SUMMATION FORMULAS ASSOCIATED WITH A CLASS OF DIRICHLET SERIES

Thesis by

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ABSTRACT

The Poisson summation formula, which gives, under suitable conditions on $f(x)$, an expression for sums of the form

$$\sum_{n=n_1}^{n_2} f(n) \qquad 1 \leq n_1 < n_2 \leq \infty$$

can be derived from the functional equation for the Riemann zeta-function $\zeta(s)$. In this thesis a class of Dirichlet series is defined whose members have properties analogous to those of $\zeta(s)$; in particular, each series in the class, written in the form

$$\phi(s) = \sum_{n=1}^{\infty} a(n)\, \lambda_n^{-s}$$

defines a meromorphic function $\phi(s)$ which satisfies a relation analogous to the functional equation of $\zeta(s)$. From this relation an identity for sum of the form

$$\sum_{\lambda_n \leq x} a(n)\, (x - \lambda_n)^q$$

is derived. This identity in turn leads, in a quite simple fashion, to summation formulas which give expressions for sums of the form

$$\sum_{n=n_1}^{n_2} a(n)\, f(\lambda_n) \qquad 1 \leq n_1 \leq n_2$$

The summation formulas thus derived include the Poisson and other well-known summation formulas as special cases and in addition embrace many expressions that are new. The formulas are not only of interest in themselves, but also provide a tool for investigating many problems that arise in analytic number theory.

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§ 1. Introduction.

In 1904, at the beginning of a long memoir, Voronoi [1]* made the following conjecture:

Let $\tau(n)$ be a function of the positive integer n; let $f(x)$ be continuous and have at most a finite number of maxima and minima in an interval $0 < a \leq x \leq b$. Then there are analytic functions $\delta(x)$ and $\alpha(x)$, depending only on $\tau(n)$, such that

$$\frac{1}{2} \sum_{n>a}^{n\leq b} \tau(n) f(n) + \frac{1}{2} \sum_{n\geq a}^{n<b} \tau(n) f(n)$$

(1.1)

$$= \int_a^b f(x)\, \delta(x)\, dx + \sum_{n=1}^{\infty} \tau(n) \int_a^b f(x)\, \alpha(nx)\, dx.$$

For $\tau(n) = 1$, we have $\delta(x) = 1$, $\alpha(x) = 2 \cos 2\pi x$, and (1.1) reduces to a case of the well-known Poisson summation formula. Voronoi succeeded in proving (1.1) for $\tau(n) = d(n)$, the number of divisors of n; accordingly, the formula

$$\frac{1}{2} \sum_{n>a}^{n\leq b} d(n) f(n) + \frac{1}{2} \sum_{n\geq a}^{n<b} d(n) f(n) = \int_a^b f(x) (\log x + 2\gamma) dx$$

(1.2)

$$+ 2\pi \sum_{n=1}^{\infty} d(n) \int_a^b \left\{ \frac{2}{\pi} K_0(4\pi\sqrt{nx}) - Y_0(4\pi\sqrt{nx}) \right\} f(x)\, dx$$

is known as Voronoi's summation formula. In (1.2) γ is Euler's constant, while Y and K represent, as usual, the Bessel function of the second kind and modified Bessel function of the second kind, respectively.

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* Numbers in brackets refer to the bibliography.

Since the appearance of Voronoi's paper, some other special cases of (1.1) have been established. In addition, a good deal of effort has been directed toward the establishment of (1.1) for fairly general classes of functions $\tau(n)$. In particular, Ferrar [2], [3] and Guinand [4], [5], [6], [7] have made notable contributions in this direction.

All such efforts have this much in common: the numbers $\tau(n)$, or $a(n)$ as they will be referred to henceforth, appear as coefficients in a Dirichlet series

$$\sum_{1}^{\infty} a(n)\, n^{-s},$$

or, more generally,

$$\sum_{1}^{\infty} a(n)\, \lambda_n^{-s}$$

with sufficiently 'nice' properties. The form of the functions $\delta(x)$ and $\alpha(x)$ is intimately connected with the analytical behavior of this Dirichlet series in the s-plane.

In this thesis, a summation formula is associated with each member of a very extensive class of Dirichlet series. This class includes a substantial proportion of the Dirichlet series that arise in analytic number theory. The summation formula itself is of a more general form than that in (1.1) and it includes as special cases the formulas of Poisson and Voronoi, as well as those developed by Ferrar and Guinand. The achievement of this measure of generality is due to a method of approach by which the association of a summation formula with a Dirichlet series is obtained in a particularly simple and natural manner.

In section 2, the class of Dirichlet series to be considered is defined. Functions corresponding to $\delta(x)$ and $a(x)$ are defined in section 3. In section 4, an identity (equation 4.1) involving the coefficients of the series of section 2 is derived. Special cases of this identity have been intensively studied, and their relevance to summation formulas has long been recognized. In section 5 the basic identity is linked with a simple method of constructing a sufficiently well-behaved function out of a more-or-less arbitrary function. The linking leads to the summation formulas themselves, which appear as equations (5.3), (5.6), and (5.9). Section 6 includes some important special cases. In section 7, we consider various applications, some of which are believed to be novel.

Certain conventions will be used throughout this thesis: The letter s will always denote a complex variable with real part σ and imaginary part t. A bare summation sign means summation on n from 1 to ∞. A symbol such as

$$\sum_{m \leq f(n) \leq M}$$

where $f(n)$ is some function of n, means a summation over those values of n for which $f(n)$ lies in the indicated range. Where no lower bound on the value of n is indicated, it is understood that a summation begins at $n = 1$. The symbol

$$\sum_{f(n) \leq M}{}'$$

is an abbreviation for

$$\frac{1}{2}\sum_{f(1)\leq f(n)<M} + \frac{1}{2}\sum_{f(1)\leq f(n)\leq M}$$

Similarly, the symbols

$$\sum_{f(n)\geq m}{}' , \quad \sum_{m\leq f(n)\leq M}{}''$$

are abbreviations for

$$\frac{1}{2}\sum_{f(n)>m} + \frac{1}{2}\sum_{f(n)\geq m} , \quad \frac{1}{2}\sum_{m<f(n)<M} + \frac{1}{2}\sum_{m\leq f(n)\leq M}$$

respectively. The symbol

$$\int_{(r)}$$

where r is a real number, is an abbreviation for

$$\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}$$

Unless explicitly stated otherwise, an integral sign denotes a proper integral, not a Cauchy principal value.

Finally it should be noted that a symbol once defined is used with the same meaning throughout the thesis and is not necessarily redefined at every use. An index of symbols is provided at the end of the thesis.

§ 2. The Class of Z-functions.

We shall be concerned with Dirichlet series of the forms

$$\sum a(n)\,\lambda_n^{-s}, \qquad \sum b(n)\,\mu_n^{-s}$$

where the coefficients $a(n)$, $b(n)$ are complex, while the numbers λ_n, μ_n are all positive and increase (strictly) without bound for increasing n. Further, we confine ourselves to series that have finite abscissas of absolute convergence: i.e., we assume the existence of a real number a such that the series with coefficients $a(n)$ converges absolutely for $\sigma > a$ and does not converge absolutely for $\sigma < a$. Similarly, we assume the existence of a corresponding real number b for the series with coefficients $b(n)$. The equations

$$(2.1) \qquad \phi(s) = \sum a(n)\,\lambda_n^{-s}$$

$$(2.2) \qquad \phi^*(s) = \sum b(n)\,\mu_n^{-s}$$

then define $\phi(s)$ and $\phi^*(s)$ as analytic functions of s for $\sigma > a$ and $\sigma > b$ respectively.

We now proceed to set up a class of functions, which we shall call the class of Z-functions.

<u>Definition 2.1</u>: A function $\phi(s)$ of the complex variable s is a <u>Z-function</u> if the following five conditions hold:

(i) For $\sigma > a$, $\phi(s)$ is given by (2.1).

(ii) $\phi(s)$ can be continued into the half-plane $\sigma \leq a$ and is a meromorphic function in the entire s-plane.

(iii) $\phi(s)$ is of finite order in every vertical strip of finite width. This means that for every finite real interval there is a constant A (depending on the interval) such that the relation

$$\phi(s) = O(|t|^{A}) \tag{2.3}$$

holds for sufficiently large $|t|$, uniformly for σ in the interval.

(iv) There is a real constant k, such that the function $H(s)$, defined for $\sigma > b$ by

$$H(s) = \frac{\phi(k - s)}{\phi^{*}(s)} \tag{2.4}$$

where $\phi^{*}(s)$ is given by (2.2), is a meromorphic function in the entire s-plane. Equation (2.4) consequently furnishes the analytic continuation of $\phi^{*}(s)$ into the half-plane $\sigma \leq b$ and shows that it too is a meromorphic function in the entire s-plane.

(v) There is a constant $a > 0$ such that the relation

$$H(s) = O(|t|^{a(2\sigma - k)}) \tag{2.5}$$

holds, for sufficiently large $|t|$, uniformly for σ in any finite interval.

It is important to note that (2.3) and (2.5) imply that $\phi(s)$ and $H(s)$ each have at most a finite number of poles in any vertical strip of finite width. In addition, these relations put a restriction on the possible values of k and b. We state this restriction as a lemma.

Lemma 2.1: We have

$$2b - k \geq 0 \tag{2.6}$$

Proof: We use the function $\mu(\sigma, f)$ defined for a function f by

$$\mu(\sigma, f) = \inf \{ A;\ f(\sigma + it) = O(|t|^A) \} \tag{2.7}$$

The general theory of Dirichlet series (see Hardy and Riesz [8], Chapter III, section 5) implies that

$$\mu(\sigma, \phi) \geq 0$$

for all σ, and that

$$\mu(\sigma, \phi^*) = 0 \tag{2.8}$$

for $\sigma \geq b$. By (2.5) we have

$$\mu(\sigma, H) \leq \alpha(2\sigma - k) \tag{2.9}$$

for all σ, and by (2.4) we have

$$\mu(k - \sigma, \phi) = \mu(\sigma, H) + \mu(\sigma, \phi^*) . \tag{2.10}$$

Taking $\sigma < k/2$, we have $\mu(\sigma, H) < 0$. Since $\mu(\sigma, \phi)$ is non-negative we must have $\mu(\sigma, \phi^*) > 0$. But $\mu(\sigma, \phi^*) = 0$ for $\sigma \geq b$, consequently we must have $\sigma < b$. So any σ less than $k/2$ is also less than b, which is possible if and only if $k/2 \leq b$. This proves the lemma.

