HIGHER ORDER APPROXIMATIONS FOR TRANSONIC FLOW OVER SLENDER BODIES

Thesis by

N. Srinivasan

In Partial Fulfillment

of the Requirements for the Degree of

Aeronautical Engineer

California Institute of Technology
Pasadena, California

October, 1947

CONTENTS

ACKNOWLEDGMENT

The author wishes to thank Dr. Hans W. Liepmann for directing the research, Dr. Charles DePrima for valuable discussions, and Mr. Julian Cole for essential assistance in the preparation of this work.

I. INTRODUCTION

The purpose of this paper is to discuss an approximate method for obtaining the velocity potential for a two-dimensional compressible fluid flow. The effect of wind tunnel walls is also to be investigated.

The fundamental equation of motion governing the motion of a compressible fluid in two-dimensions is:

$$(a^2 - u^2)\frac{\partial^2 \Phi}{\partial x^2} - 2uv\frac{\partial^2 \Phi}{\partial x \partial y} + (a^2 - v^2)\frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad (1)$$

Where: a = the local velocity of sound

u = the stream velocity along the x-axis

v = the stream velocity along the y-axis

$\Phi(x,y)$ = the velocity potential.

The above equation of motion is non-linear. It is hyperbolic in type if $u^2+v^2 > a^2$ and elliptic if $u^2+v^2 < a^2$. The equation has not been solved in general, essentially due to its non-linearity.

The equation becomes linear by transformation from the physical plane to a hodograph plane. This transformation leads to a linear equation with the velocity components as the independent variables instead of the usual physical space co-ordinates. This simplification is achieved however, at the cost of more difficult boundary conditions. The linearity of these hodograph equations in incompressible fluid theory make them very valuable for the solution of certain problems.

Exact solutions to the hodograph equations have been given in several papers e.g., Tsien and Kuo (Ref. 1) (but a paper soon to be published by Guderley doubts even these results). The hodograph method has also been used as the basis for the approximate method of Karman and Tsien (Ref. 2) which is accurate for low subsonic speeds, but breaks down as soon as the flow becomes locally supersonic.

Garrick and Kaplan (Ref. 3) have developed a "velocity correction method" to give an approximate correspondence between the compressible and incompressible flow conditions. The known approximate results of Karman and Tsien, Temple and Yarwood, and Prandtl and Glauert are unified in the above reference. Tables and figures giving velocity and pressure coefficient correction factors are presented in order to facilitate the practical application of the results.

Though the exact non-linear equation has not, as yet, been solved in the physical plane and the linear hodograph equations offer the difficulties mentioned above, several approximate methods of solution for equation (1) have been developed by various authors. One of these methods is the development of the velocity-potential in powers of a thickness parameter.

The Prandtl-Glauert method of small perturbations can be regarded as the first term in such an expansion. Hantzsche and Wendt (Ref. 4) have considered further terms and Kaplan has used a similar iteration method to obtain the flow past a curved surface. Gortler (Ref. 5) also used an iteration method to obtain higher approximations to flow past a wave-shaped wall.

A series expansion is used in this paper and hence is given in some detail later.

Another approximate method which has been used is the Rayleigh-Janzen method. The velocity potential Φ is expanded in a series of the powers of the free-stream Mach number.

$$\Phi = \Phi_0 + \Phi_1 M^2 + \Phi_2 M^4 + \Phi_3 M^6 + \cdots\cdots$$

Where $M = \frac{u_\infty}{a_\infty}$, and Φ_0 is the velocity potential of the incompressible flow. On substituting this value of Φ in the exact equation (1), the coefficients of the powers of M give sets of equations for Φ_1, Φ_2, Φ_3 etc. It assumes that the velocity potential of the incompressible flow over the body is known and successive approximations correct that potential for compressibility effects. The solution converges quickly enough only for bluff bodies. Kaplan (Ref. 6) applied this method to determine the effect of compressibility for the flow over the surface of an elliptic cylinder.

All the first order approximations are valid only for flows at low subsonic Mach numbers. In the Prandtl-Glauert method the assumption of small disturbances is not always satisfied; e.g., the theory breaks down at the stagnation point. Also the approximation deteriorates as the free stream velocity increases and as the thickness or the camber of the body increases. However, as a first approximation those methods help to solve the many practical problems.

