1)}$ whose negative value is the strain energy contained in the plate corresponding to the true solution; moreover $\Pi^{(1)}$ is finite. Because of the fact that $f_{1c}(x)$ satisfies the conditions given in section 4.1, we can show further that

$$\lim_{n\to\infty} \Pi_n^{(1)} = \Pi^{(1)}\;. \qquad (5.2,25)$$

Combining the results obtained so far, we may conclude that if $f_{1c}(x)$ is real and even, satisfies the conditions given in section 4.1, then (i) the approximate solution as represented by (5.2,18) is uniquely determined for every $n \geqslant 1$ and (ii)

$$\lim_{n \to \infty} \Pi_n^{(1)} = \Pi^{(1)} .$$

It is worthwhile to remark here that the problem of whether the approximate solution tends to the true solution as $n \to \infty$ is also interesting. However, a proof for this has not yet been found.

We shall work out a specific example by using the variational method. We consider that the plate is deformed by the action of a constant bending moment M_0 per unit length uniformly distributed around the periphery of a circle centered at origin with infinite radius. The boundary condition along the crack for the reduced problem will then be $f_{1c} = M_0 = \text{const.}$, $f_{2c} = f_{3c} = 0$. We shall apply the variational method to this reduced problem.

For the one-term approximation, we put

$$\frac{D}{c}\beta_{y(1)}^{(1)}(x,0) = \frac{1}{\pi}\; c_1^{(1)}(1-x^2)^{1/2} , \quad |x| \leqslant 1. \qquad (5.2,26)$$

From (5.2,20), we find

$$C_1^{(1)} = 2\pi M_0/(1-\nu)\,(3+\nu)\,(1 - \frac{2I_1(\epsilon)}{3+\nu}) \qquad (5.2,27)$$

where

$$I_1(\epsilon) = 4\epsilon^2 \int_0^\infty (\sqrt{\alpha^2+1/\epsilon^2} - \alpha)\,[J_1(\alpha)]^2\, d\alpha . \qquad (5.2,28)$$

The integration in (5.2,28) can be carried out explicitly § to give a series representation for $I_1(\epsilon)$ as

$$I_1(\epsilon) = 1 + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{1}{2})^2}{\Gamma(n)\Gamma(n+1)\Gamma(n+2)^2} \left(\frac{1}{\epsilon^2}\right)^n \cdot$$

$$\cdot \{ \ln\left(\frac{1}{\epsilon^2}\right) + [2\gamma - 4\ln 2 - \frac{n-1}{n(n+1)} - 4 \sum_{r=1}^{n} \frac{(r-1)}{r(2r-1)}] \} \qquad (5.2,29)$$

where γ is the Euler's constant = 0.5772... . Figure 3 shows the plot of $I_1(\epsilon)$ against ϵ.

As $\epsilon \to 0$, the asymptotic representation for $I_1(\epsilon)$ is found to be

$$I_1(\epsilon) = \frac{4}{\pi} \epsilon \ \ell n \frac{1}{\epsilon} + O(\epsilon) \ . \qquad (5.2,30)$$

Hence, the one term approximation (5.2,26) for thin plates will be

$$\frac{D}{c} \beta_{y(1)}^{(1)}(x,0) = \frac{2M_0}{(1-\nu)(3+\nu)} (1-x^2)^{1/2} + O\left(\epsilon \ell n \frac{1}{\epsilon}\right), \quad |x| \leq 1 \quad \text{as } \epsilon \to 0. \qquad (5.2,31)$$

In order to estimate this result, we compute $u_0(x)$ by (4.2,7) with $f_1 = M_0$ and find

$$\frac{D}{c} \beta_y^{(1)}(x,0) \equiv u_0(x) = \frac{2M_0}{(1-\nu)(3+\nu)} (1-x^2)^{1/2}, \quad |x| \leq 1. \qquad (5.2,32)$$

A comparison between (5.2,31) and (5.2,32) shows that when f_1 = const. the one term approximation tends as $\epsilon \to 0$ to the approximate solution of the integral equation obtained in Part IV for the

§ A contour integration procedure suggested on p. 436 in [19] can be used to evaluate $I_1(\epsilon)$.

first order term of the thin plate solution.

The stress distribution near the vertex x = 1, y = 0 can be computed by using the one term approximation (5.2,26) in formula (3.3,7). We find that as $r \to 0$ for fixed θ, $|\theta| < \pi$

$$\begin{pmatrix} M_x^{(1)} \\ M_y^{(1)} \\ M_{xy}^{(1)} \end{pmatrix} \sim \frac{(1+\nu)}{(3+\nu)\left(1-\frac{2I_1(\epsilon)}{3+\nu}\right)} M_0 (2r)^{-1/2} \begin{pmatrix} \frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2} \\ \frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{5\theta}{2} \\ -\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{5\theta}{2} \end{pmatrix} \tag{5.2,33}$$

where r, θ are the local polar coordinates centered at x = 1, y = 0 as in section 4.2.

The asymptotic behavior of $I_1(\epsilon)$ for large ϵ can be obtained directly from (5.2,29). We have

$$I_1(\epsilon) = 1 + O\left(\frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2}\right) \qquad \text{as} \quad \epsilon \to \infty . \tag{5.2,34}$$

Hence, the one term approximation (5.2,26) becomes

$$\boxed{\frac{D}{c}\beta_{1(1)}^{(1)}(x,0) = \frac{2M_0}{(1-\nu^2)}(1-x^2)^{1/2} + O\left(\frac{1}{\epsilon^2}\ln\frac{1}{\epsilon^2}\right), \quad |x| \leq 1 \quad \text{as} \quad \epsilon \to \infty .} \tag{5.2,35}$$

Again, it can be shown directly by perturbation methods that the solution of the integral equation (4.1,1) for $\epsilon \to \infty$ agrees with (52,35).

We finally conclude that for this special case ($f_{1c}=M_0=$const.) the one term approximate solution based on the variational principle agrees with the approximate solution of the integral equation (4.1,1) for small ϵ and for large ϵ.

However, large ϵ means physically a plate whose thickness is large in comparison with the length of the crack. In such a case, the differential equations may no longer be accurate near the vicinity of the crack. Thus while the physical validity of the approximate solution for large ϵ is doubtful, it is still useful to observe that the variational approximation agrees for large ϵ with a limiting solution obtained directly from the integral equation by perturbation methods as $\epsilon \to \infty$. This suggests a reasonably wide range of usefulness for the variational approximation.

