CARLEMAN INEQUALITIES AND UNIQUE CONTINUATION FOR HIGHER ORDER ELLIPTIC DIFFERENTIAL OPERATORS

Thesis by
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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology

Pasadena, California

1993

(Submitted May 25, 1993)

To my wife, Jing

Acknowledgments

It is my pleasure to express my sincere thanks to my teacher and advisor, Professor Tom Wolff who patiently introduced me to the field of Analysis and PDE. During the time in which I studied with and worked for him, I have greatly benefited from his mathematical ideas, his spirit of perfectionism, and his intricate mathematical techniques. He has always been willing to give me detailed explanation of his own and others work. Without his constant encouragement, this thesis would not have been possible.

I would like to thank Professor Alice Chang for her concern and help in my mathematical career. I would like to thank Professors John Garnett, Alexander Kechris, Wilhelmus Luxemburg, Nikolai Makarov, and Christopher Sogge for their interest in this work. I would also like to take this opportunity to thank Professors Mingde Cheng, Donggao Deng, Yongsheng Han, and Lizhong Peng, from whom I learned Harmonic Analysis when I was in China.

I wish to thank the Department of Mathematics at Caltech for providing financial support and a wonderful studying environment during my graduate career. I also wish to thank the Departments of Mathematics at UCLA and UC Berkeley for providing partial financial support (UCLA) and nice working facilities when I was visiting these schools.

This work was supported in part by the Alfred P. Sloan Doctoral Dissertation Fellowship.

Abstract

In this thesis, we study the weak unique continuation property for higher order elliptic differential operators with real coefficients via Carleman inequalities. We get several Carleman inequalities with sharp gaps for operators in a reasonable class, which lead eventually to the weak unique continuation property for differential inequalities with optimal conditions on potentials. We also get some Carleman inequalities for general operators with simple or double characteristics. The gaps here are not as good as in the first case. But we may prove the gaps in these inequalities are sharp in general. Actually we will provide counterexamples to prove such gaps are sharp in Carleman inequalities for operators in some subclasses of simple or double characteristics class. In particular, we prove that there is no Carleman inequality with positive gap for the highest order term for any operator whose symbol has double characteristics.

Contents

0.1 Introduction

It is well known that if $\boldsymbol{P}(x,D)$ is an elliptic differential operator with real analytic coefficients, then any solution u of $\boldsymbol{P}(x,D)u=0$ in an open connected set $V\subset \boldsymbol{R}^d$ is real analytic. So $\boldsymbol{P}(x,D)$ has the so-called unique continuation property (u.c.p.), i.e., if a solution u of $\boldsymbol{P}(x,D)u=0$ in V vanishes in an open subset V' of V, then $u\equiv 0$ in V. In many situations, the analyticity assumption can be dropped. In 1958 A.P. Calderon proved the following u.c.p. result [3, 10]:

Theorem([Ni]): *Let $\boldsymbol{P}(x,D)$ be a linear partial differential operator of order m with smooth coefficients defined in a neighborhood V of the origin in R^d. Suppose for each fixed $x\in V$ the corresponding polynomial $P(x,\cdot)$ satisfies the double characteristic condition* (see below). *Assume that the plane $x^d=0$ is non-characteristic at the origin (i.e., $P(0,e_d)\neq 0$). Then if a C^∞ function u in V satisfies*

$$(*)\qquad |\boldsymbol{P}u(x)|\le \sum_{|\alpha|\le m-1}|V_\alpha||D^\alpha u(x)|,\ x\in V\quad \textit{and}\quad u\equiv 0\,\textit{in}\,V\cap\{x_d\le 0\}$$

then $u\equiv 0$ in V if all $V_\alpha\in L^\infty_{loc}$.

One will see the definitions of the simple/double characteristics conditions later (or see [10] or [13]). The essential point in the proof of this theorem is a so-called Carleman inequality, the first version of which was introduced by T. Carleman in 1939. A version given in [10] states that:

$$\sum_{|\alpha|\le m} ||e^{t\phi}D^\alpha u||_{L^2(\boldsymbol{R}^{d-1}\times(0,T))} \le C(t^{-1}+T^2)||e^{t\phi}\boldsymbol{P}u||_{L^2(\boldsymbol{R}^{d-1}\times(0,T))}$$

$\forall u \in C_0^\infty$, $\forall t > 0$, where $\phi(x) = \phi(x_d) = (T - x_d)^2$ and $T > 0$.

Later when people continue to study the u.c.p. for differential inequalities under different conditions, the Carleman type inequalities still play an important role. We will be concerned with questions where $V_\alpha \in L^r_{loc}$ with $r < \infty$, and then one needs inequalities $||e^{t\phi}D^\alpha u||_q \le C||e^{t\phi}\boldsymbol{P}u||_p$, where what is important is the size of the gap $\frac{1}{p} - \frac{1}{q} > 0$. For example, with certain convex smooth functions ϕ, the following Carleman inequalities (see [1, 9, 12])

$$||e^{t\phi}u||_{L^{p'}(\boldsymbol{R}^d)} \le C||e^{t\phi}\Delta u||_{L^p(\boldsymbol{R}^d)}$$

$$||e^{t\phi}\nabla u||_{L^2(\boldsymbol{R}^d)} \le C||e^{t\phi}\Delta u||_{L^{p_1}(\boldsymbol{R}^d)}$$

$\forall u \in C_0^\infty$, $\forall t > 0$, hold with positive gaps $\frac{1}{p} - \frac{1}{p'} = \frac{2}{d}$ and $\frac{1}{p_1} - \frac{1}{2} = \frac{2}{3d-2}$, and give u.c.p. for $(*)$ with $\boldsymbol{P} = \Delta$ and $V_1 \in L^{\frac{d}{2}}_{loc}$ and $V_2 \in L^{\frac{3d-2}{2}}_{loc}$. The proofs of these inequalities depend on estimates for oscillatory integrals in harmonic analysis. One such approach is to study the restriction of the Fourier transform to hypersurfaces $S_k = \{\xi \in \boldsymbol{R}^d : Re(\sum_{j=1}^d(\xi_j + ik_j)^2) = 0\}$ which contains the real variety $N_k = \{\xi \in \boldsymbol{R}^d : \sum_{j=1}^d(\xi_j + ik_j)^2 = 0\}$, where $k \in S^{d-1}$. In this case, S_k is the unit sphere which has nonzero Gaussian curvature so that the Fourier transform of the surface measure decays fast at ∞ and this leads eventually to weighted Sobolev inequalities with good gap conditions as mentioned above.

In this paper, we are interested in the unique continuation property for higher order elliptic differential operators with real coefficients whose symbols are homogeneous polynomials via Carleman type inequalities. Except when $p = q = 2$, such inequalities were known previously only for the operators $\Delta^{\frac{m}{2}}$ and in that case only for the zero-order term in the left-hand side [7]. As usually, it is natural to study oscillatory integrals which are related to the set $S_k = \{\xi \in \boldsymbol{R}^d : \mathrm{re}P(\xi + ik) = 0\}$ for $k \in S^{d-1}$. When the set S_k satisfies suitable conditions, we will have an appropriate decay of the Fourier transform of the surface measure at ∞. For example, if a surface satisfies Stein's finite type of order m, then the decay of the Fourier transform is of order $\frac{1}{m}$ (see [11]). If a surface is convex and satisfies Bruna-Nagel-Wainger's finite type of order m, then there is a sharp rate $\frac{d-1}{m}$ of decay of the Fourier transform (see [2]). In order to apply these results, the problems are that in one hand we don't know how to get a sharp Carleman inequality from the first weaker estimate of the decay of the Fourier transform and on the other hand our surface will satisfy Bruna-Nagel-Wainger's finite type whenever it is a submanifold but in general will not be convex. So just like in studying of oscillatory integrals, it suggests to narrow the operator $\boldsymbol{P}$ into some class so that the set S_k satisfies some curvature conditions. The natural one is to hope S_k having nonzero Gaussian curvature. But the class of $\boldsymbol{P}$ with this kind condition is too small to contain our model operator $\frac{d^m}{dx_1^m} + \cdots + \frac{d^m}{dx_d^m}$ when $m \geq 4$. What is new here is that instead of studying the hypersurface S_k, we study the real variety $N_k^P = \{\xi \in \boldsymbol{R}^d : P(\xi + ik) = 0\}$ such that locally N_k^P may be contained into some hypersurface with nonzero Gaussian curvature, if P satisfies some conditions which is easy to check. We always use P to denote a polynomial and use $\boldsymbol{P}$ to denote the corresponding operator. Here is one of our crucial results:

Proposition 0.1 *If $P \in G$ (see definition in Section 1), then there is an open set K of S^{d-1} such that for each $k \in K$ and any $\xi \in \pi_k(N^P)$, i.e., $P(\xi+ik) = 0$, there is a hypersurface in $\boldsymbol{R}^d$ with nonzero Gaussian curvature containing $P(\cdot+ik)^{-1}(0)$ locally near ξ. Moreover, $\pi_k(N^P)$ is a submanifold of codimension 2 for each such $k \in K$.*

Remark. Actually the conclusion in the above Proposition is what is needed in the proofs of the following theorems 0.2, 0.3 and 0.4. The condition G is a sufficient condition which is easy to state and to verify in examples.

Our main results are weak u.c.p. with what are expected to be sharp conditions on the potentials for operators in class G. Here are two of them:

Theorem 0.2 *Suppose $P \in G$ is of order m. Let $2 \leq \mu \leq m$ be an integer. Suppose $u \in W^{m,p}$ has compact support and satisfies that $|\boldsymbol{P}u| \leq V|\nabla^{m-\mu}u|$ with $V \in L^{r_\mu}$. Then $u \equiv 0$ in $\boldsymbol{R}^d$ if*

(1) $r_\mu = \frac{d}{\mu}$ and p is such that $\frac{1}{p} < \frac{1}{s} + \frac{(d-1)(\frac{\mu}{d}-\frac{2}{d+1})}{d-3}$, where $s = \frac{2(d+1)}{d+3}$, if $\mu \leq \frac{d}{2}$;

(2) $r_\mu = \frac{d}{\mu}$ and $p = 1$, if $\frac{d}{2} < \mu < d$;

(3) $r_\mu > 1$ and $p = 1$, if $\mu \geq d$.

Theorem 0.3 *Suppose $P \in G$ is of order $m < \frac{d}{s}$ with $s = \frac{2(d+1)}{d+3}$. If a function $u \in W^{m,s}$ has compact support and satisfies $|\boldsymbol{P}u| \leq \sum_{1\leq\mu\leq m} V_\mu|\nabla^{m-\mu}u|$ with $V_\mu \in L^{\frac{d}{\mu}}$ for all $1 \leq \mu \leq m$, then $u \equiv 0$ in $\boldsymbol{R}^d$.*

Here in Theorem 0.3, we have a restriction on the degree m of P. In section 3, we will see that we may take m a slightly larger and $p = s$ slightly smaller.

As we mentioned before, the proof of these theorems are based on several Carleman inequalities with sharp gaps. For P is in the class G which will be defined in Section 1, we have the following sharp Carleman inequality.

Theorem 0.4 *Suppose $P \in G$ is of degree m and $k \in S^{d-1}$ is as in* Proposition 0.1. *For any integer $2 \leq \mu \leq m$, let (p, q) be such that $(\frac{1}{p}, \frac{1}{q}) \in A \cap A_\mu$ (*see definitions of the sets A and A_μ at the beginning of Section 2*). Then there is a constant C such that*

$$\|e^{k\cdot x}\nabla^{m-\mu}u\|_q \leq C\|e^{k\cdot x}\boldsymbol{P}u\|_p$$

for all $u \in W^{m,p}$ with compact support.

On the other hand, we are interested in Carleman inequalities for operators in some more general classes S and D of simple/double characteristics which will be also defined in Section 1. Here is our main result:

Theorem 0.5 *(1) Suppose that $P \in D$ is of degree m. Then there are an open subset K of S^{d-1} and a small constant β and a big constant C such that for each $k \in K$, with $\phi(x) = \phi_k(x) = k \cdot x + \beta|x|^2$ we have for each integer $1 \leq \mu \leq m$,*

$$\|e^{t\phi}\nabla^{m-\mu}u\|_{L^q(R^d)} \leq C\|e^{t\phi}\boldsymbol{P}u\|_{L^2(R^d)}$$

for all $u \in W^{m,2}$ with compact support, where $q \in [2, \infty]$ is such that $0 \leq \frac{1}{2} - \frac{1}{q} \leq min(\frac{\mu-1}{d-1}, \frac{\mu}{d})$.

(2) If $P \in S$, the same inequalities hold with better gaps $0 \leq \frac{1}{2} - \frac{1}{q} \leq$ *min*$(\frac{\mu - \frac{1}{2}}{d-1}, \frac{\mu}{d})$. *If* $q = \infty$, *assume* $\mu > \frac{d}{2}$.

The methods used there are only real analysis and general properties of the Fourier transform and so the gaps obtained are not sharp as in Theorem 2.2. But we will prove that our gap conditions are sharp for general P. In fact, we will prove in section 5, for general μ, that the gaps $\frac{\mu - \frac{1}{2}}{d-1}$ and $\frac{\mu - 1}{d-1}$ are sharp in the Carleman inequalities for operators in some subclasses of S and D respectively. One may see Proposition 5.3 and 5.2. Furthermore, for $\mu = 1$ we will show in section 5 that there is no Carleman inequality with positive gap for any operator in the double characteristics class (see Proposition 5.1).

We will discuss some basic geometric properties and provide some examples of operators in class G and then prove Proposition 0.1 in Section 1. Then we may prove some Carleman type inequalities with sharp gaps, including Theorem 0.4, in Section 2. As application, in Section 3, we will prove Theorem 0.2 and Theorem 0.3. In Section 4, we study Carleman inequalities for the operators in wider classes S and D (see definitions in Section 1) and prove Theorem 0.5. In Section 5, we give several counterexamples above Carleman inequalities for general operators. Finally we would like to give some general discussions about the simple characteristics and state some further questions in Section 6.

1 Several classes of higher order elliptic differential operators

Let P be a homogeneous polynomial of degree m in d variables. We call $N^P = \{z = \xi + ik \in \boldsymbol{C}^d \setminus \{0\} : P(z) = 0\}$ the characteristic variety. For any fixed k, we denote by $\pi_k(N^P) = \{\xi \in \boldsymbol{R}^d : \xi + ik \in N^P\}$ the zero set of $P(\cdot + ik)$ in $\boldsymbol{R}^d$. In this paper, we are always interested in the case when P has real coefficients and the related differential operator $\boldsymbol{P}$ is elliptic, i.e., $N^P \cap (R^d + i0) = \emptyset$. We call such P an elliptic homogeneous polynomial. Let's first define two classes of such polynomials (operators) which we will study later.

Definition 1.1 *(1) Let S be the set of all elliptic homogeneous polynomials P satisfying $\frac{dP}{dz} = (\frac{dP}{dz_1}, \cdots, \frac{dP}{dz_d}) \neq 0$ on N^P. S is called the simple characteristic class and if $P \in S$, we say P has simple characteristics.*
(2) Let D be the set of all elliptic homogeneous polynomials P such that there is a nonempty open subset K of $\boldsymbol{R}^d \setminus 0$ and a constant C such that the following hold:
(D1) $|P(\xi + ik)| \geq C\, dist\, (\xi, \pi_k(N^P))^2$ *for all* ξ and all $k \in K$.
(D2) $N_K^P = \{(\xi, k) \in \boldsymbol{R}^d \times K : P(\xi + ik) = 0\}$ is a submanifold of codimension 2 in $\boldsymbol{R}^d \times \boldsymbol{R}^d$ and N_K^P and $\Pi_k = \{(\xi, l) \in \boldsymbol{R}^d \times \boldsymbol{R}^d : l = k\}$ are transverse in $\boldsymbol{R}^d \times \boldsymbol{R}^d$ for all $k \in K$. D is called the double characteristic class and if $P \in D$, we say P has double characteristics.