In addition to (2.5), $H(s)$ may also satisfy a relation of the form

$$1/H(s) = O(|t|^{\alpha^*(k - 2\sigma)}) \tag{2.11}$$

for some $a^* > 0$. In this case $\phi^*(s)$ is also a Z-function and consequently we have

$$(2.12) \qquad 2a - k \geq 0.$$

Combining (2.6) and (2.12) we obtain

$$(2.13) \qquad a + b - k \geq 0.$$

It is convenient to define two numbers c and c^* by

$$(2.14) \qquad c = 2b - k, \qquad c^* = 2a - k.$$

Formulas (2.6), (2.12), and (2.13) then appear, respectively, as $c \geq 0$, $c^* \geq 0$, $1/2(c + c^*) \geq 0$.

Formula (2.11) and consequently (2.13), will hold in all the special cases considered later on. Indeed, we know of no example of a Z-function for which (2.13) does not hold, and with <u>strict</u> inequality at that. If (2.13) holds with strict inequality, then there is a strip $k - b < \sigma < a$ in which neither the series for $\phi(s)$ nor that for $\phi^*(s)$ is absolutely convergent. This strip is called the <u>critical strip</u> of the function $\phi(s)$.

The class of Z-functions includes a large proportion of the Dirichlet series of interest in analytic number theory. The Riemann zeta-function is a Z-function, as are all primitive Dirichlet L-series, all Hecke-series (see Hecke [9], Apostol and Sklar [10]), and all series of the form $\zeta(s)\zeta(s - r)$ or $\zeta^p(s)$ for real r and positive integral p. The very general class of functions considered by Landau in connection with the summatory functions of their coefficients

([11], Hauptsatz) is included in the class of Z-functions, as is the class of functions for which H.S.A. Potter has proved mean-value theorems ([12], Theorem 3). The outstanding example of a function that is <u>not</u> a Z-function is the function $1/\zeta(s)$. This obviously has too many singularities in the strip $0 < \sigma < 1$. Other, more recondite, examples are: (i) the function $\sum p_n^{-s}$, where p_n denotes the n^{th} prime, which has the imaginary axis as a natural boundary; and (ii) the function $\prod(1 - n^{-s})^{-1}$, where the product runs over all positive integers, which has a branch-point at $\sigma = 1$.

§ 3. The Functions $R_q(x)$ and $L_q(x)$.

In this section we consider two functions corresponding to the $\delta(x)$ and $a(x)$ of section 1. These functions are defined immediately below; after the definitions come two theorems giving properties of the functions that are useful for our purposes.

<u>Definition 3.1</u>: Let $\phi(s)$ be a Z-function. Let q be a complex number and x a complex number with positive real part. Then for $k - b \leq \max(0, a) = a^*$ (a weaker form of (2.13)), $R_q(x)$ is defined to be the sum of the residues of the function

$$x^s \phi(s) \frac{\Gamma(s)}{\Gamma(s + q + 1)}$$

in the strip $k - b \leq \sigma \leq a^*$. For $k - b > a^*$ we define $R_q(x)$ to be identically zero for all x and q.

<u>Definition 3.2</u>: Let $H(s)$ and α be as defined in Def. 2.1. Then for $\operatorname{re} x > 0$ and for complex q, $L_q(x)$ is defined by

$$(3.1) \qquad L_q(x) = \int_{(b+\varepsilon)} x^{-s} H(s) \frac{\Gamma(k - s)}{\Gamma(k - s + q + 1)} \, ds$$

for $\operatorname{re} q > \alpha c$ and by analytic continuation in the q-plane for other values of q. In (3.1), $\varepsilon > 0$ is chosen so that no singularities of the integrand lie in the strip $b < \sigma \leq b + \varepsilon$.

<u>Theorem 3.1</u>: For each fixed x in the half-plane $\operatorname{re} x > 0$, $R_q(x)$ is an entire function of q. For each fixed complex q, $R_q(x)$ is an analytic function of x in the half-plane $\operatorname{re} x > 0$. Furthermore, for $\operatorname{re} x > 0$ and any fixed complex q, we have

$$(3.2) \qquad \frac{d}{dx}\left\{x^q R_q(x)\right\} = x^{q-1} R_{q-1}(x),$$

while for $x > 0$, we have

$$(3.3) \qquad x^q R_q(x) = \frac{1}{\Gamma(q-p)} \int_0^x v^p R_p(v)\,(x-v)^{q-p-1}\,dv.$$

Formula (3.3) is valid in any region of the (p,q)-space in which the integral on the right is uniformly convergent. Another way of stating (3.3) is that $R_q(x)$ is the Riemann-Liouville fractional integral, of order $q - p$, of $R_p(x)$.

<u>Proof</u>: We need only consider the case $k - b \leq a^*$, the theorem holding vacuously otherwise. Since $\phi(s)$ and $\Gamma(s)$ have at most a finite number of poles in a vertical strip of finite width, while x^s and $1/\Gamma(s+q+1)$ are entire functions, $R_q(x)$ is the sum of a finite number of terms. Each term is an entire function of q for fixed x and an analytic function of x for fixed q, so $R_q(x)$ has the same property. The remaining properties will be obtained from an identity for $R_q(x)$ (formula (3.4) below). To derive this identity, we begin by using Cauchy's theorem to write $R_q(x)$ in the form

$$R_q(x) = \frac{1}{2\pi i} \int_C x^s \phi(s) \frac{\Gamma(s)}{\Gamma(s+q+1)}\,ds.$$

Here C is the contour consisting of the 4 segments

$$C_1 = (a^* + \delta - iT,\ a^* + \delta + iT)$$
$$C_2 = (a^* + \delta + iT,\ k - b - \varepsilon + iT)$$
$$C_3 = (k - b - \varepsilon + iT,\ k - b - \varepsilon - iT)$$
$$C_4 = (k - b - \varepsilon - iT,\ a^* + \delta - iT)$$

where $\delta > 0$, $\varepsilon > 0$, $T > 0$ are chosen so that no singularities of the integrand lie in the strips $a^* < \sigma \leq a^* + \delta$, $k - b - \varepsilon \leq \sigma < k - b$, or in the half-strips $k - b \leq \sigma \leq a^*$, $|t| \geq T$. Now we again use the function $\mu(\sigma, \phi)$ defined by (2.7). The general theory of Dirichlet series implies that $\mu(\sigma, \phi)$ is a non-increasing function of σ. Accordingly, for $\sigma \geq k - b - \varepsilon$, we have $\mu(\sigma, \phi) \leq \mu(k-b-\varepsilon, \phi)$. Combining (2.8), (2.9), and (2.10), we obtain

$$\mu(k - b - \varepsilon, \phi) \leq \alpha(2b - k + 2\varepsilon) < \alpha(2k - k + 3\varepsilon) = \alpha(c + 3\varepsilon).$$

It follows that uniformly on C_2 and C_4 we have

$$x^s \phi(s) \frac{\Gamma(s)}{\Gamma(s + q + 1)} = O(T^{\alpha(c+3\varepsilon)} \left| \frac{\Gamma(s)}{\Gamma(s + q + 1)} \right|)$$

$$= O(T^{\alpha(c+3\varepsilon) - \mathrm{re}\, q - 1})$$

Since ε can be arbitarily small, we can make the quantity $\alpha(c + 3\varepsilon) - \mathrm{re}\, q$ negative if we have $\mathrm{re}\, q > \alpha c$. Letting $T \to \infty$, the integrals along C_2 and C_4 vanish, while the ones along C_1 and C_3 are absolutely convergent. This proves the identity

$$(3.4) \qquad R_q(x) = \left(\int_{(a^*+\delta)} - \int_{(k-b-\varepsilon)} \right) x^s \phi(s) \frac{\Gamma(s)}{\Gamma(s + q + 1)} \, ds$$

for $\mathrm{re}\, q > \alpha c$. (A similar argument shows that the identity (3.4) also holds, for $\mathrm{re}\, q > 0$, in case $k - b > a^*$. Since $\alpha c \geq 0$ by Def. 1.1(v) and (2.6), it follows that (3.4) holds for $\mathrm{re}\, q > \alpha c$ in any case.)

It is now easy to complete the proof of Theorem 3.1. For $\operatorname{re} q > \alpha c + 1$, we can multiply (3.4) by x^q and differentiate under the integral signs. This yields (3.2) for $\operatorname{re} q > \alpha c + 1$ and we can extend the range of (3.2) to all q by analytic continuation. As for (3.3), we consider the standard Beta-function formula (cf. Erdélyi [13], formula (6.2.31))

$$\frac{\Gamma(s+p+1)}{\Gamma(s+q+1)} = \frac{x^{-s-q}}{\Gamma(q-p)} \int_0^x v^{s+p} (x-v)^{q-p-1}\, dv$$

valid for $\operatorname{re} q > \operatorname{re} p$, $\operatorname{re}(s+p+1) > 0$. Using this, we have by (3.4):

$$R_q(x) = \frac{x^{-q}}{\Gamma(q-p)} \Big(\int_{(a^*+\delta)} - \int_{(k-b-\varepsilon)} \Big) \int_0^x v^{s+p}(x-v)^{q-p-1} \frac{\phi(s)\Gamma(s)}{\Gamma(s+p+1)}\, dv\, ds$$

$$= \frac{x^{-q}}{\Gamma(q-p)} \int_0^x v^p (x-v)^{q-p-1} \Big(\int_{(a^*+\delta)} - \int_{(k-b-\varepsilon)} \Big) v^s \phi(s) \frac{\Gamma(s)}{\Gamma(s+p+1)}\, ds\, dv$$

$$= \frac{x^{-q}}{\Gamma(q-p)} \int_0^x v^p (x-v)^{q-p-1} R_p(v)\, dv.$$

The interchange of orders of integration is justified by absolute convergence for $\operatorname{re} p > \alpha c$. This proves (3.3) for $\operatorname{re} p > \max(\alpha c, 1+b-k, 1-a)$. Again we appeal to analytic continuation to extend the range of validity of (3.3). This completes the proof of Theorem 3.1.

