A STUDY OF DISPERSION IN PLASMAS

Thesis by

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In Partial Fulfillment of the

Requirements for the Degree

of Doctor of Philosophy

California Institute of Technology

Pasadena, California

1962

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Dr. D. G. Dow and Professor R. W. Gould for their help and guidance in the formation of this thesis.

Many of the computations were done on the IBM 7090 computer at the Western Data Processing Center. The use of its facilities is gratefully acknowledged.

This work was done under Contract DA36-039 sc-85317, Department of the Army Task No. 3A99-13-001-05.

ABSTRACT

The wave propagation characteristics in a stationary unbounded plasma in a static arbitrary magnetic field are reviewed. If the plasma is drifting, a Lorentz transformation can be made to obtain the ω-k relations in the reference frame where the plasma is moving. The transformation is a simple one for the slow waves with non-relativistic drift velocities. The modified waves for the drifting plasma have an important bearing on the instability of the transverse modes in a system of a drifting plasma in a plasma.

The dispersion for bounded (cylindrical geometry) plasmas is studied, where the propagation vector is along the z-axis. The salient features of these waves are obtained by studying the circularly symmetric mode. The general features predicted by the quasi-static approximation are verified and the exact and quasi-static solutions are compared. The range of validity of the quasi-static approximation is determined and the h-values are plotted. As the radius becomes larger these modes evolve into the plane wave cases of the unbounded plasma.

When a fast beam of charged particles, electrons, ions or a plasma, traverses a stationary plasma in a magnetic field, there is a possibility of unstable transverse modes of propagation. If the drifting particles are either electrons or ions, the circularly polarized waves exhibit an instability over a very narrow frequency range near the ion and electron cyclotron frequencies respectively. When a plasma drifts through a plasma, in addition to the instabilities noted above, there can be an instability near zero frequency, and the growth condition is determined. For typical parameter values these transverse modes have greater growth constants than the longitudinal mode. These unstable transverse modes have possible applications in the generation of high frequencies, and are possible explanations for various instabilities in the ionosphere.

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I WAVES IN AN UNBOUNDED PLASMA

1.0 Introduction

The first important study of plasma oscillations was done in 1929 by Langmuir and Tonks (1). Noting the similarity of certain newly discovered oscillations in ionized gases to the oscillations of a jelly plasma, they christened the new oscillations, "plasma-electron oscillations" and also gave the name "plasma" to the nearly neutral part of an ionized gas. Since this initial impetus, much research has been done in the field. A history of plasma oscillations is found in (2).

This paper will deal with one aspect of plasmas, that is, the study of wave propagation in a plasma medium with emphasis on the case where an external static magnetic field is present, and also taking into account the effects of drifting charged particles for the unbounded case. Before these topics are discussed, the simple cases of wave propagation in a plasma will be reviewed.

1.1 Basic Relations

The different types of waves to be discussed are based on Maxwell's equations, a definition of current density, and the macroscopic equations of motion. In the analysis to follow, the equations will be linearized and waves of the form $e^{j(\omega t - k\cdot r)}$ will be assumed. The subscripts o and 1 denote the zero order and first order a.c. terms, respectively. Maxwell's equations are

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad (1.1a)$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \qquad (1.1b)$$

$$\nabla \cdot \bar{D} = \rho \tag{1.1c}$$

$$\nabla \cdot \bar{B} = 0 \tag{1.1d}$$

and the current density is defined for single velocity charged particles

$$\bar{J}_1 = \rho_o \bar{u}_1 + \rho_1 \bar{u}_o \quad . \tag{1.2}$$

The plasma effects are determined by an equation of motion. It should be mentioned at the outset that for most of the problems considered in this paper, a cold, collisionless plasma will be assumed. However, if the finite temperature of the plasma is to be taken into account, the Boltzmann equation (3) will be considered to be valid. This is

$$\frac{\partial f}{\partial t} + \bar{u} \cdot \nabla f + \frac{e}{m}(\bar{E} + \bar{u} \times \bar{B}) \cdot \nabla_u f = \left(\frac{\partial f}{\partial t}\right)_{coll.} \tag{1.3}$$

where f is the distribution function and E and B are the total fields. Pressure and gravitational effects will be considered negligible. The collision term on the right hand side bears some comment. It is difficult to formulate the collision term since a collision implies an interaction between at least two particles and a proper treatment requires the introduction of at least the two-body function. For the type of phenomena which will be studied, the collision term is negligible. If the effects of the collision have to be included, it will be sufficient to account for them in an approximate way. This can be done by assuming a mean collision frequency, and that each collision redistributes the velocities in a random way. This introduces a slight damping of the waves. For the plasma electrons, with no drift velocity

$$\bar{J}_1 = - ne \int \bar{u} f_1 du \tag{1.4}$$

where f_1 is the first order distribution function obtained by linearizing equation 1.3. It is seen that the plasma effects enter into the second of the listed Maxwell equations.

1.2 Dispersion Relations

Since there is primary concern with waves, the propagation constant will be obtained by solving the wave equation. The procedure to be used is to express the plasma as an anisotropic dielectric medium and then to solve the wave equation. This is done by defining

$$\bar{J} + \frac{\partial \bar{D}}{\partial t} = j\omega \underline{\epsilon} \cdot \bar{E} \quad . \tag{1.5}$$

With this definition the dielectric tensor can be written from the preceding equations. With no loss of generality k_y can be set equal to zero. Then, for the dielectric tensor of a thermal plasma with no magnetic field, assuming a Maxwellian distribution of velocities,

$$\underline{\underline{\epsilon}} = \epsilon_o \, \epsilon_{mn} \qquad m,n = 1,2,3 \tag{1.6}$$

where

$$\epsilon_{11} = 1 - \frac{\omega_e^2}{\omega^2} \left[1 + \frac{V_T^2}{3\omega^2} (3k_x^2 + k_z^2) \right]$$

$$\epsilon_{12} = \epsilon_{21} = 0$$

$$\epsilon_{13} = \epsilon_{31} = - \frac{2}{3} \frac{\omega_e^2}{\omega^4} V_T^2 \, k_x k_z$$

$$\epsilon_{22} = 1 - \frac{\omega_e^2}{\omega^2}\left[1 + \frac{V_T^2}{3\omega^2}(k_x^2 + k_z^2)\right]$$

$$\epsilon_{23} = \epsilon_{32} = 0$$

$$\epsilon_{33} = 1 - \frac{\omega_e^2}{\omega^2}\left[1 + \frac{V_T^2}{3\omega^2}(k_x^2 + 3k_z^2)\right]$$

$\omega_e^2 = \frac{\rho_o}{\epsilon_o}\frac{e}{m}$, the plasma frequency; $V_T^2 = \frac{3kT}{m}$, the mean thermal speed of the electrons. In the calculation of the current densities, an asymptotic expansion (4) for the integrand in equation 1.4 was made, and assuming waves whose phase velocities are much greater than the mean thermal velocity, the first two terms of the expansion were used. The dispersion is then correct to order $(u \cdot k/\omega)^2$. There is no Landau damping for this formulation. If a finite magnetic field is included, a differential equation for f_1 appears. The calculation for this case will not be done here, but is available in the literature (5).

The dispersion equation can now be formulated. The curl of equation 1.2a is taken, and using 1.2b and 1.5

$$\nabla \times \nabla \times \bar{E} - \omega^2 \mu \underline{\underline{\epsilon}} \cdot \bar{E} = \underline{\underline{L}} \cdot \bar{E} = 0 \qquad (1.7)$$

No drift velocities have been assumed and including any would have put additional terms into the dielectric tensor. The effects of ions can be added quite easily, proceeding in the identical manner as above. Later a beam and magnetic field will be added. Various cases will be studied in the following sections.

1.3 Waves in a Plasma

1.3.1 No magnetic field. For zero magnetic field the dispersion for the transverse wave is

$$k^2 = \frac{1}{\frac{1}{k_o^2} + \frac{\omega_e^2 V_T^2}{3\omega^4}} \left(1 - \frac{\omega_e^2}{\omega^2}\right) \tag{1.8}$$

where $k_o^2 = \omega^2/c^2$. This is a plane wave propagating in a dispersive medium. The characteristics can be seen in an ω-k diagram, see Fig. 1.1. For the longitudinal mode for low electron temperatures

$$k^2 = \frac{1}{V_T^2} (\omega^2 - \omega_e^2) \quad . \tag{1.9}$$

This ω-k diagram is shown in Fig. 1.2. It is seen that in the limit of very low temperatures, the electrons can have only one oscillation frequency regardless of the wavelength.

There are different types of electrostatic waves and a more complete discussion is found in (6).

1.3.2 Finite magnetic field. The inclusion of a magnetic field in the z-direction into the analysis introduces transverse modes which are of the most interest. The dielectric tensor, including ion effects but not temperature effects, is

$$\underline{\underline{\epsilon}} = \epsilon_o \, \epsilon_{mn} \qquad m,n = 1,2,3 \tag{1.10}$$

where

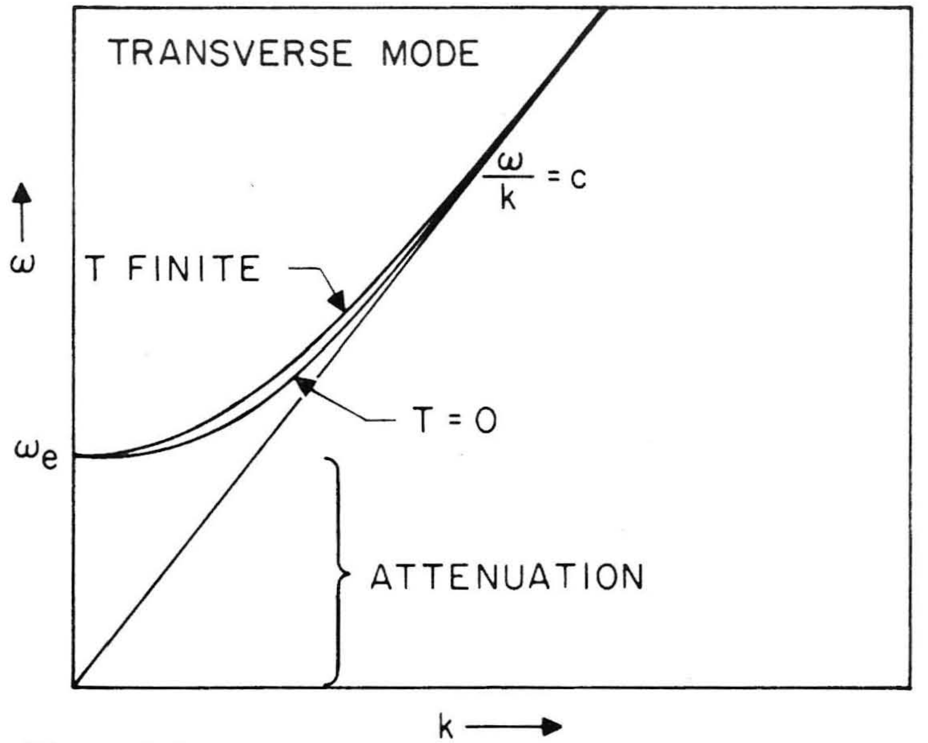


Figure 1.1

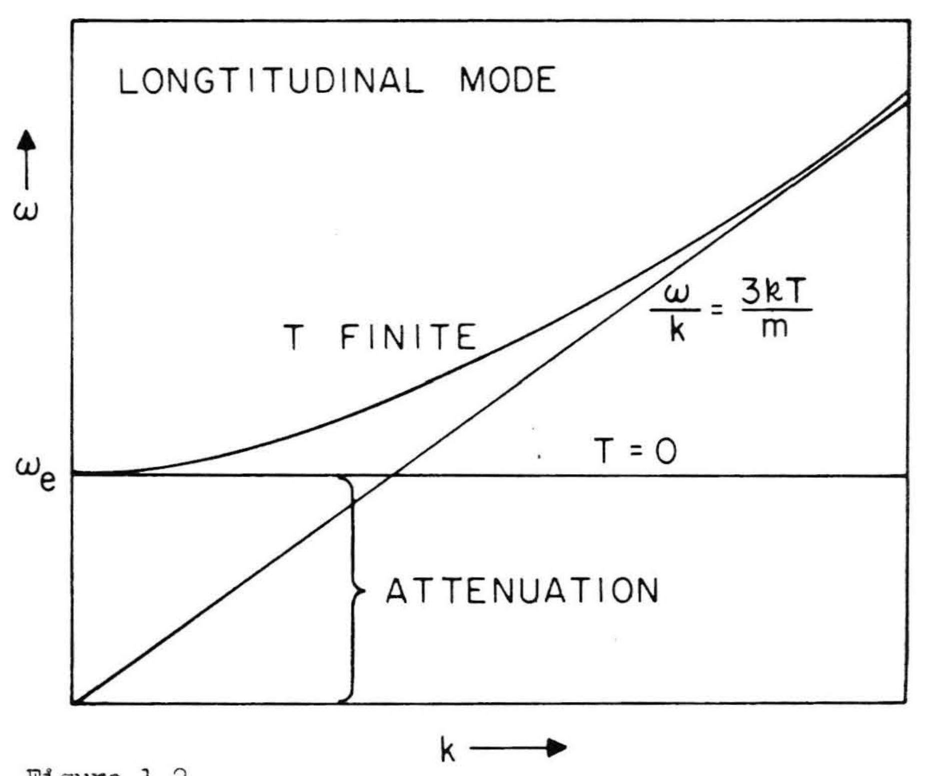


Figure 1.2

$$\epsilon_{11} = \epsilon_{22} = 1 - \frac{\omega_e^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_i^2}{\omega^2 - \omega_{ci}^2}$$

$$\epsilon_{12} = \epsilon_{21}^* = -j\left[\frac{\omega_e^2\,\omega_{ce}}{\omega(\omega^2 - \omega_{ce}^2)} - \frac{\omega_i^2\,\omega_{ci}}{\omega(\omega^2 - \omega_{ci}^2)}\right]$$

$$\epsilon_{13} = \epsilon_{31} = \epsilon_{23} = \epsilon_{32} = 0$$

$$\epsilon_{33} = 1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_i^2}{\omega^2} \quad .$$

Equation 1.10 was obtained from equations 1.3 and 1.5, setting the temperature equal to zero. The dispersion for a wave propagating in an arbitrary direction is

$$\left[\frac{c^2k_z^2}{\omega^2} - (1 - \frac{\omega_e^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_i^2}{\omega^2 - \omega_{ci}^2})\right]\left[\frac{c^2k^2}{\omega^2} - (1 - \frac{\omega_e^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_i^2}{\omega^2 - \omega_{ci}^2})\right] \times$$

$$\left[\frac{c^2k_x^2}{\omega^2} - (1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_i^2}{\omega^2})\right] - \frac{c^4}{\omega^4} k_x^2 k_z^2 \left[\frac{c^2k^2}{\omega^2} - (1 - \frac{\omega_e^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_i^2}{\omega^2 - \omega_{ci}^2})\right]$$

$$-\left[\frac{\omega_e^2\,\omega_{ce}}{\omega(\omega^2 - \omega_{ce}^2)} - \frac{\omega_i^2\,\omega_{ci}}{\omega(\omega^2 - \omega_{ci}^2)}\right]^2 \left[\frac{c^2k_x^2}{\omega^2} - (1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_i^2}{\omega^2})\right] = 0 \qquad (1.11)$$

The dispersion features four waves. The geometry is shown in Fig. 1.3. Waves are considered to propagate at an arbitrary angle to the magnetic field with the k-vector in the xz-plane. The waves in equation 1.11 have been studied extensively in the literature, see (7,8,9).

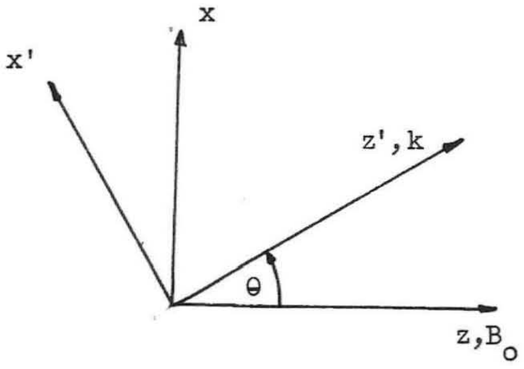


Figure 1.3.

<u>1.3.21 Propagation in the z-direction ($\theta = 0$).</u> For this case the dispersion becomes

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_e^2}{\omega(\omega \pm \omega_{ce})} - \frac{\omega_i^2}{\omega(\omega \mp \omega_{ci})} \tag{1.12}$$

The top sign applies if the ions gyrate in the same sense as the field, and the bottom sign applies if the electrons gyrate in the same sense as the field. In the very low frequency limit equation 1.12 reduces to

$$k^2 = \frac{\omega^2}{c^2}\left(1 + \frac{\omega_i^2}{\omega_{ci}^2}\right) \qquad \text{low frequency limit} \tag{1.13}$$

This gives the Alfvén waves when $\omega_i^2 \gg \omega_{ci}^2$. In the high frequency limit the ions can be assumed to be stationary, and their effects can be neglected. This will be valid for $\omega \gg \sqrt{\omega_{ce}\,\omega_{ci}}$. Then,

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_e^2}{\omega^2}\,\frac{1}{1 \mp \frac{\omega_{ce}}{\omega}}\right) \quad \text{high frequency limit .} \tag{1.14}$$

An ω-k plot is shown in Fig. 1.4 for a hydrogen plasma for different values of the ratio ω_i/ω_{ci} .

It is instructive to see the amount of Faraday rotation a linearly polarized wave can undergo in traversing a plasma. Below are tabulated a few values. The rotation is

$$2\theta = (k_+ - k_-)z \tag{1.15}$$

TABLE 1

FARADAY ROTATION IN A HYDROGEN PLASMA

2θ in radians per meter per gauss

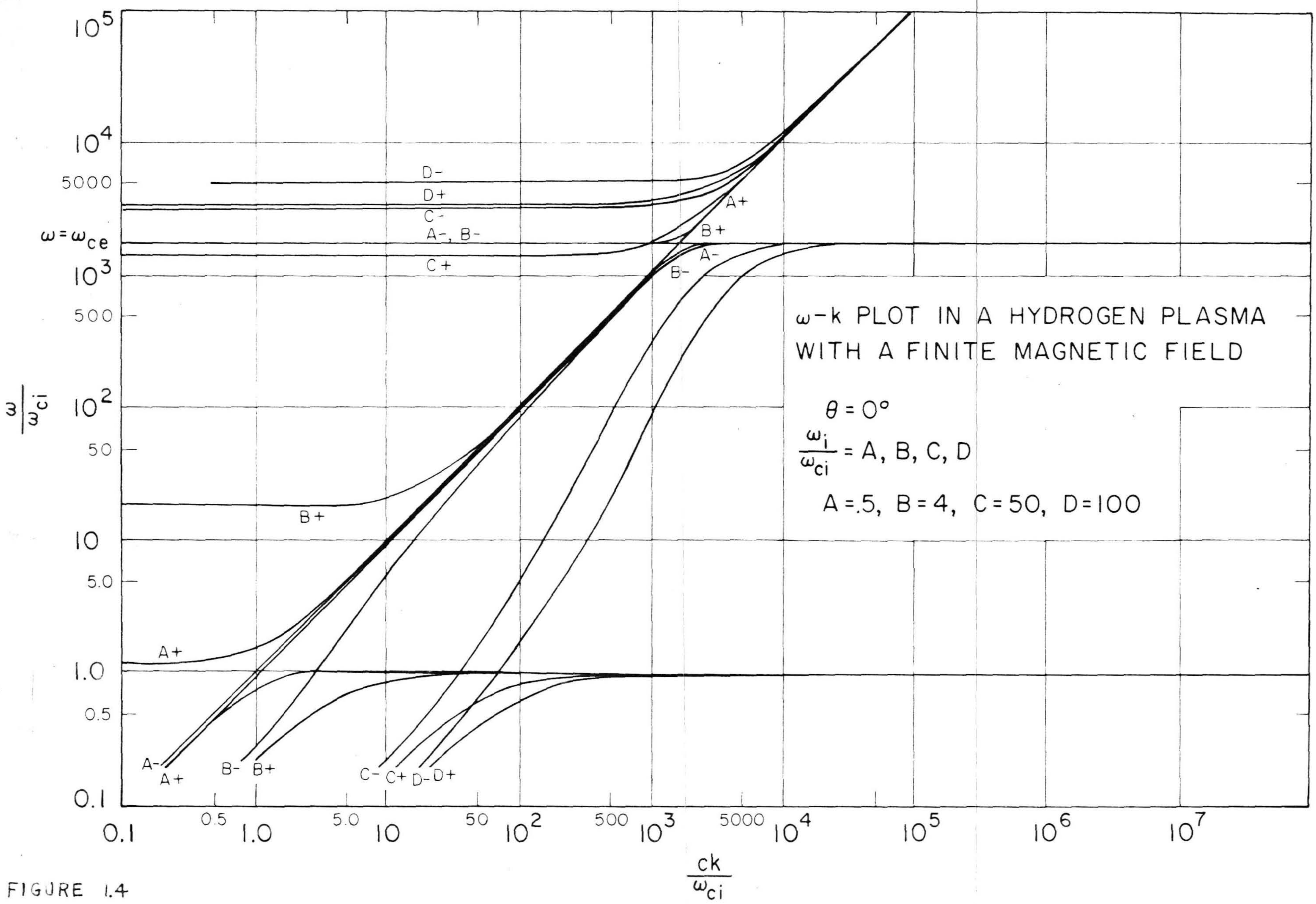
$\frac{\omega}{\omega_{ci}}$	$\omega_i/\omega_{ci} = 4$	$\omega_i/\omega_{ci} = 50$	$\omega_i/\omega_{ci} = 100$
.2	5.1×10^{-6}	6.6×10^{-5}	1.3×10^{-4}
.8	1.49×10^{-4}	1.86×10^{-3}	3.72×10^{-3}
26	-5.42×10^{-4}	*	*
1500	-1.14×10^{-3}	*	*
1900	1.14×10^{-4}	*	*
4000	5.8×10^{-8}	2.69×10^{-3}	*
10000	negligible	2.53×10^{-3}	1.24×10^{-2}
15000	negligible	1.43×10^{-3}	5.25×10^{-3}

* There is no propagation for one of the waves at these frequencies, see Fig. 1.4.

1.3.22 Propagation perpendicular to the magnetic field ($\theta = \pi/2$).

The dispersion is

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_i^2}{\omega^2} \tag{1.16a}$$



ω-k PLOT IN A HYDROGEN PLASMA
WITH A FINITE MAGNETIC FIELD
$\theta = 0°$
$\frac{\omega_i}{\omega_{ci}} = A, B, C, D$
A=.5, B=4, C=50, D=100
$\frac{\omega}{\omega_{ci}}$
$\frac{ck}{\omega_{ci}}$
$\omega = \omega_{ce}$
D-
D+
C-
A-, B-
C+
A+
B+
A-
B-
A+
B+
A-
A+
B-
B+
C-
C+
D-
D+
10^5
10^4
5000
10^3
500
10^2
50
10
5.0
1.0
0.5
0.1
0.1
0.5
1.0
5.0
10
50
10^2
500
10^3
5000
10^4
10^5
10^6
10^7

FIGURE 1.4

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_e^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_i^2}{\omega^2 - \omega_{ci}^2} - \frac{\left[\frac{\omega_e^2\,\omega_{ce}}{\omega(\omega^2 - \omega_{ce}^2)} - \frac{\omega_i^2\,\omega_{ci}}{\omega(\omega^2 - \omega_{ci}^2)}\right]^2}{1 - \frac{\omega_e^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_i^2}{\omega^2 - \omega_{ci}^2}} \quad . \tag{1.16b}$$

The first equation is just the dispersion for the ordinary electromagnetic wave in a plasma and has been discussed previously. See Fig. 1.5 for the ω-k diagram for $\theta = \pi/2$.

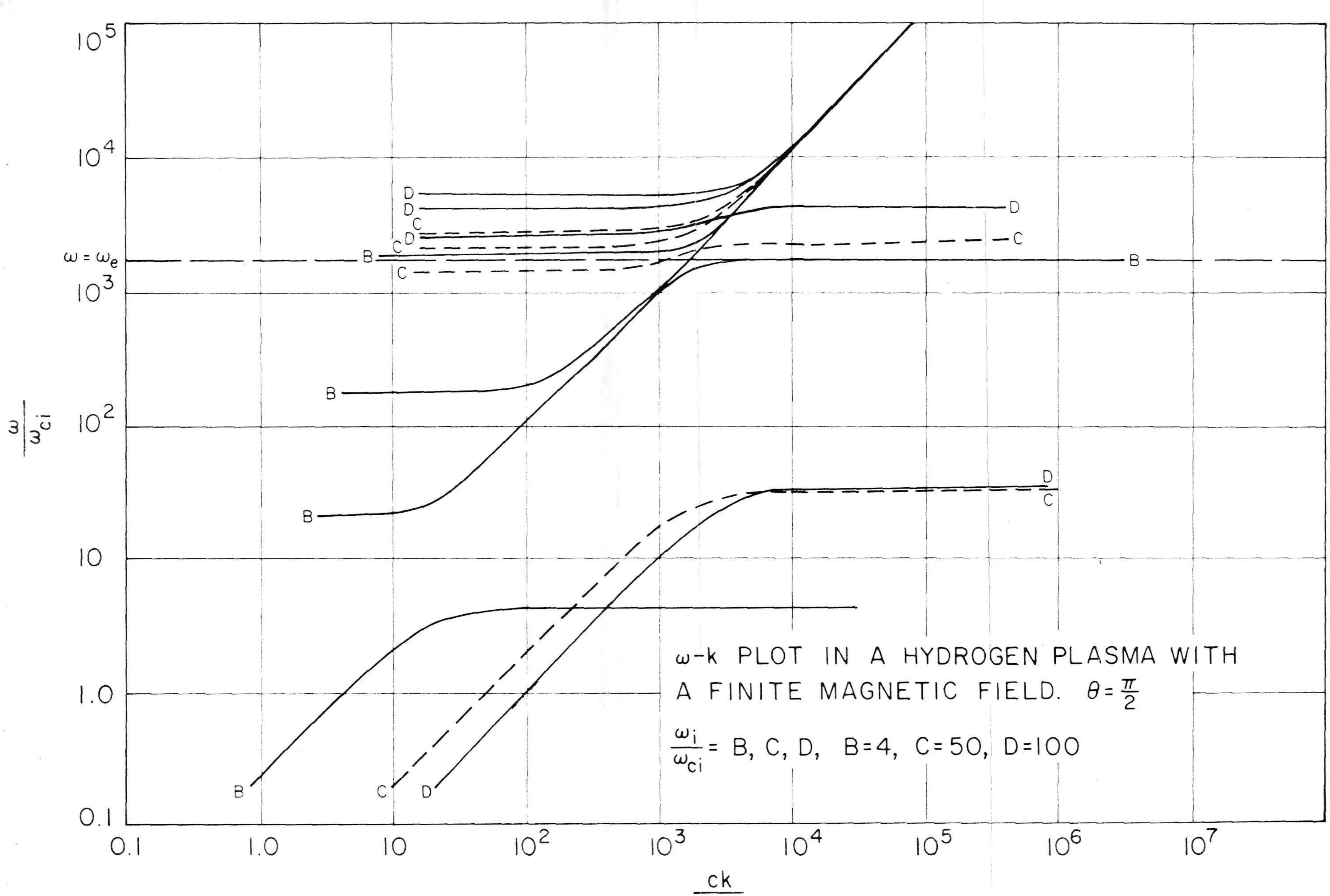
1.3.23 Propagation at an arbitrary angle. For this general case the dispersion is

$$\frac{c^2k^2}{\omega^2} = \frac{1}{2(\epsilon_{33}\cos^2\theta + \epsilon_{11}\sin^2\theta)} \left\{2\epsilon_{11}\epsilon_{33} - (\epsilon_{11}\epsilon_{33} - \epsilon_{11}^2|\epsilon_{12}|^2)\sin^2\theta \right.$$

$$\left. \pm\left[4\epsilon_{33}^2\,|\epsilon_{12}|^2\cos^2\theta + (\epsilon_{11}\epsilon_{33} - \epsilon_{11}^2 + |\epsilon_{12}|^2)\sin^4\theta\right]^{1/2}\right\} \tag{1.17}$$

Equation 1.17 describes a pair of oppositely elliptically polarized waves. An ω-k plot is shown in Fig. 1.6.

In isotropic media an elliptically polarized wave can always be separated into two linearly polarized waves, but cannot be so separated in anisotropic media, since the E and H vectors are constantly changing directions. At a boundary the conditions are that the tangential fields must be continuous. To do this in an anisotropic medium, one must take into account two components of tangential E at right angles to each other, and two components of tangential H . Thus for the general case, four waves are needed. These are available since the plasma can



ω-k PLOT IN A HYDROGEN PLASMA WITH
A FINITE MAGNETIC FIELD. $\theta = \frac{\pi}{2}$
$\frac{\omega_i}{\omega_{ci}}$ = B, C, D, B=4, C=50, D=100
$\omega = \omega_e$
$\frac{\omega}{\omega_{ci}}$
$\frac{ck}{\omega_{ci}}$
0.1
1.0
10
10^2
10^3
10^4
10^5
10^6
10^7

FIGURE 1.5

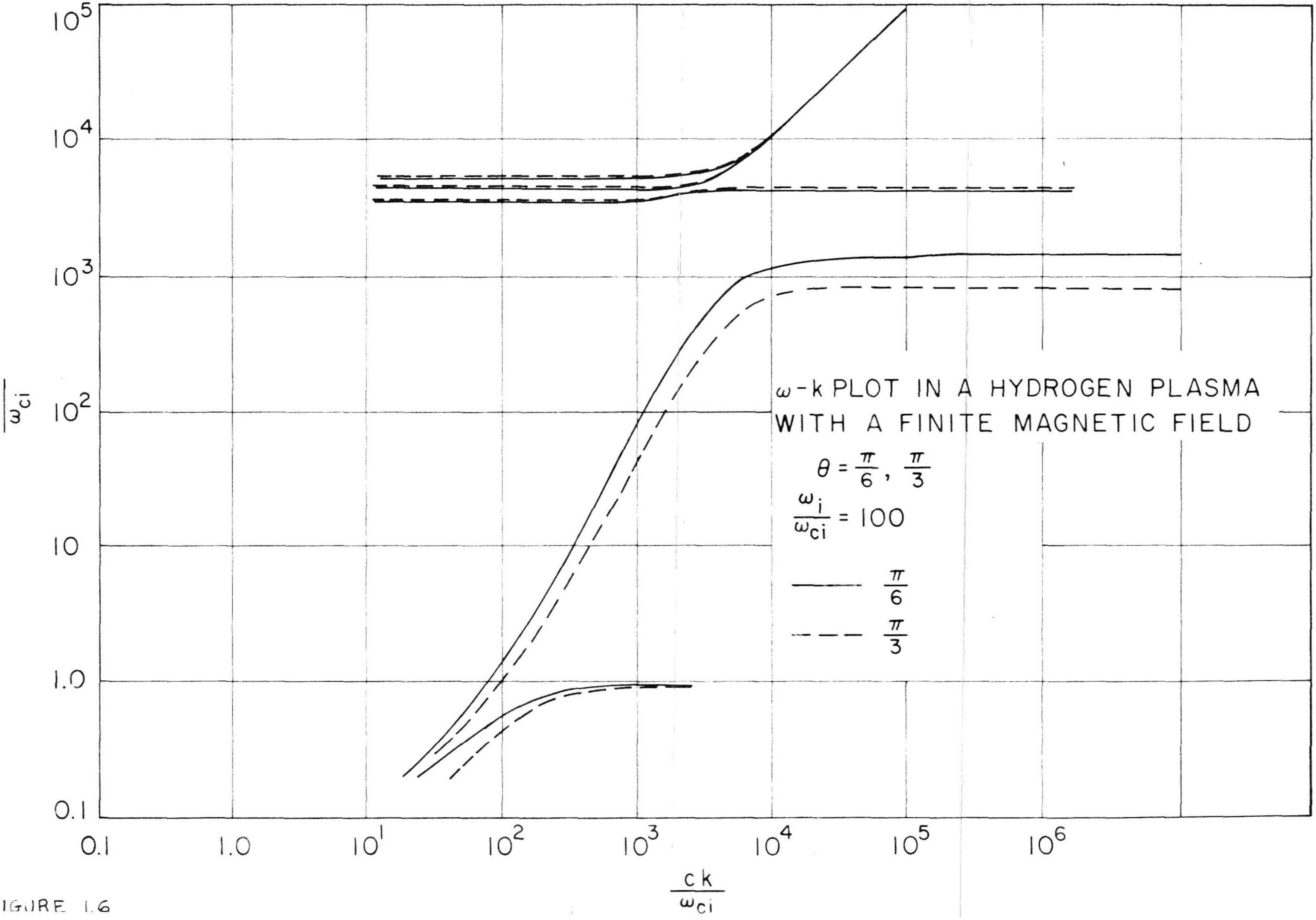

ω-k PLOT IN A HYDROGEN PLASMA
WITH A FINITE MAGNETIC FIELD
$\theta = \frac{\pi}{6}, \frac{\pi}{3}$
$\frac{\omega_i}{\omega_{ci}} = 100$
$\frac{\pi}{6}$
$\frac{\pi}{3}$
$\frac{\omega}{\omega_{ci}}$
$\frac{ck}{\omega_{ci}}$
10^5
10^4
10^3
10^2
10
1.0
0.1
1.0
10^1
10^2
10^3
10^4
10^5
10^6

FIGURE 1.6

support two pairs of waves. It is seen then that for the general case an incident wave produces two reflected waves in its own medium and two transmitted waves in the boundary medium.