5.3 Approximate Solution for Case II

By the same procedures used for Case I, we shall apply the variational principle to Case II in order to seek an approximate solution. We shall omit the details and list only the results.

We select $M_y^{(2)}(x,0)$, $\frac{D}{c}\beta_x^{(2)}(x,0)$ and $\frac{D}{c^2}w_t^{(2)}(x,0)$ as the admissible set to minimize $\Pi^{(2)}$, (5.1,13). These quantities vanish for $|x| > 1$, thus their Fourier transform will be

$$\begin{pmatrix} F(\alpha) \\ \Omega(\alpha) \\ \omega(\alpha) \end{pmatrix} = \int_{-1}^{1} \begin{pmatrix} M_y^{(2)}(x,0) \\ \frac{D}{c}\beta_x^{(2)}(x,0) \\ \frac{D}{c^2}w_t^{(2)}(x,0) \end{pmatrix} e^{-i\alpha x}\,dx \quad \text{all } \alpha. \quad (5.3,1)$$

With the aid of formulas (3.2,1) to (3.2,18), $\frac{D}{c}\beta_y^{(2)}, M_{xy}^{(2)}, Q_y^{(2)}$ can also be expressed in terms of $F(\alpha)$, $\Omega(\alpha)$ and $\omega(\alpha)$. We substitute these results into $\Pi^{(2)}$, (2.1,13). By minimizing $\Pi^{(2)}$, we obtain

$$F(\alpha) = 0 \tag{5.3,2}$$

and we again recover the dual integral equation system (3.2,50) and (3.2,51).

Using (5.3,2), $\Pi^{(2)}$ of (5.1,13) can be reduced into the following form:

$$\begin{aligned}
\Pi^{(2)} &= \\
&= \frac{c^2(1-\nu)}{D^2\pi} \int_{-\infty}^{\infty} \Big\{ \Omega(\alpha)\overline{\Omega(\alpha)} \Big[\frac{1}{2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}} + \Big(\frac{2\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}} - \frac{1-\nu}{2}|\alpha|\Big) + \\
&\qquad + 2\epsilon^2\Big(\frac{\alpha^4}{\sqrt{\alpha^2+1/\epsilon^2}} - |\alpha|\,\alpha^2\Big)\Big] + \\
&\quad + i\omega(\alpha)\overline{\Omega(\alpha)}\Big[\frac{\alpha}{2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}} + \Big(\frac{\alpha^3}{\sqrt{\alpha^2+1/\epsilon^2}} - \alpha|\alpha|\Big)\Big] - \\
&\quad - i\overline{\omega(\alpha)}\Omega(\alpha)\Big[\frac{\alpha}{2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}} + \Big(\frac{\alpha^3}{\sqrt{\alpha^2+1/\epsilon^2}} - \alpha|\alpha|\Big)\Big] + \\
&\quad + \omega(\alpha)\overline{\omega(\alpha)}\,\frac{\alpha^2}{2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}}\Big\}\,d\alpha - \\
&\quad - 2c\int_{-1}^{1} \mathrm{Re}\,\{f_{2c}\,\overline{\beta_x^{(2)}} + f_{3c}\,\overline{w_t^{(2)}}\}\,dx \;.
\end{aligned} \tag{5.3,3}$$

It is this form which we shall employ to obtain an approximate solution.

Without loss of generality, we may assume that both f_{2c} and f_{3c} are real, and

$$f_{2c}(-x) = f_{2c}(x) \quad ,$$
$$f_{3c}(-x) = - f_{3c}(x) \quad . \tag{5.3,4}$$

It then follows from integral equations (4.4,1) and (4.4,5) that both $\beta_x^{(2)}$ and $w_t^{(2)}$ are real and

$$\beta_x^{(2)}(-x,0) = \beta_x^{(2)}(x,0) \ ,$$
$$w_t^{(2)}(-x,0) = - w_t^{(2)}(x,0) \ . \tag{5.3,5}$$

It is found that an appropriate sequence of minimizing functions for $\beta_x^{(2)}(x,0)$ is

$$\sum_{k=1}^{n} b_k^{(n)}(1-x^2)^{k-1/2} \ , \qquad n \geq 1 \ .$$

Similarly, we find for $w_t^{(2)}(x,0)$

$$\sum_{\ell=1}^{n} c_\ell^{(n)} x(1-x^2)^{\ell-1/2} \ , \qquad n \geq 1.$$

The n-term approximate solution will then be as follows:

$$\frac{D}{c}\,\beta^{(2)}_{x(n)}(x,0) = \sum_{k=1}^{n} b_k^{(n)} \frac{(1/2)^k}{\Gamma(\frac{1}{2})\Gamma(k+\frac{1}{2})}(1-x^2)^{k-\frac{1}{2}}, \ |x| \leq 1 \tag{5.3,6}$$

and

$$\frac{D}{c^2}\,w^{(2)}_{t(n)}(x,0) = \sum_{\ell=1}^{n} c_\ell^{(n)} \frac{(\frac{1}{2})^\ell}{\Gamma(\frac{1}{2})\Gamma(\ell+\frac{1}{2})} x(1-x^2)^{\ell-\frac{1}{2}}, \ |x| \leq 1 \tag{5.3,7}$$

where the coefficients are arranged for the sake of convenience. We intend to use the variational principle to optimize the choice of b's and c's.

From (5.3,1), we find that the Fourier transforms of (5.3,6) and (5.3,7) are

$$\Omega^{(n)}(\alpha) = \sum_{k=1}^{n} b_k^{(n)} \frac{J_k(\alpha)}{\alpha^k} \tag{5.3,8}$$

and

$$i\omega^{(n)}(\alpha) = \Sigma\, c_k^{(n)} \frac{J_{k+1}(\alpha)}{\alpha^k} \quad . \tag{5.3,9}$$