Our class S is also called Hormander's nonsingular characteristics class, which contains the elliptic homogeneous polynomials which have the Calderon's simple characteristics (see Definition 1.8). In the last section, we will prove the fact that generic elliptic homogeneous polynomials are in class S (see Proposition 6.2). When $P \in S$, we have a property that $\nabla_\xi \mathrm{re}P(\xi + ik)$ and $\nabla_\xi \mathrm{im}P(\xi + ik)$ are linearly independent on $\pi_k(N^P)$ for generic k, which implies that $\pi_k(N^P)$ is a submanifold of codimension 2. This is a direct conclusion of Cauchy-Riemann equation and the transversality theorem (see page 68 in [4]) because $P(\xi + ik)$ is harmonic in (ξ, k) variables. We also will give a short proof for this in the last section (see Proposition 6.4). In fact it is also very easy to check that the set of k which satisfies the above property is open. An example in the class S is $\frac{d^m}{dx_1^m} + \cdots + \frac{d^m}{dx_d^m}$.

We now discuss the conditions in the definition of D, In the last section, we will show that if $P \in S$ then the submanifolds N^P and Π_k are transverse in $R^d \times R^d$ for generic $k \in R^d$ (see Proposition 6.3). So if $P \in S$ with order $\frac{m}{2}$, then $P^2 \in D$ with order m. And in this case, both (D1) and (D2) hold for generic k. A typical example is the bi-Laplacian operator $\Delta^2 \in D$ with order 4.

As we mentioned in the introduction, in order to get a sharp Carleman type inequality for a higher order elliptic differential operator $\boldsymbol{P}$, one needs some curvature conditions on the intersection $\pi_k(N^P)$ of the characteristic variety. We know that for generic k, $\pi_k(N^P)$ is a $d-2$ dimensional submanifold. There are two natural $d-1$ dimensional hypersurfaces $\mathrm{re}P(\cdot + ik)^{-1}(0)$ and

$\mathrm{im}P(\cdot + ik)^{-1}(0)$ which contain $\pi_k(N^P)$. In general these two hypersurfaces don't satisfy good curvature conditions, say nonzero Gaussian curvature. But on the other hand, the singular points of oscillatory integral related P which we are interested in only occur on $\pi_k(N^P)$ so that we may look for a third hypersurface other than those special two, which may have a nice curvature condition and contains $\pi_k(N^P)$. This is the role of Proposition 0.1.

Now let's start with some basic facts from geometry and algebra which we will use to find a class of operators for which we will be able to prove sharp Carleman inequalities in the next section.

Lemma 1.2 *Suppose f is a real smooth function on $\boldsymbol{R}^d$ with $f(a) = 0$ and $\nabla f(a) \neq 0$ for a point $a \in \boldsymbol{R}^d$. Let $S = f^{-1}(0)$. Then the following are equivalent:*

(1) S has nonzero Gaussian curvature at a.

(2) $PH_f(a)|_T$ has rank $d-1$,

where T is the linear subspace $(\nabla f(a))^{\perp}$, P is the orthogonal projection onto T and H_f is the Hessian matrix of f at a.

Proof: First we need to show that (2) is independent of f, i.e., if we have another function g with $\nabla g(a) \neq 0$ such that $f = 0$ if and only if $g = 0$, then we need to prove (2) is true for g if it is true for f. In fact, by the following Lemma , which is independent of this lemma, there is a function h with $h(a) \neq 0$ such that $g = h \cdot f$. It is easy to see that the linear subspace $T_g = (\nabla g(a))^{\perp}$ is same as T. And the Hessian matrix of g at a is $H_g(a) = h(a)H_f(a) + \nabla f(a) \otimes \nabla h(a) + \nabla h(a) \otimes \nabla f(a)$. Where $\otimes$ is as usual defined as following. If x, y and are three real vectors in $\boldsymbol{R}^d$, then $x \otimes y$ is a matrix such that $(x \otimes y)z =< z, y > x$. So $P_{T_g} H_g(a)|_{T_g} = PH_f(a)|_T$.

Now let's show our lemma. After a rotation and translation, we may assume that $a = 0$ and with $\xi = (\xi_1, \bar{\xi})$, $f(\xi) = \xi_1 - \xi_1(\bar{\xi})$ where $\xi_1(0) = \frac{d}{d\bar{\xi}}\xi_1(0) = 0$. Hence locally S is the graph of function $\xi_1(\bar{\xi})$. We know S having nonzero Gaussian curvature at $a = 0$ is equivalent to Hessian matrix of $\xi_1(\bar{\xi})$ having maximum rank at $\bar{\xi} = 0$. So (2) is true if and only (1) is true.

♯

Lemma 1.3 *Suppose that f_1 and f_2 are two smooth functions on a neighborhood U of some point $a \in \boldsymbol{R}^d$ with $f_1(a) = 0$ and $f_2(a) = 0$ such that $\bigtriangledown f_1(a)$ and $\bigtriangledown f_2(a)$ are linearly independent. If a function f with $f(a) = 0$ is such that $f^{-1}(0)$ contains $f_1^{-1}(0) \cap f_2^{-1}(0)$ locally near a, then there are two smooth functions g_1 and g_2 on another neighborhood U' of a such that $f = g_1 f_1 + g_2 f_2$ near a.*

Remark: This lemma is true in general for any number of functions instead of two functions.

Proof: Let's first prove the lemma in the simple case where $a = 0$, $f_1(\xi) = \xi_1$ and $f_2(\xi) = \xi_2$. Actually one may write

$$f(\xi) = \int_0^1 \frac{d}{dt} f(t\xi_1, t\xi_2, \xi_3, \cdots, \xi_d)dt$$

$$= \xi_1 \cdot \int_0^1 \frac{df}{d\xi_1}(t\xi_1, t\xi_2, \xi_3, \cdots, \xi_d)dt + \xi_2 \cdot \int_0^1 \frac{df}{d\xi_2}(t\xi_1, t\xi_2, \xi_3, \cdots, \xi_d)dt$$

on a small ball centered at 0.

In the general case, there are smooth functions $f_3, \cdots, f_d$ such that $\nabla f_1(a)$, $\cdots$, $\nabla f_d(a)$ are linearly independent and $f_3(a) = \cdots = f_d(a) = 0$. So $F = (f_1, \cdots, f_d)$ is a diffeomorphism from a neighborhood of a to a neighborhood of 0 by the inverse function theorem. Let $G = F^{-1}$ and let $\phi = f \circ G$. Note that $f_i \circ G(\xi) = \xi_i$. Then if $\xi_1 = \xi_2 = 0$ then $f_1(G(\xi)) = f_2(G(\xi)) = 0$ so that $f(G(\xi)) = 0$, i.e., $\phi(\xi) = 0$. So by the simple case, $\phi(\xi) = \xi_1\psi_1(\xi) + \xi_2\psi_2(\xi)$ with smooth functions ψ_i and therefore $f(\xi) = \phi(F(\xi)) = f_1(\xi)\psi_1(F(\xi)) + f_2(\xi)\psi_2(F(\xi))$. We finish the proof by taking $g_i(\xi) = \psi_i(F(\xi))$. $\sharp$

Lemma 1.4 *Let V is a real linear space of dimension r. Suppose K is an $r \times r$ symmetric matrix of rank $r-1$ on V. Let e be a vector of V. For $x \in V$, define $A_x = K + x \otimes e + e \otimes x$. Then*

(1) If $e \notin \mathrm{Im}K$, then there is an x such that $\mathrm{rank}A_x = r$.

(2) If $e \in \mathrm{Im}K$ and $e = Kf$ for some $f \in V$, then

$\mathrm{rank}A_x \leq r-1$ *for any $x \in V$, when* $< f, Kf >= 0$.

$\mathrm{rank}A_x = r$ *for some $x \in V$, when* $< f, Kf >\neq 0$.

Note: We notice that the element f in (2) is not unique. But it is easy to see the inner product $< f, Kf >$ is independent of the choice of f. In fact, suppose g is another element such that $e = Kg$. Then $< f, Kf >=< g, Kg > + < f-g, Kg >=< g, Kg >$ since $f - g \in \mathrm{Ker}K$.

Proof: (1) Without loss of generality, we may assume that under an orthogonal bases $\{e_1, \cdots, e_r\}$, $\{Ke_1, \cdots, Ke_r\} = \{0, c_2e_2, \cdots, c_re_r\}$ with $c_2, \cdots, c_r$ nonzeros since K is assumed to be of rank $r-1$. The assumption $e \notin \mathrm{Im}K$

means $< e, e_1 > \neq 0$. Let $x = e_r$ Then $A_{e_r}\{e_1, e_2, \cdots, e_{r-1}, e_r\}$ is

$$\{< e, e_1 > e_r, \quad c_2e_2 + < e, e_2 > e_r, \quad \cdots, \quad c_{r-1}e_{r-1} + < e, e_{r-1} > e_r,$$

$$(c_r + 2 < e, e_r >)e_r + \sum_{j=2}^{r-1} < e, e_j > e_j + < e, e_1 > e_1\}$$

which forms another basis and hence A_{e_r} has rank r.

(2) Now let's assume $e = Kf$.

Case 1: $< f, Kf >= 0$. Let $a \in (\mathrm{Im}K)^{\perp}$.

Claim: a and f are linearly independent.

Proof: Suppose for some constants s and t, $sa+tf = 0$. Then $sKa+tKf = 0$. Since K is symmetric, $Ka = 0$. Hence $t = 0$ and so $s = 0$.

Now let's compute A_xa and A_xf for any fixed $x \in V$.

$$\begin{aligned} A_xa &= Ka + < e, a > x + < x, a > e \\ &= 0 + < Kf, a > x + < x, a > Kf \\ &= < x, a > Kf \end{aligned}$$

$$\begin{aligned} A_xf &= Kf + < e, f > x + < x, f > e \\ &= (1 + < x, f >)Kf \end{aligned}$$

since $< e, f >=< Kf, f >= 0$. So for any given x there are nonzero numbers s and t such that $A_x(sa + tf) = 0$. Since $sa + tf \neq 0$ by claim, this means $\mathrm{rank}A_x \leq r - 1$.

Case 2: $< f, Kf > \neq 0$. Let $a \in (\mathrm{Im}K)^{\perp}$ and choose $x = a$. Let's denote $A = A_a$. Suppose that $\{u_3, \cdots, u_r\}$ together with $\{a, Kf\}$ is a basis of V and $u_3, \cdots, u_r \in (\mathrm{Span}\{a, Kf\})^{\perp}$.

Claim: $\{a,\ f,\ u_3, \cdots, u_r\}$ is a basis of V.

Proof: Suppose there are constants $s,\ t,\ s_3, \cdots, s_r$ such that $sa + tf + \sum_{j=3}^{r} s_j u_j = 0$. By the same reason as in case 1, $Ka = 0$. So $tKf + \sum s_j K u_j = 0$. Consider the inner product of $tKf + \sum s_j K u_j$ and f. Since $< f, Ku_j >= 0$ by the choice of u_j, $t < f, Kf >= 0$ which implies $t = 0$. As $a,\ u_3, \cdots, u_r$ are linearly independent by the choice of u_j's, $s = s_3 = \cdots = s_r = 0$. This proves claim.

Now let's compute the image of another basis $\{a,\ f - \frac{1+<a,f>}{<a,a>}a,\ u_3, \cdots, u_r\}$ under A. Remember $e = Kf$, $< u_j, a >=< u_j, Kf >= 0$ and $< a, Kf >= 0$.

$$
\begin{aligned}
Aa &= Ka + < e, a > a + < a, a > e &&=< a, a > Kf \\
Au_3 &= Ku_3 + < u_3, a > Kf + < u_3, Kf > a &&= Ku_3 \\
&\cdots \\
Au_r &= Ku_3 + < u_r, a > Kf + < u_r, Kf > a &&= Ku_r
\end{aligned}
$$

On the other hand, we have $A(f - \frac{1+<a,f>}{<a,a>}a) = Kf + < e, f > a + < a, f > e - (1 + < a, f >)Kf =< Kf, f > a$. By the assumption, $\mathrm{Span}\{Kf,\ 0 = Ka,\ Ku_3, \cdots, Ku_r\} = \mathrm{Im}K$ has dimension $r - 1$. Since $a \in (\mathrm{Im}K)^{\perp}$, we have

$$\mathrm{Span}\{\mathrm{Im}K, a\} = V$$

So because $< a, a >\neq 0$ and $< Kf, f >\neq 0$, we have

$$\mathrm{Span}\{Aa,\ A(f - \frac{1+ < a, f >}{< a, a >}a),\ Au_3, \cdots, Au_r\} = V$$

This means $\mathrm{rank}A = r$. $\sharp$

Let's now come back to discuss some geometric properties of zero sets of elliptic homogeneous polynomials. In the rest of this section, we will assume that P is an elliptic homogeneous polynomial. We use $\bigtriangledown$ to denote the gradient operator for functions on $\boldsymbol{R}^d$ with respect to the real part variable ξ of $z = \xi + ik \in \boldsymbol{C}^d$ while k is fixed. Let $z^0 = \xi^0 + ik^0$ be a zero point of P. Let's further assume that $\sum_{j=1}^{d} \frac{dP}{dz_j}(z^0) \cdot \mathrm{Im} z_j^0 \neq 0$, which says that P satisfies Calderon's simple characteristic condition at z^0, and the complex Hessian matrix $H_{\boldsymbol{C},P} = (\frac{d^2}{dz_j dz_l} P)$ of P is nonsingular at z^0. The Calderon's condition above immediately implies that $\bigtriangledown \mathrm{re} P(z^0)$ and $\bigtriangledown \mathrm{im} P(z^0)$ are linearly independent because P is homogeneous polynomial.

We will usually omit z^0 in what follows and always keep in mind that every function will be evaluated at z^0. Finally we introduce some notations. For $t \in \boldsymbol{C}$, we denote $H = H(t) = H_{\mathrm{re}P(\cdot+ik^0)+t\mathrm{im}P(\cdot+ik^0)}(\xi^0) = H_{\mathrm{re}P(\cdot+ik^0)}(\xi^0) + tH_{\mathrm{im}P(\cdot+ik^0)}(\xi^0)$, the Hessian matrix of $\mathrm{re}P + t\mathrm{im}P$ with respect to the ξ variable at z^0. By the Cauchy-Riemann equations, we know $H(i) = H_{\boldsymbol{C},P}(z^0)$. Let $T = (\bigtriangledown \mathrm{re} P)^{\perp} \cap (\bigtriangledown \mathrm{im} P)^{\perp}$ be the (real) linear subspace of codimension 2 in $\boldsymbol{R}^d$ and $T_1 = (\bigtriangledown \mathrm{re} P + t \bigtriangledown \mathrm{im} P)^{\perp}$ a linear subspace of codimension 1 in $\boldsymbol{R}^d$ when t is fixed in $\boldsymbol{R}$. P_{T_1} is the orthogonal projection onto T_1.