Examining the limiting cases in Figs. 1.4 and 1.5 one can visualize the transition of the ω-k diagram as Θ varies from 0 to $\pi/2$. As the angle Θ increases, the cutoff at $\omega = \omega_{ci}$ decreases until at $\Theta = \pi/2$ it becomes zero. The cutoff at $\omega = \omega_{ce}$ decreases with increasing angle until its limiting value for the case when $\Theta = \pi/2$. The upper cutoff emerges when $\Theta \neq 0$ and remains even in the limiting case when $\Theta = \pi/2$.

1.3.24 Polarization of the fields. It has been mentioned that the waves propagating in the z-direction are right and left circularly polarized, and the waves at an arbitrary angle are right and left elliptically polarized. The relations will be derived below.

Solving for the electric fields, one obtains (see Appendix I)

$$E_x = j \frac{E_o d k^2 \sin\Theta \cos\Theta}{\omega} \tag{1.18a}$$

$$E_y = - E_o b\, k^2 \sin\Theta \cos\Theta \tag{1.18b}$$

$$E_z = j \frac{E_o g\, k^2 \cos^2\Theta}{\omega} \tag{1.18c}$$

where the terms a,b,g are defined in Appendix I, E_o is an arbitrary amplitude, the factor $e^{j(\omega t - jk \cdot r)}$ is understood, and k is the propagation constant defined in equation 1.17. In the coordinate system where k is along the z axis (see Fig. 1.3), using the transformation

$$E'_x = E_x \cos\theta - E_z \sin\theta$$

$$E'_y = E_y$$

$$E'_z = -E_x \sin\theta + E_z \cos\theta$$

the fields in $x'y'z'$ system become,

$$E'_x = j\,\frac{E_o k^2 \cos^2\theta \sin\theta (d-g)}{\omega} \qquad (1.19a)$$

$$E'_y = -E_o b\, k^2 \sin\theta \cos\theta \qquad (1.19b)$$

$$E'_z = -j\,\frac{E_o k^2 \cos\theta\,(d \sin^2\theta - g \cos^2\theta)}{\omega} \qquad (1.19c)$$

The nature of the polarization can be determined by examining the ratio E'_x/E'_y . This is

$$\frac{E'_x}{E'_y} = -\frac{j\cos\theta}{P}\left[(M-M_3)\tan^2\theta \pm \frac{1}{\cos^2\theta}\sqrt{4P^2\cos^2\theta + (M-M_3)^2\sin^4\theta}\right] \qquad (1.20)$$

The quantities M, M_3 and P are defined

$$M = \frac{\epsilon_{11}}{\epsilon_{11}^2 - \epsilon_{12}^2}$$

$$M_3 = \frac{1}{\epsilon_{33}}$$

$$P = -\frac{|\epsilon_{12}|}{\epsilon_{11}^2 - \epsilon_{12}^2}$$

where the ϵ's are obtained from equation 1.10. The amplitudes of E'_x and E'_y are not, in general, equal. However, they are $\pm\ \pi/2$ radians out of phase and this means the wave is elliptically polarized. Note that if $\theta = 0$

$$\frac{E'_x}{E'_y} = \pm\ j \tag{1.21}$$

and the waves are circularly polarized.

Note that the polarization in equation 1.20 changes polarity for the + and - signs, so the waves are indeed oppositely elliptically polarized. Notice also that these waves have a field component in the direction of propagation, arising from the tensor properties of the medium.

Now consider the wave which propagates normal to the magnetic field. For equation 1.16a the wave is the ordinary electromagnetic wave of the plasma where the magnetic field has no effect and is linearly polarized. For the other case (equation 1.16b), the wave has a transverse and a longitudinal component of field, and these have a phase difference of $\pi/2$. From equation 1.18

$$\frac{E_x}{E_y} = j\,\frac{d}{b\omega} \quad . \tag{1.22}$$

The E vector rotates in an ellipse, but does so in the xy-plane where k is along the x-axis.

1.4 Drifting Plasma in a Magnetic Field

If the plasma has a drift velocity the phase characteristics become modified and it will be seen that under certain conditions a drifting plasma in a plasma will support a growing wave for both the right and left circularly polarized waves. These unstable waves will be discussed in Chapter III. Consider the ω-k relation of a stationary plasma in a magnetic field where the direction of propagation is along the magnetic field. The ω-k relation in the system in which the plasma drifts at a velocity u_o in the direction of propagation can be obtained by a Lorentz transformation. If the primed system is the one in which the plasma is stationary, and the unprimed system is the one in which the plasma is drifting,

$$k_z = \frac{1}{\sqrt{1 - \frac{u_o^2}{c^2}}} \left[k_z' + \frac{u_o}{c} \frac{\omega'}{c} \right] \tag{1.23a}$$

$$k_x = k_x' = 0 \tag{1.23b}$$

$$k_y = k_y' = 0 \tag{1.23c}$$

$$\omega = \frac{1}{\sqrt{1 - \frac{u_o^2}{c^2}}} \left[\omega' + u_o k_z' \right] \tag{1.23d}$$

The term u_o/c will be assumed to be small. In the previous derivations relativistic effects were not taken into account. Then the term $1/\sqrt{1 - (u_o^2/c^2)}$ is approximately unity.

For the slow waves under discussion the second term on the right hand side of equation 1.23a can be neglected, since $\frac{u_o}{c} \frac{V_{ph}'}{c} \ll 1$. Then the ω-k relations for the drifting plasma are approximately

$$\omega = \omega' + u_o k' \tag{1.24a}$$

$$k \approx k' \tag{1.24b}$$

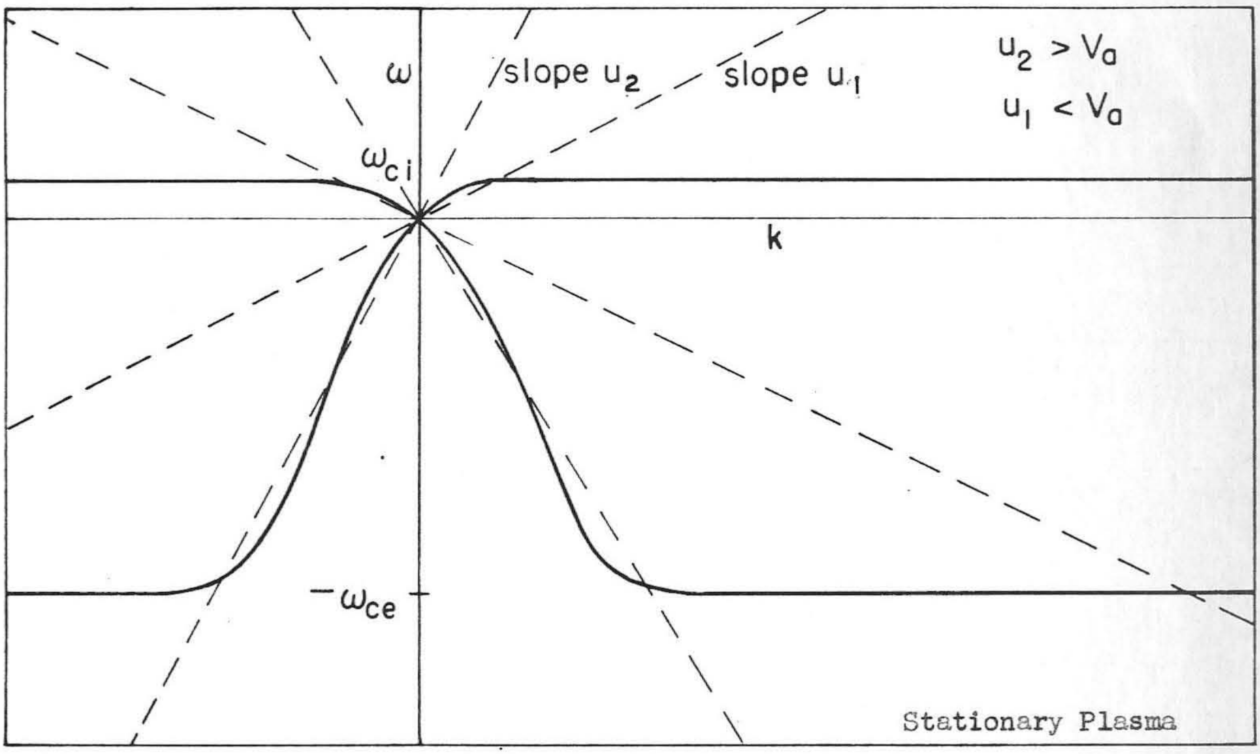
The longitudinal wave has been taken up elsewhere (10) and only the transverse waves will be discussed here. The right and left circularly polarized waves exhibit slightly different properties and will be discussed separately. The dispersion (see Appendix II) for a drifting plasma is

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_e^2(\omega - u_o k)}{\omega^2(\omega - u_o k \pm \omega_{ce})} - \frac{\omega_i^2(\omega - u_o k)}{\omega^2(\omega - u_o k \mp \omega_{ci})} \tag{1.25}$$

The top sign applies if the ions gyrate in the same direction as the E-vector and the bottom sign applies if the plasma electrons gyrate in the same direction as the field. These two cases will now be examined in some detail.

<u>Ions Gyrate in the Same Sense as the Field.</u>

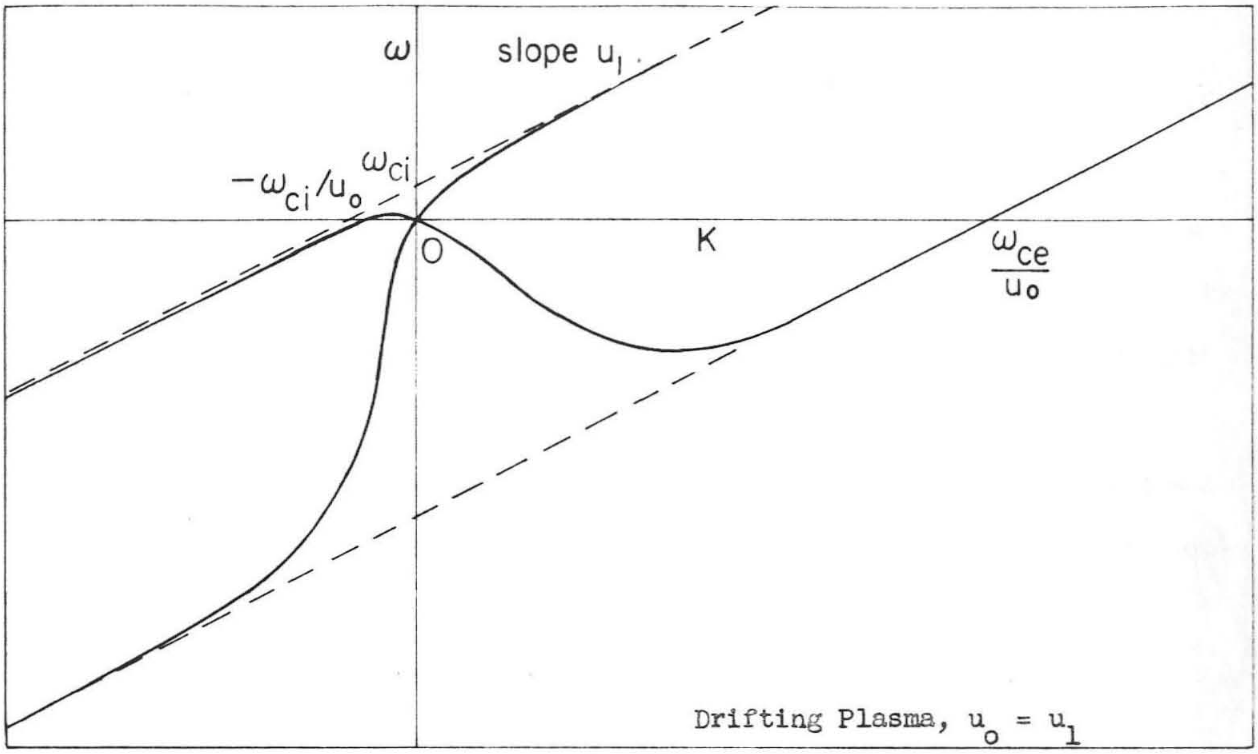
Consider the ω-k relation of the wave whose field gyrates in the same sense as the ions for positive frequencies. The case for the stationary plasma is shown in Fig. 1.7a. If a drift velocity is introduced using the relations in equations 1.37 and 1.38 one can visualize the new ω-k curve by adding $u_o k$ to ω for the stationary plasma for a given ω . This new curve is approximately a rotation of Fig. 1.7a through an angle θ , where θ is the angle whose arc tangent is the drift velocity. It should be remembered that this construction by rotation is not precise, except for the cutoff frequencies, and is just a qualitative way of visualizing the ω-k relation for the drifting plasma. This



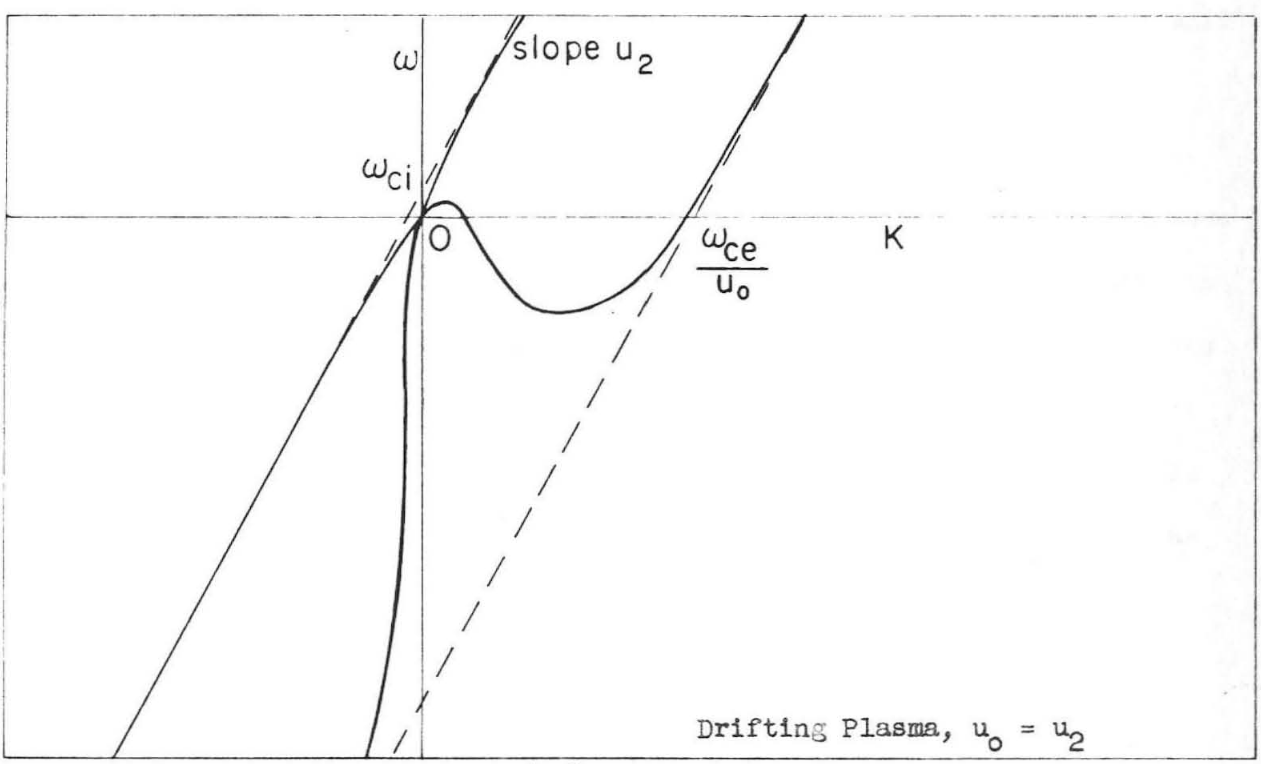
(a)

Figure 1.7. ω-k diagrams for stationary and drifting plasmas

Figure 1.7 (continued)



(b)



(c)

is shown in Fig. 1.7b. Now, as the drift velocity is increased, there is a value at which the drift velocity becomes equal to the maximum phase velocity of the positive frequency branch of the plasma wave. If the drift velocity is increased further, the ω-k curves become those shown in Fig. 1.7c. It is seen that there is a new slow wave for low frequencies. The important characteristic of this slow wave is that it is a negative energy carrying wave and later on it will be found that it is this wave which interacts with the slow wave of a stationary plasma to give instability. The curves at the right in Figs. 1.7b and 1.7c are associated with the slow cyclotron waves of the drifting electrons and these are not affected very much as the drift velocity is varied. This branch of the ω-k curve is approximately given by

$$k \approx \frac{\omega + \omega_{ce}}{u_o} \tag{1.26}$$

This wave will henceforth be neglected in the ensuing analysis, as will the wave which propagates at higher frequencies.

A plot of the slow waves for a given value of the ratio $\frac{\omega_i}{\omega_{ci}}$, for several values of drift velocity are shown in Fig. 1.8. It is seen that at a certain drift velocity the slow wave described above appears in the first quadrant of the ω-k diagram. Where this transition takes place will be calculated. It is sufficient to examine the dispersion for zero frequency. For $\omega = 0$ there are two roots for which k is zero, k_{01} and k_{02}. There are also two others, given by the equation

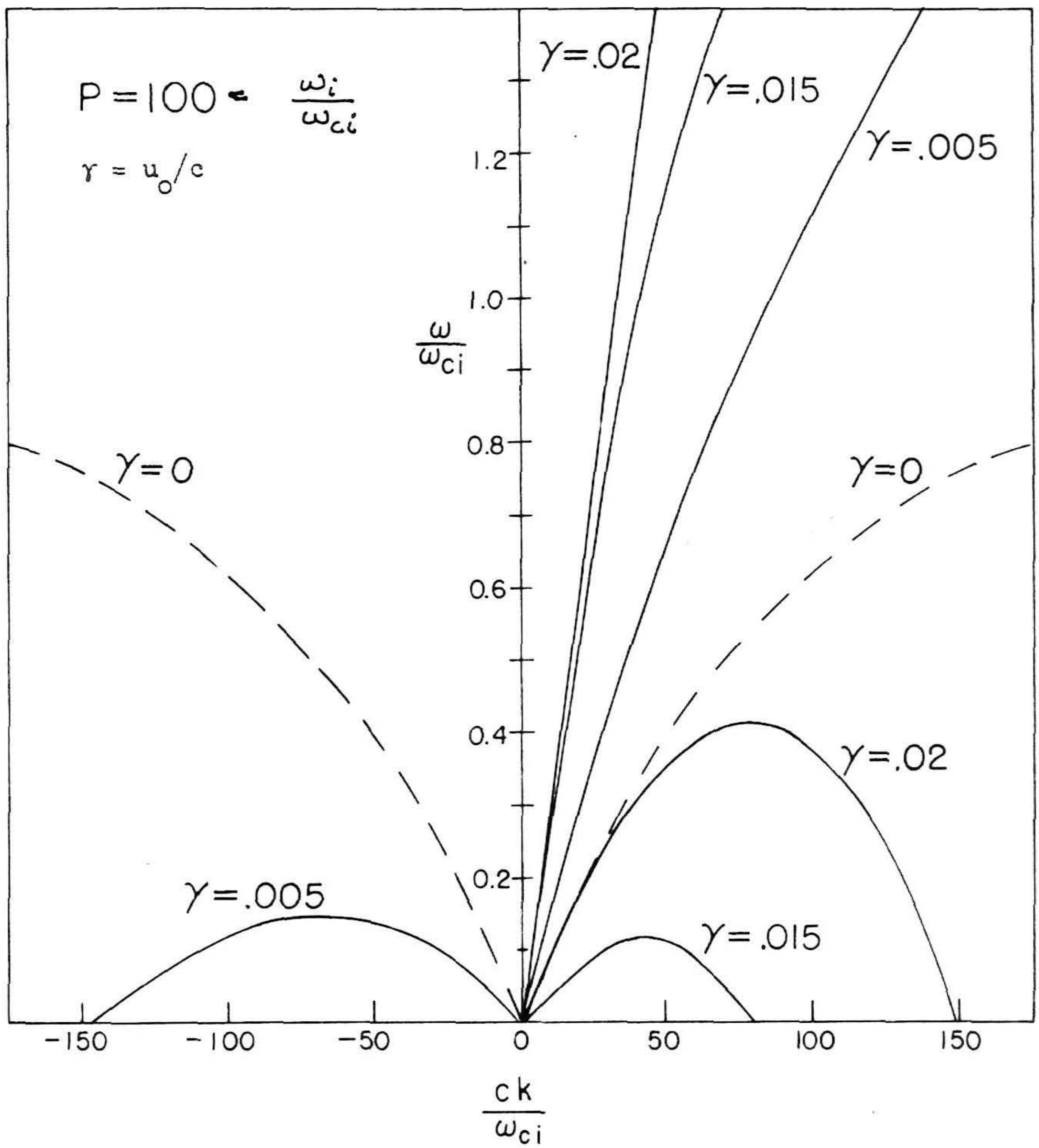


Figure 1.8. ω-k Diagram for a drifting plasma in the vicinity of the origin, where the ions gyrate in the same sense as the field.

$$k_{03}, k_{04} = \frac{\omega_{ce} - \omega_{ci}}{2u_o} \pm \sqrt{\left(\frac{\omega_{ce} - \omega_{ci}}{2u_o}\right)^2 + \frac{\omega_{ce}\,\omega_{ci}}{u_o^2}\left(1 - \frac{u_o^2}{V_a^2}\right)} \qquad (1.27)$$

where V_a is the Alfvén velocity $\left(V_a = \frac{c\omega_{ci}}{\omega_i}\right)$. The plus sign will give the value of k corresponding to the wave of equation 1.26. The wave corresponding to the minus sign is the one which can have the branches in the first or second quadrant. On examining equation 1.27 the determining condition can be readily calculated. This is

$$u_o > V_a \qquad \text{branch in first quadrant} \qquad (1.28)$$

$$u_o < V_a \qquad \text{branch in second quadrant} \qquad (1.29)$$

The condition of equation 1.28 is an important one for instability of a drifting plasma in a plasma. For an interaction one would want two waves near synchronism. The curves in Figure 1.8 could be qualitatively thought of as varying P instead of the drift velocity. As P goes from a small value to a large value, the wave starts with a branch in the first quadrant. Incidentally, if the drift velocity becomes high enough the roots in equation 1.27 become complex. This means that if the drift velocity is greater than the maximum $|V_{ph}|$ of the branch in the fourth quadrant, with the transformation, this branch appears entirely in the first quadrant of the ω-k plane.

<u>Electrons Gyrate in the Same Sense as the Field</u>

The case in which the electrons gyrate in the same sense as the field for positive frequencies can be visualized in the same way as the previous case. There is a modification to be made, however. For the stationary plasma the cutoff for positive frequency now occurs at $\omega = \omega_{ce}$ and at $\omega = -\omega_{ci}$ for negative frequency. The evolution of the ω-k curves as the drift velocity varies for this polarization can be seen in Figure 1.7 if one makes the conversion $\omega \to -\omega$, $k \to -k$. Thus one just has to rotate the curves in Figure 1.7 through π degrees to see the behavior for this polarization.

For small drift velocities the ω-k curves look like the solid ones in Figure 1.9. As the drift velocity increases, one visualizes the rotation in the ω-k plane and the ω-k diagram finally becomes like the dashed curves in Figure 1.9. It is seen that for this polarization there is no slow wave branch in the first quadrant as exhibited by the wave in the previous section, unless of course, a plasma could be generated in which the negatively charged particle has a charge to mass ratio less than that of the positively charged particle.

In Figure 1.9 it is seen that there are two possible ways in which the curves approach the k-axis. If u_o or p is large enough the dashed curves apply, otherwise the solid curves apply. The curve at the right corresponds to the slow cyclotron wave related to the drifting ions of the plasma. This is a negative energy-carrying wave. It is quite important whether this wave originates at the origin or at finite k. It will be shown that if it starts at the origin it is possible for this wave to interact with a stationary plasma wave and introduce an instability. This is a necessary but not sufficient condition for growing waves of a drifting plasma in a plasma.

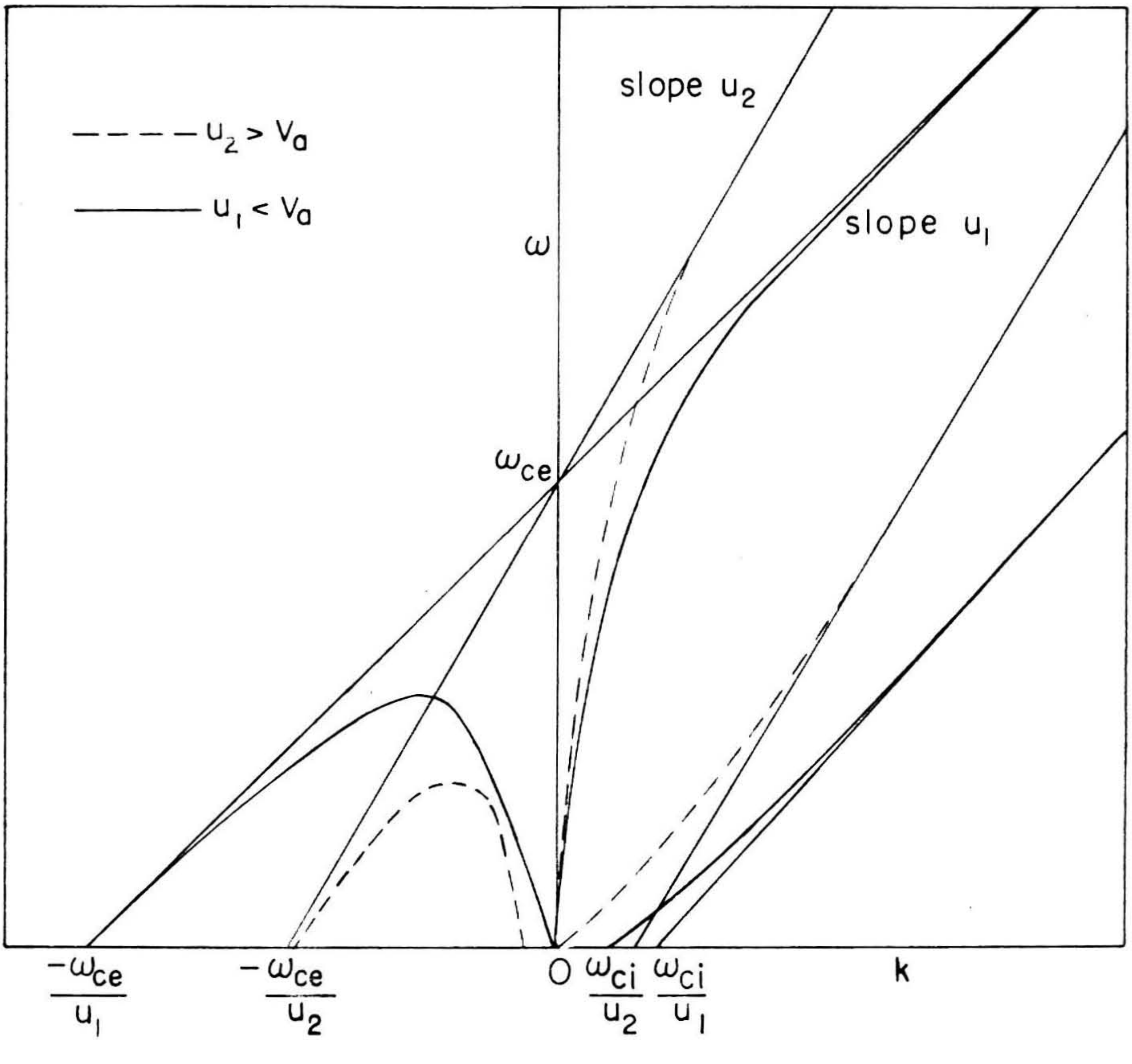


Figure 1.9. ω-k diagram for drifting plasma in the vicinity of the origin, where the electrons gyrate in the same sense as the field, for drift velocities greater and less than the Alfvén velocity.

The condition as to whether the slow wave described above starts at the origin or finite k can readily be determined. This is done by again examining the dispersion for $\omega = 0$. As before, there are two roots for which $k = 0$. Call these k_{o1}, k_{o2}. There are two finite roots given by the equation

$$k_{o3}, k_{o4} = -\frac{\omega_{ce} - \omega_{ci}}{2} \mp \sqrt{\left(\frac{\omega_{ce} - \omega_{ci}}{2}\right)^2 + \frac{\omega_{ce}\,\omega_{ci}}{u_o^2}\left(1 - \frac{u_o^2}{V_a^2}\right)} \qquad (1.30)$$

It is seen that there is always a negative root in the vicinity of $k = -\omega_{ce}/u_o$ for reasonable values of u_o and P. This is the wave corresponding to the fast cyclotron wave of the drifting electrons in a plasma. Now look at the value of k_{o4}. This can be seen to be positive or negative as u_o is smaller or larger than V_a. The condition can now be written

$$u_o > V_a \qquad k_{o4} \text{ negative for } \omega = 0 \qquad (1.31)$$

$$u_o < V_a \qquad k_{o4} \text{ positive for } \omega = 0 \,. \qquad (1.32)$$

For the second case the negative energy carrying wave starts at finite k and will not produce a growing wave if a stationary plasma is present. For the first case the negative energy carrying wave starts at the origin and a growing wave condition is possible if a stationary plasma is present.

II. SLOW WAVES IN BOUNDED PLASMAS, HIGH FREQUENCY CASE

2.0 Introduction

In the previous chapter waves in an unbounded plasma medium were studied. It will be instructive to see the nature of the waves in bounded cylindrical plasmas and note the evolution to the plane wave cases of Chapter I as the radius becomes larger. A distinguishing feature of the bounded plasma is the emergence of a slow, low frequency wave and, when a finite magnetic field is present, a backward wave. These waves have been studied by Gould and Trivelpiece (10,11) using a quasi-static approximation for several of the cases. Here the dispersion will be solved for exactly, and the results compared with the quasi-static solution (11). The emphasis will be on a plasma-filled waveguide in a magnetic field.

2.1 General Formulation

The derivation to be used is the same as before (Section 1.2). The plasma will be expressed as a dielectric tensor and this is inserted into the wave equation. This paper is primarily interested in high frequency waves and ion effects will be ignored. The frequency condition is the same as that specified previously, $\omega \gg \sqrt{\omega_{ce}\,\omega_{ci}}$. However, if one is interested in low frequencies, the dielectric tensor can be modified quite easily to include the ion effects. In cylindrical geometry with the conversion $1 \to r$, $2 \to \Theta$, $3 \to z$, the dielectric tensor is, for propagation in the z-direction, which is also the direction of the B-field,

$$\underline{\underline{\epsilon}} = \epsilon_o \begin{bmatrix} \epsilon_{11} & j\epsilon_{12} & 0 \\ -j\epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \tag{2.1}$$

where the elements of the tensor have been defined in equation 1.10, except that now the ion effects will be ignored. The reciprocal dielectric tensor is defined

$$\underline{\underline{\epsilon}}^{-1} = \frac{1}{\epsilon_o} \begin{bmatrix} M & jP & 0 \\ -jP & M & 0 \\ 0 & 0 & M_3 \end{bmatrix} \tag{2.2}$$

where

$$M = \frac{\epsilon_{11}}{\epsilon_{11}^2 - \epsilon_{12}^2}$$

$$P = - \frac{\epsilon_{12}}{\epsilon_{11}^2 - \epsilon_{12}^2}$$

$$M_3 = \frac{1}{\epsilon_{33}}$$

Using the first two of Maxwell's equations, the wave equation is

$$\nabla \times (\underline{\underline{\epsilon}}^{-1} \cdot \nabla \times H) = k_o^2 H \tag{2.3}$$

The required mathematics for this problem has been treated by Suhl and Walker (12), Gamo (13), Van Trier (14) and Epstein (15). In the analysis to follow the wave equation will be solved in an analogous manner to that done by Epstein for a gyromagnetic medium. This is worked out in Appendix I. It is seen that there are six equations for the field quantities and, in general, all of them are non-zero so there are no pure E and H modes except for special cases. Before the general case is investigated, the two simple cases will be studied; that of the infinite and the zero magnetic field.