The effect of the higher order terms in the power series expansion in terms of thickness ratio is investigated here. A simple case is worked out fully and the effect of wind tunnel walls is also shown. The method is also indicated for bodies of arbitrary shape.

II. THE APPROXIMATE PROBLEM

The exact equation for a steady, isentropic two-dimensional compressible fluid flow is:

$$\left(a^2-u^2\right)\frac{\partial^2\Phi}{\partial x^2} - 2uv\frac{\partial^2\Phi}{\partial x\partial y} + \left(a^2-v^2\right)\frac{\partial^2\Phi}{\partial y^2} = 0 \tag{1}$$

Where: a = the local velocity of sound

$u = \frac{\partial\Phi}{\partial x}$ = the local stream velocity along x-axis

$v = \frac{\partial\Phi}{\partial y}$ = the local stream velocity along y-axis

$\Phi(x,y)$ = the velocity potential.

The energy equation is:

$$\frac{W^2}{2} + \frac{a^2}{\gamma-1} = \frac{U^2}{2} + \frac{a_\infty^2}{\gamma-1} \tag{2}$$

Where: W = the local stream velocity

a_∞ = the velocity of sound in the free-stream

U = the free-stream velocity taken parallel to the x-axis

γ = the ratio of specific heats.

Substituting for a^2 from (2) in (1)

$$\left\{a_\infty^2 + \frac{\gamma-1}{2}\left[U^2-\left(\Phi_x^2+\Phi_y^2\right)\right] - \Phi_x^2\right\}\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + \left\{a_\infty^2 + \frac{\gamma-1}{2}\left[U^2-\left(\Phi_x^2+\Phi_y^2\right)\right] - \Phi_y^2\right\}\Phi_{yy} = 0 \tag{3}$$

It is assumed that this exact equation must be solved with the following "exact" boundary conditions:

i) The local velocity is tangential to the body at any point; i.e., $\Phi_y(x,\tau f(x)) = \Phi_x(x,\tau f(x))\cdot \tau f'(x)$ where the shape of the body is given by $y = \tau f(x)$ and τ is a dimensionless parameter depending on the thickness of the body.

ii) No disturbances at infinity.

Let the velocity potential Φ be made up of a uniform rectilinear flow and perturbation potentials $\tau\phi_1, \tau^2\phi_2, \tau^3\phi_3 \text{-----} \tau^n\phi_n$.

$$\Phi = Ux + \sum_{n=1}^{\infty} \tau^n \phi_n \tag{4}$$

So that as $\tau \longrightarrow 0$, $\Phi \longrightarrow Ux$.

Equation (4) is assumed to converge for small enough values of τ.

Substituting for the derivatives of Φ in (3) by using (4)

$$\Big\{a_\infty^2 + \frac{\gamma-1}{2}\Big[U^2 - \big\{U^2 + \tau^2\phi_{1x}^2 + 2U\tau\phi_{1x} + \tau^2\phi_{2x} + \cdots\big\}\Big] - \Big[U^2 + \tau^2\phi_{1x}^2 + 2U\tau\phi_{1x} + \tau^2\phi_{2x}2U + \tau^2\phi_{1y}^2 + \cdots\Big]\Big\}\big(\tau\phi_{1xx} + \tau^2\phi_{2xx} + \cdots\big)$$

$$-\,2\big\{U + \tau\phi_{1x} + \tau^2\phi_{2x} + \cdots\big\}\big\{\tau\phi_{1y} + \tau^2\phi_{2y} + \cdots\big\}\big\{\tau\phi_{1xy} + \tau^2\phi_{2xy} + \cdots\big\}$$

$$+\Big\{a_\infty^2 + \frac{\gamma-1}{2}\Big[U^2 - \big\{U^2 + \tau^2\phi_{1x}^2 + 2U\tau\phi_{1x} + \tau^2\phi_{1y}^2 + 2U\tau^2\phi_{2x} + \cdots\big\}\Big] - \Big[\tau^2\phi_{1y}^2 + \cdots\Big]\Big\}\big(\tau\phi_{1yy} + \tau^2\phi_{2yy} + \cdots\big) = 0 \tag{5}$$