<u>Theorem 3.2</u>: For each fixed x in the half-plane $\operatorname{re} x > 0$, $L_q(x)$ is an entire function of q. For each fixed complex q, $L_q(x)$ is an

analytic function of x in the half-plane $\operatorname{re} x > 0$. Furthermore, for $\operatorname{re} x > 0$ and any fixed complex q and positive μ, we have

$$(3.5) \qquad \frac{d}{dx}\left\{x^{k+q} L_q(\mu x)\right\} = x^{k+q-1} L_{q-1}(\mu x),$$

while for $x > 0$, we have

$$(3.6) \qquad x^{k+q} L_q(\mu x) = \frac{1}{\Gamma(q-p)} \int_0^x v^{k+p} L_p(\mu v)\,(x-v)^{q-p-1}\,dv.$$

Formula (3.6) is valid in any region of the (p,q)-space in which the integral on the right is uniformly convergent.

Proof: The proof of Theorem 3.2 uses virtually the same arguments as the proof of Theorem 3.1, the only essential difference being that a special argument is needed to establish the continuation of $L_q(x)$ over the entire q-plane. This continuation can be obtained as follows: Let q be any complex number and w be such that $\operatorname{re} w \geq 0$, $\operatorname{re}(q+w) > \alpha c$. Then by (3.1) we can write

$$L_{q+w}(x) = \int_{(b+\varepsilon)} x^{-s} H(s) \frac{\Gamma(k-s)}{\Gamma(k-s+q+w+1)}\,dw.$$

By virtue of (2.5) we can use Cauchy's theorem to shift the contour any distance to the left. We have, in particular,

$$L_{q+w}(x) = \int_{(b^*)} x^{-s} H(s) \frac{\Gamma(k-s)}{\Gamma(k-s+q+w+1)}\,dw + \Psi_{q+w}(x)$$

where b^* satisfies $b^* < 1/2(k + \operatorname{re} q/\alpha)$ and $\Psi_{q+w}(x)$ is the sum of the residues of the integrand in the strip $b^* < \sigma < b+\varepsilon$.

The same argument as that used in the beginning of the proof of Theorem 3.1 shows that $\Psi_{q+w}(x)$ is an entire function of w, while the integral along the line $\sigma = b^*$ is absolutely and uniformly convergent for $\text{re } w \geq 0$. Therefore we may set $w = 0$ to obtain

$$L_q(x) = \int_{(b^*)} x^{-s} H(s) \frac{\Gamma(k - s)}{\Gamma(k - s + q + 1)} \, dw + \Psi_q(x).$$

This provides the required continuation of $L_q(x)$ over the q-plane, and the remainder of the proof is, as remarked, entirely analogous to the proof of Theorem 3.1 and need not be repeated.

§ 4. A Basic Identity.

We are now in a position to establish the following theorem:

<u>Theorem 4.1</u>: Let $\phi(s)$ be a Z-function, as defined in Def. 2.1, with the associated numbers $a(n)$, λ_n, $b(n)$, μ_n, a, b, k, α. For x positive and q complex, let $R_q(x)$ and $L_q(x)$ be as defined in Defs. 3.1 and 3.2, respectively. Then we have

$$(4.1)\quad \frac{1}{\Gamma(q+1)} \sum_{\lambda_n \leq x}{}' a(n)(x-\lambda_n)^q = x^q R_q(x) + x^{k+q} \sum b(n) L_q(\mu_n x)$$

in any region of the q-plane in which the series on the right is uniformly convergent. In particular, (4.1) holds for $\operatorname{re} q > \alpha(2b-k) = \alpha c$, the series being absolutely convergent in this case.

<u>Proof</u>: By (3.4), we have for $\operatorname{re} q > \alpha c$,

$$\int_{(a^*+\delta)} x^s \phi(s) \frac{\Gamma(s)}{\Gamma(s+q+1)}\, ds = R_q(x) + \int_{(k-b-\varepsilon)} x^s \phi(s) \frac{\Gamma(s)}{\Gamma(s+q+1)}\, ds$$

where the integrals are absolutely convergent. Here $a^* = \max(0, a)$, and the numbers $\delta > 0$, $\varepsilon > 0$ are chosen so that no singularities of the integrands lie in the stripes $a^* < \sigma \leq a^* + \delta$, $k - b - \varepsilon \leq \sigma < k-b$. After expressing $\phi(s)$ by (2.1), absolute convergence enables us to interchange integration and summation and obtain, for the integral on the left

$$\int_{(a^*+\delta)} = \sum a(n) \int_{(a^*+\delta)} (\lambda_n/x)^{-s} \frac{\Gamma(s)}{\Gamma(s+q+1)}\, ds$$

$$= \frac{1}{\Gamma(q+1)} \sum_{\lambda_n \leq x}{}' a(n)(1-\lambda_n/x)^q$$

by formula (7.3.20) in [13]. In the integral on the left, we use (2.4) and then (2.2) to obtain

$$\int_{(k-b-\varepsilon)} = \int_{(k-b-\varepsilon)} x^s \phi^*(k-s)\, H(k-s)\, \frac{\Gamma(s)}{\Gamma(s+q+1)}\, ds$$

$$= \sum b(n) \mu_n^{-k} \int_{(k-b-\varepsilon)} (\mu_n x)^s\, H(k-s)\, \frac{\Gamma(s)}{\Gamma(s+q+1)}\, ds$$

$$= x^k \sum b(n) \int_{(b+\varepsilon)} (\mu_n x)^{-s}\, H(s)\, \frac{\Gamma(k-s)}{\Gamma(k-s+q+1)}\, ds$$

$$= x^k \sum b(n)\, L_q(\mu_n x)$$

by (3.1). The interchange of integration and summation in the above argument is justified by absolute convergence. This proves (4.1) for $\operatorname{re} q > \alpha c$, and, as usual, we appeal to analytic continuation to extend the range of q. This is possible since, by Theorems 3.1 and 3.2, $R_q(x)$ and $L_q(x)$ are entire functions of q. The right-hand side of (4.1) is therefore an analytic function of q in any region of the q-plane in which the series $\sum b(n)\, L_q(\mu_n x)$ is uniformly convergent. This completes the proof of the theorem.

It should be noted that if we have x equal to one of the λ_n's, say λ_N, and $a(N)$ is not zero, then the left-hand side of (4.1) is discontinuous in q at $q = 0$. Consequently the series $\sum b(n)\, L_q(\mu_n x)$ cannot be uniformly convergent in any region of the q-plane containing the point $q = 0$. This by itself has no bearing on the validity of (4.1) for $q = 0$ except to show that it will usually require a special argument to establish such validity.

Very many special cases of (4.1) are to be found in the literature. Leaving aside the case $a(n) = 1$, $\lambda_n = n$, which is a well-known Fourier series expansion, the first proof of an instance of (4.1) appears in Voronoi's paper [1]. The case considered there is $\phi(s) = \sum d(n)\, n^{-s}$, q a non-negative integer, where $d(n)$ is the number of divisors of n. Wigert [14] treated the case $\phi(s) = \sum \sigma_{-1}(n)\, n^{-s}$, where $\sigma_r(n)$ is the sum of the r^{th} powers of the divisors of n, for q a positive integer. Landau [11] established (4.1) for a large subclass of the class of Z-functions, for the particular value of q corresponding to $q = [\alpha c] + 2$, where $[x]$ denotes the greatest integer function. The case $a(n) = r_2(n)$, where $r_p(n)$ denotes the number of representations of n as a sum of p squares, was considered by Hardy, in [15] for $q = 0$ and in [16] for q equal to a non-negative integer. Wilton [17] proved (4.1) for $a(n) = \tau(n)$, where $\tau(n)$ is Ramanujan's function, $q > 0$, and Hardy [18] extended this case to $q = 0$. Oppenheim [19] treats the cases $a(n) = \sigma_r(n)$ and $a(n) = r_p(n)$ for real r and $p \geq 2$, for $q = 0$ and 1. He discusses summability as well as convergence of the infinite series. Wilton [20] effectively disposes of the case $a(n) = \sigma_r(n)$, treating it for general complex r and q and discussing the absolute convergence, ordinary convergence, summability (by Riesz means) and non-summability of the infinite series. Apostol [21] establishes (4.1) for Hecke series for integral q satisfying $q > c - \frac{1}{2}$. Other cases are to be found in Cramér [22], Walfisz [23], [24], [25] and Guinand [7].

§ 5. The Summation Formula.

The main object of this section is the association of a summation formula with every Z-function. Before this association can be achieved, some preliminary definitions are necessary.

Definition 5.1: For every real-valued function $f(x)$ of the real variable x and every positive integer p, we define a function $f_{(p)}(x)$ by the following recursion:

(i) We set $f_{(1)}(x)$ equal to $f'(x)$ wherever $f'(x)$ exists. If f' does not exist at x but does exist in a deleted neighborhood of x, we define

$$f_{(1)}(x)=\frac{1}{2}\lim_{\varepsilon\to 0}\left\{f'(x+\varepsilon)+f'(x-\varepsilon)\right\}$$

provided this limit exists. In all other cases we set $f_{(1)}(x)$ equal to zero.

(ii) $f_{(p+1)}(x) = \left\{f_{(p)}\right\}_{(1)}(x)$.

Definition 5.2: For every complex-valued function $f(x)$ of the real variable x, and every positive integer p, $f_{(p)}(x)$ is defined by

$$f_{(p)}(x) = \left\{\mathrm{re}\, f\right\}_{(p)}(x) + i\left\{\mathrm{im}\, f\right\}_{(p)}(x).$$

It should be noted that on any interval in which the (real-or complex-valued) function f has a p^{th} derivative we have $f_{(p)} = f^{(p)}$.

Definition 5.3: Let p be a non-negative integer and let $f(x)$ be a function such that $f_{(p+1)}$ is integrable on $[m,M]$, $m \leq M < \infty$. We define $f_M(x)$ by the equation

$$(5.1)\qquad f_M(x) = \begin{cases} \dfrac{(-1)^{p+1}}{\Gamma(p+1)} \displaystyle\int_x^M (v-x)^p f_{(p+1)}(v)\,dv & m \leq x \leq M \\ 0 & x \geq M. \end{cases}$$

In other words, $f_M(x)$ is $(-1)^{p+1}$ times the incomplete Weyl integral, of order $p+1$, of $f_{(p+1)}(x)$. If the complete Weyl integral exists, we denote it by $(-1)^{p+1} f^*(x)$: i.e., we define

$$(5.2)\qquad f^*(x) = \frac{(-1)^{p+1}}{\Gamma(p+1)} \int_x^\infty (v-x)^p f_{(p+1)}(v)\,dv,$$

provided the integral exists.