2.2 Infinite Magnetic Field

For this case $\epsilon_{11} = \epsilon_{22} = 1$, $\epsilon_{12} = 0$, and $\epsilon_{33} = 1 - \frac{\omega_e^2}{\omega^2}$. The plasma becomes a uniaxial dielectric with principal capacitivities ϵ_o, ϵ_o, $\epsilon_o(1 - \frac{\omega_e^2}{\omega^2})$. The properties of such a medium are well known and the basic equations in them have two solutions. For a derivation see Appendix I for the case where $\epsilon_{12} = 0$.

For the system of H-modes the wave equation becomes

$$\nabla_T^2 H_z + (k_o^2 - k_z^2)\, H_z = 0 \tag{2.4}$$

and for the E-modes

$$\nabla_T^2 E_z + (k_o^2 - k_z^2)(1 - \frac{\omega_e^2}{\omega^2})\, E_z = 0 \quad . \tag{2.5}$$

∇_T^2 is the transverse Laplacian in cylindrical coordinates

$$\nabla_T^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \quad .$$

Letting $h_1^2 = k_o^2 - k_z^2$, $h_2^2 = (k_o^2 - k_z^2)(1 - \frac{\omega_e^2}{\omega^2})$, the solutions to equations 2.4 and 2.5 are

$$H_z = H_o C_n(h_1 r)$$

$$E_z = E_o C_n(h_2 r) \tag{2.6}$$

where $C_n(hr)$ is finite at the origin. From Maxwell's equations the other field vectors are obtained.

$$E_r = -\frac{1}{h_1^2}\left[jk_z \frac{\partial E_z}{\partial r} + \frac{j\omega\mu}{r}\frac{\partial H_z}{\partial \theta}\right]$$

$$E_\theta = \frac{1}{h_1^2}\left[-j\frac{k_z}{r}\frac{\partial E_z}{\partial \theta} + j\omega\mu\frac{\partial H_z}{\partial r}\right]$$

$$H_r = \frac{1}{h_1^2}\left[j\frac{\omega\epsilon_o}{r}\frac{\partial E_z}{\partial \theta} - jk_z\frac{\partial H_z}{\partial r}\right]$$

$$H_\theta = -\frac{1}{h_1^2}\left[j\omega\epsilon_o\frac{\partial E_z}{\partial r} + jk_z\frac{\partial H_z}{\partial \theta}\right] \tag{2.7}$$

A few observations can be made. In general both E and H modes are needed to satisfy the boundary conditions. However, there is a special case when free space surrounds a plasma column. Pure E or H mode formulation is possible since the transverse dielectric constant is the same as that of free space. This can be seen by examining equation 2.7. Consider the E-mode of a dielectric rod. The dispersion is obtained by setting $E_{z1} = E_{z2}$ and $H_{\theta 1} = H_{\theta 2}$ at the boundary. The subscripts 1 and 2 refer to the outer and inner regions. However, if one tries to set $E_{\theta 1} = E_{\theta 2}$ at the boundary, one finds that this is impossible for a pure E-wave, except for the circularly symmetric mode. This is so because h^2 is different inside and outside the rod. For a plasma rod in an infinite magnetic field, the term multiplying the bracketed terms in equation 2.7 is $1/h_1^2$, so it is possible to match $E_{\theta 1} = E_{\theta 2}$ for any order angular mode. Therefore for this case, E or H mode formulation is possible for modes other than the circularly symmetric mode. The dispersion for a slow E-wave propagating along the plasma column of radius a , located in free space in an infinite magnetic field, is

$$\frac{J_n(h_2 a)}{K_n(h_1 a)} = \frac{-h_2 J_{n+1}(h_2 a) + \frac{n}{a} J_n(h_2 a)}{-h_1 K_{n+1}(h_1 a) + \frac{n}{a} K_n(h_1 a)} \tag{2.8}$$

If a dielectric other than free space surrounded the plasma except for the zero order angular mode, both E and H waves would be needed to match boundary conditions. The dispersion is obtained by equating tangential fields at dielectric boundaries and setting tangential E-fields to zero at conducting boundaries. For the latter case pure E or H mode formulation is again possible. The dispersion for the H-modes is the same as that of free space, and a discussion of these free space modes can be found elsewhere (16,17).

If the plasma completely fills a cylindrical waveguide, the dispersion is

$$J_n(ha) = 0 \qquad h^2 = (k_o^2 - k_z^2)\left(1 - \frac{\omega_e^2}{\omega^2}\right)$$

$$k_z^2 = k_o^2 - \frac{\rho_{n\nu}^2}{a^2\left(1 - \frac{\omega_e^2}{\omega^2}\right)} \tag{2.9}$$

where $\rho_{n\nu}$ is the ν^{th} root of J_n. The ω-k characteristic is shown in Fig. 2.1. If a approaches infinity it is seen that there is a solution for equation 2.11, if h approaches zero. This reduces to the TEM and longitudinal waves of a plasma in a magnetic field.

2.3 Zero Magnetic Field

When the magnetic field is zero, the plasma becomes isotropic and the problem can be treated exactly as a dielectric rod (18,19) although it is one which is dispersive and can even be negative. The wave equations inside the plasma are

$$(\nabla_T^2 - h_2^2)\, E_z = 0 \tag{2.10}$$

$$(\nabla_T^2 - h_2^2)\, H_z = 0 \tag{2.11}$$

where $h_2^2 = k_z^2 - k_o^2(1 - \frac{\omega_e^2}{\omega^2})$. The dispersion is obtained from the boundary condition. For instance, consider a plasma column surrounded by free space for $\omega < \omega_e$. Inside the plasma, excluding the common factor $e^{j(\omega t - n\theta - k_z z)}$,

$$\left.\begin{aligned} E_{z2} &= A\, I_n(h_2 r) \qquad (2.12) \\ H_{z2} &= B\, I_n(h_2 r) \qquad (2.13) \end{aligned}\right\} \quad h_2^2 = k_z^2 - k_o^2(1 - \frac{\omega_e^2}{\omega^2})$$

and outside the plasma

$$\left.\begin{aligned} E_{z1} &= C\, I_n(h_1 r) + D\, K_n(h_1 r) \qquad (2.14) \\ H_{z1} &= E\, I_n(h_1 r) + F\, K_n(h_1 r) \qquad (2.15) \end{aligned}\right\} \quad h_1^2 = k_z^2 - k_o^2$$

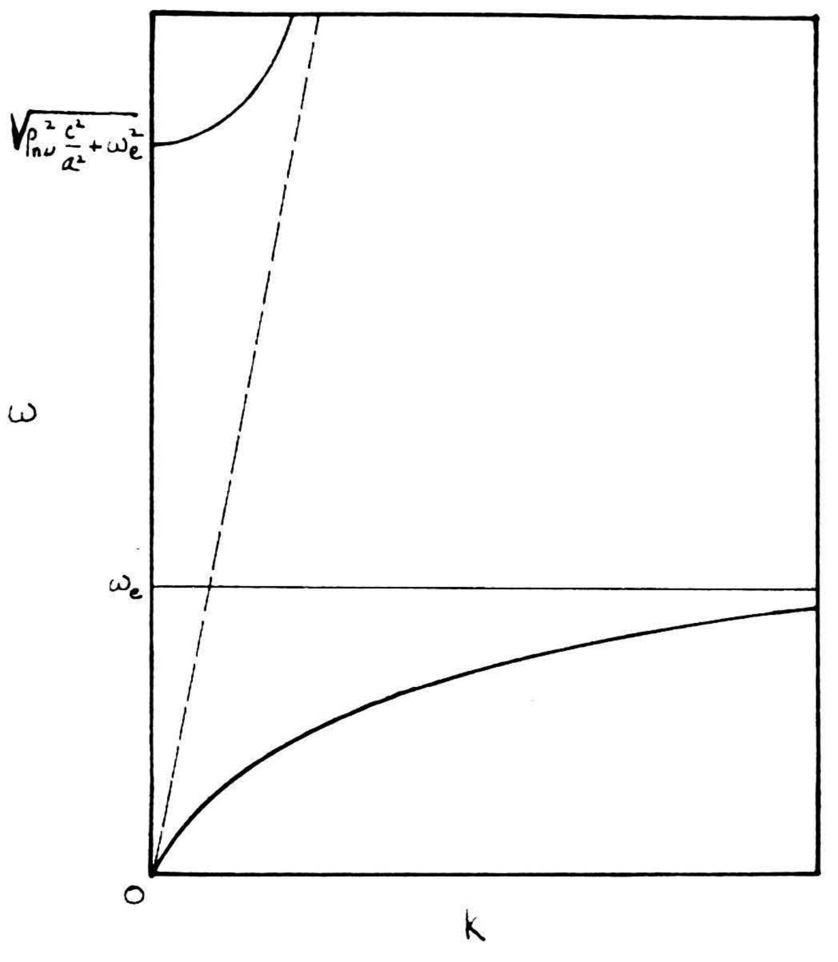


Figure 2.1. ω-k diagram of a plasma-filled waveguide in an infinite magnetic field.

The other field components can be obtained from the z fields. It is seen that there can be no slow wave for a plasma completely filling a waveguide. If free space extended to infinity, requiring the continuity of the tangential fields at the boundary, one would obtain for the dispersion

$$\left[\frac{1}{h_2}\frac{I_n'(h_2a)}{I_n(h_2a)} - \frac{1}{h_1}\frac{K_n'(h_1a)}{K_n(h_1a)}\right]\left[\frac{\mu\epsilon_1}{h_1}\frac{K_n'(h_1a)}{K_n(h_1a)} - \frac{\mu\epsilon_2}{h_2}\frac{I_n'(h_2a)}{I_n(h_2a)}\right] +$$

$$+\frac{n^2}{a^2}\left(\frac{1}{h_2^2} - \frac{1}{h_1^2}\right)\left(\frac{u\epsilon_2}{h_2^2} - \frac{u\epsilon_1}{h_1^2}\right) = 0 \qquad (2.16)$$

The dispersion for a second problem will be obtained, namely, that of a plasma column of radius a concentric to a cylindrical conductor of radius b. From the boundary conditions one can eliminate the constants and obtain

$$\left[\frac{h_1}{h_2}\frac{I_n'(h_2a)}{I_n(h_2a)} R_3 - R_4\right]\left[\frac{\mu\epsilon_2 h_1}{h_2}\frac{I_n'(h_2a)}{I_n(h_2a)} R_1 - \frac{\mu\epsilon_1 R_2}{h_1}\right]$$

$$+\frac{n^2}{a^2}\left(\frac{1}{h_2^2} - \frac{1}{h_1^2}\right)\left(\frac{u\epsilon_2}{h_2^2} - \frac{u\epsilon_1}{h_1^2}\right) h_1 R_1 R_2 = 0 \qquad (2.17)$$

where $R_1 = I_n(h_1a)\,K_n(h_1b) - I_n(h_1b)\,K_n(h_1a)$

$R_2 = I_n'(h_1a)\,K_n(h_1b) - I_n(h_1b)\,K_n'(h_1a)$

$R_3 = I_n(h_1a)\,K_n'(h_1b) - I_n'(h_1b)\,K_n(h_1a)$

$R_4 = I_n'(h_1a)\,K_n'(h_1b) - I_n'(h_1b)\,K_n'(h_1a)$.

Incidentally, if the permeabilities were different in the two media each μ accompanying each ϵ should be changed accordingly. If there are layers of dielectric, one proceeds in a similar manner, matching boundary conditions at each interface. The dispersions are quite unwieldy and it is sufficient to examine the zero order angular modes to obtain the cutoff characteristics.

2.3.1 Zero order angular modes. Setting $n = 0$, the E and H modes become uncoupled. The first bracketed term in equation 2.16 is the dispersion for the H-modes; this will be ignored. The second bracketed term gives the dispersion for the E-modes, and using recursion properties of the Bessel functions, it becomes

$$\frac{h_1}{h_2}\left(1 - \frac{\omega_e^2}{\omega^2}\right) = - \frac{I_o(h_2a)\ K_1(h_1a)}{I_1(h_2a)\ K_o(h_1a)} \tag{2.18}$$

And for the case of the plasma column concentric to a waveguide of radius b,

$$\frac{h_1}{h_2}\left(1 - \frac{\omega_e^2}{\omega^2}\right) = \frac{I_o(h_2a)}{I_1(h_2a)}\ \frac{[I_o(h_1a)\ K_o(h_1b) - I_o(h_1b)\ K_o(h_1a)]}{[I_1(h_1a)\ K_o(h_1b) + I_o(h_1b)\ K_1(h_1a)]} \tag{2.19}$$

The cutoff frequency can be obtained by letting k_z approach infinity and using asymptotic values of the modified Bessel functions for large argument. Both of the cases above have the same cutoff frequency.

$$\omega_{co} = \frac{\omega_e}{\sqrt{2}} \quad \text{for free space.} \tag{2.20}$$

If the dielectric surrounding the plasma had been other than free space the cutoff frequency would be

$$\omega_{co} = \frac{\omega_e}{\sqrt{1 + K_e}} \qquad \text{where} \quad K_e = \frac{\epsilon_1}{\epsilon_o} \tag{2.21}$$

The ω-k curve for the slow wave for equation 2.18 is shown in Fig. 2.2. Notice that there is a slow wave only in the region where the effective dielectric constant is negative. This can be shown if the expressions for h_1^2 and h_2^2 are examined. Eliminating the possibility of outwardly traveling waves (there is no line source of energy), it is seen that h_1^2 must be greater than zero. This means that $k_z > k_o$. If h_2^2 is examined, it is seen that it too is positive. The right hand side of equation 2.18 is always negative, and since h_1 and h_2 are always positive, the expression $(1 - \frac{\omega_e^2}{\omega^2})$ must be negative for a solution.

If the plasma completely fills the waveguide, one can treat this problem exactly as a dielectric filled waveguide.

2.4 Comparison with the Quasi-Static Approximation

To obtain an idea of the range of validity of the quasi-static approximation, a comparison will be made for two of the cases. In the quasi-static approximation for very slow waves, $(k_z \gg k_o)$, the curl of E is set equal to zero and then the electric field can be derived from a scalar potential. For the first case consider a plasma in an infinite magnetic field completely filling a cylindrical waveguide. The exact and quasi-static solutions are given below for $\omega < \omega_e$

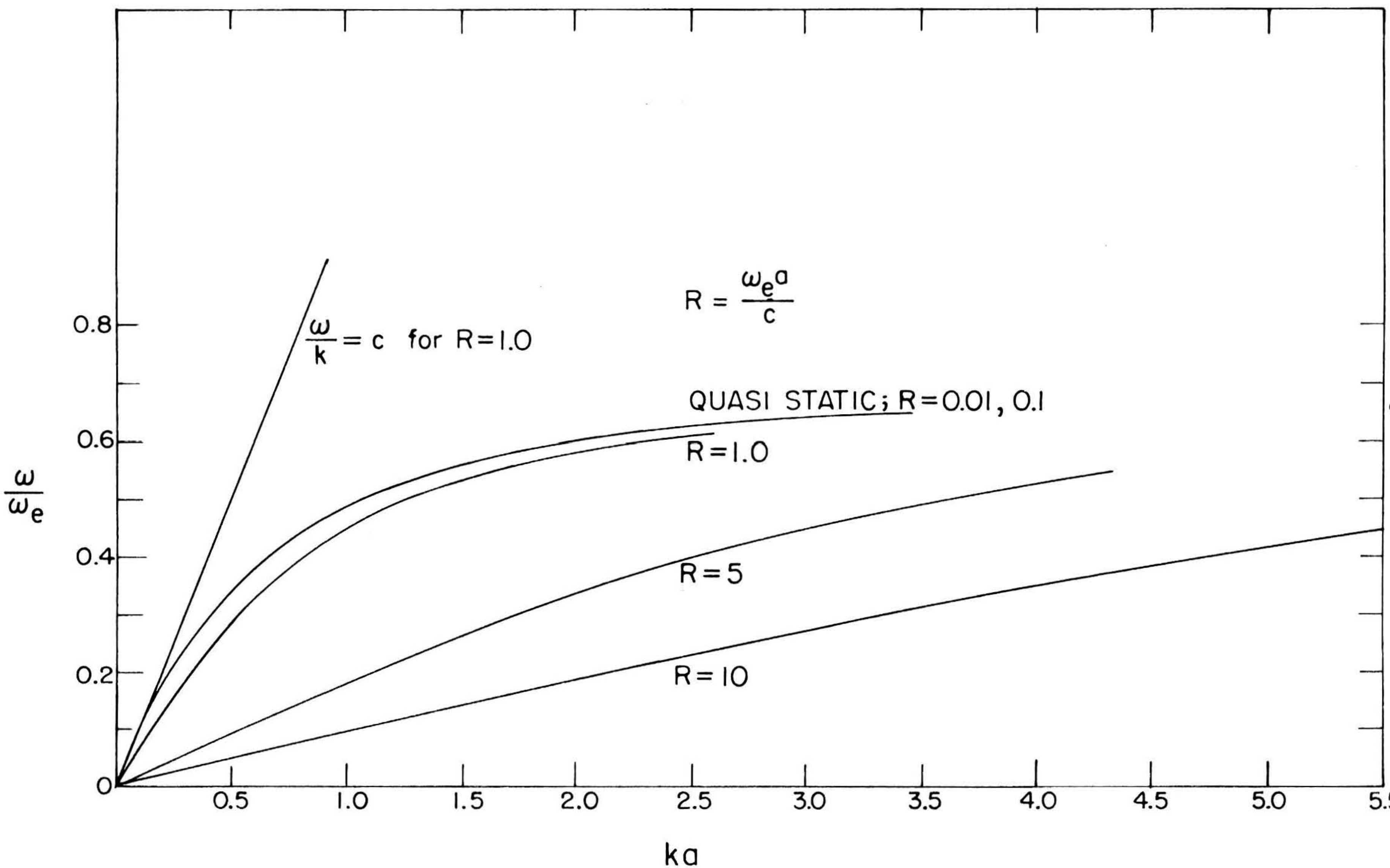


Figure 2.2. ω-k diagram for a plasma column in free space, zero magnetic field.

$$k_{z_{ex}}^2 = k_o^2 - \frac{\rho_{n\nu}^2}{a^2(1 - \frac{\omega_e^2}{\omega^2})} \qquad \text{exact} \tag{2.22}$$

$$k_{z_{qs}}^2 = - \frac{\rho_{n\nu}^2}{a^2(1 - \frac{\omega_e^2}{\omega^2})} \qquad \text{quasi-static}$$

One obtains by dividing the first equation by the second

$$\frac{k_{z_{ex}}^2}{k_{z_{qs}}^2} = 1 + \frac{1}{\rho_{n\nu}^2} \frac{\omega_e^2 a^2}{c^2} (1 - \frac{\omega^2}{\omega_e^2}) \qquad \omega < \omega_e \tag{2.23}$$

The second term of the right hand side is the error. It is seen that the exact solution is larger than the quasi-static solution and the deviation from the true solution is less for large $\rho_{n\nu}$ and ω/ω_e and small $\omega_e a/c$. If the normalized radius $\omega_e a/c$ is of the order of one or less, equation 2.23 can be written approximately

$$\frac{k_{z_{ex}}}{k_{z_{qs}}} \approx 1 + \frac{1}{2\rho_{n\nu}^2} \frac{\omega_e^2 a^2}{c^2} (1 - \frac{\omega^2}{\omega_e^2}) \ , \quad \omega < \omega_e \tag{2.24}$$

Now consider the solutions for a plasma column in free space with no magnetic field for the angularly independent mode.

$$\frac{h_1}{h_2} (1 - \frac{\omega_e^2}{\omega^2}) = - \frac{I_o(h_2 a)\, K_1(h_1 a)}{I_1(h_2 a)\, K_o(h_1 a)} \qquad \text{exact} \tag{2.25}$$

$$(1 - \frac{\omega_e^2}{\omega^2}) = - \frac{I_o(k_z a)\ K_1(k_z a)}{I_1(k_z a)\ K_o(k_z a)} \qquad \text{quasi-static} \qquad (2.26)$$

In this case the error cannot be solved for, exactly, but a few observations can be made. For low frequencies and large radii the quasi-static approximation becomes less valid. Looking at the expressions for h_1 and h_2, one can say that equation 2.26 applies if

$$\frac{k_z^2}{k_o^2} \gg 1 + \frac{\omega_e^2}{\omega^2} \quad . \qquad (2.27)$$

To obtain quantitative values the propagation constant should be calculated from both dispersions and compared. This has been done previously (10). Figure 2.2 shows the comparison between the true and quasi-static solutions for different values of a normalized radius $R (R = \omega_e a/c)$. It is seen that the discrepancy is less for smaller R, in fact the quasi-static and the true solutions for $R = .1$ and $.01$ cannot be resolved in Fig. 2.2. For larger R the waves travel at essentially the speed of light for $\omega \ll \omega_e$, see (10).

2.5 Finite Magnetic Field

2.5.1 Slow waves in a cylindrical waveguide. For this more general case the plasma becomes anisotropic and it is necessary to return to the formulation in Appendix I. As before the dispersion is obtained by matching the boundary conditions. In this section a specific problem will be solved; that of a plasma completely filling a waveguide in the presence of a longitudinal magnetic field, and a comparison will be made with the quasi-static solution.

The boundary condition to be satisfied is that at the conductor wall all tangential E-fields must be zero. Examining equation AI-27 it is seen that for a given h, this condition cannot be satisfied. However, there are two values of h; h_1 and h_2, and with a linear combination of the solutions using these values, a solution is possible. In general $h_1 \neq h_2$ since these parameters are different functions of k_z. In the subsequent analysis, the following notation will be used. Subscripts 1 and 2 will be used to denote whether the parameters belong to the system where h_1 or h_2 is used. If A_1 and A_2 are the amplitudes of systems 1 and 2, the boundary condition at $r = a$ (at the conductor) is

$$A_1 E_{1\phi} + A_2 E_{2\phi} = 0 \tag{2.28}$$

$$A_1 E_{1z} + A_2 E_{2z} = 0$$

These equations are compatible if

$$E_{1\phi} E_{2z} - E_{2\phi} E_{1z} = 0 \tag{2.29}$$

This is the needed dispersion. Written out, this is

$$-h_2^2 \tau_2 \left[b_1 h_1 C_n'(h_1 a) - \frac{nd_1}{a} C_n(h_1 a)\right] C_n(h_2 a) +$$

$$h_1^2 \tau_1 \left[b_2 h_2 C_n'(h_2 a) - \frac{nd_2}{a} C_n(h_2 a)\right] C_n(h_1 a) = 0 \tag{2.30}$$

As before $C_n'(hr)$ is the derivative with respect to the total argument. Due to the requirement of finiteness at the origin, $C_n(hr)$

can have only one form, $J_n(hr)$, where h can be pure real, pure imaginary, or even complex. Examining the expression for h one finds that if h is complex, h_1 and h_2 are complex conjugates. And from Appendix I the corresponding parameters of systems 1 and 2 become complex conjugates. Then for h complex, the dispersion of equation 2.29 becomes

$$\mathrm{Im}\left\{h_1^* \tau_1^* \left[b_1 h_1 C_n'(h_1 a) - \frac{nd_1}{a} C_n(h_1 a)\right] C_n(h_1^* a)\right\} = 0 \quad (2.31)$$

Although the dispersion is complex, there is only one condition which must be fulfilled, namely, that the imaginary part must be zero, since the real part of the dispersion is already zero.

A few observations can be made about the waves. From Appendix I it is seen that the two transverse components of the electric field are $\pi/2$ out of phase, while their absolute values are different. Therefore, the waves are elliptically polarized and the degree of ellipticity is a function of the radius. Look at the $n = \pm 1$ mode. The dispersion is different for $n = \pm 1$ so there are two k_z's which are possible, resulting in a Faraday rotation for these modes. The salient features of the dispersion can be demonstrated by considering the circularly symmetric mode $(n = 0)$. For this case the dispersion reduces to

$$-h_2 \tau_2 C_o'(h_1 a) C_o(h_2 a) + h_1 \tau_1 C_o'(h_2 a) C_o(h_1 a) = 0 \quad . \quad (2.32)$$

If h is complex, from equation 2.34, letting

$$h_1 = h_2^* = h_r + jh_i$$

$$\tau_1 = \tau_2^* = \tau_r + j\tau_i$$

$$J_o(h_1 r) = J_o^*(h_2 r) = J_{or} + jJ_{oi}$$

$$J_1(h_1 r) = J_1^*(h_2 r) = J_{1r} + jJ_{1i} \qquad (2.33)$$

where the asterisk denotes the complex conjugate and the subscripts r and i denote the real and imaginary parts respectively, the dispersion takes the form

$$h_r\tau_r J_{oi} J_{1r} + h_i\tau_r J_{or} J_{1r} + h_r\tau_i J_{or} J_{1r} - h_r\tau_r J_{or} J_{1i} +$$

$$+ h_i\tau_i J_{or} J_{1i} - h_i\tau_i J_{oi} J_{1r} + h_r\tau_i J_{oi} J_{1i} + h_i\tau_r J_{oi} J_{1i} = 0 . \qquad (2.34)$$

<u>2.5.11 Properties of the dispersion.</u> The cutoff frequencies can be obtained by letting k_z approach infinity and solving for the frequencies. In the limit of large k_z the dispersion in equation 2.35 reduces to

$$J_o(h_2 a) = 0 \quad , \qquad h_2 = k_z\left(- \frac{M^2 + P^2}{MM_3}\right)^{1/2} \qquad (2.35)$$

where M, M_3 and P are the elements of the reciprocal tensor. Since the factor $h_2 a$ must be a root of J_o, and k_z is very large, the term within the parenthesis in the above expression for h_2 must approach zero, but yet be positive. The expression above is the dispersion for the quasi-static approximation. From equations 2.1 and 2.2,

$$-\frac{M^2+P^2}{MM_3} = -\frac{(\omega^2-\omega_e^2)(\omega^2-\omega_{ce}^2)}{\omega^2(\omega^2-\omega_{ce}^2-\omega_e^2)} \tag{2.36}$$

Notice that k_z approaches infinity from below ω_{ce} and above ω_e if ω_e is greater than ω_{ce} and vice versa if ω_{ce} is greater than ω_e. This verifies the existence of the backward wave. The frequencies for which $k_z = 0$ can be obtained from the equation

$$\omega^2\left(\frac{M}{M_3}-1\right) J_1\left(\frac{k_o a}{\sqrt{M}}\right) J_o\left(\frac{k_o a}{\sqrt{M_3}}\right) = 0 \quad . \tag{2.37}$$

The term $\left(\frac{M}{M_3}-1\right)$ never goes to zero so the critical frequencies are $\omega = 0$ and those obtained from the roots of the Bessel functions. There are two ranges of frequencies for which $J_1\left(\frac{k_o a}{\sqrt{M}}\right)$ has roots. These are

$$\omega^2 < \omega_e^2 + \omega_{ce}^2 \tag{2.38a}$$

$$\omega^2 > \omega_e^2 + \frac{1}{2}\omega_{ce}^2 + \frac{1}{2}\sqrt{\omega_{ce}^4 + 4\omega_e^2\omega_{ce}^2} \tag{2.38b}$$

The first range is the backward wave region and the second is the usual fast wave, waveguide mode region. The zero order Bessel function has roots for $\omega > \omega_e$ and these correspond to waveguide waves also. In the quasi-static derivation all the zeros of the backward waves occur at the same frequency, $\omega = \sqrt{\omega_e^2 + \omega_{ce}^2}$. For the exact solution the zeros are all less than $\omega = \sqrt{\omega_e^2 + \omega_{ce}^2}$, and do not have the same value for the different modes.

It was noted previously that the arguments of the Bessel function can become complex. This depends on whether the term f^2 becomes negative or not.

$$f^2 = (M-M_3)^2(\omega^2\mu - Mk_z^2)^2 + 2(M+M_3)(\omega^2\mu - Mk_z^2)P^2k_z^2 + P^2k_z^4(P^2 + 4MM_3) \quad . \quad (2.39)$$

This can be put into the form

$$f^2 = \frac{1}{4}\left\{\left[-k_z^2\left(\frac{\epsilon_{33}}{\epsilon_{11}} - 1\right) + k_o^2\left(\epsilon_{33} - \epsilon_{11} + \frac{\epsilon_{12}^2}{\epsilon_1}\right)\right]^2 + 4k_z^2\,k_o^2\,\epsilon_{33}\frac{\epsilon_{12}^2}{\epsilon_{11}^2}\right\} \quad . \quad (2.40)$$

It is seen that f^2 can become negative only if ϵ_{33} is negative, if at all. Since $\epsilon_{33} = 1 - \frac{\omega_e^2}{\omega^2}$, this means that h can become complex for $\omega < \omega_e$ if k_z is of the proper magnitude. See Fig. 2.3 for some h-values.

The ω-k curves for the first three modes are shown in Figs. 2.4, 2.5, 2.6 and 2.7 for different values of ω_e/ω_{ce} for a given value of a normalized radius. A comparison between the exact and quasi-static solutions is seen in Fig. 2.8. The deviation is dependent mostly on the value $R = \omega_e a/c$ for relatively large values of $\Omega = \omega_e/\omega_{ce}$. At small values of Ω the deviation is frequency dependent and in the limit of infinite magnetic field, approaches the error given by equation 2.24.

The field configurations can be plotted. Although all field vectors are present, the plot cannot be distinguished from a TM_{01} plot (16).

The transition to the plane wave case is obtained by letting a approach infinity. If one examines equation 2.34, it is seen that for this condition there is a solution if h_1 or h_2 is zero. From Appendix I this means that

$$[(\omega^2\mu - Mk_z^2)(M+M_3) + P^2k_z^2]^2 = f^2 \quad . \quad (2.41)$$

Solving for k_z^2 and using the definitions of M, M_3 and P, one obtains

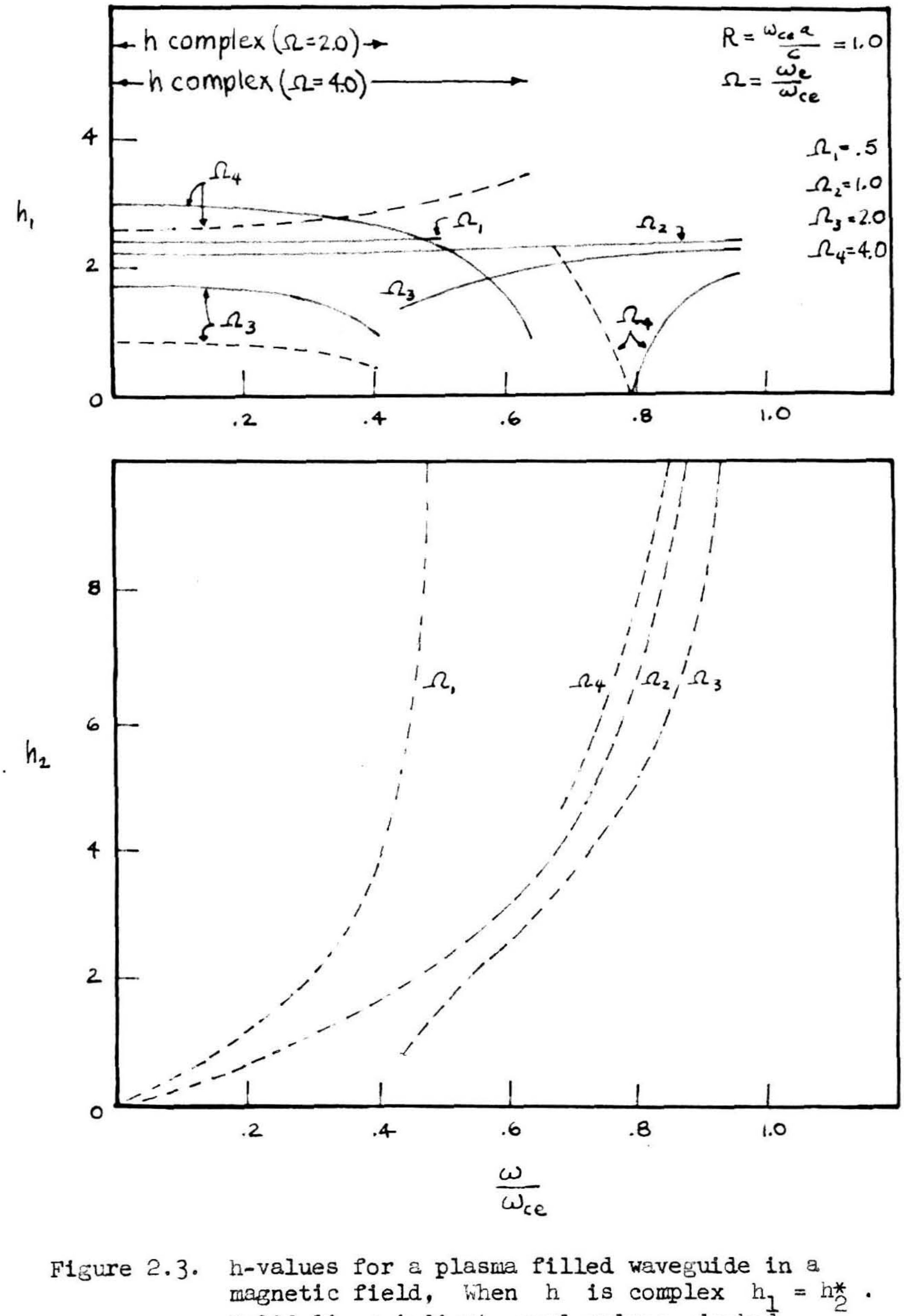


Figure 2.3. h-values for a plasma filled waveguide in a magnetic field, When h is complex $h_1 = h_2^*$. Solid lines indicate real values, dashed lines indicate imaginary values.

