Rigidity Phenomena of Group Actions on a Class of Nilmanifolds and Anosov $\mathbb{R}^n$ Actions

Thesis by

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Abstract

An action of a group Γ on a manifold M is a homomorphism ρ from Γ to Diff(M). ρ_0 is locally rigid if the nearby homomorphism ρ, $\rho(\gamma) = h \circ \rho_0(\gamma) \circ h^{-1}$ for some $h \in$Diff(M) and for all $\gamma \in \Gamma$. In other words, ρ_0 is isolated from other actions up to a smooth conjugation.

In this thesis we studied some standard group actions on a broader class of manifolds, the free, k-step nilmanifolds $N(n, k)$; we obtained that the standard $SL(n, Z)$ action on $N(n, 2)$ is locally rigid for $n = 3$, and $n \geq 5$.

We recall that $N(n, 1) = T^n$. Hence, our results are the generalization to the local rigidity result for the standard action on torus T^n.

We observed also, for the first time, that for discrete subgroups $Aut(n, 2)$ of a Lie group, which is not even reductive, the action on $N(n, 2)$ is deformation-rigid for $n = 3$, and $n \geq 5$.

We also investigated the dynamics of Anosov R^n actions and obtained a number of results parallel to those of Anosov diffeomorphisms and flows. E.g., the strong stable (unstable) manifold for a regular element is dense iff the action is weakly mixing (for a volume-preserving action); an Anosov action with no dense, strong stable (unstable) manifold can always be reduced to suspension of the action mentioned above; there are two compatible measures to the Anosov actions.

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PART I: Rigidity Phenomena of Group Actions on a Class of Nilmanifolds

1.Introduction

As observed by R. Zimmer, A. Katok, S. Hurder, R. Spatzier, J. Lewis and other authors, large groups often act on manifolds in a rigid way.

Let G be a connected semisimple Lie group with finite center and without compact factors, $\Gamma \subset G$ a lattice. Γ is said to be *irreducible* in G if Γ projects densely into G/N for every non-central, normal subgroup N. Also, if $G = KAN$ is the Iwasawa decomposition for G, then we define the *real rank* of G, $\mathbb{R}$-rank(G)=dim(A); we say G has *higher rank* if $\mathbb{R}$-rank(G)≥ 2. The celebrated Margulis superrigidity theorem (see [M1],[Z2]) reflects the rigidity of Γ in G when Γ is a higher-rank lattice in G.

If Γ is any finitely generated discrete group and G is any topological group, we denote $R(\Gamma, G)$ the set of all homomorphisms from Γ to G with the compact/open topology. The topology can also be described as follows (see [Ra]). Fix generators $\gamma_1, \ldots, \gamma_k$ for Γ and identify $R(\Gamma, G)$ with a closed subset of G^k via $\rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_k))$; then the topology on $R(\Gamma, G)$ is simply the subspace topology inherited from G^k. Note that G acts naturally on $R(\Gamma, G)$ by conjugation. A homomorphism ρ_0 in $R(\Gamma, G)$ is said to be *locally rigid* if its orbit under the action of conjugation in $R(\Gamma, G)$ is open. In other words, ρ_0 is locally rigid if it is isolated up to a conjugation. So for locally rigid action ρ_0 and any nearby homomorphism $\rho \in R(\Gamma, G)$, there exists a $g \in G$ such that $\rho(\gamma) = g\rho_0(\gamma)g^{-1}$ for every $\gamma \in \Gamma$.

There are a number of classical local rigidity results for homomorphisms of Γ to G in case that G is a finite-dimensional Lie group. A. Weil [W] observed that if $\rho \in R(\Gamma, G)$ such that $H^1(\Gamma, Ad_G \circ \rho) = 0$, then ρ is locally rigid. Margulis' result (see [M1]), that $H^1(\Gamma, \rho) = 0$ for every homomorphism ρ of Γ to $GL(N, \mathbb{R})$, where Γ is irreducible in a higher-rank, connected, semisimple algebraic $\mathbb{R}$-group without compact factors, gives a practical criterion of the local rigidity.

Suppose the group G is Diff(M) for a compact manifold M; then we call a homomorphism $\rho \in R(\Gamma,\text{Diff}(M))$ an *action* of Γ on manifold M.

R. Zimmer [Z1] initiated a program of understanding the action of Γ on compact manifolds. The guiding philosophy is that the finite-dimensional, local rigidity phenomena should be reflected in the context of actions on manifolds. Zimmer raised the question of local rigidity for the action of $SL(n, \mathbb{Z})$ on torus $\mathbb{T}^n$, $n \geq 3$, during the 1984 M.S.R.I. workshop on ergodic theory, Lie group, and geometry, and again in his 1986 address to the I.C.M. [Z1]. Several recent results have been obtained by Hurder, Katok-Lewis, Hurder-Katok-Lewis-Zimmer as follows.

THEOREM (HURDER[HU2]) 1.1. *Let $\Gamma = SL(n, \mathbb{Z})$ or any subgroup of finite index, $n \geq 3$. Let $\rho_t \in R(\Gamma,\text{Diff}(\mathbb{T}^n))$ be a continuous path based at $\rho_0 =$ the standard action by automorphisms. Then there exists a continuous path $g_t \in$Diff$(\mathbb{T}^n)$ such that $\rho_t(\gamma) = g_t \rho_0(\gamma) g_t^{-1}$ for all small t and $\gamma \in \Gamma$.* □

THEOREM (KATOK-LEWIS[K-L2]) 1.2. *Let $\Gamma = SL(n, \mathbb{Z})$ $(Sp(n, \mathbb{Z}))$ or any subgroup of finite index, $n \geq 4$ $(n \geq 3)$. Then the standard action of Γ on $\mathbb{T}^n$ $(\mathbb{T}^{2n})$ is locally rigid.* □

THEOREM (HURDER-KATOK-LEWIS-ZIMMER[H-K-L-Z]) 1.3. *Let $\Gamma = SL(n, \mathbb{Z})$ or any subgroup of finite index, $n \geq 3$. Then the standard action of Γ on $\mathbb{T}^n$ is locally rigid.* □

In his paper [Z1], R. Zimmer also posted the question of whether every action of a higher-rank lattice of a semisimple Lie group on a compact manifold, which preserves a smooth volume form, comes from algebra in origin; in other words, whether or not the following examples with simple algebraic constructions exhaust all the possibilities:

(1) Isometric actions

(2) Γ acts on $M = H/\Lambda$ via ρ, where $\Gamma \subset G$ and $\Lambda \subset H$ are lattices, with Λ co-compact, and $\rho : G \to H$ is a homomorphism.

(3) Γ acts on $M = H/\Lambda$, where Λ is a (necessarily co-compact) lattice in a nilpotent Lie group N, and Γ is a lattice of G, where G is a semisimple group of automorphisms of N, such that Γ preserves Λ.

This is the "global rigidity problem." This question has been answered affirmatively in recent papers by Katok-Lewis [K-L1], and by [H-K-L-Z] for some special classes of lattice actions on high-dimensional tori.

THEOREM (KATOK-LEWIS [K-L1]) 1.4. *Let Γ be a subgroup of finite index in $SL(n,\mathbb{Z})$, $n \geq 4$, $M = \mathbb{T}^n$, and $\rho \in R(\Gamma,Diff(M))$ such that*

(1) *there exists a fixed point; i.e., there exists $x_0 \in M$ such that $\rho(\gamma)x_0 = x_0$ for every $\gamma \in \Gamma$,*

(2) *there exists a direct-sum decomposition of $\mathbb{Q}^n$ as a vector space over $\mathbb{Q}$,*

$$\mathbb{Q}^n = V_1 \oplus V_2,\ V_1 \cong \mathbb{Q}^k,\ V_2 \cong \mathbb{Q}^l,\ k+l=n,\ k,l \geq 2,$$

and an element $\lambda_0 \in \Lambda = \{\gamma \in \Gamma | \gamma V_i = V_i,\ i = 1,2\}$ such that the diffeomorphism $\rho(\lambda_0)$ is Anosov.

Let $\rho_* : \Gamma \to GL(n,\mathbb{Z})$ denote the homomorphism corresponding to the action on $H^1(M,\mathbb{Z}) \cong \mathbb{Z}^n$. Then ρ is smoothly equivalent to the linear action corresponding to ρ_*; i.e., there exists a diffeomorphism h of M, homotopic to the identity, such that $\rho(\gamma) = h\rho_*(\gamma)h^{-1}$ for every $\gamma \in \Gamma$. $\square$

THEOREM (HURDER-KATOK-LEWIS-ZIMMER [H-K-L-Z]) 1.5. Cartan actions *of higher-rank lattice* Γ *on* $\mathbb{T}^n$ *($n \geq 3$) are globally rigid if the invariant 1-dimensional foliations for a suitable Abelian subgroup* $\mathcal{A} \subset \Gamma$ *are the intersections of stable foliations for some Anosov elements.* □

In [K-L1], a remarkable example has been constructed that shows that non-algebraic lattice actions exist for lattice actions. This example serves as a counterexample to the conjecture that every action of a higher-rank lattice of a semisimple Lie group on a compact manifold, which preserving a smooth volume form, should come from algebra in origin.

We remark that all the rigidity results mentioned above deal with the actions on high-dimensional tori. Hurder, Katok, Zimmer and other authors conjectured that the rigidity phenomena should also appear for actions on other classes of manifolds.

Our work confirm the conjecture for a class of algebraic actions on some nilmanifolds, as we call them $N(n, 2)$. The nilmanifolds considered are a factor space of 2-step, free nilpotent groups associated with n-dimensional vector spaces. The restriction of freeness is not essential. We consider this class of nilmanifolds mainly because we want the groups of automorphisms to be "big" groups in the sense that they contain higher-rank lattices, as well as Anosov elements. We can construct other nilmanifolds of step 2 and the same argument may produce rigidity phenomena there.

Our main results are the following two theorems.

THEOREM A. *Let* $\Gamma = SL(n, \mathbb{Z})$ *or any subgroup of finite index. Then the standard diagonal-block action of* Γ *on* $N(n, 2)$, $n \geq 3$, $n \neq 4$ *is locally rigid.* □

Our approach to the problem combines the structure theory for the lattices in higher-rank groups, especially a theorem by Prasad and Raghunathan [Pr-R] about the relation of Cartan subgroups and lattices in semisimple groups; Katok-Lewis's non-stationary Sternberg linearization [K-L2]; a theory of Hölder continuous linearization developed by Hurder-Katok-Lewis-Zimmer [H-K-L-Z]; and the property that there exists an Abelian subgroup in Γ generated by Anosov elements, which has enough 1-dimensional foliations. As a corollary of our approach, we constructed a new class of examples of locally rigid group actions on tori. The construction is based on some simple facts in multilinear algebras.

We also mention that we have "topological deformation rigidity" (in the sense of Hurder, see Theorem 1, with g_t being a continuous family of homeomorphisms) of the standard action of higher-rank lattice Γ on $N(n,k)$ for all $n \geq 3$, $k \leq n-1$. Some of them are smoothly rigid. We hope we can prove the "smooth deformation rigidity" and then local rigidity for all cases in the future.

In literature, the only deformation-rigidity phenomena are observed for higher-rank lattices in semisimple Lie groups. We obtain the first example of the deformation-rigid action of a discrete group in a Lie group, which is not even reductive.

THEOREM B. *$Aut(n,2)$ action on $N(n,2)$ is deformation-rigid for $n \geq 3$, $n \neq 4$.* □

The proof of this fact is based on a criterion of topological deformation rigidity of an action by Hurder [Hu3], which reduces the question to the calculation of the first cohomology group for the action together with the deformed action. We first show that $Aut(n,2)$ is generated by several copies of $SL(n,\mathbb{Z})$. Then we prove that a cocycle for the whole group action has to be a coboundary because this cocycle, restricted to each copy of $SL(n,\mathbb{Z})$, is a coboundary.

We remark that local rigidity for the rigidity of $Aut(n,2)$ action on $N(n,2)$ should also be true, as conjectured by Katok. We hope to solve this problem soon.

We would like to mention that not all nilmanifolds support Anosov diffeomorphisms. For example, Heisenberg nilmanifolds do not have Anosov diffeomorphisms. Yet, it is conjectured by Katok that some natural, higher-rank lattice actions on such nilmanifolds should also be rigid.

2. A Class of Nilmanifolds and Their Automorphisms

2.1 Nilmanifolds $N(n,k)$. Let $\mathcal{N}$ be a Lie group and let Λ be a discrete subgroup such that $\mathcal{N}/\Lambda$ has a finite Haar measure. We call such a discrete subgroup a *lattice*. In the case that $\mathcal{N}/\Lambda$ is compact, we call the lattice *uniform*.

Let $A : \mathcal{N} \to \mathcal{N}$ be an automorphism of $\mathcal{N}$ whose restriction to Λ is an automorphism of Λ. Then the automorphism A induces a map on $\mathcal{N}/\Lambda$. We call the induced map an *automorphism* of $\mathcal{N}/\Lambda$.

In previous work by Hurder [Hu2],[Hu3] and Katok, Lewis [K-L1],[K-L2], it is clear that the existence of hyperbolicity, especially the existence of plenty of Anosov elements plays an important role in the rigidity phenomena. We recall a conjecture that the only compact manifolds supporting Anosov diffeomorphisms are tori, some nilmanifolds, and some infranilmanifolds [F]. Therefore, it is natural to ask whether the rigidity phenomena can be observed for nilmanifolds.

A connected, simply connected nilpotent Lie group may be identified with its Lie algebra via the exponential map. The group operation is then given in terms of the operation in the Lie algebra by the Baker-Campbell-Hausdorff formula, and the exponential map is the identity map. We recall the Baker-Campbell-Hausdorff formula as follows (see [V]).

Let $\mathcal{G}$ be a Lie algebra, let $X, Y \in \mathcal{G}$. We define

$$C(X:Y) = \sum_{n=1}^{\infty} c_n(X:Y),$$

where

$$c_1(X:Y) = X + Y$$

$$c_2(X:Y) = \frac{1}{2}[X,Y]$$

$$c_3(X:Y) = \frac{1}{12}[[X,Y],Y] - \frac{1}{12}[[X,Y],X]$$

$$c_4(X:Y) = -\frac{1}{48}[Y,[X,[X,Y]]] - \frac{1}{48}[X,[Y,[X,Y]]]$$

$$\begin{aligned}(n+1)c_{n+1}(X:Y) &= \frac{1}{2}[X-Y, c_n(X:Y)]+\\ &+ \sum_{p\geq 1, 2p\leq n} K_{2p} \sum_{\substack{k_1,\dots,k_{2p}>0\\ k_1+\dots+k_{2p}=n}} [c_{k_1}(X:Y),[\dots[c_{k_{2p}}(X:Y), X+Y]\dots],\end{aligned}$$

where K_{2p} are some rational numbers. It is clear that when one calculates the $c_n(X:Y)$ by the recursion formula, one finds that each $c_n(X:Y)$ is a linear combination of the commutators of the form $[Z_1,[Z_2,[\dots[Z_{n-1},Z_n]\dots]]]$ with $Z_i \in \{X,Y\}$. Therefore, for nilpotent Lie algebra, the right-hand side of Baker-Campbell-Hausdorff formula is a finite sum. For example, for Abelian Lie algebra

$$C(X:Y) = X+Y;$$

for 2-step nilpotent Lie algebra

$$C(X:Y) = X+Y+\frac{1}{2}[X,Y];$$

for 3-step nilpotent Lie algebra

$$C(X:Y) = X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[[X,Y],Y]-\frac{1}{12}[[X,Y],X];$$

for 4-step nilpotent Lie algebra

$$\begin{aligned}C(X:Y) = X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[[X,Y],Y]-&\\ -\frac{1}{12}[[X,Y],X]-\frac{1}{48}[Y,[X,[X,Y]]]-\frac{1}{48}[X,[Y,[X,Y]]].&\end{aligned}$$

We also remark that after identification of the connected, simply connected nilpotent Lie group with its Lie algebra, the automorphism of the Lie group and the automorphism of the Lie algebra are the same. (see [A-S])

Now we want to construct a nilpotent Lie algebra with the property that the group of its automorphisms is sufficiently large.

Let V be an n-dimensional vector space over $\mathbb{R}$. Let $\{x_1, x_2, ...x_n\}$ be a basis. We define $N_k(V)$ to be the k-step, free nilpotent Lie algebra associated with V (see [A-S]). We let

$$C_0 := \{\underbrace{[[[[x_i, x_j], x_k], ...], x_l]}_{k-1 brackets}\}$$

and let C:= $\mathbb{Z}$-span of C_0. We may also view $N_k(V)$ as a simply connected, connected nilpotent group. We recall an easy result.

PROPOSITION (SEE [A-S]) 2.1.1. *There exists an integer $m \in \mathbb{Z}$ such that mC is a uniform lattice for nilpotent group $N_k(V)$.*

We point out that for $k = 1, 2, 3, 4$, we may take $m = 1, 2, 12, 48$, respectively.

We will denote the lattice $exp(mC)$ by Λ , denote the factor space $N_k(V)/\Lambda$ by $N(n, k)$, and denote the group of automorphisms of $N(n, k)$ by $Aut(n, k)$.

Our main objective is to investigate the group actions on $N(n, k)$.

2.2 Automorphisms of $N(n, k)$**.** Recall that $A : N(n, k) \to N(n, k)$ is an automorphism if A induces a Lie algebra automorphism $A : N_k(V) \to N_k(V)$ preserving the lattice group Λ. Namely, A is a linear-space automorphism preserving the bracket operation and $A(\Lambda) = \Lambda$. The following proposition gives a full description of what an automorphism of $N(n, k)$ should look like.

PROPOSITION 2.2.1.

$$Aut(n,k) = \left\{ \begin{pmatrix} A_0^{(0)} & 0 & 0 & \dots & 0 \\ A_1^{(0)} & A_0^{(1)} & 0 & \dots & 0 \\ A_2^{(0)} & A_1^{(1)} & A_0^{(2)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{k-1}^{(0)} & A_{k-2}^{(1)} & A_{k-3}^{(2)} & \dots & A_0^{(k-1)} \end{pmatrix} \right\},$$

where $A_0^{(0)} \in SL(n,\mathbb{Z}), A_i^{(0)} \in M(n_i \times n,\mathbb{Z})$, $(n_i = dimV^{(i)} = dim \underbrace{[[[[V,V],V],\ldots].V]}_{\text{i brackets}})$ *can be any matrices, and* $A_i^{(l)}$ *is determined by* $A_j^{(0)}$ *for* $j < i$ *by the formula* $A[v,w] = [Av, Aw]$. *When* $A_i^{(l)} = 0$ *for* $i \geq 1$, *the automorphism is said to be in diagonal block.*

PROOF. Take $A \in Aut(n,k)$; A is lower block-triangular because $V^{(i)} \oplus V^{(i+1)} \oplus \ldots \oplus V^{(k-1)}$ is invariant under A. A straightforward computation shows that $A_i^{(l)}$ is determined by $A_j^{(0)}$, $j \leq i$. □

We next establish the fact that all "rational points" are actually periodic points for the automorphisms.

LEMMA 2.2.2. *Let* $A \in Aut(n,k)$, $x \in N_k(n)$ *be a rational point with respect to a basis chosen from* C_0; *i.e., under that basis,* $x = (\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_N}{q_N})$. *Let the least common multiple of* $q_i's$ *be* q. *Let* m *be the constant that appeared in Theorem 2.1.1. Then there exists a positive integer* t, *such that* $A^t(x) = x + qm^2 z$, $z = (z_1, \ldots, z_N) \in \mathbb{Z}^N$. *In other words, the point* x *considered as a point on* $N(n,k)$ *is a periodic point for* A.

PROOF. In fact, the orbit of $\frac{x}{qm^2}$ under A^n with $n \geq 1$ is finite (mod $\mathbb{Z}^N$). So there exist $t_1 > t_2 \geq 1$ such that $A^{t_1}(\frac{x}{qm^2}) = A^{t_2}(\frac{x}{qm^2})$ mod$(\mathbb{Z}^N)$. But A as well as A^{-1} preserves $\mathbb{Z}^N$, so $A^{t_1 - t_2}(\frac{x}{qm^2}) = \frac{x}{qm^2}$ mod$(\mathbb{Z}^N)$. Therefore, our first assertion follows.

To prove that the point x considered as a point on $N(n,k)$ is a periodic point for A, we notice that $C(x + qm^2 z, -x) = \sum_{n=1}^{\infty} c_n(x + qm^2 z, -x)$, and also that

$$c_1(x + qm^2 z, -x) = qm^2 z \in m(\mathbb{Z}\text{-span of } C_0);$$

$$c_2(x + qm^2 z, -x) = \frac{1}{2}[x + qm^2 z, -x] = \frac{m}{2} m[z, -qx] \in m(\mathbb{Z}\text{-span of } C_0),$$

so for 2-step nilmanifold the assertion is true.

For higher-step nilmanifolds, we can prove this assertion by proving that $c_n(x+qm^2z,-x) \in m(\mathbb{Z}\text{-span of } C_0)$, using induction on n. $\square$

3. Anosov Elements Generate Diagonal-block $SL(n, \mathbb{Z})$ Actions and $Aut(n, k)$ Actions

In view of Katok-Lewis [K-L1] [K-L2] (see Theorem 1.4), the existence of Anosov elements is important to get a global rigidity result. Also, various known approaches to obtain the rigidity results use heavily the hyperbolicity of certain elements in the groups. For a general nilmanifold, one cannot expect the existence of an Anosov element in the group automorphisms. For example, we can prove that any nilmanifold with dimension ≤ 5 does not have Anosov automorphism. Some nilmanifolds of dimension ≥ 5, do not have an Anosov element either.

We summarize some easy facts below.

PROPOSITION 3.1. *The following nilmanifolds do not support Anosov Diffeomophisms:*

(1) *Nilmanifolds with dimension ≤ 5 that are not tori;*

(2) *$N(n, k)$ with $n \leq k$;*

(3) *The example constructed in [D], which has dimension 9.*

PROOF. (1) is a straightforward computation because the corresponding Lie algebras do not support Anosov diffeomorphisms that preserves a lattice.

(2) is true because any automorphisms of the nilmanifolds have eigenvalue 1.

(3) the nilpotent Lie group considered in [D] is a nine-dimensional simply connected, connected group having a unipotent group as its automorphism group. It is easy to see that it has a uniform lattice. Hence, there exists no Anosov diffeomorphism for any nilmanifolds obtained from the group as a factor space by a uniform lattice. □

Nevertheless, we can prove that for our diagonal-block action, there are plenty of Anosov elements for $n > k$. Actually, the diagonal-block action is a Cartan action when $k = 2$.

First of all, we give a definition.