Some immediate consequences of Def. 5.3 are embodied in the following lemma:

Lemma 5.1: $f_M(x)$ is of class C^p on (m,∞), and $f_M(x)$ and its first p derivatives vanish for $x \geq M$. $f_M(x)$ has a $(p+1)^{st}$ derivative on (m,M) almost everywhere equal to $f_{(p+1)}(x)$. $f^*(x)$ has corresponding properties.

Thus we have constructed out of $f(x)$ a function $f_M(x)$ (or $f^*(x)$) which is very well-behaved. It is this function, rather than $f(x)$ itself, which will appear in the summation formula. f and f_M may be quite different in appearance; but if f is itself sufficiently well-behaved, then there is a simple relationship between f and f_M:

<u>Lemma 5.2</u>: If $f(x)$ is of class C^{p+1} on an open interval (x_1,x_2) contained in (m,M), then on (x_1,x_2) the difference $f(x) - f_M(x)$ is a polynomial of degree not greater than p. If in addition there is a point x_0 in the closed interval $[x_1,x_2]$ such that

$$\lim_{x\to x_0} f^{(j)}(x) = \lim_{x\to x_0} f_M^{(j)}(x)$$

for each $j = 0, 1, \ldots, p$, then $f(x) = f_M(x)$ on (x_1,x_2).

The proof of the first statement of the lemma consists simply in the observation that the $(p+1)^{st}$ derivative of the difference $f(x) - f_M(x)$ vanishes identically on (x_1,x_2). The second sentence of the lemma follows directly from the first and the fact that a polynomial of degree not greater than p, with a $(p+1)$-fold root must vanish identically.

We come now to the summation formula itself, which we state in 2 forms, corresponding respectively to $m > 0$ and $m = 0$ in Def. 5.3.

<u>Theorem 5.1</u>: Let $\phi(s)$ be a Z-function, as defined in Def. 2.1, with the associated numbers $a(n)$, $b(n)$, λ_n, μ_n, a, b, k, α. Let $R_q(x)$ and $L_q(x)$ be as defined in Definitions 3.1 and 3.2, and let m and M be constants satisfying $0 < m \leq M < \infty$. Let p be a non-negative integer such that (4.1) holds for $q = p$, and assume that the series $\sum b(n) L_p(\mu_n x)$ is boundedly convergent for x in $[m,M]$. (In particular this will always hold for $p > \alpha(2b-k) = \alpha c$.) Then for any function $f(x)$ such that $f_M(x)$ exists on $[m,M]$ we have

$$\sum_{m \le \lambda_n \le M}{}'' a(n)\, f_M(\lambda_n) = Q(p;m,M) + (-1)^p \int_m^M v^{p-1} R_{p-1}(v)\, f_M^{(p)}(v)\, dv$$

$$+ (-1)^p \sum b(n) \int_m^M v^{k+p-1} L_{p-1}(\mu_n v)\, f_M^{(p)}(v)\, dv \tag{5.3}$$

where

$$Q(p;m,M) = \frac{(-1)^{p-1}}{\Gamma(p)} \sum_{\lambda_n \le m}{}' a(n) \int_m^M (v-\lambda_n)^{p-1} f_M^{(p)}(v)\, dv \tag{5.4}$$

for $p > 0$, and $Q(0;m,M) = 0$.

<u>Theorem 5.2</u>: If the hypotheses of Theorem 5.1 hold for all $m > 0$ and in addition the following conditions hold,

(i) $v^p R_p(v) f_{(p+1)}(v)$ is integrable on $(0,M)$,

(ii) $v^{k+p} L_p(\mu_n v) f_{(p+1)}(v)$ is integrable on $(0,M)$ for all μ_n,

(iii) $\sum b(n) \int_0^M v^{k+p} L_p(\mu_n v) f_{(p+1)}(v)\, dv$ converges,

(iv) $\lim_{m \to 0+} \sum b(n) \int_0^m v^{k+p} L_p(\mu_n v) f_{(p+1)}(v)\, dv = 0$,

then we have

$$\sum_{\lambda_n \le M} a(n)\, f_M(\lambda_n) = (-1)^{p+1} \int_0^M v^p R_p(v)\, f_M^{(p+1)}(v)\, dv$$

$$+ (-1)^{p+1} \sum b(n) \int_0^M v^{k+p} L_p(\mu_n v)\, f_M^{(p+1)}(v)\, dv. \tag{5.5}$$

If (5.5) holds and in addition the following limits exist,

$$\lim_{v\to 0+} f_M^{(p)}(v) = f_M^{(p)}(0+),$$

$$\lim_{v\to 0+} v^p R_p(v) = R_p^* ,$$

$$\lim_{v\to 0+} v^{k+p} L_p(v) = L_p^* ,$$

then we have

$$\sum_{\lambda_n\le M} a(n)\, f_M(\lambda_n) = (-1)^p R_p^* f_n^{(p)}(0+) + (-1)^p \int_0^M v^{p-1} R_{p-1}(v) f_M^{(p)}(v)dv$$

(5.6)
$$+ (-1)^p \sum b(n) \Big\{ \mu_n^{-k-p} L_p^* f_M^{(p)}(0+)$$

$$+ \int_0^M v^{k+p-1} L_{p-1}(\mu_n v)\, f_M^{(p)}(v)\, dv \Big\} .$$

<u>Proof of Theorem 5.1</u>: By (5.1) we have

$$\sum_{m\le\lambda_n\le M}{}'' a(n)\, f_n(\lambda_n) = \frac{(-1)^{p+1}}{\Gamma(p+1)} \sum_{m\le\lambda_n\le M}{}'' a(n) \int_{\lambda_n}^M (v-\lambda_n)^p f_{(p+1)}(v)\, dv$$

$$= \frac{(-1)^{p+1}}{\Gamma(p+1)} \int_m^M \Big\{ \sum_{m\le\lambda_n\le v}{}'' a(n)(v-\lambda_n)^p \Big\} f_{(p+1)}(v)\, dv$$

$$= - \frac{(-1)^{p+1}}{\Gamma(p+1)} \int_m^M \Big\{ \sum_{\lambda_n\le m}{}' a(n)(v-\lambda_n)^p \Big\} f_{(p+1)}(v)\, dv$$

$$+ \frac{(-1)^{p+1}}{\Gamma(p+1)} \int_m^M \Big\{ \sum_{\lambda_n\le v}{}' a(n)(v-\lambda_n)^p \Big\} f_{(p+1)}(v)\, dv$$

In the last term we apply (4.1) and obtain

$$\sum_{m\le\lambda_n\le M}'' a(n)f_M(\lambda_n) = \frac{(-1)^p}{\Gamma(p+1)} \int_m^M \Big\{ \sum_{\lambda_n\le m}' a(n)(v-\lambda_n)^p \Big\} f_{(p+1)}(v)\,dv$$

$$+ (-1)^{p+1} \int_m^M \Big\{ v^p R_p(v) +$$

$$+ v^{k+p} \sum b(n)\, L_p(\mu_n v) \Big\} f_{(p+1)}(v)\,dv$$

$$= \frac{(-1)^p}{\Gamma(p+1)} \sum_{\lambda_n\le m}' a(n) \int_m^M (v-\lambda_n)^p f_{(p+1)}(v)\,dv$$

$$+ (-1)^{p+1} \int_m^M v^p R_p(v) f_{(p+1)}(v)\,dv$$

$$+ (-1)^{p+1} \int_m^M v^{k+p} \Big\{ \sum b(n) L_p(\mu_n v) \Big\} f_{(p+1)}(v)\,dv$$

In the last term the assumption of bounded convergence enables us to integrate the series term-by-term. The result is

$$\sum_{m\le\lambda_n\le M}'' a(n)f_M(\lambda_n) = \frac{(-1)^p}{\Gamma(p+1)} \sum_{\lambda_n\le m}' a(n) \int_m^M (v-\lambda_n)^p f_{(p+1)}(v)\,dv$$

(5.7) $$+ (-1)^{p+1} \int_m^M v^p R_p(v) f_{(p+1)}(v)\,dv$$

$$+ (-1)^{p+1} \sum b(n) \int_m^M v^{k+p} L_p(\mu_n v) f_{(p+1)}(v)\,dv$$

The next step is to integrate by parts in all the integrals on the right of (5.7). By Lemma 5.1, terms containing $f_M^{(p)}(M)$ vanish, and the result is

$$\sum_{m \le \lambda_n \le M}'' a(n)\, f_M(\lambda_n) = -f_M^{(p)}(m) \frac{(-1)^p}{\Gamma(p+1)} \sum_{\lambda_n \le m}' a(n)(m-\lambda_n)^p + Q(p;m,M)$$

$$(5.8) \quad + (-1)^p f_M^{(p)}(m)\, m^p\, R_p(m) + (-1)^p \int_m^M v^{p-1} R_{p-1}(v) f_M^{(p)}(v)\,dv$$

$$+ (-1)^p \sum b(n) \Big\{ m^{k+p}\, L_p(\mu_n m)\, f_M^{(p)}(m)$$

$$+ \int_m^M v^{k+p-1}\, L_{p-1}(\mu_n v)\, f_M^{(p)}(v)\,dv \Big\}$$

By hypothesis the series $\sum b(n)\, L_p(\mu_n m)$ converges, so the last term on the right of (5.8) can be written as

$$(-1)^p\, f_M^{(p)}(m)\, m^{k+p} \sum b(n)\, L_p(\mu_n m)$$

$$+ (-1)^p \sum b(n) \int_m^M v^{k+p+1}\, L_{p-1}(\mu_n v)\, f_M^{(p)}(v)\,dv$$

We now use (4.1) to obtain

$$(-1)^p\, f_M^{(p)}(m) \Big\{ -\frac{1}{\Gamma(p+1)} \sum_{\lambda_n \le m}' a(n)(m-\lambda_n)^p + m^p\, R_p(m)$$

$$+ m^{k+p} \sum b(n)\, L_p(\mu_n m) \Big\} = 0$$

Substitution of this into (5.8) yields (5.3) and completes the proof of Theorem 5.1.