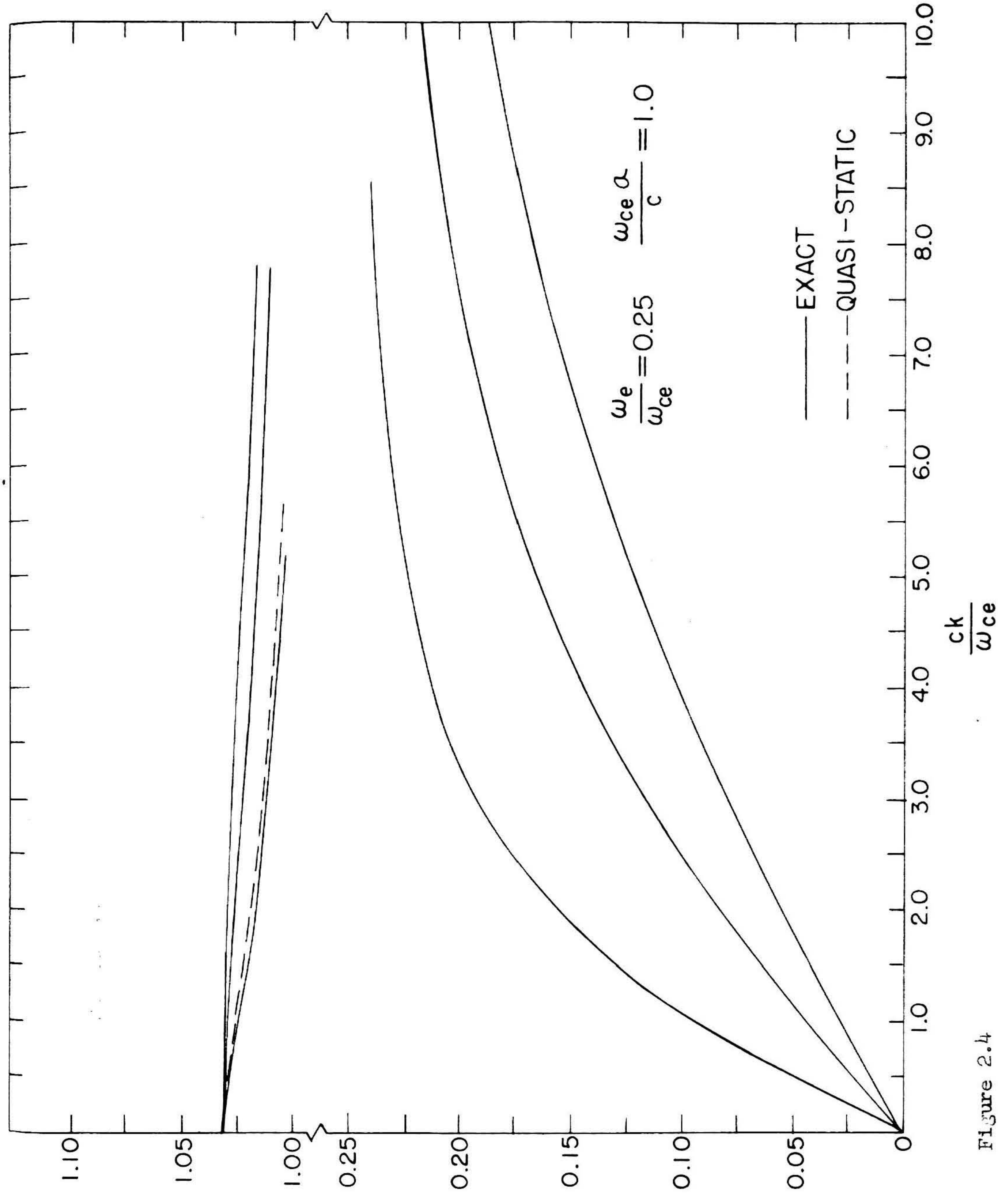
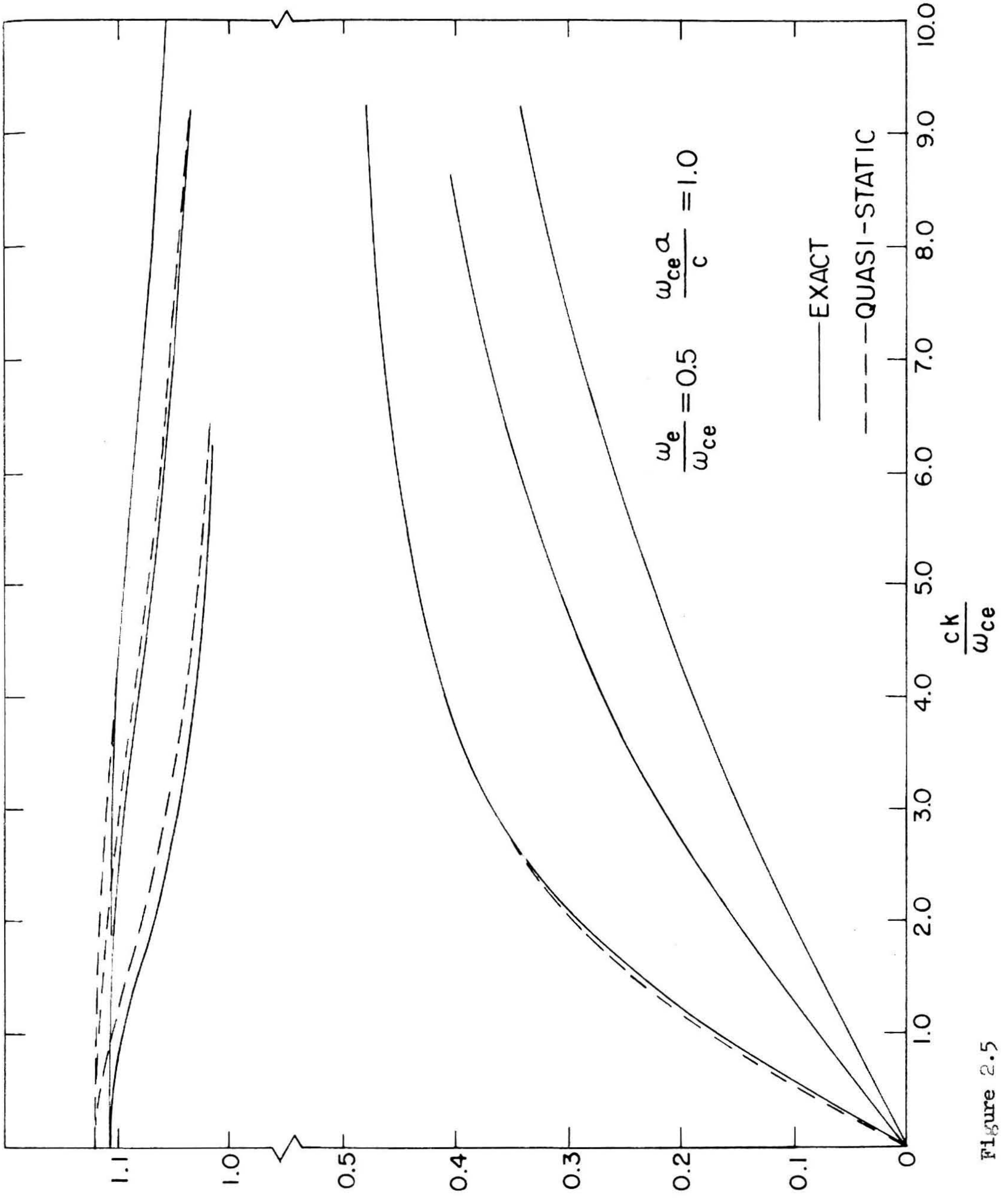
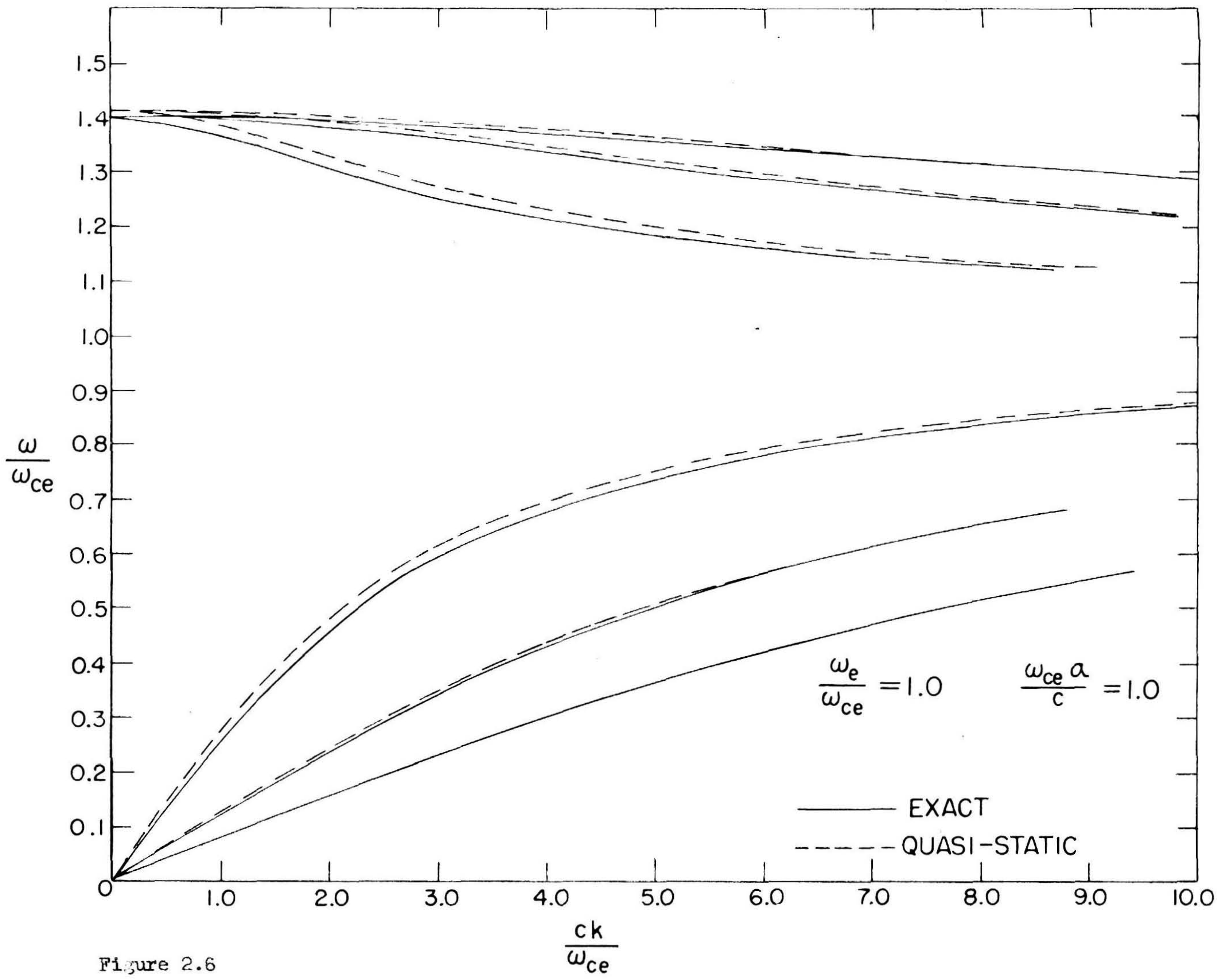

1.10
1.05
1.00
0.25
0.20
0.15
0.10
0.05
0
$\frac{\omega}{\omega_{ce}}$
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
$\frac{ck}{\omega_{ce}}$
$\frac{\omega_e}{\omega_{ce}} = 0.25$
$\frac{\omega_{ce}\, a}{c} = 1.0$
EXACT
QUASI-STATIC

Figure 2.4



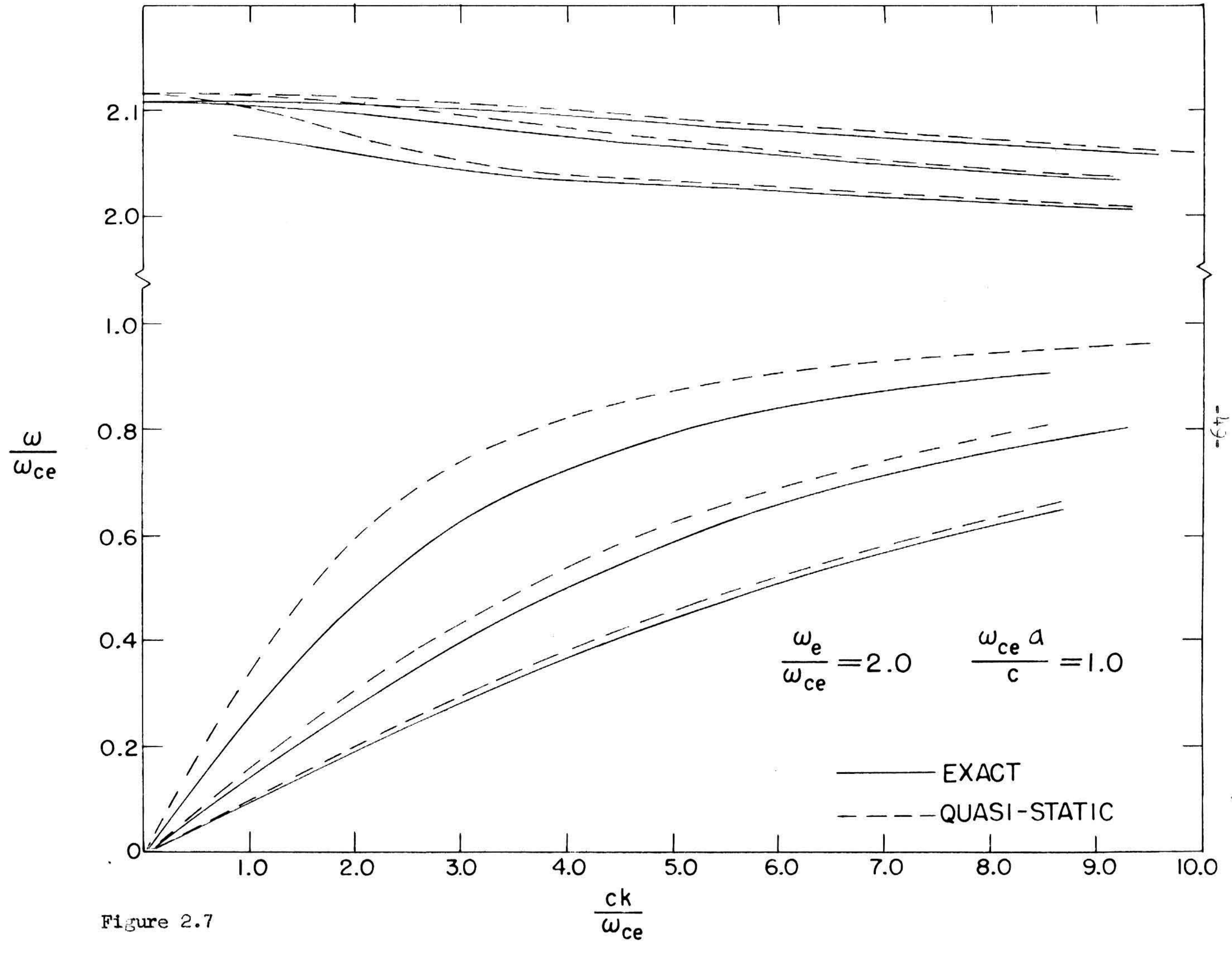
$\frac{\omega_e}{\omega_{ce}} = 0.5 \quad \frac{\omega_{ce} a}{c} = 1.0$
EXACT
QUASI-STATIC
$\frac{ck}{\omega_{ce}}$
$\frac{\omega}{\omega_{ce}}$
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
0.1
0.2
0.3
0.4
0.5
1.0
1.1

Figure 2.5



$\frac{\omega}{\omega_{ce}}$
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
$\frac{ck}{\omega_{ce}}$
$\frac{\omega_e}{\omega_{ce}} = 1.0$
$\frac{\omega_{ce} a}{c} = 1.0$
EXACT
QUASI-STATIC

Figure 2.6



$\frac{\omega}{\omega_{ce}}$
2.1
2.0
1.0
0.8
0.6
0.4
0.2
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
$\frac{ck}{\omega_{ce}}$
$\frac{\omega_e}{\omega_{ce}} = 2.0$
$\frac{\omega_{ce} a}{c} = 1.0$
EXACT
QUASI-STATIC

Figure 2.7

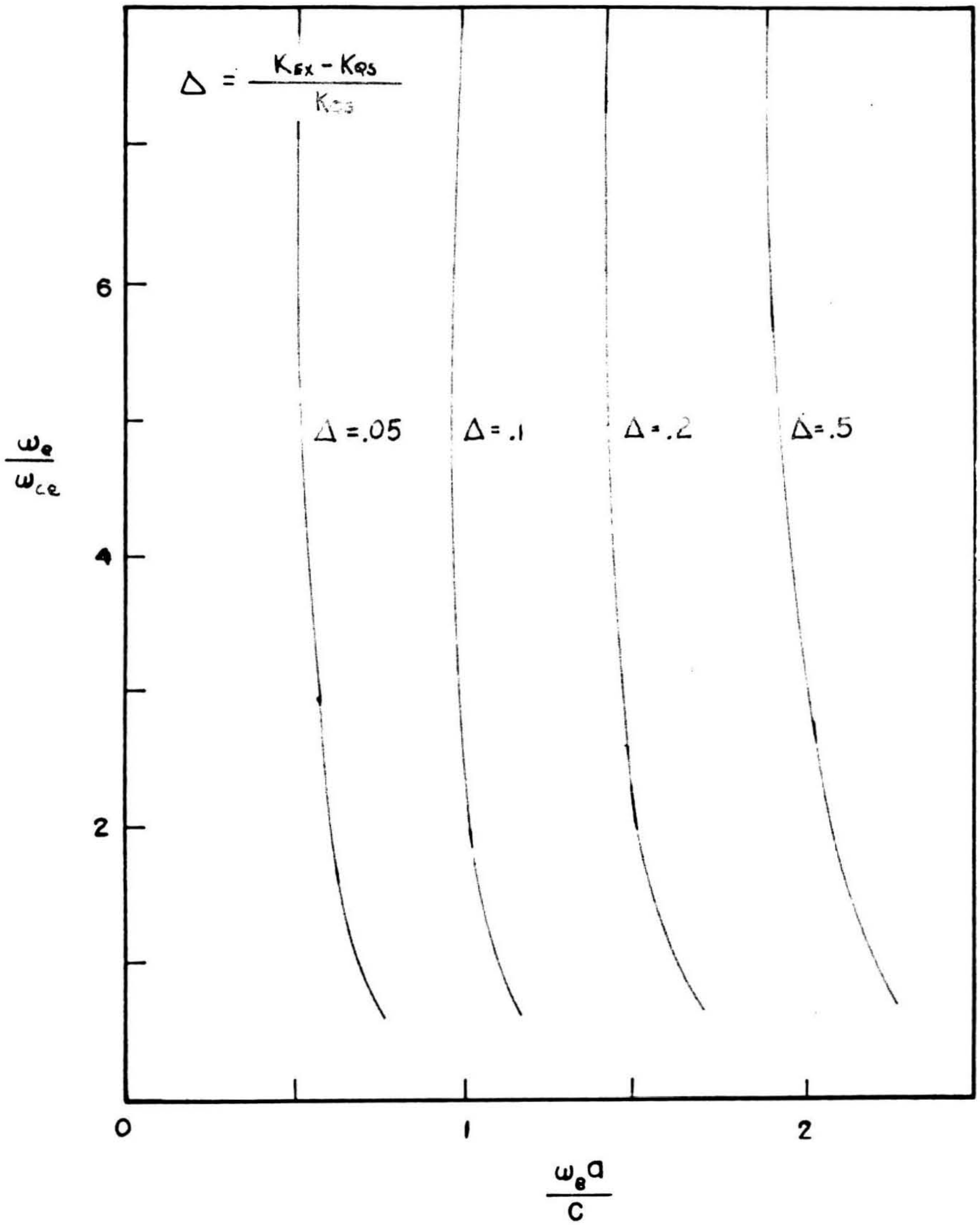


Figure 2.8 A comparison of the exact and quasi-static solutions

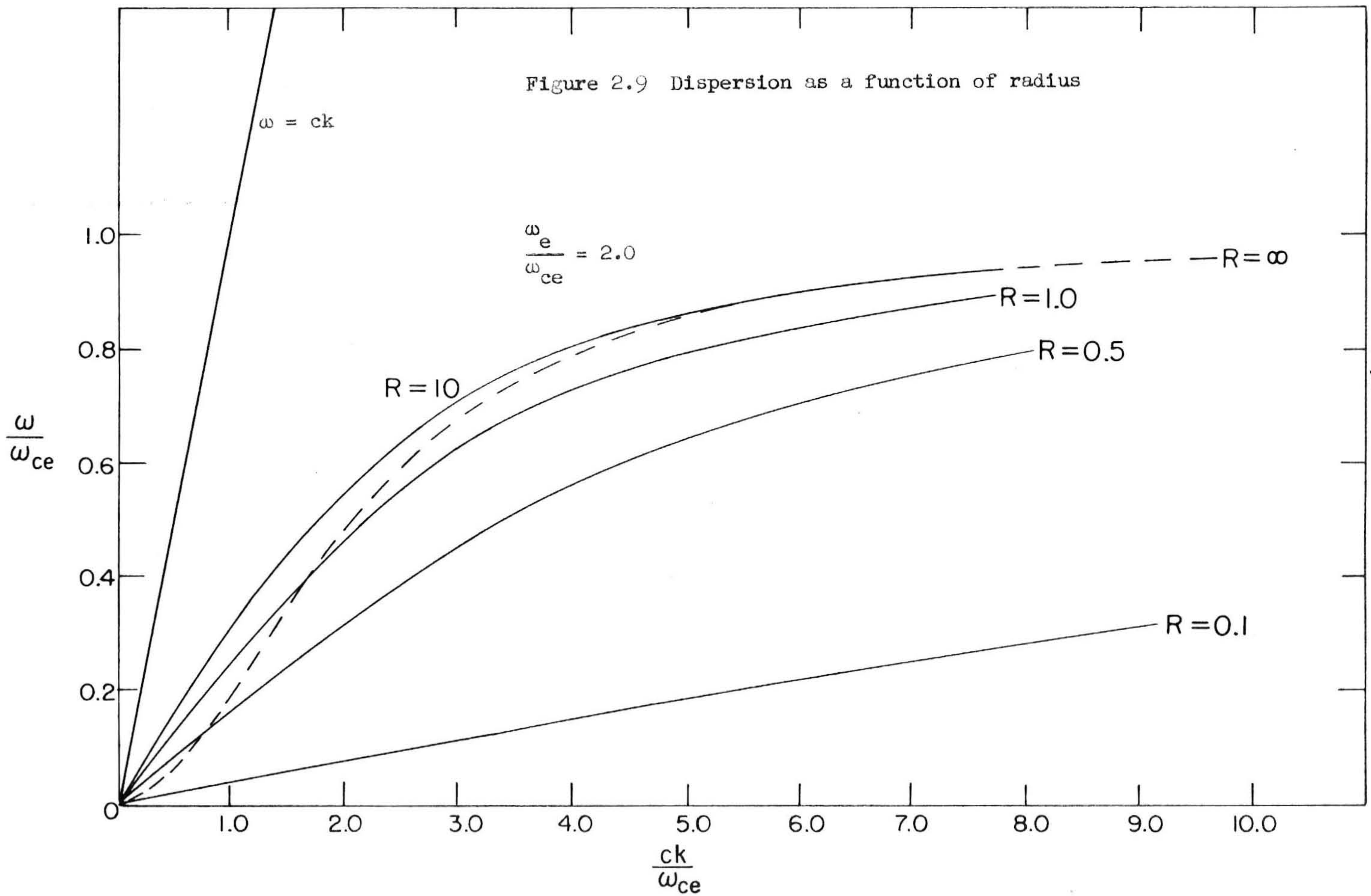


Figure 2.9 Dispersion as a function of radius

$$k_z^2 = \frac{\omega^2}{c^2}(\epsilon_{11} \pm \epsilon_{12}) \tag{2.42}$$

the expected dispersion, see Fig. 2.9.

The average power flow is given by

$$P = \frac{1}{2} \operatorname{Re} \int_S (E \times H^*) dS \quad . \tag{2.43}$$

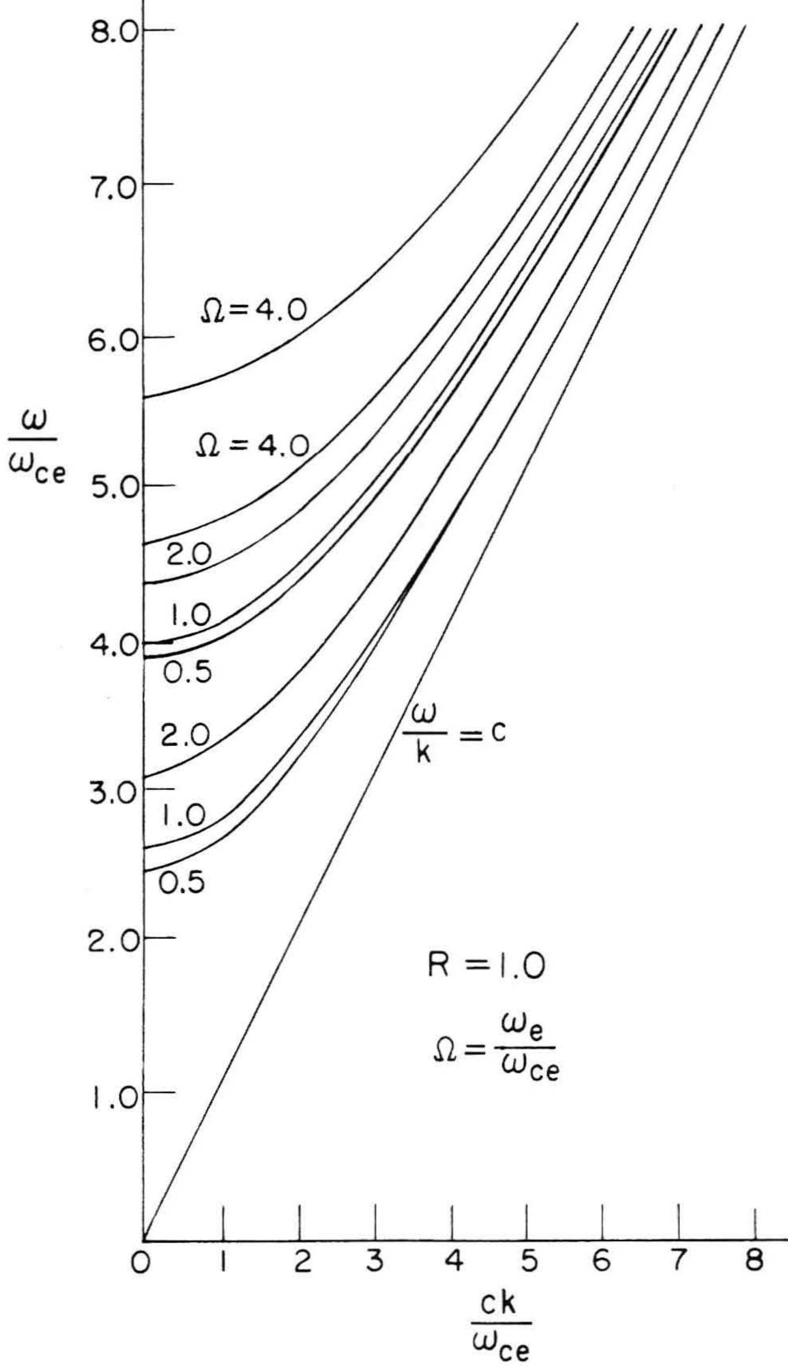
Using known integrals of Bessel functions, this becomes

$$\begin{aligned} P = \operatorname{Re} \pi_a^2 k_z \Big\{ & A_1^2(d_1\tau_1 + b\sigma) \frac{h_1^2 a^2}{2} \left[J_1^2(h_1 a) + J_o(h_1 a)^2 \right] + \\ & A_2^2(d_2\tau_2 + b\sigma) \frac{h_2^2 a^2}{2} \left[J_1^2(h_1 a) + J_o^2(h_2 a) \right] - \\ & A_1 A_2(d_1\tau_2 + d_2\tau_1 + 2b\sigma) \frac{h_1^2 \, h_2 a}{h_1^2 - h_2^2} J_1(h_2 a)(1 - \frac{\tau_1}{\tau_2}) \Big\} \quad . \end{aligned} \tag{2.44}$$

The equation above is valid for h_1^2 and h_2^2 real, negative, or complex.

2.5.2 Fast waveguide modes. In addition to the slow waves described previously, there also exist fast waves corresponding to the usual waveguide modes. These waves are not predicted by the quasi-static approximation, since an assumption of the quasi-static solution is that $k_z \gg k_o$. These waves, due to the anisotropy of the plasma, are hybrid modes. The low frequency cutoffs for these modes are given by equation 2.37. In Fig. 2.10 are plotted the first two modes for different values of the ratio ω_e/ω_{ce} for $\omega_{ce} a = 1$. The lower curves for a given ω_e/ω_{ce} evolve to the E-modes of the limiting cases of $\omega_{ce} = 0,\infty$; and the upper curves to the H-modes.

Figure 2.10 Fast wave modes in a plasma-filled waveguide in a magnetic field



III. INTERACTION OF DRIFTING CHARGED PARTICLES AND A STATIONARY PLASMA

3.0 Introduction

Traveling waves in a region where both an electron beam and a plasma are present, where the propagation and beam direction are the same, have been studied by a few authors in recent years (20,21,22,23). The presence of the electron beam introduces an instability, and it is this instability that merits the most interest. Haeff (20) and Pierce and Hebenstreit (21) have described the principles of double stream amplification. In the double stream amplifier two parallel electron beams of different velocities travel in the same region of space, and the same electric fields act on the electrons of both beams. When the currents in the two beams are sufficiently large, or when the frequency is sufficiently low, applying space charge wave theory, an exponentially increasing wave is possible. The dispersion for this type of amplifier is, for all velocities parallel to $\overline{e_z}$,

$$1 = \frac{\omega_{e1}^2}{(\omega - u_{01}k)^2} + \frac{\omega_{e2}^2}{(\omega - u_{02}k)^2} \tag{3.1}$$

where waves of the form $e^{j(\omega t - k_z z)}$ have been assumed, and the subscripts 1 and 2 refer to beams 1 and 2 respectively. Solving equation 3.1 for k for various ranges of ω, u_{01}, u_{02}, ω_{e1}, and ω_{e2} , one finds a growing wave. This theory can be easily extended to multiple streams,

$$1 = \sum_{i=1}^{N} \frac{\omega_{ei}^2}{(\omega - u_{0i}k)^2} \tag{3.2}$$

and to streams with a continuous distribution of velocities

$$1 = \omega_e^2 \int \frac{f(u)\ du}{(\omega - uk)^2} \tag{3.3}$$

or to a combination of the above. The beam-plasma problem can be considered a two-stream system in which one of the streams, electrically neutral, containing electrons and ions, is stationary except for thermal fluctuations.

Bohm and Gross (23) have proposed a theory of electron oscillations with a different approach, and their dispersion which determines the frequency of oscillation is identical with equation 3.2. There are no organized oscillations for wavelengths less than the Debye length. Whenever the plasma contains a beam of particles of well defined velocity which is greater than the mean thermal speed of the plasma electrons, the system can become unstable and it is possible for an oscillation to grow until limited by nonlinear effects. There have been various experiments (24,25,26,27) to verify the spatially growing wave.

The waves described above are longitudinal in character where the restoring force is electrostatic. In addition to these longitudinal modes, there can be transverse modes which are unstable, and these will also be studied.