Lemma 1.5 *For all t in $\boldsymbol{R}$ except finitely many points, $H(t)$ is nonsingular.*

Proof: $\det H(t)$ is a polynomial in the t variable and $\det H(i) = \det H_{\boldsymbol{C},P}(z^0) \neq 0$. So by the fundamental theorem of algebra, $\det H(t)$ has only finitely many zeros. $\sharp$

Lemma 1.6 *Let $a(t)$ and $b(t)$ be any two rational functions on $\boldsymbol{R}$ with nonzero real coefficients. Then for any t in some dense subset of $\boldsymbol{R}$,*

$$a(t) \bigtriangledown \mathrm{re}P + b(t) \bigtriangledown \mathrm{im}P \notin \mathrm{Im}H_1|_T$$

where $H_1 = H_1(t) = P_{T_1} H(t)|_{T_1}$.

Proof: Let $P' = \frac{dP}{dz}$ which is $\bigtriangledown \mathrm{re}P + i \bigtriangledown \mathrm{im}P$, and let $\bar{P}' = \bigtriangledown \mathrm{re}P - i \bigtriangledown \mathrm{im}P$. We use $<\ ,\ >$ to denote the inner product in C^d. Since P and P' are homogeneous functions, we have $\sum \frac{dP}{dz_j} \cdot z_j^0 = mP(z^0)$ which is zero, and $H(i)z^0 = \left(\sum_{j=1}^d \frac{d}{dz_j}(\frac{dP}{dz_1}) \cdot z_j^0, \cdots, \sum_{j=1}^d \frac{d}{dz_j}(\frac{dP}{dz_d}) \cdot z_j^0\right) = (m-1)(\frac{dP}{dz_1}, \cdots, \frac{dP}{dz_d})$ $= (m-1)P'$. So we have the following two formulas:

$$(*) \qquad \sum_{j=1}^{d} \frac{dP}{dz_j} \cdot z_j^0 = 0$$

$$(**) \qquad H(i)^{-1}P' = \frac{1}{m-1} z^0.$$

Now let's assume our conclusion is false. That means for each t in some nonempty open subset of R, there is a $u \in T$ such that $a(t) \bigtriangledown \mathrm{re}P + b(t) \bigtriangledown \mathrm{im}P = H_1 u$, i.e.,

$$a(t) \bigtriangledown \mathrm{re}P + b(t) \bigtriangledown \mathrm{im}P =$$

$$= Hu - \frac{< Hu, \bigtriangledown \mathrm{re}P + t \bigtriangledown \mathrm{im}P >}{< \bigtriangledown \mathrm{re}P + t \bigtriangledown \mathrm{im}P, \bigtriangledown \mathrm{re}P + t \bigtriangledown \mathrm{im}P >}(\bigtriangledown \mathrm{re}P + t \bigtriangledown \mathrm{im}P).$$

This shows that u is a solution of a linear system which depends on t in polynomial sense. In other word, we may write the above equations into a usual form, $\boldsymbol{A}(t)u = \boldsymbol{f}(t)$, where $\boldsymbol{A}(t)$ and $\boldsymbol{f}(t)$ are corresponding matrix and vector, respectively from the above system, which elements are rational functions of the t. So the basic linear algebra theory tells us there is another u

which elements are rational functions of t such that the above system satisfies also, i.e.,

$$a(t) \bigtriangledown \mathrm{re}P + b(t) \bigtriangledown \mathrm{im}P = Hu + \alpha(t)(\bigtriangledown \mathrm{re}P + t \bigtriangledown \mathrm{im}P)$$

where $\alpha(t) = -\frac{<Hu, \bigtriangledown \mathrm{re}P + t\bigtriangledown \mathrm{im}P>}{<\bigtriangledown \mathrm{re}P + t\bigtriangledown \mathrm{im}P, \bigtriangledown \mathrm{re}P + t\bigtriangledown \mathrm{im}P>}$ is a rational function of t. Let $R(t)$ be the covariant matrix of $H(t)$, which depends on t in polynomial sense. (Notice that when we substitute t by i, $R(i) = \det H(i) \cdot H(i)^{-1}$). So the above formula becomes

$$R(t)[(a(t) - \alpha(t)) \bigtriangledown \mathrm{re}P + (b(t) - t\alpha(t)) \bigtriangledown \mathrm{im}P] = \det H(t) \cdot u.$$

If $a(t) - \alpha(t)$ and $b(t) - t\alpha(t)$ are the zero functions, then we have $a(t) \bigtriangledown \mathrm{re}P + b(t) \bigtriangledown \mathrm{im}P = \alpha(t)(\bigtriangledown \mathrm{re}P + t \bigtriangledown \mathrm{im}P) \in T_1^{\perp}$ which leads to the conclusion of Lemma 1.6 trivially. So we assume one of such two functions is not the zero function. Let's say $a(t) - \alpha(t)$ is not zero function. So we have

$$R(t)(\bigtriangledown \mathrm{re}P + c(t) \bigtriangledown \mathrm{im}P) = \frac{\det H(t)}{a(t) - \alpha(t)} \cdot u$$

where $c(t) = \frac{b(t) - t\alpha(t)}{a(t) - \alpha(t)}$ is a rational function of t. Since $u \in T$, we have the following two formulas by the above:

$$(***)_{\pm} \quad < R(t)(\bigtriangledown \mathrm{re}P + c(t) \bigtriangledown \mathrm{im}P), \bigtriangledown \mathrm{re}P \pm i \bigtriangledown \mathrm{im}P >= 0.$$

These two formulas are true for all t in a smaller nonempty open subset of R so that they will be extended to be true for all t in $\boldsymbol{C}$ except for poles of $c(t)$. Let's discuss the following possible cases.

Case 1: $c(i) \neq \infty, \neq i$. Then let $t \to i$ in $(***)_-$. We get

$$< R(i)(\bigtriangledown \mathrm{re}P + c(i) \bigtriangledown \mathrm{im}P), \bigtriangledown \mathrm{re}P - i \bigtriangledown \mathrm{im}P >= 0.$$

Substitute $\bigtriangledown \mathrm{re}P = \frac{1}{2}(P' + \bar{P'})$, $\bigtriangledown \mathrm{im}P = \frac{1}{2i}(P' - \bar{P'})$ into the above formula and notice that $R(i) = \det H(i) \cdot H(i)^{-1}$ with $\det H(i) \neq 0$. We have

$$\left(\frac{1}{2} + \frac{c(i)}{2i}\right) < H(i)^{-1}P', \bar{P'} > + \left(\frac{1}{2} - \frac{c(i)}{2i}\right) < H(i)^{-1}\bar{P'}, \bar{P'} >= 0.$$

Since $< H(i)^{-1}P', \bar{P'} >=< \frac{1}{m-1}z^0, \bar{P'} > = \frac{1}{m-1}\sum \frac{dP}{dz_j} \cdot z_j^0 = 0$ by $(*)$ and $(**)$, the above formula becomes

$$< H(i)^{-1}\bar{P'}, \bar{P'} >= 0$$

as $\frac{1}{2} - \frac{c(i)}{2i} \neq 0$. $H(i)^{-1}$ is symmetric, so using $(**)$ we have $< H(i)^{-1}\bar{P'}, \bar{P'} >=$ $< \bar{P'}, \bar{H(i)}^{-1}\bar{P'} >=< \bar{P'}, \frac{1}{m-1}\bar{z^0} >$. Then

$$\sum \frac{dP}{dz_j} \cdot \bar{z_j^0} = 0.$$

Combine this formula with $(*)$, we get

$$\sum_{j=1}^{d} \frac{dP}{dz_j} \cdot \mathrm{Im} z_j^0 = 0.$$

This is a contradiction with the assumption.

Case 2: $c(i) = i$. Let $t \to i$ in $(***)_+$. By the same process as in case 1, we may get

$$\left(\frac{1}{2} + \frac{c(i)}{2i}\right) < H(i)^{-1}P', P' > + \left(\frac{1}{2} - \frac{c(i)}{2i}\right) < H(i)^{-1}\bar{P'}, P' >= 0,$$

i.e., $< H(i)^{-1}P', P' >= 0$. This is, by using $(**)$,

$$\sum \frac{dP}{dz_j} \cdot \bar{z_j^0} = 0$$

which will lead to a contradiction as in case 1.

Case 3: $c(i) = \infty$. As c is a rational function, we may write $c(t) = (t-i)^{-k}d(t)$

with some integer $k \geq 1$ and another rational function $d(t)$ which is finite at i. So multiply $(t-i)^k$ in the both sides of $(***)_-$ and let $t \to i$. The result is

$$< R(i) \bigtriangledown \mathrm{im}P, \bar{P'} >= 0$$

or

$$\frac{1}{2} < R(i)P', \bar{P'} > -\frac{1}{2i} < R(i)\bar{P'}, \bar{P'} >= 0.$$

This will lead to a contradiction as in case 1.

So we prove our lemma. $\sharp$

Lemma 1.7 *Let's keep the assumptions as before. Let* $e(t) = t \bigtriangledown \mathrm{re}P - \bigtriangledown \mathrm{im}P - \frac{<t\bigtriangledown \mathrm{re}P - \bigtriangledown \mathrm{im}P, \bigtriangledown \mathrm{re}P + t\bigtriangledown \mathrm{im}P>}{<\bigtriangledown \mathrm{re}P + t\bigtriangledown \mathrm{im}P, \bigtriangledown \mathrm{re}P + t\bigtriangledown \mathrm{im}P>}(\bigtriangledown \mathrm{re}P + t \bigtriangledown \mathrm{im}P)$ *be the projection of vector* $t \bigtriangledown \mathrm{re}P - \bigtriangledown \mathrm{im}P$ *onto the subspace* $(\mathrm{Span}\{\bigtriangledown \mathrm{re}P + t \bigtriangledown \mathrm{im}P\})^{\perp}$. *Then for any t in $\boldsymbol{R}$ except finitely many points, there is an $x \in T_1$ such that*

$$\mathrm{rank}(P_{T_1}H(t) + e(t) \otimes x + x \otimes e(t))|_{T_1} = d - 1.$$

Proof: First notice that for any $t \in \boldsymbol{R}$, $e(t) \in T_1$. Moreover, $e(t) \notin T$ because $\bigtriangledown \mathrm{re}P$ and $\bigtriangledown \mathrm{im}P$ are linearly independent. So we have $T_1 = \mathrm{Span}(T, e(t))$.

As $e(t)$ is a linear combination of $\bigtriangledown \mathrm{re}P$ and $\bigtriangledown \mathrm{im}P$ with rational functions, which are not identity zero, as coefficients, Lemma 1.6 applies to $e(t)$. So let $t \in \boldsymbol{R}$ be such that Lemma 1.6 and Lemma 1.5 hold. We have $e(t) \notin \mathrm{Im}H_1|_T$ and $\mathrm{rank}H = d$, where $H_1 = P_{T_1}H(t)|_{T_1}$ and $H = H(t)$. By the definition of H_1, $\mathrm{rank}H_1 \geq d-2$ since $\mathrm{rank}H = d$. If $\mathrm{rank}H_1 = d-1$, then we are done by taking $x = 0$. Now let's assume $\mathrm{rank}H_1 = d-2 = (d-1)-1$. We will apply Lemma 1.4 to $V = T_1$, $K = H_1$, $r = d-1$ and $e = e(t) \in T_1$.

Case 1: $e(t) \notin \mathrm{Im}H_1$. Then part (1) of Lemma 1.4 implies our conclusion is true, i.e., there is an $x \in T_1$ such that

$$\mathrm{rank}(P_{T_1}H(t) + e(t) \otimes x + x \otimes e(t))|_{T_1} = d - 1.$$

Case 2: $e(t) \in \mathrm{Im}H_1$. So we may assume there are a $u \in T$ and a constant $c \in R$ such that

$$e(t) = H_1(u + ce(t))$$

since $T_1 = \mathrm{Span}(T, e(t))$. As $u \in T$, $u \perp e(t)$. So $< e(t), u + ce(t) >= c < e(t), e(t) >$ which is zero if and only if $c = 0$. If $c \neq 0$, then part (2) of Lemma 1.4 implies our conclusion is true. If $c = 0$, then $e(t) = H_1 u \in \mathrm{Im}H_1|_T$ which is a contradiction with $e(t) \notin \mathrm{Im}H_1|_T$. This proves the lemma. ♯

Now let's define our class of operators which we will study in the next section. Notice that the assumption $\sum_{j=1}^{d} \frac{dP}{dz_j}(z^0) \cdot \mathrm{im} z_j^0 \neq 0$ is nothing but the Calderon's simple characteristic condition ([3, 5]). So we have the following natural definition.

Definition 1.8 *Let G be the set of all such elliptic homogeneous polynomials P that there is a $k \in S^{d-1}$ such that for any $\xi \in \pi_k(N^P)$ the following conditions are satisfied:*
(G1) (Calderon's simple characteristic condition) $\sum_{j=1}^{d} k_j \frac{dP}{dz_j}(\xi + ik) \neq 0$
(G2) (Curvature condition) $D(P)(\xi + ik) \neq 0$.
where $D(P)$ is the determinant of the complex Hessian matrix of P, which is also a homogeneous polynomial.

Remark 1.1 We notice that the conditions (G1) and (G2) have open property, i.e., there is an open neighborhood K of k in S^{d-1} such that for any other $k' \in K$, (G1) and (G2) hold too. This is because P is elliptic. On the other hand, the condition (G1) implies that for such k in (G1), $\nabla \mathrm{re}P(\xi+ik)$ and $\nabla \mathrm{im}P(\xi+ik)$ are linearly independent for all $\xi \in \pi_k(N^P)$ since P is homogeneous. This property is also open. So we conclude that if $P \in G$ with some $k \in S^{d-1}$, then there is a neighborhood K of k in S^{d-1} such that for each $k' \in K$, (G1) and (G2) hold and $\pi_{k'}(N^P)$ is a smooth submanifold of dimension $d-2$.

Remark 1.2 The condition (G2) is not trivial. When $d \geq 3$ we know, by a basic fact of algebraic geometry (Lemma 1.3 on p29 in [8]), that $N^P \cap N^Q \neq \emptyset$ for any two homogeneous polynomials P and Q. So it is impossible for (G2) to hold for all $k \in S^{d-1}$, (G2) holds.

We have a lot of homogeneous polynomials in the class G. In particular, if a P satisfies the Calderon's simple characteristic condition and the Hessian matrix $(\frac{d^2 \mathrm{re}P(\xi+ik)}{d\xi_j \xi_l})$ of the real part of P has full rank for all $\xi \in \pi_k(N^P)$ and for some k, then $P \in G$. on the other hand, there are also a lot of operators, for which we don't know if the above condition works, which are in G.

Examples 1.3 (1) $\frac{d^m}{dx_1^m} + \cdots + \frac{d^m}{dx_d^m} \in G$. Its symbol is $P = z_1^m + \cdots + z_d^m$. When $k = e_1 = (1, 0, \cdots, 0)$, $\sum_{j=1}^d \frac{dP}{dz_j} \cdot k_j = m(\xi_1 + i)^{m-1} \neq 0$ for all $\xi \in \boldsymbol{R}^d$. So (G1) holds for this k. When $k = (k_1, \cdots, k_d)$ with $k_j \neq 0$, $j = 1, \cdots, d$,

$D(P)(\xi + ik) \neq 0$ for any $\xi \in \boldsymbol{R}^d$. Then by Remark 1.1 it is easy to show that there is a small open subset K of S^{d-1} close to e_1 ($e_1 \notin K$) such that (G1) and (G2) hold and $\pi_k(N^P)$ is smooth submanifold of dimension $d-2$ for all $k \in K$.