Substituting these expressions into (5.3,3) and minimizing $\Pi^{(2)}$ by varying the $b_k^{(n)}$'s and $c_\ell^{(n)}$'s, leads to a system of 2n x 2n simultaneous equations:

$$\sum_{\ell=1}^{2n} a_{k\ell}\, t_\ell^{(n)} = \gamma_k \qquad k = 1,2,\ldots.2n \tag{5.3,10}$$

where

$$a_{k\ell} = \int_0^\infty \frac{J_k J_\ell}{\alpha^{k+\ell}} \left[\frac{1}{2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}} + \left(\frac{2\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}} - \frac{(1-\nu)}{2}\alpha\right) + 2\epsilon^2\left(\frac{\alpha^4}{\sqrt{\alpha^2+1/\epsilon^2}} - \alpha^3\right)\right] d\alpha$$

as $1 \leq k,\ \ell \leq n$

$$= \int_0^\infty \frac{J_k J_{\ell-n+1}}{\alpha^{k+\ell-n-1}} \left[\frac{1}{2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}} + \left(\frac{\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}} - \alpha\right)\right] d\alpha$$

as $1 \leq k \leq n, \quad n < \ell \leq 2n$

$$= \int_0^\infty \frac{J_{k-n+1} J_\ell}{\alpha^{k-n+\ell-1}} \left[\frac{1}{2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}} + \left(\frac{\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}} - \alpha\right)\right] d\alpha$$

as $n < k \leq 2n, \quad 1 \leq \ell \leq n$

$$= \int_0^\infty \frac{J_{k+1}\, J_{\ell+1}}{\alpha^{k+\ell-2n-2}} \cdot \frac{1}{2\epsilon^2 \sqrt{\alpha^2+1/\epsilon^2}}\, d\alpha \qquad (5.3,11)$$

as $n < k,\ \ell \leqslant 2n$,

$$t_\ell^{(n)} = b_\ell^{(n)} \qquad \text{as } 1 \leqslant \ell \leqslant n$$

$$= c_\ell^{(n)} \qquad \text{as } n < \ell \leqslant 2n \qquad (5.3,12)$$

and

$$\gamma_k = \frac{\pi\ (1/2)^k}{(1-\nu)\Gamma(1/2)\Gamma(k+1/2)} \int_{-1}^{1} f_{2c}(x)\ (1-x^2)^{k-1/2}\, dx$$

as $1 \leqslant k \leqslant n$

$$= \frac{\pi c\ (1/2)^{k-n}}{(1-\nu)\Gamma(1/2)\Gamma(k-n+1/2)} \int_{-1}^{1} f_{3c}(x) x (1-x^2)^{k-n-1/2}\, dx$$

as $n < k \leqslant 2n$. (5.3,13)

It can be shown that the matrix $(a_{k\ell})$ in (5.3,10) is positive definite and, thus, non-singular. Hence the solution to (5.3,10) exists and is unique if γ_k's are finite for all k, k = 1,... 2n. Since f_{2c} and f_{3c} satisfy the conditions given in section 4.4, γ's are seen to be finite from (5.3,13).

5.4 Approximate Solution for Case III

In order to apply the variational method to Case III, we shall again use $M_y^{(2)}(x,0)$, $\frac{D}{c}\beta_x^{(2)}(x,0)$ and $\frac{D}{c^2} w_t^{(2)}(x,0)$ as given among formulas (3.2,1) to (3.2,18) as the admissible set. The proper energy expression to be varied is $\mathcal{E}^{(2)}$, (5.1,18). We shall omit the details of reduction here.

The final form of $\mathcal{E}^{(2)}$ becomes

$$\mathcal{E}^{(2)} = \frac{c}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\,\overline{F(\alpha)}\,\Big[\frac{1}{2|\alpha|} + \frac{2\epsilon^2\alpha^2}{1-\nu}\Big(\frac{1}{|\alpha|} - \frac{1}{\sqrt{\alpha^2+1/\epsilon^2}}\Big)\Big]\,d\alpha\; -$$

$$- 2c\int_{-1}^{1} \mathrm{Re}\,\{f_{2r}\,\overline{M_y^{(2)}}\}\,dx \qquad (5.4,1)$$

where

$$F(\alpha) = \int_{-1}^{1} M_y^{(2)}(x,0)\,e^{-i\alpha x}\,dx \qquad \text{all } \alpha \qquad (5.4,2)$$

and $f_{2r}(x)$ is defined in (3.1,24).

Let us assume that f_{2r} is real and odd in x, so that $M_y^{(2)}(x,0)$ is also real and odd from the integral equation (4.7,1).

The sequence $\{\sum_{k=1}^{n} c_k^{(n)}\, x(1-x^2)^{k-3/2}\}$ is found to be proper as the minimizing functions for $M_y^{(2)}(x,0)$.

The n-term approximation for $M_y^{(2)}(x,0)$ will then be

$$M_{y(n)}^{(2)} = \sum_{k=1}^{n} - c_k^{(n)}\,\frac{(1/2)^{k-1}}{\Gamma(1/2)\Gamma(k-1/2)}\, x(1-x^2)^{k-3/2}\,,$$

$$|x| < 1. \qquad (5.4,3)$$

We can easily compute the Fourier transform of (5.4,3), which is

$$F^{(n)}(\alpha) = i\sum_{k=1}^{n} c_k^{(n)}\,\frac{J_k(\alpha)}{\alpha^{k-1}}\,. \qquad (5.4,4)$$

Substituting these results into (5.4,1) and minimizing $\mathcal{E}^{(2)}$ with respect to c_k's, we find again a system of n x n simultaneous equations:

$$\sum_{\ell=1}^{n} a_{k\ell}\, c_\ell^{(n)} = b_k \;, \qquad k = 1, \dots n \qquad (5.4,5)$$

where

$$a_{k\ell} = \int_0^\infty \frac{J_k(\alpha)J_\ell(\alpha)}{\alpha^{k+\ell-2}} \left[\frac{1}{2\alpha} + \frac{2\epsilon^2\alpha^2}{(1-\nu)} \left(\frac{1}{\alpha} - \frac{1}{\sqrt{\alpha^2 + 1/\epsilon^2}} \right) \right] d\alpha \quad (5.4,6)$$

and

$$b_k = - \frac{\pi (1/2)^{k-1}}{\Gamma(1/2)\Gamma(k-1/2)} \int_{-1}^{1} f_{2r}(x) x (1-x^2)^{k-3/2}\, dx \,. \qquad (5.4,1)$$

The matrix $(a_{k\ell})$ in (5.4,4) is positive definite, thus the equation (5.4,5) has unique solution for every finite b_k's. Again, the b_k's are seen to be finite since $f_{2r}(x)$ satisfies the conditions given in section 4.7.