3.1 General Dispersion Relation

The dispersion will be obtained in the same way as has been done previously. The plasma-beam medium will be expressed as a dielectric tensor and this will be incorporated into the wave equation. The inclusion of the beam complicates the formulation somewhat and this will be worked out in Appendix II. The dispersion is written for an electron beam in a plasma

$$\underline{\underline{L}} \cdot E = 0 \tag{3.4}$$

where $L = \ell_{mn}$; $m,n = 1,2,3$. The components of the operator L are

$$\ell_{11} = \frac{c^2}{\omega^2}(k^2 - k_x^2) - \left[1 - \frac{\omega_e^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_i^2}{\omega^2 - \omega_{ci}^2} - \frac{\omega_{ed}^2}{\omega^2}\frac{(\omega - u_o k_z)^2}{(\omega - u_o k_z)^2 - \omega_{ce}^2}\right]$$

$$\ell_{12} = \ell_{21}^* = -\frac{c^2 k_x k_y}{\omega^2} + j\left[\frac{\omega_e^2 \omega_{ce}}{\omega(\omega^2 - \omega_{ce}^2)} - \frac{\omega_i^2 \omega_{ci}}{\omega(\omega^2 - \omega_{ci}^2)} + \frac{\omega_{ed}^2 \omega_{ce}}{\omega^2}\frac{(\omega - u_o k_z)}{(\omega - u_o k_z)^2 - \omega_{ce}^2}\right]$$

$$\ell_{13} = \ell_{31}^* = -\frac{c^2 k_x k_z}{\omega^2} + \frac{\omega_{ed}^2 u_o}{\omega}\left[\frac{k_x(\omega - u_o k_z) + j\omega_{ce} k_y}{(\omega - u_o k_z)^2 - \omega_{ce}^2}\right]$$

$$\ell_{22} = \frac{c^2}{\omega^2}(k^2 - k_y^2) - \left[1 - \frac{\omega_e^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_i^2}{\omega^2 - \omega_{ci}^2} - \frac{\omega_{ed}^2}{\omega^2}\frac{(\omega - u_o k_z)^2}{(\omega - u_o k_z)^2 - \omega_{ce}^2}\right]$$

$$\ell_{23} = \ell_{32}^{*} = -\frac{c^2}{\omega^2} k_y k_z + \omega_{ed}^2 \frac{u_o}{\omega}\left[\frac{k_y(\omega - u_o k_z) - j\omega_{ce} k_x}{(\omega - u_o k_z)^2 - \omega_{ce}^2}\right]$$

$$\ell_{33} = \frac{c^2}{\omega^2}(k^2 - k_z^2) - \left[1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_i^2}{\omega^2} - \frac{u_o^2}{\omega^2}\frac{\omega_{ed}^2 (k_x^2 + k_y^2)}{\left[(\omega - u_o k_z^2) - \omega_{ce}^2\right]} - \frac{\omega_{ed}^2}{(\omega - u_o k_z)^2}\right].$$

There is no loss of generality if k_y is set equal to zero. In the above formulation B and u_o are along the z-axis.

3.2 Amplifying and Evanescent Waves

Before an investigation is made of growing waves, one should be able to distinguish between an amplifying and an evanescent wave. If one solves the dispersion relation for a system and a complex propagation constant is obtained, it is not obvious that an amplifying wave exists. Various criteria to test for the type of instability have been discussed elsewhere (28,29). The one generally used in this paper follows the ideas of Sturrock (28). The rule is this:

> "If for a given set of modes, ω is real for all real k, then any complex k for real ω denotes an evanescent wave; conversely if k is real for all real ω, then any complex ω for real k denotes nonconvective instability."

The terms "amplifying wave" and "convective instability" mean the same type of instability. The advantage of this criterion is that one can determine the character of the wave by examining the "free wave", that is, by examining the dispersion alone, which is written in the form

$$D(k,\omega) = 0 \quad . \tag{3.5}$$

Of course, if the system in question is quite similar to others which are known to have a convective instability, such as double stream and traveling wave amplifiers, one need not go through the often cumbersome procedure of satisfying the rule stated above. With the advent of high speed computers, however, the test can be made with relatively little difficulty. The dispersion of equation 3.5 in general is a complicated function of ω and k . In practice, to test for the amplifying or evanescent character of the wave in question, one plots k as a function of ω , or ω as a function of k . If the plot for real ω and k looks like Figure 3.1, the wave is amplifying. If, on the other hand, the plot looks like Figure 3.2, the wave in question is evanescent. This means that if one is looking for amplifying waves, one looks for regions in the ω-k plane in which k is complex for real ω, and ω is complex for real k . This region is the rectangle in Figure 3.1. Another hint as to the possible existence of an amplifying wave is an interaction between a negative energy carrying wave and a positive energy carrying wave which have the same sign phase velocities. It may be that the growing wave may not have a real ω,k branch and the rule stated previously must be applied.

3.3 Longitudinal Modes

In the following analysis only high frequency cases will be taken up and the ion effects will be neglected.

3.3.1 Infinite magnetic field. If there is an infinite magnetic field, there can be no transverse currents. For propagation

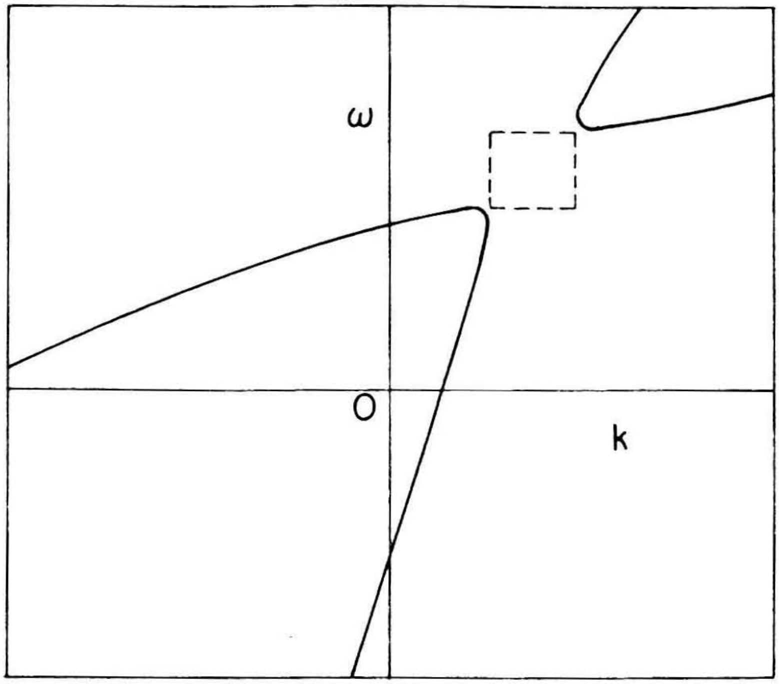


Figure 3.1

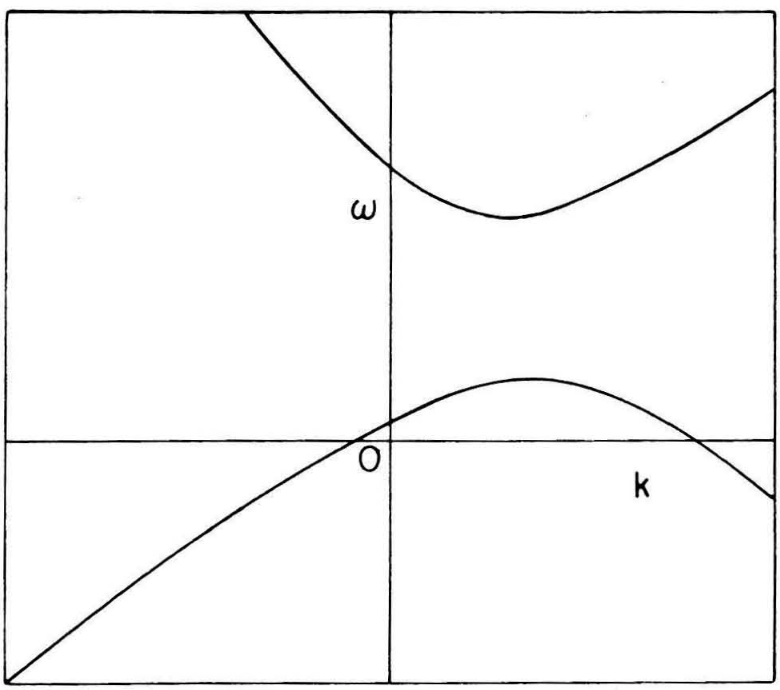


Figure 3.2

parallel to the field the dispersion is readily obtained by setting $k_x, k_y = 0$ and $\omega_{ce} = \infty$ in equation 3.4. The result is

$$\left(\frac{c^2}{\omega^2} k_z^2 - 1\right) \left[1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_{ed}^2}{(\omega - u_o k_z)^2}\right] = 0 . \tag{3.6}$$

Two types of waves emerge from the dispersion. The first is

$$k_z = \omega/c \quad . \tag{3.7}$$

This is just a transverse electromagnetic wave which travels at the velocity of light. Because of the infinite magnetic field, the electrons have no effect on the wave. The second and more interesting wave, which can be exponentially increasing, is longitudinal in character and is simple enough that temperature effects can be taken into account quite readily. Integrating Boltzmann's equation using the approximations of Section 1.2, the dispersion becomes

$$1 - \frac{\omega_e^2}{\omega^2\left(1 - \frac{V_T^2 k_z^2}{\omega^2}\right)} - \frac{\omega_{ed}^2}{(\omega - u_o k_z^2)} = 0 \tag{3.8}$$

where $V_T^2 = \frac{3kT}{m}$. This wave is the one discussed in Section 3.0 and its important features are shown in Figures 3.3, 3.4 and 3.5. Taking into account the temperature of the plasma electrons introduces two more waves which are related to the longitudinal waves discussed in Chapter I. A few effects of the finite temperature should be noticed. Unless the beam energy is very low, or alternatively, if the electron temperature is very high, the zero temperature dispersion is

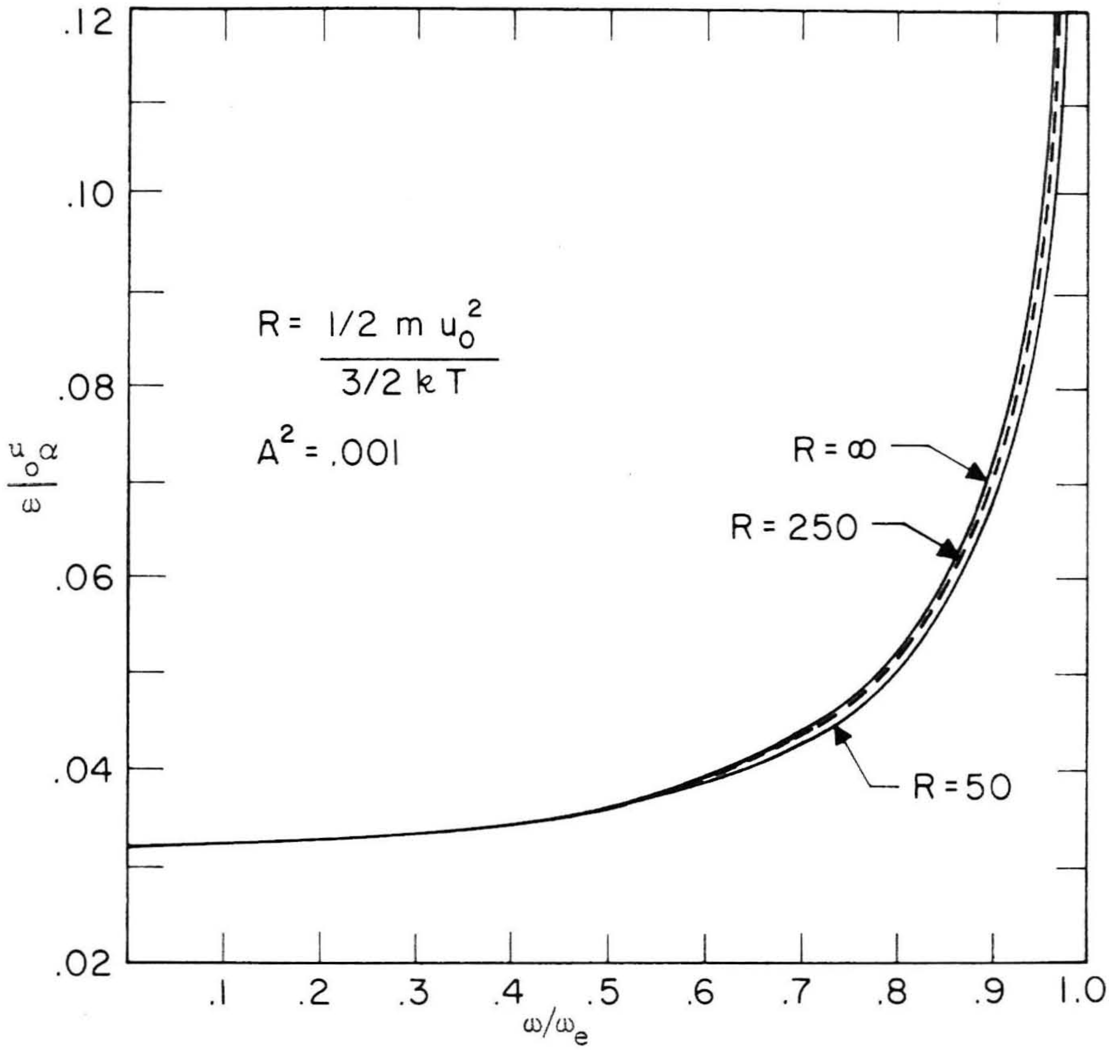


Figure 3.3 Growth Constant as a function of electron temperature or, alternatively, as a function of beam energy

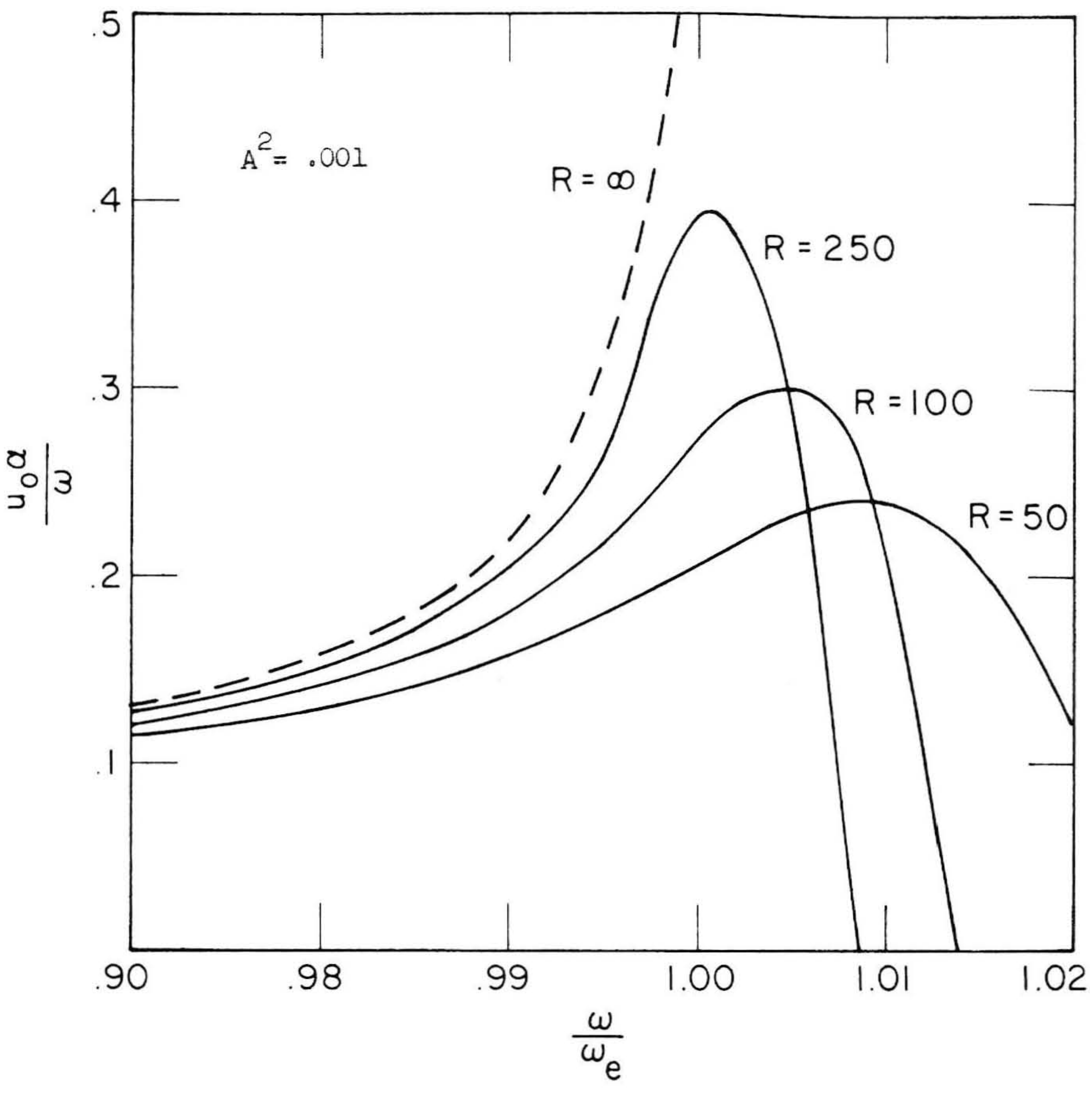


Figure 3.4 Growth Constant near the Maximum

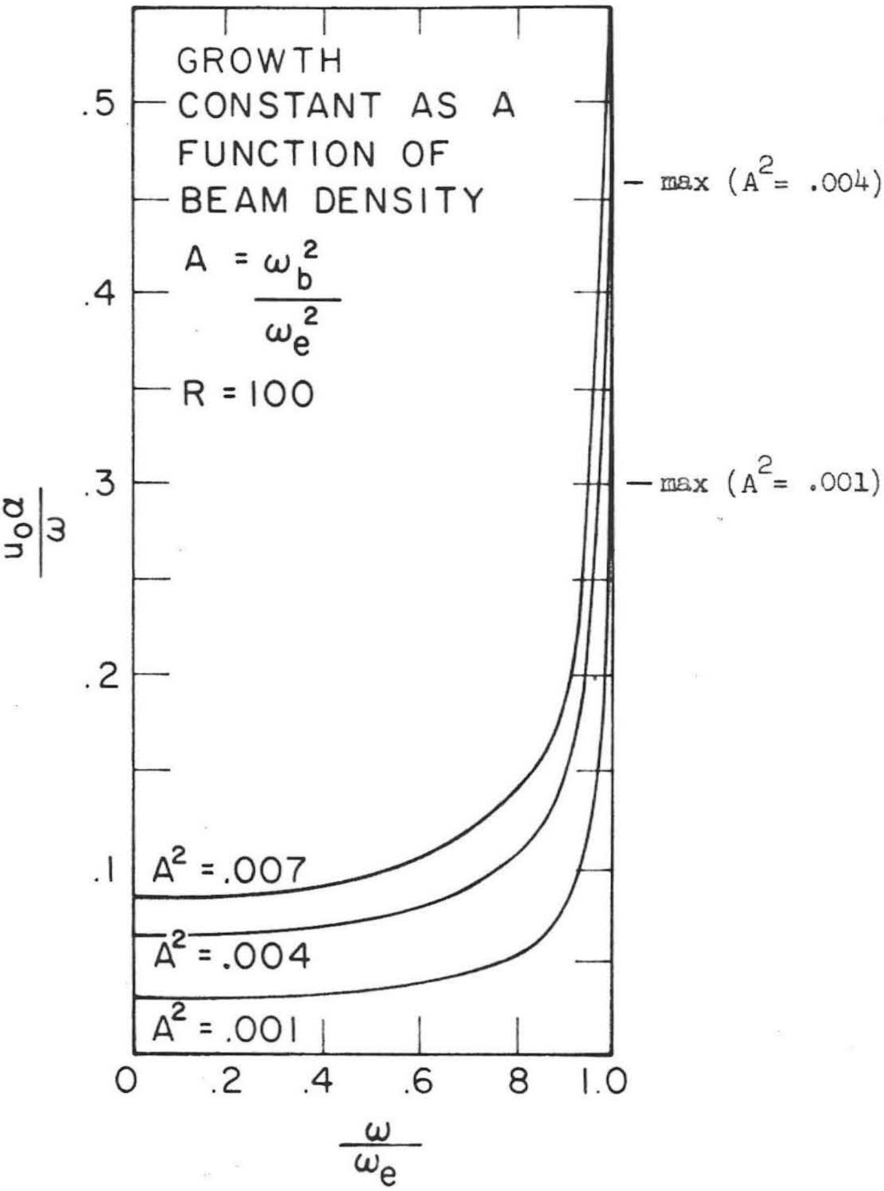


Figure 3.5 Growth Constant as a Function of Beam Density

an excellent approximation except in the vicinity of ω_e . The finite temperature extends the region of growth past the plasma frequency and the maximum growth occurs at a frequency slightly higher than the plasma frequency. There is always growth for frequencies below the plasma frequency. As expected, the effects mentioned above are more pronounced for higher electron temperatures.

3.3.2 Zero magnetic field. Taking equation 3.4 and setting $k_y, \omega_{ce} = 0$, the dispersion has, for propagation at an arbitrary angle to the direction of the electron beam, a pure transverse wave

$$\frac{c^2k^2}{\omega^2} - \left(1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_{ed}^2}{\omega^2}\right) = 0 \tag{3.9}$$

and a wave which has both transverse and longitudinal electric fields and whose propagation characteristics are determined by a pair of coupled equations

$$\ell_{11}E_x + \ell_{12}E_z = 0 \tag{3.10}$$

$$\ell_{21}E_x + \ell_{22}E_z = 0 \tag{3.11}$$

where the ℓ's are defined in equation 3.4. Equation 3.9 has been discussed previously in Chapter I. The geometry for the second wave is shown in Figure 3.6.

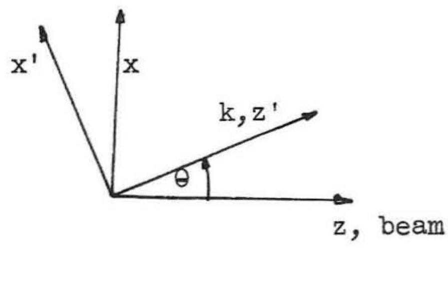


Fig. 3.6

It will be convenient to write out the coupled equations,

$$\underline{\underline{L}} \cdot E = 0 \tag{3.12}$$

$$\begin{bmatrix} \frac{c^2k^2}{\omega^2} - \left(1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_{ed}^2}{\omega^2}\right) & -\frac{c^2k^2}{\omega^2}\cos\theta\sin\theta + \frac{\omega_{ed}^2\, u_o k \sin\theta}{\omega^2(\omega - u_o k\cos\theta)} \\ \frac{c^2k^2}{\omega^2}\cos\theta\sin\theta + \frac{\omega_{ed}^2\, u_o k\sin\theta}{\omega^2(\omega - u_o k\cos\theta)} & k^2\sin^2\theta\left[\frac{c^2}{\omega^2} + \frac{\omega_{ed}^2\, u_o^2}{\omega^2(\omega - u_o k\cos\theta)^2}\right] - \left[1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_{ed}^2}{(\omega - u_o k\cos\theta)^2}\right] \end{bmatrix} \begin{bmatrix} E_x \\ E_z \end{bmatrix} = 0$$

If the coordinate system were rotated such that the new z-axis coincided with the direction of propagation, one would obtain for the dispersion, as did Neufeld and Doyle (30)

$$\underline{\underline{L}}' \cdot E' = 0 \tag{3.13}$$

$$\begin{bmatrix} \frac{c^2k^2}{\omega^2} - \left[1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_{ed}^2}{\omega^2} - \frac{\omega_{ed}^2\, u_o^2 k^2\sin^2\theta}{\omega^2(\omega - u_o k\cos\theta)^2}\right] & \frac{\omega_{ed}^2\, u_o k\sin\theta}{\omega(\omega - u_o k\cos\theta)^2} \\ \frac{\omega_{ed}^2\, u_o k\sin\theta}{\omega(\omega - u_o k\cos\theta)^2} & -\left[1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_{ed}^2}{(\omega - u_o k\cos\theta)^2}\right] \end{bmatrix} \begin{bmatrix} E'_x \\ E'_z \end{bmatrix} = 0$$

Although equations 3.12 and 3.13 exhibit waves which have both transverse and longitudinal field vectors, it will be shown that the transverse and longitudinal modes are weakly coupled. The propagation constant can be obtained by setting the determinant of the operator L or L' equal to zero, and if the components of the determinant are multiplied out, a dispersion polynomial results

$$C_4\Gamma^4 + C_3\Gamma^3 + C_2\Gamma^2 + C_1\Gamma + C_o = 0 \tag{3.14}$$

where

$$C_4 = \cos^2\theta(\Omega_e^2 - 1)$$

$$C_3 = -2\cos\theta \quad (\Omega_e^2 - 1)$$

$$C_2 = \Omega_e^2(\Omega_e^2 - A^2 - 1) - \gamma^2(\Omega_e^2 - 1)\,[\cos^2\theta(\Omega_e^2 - A^2 - 1) - A^2\sin^2\theta]$$

$$C_1 = 2\gamma^2 \quad \cos\theta(\Omega_e^2 - 1)(\Omega_e^2 - A^2 - 1)$$

$$C_o = -\gamma^2\,\Omega_e^2(\Omega_e^2 - A^2 - 1)^2$$

and normalized quantities are used, which are defined

$$\Gamma = \frac{u_o k}{\omega_e}, \quad \Omega_e = \frac{\omega}{\omega_e}, \quad A = \frac{\omega_{ed}}{\omega_e}, \quad \gamma = \frac{u_o}{c} \quad .$$

Using equation 3.14, equation 3.13 can be manipulated into the form

$$L'' \cdot E' = 0 \tag{3.15}$$

$$\begin{bmatrix} \dfrac{\omega^2}{\omega^2 - \omega_e^2}\left[\dfrac{c^2k^2}{\omega^2} - \left(1 - \dfrac{\omega_e^2}{\omega^2} - \dfrac{\omega_{ed}^2}{\omega^2}\right)\right] & \dfrac{u_o k \sin\Theta}{\omega} \\ \dfrac{u_o k \sin\Theta}{\omega} & -\dfrac{\omega^4}{\omega_e^2\omega_{ed}^2}(\omega - u_o k)^2\left[1 - \dfrac{\omega_e^2}{\omega^2} - \dfrac{\omega_{ed}^2}{(\omega - u_o k\cos\Theta)^2}\right] \end{bmatrix} \begin{bmatrix} E'_x \\ E'_z \end{bmatrix} = 0$$

If terms of the order γ^2 are ignored compared to unity, two of the roots can be written

$$\frac{u_o k \cos \Theta}{\omega} \approx 1 \pm \frac{\omega_{ed}}{\omega_e \left(\frac{\omega^2}{\omega_e^2} - 1\right)^{1/2}} \quad . \tag{3.16}$$

This is the familiar longitudinal wave which is unstable for frequencies less than the plasma frequency, and its characteristics have been discussed in Section 3.3. The only modification to be made is to let $k = k(\Theta = 0)/\cos \Theta$, where $k(\Theta = 0)$ is the propagation constant for zero angle. In effect, all that has been done is to take the component of beam velocity in the direction of propagation and solve for k . Temperature effects can be taken into account as in Section 1.2.

Again ignoring terms of the order γ^2 , the transverse propagation constant can be determined approximately

$$\frac{ck}{\omega} \approx \pm \left(1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_{ed}^2}{\omega^2}\right)^{1/2} \quad . \tag{3.17}$$

This wave is the same as that in equation 3.9. The first order correction to the propagation constants can be readily calculated. If

$$k_{1,2} = \pm \frac{\omega}{c} \left(1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_{ed}^2}{\omega^2}\right) \tag{3.18}$$

$$k_{3,4} = \frac{\omega}{u_o \cos \Theta} \left[1 \pm \frac{\omega_{ed}}{(\omega^2 - \omega_e^2)^{1/2}}\right] \tag{3.19}$$

and letting $k = k_i + \Delta_i$ $(i = 1,2,3,4)$, one obtains for the transverse mode

$$\Delta_{1,2} = \mp \frac{\gamma^2 \sin^2\theta\, k_{1,2}}{\dfrac{2\omega^2 u_o^2}{\omega_e^2\omega_{ed}^2(\omega^2 - \omega_e^2)}\,(k_{1,2} - k_3)(k_{1,2} - k_4)}, \qquad \omega^2 > \omega_e^2 + \omega_{ed}^2 \tag{3.20}$$

and for the longitudinal wave in the unstable region,

$$\Delta_{3,4} = - \frac{\gamma^2 \sin^2\theta\, k_{3,4}^2}{\dfrac{2\omega^2 u_o^2}{\omega_e^2\omega_{ed}^2(\omega^2 - \omega_e^2)}\,\dfrac{\omega}{u_o \cos\theta}\,(k_{3,4} - k_1)(k_{3,4} - k_2)}, \qquad \omega < \omega_e \tag{3.21}$$

Notice that $\omega = \omega_e$ is not a singularity and the Δ's are indeed small.

In the waves discussed above, there are both transverse and longitudinal fields and the relative magnitudes can be determined approximately. For the transverse wave for small values of the ratio ω_{ed}^2/ω_e^2,

$$\left|\frac{E'_x}{E'_z}\right| = \left|\frac{\omega^4}{\omega_{ed}^2\,\omega_e^2}\,\frac{1}{\gamma}\left(1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_{ed}^2}{\omega^2}\right)\right| \qquad \omega^2 > \omega_e^2 + \omega_{ed} \tag{3.22}$$

and for the longitudinal wave in the unstable region

$$\left|\frac{E'_z}{E'_x}\right| = \left|\frac{\omega^2}{\gamma^2(\omega^2 - \omega_e^2)}\,\frac{1}{\sin\theta\cos\theta}\right| \qquad \omega < \omega_e \tag{3.23}$$

There is some interest in the transverse field component of the unstable wave. By examining the relations above, it is seen that for its largest relative amplitude, ω should be small, γ large, and $\theta = \frac{\pi}{4}$.

It should be mentioned that there is another mechanism for instability for this zero field case for a transverse wave. This has been discussed by Weibel (31) and Fried (32). They have shown that for an anisotropic velocity distribution of electrons there is a process for which there is an instability. This mechanism can be described briefly as follows. The curvature of the electrons in the fluctuating magnetic field causes a momentum flux which in turn affects the average velocity in such a way as to reinforce the fluctuation field.

3.4 Transverse Modes

In the next two subsections transverse modes which have an instability will be studied. In these cases the propagation is in the z-direction, and there must be a magnetic field and at least one beam of charged particles for instability.

3.4.1 Electron or ion beam in a plasma. The dispersion for an electron beam in a plasma where k, u_o, and B_o have the same direction can be written as

$$\left[1 - \frac{c^2k^2}{\omega^2} - \frac{\omega_e^2}{\omega(\omega+\omega_{ce})} - \frac{\omega_i^2}{\omega(\omega-\omega_{ci})} - \frac{\omega_{ed}^2(\omega - u_o k)}{\omega^2(\omega - u_o k + \omega_{ce})}\right] \times$$

$$\left[1 - \frac{c^2k^2}{\omega^2} - \frac{\omega_e^2}{\omega(\omega-\omega_{ce})} - \frac{\omega_i^2}{\omega(\omega+\omega_{ci})} - \frac{\omega_{ed}^2(\omega - u_o k)}{\omega^2(\omega - u_o k - \omega_{ce})}\right] \times$$

$$\left[1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_{ed}^2}{(\omega - u_o k)^2}\right] = 0 \tag{3.24}$$

This is conveniently written in the form

$$D_+(k,\omega)\ D_-(k,\omega)\ D_\ell(k,\omega) = 0 \quad . \tag{3.25}$$

The third factor is the longitudinal mode and will be ignored. The first term contains the interaction of the plasma wave in which the ions gyrate in the same sense as the field and the slow cyclotron wave of the beam, and the second term contains the interaction of the plasma wave in which the electrons gyrate in the same sense as the field and the fast cyclotron wave. The first dispersion displays an instability near $\omega = \omega_{ci}$ while the second is stable. The second dispersion can exhibit an instability near $\omega = \omega_{ce}$ if a fast beam of ions is present. This case will be examined later. The instability of $D_+(k,\omega) = 0$ can be examined graphically. This can be done in two ways. First one can proceed as Dawson and Bernstein (33) by writing the dispersion in the form

$$\omega^2 = c^2k^2 + \frac{\omega\,\omega_i^2}{\omega - \omega_{ci}} + \frac{\omega\,\omega_e^2}{\omega + \omega_{ce}} + \frac{\omega_{ed}^2(\omega - u_o k)}{\omega - u_o k + \omega_{ce}} \tag{3.26}$$

The curves in Fig. 3.7 are obtained by fixing k and then plotting the left and right hand sides of Eq. 3.26 separately as a function of ω. The left hand side is ω^2 and yields simply the parabola shown in the figure. The other curve is the right hand side and has singularities at $-\omega_{ce}$, ω_{ci}, and $u_o k - \omega_{ce}$. It crosses the curve ω^2 at five places. Each of these is a possible ω for a given k.

