

I. NONLINEAR GAS OSCILLATIONS IN PIPES.

II. WAVETRAINS WITH SMALL DISSIPATION.

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Javier Jimenez.

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ABSTRACT.

In part I of this thesis we study theoretically the problem of forced acoustic oscillations in a pipe. The oscillations are produced by a moving piston in one end of the pipe, while a variety of boundary conditions ranging from a completely closed end to a completely open mouth are considered at the other end. All these boundary conditions are modelled by two parameters: a length correction and a reflection coefficient equivalent to the acoustic impedance.

The linear theory predicts large amplitudes near resonance and non-linear effects become crucially important. By expanding the equations of motion in a series of the Mach number, both the amplitude and waveform of the oscillations are predicted there.

In both the open and closed-end cases the need for shock waves in some range of parameters is found. The amplitude of the oscillation is different for the two cases, however, being proportional to the square root of the piston amplitude in the closed end case, and to the cube root in the open end.

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In part II we modify the averaged Lagrangian method used by Whitham to analyze slowly varying non-linear wavetrains to include cases with a small dissipation. To do this, we use a pseudo-variational principle introduced by Prigogine in which the

Lagrangian depends on the variable and the solution of the problem , and which can be used to describe irreversible processes.

We prove the corresponding averaged equations to all orders and describe practical ways to use them to lowest order.

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I. NONLINEAR GAS OSCILLATIONS IN PIPES.

1. Introduction.

The problem we consider is the oscillation of a gas inside a pipe of length L , and whose transverse dimension is small with respect to the length. At one end of the pipe ($x = 0$) a piston executes small harmonic oscillations with a frequency that we choose to be of the order of the resonant frequency associated with L . At the other end ($x = L$) we want to model a range of physical conditions progressing from a completely closed to a completely open tube, including different kinds of perforated end plates or other mouth configurations, thereby producing varying amounts of coupling to the room.

If the pipe we study is slender a reasonable assumption is that there exist an equivalent one-dimensional problem, approximating the actual one and characterized by "effective" cross sectional conditions. This is usually a good assumption everywhere in the pipe except near the mouth section where the matching of the flow in the tube to the three-dimensional flow in the room gives rise to local transverse effects. The classical way to model these effects in linear acoustics is the use of an equivalent impedance of the end section. (Morse & Ingard 1968) The real and imaginary part of this impedance can be considered quite separately.

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The imaginary part corresponds simply to a length correction. The effective length of the pipe is different from the real length, and this difference accounts for part of the two-dimensional effects at the mouth section. There are some classical theoretical results (Rayleigh 1945) aiming to predict the value of this correction, based on assumptions like potential flow near the pipe end, and no viscous effects. In the real world, however, these assumptions are hardly ever fulfilled except for the smallest amplitudes of the gas oscillations. In any case, as the correction arises from transverse effects, the general order of magnitude can be expected to be no more than a few pipe diameters, and this should cause no qualitative difference in the behaviour of a slender pipe.

In a similar way, the real part of the impedance can be interpreted as a partial reflection coefficient. The general idea is that, from some diameters away, the pipe end should look like a virtual plane section. A long wave running into this section from the pipe is partially reflected and partially transmitted, or somehow dissipated. The simplest model for this process, which was also proposed by Seymour & Mortell (1973), is to assume that the reflected wave is proportional to the incoming one, with a proportionality coefficient ranging between +1 and -1. In the same fashion as with the length correction there are theoretical estimates for the value of this factor, but they cannot be trusted for waves of any reasonable amplitude.

In summary, although the impedance model was created for linear oscillations, it still provides a very compact way of treating a wide range of physical cases. In fact, there have been several attempts to justify its use in the treatment of acoustically absorbant materials at high radiation intensities where nonlinear effects appear. Thus, in 1967, Ingard and Ising, using experiments with resonators, proved that the behaviour of orifices could be successfully approximated at moderated intensities by the use of an amplitude-dependent impedance coefficient.

The experiments undertaken to complement the present work (Sturtevant 1973) have been carried out at much higher intensities than in previous cases. Therefore it is of interest to check them against the predictions of the impedance model and, in that way, to judge the validity of the model itself.

Here we, therefore, develop a nonlinear theory applicable to those cases in which resonance peaks occur with sufficient amplitude that linear analysis is inadequate. The first suggestion of the importance of nonlinear effects was made by Lettau (1939) who observed experimentally the appearance of traveling shock waves near the linear resonance frequency in both closed and open tubes.

Using these experiments as a guide Betchov (1958), constructed a theoretical solution of the flow in a closed pipe, in which, with a few well chosen assumptions on the general form

of the solution, he was able to prove that the nonlinearity of the equations alone bounds the resonant amplitude away from infinity without recourse to dissipation, as well as to compute waveforms in qualitative agreement with the experimental results.

Saenger and Hudson (1960) further refined the experimental observations and attempted to account theoretically for the effects of viscous shear and heat conduction. Finally, Chester (1964) developed a consistent theory for the closed pipe in which, without any special assumption, the appearance and strength of shock waves, as well as the detailed waveforms for all frequencies, were predicted. One important result of his paper was to show that the amplitude of the pressure oscillation is $O(\delta^{\frac{1}{2}})$ whereas the piston amplitude is much smaller, $O(\delta)$. Temkin (1968) obtained still more experimental data on the closed tube and gave a simple but elegant account of the different effects present in the problem using energy balance considerations.

Wijngaarden (1968) treated the case of an open pipe at resonance using a nonlinear boundary condition in which viscous dissipation caused by flow separation at the pipe exit was assumed to dominate radiation losses and, in fact, the nonlinear behaviour of the gas in the pipe itself. This is probably true for some range of the geometrical parameters of the pipe and represents the opposite extreme to the case treated in the present work. The pressure amplitude he derives is again $O(\delta^{\frac{1}{2}})$

but this is governed primarily by the dissipation boundary condition.

Mortell (1971) attempted a straightforward generalization of Chester's analysis of the closed pipe to other cases of nonlinear oscillations, including the open pipe with a perfectly reflecting exit. It turns out, however, that the method does not generalize to this case without special precautions, and Mortell derived the wrong result. In particular, the amplitude of the oscillation was mistakenly given as $O(\delta^{\frac{1}{2}})$.

The first author to point to the correct result was Collins (1971), who studied the problem of a nonlinear wave equation applied to the vibration of a string. He correctly expanded the equation in terms of the resulting amplitude, instead of the forcing amplitude, pointed out the similarity to the open pipe, and predicted the resulting amplitude to be $O(\delta^{\frac{1}{3}})$. He did not present detailed calculations for the gas dynamics case, and, in particular, failed to point out the presence of shock waves in the solution.

A closely related class of problems was treated by Chu and Ying (1963), when they studied thermally induced oscillations in closed pipes. They used a characteristics perturbation procedure due to Lin (1954) which is very close to the one used in this work.

Finally, Seymour and Mortell (1973) describe an extension

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of Chester's method to a nearly closed pipe with radiation damping and obtain results very similar to the ones in section 4 of this work.

2. General equations.

Consider the pipe along the x axis. The passive end of the pipe is located at $x = L$. At $x = 0$ a piston oscillates according to the law

$$x = -l \cos \tilde{\omega} t$$

and this causes the gas to vibrate around an equilibrium state given by a sound speed a_0 and zero gas velocity. By making velocity and time non-dimensional with a_0 and $\pi/\tilde{\omega}$ respectively, we have the sound speed fluctuating about the value 1 and the period of the oscillation fixed for all driving frequencies and equal to 2.

When we change the driving frequency, however, the unit of length changes and the pipe has a variable length in the new coordinates, with the passive end located at

$$x = \frac{\tilde{\omega} L}{\pi a_0} = \frac{\omega}{\pi} , \quad (2.1)$$

where

$$\omega \equiv \frac{\tilde{\omega} L}{a_0} \quad (2.2)$$

is a non-dimensional measure of the forcing frequency. The motion of the piston, however, has a constant frequency, and is given by

$$x = -\frac{\delta}{\pi} \cos \pi t . \quad (2.3)$$

The parameter δ

$$\delta = \frac{\tilde{\omega} \ell}{a_0} \equiv \omega \frac{\ell}{L} \quad (2.4)$$

is now a good indicator of the strength of the forcing terms, and is in fact a Mach number for the motion of the piston and for the motion of the gas near the piston.

One might assume then that δ also measures the strength of the gas oscillation everywhere in the pipe, so that, if δ is small the linearized acoustic equations would be applicable. If one tries to do that, one gets consistent results for all values of ω except for those pipe lengths which are near resonance with the piston frequency. At these values of ω the motion of the gas is much larger, in general, than δ and it becomes necessary to include higher order nonlinear terms in the equations of motion. It should be noted, therefore, that the correct expansion parameter should be the typical Mach number of the gas, which we take to be ϵ , not the velocity parameter of the piston δ , and that one of the aims of the theory should be to find the relation between δ and ϵ .

We first write the general equations of motion, considering the gas to be ideal and isentropic, and the motion to be one-dimensional. The velocity of the gas is u and the sound speed $1+a$; x and t are Eulerian coordinates and α and β the corresponding characteristic coordinate system. Under these circumstances the equations and boundary conditions are (Cou-

rant & Friedrichs 1948)

$$\frac{\partial x}{\partial \alpha} = (u-a-1) \frac{\partial t}{\partial \alpha} , \quad (2.5a)$$

$$\frac{\partial x}{\partial \beta} = (u+a+1) \frac{\partial t}{\partial \beta} , \quad (2.5b)$$

$$\frac{2a}{\gamma-1} - u = 2f(\beta) , \quad (2.6a)$$

$$\frac{2a}{\gamma-1} + u = 2g(\alpha) . \quad (2.6b)$$

$$\text{On} \quad \alpha = \beta , \quad x = 0 \quad \text{and} \quad t = \alpha. \quad (2.7)$$

$$\text{On} \quad x = -\frac{\delta}{\pi} \cos \pi t, \quad u = g(\alpha) - f(\beta) = \delta \sin \pi t. \quad (2.8)$$

Equations (2.5) define the geometrical coordinates, x and t , in the characteristic plane, and (2.7) makes the definition unique by choosing α and β to be the time t at $x=0$. In the pipe α is constant along the C^+ characteristics and β along the C^- characteristics. Equations (2.6) define the invariants riding each family of characteristics. In particular $g(\alpha)$ can be considered as a "simple" wave going to the right, and $f(\beta)$ as the left-going reflected wave. Equation (2.8) is the boundary condition at the piston.

The only remaining equation is the boundary condition at the passive end of the pipe, and, following the discussion in the introduction, we assume it to be:

$$\text{On} \quad x = \frac{\omega}{\pi} , \quad f(\beta) = b g(\alpha) , \quad (2.9)$$

where b is a number between $+1$ and -1 , which gives the fraction of the right-going wave that is reflected back into the tube. The condition that b be bounded between $+1$ and -1 obviously means that no energy is created at the passive end, or, more specifically, that any radiated energy is transmitted from the pipe to the room, and not vice versa. It is easily seen that the value $b = 1$ corresponds to zero velocity at the end section and is equivalent to a perfectly closed pipe, while $b = -1$ implies $a = 0$ and represents an ideally open end, at which the pressure is always equal to room pressure.

Equations (2.5) to (2.9), plus the periodicity condition which says that we are looking for a steady oscillation of the same period as the piston, completely define the problem.

If we assume now that u and a are $O(\delta)$ and $\delta \ll 1$, we can neglect the second order terms on (2.5) and apply the piston condition at $x = 0$. The result is the classical linear theory. The characteristics are parallel straight lines and the solution for $f(\beta)$ is sinusoidal with amplitude

$$f_{\max} = \delta b [1 + b^2 - 2b \cos 2\omega]^{-\frac{1}{2}}. \quad (2.10)$$

When b is close to ± 1 this amplitude has sharp resonance peaks in ω . There the oscillation is much larger than $O(\delta)$, and the nonlinear effects may be expected to be important. For b far from these values, however, the linear theory is correct for δ small, as only broad resonance peaks occur and amplitudes

remain of order δ .

A particularly interesting case arises for $b=0$, when the amplitude of the oscillation is completely independent of frequency. This is, of course, because no wave is reflected from the pipe end and the oscillation is just the simple wave produced by the piston, travelling undisturbed to infinity.

Therefore, the only regions in the $b-\omega$ plane where linear theory should not be expected to hold are very small regions whose extent logically depends on the size of ϵ . Their location can be derived from the linear theory and falls into two families. Almost-closed pipes, where b is near $+1$, have resonance peaks at $\omega \sim \pi, 2\pi, 3\pi, \dots$, and almost-open pipes, with b near -1 , resonate at $\omega \sim \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$.

An asymptotic theory trying to explore these regions for small ϵ should include, then, expansions for b and ω as well as for the other quantities. We develop such a theory in the following sections, dealing primarily with the first resonance peaks for both the open and the closed cases.

3. Perturbation scheme.

3.1 Basic expansions.

The first question to be solved is the relation between ϵ , the Mach number of the gas, and δ , the piston motion. We have seen that assuming both to be of the same order leads to a first order solution for the velocity that is inadequate near resonance. This suggests that the nonlinear behaviour of the wave should be made to balance the forcing term, and as this nonlinearity can be expressed as a power series in ϵ , it is logical to expect that δ can be equated to some integer power of ϵ :

$$\delta = \epsilon^N . \quad (3.1)$$

The value of N has to be assumed at the beginning of the perturbation procedure, and the test of the assumption is the consistency of the resulting analysis. If the assumed value of N is too low, we will get essentially the linear theory ($N=1$), and if too large the solution will be identically zero to the first order, contrary to the assumption that ϵ is the order of the oscillation.

It seems to be important to work the problem directly in characteristic coordinates, as working with approximate characteristics in the $x-t$ plane gives rise to secular terms in the solution. The problem seems to be the same as with weakly nonlinear oscillators, where the period depends on the amplitude, and

the correct perturbation procedure is by Poincare's method. The use of characteristic coordinates here corresponds to the expansion of the independent variables used in that case.

If α and β are, then, considered as the independent variables, the appropriate expansions for u and a are

$$\begin{aligned} \frac{a}{\gamma-1} - \frac{u}{2} &= f(\beta) = \epsilon f_1(\beta) + \epsilon^2 f_2(\beta) + \dots, \\ \frac{a}{\gamma-1} + \frac{u}{2} &= g(\alpha) = \epsilon g_1(\alpha) + \epsilon^2 g_2(\alpha) + \dots. \end{aligned} \quad (3.2)$$

The geometrical coordinates x and t must be expanded too in powers of ϵ . We take

$$\begin{aligned} x &= x_0(\alpha, \beta) + \epsilon x_1 + \epsilon^2 x_2 + \dots, \\ t &= t_0(\alpha, \beta) + \epsilon t_1 + \epsilon^2 t_2 + \dots. \end{aligned} \quad (3.3)$$

Using these expansions in the equations for the characteristics (2.5) and separating orders, we get for all n

$$\begin{aligned} \frac{\partial}{\partial \alpha} (x_n + t_n) &= - \sum_{i=1}^n \left\{ \frac{\gamma+1}{2} f_i(\beta) + \frac{\gamma-3}{2} g_i(\alpha) \right\} \frac{\partial}{\partial \alpha} t_{n-i} \\ \frac{\partial}{\partial \beta} (x_n - t_n) &= \sum_{i=1}^n \left\{ \frac{\gamma-3}{2} f_i(\beta) + \frac{\gamma+1}{2} g_i(\alpha) \right\} \frac{\partial}{\partial \beta} t_{n-i} \end{aligned} \quad (3.4)$$

The corresponding boundary conditions (2.7) are expanded to:

$$\begin{aligned} \text{At } \alpha = \beta ; x_0 &= 0 \text{ and } t_0 = \alpha, \\ x_n &= t_n = 0 ; n \geq 1. \end{aligned} \quad (3.5)$$

The system of (3.4)-(3.5) can be solved recursively in terms of the f's and g's. The first two orders in x and t are

$$x_0 = \frac{1}{2}(\beta - \alpha) , \quad t_0 = \frac{1}{2}(\beta + \alpha) , \quad (3.6)$$

which correspond to the linear characteristics, and

$$x_1 = \frac{\gamma+1}{8} (\beta - \alpha) [f_1(\beta) + g_1(\alpha)] + \frac{\gamma-3}{8} [\psi_1(\beta) + \phi_1(\beta) - \psi_1(\alpha) - \phi_1(\alpha)] \quad (3.7)$$

$$t_1 = \frac{\gamma+1}{8} (\beta - \alpha) [f_1(\beta) - g_1(\alpha)] + \frac{\gamma-3}{8} [\psi_1(\beta) - \phi_1(\beta) - \psi_1(\alpha) + \phi_1(\alpha)] ,$$

where

$$\begin{aligned} \psi_i(\alpha) &= \int_0^\alpha g_i(\xi) d\xi \\ \phi_i(\alpha) &= \int_0^\alpha f_i(\xi) d\xi . \end{aligned} \quad (3.8)$$

Similarly, x_2 and t_2 can be computed as functions of f_1 , g_1 , f_2 and g_2 , and x_n and t_n as functions of the f_i , g_i up to n only. This fact makes for an ordered expansion procedure in which higher order terms do not "feedback" to lower orders.

It should be noted that the system (3.4)-(3.5) together with the definitions of f_i and g_i in (3.2) contain all the equations of motion of the gas, and, in particular, all the nonlinearities of the problem. All that remains now is to apply the boundary conditions to find f and g .

The first boundary condition is the one at the piston (2.8). With the assumption made above on δ , we can write it, to $O(\epsilon^N)$ as:

$$\begin{aligned}
 g_i(\alpha) &= f_i(\alpha) ; i < N \\
 g_N(\alpha) &= f_N(\alpha) + \sin \pi \alpha .
 \end{aligned}
 \tag{3.9}$$

This condition halves the number of unknown functions by eliminating the g_i . Physically, the wave in the pipe does not "see" the piston up to order N , and if $N > 1$, the solution is essentially a free standing wave, with its shape determined by the piston only through the higher order terms. As $N=1$ corresponds to the classical linear theory, near resonance we must actually have the situation $N > 1$.