We shall work out an example. Let us take $f_{2r} = - \frac{c\, M_0}{D(1-\nu)}\, x$, which is induced by a loading at infinity in the original problem described as follows:

$$\left. \begin{aligned} & M_x = M_y = Q_x = Q_y = 0 \\ & M_{xy} = M_0 = \text{const.} \end{aligned} \right\} \quad \text{at } x = \pm\infty \text{ and } y = \pm\infty \,.$$

The one term approximation can be found through (5.4,5) to be

$$M^{(2)}_{y(1)}(x,0) = \frac{2M_0}{(1-\nu)\ I_2(\epsilon)} \frac{x}{(1-x)^{1/2}} , \qquad |x| < 1 \qquad (5.4,8)$$

where

$$I_2(\epsilon) = \int_0^\infty [J_1(\alpha)]^2 \left[\frac{2}{\alpha} + \frac{8\epsilon^2}{1-\nu} \left(\alpha - \frac{\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}}\right)\right] d\alpha . \qquad (5.4,9)$$

The integral $I_2(\epsilon)$ has the following limits:

$$\lim_{\epsilon \to 0} I_2(\epsilon) = 1 , \qquad (5.4,10a)$$

$$\lim_{\epsilon \to \infty} I_2(\epsilon) = \frac{3-\nu}{1-\nu} . \qquad (5.4,10b)$$

Using (5.4,10a) and (5.4,10b), we have from (5.4,8)

$$\boxed{\lim_{\epsilon \to 0} M^{(2)}_{y(1)}(x,0) = \frac{2M_0}{(1-\nu)} \cdot \frac{x}{(1-x^2)^{1/2}} , \quad |x| < 1.} \qquad (5.4,11a)$$

and

$$\boxed{\lim_{\epsilon \to \infty} M^{(2)}_{y(1)}(x,0) = \frac{2M_0}{(3-\nu)} \cdot \frac{x}{(1-x^2)^{1/2}} , \quad |x| < 1.} \qquad (5.4,11b)$$

We shall remark here that (5.4,11a) and (5.4,11b) can also be obtained from the solution $\tau(x)$ to the integral equation (4.7,1) as $\epsilon \to 0$ and as $\epsilon \to \infty$ by perturbation procedures. To do this it is necessary to make use of the relation

$$M_y^{(2)}(x,0) = \frac{d\tau(x)}{dx} .$$

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APPENDICES

Appendix A

Statement: If u(x) vanishes at $x = \pm 1$ and is Hölder continuous with some positive Hölder index μ for all x in the closed interval $[-1, 1]$, then the stress singularity for the quantities in (3.3,7) will not be worse than $O(r^{-1+\delta})$ for some $\delta > 0$ where r is the distance measured from an interior point to one of the ends of the crack.

Proof: We prove this for $M_y^{(1)}$ since $M_x^{(1)}$, $M_{xy}^{(1)}$ have similar characters. From (3.3,7) we have

$$M_y^{(1)}(x,y) = \frac{(1-\nu)}{\pi} \int_{-1}^{1} u(\xi) \frac{H_y(x-\xi,y)}{[(x-\xi)^2 + y^2]} d\xi \tag{A,1}$$

where

$$H_y(x,y) = -\frac{(3+\nu)(y^2-x^2)}{2\rho^2} + (1-\nu)\frac{y^2(y^2-3x^2)}{\rho^4} +$$

$$+ 2\epsilon^2[-\frac{1}{\epsilon^2}K_2(\frac{\rho}{\epsilon}) + \frac{\rho}{\epsilon^3}K_3(\frac{\rho}{\epsilon}) - \frac{x^2y^2}{\epsilon^4\rho^2}K_4(\frac{\rho}{\epsilon}) -$$

$$-\frac{6(x^4+y^4-6x^2y^2)}{\rho^6}]$$

with $\rho^2 = x^2 + y^2$. (A,2)

Apparently $H_y(x,y)$ is bounded for all x, y, $\epsilon>0$ hence there exist a number $0 < H < +\infty$ such that for fixed $\epsilon > 0$

$$|H_y(x,y)| \leq H \tag{A,3}$$

for all x and y. Suppose the behavior of the integral in (A,1) near the end $x = 1$, $y = 0$ is under consideration. Then it is preferable to transform the coordinates x, y to $x = 1 + r\cos\theta$ and $y = r\sin\theta$; by doing so (A,1) takes the form

$$\frac{\pi}{(1-\nu)} M_y^{(1)}(r,\theta) = \int_{-1}^{1} [u(\xi) - u(1)] \frac{H_y(r,\theta,\xi)}{(1-\xi)^2 + 2r\cos\theta(1-\xi) + r^2} d\xi + \\ + u(1) \int_{-1}^{1} \frac{H_y(r,\theta,\xi)}{(1-\xi)^2 + 2r\cos\theta(1-\xi) + r^2} d\xi \quad . \qquad (A,4)$$

Using the Hölder continuity property of u(x), i.e.

$$|u(x_1) - u(x_2)| \leq M \; |x_1 - x_2|^{\mu} \qquad (A,5)$$

and using (A,3), we have the following estimation:

$$\left| \frac{\pi}{(1-\nu)} M_y^{(1)}(r,\theta) \right| \leq MH \int_{-1}^{1} \frac{|1-\xi|^{\mu}}{(1-\xi)^2 + 2r\cos\theta(1-\xi) + r^2} d\xi + \\ + |u(1)| \, H \int_{-1}^{1} \frac{d\xi}{(1-\xi)^2 + 2r\cos\theta(1-\xi) + r^2} \quad . \qquad (A,6)$$

If the change of variable $1 - \xi = tr$ is used, then (A,6) becomes

$$\left| \frac{\pi}{(1-\nu)} M_y^{(1)} (r,\theta) \right| \leq MH \, r^{-1+\mu} \int_0^{2/r} \frac{t^{\mu} dt}{(1 + 2t\cos\theta + t^2)} + \\ + H \, |u(1)| \, r^{-1} \int_0^{2/r} \frac{t^{\mu} dt}{(1 + 2t\cos\theta + t^2)} \quad . \qquad (A,7)$$

Suppose we fix θ so that $|\theta| < \pi$; thus we approach $x = 1$, $y = 0$

Appendix B

To verify (3.3,14), it is equivalent to show that if u $(\pm 1) = 0$ and if u'(x) exists and is Hölder continuous for all x in (-1, 1) then the following statement is true.