Now consider what happens when either the velocity of the beam or k is increased such that $u_o k - \omega$ approaches ω_{ci}. A situation like that shown in Fig. 3.8 will ultimately occur. It is seen that the diagram is very similar to Fig. 3.7. However, the region of the curve representing the right hand side of the dispersion relation and lying

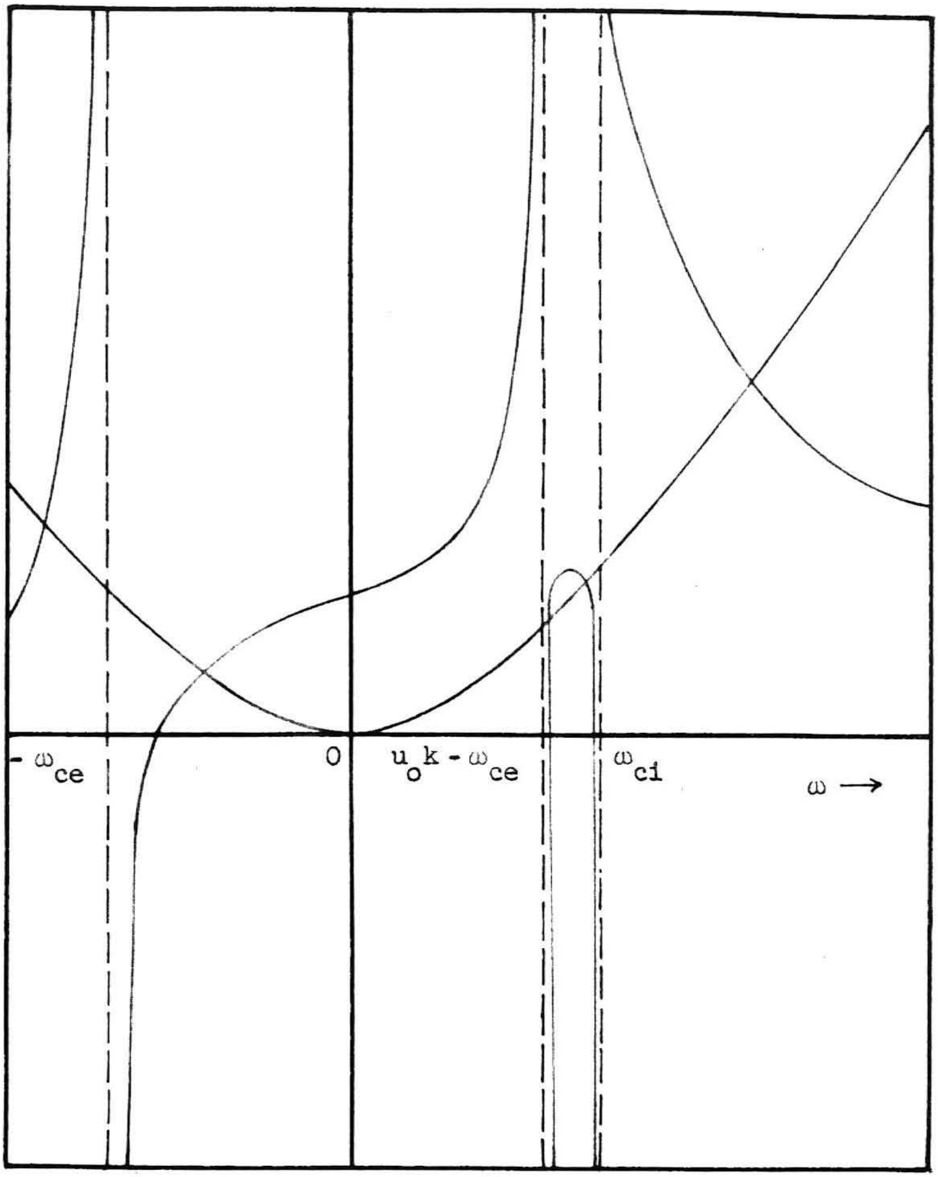


Figure 3.7 Stable case for $D_+(k,\omega) = 0$, with five real roots

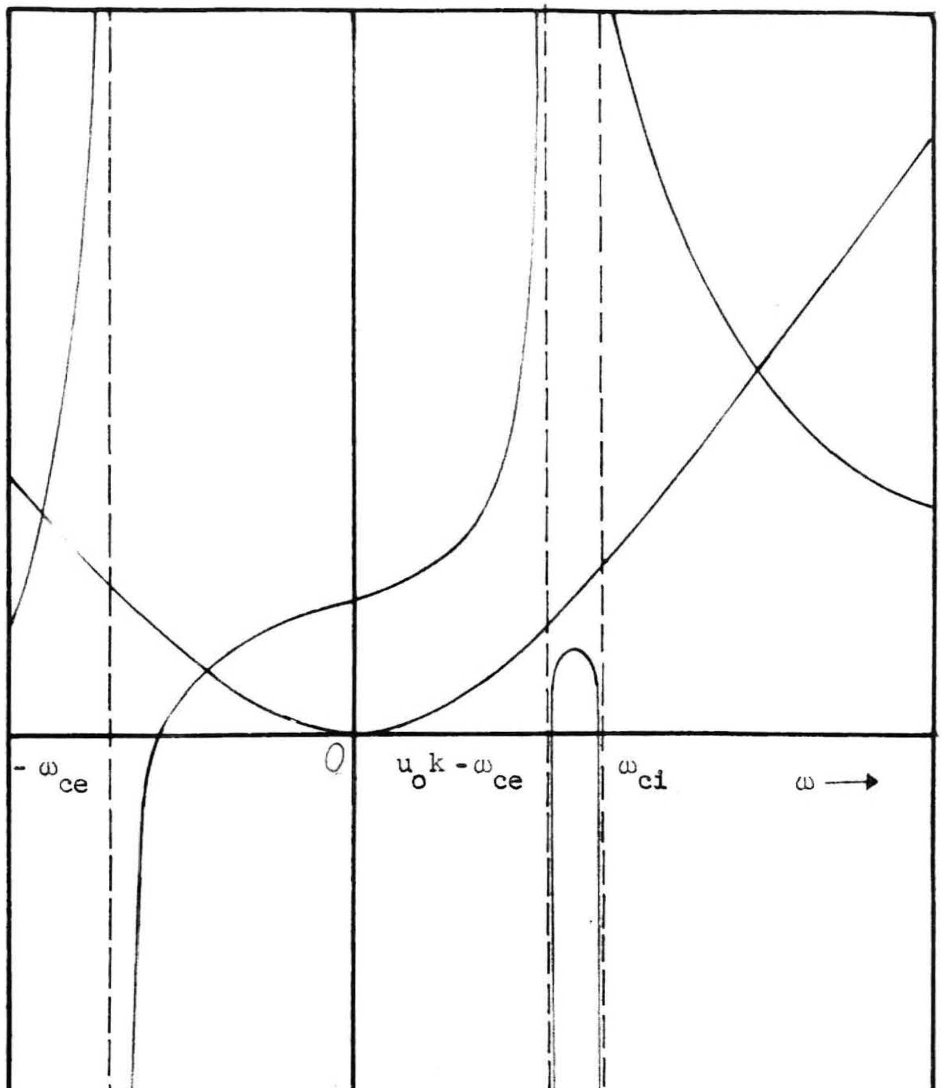


Figure 3.8 Unstable case for $D_+(k,\omega) = 0$ with three real and two complex roots

between $u_o k - \omega_{ce}$ and ω_{ci} has been compressed and its maximum has become lower. There will be a value of u_o for a given k, or a k for a given u_o for which it no longer intersects the parabola. When this happens one loses two real roots and in their places obtains two complex conjugate roots. One of these represents a growing wave, the other a damped wave. It is seen that if $u_o k - \omega_{ce}$ is larger than ω_{ci} there can still be an instability so that there is a small frequency band near ω_{ci} for which there is an instability. One merely interchanges $u_o k - \omega_{ce}$ and ω_{ci} in the diagrams.

The $D_-(k,\omega) = 0$ wave can be examined in the same manner. For this case there are poles at $-\omega_{ci}$, ω_{ce} and $u_o k + \omega_{ce}$. The corresponding plot is shown in Fig. 3.9. Examining the figure one sees that for any positive $u_o k$ there are always five real roots. If $u_o k$ is negative (either u_o or k negative) there is a possibility of the ω-roots becoming complex. This happens if $\omega_{ce} > |u_o k|$ but of comparable magnitude such that the lost term on the right hand side is for $\omega = 0$ negative but absolutely greater than $c^2 k^2$, the plot may become that shown in Fig. 3.10. However, following the discussion of Section 3.2, this is a nonconvective instability.

Another way to examine the instability is to look at the undisturbed waves and look for regions where the waves are near synchronism so that there is a possibility of an interaction and a resulting growing wave. Particularly one should look for a negative energy carrying wave interacting with a positive energy carrying wave. See Figs. 3.11 and 3.12 for the plot of the undisturbed waves of $D_+(k,\omega) = 0$ and $D_-(k,\omega) = 0$. The interacting regions are encircled. For the growing

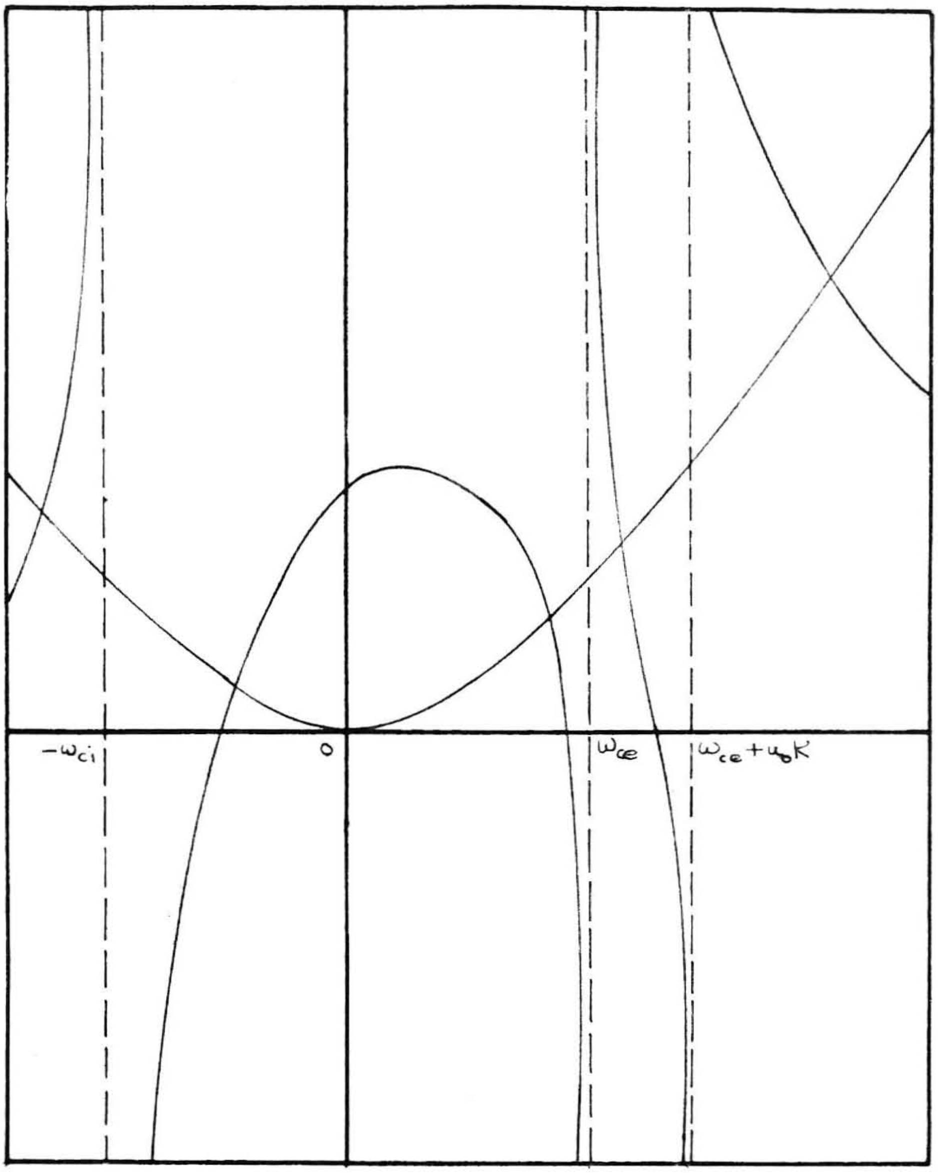


Figure 3.9 Case where $D_-(k,\omega)$ has five real roots

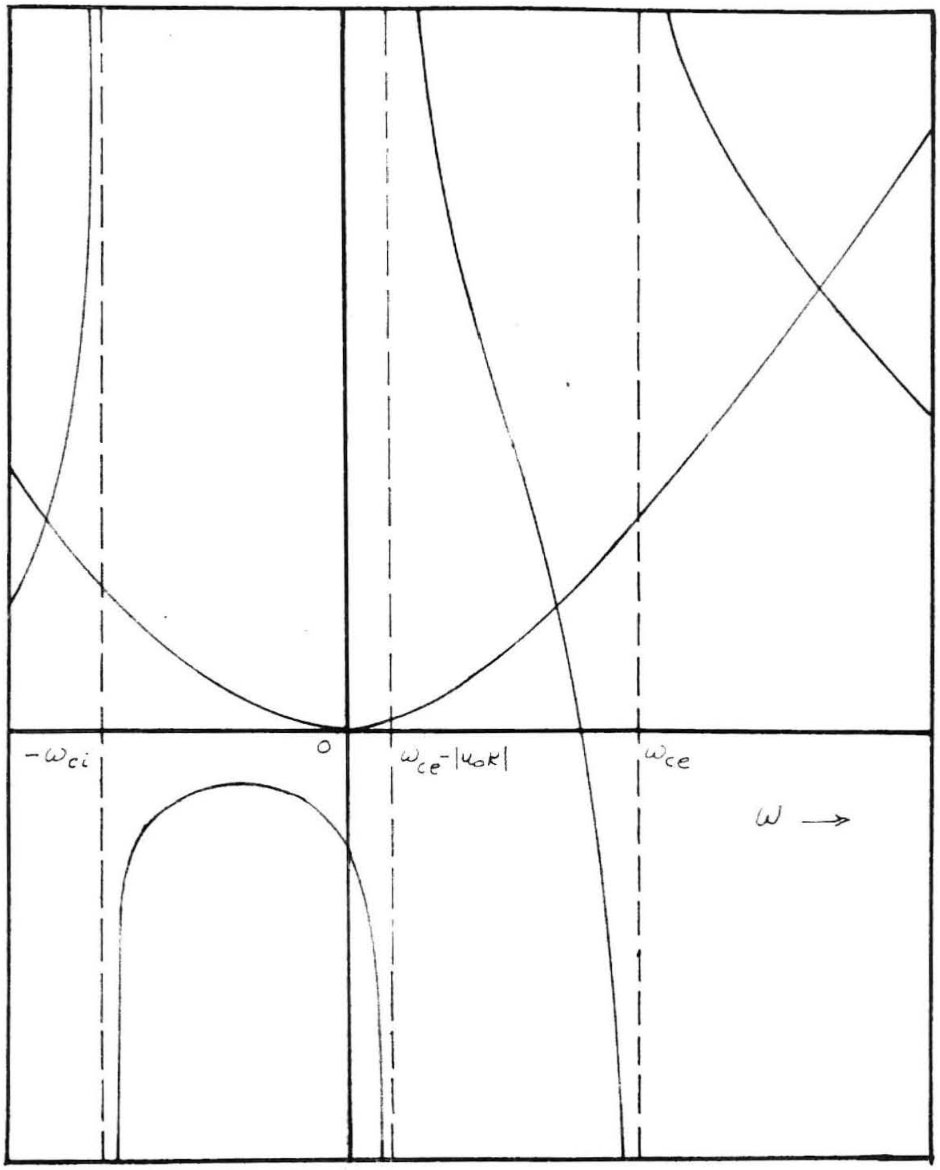


Figure 3.10 $D_{-}(k,\omega)$ has three real and two complex (wave evanescent) roots

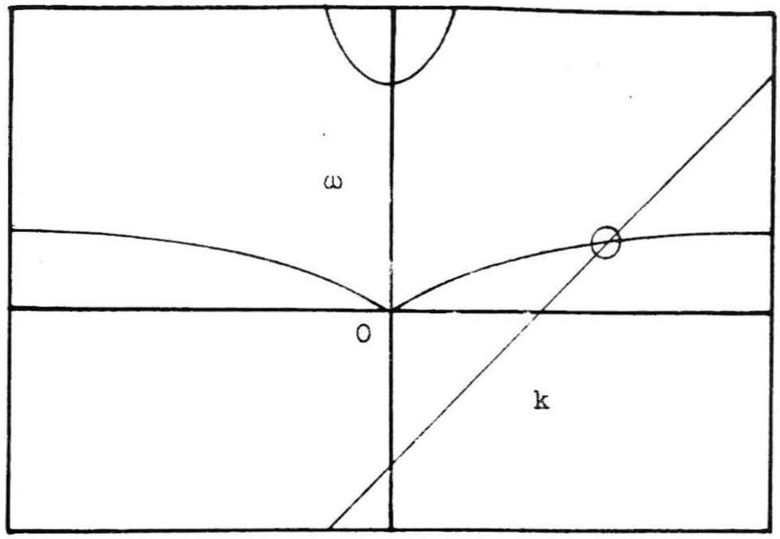


Figure 3.11 Undisturbed waves of + wave

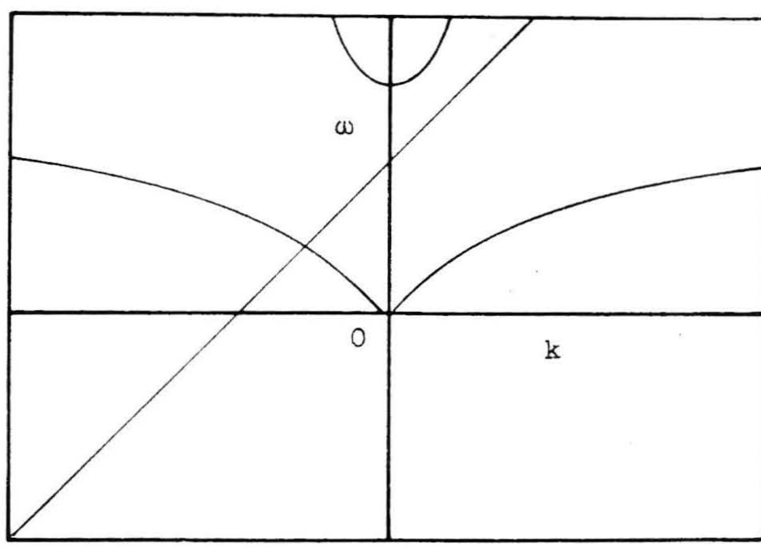


Figure 3.12 Undisturbed waves of - wave

wave it is seen that the undisturbed waves are close to synchronism near ω_{ci} and there should be strong interaction in that frequency vicinity. This will now be examined in some detail.

Instability near ω_{ci}

To examine the instability the dispersion is written in the form:

$$\left\{ \frac{c^2k^2}{\omega^2} - \left[1 - \frac{\omega_e^2}{\omega(\omega+\omega_{ce})} - \frac{\omega_i^2}{\omega(\omega-\omega_{ci})} \right] \right\} (k - \frac{\omega+\omega_{ce}}{u_o}) + \frac{\omega_{ed}^2}{\omega^2}(k - \frac{\omega}{u_o}) = 0 \tag{3.27}$$

Since interaction is expected near $\omega \approx \omega_{ci}$, k is approximately equal to ω_{ce}/u_o and the dispersion can be written approximately

$$(k^2 - k_1^2)\ (k - \beta_c - \beta_e) + \frac{\omega_{ed}^2}{c^2} k = 0 \tag{3.28}$$

where $k_1^2 \approx - \dfrac{\omega\ \omega_i^2}{c^2(\omega-\omega_{ci})}$, $\beta_c = \dfrac{\omega_{ce}}{u_o}$, $\beta_e = \dfrac{\omega}{\mu_o} \approx \dfrac{\omega_{ci}}{u_o}$.

The dispersion can be analyzed in the same manner as was done by Pierce (34). Now a few parameters are defined.

$$\frac{\omega_{ed}^2}{c^2} = 2k_1^{\ 2}A^2 \qquad\qquad k = \beta_c(1 + jA\delta)$$

$$\gamma = u_o/c \qquad\qquad k_1 = \beta_c(1 + Ab - jAd)$$

$$A^2 = \frac{\gamma^2\omega_{ed}^2}{2\omega_{ce}^2} \qquad\qquad \beta_e = \beta_c/\lambda$$

$$\lambda = \omega_{ce}/\omega_{ci}$$

$Q \approx 1/\lambda A^2$ $\quad$ Ab = differential phase velocity between k and k_1

A is the interaction parameter and corresponds to the C in Pierce's notation. A depends on the beam velocity and density and the strength of the magnetic field. From equations 3.28 and 3.29 the dispersion becomes

$$\delta^2 + \delta(jQA + jb + d) + QA(jd - b) - 1 = 0 \quad . \tag{3.30}$$

The roots for δ are

$$\delta_{1,2} = -\frac{1}{2}(jQA + jb + d) \pm \frac{1}{2}\sqrt{(jQA + jb + d)^2 - 4QA(jd - b)} \tag{3.31}$$

If $d = 0$, which means that the plasma is lossless, the equation for δ becomes

$$\delta_{1,2} = -\frac{j}{2}(QA + b) \pm \frac{1}{2}\sqrt{-(QA + b)^2 + 4(1 + QAb)} = x \pm jy \tag{3.32}$$

The gain condition can be determined from equation 3.32, and is

$$2 > |QA - b| \qquad \text{gain condition} \tag{3.33}$$

Maximum growth occurs at $b = QA$. Plots of δ in the growth region are shown in Figs. 3.15, 3.16, 3.17 and 3.18. The bandwidth for growth can be determined approximately. If $p = \omega_i/\omega_{ci}$ and $z = \omega/\omega_{ci}$, and using the fact $z \approx 1$,

$$k_1 \approx \frac{\gamma \beta_c p}{\lambda} \frac{z^{1/2}}{(1 - z)^{1/2}} \tag{3.34}$$

$$\frac{dk_1}{dz} \approx \frac{\gamma \beta_c p}{\lambda} \frac{1}{z^{1/2}(1 - z)^{3/2}} \quad . \tag{3.35}$$

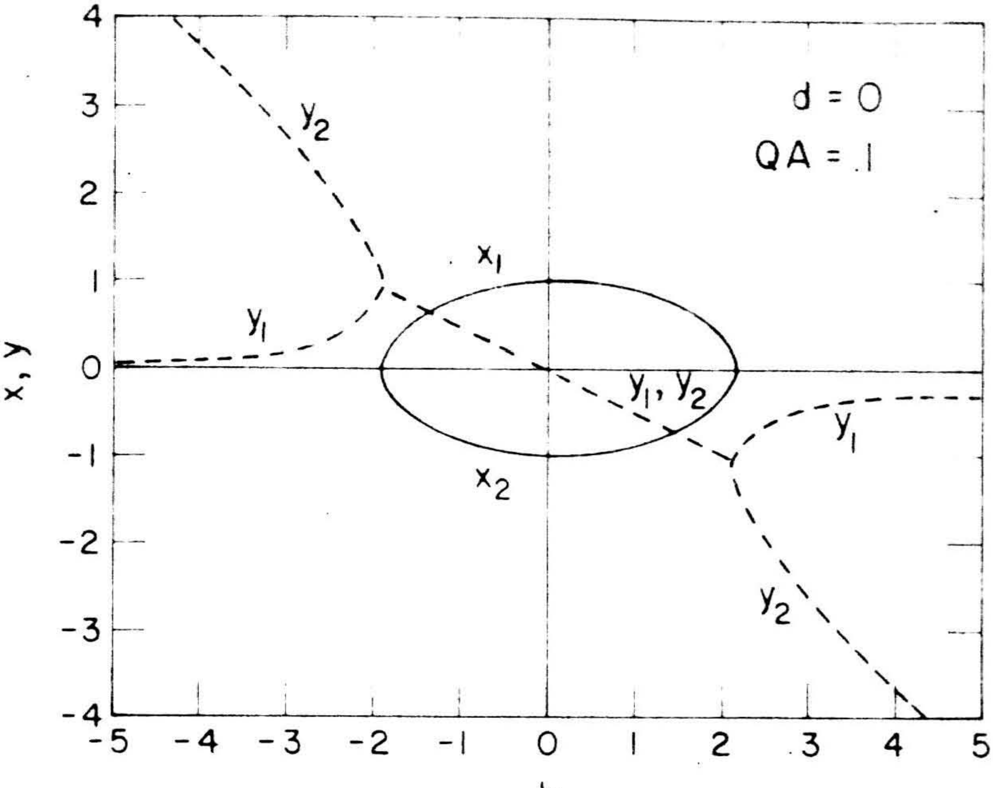


Fig. 3.15

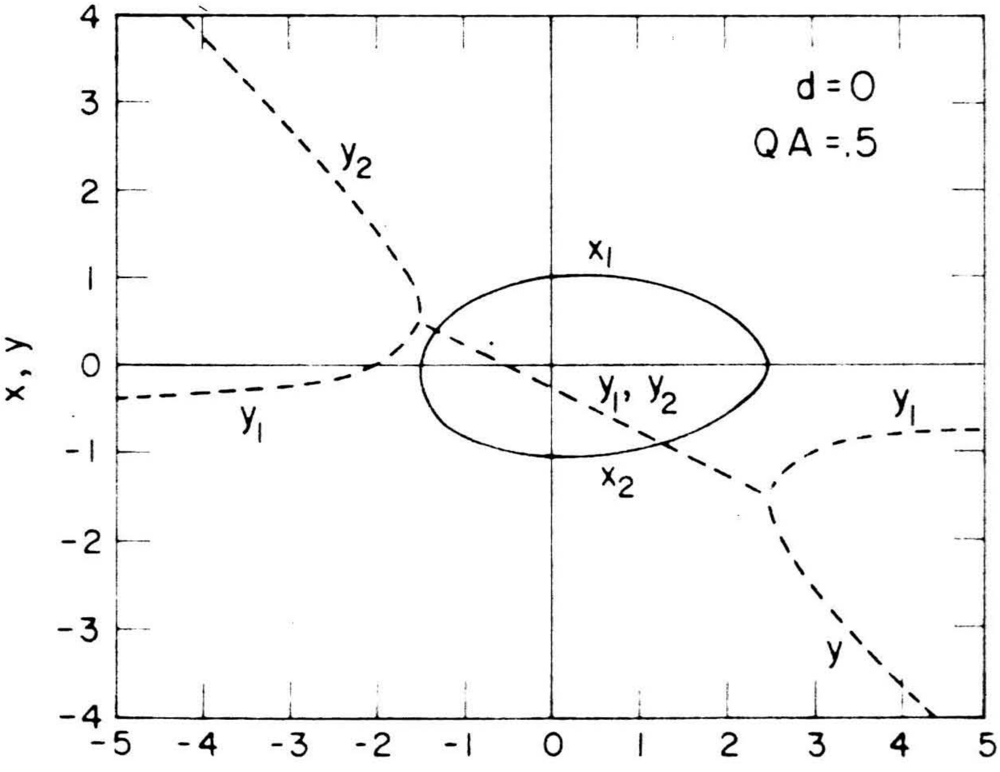


Fig. 3.16

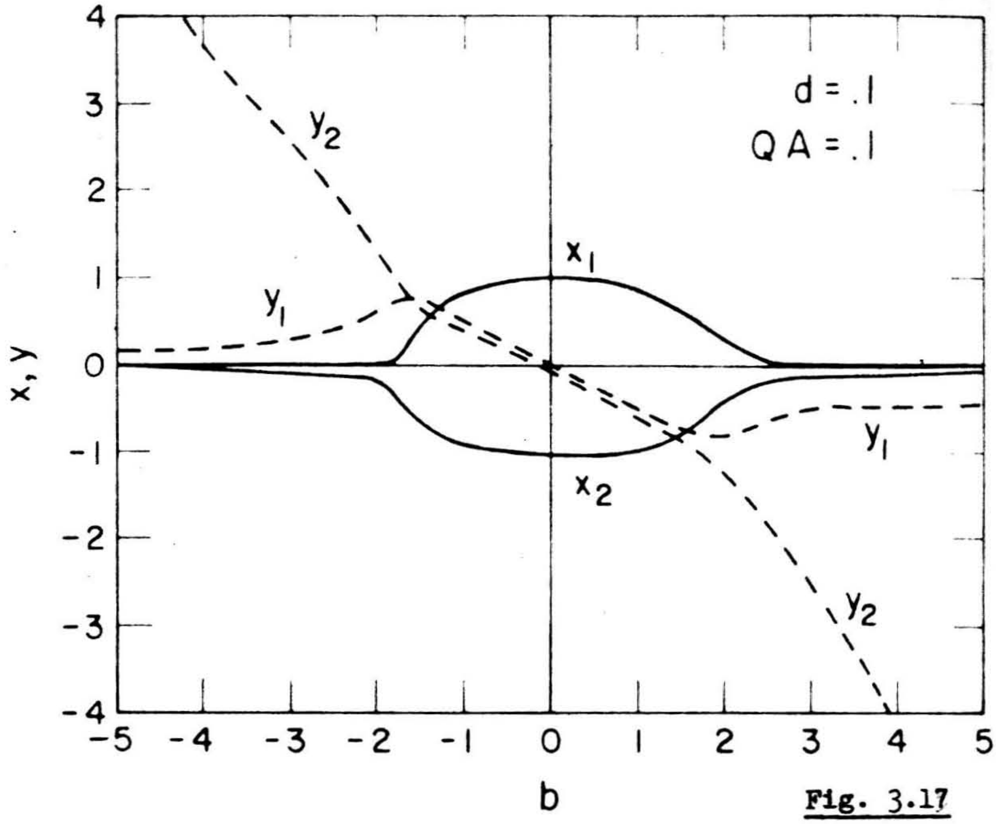


Fig. 3.17

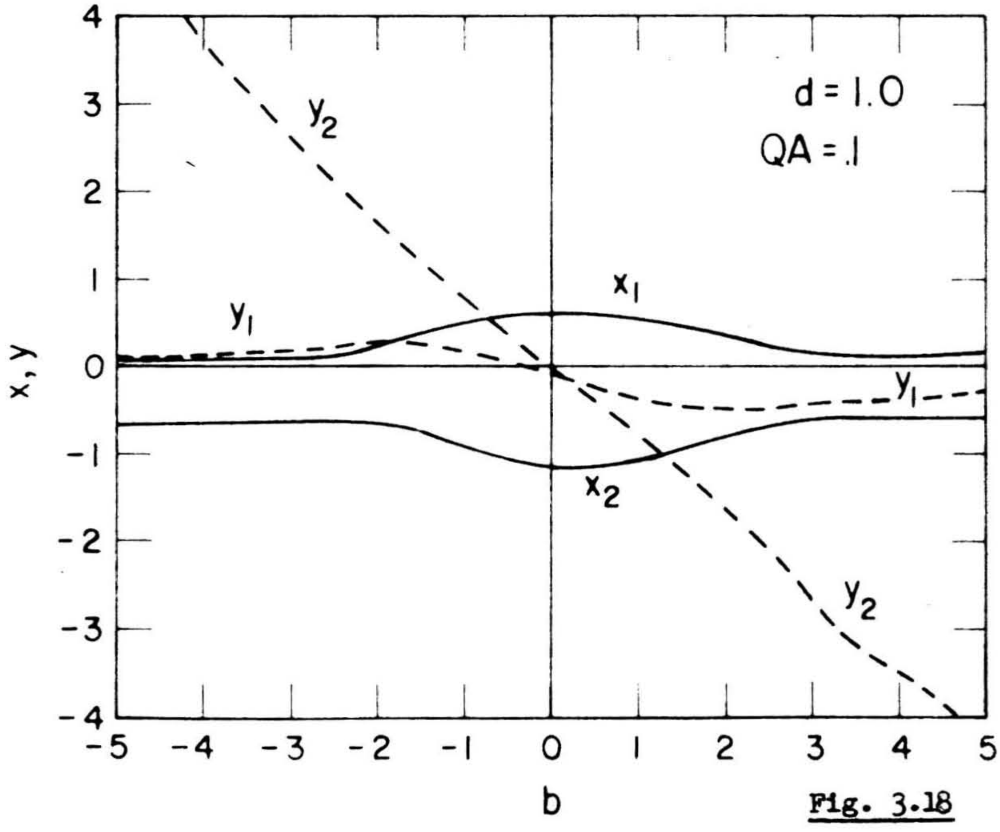


Fig. 3.18

And since at synchronism $\gamma p/\lambda \approx \dfrac{(1-z)^{1/2}}{z^{1/2}}$ one obtains

$$\frac{dk_1}{dz} \approx \beta_c \frac{1}{1-z} \quad . \tag{3.36}$$

The range over which there is growth is, using equations 3.29 and 3.33

$$\Delta k = 4\beta_c A \tag{3.37}$$

The bandwidth over which there is gain is then

$$\Delta z = 4A(1-z)$$

or

$$\Delta\omega = 2\sqrt{2}\,\frac{u_o^3}{c^3}\,\frac{\omega_{ed}\,\omega_{ci}\,\omega_i^2}{\omega_{ce}^3} \tag{3.38}$$

A sample calculation can be made to see the quantities involved. Suppose that

$$\gamma = \frac{u_o}{c} = .02\ , \qquad p = \frac{\omega_i}{\omega_{ci}} = 100\ \left(\frac{\omega_e}{\omega_{ce}} = 2.32\right)$$

$$\lambda = 1837 \text{ (hydrogen)} \qquad \sigma = \omega_{ed}^2/\omega_e^2 = .001$$

Then $(1-z) = \left(\frac{\gamma p}{\lambda}\right)^2 = 1.18 \times 10^{-6}$

$$A = \frac{2.32\,\gamma\,\sigma^{1/2}}{\sqrt{2}} = 1.04 \times 10^{-3}$$

and the bandwidth over which there is gain is

$$\Delta z = 4A(1-z) = 4.93 \times 10^{-9}$$

or $$\Delta\omega = 4.93 \times 10^{-9}\,\omega_{ci} \quad .$$

It is seen that there is gain over a very limited frequency range.