(2) $P(D) = \frac{d^m}{dx_1^m} + Q(\frac{d}{dx_2}, \cdots, \frac{d}{dx_d}) \in G$ for any homogeneous polynomial Q of degree m in $d-1$ variables such that the determinant of the Hessian matrix $\det H_Q(\bar{\xi})$ of Q in $\bar{\xi} = (\xi_2, \cdots, \xi_d)$ variable is an elliptic polynomial, i.e., Q is a strictly convex homogeneous polynomial. In fact, let $k = e_1$. $\nabla P \cdot k = m(\xi_1 + i)^{m-1} \neq 0$ for any $\xi \in R^d$. On the other hand, if $\xi \in \pi_k(N^P)$, then $\mathrm{im}(\xi_1 + i)^m = \mathrm{im}P(\xi + ik) = 0$. So $\mathrm{re}(\xi_1 + i)^m \neq 0$. Hence $Q(\bar{\xi}) \neq 0$ and so $\bar{\xi} \neq 0$ because Q is elliptic. So by the assumption, $D(P)(\xi + ik) = m(m-1)(\xi_1 + i)^{m-2} \det H_Q(\bar{\xi}) \neq 0$. Hence, $P \in G$.

Finally before we end this section, let's prove Proposition 0.1 stated in Introduction.

Proof of Proposition 0.1: Let $P \in G$. Remark 1.1 tells us that there is an open subset K of S^{d-1} such that for each $k^0 \in K$ and any $\xi^0 \in \pi_{k^0}(N^P)$, the conditions (G1) and (G2) hold and $\nabla \mathrm{re}P(\xi^0 + ik^0)$ and $\nabla \mathrm{im}P(\xi^0 + ik^0)$ are linearly independent. These conditions are all requested in Lemma 1.7. Let x and t be as in Lemma 1.7, and let $f(\xi) = \mathrm{re}P(\xi + ik^0) + t\mathrm{im}P(\xi + ik^0) + < x, \xi - \xi^0 > [t\mathrm{re}P(\xi + ik^0) - \mathrm{im}P(\xi + ik^0) - c(\mathrm{re}P(\xi + ik^0) + t\mathrm{im}P(\xi + ik^0))]$, where $c = \frac{<t\nabla \mathrm{re}P(z^0) - \nabla \mathrm{im}P(z^0), \nabla \mathrm{re}P(z^0) + t\nabla \mathrm{im}P(z^0)>}{\nabla \mathrm{re}P(z^0) + t\nabla \mathrm{im}P(z^0), \nabla \mathrm{re}P(z^0) + t\nabla \mathrm{im}P(z^0)>}$ is a constant. Then the

Hessian matrix of f at ξ^0 is

$$H_f = H_{\text{re}P+t\text{im}P} + x \otimes e(t) + e(t) \otimes x$$

where $e(t)$ is as in Lemma 1.7. Lemma 1.7 says $P_{T_1} H_f|_{T_1}$ has rank $d-1$. So by Lemma 1.2, $f^{-1}(0)$ has nonzero Gaussian curvature at ξ^0. ♯

Remark 1.4 Notice that if we define a function $f(\xi, k)$ by substituting k_0 by k in the function f in the above proof with t and x fixed, then $f(\xi, k)$ is continuous in both ξ and k and $f(\xi, k_0) = f(\xi)$. This shows that there is a small open ball U containing ξ_0 in R^d such that for any k sufficiently close to k_0, $\pi_k(N^P) \cap U$ may be contained in a hypersurface with nonzero Gaussian curvature which is bigger than or equal to half of the Gaussian curvature of $f^{-1}(0)$ at ξ_0.

2 Carleman inequalities with linear weights for the operators in class G

We will prove a sharp Carleman inequality for higher order elliptic differential operators of class G in this section. First let's introduce some notations. Let $a = (\frac{1}{s}, 0)$, $b = (1, 0)$, $c = (1, \frac{1}{2})$ and $d = (\frac{1}{s}, \frac{1}{s'})$ where s is always the number $\frac{2(d+1)}{d+3}$ with s' its conjugate number. Let A be a subset of $\boldsymbol{R}^2$ consisting of the quadrilateral $abcd$ and two sides ad and bc. Let μ be a positive integer. Let $A_\mu = \{(x, y) \in R^2 : 0 < y \leq x \leq 1, x - y \leq \frac{\mu}{d}\}$. Then we know when $\mu \geq 2$, $A \cap A_\mu$ is nonempty. Now let's state a technical lemma which will be useful in proving our theorem.

Lemma 2.1 *Let H be a piece of smooth hypersurface in $\boldsymbol{R}^d$ with nonzero Gaussian curvature and $N \subset H$ be a $d-2$ dimensional submanifold. Suppose that $m(\xi)$ is a smooth function on $\boldsymbol{R}^d$ satisfying $|m(\xi)| \leq \frac{C}{dist(\xi, N)}$ for all $\xi \in \boldsymbol{R}^d$ with some positive constant C. Then for any (p, q) with $(\frac{1}{p}, \frac{1}{q}) \in A$ and each $\xi_0 \in N$ there are a small neighborhood U of ξ_0 in $\boldsymbol{R}^d$ and a constant C such that*

$$\|T_m f\|_q \leq C\|f\|_p \qquad \forall\, f \in \mathcal{S} \text{ with } supp\hat{f} \subset U$$

where $T_m f = (m\hat{f})^\vee$.

Proof: Let n be a normal vector of H at ξ_0. By the Tubular neighborhood theorem (see page 76 in [4]), there is a neighborhood U of ξ_0 which may be written as $U = (-\delta, \delta) \times H = \bigcup_{t \in (-\delta, \delta)} H_t$ for some small $\delta > 0$, where H_t is the translation of $H \cap U$ by distance t along the direction n. Moreover, since $m(\xi)$ satisfies $|m(\xi)| \leq \frac{C}{\mathrm{dist}(\xi,N)}$ and $N \cap U$ is a $d-2$ dimensional submanifold, we have the following estimates by choosing U small enough:

$$\begin{aligned} \int_{-\delta}^{\delta} \|m\|_{L^r(H_t)} dt < \infty & \qquad \forall\, r \in (0,\infty) \\ \|m\|_{L^r(U)} < \infty & \qquad \forall\, r \in (0,2). \end{aligned}$$

In fact, after making U small enough, there is a diffeomorphism $F : U \to B(0,1) \subset \boldsymbol{R}^d = \{(x,s,t) : x \in \boldsymbol{R}^{d-2}, s \in \boldsymbol{R}, t \in \boldsymbol{R}\}$ such that $F(\xi_0) = 0$, $FH \subset \{(x,s,0) \in B(0,1)\}$, $FN \subset \{(x,0,0) : x \in \boldsymbol{R}^{d-2}\}$ and $FH_t = FH + (0,0,t)$, $FN_t = FN + (0,0,t)$. This is because $H \cap U$ is a graph when U is small enough and $N \subset H$ is a submanifold of codimension 2. We notice that when $\xi \in H_t$, i.e., $F(\xi) \in FH_t$, $\mathrm{dist}(\xi, N) \approx \mathrm{dist}(F(\xi), FN) \approx |t| + |s|$ by the triangle formula. So $\int_{-\delta}^{\delta} \|m\|_{L^r(H_t)} dt = \int_{-\delta}^{\delta} \|m \circ F^{-1}\|_{L^r(FH_t)} dt \leq C \int_{-\delta}^{\delta} (\int_{-1}^{1} (|t| + |s|)^{-r} ds)^{\frac{1}{r}} dt$ which is finite number for any $r \in (0,\infty)$. Similarly, $\|m\|_{L^r(U)} = \|m \circ F^{-1}\|_{L^r(FU)} \leq C(\int_{-\delta}^{\delta} \int_{-1}^{1} (|t| + |s|)^{-r} ds dt)^{\frac{1}{r}} < \infty$ for all $r \in (0,2)$.

Since U is small, H_t is also a piece of hypersurface with nonzero Gaussian curvature which is same as H's. Applying the dual version of the restriction theorem and interpolation, we have $\|(g d\sigma_{H_t})^{\vee}\|_{L^q(\boldsymbol{R}^d)} \leq C \|g\|_{L^{2-\rho}(H_t)}$ for all $g \in C_0^{\infty}$ and all $t \in (-\delta, \delta)$ with some positive constant C, where $s' < q < \infty$ and ρ is a (small) positive number depending only on q. Then we have for any $f \in \mathcal{S}$ with $\mathrm{supp} \hat{f} \subset U$,

$$\begin{aligned}
\|T_m f\|_q &= \|\textstyle\int_{-\delta}^{\delta}(m\hat{f}d\sigma_{H_t})^\vee dt\|_q \\
&\le \textstyle\int_{-\delta}^{\delta}\|(m\hat{f}d\sigma_{H_t})^\vee dt\|_q && \text{(Minkowski's ineq.)} \\
&\le C\textstyle\int_{-\delta}^{\delta}\|m\hat{f}\|_{L^{2-\rho}(H_t)}dt && \text{(duality restr. thm)} \\
&\le C\textstyle\int_{-\delta}^{\delta}\|\hat{f}\|_{L^2(H_t)}\|m\|_{L^{\frac{2(2-\rho)}{2}}(H_t)}dt && \text{(Holder ineq.)} \\
&\le C\|f\|_p\textstyle\int_{-\delta}^{\delta}\|m\|_{L^{\frac{2(2-\rho)}{2}}(H_t)}dt && \text{(restr. thm)} \\
&\le C\|f\|_p && \text{(estimate for } m)
\end{aligned}$$

where p is arbitrary in $[1, s]$ with $s = \frac{2(d+1)}{d+3}$.

Furthermore, by the Hausdorff-Young theorem and the second estimate of m, for any $q > 2$, $\|T_m f\|_q \le \|m\hat{f}\|_{q'} \le \|\hat{f}\|_\infty \|m\|_{q'} \le C\|f\|_1$. Combining the previous inequality and this one and using interpolation, we prove our lemma. $\sharp$

Now let's prove our Theorem 0.4 stated in Introduction.

Proof of Theorem 0.4: It is sufficient to prove the inequality in Theorem 0.4 for all $u \in C_0^\infty$. After substituting u by $e^{k\cdot x}u(x)$, it is easy to see that it is equivalent to $\|(m\hat{v})^\vee\|_q \le C\|v\|_p$ for all $v \in C_0^\infty$, where $m(\xi) = \frac{|\xi+ik|^{m-\mu}}{P(\xi+ik)}$.

By the assumption of P and Proposition 0.1, for each $\xi \in \pi_k(N^P)$ there are a small ball $D(\xi) \subset \boldsymbol{R}^d$ with center at ξ and a hypersurface H_ξ with nonzero Gaussian curvature such that $D(\xi) \cap \pi_k(N^P) \subset D(\xi) \cap H_\xi$. Since P has simple characteristics which implies that $|P(\xi + ik)| \ge C\text{dist}(\xi, \pi_k(N^P))$

with some constant C for all ξ, we have $|m(\xi)| \leq \frac{C}{\mathrm{dist}(\xi, \pi_k(N^P))}$ with another constant C for all ξ. So Lemma 2.1 says by making $D(\xi)$ smaller, one has $\|T_m f\|_q \leq C\|f\|_p$ for all $f \in \mathcal{S}$ with $\mathrm{supp}\hat{f} \subset D(\xi)$ and for (p,q) as in Lemma 2.1. By the compactness of $\pi_k(N^P)$, there is a finite cover $\{D_j\}_{j=1}^J$ of $\pi_k(N^P)$ such that for each j, $\|T_m f\|_q \leq C\|f\|_p$ for all $f \in \mathcal{S}$ with $\mathrm{supp}\hat{f} \subset 2D_j$. Let $\{\psi_j\}_{j=0}^J$ be a partition of unity for $\{2D_j\}_{j=1}^J \bigcup (\boldsymbol{R}^d \setminus \bigcup_{j=1}^J D_j)$, i.e., $\sum \psi_j = 1$ and $\psi_j \in C_0^\infty(2D_j)$ for $j \geq 1$ and $\psi_0 = 0$ on $\bigcup_{j=1}^J D_j$. Now let's decompose $\hat{v}(\xi)$ into $\hat{v}(\xi) = \sum \psi_j(\xi)\hat{v}(\xi) = \hat{v_0}(\xi) + \sum_{j=1}^J \hat{v_j}(\xi)$. Then for each $j \geq 1$,

$$\|(m\hat{v_j})^\vee\|_q \leq C\|v_j\|_p \leq C\|v\|_p$$

for all (p,q) with $(\frac{1}{p}, \frac{1}{q}) \in A$.

For v_0, since on $\mathrm{supp}\hat{v_0}$, $|P(\xi + ik)| \geq C(1 + |\xi|^2)^{\frac{m}{2}}$, we have $|m(\xi)| \leq C(1 + |\xi|^2)^{-\frac{\mu}{2}}$ on $\mathrm{supp}\hat{v_0}$. Hence, the Bessel Potential theory implies that

$$\|(m\hat{v_0})^\vee\|_q \leq C\|v_0\|_p \leq C\|v\|_p$$

for all (p,q) with $(\frac{1}{p}, \frac{1}{q}) \in A_\mu$. So combining the above two inequalities, for all all (p,q) with $(\frac{1}{p}, \frac{1}{q}) \in A \cap A_\mu$ we have

$$\|(m\hat{v})^\vee\|_q \leq C\|v\|_p$$

for all $v \in C_0^\infty$. This proves Theorem 0.4. $\sharp$

An immediate corollary is the following.

Corollary 2.2 *Suppose $P \in G$ is of order m. If $m < \frac{d}{s} = \frac{d(d+3)}{2(d+1)}$, then for any integer $2 \leq \mu \leq m$, with $p = s = \frac{2(d+1)}{d+3}$ and $\frac{1}{p} - \frac{1}{q_\mu} = \frac{\mu}{d}$, there is a constant C such that*

$$\|e^{tk\cdot x}\nabla^{m-\mu}u\|_{q_\mu} \leq C\|e^{tk\cdot x}\boldsymbol{P}u\|_p$$

for all $u \in W^{m,p}$ with compact support.

Proof: When $m \leq \frac{d}{s} = \frac{d(d+3)}{2(d+1)}$, for each $2 < \mu \leq m$, the points $(\frac{1}{s}, \frac{1}{s} - \frac{\mu}{d})$ are in the set $A \cap A_\mu$. That means the inequalities in Theorem 0.4 hold for a common $p = s$ with the corresponding q_μ's. This is the proof of Corollary 2.2. $\sharp$

Remark Let $p_0 = \frac{2d(d+1)(d-3)}{d^3-5d-4}$ which is less than $s = \frac{2(d+1)}{d+3}$. If $m < \frac{d}{p_0}$ (which is, of course, bigger than $\frac{d}{s} = \frac{d(d+3)}{2(d+1)}$), then the above inequality holds with $p = p_0$ and $q_\mu = (\frac{1}{p} - \frac{\mu}{d})^{-1}$. The proof of this is the same as the proof of Corollary 2.2.

3 Application to weak unique continuation

In this section we will use the Carleman inequalities in the last section to prove weak unique continuation theorems. A direct corollary is Theorem 0.2 stated in Introduction. In Theorem 0.2, we don't have any restriction on degree m of P. We think that if m is large, one doesn't need P having such strong curvature condition in G to get a Carleman inequality. For example, we may relax the assumption that the complex Hessian matrix $H_{C,P}$ has rank d when $m \geq d$. So we will mainly consider the case where the dimension d is greater than the degree m of P. Let's first state and prove another weak unique continuation theorem as a corollary of Theorem 0.4 as follows and then prove Theorem 0.2 because both proofs are same.