DEFINITION (CARTAN ACTION) 3.2. Let $\mathcal{A}$ be a free Abelian group with a given set of generators $\Delta = \{\gamma_1, \dots, \gamma_n\}$. (ϕ, Δ) is a Cartan C^r-action on the manifold M if

(a) $\phi : \mathcal{A} \times M \to M$ is a C^r-action on M;

(b) each $\gamma_i \in \Delta$ is ϕ-hyperbolic and $\phi(\gamma_i)$ has a 1-dimensional, strongest stable foliation $\mathcal{F}^{ss}$;

(c) the tangential distributions $E_i^{ss} = T\mathcal{F}^{ss}$ are pairwise-transversal with their internal direct sum $E_1^{ss} \oplus \cdots \oplus E_n^{ss} \cong TM$.

Let $\phi : \Gamma \times M \to M$ be a C^r-action. We say ϕ is a Cartan action if there exists a subset of commuting elements $\Delta = \{\gamma_1, \dots, \gamma_n\} \subset \Gamma$, which generate an Abelian subgroup $\mathcal{A}$, such that the restriction of ϕ to $\mathcal{A}$ is a Cartan C^r-action on M. We call $(\phi|\mathcal{A}, \Delta)$ a Cartan subaction for ϕ.

THEOREM 3.3. *Diagonal-block $SL(n, \mathbb{Z})$ Action on $N(n, 2)$ is a Cartan action, if $n \geq 3$.*

PROOF. The proof of this theorem is a consequence of Theorem 5.1.5. □

It is unfortunate, however, that for any $k > 2$, the diagonal-block $SL(n, \mathbb{Z})$ action on $N(n, k)$ is never a Cartan action.

Although the actions are generally not Cartan actions, they have plenty of Anosov automorphisms. The following theorem asserts that the diagonal-block group is actually generated by Anosov automorphisms for $k \leq n - 1$.

THEOREM 3.4. *Consider $SL(n, \mathbb{Z})$ as an action on $N(n, k)$. Then $SL(n, \mathbb{Z})$ is generated by Anosov automorphisms, if $n \geq k + 1$.*

This theorem is a generalization of a result in [K-L2], which states that $SL(n, \mathbb{Z})$

as an action on $\mathbb{T}^n$ is generated by Anosov automorphisms. The proof, though different from that in [K-L2], is inspired by the argument there.

We will first prove the following Lemma.

LEMMA 3.5. *$SL(n,\mathbb{Z})$ as a matrix group has generators $A_1,\ldots,A_p$, such that each A_i is hyperbolic with one eigenvalue > 1 and other eigenvalues having absolute values < 1.*

PROOF. We first recall that there exists a matrix $A = (a_{ij}) \in SL(n,\mathbb{Z})$ such that all eigenvalues of A are positive, real numbers with only one eigenvalue $\lambda_1 > 1$. The remaining eigenvalues are $\lambda_2,\ldots,\lambda_n < 1$ [K-L2].

It is clear then that A^l ($l \gg 1$) has one big eigenvalue λ_1^l and $n-1$ small eigenvalues. Since E_{ij} are generators of $SL(n,\mathbb{Z})$, and

$$E_{ij} = (E_{ij}A^{-l}E_{ij}^{-1})(E_{ij}A^l),$$

it is clear that if we can prove that $E_{ij}A^l$ is hyperbolic with one eigenvalue > 1 and that the rest of the eigenvalues have absolute value < 1, we are done.

We rewrite

$$\begin{aligned} E_{ij}A^l &= E_{ij}B^{-1}Diag\{\lambda_1,\ldots,\lambda_n\}B \\ &= B^{-1}(BE_{ij}B_{-1})Diag\{\lambda_1,\ldots,\lambda_n\}B; \end{aligned}$$

it is clear that it has the same eigenvalues as those of

$$C_{ij} := (BE_{ij}B_{-1})Diag\{\lambda_1,\ldots,\lambda_n\},$$

so we have only to prove that C_{ij} has one eigenvalue > 1 and that the rest of eigenvalues have absolute value < 1.

Without loss of generality, we will prove C_{12} has the desired property.

Let $B=(b_{ij})$, $E_{12}=I+e_{12}$, $B^{-1}=(k_{ij})$; then

$$\begin{aligned}
&BE_{12}B^{-1}Diag\{\lambda_1,\ldots,\lambda_n\}\\
&=B(I+e_{12})B^{-1}Diag\{\lambda_1,\ldots,\lambda_n\}\\
&=I+Be_{12}B^{-1}Diag\{\lambda_1,\ldots,\lambda_n\}\\
&=\left(I+\begin{pmatrix} b_{11}k_{21} & b_{11}k_{22} & \ldots & b_{11}k_{2n}\\ b_{21}k_{21} & b_{21}k_{22} & \ldots & b_{21}k_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1}k_{21} & b_{n1}k_{22} & \ldots & b_{n1}k_{2n}\end{pmatrix}\right)Diag\{\lambda_1^l,\ldots,\lambda_n^l\}.
\end{aligned}$$

We may assume that $b_{11}k_{21}\geq 0$. (Otherwise we consider $B(I-e_{12})B^{-1}$ instead; recall that $(I+e_{12})^{-1}=I-e_{12}$.) Let

$$BE_{12}B^{-1}=\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n}\\ f_{21} & f_{22} & \cdots & f_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ f_{n1} & f_{n2} & \cdots & f_{nn}\end{pmatrix};$$

then $f_{11}\geq 1$ and

$$\begin{aligned}
C_{12}&=BE_{12}B^{-1}Diag\{\lambda_1,\ldots,\lambda_n\}\\
&=\begin{pmatrix} f_{11}\lambda_1^l & f_{12}\lambda_2^l & \cdots & f_{1n}\lambda_n^l\\ f_{21}\lambda_1^l & f_{22}\lambda_2^l & \cdots & f_{2n}\lambda_n^l\\ \vdots & \vdots & \ddots & \vdots\\ f_{n1}\lambda_1^l & f_{n2}\lambda_2^l & \cdots & f_{nn}\lambda_n^l\end{pmatrix}.
\end{aligned}$$

Consider

$$\det(C_{12}-\lambda I)=\det\begin{pmatrix} f_{11}\lambda_1^l-\lambda & f_{12}\lambda_2^l & \cdots & f_{1n}\lambda_n^l\\ f_{21}\lambda_1^l & f_{22}\lambda_2^l-\lambda & \cdots & f_{2n}\lambda_n^l\\ \vdots & \vdots & \ddots & \vdots\\ f_{n1}\lambda_1^l & f_{n2}\lambda_2^l & \cdots & f_{nn}\lambda_n^l-\lambda\end{pmatrix}$$

$$=(\lambda_1^l)^n\det\begin{pmatrix} f_{11}-(\lambda/\lambda_1^l) & f_{12}(\lambda_2^l/\lambda_1^l) & \cdots & f_{1n}(\lambda_n^l/\lambda_1^l)\\ f_{21} & f_{22}(\lambda_2^l/\lambda_1^l)-(\lambda/\lambda_1^l) & \cdots & f_{2n}(\lambda_n^l/\lambda_1^l)\\ \vdots & \vdots & \ddots & \vdots\\ f_{n1} & f_{n2}(\lambda_2^l/\lambda_1^l) & \cdots & f_{nn}(\lambda_n^l/\lambda_1^l)-(\lambda/\lambda_1^l)\end{pmatrix}.$$

It is an easy exercise to prove that for sufficiently large l, one of the eigenvalues satisfies

$$|\lambda/\lambda_1^l - f_{11}| < f_{11}/2$$

or

$$f_{11}/2 < \lambda/\lambda_1^l < 3f_{11}/2;$$

the rest of the eigenvalues satisfy

$$|\lambda/\lambda_1^n| < f_{11}/2.$$

Furthermore, it is a straightforward calculation that those eigenvalues with $|\lambda/\lambda_1^n| < f_{11}/2$ actually satisfy $|\lambda| < 1$. Indeed,

$$\det \begin{pmatrix} f_{11}\lambda_1^l - \lambda & f_{12}\lambda_2^l & \dots & f_{1n}\lambda_n^l \\ f_{21}\lambda_1^l & f_{22}\lambda_2^l - \lambda & \dots & f_{2n}\lambda_n^l \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}\lambda_1^l & f_{n2}\lambda_2^l & \dots & f_{nn}\lambda_n^l - \lambda \end{pmatrix} =$$

$$= (-1)^n(\lambda^n - \lambda_1^l(f_{11} + \varepsilon_1(l))\lambda^{n-1} + \lambda_1^l(\varepsilon_2(l))\lambda^{n-2} + \dots + \lambda_1^l(1/\lambda_1^l));$$

suppose that $1 < |\lambda| < (f_{11}/2)\lambda_1^l$; then

$$\begin{aligned} &|\lambda^n - \lambda_1^l(f_{11} + \varepsilon_1(l))\lambda^{n-1} + \lambda_1^l(\varepsilon_2(l))\lambda^{n-2} + \dots + \lambda_1^l(1/\lambda_1^l))| \\ &\geq |\lambda^n - (f_{11}\lambda_1^l)\lambda^{n-1}| - |\lambda_1^l\varepsilon_1(l)\lambda^{n-1} + \lambda_1^l(\varepsilon_2(l))\lambda^{n-2} + \dots + \lambda_1^l(1/\lambda_1^l))| \\ &\geq |\lambda - (f_{11}\lambda_1^l)||\lambda^{n-1}| - |\lambda_1^l\varepsilon_1(l) + \lambda_1^l(\varepsilon_2(l)) + \dots + 1/\lambda_1^l))||\lambda^{n-1}| \\ &> 0. \end{aligned}$$

We get a contradiction. □

The theorem is then a corollary of this Lemma. We remark, however, that we do not have the diagonalizability of those Anosov generators. It is conceivable that we should have diagonalizable, hyperbolic generators. But we cannot prove this fact in this paper.

Remark 3.6. It is easy to prove that $Aut(n, k)$ is generated by Anosov elements, if $n \geq k + 1$. We omit the proof.

4. Local Rigidity for Diagonal-block $SL(n,\mathbb{Z})$ Actions

To prove the local rigidity for $SL(n.\mathbb{Z})$ (or any subgroup of it with a finite index) diagonal-block action, we want to utilize a powerful machinery by [H-K-L-Z], which gives that for every finitely indexed subgroup $\Gamma \subset SL(n.\mathbb{Z})$, the Cartan action can be measurably linearized up to a finite covering. Therefore, we could use the result there to conclude the rigidity of $SL(n.\mathbb{Z})$ (or any subgroup of it with finite index) diagonal-block action. We summarize the result of [H-K-L-Z] in the following theorem suitable for our purpose.

THEOREM (HURDER, KATOK, LEWIS, ZIMMER) 4.1. *Let $\Gamma \subset SL(n,\mathbb{Z})$ be a subgroup of finite index; let α be a Cartan action of Γ on M. Then there exists a normal subgroup $\Gamma' \subset \Gamma$ of finite index, a finite Galois covering $M' \to M$, a lift $\widetilde{\phi}'$ and measurable vector fields $\{w_1, \dots, w_k\}$, $k = dimM$ on M', such that the cocycle $D\alpha$ with respect to the framing $\{w_1, \dots, w_k\}$ is given almost everywhere by a representation $\rho : \Gamma' \to SL(k,\mathbb{Z})$.* □

Combining this theorem with the theorem in the next section, we can prove the following theorem.

THEOREM A. *The standard block action of finite indexed subgroup $\Gamma \subset SL(n,\mathbb{Z})$ on $N(n,2)$ is locally rigid, if $n \geq 3$, $n \neq 4$.*

PROOF. The action we consider is a Cartan action; a small perturbation will also be Cartan. Use the fact that we have an Abelian subgroup $\mathcal{A}$; the restricted action is smoothly conjugate to the original linear action (see the next section). Thus we may assume that the perturbation of the action keeps $\mathcal{A}$ intact. Then the argument of the proof of Lemma 8 in [H-K-L-Z] goes through without any modification. We conclude that the action of Γ on $N(n,2)$ is smoothly conjugate to a linear action. □

5. Smooth, Local Rigidity of Abelian Group $\mathcal{A}$ Actions

In this section, we investigate the smoothness of topological conjugacy. We will prove that for $n \geq 3$, $n \neq 4$, topological conjugacy of the diagonal-block actions on $N(n,2)$ are, in fact, smooth. This fact is actually proved by showing that for a certain Abelian group, the action is locally smoothly rigid.

5.1 Density of Exponents. Let $\Gamma \subset SL(n,\mathbb{Z})$ be a subgroup of finite index. Take an Abelian subgroup $\mathcal{A} \subset \Gamma$ such that $\mathcal{A}$ can be diagonalized over $\mathbb{R}$. The existence of such a group was established in [R-P]. We also may assume that there exists a splitting Cartan subgroup H of $SL(n,\mathbb{R})$ and $\mathcal{A} \subset H^0 \cap \Gamma$ such that $H^0/\mathcal{A}$ is compact. This result was utilized in [K-L2] to prove the smoothness of topological conjugacy. We summarize this result as a theorem below.

THEOREM 5.1.1. *There exists a splitting Cartan subgroup $H \subset SL(n,\mathbb{R})$, a subgroup $\mathcal{A} \subset H^0 \cap \Gamma$, such that*

(1) *$H^0/\mathcal{A}$ is compact;*

(2) *if we denote $\lambda_i(h)$ to be the i-th eigenvalue of $h \in H^0$ (for a fixed basis such that all $h \in H^0$ are diagonal), then*

$$\lambda(:= \lambda_1 \times \lambda_2 \times \ldots \times \lambda_n) : H^0 \to V_1(:= \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^{+^n} | x_1 x_2 \ldots x_n = 1\})$$

is an isomorphism. (Consequently, $\ln(.) : H^0 \to \ln(V_1)$ is an isomorphism. We will sometimes abuse the notation to say these two isomorphisms are the same and denote them by $\lambda : H^0 \to \mathbb{R}^{n-1} = \{y_1 + y_2 + \ldots + y_n = 0\}$).

Next, we want to prove the existence of Anosov elements in any subgroup $\Gamma \subset SL(n,\mathbb{Z})$ of finite index.

LEMMA 5.1.2. *If $A \in SL(N,\mathbb{Z})$ is diagonalizable, i.e.*

$$A \sim diag(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

then as an automorphism of $N(n,k)$, *A can be diagonalized as*

$$A \sim diag(\lambda_i, \lambda_i\lambda_j, \ldots, \lambda_1^{i_1}\lambda_2^{i_2}\ldots\lambda_n^{i_n}), \text{ with } i_1 + i_2 + \ldots + i_n \leq k.$$

PROOF:. Let $v_1, v_2, \ldots, v_n$ be different eigenvectors for $A : \mathbb{R}^n \to \mathbb{R}^n$; then $[[[[v_i, v_j], v_k], \ldots], v_l]$ spans $N_k(\mathbb{R}^n)$, which are also eigenvectors for the induced automorphism of $N(n,k)$, and our assertion then is clear. □

Remark 5.1.3. Some $\lambda_i^{i_1}\ldots\lambda_j^{j_1}$ may not be eigenvalues. For example, λ_i^2 is not an eigenvalue. But all eigenvalues of the induced automorphisms have this form. □

We will have the existence of Anosov elements in $\mathcal{A}$ considered as a subgroup of the group of automorphisms in the next theorem. In what follows in this section, we will denote by $\overline{A}$ the induced automorphism on $N(n,k)$ from a matrix $A \in SL(n,\mathbb{Z})$. We will always assume that $k \leq n-1$.

THEOREM 5.1.4. *There exists an element* $A \in \mathcal{A}$ *such that if* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are* n *different eigenvalues of* A *as a matrix in* $SL(n,\mathbb{Z})$, *then* $\lambda_i, \lambda_i\lambda_j, \ldots, \lambda_1^{i_1}\lambda_2^{i_2}\ldots\lambda_n^{i_n}$, $i_1 + i_2 + \ldots + i_n \leq k \leq n-1$ *are positive, different from 1, different from one another.*

PROOF. Using notation in Theorem 5.1.1, we construct an open set in $\mathbb{R}^{n-1}$ as follows. Let m be a positive integer; define

$$D_m := \bigcap_{\substack{|i_1|+\ldots+|i_n|\leq m \\ i_1, i_2, \ldots, i_n \text{ are not all equal}}} \{i_1y_1 + i_2y_2 + \ldots + i_ny_n \neq 0\};$$

it is the complement set of finite hyperplanes. Every connected component, say D_0, have an arbitrarily large diameter. So D_0 contains a fundamental domain for $\mathcal{A}$ in H_0 (view $\mathcal{A}$ as a lattice of $\mathbb{R}^{n-1}, H^0$ as $\mathbb{R}^{n-1}$). So $\mathcal{A} \cap D \neq \phi$. Then any element in the intersection can be taken as A. □

Fix an eigenvector v for $\overline{A}$, A as in last Theorem; v is then a common eigenvector for all $\overline{B}$, $B \in \mathcal{A}$. Denote

$$E(v) := \{\text{eigenvelues for all } \overline{B} \text{ corresponding to the eigenvector } v, B \in \mathcal{A}\};$$

we hope the following is true:

$$\overline{E(v)} = \mathbb{R}.$$

Till this writing, we can obtain only this result for $N(n,k)$ for certain n, k. We do not know whether it is true in general.

THEOREM 5.1.5. *Let $n = dim(V), k \leq n-1$ be the step of our nilpotent algebra. If $n \neq 4, 6, 8, \ldots, 2(k+1)$, then $\overline{E(v)} = \mathbb{R}$ for all common eigenvectors of $\overline{A}$ ($A \in \mathcal{A}$).*

To prove this theorem, we need two preliminary results, which have their own interest. We state them as two propositions.

PROPOSITION 5.1.6. *Let $A : \mathbb{T}^n \to \mathbb{T}^n$ be an automorphism with different eigenvalues. Let $f(\lambda)$ be the characteristic polynomial of A. Let $p(\lambda)$ be an irreducible factor of $f(\lambda)$ (over $\mathbb{Q}$). Then*

(1) *there exists a subtorus $\mathbb{T}^k$ such that $A(\mathbb{T}^k) = \mathbb{T}^k$, and the characteristic polynomial of $A|_{\mathbb{T}^k} = p(\lambda)$;*
(2) *any automorphism B such that $AB = BA$, satisfies $B(\mathbb{T}^k) = \mathbb{T}^k$.*

PROOF. (2) is clear from the fact that all the eigenvalues of A are different. To prove (1), let

$$f(\lambda) = p(\lambda)h(\lambda)$$

be a decomposition of f over $\mathbb{Q}$; then we may take $p(\lambda), h(\lambda) \in \mathbb{Z}[\lambda]$. (See, for example, T. Hungerford [H]: Algebra, p.163, Lemma 6.13). Since $p(\lambda), h(\lambda)$ do not have common zeros, we obtain

$$(p, h) = 1.$$

We choose $p', h' \in \mathbb{Z}[\lambda]$ such that

$$1 = p'p + h'h.$$

(See T. Hungerford [H], p.140, Theorem 3.11 (ii) for a proof of this fact.)

Therefore,

$$x = p'(A)p(A)(x) + h'(A)h(A)(x)$$

for all $x \in \mathbb{T}^k$. Define

$$Ker(p(A)) = \{x : p(A)x = 0\},$$

$$Ker(h(A)) = \{x : h(A)x = 0\}.$$

They are closed A-invariant subgroups, and it is clear that

$$Ker(p(A)) + Ker(h(A)) = \mathbb{T}^n.$$

Let $y \in Ker(p(A)) \cap Ker(h(A))$; then

$$y = p'(A)p(A)y + h'(A)h(A)y.$$

Hence, from the fact that $p(A)y = 0$ and $h(A)y = 0$, we have $y = 0$. In other words, $Ker(p(A)) \cap Ker(h(A)) = 0$. Therefore,

$$\mathbb{T}^n = Ker(p(A)) \oplus Ker(h(A)).$$

But then it is easy to see that the closed subgroups $Ker(p(A))$ and $Ker(h(A))$ are subtori of $\mathbb{T}^n$ and that they intersect transversely, and the intersection is only one point.

Now consider $A|_{ker(p(A))} : Ker(p(A)) \rightarrow Ker(p(A))$, an automorphism of $Ker(p(A))$, so it has integer-matrix representation for some basis. Therefore, the characteristic polynomial $\hat{p}(\lambda)$ belongs to $\mathbb{Z}[\lambda]$. Since $p(A|_{Ker(p(A))}) = 0$, it is clear then that the minimal polynomial of $A|_{Ker(p(A))}$ divides $p(\lambda)$. But $\hat{p}(\lambda)|f(\lambda)$, so there are no multiple roots for $\hat{p}(\lambda)$. So the minimal polynomial of $A|_{Ker(p(A))}$ is exactly the characteristic polynomial of $A|_{Ker(p(A))}$; hence, $\hat{p}(\lambda)|p(\lambda)$. But $p(\lambda)$ is irreducible over $\mathbb{Q}$, so

$$\hat{p}(\lambda) = p(\lambda).$$

□

To state the next proposition, we need some easy facts from multilinear algebra. We give a brief summary here. See also [N].

Let $f : M \rightarrow N$ be a linear map between vector spaces M, N, let $S(M), S(N)$ be the symmetric tensor algebras of M, N. Then f has a unique extension to a homomorphism $S(f) = \oplus S_p(f)$ of graded algebra $S(M) = \oplus S_p(M)$ to $S(N) = \oplus S_p(N)$. Moreover, if $m_1, \ldots, m_p \in M$, then

$$S(f)(m_1 \ldots m_p) = f(m_1) \ldots f(m_p).$$

If $g : K \rightarrow M$ is another linear map between real vector spaces K, M, then

$$S(f) \circ S(g) = S(f \circ g)$$

$$S_p(f) \circ S_p(g) = S_p(f \circ g).$$

It is clear then that if $M = N = K := V$ and $f : V \rightarrow V$ is an isomorphism, then $S(f)$ as well as $S_p(f)$ is an isomorphism.

In other words, any matrix subgroup of $GL(n, \mathbb{R})$ acts on $S(V)$ as well as $S_p(V)$ in a "standard way."

Let V be an n-dimensional vector space; then we may consider $S(V)$ ($S_p(V)$) as a real vector space. Fix a basis $\{e_1, \ldots, e_n\}$ of V; then $\{e_{i_1}$ $(1 \leq i_1 \leq n)$; $e_{i_1}e_{i_2}$ $(1 \leq i_1 \leq i_2 \leq n)$; $\ldots$; $e_{i_1} \ldots e_{i_n}$ $(1 \leq i_1 \leq \cdots \leq i_n \leq n)\}$ is a basis for $S(V)$, and $\{e_{i_1} \ldots e_{i_p}$ $(1 \leq i_1 \leq \cdots \leq i_p \leq n)\}$ is a basis of $S_p(V)$.

Let us take a lattice generated by the integer points with respect to the chosen basis. Then any subgroup Γ of $SL(n, \mathbb{Z})$ induces an action preserving the lattice. So it factors through a toral action.