Next it is necessary to introduce the condition (2.9) at the passive end; this condition includes the parameters ω and b . From the discussion in the last section we know that these parameters should be expanded around their values at resonance,

$$\begin{aligned}
 \omega &= \omega_o [1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots] , \\
 b &= b_o [1 - \epsilon b_1 - \epsilon^2 b_2 + \dots] ,
 \end{aligned}
 \tag{3.10}$$

where the two first resonance peaks correspond to

$$\begin{aligned}
 \text{Open end: } & b_o = -1 ; \omega_o = \pi/2 , \\
 \text{Closed end: } & b_o = +1 ; \omega_o = \pi .
 \end{aligned}
 \tag{3.11}$$

The position $x = \omega/\pi$ of the passive end will now be mapped into a certain line in the α - β plane

$$\beta = \tilde{\beta}(\alpha) = \beta_o(\alpha) + \epsilon \beta_1(\alpha) + \epsilon^2 \beta_2(\alpha) . \tag{3.12}$$

To compute it expand

$$\mathbf{x} = \frac{\omega}{\pi} = \frac{\omega_0}{\pi} (1 + \epsilon \omega_1 + \epsilon^2 \omega_2) = \mathbf{x}_0(\tilde{\beta}, \alpha) + \epsilon \mathbf{x}_1(\tilde{\beta}, \alpha) + \dots \quad (3.13)$$

Then substitute (3.12) in the right hand side of (3.13), expand the $\mathbf{x}_i(\tilde{\beta}, \alpha)$ in Taylor series, and equate like orders to solve for $\tilde{\beta}$:

$$\begin{aligned} \beta_0(\alpha) &= \alpha + \frac{2\omega_0}{\pi} \\ \beta_1(\alpha) &= \frac{2\omega_0}{\pi} \omega_1 - 2x_1(\alpha, \beta_0) \\ \beta_2(\alpha) &= \frac{2\omega_0}{\pi} \omega_2 - 2x_2(\alpha, \beta_0) - 2\beta_1(\alpha) \frac{\partial x_1}{\partial \beta}(\alpha, \beta_0) . \end{aligned} \quad (3.14)$$

Equations (3.14) can be expressed in terms of the f_i . Once again, the solution is ordered in the sense that $\beta_n(\alpha)$ does not contain terms from orders higher than n . Finally, we use all these expansions in the reflection condition (2.9), which can now be written as:

$$b [f(\alpha) + \epsilon^N \sin \pi \alpha] = f(\tilde{\beta}) .$$

Expanding the right hand side in Taylor series and separating orders:

$$O(\epsilon): \quad f_1(\alpha) - b_0 f_1(\beta_0) = 0 , \quad (3.15a)$$

$$O(\epsilon^2): \quad f_2(\alpha) - b_0 f_2(\beta_0) = b_1 f_1(\alpha) + b_0 \beta_1(\alpha) f_1'(\beta_0) - \delta_{N2} \sin \pi \alpha , \quad (3.15b)$$

$$O(\epsilon^3): \quad f_3(\alpha) - b_0 f_3(\beta_0) = \text{funct}\{ f_1, f_2 \} - \delta_{N3} \sin \pi \alpha , \quad (3.15c)$$

where δ_{NM} denotes the Kronecker delta.

This hierarchy by itself does not allow the calculation of the f_i , unless we impose some conditions on the solution, which, in fact, correspond to the initial conditions necessary in the hyperbolic problem. As we are looking for steady oscillations, we impose the condition that the solution must be periodic with the same period as the piston. But, since α and β correspond to real time at $x = 0$, periodicity in time means directly periodicity in α and β . So, the desired condition is that the $f_i(\alpha)$ be periodic with period 2, i. e.

$$f_i(\alpha) = f_i(\alpha+2) \quad \text{for all } i. \quad (3.16)$$

These conditions enable us to eliminate the left hand sides from (3.15) and get finally a set of equations for f . The details of the elimination vary slightly from the open to the closed end case, so that the two cases must be considered separately.

3.2 Closed-end case.

Doing first the closed end case, we start by assuming that $N=2$ ($N=1$ would give the classical linear theory). From (3.11) and (3.14), $\beta_0 = \alpha+2$, and the left hand sides of (3.15) are of the type

$$f_1(\alpha) - f_1(\alpha+2) ,$$

and, because of periodicity, they are all identically zero. The desired set of equations is then given by the right hand sides of (3.15) equated to zero.

The equation for $O(\epsilon)$, (3.15a), is satisfied identically and gives no information, but the second equation $O(\epsilon^2)$, gives an equation for f_1

$$2 \left[\hat{\omega}_1 - \frac{\gamma+1}{2} f_1 \right] f_1' + b_1 f_1 = \sin \pi \alpha, \quad (3.17)$$

$$\hat{\omega}_1 = \omega_1 - \frac{\gamma-3}{2} \langle f_1 \rangle ,$$

with

$$\langle f_1 \rangle \equiv \frac{1}{2} \int_0^2 f_1(\xi) d\xi \quad (3.18)$$

representing the mean value of f_1 over one period. For $b_1=0$, the completely closed end, (3.17) reduces to the equation obtained by Chester (1964). We delay the analysis of (3.17) until the next section.

3.3 Open end case.

For an open end $b_0 = -1$ and $\beta_0 = \alpha + 1$, so that the left hand sides of (3.15) are of the type:

$$f_i(\alpha) + f_i(\alpha + 1) \quad (3.19)$$

and do not vanish in general. However, if the f_i have period 2, the expressions in (3.19) have period 1, and that imposes restrictions on the right hand sides, which give the desired equations.

In particular,

$$[\text{R. H. S.}]_i(\alpha) = [\text{R. H. S.}]_i(\alpha + 1) \quad (3.20)$$

is an equation involving only the f up to order $i-1$. From the first order in (3.15) we get

$$f_1(\alpha + 1) = -f_1(\alpha) \quad (3.21)$$

so that the waveform changes sign as we advance a semiperiod. This is important because it means that any shock wave in the solution implies an expansion shock half a period away. It should be remembered that the original equations were isentropic, so expansion discontinuities are not really inconsistent in lower order approximations. On the other hand, we will see later that f_1 cannot be calculated until we include third order effects, so that the isentropic assumption breaks down with the presence of shocks of $O(\epsilon)$. We will come back later to this problem.

Assume now $N = 2$, and consider $O(\epsilon^2)$ in (3.15). Forming

the corresponding equation (3.20) we get, after some algebra,

$$b_1 f_1 + \omega_1 f_1' = \sin \pi \alpha, \quad (3.22)$$

which is linear and has no bounded solution for the case $b_1 = \omega_1 = 0$, corresponding to resonance. That means that the assumption $N=2$ was wrong and that we should go to higher order. For $N > 2$, (3.22) appears with right hand side zero. Therefore, b_1 and ω_1 have to vanish in the resonance band, as the homogeneous part of (3.22) has no non-trivial periodic solution.

The physical interpretation of this result is that the resonance band in this case is, at most, $O(\epsilon^2)$ and the piston motion $O(\epsilon^3)$. Outside this region we recover again the linear theory.

Assuming then

$$N = 3, \quad b_1 = \omega_1 = 0, \quad (3.23)$$

and repeating the process for (3.15) up to third order, we get, after a great deal of algebra, an equation for f_1

$$\begin{aligned} \left[\hat{\omega}_2 + \frac{\gamma+1}{16} (3\gamma+7) f_1^2 \right] f_1' + b_2 f_1 &= \sin \pi \alpha, \\ \hat{\omega}_2 = \omega_2 + \frac{\gamma-3}{16} (13-3\gamma) \langle f_1^2 \rangle, \end{aligned} \quad (3.24)$$

where

$$\langle f_1^2 \rangle \equiv \frac{1}{2} \int_0^2 f_1^2 d\xi = \int_0^1 f_1^2 d\xi. \quad (3.25)$$

It is, in fact, easy to prove from (3.15) that $\langle f_1^2 \rangle$ is proportional to $\langle f_2 \rangle$, and, so, to $\langle f \rangle$, as the mean value of f_1 over one

period is zero because of (3.21).

The correction to the frequency given by (3.24) in the open-end case is then of the same type as the one given in (3.17) for the closed end, and both can be interpreted as a shift in the linear resonant frequency due to the difference between the real mean pressure and the pressure defined a priori as mean.

It is interesting, in fact, to examine the validity of the separation of the pressure (sound speed) waveform into a mean value and a perturbation, particularly as this separation is usually not clearcut in nonlinear problems. In this case, however, a clear definition of $a=0$ is introduced by the boundary condition at $x=L$, as this condition is linear. In fact the reflection condition is equivalent to making $a \sim u$, except for b exactly equal to 1. With that exception, then, $a=0$ corresponds to the state at the pipe exit when $u=0$, and that can, in principle, be related to ambient conditions.

The special case $b=1$ corresponds to the completely closed pipe and the gas in the tube, having no connection with the atmosphere, does not have any clearly defined reference pressure. This is reflected in the equation (3.17) where $\langle f_1 \rangle$ can be determined for all cases except $b_1=0$. Integrating (3.17) over one period, we get

$$2\hat{\omega}_1 \langle f_1' \rangle - (\gamma+1) \langle f_1 f_1' \rangle + b_1 \langle f_1 \rangle = \langle \sin \pi \alpha \rangle .$$

The first two terms are perfect differentials and vanish because

of periodicity as does the right hand side, so we get

$$b_1 \langle f_1 \rangle = 0.$$

If $b_1 \neq 0$, f_1 has to have zero mean. If $b_1 = 0$, $\langle f_1 \rangle$ is not fixed and we know from the previous discussion that we can define it arbitrarily. To preserve continuity of the solution with b_1 we define as

$$\langle f_1 \rangle = 0 \tag{3.26}$$

for all b_1 , and use this to simplify (3.17).

4. Analysis of the closed-end case.

We now come to the problem of actually solving the equations (3.17) and (3.24) to find the response of the system near the two resonances. The first difficulty is that both equations are singular for some values of the parameters, and the effect of these singularities must be studied before we attempt a numerical treatment of the equations.

Consider first (3.17), representing the closed ended case. Following the discussion at the end of the last section, it can be simplified to

$$[2\omega_1 - (\gamma+1)f_1] f_1' + b_1 f_1 = \sin \pi\alpha , \quad (4.1)$$

with the boundary conditions

$$f_1(\alpha+2) = f_1(\alpha) , \quad (4.2)$$

$$\langle f_1 \rangle \equiv \frac{1}{2} \int_0^2 f_1(\xi) d\xi = 0 . \quad (4.3)$$

For all $b_1 \neq 0$, (4.3) is redundant and follows directly from periodicity. In numerical calculations, moreover, (4.2) proves to be much easier to use, as it reduces the order of the problem by one. When $b_1=0$, however, (4.2) is automatically satisfied and (4.3) must be used. In this case, though, the equation can be integrated exactly, and there is no need for numerical work.

It may be well at this point to remember the significance of the parameters b_1 and ω_1 . To do this we rewrite the expansions for ω and b

$$\begin{aligned}\omega &\approx \pi(1 + \epsilon\omega_1), \\ b &\approx 1 - \epsilon b_1,\end{aligned}\tag{4.4}$$

and note that ω_1 represents the distance in frequency from exact resonance, and b_1 indicates the deviation of the end condition from the perfectly reflecting closed end, or, in other words, the amount of wave radiated to the exterior. It follows that negative values of b_1 have no physical significance, and that the range of parameters to be studied is the upper-half plane in b_1 - ω_1 space.

The origin of this plane represents the perfectly closed pipe at resonance, and so, as we move away from it, we should approach the results of the linear theory. In fact, if in (4.1) we let b_1 or ω_1 grow large, the nonlinear term can be neglected and we get asymptotically the linear result

$$\begin{aligned}f_1(\alpha) &= -(4\pi^2\omega_1^2 + b_1^2)^{-\frac{1}{2}} \cos(\pi\alpha + \chi) + O(4\pi^2\omega_1^2 + b_1^2)^{-1}, \\ \chi &= \tan^{-1} \frac{b_1}{2\pi\omega_1}.\end{aligned}\tag{4.5}$$

Another useful property of the system (4.1)-(4.3) is that it is invariant to the transformation

$$\begin{aligned}f_1(\alpha) &\rightarrow -f_1(-\alpha), \\ \omega_1 &\rightarrow -\omega_1, \\ b_1 &\rightarrow b_1,\end{aligned}\tag{4.6}$$

and so, it is possible to study the solution for $\omega_1 \geq 0$ and extend

it to all frequencies by using (4.6). In what follows we always assume that ω_1 is positive.

We now solve (4.1) for the special case $b_1=0$, and, as stated before, the solution should reduce to the results given by Chester for the completely closed end. The equation can be integrated directly to

$$2\omega_1 f_1 - \frac{\gamma+1}{2} f_1^2 + \frac{1}{\pi} \cos \pi \alpha = \text{const.} \quad , \quad (4.7)$$

or

$$f_1 = \frac{2\omega_1}{\gamma+1} - \left[\frac{2}{\pi(\gamma+1)} \right]^{\frac{1}{2}} (K + \cos \pi \alpha)^{\frac{1}{2}} \quad (4.8)$$

where K is an integration constant to be determined with the help of (4.3). In attempting to do this, however, we get a transcendental equation in K involving elliptic functions which has a real root only if

$$|\omega_1| \geq \frac{2}{\pi} \left(\frac{\gamma+1}{\pi} \right)^{\frac{1}{2}} . \quad (4.9)$$

For all other values of ω_1 , then, there is no continuous solution satisfying (4.1) and (4.5). Chester (1964) interprets this fact as an indication of the appearance of shock waves in the flow, and this is confirmed by experiment.

In fact, in deriving (4.1), we only used the equations of motion up to $O(\epsilon^2)$, and to this order, a shock wave of amplitude $O(\epsilon)$ produces no change in entropy and can be treated simply as a discontinuity in the solution (Courant & Friedrichs 1948). The

speed of propagation of this discontinuity is just the arithmetic mean of the wave velocities immediately in front of and behind it. This property, and the fact that a shock travelling along one set of characteristics does not modify waves travelling along the other set, contains the complete shock relations to the order needed.

By using periodicity, the jump conditions across the shock discontinuity can now be determined.

Denote by + and - superscripts the conditions in front and behind the discontinuity and consider first a shock, as AB, travelling to the right (see fig. 2). To first order, this shock is always located at a given value of α , say α_s , and its speed is

$$V_s = 1 + \epsilon V_1 = 1 + \frac{1}{2}\epsilon[(u_1 + a_1)^+ + (u_1 + a_1)^-], \quad (4.10)$$

or, expressed in terms of f_1 ,

$$V_1 = \frac{\gamma+1}{4} [f_1^+(\alpha_s) + f_1^-(\alpha_s)] + \frac{\gamma-3}{2} f_1(\beta). \quad (4.11)$$

Therefore, along the shock

$$dx_s = (1 + \epsilon V_1) dt_s = dt_s + \epsilon \frac{V_1}{2} d\beta. \quad (4.12)$$

Integrating this equation from A to B, and using the known values of $x_s(A)$ and $x_s(B)$, we get

$$t_s(B) - t_s(A) = 1 + \epsilon \left[\hat{\omega}_1 - \frac{\gamma+1}{2} \frac{f_1^+ + f_1^-}{2} \right]. \quad (4.13)$$

Note that, here, we can not use the condition that $\langle f_1 \rangle = 0$, which was derived using the continuity of the solution. Repeating the process for the left going shock, BA' , and combining both results, we get

$$t_s(A') - t_s(A) = 2 + 2\epsilon \left[\hat{\omega}_1 - \frac{\gamma+1}{2} \frac{f_1^+ + f_1^-}{2} \right]. \quad (4.14)$$

But periodicity imposes that this difference be exactly 2, so the shock relation reduces to:

$$\frac{f_1^+ + f_1^-}{2} = \frac{2\hat{\omega}_1}{\gamma+1}. \quad (4.15)$$

This is, however, still not enough to completely determine the solution, as $\langle f_1 \rangle$ is left unknown. We can get this information by integrating (3.17), in much the same way as with the continuous case, from just in front of the shock to just behind the shock one period ahead. Then

$$\left[\hat{\omega}_1 - \frac{\gamma+1}{2} \frac{f_1^+ + f_1^-}{2} \right] [f_1^- - f_1^+] + b_1 \langle f_1 \rangle = 0, \quad (4.16)$$

and using (4.15) we have

$$\langle f_1 \rangle = 0, \quad \hat{\omega}_1 = \omega_1, \quad (4.17)$$

for all cases except $b_1 = 0$, where (4.17) can be fixed by definition as before.

We may, then, simplify the shock condition with (4.17), and use both to construct discontinuous solutions by piecing to-

gether segments of continuous solutions of (4.1) with jumps satisfying (4.15). Detailed examples of waveforms constructed in this way are given by Chester (1964) for $b_1 = 0$, so we turn our attention to the more general case where $b_1 \neq 0$.

We can expect that in those cases, too, it will be necessary to introduce shocks for some range of parameters and, so, we must study first of all the question of existence of continuous solutions; this obviously depends on the behaviour of the singularities of (4.1). We study these singularities next.

They occur in the $f_1 - \alpha$ plane, when the coefficient of f_1 vanishes, or

$$f_1 = \frac{2\omega_1}{\gamma+1} \quad (4.18)$$

and the solution $f_1(\alpha)$ is only affected by them when it has to pass through that value. For large values of ω_1 , that can only happen for very large amplitude waves. However, the asymptotic solution (4.5) suggests that the amplitude really decreases as ω_1 increases, so condition (4.18) is never realized for large ω_1 and continuous solutions should be expected for that range. Once a continuous solution has been shown to exist, it is easy to convince oneself that it is unique.