Theorem 3.1 *Suppose $P \in G$ is of degree $m < \frac{d}{s}$. Suppose a function $u \in W^{m,s}$ has compact support and satisfies $|\boldsymbol{P}u| \leq \sum_{\mu=2}^{m} V_\mu |\nabla^{m-\mu} u|$ with $V_\mu \in L^{\frac{d}{\mu}}$. Then $u \equiv 0$ in $\mathbf{R}^d$.*

Proof: Let k be a direction which is as in the assumption of G for P. Since supp$u$ is compact, there is a point ξ_0 on the boundary of supp$u$ and a hyperplane such that $u \equiv 0$ on the one side of the hyperplane and the normal vector of this hyperplane is k. So after a rotation and translation, we may assume that $0 \in \partial(\text{supp}u)$ and supp$u \subset R^d_+$ and e_d is that k. Consider $S_\rho = \{x \in \boldsymbol{R}^d : 0 \leq x_d \leq \rho\}$. Let $\rho > 0$ be chosen small enough so that $\max_{2\leq\mu\leq m} \|V_\mu\|_{L^{\frac{d}{\mu}}(S_\rho \cap \text{supp}u)} \leq \frac{1}{2mC}$ where C is the constant in Corollary 2.2. Let's denote $S^+_\rho = S_\rho \cap \text{supp}u$. Then with p and q_μ as in Corollary 2.2, by applying Corollary 2.2 and the Holder inequality, we have for $t < 0$,

$$\sum_{2\le\mu\le m} \|e^{tx_d}|\nabla^{m-\mu}u|\|_{L^{q_\mu}(S_\rho)} \le mC\|e^{tx_d}\boldsymbol{P}u\|_{L^p(\boldsymbol{R}^d)}$$

$$\le mC\left(\|e^{tx_d}\sum_{2\le\mu\le m} V_\mu|\nabla^{m-\mu}u|\|_{L^p(S_\rho)} + \|e^{tx_d}\boldsymbol{P}u\|_{L^p(S_\rho^c)}\right)$$

$$\le mC\left(\sum_{2\le\mu\le m}\|V_\mu\|_{L^{\frac{d}{\mu}}_{(S_\rho^+)}}\|e^{tx_d}|\nabla^{m-\mu}u|\|_{L^{q_\mu}(S_\rho)} + \|e^{tx_d}\boldsymbol{P}u\|_{L^p(S_\rho^c)}\right)$$

$$\le \frac{1}{2}\sum_{2\le\mu\le m}|e^{tx_d}|\nabla^{m-\mu}u|\|_{L^{q_\mu}(S_\rho)} + mCe^{t\rho}\|\boldsymbol{P}u\|_{L^p(S_\rho^c)}.$$

So,

$$\sum_{2\le\mu\le m}\|e^{t(x_d-\rho)}|\nabla^{m-\mu}u|\|_{L^{q_\mu}(S_\rho)} \le 2mC\|\boldsymbol{P}u\|_{L^p(S_\rho^c)}.$$

Let $t \to -\infty$, we get contradiction if supp$u \neq 0$. $\sharp$

Remark 3.1 Let p_0 be as in Remark in Section 2. Then the same conclusion as in Theorem 3.1 is still true if replacing s by p_0 and $m < \frac{d}{s}$ by $m < \frac{d}{p_0}$. The proof is exactly same as the previous one.

Now let's give a sketch proof of Theorem 3.1.

Sketch of Proof of Theorem 0.2: In the first case, $\mu \le \frac{d}{2}$. The Carleman inequality holds, by Theorem 0.4 in particular, for $(\frac{1}{p}, \frac{1}{q})$ in the set $A \cap A_\mu \cap \{(\frac{1}{p}, \frac{1}{q}) : \frac{1}{p} - \frac{1}{q} = \frac{\mu}{d}\}$. Let p be as small as possible in that region and then the remainder of the proof is same as the proof of Theorem 3.1. Similarly, when $\frac{d}{2} < \mu < d$, with $p = 1$ one may find a q such that $\frac{1}{p} - \frac{1}{q} = \frac{\mu}{d}$ and

$(\frac{1}{p}, \frac{1}{q}) \in A \cap A_\mu$. This leads to (2) of Theorem 0.2. When $\mu \geq d$, with $p = 1$, for any $r_\mu > 1$ there is a $q < \infty$ such that $1 - \frac{1}{q} = \frac{1}{r_\mu}$ and $(1, \frac{1}{q}) \in A \cap A_\mu = A$. So the Carleman inequality implies (3) of Theorem 0.2. $\sharp$

We already noticed that we didn't involve the highest order term, $\nabla^{m-1}u$, neither on the right-hand side of a differential inequality as in Theorem 3.1 nor on the left hand side of a Carleman inequality as in Theorem 0.2. In fact one cannot expect such Carleman inequality for the highest order term with the gap $\frac{1}{p} - \frac{1}{q} = \frac{1}{d}$, just as in the case of the Laplacian operator. But by using the technique in [12], we may also prove a weak unique continuation theorem for a differential inequality having the highest order term with a nice L^d- condition on the coefficient. Let's first state a Carleman-Wolff type inequality.

Lemma 3.2 *Suppose $P \in G$ is of order m and the open set K of S^{d-1} is as in* Proposition 0.1. *Assume $m < \frac{d}{s}$ and $p = s$. Then there is a constant $\theta = \theta(p) < \frac{1}{d}$ such that for any $t \in \boldsymbol{R}^1 \setminus \{0\}$ and any set $E \subset \boldsymbol{R}^d$ with $|E| \geq |t|^{-d}$, we have*

$$\|e^{tk\cdot x}\nabla^{m-1}u\|_{L^q(E)} \leq C_\theta(|t|^d|E|)^\theta\|e^{tk\cdot x}\boldsymbol{P}u\|_p$$

for all $u \in W^{m,p}$ with compact support and k in any fixed compact subset of K, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$.

Proof: Let's only prove the above inequality for all $u \in C_0^\infty$. Fix any $k \in K$ and let $t = 1$. The proof of Theorem 0.4 shows that one may decompose the multiplier $m(\xi) = \frac{|\xi+ik|^{m-1}}{P(\xi+ik)}$ into two parts $m(\xi) = m_1(\xi) + m_2(\xi)$ where m_1 has compact support and m_2 is bounded by $(1+|\xi|^2)^{-\frac{1}{2}}$. Moreover, one has for all $v \in \mathcal{S}$

$$||T_{m_1}v||_{q_1} \leq C||v||_p$$

for all (p, q_1) with $(\frac{1}{p}, \frac{1}{q_1}) \in A$ and

$$||T_{m_2}v||_q \leq C||v||_p$$

for all (p, q) with $(\frac{1}{p}, \frac{1}{q}) \in A_1$, where A and A_1 are defined in the beginning of Section 2.

Let $p = s$, q be such that $\frac{1}{p} - \frac{1}{q} = \frac{1}{d}$ and q_1 be very close to s'. Combine those two inequalities and use the Holder inequality. For any set E, we have

$$||T_m v||_{L^q(E)} \leq C|E|^{\frac{1}{q}-\frac{1}{q_1}}||v||_p.$$

Let $\theta = \frac{1}{q} - \frac{1}{q_1}$. Computing out, we have $\theta < \frac{1}{d}$.

Notice that Remark 1.4 implies the above process is actually true uniformly in $k' \in S^{d-1}$ near that fixed k. So by a compactness argument and scaling in k, we prove this lemma. $\sharp$

Remark 3.2 (1) Again as before, with $p = p_0$ and $m < \frac{d}{p_0}$ as in Remark 3.1, the same conclusion as in Lemma 3.2 is still true with some other $\theta < \frac{1}{d}$.
(2) From the above Carleman-Wolff type inequality, one may get the Carleman inequality with some convex weight instead of linear weight $k \cdot x$ there. See Remark in the next section.

Now we are ready to prove Theorem 0.3 stated in Introduction.

Proof of Theorem 0.3: The proof is very similar to the one in Theorem 1 of [12]. Because of homogeneity of P, after a change of scale and rotation and translation, we may reduce to the following case: (1) $\mathrm{supp}u \subset \boldsymbol{R}^d_-$; (2) there is a cone $\Gamma_\alpha = \{k \in R^d \setminus \{0\} : k_1^2 + \cdots + k_{d-1}^2 \leq \alpha k_d^2\}$ for some $\alpha > 0$ such that the inequality in Lemma 3.2 is true for all $k \in \Gamma_\alpha$ and $|E| \geq |k|^{-d}$ (with $t = 1$ there). Since $\mathrm{supp}u$ is compact, with C being its convex hull, we may choose a C_0^∞ function $\phi : \boldsymbol{R}^d \to \boldsymbol{R}$ such that $\phi = 1$ on a neighborhood of ∂C and $\sum_{1 \leq \mu \leq m} \|V_\mu\|^{\frac{d}{\mu}}_{L^{\frac{d}{\mu}}(\mathrm{supp}\phi)} < \beta$, where β is a sufficiently small positive number depending only on d and m to be chosen later. Let $v = \phi u$. Then by a simple calculation

$$|\boldsymbol{P}v| \leq \sum_{1 \leq \mu \leq m} V_\mu |\nabla^{m-\mu} v| + \chi \tag{$\dagger$}$$

where $\chi \in L^p$ and $\mathrm{supp}\chi \subset C \cap \mathrm{supp} \nabla \phi$. After making α small enough depending on the diameter of C and ϕ, i.e., on u and d and m, the same proof as Lemma 7.1 in [12] shows that if $k \in \Gamma_\alpha$ and $|k|$ is sufficiently large, we have an estimate

$$\|e^{k \cdot x} \chi\|_p \leq \|e^{k \cdot x} \sum_{1 \leq \mu \leq m} V_\mu |\nabla^{m-\mu} v|\|_p. \tag{$\ddagger$}$$

Now we may apply Wolff's measure lemma in [12] to the measure

$$\Big(\sum_{1 \leq \mu \leq m} V_\mu |\nabla^{m-\mu} v|\Big)^p dx.$$

Let M be large enough so that $B(pMe_d, \frac{pM}{1000}) \subset \Gamma_\alpha$. Then the Wolff's lemma says that there are $\{k_j\}$ and disjoint convex sets $\{E_j\}$ such that

$$(\dagger\dagger) \quad \begin{cases} \frac{M}{2} < |k_j| < 2M, \quad k_j \in B(pMe_d, \frac{pM}{1000}) \subset \Gamma_\alpha \\ \|e^{k_j \cdot x} \sum_{1 \le \mu \le m} V_\mu |\nabla^{m-\mu} v|\|_{L^p(E_j)} \ge 2^{-\frac{1}{p}} \|e^{k_j \cdot x} \sum_{1 \le \mu \le m} V_\mu |\nabla^{m-\mu} v|\|_p \\ \sum |E_j|^{-1} \ge C^{-1} M^d \quad \text{and} \quad |E_j| \ge M^{-d} \ \text{ for each } j \end{cases}$$

with an absolute constant C which is independent of M and $\{E_j\}$.

Now let's denote $E_j^+ = E_j \cap \operatorname{supp}\phi$. By the Holder inequality, we have

$$\|e^{k_j \cdot x} \sum_{1 \le \mu \le m} V_\mu |\nabla^{m-\mu} v|\|_{L^p(E_j)}$$

$$\le C\Big(\sum_{2 \le \mu \le m} \|V_\mu\|_{L^{\frac{d}{\mu}}(E_j^+)} \|e^{k_j \cdot x} |\nabla^{m-\mu} v|\|_{q_\mu} + \|V_1\|_{L^d(E_j^+)} \|e^{k_j \cdot x} |\nabla^{m-1} v|\|_{L^q(E)}\Big).$$

For the first terms above, we apply Corollary 2.2. For the second term above, we apply Lemma 3.2. Then the above is bounded by

$$C\Big(\sum_{2 \le \mu \le m} \|V_\mu\|_{L^{\frac{d}{\mu}}(E_j^+)} + (M^d |E_j|)^\theta \|V_1\|_{L^d(E_j^+)}\Big) \cdot \|e^{k_j \cdot x} \boldsymbol{P} v\|_{L^p}.$$

Because of ($\dagger$) and ($\ddagger$), we have

$$\|e^{k_j \cdot x} P v\|_{L^p} \le C \|e^{k \cdot x} \sum_{1 \le \mu \le m} V_\mu |\nabla^{m-\mu} v|\|_p.$$

Finally using the second fact in ($\dagger\dagger$), we get

$$\|e^{k_j \cdot x} \sum_{1 \le \mu \le m} V_\mu |\nabla^{m-\mu} v|\|_{L^p(E_j)}$$

$$\le C \left(\sum_{2 \le \mu \le m} \|V_\mu\|_{L^{\frac{d}{\mu}}(E_j^+)} + (M^d |E_j|)^\theta \|V_1\|_{L^d(E_j^+)} \right) \times$$

$$\times \left\| e^{k_j \cdot x} \sum_{1 \le \mu \le m} V_\mu |\nabla^{m-\mu} v| \right\|_{L^p(E_j)} .$$

Hence,

$$\sum_{2 \le \mu \le m} \|V_\mu\|_{L^{\frac{d}{\mu}}(E_j^+)} + (M^d |E_j|)^\theta \|V_1\|_{L^d(E_j^+)} \ge C^{-1}$$

for some positive constant C. By the assumptions on the functions V_μ and choosing β small enough which depends only on d and m, we have by the third fact of (††), for a new constant C,

$$\|V_1\|_{L^d(E_j^+)} \ge C^{-1}(M^d|E_j|)^{-\theta} \ge C^{-1}(M^d|E_j|)^{-\frac{1}{d}}$$

since $\theta < \frac{1}{d}$ by Lemma 3.2. So raise to the d^{th} power and sum over j obtaining

$$\beta \ge \|V_1\|^d_{L^d(\text{supp}\phi)} \ge C^{-1} \sum_j (M^d|E_j|)^{-1} \ge C^{-1}$$

because of the third fact of (††) in for the last step above. This is contradiction if β is small enough. $\sharp$

Remark 3.3 The same conclusion as in Theorem 0.3 is true after replacing s by p_0 and $m < \frac{d}{s}$ by $m < \frac{d}{p_0}$, because of the proof of Theorem 0.3 and (1) of Remark 3.2.

4 Carleman inequalities for operators in classes S and D

In this section, we continue to study Carleman inequalities for a wider class S or D which doesn't have any curvature assumption for operators. In fact, most results here are improvements of results in [12]. The proof of the Carleman inequalities for an operator $\boldsymbol{P}$ in class S is very similar to the one for $\boldsymbol{P}$ in class D. So we will give a detail proof of the result when $\boldsymbol{P}$ in class D and only mention the result for $\boldsymbol{P}$ in class S. Let's start with several simple lemmas.

Lemma 4.1 *Suppose $\phi \in C_0^\infty([0,1))$ with $\phi(0) = \phi'(0) = 0$. Then*

$$\int_0^1 \frac{\phi(t)^2}{t^3} dt \leq C\left(\int_0^1 \phi(t)^2 t dt + \int_0^1 \phi'''(t)^2 t dt\right)$$

with some universal constant C.