It is clear that if $\mathcal{A}$ is a diagonalizable subgroup, then the $S(A)$ is also diagonalizable; if A is an element in $\mathcal{A}$ with n different eigenvalues (A as a matrix in $SL(n, \mathbb{Z})$) $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\lambda_i, \lambda_i\lambda_j, \ldots, \lambda_1^{i_1}\lambda_2^{i_2}\ldots\lambda_n^{i_n}$, $1 \leq i_1 + i_2 + \ldots + i_n \leq k, i_j \geq 0$ are all eigenvalues for $S(A)$. For some special A as in Theorem 5.1.4, they are positive, different from 1, different from one another.

PROPOSITION 5.1.7. *Let $\{v_1, \ldots, v_n\}$ be common eigenvectors for $\mathcal{A}$ as a matrix subgroup of $SL(n, \mathbb{Z})$. Let v be a common eigenvector for $\mathcal{A}$ as a subgroup of induced, nipotent algebra automorphisms. Let v be obtained by bracket operations from $\{v_1, \ldots, v_n\}$. Then $\overline{E(v)} = \mathbb{R}$, if $n \neq 4, 6, 8, \ldots, 2k$.*

PROOF. We actually prove more. We will prove that for any common eigenvector v of $S_p(\mathcal{A})$, $v = v_1^{p_1} \ldots v_q^{p_q}$, $\overline{E(v)} = \mathbb{R}$, if every irreducible factor of the characteristic polynomial of $S_p(A)$ has a degree ≥ 3.

Take A as in Theorem 5.1.4; let

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$$

be the eigenvalues of A. Take a $\mathbb{Q}$-irreducible factor $p(\lambda)$ of the characteristic polynomial of $S_p(A)$. Let the corresponding invariant space be V^{irr}, which has the dimension $r = \dim(p(\lambda))$. Also let the corresponding torus be $\mathbb{T}^r$.

Let

$$0 < \eta_1^- < \eta_2^- < \cdots < \eta_{r_1}^- < 1 < \eta_1^+ < \eta_2^+ < \cdots < \eta_{r_2}^+$$

be eigenvalues of $S_p(A)|_{V^{irr}}$. It is then clear that

$$\eta_j^+ = \lambda_1^{i_{j1}^+} \lambda_2^{i_{j2}^+} \dots \lambda_n^{i_{jn}^+}$$

$$\eta_j^- = \lambda_1^{i_{j1}^-} \lambda_2^{i_{j2}^-} \dots \lambda_n^{i_{jn}^-}$$

$\sum i_n^\pm \leq r$ (because V^{irr} has dimension r).

Go back to the notation of Theorem 5.1.2; we know that

$$D = (\cap_j \{\sum_{t=1}^{n} i_{jt}^+ y_t > 0\}) \cap (\cap_j \{\sum_{t=1}^{n} i_{jt}^- y_t < 0\}) \neq \phi$$

and that D is an open set of $\mathbb{R}^{n-1}$. $A^n \in D \Rightarrow D$ is unbounded. Take a connected domain $D(A)$ containing A; take an unbounded side L of $D(A)$ that does not intersect any other sides. It is in fact a part of a hyperplane (codimension 1, affine subspace).

<u>*Claim :*</u> If $L = \{\sum_t i_{jt}^\pm y_t = 0\}$, then for all $s \in L - \{\text{other sides}\}$, there exists a neighborhood $U(s)$ of s, and $s^\pm \in U(s)$, such that s^+ belongs to $\sum_t i_{jt}^+ y_t \geq 0$; and s^- belongs to $\sum_t i_{jt}^- y_t \leq 0$.

(This is a obvious.)

The consequence of this is that $\mathcal{A}$ contains at least two elements A^+, A^-, as automorphisms of $\mathbb{T}^r$, the signs of the logarithm of the ordered eigenvelues of which differ in the following fashion:

$$(*) \qquad \begin{array}{c} + + \cdots + + - - \cdots - - \\ + + \cdots + - - - \cdots - - \end{array}$$

We claim also that if A is taken as in Theorem 5.1.2, then every 1-dimensional eigenspace of $S_p(A)|_{V^{irr}}$ project densely to $\mathbb{T}^r$. This is because $p(\lambda)$ is irreducible.

Indeed, if it is not the case, then the closure of the projection of the 1-dimensional eigenspace of V^{irr} would contain an A-invariant subtorus; the characteristic polynomial of the restriction of $S_p(A)$ to that smaller torus will then divide $p(\lambda)$.

Now we can prove the density of $E(v)$ in $\mathbb{R}$. Otherwise, $\overline{E(v)} \cap \mathbb{R}^+$ is a infinite cyclic subgroup of $\mathbb{R}^+$. This implies that $S_p(A^+)^n(v) = S_p(A^-)^m(v)$ for some integers m, n. Because that every 1-dimensional eigenspace of $S_p(A)|_{V^{irr}}$ projects densely to $\mathbb{T}^r$, we know that the above identity holds for all $t \in \mathbb{T}^r$; i.e., $S_p(A^+)^n(t) = S_p(A^-)^m(t)$. this will contradict the pattern in $(*)$.

It is clear that when $n \neq 4, 6, 8, \ldots, 2k$, the characteristic polynomial of $S_p(A)|_{S_p(V)}$ cannot have factors with a degree < 3. (This is because the constant term of the polynomial has to be 1, and any product of two eigenvalues cannot be 1.) We thus finish our proof. $\square$

5.2 Smoothness of topological conjugacy. In this section, we will abuse the notation not to distinguish the elements of $SL(n, \mathbb{Z})$ and the induced automorphisms on $N(n, k)$.

Using Katok-Lewis's non-stationary Sternberg linearization theorem and a similar argument in [K-L2], we may prove the following theorem.

THEOREM 5.2.1. *$\mathcal{A}$ action on $N(n, 2)$ is locally rigid, if $n \neq 4$, $n \geq 3$.*

COROLLARY 5.2.2. *Topological conjugacies for small perturbations of group actions Γ on $N(n, 2)$ are smooth as long as Γ contains a Cartan subgroup $\mathcal{A}$ as in Theorem 5.1.2.*

We are going to follow the idea in [K-L2] to prove the local rigidity of the $\mathcal{A}$ action on $N(n, 2)$. First of all, we give a proposition that provides simultaneous diagonalization of a small perturbation of the $\mathcal{A}$ action. The same result for the action on torus is essentially given by [Pa-Y], which is utilized in [K-L2]; their result is a little bit stronger than ours, but for the purpose of proving local rigidity, the result below is sufficient. We will modify their argument to fit our situation.

PROPOSITION 5.2.3. *Let $\mathcal{A}$ be an Abelian subgroup of $SL(n,\mathbb{Z})$ acting on $N(n,k)$ with $n \geq k+1$. Assume that there exists an Anosov element A in $\mathcal{A}$. Let $\rho(\mathcal{A})$ be a small perturbation of the action such that there exists a conjugacy f (close to identity) between A and $\rho(A)$. If the perturbation is small enough, then f is a conjugacy between all $B \in \mathcal{A}$ and $\rho(B)$.*

PROOF. Since

$$f^{-1}\rho(A)f = A,$$

we have

$$f^{-1}\rho(A)\rho(B)f = Af^{-1}\rho(B)f$$

for all $B \in \mathcal{A}$. Let us fix n_0 generators $A_1, A_2, \ldots, A_{n_0}$ for $\mathcal{A}$; fix a Remannian metric d on $N(n,k)$. Assume that the perturbation is so small that

$$d(A_i^{-1}f^{-1}\rho(A_i)f(x), x) < c_0,$$

where c_0 is the expansive constant for A. From

$$f^{-1}\rho(A_i)\rho(A)f = f^{-1}\rho(A_i)fA,$$

we obtain

$$AA_i^{-1}f^{-1}\rho(A_i)f = A_i^{-1}f^{-1}\rho(A_i)fA.$$

Let $C_i := A_i^{-1}f^{-1}\rho(A_i)f$; then

$$d(C_i(x), x) < c_0$$

for all $x \in N(n, k-1)$; hence

$$d(A^m(C_i(x)), A^m(x)) = d(C_i(A^m(x)), A^m(x)) < c_0$$

for all $x \in N(n, k-1)$ and $m \in \mathbb{Z}$. This forces

$$C_i(x) = x.$$

So

$$A_i^{-1} f^{-1} \rho(A_i) f = id_{N(n,k)};$$

hence,

$$A_i = f^{-1} \rho(A_i) f.$$

□

REMARK. Our argument here has the advantage that we actually can prove more; i.e., we can prove that for any finitely generated, discrete Abelian group action on a compact manifold with one expansive element, a small conjugacy of this element is the conjugacy of the action.

Now we will apply Katok-Lewis's Non-stationary Sternberg Linearization Theorem [K-L2] to conclude that the conjugacy is actually smooth along the 1-dimensional foliations.

THEOREM(KATOK-LEWIS' NON-STATIONARY STERNBERG LINEARIZATION) 5.2.4. *Let M be a compact manifold, $\mathcal{L} = M \times \mathbb{R}$, the trivial. real line bundle over M, and let*

$$F : \mathcal{L} \to \mathcal{L}, (x, t) \mapsto (f(x), F_x(t)),$$

with F_x a C^∞ diffeomorphism of $\mathbb{R}$ for each $x \in M$; satisfy

(1) *$F_x(0) = 0$ for every $x \in M$ (F preserves the zero section);*

(2) *$0 < F_x' < 1$ for every $x \in M, t \in \mathbb{R}$, and*

(3) *$x \mapsto F_x$ is a continuous map $M \to C^\infty(\mathbb{R})$.*

Then there exists a unique reparameterization

$$G : \mathcal{L} \to \mathcal{L},\ (x, y) \mapsto (x, G_x(t)),$$

such that

(1) *each* G_x *is a* C^∞ *diffeomorphism of* $\mathbb{R}$;

(2) $G_x(0) = 0,\ G'_x(0) = 1$ *for every* $x \in M$;

(3) $x \mapsto G_x$ *is a continuous map* $M \to C^\infty(\mathbb{R})$, *and*

(4) $GFG^{-1}(x,t) = (f(x), F'_x(0)t)$ *for every* $x \in M,\ t \in \mathbb{R}$.

Fix an $A \in \mathcal{A}$, such that all the eigenvalues of A as an automorphism of $N(n,k)$ are different. Fix a 1-dimensional foliation $\mathcal{F}$ of $N(n,2)$ corresponding to an eigenvector v of A; hence v is a common eigenvector for $\mathcal{A}$. Hence the 1-dimensional foliation $\mathcal{F}$ of $N(n,2)$ is an invariant foliation for $\mathcal{A}$. It will be proved in Lemma 5.2.5 that this 1-dimensional foliation is the intersection of several stable foliations for some Anosov automorphisms. As an important corollary of this lemma, if we take a small perturbation of the action $\mathcal{A}$ on $N(n,2)$, then the foliation $\mathcal{F}$ persists, and also the individual leaf of the perturbed foliation $\mathcal{F}'$ is C^∞ manifold. The perturbed foliation $\mathcal{F}'$ is Hölder foliation and varies continuously (because the foliation is strong stable foliation for some Anosov element, then the results follow from [B-P],[Sh]). The topological conjugacy f as in Proposition 5.2.3 carries the leaves of $\mathcal{F}$ to those of $\mathcal{F}'$.

LEMMA 5.2.5. *The 1-dimensional foliation* $\mathcal{F}$ *of* $N(n,2)$ *with* $n \geq 3$ *is the intersection of several stable foliations for some Anosov automorphisms in* $\mathcal{A}$ *of* $N(n,2)$. *The action of* $\mathcal{A}$ *on* $N(n,2)$ *is a Cartan action.*

PROOF. It is clear that leaves of $\mathcal{F}$ are the images under π of lines in the universal covering $\mathbb{R}^N$ of $N(n,2)$ parallel to v_λ, which is the eigenvector corresponding to the eigenvalue λ. Therefore, it is sufficient to prove that in the universal cover, or in the Lie algebra considered as a linear vector space over $\mathbb{R}$, every 1-dimensional, common eigenspace space of $\mathcal{A}$ is the intersection of several stable vector spaces of Anosov elements in $\mathcal{A}$.

Let the Lie algebra $\mathfrak{g}$ be associated with n-dimensional vector space V. Let $\mathcal{A}$ have n-different, common 1-dimensional eigenspaces. Take an ordered basis $\{v_1, \ldots, v_n\}$; each v_i generates an eigenspace.

It is clear that all 1-dimensional, common eigenspaces of $\mathcal{A}$ are generated by v_i; $[v_j, v_k]$ for $i, j, k = 1, \ldots, n, j \neq k$.

We first show that v_i is the intersection of several stable manifolds of Anosov elements in $Aut(n, 2)$. Without loss of generality, we let $i = 1$. Take $A \in \mathcal{A}$ such that as a matrix in $SL(n, \mathbb{Z})$, it has one eigenvalue (corresponding to eigenvector v_1) $\lambda_1 < 1$, and other eigenvalues (corresponding to eigenvectors v_i) $\lambda_i > 1$. In other words, A has ordered eigenvalues $\lambda_1, \ldots, \lambda_n$, satisfying

$$0 < \lambda_1 < 1 < \lambda_i, \ i \neq 1.$$

Therefore, the stable vector space $W^s(A)$ for A as an automorphism on $\mathfrak{g}$ is spanned by $v_1; [v_1, v_2], \ldots, [v_1, v_n]$. Next, we take $A' \in \mathcal{A}$ such that as a matrix in $SL(n, \mathbb{Z})$, A' has ordered eigenvalues $\lambda_1', \lambda_2', \ldots, \lambda_n'$, satisfying

$$0 < \lambda_1', \lambda_2' < 1 < \lambda_i', \ i \neq 1, 2.$$

The existence of such A' is a corollary of Theorem 5.1.1. We may assume also that $\lambda_2'\lambda_i' > 1, \lambda_1'\lambda_i' < 1, \ i \neq 1, 2$. (For example, we may consider $A^p(A')^q$ to be our new A'.) Hence, the stable vector space $W^s(A')$ for A as an automorphism on $\mathfrak{g}$ is spanned by $v_1, v_2; [v_1, v_2], \ldots, [v_1, v_n]$. Similarly, we may find $B \in \mathcal{A}$, such that the stable vector space $W^s(B)$ for B as an automorphism on $\mathfrak{g}$ is spanned by $v_1, v_2; [v_2, v_1], [v_2, v_3], \ldots, [v_2, v_n]$. So $W^s(A) \cap W^s(A') \cap W^s(B)$ is spanned by $v_1, [v_1, v_2]$. The same argument may imply that the space spanned by $v_1, [v_1, v_3]$ is the intersection of some stable vector spaces for some elements in $\mathcal{A}$. So the space spanned by v_1 is the intersection of some stable vector spaces for some elements in $\mathcal{A}$.

Next, we will show that $[v_i, v_j]$, $i \neq j$ is the intersection of several stable manifolds of Anosov elements in $Aut(n, 2)$. Without loss of generality, we let $i = 1, j = 2$. We already know that the space spanned by $v_1; [v_1, v_2], \ldots, [v_1, v_n]$ is the stable vector space $W^s(A)$ for some $A \in \mathcal{A}$. Similarly, the space spanned by $v_2; [v_2, v_1], [v_2, v_3], \ldots, [v_2, v_n]$ is the stable vector space $W^s(C)$ for some $C \in \mathcal{A}$. So the space spanned by $[v_1, v_2]$ is the intersection of several stable vector spaces of Anosov elements in $Aut(n, 2)$.

The other statement in the lemma is clear. $\square$

The leaves of both $\mathcal{F}$ and $\mathcal{F}'$ inherit natural Riemannian metrics as submanifolds of $N(n, 2)$. Let v_λ be the vector in $\mathfrak{g}$ that determines $\mathcal{F}$. For each $x \in N(n, 2)$, let $\phi_x : \mathbb{R} \to \mathcal{F}(x)$ denote the arc-length parameterization based at x, oriented so that v_λ points in the positive direction; i.e., $\phi_x(0) = x$, the distance along $\mathcal{F}(x)$ between x and $\phi_x(t) \in \mathcal{F}(x)$ is t, and $\langle v_\lambda, (\phi_x)_*(d/dt)\rangle > 0$ (standard inner product on $T_x(N(n, 2)) \cong \mathbb{R}^N$). Define $\widetilde{\phi}_x : \mathbb{R} \to \widetilde{\mathcal{F}}(x)$ similarly, oriented so that $\widetilde{\phi}^{-1}_{f(x)} \circ f \circ \phi_x : \mathbb{R} \to \mathbb{R}$ is an orientation-preserving homeomorphism.

Recall that a family $\{W_x\}_{x\in M}$ of k-dimensional C^∞ submanifolds of M is said to *vary continuously* if for each $x \in M$ there exists a neighborhood U of x in M and a continuous map $\phi : M \to C^\infty(D^k, M)$ such that ϕ_x maps D^k diffeomorphically onto a neighborhood centered at x in W_x, where D^k denotes the unit disk in $\mathbb{R}^k$. It is well-known (see Shub [Sh]) that the strong, stable foliation varies continuously.

By construction, $\phi_x : \mathbb{R} \to \mathcal{F}(x)$ and $\widetilde{\phi}_x : \mathbb{R} \to \widetilde{\mathcal{F}}$ are diffeomorphisms for every $x \in N(n, 2)$. Let $\mathcal{L} \times \mathbb{R}$ denote the trivial line bundle over $N(n, 2)$. It follows easily from the continuous varying of the two foliations that $\phi : \mathcal{L} \to N(n, 2), (x, t) \mapsto \phi_x(t)$ and $\widetilde{\phi} : \mathcal{L} \to N(n, 2), (x, t) \mapsto \widetilde{\phi}_x(t)$ are continuous, and that $x \mapsto \phi_x$, $x \mapsto \widetilde{\phi}_x$ are continuous maps $N(n,2) \to C^\infty(\mathbb{R}, N(n,2))$.

Let $f = \rho(A) \in$Diff$(N(n,2))$. Extend f and h to transformations on $\mathcal{L}$ in the obvious way

$$F : \mathcal{L} \to \mathcal{L},\ (x,t) \mapsto (f(x), F_x(t))$$

and

$$H : \mathcal{L} \to \mathcal{L},\ (x,t) \mapsto (h(x), H_x(t)),$$

so that

$$\widetilde{\phi}(F(x,t)) = f(\widetilde{\phi}(x,t)) \text{ and } \widetilde{\phi}(H(x,t)) = h(\phi(x,t)).$$

Then F and H are continuous, $F_x \in C^\infty(\mathbb{R})$ for each $x \in N(n,2)$, $0 < F_x'(t) < 1$ for every $x \in N(n,2)$, $t \in \mathbb{R}$, and $x \mapsto F_x$ is a continuous map $N(n,2) \to C^\infty(\mathbb{R})$. In fact, $F_x(t)$ is the length between $f(x)$ and $f(\widetilde{\phi}_x(t))$.

Next we will show that $H_x \in C^\infty(\mathbb{R})$ and $x \mapsto H_x$ are continuous. The argument is identical to that of [K-L2]. For the sake of completeness, we will repeat the argument below. And also we remark that this is the only place that we need the density of the exponents.

By Theorem 5.2.4, there exists a unique continuous linearization

$$G : \mathcal{L} \to \mathcal{L},\ (x,t) \mapsto (x, G_x(t))$$

such that

(1) $G_x \in C^\infty(\mathbb{R})$ with $G_x'(0) = 1$ for every $x \in N(n,2)$;

(2) $N(n,2) \to C^\infty(\mathbb{R})$, $x \mapsto G_x$ is continuous;

(3) $GFG^{-1}(x,t) = (f(x), F_x'(0)t)$ for every $x \in N(n,2)$ and $t \in \mathbb{R}$.

LEMMA 5.2.6. *Suppose that $p \in N(n,2)$ is rational (hence a periodic point for the standard action of every element the $BSL(n,\mathbb{Z})$). Then $G_{h(p)} \circ H_p|_{\mathbb{R}^+} : \mathbb{R}^+ \to \mathbb{R}^+$ has the form $G_{h(p)} \circ H_p(t) = c_p t^{\nu_p}$ for some $c_p, \nu_p > 0$.*

PROOF. Since $\mathcal{A}$ is Abelian, it follows from the uniqueness of G that G simultaneously linearizes the transformations on $\mathcal{L}$ corresponding to $\rho(A)$ for each $A \in \mathcal{A}$. From the fact that the eigenvalue set $E(v)$ is dense in $\mathbb{R}$, we can find $B, C \in \mathcal{A}$ such that $\lambda_v(B) = \beta, \lambda_v(C) = \gamma$ with $\beta, \gamma > 1$, such that β, γ generate a dense subgroup in $\mathbb{R}^+$. By replacing B and C with appropriate powers, we may assume that p is a fixed point for the action of both B and C.

Let $r = \rho(B), s = \rho(C)$, and define corresponding $R, S : \mathcal{L} \to \mathcal{L}$ as before, so that $\widetilde{\phi} R = r\widetilde{\phi}$ and $\widetilde{\phi} S = s\widetilde{\phi}$. Then

$$GRG^{-1}(x,t) = (r(x), \widetilde{\beta}_x t), GSG^{-1}(x,t) = (s(x), \widetilde{\gamma}_x t),$$

where $\widetilde{\beta}_x = R'_x(0),\ \widetilde{\gamma}_x = S'_x(0)$. In particular, since $h(p)$ is fixed by r and s,

$$G_{h(p)} \circ R_{h(p)} = \widetilde{\beta} G_{h(p)} \text{ and } G_{h(p)} \circ S_{h(p)} = \widetilde{\gamma} G_{h(p)},$$

with $\widetilde{\beta} = \widetilde{\beta}_{h(p)}, \widetilde{\gamma} = \widetilde{\gamma}_{h(p)}$. Also, since h intertwines ρ and the standard action,

$$R_{h(p)} \circ H_p(t) = H_p(\beta t) \text{ and } S_{h(p)} \circ H_p(t) = H_p(\gamma t).$$

Let

$$\psi = G_{h(p)} \circ H_p|_{\mathbb{R}+} : \mathbb{R}^+ \to \mathbb{R}^+.$$

Then we have shown that for every $t \in \mathbb{R}^+$,

$$\psi(\beta t) = \widetilde{\beta}\psi(t) \text{ and } \psi(\gamma t) = \widetilde{\gamma}\psi(t).$$

By construction, ψ is an orientation-preserving homeomorphism.