<u>Proof of Theorem 5.2</u>: Taking $m < \lambda_1$ in (5.7) we have

$$\sum_{\lambda_n \le M} a(n)\, f_M(\lambda_n) = (-1)^{p+1} \int_m^M v^p\, R_p(v)\, f_{(p+1)}(v)\,dv$$

$$+ (-1)^{p+1} \sum b(n) \int_m^M v^{k+p} L_p(\mu_n v)\, f_{(p+1)}(v)\,dv.$$

The hypotheses of Theorem 5.2 permit us to replace m by 0 everywhere in this expression. We may then replace $f_{(p+1)}(v)$ by $f_M^{(p+1)}(v)$ since, by Lemma 5.1, the two functions can differ at most on a set of measure zero on the interval $(0,M)$. This yields (5.5). Finally, (5.6) follows immediately from (5.5) on integration by parts.

For $p = 0$, (5.3) reduces to

$$\sum_{m \leq \lambda_n \leq M}'' a(n)\, f_M(\lambda_n) = \int_m^M R(v)\, f_M(v)\, dv$$

$$+ \sum b(n) \int_m^M v^{k-1} L(\mu_n v)\, f_M(v)\, dv \tag{5.9}$$

where $R(v) = v^{-1} R_{-1}(v)$ and $L(v) = L_{-1}(v)$. Formula (5.9) will also be valid for $p > 0$ in case we have

$$f_M(m) = f_M'(m) = \cdots = f_M^{(p-1)}(m) = 0. \tag{5.10}$$

That this is so can be seen by integrating by parts p times in the integrals on the right of (5.3). We state the results of this paragraph as a theorem:

<u>Theorem 5.3</u>: Sufficient conditions for the validity of (5.9) are: the validity of the hypotheses of Theorem 5.1 for $p = 0$, or the validity of the hypotheses of Theorem 5.1 for $p > 0$ and in addition the validity of (5.10).

Equation (5.9) should be compared with equation (1.1).

If we formally replace M by ∞ and f_M by f^* in (5.3) we obtain

$$\sum_{\lambda_n \geq m}{}' a(n)\, f^*(\lambda_n) = Q^*(p;m) + (-1)^p \int_m^\infty v^{p-1} R_{p-1}(v)\, f^{*(p)}(v)\, dv$$

$$(5.11) \qquad + (-1)^p \sum b(n) \int_m^\infty v^{k+p-1} L_{p-1}(\mu_n v)\, f^{*(p)}(v)\, dv$$

where

$$(5.12) \quad Q^*(p;m) = \frac{(-1)^{p-1}}{\Gamma(p)} \sum_{\lambda_n \leq m}{}' a(n) \int_m^\infty (v - \lambda_n)^{p-1} f^{*(p)}(v)\, dv$$

if $p > 0$, and $Q^*(0;m) = 0$. Similarly, the formula corresponding to (5.6) is

$$a(n)\, f^*(\lambda_n) = (-1)^p R_p^* f^{*(p)}(0+) + (-1)^p \int_0^\infty v^{p-1} R_{p-1} f^{*(p)}(v)\, dv$$

$$(5.13) \qquad + (-1)^p \sum b(n) \Big\{ \mu_n^{-k-p} L_p^* f^{*(p)}(0+)$$

$$+ \int_0^\infty v^{k+p-1} L_{p-1}(\mu_n v)\, f^{*(p)}(v)\, dv \Big\}.$$

It would not be difficult to give sufficient conditions for the validity of (5.11) or (5.13). In practice, however, it is usually simpler to derive (5.11) or (5.13) directly from (5.3) or (5.6) in any particular case.

§ 6. Some Special Cases.

a) The Riemann Zeta-function.

For $\phi(s) = r^{-s}\zeta(s)$, where r is positive and $\zeta(s)$ is the Riemann zeta-function, we have

$$a(n) = b(n) = 1 \quad \text{for all } n$$

$$\lambda_n = rn, \quad \mu_n = \pi n/r$$

$$a = b = k = c = 1$$

$$\alpha = \frac{1}{2}$$

$$H(s) = \frac{\sqrt{\pi}}{r}\,\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} = \frac{2}{r}\,2^{-s}\cos\frac{\pi s}{2}\,\Gamma(s)$$

$$R_q(x) = \frac{1}{\Gamma(q+1)}\left\{\frac{x}{r(q+1)} - \frac{1}{2}\right\}$$

$$L(x) = L_{-1}(x) = \frac{2}{r}\cos 2x$$

$$(6.0)\quad L_q(\mu x) = \frac{2x^{-q-1}}{r\,\Gamma(q+1)}\int_0^x \cos(2\mu v)(x-v)^q\,dv \qquad \text{for } \operatorname{re} q > -1.$$

In this case, (4.1) appears in the form

$$(6.1)\quad \sum_{n\leq x/r}{}' (x-rn)^q = \frac{x^{q+1}}{r(q+1)} - \frac{x^q}{2} + \frac{2}{r}\sum\int_0^x \cos(2\pi nv/r)(x-v)^q\,dv$$

valid for $q = 0$ and for $\operatorname{re} q > 0$. Accordingly, (5.3) is valid with $p = 0$ in the form

$$(6.2) \quad \sum_{m/r \le n \le M/r}'' f_M(rn) = \frac{1}{r} \int_m^M f_M(v)\, dv + \frac{2}{r} \sum \int_m^M \cos(2\pi nv/r)\, f_M(v)\, dv,$$

and if $f_M(0+)$ exists, then (5.6) is valid with $p = 0$ in the form

$$(6.3) \quad \sum_{n \le M/r} f_M(rn) = -\frac{1}{2} f_M(0+) + \frac{1}{r} \int_0^M f_M(v)\, dv$$

$$+ \frac{2}{r} \sum \int_0^M \cos(2\pi nv/r)\, f_M(v)\, dv.$$

The equations corresponding to (5.11) and (5.13) are, respectively,

$$(6.4) \quad \sum_{n \ge m/r}' f^*(rn) = \frac{1}{r} \int_m^\infty f^*(v)\, dv + \frac{2}{r} \sum \int_m^\infty \cos(2\pi nv/r)\, f^*(v)\, dv.$$

$$(6.5) \quad \sum f^*(rn) = -\frac{1}{2} f^*(0+) + \frac{1}{r} \int_0^\infty f^*(v)\, dv + \frac{2}{r} \sum \int_0^\infty \cos(2\pi nv/r) f^*(v) dv.$$

All of these equations are, of course, forms of the well-known Poisson summation formula.

b) Generalized Hecke-series.

By a "generalized Hecke-series" we mean a Z-function for which $H(s)$ is given by

$$(6.6) \qquad H(s) = h \frac{\Gamma(s)}{\Gamma(k-s)}$$

where h is a constant. In this case (3.1) yields, by formula (7.3.23) of [13],

$$(6.7) \qquad L_q(x) = hx^{-\frac{1}{2}(k+q)} J_{k+q}(2\sqrt{x})$$

where J_ν is the usual Bessel-function of order ν. From the relation

$$J_\nu(x) = O(x^{-\frac{1}{2}})$$

for $x > 0$, it is apparent that the series

$$\sum b(n)\, L_q(\mu_n x) = x^{-\frac{1}{2}(k+q)}\, h \sum b(n) \mu_n^{-\frac{1}{2}(k+q)}\, J_{k+q}(2\sqrt{(\mu_n x)})$$

is absolutely and uniformly convergent for any set of values of x and q for which x is in a compact interval of the positive real axis and q is in any half-plane $\operatorname{re} q \geq q_o > 2b - k - \frac{1}{2} = c - \frac{1}{2}$. Accordingly, (4.1) is valid for $\operatorname{re} q > c - \frac{1}{2}$, and for any such q the left side of (4.1) must be a continuous function of x for x positive. But the left side of (4.1) is a discontinuous function of x for $q = 0$; therefore the point $q = 0$ cannot fall in the half-plane $\operatorname{re} q > c - \frac{1}{2}$. This means that the relation

(6.8) $\quad c = 2b - k \geq \frac{1}{2}$

must hold for any generalized Hecke-series.

Since the form of (6.6) guarantees that $\phi^*(s)$ is also a generalized Hecke-series, we have the corresponding relation,

(6.9) $\quad c^* = 2a - k \geq \frac{1}{2}$.

Combining (6.8) and (6.9) yields

(6.10) $\quad a + b - k \geq \frac{1}{2}$.

Since $a + b - k$ is the width of the critical strip, (6.10) can be restated in the form: every generalized Hecke-series has a critical strip whose width is at least $\frac{1}{2}$. That equality can hold in (6.8), (6.9) and (6.10) is shown by the example $\phi(s) = \zeta(2s)$ where we have $a = b = c = c^* = k = \frac{1}{2}$.

The class of Hecke-series proper (see [9]) is distinguished by the following characteristics in addition to (6.6):

$$b(n) = a(n),$$
$$\lambda_n = n, \ \mu_n = 4\pi^2 n/\lambda^2, \text{ where } \lambda \text{ is a positive constant,}$$
$$k > 0, \quad h = \gamma\,(2\pi/\lambda)^k, \text{ where } \gamma = \pm 1,$$

and the requirement that $(s-k)\,\phi(s)$ be an entire function, i.e., $\phi(s)$ can have no singularities in the finite s-plane other than (at worst) a simple pole at $s = k$. For a Hecke-series, (2.4) takes the form

$$(6.11) \qquad (\lambda/2\pi)^s\,\Gamma(s)\,\phi(s) = \gamma\,(\lambda/2\pi)^{k-s}\,\Gamma(k-s)\,\phi(s)$$

This functional equation shows that $\phi(-p) = 0$ for any positive integer p and that

$$(6.12) \qquad \phi(0) = -\gamma\,(\lambda/2\pi)^k\,\Gamma(k)\,\rho$$

where ρ is the (possibly zero) residue of $\phi(s)$ at $s = k$. If ρ is not zero, then we have $b \geq k$, and $R_q(x)$ is given by

$$(6.13) \qquad R_q(x) = \rho\,\frac{\Gamma(k)}{\Gamma(k+q+1)}\,x^k + \frac{\phi(0)}{\Gamma(q+1)}\,.$$

If $\rho = 0$, then $R_q(x)$ is identically zero, so (6.13) holds in all cases.