Proof: If $\phi''(0) = 0$, then we have

$$\begin{aligned} \phi(t) = &\ \tfrac{1}{2}\textstyle\int_0^t (t-s)^2 \phi'''(s) ds \\ = &\ \tfrac{1}{2}\textstyle\int_0^t (t^2 + s(s-2t))\phi'''(s) ds \\ = &\ \tfrac{1}{2}t^2 \textstyle\int_0^t \phi'''(s) ds + \int_0^t (\tfrac{1}{2}s - t)\phi'''(s) s ds, \end{aligned}$$

i.e.,

$$\phi(t) = \frac{1}{2}t^2\phi''(t) + \int_0^t (\frac{1}{2}s - t)\phi'''(s) s ds.$$

If $\phi''(0) \neq 0$, let's apply the above formula to $\phi(t) - \frac{1}{2}\phi''(0)t^2$. We get

$$\phi(t) - \frac{1}{2}\phi''(0)t^2 = \frac{1}{2}t^2(\phi''(0) - \frac{1}{2}\phi''(0)\cdot 2) + \int_0^t (\frac{1}{2}s - t)\phi'''(s)sds.$$

This is the same as above. So for any $\phi \in C_0^\infty$ with $\phi(0) = \phi'(0) = 0$ we have the following inequality

$$\phi(t)^2 \leq t^4\phi''(t)^2 + t^4\int_0^t \phi'''(s)^2 sds$$

by the Holder inequality. Divide by t^3 and integrate the above inequality. We have

$$\int_0^1 \frac{\phi(t)^2}{t^3}dt \leq \int_0^1 t\phi''(t)^2 dt + \int_0^1 t\int_0^t \phi'''(s)^2 sds.$$

The second integral on the right-hand side is bounded by $\int_0^1 \phi'''(s)^2 sds$. So let's compute the first one. Notice that $t\phi''(t)^2 = [(t\phi'(t))' - \phi'(t)]\cdot\phi''(t)$. So by integration by parts, the first integral in the right-hand side is

$$\begin{aligned}
\int_0^1 t\phi''(t)^2 dt &= \int_0^1((t\phi')' - \phi')\cdot\phi'' dt \\
&= t\phi'\cdot\phi''|_0^1 - \int_0^1 t\phi'\cdot\phi''' dt - \int_0^1 \phi'\cdot\phi'' dt \\
&= 0 - \int_0^1 t\phi'\cdot\phi''' dt - \tfrac{1}{2}\int_0^1(\phi^2)' dt \\
&= -\int_0^1 t\phi'\cdot\phi''' dt \\
&\leq \int_0^1 t\phi'(t)^2 dt + \int_0^1 \phi'''(t)^2 t dt
\end{aligned}$$

here we used $\phi(0) = \phi'(0) = 0$ and $\phi(1) = 0$, and the triangle inequality in the last step. Once again using $t\phi' = (t\phi)' - \phi$,

$$\begin{aligned}
\textstyle\int_0^1 t\phi'(t)^2 dt = &\ \textstyle\int_0^1 (t\phi)' \cdot \phi' dt - \int_0^1 \phi \cdot \phi' dt \\
= &\ \textstyle t\phi \cdot \phi'|_0^1 - \int_0^1 t\phi \cdot \phi'' dt - \frac{1}{2}\int_0^1 (\phi^2)' dt \\
= &\ \textstyle -\int_0^1 t\phi \cdot \phi'' dt \\
\leq &\ \textstyle \frac{1}{10}\int_0^1 t\phi''(t)^2 dt + 10\int_0^1 \phi(t)^2 t dt
\end{aligned}$$

by the triangule inequality. So we have proved

$$\int_0^1 t\phi''(t)^2 dt \leq \frac{1}{10}\int_0^1 t\phi''(t)^2 dt + 10\int_0^1 \phi(t)^2 t dt + \int_0^1 \phi'''(t)^2 t dt.$$

Absorb the 1st term on the right-hand side into the left-hand side,

$$\int_0^1 t\phi''(t)^2 dt \leq 20\int_0^1 \phi(t)^2 t dt + 2\int_0^1 \phi'''(t)^2 t dt.$$

On account of the above calculation, we have

$$\int_0^1 \frac{\phi(t)^2}{t^3} dt \leq 20\int_0^1 \phi(t)^2 t dt + 2\int_0^1 \phi'''(t)^2 t dt. \qquad \sharp$$

Lemma 4.2 *Suppose $\phi \in C_0^\infty(D(0,1))$ where $D(0,1)$ is the unit ball of $\boldsymbol{R}^2$. Then for $2 \leq q \leq \infty$ and $R \geq 1$, we have*

$$||\phi||_{q'} \leq 100R^{1+\frac{2}{q}}(||\psi||_2 + R^{-3}||\nabla^3\psi||_2)$$

where $\psi(x) = |x|^2\phi(x) = r^2\phi(x)$.

Proof: Let's first prove the inequality with $R = 1$. Since $q' \leq 2$, the Holder inequality implies $||\phi||_{q'}^2 \leq ||\phi||_2^2$ which is $\int_{S^1}\int_0^1 \frac{(r^2\phi(rx'))^2}{r^3} dr d\sigma(x')$. Let's apply

Lemma 4.1 to the function $r^2\phi(rx') = \psi(rx')$ in the r variable. So the inner integral is bounded by

$$20\int_0^1 \psi(rx')^2 r dr + 2\int_0^1 \left(\frac{d^3}{dr^3}\psi(rx')\right)^2 r dr.$$

As $|\frac{d^3}{dr^3}\psi(rx')| \leq |\bigtriangledown^3 \psi(rx')|$, the above estimates give $||\phi||_{q'}^2 \leq 20||\psi||_2^2 + 2||\bigtriangledown^3 \psi||_2^2$ for all $2 \leq q \leq \infty$. So we prove the inequality with $R = 1$.

Now let $R \geq 1$ and define $\phi_R(y) = \phi(R^{-1}y)$ and $\psi_R(y) = |y|^2\phi_R(y) = R^2\psi(\frac{y}{R})$. Then

$$\left(\int_{D(0,1)} |\phi(x)|^{q'} dx\right)^{\frac{1}{q'}} \leq \left(\int_{D(0,R^{-1})} |\phi(x)|^{q'} dx\right)^{\frac{1}{q'}} + \left(\int_{D(0,1)\setminus D(0,R^{-1})} |\phi(x)|^{q'} dx\right)^{\frac{1}{q'}}$$

$$= \left(\int_{D(0,1)} |\phi_R(y)|^{q'} R^{-2} dy\right)^{\frac{1}{q'}} + \left(\int_{D_1\setminus D_{R^{-1}}} (|\psi(x)|\cdot|x|^{-2})^{q'} dx\right)^{\frac{1}{q'}}.$$

The second is bounded by $(\int|\psi(x)|^2)^{\frac{1}{2}}(\int_{D_1\setminus D_{R^{-1}}} |x|^{-2q'\cdot\frac{2}{2-q'}} dx)^{\frac{2-q'}{2q'}}$ by the Holder inequality, which is less than $10R^{1+\frac{2}{q}}\|\psi\|_2$. Apply our inequality with $R = 1$ to the function $\phi_R(y)$, then $(\int_{D(0,1)} |\phi_R(y)|^{q'} R^{-2} dy)^{\frac{1}{q'}} \leq R^{-\frac{2}{q'}}(20\|\psi_R\|_2 + \|\bigtriangledown^3\psi_R\|_2)$ which is less than or equal to $R^{-\frac{2}{q'}}(20R^3\|\psi\|_2 + \|\bigtriangledown^3\psi\|_2)$. So combine these estimates, we get

$$\left(\int_{D(0,1)} |\phi(x)|^{q'} dx\right)^{\frac{1}{q'}} \leq 100R^{1+\frac{2}{q}}(\|\psi\|_2 + R^{-3}\|\bigtriangledown^3\psi\|_2).$$

This proves the lemma. ♯

Lemma 4.3 *Let N be a submanifold of codimension 2. Suppose that $G : U \to B^{d-2} \times D(0,1)$ is a diffeomorphism for some open set U in $\boldsymbol{R}^d$, where B^{d-2} is the unit ball of dimension $d-2$ and $D(0,1)$ is the unit disk, such that*

$G(N) \subset B^{d-2}$. Then for any $2 \le q \le \infty$ and any function $\varphi \in C_0^\infty(U)$ there is a constant C depending on q, φ, upper bound of finitely many derivatives of G and lower bound of the gradient of G such that for all $R \ge 1$, the following inequality

$$\|\phi_1 * \varphi^\vee\|_q \le CR^{1+\frac{2}{q}}\|(1+\frac{|x|}{R})^3\phi_2\|_2$$

holds for all Schwartz functions ϕ_1 and ϕ_2 such that $supp\hat{\phi}_1 \cup supp\hat{\phi}_2 \subset B$ and $|\hat{\phi}_1(\xi)| \le (dist(\xi, N))^{-2} \cdot |\hat{\phi}_2(\xi)|$ for all ξ.

Proof: Let's denote by z, x the points in B^{d-2}, $D(0,1)$ respectively. As G is a diffeomorphism, $|x| \le C\mathrm{dist}(G(z,x), N)$. Now let $\psi = \frac{\hat{\phi}_2(G(z,x))}{|x|^2}$. Then by using Lemma 4.2,

$$\|\psi\|_{L^{q'}(U)}^{q'} \le C\int_{B^{d-2}}\int_{D(0,1)}|\psi(G(z,x))|^{q'}dxdz$$

$$\le CR^{q'(1+\frac{2}{q})}\int_{B^{d-2}}\left(\int_{D(0,1)}\left(\left||x|^2\psi(G(z,x))\right|^2 + R^{-6}\left|\nabla_x^3(|x|^2\psi(G(z,x)))\right|^2\right)dx\right)^{\frac{q'}{2}}dz$$

$$= CR^{q'(1+\frac{2}{q})}\int_{B^{d-2}}\left(\int_{D(0,1)}|\hat{\phi}_2(G(z,x))|^2 + R^{-6}|\nabla_x^3\,\hat{\phi}_2(G(z,x))|^2dx\right)^{\frac{q'}{2}}dz$$

$$\le CR^{q'(1+\frac{2}{q})}\left(\|\hat{\phi}_2\|_2 + R^{-3}\|\nabla^3\,\hat{\phi}_2\|_2\right)^{q'}$$

where we used Holder inequality, the change of variables, and the interpolation in the last step.

Now by the Hausdorff-Young inequality,

$$\|\phi_1 * \varphi^\vee\|_q \le C\|\hat{\phi}_1\cdot\varphi\|_{q'} \le C\|\frac{|x|^2}{\mathrm{dist}(\xi, N)}\psi\cdot\varphi\|_{q'}$$

$$\le C\|\psi\|_{L^{q'}(U)} \le CR^{(1+\frac{2}{q})}\left(\|\hat{\phi}_2\|_2 + R^3\|\nabla^3\,\hat{\phi}_2\|_2\right).$$

This is the proof of the lemma. $\sharp$

Now let's start our main lemma in this section.

Lemma 4.4 *(1) Suppose $P \in D$ is of order m. Then there is an open cone Γ with vertex at 0 such that for any $q \in [2, \infty)$ and any integer $0 < \mu \leq m$ with $\frac{1}{2} \geq \frac{1}{q} \geq \frac{1}{2} - \frac{\mu}{d}$, or for $q = \infty$ and $\mu > \frac{d}{2}$, the following inequality*

$$\|e^{k\cdot x}\nabla^{m-\mu}u\|_q \leq C_{P,q}|k|^{d(\frac{1}{2}-\frac{1}{q})-\mu}(|k|R)^{1+\frac{2}{q}}\|e^{k\cdot x}(1+\frac{|x|}{R})^3 \boldsymbol{P}u\|_2$$

holds for all $u \in W^{m,2}$ with compact support, $\forall R \geq |k|^{-1}$ and all $k \in \Gamma$. If P is of form Q^2 for some $Q \in S$ of degree $\frac{m}{2}$, then the above inequality holds for almost all $k \in \boldsymbol{R}^d$.

(2) If $P \in S$, then with the same notations as above, we have

$$\|e^{k\cdot x}\nabla^{m-\mu}u\|_q \leq C_{P,q}|k|^{d(\frac{1}{2}-\frac{1}{q})-\mu}(|k|R)^{\frac{2}{q}}\|e^{k\cdot x}(1+\frac{|x|}{R})^2 \boldsymbol{P}u\|_2$$

for almost all $k \in \boldsymbol{R}^d$, $\forall R \geq |k|^{-1}$ and $u \in W^{m,2}$ with compact support.

Proof: The proof of part (2) is very similar to the one of part (1). The similar proof may be also found in [12]. So we will only prove part (1) for $u \in C_0^\infty$. In the following proof, we use C to denote a constant depending only P, p, d.

Let's fix a $k_0 \in S^{d-1} \cap K$ where K is the open subset of $\boldsymbol{R}^d \setminus 0$ as in the definition of D for P. We first prove the following.

Claim: *For any $\xi_0 \in \pi_{k_0}(N^P)$ there are a neighborhood U of ξ_0 and a constant $s > 0$ such that for any function $\varphi \in C_0^\infty(U)$ and $k \in S^{d-1}$ with*

$|k-k_0| \le s$, *the following inequality*

$$||(\varphi(e^{k\cdot x} \bigtriangledown^{m-\mu} u)^\wedge)^\vee|| \le CR^{1+\frac{2}{q}}||e^{k\cdot x}(1+\frac{|x|}{R})^3 \boldsymbol{P}u||_2$$

holds for the same u, R *as in the lemma.*

By the assumption, N_K^P is a submanifold of codiemsion 2 in $\boldsymbol{R}^{2d}$ and N_K^P and Π_k are transverse for $k \in K$. By Proposition 6.5 in the appendix, there are a neighborhood U of ξ_0 in $\boldsymbol{R}^d$ and a number $s > 0$ such that for any $k \in S^{d-1} \cap K$ with $|k-k_0| \le s$ there is a diffeomorphism $G_k : U \to B^{d-2} \times D(0,1)$ satisfying that $G_k(\pi_k(N)) \subset B^{d-2} \times \{0\}$ and $|\bigtriangledown G_k| \ge C^{-1}$ for all $\xi \in U$. Now let $\phi_1 = (e^{k\cdot x} \bigtriangledown^{m-\mu} u)^\wedge$ and $\phi_2 = e^{k\cdot x}\boldsymbol{P}u$. Then we have $|\hat{\phi}_1(\xi)| \le C\text{dist}(\xi, \pi_k(N))^{-2}|\hat{\phi}_2(\xi)|$ by the assumption $P \in D$ in definition 1.1. So by applying Lemma 4.3 to ϕ_1 and ϕ_2, we prove the claim.

Now since $\pi_{k_0}(N)$ is a compact submanifold, there are a number s and finitely many open sets $\{U_j\}_{j=1}^J$ and a partition of unity $\{\varphi_j\}_{j=0}^J$ with $\varphi_j \in C_0^\infty(U_j)$ for $j \ge 1$ such that the inequality in the claim is true for each φ_j and $\text{supp}\varphi_0 \cap (\cup_{k\in S^{d-1}, |k-k_0|\le s}\pi_k(N^P)) = \emptyset$. Let's write

$$e^{k\cdot x} \bigtriangledown^{m-\mu} u = \sum_{j=1}^J (\varphi_j(e^{k\cdot x} \bigtriangledown^{m-\mu} u)^\wedge)^\vee + (\varphi_0(e^{k\cdot x} \bigtriangledown^{m-\mu} u)^\wedge)^\vee.$$

For the last term, since $|\varphi_0(\xi)\frac{|\xi+ik|^{m-\mu}}{P(\xi+ik)}| \le C(1+|\xi|)^{-\mu}$, by using the Bessel potential and Plancherel theorems we have

$$||(\varphi_0(e^{k\cdot x} \bigtriangledown^{m-\mu} u)^\wedge)^\vee||_q \le C||e^{k\cdot x}\boldsymbol{P}u||_2$$

for all q with $\frac{1}{2} \ge \frac{1}{q} \ge \frac{1}{2} - \frac{\mu}{d}$ if $\mu \le \frac{d}{2}$ and $q = \infty$ if $\mu > \frac{d}{2}$. For the other terms, by using the claim, we finally get

$$||e^{k\cdot x}D^\alpha u||_q \le \sum_{j=1}^J ||(\varphi_j(e^{k\cdot x} \bigtriangledown^{m-\mu} u)^\wedge)^\vee||_q + ||(\varphi_0(e^{k\cdot x} \bigtriangledown^{m-\mu} u)^\wedge)^\vee||_q$$

$$\leq C(J+1)R^{1+\frac{2}{q}}\|e^{k\cdot x}(1+\frac{|x|}{R})^3 \boldsymbol{P}u\|_2$$

for all $k \in K$ with $|k-k_0| \leq s$. Finally by scaling, we prove our inequality. $\sharp$

By using this lemma, we may prove our Carleman inequalities with weaker gaps for $P \in D$ or S which is stated in Theorem 0.5 in Introduction.