As ω_1 approaches zero, though, the wave amplitude increases and (4.18) decreases, so that, at some sufficiently small frequency the solution will touch the singularity at some point. We need to consider, then the behaviour of (4.1) near those points,

and, to do this, we expand the equation for small displacements around (4.18). In particular, let

$$f_1 = \frac{2\omega_1}{\gamma+1} + g$$

$$\alpha = \Phi + z$$
(4.19)

where Φ is some general value of α in the neighborhood of which we want to study the solution. Substituting in (4.1), we have

$$(\gamma+1)gg' - b_1g = \left(\frac{2}{\gamma+1} b_1\omega_1 - \sin\pi\Phi\right) - (\pi \cos\pi\Phi)x + \dots$$
(4.20)

There are two possible cases, depending on the Φ chosen. In most instances the constant on the right hand side of (4.20) is not zero, and the behaviour of the solution is then given by

$$(\gamma+1)gg' \approx \left(\frac{2}{\gamma+1} b_1\omega_1 - \sin\pi\Phi\right),$$

$$g \approx \frac{2}{\gamma+1} \left[\left(b_1\omega_1 - \frac{\gamma+1}{2} \sin\pi\Phi\right)x \right]^{\frac{1}{2}}.$$
(4.21)

The exact shape of g depends on the sign of the constant in brackets, but it always includes a branch point at $x = 0$, giving two-valued solutions which are inadmissible from a physical point of view (see table I).

If, however $b_1\omega_1 \leq \frac{\gamma+1}{2}$, there are two points in every cycle in which the constant term in (4.19) vanishes. Name these point as

$$\Phi^+ = \sin^{-1} \frac{2b_1\omega_1}{\gamma+1},$$
(4.22a)





$\sin \varphi \neq \frac{2b_1 \omega_1}{(\gamma+1)}$	$\omega_1 < \frac{\gamma+1}{2b_1} \left\{ 1 - \frac{b_1^4}{16\pi^2(\gamma+1)^2} \right\}^{\frac{1}{2}}$	 <p>I:</p>
	$\omega_1 > \frac{\gamma+1}{2b_1} \left\{ 1 - \frac{b_1^4}{16\pi^2(\gamma+1)^2} \right\}^{\frac{1}{2}}$	 <p>II:</p>
$b_1 \omega_1 \leq \frac{\gamma+1}{2}$	$\varphi^+ = \sin^{-1} \frac{2b_1 \omega_1}{(\gamma+1)}$	 <p>III:</p>
	$\varphi^+ = \pi - \varphi^-$	 <p>IV:</p>

TABLE I

$$\Phi^- = \pi - \Phi^+ \quad . \quad (4.22b)$$

The leading terms of (4.20) near those points are

$$(\gamma+1)gg' - b_1 g + (\pi \cos \pi \Phi) x = 0 \quad . \quad (4.23)$$

This is now a bilinear equation, which can be solved exactly (Birkhoff & Rota 1969). The type of the solution depends on the behaviour of a particular quadratic algebraic equation involving the coefficients of (4.23). As the value of Φ in (4.23) can be expressed through (4.22) in terms of b_1 and ω_1 , the nature of the singular points can be classified in terms of these parameters alone.

A summary of the most important results of this classification is given in table I. From this table it is clear that if a continuous solution is to cross the singular line (4.18) anywhere, it has to do it through one point of type III or IV. In fact, if it crosses the line at all, it has to cross it twice, once at Φ^+ and another at Φ^- , and because of the shape of type III, it has to cross at Φ^+ going up, and come back at Φ^- going down. If now, maintaining the same b_1 , we start decreasing ω_1 , the point at Φ^+ starts to "roll-up" from type III to a spiral point of type II. At the moment that this happens, and Φ^+ becomes of type II, the solution is no longer able to cross the singular line through that point, although it is still able to get back through Φ^- . At this moment an incipient shock develops at Φ^+ and it grows bigger and bigger as the spiral rolls tighter with ω_1 approaching zero.

Thus, the nature of the singular points divides the $b_1-\omega_1$ plane into regions, and in each region the possible types of solution are different. This classification is presented in figures 3 and 4. These, and all subsequent figures are drawn for air ($\gamma = 1.4$), although a simple change in scale will adapt them for other values of γ . The waveforms given in figure 4, are those of the possible singular solutions in each region. In each case it is possible, in principle to have, besides, a continuous solution which never crosses the singular line, like that in region α . The existence of this solution can be best decided by trying to integrate the equations numerically to find it.

The process used by us was a second order "shooting" scheme (Keller 1968), starting from large ω_1 for a fixed b_1 with the asymptotic solution (4.5) and working inwards keeping b_1 constant and decreasing ω_1 , until the solution touched the singular line. The points in the $b_1-\omega_1$ plane where the solution first touched this line are given by OO' in figure 3. To the right of that line the solution is continuous and nonsingular. To the left, the solution has to cross the singular line and the best way to find it numerically is to start integrating from Φ^+ with the slope given by the analysis of the singular points and integrate forward and backwards until the solution crosses the singular line again. A shock can then be fitted, if needed, using (4.15).

The calculations show that, in the regions α and β , it is always possible to find a continuous solution for f_1 , although the

derivative may be discontinuous at $\bar{\phi}^-$ in some cases. The region of the $b_1-\omega_1$ plane where the solutions contains shocks is then only the part OCA of γ lying to the left of OO'.

All this discussion belongs, of course, only to the case of $\omega_1 > 0$. For negative ω_1 the results are completely symmetric, according to the transformation (4.6).

Finally, figure 5 gives some examples of waveforms computed for three different values of b_1 and several ω_1 , using the method outlined above. Figure 6 is a plot of wave amplitudes as a function of ω_1 for various values of b_1 . The four quantities represented in each plot are respectively the maximum and minimum values of f_1 , and the values at the top and foot of the shock.

It is to be noted that, although a shock is present in the solution for relatively large values of b_1 , its strength becomes very small for much lower values of the radiation coefficient, to the point of being practically negligible for the larger b_1 's.

5. Results for the open end case.

We now turn our attention to the pipe with the open end. The pertinent equation is (3.24)

$$\left[\hat{\omega}_2 + \frac{(\gamma+1)(3\gamma+7)}{16} f_1^2 \right] f_1' + b_2 f_1 = \sin \pi \alpha , \quad (5.1)$$

$$\hat{\omega}_2 = \omega_2 + \frac{3-\gamma}{16} (13-3\gamma) \langle f_1^2 \rangle ,$$

and is to be integrated in the interval $(0, 1)$ subject to the condition

$$f_1(1) = -f_1(0) . \quad (5.2)$$

The results can be extended to the full period $(0, 2)$ by using

$$f_1(\alpha+1) = -f_1(\alpha) . \quad (5.3)$$

The significance of the parameters b_2 and ω_2 is similar to the corresponding ones in the previous section, and, here too, b_2 is essentially a non-negative number.

The whole problem is mathematically very similar to the closed end case, and most of the analysis carries through directly to (5.1). The singularities in this case are located at

$$f_1 = \pm \left\{ - \frac{\hat{\omega}_2}{(\gamma+1)(3\gamma+7)} \right\}^{\frac{1}{2}} , \quad (5.4)$$

so they form two singular lines, instead of one. From (5.4), too, these lines are only real when $\hat{\omega}_2 \leq 0$. For positive ω_2 , we can expect no trouble with singularities and the solutions are

continuous and well behaved; this is of course also true for $\hat{\omega}_2$ sufficiently negative.

For intermediate values of the frequency, however, the solution crosses the singular lines and exhibits the same kind of phenomena as the closed end solution. The analysis of the singular points runs exactly parallel to the one there, and, in fact, the approximate equation near (5.4) is also bilinear in this case, the only difference being that the roles of the points Φ^+ and Φ^- are interchanged. The fact that there are two singular lines instead of one does not affect the results much, because it turns out that each line interacts with only one half period of the waveform, as could be suspected from (5.3). In particular, the interval (0,1) is only involved with the plus sign in (5.4).

There are, however, several important differences between this and the closed end case. The first one is the existence of a shift between the effective frequency $\hat{\omega}_2$, and the physical quantity ω_2 . This shift vanished in the closed pipe, as we were able to show that $\langle f_1 \rangle$ was always zero. In (5.1), however, the shift depends on $\langle f_1^2 \rangle$ which is a strictly positive number.

The significance of this shift was discussed in section 3, and its effect in the system is to tilt the resonance peak toward lower frequencies, this effect being more pronounced as b_2 becomes smaller and the wave amplitudes grow larger.

Numerically, of course, all the work is done first using

$\hat{\omega}_2$ as a parameter, and after the solution is found, ω_2 is computed using (5.1). The results presented in this section are for $\gamma=1.4$, but, due to (5.1), the change to other gases no longer corresponds to a simple change in scale.

The division of the $b_2-\hat{\omega}_2$ plane according to the type of singularities is plotted in figure 7, where the name of the regions correspond roughly to the descriptions given in section 4. The solution passes through the singular lines in the region between the OO' line and the b_2 axis. The only region qualitatively different from the closed case is the one to the right of the ordinate axis, where no singularities exist and the solution is always continuous. The shock region is given by the area of region γ to the right of OO' and, in figure 8, it is plotted in "physical" $b_2-\omega_2$ coordinates.

A very important difference with section 4 occurs, however, in the behaviour of the solution within the shock region. The main reason for it lies in equation (5.3), for this equation assures that anything that happens in one semiperiod, will happen with opposite sign half a period later. In particular, any compression shock in the wave must be followed by an expansion shock of the same strength, which is physically quite unlikely.

Even more important is the fact that (5.1) really derives from the equations of motion used up to $O(\epsilon^3)$, and assumed isentropic. Now, the entropy production of a shock of amplitude

of $O(\epsilon)$ first appears in the equations at $O(\epsilon^3)$, so that the assumption of isentropic flow is inconsistent with the existence of shocks. This is reflected mathematically in the impossibility of finding any condition for a discontinuity in the solution to represent a shock, equivalent to (4.15) for the closed case. In fact, as we try to repeat the process in section 4, to get this kind of condition, we run into the difficulty that any effect produced by the shock is cancelled by the opposite expansion shock somewhere during the period.

The influence of entropy production on the solution has another effect on the attempt to formulate a physical model including this influence. The flow in the pipe is supposed to be periodic, so that the entropy produced by one passage of the shock at one point has to be removed somehow before the next passage. The way this entropy is removed is, of course, by cooling the fluid through the walls of the tube, and the modelling of this cooling depends on the exact experimental set up, and introduces new parameters in the problem.

To avoid these complications, and in view of the fact that the reflection condition is probably not very good for shocks at an open end, we decided to abandon the attempt to compute discontinuous waveforms in this case. The boundary of shock formation, plotted in figure 8, should remain valid, however, as it is essentially a negative result establishing the impossibility of continuous solutions.

Some representative waveforms computed for three different values of b_2 and several frequencies are given in figure 9. For the first two values of b_2 the solution cuts across the shock region so that only results for the frequencies on either side of the boundary are shown; in those cases, the tendency for shocks to form can already be seen quite clearly. The third value of b_2 is above the shock boundary and so the waveforms can be computed for all frequencies and is always continuous. In figure 10, the half amplitude of the wave is plotted versus frequency with b_2 as a parameter. The gaps in the curves correspond to regions with shocks.

The most important result in connection with the open end case is, however, connected with orders of magnitude. Going back to the definition of δ , as measuring the amplitude of the piston motion and ϵ as measuring the strength of the gas motion, we see that in the closed end $\epsilon = \delta^{\frac{1}{2}}$, while in the open end $\epsilon = \delta^{\frac{1}{3}}$, and the oscillation is in fact stronger when the end is open.

This result, which may seem somewhat surprising at first sight, is however easily explained. The effect of a closed end on an incoming wave is to reflect it with the same sign in pressure. Thus, a compression wave is reflected as a compression wave, and never changes sign. Consider now a small pressure step produced, say, at the piston in a closed pipe. If the step was initially a compression, it remains a compression forever, and the steepening of the wave, that is, the interaction of the wave

with itself, acts continually and becomes eventually important. This nonlinear effect appears in the equation at $O(\epsilon^2)$ and helps "kill" the linear resonance.

For an open end the sign of the wave in the pipe changes every time it is reflected at the open section, and so any particular signal is a compression half of the time and an expansion the other half. So, the steepening by interaction of the wave with itself never accumulates, and it is only after third-order interactions come into play that the linear resonance can be limited.

It is important to realize on the other hand that, from the point of view of the energy, an ideal open end is as closed as a rigid wall. In fact the energy flow out of the end section is given by

$$\dot{E} = \oint p dV \quad (5.5)$$

where V is the volume of the gas originally in the tube. For the closed pipe $dV=0$ at all time and the energy flow vanishes. But for the open pipe the pressure is constant at the exit, so that

$$\dot{E} = p \oint dV = p \Delta V = 0 \quad , \quad (5.6)$$

by periodicity, and the energy flow vanishes too.

Therefore, the only remaining factor to decide the strength of the wave is the order of the nonlinearity, and a weaker effect, like the one in the open end will not limit the resonance peak

until the higher amplitudes necessary to make the nonlinearity important are attained.

In the real world, of course, open ends do radiate a lot of energy, which means that the perfect open end, $b_2=0$, is probably a limiting case with no physical reality.

6. Conclusions

The breakdown of linear acoustic theory at resonance in both closed and open pipes can be remedied by appeal to higher order non-linear effects. A consistent perturbation analysis of the non-linear equations is presented for the case of oscillations produced by the sinusoidal motion of a piston in one end of the pipe. At the other end of the pipe, it is supposed that the wave profile reaching that end is reflected with a factor b . This reflection coefficient ranges from $b = 1$ for the completely closed end to $b = -1$ for the "ideal" open end used in acoustics. Resonance occurs for b in the neighborhood of $b = 1$ and $b = -1$. Particularly for the near open end, this boundary condition is obviously a severe simplification of a complicated situation. In fact, the reflection characteristics may depend on the frequency, the shape of the particular wave profile and so on. The attitude here is not, however, to insist that the reflection is independent of these influences, but rather to learn about the "equivalent b " by comparison of the results with experiment. As reported by Sturtevant (1973), it was possible to correlate theory and experiment in this way, and indeed the theoretical results were invaluable in developing a correct rational interpretation of the experiments for a variety of input conditions and end conditions. This information on the effective reflection coefficient, its dependence on frequency etc. should be valuable in other situations.

For the open and near-open pipes the amplitude of the oscillations in the pipe are $O(\delta^{\frac{1}{3}})$, where δ is the piston amplitude. This is in marked contrast to the result $O(\delta^{\frac{1}{2}})$ obtained by earlier investigators for the closed end case. In the closed or nearly closed cases, $b \approx 1$, the result stems from a balance between non-linear steepening and forcing by the piston. For the open end cases $b \approx -1$, however, the second order distortion effect alternates in sign for the successive runs up and down the tube, and the forcing can only be balanced by third order terms in the gas amplitude. The resulting amplitude proportional to $\delta^{\frac{1}{3}}$ is then higher than in the closed end case. This is confirmed by experiment for appropriate ranges of the parameters. The detailed comparison of this and other predictions with the experimental observations is given by Sturtevant (1973).

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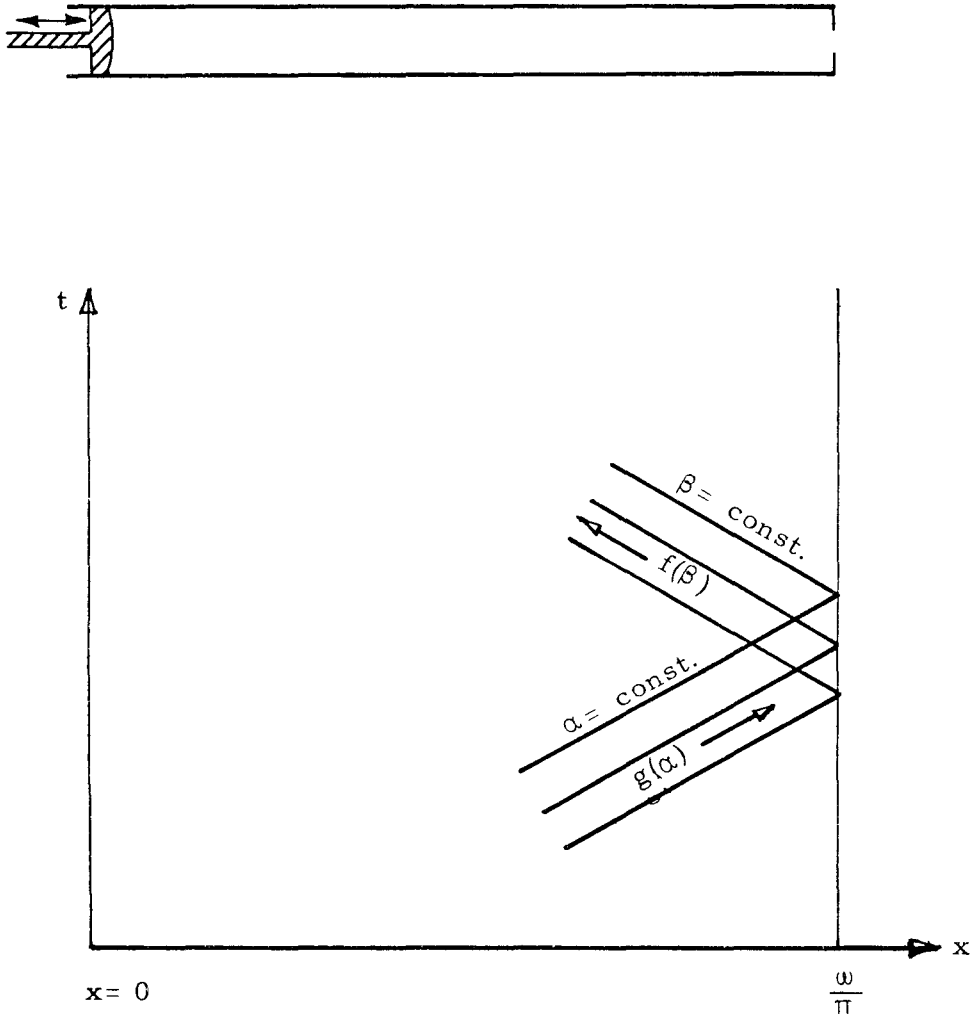


Figure 1. Pipe configuration and non-dimensional coordinates.