Let $c = \psi(1)$. Then $\psi(\beta^k \gamma^l) = c\widetilde{\beta}^k \widetilde{\gamma}^l$ for every $k, l \in \mathbb{Z}$. Hence,

$$\{\beta^k \gamma^l | k, l \in \mathbb{Z}\} \to \{\widetilde{\beta}^k \widetilde{\gamma}^l | k, l \in \mathbb{Z}\}, \beta^k \gamma^l \mapsto \widetilde{\beta}^k \widetilde{\gamma}^l$$

is an order-preserving, continuous map between these two subsets of $\mathbb{R}^+$. If we denote $\widetilde{\beta} = \beta^{\alpha_1}$ and $\widetilde{\gamma} = \gamma^{\alpha_2}$, then it is easy to see that $\alpha_1 = \alpha_2 := \nu$. Hence

$$\psi(t) = ct^\nu \text{ for every } \ t \in \{\beta^k \gamma^l\}.$$

But this set is dense in $\mathbb{R}^+$ and ψ is continuous; hence $\psi(t) = ct^\nu$ for every $\theta \in \mathbb{R}^+$. □

Now for each $x \in N(n,2)$, set $\psi_x = G_{h(x)} \circ H_x|_I : I \to \mathbb{R}^+$, $I = [0,1]$. Since I is compact and $G \circ H|_{N(n,2)\times I}$ is continuous, it follows that $N(n,2) \to C^0(I)$, $x \mapsto \psi_x$ is continuous with respect to the uniform topology on $C^0(I)$. But $\psi_p(t) = c_p t_p^\nu$ for a dense set of $p \in N(n,2)$, and $\nu_p = \frac{\log \beta_p}{\log \widetilde{\beta}_p}$, which is continuous in p on a dense subset, so $p \mapsto c_p$, and $p \mapsto \nu_p$ must extend to continuous functions $N(n,2) \to \mathbb{R}^+$ such that $\psi_x(t) = c_x t^{\nu_x}$ for every $x \in N(n,2)$. An entirely analogous argument works with $-I = [-1,0]$ in place of I and $\mathbb{R}^-$ in place of $\mathbb{R}^+$. Also, we can replace I with any compact interval $[0,T]$.

Thus, we have proved the following:

LEMMA 5.2.7. *There exist continuous functions $c^\pm, \nu^\pm : N(n,2) \to \mathbb{R}^+$ such that for every $x \in N(n,2)$, $G_{h(x)} \circ H_x : \mathbb{R} \to \mathbb{R}$ has the form*

$$G_{h(x)} \circ H_x(t) = \begin{cases} c_x^+ t^{\nu_x^+} & t \geq 0 \\ -c_x^- |t|^{\nu_x^-} & t \leq 0. \end{cases}$$

Now for each $x \in N(n,2)$, $G_{h(x)} \circ H_x$ is smooth away from 0, and $G_{h(x)}$ is a C^∞ diffeomorphism; hence H_x is smooth away from 0. But ϕ maps $N(n,2) \times (\mathbb{R} - \{0\})$ onto $N(n,2)$, so this implies that h is C^∞ along each leaf of $\mathcal{F}$; more precisely, $h|_{\mathcal{F}(x)} : \mathcal{F}(x) \to \widetilde{\mathcal{F}}(h(x))$ is C^∞ for every $x \in N(n,2)$. Thus, $G_{h(x)} \circ H_x$ must be smooth at 0. So $c_x^+ = c_x^-$, and $\nu_x^+ = \nu_x^- = 1$ for every $x \in N(n,2)$. We have shown that $x \to G_{h(x)} \circ H_x$ defines a continuous map $N(n,2) \to C^\infty(\mathbb{R})$. The same is true for $x \to G_{h(x)}$, and each $G_{h(x)}$ is a diffeomorphism. Since the diffeomorphisms of

$\mathbb{R}$ form a topological group with respect to the subspace topology inherited from $C^\infty(\mathbb{R})$, we conclude that $N(n,2) \to C^\infty(\mathbb{R})$, $x \mapsto H_x = G_{h(x)}^{-1} \circ (G_{h(x)} \circ H_x)$ is continuous.

Now we get the result that the conjugacy h is smooth along all the 1-dimensional foliations that are invariant foliations for the action of the group $\mathcal{A}$. Recall that the foliations are strong, stable foliations for some Anosov elements, so they are Hölder foliations (see, for example, [B-P]).

An application of Journé's theorem [J] (see also [K-L2]), which is again an identical argument as in [K-L2], implies that the conjugacy h is smooth.

6. Deformation Rigidity of Group Actions

6.1 Topological and Smooth Deformation Rigidity of Diagonal-block $SL(n,\mathbb{Z})$ Action. We will first mention that $SL(n,\mathbb{Z})$ action on $N(n,k)$ for $n \geq 3$, $k \leq n-1$ is topologically deformation-rigid. This follows from a general result in [Hu3]. This, together with a general philosophy that a topological conjugacy for a large group should be a smooth conjugacy, support the conjecture that $SL(n,\mathbb{Z})$ action on $N(n,k)$ is smoothly deformation-rigid. In this writing, we cannot confirm it for $k \geq 3$. The smooth deformation rigidity for $k=2, n \geq 3, n \neq 4$ is a corollary of Theorem A.

First let us give some definitions that first appeared in [Hu3]. Let Γ be a finitely generated group, X a compact Riemannian manifold without boundary and $\phi : \Gamma \times X \to X$ a smooth action. A *deformation* of action ϕ is a continuous 1-parameter family of smooth actions

$$\{\phi_t : \Gamma \times X \to X | 0 \leq t \leq 1\},$$

so that $\phi_0 = \phi$.

Fix a set of generators $\{\delta_1, \ldots, \delta_d\}$ of Γ. For $\epsilon > 0$, an ϵ-perturbation of ϕ is an action $\phi_1 : \Gamma \times X \to X$ such that for each generator δ_i, the diffeomorphism $\phi_1(\delta_i)$ of X is ϵ-close to $\phi(\delta_i)$. So for a fixed deformation of ϕ and sufficiently small $t > 0$, ϕ_t gives an ϵ-perturbation of ϕ.

DEFINITION 6.1.1. *An action ϕ is topologically deformation-rigid if for any deformation $\{\phi_t\}$ of the action ϕ, there exists an $\epsilon > 0$, and a continuous 1-parameter family of homeomorphisms $H_t : X \to X$, such that*

$$H_t^{-1} \circ \phi_t(\gamma) \circ H_t = \phi(\gamma)$$

$$H_0 = Id_X$$

for any $\gamma \in \Gamma$ and $0 \leq t < \epsilon$. In case that H_t are diffeomorphisms, we say the action ϕ is smoothly deformation-rigid or simply deformation-rigid.

The approach to attack the deformation rigidity utilizes the theory of Stowe [St], where a criterion is given for the persistence of a fixed point of a group action.

Let Γ act on a manifold M (denote the action by α), and $p \in M$ be a fixed point for the action. Let α_0 be the induced linear action on T_pM, the tangent space to M at p. We denote by $H^1(\Gamma, T_pM)$ the ordinary group cohomology with coefficients in this representation.

PROPOSITION 6.1.2. *If $H^1(\Gamma, T_pM) = 0$, then p is stable under perturbation of α; i.e., given any neighborhood U of $p \in M$, there exists a neighborhood V of $\alpha \in R(\Gamma, Diff^1(M))$, such that each $\beta \in V$ has a fixed point in U.*

In light of this result, and also given that our $SL(n, \mathbb{Z})$ action has dense periodic points, if the dense set of periodic points persists, then the topological conjugacy is defined by sending periodic points to the perturbed periodic points. To carry out this idea, we need several definitions [Hu3], and also a cohomology-vanishing theorem of Margulis [M1].

DEFINITION 6.1.3. *Let E be a finite dimensional, real vector space, $\widetilde{\Gamma}$ a finitely generated group.*

A representation $\rho : \widetilde{\Gamma} \to GL(E)$ is infinitesimally rigid if

(1) *the linear action of $\rho(\widetilde{\Gamma})$ on E has 0 as the unique fixed point;*
(2) *the first cohomology group of $\widetilde{\Gamma}$ with coefficients in the $\widetilde{\Gamma}$-module E is trivial; i.e., $H^1(\widetilde{\Gamma}, E_\rho) = 0$.*

A representation $\rho : \widetilde{\Gamma} \to GL(E)$ is strongly infinitesimally rigid if

(1) *the linear action of $\rho(\widetilde{\Gamma})$ on E is hyperbolic for some $\gamma \in \widetilde{\Gamma}$;*

(2) *for all ϵ-perturbation $\widetilde{\rho}$ of action ρ, the first cohomology group of $\widetilde{\Gamma}$ with coefficients in the $\widetilde{\Gamma}$-module E is trivial; i.e., $H^1(\widetilde{\Gamma}, E_{\widetilde{\rho}}) = 0$.*

The strong infinitesimally rigid action is clearly stable under small perturbation and the perturbed action is also infinitesimally rigid.

DEFINITION 6.1.4. *An action ϕ of Γ on X is (strongly) infinitesimally rigid at a periodic point $x \in \Lambda$ if the isotropy representation*

$$\rho_x = D_x\phi : \Gamma_x \to GL(T_xX)$$

is (strongly) infinitesimally rigid.

The following theorem of Hurder gives a criterion for the topological rigidity of a group action.

THEOREM (HURDER [HU3]) 6.1.5. *Let ϕ be an Anosov action such that*

(1) *the periodic points Λ are dense in X;*

(2) *ϕ is strongly infinitesimally rigid at each periodic point $x \in \Lambda$.*

Then ϕ is topologically deformation-rigid.

From this theorem, the proof of the topological rigidity of our $SL(n, \mathbb{Z})$ action reduces to the proof of the density of periodic points and the strongly infinitesimally rigid at each periodic point $x \in \Lambda$. The first is the consequence that all rational points are periodic points (see Lemma 2.2.2), and the second is the consequence of a powerful theorem of Margulis [M1], see also [Hu3].

THEOREM (MARGULIS) 6.1.6. *Let $\Gamma \subset G$ be an irreducible lattice in a connected, semisimple algebraic $\mathbb{R}$-group of higher rank, G, and assume that $G^0_{\mathbb{R}}$ has no compact factors. Then $H^1(\Gamma, \mathbb{R}^N_\rho) = 0$ for every representation $\rho : \Gamma \to GL(N, \mathbb{R}), N > 1$.*

Combining the results above and those in Section 5, we obtain

THEOREM 6.1.7. *Let $\Gamma = SL(n, \mathbb{Z})$ or any subgroup of finite index. Then the action of Γ on $N(n, k)$ is topologically deformation-rigid for $k \leq n-1, n \geq 3$. If $k = 2$, the action is smoothly deformation-rigid for $n = 3$,and $\neq 4$.*

6.2 Deformation Rigidity of $Aut(n, 2)$ Action. We now consider the whole automorphism group action on $N(n, 2)$. Recall that a criterion of Hurder (Theorem 6.1.5) reduces the question to the calculation of the first group cohomology. We therefore introduce some easy facts about the first group cohomology.

LEMMA 6.2.1. *Let $\rho : \Gamma \to GL(N, \mathbb{R})$ be a homomorphism, Γ_1 be a finitely-indexed subgroup of Γ, and $H^1(\Gamma_1, \rho|_{\Gamma_1}) = 0$. Let $\rho(\Gamma)$ be generated by Anosov automorphisms $\rho(\gamma_1), \ldots, \rho(\gamma_k)$. Then $H^1(\Gamma, \rho) = 0$.*

PROOF. Let $\phi : \Gamma \to \mathbb{R}^n$ be a cocycle; then $f|_{\Gamma_1}$ is a coboundary by the assumption; i.e., there exists an element $v \in \mathbb{R}^N$, such that for all $\gamma \in \Gamma_1$,

$$f(\gamma) = v - \rho(\gamma)v.$$

Let $\gamma_i^p \in \Gamma_1$. Since f is a cocycle, we have

$$\begin{aligned} f(\gamma_i^p) &= f(\gamma_i) + \rho(\gamma_i) f(\gamma_i^{p-1}) \\ &= \ldots \\ &= (I + \rho(\gamma_i) + \cdots + \rho(\gamma_i^{p-1})) f(\gamma_i). \end{aligned}$$

Because $\gamma_i^p \in \Gamma_1$, we have

$$\begin{aligned} f(\gamma_i^p) &= v - \rho(\gamma_i^p) v \\ &= (I - \rho(\gamma_i))(I + \rho(\gamma_i) + \cdots + \rho(\gamma_i^{p-1}))v, \end{aligned}$$

and also since

$$\det(I - \rho(\gamma_i^p)) \neq 0,$$

we obtain

$$\det(I + \rho(\gamma_i) + \cdots + \rho(\gamma_i^{p-1})) \neq 0.$$

Therefore,

$$f(\gamma_i) = v - \rho(\gamma_i)v.$$

Now for any $\gamma \in \Gamma$, γ can be expressed as

$$\gamma = \gamma_{i_1} \cdots \gamma_{i_q}.$$

Since f is a cocycle, we obtain that

$$f(\gamma) = v - \rho(\gamma)v.$$

□

It is clear that this Lemma can be strengthened; e.g., we may assume that $\rho(\Gamma)$ is generated by Γ_1 together with some matrices that do not have eigenvalues of roots of unity. But we are satisfied with the weaker form of Lemma 6.2.1. We define $\Gamma_m = \{\gamma \in \Gamma : \gamma = I (\text{mod } m)\}$ for any group $\Gamma \subset GL(N, \mathbb{R})$. We have the following result.

LEMMA 6.2.2. *Let $\Gamma = Aut(n,2)$, $BSL(n,\mathbb{Z})$ the diagonal copy of $SL(n,\mathbb{Z})$. Then $\Gamma_m = N_m BSL(n,\mathbb{Z})_m$, where $N = \left\{ \begin{pmatrix} I & 0 \\ M & I \end{pmatrix} : M \text{ is the integer matrix} \right\}$.*

PROOF. It is a straightforward calculation. □

COROLLARY 6.2.3. *$\Gamma_m = Aut(n,2)_m$ is generated by $\begin{pmatrix} I & 0 \\ mE_{ij} & I \end{pmatrix}$ and $BSL(n,\mathbb{Z})_m$.*

PROOF. It is obvious. □

LEMMA 6.2.4. *Let $\Gamma = Aut(n,2)$. Then Γ_m is generated by Anosov elements.*

PROOF. We first prove that $BSL(n,\mathbb{Z})_m$ is generated by Anosov elements by an argument similar to Theorem 3.4; then our result follows easily. □

In the next several Lemmas, we will prove that the first cohomology of $Aut(n,2)_m$ is trivial for each m. The approach is simple. We first show that the whole group $Aut(n,2)_m$ is generated (up to the finite index) by several subgroups with vanishing cohomologies; then we use these data to show that the first cohomology of the group $Aut(n,2)_m$ itself vanishes.

For the sake of transparency, we first prove the statement for $n=3$. We will assume that $\Gamma = Aut(3,2)$ for Lemma 6.2.5–Lemma 6.2.9.

LEMMA 6.2.5. $\Gamma_{m^2} \subset \Gamma_{(m,m^2)} := < \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix}^{-1} SL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix},$

$$\begin{pmatrix} I & 0 \\ mE_{12} & I \end{pmatrix}^{-1} SL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{12} & I \end{pmatrix},$$
$$\begin{pmatrix} I & 0 \\ mE_{21} & I \end{pmatrix}^{-1} SL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{21} & I \end{pmatrix},$$
$$\begin{pmatrix} I & 0 \\ mE_{23} & I \end{pmatrix}^{-1} SL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{23} & I \end{pmatrix},$$
$$\begin{pmatrix} I & 0 \\ mE_{32} & I \end{pmatrix}^{-1} SL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{32} & I \end{pmatrix},$$
$$\begin{pmatrix} I & 0 \\ mE_{33} & I \end{pmatrix}^{-1} SL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{33} & I \end{pmatrix} > \subset \Gamma_m.$$

PROOF. Take $A = \begin{pmatrix} 1 & 0 & 0 \\ m & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$; then the induced automorphism A' on $[V,V]$ will be $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m & 1 \end{pmatrix}$. A straightforward calculation gives

$$\begin{pmatrix} I & 0 \\ mE_{12} & I \end{pmatrix}^{-1} \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \begin{pmatrix} I & 0 \\ mE_{12} & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ -m^2E_{11} & I \end{pmatrix}.$$

Similarly, we may get $\begin{pmatrix} I & 0 \\ -m^2E_{ij} & I \end{pmatrix}$ in this way by changing A, and using suitable $E = \begin{pmatrix} I & 0 \\ mE_{st} & I \end{pmatrix}$ that appeared in the formula:

To get $\begin{pmatrix} I & 0 \\ -m^2E_{12} & I \end{pmatrix}$, we use $A = \begin{pmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix}$;

To get $\begin{pmatrix} I & 0 \\ -m^2E_{13} & I \end{pmatrix}$, we use $A = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} I & 0 \\ mE_{33} & I \end{pmatrix}$;

To get $\begin{pmatrix} I & 0 \\ m^2E_{21} & I \end{pmatrix}$, we use $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} I & 0 \\ mE_{23} & I \end{pmatrix}$;

To get $\begin{pmatrix} I & 0 \\ m^2E_{22} & I \end{pmatrix}$, we use $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m & 1 \end{pmatrix}$, $E = \begin{pmatrix} I & 0 \\ mE_{23} & I \end{pmatrix}$;

To get $\begin{pmatrix} I & 0 \\ -m^2E_{23} & I \end{pmatrix}$, we use $A = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} I & 0 \\ mE_{21} & I \end{pmatrix}$;

To get $\begin{pmatrix} I & 0 \\ -m^2E_{31} & I \end{pmatrix}$, we use $A = \begin{pmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} I & 0 \\ mE_{32} & I \end{pmatrix}$;

To get $\begin{pmatrix} I & 0 \\ -m^2E_{32} & I \end{pmatrix}$, we use $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m & 1 \end{pmatrix}$, $E = \begin{pmatrix} I & 0 \\ mE_{23} & I \end{pmatrix}$;

To get $\begin{pmatrix} I & 0 \\ -m^2E_{33} & I \end{pmatrix}$, we use $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & m \\ 0 & m & 1 \end{pmatrix}$, $E = \begin{pmatrix} I & 0 \\ mE_{32} & I \end{pmatrix}$. □

COROLLARY 6.2.6. *If* $H^1(\Gamma_{(m,m^2)}, \mathbb{R}^6) = 0$, *then* $H^1(\Gamma_m, \mathbb{R}^6) = 0$.

PROOF. Since $\Gamma_{m^2} \subset \Gamma_{(m,m^2)} \subset \Gamma_m$, it is clear that $\Gamma_{(m,m^2)}$ is a finitely indexed subgroup of Γ_m.

Recall that Γ_m is generated by Anosov elements; our result then follows. □

Next we want to prove that $H^1(\Gamma_{(m,m^2)}, \mathbb{R}^6) = 0$. We need some facts about the intersections of those copies of $BSL(3,2)_m$ with $BSL(3,2)_m$ itself.

LEMMA 6.2.7. *If* $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, *then the induced action on* $[\mathbb{R}^3, \mathbb{R}^3]$ *is*

$$A' = \begin{pmatrix} ae - bd & af - de & bf - ce \\ ah - bg & ai - cg & bi - ch \\ dh - eg & di - fg & ei - fh \end{pmatrix}.$$

PROOF. It is a straightforward calculation. □

LEMMA 6.2.8. *For all integers* b, c, d, f, g, h,

$$\begin{pmatrix} 1 & 0 & 0 \\ md & 1 & mf \\ 0 & 0 & 1 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix} \cap BSL(3,\mathbb{Z})_m;$$

$$\begin{pmatrix} 1 & mb & mc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ mE_{12} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{12} & I \end{pmatrix} \cap BSL(3,\mathbb{Z})_m;$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ mg & mh & 1 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ mE_{21} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{21} & I \end{pmatrix} \cap BSL(3,\mathbb{Z})_m;$$

$$\begin{pmatrix} 1 & mb & mc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ mE_{23} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{23} & I \end{pmatrix} \cap BSL(3,\mathbb{Z})_m;$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ mg & mh & 1 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ mE_{32} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{32} & I \end{pmatrix} \cap BSL(3,\mathbb{Z})_m;$$

$$\begin{pmatrix} 1 & 0 & 0 \\ md & 1 & mf \\ 0 & 0 & 1 \end{pmatrix} \in \begin{pmatrix} I & 0 \\ mE_{33} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{33} & I \end{pmatrix} \cap BSL(3,\mathbb{Z})_m.$$

PROOF. Let us prove the first formula.

Let $A \in SL(3,\mathbb{Z})$; then $\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \in BSL(3,\mathbb{Z})$. So

$$\begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix}^{-1} \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ -mE_{11}A + mA'E_{11} & A' \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix}^{-1} \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix} \in BSL(3,\mathbb{Z})$$

if and only if

$$-mE_{11}A + mA'E_{11} = 0.$$

This condition forces

$$\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} ae-bd & 0 & 0 \\ ah-bg & 0 & 0 \\ dh-eg & 0 & 0 \end{pmatrix}.$$

Then what is left is a straightforward calculation. $\square$

Having had these Lemmas in hand, we are in a position to prove the vanishing of the first cohomology of $H^1(\Gamma_{(m,m^2)}, \mathbb{R}^6)$.

PROPOSITION 6.2.9. $H^1(\Gamma_{(m,m^2)}, \mathbb{R}^6) = 0$.

PROOF. Let $f : \Gamma_{(m,m^2)} \to \mathbb{R}^6$ be a cocycle. By a theorem of Margulis, $f|BSL(3,\mathbb{Z})$ as well as $f|\begin{pmatrix} I & 0 \\ mE_{ij} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z}) \begin{pmatrix} I & 0 \\ mE_{ij} & I \end{pmatrix}$ are coboundaries.

Let

$$f|BSL(3,\mathbb{Z})(\gamma) = v - \rho(\gamma)v,$$
$$f|\begin{pmatrix} I & 0 \\ mE_{ij} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z}) \begin{pmatrix} I & 0 \\ mE_{ij} & I \end{pmatrix}(\gamma) = v_{ij} - \rho(\gamma)v_{ij}.$$

We want to prove that

$$v_{ij} = v$$

for all i, j.

For example, let us show that $v_{11} = v$.

In fact,

$$\gamma_{d,f} = \begin{pmatrix} 1 & 0 & 0 \\ md & 1 & mf \\ 0 & 0 & 1 \end{pmatrix}$$
$$\in \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix} \cap BSL(3,\mathbb{Z})_m;$$

hence, $v - \rho(\gamma_{d,f})v = v_{11} - \rho(\gamma_{d,f})v_{11}$, or

$$v - v_{11} = \rho(\gamma_{d,f})(v - v_{11}).$$

This is to say that $(v - v_{11})$ is a common eigenvalue for all $\rho(\gamma_{d,f})$ with $d, f \in \mathbb{Z}$ corresponding to eigenvalue $+1$.