Similarly substituting for the boundary condition (i) of (3)

$$\tau\phi_{1y}(x,y) + \tau^2\phi_{2y}(x,y) + \tau^3\phi_{3y}(x,y) + \cdots = \Big[U + \tau\phi_{1x}(x,y) + \tau^2\phi_{2x}(x,y) + \tau^3\phi_{3y}(x,y) + \cdots\Big]\tau f'(x)$$

Expanding the derivatives of Φ by Taylor Series and substituting for $y = \tau f(x)$ the general boundary condition of the problem is:

$$\tau\left[\phi_{1y}(x,0)+\tau f(x)\,\phi_{1yy}(x,0)+\frac{\tau^2 f^2(x)}{2!}\phi_{1yyy}(x,0)+\cdots\right]+\tau^2\left[\phi_{2y}(x,0)+\tau f(x)\,\phi_{2yy}(x,0)+\frac{\tau^2 f^2(x)}{2!}\phi_{2yyy}(x,0)+\cdots\right]$$

$$+\tau^3\left[\phi_{3y}(x,0)+\tau f(x)\,\phi_{3yy}(x,0)+\cdots\right]=\Big\{U+\tau\left[\phi_{1x}(x,0)+\tau\phi_{1xy}(x,0)f(x)+\frac{\tau^2 f^2(x)}{2!}\phi_{1xyy}(x,0)+\cdots\right]$$

$$+\tau^2\left[\phi_{2x}(x,0)+\tau f(x)\,\phi_{2xy}(x,0)+\cdots\right]+\tau^3\left[\phi_{3x}(x,0)+\cdots\right]\Big\}\,\tau f'(x). \tag{6}$$

Collecting the coefficients of the powers of τ in (5) and (6), the three sets of equations for the first, second, and third order perturbation potentials with their respective boundary conditions are:

(A)

$$(1-M_\infty^2)\,\phi_{1xx}+\phi_{1yy}=0 \qquad \text{(i)}$$

$$\phi_{1y}(x,0)=U f'(x) \qquad \text{(ii)} \qquad \text{(A)}$$

$$\phi_{1y}(\infty)=\phi_{1x}(\infty)=0 \qquad \text{(iii)}$$

(B)

$$(1-M_\infty^2)\,\phi_{2xx}+\phi_{2yy}=\frac{2M_\infty}{a_\infty}\left[\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\phi_{1x}\phi_{1xx}+\phi_{1y}\phi_{1xy}\right] \qquad \text{(i)}$$

$$\phi_{2y}(x,0)=\phi_{1x}(x,0)f'(x)-\phi_{1yy}(x,0)f(x). \qquad \text{(ii)} \qquad \text{(B)}$$

$$\phi_{2y}(\infty)=\phi_{2x}(\infty)=0. \qquad \text{(iii)}$$

(C)

$$(1-M_\infty^2)\,\phi_{3xx}+\phi_{3yy}=\frac{1}{a_\infty^2}\Bigg[\frac{\gamma-1}{2}\Big\{(1+m^2)\left[\phi_{1x}^2\phi_{1xx}+2U\phi_{2x}\phi_{1xx}\right]+2U\phi_{1x}\{\phi_{2xx}+\phi_{2yy}\}\Big\}+2U\left[\phi_{1x}\phi_{2xy}+\phi_{2y}\phi_{1xy}+\frac{1}{U}\phi_{1x}^2\phi_{1xy}\right]$$

$$+\left[\phi_{1x}^2\phi_{1xx}+\phi_{1y}^2\phi_{1yy}+\frac{\gamma+1}{2}\phi_{1y}^2\phi_{1xx}\right]\Bigg] \qquad \text{(i)}$$

$$\phi_{3y}(x,0)=\phi_{2x}(x,0)f'(x)+\phi_{1xy}(x,0)f(x)f'(x)-\phi_{1yyy}(x,0)\frac{f^2(x)}{2!}-\phi_{2yy}(x,0)f(x). \qquad \text{(ii)} \qquad \text{(C)}$$

$$\phi_{3x}(\infty)=\phi_{3y}(\infty)=0. \qquad \text{(iii)}$$

The solutions of the above set of equations with the corresponding boundary conditions for an arbitrary body shape, give the values of the perturbation potentials ϕ_1, ϕ_2, ϕ_3 etc.