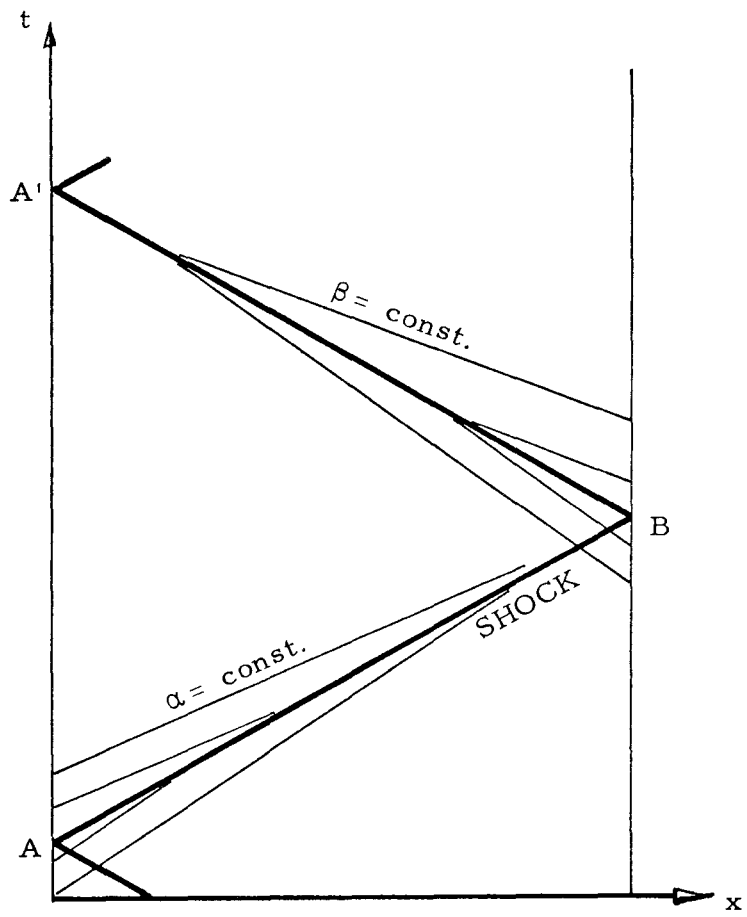


Figure 2. Shock geometry in the x - t plane.

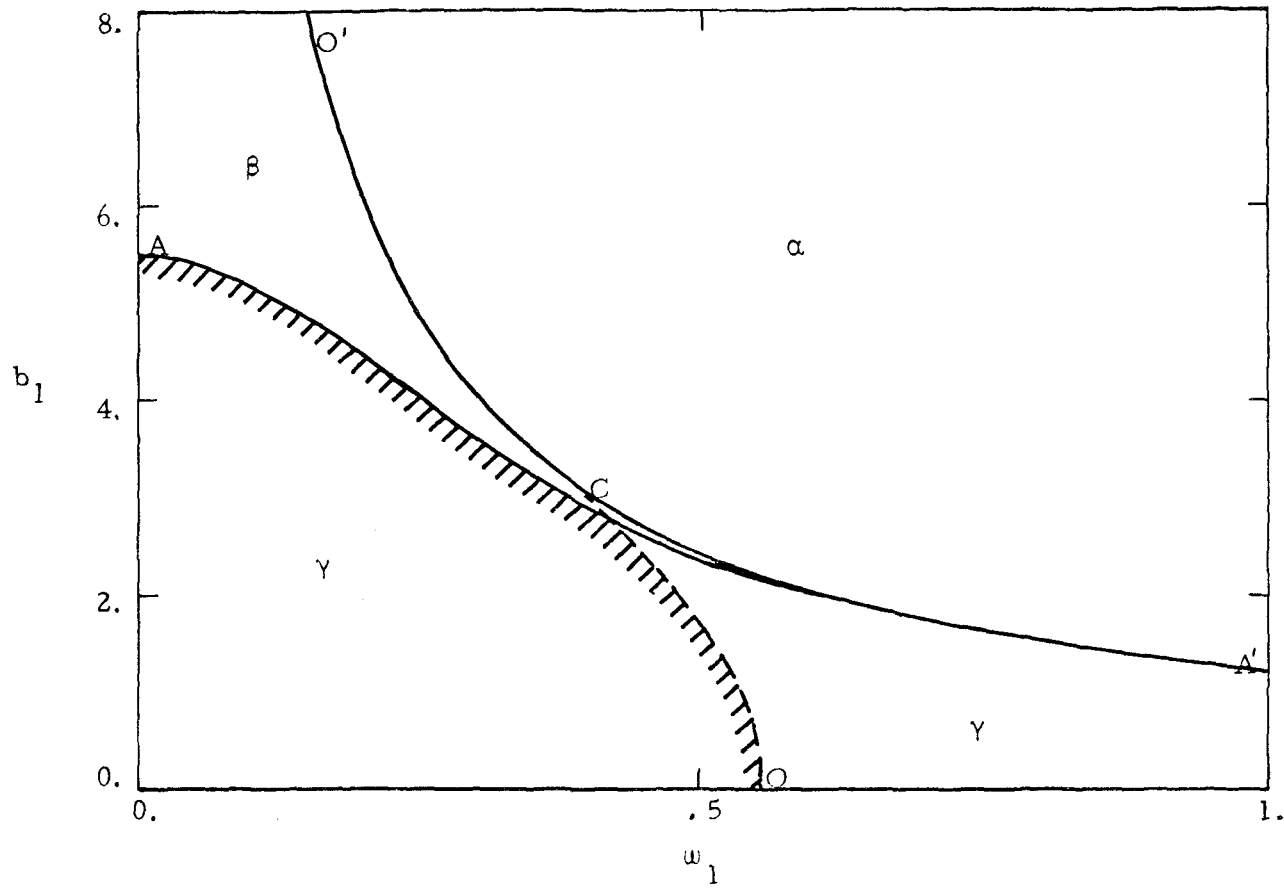


Figure 3 Division of the b_1 - ω_1 plane with regard to the nature of the singular points in the equation for the closed end pipe.

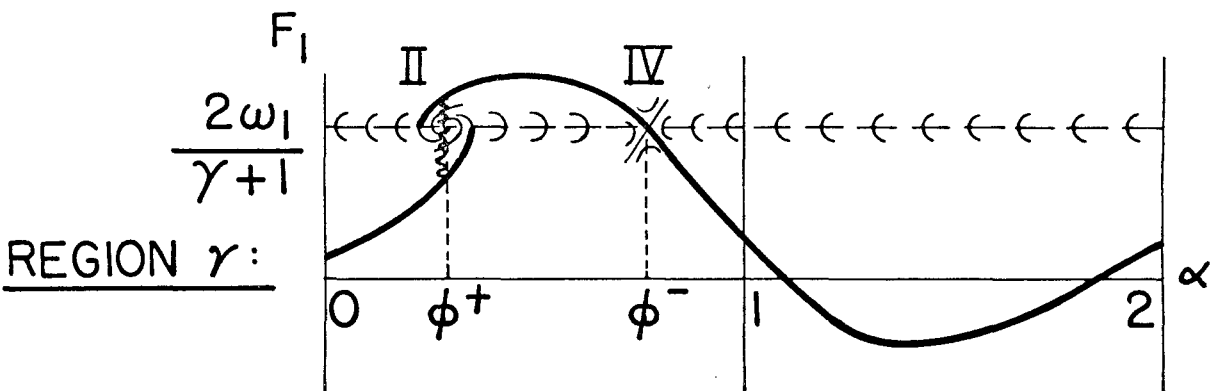
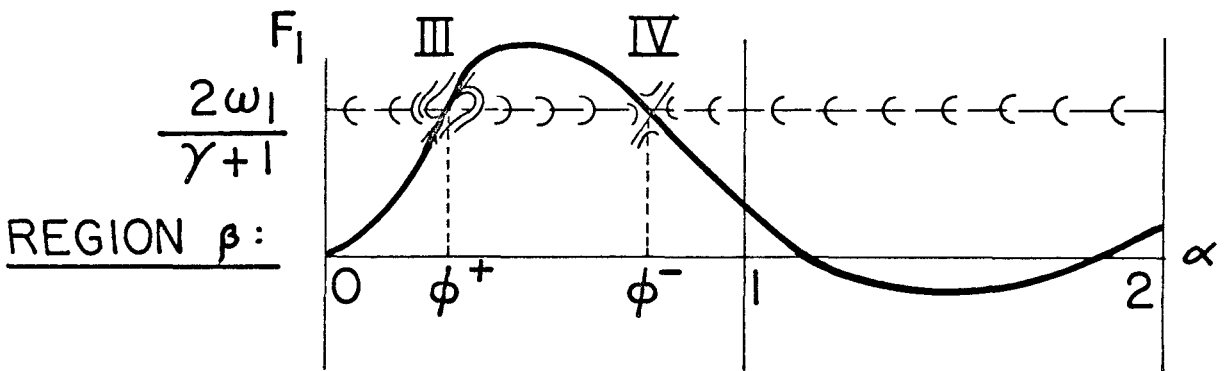
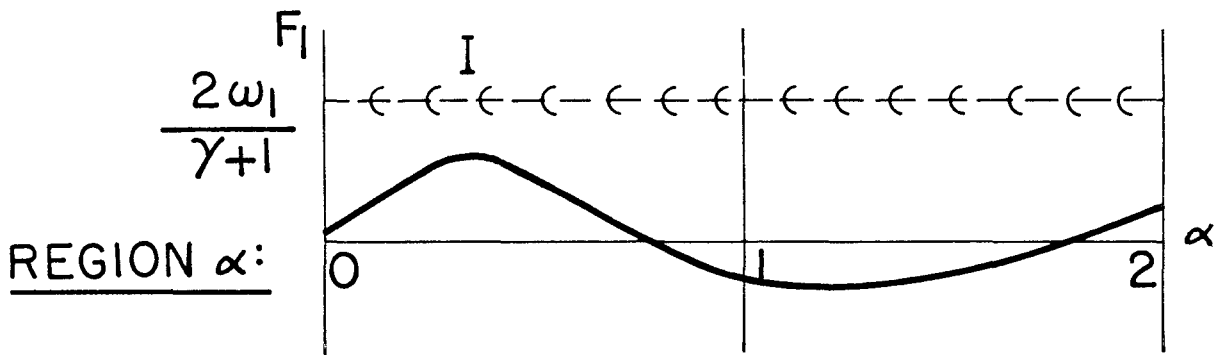


Figure 4. Character of the solution of equation (4.1) depending on the type of the singular points.

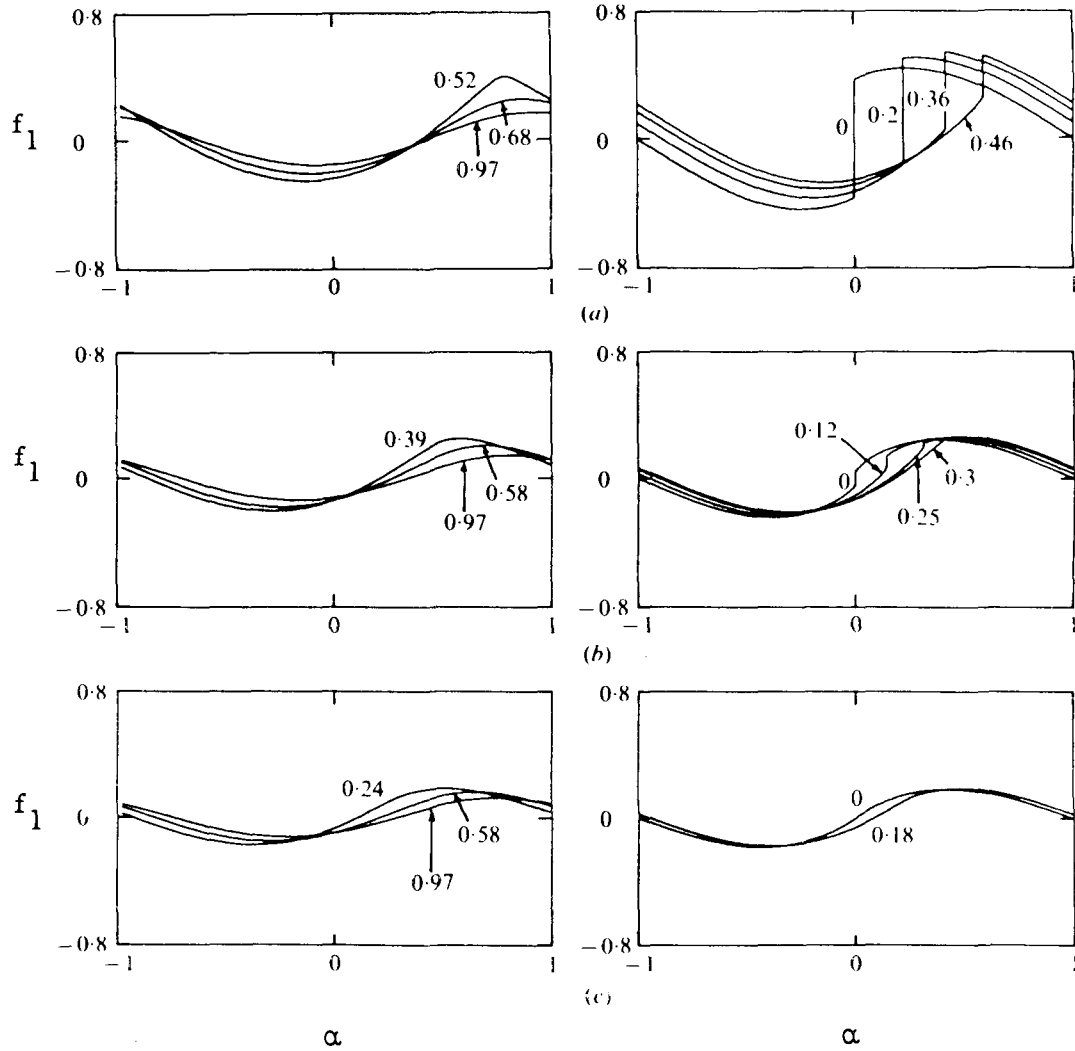


Figure 5. Typical waveforms computed for the closed-pipe case. (a) $b_1=1.55$, (b) $b_1=3.87$, (c) $b_1=5.42$.

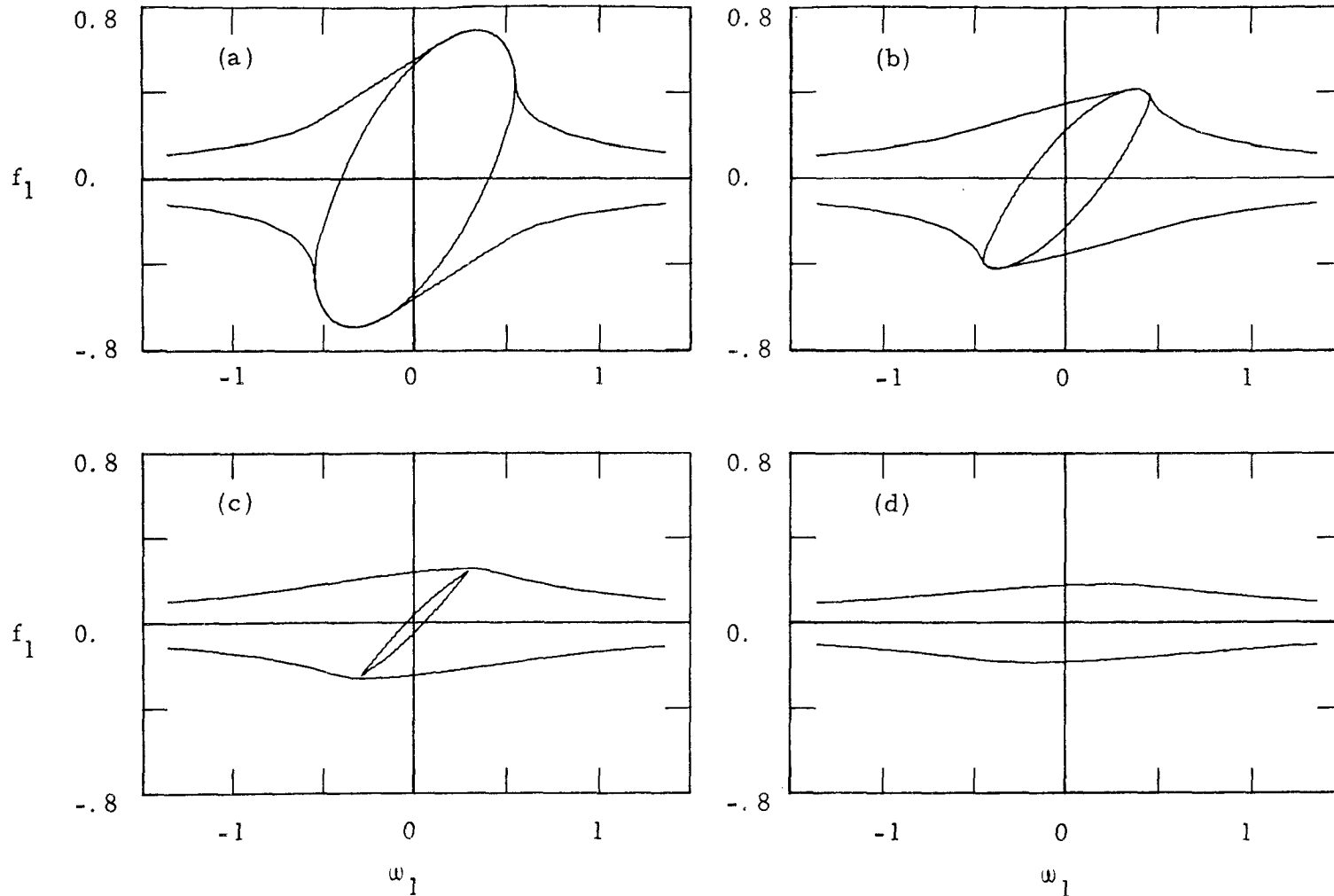


Figure 6. Wave amplitude for the closed pipe computed for several values of b_1 as a function of ω_1 . (a) $b_1 = 0.77$, (b) $b_1 = 2.32$, (c) $b_1 = 3.87$, (d) $b_1 = 5.42$.