Let

$$v - v_{11} = (w_1^{(11)}, w_2^{(11)}, w_3^{(11)}, w_4^{(11)}, w_5^{(11)}, w_6^{(11)})^t;$$

notice that

$$\rho(\gamma_{d,f})(v - v_{11}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ md & 1 & mf & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & mf & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & md & 1 \end{pmatrix} \begin{pmatrix} w_1^{(11)} \\ w_2^{(11)} \\ w_3^{(11)} \\ w_4^{(11)} \\ w_5^{(11)} \\ w_6^{(11)} \end{pmatrix} = \begin{pmatrix} w_1^{(11)} \\ w_2^{(11)} \\ w_3^{(11)} \\ w_4^{(11)} \\ w_5^{(11)} \\ w_6^{(11)} \end{pmatrix};$$

we have that

$$\begin{pmatrix} w_1^{(11)} \\ mdw_1^{(11)} + w_2^{(11)} + mfw_3^{(11)} \\ w_3^{(11)} \\ w_4^{(11)} + mfw_5^{(11)} \\ w_5^{(11)} \\ mdw_5^{(11)} + w_6^{(11)} \end{pmatrix} = \begin{pmatrix} w_1^{(11)} \\ w_2^{(11)} \\ w_3^{(11)} \\ w_4^{(11)} \\ w_5^{(11)} \\ w_6^{(11)} \end{pmatrix}$$

for all $f, d \in \mathbb{Z}$; it forces $w_1^{(11)} = w_3^{(11)} = w_5^{(11)} = 0$.

Therefore,

$$v - v_{11} = (0, w_2^{(11)}, 0, w_4^{(11)}, 0, w_6^{(11)})^t.$$

Similar consideration yields

$$v - v_{12} = (w_1^{12}, 0, 0, w_4^{(12)}, w_5^{(12)}, 0)^t$$

$$v - v_{21} = (0, 0, w_3^{(21)}, 0, w_5^{(21)}, w_6^{(21)})^t$$

$$v - v_{23} = (w_1^{(23)}, 0, 0, w_4^{(23)}, w_5^{(23)})^t$$

$$v - v_{32} = (0, 0, w_3^{(32)}, w_4^{(32)}, w_5^{(32)}, 0)^t$$

$$v - v_{33} = (0, w_2^{(33)}, 0, w_4^{(33)}, 0, w_6^{(33)})^t.$$

Let us then show that $w_k^{ij} = 0$ for all $i, j, k = 1, 2, 3$. For example, $w_k^{11} = 0$.

Consider

$$G_1 := \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix}$$
$$\cap \begin{pmatrix} I & 0 \\ mE_{12} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{12} & I \end{pmatrix};$$

a straightforward calculation shows that

$$\begin{pmatrix} B & 0 \\ -mE_{12}B + mB'E_{12} & B' \end{pmatrix} \in G_1$$

with

$$B = \begin{pmatrix} 1+ma & mb & mc \\ md & 1+me & mf \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$B' = \begin{pmatrix} 1 & (1+ma)(mf)-(md)(mc) & (mb)(mf) \\ 0 & 1+ma & mb \\ 0 & md & 1+me \end{pmatrix},$$

such that the following conditions are satisfied:

$$(1+ma)(1+me)-(mb)(md) = 1$$
$$1+ma-md = 1$$
$$1+me-mc = 1,$$
$$f = c.$$

(For example, $B = \begin{pmatrix} 1+mk & -mk & ml \\ mk & 1-mk & ml \\ 0 & 0 & 1 \end{pmatrix}$). Given this fact, together with a similar argument for the assertion of the possible forms of $v - v_{ij}$, we conclude that

$$v_{11} - v_{12} = (0, 0, 0, w_4^{11} - w_4^{12}, 0, 0)^t,$$

and

$$w_2^{11} = w_6^{11} = w_1^{12} = w_5^{12} = 0.$$

To conclude that $w_4^{11} = 0$, we may consider

$$G_2 := \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{11} & I \end{pmatrix}$$
$$\cap \begin{pmatrix} I & 0 \\ mE_{21} & I \end{pmatrix}^{-1} BSL(3,\mathbb{Z})_m \begin{pmatrix} I & 0 \\ mE_{21} & I \end{pmatrix}$$

and do the same analysis.

A similar argument can be used to prove that $w_k^{ij} = 0$ for other $i, j, k \in \mathbb{Z}$.

This proves that when f is restricted to the specified copies of $BSL(3,2)$, we get $f(\gamma) = v - \rho(\gamma)v$. But our original group is generated by these copies; this proves that f is a cocycle. $\square$

To prove the vanishing for general $n \geq 3$ with the coefficient in the standard representation ρ_0, the argument we had goes through. So we have

PROPOSITION 6.2.10. *$H^1(\Gamma_m, \mathbb{R}^N) = 0$ for $\Gamma = Aut(N(2,n))$, $dim(N(2,n)) = N$.*

To prove that the action ρ_0 is strongly infinitesimally rigid, we need to prove that the first cohomologies are also vanishing for the coefficient in the perturbed representation.

PROPOSITION 6.2.11. *$H^1(\Gamma_{(m,m^2)}, \rho_t) = 0$ hence $H^1(\Gamma_m, \rho_t) = 0$ for sufficiently small t and $n \geq 3, n \neq 4$.*

PROOF. Recall that the diagonal-block action is locally rigid. Hence for sufficiently small t, $\rho_t|_{BSL(n,\mathbb{Z})} = h_t^{-1}\rho_0 h_t$ with h_t smoothly close to identity. It is clear that other copies of $SL(n,\mathbb{Z})$ also act on the manifold in a locally rigid way, so $\rho_t|_{\begin{pmatrix} I & 0 \\ mE_{ij} & I \end{pmatrix}^{-1} BSL(n,\mathbb{Z}) \begin{pmatrix} I & 0 \\ mE_{ij} & I \end{pmatrix}} = {h_{ij}}_t^{-1}\rho_0 {h_{ij}}_t$ with ${h_{ij}}_t$ smoothly close to identity.

We may go over the proof of Proposition 6.2.9 again with necessary modification, which gives the proof of this proposition. $\square$

Now we are ready to prove the deformation rigidity for the $Aut(n,2)$ action on the nilmanifold.

THEOREM 6.2.12. *$Aut(n,2)$ actions on nilmanifold $N(n,2)$ are smoothly deformation-rigid if* $n = 3,\ n = 5, 6, \ldots$.

PROOF. The proof of this theorem uses a criterion of [Hu3]. See Theorem 6.1.5. The only thing needing proof is that $Aut(n,2)_x$ (which is the subgroup of the $Aut(n,2)$ fixing point $x \in N(n,2)$) contains $Aut(n,2)_m$ for x, a rational point, and for some integer m. But it is clear. $\square$

Remark 6.2.13. It is easy to show that for any finitely indexed subgroup $\Gamma \subset Aut(n,2)$, the action of Γ on $N(n,2)$ (with $n \geq 3, n \neq 4$) is also smoothly deformation-rigid. We will sketch the argument below.

(1) $\Gamma \cap BSL(n,\mathbb{Z})$ is a finitely indexed subgroup of $BSL(n,\mathbb{Z})$. Indeed, for all $A \in BSL(n,\mathbb{Z})$, $A^d \in \Gamma$ for $d =$the index.

(2) The same reason implies that $\Gamma \cap UN(UN := \left\{\begin{pmatrix} I & 0 \\ M & I \end{pmatrix}\right\})$ is a finitely indexed subgroup of UN.

So we will have the following.

(3) There exists an m such that

$$BSL(n,\mathbb{Z})_m \subset \Gamma.$$

(4) There exists an m' such that

$$UN_{m'} \subset \Gamma.$$

Therefore, we have $Aut(n,2)_{mm'} \subset \Gamma$. The computation of cohomologies goes through. So we will have smooth deformation rigidity. $\square$

7. New Examples of Cartan Actions and Other Anosov Actions on Tori and Limitation of Our Method

In this section, we will give a complete list of linear $SL(n,\mathbb{Z})$ actions on tori for fixed n. Among them, Cartan actions and Anosov actions are specified; also, the actions with "complete 1-dimensional splitting foliations" are singled out. Using the local rigidity theorem for Cartan actions in [H-K-L-Z], we then have a number of locally rigid $SL(n,\mathbb{Z})$ actions on tori. These examples are new; the construction is natural and simple. V. Nitica informed me of the classical results about Young tableaux and a recent survey. I thank him for his generous help.

We remark that the examples of higher-rank actions on tori appearing in the literature are $SL(n,\mathbb{Z})$ actions on $\mathbb{T}^n$ (locally rigid if $n \geq 3$), $Sp(n,\mathbb{Z})$ actions on $\mathbb{T}^{2n}$ (locally rigid if $n \geq 2$), and also Hurder's examples using the trick of A. Weil, i.e.,"restriction of scales." The latter examples can be described as follows. Take an algebraic number field $\mathbf{k}$ of degree d over $\mathbb{Q}$; let $\mathcal{O}(\mathbf{k})$ be the ring of integers for the field and let $SL(n,\mathcal{O}(\mathbf{k}))$ be the subgroup of $SL(n,\mathbf{k})$ with the entries from $\mathcal{O}(\mathbf{k})$. Then take any $\Gamma \subset SL(n,\mathcal{O}(\mathbf{k}))$; there exists an analytic "standard " action of Γ on $\mathbb{T}^{dn}$.

In our list, the Cartan actions are those obtained from exterior tensor product representations; the Anosov actions are those obtained from decomposing the k-fold tensor product representations into irreducible ones with $k \neq 0(mod n)$; the actions with "complete 1-dimensional splitting foliations" are those obtained from symmetric tensor representations.

Let us first look at some examples. Instead of considering the automorphisms on free k-step nilpotent algebra (we remark that the construction of automorphisms of the free, k-step, nilpotent Lie algebra also gives the automorphisms of vector-space automorphisms and hence may factor through to toral automorphisms) we consider the automorphisms on the exterior algebra $E(V)$ of a real vector space V. We first quote some basic facts from the theory of exterior algebra. (See D. G. Northcott: Multilinear Algebra [N] for a detailed treatment. For our purpose we will assume all modules in [N] to be real vector spaces).

Let $f : M \to N$ be a linear map between vector spaces M, N, let $E(M), E(N)$ be the exterior algebras. Then f has a unique extension to a homomorphism $E(f) = \oplus E_p(f)$ of graded algebra $E(M) = \oplus E_p(M)$ to $E(N) = \oplus E_p(N)$. Moreover, if $m_1, \ldots, m_p \in M$, then

$$E(f)(m_1 \wedge \cdots \wedge m_p) = f(m_1) \wedge \cdots \wedge f(m_p).$$

If $g : K \to M$ is another linear map between real vector spaces K, M, then

$$E(f) \circ E(g) = E(f \circ g)$$

$$E_p(f) \circ E_p(g) = E_p(f \circ g).$$

It is clear then that if $M = N = K := V$ and $f : V \to V$ is an isomorphism, then $E(f)$ as well as $E_p(f)$ is an isomorphisms.

In other words, any matrix subgroup of $GL(n, \mathbb{R})$ acts on $E(V)$ as well as $E_p(V)$.

Let V be an n-dimensional vector space; then we may consider $E(V)$ $(E_p(V))$ as a real vector space. Fix a basis $\{e_1, \ldots, e_n\}$ of V; then $\{e_{i_1}\ (1 \leq i_1 \leq n); e_{i_1} \wedge e_{i_2}\ (1 \leq i_1 < i_2 \leq n); \ldots; e_{i_1} \wedge \cdots \wedge e_{i_n}\ (1 \leq i_1 < \cdots < i_n \leq n)\}$ is a basis for $E(V)$, and $\{e_{i_1} \wedge \cdots \wedge e_{i_p}\ (1 \leq i_1 < \ldots \cdots < i_p \leq n)\}$ is a basis of $E_p(V)$.

Let us take a lattice generated by the integer points with respect to the chosen basis. Then any subgroup Γ of $SL(n, \mathbb{Z})$ induces an action preserving the lattice. So it factors through a torus action.

Example 7.1. Let $\Gamma \subset SL(n, \mathbb{Z})$ be a subgroup of finite index with $n \geq 3$; then it acts on $\mathbb{T}^{\binom{n}{p}}, 1 \leq p \leq n-1$ through the action of Γ on $E_p(V)$. The action is a Cartan action and is locally rigid. $\square$

It is worth mentioning that the actions constructed above and also examples 7.5 are actually irreducible by an easy argument, using the well-known Schur's Lemma. On the other hand, we cannot say anything about the irreducibility of the action constructed in the same manner, using automorphisms of free, k-step, nipotent Lie algebra.

Example 7.2. We may group certain actions above together; in other words, we may consider the actions Γ on the spaces of the form $\oplus_{i \in S} E_i(V)$ factors a lattice, where $S \subset I = \{1, 2, \ldots, n-1\}$. The actions then are Cartan actions. The actions are also rigid if $n \geq 3$. $\square$

It looks as if the actions here are product actions, but they are not. They are "diagonal actions" of some product actions.

For Examples 7.1 and 7.2, we have enough 1-dimensional foliations for an Abelian subgroup to trellis the tori, and each of the foliations is the intersection of the stable foliations of several Anosov elements, and these foliations are also the strongest foliations for some Anosov elements in that Abelian subgroup. We therefore can use the arguments in [H-K-L-Z] and prove that the actions are locally rigid.

If we consider other tensor algebras, for example, tensor algebra $T(V), T_p(V)$ or symmetric tensor algebra $S(V), S_p(V)$, the latter having been already introduced in Section 5, we may construct other examples of actions on tori.

Example 7.3. Let $\Gamma \subset SL(n, \mathbb{Z})$ be a subgroup of finite index with $n \geq 3$; then it acts on $\mathbb{T}^{n^p}, p \neq 0(mod n)$, through the action of Γ on $T_p(V)$. The actions are Anosov actions and do not have enough 1-dimensional foliations for a suitable Abelian subgroup in Γ. $\square$

Example 7.4. We may group certain actions above together; in other words, we may consider the actions Γ on the spaces of the form $\oplus_{i \in S} T_i(V)$ factors a lattice, where $S \subset I = \{k \in \mathbb{Z}^+; k \neq 0(mod n)\}$. The actions then are Anosov actions. $\square$

Example 7.5. Let $\Gamma \subset SL(n, \mathbb{Z})$ be a subgroup of finite index with $n \geq 3$; then it acts on $\mathbb{T}^{d(p)}, p \neq 0(mod n)$, $d(p) = \binom{n+p-1}{p}$ through the action of Γ on $S_p(V)$. The actions are Anosov actions and do have enough 1-dimensional foliations for a suitable Abelian subgroup in Γ to trellis the tori, but some of the foliations can never be realized as strong stable foliations for some Anosov elements in Γ. Nevertheless, the foliations are intersections of stable foliations for some Anosov elements and therefore persist under small perturbation. $\square$

Example 7.6. We may group certain actions above together; in other words, we may consider the actions Γ on the spaces of the form $\oplus_{i \in S} S_i(V)$ factors a lattice, where $S \subset I = \{k \in \mathbb{Z}; k \neq 0(mod n)\}$, $k_1 \neq k_2 \pmod{n}$ for all $k_1, k_2 \in S$, and S is finite. The actions then are Anosov actions if $n \geq 3$. The actions have enough 1-dimensional foliations for a suitable Abelian subgroup in Γ to trellis the tori, and the foliations are intersections of stable foliations for some Anosov elements, and therefore persist under small perturbation. $\square$

We cannot use the Cartan action theory to attack the rigidity of the actions in example 7.3-7.6. But it is clear that the possessing and persistence of enough 1-dimensional foliations in example 7.5, 7.6 make it possible to generalize the Cartan action theory to get the rigidity for these cases.

It is a classic result that all irreducible, rational representations of $SL(n, \mathbb{Z})$ on finite-dimensional vector spaces are polynomial representations, and they are obtained by decomposing the representation in Example 7.3 into irreducible representations. In fact, we have the following parameterization for polynomial representations by Young tableaux [Sun]. We will omit the definitions of Young tableaux and their properties.

We denote by λ a Young tableau; we let ρ_λ be the irreducible polynomial representation corresponding to it. We say that the length of λ is p if ρ_λ is contained in $SL(n,\mathbb{Z}) \times T_p(V) \to T_p(V)$.

PROPOSITION 7.7. *Fix a Young tableau λ; we have the following:*

(1) *ρ_λ preserves integer points with respect to a "standard basis," and hence factors through a toral automorphism also denoted by ρ_λ;*

(2) *the toral action ρ_λ is Cartan iff ρ_λ is one of the actions listed in Example 7.1;*

(3) *the toral action ρ_λ is Anosov iff the length of λ is not equal to $0(mod n)$;*

(4) *the toral action ρ_λ has 1-dimensional splitting foliations iff ρ_λ is one of the actions listed in Example 7.1 and in Example 7.3.*

The proof of this proposition is easy. We omit the proof. Next we will give a description of all the linear toral $SL(n,\mathbb{Z})$ actions.

PROPOSITION 7.8. *Any linear $SL(n,\mathbb{Z})$ toral action on $\mathbb{T}^N$ induces a homomorphism between $SL(n,\mathbb{Z})$ and $SL(N,\mathbb{Z})$. For $n \geq 3$, the homomorphism is a polynomial homomorphism restricted to a subgroup of finite index.*

This is proved in [Ste]. The two propositions above describe all the linear $SL(n,\mathbb{Z})$ toral actions in terms of Cartan action, Anosov action and action with "complete 1-dimensional splitting foliations."

Although we may have a number of other actions on tori, for example, we consider V with dimension $2n$; we consider $Sp(n,\mathbb{Z})$ on the factored tori; we cannot get the rigidity results using the Cartan action theory. We hope we may prove the rigidity result for this class of actions using another approach that is well-developed in [K-L]. But we also have to exclude those actions with length $0(mod n)$ of the Young tableau.

We would like to ask the following question. Fix an integer $n \geq 3$. What is the torus of the biggest dimension on which $SL(n, \mathbb{Z})$ could act locally rigidly? If, by any chance, that action is algebraic, we have one of the above examples as an answer. So it is clear that either the torus is obtained as in Example 7.2 or such a torus does not exist.

It is worth mentioning that our method of using the Cartan action theory to prove rigidity is impossible to be carried out in the case of the nilmanifolds $N(n, k)$ of higher steps, i.e., $k \geq 3$. The reason is simple: We do not have an Anosov element at all for $k \geq n$, and also we do not have enough 1-dimensional foliations for a suitable Abelian subgroup to trellis the space. This is the limitation of the method we use. Some progress has been made by A. Katok that for certain cases the density of the orbit of the abelian group in the higher-dimensional foliation is enough to prove the rigidity. But this is not the case for our nilmanifolds $N(n, k)$ with $k \geq 3$, because the orbit is never dense.

We want to state a conjecture to finish the first part of our paper.

Conjecture. (1) The $SL(n, \mathbb{Z})$ action on $N(n, k)$ is locally rigid if $n \geq 3,\ k \leq n-1$.

(2) The group $Aut(n, 2)$ of automorphisms of $N(n, 2)$ is locally rigid.

We remark that our results in the paper support this conjecture.

PART II: Anosov $\mathbb{R}^n$ Actions

1. Introduction

Hyperbolic behavior in dynamical systems has been investigated extensively, especially for Anosov diffeomorphisms and Anosov flows. See, for example, [A] [Pl] and references there.

Anosov diffeomorphism on a compact manifold possesses nice structures and properties. It foliates the manifold by stable foliation and unstable foliation. It is structurally stable in the sense that any perturbation of the Anosov diffeomorphism is still an Anosov diffeomorphism, and they are topologically conjugate. It is ergodic provided that it preserves a volume. And it has two naturally associated measures, the SBR measure and the Margulis measure. For these two measures, the diffeomorphism is just a Bounoulli shift and hence ergodic, weakly mixing and strongly mixing.

For the Anosov flow on a compact manifold, the development is parallel to the Anosov diffeomorphism case. We have the same properties for the Anosov flows as those for Anosov diffeomorphisms with minor modifications.

Two generalizations of the Anosov diffeomorphisms and flows have been made; one is called normally hyperbolic dynamical systems; the other is called Anosov group actions. We call a group action Anosov if the group contains at least one Anosov element [P-S1]. In our work, we are interested in a special kind of Anosov group actions, namely, the Anosov $\mathbb{R}^n$ actions.

We now give the definition and some examples of Anosov $\mathbb{R}^n$ actions. Let M be a compact manifold without boundary. Let $\Phi : \mathbb{R}^n \times M \to M$ be a smooth map such that $\Phi_t : M \to M,\ x \mapsto \Phi(t, x)$ is a diffeomorphism. We call Φ an $\mathbb{R}^n$ *action* if $\pi : \mathbb{R}^n \to \mathrm{Diff}(M),\ t \mapsto \Phi_t$ is a group homomorphism.

In this work, we will assume that π is locally free; i.e., for any $x \in M$, there exists a neighborhood U of origin in $\mathbb{R}^n$, such that $\pi(u)x \neq x$ for all $u \in U - \{0\}$. It is clear that the orbits of the locally free $\mathbb{R}^n$ action define a smooth foliation $\mathcal{R}^n$.

DEFINITION 1.1. The locally free $\mathbb{R}^n$ action Φ is called an Anosov $\mathbb{R}^n$ action if there exists an Anosov element; i.e., there exists an element $r \in \mathbb{R}^n$ such that $f = \Phi_r : M \to M$ is hyperbolic at $\mathcal{R}^n$, or in other words, $Tf : M \to M$ leaves invariant a splitting

$$E^u \oplus T\mathcal{R}^n \oplus E^s = TM,$$

contracting E^s more sharply than $T\mathcal{R}^n$, expanding E^u more sharply than $T\mathcal{R}^n$.

EXAMPLES 1.2. The following are a list of currently known examples of Anosov $\mathbb{R}^n$ actions.

(1) The standard actions of $\mathbb{R}^n$ on $\mathbb{T}^n$;

(2) Anosov flows;

(3) Suspensions of $\mathbb{Z}^k$ Anosov actions ($\mathbb{Z}^k$ Anosov actions are actions generated by Anosov diffeomorphisms); i.e., let the action of $\mathbb{Z}^k$ on N be anosov; let $\mathbb{Z}^k$ be embedded in $\mathbb{R}^k$ as a lattice. Consider the action of $\mathbb{Z}^k$ on $\mathbb{R}^k \times N$ by $z(x,n) = (x - z, zn)$ and form the quotient

$$M = (\mathbb{R}^k \times N)/\mathbb{Z}^k.$$

Note that the action of $\mathbb{R}^k$ on $\mathbb{R}^k \times N$ by $r(x,n) = (r + x, n)$ commutes with the $\mathbb{Z}^k$-action and therefore descends to an $\mathbb{R}^k$-action on M.

(4) Weyl chamber flows; i.e., let G be a real connected semisimple Lie group of the non-compact type, and Γ be a cocompact lattice, A being a splitting Cartan subgroup. Recall that the centralizer of A splits as a product MA, where M is

a compact group. It is clear then that A acts on $M\backslash G/\Gamma$ from the left; it is an Anosov action [I];

(5) Assume the notations of the last example. Let $\rho : \Gamma \to SL(n,\mathbb{Z})$ be a representation of Γ. Suppose that Γ acts on a compact manifold N via ρ such that A action on N via ρ is Anosov (e.g., see § 7 of the first part of my thesis). Then Γ acts on $M\backslash G \times N$ via

$$\gamma(x,t) = (x\gamma^{-1}, \gamma t).$$

Let $V := M\backslash G \times_\Gamma N := (M\backslash G \times N)/\Gamma$ be the quotient of this action. As the action of A on $M\backslash G \times N$ given by $a(x,t) = (ax,t)$ commutes with the Γ-action, it induces an Anosov action of A on V.