Statement: Given $\eta > 0$ there exists $|y_0| > 0$ such that

$$\left| \frac{\pi}{(1-\nu)} M_y^{(1)}(x,y) - \frac{d}{dx} \int_{-1}^{1} u(\xi) \left\{ - \frac{(3+\nu)}{2(x-\xi)} + \right.\right.$$
$$\left.\left. + 2\epsilon^2 \left[\frac{2}{(x-\xi)^3} - \frac{K_2(\frac{|x-\xi|}{\epsilon})}{\epsilon^2 (x-\xi)} \right] \right\} d\xi \right| < \eta \qquad \text{(B,1)}$$

whenever $0 \leq |y| \leq |y_0|$ for fixed x in (-1, 1).

Proof: Using the assumptions that u'(x) exists and that $u(\pm 1) = 0$, we can deduce the following result from (3.3,4) by integration by parts.

$$\frac{\pi}{(1-\nu)} M_y^{(1)}(x,y) = \int_{-1}^{1} u'(\xi) \frac{(x-\xi)}{R^2} S(x-\xi, y) \, d\xi \qquad \text{(B,2)}$$

where

$$S(x,y) = - \frac{3+\nu}{2} + (1-\nu) \frac{y^2}{\rho^2} + 2\epsilon^2 \left[- \frac{K_2(\frac{\rho}{\epsilon})}{\epsilon^2} + \right.$$
$$\left. + \frac{y^2 K_3(\frac{\rho}{\epsilon})}{\epsilon^3 \rho} - 2 \frac{(3y^2 - x^2)}{\rho^4} \right] \qquad \text{(B,3)}$$

and $R^2 = (x-\xi)^2 + y^2$, $\rho^2 = x^2 + y^2$ as before. By suitable integration by parts, it may also be verified that

$$\frac{d}{dx}\int_{-1}^{1} u(\xi)\ \{-\frac{(3+\nu)}{2(x-\xi)} + 2\epsilon^2[\frac{2}{(x-\xi)^3} - \frac{K_2(\frac{|x-\xi|}{\epsilon})}{\epsilon^2(x-\xi)}]\ \}\ d\xi =$$

$$= \int_{-1}^{1} u'(\xi)\ \{-\frac{(3+\nu)}{2(x-\xi)} + 2\epsilon^2[\frac{2}{(x-\xi)^3} - \frac{K_2(\frac{|x-\xi|}{\epsilon})}{\epsilon^2(x-\xi)}]\}\ d\xi \quad . \quad (B,4)$$

Through (B,2) and (B,4), the left side of (B,1) can be written as a sum of two integrals $|I_1 + I_2|$ with

$$I_1 = \int_{-1}^{1}[u'(\xi)-u'(x)]\ [\frac{(x-\xi)}{R^2} S(x-\xi,y) - \frac{S_0(x-\xi)}{(x-\xi)}]\ d\xi \qquad (B,5)$$

and

$$I_2 = u'(x)\int_{-1}^{1}[\frac{(x-\xi)}{R^2} S(x-\xi,y) - \frac{S_0(x-\xi)}{(x-\xi)}]\ d\xi\ , \qquad (B,6)$$

where

$$S_0(x) = S(x,0)\ . \qquad (B,7)$$

Let $\delta > 0$ and small. Then write

$$I_1 = I_{11} + I_{12} \qquad (B,8)$$

where

$$I_{11} = \int_{x-\delta}^{x+\delta} [u'(\xi) - u'(x)]\ [\frac{(x-\xi)}{R^2} S(x-\xi,y) - \frac{1}{(x-\xi)} S_0(x-\xi)]\ d\xi \qquad (B,9)$$

and

$$I_{12} = \left(\int_{-1}^{x-\delta} + \int_{x+\delta}^{1}\right) [u'(\xi)-u'(x)] \left[\frac{(x-\xi)}{R^2} S(x-\xi,y) - \frac{S_0(x-\xi)}{(x-\xi)}\right] d\xi . \quad (B,10)$$

By changing the integration variable ξ in (B,9) to $t = \xi - x$, I_{11} takes the form

$$I_{11} = \int_{-\delta}^{\delta} \frac{u'(x+t)-u'(x)}{t} \left[- \frac{t^2}{R^2} S(t,y) + S_0(t)\right] dt. \quad (B,11)$$

It is obvious that for all values of t and y

$$\left| S_0(t) - \frac{t^2}{R^2} S(t,y) \right| \quad (B,12)$$

is bounded, say less than or equal to $\frac{K}{2}$. From the given condition on $u'(x)$, we have the Hölder inequality

$$| u'(x_1) - u'(x_2) | \leqslant M' |x_1-x_2|^{\mu} \quad (B,13)$$

for some $0 < \mu < 1$ and for every pair of x_1, x_2 in $(-1,1)$. Using (B,12) and (B,13) we obtain the following estimate for I_{11}:

$$| I_{11} | \leqslant \frac{KM'}{\mu} \delta^{\mu} \quad (B,14)$$

From (B,14) δ is then chosen such that

$$\frac{KM'}{\mu} \delta^{\mu} \leqslant \frac{\eta}{3} \quad (B,15)$$

with the restriction $-1 < x - \delta < x + \delta < 1$.

The function

$$\frac{(x-\xi)}{R^2} S(x-\xi,y) - \frac{1}{(x-\xi)} S_0(x-\xi)$$

in (B,9) is continuous with respect to y as ξ is in the intervals $-1 \leq \xi \leq x - \delta$, $x + \delta \leq \xi \leq 1$ (δ now fixed) and moreover tends to zero uniformly in that interval as $|y| \to 0$. Since $u'(\xi)$ is integrable from -1 to 1, there exists $|y_1| > 0$ such that

$$|I_{12}| < \frac{\eta}{3} \tag{B,16}$$

whenever $|y| \leq |y_1|$.

With δ fixed by (B,15), I_2 can be written as a sum of two integrals

$$I_2 = I_{21} + I_{22} \tag{B,17}$$

where

$$I_{21} = u'(x)\int_{x-\delta}^{x+\delta} [\frac{x-\xi}{R^2} S(x-\xi,y) - \frac{1}{(x-\xi)} S_0(x-\xi)]\, d\xi \tag{B,18}$$

and

$$I_{22} = u'(x)(\int_{-1}^{x-\delta} + \int_{x+\delta}^{1}) [\frac{x-\xi}{R^2} S(x-\xi,y) - \frac{1}{(x-\xi)} S_0(x-\xi)]\, d\xi \ . \tag{B,19}$$

I_{21} vanishes on account of oddness of the integrand and there exists $|y_2| > 0$ such that

$$|I_{22}| < \frac{\eta}{3} \tag{B,20}$$

whenever $|y| \leq |y_2|$ by an analysis similar to that for I_{12}.