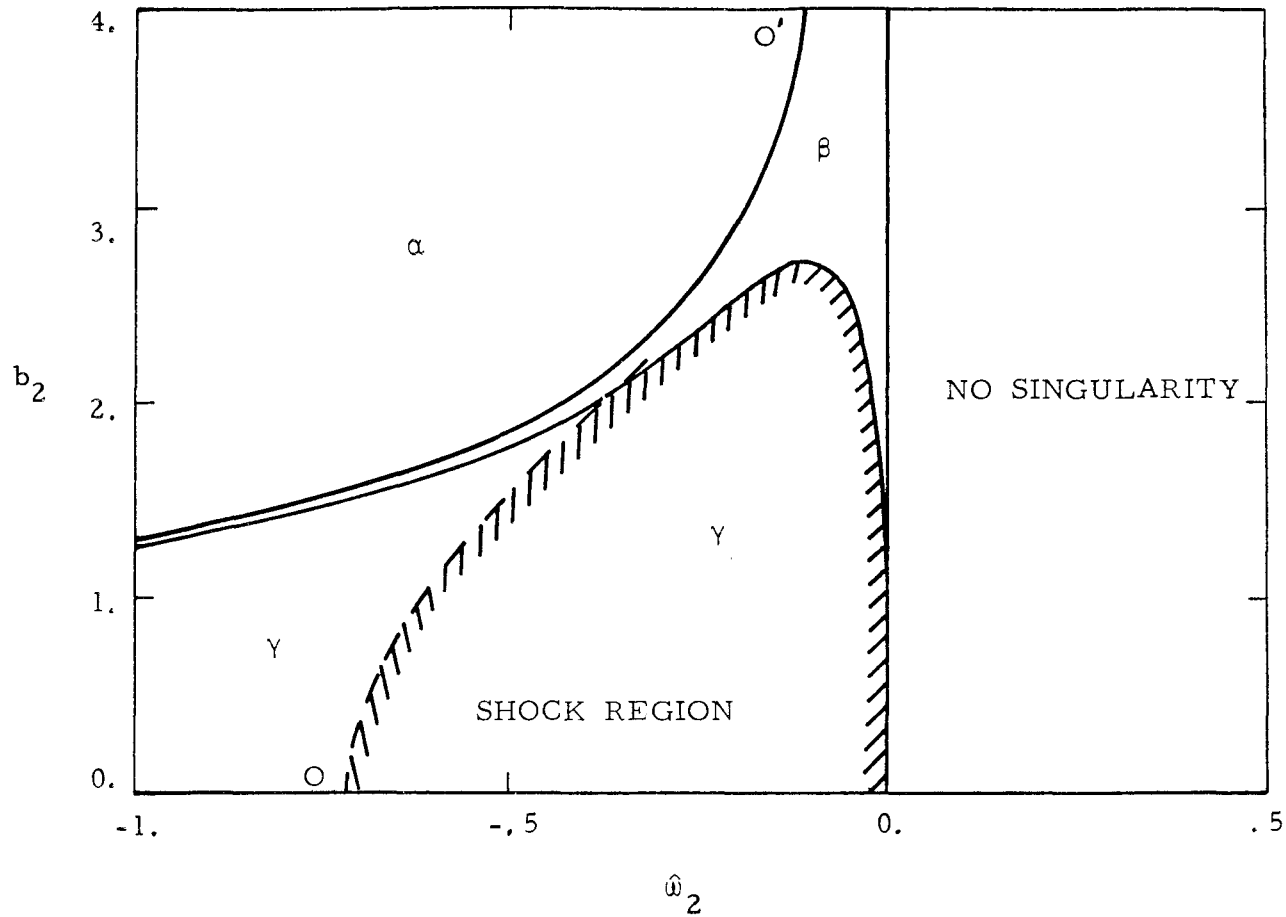


Figure 7. Division of the b_2 - $\hat{\omega}_2$ plane with regard to the nature of the singular points in equation (5.1).

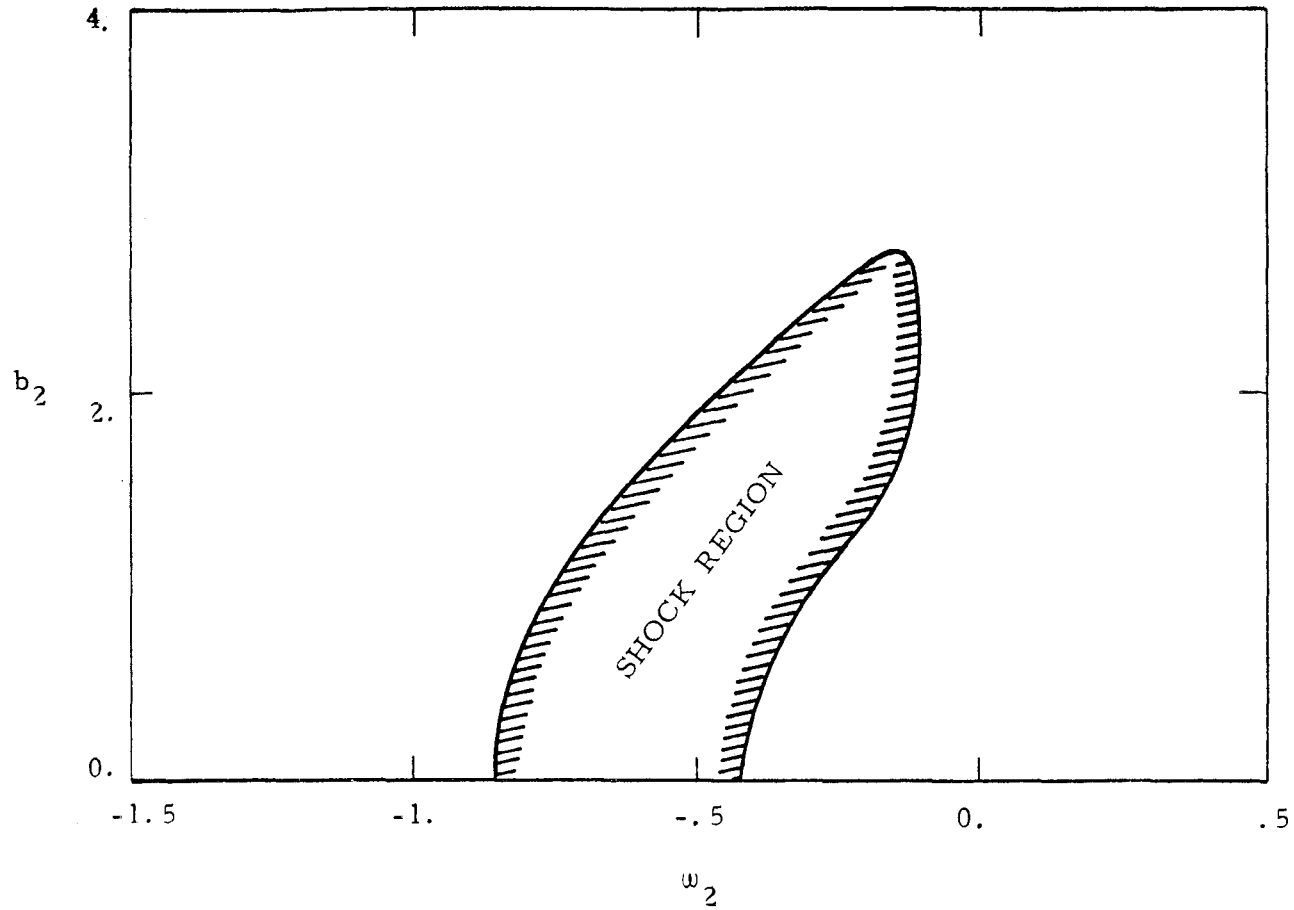


Figure 8. Extent of the region containing shocks in the solution of equation (5.1), in the b_2 - ω_2 plane.

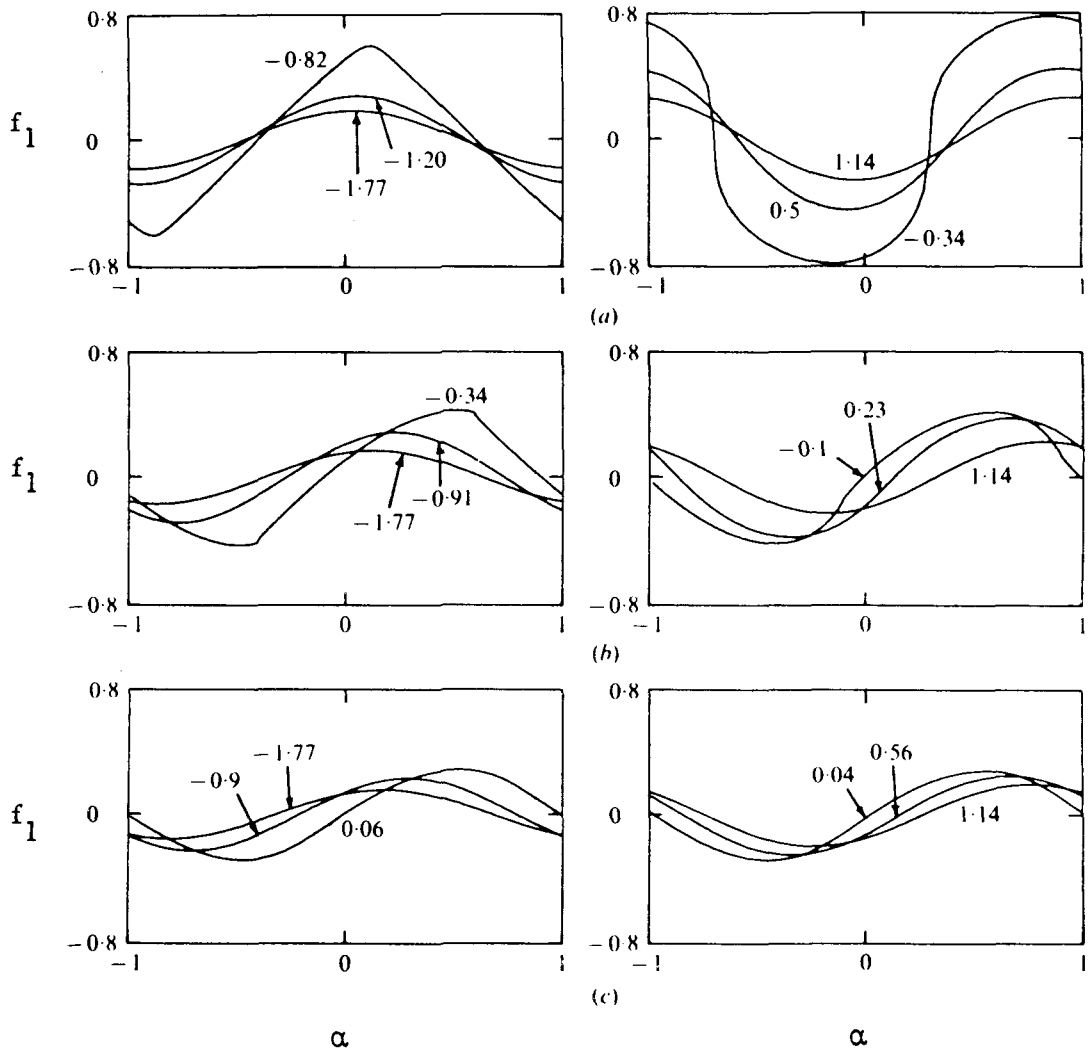


Figure 9. Typical waveforms computed for the open-pipe case. (a) $b_2=0.59$, (b) $b_2=2.34$, (c) $b_2=3.51$.

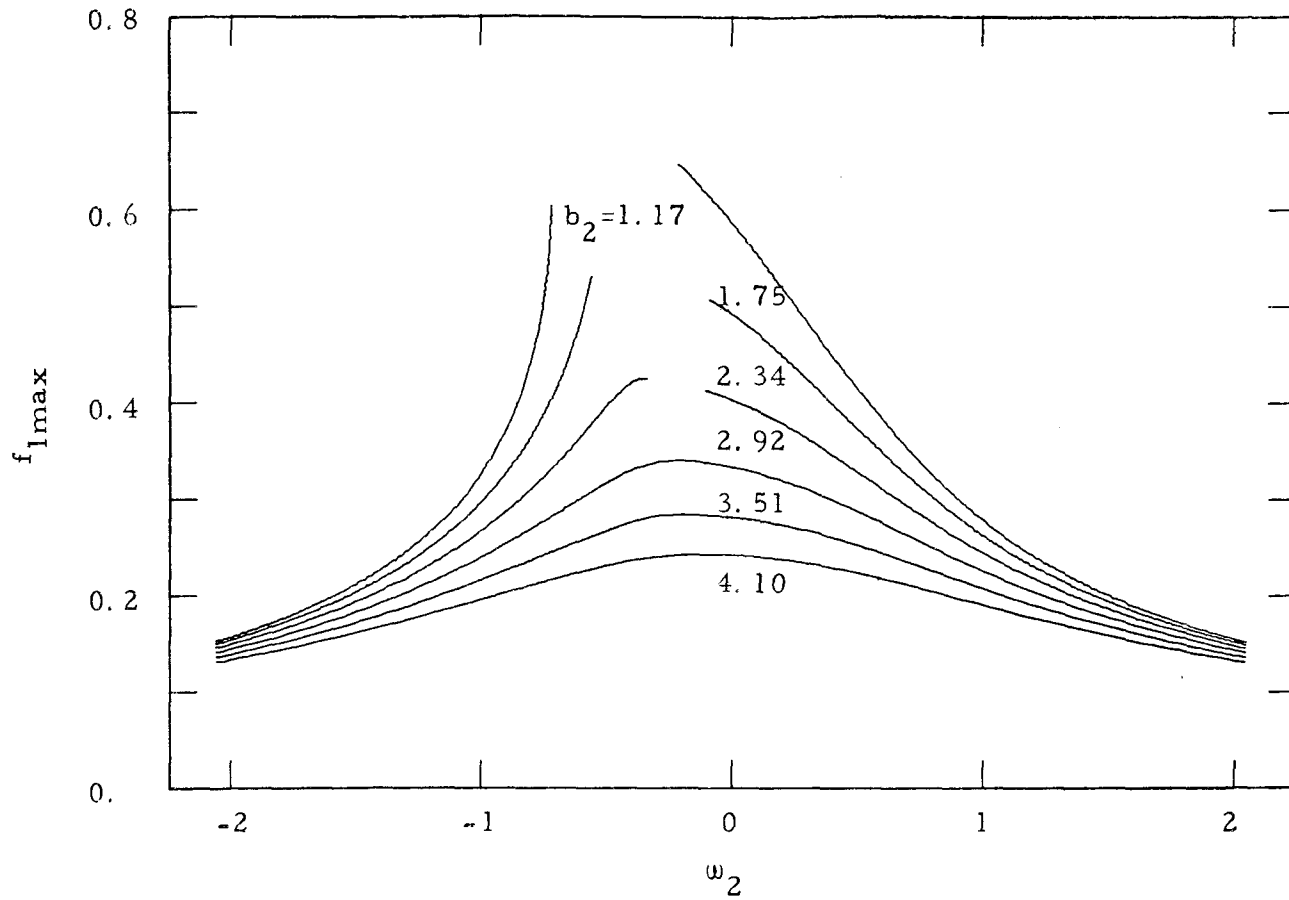


Figure 10. Wave amplitude for the open pipe computed for several values of the radiation coefficient as a function of w_2 .

II. WAVETRAINS WITH SMALL DISSIPATION.

1. Introduction.

The concept of a stationary wavetrain is a very useful one in mathematical physics. For the simple case of a linear partial differential equation, wavetrains have the form

$$u = a e^{i\theta} \quad ; \quad \theta = kx - \omega t \quad , \quad (1.1)$$

where ω and k are related by a dispersion relation

$$\omega = \omega(k) \quad (1.2)$$

which can be derived from the original equation. More general solutions can, then, be constructed by Fourier superposition of wavetrains with different values of k .

The best known example of approximate wavetrains comes from the theory of linear dispersive waves (Brillouin 1960). In these cases the dispersion relation (1.2) is assumed to be real with $\omega''(k) \neq 0$. It can, then, be shown that for appropriate (localized) initial conditions the solution tends, after a long time, to a form

$$u = a e^{i\theta} \quad (1.3)$$

with

$$\omega = -\theta_t \quad , \quad k = \theta_x \quad (1.4)$$

and the amplitude, a , no longer constant but depending on x and

t. This dependence, however, is slow, in the sense that the characteristic scales for the variation of ω and k are much larger than the period of the oscillation (1.3). In this sense, the solution can be considered locally as a wavetrain with parameters varying slowly over a much longer scale.

The rigorous way of showing this is to use Fourier analysis and study the asymptotic behaviour of the solution as $t \rightarrow \infty$. But a simpler way of studying the final asymptotic state is to use the WKB method in which we look specifically for solutions of the type (1.3) with slowly varying ω , k , and a . In this method we define slow variables

$$X = \epsilon x, \quad T = \epsilon t, \quad (1.5)$$

and look for solutions of the form

$$u = a(X, T) e^{i \frac{\theta(X, T)}{\epsilon}}, \quad (1.6)$$

where θ has been written as $\epsilon^{-1} \theta(X, T)$ and

$$\omega = -\theta_t = -\theta_T, \quad k = \theta_x = \theta_X \quad (1.7)$$

are functions of X and T . Substituting this form in the original equation and separating different orders in ϵ we are able to get the "modulation" equations satisfied by ω , k , and a .