(6) Suspension of the above $\mathbb{R}^k$ actions; i.e., let $\Phi : \mathbb{R}^k \times M \to M$ be an Anosov $\mathbb{R}^k$ action on M, let $\mathbb{Z} =< g_1, \ldots, g_n >$ act on M, such that g_i commutes with the $\mathbb{R}^k$ action (in other words, $g_i\Phi_t = \Phi_t g_i$ for all $t \in \mathbb{R}^n$). Let $\mathbb{Z}^n$ be embedded in $\mathbb{R}^n$. Let $\mathbb{Z}$ act on $M \times \mathbb{R}^n$ via

$$z(m,r) = (zm, z - r).$$

Let $V = (M \times \mathbb{R}^n)/\mathbb{Z}^n$ be the quotient space. Then the action of $\mathbb{R}^n$ on $M \times \mathbb{R}^n$ via

$$(s,t)(m,r) = (sm, t + r)$$

commutes with $\mathbb{Z}^n$ action on $M \times \mathbb{R}^n$ and hence descends to an $\mathbb{R}^{n+k}$ action. It is clear that this action is an Anosov action.

(7) Any product actions of the above. $\square$

The definition of Anosov actions first appeared in [P-S1] in a more general form in 1972. In that paper, ergodicity of the action was considered. In [P-S2], they also proved that for an ergodic group action, almost every element in the group is ergodic.

In my work, we will consider the individual elements of the group $\mathbb{R}^n$ and prove that every Anosov element is ergodic. The unsolved interesting problem is whether or not the Anosov elements form a dense set.

We remark that although there are plenty of examples of Anosov diffeomorphisms and Anosov flows (for the Anosov flows, even non-algebraic examples have been constructed), there are relatively few examples for the higher-rank non-product Anosov R^n-actions. Conjecturally, there are "too many" structures inside such actions that inhibit the non-trivial perturbations. On the based of this observation, A. Katok, R. Spartzier conjectured that such actions should be locally rigid. This is confirmed in a recent paper [K-S] for Weyl chamber flows, suspensions of Anosov $\mathbb{Z}^k$-actions on tori, and "twisted Weyl chamber flows" (Example 1.2(5) with $N = \mathbb{T}^k$).

This work is inspired by a talk of R. Spartzier at Penn State in March 1991. The examples listed above are due to R. Spartzier and A. Katok. Our work is an attempt to understand the dynamics of Anosov $\mathbb{R}^n$ actions, reveal the rich structures of the actions, and it is hoped, to give additions to the rigidity investigation.

2. Suspension Construction of Anosov $\mathbb{R}^n$ Actions

Let $\mathbb{R}^n \times M \xrightarrow{\Phi} M$ be a volume-preserving Anosov action. We call an element in $\mathbb{R}^n$ a *regular element* if it is an Anosov element. Similarly, any 1-parameter subgroup containing a regular element (hence every non-trivial element is regular) is called regular. Fix a regular element $r \in \mathbb{R}^n$ and let $f = \Phi_r$. Let W_x^{ss}, W_x^{uu} be strong stable and strong unstable manifolds at x, W_x^s, W_x^u be stable and unstable manifolds for f at x. We will later abuse the notation not to distingush r and Φ_r.

LEMMA 2.1.

(1) $\forall g \in \mathbb{R}^n$,

$$gW_x^{ss} = W_{gx}^{ss},$$

$$gW_x^{uu} = W_{gx}^{uu}.$$

(2) *If x, y are in the same $\mathbb{R}^n$ orbit, then*

$$W_x^{ss} = gW_y^{ss},$$

$$W_x^{uu} = gW_y^{uu}.$$

for some $g \in \mathbb{R}^n$.

PROOF. (1) Since

$$W_x^{ss} = \{y \in M : \lim_{n\to\infty} d(f^n y, f^n x) = 0\},$$

$$W_{gx}^{ss} = \{u \in M : \lim_{n\to\infty} d(f^n u, f^n gx) = 0\}$$

$$= \{u := gu_0 : \lim_{n\to\infty} d(gf^n u_0, gf^n x) = 0\}.$$

But g is a diffeomorphism, so the above is

$$g\{u_0 : \lim_{n\to\infty} d(f^n u_0, f^n x) = 0\}.$$

So (1) is true.

(2) is a corollary of (1). □

We will call $x \in M$ a *periodic point* if $\mathbb{R}^n x$ is compact. We denote the set of all the periodic points by Ω. Then Burns and Spatzier [B-S] proved that $\overline{\Omega} = M$.

We recall a result that is called the "Product Neighborhood Theorem." We fix a metric d on M. Let d_u, d_s, d_{uu}, d_{ss} be the induced metrics on the leaves of foliations $\mathcal{F}^u, \mathcal{F}^s, \mathcal{F}^{uu}, \mathcal{F}^{ss}$, respectively. We define for $x \in M, \delta > 0$

$$B_x(\delta) = \{y \in M : d(x,y) < \delta\}$$

$$B_x^u(\delta) = \{y \in W_x^u(\delta) : d_u(x,y) < \delta\}$$

$$B_x^s(\delta) = \{y \in W_x^s(\delta) : d_s(x,y) < \delta\}$$

$$B_x^{uu}(\delta) = \{y \in W_x^{uu}(\delta) : d_{uu}(x,y) < \delta\}$$

$$B_x^{ss}(\delta) = \{y \in W_x^{ss}(\delta) : d_{ss}(x,y) < \delta\}.$$

Then, there exists $\delta_0 > 0$ independent of $x \in M$ such that for $\delta \leq \delta_0$ the maps

$$G : B_x^s(\delta) \times B_x^{uu}(\delta) \to M$$

$$H : B_x^{ss}(\delta) \times B_x^u(\delta) \to M$$

given by

$$G(y,z) = B_z^s(2\delta) \cap B_y^{uu}(2\delta)$$

$$H(y,z) = B_z^{ss}(2\delta) \cap B_y^u(2\delta)$$

are unambiguously defined and are homeomorphisms onto their respective images, which are called product neighborhoods of x. This result is proved by Hirsch, Pugh and Shub for the flow case in [H-P-S] and is a straightforward generalization for general Anosov actions. This result is used in [P-S1].

LEMMA 2.2. *$\overline{W_x^u} = M$, $\overline{W_x^s} = M$ for all $x \in M$.*

PROOF. If we can prove that $\overline{W_x^u}$ is open, then from the fact that M is connected, we are done.

Let $z \in \overline{W_x^u}$. Take a product neighborhood N of z. Take $p \in N$, a periodic point for the action. We may choose so that p is a periodic for a one-parameter subgroup g_t, which is a regular one-parameter subgroup of $\mathbb{R}^n$, and it has the same stable, strong stable, unstable, strong unstable manifolds at any points $x \in M$ as f has.

Let $g_{t_0}p = p$ for some $t_0 \in \mathbb{R}^+$. Now W_p^{ss} intersects W_z^u at $q \in W_z^u$, so it will intersect W_x^u at $q' \in W_x^u$, so we have

$$\lim_{n\to\infty} g_{nt_0}q' = p, \text{ and } g_{nt_0}p = p,$$

but $g_{nt_0}q' \in W_x^u$, so $p \in \overline{W_x^u}$. □

THEOREM 2.3. *If $\overline{W_x^{uu}} \neq M$ for some $x \in M$, then for all $y \in M$, $\overline{W_y^{uu}} \neq M$.*

PROOF. We first prove that if $\overline{W_x^{uu}} \neq M$ for a periodic point $x \in M$. Then $\overline{W_y^{uu}} \neq M$ $\forall y \in M$; secondly, we prove that if $\overline{W_x^{uu}} = M$ for all periodic points, then $\overline{W_y^{uu}} = M$ $\forall y \in M$.

Let $K = \overline{W_x^{uu}}$ for x a periodic point in compact orbit $CO(x)$; then

$$\bigcup_{y\in CO(x)} \overline{W_y^{uu}} = \bigcup_{t\in \text{ some compact set of } \mathbb{R}^n} \phi_t\overline{W_x^{uu}} \supset W_x^u,$$

the middle term here, is actually the image of a compact set under the continuous map

$$\Phi : \mathbb{R}^n \times M \to M,$$

so $\bigcup_{y\in CO(x)} \overline{W_y^{uu}}$ is compact, but it contains a dense set W_x^u, so it is M.

Now take any $y \in M$; let $y \in \phi_{t_1}K$ for some $t_1 \in \mathbb{R}^n$; we claim $W_y^{uu} \subset \phi_{t_1}K$. Indeed, Lemma 2.1 implies that $\phi_{t_1}^{-1}W_y^{uu} \subset K$ So $W_y^{uu} \subset \phi_{t_1}K$, so $\overline{W_y^{uu}} \neq M$.

In the next several lemmas, we will prove that if $\overline{W_x^{uu}} = M$ for all periodic points $x \in M$, then it is also true for all $x \in M$. $\square$

LEMMA 2.4. *Let $\mathbb{T}_1 = \mathbb{R}^n/\mathbb{Z}_1^n, \ldots, \mathbb{T}_k = \mathbb{R}^n/\mathbb{Z}_k^n$ be n-tori, $\mathbb{Z}_i^n \cong \mathbb{Z}^n$. Let $\mathbb{R}^n$ act on $\mathbb{T}_1 \times \cdots \times \mathbb{T}_k$ by translation on each factor. Then for any 1-parameter subgroup g_t of $\mathbb{R}^n$, for any $\varepsilon > 0$, and for a dense set of $(p_1, \ldots, p_k) \in \mathbb{T}_1 \times \cdots \times \mathbb{T}_k$, there exists a sequence $\{t_i\}$ with $t_i \to \infty$, such that*

$$d(g_{t_i}p_j, p_j) < \varepsilon, \text{ for all } j = 1, 2, \ldots, k.$$

PROOF. Since every element of $\mathbb{R}^n$ acts on $\mathbb{T}_1 \times \cdots \times \mathbb{T}_k$ by a volume-preserving map, the result then follows from Poincaré's recurrence theorem. $\square$

LEMMA 2.5. *Let $\overline{W_p^{uu}} = M$ for all periodic points p; then $\overline{W_y^{uu}} = M$ for all $y \in M$.*

PROOF. Let $x \in M$ be any point in M, and $B_\varepsilon(x)$ be a ε-neighborhood of x. We want to prove that any W_y^{uu} intersects this neighborhood for any $y \in M$.

Take a finite $\varepsilon/2$-net, the center of which, $p_i, i = 1, 2, \ldots, k$, are periodic points. Let δ be small enough such that if we take another $\varepsilon/2$-net centering at q_i with $d(q_i, p_i) < \delta$, it still covers M. Let $p_i \in C_i$, which is the compact orbit containing p_i. Using the fact that $\mathbb{R}^n$ acts transitively on C_i and that every orbit is an n-dimensional manifold, we get that $C_i \cong \mathbb{T}_i = \mathbb{R}^n/\mathbb{Z}_i^n$ and the action is the usual translation.

Now take a 1-parameter subgroup g_t containing f. We claim that there exists a $t_0 > 0$, such that for a dense set of q_i in a small neighborhood of p_i, $g_{t_0}(W_{q_i,\varepsilon/2}^{uu}) \cap B_{\varepsilon/2}(x) \neq \phi$. Indeed, $W_{p_i,T}^{uu} \cap B_{\varepsilon/2}(x) \neq \phi$ for sufficiently large T, so there exists a small number $c > 0$, such that if $|t| < c$, and q_i sufficiently close to p_i $g_t(W_{q_i,T}^{uu}) \cap B_{\varepsilon/2}(x) \neq \phi$. From Lemma 2.4, we know that for a dense set q_i close to p_i and

$t_0 \gg T$, such that $d(g_{t_0}q_i, q_i)$ is very small and hence $g_{t_0}(W^{uu}_{q_i,T}) \cap B_{\varepsilon/2}(x) \neq \phi$. But it is clear that $W^{uu}_{g_{t_0}q_i,T} \subset g_{t_0}(W^{uu}_{q_i,\varepsilon/2})$ if t_0 is sufficiently large, so $g_{t_0}(W^{uu}_{q_i,\varepsilon/2}) \cap B_{\varepsilon/2}(x) \neq \phi$.

This claim tells us that we may find $q \in W^{uu}_{q_i} \cap g_{-t_0}(B_{\varepsilon/2}(x))$ such that $q \in W^{uu}_{q_i,\varepsilon/2}$. Now $g_{-t_0}(W^{uu}_y)$ intersects with some $B_{\varepsilon/2}(q_i)$ (this is because all $B_{\varepsilon/2}(q_i)$ cover M); take $q \in W^{uu}_{q_i} \cap g_{-t_0}(B_{\varepsilon/2}(x))$ (the non-emptiness is proved in the above argument); let us assume that $q \in W^{uu}_{q_i,\varepsilon/2}$. Take $r \in g_{-t_0}(W^{uu}_y) \cap W^s_q$. It is clear from the fact that $g_{-t_0}(W^{uu}_y) \cap B_{\varepsilon/2}(q_i) \neq \phi$, that we may assume that $d(q,r) < \varepsilon/2$, so

$$d(x, g_{t_0}(r)) \leq d(x, g_{t_0}q) + d(g_{t_0}(q), g_{t_0}(r)) \leq \varepsilon.$$

But $g_{t_0}(r) \in W^{uu}_y$, so we are done. □

DEFINITION 2.6. An Anosov $\mathbb{R}^n$ action is called irreducible if $\overline{W^{uu}_x} = M$ with respect to some regular element (hence for all regular elements) for some $x \in M$ (hence for all $x \in M$). Otherwise, the action is called reducible. □

THEOREM 2.7. *If $\overline{W^{uu}_x} \neq M$ for some $x \in M$, then there exists a compact set $K \subset M$ such that*

(1) *$\exists \mathbb{R}^k \subset \mathbb{R}^n$ such that $\mathbb{R}^k K = K$;*

(2) *let $\mathbb{R}^{n-k}$ be a complement of $\mathbb{R}^k \subset \mathbb{R}^n$; then the action is the suspension of $(K, \mathbb{R}^k)$ by $n-k$ homeomorphisms in $\mathbb{R}^{n-k}$;*

(3) *K is a C^1 submanifold of M. (Hence the homeomorphisms in (2) are actually C^1 diffeomorphisms.)*

PROOF. Since $\overline{W^{uu}_x} \neq M$, we may assume that for some periodic point $p \in M$, $K := \overline{W^{uu}_p} \neq M$.

(1) Let $G = \{g \in \mathbb{R}^n : (\mathbb{R}g)K \subset K\}$. It is clear that G is a group and it has to be $\mathbb{R}^k$ for some k.

(2) Let $\mathbb{R}^{n-k}$ be a complement of $\mathbb{R}^k$. Let $\mathbb{R}^{n-k} = \underbrace{\mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \cdots \oplus \mathbb{R}e_{n-k}}_{n-k \text{ copies}}$; it is easy to see that for each $e_i, \exists t_i$ such that $t_i e_i K = K$, and if $-t_i < t < t_i$, $te_i K \neq K$.

Indeed, we denote te_1 by $g_t^{(1)}$; let s be the smallest, positive, real number such that $K \cap g_t^{(1)}(K) \neq \varnothing$. That such an s exists may be seen as follows. Suppose there exists arbitrarily small positive $t \in \mathbb{R}$ such that $K \cap g_t^{(1)}(K) \neq \varnothing$. This implies that $K = g_n^{(1)}t(K)$ for arbitarily small t and all $n \in \mathbb{Z}$. This means that $\{t \in \mathbb{R} : K = g_t^{(1)}(K)\}$ is dense in $\mathbb{R}$. Thus, K has to be invariant under $g_t^{(1)}$, which is a contradiction.

Without loss of generality, let $e_i K = K$, $tK \neq K$ for $0 \neq |t| < 1$. Then our action is obviously the $\mathbb{R}^k$ action on K suspended (see §1 Example 6) by $e_1, \ldots, e_{n-k}$.

We use a fact here that $\phi_t K$, $t \in \mathbb{R}^n$ foliates the manifold M. We will prove this fact in the appendix.

(3) The proof is basically the copy of Anosov's argument. We will produce the proof in the following Lemmas. □

We remark that this theorem actually asserts that any Anosov action can be "decomposed" into the suspension of an irreducible C^1 action.

Following Plante [Pl], we call a set $S \subset M$ $\mathcal{F}$-saturated for a foliation $\mathcal{F}$ if it is a union of leaves of $\mathcal{F}$. It is clear that that the closure and complement of an $\mathcal{F}$-saturated set are again $\mathcal{F}$-saturated.

LEMMA 2.8. *Let $\mathcal{F}^{ss}, \mathcal{F}^{uu}, \mathcal{R}^k$ be the strong stable, strong unstable and $\mathbb{R}^k$ orbit foliations with dimension m, l, k, respectively. Then*

(1) *K is $\mathcal{R}^k$ saturated;*

(2) *K is $\mathcal{F}^{uu}$ saturated;*

(3) *K is $\mathcal{F}^{ss}$ saturated.*

PROOF. (1) and (2) are clear.

Suppose that (3) is not true; let $p \in K$ and $q \in W_p^{ss}$, $q \notin K$.

Let $q = \phi_s p_0$ for $p_0 \in K$. Then $K \cap \phi_s K = \phi$. Take t_0 to be a regular element close enough to f, such that $\phi_{t_0} K = K$, $\phi_{t_0}\phi_s K = \phi_s K$. Then $\phi_{nt_0} K = K$, $\phi_{nt_0}\phi_s K = \phi_s K$. So $\lim_{n\to\infty} d(\phi_{nt_0} q, \phi_{nt_0} p) = 0$, so two compact sets K, $\phi_s K$ have distance 0, so they intersect. We get a contradiction. □

LEMMA 2.9. *Let $w_0 \in M$ and $w_1 \in W_{w_0}^{ss}$. Consider $H : p \in W_{w_0}^u \to q \in W_{w_1}^u$ by sliding p along W_p^{ss} to hit $W_{w_1}^u$ at q. Then H maps $\mathbb{R}^k W_p^{ss}$ to $\mathbb{R}^k W_q^{ss}$.*

PROOF. Otherwise, if we choose w_1 close enough to w_0, we know that there exists a small neighborhood of 0 in $\mathbb{R}^{n-k}$ such that there exists t_1 in it, and $\phi_{t_1} K = K$, which is a contradiction. □

The foliations $\mathcal{R}^k$ $\mathcal{F}^{uu}$ $\mathcal{F}^{ss}$ are said to be jointly integrable if there exists a C^1 foliation $\mathcal{F}$ such that $\dim \mathcal{F} = m + l + k$ and any leaf of $\mathcal{R}^k$, $\mathcal{F}^{uu}$ or $\mathcal{F}^{ss}$ is entirely contained in the leaf of $\mathcal{F}$.

LEMMA 2.10. *The foliations $\mathcal{R}^k$, $\mathcal{F}^{uu}$ and $\mathcal{F}^{ss}$ are jointly integrable. So K is a C^1 manifold.*

PROOF. We now use Anosov's argument. It is easy to see that K is C^1. We are not going to repeat the argument. We refer the reader to [A]. □

Appendix 2.11. We will give a proof that if for a periodic point $p \in M$ $K = \overline{W_p^{uu}} \neq M$, then $\phi_t K$ form a partition of M. In other words, either $K \cap \phi_t K = \phi$ or $K = \phi_t K$.

Following Theorem 2.7, we let $\mathbb{R}^k$ be the maximal subgroup in $\mathbb{R}^n$ such that K is invariant under $\mathbb{R}^k$. Choose $n - k$, regular 1-parameter subgroups $g_t^{(i)}$ from the complement of $\mathbb{R}^k$ in $\mathbb{R}^n$, which generate $\mathbb{R}^{n-k}$, such that the stable, unstable, strong stable and strong unstable foliations are the same as those of ϕ, and p is a periodic point of $g_t^{(i)}$ with period r_i. It is clear that K cannot be invariant under any $g_t^{(i)}$. Let $K_0 \subset \overline{W_p^{uu}}(K_0 \neq \varnothing)$ be a minimal set with respect to the following conditions:

(1) K_0 is closed in M;

(2) K_0 is $\mathcal{F}^{uu}$-saturated and invariant under $\mathbb{R}^k$;

(3) $g_{r_i}^{(i)}(K_0) = K_0$.

Such a set exists by Zorn's Lemma. We claim that $K_0, \phi_t K_0$ either disjoint or coincide. Indeed, $K_0 \cap \phi_t K_0$ satisfies (1),(2),(3), so our result follows. So it is then clear that $\phi_s K_0, \phi_t K_0$ either disjoint or coincide. Remember also that $\cup_{t \in \mathbb{R}^n} \phi_t K_0 = M$ because the set contains an unstable manifold and is also compact, so $\phi_t K_0 = M, t \in \mathbb{R}^n$ is a partition of M. Because W_p^{uu} is in $\phi_t K_0$ for some $t \in \mathbb{R}^n$, this forces $K = K_0$. $\square$

3. Irreducible, Weakly Mixing and Continuous Spectrum

In [P-S1], the ergodicity of a general measure preserving Anosov actions is established. Our effort here is to show that the structure of irreducible Anosov $\mathbb{R}^n$ actions are richer than being ergodic. It is actually weakly mixing, which is equivalent to the fact that the induced unitary representation does not have measurable eigenfunctions except the constant eigenfunctions. Moreover, the action is metrically transitive. The latter will be proved in the next section.

We first recall some definitions.

DEFINITION 3.1. A $\mathbb{R}^n$ action on M is ergodic iff it is measure-preserving and all measurable, invariant functions are constant; it is weakly mixing iff

$$\lim_{r\to\infty} \frac{1}{m(B(r))} \int_{B(r)} |(U^t f, g) - (f,1)(g,1)| dt = 0$$

for all integrable functions f, g; it has continuous spectrum iff it has no measurable eigenfunctions other than the constants. □

In this section, we will prove the equivalence of the concepts of irreducibility, weakly mixing and continuous spectrum.

THEOREM 3.2. *The following statements are equivalent:*

(1) *Anosov $\mathbb{R}^n$ action is weakly mixing:*

(2) *the action is irreducible;*

(3) *the action has continuous spectrum.*

□

We will first prove the equivalence of (1) and (3).