In a similar way one may find an expression for Mach number, which up to terms of τ^2 is

$$M^2(x,y) = M_\infty^2\left[1+\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\left\{\frac{2}{U}\tau\phi_{1x}+\frac{2}{U}\tau^2\phi_{2x}+\frac{\tau^2}{U^2}\left[\phi_{1y}^2+\phi_{1x}^2\left\{1+2(\gamma-1)M_\infty^2\right\}\right]\right\}\right] \quad (7)$$

and M^2 on the body

$$M^2(x,0) = M_\infty^2\left[1+\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\left\{\frac{2}{U}\tau\phi_{1x}(x,0)+\frac{2\tau^2}{U}\left[\phi_{2x}(x,0)+f(x)\phi_{1xy}(x,0)\right]\right.\right.$$
$$\left.\left.+\frac{\tau^2}{U^2}\left[\phi_{1y}^2(x,0)+\phi_{1x}^2(x,0)\left\{1+2(\gamma-1)M_\infty^2\right\}\right]\right\}\right] \quad (8)$$

To the same order the expression for C_p the pressure coefficient

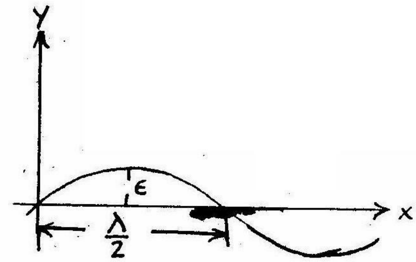
$$C_p(x,0) = \frac{p-p_\infty}{\frac{1}{2}\rho U^2}$$

$$C_p(x,0) = -\frac{2\tau}{U}\phi_{1x}(x,0)-\frac{2\tau^2}{U}\left[\phi_{2x}(x,0)+f(x)\phi_{1xy}(x,0)\right]-\frac{\tau^2}{U^2}\left[\phi_{1y}^2(x,0)+\phi_{1x}^2(x,0)\left\{1+2(\gamma-1)M_\infty^2\right\}\right] \quad (9)$$

The derivativions of the above expressions for M^2 and C_p are shown in the appendix.

III. SOLUTION OF SIMPLE PROBLEM

a) A Wave-Shaped Wall in Free-Stream



Let $f(x) = \frac{\pi}{\alpha} \sin \alpha x$ where $\alpha = \frac{2\pi}{\lambda}$ and $\tau = \frac{\epsilon\alpha}{\pi}$

or $y = \tau f(x) = \epsilon \sin \alpha x$.

The flow equations and the boundary conditions for Φ are from (A)

$$(1 - M_\infty^2)\phi_{1xx} + \phi_{1yy} = 0 \qquad \text{(i)}$$

$$\phi_{1y}(x,0) = U\pi \cos \alpha x \qquad \text{(ii)} \qquad (A_1)$$

$$\phi_{1x}(\infty) = \phi_{1y}(\infty) = 0 \qquad \text{(iii)}$$

A solution for $\phi_1(x,y)$ is expressed as

$$\phi_1(x,y) = \begin{Bmatrix} \sin \chi x \\ \cos \chi x \end{Bmatrix} \begin{Bmatrix} e^{-m\chi y} \\ e^{+m\chi y} \end{Bmatrix}$$

where χ is an arbitrary constant and $m = \sqrt{1 - M_\infty^2}$

and of course any linear combination of such solutions could be used to satisfy the boundary values.

For the simple case, only one term is needed to satisfy the boundary conditions (ii and iii), and putting $\chi = \alpha$ we write

$$\phi_1(x,y) = -\frac{U\pi}{m\alpha} \cos \alpha x \, e^{-m\alpha y}. \qquad (10)$$

Substituting for the derivatives of ϕ_1 in (B i and ii)

$$m^2\phi_{2xx} + \phi_{2yy} = \frac{\gamma+1}{2}\frac{M_\infty^4\pi^2}{m^2}\alpha U \sin 2\alpha x\, e^{-2m\alpha y} \qquad \text{(i)}$$

$$\phi_{2y}(x,0) = \frac{U\pi^2(1+m^2)}{2m}\sin 2\alpha x\,. \qquad \text{(ii)} \qquad (B_2)$$

$$\phi_{2x}(\infty) = \phi_{2y}(\infty) = 0\,. \qquad \text{(iii)}$$

A particular solution of ϕ_2 is easily obtained as

$$\phi_2(x,y) = -\frac{\gamma+1}{8}\frac{M_\infty^4\pi^2}{m^3}U y \sin 2\alpha x\, e^{-2m\alpha y} \qquad (11)$$

The complete solution which damps out at ∞ is written as

$$\phi_2(x,y) = \sum\left(a_n \sin n\alpha x + b_n \cos n\alpha x\right)e^{-nm\alpha y} - \frac{\gamma+1}{8}\frac{M_\infty^4\pi^2}{m^3}U y \sin 2\alpha x\, e^{-2m\alpha y}$$

where the a_n and b_n are still undetermined.