In nonlinear PDE's we also find stationary wavetrains, although in this case we can not use them to generate other solutions by superposition. However, the concept of slowly varying wavetrains, not depending directly on the additivity of the

solutions, is still useful, and is introduced in much the same way as in the linear case, with the appropriate form for the wavetrain substituted instead of (1.6).

The methods used to study these problems are usually called two-timing methods, because a solution of the type (1.6) can be considered as depending in x and t in two different scales: a fast oscillation in $\theta = \epsilon^{-1}\Theta$ with period of $O(1)$, and a slow modulation in X and T , in which the relevant times are $O(\epsilon^{-1})$.

The existence of two widely separated scales presumes the existence of some small effect forcing the slow variation on the fundamental wavetrain. In the case of the dispersive waves the small effect is the dispersion, which acts slowly once the wavetrain is strongly dispersed after a long time and each point in space contains essentially a single wave number.

In this part of the thesis we study the case in which the slow variation is induced by a small dissipation term in the equation. The perturbation method we use is Whitham's averaged Lagrangian technique.

The averaging two-timing methods were introduced by Kuzmak (1959) for ordinary differential equations. The general idea is to integrate over a few periods of the wavetrain so as to "smooth out" the fast oscillations and recover the equations satisfied by the large scale variations of k , ω , and a .

Whitham (1965) generalized the method to dispersive

partial differential equations by averaging on the conservation equations of the system and, later (1970), developed a way of treating the problem by averaging directly on the Lagrangian of the system. This last version of the method offers a remarkably compact way of deriving the slow equations for the problem, and has the advantage of being as effective in nonlinear as in linear problems. One drawback is that it can not be used in its original form if the system under study does not derive from a known variational principle, as is the case with most dissipative systems.

In this work we remove this limitation. To do that we use a pseudo-variational principle introduced by Glansdorff and Prigogine (1954) for irreversible systems. This principle is discussed in §2.

In §3 we present the original averaged Lagrangian method and illustrate its generalization to irreversible systems with a few simple examples. The general method is presented and justified in §4.

2. Variational principles for irreversible systems.

2.1 Reversible and conservative systems.

The classical use of variational principles has been in mechanics, in processes which are normally characterized as reversible.

In fact, it has long been maintained that a Lagrangian formulation of the usual type does not exist for irreversible systems, although, while probably true, this has never been rigorously proved.

Part of the difficulty seems to be the lack of a proper definition for irreversibility. A reversible system is usually thought of as one for which the internal production of entropy is zero. Entropy, however, is not a quantity clearly defined for all cases. Thus, for instance, although most people would agree that a damped oscillator is irreversible, it is quite difficult to argue about its entropy production unless it is considered part of a much larger system including a heat sink, etc.

When the problem at hand is to decide if a given equation, which physically "looks" irreversible, is going to derive from a Lagrangian formulation or not, the entropy argument seems fairly hopeless.

The other usual definition of irreversibility, namely that the equation be invariant under the change of t into $-t$, while

very useful in mechanical systems, loses much of its utility when used in partial differential equations. In particular, to recover the original equation from a "physically reversible" PDE, we need sometimes to reverse not only the time, but the space coordinates, or, in nonlinear cases, the sign of the dependent variable.

On the other hand, many irreversible systems can be reversed by changing the sign of "too many" variables, and, in any given case, the number of variables that we are allowed to change is very much determined by the physical model.

in fact, the feeling that the damped oscillator is not reversible comes mainly from the fact that it is not conservative. Conservative systems are usually defined as those which conserve energy. The concept can be made more appropriate to PDE's by requiring them to conserve all components of the energy tensor. For mechanical systems these include, of course, momentum.

Irreversible systems, at least mechanical ones, are usually not conservative. The converse is, however, not true, and there is a wide class of systems which do not conserve energy but are perfectly reversible. These are all those which interact with variable external fields or moving constraints.

The feeling that those systems are reversible stems from the fact that, in reversing the time, we are allowed also

to reverse the evolution of the external conditions, and the behaviour of the new system is qualitatively the same as before the change in time direction. What this means is that the non-conservation of energy does not worry us because we know where it comes from, and its source is simple enough for us to recognize explicitly in our calculations.

The problem of finding variational principles for these systems is generally not more complicated than for the equivalent systems with constant external conditions. The only difference is that we get Lagrangians depending explicitly on time.

There is a close relationship between Lagrangian representation and conservative systems. It is well known (see Gelfand & Fomin 1963) that any system possessing a time-independent Lagrangian satisfies an energy conservation law. Additional invariances in the Lagrangian induce in turn corresponding conserved quantities.

The point is that, if a system derives from a variational principle and does not conserve energy, the only possible reason is an explicit dependence on time within the Lagrangian. And this dependence is physically allowable only if it is explicit in the system itself. All these dependences are usually grouped under the generic name of external conditions.

The non-conservation of energy in a damped oscillator is of a different type. In this case, also, the energy lost by the system is lost because of interaction with the external medium,

but now this interaction is not so simple as before, and we can not expect to reverse it easily by manipulating the external world.

Thus, a pendulum running backwards will still eventually stop. A profound enough manipulation of its bearings might enable us to produce a negative friction coefficient, in which case the system might be considered reversible. But we know experimentally that this manipulation has to be drastic enough to produce what must be considered a different physical system.

Irreversibility then arises from a failure to analyze properly the external universe. For example, although it is possible to find a time dependent Lagrangian for the damped oscillator, this dependence derives from artificial transformations instead of from any intrinsic understanding of the interactions with the outside world, and we do not consider it physically relevant.

In fact, it is well known that any Sturm-Liouville system can be put into self-adjoint form by a simple change of variable. But this transformation involves the time in a way that has nothing to do with the forces acting on the system.

As noted above, it is impossible to find a time independent Lagrangian for a non-conservative system.

The problem is very much the same as the distinction between heat and work in thermodynamics, with work being defined as the effect of external fields, ordered and controllable,

and heat as the rest of the interactions with the exterior, which are not counted as fields, and are not completely described.

2.2 The beauty of conservation laws.

The easiest cases of conservation laws appear in mechanical systems depending on a single variable. In these cases, conservation of a quantity simply means that this quantity is an integral of motion. This is the result of integrating an equation of the type

$$\frac{dE}{dt} = 0. \quad (2.1)$$

In systems described by PDE's the form of a conservation law is somewhat more complicated. A typical example is

$$\frac{\partial e}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad (2.2)$$

where e is the density of the conserved quantity, and q is called its flux. Assuming appropriate boundary conditions, equation (2.2) can be integrated over all space to give

$$\frac{d}{dt} \int e \, dx = 0 \quad (2.3)$$

which defines again a global conserved quantity as in (2.1).

The great significance of these quantities is that they relate initial and final states of a system without any regard to the details of the motion in between. In this way it becomes possible to characterize some aspects of the entire evolution of the system by a single number, instead of a complete function of time, and conservation laws occupy a central position among

the laws of nature.

The slowly varying wavetrains that we study in this work can be considered as almost conservative systems. In fact, any periodic motion is conservative in some sense, as it repeats itself every cycle, and the idea of the two-timing method in mechanics is precisely to explore the slow modulation of the integrals of motion (see Cole 1968).

It is not surprising, then, that Whitham (1965) extended the method to PDE's by applying the two-timing to the conservation laws, and later (1970), using the relation between these and the Lagrangian formulation, was able to treat the whole slow-modulation problem directly from the variational principle.

The great simplification of using the two-timing on conservation laws can be seen from the following example. Assume that E is a quantity associated with some slowly changing oscillation, and so is periodic on some phase θ and modulated over some slow time T , defined in such a way that

$$\frac{dE}{dt} = \omega(T) \frac{\partial E}{\partial \theta} + \epsilon \frac{\partial E}{\partial T} . \quad (2.4)$$

Assume first that E is a conserved quantity, so that

$$\frac{dE}{dt} = 0 . \quad (2.5)$$

The two-timing of (2.5) is straightforward as, using (2.4) and integrating over one period in θ , we arrive at

$$\frac{d}{dT} \oint E d\theta = 0. \quad (2.6)$$

Assume, however, that E satisfies some other equation, not in conservation form, such as

$$\frac{dE}{dt} + E^2 = 0 \quad (2.7)$$

When we try to use the same averaging procedure on this equation as we used on (2.5) we find that, not only we get two different integrals mixed in the resulting equation

$$\epsilon \frac{d}{dT} \oint E d\theta + \oint E^2 d\theta = 0, \quad (2.8)$$

but this equation contains terms of several orders in ϵ , and we have to continue working on it before we get useful results.

In general, the "averaging" versions of two-timing, like the one used above, although they offer great simplicity of computation, can only be used on conservation equations.

The next step in simplification is Whitham's averaged Lagrangian method, in which the two-timing is carried directly on the variational principle. We will talk about it in section 3, but first we have to look into the problem of finding variational principles for a given system.

2.3 Variational principles for irreversible systems.

In section 2.1 we discussed which physical systems can be expected to be described by a variational principle. The mathematical problem of finding the Lagrangian given the equations of motion of the system is much more complicated. In fact, we do not even know the existence conditions for these principles.

It is well known (Vainberg 1964) that a necessary and sufficient condition for a system of equations to derive from a variational principle is that the linearized system be self-adjoint. And, given a system of this kind, it is easy to compute the corresponding Lagrangian.

The property of self-adjointness, though, depends critically on the exact way the equations are written, and in many cases a simple change of variables will convert an operator which is not self-adjoint in one which is.

There have been many attempts to characterize which systems of equations can be thrown into the right form by some suitable trick, and recently (Seliger & Whitham 1968, Seliger 1968, Myers 1972) some progress has been made in that direction.

A different problem is to find some kind of useful extremum principle for systems for which we do not expect to find a classical Lagrangian. These principles should have for these systems as many as possible of the useful properties that clas-

sical Lagrangians have for the conservative ones.

In particular we are interested here in a pseudo-Lagrangian that is useful for approximate calculations and gives us the equations of energy, momentum, etc.

Suppose that we want to derive a real, arbitrary set of equations

$$Mu = 0 \quad (2.9)$$

from an extremum principle. Perhaps the simplest way is to minimize the functional

$$J[u] = \int (Mu)^2 dt = (Mu, Mu) , \quad (2.10)$$

where $(.,.)$ is a suitable inner product. Obviously the solutions of (2.9) make J minimum, but the class of minimizing solutions for (2.10) is much wider. Assume, in fact, that M is linear and carry out the variation in u :

$$\delta J = \delta (Mu, Mu) = (M\delta u, Mu) + (Mu, M\delta u) = 2(M^* Mu, \delta u) , \quad (2.11)$$

where M^* is the adjoint operator for M . The Euler equation for (2.10) is then

$$M^* Mu = 0. \quad (2.12)$$

The interesting things about (2.12) is that the extra solutions besides those of (2.9) are associated with the operator M^* , and that, generally, the dissipative properties of M^* are opposite to the ones of M . So, for a damped oscillator

$$M = \frac{d^2}{dx^2} + 2\epsilon \frac{d}{dx} + \beta^2, \quad (2.13)$$

$$M^* = \frac{d^2}{dx^2} - 2\epsilon \frac{d}{dx} + \beta^2,$$

and M^* is a negatively damped system which gives unstable solutions.

The reason is that (2.12), coming from a Lagrangian, has to be conservative and the energy dissipated by M has to be transferred somewhere within the system. In this example it appears in M^* as a negative dissipation.

This is quite a general argument, and in all variational formulations of dissipative processes we can expect a "ghost" system where the dissipated energy is fed, having no physical reality whatsoever. The main problem with the form (2.10) is that the physical and conjugate systems are completely inter-mixed in the Euler equations and can not be separated effectively.

Some ways to circumvent this difficulty have been investigated, specially in the field of irreversible thermodynamics, and, in particular, Glansdorff & Prigogine (1954) developed a pseudo-Lagrangian formulation that suits our problem quite well.

They point out that practically any system, like (2.9), can be derived from a variational principle if the dependent variable is allowed in the Lagrangian in two forms, u and \bar{u} , and in executing the variation only one of them, u , is varied, while \bar{u} is considered fixed.

Only after the Euler equations are obtained in this way do we drop the distinction between the two varieties of u , and set $u = \bar{u}$.

For example, a trivial pseudo-Lagrangian for (2.9) might be

$$J[u, \bar{u}] = \int u M \bar{u} dt, \quad (2.14)$$

whose Euler equation after variation of u is

$$M \bar{u} = 0; \quad (2.15)$$

and letting now

$$u = \bar{u} \quad (2.16)$$

we recover (2.9).

It is easy to see where the conjugate divergent system is hidden in this method, for, if M is linear

$$J[u, \bar{u}] = (u, M \bar{u}) = (\bar{u}, M^* u), \quad (2.17)$$

and, upon variation of \bar{u} we recover the adjoint equation for (2.9)

$$M^* u = 0. \quad (2.18)$$

Although this method may look artificial at first sight, its main usefulness resides in the clear separation between the physical system (2.9) and the conjugate one (2.18). In fact, it can be argued that it represents a straight generalization of the classical variational principles.

Consider a classical Lagrangian

$$J[u] = \int L(u) dt = (L(u), 1) , \quad (2.19)$$

where $L(u)$ is not to be considered as just a function but as the result of an appropriate operator on u . For instance, most Lagrangians are functions of u and u_t at least.

The operation of varying u in (2.19) can be considered a differentiation in function space of the functional $J[u]$ (see Vainberg 1964, Ch. II). In fact

$$\delta J[u] = \delta(L, 1) = \left(\frac{DJ}{Du}, \delta u \right) = \left(\frac{\delta L}{\delta u}, \delta u \right) = 0 \quad (2.20)$$

where $\frac{DJ}{Du}$ represents the gradient of J with respect to u , and $\frac{\delta L}{\delta u}$ represents the variational derivative in the Euler equation for L under variation of u . Vainberg called operators that can be written as a gradient of a functional, potential operators, and they correspond to the ones that can be derived directly from a variational principle.

Not all operators are potential, and a natural step might be to explore forms of the type

$$\delta' J = (Mu, \delta u) = 0 \quad (2.21)$$

where M is not potential.

In fact, any equation $Mu=0$ can be derived directly from (2.21), and the problem of finding a Lagrangian for it is to reduce $\delta' J$ to a 'perfect differential' form like (2.20). This sounds a lot like the reduction of Pfaffian forms, and it would be interesting to look into this idea further.

An earlier connection between the problems of finding a Lagrangian and reducing a Pfaffian was suggested by Seliger & Whitham (1968).

In this sense, pseudo-Lagrangians like (2.17) can be seen as giving rise to forms of the type (2.21) that can not be reduced to exact potentials.

In choosing a pseudo-Lagrangian we have much more freedom than in the classical case. Besides being only determined up to the addition of any divergence term, it is clear that we can add to it any term depending only on \bar{u} , as it does not influence the Euler equations.

Moreover, we have the choice of retrieving the final operator in terms of u , \bar{u} or both. For instance, the two Lagrangians

$$uM\bar{u} \quad \text{and} \quad uMu - \bar{u}Mu$$

are perfectly equivalent representations of (2.9), but their Euler equations come out respectively as

$$M\bar{u}=0 \quad \text{and} \quad Mu + M^*u - M^*\bar{u} = 0,$$

which reduce to the same equation once we make $u=\bar{u}$. This is just a particular example of the equivalence between

$$L(u, \bar{u}) \quad \text{and} \quad L(u, u) - L(\bar{u}, u).$$

The question of which form to choose for a particular case depends on the application desired. In our case, in which the important thing to be emphasized is the proximity of the system to a conservative one, it is convenient to display this

in the Lagrangian. Assume, in fact, that our system is described by

$$M(u, \epsilon) = 0 \quad (2.22)$$

where $M(u, 0)$ is conservative, and derives from the regular Lagrangian $L(u)$, and $M(u, \epsilon) - M(u, 0)$ is $o(1)$. Then, the pseudo-Lagrangian for (2.22) can be written as

$$A(u, u) = L(u) + I(u, \bar{u}) , \quad (2.23)$$

where $I(u, \bar{u})$ is of $o(1)$ and represents the irreversible part of $M(u, \epsilon)$. From now on we will always use a form like (2.23).

Other forms have been developed for the original application of this method in irreversible thermodynamics and numerical calculations. An extensive review of these applications can be found in Donnelly (1966).

3. The averaged Lagrangian technique.

3.1 Reversible systems.

We will review briefly Whitham's technique of the averaged Lagrangian before we try to apply it to dissipative systems. A more extensive account can be found in Whitham's own papers (1970,1971).

The method was developed primarily to treat slowly changing dispersive wavetrains, and it is in partial differential equations where it finds its main use. However, all the essential features can be illustrated using ODE's, and this permits a considerable reduction in computation. The extension to several independent variables is straightforward and we will present later an example of that use.

Consider now a perturbed oscillator described by a Lagrangian $L(u, u_t, T)$, where $T = \epsilon t$ represents an explicit time dependence in L , but only in a time scale long with respect to the characteristic period of the system. The small parameter ϵ measures the ratio between fast and slow time scales.

For $\epsilon = 0$ the time dependence disappears, and the motion of the oscillator is periodic in time, with some characteristic amplitude and frequency. For $\epsilon \neq 0$ we can use a two-time representation for u

$$u = U(\theta, T; \epsilon), \quad (3.1)$$

where U is assumed to have period 2π in θ , and

$$\theta = \frac{\Theta(T)}{\epsilon}, \quad \theta_t = \Theta_T = \omega(T), \quad T = \epsilon t. \quad (3.2)$$

In this representation, the instantaneous angular frequency ω , and the amplitude implicit in U are considered functions only of T . The two scales of the motion are, then, θ and T .