Proof of Theorem 0.5: We only prove (1) because the proof of (2) is very similar. Since $P \in D$, there is a cone Γ such that for each $k \in 2\Gamma$, the inequalities in Lemma 4.4 (1) hold. Let $K = \Gamma \cap S^{d-1}$, $k_0 \in K$ and $\phi(x) = k_0 \cdot x + \beta|x|^2$. Divide the unit ball into about $t^{\frac{1}{2}d}$'s disjoint little cubes B_j of radius $t^{\frac{1}{2}d}$, which are paralleled to the coordinate system and centered at a_j. Let $v(x) = u(x-a_j) \in C_0^\infty(B_1)$. Now apply Lemma 4.4 to v, we have by taking a change of variable $x \to x - a_j$, with (p,q) as in Lemma 4.4,

$$\|e^{k\cdot x}\bigtriangledown^{m-\mu}v\|_q \leq C|k|^{d(\frac{1}{2}-\frac{1}{q})-\mu}(|k|R)^{1+\frac{2}{q}}\|e^{k\cdot x}(1+\frac{|x|}{R})^3 \boldsymbol{P}v\|_2$$

for all $k \in 2\Gamma$. Hence for all such k,

$$\|e^{k\cdot x}\bigtriangledown^{m-\mu}u\|_{L^q(B_j)} \leq C|k|^{d(\frac{1}{2}-\frac{1}{q})-\mu}(|k|R)^{1+\frac{2}{q}}\|e^{k\cdot x}(1+\frac{|x-a_j|}{R})^3 \boldsymbol{P}u\|_2.$$

Now choose β to be small enough such that for each j, $t \bigtriangledown \phi(a_j) \in 2\Gamma$. Let's substitute k by $t \bigtriangledown \phi(a_j)$ and R by $t^{-\frac{1}{2}}$ in the above inequality. Notice that when $x \in B_j$, $|t \cdot O(|x-a_j|^2)|$ is bounded by a universal constant. So the above inequality implies that

$$\|e^{t\phi(x)}\bigtriangledown^{m-\mu}u\|_{L^q(B_j)} = \|e^{t[\phi(a_j)+\bigtriangledown\phi(a_j)\cdot(x-a_j)+O(|x-a_j|^2)]}\bigtriangledown^{m-\mu}u\|_{L^q(B_j)}$$

$$\leq Ce^{t\phi(a_j)-t\nabla\phi(a_j)\cdot a_j}|t\nabla\phi(a_j)|^{d(\frac{1}{2}-\frac{1}{q})-\mu}(t\nabla\phi(a_j)\cdot t^{-\frac{1}{2}})^{1+\frac{2}{q}}\cdot\|e^{t\phi(x)}A_j\boldsymbol{P}u\|_2,$$

where $A_j = e^{-t[\phi(x)-\phi(a_j)-(t\nabla\phi(a_j))(x-a_j]}(1+t^{\frac{1}{2}}|x-a_j|)^3$. Since $|\nabla\phi(x)|\approx 1$ and $\phi(a_j)-\nabla\phi(a_j)\cdot a_j\leq 0$ because ϕ is convex, the above inequality becomes

$$\|e^{t\phi}\nabla^{m-\mu}u\|_{L^q(B_j)}$$

$$\leq Ct^{(d-1)(\frac{1}{2}-\frac{1}{q})-(\mu-1)}\|e^{t\phi}\cdot A_j\cdot\boldsymbol{P}u\|_2$$

$$\leq C\|e^{t\phi}\cdot A_j\cdot\boldsymbol{P}u\|_2$$

if $\frac{1}{2}-\frac{1}{q}\leq\frac{\mu-1}{d-1}$. Take q^{th} power to both sides and sum over j,

$$\|e^{t\phi}\nabla^{m-\mu}u\|_q^q = \sum_j\|e^{t\phi}\nabla^{m-\mu}u\|_{L^q(B_j)}^q$$

$$\leq C\sum_j\|e^{t\phi}\cdot A_j\cdot\boldsymbol{P}u\|_2^q$$

$$\leq C[\int\sum_j|A_j(x)|^2(e^{t\phi}\boldsymbol{P}u)^2dx]^{\frac{q}{2}}$$

by Minkowski inequality since $q\geq 2$. Now we need to estimate $\sum_j A_j(x)^2$ pointwisely. We can assume $x=0$. Let $C_l=\{B_j : |a_j|\approx lt^{-\frac{1}{2}}\}$. Then $\#C_l\approx\frac{(lt^{-\frac{1}{2}})^{d-1}}{t^{-\frac{1}{2}(d-1)}}=l^{d-1}$. So

$$\sum_j A_j^2 = \sum_l\sum_{j\in C_l}[e^{-t[\phi(0)-\phi(a_j)-\nabla\phi(a_j)(0-x)]}(1+t^1 2|0-a_j|)^3]$$

$$\leq\sum_j A_j^2 = \sum_l\sum_{j\in C_l}e^{-ct|a_j|^2}(1+t^3 2|a_j|^3)$$

$$\leq C\sum_l l^{d-1}e^{-cl^2}(1+l^3)\leq C<\infty.$$

This proves that $\|e^{t\phi}\nabla^{m-\mu}u\|_q\leq C\|e^{t\phi}\boldsymbol{P}u\|_2$ for all $u\in C_0^\infty(B(0,1))$. ♯

Remark (1) The Carleman-Wolff type inequalities in Lemma 4.4 may imply some weak u.c.p. theorem as we did in Theorem 0.3 with potentials in L^{r_μ}, where r_μ^{-1} is very close to the gaps in the above. And here we may use the Carleman inequality in Theorem 0.5 to prove some weak u.c.p. theorems by a simple proof as we did in Theorem 0.2.

(2) If $P \in G$ is of order $m < \frac{d}{s}$, then by the Carleman-Wolff inequality in Lemma 3.2, with the same function ϕ as in Theorem 0.5 we have

$$\|e^{t\phi}\nabla^{m-1}u\|_{L^q(\boldsymbol{R}^d)} \leq C\|e^{t\phi}\boldsymbol{P}u\|_{L^s(\boldsymbol{R}^d)}$$

for all $u \in W^{m,s}$ with compact support, where $q \in [s, \infty)$ is such that $0 \leq \frac{1}{s} - \frac{1}{q} \leq \frac{2d}{(2d-1)(d+1)}$ which is better than the gap in (2) of Theorem 0.5.

5 Some counterexamples and application to UCP for operators in D

In Section 4, we only used real analysis and general properties of Fourier analysis to obtain some Carleman inequalities in Theorem 0.5. One may already see that we didn't get Carleman inequality for the highest order term in the left-hand side ($\mu = 1$) with positive gap when $P \in D$. In fact, the following proposition will tell us that there is no such inequality with positive gap. Moreover, for the other terms with $\mu \geq 2$, we will point out such gap conditions in the Carleman inequalities in Theorem 0.5 are also sharp in some sense.

Proposition 5.1 *Suppose that P is a homogeneous polynomial of order m with real coefficients. Assume that $P(z_0) = 0$ and $\nabla P(z_0) = 0$ for some nonzero $z_0 \in C^d$. Then with any smooth function ϕ of one variable which has nonzero derivative at 0 the following inequality with $k = rez_0$ and some constant C*

$$\|e^{t\phi(k\cdot x)}\nabla^{m-1}u\|_q \leq C\|e^{t\phi(k\cdot x)}\boldsymbol{P}u\|_p$$

for all $u \in C_0^\infty(B(0,1))$, $\forall t > 0$ will imply that $q \leq p$.

Proof: We may assume that $\phi(0) = 0$ and $\phi'(0) = 1$. Let $z_0 = k + il$. By assumption, $P(k+il) = \nabla P(k+il) = 0$, i.e., $\boldsymbol{P}(e^{-(k+il)\cdot x}) = 0$ and $(\nabla P)(D)(e^{-(k+il)\cdot x}) = 0$. Let $R = B(0, (\max(|k|,|l|))^{-\frac{1}{2}})$. Define $\psi \in C_0^\infty$ such that $\psi = 1$ on $\frac{1}{2}R$, $= 0$ on R^c and $|\nabla^j\psi| \leq C|k+il|^{\frac{1}{2}j}$ for some absolute constant C. Then

$$|\boldsymbol{P}(e^{-(k+il)\cdot x}\psi)| \leq Ce^{-k\cdot x}|k+il|^{m-1}\chi_R$$

and

$$|\nabla^{m-1}(e^{-(k+il)\cdot x}\psi)| \geq C^{-1}e^{-k\cdot x}|k+il|^{m-1}\chi_{\frac{1}{2}R}.$$

Apply the inequality in the statement to $e^{-(k+il)\cdot x}\psi$. We have $|\frac{1}{2}R|^{\frac{1}{q}} \leq C|R|^{\frac{1}{p}}$. Letting $|k|$ or $|l| \to \infty$ (this is possible because P is homogeneous), we obtain $q \leq p$. $\sharp$

The idea of the proof above proposition comes from [6] and [1]. A typical example of such a polynomial operator is the bi-Laplacian Δ^2.

When $\mu \leq 2$, we provide the following result to say the gap $\frac{\mu-1}{d-1}$ in Theorem 0.5 is sharp in some sense.

Proposition 5.2 *Fix μ to be a positive integer with $\mu < d$. Then for any smooth function ϕ with $\nabla\phi(0) \neq 0$, the following inequalities with gap $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\mu-1}{d-1}$*

$$\|e^{t\phi}\nabla^{m-\mu}u\|_q \leq \|e^{t\phi}\boldsymbol{P}u\|_p, \qquad \forall u \in C_0^\infty(B(0,1)),\ \forall t > 0$$

are sharp in the class of $P \in D$ with $P(\nabla\phi(0)+il) = (\nabla P)(\nabla\phi(0)+il) = 0$ for some $l \in \boldsymbol{R}^d$.

Proof: After a rotation, we may assume that $\phi(x) = x_1 + O(|x|^2)$. So consider $P = P_0(\xi_1, \xi_2) + Q(\xi_3, \cdots, \xi_d) \in D$ be of degree m such that P_0 is a two variables elliptic homogeneous operator having double characteristics and Q is another $d-2$ variables operator. Choose P_0 such that $P(e^{t(x_1+ix_2)}) = \nabla P(e^{t(x_1+ix_2)}) = 0$. We should mention such operators exist. For example, we may construct such P from $(\xi_1^{\frac{m}{2}}+\xi_2^{\frac{m}{2}})^2+\cdots$ by a rotation. Let $\Gamma = \Gamma(x_1, x_2) = e^{-t(x_1+ix_2)}$ and $\varphi(x_1, x_2) = \varphi_0(t^{\frac{1}{2}}x_1, t^{\frac{1}{2}}x_2)$ where $\varphi_0 \in C_0^\infty(D(0,1))$ and $\varphi_0 = 1$ on $D(0, \frac{1}{2})$. Let $\psi = \psi(x_3, \cdots, x_d) = \psi_0(t^{\frac{m-1}{m}}x_3, \cdots, t^{\frac{m-1}{m}}x_d)$ for some $\psi \in C_0^\infty(B(0,1))$. Then

$$|P(\Gamma\varphi \cdot \psi)| = |P_0(\Gamma\varphi)\cdot\psi + \Gamma\varphi\cdot Q(\psi)|$$

$$\leq Ce^{-tx_1}t^{m-1}\chi_R$$

and

$$|\nabla^{m-\mu}(\Gamma\varphi\cdot\psi)| \geq |\frac{d^{m-\mu}}{dx_1^{m-\mu}}\Gamma\varphi\cdot\psi| \geq C^{-1}e^{-tx_1}t^{m-\mu}\chi_{\frac{1}{2}R}$$

where $R = (-t^{-\frac{1}{2}}, t^{-\frac{1}{2}})^2 \times (-t^{-\frac{m-1}{m}}, t^{-\frac{m-1}{m}})^{d-2}$. So applying the Carleman inequality for $\Gamma\varphi\cdot\psi$, we get $t^{m-\mu}|R|^{\frac{1}{q}} \leq Ct^{m-1}|R|^{\frac{1}{p}}$. As t may be large, $\frac{1}{p} - \frac{1}{q} \leq \frac{\mu-1}{d-1-\frac{d-2}{m}}$. That means the gap $\frac{1}{p}-\frac{1}{q} \leq \frac{\mu-1}{d-1}$ is sharp in the sense of $m \to \infty$. ♯

It is interesting to point out that for the class S of simple characteristics, we may also prove the gap in Theorem 0.5 is sharp.

Proposition 5.3 *Let μ and ϕ be as in* Proposition 5.2. *Then with the gap $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\mu-\frac{1}{2}}{d-1}$, the following inequalities*

$$\|e^{t\phi}\nabla^{m-\mu}u\|_q \leq \|e^{t\phi}\boldsymbol{P}u\|_p, \qquad \forall u \in C_0^\infty(B(0,1)),\ \forall t > 0$$

are sharp in the class of $P \in S$ with $P(\nabla\phi(0) + il) = 0$ for some $l \in \boldsymbol{R}^d$.

Remark: We may replace the class S by the class of polynomials which have Calderon's simple characteristics (see Definition 1.8). This is because the class of Calderon's simple characteristics is invariant under rotation. So we may always assume $\nabla\phi(0) = e_1$ and test the polynomial function $\xi_1^m + \xi_2^m + \cdots + \xi_d^m$ which satisfies the condition $(1 + il_1)^m + (ij_2)^m = 0$ for some $l = (l_1, l_2, 0, \cdots, 0)$.

Proof: As in the proof of Proposition 5.2 and the discussion in the above Remark, we may assume $\phi(x) = x_1 + O(|x|^2)$ and choose a $P = P_0(\xi_1, \xi_2) + Q(\xi_3, \cdots, \xi_d)$ of degree m such that $P(e^{t(x_1+ix_2)}) = 0$. Let Γ, φ and ψ be as in the proof of Proposition 5.2. Then we have

$$|P(\Gamma\varphi \cdot \psi)| = |P_0(\Gamma\varphi) \cdot \psi + \Gamma\varphi \cdot Q(\psi)|$$

$$\leq Ce^{-tx_1} t^{m-\frac{1}{2}} \chi_R$$

since the main contribution comes from $\nabla^{m-1}\Gamma \cdot \nabla\varphi$ because $P_0(\Gamma) = 0$. On the other hand,

$$|\nabla^{m-\mu}(\Gamma\varphi \cdot \psi)| \geq |\frac{d^{m-\mu}}{dx_1^{m-\mu}} \Gamma\varphi \cdot \psi| \geq C^{-1} e^{-tx_1} t^{m-\mu} \chi_{\frac{1}{2}R}$$

where R is as in before. Then by the same argument as in the proof of Proposition 5.2, we obtain $\frac{1}{p} - \frac{1}{q} \leq \frac{\mu - \frac{1}{2}}{d-1}$. ♯

This proposition says in particular that there is no Carleman inequality with gap $\frac{\mu}{d}$ for all $P \in S$ if we don't add any curvature condition.