To satisfy the boundary condition (B_2 iii)

$b_n = 0$;

$a_n = 0$ for all values of n except for $n = 2$.

When $n = 2$, a_2 can be found from (B_2 ii) which is

$$\phi_{2y}(x,0) = \left[-2m\alpha a_2 - \frac{\gamma+1}{8}\frac{M_\infty^4\pi^2}{m^3}U\right]\sin 2\alpha x = \frac{\pi^2 U(1+m^2)}{2m}\sin 2\alpha x$$

so that

$$a_2 = -\frac{\pi^2 U}{4m^2\alpha}\left[1 + m^2 + \frac{\gamma+1}{4}\frac{M_\infty^4}{m^2}\right] \qquad (12)$$

Then

$$\phi_2(x,y) = \frac{-\pi^2 U}{4m^2\alpha}\left[1+m^2+\frac{\gamma+1}{4}\frac{M_\infty^4}{m^2}+\frac{\gamma+1}{2}\frac{M_\infty^4}{m}\alpha y\right]\sin 2\alpha x\, e^{-2m\alpha y} \tag{13}$$

The velocity potential $\Phi(x,y)$ up to terms of τ^2 is expressed as:

$$\Phi(x,y) = Ux - \tau\frac{U\pi}{m\alpha}\cos\alpha x\, e^{-m\alpha y} - \tau^2\frac{U\pi^2}{4m^2\alpha}\left[1+m^2+\frac{\gamma+1}{4}\frac{M_\infty^4}{m^2}+\frac{\gamma+1}{2}\frac{M_\infty^4}{m}\alpha y\right]\sin 2\alpha x\, e^{-2m\alpha y} \tag{14}$$

Formulas for the velocity components in space and on the body are:

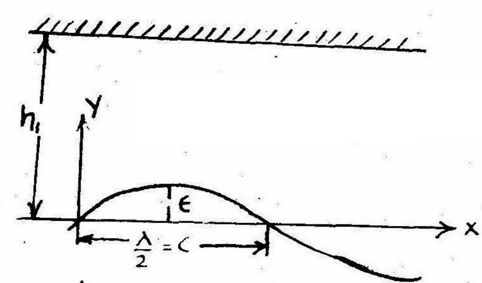
$$u(x,y) = U + \tau\cdot\frac{\pi U}{m}\sin\alpha x\, e^{-m\alpha y} - \tau^2\frac{\pi^2 U}{2m^2}\left[1+m^2+\frac{\gamma+1}{4}\frac{M_\infty^4}{m^2}+\frac{\gamma+1}{2}\frac{M_\infty^4}{m}\alpha y\right]\cos 2\alpha x\, e^{-2m\alpha y}$$

$$v(x,y) = \tau\cdot\pi U\cos\alpha x\, e^{-m\alpha y} + \tau^2\frac{\pi^2 U}{2m}\left[1+m^2+\frac{\gamma+1}{4}\frac{M_\infty^4}{m^2}+\frac{\gamma+1}{2}\frac{M_\infty^4}{m}\alpha\,(y-2m\alpha)\right]\sin 2\alpha x\, e^{-2m\alpha y}$$

$$u(x,0) = U + \tau\frac{\pi U}{m}\sin\alpha x - \tau^2\frac{\pi^2 U}{2m^2}\left[1+m^2+\frac{\gamma+1}{4}\frac{M_\infty^4}{m^2}\right]\cos 2\alpha x - \tau^2 f(x)\,\pi U\alpha\sin\alpha x.$$

$$v(x,0) = \tau\,\pi U\cos\alpha x - \frac{\tau^2 U\pi^2}{2m}\left[1+m^2+\frac{\gamma+1}{4}\frac{M_\infty^4}{m^2}\right]\sin 2\alpha x - \tau^2 f(x)\,U\pi m\alpha\cos\alpha x.$$