Combining these estimates all together, we have

$$\left| \frac{\pi}{(1-\nu)} M_y^{(1)}(x,y) - \int_{-1}^{1} u'(\xi) \left\{ -\frac{3+\nu}{2(x-\xi)} + 2\epsilon^2 \left[\frac{2}{(x-\xi)^3} - \frac{K_2(\frac{|x-\xi|}{\epsilon})}{\epsilon^2 (x-\xi)} \right] \right\} d\xi \right| < \eta$$

whenever $|y| \leqslant |y_0|$ for fixed x in (-1,1) where $|y_0|$ is the lesser of $|y_1|$ and $|y_2|$.

Appendix C

Uniqueness of solutions of dual integral equations

We shall consider the uniqueness of solutions to the various cases of present problems. We illustrate for example the anti-symmetric case of the crack problem. Multiplying (3.2,50) with v(x) of (3.3,16) and integrating from x = -1 to x = 1, we obtain §

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[\,C_{11}(\alpha)\Omega(\alpha)\overline{\Omega(\alpha)} + iC_{12}(\alpha)\omega(\alpha)\overline{\Omega(\alpha)}\,]\,d\alpha = \frac{1}{(1-\nu)}\int_{-1}^{1} f_{2c}v\,dx \qquad (C,1)$$

where we have abbreviated by writing

$$C_{11}(\alpha) = \frac{2\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}} - \frac{(1-\nu)}{2}|\alpha| + \frac{1/2\,\epsilon^2}{\sqrt{\alpha^2+1/\epsilon^2}} - 2\epsilon^2\alpha^2\left(|\alpha| - \frac{\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}}\right), \qquad (C,2)$$

$$C_{12}(\alpha) = \alpha\left[\frac{1/2\,\epsilon^2}{\sqrt{\alpha^2+1/\epsilon^2}} - \left(|\alpha| - \frac{\alpha^2}{\sqrt{\alpha^2+1/\epsilon^2}}\right)\right]. \qquad (C,3)$$

Multiplying (3.2,51) with w(x) of (3.3,17) and integrating from x = -1 to x = 1, we obtain§

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[\,-iC_{12}(\alpha)\,\overline{\omega(\alpha)}\,\Omega(\alpha) + C_{22}(\alpha)\,\omega(\alpha)\overline{\omega(\alpha)}\,]\,d\alpha = \frac{c}{(1-\nu)}\int_{-1}^{1} f_{3c}w\,dx \qquad (C,4)$$

where $C_{12}(\alpha)$ is given by (C,3) and

$$C_{22}(\alpha) = \frac{\alpha^2}{2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}}. \qquad (C,5)$$

Adding (C,1) and (C,4), we have the following expression:

§ An interchange of orders of integration has been performed in obtaining (C.1) and (C,4). Thus our subsequent argument applies only to those solutions for which this interchange is valid.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[C_{11}(\alpha)\Omega(\alpha)\,\overline{\Omega(\alpha)} + iC_{12}(\alpha)\omega(\alpha)\,\overline{\Omega(\alpha)} - iC_{12}(\alpha)\overline{\omega(\alpha)}\,\Omega(\alpha) +$$

$$+ C_{22}(\alpha)\omega(\alpha)\overline{\omega(\alpha)}\Big]\,d\alpha = \frac{1}{(1-\nu)}\int_{-1}^{1}[\,f_{2c}v + cf_{3c}w]\,dx. \qquad (C,6)$$

It can be easily shown that the integrand in the left hand side of (C, 6) is positive definite for almost all α since

$$C_{11}(\alpha) = \frac{(1+\nu)}{2}|\alpha| + \frac{(\alpha^2 + \frac{1}{2\epsilon^2})}{2\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}\,[\,\alpha^2 + \frac{1}{2\epsilon^2} + |\alpha|\sqrt{\alpha^2+\frac{1}{\epsilon^2}})}$$

$$> 0 \text{ for all } \alpha,$$

$$C_{22}(\alpha) > 0 \text{ for all } \alpha \neq 0$$

$$= 0 \text{ for } \alpha = 0$$

and

$$C_{11}(\alpha)C_{22}(\alpha) - [\,C_{12}(\alpha)]^2 = \frac{(1+\nu)\,|\alpha|^3}{4\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}} + \frac{|\alpha|^3}{4\epsilon^2\sqrt{\alpha^2+1/\epsilon^2}\,(\alpha^2 + \frac{1}{2\epsilon^2} + |\alpha|\sqrt{\alpha^2+\frac{1}{\epsilon^2}})}$$

$$> 0 \text{ for all } \alpha \neq 0$$

$$= 0 \text{ for } \alpha = 0.$$

Hence, $\Omega(\alpha) = \omega(\alpha) = 0$ if $f_{2c} = f_{3c} = 0$, so that if the solution to this system exists, it is unique.

The uniqueness for the other cases can be similarly established by considering the appropriate dual integral equations.