Before we end this section, we would like to give an application to weak UCP by using Wolff's version Carleman inequality in Lemma 4.4 for $P \in D$.

Theorem 5.4 *Let $d \geq 2$ and $\mu \geq 0$ an integer. Let $r_\mu = \frac{d-\frac{3}{2}}{\mu-1}$ if $\mu \leq \frac{d}{2}+\frac{1}{4}$, $r_\mu = 2$ if $\mu > \frac{d}{2}+\frac{1}{4}$. Suppose $P \in D$ is of order m. If a function $u \in W^{m,2}$ with compact support and satisfies that $|\boldsymbol{P}u| \leq \sum_{2\leq\mu\leq m} A_\mu |\nabla^{m-\mu}u|$, then u vanishes identically if $A_\mu \in L^{r_\mu}_{loc}$ for all $2 \leq \mu \leq m$.*

The proof of Theorem 5.4 is exactly same as the proof of Theorem 4 in [Wo] by using Lemma 4.4. Or one may look at the proof of Theorem 0.3. So we omit it.

Remark Theorem 0.5 may also give a u.c.p. theorem directly. But our index r_μ here is smaller than the one got directly from Theorem 0.5.

6 Appendix and some further questions

First we would like make a little remark about our class G in Section 1. We have already given a lot of examples of elliptic homogeneous polynomials in G. A natural question is to ask how large this class G is in the universal class of elliptic homogeneous polynomials. Professor Wolff tells me the following result:

Proposition 6.1 *For generic homogeneous polynomials $P_1, \cdots, P_d$, their common zero set is only $\{0\}$.*

Remark If we assume the degree of P_j is m_j and write $P_j = \sum_{|\alpha|=m_j} a_{j,\alpha} z^\alpha$, then the "generic" means almost all of $\{a_{j,\alpha}\}_{|\alpha|=m_j, j=1,\cdots,d}$.

Proof: We believe that one may find a proof of this proposition in any regular text book of algebraic geometry such as in [8]. But we would like to give a short proof here instead of finding the exact reference. First let's make a claim.

Claim: *Suppose $P_1, \cdots, P_d$ are homogeneous polynomials of degrees $m_1, \cdots, m_d$ respectively. If the map $\Psi : z \in \mathbf{C}^d \setminus \{0\} \mapsto (P_1, \cdots, P_d) \in \mathbf{C}^d$ is transverse to 0, then there is no common zero with $P_1, \cdots, P_d$ except $\{0\}$.*

Proof: The assumption says that $D\Psi$ is surjective for all z of $\Psi(z) = 0$. Now assume that there is at least a nonzero z such that $\Psi(z) = 0$. Since P_j's are homogeneous, $\nabla P_j \cdot z = m_j P_j(z) = 0$, here $a \cdot b = \sum a_j b_j$. Hence $D\Psi(z) \cdot z = (0, \cdots, 0)$ which means $\det D\Psi(z) = 0$. This is a contradiction with $D\Psi$ being surjective. Now let's prove our proposition.

Proof of Proposition 6.1: let $A = \{a_{j,\alpha}\}_{|\alpha|=m_j, j=1,\cdots,d}$ be in $\boldsymbol{R}^N$ for some large number N. Consider $\Psi : \mathbf{C}^d \backslash \{0\} \times \mathbf{R}^N \to \mathbf{C}^d$ by $(z, A) \mapsto (P_1, \cdots, P_d)$. We claim that $D\Psi(z, A)$ is surjective. In fact, $D\Psi(z, A)$ is nothing but $(D_z\Psi, D_{\{\alpha\}}\Psi, D_{\{j\}}\Psi)$ which has a $z^\alpha I_{d\times d}$ submatrix from the last part. So $D\Psi$ is surjective. Now applying the transversality theorem (see [4]) to Ψ, we prove the conclusion. $\sharp$

By the same proof, we have the following result.

Proposition 6.2 *For generic homogeneous polynomials P of degree m, $\nabla P \neq 0$.*

Proof: We use almost the same notations as before. Write $P = \sum_{|\alpha|=m} a_\alpha z^\alpha$. Then $\nabla P = (\cdots, \sum_{|\alpha|=m} a_\alpha \alpha_j z^{\alpha - e_j}, \cdots)$, where $\alpha = (\alpha_1, \cdots, \alpha_d)$ and $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ hence $\alpha = \sum \alpha_j e_j$. Let $A = \{a_\alpha\}_{|\alpha|=m}$ be in some $\mathbf{R}^N$. By the claim in proof of proposition 6.1, we need to only show that

$$\Psi : \mathbf{C}^d \setminus \{0\} \times \mathbf{R}^N \to \mathbf{C}^d$$

by

$$(z, A) \mapsto (\cdots, \sum_{|\alpha|=m} a_\alpha \alpha_j z^{\alpha - e_j}, \cdots)$$

is transverse to 0. When $z \neq 0$, we may assume $z_1 \neq 0$. Consider the terms z_1^m, $z_1^{m-1}z_2$, $\cdots$, $z_1^{m-1}z_d$. We know that $\Psi(z, A) = (a_{\alpha^1} m z_1^{m-1} + \cdots, a_{\alpha^2} z_1^{m-1} + \cdots, \cdots, a_{\alpha^d} z_1^{m-1} + \cdots)$, where $\alpha^1 = me_1 = (m, 0, \cdots, 0)$, $\alpha^j = (m-1)e_1 + e_j = (m-1, 0, \cdots, 0, 1, 0, \cdots, 0)$. So in $D\Psi(z, A)$, there is a $d \times d$ diagonal submatrix where elements are mz_1^{m-1}, z_1^{m-1}, $\cdots$, z_1^{m-1} respectively. Of course it is invertible and hence $D\Psi(z, A)$ is surjective. Therefore, $\Psi(z, A)$ is transverse to zero. $\sharp$

This fact tells us that the generic elliptic homogeneous polynomials are in our class S.

Our other remark is about the class D. In order to see the class P is a subset of D, the following proposition will be helpful.

Proposition 6.3 *If $P \in S$, then for generic $k \in \boldsymbol{R}^d$ the sets $N = \{(x,y) \in \boldsymbol{R}^d \times \boldsymbol{R}^d \setminus (0,0) : P(x+iy) = 0\}$ and $\Pi_k = \{(x,y) \in \boldsymbol{R}^d \times \boldsymbol{R}^d : y = k\}$ are transversal.*

Proof: $\nabla_z P \neq 0$ on $\{z \in \boldsymbol{C}^d \setminus 0 : P(z) = 0\}$ says that N is a submanifold of codimension 2 in $\boldsymbol{R}^d \times \boldsymbol{R}^d$ and for generic $k \in \boldsymbol{R}^d$, $\nabla_x \mathrm{re}P(x+ik)$ and $\nabla_x \mathrm{im}P(x+ik)$ are linearly independent on the intersection $N \cap \Pi_k$ by the transversality theorem in [4] (or one may see Proposition 6.4 and the proof of Proposition 6.5 below). Let (x,k) be a point on $N \cap \Pi_k$. At (x,k), the tangent space of N in $\boldsymbol{R}^d \times \boldsymbol{R}^d$ is

$$T = (\nabla_x \mathrm{re}P, -\nabla_x \mathrm{im}P)^{\perp} \bigcap (\nabla_x \mathrm{im}P, \nabla_x \mathrm{re}P)^{\perp}$$

by the Cauchy-Riemann equations. The tangent space of Π_k at (x,k) in $\boldsymbol{R}^d \times \boldsymbol{R}^d$ is

$$S = \mathrm{Span}\,((e_1,0), \cdots, (e_d,0))$$

where $\{e_j\}$'s are the standard bases of $\boldsymbol{R}^d$. We want to show $T+S = \boldsymbol{R}^d \times \boldsymbol{R}^d$. We only need show both vectors $(\nabla_x \mathrm{re}P, -\nabla_x \mathrm{im}P)$ and $(\nabla_x \mathrm{im}P, \nabla_x \mathrm{re}P)$ are in $T+S$ because T is of dimension $2d-2$. Since $\nabla_x \mathrm{re}P$ and $\nabla_x \mathrm{im}P$ are linearly independent, there are two vectors a and b in $\boldsymbol{R}^d$ such that $(a, -\nabla_x \mathrm{im}P) \in T$ and $(b, \nabla_x \mathrm{re}P) \in T$. So we have $(\nabla_x \mathrm{re}P, -\nabla_x \mathrm{im}P) - (a, -\nabla_x$

$\mathrm{im}P) \in S$ and $(\nabla_x \mathrm{im}P, \nabla_x \mathrm{re}P) - (b, \nabla_x \mathrm{re}P) \in S$ and hence $(\nabla_x \mathrm{re}P, -\nabla_x \mathrm{im}P)$ and $(\nabla_x \mathrm{im}P, \nabla_x \mathrm{re}P)$ are in $T + S$. ♯

Now let's prove a proposition which we mentioned in section 1.

Proposition 6.4 *If $P \in S$, then $\nabla_\xi \mathrm{re}P(\xi + ik)$ and $\nabla_\xi \mathrm{im}P(\xi + ik)$ are linearly independent on $\pi_k(N^P)$ for almost all $k \in \boldsymbol{R}^d$*

Proof: Let $P \in S$. Then $\frac{d}{dz}P \neq 0$ on N^P which says for any $z \in N^P$ there at least is a j such that one of the following is not zero at z: $\frac{d}{d\xi_j}\mathrm{re}P$, $\frac{d}{dk_j}\mathrm{re}P$, $\frac{d}{d\xi_j}\mathrm{im}P$, $\frac{d}{dk_j}\mathrm{im}P$. On the other hand, we know that $\frac{d}{d\xi_j}\mathrm{re}P = \frac{d}{dk_j}\mathrm{im}P$ and $\frac{d}{dk_j}\mathrm{re}P = -\frac{d}{d\xi_j}\mathrm{im}P$ by the Cauchy-Riemann equation since P is analytic. So this says the matrix

$$\begin{pmatrix} \nabla_\xi \mathrm{re}P & \nabla_\xi \mathrm{im}P \\ \nabla_k \mathrm{re}P & \nabla_k \mathrm{im}P \end{pmatrix}$$

has rank 2 for all $\xi + ik \in N^P$. If we consider the map $(\mathrm{re}P, \mathrm{im}P) : \boldsymbol{R}^d \times \boldsymbol{R}^d \to \boldsymbol{R} \times \boldsymbol{R}$, the above fact tells us that this map is transverse to $0 \in \boldsymbol{R} \times \boldsymbol{R}$. Hence by the transversilty theorem (see page 68 in [4]), for almost all $k \in \boldsymbol{R}^d$ the map $(\mathrm{re}P(\cdot + ik), \mathrm{im}P(\cdot + ik)) : \boldsymbol{R}^d \to \boldsymbol{R}$ is transverse to $0 \in \boldsymbol{R}$. By the definition of transversilty, $\nabla_\xi \mathrm{re}P(\cdot + ik)$ and $\nabla_\xi \mathrm{im}P(\cdot + ik)$ are linearly independent on $\pi_k(N^P)$ for generic k. This proves the proposition. ♯

Now we would like to prove a geometric proposition which we used in the proof of Lemma 4.4 in section 4.

Proposition 6.5 *Suppose N is a submanifold of codimension 2 in $\boldsymbol{R}^{2d}$. Suppose N and $\Pi_k = \{(x,l) \in \boldsymbol{R}^{2d} : l = k\}$ are transverse for some $k \in S^{d-1}$. Then for any $\xi_0 \in \pi_k(N) = N \cap \Pi_k$ there are a neighborhood U of ξ_0 in $\boldsymbol{R}^d$ and a small number $s > 0$ such that for any $l \in S^{d-1}$ with $|l-k| \leq s$ there is a diffeomorphism $G_l : U \to B^{d-2} \times D(0,1)$ satisfying the following properties. $G_l(\pi_l(N)) \subset B^{d-2} \times \{0\}$ and $|\bigtriangledown G_l(\xi)| \geq C^{-1}$ for all $\xi \in U$. Where B^{d-2} is the unit ball of dimension $d-2$ and $D(0,1)$ is the unit disk.*

Proof: Since N is submanifold of codimension 2, for $(\xi_0, k) \in N$, there are a neighborhood M of (ξ_0, k) in $\boldsymbol{R}^{2d}$ and two smooth functions f and g such that $f^{-1}(0) \cap g^{-1}(0) \cap M = N \cap M$ and

$$\begin{pmatrix} \bigtriangledown_\xi f & \bigtriangledown_k f \\ \bigtriangledown_\xi g & \bigtriangledown_k g \end{pmatrix}$$

has rank 2 on M. Now we claim that

$$\begin{pmatrix} \bigtriangledown_\xi f \\ \bigtriangledown_\xi g \end{pmatrix}$$

has rank 2 on M.

Suppose not. Then for some point $(\xi, l) \in M$ there are two numbers a and b with $a^2 + b^2 \neq 0$ such that $a \bigtriangledown_\xi f + b \bigtriangledown_\xi g = 0$. We know that

$(\nabla_\xi f, \nabla_k f)$ and $(\nabla_\xi g, \nabla_k g)$ are two normal vectors to N. So $a(\nabla_\xi f, \nabla_k f) + b(\nabla_\xi g, \nabla_k g) = (0, a \nabla_k f + b \nabla_k g) \perp T_N$. On the other hand, it is obvious that $(0, a \nabla_k f + b \nabla_k g) \perp T_{\Pi_l}$. Since N and Π_l are also transverse when l is too close to k, this shows that $a(\nabla_\xi f, \nabla_k f) + b(\nabla_\xi g, \nabla_k g) = 0$ which is a contradiction.

So now we may assume without loss of generality that $\xi_{d-1} = f_1(\bar{\xi}, l)$ and $\xi_d = g_1(\bar{\xi}, l)$ define the submanifold $\pi_l(N) \cap U$ for some neighbohood U of ξ_0 in $\boldsymbol{R}^d$, where $\bar{\xi} \in \boldsymbol{R}^{d-2}$. Let $F_l(\xi) = (\bar{\xi}, \xi_{d-1} - f_1(\bar{\xi}), \xi_d - f_2(\bar{\xi}), l)$. Then we may see that $F_l(\pi_l) \subset \boldsymbol{R}^{d-2} \times \{0\}$ and $|\nabla F_l(\xi)| \geq 2$. Finally since π_l is compact, we may construct G_l as a composition of F_l with a certain dilation in the variables of $\boldsymbol{R}^{d-2}$. $\sharp$

Finally let's state some further questions in which we are interested to end this paper.

(1) Are generic elliptic homogeneous polynomials in the class G? Since we are interested in Carleman inequality, we may also ask another question like the following. Does Carleman inequality hold with a sharp gap as in Theorem 0.4 for generic elliptic homogeneous polynomials?

(2) In Theorem 0.2 and Theorem 0.3 we have a weak u.c.p. theorem with the reasonable condition that $V_\mu \in L^{\frac{d}{\mu}}$ if $m \leq \frac{d}{s}$. Do we have the same result with the condition on V_ν for m is close to d?

(3) Can we have a Carleman type inequality for an operator in G such that which may be directly used to prove a u.c.p. theorem, instead of a weak

u.c.p. theorem as in Section 3? If it is not possible, what is a reasonable condition for the operator? In [12], T. Wolff gets a u.c.p. theorem for $P \in S$.

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