Some examples of Hecke-series are: the Riemann zeta-function in the form $\zeta(2s)$; Dirichlet L-series whose coefficients are real primitive characters; zeta-functions of imaginary quadratic fields; the function whose coefficients are Ramanujan's function $\tau(n)$; and the functions $\zeta(s)\zeta(s-2p-1)$ where p is a positive integer. Generalized Hecke-series that are <u>not</u> Hecke-series proper include Dirichlet L-series whose coefficients are non-real primitive characters, and also the function $\zeta(s)\ \zeta(s-1)$ whose coefficients are $\sigma(n)$, the sum of the divisors of n. This last function has two simple poles.

For a generalized Hecke-series, (5.3) assumes the form

$$\sum_{m\leq\lambda_n\leq M}'' a(n)\, f_M(\lambda_n) = Q(p;m,M) + (-1)^p \int_m^M v^{p-1} R_{p-1}(v)\, f_M^{(p)}(v)\, dv$$

$$(6.14) \qquad + (-1)^p \sum b(n) \mu_n^{-\frac{1}{2}(k+p-1)} \int_m^M v^{\frac{1}{2}(k+p-1)} J_{k+p-1}(2\sqrt{\mu_n v}) f_M^{(p)}(v)\, dv$$

and is valid for $p = [c+\frac{1}{2}]$, where $[x]$ is the greatest integer function. Formula (6.14) may of course be valid for smaller values of p, depending on the range of validity of (4.1). Special cases of (6.14) may be found in Landau ([26], Satz. 559), Mordell [27], Koshliakov [28], Olevsky [29], Guinand [7].

c) Products of Two Zeta-functions.

For the functions $\phi(s) = \zeta(s)\zeta(s-r)$, where r is any real number, we have

$a(n) = b(n) = \sigma_r(n)$, the sum of the r^{th} powers of the divisors of n

$\lambda_n = n$, $\mu_n = \pi^2 n$,

$a = b = \max(1, 1+r)$, $k = 1 + r$,

$c = c^* = 1 + |r|$,

$\alpha = 1$,

$$H(s) = \pi^{r+1} \frac{\Gamma(\frac{s}{2})\,\Gamma(\frac{s-r}{2})}{\Gamma(\frac{1+r-s}{2})\,\Gamma(\frac{1-s}{2})} \quad .$$

It is customary to write $d(n)$ for $\sigma_0(n)$, $\sigma(n)$ for $\sigma_1(n)$, and $\sigma(n)/n$ for $\sigma_{-1}(n)$.

The form of $R_q(x)$ depends on the value of r. For $r > -1$, $r \neq 0$, we have

$$(6.15)\quad R_q(x) = \frac{\Gamma(r+1)\,\zeta(r+1)}{\Gamma(r+q+2)}\, x^{r+1} + \frac{\zeta(1-r)}{\Gamma(q+2)}\, x - \frac{\zeta(-r)}{2\Gamma(q+1)}$$

For $r = 0$, we have

$$(6.16)\quad R_q(x) = \frac{x}{\Gamma(q+2)} \left\{ \log x - \psi(q+2) + \gamma \right\} + \frac{1}{4\Gamma(q+1)}$$

where γ is Euler's constant and $\psi(x) = \Gamma'(x)/\Gamma(x)$. For $r = -1$, we have

$$(6.17)\quad R_q(x) = \frac{\pi^2 x}{6\Gamma(q+2)} - \frac{\log 2\pi x - \psi(q+1)}{2\Gamma(q+1)} - \frac{1}{24\Gamma(q)x}$$

while for other r's the expression is more complicated.

The expression for $L_q(x)$ is also complicated and is given here only for integral q. In that case we have

$$L_q(x) = -(2\pi)^{r+1}(4x)^{-(r+q+1)/2}\left\{\sin\frac{\pi r}{2} J_{r+q+1}(4\sqrt{x})\right.$$

(6.18)

$$\left. + \cos\frac{\pi r}{2}\left\{Y_{r+q+1}(4\sqrt{x}) + (-1)^q \frac{2}{\pi} K_{r+q+1}(4\sqrt{x})\right\}\right\}$$

where J, Y, and K represent, respectively, the Bessel functions of the first and second kind, and the modified Bessel functions of the second kind.

Oppenheim [19] and Wilton [20] have shown that (4.1) holds for $\operatorname{re} q \geq \max(0, |r| - \frac{1}{2} + \varepsilon)$ for any $\varepsilon > 0$. For $\operatorname{re} q > |r| + \frac{1}{2}$ (4.1) holds with absolute convergence. Accordingly, for $r = 0$ we may set $p = 0$ in (5.3) and obtain

$$\sum_{m \leq n \leq M}'' d(n)\, f_M(n) = \int_m^M (\log v + 2\gamma)\, f_M(v)\, dv$$

(6.19)

$$+ 2\pi \sum d(n) \int_m^M \left\{\frac{2}{\pi} K_0(4\pi\sqrt{nv}) - Y_0(4\pi\sqrt{nv})\right\} f_M(v)\, dv.$$

This is Voronoi's summation formula (cf. (1.2)).

For $r = -1$ we may set $p = 1$ in (5.3) and obtain

$$\sum_{m\le n\le M}'' \frac{\sigma(n)}{n} f_M(n) = - f_M(m) \sum_{n\le m}' \frac{\sigma(n)}{n} - \int_m^M \left(\frac{\pi^2}{6} v - \frac{\log 2\pi v + \gamma}{2}\right) f_M'(v)\,dv$$

$$- \sum \frac{\sigma(n)}{n} \int_m^M J_0(4\pi\sqrt{nv})\, f_M'(v)\, dv$$

$$(6.20) \qquad = f_M(m) \left\{ \frac{\pi^2}{6} m - \frac{\log 2\pi m + \gamma}{2} - \sum_{n\le m}' \frac{\sigma(n)}{n} \right\}$$

$$+ \int_m^M \left(\frac{\pi^2}{6} - \frac{1}{2v}\right) f_M(v)\, dv$$

$$+ \sum \frac{\sigma(n)}{n} \left\{ J_0(4\pi\sqrt{mn})\, f_M(m) \right.$$

$$\left. - 2\pi\sqrt{n} \int_m^M \frac{J_1(4\pi\sqrt{nv})}{\sqrt{v}} f_M(v)\, dv \right\}.$$

For $r = 1$, we may again take $p = 1$ in (5.3) and obtain

$$\sum_{m\le n\le M}'' \sigma(n)\, f_M(n) = - f_M(m) \sum_{n\le m}' \sigma(n) - \int_m^M \left(\frac{\pi^2}{12} v^2 - \frac{v}{2} + \frac{1}{24}\right) f_M'(v)\, dv$$

$$+ \sum \sigma(n) \int_m^M \frac{v}{n} J_2(4\pi\sqrt{nv})\, f_M'(v)\, dv$$

$$= f_M(m) \left\{ \frac{\pi^2}{12} m^2 - \frac{m}{2} + \frac{1}{24} - \sum_{n\le m}' \sigma(n) \right\}$$

$$(6.21) \qquad + \int_m^M \left(\frac{\pi^2}{6} v - \frac{1}{2}\right) f_M(v)\, dv$$

$$- \frac{\sigma(n)}{n} \left\{ m J_2(4\pi\sqrt{mn}\, f_M(m) \right.$$

$$\left. + 2\pi\sqrt{n} \int_m^M \sqrt{v}\, J_1(4\pi\sqrt{nv})\, f_M(v)\, dv \right\}.$$

Formulas for other special values of r can naturally be obtained without difficulty. Guinand [30] and Anferteva [31] give examples of summation formulas connected with special values of r.

d) Powers of the Riemann Zeta-function.

For $\phi(s) = \zeta^j(s)$ where j is a positive integer, we have

$$a(n) = b(n) = d_j(n)$$

where $d_j(n)$ is the number of ways the positive integer n can be written as an ordered product of j positive integers;

$$\lambda_n = n, \quad \mu_n = \pi^j n,$$
$$a = b = c = c^* = k = 1,$$
$$\alpha = j/2,$$

$$H(s) = \left\{ \sqrt{\pi} \, \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} \right\}^j = \left\{ 2^{1-s} \cos \frac{\pi s}{2} \, \Gamma(s) \right\}^j.$$

The function $R_q(x)$ has the form

$$R_q(x) = x \, P(\log x; \, j-1, \, q) + \frac{(-1)^j}{2^j \, \Gamma(q+1)}$$

where $P(\log x; \, j-1, \, q)$ is a polynomial of degree $j-1$ in $\log x$, whose coefficients are functions of j and q. The leading coefficient of P is

$$\frac{1}{(j-1)! \, \Gamma(q+2)}.$$

The form of $L_q(x)$ likewise depends on j and becomes more complicated with increasing j. Instead of determining $L_q(x)$ explicitly, we give formulas that define $L_q(x)$ recursively in j. To emphasize the dependence of $L_q(x)$ on j, we shall write it as $L_q(x|j)$. The following lemma relates $L_q(x|j+1)$ to $L_q(x|j)$:

<u>Lemma 6.1</u>: $L_q(x|j+1)$ is the Fourier cosine transform of $2/v\ L_q(2/v|j)$; i.e., we have

$$(6.22)\qquad L_q(x|j+1) = \int_0^\infty \frac{2}{v} L_q\left(\frac{2}{v}\,\middle|\, j\right) \cos (xv)\, dv$$

for all positive integral j, in any region of the q-plane in which the integral is uniformly convergent.