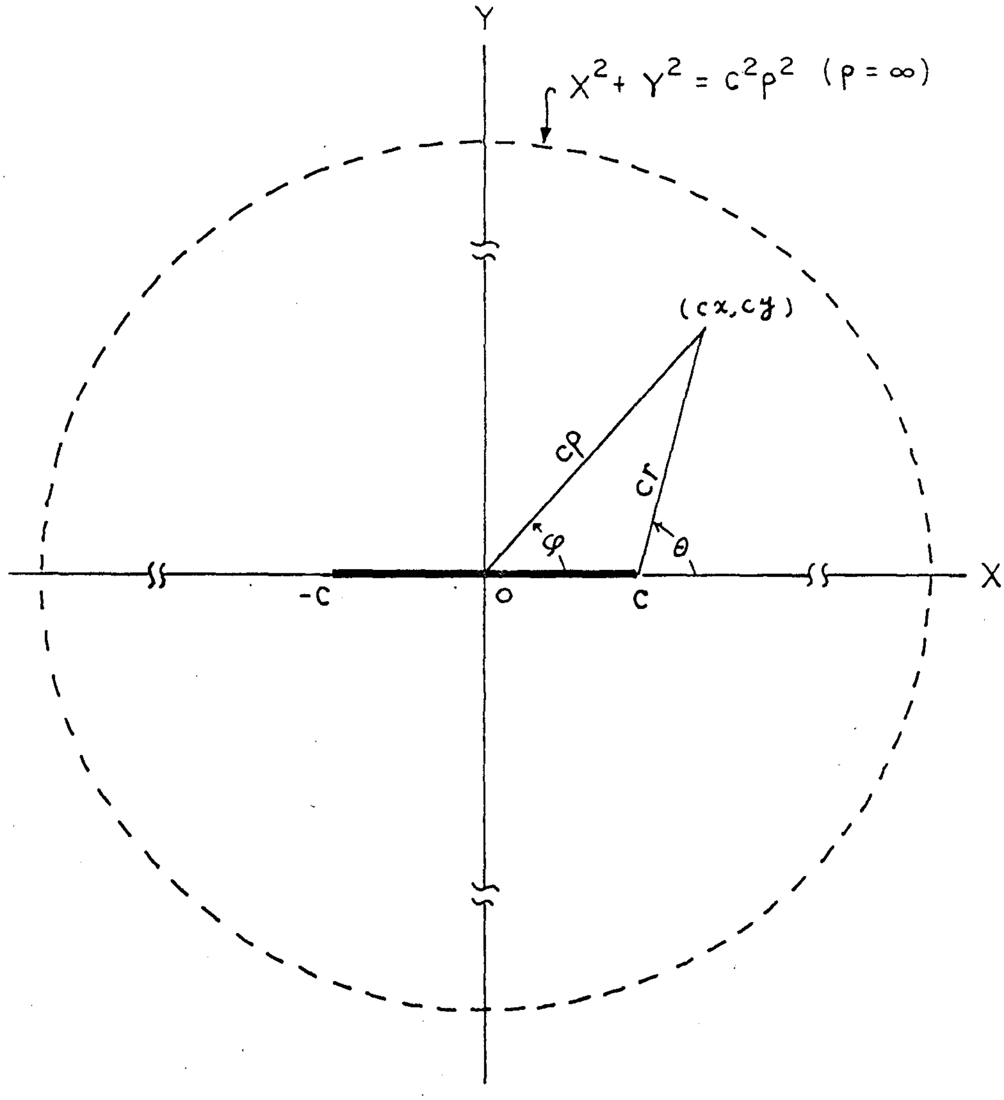

Figure 1a. Midplane of an elastic plate containing a line discontinuity

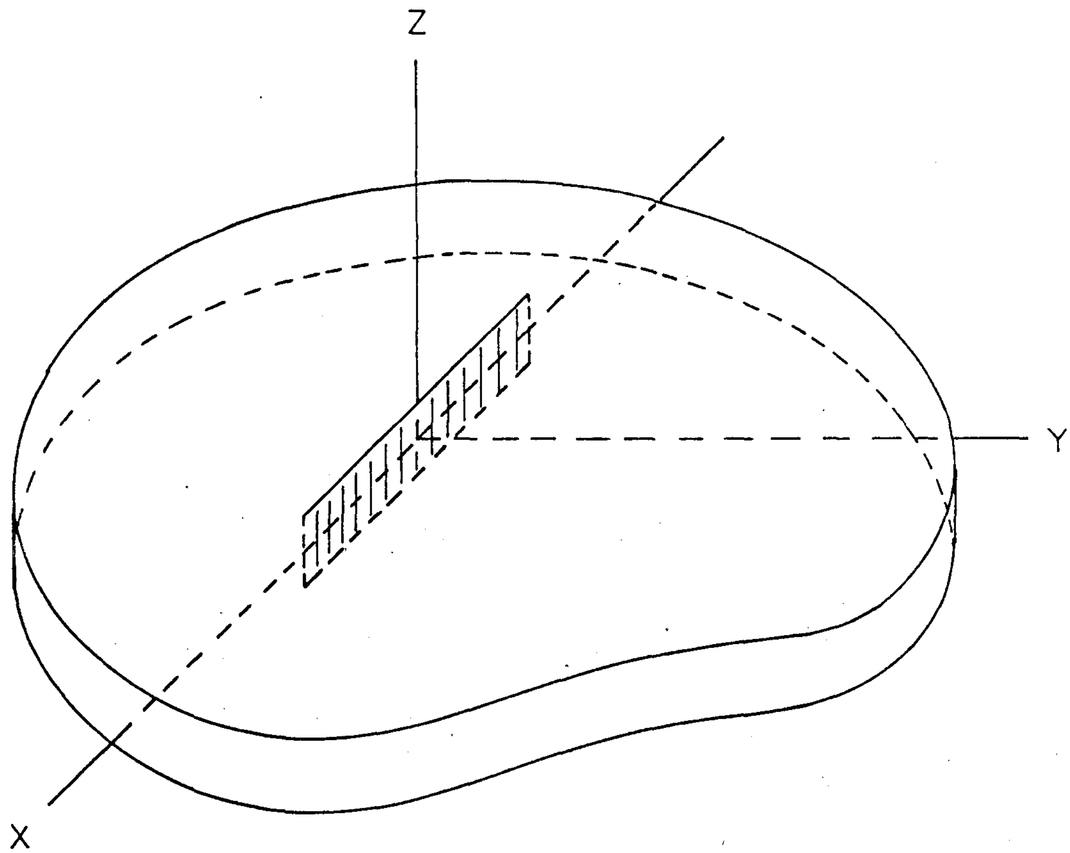

Figure 1b. Three dimensional view of the plate containing a surface of discontinuity