Instability near ω_{ce}, Fast Ion Beam

If there were a fast beam of ions instead of electrons, one would find that it is the $D_-(k,\omega) = 0$ dispersion which is unstable. The slow cyclotron wave of the ion beam interacts with the plasma wave in which the electrons gyrate in the same sense as the field. The dispersion can be solved for in the same manner as before, substituting the ion beam plasma frequency in place of the electron beam plasma frequency. It is noticed that although the interaction takes place at different frequencies the β's are approximately the same.

3.4.2 Drifting plasma in a plasma. If there is a drifting plasma in a plasma it will be shown that under certain conditions it is possible for an unstable wave to exist. This instability was first recognized by Dokuchaev (35). Here the analysis will be carried out more fully and the nature of the instability will be investigated as an interaction of the slow drifting plasma wave and the plasma wave. As is usual there are two transverse normal modes, a right and a left circularly polarized wave. The dispersion can be written

$$D_+(k,\omega)\, D_-(k,\omega) = 0 \tag{3.39}$$

where

$$D_+(k,\omega) = \frac{c^2k^2}{\omega^2} - \left[1 - \frac{\omega_e^2}{\omega(\omega+\omega_{ce})} - \frac{\omega_i^2}{\omega(\omega-\omega_{ci})} - \frac{\omega_{ed}^2(\omega - u_ok)}{\omega^2(\omega - u_ok + \omega_{ce})} - \frac{\omega_{id}^2(\omega - u_ok)}{\omega^2(\omega - u_ok - \omega_{cid})}\right] \tag{3.40a}$$

$$D_-(k,\omega) = \frac{c^2k^2}{\omega^2} - \left[1 - \frac{\omega_e^2}{\omega(\omega-\omega_{ce})} - \frac{\omega_i^2}{\omega(\omega+\omega_{ci})}\right.$$

$$\left. - \frac{\omega_{ed}^2(\omega-u_ok)}{\omega^2(\omega-u_ok-\omega_{ce})} - \frac{\omega_{id}^2\,(\omega-u_ok)}{\omega^2(\omega-u_ok+\omega_{cid})}\right] \qquad (3.40b)$$

The plus sign applies when the ions gyrate in the same sense as the field, and the minus sign applies when the electrons gyrate in the same sense as the field. Both of these dispersions can exhibit a growing wave. The cases will be studied individually.

$D_+(k,\omega) = 0$:

It was mentioned in Chapter I that in order to have growing waves in a system of a drifting plasma in a plasma, the undisturbed drifting plasma must have a slow negative energy wave for real ω,k. This is in accordance with the idea in Section 3.2 that a negative energy wave and a positive energy wave be near synchronism for instability. It was stated that a necessary condition was

$$u_o > V_{ad} \qquad (3.41)$$

where V_{ad} was the Alfvén velocity if the plasma were stationary. The other condition, of course, is on the density of the stationary plasma. An analysis will now be carried out which is valid near the origin. The dispersion is written

$$k^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_e^2}{\omega(\omega+\omega_{ce})} - \frac{\omega_i^2}{\omega(\omega-\omega_{ci})}\right]$$

$$- \frac{\omega_{ed}^2}{c^2\omega_{ce}}(\omega-u_ok)\left[\frac{1}{\frac{\omega}{\omega_{ce}} - \frac{u_ok}{\omega_{ce}} + 1} + \frac{1}{\frac{\omega}{\omega_{cid}} - \frac{u_ok}{\omega_{cid}} - 1}\right] \qquad (3.42)$$

For the case $\omega_{cid}/k \gg u_o > \omega/k$ and using the relations ω_{ce}/ω_{ci} , $\omega_{ce}/\omega_{cid} \gg 1$, equation 3.42 can be put in the form by an expansion

$$k^2 = \frac{\omega^2 \omega_e^2}{c^2 \omega_{ce} \omega_{ci}} + \frac{\omega_{ed}^2 (\omega - u_o k)^2}{c^2 \omega_{ce} \omega_{cid}} \tag{3.43}$$

Defining, as before

$$p_s^2 = \frac{\omega_i^2}{\omega_{ci}^2} = \frac{c^2}{V_{as}^2} \quad , \quad p_d^2 = \frac{\omega_{id}^2}{\omega_{cid}^2} = \frac{c^2}{V_{ad}^2}$$

one obtains for the dispersion

$$k^2 = \frac{\omega^2}{c^2} p_s^2 + \frac{p_d^2}{c^2} (\omega - u_o k)^2 \quad . \tag{3.44}$$

The propagation constant can now be solved for, and the result is

$$k = \frac{\omega}{u_o^2 - V_{ad}^2} \left[u_o \pm \frac{V_{ad}}{V_{as}} \sqrt{(V_{ad}^2 + V_{as}^2) - u_o^2} \right] . \tag{3.45}$$

The condition for growth is

$$u_o^2 > V_{as}^2 + V_{ad}^2 \quad . \tag{3.46}$$

Compare this with the condition given by equation 3.41. This derivation was carried out under the assumption that $\omega \ll \omega_{ci}, \omega_{cid}$. To determine the propagation constant over all ranges of frequency, the full dispersion in equation 3.42 must be solved. Figures 3.19 and 3.20 show the real and imaginary parts in the unstable region as a function

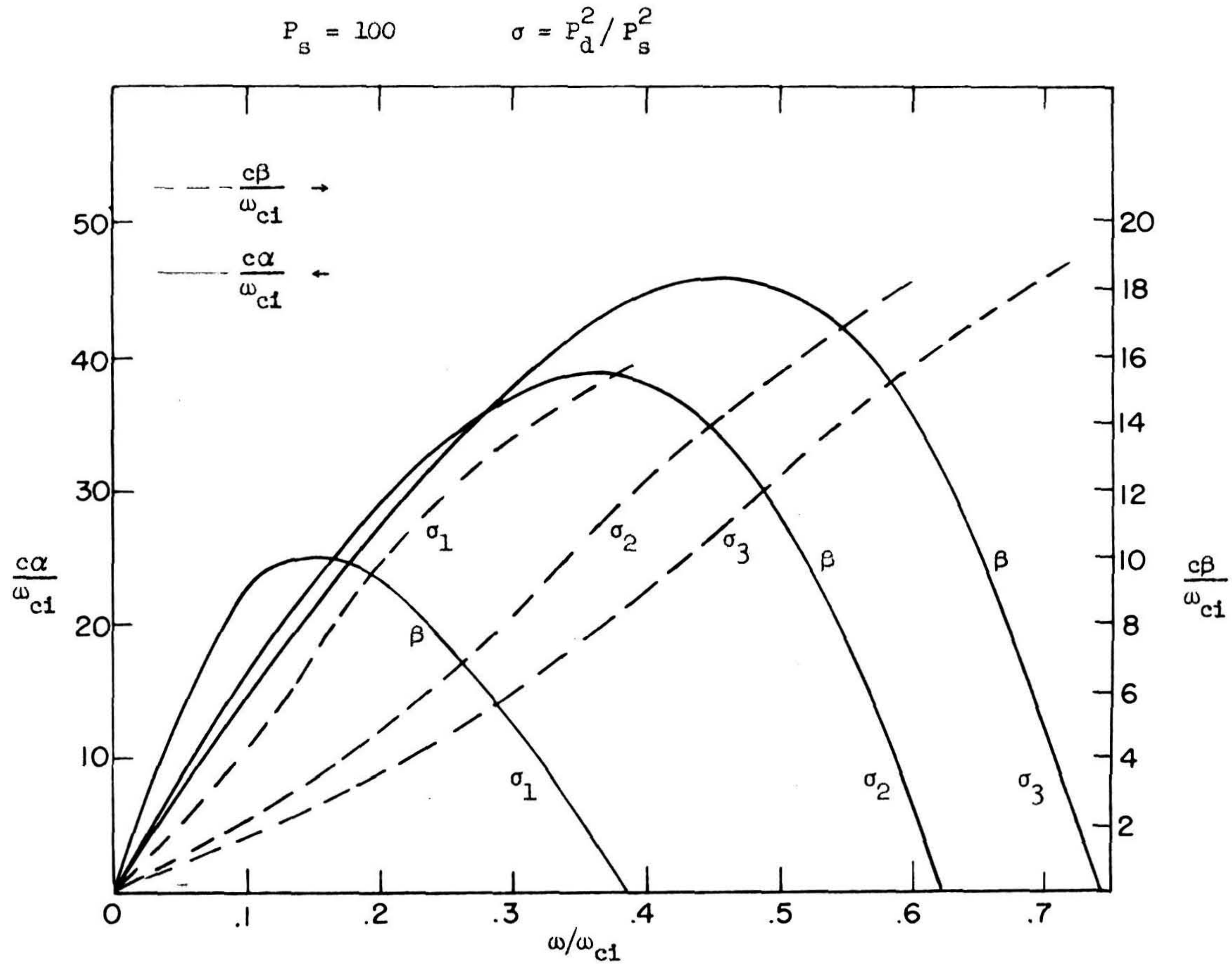


Figure 3.19 Real and imaginary values of k for the + wave

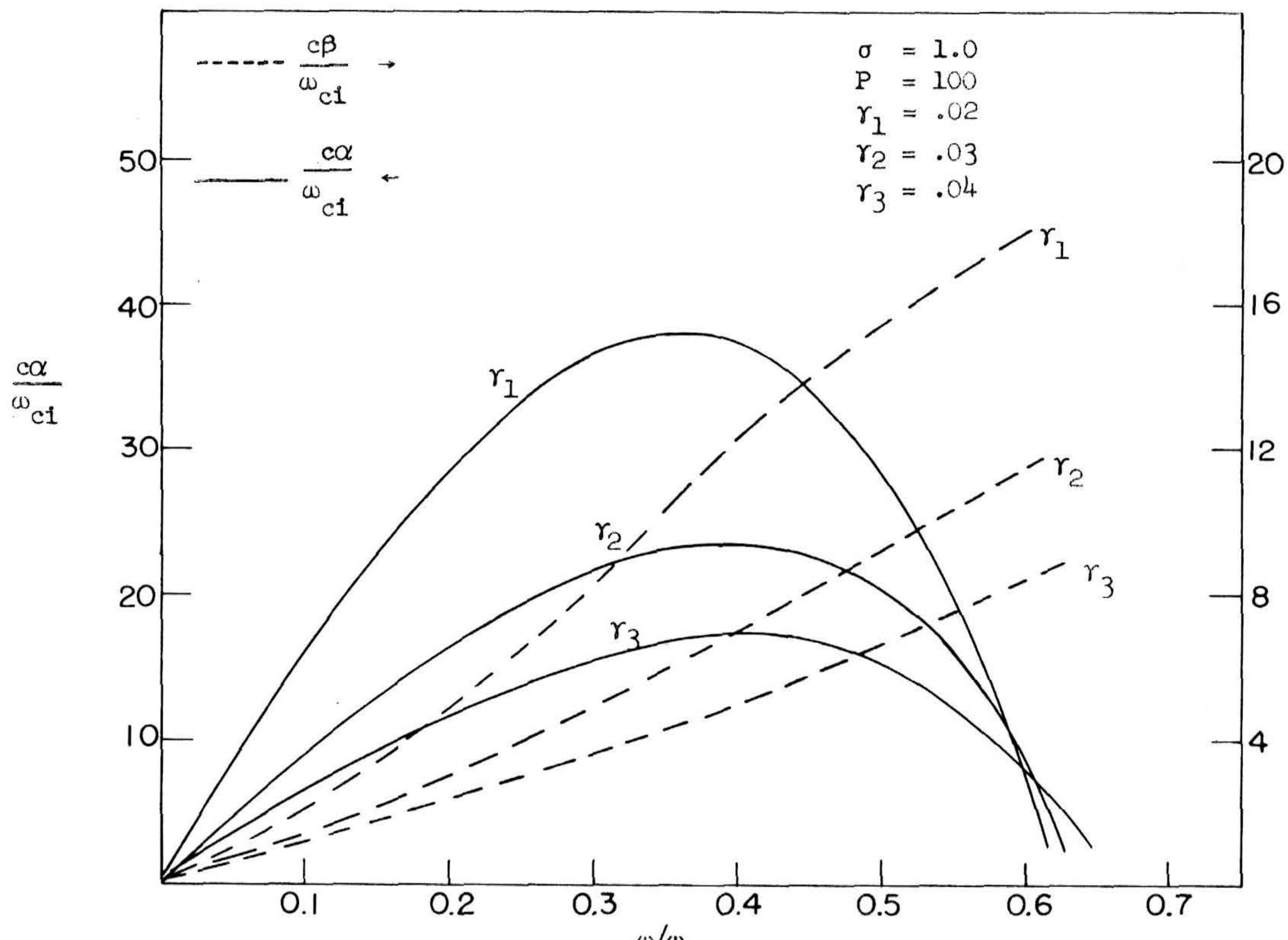


Figure 3.20 Real and imaginary values of **k** for the + wave

of two dimensionless variables, γ and σ. Incidentally, for this case there is still the interaction of the slow electron wave and the plasma wave near $\omega = \omega_{ci}$, which was worked out in the previous section.

$D_-(k,\omega) = 0$:

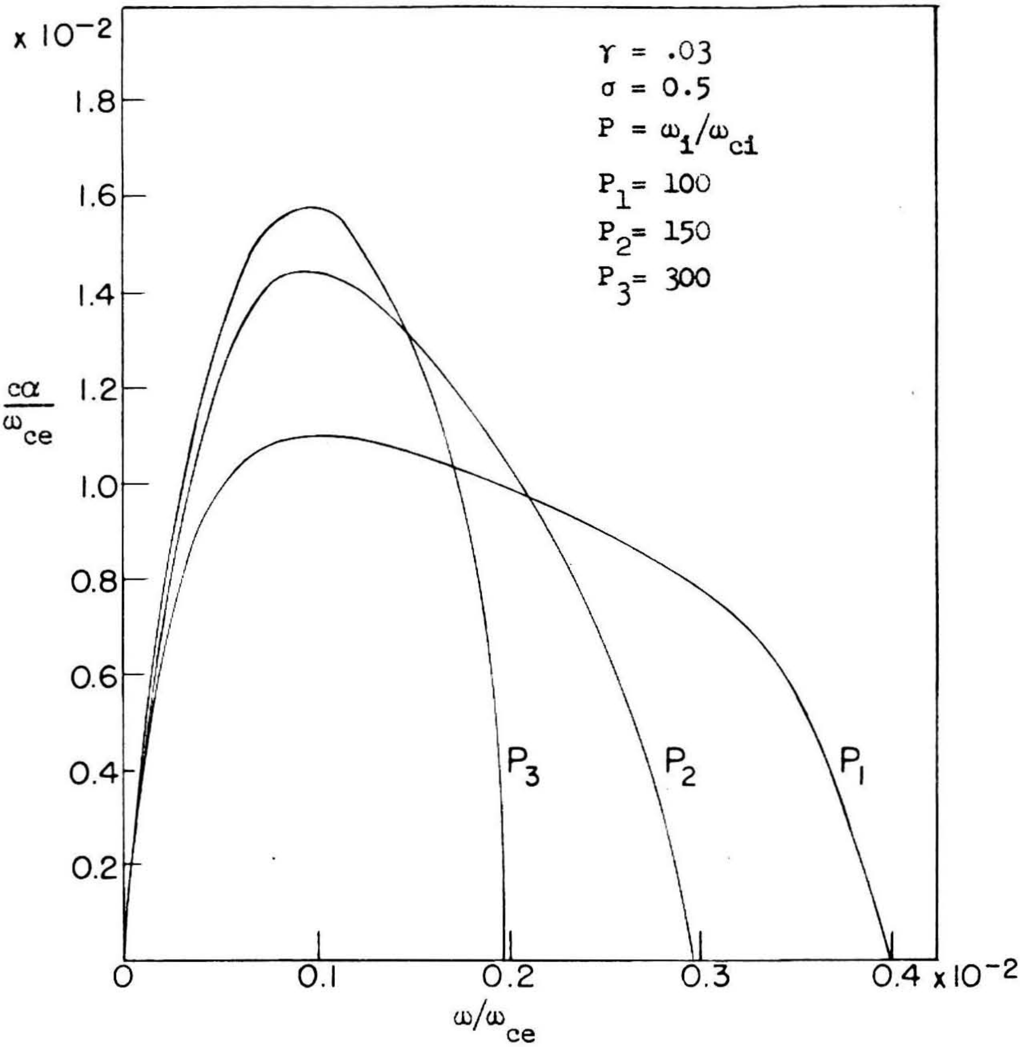
In the same way, the condition for a growing wave can be determined by examining the dispersion near zero frequency. If the expansion is carried out under the same assumptions as previously, the dispersion is the same as equation 3.44. Therefore the gain condition is the same as before, that is

$$u_o^2 > V_{as}^2 + V_{ad}^2 \tag{3.47}$$

In the $D_+(k,\omega) = 0$ dispersion there is no amplification for $\omega > \omega_{ci}$ but the $D_-(k,\omega) = 0$ dispersion can be unstable for frequencies above ω_{ci}. Plots of k in the gain region are shown in Fig. 3.21 and 3.22 as a function of P and σ. In addition there is still the interaction near ω_{ce} between the slow wave of the drifting ions and the plasma wave.

3.5 Comparison of the Growth Rates of Transverse and Longitudinal Modes

Suppose that there is a region where both the transverse and longitudinal waves can exhibit instability. These modes have different rates of growth, and growth occurs over different frequency ranges. For example, the longitudinal mode is unstable from zero frequency to



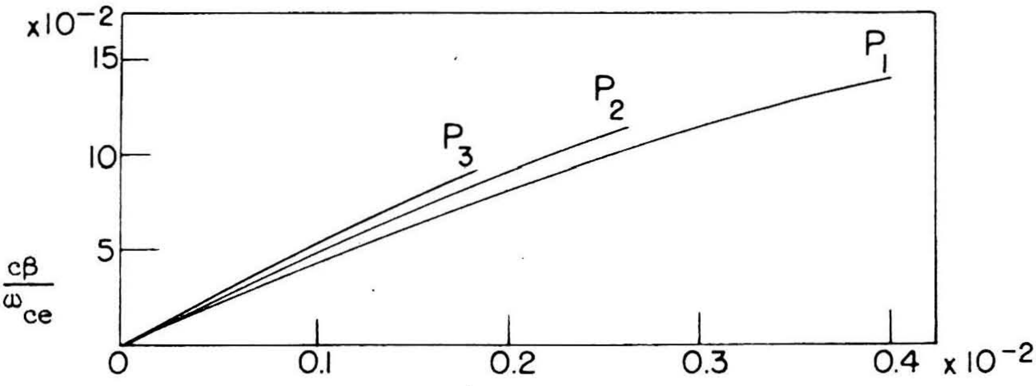


Figure 3.21 Plot of k for - wave

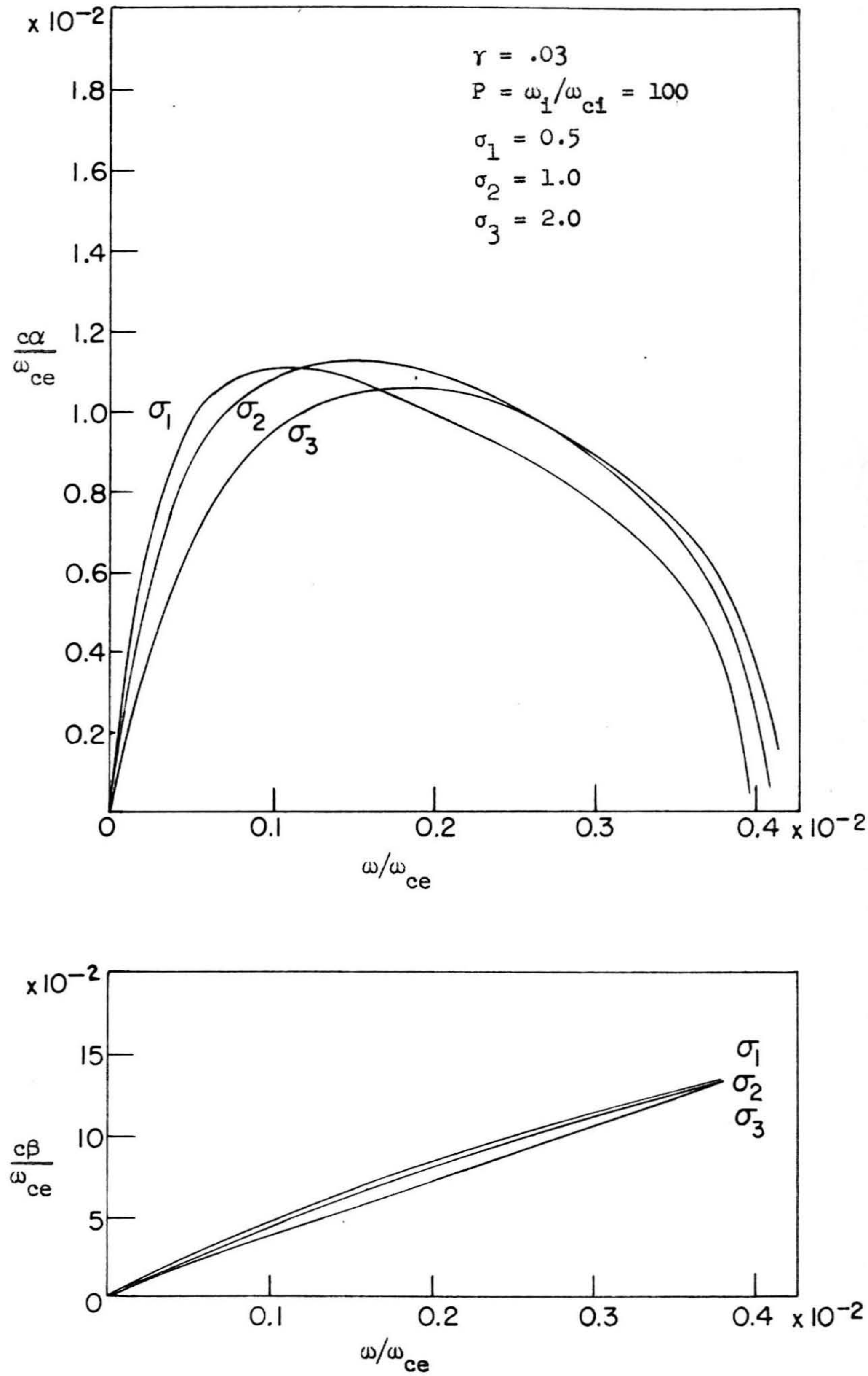


Figure 3.22 **Plot of k for - wave**

the plasma frequency, with maximum growth at the plasma frequency. For typical parameter values the + wave of the drifting plasma in a plasma system can be unstable over a frequency band between zero frequency and several tenths of the ion cyclotron frequency with maximum growth occurring near mid-frequency, and the - wave can be unstable from zero frequency to several times the ion cyclotron frequency, with maximum growth occurring at a frequency somewhat below mid-frequency. In the system of a drifting electron or ion beam through a plasma, there is an instability only in the immediate vicinity of the ion and electron frequency, respectively.

The propagation constant for the longitudinal mode is

$$k_\ell = \frac{\omega}{u_o}\left[1 \pm \frac{\omega_{ed}/\omega_e}{\sqrt{\frac{\omega^2}{\omega_e^2} - 1}}\right] . \qquad (3.48)$$

For the + and - waves of the drifting plasma in a plasma system valid near zero frequency

$$k_{+,-} = \frac{\omega}{u_o\left(1 - \frac{V_{ad}^2}{u_o^2}\right)}\left[1 \pm \frac{V_{ad}}{V_{as}}\sqrt{\frac{(V_{as}^2 + V_{ad}^2)}{u_o^2} - 1}\right] \qquad (3.49)$$

and at maximum growth for the drifting electron beam

$$k_e = \frac{\omega_{ce}}{u_o}\left[1 \pm \frac{j}{\sqrt{2}}\frac{u_o}{c}\frac{\omega_{ed}}{\omega_{ce}}\right] \qquad (3.50)$$

and for the drifting ion beam

$$k_i = \frac{\omega_{ce}}{u_o}\left[1 \pm \frac{j}{\sqrt{2}}\frac{u_o}{c}\frac{\omega_{id}}{\omega_{ce}}\right] . \tag{3.51}$$

Now the growth rates can be compared. Near zero frequency

$$\frac{\alpha_\ell}{\alpha_{+,-}} \approx \frac{\frac{V_{as}}{V_{ad}}\frac{\omega_{ed}}{\omega_e}\left(1 - \frac{V_{ad}^2}{u_o^2}\right)}{\sqrt{1 - \frac{V_{as}^2 + V_{ad}^2}{u_o^2}}\sqrt{1 - \frac{\omega^2}{\omega_e^2}}} \tag{3.52}$$

Upon examining equation 3.52 it is seen that depending on parameter values, the transverse growth can be greater than the longitudinal growth. For example, suppose that the electron beam densities of the two systems and the ion masses of the drifting and stationary plasmas were the same. Then for $\omega_e \gg \omega$,

$$\frac{\alpha_\ell}{\alpha_{+,-}} \approx \frac{\left(\frac{\omega_{ed}}{\omega_e}\right)^2\left(1 - \frac{V_{ad}^2}{u_o^2}\right)}{\sqrt{1 - \frac{V_{as}^2 + V_{ad}^2}{u_o^2}}} . \tag{3.53}$$

For $\omega_{ed}/\omega_e = 1.0$, .1, and .01; and $\frac{V_{as}^2 + V_{ad}^2}{u_o^2} = .8$; $\alpha_\ell/\alpha_{+,-} = 1.34$, .0134, and .000134, respectively.

Now compare the two other cases. First, for the drifting electron beam

$$\frac{\alpha_\ell}{\alpha_e} \approx \sqrt{2} \quad \frac{\frac{\omega_{ci}}{\omega_e}}{u_o/c} \qquad \omega \approx \omega_{ci} \tag{3.54}$$

To take a numerical example, consider an electron beam with a velocity

$u_o = .02\,c$ traveling through a hydrogen plasma in which the electron plasma and cyclotron frequency are approximately the same. Then at the maximum growth of the transverse wave

$$\frac{\alpha_\ell}{\alpha_e} \approx 3.8 \times 10^{-2} . \tag{3.55}$$

For the case of the drifting ion beam in a plasma

$$\frac{\alpha_\ell}{\alpha_i} \approx \sqrt{2}\ \frac{\omega_{ce}/\omega_e \ \times \ \omega_{ed}/\omega_{id}}{u_o/c} \tag{3.56}$$

For an ion and electron beam with equal number density traveling through a hydrogen plasma at $u_o = .02\,c$ with the electron plasma and cyclotron frequency equal

$$\frac{\alpha_\ell}{\alpha_i} \approx 303 \quad . \tag{3.57}$$

It is seen that here the longitudinal growth rate is much larger than the transverse rate.

3.6 Propagation at an Arbitrary Angle

For the case of propagation at an arbitrary angle in a plasma, with a z-directed electron beam and a magnetic field, the dispersion can be written as an eight degree polynomial

$$A_8\Gamma^8 + A_7\Gamma^7 + A_6\Gamma^6 + A_5\Gamma^5 + A_4\Gamma^4 + A_3\Gamma^3 + A_2\Gamma^2 + A_1\Gamma + A_o = 0 \tag{3.58}$$

where the coefficients and the normalization are given in Appendix III. For this more general case, the transverse and longitudinal modes cannot be separated and all field vectors are present. It is seen that the dispersion is quite complicated and the computation a tedious process. From previous results the solution for two limiting cases are known: (a) propagation at an arbitrary angle for zero magnetic field, and (b) propagation at zero angle for a finite magnetic field. So a perturbation technique can be used for cases in which there is a small deviation from the two cases above. For example, if one is interested in the growing waves, one would find for small magnetic fields, the wave described in Section 3.32 , the zero order result. Likewise, for small angles, there would be an interaction near the ion-cyclotron frequency. For the general case, however, the full dispersion must be solved.

3.7 Traveling-Wave Amplification in the Ionosphere

Bell and Helliwell (36) have proposed a theory of a traveling wave type of amplification of electromagnetic waves in the ionosphere. From the analysis of the previous sections for propagation along the magnetic field, the only TEM modes of instability for typical conditions in the ionosphere occur over a narrow band of frequencies near ω_{ci} if a fast beam of electrons is present, and near ω_{ce} if a fast beam of ions is present. If a plasma drifts through a plasma in addition to the instabilities above, there is an instability from zero frequency to a fraction of ω_{ci} for the + wave and from zero frequency to several times ω_{ci} for the - wave. For propagation at an angle to the direction of the beam and magnetic field the waves are not purely longitudinal or transverse. However, for small angles one should expect the longitudinal and transverse modes to be very weakly coupled and the dispersions analyzed previously should be approximately correct. In view of the previous analyses, the only wave that is unstable for the assumptions made by (36) for the frequency under consideration is the one that is mainly longitudinal in character. Although there is a transverse component of electric field for propagation at an angle to the magnetic field, it is small compared to the longitudinal component.

3.8 Plasma-Beam Devices

Tubes have been built employing the electron beam - plasma interaction and large gains have been produced over a narrow band of frequencies centered about the electron plasma frequency of the stationary plasma (26,37). These devices utilize the longitudinal instability described in Section 3.3. Presumably a device could be built using the transverse instability with amplification possible over a narrow band of frequencies centered about the electron plasma cyclotron frequency in case the beam consisted of ions, or centered about the ion cyclotron frequency in case the beam consisted of electrons. The disadvantage here is that high magnetic fields are needed for high frequency generation. If the beam is a neutral plasma, a low frequency instability exists, but this is not of interest for microwave generation.

IV. SUMMARY AND SUGGESTIONS FOR FURTHER WORK

The types of waves in a plasma are quite varied, and further investigation into these will certainly be fruitful. The first chapter is mainly a review of waves in a plasma with emphasis on the transverse plasma waves in a magnetic field. For waves along the magnetic field the normal modes are a right and a left circularly polarized wave, with frequency cutoffs at the ion and electron cyclotron frequencies. If the direction of propagation is at an angle to the magnetic field, the normal modes are a right and a left elliptically polarized wave.

The problem of propagation in a drifting plasma can be solved by applying a Lorentz transformation to the solution of the system in which the charged particles are at rest. For the slow waves under investigation for non-relativistic drift velocities, the transformation is a simple one, $\omega \approx \omega' + u_o k'$, $k \approx k'$; where the primed system is the one in which the plasma is stationary, and the unprimed system is the one in which the plasma is drifting. If the drift velocity or the plasma density is large enough, a slow wave emerges for positive ω, k . This has important consequences in the problem of instability in a system in which there is a drifting plasma in a plasma.

In the second chapter the problem of finite plasmas in cylindrical geometry is studied. The general features of the dispersion predicted by the quasi-static approximation are verified, and the exact and quasi-static solutions are compared. A guide is given as to the range of validity of the quasi-static approximation.