<u>Proof</u>: For $\operatorname{re} q > \alpha c + 1$, $0 < \varepsilon < 1$, we have by (3.1),

$$L_q(x|j+1) = 2^{j+1} \int_{(1+\varepsilon)} (2^{j+1} x)^{-s} \left\{ \Gamma(s) \cos \frac{\pi s}{2} \right\}^{j+1} \frac{\Gamma(1-s)}{\Gamma(q+2-s)}\, ds$$

$$(6.23)\qquad = 2^{j+1} \int_{(\varepsilon)} (2^{j+1} x)^{-s} \left\{ \Gamma(s) \cos \frac{\pi s}{2} \right\}^{j+1} \frac{\Gamma(1-s)}{\Gamma(q+2-s)}\, ds,$$

where the shifting of the contour can be justified by the usual arguments. By formula (6.5.21) of [13], we have

$$2^{-s}\, \Gamma(s) \cos \frac{\pi s}{2} = \int_0^\infty y^{s-1} \cos 2y\, dy,$$

for $0 < \sigma < 1$. Integrating by parts, we obtain

$$2^{-s}\, \Gamma(s) \cos \frac{\pi s}{2} = \frac{1}{2}(1-s) \int_0^\infty y^{s-2} \sin 2y\, dy,$$

where the integral is absolutely convergent. Substitution of this into (6.20) yields

$$L_q(x|j+1) = 2^j \int_{(\varepsilon)} \int_0^\infty y^{s-2} \sin 2y \, (2^j x)^{-s} \left\{ \Gamma(s) \cos \frac{\pi s}{2} \right\}^j \frac{\Gamma(2-s)}{\Gamma(q+2-s)} \, dy ds$$

$$= 2^j \int_0^\infty \int_{(\varepsilon)} y^{s-2} \sin 2y \, (2^j x)^{-s} \left\{ \Gamma(s) \cos \frac{\pi s}{2} \right\}^j \frac{\Gamma(2-s)}{\Gamma(q+2-s)} \, ds \, dy$$

$$= \int_0^\infty \sin 2y \; 2^j \int_{(\varepsilon)} y^{s-2} (2^j x)^{-s} \left\{ \Gamma(s) \cos \frac{\pi s}{2} \right\}^j \frac{\Gamma(2-s)}{\Gamma(q+2-s)} \, ds \, dy$$

The interchange of orders of integration is justified by absolute convergence. Another integration by parts now yields

$$L_q(x|j+1) = \int_0^\infty 2 \cos 2y \; 2^j \int_{(\varepsilon)} y^{s-1} (2^j x)^{-s} \left\{ \Gamma(s) \cos \frac{\pi s}{2} \right\}^j \frac{\Gamma(1-s)}{\Gamma(q+2-s)} \, ds dy$$

$$= \int_0^\infty 2 \cos 2y \; y^{-1} L_q\left(\frac{x}{y}\Big| j\right) dy$$

$$= \int_0^\infty \frac{2}{v} L_q\left(\frac{2}{v}\Big| j\right) \cos (xv) \, dv$$

upon making the substitution $y = vx/2$. We now extend the range of q by analytic continuation. This completes the proof of the lemma.

The recursion is completed by noting that $L_q(x|1)$ is given by (6.0).

The summation formulas corresponding to $\zeta^j(s)$ are cases of Poisson's formula for $j = 1$ and cases of Voronoi's formula for $j = 2$. Summation formulas for larger values of j seem to be nonexistent in the literature.

§ 7. Applications.

We consider first the case $f(x) = x^{-s}$, where σ is greater than $a^* = \max(a,0)$. A straightforward calculation gives

$$(7.1)\quad f_M(x) = x^{-s} - \sum_{j=0}^{p} \frac{\Gamma(s+j)}{\Gamma(s)\,\Gamma(j+1)} M^{-s-j}(M-x)^j \qquad (x \leq M)$$

Accordingly, (5.3) assumes the form

$$(7.2)\quad \begin{aligned} \sum_{m\leq \lambda_n\leq M}'' a(n)\lambda_n^{-s} = {} & \sum_{j=0}^{p} \frac{\Gamma(s+j)}{\Gamma(s)\,\Gamma(j+1)} M^{-s-j} \sum_{m\leq\lambda_n\leq M}'' a(n)(M-\lambda_n)^j \\ & - \frac{\Gamma(s+p)}{\Gamma(s)\,\Gamma(p)} \sum_{\lambda_n\leq m}' a(n)\int_m^M (v-\lambda_n)^{p-1}\,(v^{-s-p}-M^{-s-p})\,dv \\ & + \frac{\Gamma(s+p)}{\Gamma(s)} \int_m^M v^{p-1} R_{p-1}(v)(v^{-s-p}-M^{-s-p})\,dv \\ & + \frac{\Gamma(s+p)}{\Gamma(s)} \sum b(n) \int_m^M v^{p+p-1} L_{p-1}(\mu_n v)(v^{-s-p}-M^{-s-p})\,dv \end{aligned}$$

for $p > 0$, while for $p = 0$ we have

$$(7.3)\quad \begin{aligned} \sum_{m\leq\lambda_n\leq M} a(n)\lambda_n^{-s} = {} & M^{-s} \sum_{m\leq\lambda_n\leq M}' a(n) + \int_m^M v^{-1} R_{-1}(v)(v^{-s}-M^{-s})\,dv \\ & + b(n)\int_m^M v^{k-1} L_{-1}(\mu_n v)\,(v^{-s}-M^{-s})\,dv. \end{aligned}$$

From the general theory of Dirichlet series we know that

$$\sum_{m\leq\lambda_n\leq M}'' |a(n)| = O(M^{a^*+\varepsilon})$$

for every $\varepsilon > 0$. Taking $\varepsilon < \sigma - a^*$ it follows that we have

$$M^{-s-j} \sum_{m \leq \lambda_n \leq M} a(n)\,(M - x)^j = O(M^{-\sigma} \sum_{m \leq \lambda_n \leq M} |a(n)|)$$

$$= O(M^{a^* - \sigma + \varepsilon}) = o(1) \quad \text{as} \quad M \to \infty .$$

Since the series $\sum a(n)\lambda_n^{-s}$ is absolutely convergent for $\sigma > a^*$ we may let M go to infinity in (7.2) and (7.3) and obtain, respectively,

$$\sum_{\lambda_n \geq m}{}' a(n)\lambda_n^{-s} = -\frac{\Gamma(s+p)}{\Gamma(s)\,\Gamma(p)} \sum_{\lambda_n \leq m}{}' a(n) \int_m^\infty (v-\lambda_n)^{p-1}\, v^{-s-p}\, dv$$

$$(7.4) \qquad + \frac{\Gamma(s+p)}{\Gamma(s)} \lim_{M\to\infty} \Big\{ \int_m^M v^{p-1}\, R_{p-1}(v)(v^{-s-p} - M^{-s-p})\, dv$$

$$+ \sum b(n) \int_m^M v^{k+p-1}\, L_{p-1}(\mu_n v)(v^{-s-p} - M^{-s-p})\, dv \Big\}$$

for $p > 0$, and

$$\sum_{\lambda_n \geq m}{}' a(n)\lambda_n^{-s} = \lim_{M\to\infty} \Big\{ \int_m^M v^{-1}\, R_{-1}(v)\,(v^{-s} - M^{-s})\, dv$$

$$(7.5) \qquad + \sum b(n) \int_m^M v^{k-1}\, L_{-1}(\mu_n v)(v^{-s} - M^{-s})\, dv \Big\}$$

for $p = 0$.

Since $\phi(s) = \sum a(n)\lambda_n^{-s}$ for $\sigma > a$, we may put (7.4) and (7.5) in the respective forms

$$\phi(s) = \sum_{\lambda_n \leq m}{}' a(n)\lambda_n^{-s} - \frac{\Gamma(s+p)}{\Gamma(s)\Gamma(p)} \sum_{\lambda_n \leq m}{}' a(n) \int_m^\infty (v-\lambda_n)^{p-1} v^{-s-p}\, dv$$

$$(7.6) \qquad + \frac{\Gamma(s+p)}{\Gamma(s)} \lim_{M\to\infty} \Big\{ \int_m^M v^{p-1} R_{p-1}(v)\,(v^{-s-p} - M^{-s-p})\, dv$$

$$+ \sum b(n) \int_m^M v^{k+p-1} L_{p-1}(\mu_n v)\,(v^{-s-p} - M^{-s-p})\, dv \Big\},$$

$$\phi(s) = \sum_{\lambda_n \leq m}{}' a(n)\lambda_n^{-s} + \lim_{M\to\infty} \Big\{ \int_m^M v^{-1} R_{-1}(v)(v^{-s} - M^{-s})\, dv$$

$$(7.7) \qquad + \sum b(n) \int_m^M v^{k-1} L_{-1}(\mu_n v)\,(v^{-s} - M^{-s})\, dv \Big\}.$$

We have thus obtained formulas for $\phi(s)$ that are valid in the half-plane $\sigma > a^*$. In many instances the right-hand sides of (7.6) and (7.7) reduce to expressions that are meaningful in a larger region; the formulas then furnish an explicit analytic continuation of $\phi(s)$ into this larger region.

For example, in the case of Hecke-functions (7.6) reduces to

$$\phi(s) = \sum_{n \leq m}{}' a(n)\, n^{-s} - \frac{\Gamma(s+p)}{\Gamma(s)\Gamma(p)} \sum_{n \leq m}{}' a(n) \int_m^\infty (v-n)^{p-1} v^{-s-p}\, dv$$

$$(7.8) \qquad + \frac{\Gamma(s+p)\,\Gamma(k)}{(s-k)\,\Gamma(s)\,\Gamma(k+p)}\, m^{k-s} + \phi(0)\, \frac{\Gamma(s+p)}{\Gamma(p)\,\Gamma(s+1)}\, m^{-s}$$

$$+ \gamma \left(\frac{\lambda}{2\pi}\right)^{p-1} \frac{\Gamma(s+p)}{\Gamma(s)} \sum a(n)\, n^{-(k+p-1)/2} \times$$

$$\times \int_m^\infty v^{(k-p-1-2s)/2}\, J_{k+p-1}\!\left(\frac{4\pi}{\lambda}\sqrt{nv}\right) dv,$$

where ρ, γ, λ are as given in § 6, b). This identity, valid for $\sigma > \frac{1}{2}(k - p - \frac{1}{2})$, is equivalent to the identity ((3.2) in [10]) used by Apostol and Sklar in deriving approximate functional equations for Hecke-series.