The trick now is to consider U as a function of these two variables. Then,

$$\frac{d}{dt} = \omega \frac{\partial}{\partial \theta} + \epsilon \frac{\partial}{\partial T}, \quad (3.3)$$

and

$$u_t = \omega U_\theta + \epsilon U_T. \quad (3.4)$$

The Euler equation for the oscillator is

$$\frac{d}{dt} L_{u_t} - L_u = 0, \quad (3.5)$$

and using in it the transformation (3.3) we get

$$\frac{\partial}{\partial \theta} (\omega L_2) - L_1 + \epsilon \frac{\partial}{\partial T} L_2 = 0, \quad (3.6)$$

where L_1 and L_2 are derivatives of L with respect to its first and second arguments respectively. But the Lagrangian now has the form

$$L(U, \omega U_\theta + \epsilon U_T; \epsilon), \quad (3.7)$$

so that (3.6) can be written

$$\frac{\partial}{\partial \theta} (L_{U_\theta}) + \frac{\partial}{\partial T} (L_{U_T}) - L_U = 0, \quad (3.8)$$

which is just the Euler equation for the two-variable variational principle

$$\delta \int \int_0^{2\pi} L(U, \omega U_\theta + \epsilon U_T; \epsilon) d\theta dT = 0, \quad (3.9)$$

where the integral in θ is taken over one period.

To obtain the modulation in the slow time T we can use the average Lagrangian

$$\mathcal{L} = \frac{1}{2\pi} \int_0^{2\pi} L(U, \omega U_\theta + \epsilon U_T; \epsilon) d\theta \quad (3.10)$$

in

$$\delta \int \mathcal{L} dT = 0. \quad (3.11)$$

Equation (3.11) is the expression of the averaged variational principle.

This principle is exact, as no approximations are involved in its derivation. In practice, though, to use it as such would mean that we knew the exact form (3.1) of U , and that is equivalent to solving completely the Euler equation.

Using it as part of an asymptotic expansion, however, we can in principle solve the Euler equation (3.6) in θ to any order needed, while solving for the modulation from (3.11). The zeroth order solution is particularly simple, because (3.6) becomes independent of T and the solution is just that of the unperturbed

system.

Consider, for example, the linear oscillator with a variable spring constant

$$u_{tt} + \beta^2(T)u = 0, \quad (3.12)$$

whose Lagrangian is

$$L = \frac{1}{2}(u_t^2 - \beta^2 u^2). \quad (3.13)$$

Using the change of variables (3.2) the Euler equation becomes, to zeroth order,

$$\omega^2 U_{\theta\theta} + \beta^2 U = 0. \quad (3.14)$$

If we force U to have period 2π , we need $\omega = \beta$, and the solution of (3.14) is

$$U = \alpha \sin \theta. \quad (3.15)$$

Substituting in the averaged Lagrangian we have, to the same approximation,

$$\mathcal{L} = \frac{1}{4\pi} \int_0^{2\pi} (\omega^2 U_{\theta}^2 - \beta^2 U^2) d\theta = \frac{1}{4}\alpha^2(\omega^2 - \beta^2). \quad (3.16)$$

Here the "constants" α and ω are functions of T and become the variables in \mathcal{L} . When we use (3.16) in the variational principle, the functions that we have to vary are α , and ω , which is now present thru ω .

Varying α we get

$$\frac{\partial \mathcal{L}}{\partial \alpha} = 0 \quad \text{i. e.} \quad \omega = \beta, \quad (3.17)$$

which gives the correct dispersion relation, and, varying Θ ,

$$\frac{d}{dT} \left(\frac{\partial \mathcal{L}}{\partial \omega} \right) = 0 \quad \text{i. e.} \quad \alpha^2 \omega = \text{const.} \quad (3.18)$$

This last expression is nothing else than the adiabatic invariant for the oscillator.

We will now see the effect of introducing dissipation in the system.

3.2 Irreversible systems.

Consider a slightly damped linear oscillator

$$u_{tt} + \epsilon u_t + u = 0. \quad (3.19)$$

If ϵ is small, its solution is clearly a steady oscillation dying away slowly in time. For $\epsilon=0$ we recover the conservative case. It should be possible, therefore, to treat it as a two-timing perturbation problem, but, when we try to apply the averaged Lagrangian technique, we run into the difficulty that (3.19), being irreversible, has no Lagrangian of the classical type. We can try, however, to use the pseudo-Lagrangian introduced in section 2.3.

A pseudo-Lagrangian for (3.19) is

$$A = \frac{1}{2}(u_t^2 - u^2) - \epsilon \bar{u}_t u. \quad (3.20)$$

In order to understand the problem involved, we proceed first in a flexible way without the formalities of two-timing. To lowest order, the solution for u will be a modulated sinusoid of the form

$$U = \alpha \sin \theta, \quad (3.21)$$

where α and $\omega = \theta_t$ are slowly varying functions of time, in the sense that changes in one period are $O(\epsilon)$. In applying (3.20) we must also take

$$\bar{U} = \bar{\alpha} \sin \bar{\theta}. \quad (3.22)$$

Then

$$A \approx \frac{1}{2} \omega^2 \alpha^2 \cos^2 \theta - \frac{1}{2} \alpha^2 \sin^2 \theta - \epsilon \alpha \bar{\alpha} \bar{\omega} \sin \theta \cos \bar{\theta}. \quad (3.23)$$

This quantity has oscillations with respect to t on the scale of the period and slow variations over the scale ϵ^{-1} . The original averaging method eliminated the former by integrating over a few periods assuming that α and ω could be taken as approximately constant over this scale. With this in view, the expression for A is first written

$$A \approx \frac{1}{4} (\omega^2 - 1) \alpha^2 + \frac{1}{4} (\omega^2 + 1) \alpha^2 \cos 2\theta - \frac{1}{2} \epsilon \alpha \bar{\alpha} \bar{\omega} \{ \sin(\theta + \bar{\theta}) + \sin(\theta - \bar{\theta}) \} \quad (3.24)$$

Now, the average value of $\cos 2\theta$ and $\sin(\theta + \bar{\theta})$ over a few periods are zero. However, $\theta - \bar{\theta}$ represents what will eventually be the variation of θ , and can be made to have a period as long as we want. Accordingly, the term in $\theta - \bar{\theta}$ is retained, and the averaged variational principle is written

$$\delta \int_{t_1}^{t_2} \left\{ \frac{1}{4} (\omega^2 - 1) \alpha^2 - \frac{1}{2} \epsilon \alpha \bar{\alpha} \bar{\omega} \sin(\theta - \bar{\theta}) \right\} dt = 0. \quad (3.25)$$

Then, the variation of α gives

$$\frac{1}{2} (\omega^2 - 1) \alpha - \bar{\alpha} \bar{\omega} \sin(\theta - \bar{\theta}) = 0, \quad (3.26)$$

and the one of θ gives

$$\frac{d}{dt} (\omega \alpha^2) + \epsilon \alpha \bar{\alpha} \bar{\omega} \cos(\theta - \bar{\theta}) = 0. \quad (3.27)$$

Letting now $\alpha = \bar{\alpha}$ and $\theta = \bar{\theta}$, we get from (3.26)

$$\omega = 1 \quad (3.28)$$

which is the dispersion relation, and from (3.27)

$$\frac{d}{dt} (\omega \alpha^2) + \epsilon \omega \alpha^2 = 0, \quad (3.29)$$

which can be integrated to

$$\omega \alpha^2 = \text{const.} \times e^{-\epsilon t}. \quad (3.30)$$

These are, of course, correct results as can be seen from the exact solution of the problem.

In the two-timing approach we should expect to define an average Lagrangian in terms of A by

$$\mathcal{A} = \frac{1}{2\pi} \int_0^{2\pi} A(U, \bar{U}) d\theta, \quad (3.31)$$

and, substituting in it the values for U and \bar{U} we get

$$\mathcal{A} = \frac{1}{4} \alpha^2 (\omega^2 - 1) - \frac{\epsilon}{2\pi} \alpha \bar{\alpha} \bar{\omega} \int_0^{2\pi} \sin \theta \cos \bar{\theta} d\theta. \quad (3.32)$$

The trouble arises in the second term of (3.32), which is precisely the one representing the irreversibility. The problem is that, to compute the integral in that term, we have to make up our minds about the relationship between θ and $\bar{\theta}$.

If we consider them independent of one another the integral vanishes. Moreover, this choice does not seem a good one since we know that the two variables will eventually be numerically equal.

On the other hand, if we make them equal and integrate,

the integral vanishes again, and we commit the error of using the condition $\theta = \bar{\theta}$ before carrying out the variations.

The problem is to find a way of using the fact that θ and $\bar{\theta}$ are eventually the same thing, but maintaining their individualities until the Euler equations are computed.

We may argue that, θ and $\bar{\theta}$ being both functions of time, we may consider them as functions of one another, even if this functional dependence is not a fixed one but changes for each particular choice of θ and $\bar{\theta}$. After the variational principle is carried out, we will set this function to the identity, so it seems useful to express θ as

$$\theta = \bar{\theta} + \psi(\bar{\theta}) \quad (3.33)$$

where ψ is a new variable whose eventual value will be zero. Introducing (3.33) in the averaged Lagrangian, the value of the integral depends clearly on ψ . But, in carrying out the variational principle, we are only interested in the value of the functional and its derivatives at $\psi = 0$, so that we only need the value of \mathcal{A} up to $O(\psi)$. Moreover, ψ is arbitrary, so that we can make it as smooth as we want and take it out of the integral. Accordingly, the integral in (3.32) can be expanded to $O(\psi)$ as

$$\begin{aligned} \int_0^{2\pi} \sin \theta \cos \bar{\theta} \, d\theta &= \int_0^{2\pi} \sin(\bar{\theta} + \psi) \cos \bar{\theta} \, d\theta \approx \\ &\approx \int_0^{2\pi} (\sin \bar{\theta} + \psi \cos \bar{\theta}) \cos \bar{\theta} \, d\bar{\theta} \approx \pi \psi + O(\psi^2). \end{aligned} \quad (3.34)$$

Using this result in (3.32), we have

$$\mathcal{A} = \frac{1}{4}\alpha^2 (\omega^2 - 1) - \frac{\epsilon}{2}\alpha \bar{\alpha} \bar{\omega} \psi \quad (3.35)$$

and we can now use this expression in the variational principle.

The variation of α is straightforward,

$$\frac{\partial \mathcal{A}}{\partial \alpha} = 0. \quad (3.36)$$

Varying θ , however, is a little different from the reversible case.

In that case, θ was present in the averaged Lagrangian only through the frequency $\omega = \theta_t$, and, so, the Euler equation was (3.18). But now the phase function is present itself through ψ , as

$$\delta \psi = \delta(\theta - \bar{\theta}) = \delta \theta. \quad (3.37)$$

The corresponding Euler equation then becomes

$$\frac{d}{dt} \left(\frac{\partial \mathcal{A}}{\partial \omega} \right) - \frac{\partial \mathcal{A}}{\partial \psi} = 0. \quad (3.38)$$

After we have computed these equations, it is time to make $\alpha = \bar{\alpha}$, $\omega = \bar{\omega}$ and $\psi = 0$. If we carry out all these operations on the expression for \mathcal{A} in (3.35), we get exactly the same equations, (3.28) and (3.29), for the damped oscillator.

Encouraged by these results we may now try a more complicated, nonlinear, case. Consider the system

$$u_{tt} + V_u(u) + \epsilon f(u)u_t = 0, \quad (3.39)$$

which represents a nonlinear oscillator with a nonlinear dissipation term, and was proposed, and solved, by Kuzmak (1959)

as a model problem for developing two-timing methods.

A pseudo-Lagrangian for it is

$$A = \frac{1}{2} u_t^2 - V(u) - \epsilon \bar{f} \bar{u}_t u ; \quad (3.40)$$

on introducing the two-timed expressions for u and \bar{u} , it becomes

$$A = \frac{1}{2} \omega^2 U_\theta^2 - V(U) - \epsilon \bar{\omega} \bar{f} \bar{U}_\theta U. \quad (3.41)$$

Before going any further we anticipate the trouble in the last term of A and introduce the variable ψ , and the corresponding expansion of U ,

$$U(\theta) = U(\bar{\theta} + \psi) \approx U(\bar{\theta}) + \psi U_\theta(\bar{\theta}) \equiv \bar{U} + \psi \bar{U}_\theta . \quad (3.42)$$

Introducing this expansion in A and dropping terms depending only on \bar{U} , we get

$$A = \frac{1}{2} \omega^2 U_\theta^2 - V(U) - \epsilon \bar{\omega} \bar{f} \bar{U}_\theta^2 \psi . \quad (3.43)$$

To find the expression for U , we change to θ variables in (3.39) and integrate once to get the energy equation (Whitham 1970). Solving in it for U_θ , we get, to zeroth order

$$U_\theta^2 = \frac{\alpha^2 - 2V(U)}{\omega^2} , \quad (3.44)$$

where α is an energy "constant" and is related to the amplitude. Using this result, the averaged Lagrangian can be brought into the form

$$\mathcal{A} = \frac{1}{2\pi} \int_0^{2\pi} \omega^2 U_\theta^2 d\theta - \frac{\alpha}{2} - \frac{\epsilon}{2\pi} \bar{\omega} \psi \int_0^{2\pi} f(\bar{U}) \bar{U}_\theta^2 d\bar{\theta} , \quad (3.45)$$

or

$$\mathcal{A} = \frac{\omega}{2\pi} \oint (\alpha - 2V)^{\frac{1}{2}} dU - \frac{\alpha}{2} - \frac{\epsilon}{2\pi} \psi \oint (\alpha - 2V)^{\frac{1}{2}} f dU . \quad (3.46)$$

In this form we can already compute the Euler equations

$$\delta\alpha: \quad \left. \frac{\partial \mathcal{A}}{\partial \alpha} \right|_{\psi=0} = 0, \quad (3.47)$$

and

$$\delta\Theta: \quad \frac{d}{dT} \left(\frac{\partial \mathcal{A}}{\partial \omega} \right) - \left. \frac{\partial \mathcal{A}}{\partial \psi} \right|_{\psi=0} = 0 . \quad (3.48)$$

The final results are

$$\omega^{-1} = \frac{1}{2\pi} \oint (\alpha - 2V)^{-\frac{1}{2}} dU, \quad (3.49)$$

and

$$\frac{d}{dT} \oint (\alpha - 2V)^{\frac{1}{2}} dU + \oint f(U) (\alpha - 2V)^{\frac{1}{2}} dU = 0, \quad (3.50)$$

which are equivalent to the equations given by Kuzmak in his paper.

4. The general method.

4.1 The zeroth order approximation.

After the examples in 3.2, we will now summarize the general application of the averaged Lagrangian method to irreversible systems. The formal justification will be postponed until the next section, where the extension of the formulas to all orders will also be given. In this section we will only attempt to get results to zeroth order in ϵ , and, again, we will confine ourselves to ordinary differential equations.

Assume that we have an irreversible system described by the pseudo-Lagrangian

$$A = L(u, u_t) + \epsilon I(u, u_t; \bar{u}, \bar{u}_t). \quad (4.1)$$

We define the slow time $T = \epsilon t$, as well as the phase function θ and angular frequency ω as in section 3.

We add the appropriate form for u , which, to lowest order is

$$u = U(\theta, T), \quad (4.2)$$

and substitute it into (4.1). To that same approximation

$$A = L(U, \omega U_\theta) + \epsilon I(U, \omega U_\theta; \bar{U}, \bar{\omega} \bar{U}_\theta). \quad (4.3)$$

Now we introduce the variable

$$\psi = \theta - \bar{\theta}, \quad (4.4)$$

and expand U around $\bar{\theta}$. Then

$$U(\theta) = U(\bar{\theta} + \psi) = \bar{U} + \psi \bar{U}_\theta , \quad (4.5)$$

and, similarly,

$$U_\theta = \bar{U}_\theta + \psi \bar{U}_{\theta\theta} . \quad (4.6)$$

Next, we substitute these expansions in the Lagrangian and expand again in ψ . We are only interested in keeping those terms that will give lowest order contributions to the Euler equations (3.47) and (3.48). These are all the zeroth order terms, plus those linear in ψ and of order ϵ . Carrying out the expansion, we have

$$A = L(U, \omega U_\theta) + \epsilon (\bar{I}_1 \bar{U}_\theta + \omega \bar{U}_{\theta\theta} \bar{I}_2) \psi , \quad (4.7)$$

where

$$\begin{aligned} \bar{I}_1 &= \frac{\partial I}{\partial u} (\bar{U}, \omega \bar{U}_\theta; \bar{U}, \bar{\omega} \bar{U}_\theta) , \\ \bar{I}_2 &= \frac{\partial I}{\partial u_t} (\bar{U}, \omega \bar{U}_\theta; \bar{U}, \bar{\omega} \bar{U}_\theta) . \end{aligned} \quad (4.8)$$

A word about notation may be in order here. All through this work, for a function $F(u, \bar{u})$,

$$\frac{\partial}{\partial u} \{ F(u, u) \} \quad (4.9)$$

is intended to mean that both arguments on F are first substituted by u , and then the derivative is computed, while in

$$\frac{\partial F}{\partial u} (u, u) , \quad (4.10)$$

the partial is taken first and then both arguments are made equal to u .

From (4.7) we define the averaged Lagrangian

$$\mathcal{H} = \frac{1}{2\pi} \int_0^{2\pi} \bar{L} d\theta + \frac{\epsilon}{2\pi} \int_0^{2\pi} (\bar{I}_1 U_\theta + \omega \bar{I}_2 U_{\theta\theta}) d\theta, \quad (4.11)$$

where the bars inside the second integral can be dropped to zeroth order. The only problem remaining is to find the form of $U(\theta)$ to use in computing the integrals.