In the analyses above a cold uniform plasma was assumed. In practical problems, however, the variation in density and the finite temperature may be important factors. Therefore in a few of the problems considered, it would be worthwhile to take into account the finite temperature and the variation in density.

Whenever there is a fast beam of charged particles in a plasma, there exists the possibility of a growing wave. These waves can be either transverse or longitudinal, or both. The longitudinal growing wave has been studied extensively in the literature, whereas the transverse modes have been scarcely studied at all. These transverse modes are investigated in some detail in Chapter III. For these transverse modes, if the fast beam contains only one kind of charged particle, there is an instability over a narrow band near the electron or ion cyclotron frequency depending on whether the drifting particles are positive ions or electrons. When there is a drifting plasma in a plasma, there can be growth at lower frequencies. These transverse instabilities are possible in the earth's ionosphere, and the results in this chapter give a guide to the types of instabilities possible and their expected frequency ranges. A possible application of these unstable modes is their utilization in a plasma-ion beam interaction tube for the generation of high frequencies. The principal disadvantages of this type of device are its narrow frequency range of operation and the high magnetic fields required.

Bogdanov, Kislov and Tchernov (37) have analyzed the dispersion of a finite electron beam in an unbounded plasma in an infinite magnetic field. Perhaps it would be possible to carry on further and solve the dispersion for a finite beam in a bounded plasma with zero and finite magnetic fields.

In all of these problems dealing with the unstable modes it would be a worthwhile contribution to solve for the dispersion where the plasma density is a function of position. Sumi (38) has considered the problem of an electron beam in a plasma, assuming zero temperature and using a WKB approximation. However, the more general variations have not been solved as yet.

Partial List of Symbols

α	imaginary part of propagation constant
β	real part of propagation constant
γ	u_o/c , ratio of drift velocity to velocity of light
k	propagation constant
k_o	$\omega\sqrt{\mu\epsilon_o}$, wave number of free space
T	temperature
u_i	drift velocity
V_T	$\sqrt{3kT/m}$, mean thermal speed of electrons
V_a	$c\,\omega_{ci}/\omega_i$, Alfvén velocity
V_{ph}	ω/k , phase velocity
ω	angular frequency
ω_e	electron plasma frequency
ω_{ed}	electron plasma frequency of drifting electrons
ω_i	ion plasma frequency
ω_{id}	ion plasma frequency of drifting ions
ω_{ce}	electron cyclotron frequency
ω_{ci}	ion cyclotron frequency
ω_{cid}	ion cyclotron frequency of drifting ions

APPENDIX I

WAVE PROPAGATION IN A GYROELECTRIC MEDIUM

The problem of waves in a gyromagnetic medium has been treated by several authors and the problem of waves in a gyroelectric medium will be solved by the method used by Epstein (15). The wave equation is

$$\nabla \times [\underline{\underline{\epsilon}}^{-1} \cdot \nabla \times H] = \omega^2 \mu H \tag{AI-1}$$

where $\underline{\underline{\epsilon}}^{-1}$ is the reciprocal dielectric tensor and has been defined in Chapter II. The components of $\underline{\underline{\epsilon}}^{-1}$ are

$$\underline{\underline{\epsilon}}^{-1} = \begin{bmatrix} M & jP & 0 \\ -jP & M & 0 \\ 0 & 0 & M_3 \end{bmatrix} \tag{AI-2}$$

The wave equation will be first solved in Cartesian coordinates and the conversion to cylindrical coordinates can be made quite readily. AI-1 is a set of three equations involving the three magnetic field quantities, H_x, H_y and H_z. The equations have the form

$$[\omega^2 \mu + M_3 \nabla^2 - (M_3 - M)\frac{\partial^2}{\partial z^2}]\, H_x + jP\frac{\partial^2 H_y}{\partial z^2} + \frac{\partial}{\partial z}[(M_3 - M)\frac{\partial H_z}{\partial x} - jP\frac{\partial H_z}{\partial y}] = 0 \tag{AI-3a}$$

$$[\omega^2 \mu + M_3 \nabla - (M_3 - M)\frac{\partial^2}{\partial z^2}]\, H_y - jP\frac{\partial^2 H_x}{\partial z^2} + \frac{\partial}{\partial z}[(M_3 - M)\frac{\partial H_z}{\partial y} + jP\frac{\partial H_z}{\partial x}] = 0 \tag{AI-3b}$$

$$(\omega^2 \mu + M\nabla^2)\, H_z - jP\, \frac{\partial}{\partial z}(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}) = 0 \quad . \tag{AI-3c}$$

In addition to the wave equation, one more relation is needed. This is

$$\nabla \cdot B = 0 \tag{AI-3d}$$

At this point, two linear variables are introduced

$$Q_{1,2} = H_x \pm H_y \tag{AI-4}$$

where the subscripts 1,2 refer to the upper and lower signs, respectively. Now the equations go through a series of manipulations. AI-3b is multiplied by $\pm j$ and is added to AI-3a . To AI-3c is added

$$\mp P \frac{\partial}{\partial z} \nabla \cdot H = 0 \; .$$

The z-direction is chosen as the direction of the B-field, and writing

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla_T^2 + \frac{\partial^2}{\partial z^2} \tag{AI-5}$$

the equations in AI-3 take the form

$$\omega^2\mu + M_3\nabla_T^2 + (M \pm P)\frac{\partial^2}{\partial z^2} \quad Q_{1,2} = (M - M_3 \pm P)\frac{\partial}{\partial z}(\frac{\partial}{\partial x} \pm j\frac{\partial}{\partial y}) H_z \tag{AI-6a}$$

$$[\omega^2\mu + M\nabla_T^2 + (M \mp P) \frac{\partial^2}{\partial z^2}] \; H_z = \pm P \frac{\partial}{\partial z}(\frac{\partial}{\partial x} \mp \frac{\partial}{\partial y}) \; Q_{1,2} \tag{AI-6b}$$

where the upper sign refers to Q_1 and the lower to Q_2 . Now H_z and $Q_{1,2}$ are eliminated from the above equations. Operating on AI-6b with

$$(M - M_3 \pm P.) \frac{\partial}{\partial x} \pm j \frac{\partial}{\partial y})$$

one finds

$$LQ_{1,2} = 0 \tag{AI-7}$$

where L is the operator

$$L = \left\{\omega^4\mu^2 + \omega^2\mu[(M+M_3)\,\nabla_T^2 + 2M\frac{\partial^2}{\partial z^2}] + MM_3\,\nabla_T^4 + [M(M+M_3) - P^2]\,\nabla_T^2\,\frac{\partial^2}{\partial z^2} + (M^2 - P^2)\,\frac{\partial^4}{\partial z^4}\right\} \tag{AI-8}$$

Operating with $\mp P\frac{\partial}{\partial z}\,(\frac{\partial}{\partial x} \mp \frac{\partial}{\partial y})$ on AI-6a leads to an expression of exactly the same form for H_z .

$$LH_z = 0 \tag{AI-9}$$

It should be noticed that while AI-6 is different for both Q_1 and Q_2 , the operator L is the same for both, and they satisfy the same equation of the fourth order. Since H_x and H_y are linear combinations of Q_1 and Q_2, summarized in the vector equations

$$L\overline{H} = 0 \tag{AI-10a}$$

$$\nabla \cdot \overline{H} = 0 \tag{AI-10b}$$

This new system contains the same number of equations as AI-3, but AI-10a is of fourth order and AI-3 is of second order. Therefore AI-10 contains spurious solutions, in addition to the correct ones, and only solutions satisfying AI-3 and AI-6 can be used. One can use as a test, a subsidiary condition

$$\frac{\partial}{\partial x}\,[(\omega^2\mu + M\nabla^2)\,H_x + jP\,\frac{\partial^2}{\partial z^2}\,H_z] + \frac{\partial}{\partial y}\,[-jP\,\frac{\partial^2}{\partial z^2}\,H_x + (\omega^2\mu + M\,\nabla^2)H_y] = 0 \tag{AI-11}$$

This equation was obtained by differentiating AI-6b with respect to z and using AI-10b.

The operator ∇_T^2 is independent of the coordinate z and can be applied as the transverse operator in cylindrical coordinates as well. If wave propagation is along the z-axis $(\frac{\partial}{\partial z} = -jk_z)$, the operator L can be written as

$$L = \left\{MM_3\nabla_T^4 + [(M+M_3)(\omega^2\mu - Mk_z^2) + P^2k_z^2]\,\nabla_T^2 + [(\omega^2\mu - Mk_z^2)^2 - P^2k_z^4]\right\} \tag{AI-12}$$

and this can be split into two factors

$$L = MM_3\,[\nabla_T^2 + h_1^2]\cdot[\nabla_T^2 + h_2^2] \tag{AI-13}$$

where

$$h_{1,2}^2 = \frac{[(\omega^2\mu - Mk_z^2)(M+M_3) + P^2k_z^2 \pm f]}{2MM_3} \tag{AI-14}$$

and f is defined

$$f = [(M-M_3)^2(\omega^2\mu - Mk_z^2)^2 + 2(M+M_3)(\omega^2\mu - Mk_z^2)P^2k_z^2 + P^2k_z^4(P^2 + 4MM_3)]^{1/2}.$$

Since h_1 and h_2 are, in general, not equal, the general solution is

$$\phi = \phi_1 + \phi_2 \tag{AI-15}$$

where the two terms satisfy two ordinary wave equations

$$(\nabla_T^2 + h_1^2)\phi_1 = 0 \qquad (\nabla_T^2 + h_2^2)\phi_2 = 0 \tag{AI-16}$$

In view of AI-15, H is divided into two vector parts $\bar{H} = \bar{H}_1 + \bar{H}_2$. For a given frequency ω, the two-part vectors have different wave numbers (h_1 and h_2) and so AI-10b must become

$$\nabla \cdot H_1 = 0 , \qquad \nabla \cdot H_2 = 0 \tag{AI-17}$$

The same is true with respect to the subsidiary condition. The two partial vectors $\bar{H}_1$ and $\bar{H}_2$ are independent as long as they propagate in open space, but the independence ceases as soon as they are incident on a boundary, and the boundary conditions, in general, establish a linkage between H_1 and H_2.

The general field expressions can be obtained from AI-10 and AI-11. AI-11 can be written as

$$\frac{\partial}{\partial x}\,[\tau H_x + j\,\sigma H_y] + \frac{\partial}{\partial y}\,[-j\,\sigma H_x + \tau\,H_y] = 0 \tag{AI-18}$$

with the abbreviations

$$\tau = M(h^2 + k_z^2) - \omega^2\mu \quad , \qquad \sigma = Pk_z^2 \; .$$

AI-18 points to the existence of a Hertzian function from which the two bracket expressions may be derived by differentiation. Since this function is indeterminate to an arbitrary constant, it may be denoted by $(\tau^2 - \sigma^2)Z$,

$$\tau H_x + j\sigma\,H_y = (\tau^2 - \sigma^2)\,\frac{\partial Z}{\partial y} \tag{AI-19a}$$

$$-j\sigma H_x + \tau\,H_y = -(\tau^2 - \sigma^2)\,\frac{\partial Z}{\partial x} \tag{AI-19b}$$

Then

$$H_x = j\sigma \frac{\partial Z}{\partial x} + \tau \frac{\partial Z}{\partial y} \tag{AI-20a}$$

$$H_y = -\tau \frac{\partial Z}{\partial x} + j\sigma \frac{\partial Z}{\partial y} \tag{AI-20b}$$

By substituting into AI-10b one obtains

$$\frac{\partial H_z}{\partial z} = -j\sigma \nabla_T^2 Z \quad . \tag{AI-21}$$

Since Z must satisfy the same equation as its derivatives, it follows that

$$(\nabla_T^2 + h^2)Z = 0 \tag{AI-22}$$

where h^2 is either h_1^2 or h_2^2. Then

$$H_z = -\frac{\sigma h^2}{k_z} Z \quad . \tag{AI-23}$$

The results of AI-20 and AI-23 can be summarized in the tensor equation

$$H = \underline{\underline{S_H}} \cdot \nabla Z \tag{AI-24}$$

$$\underline{\underline{S_H}} = \begin{bmatrix} j\sigma & \tau & 0 \\ -\tau & j\sigma & 0 \\ 0 & 0 & \dfrac{-j\sigma h^2}{k_z^2} \end{bmatrix}$$

The electric field is obtained with the aid of Maxwell's equations and this leads to another tensor equation

$$E = k_z \underline{\underline{S_E}} \cdot \nabla Z \tag{AI-25}$$

$$\underline{\underline{S_E}} = \begin{bmatrix} d & jb & 0 \\ -jb & d & 0 \\ 0 & 0 & g \end{bmatrix}$$

where

$$d = \frac{1}{\omega}\,[M\tau - P^2(h^2 + k_z^2)]\ , \qquad b = -P\omega\mu\ , \qquad g = \frac{-M_3 h^2 \tau}{\omega k_z^2}$$

From AI-22, AI-24 and AI-25 , it follows in cylindrical coordinates

$$Z = C_n(hr)\, e^{-j(n\Theta + k_z z)} \tag{AI-26}$$

where $C_n(hr)$ is the general cylindric function of order n , and h can have two values h_1 and h_2 , and

$$H_r = j[\sigma \frac{\partial}{\partial r} C_n(hr) + \frac{n\tau}{r} C_n(hr)] \tag{AI-27a}$$

$$H_\Theta = -[\tau \frac{\partial}{\partial r} C_n(hr) + \frac{n\sigma}{r} C_n(hr)] \tag{AI-27b}$$

$$H_z = \frac{h^2\sigma}{k_z} C_n(hr) \tag{AI-27c}$$

$$E_r = k_z\,[d \frac{\partial}{\partial r} C_n(hr) - \frac{nb}{r} C_n(hr)] \tag{AI-27d}$$

$$E_\Theta = -jk_z\,[b \frac{\partial}{\partial r} C_n(hr) - \frac{nd}{r} C_n(hr)] \tag{AI-27e}$$

$$E_z = jk_z^2\, g\, C_n(hr) \tag{AI-27f}$$

Case for $\epsilon_{12} = 0$

Consider the case when the plasma becomes uni-axial. For this case it is easier to obtain the wave equations directly than to use the general formulation described previously. Maxwell's equations become, inserting plasma effects into the dielectric tensor and assuming a variation $e^{j(\omega t - k_z z)}$,

$$\nabla \times \overline{E} = - j\omega u \overline{H} \qquad \text{(AI-28)}$$

$$\nabla \times H = j\omega \underline{\underline{\epsilon}} \cdot E \qquad \text{(AI-29)}$$

$$\nabla \cdot (\underline{\underline{\epsilon}} \cdot E) = 0 \quad , \quad \nabla \cdot B = 0 \qquad \text{(AI-30)}$$

and

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}$$

Since all other field components can be obtained from E_z and H_z, it will be sufficient to derive the wave equation in terms of these field vectors. Taking the curl of equation AI-28, one gets

$$\nabla \times \nabla \times \overline{E} = - j\omega u (\nabla \times \overline{H}) \qquad \text{(AI-31)}$$

Noting that

$$\nabla \times \nabla \times \overline{E} = - \nabla^2 \overline{E} + \nabla(\nabla \cdot \overline{E}) \qquad \text{(AI-32)}$$

$$\nabla \cdot \overline{E} = \frac{1}{\epsilon_{11}} \nabla \cdot (\underline{\underline{\epsilon}} \cdot \overline{E}) - \left[\frac{\epsilon_{33}}{\epsilon_{11}} - 1 \right] \frac{\partial E_z}{\partial z} \qquad \text{(AI-33)}$$

Substituting equations AI-32 and AI-33 into equation AI-31 and taking a dot product with the unit vector $\overline{a}_z$ in the z-direction, one obtains for the system of E modes

$$\nabla_T^2 E_z - (\frac{\epsilon_{33}}{\epsilon_{11}} k_z^2 - \omega^2 u \epsilon_{33}) E_z = 0 \qquad \text{(AI-34)}$$

where ∇_T^2 is the transverse Laplacian.

Noting that

$$\nabla \times (\underline{\underline{\epsilon}} \cdot \overline{E}) = \epsilon_{11} \nabla \times \overline{E} + \overline{a}_x (\epsilon_{33} - \epsilon_{11}) \frac{\partial E_z}{\partial y} - \overline{a}_y (\epsilon_{33} - \epsilon_{11}) \frac{\partial E_z}{\partial x}$$

(AI-35)

One takes the curl of equation AI-29, and taking the dot product with $\overline{a}_z$, one gets

$$\nabla_T^2 H_z - (k_z^2 - \omega^2 u \epsilon_{11}) H_z = 0 \quad . \qquad \text{(AI-36)}$$

APPENDIX II

EQUIVALENT DIELECTRIC CONSTANT OF AN ELECTRON BEAM IN A MAGNETIC FIELD

To calculate the beam contribution to the dielectric tensor several relations are needed. The waves are assumed to vary as $e^{j(\omega t - k \cdot r)}$. The current density is defined

$$\overline{J}_e = \rho_o \overline{u}_1 + \rho_1 \overline{u}_o \quad . \tag{AII-1}$$

Also needed are the continuity, force, and Maxwell's equations.

$$\nabla \cdot \overline{J}_e = - \frac{\partial \rho_1}{\partial t} \tag{AII-2}$$

$$\frac{\partial \overline{u}_1}{\partial t} + u_o \cdot \nabla \overline{u}_1 = - \frac{e}{m}(\overline{E}_1 + \overline{u}_o \times \overline{B}_1 + \overline{u}_1 \times B_o) \tag{AII-3}$$

$$\nabla \times \overline{E}_1 = - \frac{\partial \overline{B}_1}{\partial k} \tag{AII-4}$$

$$\nabla \times \overline{H}_1 = \overline{J}_e + \frac{\partial \overline{D}_1}{\partial t} \tag{AII-5}$$

First the velocity is expressed in terms of the electric field. This can be obtained by combining equations AII-2, AII-3, and AII-5. The result is

$$\overline{u}_1 \, [j(\omega - u_o k_z) + X \frac{e}{m} B_o] = - \frac{e}{m} \; \overline{E}_1 + \frac{1}{j\omega} \, [\nabla(\overline{u}_o \cdot \overline{E}) - u_o \cdot \nabla \overline{E}] \tag{AII-6}$$

The velocity can be expressed in terms of the current density from AII-1 and AII-2

$$\overline{J}_e = \rho_o \overline{u}_1 - \frac{\overline{u}_o}{j\omega} \nabla \cdot \overline{J}_e \quad . \tag{AII-7}$$

The dielectric tensor is defined

$$j\omega\underline{\underline{\epsilon}} \cdot \overline{E} = \overline{J} + \overline{\frac{\partial D}{\partial t}} \qquad \text{(AII-8)}$$

If one includes the stationary plasma effects in the current density, using equations AII-6, AII-7 and AII-8, and the known effects of the stationary plasma, one obtains for the total dielectric tensor

$$\underline{\underline{\epsilon}} = [\epsilon_{ij}] \qquad i,j = 1,2,3 \qquad \text{(AII-9)}$$

where

$$\epsilon_{11} = 1 - \frac{\omega_e^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_{ed}^2}{\omega^2} \frac{(\omega - u_o k_z)^2}{(\omega - u_o k_z)^2 - \omega_{ce}^2}$$

$$\epsilon_{12} = -j \frac{\omega_{ce}\omega_{ed}^2}{\omega^2} \frac{(\omega - u_o k_z)}{(\omega - u_o k_z)^2 - \omega_{ce}^2} - j \frac{\omega_{ce}\omega_e^2}{\omega(\omega^2 - \omega_{ce}^2)}$$

$$\epsilon_{13} = -\frac{\omega_{ed}^2 u_o}{\omega} \left[\frac{k_x(\omega - u_o k_z) + j\omega_{ce} k_y}{(\omega - u_o k_z)^2 - \omega_{ce}^2} \right]$$

$$\epsilon_{21} = \epsilon_{12}^*$$

$$\epsilon_{22} = \epsilon_{11}$$

$$\epsilon_{23} = -\frac{\omega_{ed}^2 u_o}{\omega} \left[\frac{k_y(\omega - u_o k_z) - j\omega_{ce} k_x}{(\omega - u_o k_z)^2 - \omega_{ce}^2} \right]$$

$$\epsilon_{31} = \epsilon_{13}^*$$

$$\epsilon_{32} = \epsilon_{23}^*$$

$$\epsilon_{33} = 1 - \frac{\omega_e^2}{\omega^2} - \frac{\omega_{ed}^2}{(\omega - u_o k_z)^2} - \frac{\omega_{ed}^2\, u_o^2 (k_x^2 + k_y^2)}{\omega[(\omega - u_o k_z)^2 - \omega_{ce}^2]}$$

If one has to take into account the effect of drifting ions, one proceeds in the same manner as before, adding a term for every drifting electron effect with the conversion

$$\omega_{ed}^2 \rightarrow \omega_{id}^2$$

$$\omega_{ce} \rightarrow -\omega_{ci}$$

$$u_o \rightarrow u_i$$

APPENDIX III

PROPAGATION AT AN ARBITRARY ANGLE IN A SYSTEM WITH AN ELECTRON BEAM IN A PLASMA IN A MAGNETIC FIELD

The dispersion for the case of propagation at an arbitrary angle in a plasma with a z-directed electron beam and magnetic field is obtained by incorporating the beam and plasma effects into an equivalent dielectric constant, and substitutingthis into the wave equation which has the form

$$\underline{\underline{L}} \cdot \overline{E} = 0 \quad . \tag{AII-1}$$

The dispersion is obtained by setting the determinant of the operator L equal to zero. This results in a polynomial

$$A_8\Gamma^8 + A_7\Gamma^7 + A_6\Gamma^6 + A_5\Gamma^5 + A_4\Gamma^4 + A_3\Gamma^3 + A_2\Gamma^2 + A_1\Gamma + \Gamma_o = 0 \tag{AII-2}$$

and the parameters below are defined

$$\Gamma = \frac{u_o k}{\omega} \quad , \qquad \gamma = \frac{u_o}{c}$$

$$M^2 = \frac{\omega_e^2}{\omega_{ce}^2(\frac{\omega^2}{\omega_{ce}^2} - 1)} + \frac{\omega_i^2}{\omega_{ci}^2(\frac{\omega^2}{\omega_{ci}^2} - 1)}$$

$$\Omega_e^2 = \frac{\omega_e^2}{\omega^2} + \frac{\omega_i^2}{\omega^2}$$

$$N^2 = \frac{\omega_{ce}^2\ \omega_e^2}{\omega(\omega^2 - \omega_{ce}^2)} - \frac{\omega_{ci}^2\ \omega_i^2}{\omega(\omega^2 - \omega_{ci}^2)}$$

and the coefficients are defined

$$A_8 = -\cos^4\theta\ [\sin^2\theta(1-M^2) + \cos^2\theta(1-\Omega_e^2)]$$

$$A_7 = 4\cos^3\theta\ [\sin^2\theta(1-M^2) + \cos^2\theta(1-\Omega_e^2)]$$

$$\begin{aligned}
A_6 = \Big\{ & \Omega_b^2 \sin^2\theta\ \cos^2\theta - (1-\Omega_c^2)\cos^2\theta[\sin^2\theta(1-M^2) + \cos^2\theta(1-\Omega_e^2)] \\
& - 4\cos^2\theta[\sin^2\theta(1-M^2) + \cos^2\theta(1-\Omega_e^2)] - \cos^2\theta[\sin^2\theta(1-M^2) + \\
& + \cos^2\theta(1-\Omega_e^2 - \Omega_b^2)]\Big\} + \gamma^2 \Big\{ -\Omega_b^2 \sin^2\theta\ \cos^2\theta(1+\cos^2\theta)(1-M^2) - \\
& - \Omega_b^2 \cos^4\theta(1+\cos^2\theta)(1-\Omega_e^2) + \cos^4\theta(1-M^2)\ [\sin^2\theta(1-M^2) + \cos^2\theta - \Omega_e^2)] \\
& + \cos^4\theta(1-M^2)(1-\Omega_e^2) - N^4\cos^4\theta\ \sin^2\theta \Big\}
\end{aligned}$$

$$\begin{aligned}
A_5 = \Big\{ & -2\Omega_b^2 \sin^2\theta\ \cos\theta + 2\cos\theta(1-\Omega_c^2\ [\sin^2\theta(1-M^2) + \cos^2\theta(1-\Omega_e^2)] \\
& + 2\cos\theta\ [\sin^2\theta(1-M^2) + \cos^2\theta(1-\Omega_e^2 - \Omega_b^2)]\Big\} + \\
& + \gamma^2 \Big\{ 2\Omega_b^2 \sin^2\theta\ \cos^3\theta(1-M^2) + 2\Omega_b^2 \sin^2\theta\ \cos\theta(1+\cos^2\theta)(1-M^2) + \\
& + 4\Omega_b^2 \cos^3\theta(1+\cos^2\theta)(1-\Omega_e^2) - 4\cos^3\theta(1-M^2)[\sin^2\theta(1-M^2) + \\
& + \cos^2\theta(1-\Omega_e^2)] - 4\cos^3\theta(1-M^2)(1-\Omega_e^2) + 4N^4\sin^2\theta\cos^3\theta \Big\}
\end{aligned}$$

$$A_4 = \left\{\Omega_b^2 \sin^2\theta - (1-\Omega_c^2)\left[\sin^2\theta(1-M^2) + \cos^2\theta(1-\Omega_e^2-\Omega_b^2)\right]\right\}$$
$$+ \gamma^2\left\{-7\Omega_b^2\sin^2\theta\cos^2\theta(1-M^2) - \Omega_b^2\sin^2\theta(1-M^2) - 5\Omega_b^2\cos^2\theta(1-\Omega_e^2)\right.$$
$$+ \Omega_b^4\sin^2\theta\cos^2\theta - \Omega_b^2\cos^2\theta(1+\cos^2\theta)(1-\Omega_e^2-\Omega_b^2)$$
$$+ (1-\Omega_e^2)(1-M^2)\cos^2\theta\left[\sin^2\theta(1-M^2) + \cos^2\theta(1-\Omega_e^2)\right]$$
$$+ 4\cos^2\theta(1-M^2)\left[\sin^2\theta(1-M^2) + \cos^2\theta(1-\Omega_e^2)\right]$$
$$+ \cos^2\theta(1-M^2)\left[\sin^2\theta(1-M^2 + \cos^2\theta(1-\Omega_e^2-\Omega_b^2)\right] + (1-\Omega_c^2)(1-M^2)(1-\Omega_e^2)\cos^2\theta$$
$$+ 4\cos^2\theta(1-M^2)(1-\Omega_e^2) + \cos^2\theta(1-M^2)(1-\Omega_e^2-\Omega_b^2) - 2N^2\Omega_c\Omega_b^2\sin^2\theta\cos^2\theta$$
$$\left. - (1-\Omega_c^2)N^4\sin^2\theta\cos^2\theta - 5N^4\sin^2\theta\cos^2\theta\right\}$$
$$+ \gamma^4\left\{-\Omega_b^4\sin^2\theta\cos^2\theta(1-M^2) - \Omega_b^4\cos^4\theta(1-\Omega_b^2\sin^2\theta\cos^2\theta(1-M^2)^2\right.$$
$$+ 2\Omega_b^2\cos^4\theta(1-M^2)(1-\Omega_e^2) - \cos^4\theta(1-M^2)^2(1-\Omega_e^2)$$
$$\left. - \Omega_b^2 N^4\sin^2\theta\cos^2\theta + N^4\cos^4\theta(1-\Omega_e^2)\right\}$$

$$A_3 = \gamma^2\left\{6\Omega_b^2\sin^2\theta\cos\theta(1-M^2) + 2\Omega_b^2\cos\theta(1+\cos^2\theta)(1-\Omega_e^2) - 2\Omega_b^4\sin^2\theta\cos\theta\right.$$
$$+ 2\Omega_b^2\cos\theta(1+\cos^2\theta)(1-\Omega_e^2-\Omega_b^2) - 2\cos\theta(1-\Omega_c^2)(1-M^2)\left[\sin^2\theta(1-M^2)\right.$$
$$\left. + \cos^2\theta(1-\Omega_e^2)\right] - 2\cos\theta(1-M^2)\left[\sin^2\theta(1-M^2) + \cos^2\theta(1-\Omega_e^2-\Omega_b^2)\right]$$
$$- 2\cos\theta(1-\Omega_c^2)(1-M^2)(1-\Omega_e^2) - 2\cos\theta(1-M^2)(1-\Omega_e^2-\Omega_b^2)$$
$$\left. + 2N^4\sin^2\theta\cos\theta + 2N^4(1-\Omega_c^2)\sin^2\theta\cos\theta + 4N^2\Omega_c\Omega_b^2\sin^2\theta\cos\theta\right\}$$
$$+ \gamma^4\left\{2\Omega_b^4\sin\theta\cos\theta(1-M^2) + 4\Omega_b^4\cos^3\theta(1-\Omega_e^2) - 2\Omega_b^2\sin^2\theta\cos\theta(1-M^2)^2\right.$$
$$- 8\Omega_b^2\cos^3\theta(1-M^2)(1-\Omega_e^2) + 4\cos^3\theta(1-M^2)^2(1-\Omega_e^2)$$
$$\left. + 2\Omega_b^2 N^2\sin^2\theta\cos\theta - 2N^2\Omega_c\Omega_b^2\cos^3\theta(1-\Omega_e^2) - 4N^4\cos^3\theta(1-\Omega_e^2)\right\}$$

$$A_2 = \gamma^2 \Big\{ \Omega_b^4 \sin^2\theta - 2\Omega_b^2 \sin^2\theta(1-M^2) - \Omega_b^2(1+\cos^2\theta)(1-\Omega_e^2-\Omega_b^2)$$
$$+ (1-\Omega_c^2)(1-M^2)\left[\sin^2\theta(1-M^2)+\cos^2\theta(1-\Omega_e^2-\Omega_b^2)\right]$$
$$+ (1-\Omega_c^2)(1-M^2)(1-\Omega_e^2-\Omega_b^2) - N^4\sin^2\theta(1-\Omega_c^2) - 2N^2\Omega_c\Omega_b^2\sin^2\theta\Big\}$$
$$+ \gamma^4\Big\{-\Omega_b^4\sin^2\theta(1-M^2) - 5\Omega_b^4\cos^2\theta(1-\Omega_e^2) + \Omega_b^2\sin^2\theta(1-M^2)^2$$
$$+ 10\Omega_b^2\cos^2\theta(1-M^2)(1-\Omega_e^2) - \Omega_b^4\cos^2\theta(1-\Omega_e^2-\Omega_b^2)$$
$$+ 2\Omega_b^2\cos^2\theta(1-M^2)(1-\Omega_e^2-\Omega_b^2) - \cos^2\theta(1-\Omega_c^2)(1-M^2)(1-\Omega_e^2)$$
$$- 4\cos^2\theta(1-M^2)^2(1-\Omega_e^2) - \cos^2\theta(1-M^2)^2(1-\Omega_e^2-\Omega_b^2) - N^4\Omega_b^2\sin^2\theta$$
$$+ 6N^2\Omega_c\Omega_b^2(1-\Omega_e^2)\cos^2\theta + N^4\cos^2\theta(1-\Omega_e^2-\Omega_b^2) + 4N^4\cos^2\theta(1-\Omega_e^2)$$
$$+ N^4\cos^2(1-\Omega_c^2)(1-\Omega_e^2)\Big\}$$

$$A_1 = \gamma^4\Big\{2\Omega_b^4\cos\theta(1-\Omega_e^2) - 4\Omega_b^2\cos\theta(1-M^2)(1-\Omega_e^2) + 2\Omega_b^4\cos\theta(1-\Omega_e^2-\Omega_b^2)$$
$$- 4\Omega_b^2\cos\theta(1-M^2)(1-\Omega^2)(1-\Omega_e^2-\Omega_b^2) + 2\cos\theta(1-\Omega_c^2)(1-M^2)(1-\Omega_e^2)$$
$$+ 2\cos\theta(1-M^2)^2(1-\Omega_e^2-\Omega_b^2) - 2N^2\Omega_c\Omega_b^2) - 2N^2\Omega_c\Omega_b^2\cos\theta(1-\Omega_e^2-\Omega_b^2)$$
$$- 4N^2\Omega_c\Omega_b^2(1-\Omega_e^2)\cos\theta - 2N^4\cos\theta(1-\Omega_e^2-\Omega_b^2) - 2N^4\cos\theta(1-\Omega_c^2)(1-\Omega_e^2)\Big\}$$

$$A_0 = \gamma^4\Big\{-\Omega_b^4(1-\Omega_e^2-\Omega_b^2) + 2\Omega_b^2(1-M^2)(1-\Omega_e^2-\Omega_b^2) - (1-\Omega_c^2)(1-M^2)^2(1-\Omega_e^2-\Omega_b^2)$$
$$+ (1-\Omega_e^2-\Omega_b^2)\left[2N^2\Omega_c\Omega_b^2 + (1-\Omega_c^2)N^4\right]\Big\} \quad .$$

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