An interesting application of (5.5) is afforded by taking $\phi(s)$ to be a Hecke-series and letting $f(v)$ be defined by

$$f(v) = v^{-\frac{k+q}{2}} J_{k+q}\left(\frac{4\pi}{\lambda}\sqrt{xv}\right) \tag{7.9}$$

where q is complex and x is positive. In this case we have

$$f_M(v) = f(v) - \sum_{j=o}^{p} \left(\frac{2\pi}{\lambda}\sqrt{x}\right)^j M^{-\frac{k+q+j}{2}} J_{k+q+j}\left(\frac{4\pi}{\lambda}\sqrt{xM}\right) \frac{1}{j!}(M-v)^j \tag{7.10}$$

Substituting into (5.3), we obtain

$$\begin{aligned}
&\sum_{n\leq M} a(n)\, n^{-\frac{k+q}{2}} J_{k+q}\left(\frac{4\pi}{\lambda}\sqrt{xn}\right) \\
&= \sum_{j=o}^{p} \left(\frac{2\pi}{\lambda}\sqrt{x}\right)^j M^{-\frac{k+q+j}{2}} J_{k+q+j}\left(\frac{4\pi}{\lambda}\sqrt{xM}\right) \frac{1}{j!} \sum_{n\leq M} a(n)(M-n)^j \\
&+ \rho \frac{\Gamma(k)}{\Gamma(k+p+1)} \left(\frac{2\pi}{\lambda}\sqrt{x}\right)^{p+1} \int_o^M v^{\frac{k+p-q-1}{2}} J_{k+q+p+1}\left(\frac{4\pi}{\lambda}\sqrt{xn}\right) dv \\
&+ \frac{\phi(0)}{\Gamma(p+1)} \left(\frac{2\pi}{\lambda}\sqrt{x}\right)^{p+1} \int_o^M v^{\frac{p-q-k-1}{2}} J_{k+q+p+1}\left(\frac{4\pi}{\lambda}\sqrt{xv}\right) dv \\
&+ \gamma \frac{2\pi}{\lambda} x^{\frac{p+1}{2}} \sum a(n)\, n^{-\frac{k+p}{2}} \int_o^M v^{-\frac{q+1}{2}} J_{k+q}\left(\frac{4\pi}{\lambda}\sqrt{nv}\right) J_{k+q+p+1}\left(\frac{4\pi}{\lambda}\sqrt{xv}\right) dv
\end{aligned} \tag{7.11}$$

Integration by parts in all the integrals on the right of (7.11), followed by use of (4.1) yields

$$\sum_{n\leq M} a(n)\, n^{-\frac{k+q}{2}} J_{k+q}\left(\frac{4\pi}{\lambda}\sqrt{xn}\right)$$

$$= \sum_{j=0}^{p-1} \left(\frac{2\pi}{\lambda}\sqrt{x}\right)^j M^{-\frac{k+q+j}{2}} J_{k+q+j}\left(\frac{4\pi}{\lambda}\sqrt{xM}\right) \frac{1}{j!} \sum_{n\leq M} a(n)(M-n)^j$$

$$+ \rho \frac{\Gamma(k)}{\Gamma(k+p)} \left(\frac{2\pi}{\lambda}\sqrt{x}\right)^p \int_0^M v^{-\frac{k+p-q}{2}-1} J_{k+q+p}\left(\frac{4\pi}{\lambda}\sqrt{xv}\right) dv$$

$$+ \frac{\phi(0)}{\Gamma(p)} \left(\frac{2\pi}{\lambda}\sqrt{x}\right)^p \int_0^M v^{\frac{p-q-k}{2}-1} J_{k+q+p}\left(\frac{4\pi}{\lambda}\sqrt{xv}\right) dv$$

$$+ \gamma x^{\frac{p}{2}} \sum a(n) n^{-\frac{k+p-1}{2}} \int_0^M v^{-\frac{q+1}{2}} J_{k+p-1}\left(\frac{4\pi}{\lambda}\sqrt{nv}\right) J_{k+q+p}\left(\frac{4\pi}{\lambda}\sqrt{xv}\right) dv$$

The left-hand side of (7.12) is a partial sum of the series that appears on the right of (4.1). Since the value of q in (7.12) is unrestricted, (7.12) can be used to study the behavior of the series of (4.1) outside the region of absolute convergence. In many cases it is possible in this way to establish the validity of (4.1) for certain values of q outside the half-plane $\operatorname{re} q > c - \frac{1}{2}$. Such use of (7.12) is a generalization of the methods used by Hardy and Landau [32], [33], [26] (Achter Tiel, Kap. 4,5) in proving the validity of (4.1) for $q = 0$ in the case $\phi(s) = \sum r_2(n)\, n^{-s}$ and by Hardy [18] in the case $\phi(s) = \sum \tau(n)\, n^{-s}$. Here $r_2(n)$ is the number of representations of n as the sum of two squares, while $\tau(n)$ is Ramanujan's function.

The identity (7.12) is complicated and it would be desirable to simplify it. We shall outline a method for accomplishing this.

We begin by defining some convenient functions:

<u>Definition 7.1</u>: Let x, x_1, and x_2 be real and $x_1 < x_2$. Let ε be positive. Then $g(x)$ and $g_1(x)$ are defined by

$$(7.13) \qquad g(x) = g(x|x_1,x_2,\varepsilon) = \begin{cases} 0 & x \leq x_1 \\ \exp\left\{\frac{4\varepsilon}{x_2-x_1} - \frac{\varepsilon}{x-x_1} - \frac{\varepsilon}{x_2-x}\right\} & x_1 < x < x_2 \\ 0 & x \geq x_2 \end{cases}$$

$$(7.14) \qquad g_1(x) = g_1(x|x_1,x_2,\varepsilon) = \frac{\int_{-\infty}^{x} g(v|x_1,x_2,\varepsilon)\, dv}{\int_{-\infty}^{\infty} g(v|x_1,x_2,\varepsilon)\, dv}$$

Thus $g(x)$ and $g_1(x)$ are each infinitely differentiable, and all of their derivatives vanish outside the open interval (x_1,x_2). The function $g(x)$ itself also vanishes outside the open interval (x_1,x_2), while $g_1(x)$ vanishes for $x \leq x_1$ and has the constant value 1 for $x \geq x_2$.

If we repeat the derivation of (7.12), beginning however with the function

$$\bar{f}(v) = g_1(v|M-\delta,M,\varepsilon)\, v^{-\frac{k+q}{2}}\, J_{k+q}\left(\frac{4\pi}{\lambda}\sqrt{xv}\right) \qquad (\delta < M)$$

in place of (7.9), we then have $\bar{f}_M(v) = \bar{f}(v)$, and the analog of (7.11) is

$$\sum_{n\leq M} a(n)\, n^{-\frac{k+q}{2}} J_{k+q}\left(\frac{4\pi}{\lambda}\sqrt{xn}\right) g_1(n)$$

$$= (-1)^{p+1} \rho \frac{\Gamma(k)}{\Gamma(k+p+1)} \int_0^M v^{k+p}\, \bar{f}^{(p+1)}(v)\, dv$$

$$+ (-1)^{p+1} \frac{\phi(0)}{\Gamma(p+1)} \int_0^M v^p\, \bar{f}^{(p+1)}(v)\, dv$$

$$+ (-1)^{p+1} \gamma \left(\frac{\lambda}{2\pi}\right)^p \sum a(n) n^{-\frac{k+p}{2}} \int_0^M v^{\frac{k+p}{2}} J_{k+p}\left(\frac{4\pi}{\lambda}\sqrt{nv}\right) \bar{f}^{(p+1)}(v)\, dv$$

Integrating by parts $p+1$ times in every integral on the right we obtain

$$\sum_{n\leq M} a(n)\, n^{-\frac{k+q}{2}} J_{k+q}\left(\frac{4\pi}{\lambda}\sqrt{xn}\right) g_1(n)$$

$$= \rho \int_0^M v^{\frac{k-q}{2}-1} J_{k+q}\left(\frac{4\pi}{\lambda}\sqrt{xv}\right) g_1(v)\, dv + \phi(0) \left(\frac{4\pi}{\lambda}\sqrt{x}\right)^{k+q}$$

$$(7.15) \qquad + \gamma \frac{2\pi}{\lambda} \sum a(n) n^{-\frac{k-1}{2}} \int_0^M v^{-\frac{q+1}{2}} J_{k-1}\left(\frac{4\pi}{\lambda}\sqrt{nv}\right) J_{k+q}\left(\frac{4\pi}{\lambda}\sqrt{xv}\right) g_1(v)\, dv$$

By taking δ so small that no integer lies in the open interval $(M-\varepsilon, M)$, the left-hand side of (7.15) becomes a partial sum of the series on the right of (4.1). Therefore (7.15) may be used in place of (7.12) to investigate the behavior of the series in (4.1).

As a final application of our summation formulas we consider the function

$$f(x) = f_M(x) = g(x|m, M, \varepsilon)$$

where g is defined by (7.13). Theorem 5.3 applies for this function, so we may use (5.9) and obtain

$$\sum_{m \leq \lambda_n \leq M} a(n)\, g(\lambda_n) = \int_m^M R(v)\, g(v)\, dv$$

(7.16)
$$+ \sum b(n) \int_m^M v^{k-1} L(\mu_n v)\, g(v)\, dv$$

As an example of (7.16) we consider the case $\phi(s) = \zeta(s)\,\zeta(s-1)$. We can set $m = 0$ and obtain

$$\sum_{n \leq M}{}' \sigma(n)\, g(n) = \int_0^M \left(\frac{\pi^2}{6} v - \frac{1}{2}\right) g(v)\, dv$$

(7.17)
$$- 2\pi \sum \frac{\sigma(n)}{\sqrt{n}} \int_0^M \sqrt{v}\, J_1(4\pi \sqrt{nv})\, g(v)\, dv.$$

Allowing ε to approach zero in (7.17) yields

$$\sum_{n < M} \sigma(n) = \frac{\pi^2}{12} M^2 - \frac{M}{2}$$

(7.18)
$$- 2\pi \lim_{\varepsilon \to 0} \left\{ \sum \frac{\sigma(n)}{\sqrt{n}} \int_0^M \sqrt{v}\, J_1(4\pi \sqrt{nv})\, g(v \mid 0, M, \varepsilon)\, dv \right\}.$$

This should be compared with

(7.19) $$\sum_{n \leq M}{}' \sigma(n) = \frac{\pi^2}{12} M^2 - \frac{M}{2} + \frac{1}{24} - M \sum \frac{\sigma(n)}{n} J_2(4\pi \sqrt{nM})$$

given by Walfisz [23], where, however, the series on the right is not convergent, but is summable (C,1) or $(R, n, x > \frac{1}{2})$. Since we have

$$\lim_{\varepsilon \to 0} g(v|0, M, \varepsilon) = 1 \qquad (0 < v < M),$$

and

$$\frac{2\pi}{\sqrt{n}} \int_0^M \sqrt{v} \; J_1(4\pi\sqrt{nv})\, dv = \frac{M}{n} J_2(4\pi\sqrt{nM}),$$

equation (7.18) may be looked upon as providing a new method for summing the series on the right of (7.19).

Index of Symbols

Symbols appearing only in the introduction are not listed, nor are symbols used only as real or complex variables, variable integers, etc. The pages listed are those on which the corresponding symbols are first defined.

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