The best way to do this was shown by Whitham (1970) for reversible systems, and is only sketched here.

To lowest order the system is described by the Lagrangian $L(U, \omega U_\theta)$, and it is clear that this Lagrangian does not depend explicitly on θ . So, U obeys an energy conservation law

$$U_\theta \frac{dL}{dU_\theta} - L = \alpha. \quad (4.12)$$

The steps from here on are the same as in the Hamiltonian transformation in mechanics. A momentum is defined

$$\mathcal{H} = \frac{dL}{dU_\theta}, \quad (4.13)$$

from which we solve for U_θ ,

$$U_\theta = U_\theta(U, \mathcal{H}). \quad (4.14)$$

From (4.12) we then find

$$\mathcal{H} = \mathcal{H}(U, \alpha), \quad (4.15)$$

$$U_\theta = F(U, \alpha), \quad (4.16)$$

and

$$U_{\theta\theta} = F_U(U, \alpha) U_\theta. \quad (4.17)$$

Using these formulas in (4.11)

$$\mathcal{A} = \frac{1}{2\pi} \int_0^{2\pi} (\pi U_\theta - \alpha) d\theta + \frac{\varepsilon}{2\pi} \psi \int_0^{2\pi} (I_1 + \omega I_2 F_U) U_\theta d\theta, \quad (4.18)$$

or

$$\mathcal{A} = \frac{1}{2\pi} \oint \pi dU - \alpha + \frac{\varepsilon}{2\pi} \psi \oint (I_1 + \omega I_2 F_U) dU, \quad (4.19)$$

where everything can be expressed as a function of U and α .

The Euler equations, from section 3.2, are

$$\delta\alpha: \quad \frac{1}{2\pi} \oint \frac{\partial \pi}{\partial \alpha} dU = 1, \quad (4.20)$$

$$\delta\omega: \quad \frac{d}{dT} \oint \frac{\partial \pi}{\partial \omega} dU - \oint (I_1 + \omega I_2 F_U) dU = 0. \quad (4.21)$$

The first of these equations is the dispersion relation, and is not changed, to this approximation, by the dissipation. The second one gives the decay of the adiabatic invariant of the system. It is here that the effect of the dissipation appears.

4.2 Formal justification of results.

Up to now the theory has been developed at a purely heuristic level, and its only justification is that it seems to work in all cases in which it has been tried.

A particularly worrisome problem is the lack of assurance that the equation (4.21) is invariant under all possible choices of the irreversible term in the pseudo-Lagrangian. Also, although the averaged Lagrangian method is known to be accurate to all orders, we have only been able to implement it here to the lowest approximation.

In this section we discuss all those problems. First we establish the rigor of the method. The way to do this is very similar to the one used for reversible systems (Whitham 1970), but in this case there are some more subtleties involved. What we want to prove are the equations (4.20) and (4.21) for α and Θ .

In the simplest cases u can be expressed as $u=U(\theta,\alpha)$, where the dependence on a parameter α is taken to contain the explicit dependence on $T=\epsilon t$. The crucial condition in the method is that $U(\theta,\alpha)$ be periodic in θ . (Without loss of generality the period may be taken to be 2π). One way to introduce the periodicity is to express U as a Fourier series

$$U(\theta,\alpha) = \sum_n U_n(\alpha) e^{in\theta}, \quad (4.22)$$

and use a similar expansion for \bar{U} ,

$$\bar{U}(\bar{\theta}, \bar{\alpha}) = \bar{U}_n(\bar{\alpha}) e^{in\bar{\theta}}, \quad (4.23)$$

where we adopt the convention that repeated indices mean summation from $-\infty$ to $+\infty$.

In any of these expansions we are trying to force a periodic behaviour with respect to the explicit dependence on θ . In this sense θ is considered as independent of T , and the Fourier coefficients are defined as

$$U_n(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} U(\xi, \alpha) e^{-in\xi} d\xi. \quad (4.24)$$

This is still so in more complicated cases in which the variable depends on θ explicitly, and through θ_t . For example, assume that the system derives from a pseudo-Lagrangian $A(u, \bar{u})$, where the arguments of A include in general u_t and \bar{u}_t , even if they are not explicitly indicated. Substituting in A the expressions U and \bar{U} , we get a function $A(\theta, \bar{\theta}; \omega, \alpha, \alpha_t, \bar{\omega}, \bar{\alpha}, \bar{\alpha}_t)$ which we want to be periodic in θ and $\bar{\theta}$, but with ω and $\bar{\omega}$ considered independent of them and included in the slow dependence in T . So, the corresponding expansion is

$$A = A_{mn} e^{i(m\theta + n\bar{\theta})}, \quad (4.25)$$

with

$$A_{mn}(\omega, \alpha, \alpha_t, \bar{\omega}, \bar{\alpha}, \bar{\alpha}_t) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} A(\xi, \eta; \omega, \alpha, \alpha_t, \bar{\omega}, \bar{\alpha}, \bar{\alpha}_t) \times \\ \times e^{-i(m\xi + n\eta)} d\xi d\eta. \quad (4.26)$$

Once the Fourier expansions are carried out, however, we always take $\theta = \frac{\Theta}{\epsilon}$ and $\alpha = \alpha(T)$ as functions of time, with no attempt to consider them independent variables.

The notations

$$A_{mn}, [A]_{mn} \quad \text{or} \quad [A]_{m,n} \quad (4.27)$$

represent the mn -th Fourier coefficient of the double series for $A(\theta, \bar{\theta})$, while

$$[A(\theta, \theta)]_m \quad (4.28)$$

is the m -th coefficient of the simple series for $A(\theta, \bar{\theta})$ when we let $\theta = \bar{\theta}$ before the series is computed. Finally

$$A_{mn;\omega} = \frac{\partial}{\partial \omega} A_{mn}. \quad (4.29)$$

The variational principle can now be written as

$$\delta \int A_{mn} e^{i(m\theta + n\bar{\theta})} dT = 0. \quad (4.30)$$

We can perform the variations of Θ and α in it, and extract the information we want.

Varying Θ , and remembering that $\omega = \Theta_T$, the Euler equation is

$$\frac{im}{\epsilon} A_{mn} e^{i(m\theta + n\bar{\theta})} - \frac{d}{dT} \left\{ A_{mn;\omega} e^{i(m\theta + n\bar{\theta})} \right\} = 0, \quad (4.31)$$

which can be expanded to

$$\left\{ \frac{im}{\epsilon} A_{mn} - \frac{d}{dT} A_{mn;\omega} - \frac{i(m\omega + n\bar{\omega})}{\epsilon} A_{mn;\omega} \right\} e^{i(m\theta + n\bar{\theta})} = 0. \quad (4.32)$$

Now, we let $\theta = \bar{\theta}$, $\omega = \bar{\omega}$, $\alpha = \bar{\alpha}$, and (4.32) collapses into a simple Fourier series ($\ell = m+n$)

$$\left\{ \frac{im}{\epsilon} A_{m, \ell-m} - \frac{d}{dT} A_{m, \ell-m}; \omega - \frac{i\ell\omega}{\epsilon} A_{m, \ell-m}; \omega \right\} e^{i\ell\theta} = 0; \quad (4.33)$$

this is the Euler equation corresponding to the variation of Θ . It is still in the form of a doubly infinite series and is not very practical. It turns out, however, that we can sum one of the series.

First we note (see Appendix) that

$$A(\theta, \theta)_{m, \ell-m} = [A(\theta, \theta)]_{\ell} \quad (4.34)$$

for any Fourier series, that

$$imA_{mn} = \left[\frac{\partial A}{\partial \theta} \right]_{mn}, \quad (4.35)$$

and

$$\frac{d}{dT} A_{mn}; \omega = \left[\frac{\partial A}{\partial T} \right]_{mn}. \quad (4.36)$$

Equation (4.33) can then be written as

$$\left[\frac{1}{\epsilon} \frac{\partial A}{\partial \theta}(\theta, \theta) - \frac{\partial}{\partial T} A_{\omega}(\theta, \theta) - \frac{\omega}{\epsilon} \frac{\partial}{\partial \theta} \{ A_{\omega}(\theta, \theta) \} \right]_{\ell} e^{i\ell\theta} = 0. \quad (4.37)$$

This last expression is formally the Fourier expansion of the expression inside the square bracket. So, for functions which are regular enough, every coefficient has to vanish independently, and, for every ℓ ,

$$\frac{1}{\epsilon} \left[\frac{\partial A}{\partial \theta} (\theta, \theta) \right]_{\ell} - \frac{d}{dT} [A_{\omega} (\theta, \theta)]_{\ell} - \frac{1}{\epsilon} i \ell \omega [A_{\omega} (\theta, \theta)]_{\ell} = 0. \quad (4.38)$$

In particular, this has to be true for $\ell=0$, in which case the last term drops out, and the Fourier coefficients turn into simple averages over one period. We conclude that

$$\frac{1}{\epsilon} \left[\frac{\partial A}{\partial \theta} (\theta, \theta) \right]_0 - \frac{d}{dT} [A_{\omega} (\theta, \theta)]_0 = 0, \quad (4.39)$$

which is the averaged Euler equation for the variation of Θ . For $\ell \neq 0$ we get information on the higher Fourier coefficients of A , and, in principle, solving (4.38) for all ℓ , we could get the complete series. These equations, however, do not modify (4.39).

Carrying out a similar derivation for α , we arrive at an equation

$$\frac{1}{\epsilon} \left[\frac{\partial A}{\partial \alpha} (\theta, \theta) \right]_0 - \epsilon \frac{d}{dT} [A_{\alpha_t} (\theta, \theta)]_0 = 0 \quad (4.40)$$

where the extra ϵ in the second term arises because A depends explicitly on $\alpha_t = \epsilon \alpha_T$, as opposed to $\omega = \Theta_T$.

Equations (4.39) and (4.40) are the required Euler equations for the averaged principle, and they are exact, independent of the size of ϵ . In the practical use of them we work to the lowest order approximation only.

We now show that these equations are equivalent, in that order, to the ones derived in section 4.1. Assume

$$A = L(U) + \epsilon I(U, \bar{U}). \quad (4.41)$$

The reversible part, L , depends only on $\dot{\theta}$, not on θ , so that $\frac{\partial L}{\partial \theta}$ becomes a total derivative,

$$\frac{\partial A}{\partial \theta}(\theta, \dot{\theta}) = \frac{dL}{d\theta} + \epsilon (I_U U_{\dot{\theta}} + I_{U_{\dot{\theta}}} U_{\theta\dot{\theta}}), \quad (4.42)$$

and, in integrating over one period, the term in L cancels because of periodicity, and

$$\left[\frac{\partial A}{\partial \theta}(\theta, \dot{\theta}) \right]_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial A}{\partial \theta}(\theta, \dot{\theta}) d\theta = \frac{\epsilon}{2\pi} \int_0^{2\pi} (I_U U_{\dot{\theta}} + I_{U_{\dot{\theta}}} U_{\theta\dot{\theta}}) d\theta. \quad (4.43)$$

When we compute the second term of (4.39), however, it is only the part depending on L that remains to lowest order. The final equation is

$$\frac{d}{dT} \int_0^{2\pi} L_w d\theta - \int_0^{2\pi} (I_U U_{\dot{\theta}} + I_{U_{\dot{\theta}}} U_{\theta\dot{\theta}}) d\theta = 0, \quad (4.44)$$

which is easily seen to be equivalent to (4.21). The equation for the variation of α comes out with equal ease.

The derivation above also answers the question of uniqueness of the averaged equations, as it shows that they are just a consequence of the Euler equation, and, for all forms of the pseudo-Lagrangian resulting in a given Euler equation, the averaged equations will also be unique.

The only question remaining is if it is allowable to change the dependent variable in a variational principle. Specifically, if a system is described by $L(u, \dot{u})$ and we let u be a given function

of a new variable $\theta(t)$, we want to know if we can vary θ instead of u . It is easy to see that this is indeed the case.

Assume there are no more variables. The Euler equation derived from varying u is

$$L_u(u, u) - \frac{d}{dt} L_{u_t}(u, u) = 0, \quad (4.45)$$

while that from θ is

$$L_\theta(u, u) - \frac{d}{dt} L_{\theta_t}(u, u) = 0. \quad (4.46)$$

But

$$u = u(\theta), \quad u_t = u_\theta \theta_t, \quad (4.47)$$

so that

$$L_\theta = L_u u_\theta + L_{u_t} u_{\theta\theta} \theta_t, \quad (4.48)$$

and

$$L_{\theta_t} = L_{u_t} u_\theta. \quad (4.49)$$

Multiplying now (4.45) by u_θ and rearranging terms, we recover (4.46). It is this fact that allows us to vary θ and α in the averaged principle instead of u .

4.3 Partial differential equations.

The application of the averaged Lagrangian method to partial differential equations is well known from the work on reversible systems (Whitham 1970), and involves little more than the substitution of the time variable by a vector (\underline{x}, t) , and of the corresponding derivatives by gradients and divergences. We still retain a single phase function, θ , which characterizes the wavetrain, but the local frequency splits into a wave number vector and a scalar frequency, in such a way that

$$\theta_{x_i} = k_i \quad , \quad \theta_t = -\omega. \quad (4.50)$$

The averaged Lagrangian becomes a function of ω , k and the energy. Assuming only one space dimension and a system with only one energy constant, α , we have

$$\mathcal{L} = \mathcal{L}(\omega, k, \alpha). \quad (4.51)$$

The variation in α results, as before, in

$$\mathcal{L}'_{\alpha} = 0 \quad , \quad (4.52)$$

which is the dispersion relation, giving $\omega = \omega(k, \alpha)$. On the other hand, the variation in θ results in

$$\frac{\partial}{\partial t} \mathcal{L}'_{\omega} - \frac{\partial}{\partial x} \mathcal{L}'_{k} = 0. \quad (4.53)$$

An extra condition is the consistency relation between k and ω

$$k_t + \omega_x = 0. \quad (4.54)$$

The addition of dissipation does not introduce any essentially new features. To first order, the averaged pseudo-Lagrangian defined in section 4.1 will be now

$$\mathcal{A} = \mathcal{L}(\omega, \mathbf{k}, \alpha) + \epsilon \psi \mathcal{J}(\omega, \mathbf{k}, \alpha), \quad (4.55)$$

and the equations (4.52) and (4.54) remain unchanged, but equation (4.53) gets a new term representing the dissipation,

$$\frac{\partial}{\partial t} \mathcal{L}_\omega - \frac{\partial}{\partial \mathbf{x}} \mathcal{L}_\mathbf{k} + \epsilon \mathcal{J} = 0. \quad (4.56)$$

As a simple example consider a linear system. In those systems the averaged Lagrangian turns out to be linear in α , and, as a first approximation, we can consider the dissipation term to be also linear,

$$\mathcal{A} = \alpha G(\omega, \mathbf{k}) + \epsilon \psi \alpha D(\omega, \mathbf{k}). \quad (4.57)$$

The dispersion relation (4.52) becomes independent of α

$$G(\omega, \mathbf{k}) = 0, \quad (4.58)$$

so that we can write $\omega = \omega(\mathbf{k})$, and (4.54) becomes

$$k_t + c_g k_x = 0, \quad (4.59)$$

where

$$c_g = \frac{d\omega}{d\mathbf{k}} = -\frac{G_\mathbf{k}}{G_\omega} \quad (4.60)$$

is the group velocity. Using these relations it is easy to transform equation (4.56) into the form of an energy equation,

(Whitham 1967)

$$\alpha_t + (c_g \alpha)_x = -\epsilon \alpha \lambda(k), \quad (4.61)$$

with

$$\lambda(k) = \frac{D}{G_w}. \quad (4.62)$$

For a fixed wave number k , it is clear that α has solutions of the type $e^{-\epsilon \lambda t} F(x - c_g t)$, so that λ is a measure of the energy decay rate.

The equation (4.61) was derived in a somewhat more heuristic way by Davey(1972).

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Appendix

In the first place we prove equation (4.36), which we do, for simplicity, for a simple Fourier series. Consider

$$\begin{aligned} \frac{d}{dT} A_n(T) &= \frac{1}{2\pi} \frac{d}{dT} \int_0^{2\pi} A(\xi, T) e^{-in\xi} d\xi = \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{dT} A(\xi, T) e^{-in\xi} d\xi. \end{aligned} \quad (1)$$

In the integrand of (1) the only dependence on T is on the second argument of A , as ξ is a dummy variable of integration, so that

$$\frac{d}{dT} A_n = \left(\frac{\partial A}{\partial T} \right)_n, \quad (2)$$

where the partial derivative means that θ has to be considered an independent variable, and the derivative is only taken with respect to the rest of the dependance in T . Equation (2) is the result we wanted.

We turn now to prove equation (4.34). We use the expression

$$A(\theta, \bar{\theta})_{m, l-m} = \frac{1}{4\pi} \int \int_0^{2\pi} A(\xi, \eta) e^{-i[lm\xi + (l-m)\eta]} d\xi d\eta, \quad (3)$$

and rearrange terms inside the integral

$$A(\theta, \bar{\theta})_{m, l-m} = \frac{1}{4\pi} \int \int_0^{2\pi} A(\xi, \eta) e^{-il\eta} \sum_m e^{im(\eta-\xi)} d\xi d\eta. \quad (4)$$

Now we use the formula

$$\frac{1}{2\pi} \sum_m e^{imx} = \delta(x) , \quad (5)$$

where $\delta(x)$ is the Dirac delta, and integrate in (4) over ξ

$$\begin{aligned} A(\theta, \bar{\theta})_{m, \ell-m} &= \frac{1}{2\pi} \iint_0^{2\pi} A(\xi, \eta) e^{-i\ell\eta} \delta(\eta-\xi) d\xi d\eta = \\ &= \frac{1}{2\pi} \int_0^{2\pi} A(\eta, \eta) e^{-i\ell\eta} d\eta = \\ &= [A(\theta, \theta)]_{\ell} , \end{aligned} \quad (6)$$

which is the desired result.