LEMMA 3.3. *Let f be a bounded, measurable function. Then*

$$\lim_{r\to\infty} \frac{1}{m(B(r))} \int_{B(r)} |f(t)| dt = 0$$

iff there exists a subset $J \subset \mathbb{R}^n$, $\lim_{r\to\infty} \frac{m(J\cap B_r)}{m(B_r)} = 0$ *such that*

$$\lim_{\substack{|t|\to\infty \\ t\notin J}} |f(t)| = 0.$$

Hence,

$$\lim_{r\to\infty} \frac{1}{m(B(r))} \int_{B(r)} |f(t)|dt = 0$$

iff

$$\lim_{r\to\infty} \frac{1}{m(B(r))} \int_{B(r)} |f(t)|^2 dt = 0.$$

PROOF. The argument is standard. We first prove that sufficiency. For any $\varepsilon > 0$, take $r_0 > 0$, such that if $|t| \geq r_0, t \notin J$, $|f(t)| < \varepsilon/3$. Now

$$\begin{aligned}
&\frac{1}{m(B(r))} \int_{B(r)} |f(t)|dt \\
&= \frac{1}{m(B(r))} \int_{B(r)-J} |f(t)|dt + \frac{1}{m(B(r))} \int_J |f(t)|dt \\
&\leq \frac{1}{m(B(r))} \int_{B(r_0)-J} |f(t)|dt + \frac{1}{m(B(r))} \int_{B(r)-B(r_0)-J} |f(t)|dt \\
&\qquad + \frac{1}{m(B(r))} \int_J |f(t)|dt \\
&\leq \varepsilon/3 + \frac{1}{m(B(r))} \int_{B(r)-B(r_0)-J} |f(t)|dt + \frac{1}{m(B(r))} \int_J |f(t)|dt.
\end{aligned}$$

If we take r sufficiently large, we get the last two terms $\leq \varepsilon/3 + \varepsilon/3$. So we proved that

$$\lim_{r\to\infty} \frac{1}{m(B(r))} \int_{B(r)} |f(t)|dt = 0.$$

Next we prove the necessary part.

Let $J_k = \{t \in \mathbb{R}^n,\ |f(t)| \geq \frac{1}{k}\}$; then it is clear that $J_1 \subset J_2 \subset \cdots \subset J_k \subset J_{k+1} \subset \ldots$. We also have

$$\frac{1}{m(B(n))} \int_{B(n)} |f(t)|dt \geq \frac{1}{m(B(n))} \frac{1}{k} m(J_k \cap B(n)),$$

so $\lim_{n\to\infty}\frac{m(J_k\cap B(n))}{m(B(n))}=0$. So we may find a sequence

$$0<l_1<l_2<\cdots<l_k<l_{k+1}<\cdots\to\infty$$

such that $\frac{m(J_k\cap B(n))}{m(B(n))}<\frac{1}{k}$ for all $n\ge l_k$.

Let $J=\cup_{k=1}^{\infty}[J_{k+1}\cap(B(l_{k+1})-B(l_k))]$; then if $l_k\le n<l_k+1$, we will have

$$J\cap B(n)=(J\cap B(l_k))\cup(J\cap B(n)-B(l_k))\subset(J_k\cap B(l_k))\cup(J_{k+1}\cap B(n)).$$

So

$$\begin{aligned}&\frac{m(J\cap B(n))}{m(B(n))}\\ &\le\frac{m(J_k\cap B(n))}{m(B(n))}+\frac{m(J_{k+1}\cap B(n))}{m(B(n))}\\ &\le\frac{1}{k}+\frac{1}{k+1}\\ &\to 0.\end{aligned}$$

Now we show that $\lim_{\substack{t\to\infty\\ t\notin J}}|f(t)|=0$. Indeed, we removed from $B(l_1)$ those $t's$ such that $|f(t)|>1$; we removed from $B(l_2)-B(l_1)$ those $t's$ such that $|f(t)|>\frac{1}{2}$; we removed from $B(l_{k+1})-B(l_k)$ those $t's$ such that $|f(t)|>\frac{1}{k+1}$; hence it is clear that if $l_k\le|t|<l_{k+1}$, then $|f(t)|<\frac{1}{k+1}$. This gives our result. □

LEMMA 3.4. *Let σ be a finite measure on $\mathbb{R}^n$, let $b_t=\int_{\mathbb{R}^n}e^{i<\theta,t>}d\sigma(\theta),t\in\mathbb{R}^n$, and let $\{\theta_i\}$, $i\in\mathbb{Z}$ be the atoms of σ. Then*

$$\lim_{r\to\infty}\frac{1}{m(B_r)}\int_{B(r)}|b_t|^2dt=\Sigma_{i\in\mathbb{Z}}\sigma^2(\theta_i).$$

PROOF. Since

$$\|b_t\|^2=b_t\overline{b_t}=\int_{\mathbb{R}}^{n}e^{i<\theta,t>}d\sigma(\theta)\overline{\int_{\mathbb{R}}^{n}e^{i<\lambda,t>}d\sigma(\lambda)}=$$

$$=\int_{\mathbb{R}^n\times\mathbb{R}^n}e^{i<\theta,t>-i<\lambda,t>}d\sigma(\theta)\times d\sigma(\lambda),$$

we have

$$\frac{1}{m(B_r)}\int_{B_r}\|b_t\|^2 dt = \int_{\mathbb{R}^n\times\mathbb{R}^n}\left(\frac{1}{m(B_r)}\int_{B_r} e^{i<\theta-\lambda,t>}dt\right) d\sigma(\theta)\times d\sigma(\lambda).$$

Now $\frac{1}{m(B_r)}\int_{B_r} e^{i<\theta-\lambda,t>}dt$ is bounded, so we may use the Lebesgue Dominant Convergence Theorem to conclude that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(\frac{1}{m(B_r)}\int_{B_r} e^{i<\theta-\lambda,t>}dt\right) d\sigma(\theta)\times d\sigma(\lambda)\to \Sigma^2_{i\in\mathbb{Z}}(\theta_i).$$

□

Next we introduce some standard facts for the unitary representation for $\mathbb{R}^n$. For detailed treatment, see [Su].

PROPOSITION 3.5. *(1) (**Stone**) If U is a unitary representation of $\mathbb{R}^n$ on a separable Hilbert space, then there exists a spectral measure E in H on the set of all Borel sets in $\mathbb{R}^n$ such that*

$$U_t = \int_{\mathbb{R}^n} e^{i<t,\theta>}dE(\theta) \text{ for all } t\in\mathbb{R}^n.$$

*(2) (**Bochner**) If ϕ is a positive definite function on $\mathbb{R}^n$, then there exists a unique positive Radon measure σ with total measure $\sigma(\mathbb{R}^n)=\phi(0)$ such that*

$$\phi(t) = \int_{\mathbb{R}^n} e^{i<t,\theta>}d\sigma(\theta) \text{ for all } t\in\mathbb{R}^n.$$

□

We remark that the above two theorems are equivalent in the sense that one theorem can be easily proved, assuming that the other is true. We also remark that if we fix $f\in H$, then (U_tf,f) is a positive definite function on $\mathbb{R}^n$ for a unutary representation U of $\mathbb{R}^n$ on H. We let $(U_tf,f)=\int_{\mathbb{R}^n} e^{i<t,\theta>}d\sigma_f(\theta)$.

LEMMA 3.6. *$f \in L^2(M,m)$ is an eigenfunction for $\{U_t\}$ corresponding to the eigenvalue $e^{i<\theta_0,t>}$ iff σ_f is concentrated at the point θ_0; i.e.,*

$$\sigma_f = \|f\|^2 \delta(\theta - \theta_0).$$

PROOF. Since

$$(U_t f, f) = e^{i<\theta_0,t>}(f,f) = \|f\|^2 \int_{\mathbb{R}^n} e^{i<\theta,t>} d\delta(\theta - \theta_0),$$

so from the uniqueness part of Proposition 3.5.(2) we obtain

$$\sigma_f = \|f\|^2 \delta(\theta - \theta_0).$$

Conversely,

$$(U_t f, f) = \int_{\mathbb{R}^n} e^{<\theta,t>} d\sigma(\theta) = \|f\|^2 e^{i<\theta_0,t>},$$

so

$$|(U_t f, f)| = \|f\|^2 = \|U_t f\| \|f\|,$$

so the Cauchy inequality becomes equality. Therefore,

$$U_t f = c_t f$$

for some constant c_t. But

$$(U_t f, f) = \|f\|^2 e^{i<\theta_0,t>},$$

so $c_t = e^{i<\theta_0,t>}$. □

LEMMA 3.7. *Supose that the $\mathbb{R}^n$ Anosov action has a continuous spectrum. Let $f \in L^2(M)$, with $\int f dm = 0$. Then σ_f has no atom.*

PROOF. Otherwise, $\sigma_f(\theta_0) > 0$ for some $\theta_0 \in \mathbb{R}^n$. But $\delta(\theta - \theta_0)$ is absolutely continuous with respect to σ_f; we know that there exists a function $h \in L^2(\mathbb{R}^n, \sigma_f)$

such that $\delta(\theta - \theta_0) = h(\theta)\sigma_f(\theta_0)$. Recall a result in [Su], p.141, that we can find a $g \in < f >$ (a subspace in $L^2(M)$ generated by f) such that $\sigma_g = \delta(\theta - \theta_0)$ (see for example [C-F-S], Appendix 2 for the csae of Anosov diffeomorphisms and Anosov flows). Lemma 3.6 implies that g is a non-constant eigenfunction. Contradiction. □

We are now able to prove the equivalence of (1) and (3) of Theorem 3.2.

PROPOSITION 3.8. *An $\mathbb{R}^n$ action has a continuous spectrum iff*

$$\lim_{r\to\infty} \frac{1}{m(B(r))} \int_{B(r)} |(U^t f, g) - (f,1)(g,1)| dt = 0.$$

PROOF. " $\Rightarrow$ ": A standard polarization trick makes it sufficient to prove

$$\lim_{r\to\infty} \frac{1}{m(B_r)} \int_{B_r} |(U^t f, f) - (f,1)(1,f)| dt = 0.$$

Moreover, we may assume that $(f,1) = 0$ by replacing f with $f - (f,1)$. So it is sufficient to show

$$\lim_{r\to\infty} \frac{1}{m(B_r)} \int_{B_r} |(U^t f, f)| dt = 0.$$

By Lemma 3.3, we know that it is sufficient to prove that

$$\lim_{r\to\infty} \frac{1}{m(B_r)} \int_{B_r} |(U^t f, f)|^2 dt = 0.$$

But

$$(U^t f, f) = \int e^{i<\theta,t>} d(E(\theta)f, f) = \int e^{i<\theta,t>} d\sigma_f(\theta);$$

applying Lemma 3.4 to σ_f, and noticing that the above quantity is exactly b_t in Lemma 3.4, we conclude our result.

The other direction is trivial. □

We remark that we have actually proved that the equivalence of (1) and (3) is true for general $\mathbb{R}^n$ actions, not necessarily Anosov. The proof here is essentially the same as the proof of the same results for diffeomorphisms and for flows, except more complicated technicality.

Now we will prove the equivalence of (2) and (1) or (3). First of all we will prove that an irreducible Anosov action cannot have non-constant, measurable eigenfunctions. Recall that $\mathbb{R}^n$ Anosov action on M is ergodic [P-S1]. So by [P-S2], almost all $t \in \mathbb{R}^n, t$ is an ergodic diffeomorphism. Take $t_1, \ldots, t_n \in M$ ergodic, regular with the same foliations as f has and spans $\mathbb{R}^n$, so $\mathbb{R}^n = \oplus \mathbb{R} t_i$. Suppose that H is an eigenfunction for induced, unitary representation. Using a well-known result of Rohlin, we may assume that the following equality is true for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$ and for almost every $x \in M$. $U_{(t_1,\ldots,t_n)}H = e^{i<t_1,\ldots,t_n|\lambda_1,\ldots,\lambda_n>}H$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

LEMMA 3.9. *For a.e. $x \in M$, H is constant on W_x^{uu} a.e.*

PROOF. The proof is a standard one. There are two approaches to this kind of statement. One is originally used in [A]. We will use the other approach. Since $U_{t_1}H = e^{it_1\lambda_1}H$, without loss of generality we assume that $\lambda \neq 0$. Then $U_{(2\pi t_1/l_1)} = H$, so H is invariant under the regular element $g = 2\pi t_1/l_1$. We define $Inv(g)$ to be the set of all integrable, g-invariant functions $M \to \mathbb{R}$. We are trying to prove that every function in it is constant almost everywhere on almost every unstable leaf.

We define a projection $I_g : L^1(M) \to Inv(g)$ by

$$I_g\phi(x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \phi(g^k x);$$

then the limit exists almost everywhere and is integrable, and $\phi \to I_g\phi$ is a continuous linear map onto the $Inv(g)$. Moreover, the limit

$$I_g^{\pm}\phi(x) = \lim_{n\to\pm\infty} \frac{1}{|n|+1} \sum_{k=0}^{n} \phi(g^k x)$$

exists almost everywhere and $I_g^{\pm}\phi(x) = I_g\phi(x)$ for almost all x. That is, $I_g^+ = I_g^- = I_g$ as maps $L^1(M) \to Inv(g)$.

Since the continuous functions are dense in $L^1(M)$, their I_g-images are dense in $Inv(g)$. We now prove that for the continuous function ϕ, $I_g\phi$ is essentially constant along $\mathcal{F}^{uu}, \mathcal{F}^{ss}$.

For any $x, y \in W_p^{uu}$ and any continous $\phi : M \to \mathbb{R}$, it is clear that either both $I_g^-(x), I_g^-(y)$ are defined, or neither, and if defined they are equal. Since $I_g^-\phi$ is defined almost everywhere, $I_g^-\phi$ is defined and constant on almost all $\mathcal{F}^{uu}$ leaves. Since $\mathcal{F}^{uu}$ is absolutely continuous [P-S1] and $I_g^-\phi = I_g\phi$ almost everywhere, $I_g\phi$ is essentially constant on almost every $\mathcal{F}^{uu}$ leaf. Similarly, for $\mathcal{F}^{ss}$.

The density of the image of continuous functions implies that our function H is essentially constant along $\mathcal{F}^{uu}$, $\mathcal{F}^{ss}$. □

LEMMA 3.10. *H is continuous on a.e.* W_x^u, W_x^s.

PROOF. From $U_{(t_1,\dots,t_n)}H = e^{i<t_1,\dots,t_n|\lambda_1,\dots,\lambda_n>}H$, it is clear. □

LEMMA 3.11. *H is a constant function almost everywhere.*

PROOF. We claim that H almost everywhere coincides with some continuous function. The approach is exactly the same as [A] p169. (We do not copy the argument from there; instead, we sketch the idea below. Since H "is" continuous on "every" $\mathcal{F}^u$ leaf and constant on "every" $\mathcal{F}^{ss}$ leaf, hence H "is" continuous.) Now let H' be that continuous function; then it is clear that H' is constant along the leaves of foliation $\mathcal{F}^{uu}$. Hence, H' is constant. □

Combining the above Lemmas, we proved that (2) implies (1) and (3). Next, we will prove that the other direction is also true.

LEMMA 3.12. *If Anosov* $\mathbb{R}^n$ *action has a continuous spectrum, then the action is irreducible.*

PROOF. Otherwise, let p be a periodic point, $K = \overline{W_p^{uu}} \neq M$. Fix a 1-parameter subgroup $g_t \in \mathbb{R}^n$ that does not fix K for some t, and $g_{t_0}K = K$. Choose $n-1$ other elements $t_1, \ldots, t_{n-1}$ in $\mathbb{R}^n$ such that p is a common fixed point for them, and together with t they span $\mathbb{R}^n$. It is clear that if we denote by $\mathbb{R}^{n-1}$ the space spanned by $t_1, \ldots, t_{n-1}$, then $K' = \mathbb{R}^{n-1}K$ is compact and is a proper subset of M. Let a function H be defined such that $H|_{K'} = 1$ $H|_{g_tK'} = e^{2\pi i(t/t_0)}$. Then H is a non-constant eigenfunction for U_t. Contradiction. $\square$

4. Metric Transitivity

In this section, we will prove the metric transitivity of Anosov $\mathbb{R}^n$ actions, or roughly speaking, for any regular element, the strong stable and strong unstable foliations are measurably indecomposable. There are several consequences of this result. The interesting one is that we obtain that ergodicity of regular individual elements. Different from the result of [P-S1], where every element off countably many hyperplanes is ergodic for Anosov actions, we can only have the ergodicity for regular elements. It seems to be a difficult problem to prove that the regular elements are dense even in $\mathbb{R}^n$. We remark that the problem raised by A. Katok fits well into the investigation of rigidity of $\mathbb{R}^n$ actions and is one of the motivations for me to study the dynamics of $\mathbb{R}^n$ actions.

Our method follows closely that of Anosov. Some results of Anosov can be used directly without any change, while others may be proved using his idea. The only difference is that we fit the $\mathbb{R}^n$ situation well into the original argument. First we will give the precise definition of metric transitivity.

DEFINITION 4.1. A foliation $\mathcal{F}$ is called metrically transitive if for an arbitrary measurable $\mathcal{F}$-saturated set $A \subset M$, either $m(A) = 0$ or $m(M - A) = 0$. □

The idea of the proof of metric transitivity is simple. We assume that we have an intermediate saturated set A and prove that the union of all short orbits starting from A is still an intermediate saturated set. Then we use a result that when you iterate A "backwards" sufficiently far away, "all" stable leaves will be contained in the iterated set. Hence, the iterated set has to "have" full measure. But our action is a measure-preserving action; hence A itself "has" full measure. This presents a contradiction.

Lemma 4.2. *Let A be an $\mathcal{F}^{uu}$-saturated Borel set. Let*

$$m_\tau(w) = \frac{1}{m(B(\tau))} m\{t : |t| \le \tau, \Phi^t w \in A\}.$$

Then it is measurable, constant on the leaves of $\mathcal{F}^{uu}$ and

$$\lim_{\tau \to 0} \int_A |m_\tau(w) - 1| dw = 0.$$

Proof. Measurability follows from the Fubini theorem, since the set $\{t : |t| \le \tau, \Phi^t w \in A\}$ is an intersection of the pre-image of the Borel set A under the smooth map

$$M \times B(\tau) \to M, \ (w,t) \mapsto \Phi_t w,$$

with $\{w\} \times B(\tau)$. Since A is $\mathcal{F}^{uu}$-saturated and since Φ_t permutes the leaves in A, it is clear that $m_\tau(w)$ is constant on the leaves of $\mathcal{F}^{uu}$.

It is clear that

$$m_\tau(w) = \frac{1}{m(B(\tau))} \int_{B(\tau)} \chi_A(\Phi_t w) dt,$$

where χ_A is the characteristic function of the set A. We have

$$\begin{aligned} \int_A m_\tau(w) &= \frac{1}{m(B(\tau))} \int_M \left(\int_{B(\tau)} \chi_A(\Phi_t w) \chi_A(w) dt \right) dw \\ &= \frac{1}{m(B(\tau))} \int_{B(\tau)} \left(\int_M \chi_A(\Phi_t w) \chi_A(w) dw \right) dt. \end{aligned}$$

We show that $\lim_{t \to 0} \int_M \chi_A(\Phi_t w)\chi_A(w) dw = m(A)$. Indeed, for any function $f \in L^2(M)$, we have $\lim_{t \to 0} \|\Phi_t f - f\|_{L^2} = 0$; hence,

$$\lim_{t \to 0} \int_M \chi_A(\Phi_t w) \chi_A(w) dw = \lim_{t \to 0} (\chi_A, \Phi_t \chi_A) = (\chi_A, \chi_A) = m(A).$$

Hence,

$$\lim_{\tau \to 0} \int_A m_\tau(w) = m(A).$$

But obviously $0 \leq m_\tau(w) \leq 1$, so our result follows. □

We remark that we may choose a smooth metric on M, such that the $\mathbb{R}^n$ action on an individual orbit is an isometric action. Indeed, pick n 1-dimensional parameter subgroups of $\mathbb{R}^n$ that generate $\mathbb{R}^n$; these subgroups give n vector fields on M. These vector fields are linearly independent at every point on M. We define a metric on M such that these vector fields are pairwisely orthogonal and have norm 1. It is then clear that $\mathbb{R}^n$ action on the individual orbit is isometric action.

LEMMA 4.3. *Let $w, w' \in M$ such that $w' = \Phi_s w$, $s \leq \tau$. Define*

$$G_\tau(w) = \{x \in M, x = \Phi_r w, r \in \mathbb{R}^n, |r| \leq \tau\}$$

$$G_\tau(w') = \{x \in M, x = \Phi_r w', r \in \mathbb{R}^n, |r| \leq \tau\}.$$

Let $G_{\tau,s} = G_\tau(w) \cap G_\tau(w')$ and $T_{\tau,s} = \{r \in \mathbb{R}^n, |r| \leq \tau, \Phi_r w \in G_{\tau,s}\}$. Then

$$\lim_{\tau \to 0} \frac{m(T_{\tau,s})}{m(B_\tau)} > c > 0,$$

which is independent of w, w'.

PROOF. Using remark above we may assume that we are working on space $M = \mathbb{R}^n$ with a flat metric, which is the orbit (up to covering) containing w, w'. The action is the usual translation. Then everything is simple. We draw two τ-balls around w and w', the intersection is $T_{\tau,s}$, which contains $T_{\tau,\tau}$. What is left is an exercise for calculus.

LEMMA 4.4. *If A is an $\mathcal{F}^{uu}$-saturated, measurable set of M with an intermediate measure, i.e., $0 < m(A) < m(M)$, then there exists an $\mathcal{F}^{uu}$-saturated Borel set $B \subset A$ with an intermediate measure such that*

$$C = \cup_{|t| \leq \tau} \Phi^t B$$

is also an $\mathcal{F}^{uu}$-saturated, measurable set with an intermediate measure for some small τ.

PROOF. First of all we may assume that A itself is a Borel set. In other words, A contains a Borel subset of the same measure, such that the subset is also $\mathcal{F}^{uu}$-saturated. This fact is proved in [A].

Let $D = M - A$. We define $m_\tau(w)$ as in the last lemma and define $n_\tau(w)$ as

$$n_\tau(w) = \frac{1}{m(B(\tau))} m\{t : |t| \leq \tau, \Phi_t w \in D\}.$$

Lemma 4.2 implies that for sufficiently small $\tau > 0$, and c as in the last lemma, the sets

$$B = \{w : w \in A, m_\tau(w) > 1 - c/10\},$$

$$E = \{w : w \in D, n_\tau(w) > 1 - c/10\}$$

have a positive measure and are $\mathcal{F}^{uu}$-saturated.

We show that $E \cap C = \phi$. Otherwise, there would exist a point $w \in M$ and $s \in \mathbb{R}^n$ such that

$$|s| \leq \tau, \; w \in B, \; w' = \Phi_s \in E.$$

But this is impossible. Indeed, let us calculate the measure of $T_{\tau,s}$ as defined in the last lemma. Lemma 4.2 tells us that

$$m(\{t \in T_{\tau,s}, \Phi_t w \in D\}) \leq m(\{t \in B(\tau), \Phi_t w \in D\}) \leq \frac{c}{10} m(B(\tau)).$$

For the same reason we get

$$m(\{t \in T_{\tau,s}, \Phi_t w' \in A\}) \leq m(\{t \in B(\tau), \Phi_t w' \in A\}) \leq \frac{c}{10} m(B(\tau)).$$

But $A \cup D = M$, so for every $t \in T_{\tau,s}$, $\Phi_t w = \Phi_{t-s} \Phi_s w'$ is either in A or in D. So the total measure of $T_{\tau,s}$ is less than $2\frac{c}{10} m(B(\tau)) \leq \frac{1}{5} c m(B(\tau))$, and contradicts Lemma 4.3. So $C \cap E = \phi$.