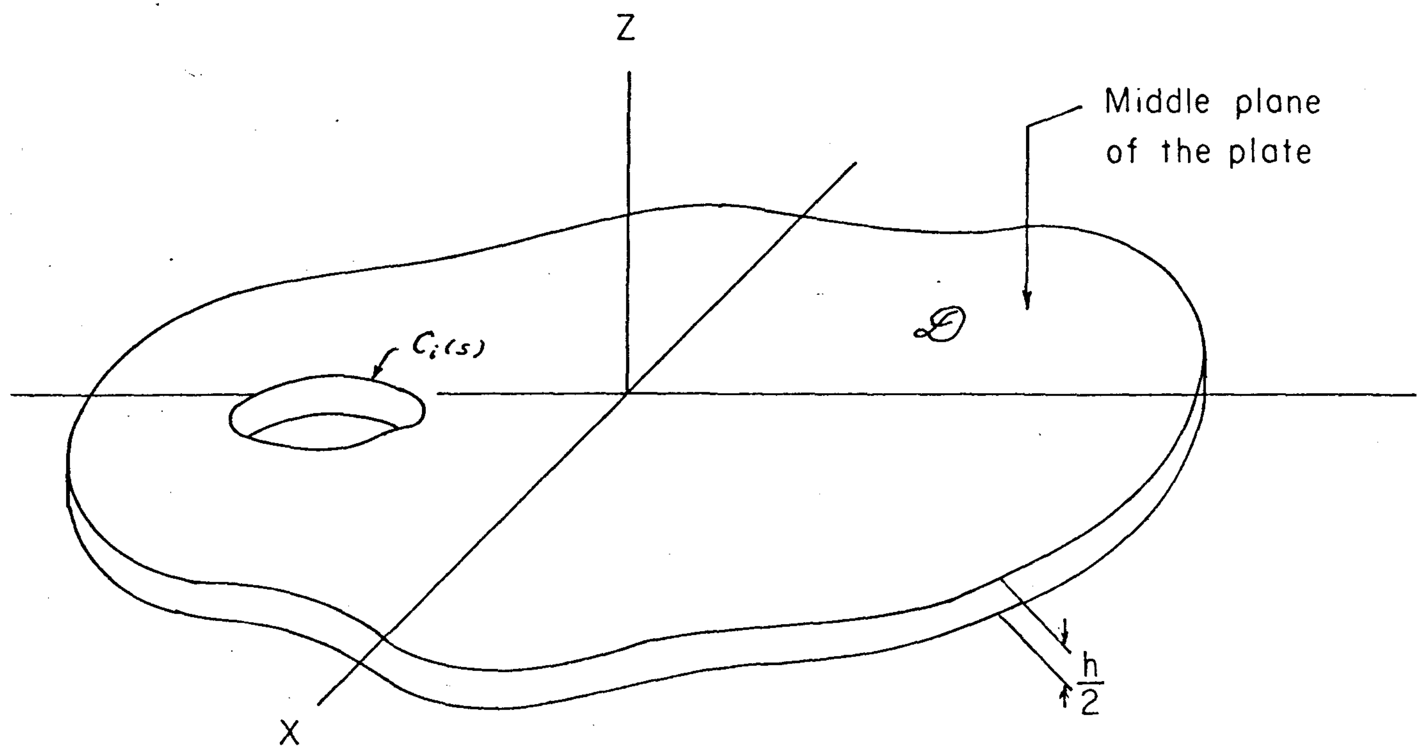

Figure 2. Coordinate system attached to a plate of constant thickness h

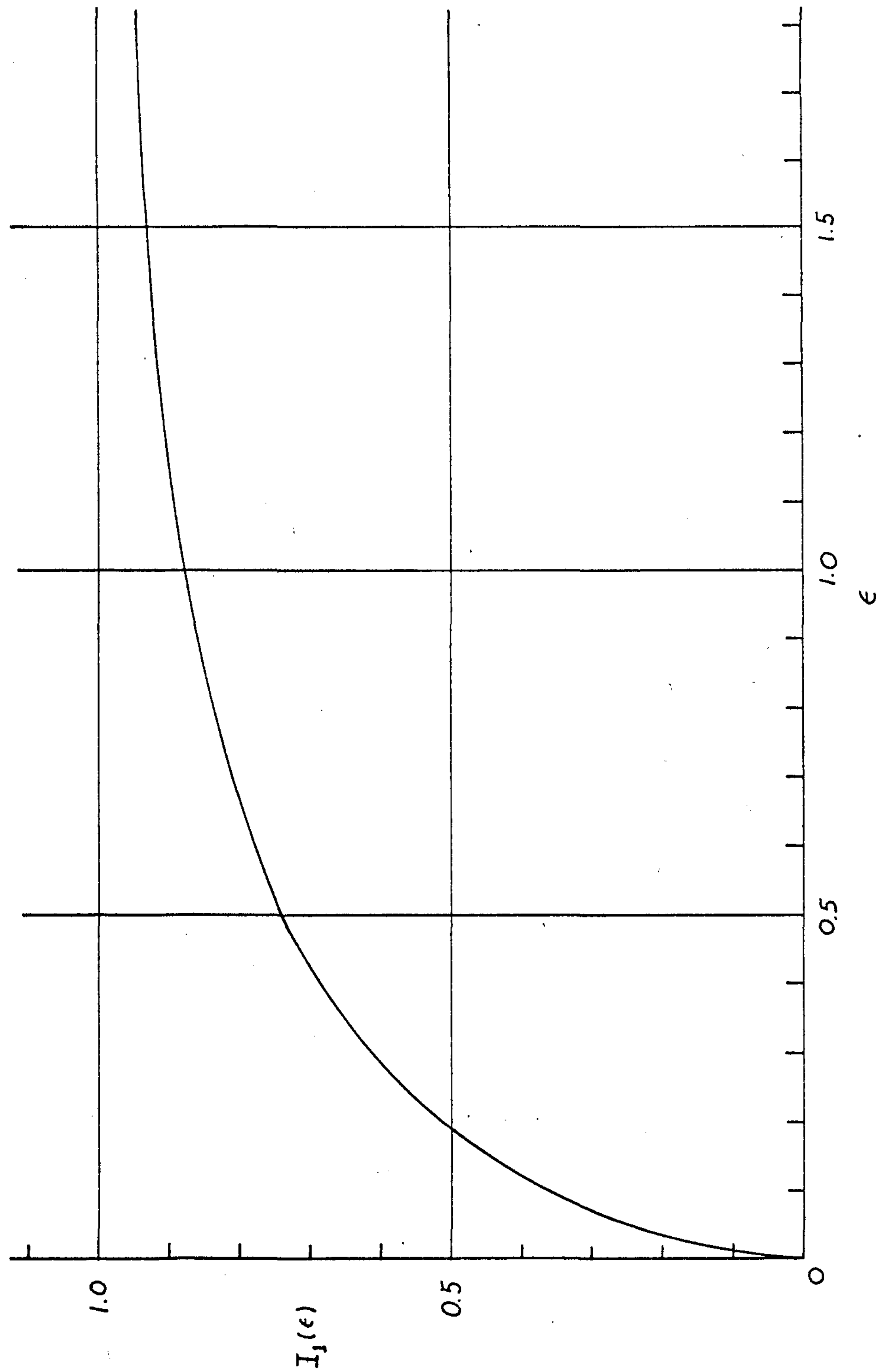

Figure 3. Plot of $I_1(\epsilon)$ against ϵ