After possibly removing from B some set of measure zero, we may assume that B is a Borel set. Then C is measurable because it is the image of the Borel set $B \times B(\tau)$ under a smooth map. It is clear that C is $\mathcal{F}^{uu}$-saturated, and has an intermediate measure. □

We quote a modified result of Anosov [A].

LEMMA 4.5. *Fix a regular element $f \in \mathbb{R}^n$; let g_t be the 1-parameter subgroup containing f with $g_{t_0} = f, t_0 > 0$. Let A be an arbitrary, measurable set. For any $r > 0, \varepsilon > 0$ and sufficiently large $t > 0$, there exists a set M_ε^t such that $m(M_\varepsilon^t) < \varepsilon$ and any two points $w, w' \in M - M_\varepsilon^t$, which lie on the same leaf of the strong stable foliation $\mathcal{F}^{ss}$ at distance $< r$ from each other; either both belong or both do not belong to the set $g_{-t}A$.*

We remark that the proof in [A] uses only the exponential contraction property of the foliation $\mathcal{F}^{ss}$. So all the proof goes through without any modification. Moreover, the proof is clearly true if we replace the 1-parameter subgroup g_t by a small cone centering around g_t. We require only that the cone be small enough to have a uniform estimate for the exponential contraction.

We need one more technique lemma.

LEMMA 4.6. *Let $\mathbb{R}^n$ Anosov action be irreducible. Fix measurable sets $B, U_1, \dots, U_N$ of positive measure and also fix a small cone around g_t as in Lemma 4.5. We call the half-cone containing f the positive half-cone, and denote it by T^+. Then there exists a sequence $t_i = (t_1^{(i)}, \dots, t_n^{(i)}) \in T^+$ with $|t_i| \to \infty$ and a $\delta > 0$ such that for all t_n*

$$m(\Phi_{-t_n} B \cap U_i) > \delta.$$

PROOF. From Proposition 3.8 it is easy to see that for each U_i there exists a

$J_i \subset \mathbb{R}^n$ of zero density such that $\lim_{\substack{|t|\to\infty \\ t\notin J_i}} m(\Phi_{t_i} \cap U_i) = m(B)m(U_i)$. We take $J = J_1 \cup \cdots \cup J_N$; then this set has zero density, and our result follows easily. $\square$

THEOREM 4.7 (METRIC TRANSITIVITY). *Weakly mixing Anosov $\mathbb{R}^n$ action is metrically transitive.*

PROOF. We wish to arrive at a contradiction on assuming the existence of the sets B and C and the number τ of the Lemma 4.4. It is clear that under this assumption there exists an $r > 0$ such that if $w \in B$, then the r-neighborhood of the point w on the leaf of the foliation $\mathcal{F}^u$ passing through this point is entirely contained in C. This r is dependent on τ, but not on B; hence for the sets $\Phi_t B$ and $\Phi_t C$, which obviously have the same properties as B and C, we can use this same number r (since τ is the same for these).

We consider a finite atlas $\{(U_i, \phi_i)\}, i = 1, 2, \dots, N$, of the manifold M. Each coordinate neighborhood $U_i = \phi_i^{-1}(X \times Y)$ is sufficiently small such that an r-neighborhood of any point $w \in U_i$ on the leaf of the foliation $\mathcal{F}^u$ passing through it contains the connected component N_w^u of the intersection of this leaf with U_i. Note that if w has coordinates $\phi_i(w) = (x, y)$, then $\phi(N_w^u) = x \times Y$. The ϕ_i is absolutely continuous. The existence of these kinds of product neighborhoods is proved in [A] for any two foliations that have smooth individual leaves and that are absolutely continuous, transversal to each other.

We apply Lemma 4.7 to the sets $B, U_1, \dots, U_N$. This lemma and the absolute continuity of ϕ_i guarantee the existence of a sequence $t_n \to \infty$, $t \in T^+$ and a number $\Delta > 0$ such that $m(\phi_i(\Phi_{-t_n} B \cap U_i)) > \delta$, $i = 1, \dots, N$. This and the properties of the set C imply that for some $\delta > 0$

$$m\{x : x \times Y \subset \phi_i(\Phi_{-t_n} C \cap U_i)\} > \delta, i = 1, \dots, N.$$

We now use Lemma 4.5 and the remark after it, replacing A by C and assuming that the number r in this lemma exceeds the size of leaves of the foliation $\mathcal{F}^{ss} \cap U_i$. For sufficiently large $|t_n|$ the measure of the "exceptional" set $M_\varepsilon^{t_n}$ of the Lemma 4.5 will be arbitrarily small. That means that the measure of the set $\phi_i(M_\varepsilon^{t_n} \cap U_i)$ can also be considered arbitrarily small, significantly smaller than δ. Hence, for each of these sets, if $|t_n|$ is sufficiently large, there exists an x_n such that

$$x_n \times Y \subset \phi_i(\Phi_{-t_n} \cap U_i)$$

and $m(\phi_i(M_\varepsilon^{t_n} \cap U_i) \cap (x_n \times Y))$ is small. Now if the point (x, y) lies outside the set

$$\phi_i(M_\varepsilon^{t_n} \cap U_i) \cup \{X \times [\phi_i(M_\varepsilon^{t_n} \cap U_i) \cap (x_n \times Y)]\},$$

then it belongs or does not belong to $\phi_i(\Phi_{-t_n} \cap U_i)$ simultaneously with the points (x_n, y), and the last one belongs to $\phi_i(\Phi_{-t_n} \cap U_i)$. Therefore,

$$m[(X \times Y) - \phi_i(\Phi_{-t_n} \cap U_i)] \to 0 \text{ as } n \to \infty,$$

and hence also $m(U_i - \Phi_{-t_n} C) \to 0$ as $n \to \infty$. Therefore,

$$m(M - C) = m(M - \Phi_{-t_n} C) \leq \sum_i m(U_i - \Phi_{-t_n} C) \to 0.$$

But this contradicts the fact that the measure of C is intermediate. □

COROLLARY 4.8. *Let the $\mathbb{R}^n$ action be irreducible. Then*

(1) *every regular elements are ergodic;*

(2) *every 1-parameter regular subgroup g_t is weakly mixing.*

PROOF. (1) Let $Uf = F$ or $f(gx) = f(x)$ for a regular element and a.e $x \in M$. Consider $f_1 = Re(f)$; it is clear that $f_1(gx) = f_1(x)$ a.e. Then $\omega_\alpha := \{x : f(x) \geq \alpha\}$ is ϕ invariant. But it is easy to see that f_1 is constant on a.e. $\mathcal{F}^{uu}$-leaf, so ω_α is $\mathcal{F}^{uu}$-saturated. So $\omega_\alpha = M$ or ϕ. This forces f_1 to be essentially constant.

The same argument applies to the imaginary part of f. Finally, we get f as essentially constant.

(2) $f(g_t x) = e^{i\lambda t} f(x)$ implies that $f(g_{2\pi/\lambda} x) = f(x)$, but $g_{2\pi/\lambda}$ is regular. Our result follows from (1). $\square$

5. Measures Compatible with $\mathbb{R}^n$ Actions

There are two natural measures compatible with Anosov diffeomorphisms and Anosov flows, which are called the Sinai-Bowen-Ruelle measure and the Bowen-Margulis measure. These two measures are considered extensively by various authors, and it is clear that they play an important role in the rigidity investigation. They will be proved to exist for Anosov $\mathbb{R}^n$ actions as well.

THEOREM (THE EXISTENCE OF THE SBR MEASURE) 5.1. *Let the $\mathbb{R}^n$ action be irreducible, hence weakly mixing. Fix a regular element and then the corresponding strong unstable foliation $\mathcal{F}^{uu}$. Then there exists an invariant measure μ such that the conditional measure with respect to $\mathcal{F}^{uu}$ is absolutely continuous with respect to the Lebesgue measure.*

PROOF. Fix a regular element $g \in \mathbb{R}^n$; then g expands the foliation $\mathcal{F} = \mathcal{F}^{uu}$, the leaves of which are smooth. Fix a positive integer r.

Choose a covering of M by a finite number of charts $D^{m-k} \times D^k$ such that the leaves of $\mathcal{F}$ are of the form $D^{m-k} \times \{v\}$. Let a probability measure ρ on M have, to each of these charts, a restriction of the form

$$\rho^0(u,v)du\mu(dv),$$

where du is the Lebesgue measure on D^{n-k} and $\mu(dv)$ some positive measure on D^k. Assume that for μ-almost all v, the function $u \to \rho^0(u,v)$ is strictly positive, and its logarithm has derivatives up to r which are Lipschitz, with Lipschitz constant $\leq l$. Let $\mathcal{K}$ be the set of such measures ρ, with l fixed, but $\mu(dv)$ are allowed to vary. Define

$$\mathcal{K}_+ = \cup_{l \geq 0} \cap_{n \geq 0} g^n \mathcal{K}(l).$$

It is easy to verify that $\mathcal{K}_+$ does not depend on the choice of the charts used to define $\mathcal{K}(l)$.

Then we use the theorem in [R1], obtaining the following:

(a) $\mathcal{K}$ is vaguely compact; it is a Choquet simplex, and the conditional measure $\rho^0(u,v)du$ of $\rho \in \mathcal{K}_+$ on a leaf of $\mathcal{F}$ is independent of ρ (up to normalization).

(b) $\mathcal{K}_+$ is non-empty.

(c) Let $\mathcal{K}_g$ be the set of g-invariant elements of $\mathcal{K}_+$; then $\mathcal{K}_g$ is a simplex.

It is also straightforward that the following statement is true.

Claim: (d) if $\mu \in \mathcal{K}_+$, then $f\mu \in \mathcal{K}$ for any $\phi \in \mathbb{R}^n$. (e) $\mathcal{K}_+$ is convex.

Indeed, in a coordinate chart, let

$$f^{-1} = (f_1(u,v), f_2(v));$$

then

$$\|D_u f_1(u,v)\| \leq \alpha$$

for some $\alpha \geq 0$. So the density of the conditional measure corresponding to $f\mu$ on $D^{m-k} \times \{v\}$ is

$$\sigma^1 = K\sigma^0(f_1(u,v), f_2(v))|\det(D_u f_1(u,v))|,$$

and so

$$D_u \log \sigma^1(u,v) = D_u \log \sigma^0(f_1(u,v), f_2(v)) D_u f_1(u,v) + D_u \log|\det(Df_1(u,v))| \leq$$

$$\leq \alpha D_u \log \sigma^0(f_1(u,v), f_2(v)) + C.$$

So for μ-almost all v,

$$|D_u \log \sigma^1(f_1(u,v), f_2(v))| \leq \alpha l + C.$$

Same calculations (as those in [R1]) conclude that there exists l' such that $f\mu \in \mathcal{K}_{l'}$. So $f\mu \in \cap f g^n \mathcal{K}(l) = \cap g^n(f\mathcal{K}(l)) \subset \cap g^n(\mathcal{K}(l'))$. So $f\mu \in \mathcal{K}_+$. This proves (d). (e) is clear.

Because (a)–(e), we may use first use an average method in $\mathcal{K}_+$ to get that $\mathcal{K}_g$ is non-empty, and then to use the average method to get that in there exists an $\mathbb{R}^n$-invariant measure in $\mathcal{K}_g$. (e) and (a) guarantee that they are in the right class of measures. $\square$

Margulis considered an invariant measure for anosov flows whose conditional measure restricted to $\mathcal{F}^{uu}$,$\mathcal{F}^{ss}$ have uniform expansion (contraction) property. Moreover, the coefficients are both topological entropy for the flow. There are also several other constructions, e.g., Sinai [Si] for toral diffeomorphisms, Hasselblatt [Ha] for Anosov flows, Hamenstädt [Ham] for geodesic flows for negatively curved manifolds, which are all interesting. In this paper we will give a similar fact about this kind of measure, and we will call it Margulis measure.

DEFINITION 5.2. Fix a regular element $f \in \mathbb{R}^n$.We call an invariant measure μ a Margulis measure if

(1) $g^*\mu^{uu} = \lambda(g)\mu^{uu}$, for all $g \in \mathbb{R}^n$, some $\lambda(g)$.

(2) μ^{uu} is honolomy-invariant.

The same formulas hold for μ^{ss}.

THEOREM (THE EXISTENCE OF THE MARGULIS MEASURE) 5.3. *For any irreducible Anosov $\mathbb{R}^n$ action, there exists at least one Margulis measure.*

PROOF. A similar argument in [R2] establishes the existence of transversal holonomy-invariant measure with uniform expansion in the $\mathcal{F}^{uu}$-direction. For completeness, we will reproduce the proof with necessary modification.

Let $\mathcal{F} = \mathcal{F}^{uu}$, with $codim(\mathcal{F}) = k$. Let $\mathcal{S}$ denote the set of open submanifolds of dimension k transversal to $\mathcal{F}$. Let

$$\mathcal{J} = \{\rho : \rho \text{ is a signed transversal-invariant measure for } f\}.$$

(A transversal-invariant measure is a collection (ρ_Σ) of real measures on the set $\Sigma \in \mathcal{S}$ such that under the canonical isomorphisms they coincide.) We define *vague topology* as the topology defined on $\mathcal{J}$ by the seminorms:

$$\rho \to |\rho_\Sigma(\phi)|,$$

where ϕ is a real, continuous function with compact support in $\Sigma \in \mathcal{S}$. Call $\rho \geq 0$ if $\rho_\Sigma \geq 0$ for all $\Sigma \in \mathcal{S}$. Let C be the cone of positive measures in $\mathcal{J}$. Then we have (see [R2])

(a) $g\mathcal{J} = \mathcal{J}$, $gC = C$;

(b) there exists a $\lambda_0 \geq 0$, such that $C(\lambda_0, g) = \{\rho \in C : g\rho = \lambda_0 \rho\}$ is a non-empty, closed subset of C.

Let us take a 1-parameter subgroup g_t in $\mathbb{R}^n$ with $g_1 = g$ and show that the set $C(\lambda_0, \{g_t\}) = \{\rho \in C : g\rho = \lambda_0^t \rho \text{ for all } t\}$ is non-empty and g-invariant.

Indeed, $C(\lambda_0, g_{1/2}) \subset C(\lambda_0, g)$, and g-invariance is obvious. Then repeating the argument again and again, we may prove that $C(\lambda_0, g_{1/2^{n+1}}) \subset C(\lambda_0, g_{1/2^n})$ and is also g-invariant. Local compactness of C implies that the intersection C_0 of these sets is non-empty and is g-invariant. Now the continuity of $g_t\mu$ in t with $\mu \in C_0$ implies our result. So we have that

(c) $C(\lambda_0, \{g_t\}) = \{\rho \in C : g\rho = \lambda_0^t \rho \text{ for all } t\}$ is a non-empty and g-invariant set.

It is clear that C_0 is a closed subcone of C, and for all $f \in \mathbb{R}^n$ $fC_0 = fC_0$. So using the argument of [R2], we can get that $C(\lambda_1, f) = \{\rho \in C_0 : f\rho = \lambda_1 \rho\}$ is a non-empty, closed subset of C_0 for f sufficiently close to g such that f is regular and expands $\mathcal{F}$. Then we may have a statement similar to (c). Continuing to do this, we will obtain a measure that is a transversal holonomy-invariant measure with uniform expansion property for all elements for $\mathbb{R}^n$. Using Margulis' construction of Margulis measure for Anosov flow, we will get a Margulis measure for our Anosov $\mathbb{R}^n$ action. $\square$

Remark 5.4. The argument for the two above theorems is almost the duplication of those in [R1] [R2]; we do not have uniqueness automatically from the argument of Ruelle [R1] [R2]. Yet, we do have the uniqueness of the uniform expansion coefficient for Margulis measure. This fact can be proved using either Margulis' original argument [M2] or Hasselblatt's construction for the Margulis measure [Ha], which we do not discuss here. We also remark that the uniqueness of the Margulis measure might be important for the proof of the rigidity of $\mathbb{R}^n$ actions, because the Margulis measure is carried over by topological conjugacy.

References

[A] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, (English translation, A.M.S.,Providence,RI, 1969), Proceedings of the Steklov Institute of Mathematics **90** (1967), 1–235.

[A-S] L. Auslander and J. Scheuneman, *On certain automorphisms of nilpotent Lie groups*, in A.M.S. Proceedings of Symposia in pure mathematics, vol. 14, 1970, pp. 9–15.

[B-P] M.I. Brin and Ja.B. Pesin, *Partially hyperbolic dynamical systems*, Math. USSR Izvestija **8** (1974), 177–218.

[B-S] K.Burns and R.Spatzier, unpublished.

[C-F-G] I.P.Cornfeld, S.V.Fomin and Ya.G.Sinai, *Ergodic theory*, Springer-Verlag, New York, 1982.

[D] J. L. Deyer, *A nilpotent Lie algebra with nilpotent automorphism group*, Bulletin of the American Mathematical Society **76** (1970), 52–60.

[F] J. Franks, *Anosov diffeomorphisms*, in A.M.S. Proceedings of Symposia in pure mathematics, vol. 14, 1970, pp. 61–93.

[Ham] U. Hamenstädt, *A new description of the Bowen-Margulis measure*, Ergod. Th. and Dynam. Sys. **9** (1989), 455–464.

[Ha] B. Hasselblatt, *A new construction of the Margulis measure for Anosov flows*, Ergod. Th. and Dynam. Sys. **9** (1989), 465–468.

[H-P-S] M. W. Hirsch, C. C. Pugh and M. Shub, *Invariant manifolds*, Bulletin of the American Mathematical Society **76** (1970), no. 5, 1015–1019.

[H] T.W. Hungerford, *Algebra*, Springer-Verlag, New York, 1987.

[Hu1] S. Hurder, *Problems on rigidity of group actions and cocycles*, Ergodic Theory and Dynamical Systems **5** (1985), 437–484.

[Hu2] S. Hurder, *Deformation rigidity for subgroups of $SL(n,\mathbb{Z})$ acting on the n-torus*, Bulletin of the American Mathematical Society **23** (1990), 107–113.

[Hu3] S. Hurder, *Rigidity for Anosov actions of higher-rank lattices*, preprint (1991).

[H-K-L-Z] S.Hurder, A.Katok, J.Lewis and R.Zimmer, *Rigidity for Cartan actions of higher-rank lattices*, preprint (1991).

[I] H.-C. Im Hof, *An Anosov action on the bundle of Weyl chambers*, Ergod. Th. and Dynam. Sys. **5** (1985), 455-464.

[J] J-L. Journé, *On a regularity problem occurring on connection with Anosov diffeomorphisms*, Communication in Mathematical Physics **106** (1986), 345–351.

[K-L1] A. Katok and J. Lewis, *Global rigidity results for lattice actions on tori and new examples of volume-preserving actions*, to appear. (1991).

[K-L2] A. Katok and J. Lewis, *Local rigidity for certain groups of toral automorphisms*, Israel Math. Jour. **75** (1991), 203–241.

[K-S] A. Katok and R. Spatzier, *Differentiable rigidity of Abelian Anosov actions*, preprint (1991).

[K] N. Koppel, *Commuting diffeomorphisms*, in A.M.S. Proceedings of symposia in pure mathematics, vol. 14, 1970, pp. 165–184.

[L] J. Lewis, *Infinitesimal rigidity for the action of $SL(n,\mathbb{Z}^n)$ on $\mathbb{T}^n$*, Transactions of the American Mathematical Society **324** (1991), 421–445.

[M1] G.A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, New York, 1991.

[M2] G.A. Margulis, *Certain measures associated with U-flows on compact manifolds*, Func. Anal. and Appl. **4** (1970), 55–67.

[N] D.G. Northcott, *Multilinear algebra*, Cambridge University Press, New York, 1984.

[Pa-Y] J. Palis and J.C. Yoccoz, *Centralizers of Anosov diffeomorphisms on tori*, Ann. Scient. Èc.Norm. Sup. **22** (1989), 99–108.

[Pl] J. Plante, *Anosov flows*, American J. of Math. **94** (1972), 729–754.

[Pr-R] G.Prasad and M.S.Raghunathan, *Cartan subgroups and lattices in semisimple groups*, Ann. Math. **96** (1972), 296–317.

[P-S1] C. Pugh and M. Shub, *Ergodicity of Anosov actions*, Inventiones Math. **15** (1972), 1–23.

[P-S2] C. Pugh and M. Shub, *Ergodic elements of ergodic actions*, Composito Mathematica **23** (1971), 115–122.

[Ra] M.S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, New York, 1972.

[Ro] V.A.Rohlin, *On the fundamental ideas in measure theory*, A.M.S. Transl. 1 **10** (1962), 1–54.

[R1] D. Ruelle, *Integral representation of measures associated with a foliation*, Publ. IHES **181** (1977), 127–132.

[R2] D. Ruelle, *Invariant measures for a diffeomorphism which expands the leaves of a foliation*, Publ. IHES **181** (1977), 133–135.

[Sh] M. Shub, *Global stability of dynamical systems*, Springer-Verlag, New York, 1987.

[Si] Ya. Sinai, *Markov partitions and C-diffeomorphisms*, Funct. Anal. Appl. **2** (1968), 61–82.

[Sm] S. Smale, *Differentiable dynamical systems*, Bulletin of the American Mathematical Society **73** (1967), 747–817.

[Ste] R. Steinberg, *Some consequences of the elementary relations in SL_n*, in Contemporary Mathematics, vol. 45, 1985, pp. 335–350.

[St] D. Stowe, *The stationary set of a group action*, Proceedings of the American Mathematical Society **79** (1980), 139–146.

[Su] M. Sugiura, *Unitary representations and harmonic analysis: an introduction*, North-Holland/Kodansa, New York, 1990.

[Sun] S. Sundaram, *Tableaux in the representation theory of the classical Lie groups*, in Invariant theory and tableaux, Springer, New York, 1990, pp. 191–225.

[T] P. Tomter, *Anosov flows on infrahomogeneous spaces*, in Proceedings of symposia in pure mathematics, vol. 14, 1970, pp. 299–328.

[V] V.S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Springer-Verlag, New York, 1984.

[W] A. Weil, *Remarks on the cohomology of groups*, Ann. of Math. **80** (1964), 149–157.

[Z1] R. Zimmer, *Actions of semisimple groups and discrete subgroups*, in Proc. Internat. Congr. Math. (Berkeley, Calif., 1986), A.M.S., Providence, RI., 1987, pp. 1247–1258.

[Z2] R. Zimmer, *Ergodic theory and semisimple groups*, Birkhauser, Boston, 1984.