b) <u>A Wave-Shaped Wall in a Closed Tunnel</u>



$$y(x) = \frac{\pi}{\alpha}\sin\alpha x$$

and the height of the tunnel wall from the x-axis is $= h_1$

The boundary condition (A ii) in this case is

$$\phi_{1y}(x, h_1) = 0 \tag{A_3 ii}$$

The solution of (A i) is easily obtained as

$$\phi_1(x,y) = -\frac{U\pi}{m\alpha\, sh(m\alpha h_1)} \cos\alpha x \cosh m\alpha(h_1-y) \tag{15}$$

Substituting for the derivatives of ϕ in (B i and ii)

$$m^2\phi_{2xx} + \phi_{2yy} = \frac{M_\infty^2 \pi^2}{m^2 sh^2(m\alpha h_1)} \alpha U \left[\frac{\gamma+1}{4} M_\infty^2 \cosh 2m\alpha(h_1-y) + \left(\frac{\gamma-3}{4} M_\infty^2 + 1\right)\right] \sin 2\alpha x \quad \text{(i)}$$

$$\phi_{2y}(x,0) = \frac{U\pi^2(1+m^2)}{2m} \coth(m\alpha h_1) \sin 2\alpha x \quad \text{(ii)} \qquad (B_3)$$

$$\phi_{2y}(\infty) = 0. \quad \text{(iii)}$$

A particular solution of (B_3 i) is

$$\phi_2(x,y) = \left[A_1 y\, sh\, 2m\alpha(h_1-y) + B_1 sh^2 m\alpha(h_1-y)\right] \sin 2\alpha x$$

where

$$A_1 = -\frac{\gamma+1}{16} \frac{M_\infty^4 \pi^2}{m^3 sh^2(m\alpha h_1)} U$$

and

$$B_1 = -\left(\frac{\gamma-3}{4} M_\infty^2 + 1\right) \frac{M_\infty^2 \pi^2}{2m^4 \alpha\, sh^2(m\alpha h_1)} U \tag{16}$$

The complete solution for ϕ_2 can be written in the form

$$\phi_2(x,y) = \sum \left(a_n' \sin n\alpha x + b_n' \cos n\alpha x\right) sh\, 2m\alpha(h_1-y)$$
$$+ \left[A_1 y\, sh\, 2m\alpha(h_1-y) + B_1 sh^2 m\alpha(h_1-y)\right] \sin 2\alpha x. \tag{17}$$

where a'_n and b'_n are undetermined.

To satisfy the boundary condition (B_3 ii)

$b'_n = 0$

$a'_n = 0$ for all values of n, except for $n = 2\alpha$

When $n = 2\alpha$

$$\phi_{2Y}(x,Y) = \left[-2m\alpha a'_2 \cosh 2m\alpha(h_1-Y) + A_1 \sinh 2m\alpha(h_1-Y) - B_1 m\alpha \sinh 2m\alpha(h_1-Y)\right] \sin 2\alpha x$$

$$= \frac{U\pi^2(1+m^2)}{2m} \coth m\alpha h_1 \sin 2\alpha x; \quad \text{for } Y=0.$$

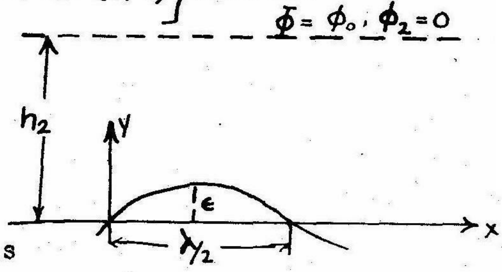
$$a'_2 = U\frac{M_\infty^2 \pi^2}{4m^4\alpha}\left(1+\frac{\gamma-7}{8}M_\infty^2\right)\frac{\tanh 2m\alpha h_1}{\sinh^2 m\alpha h_1} - U\frac{1+m^2}{4m^2\alpha}\pi^2\frac{\coth m\alpha h_1}{\cosh 2m\alpha h_1} \tag{18}$$

$$\phi_2(x,Y) = \left[(a'_2 + A_1 Y)\sinh 2m\alpha(h_1-Y) + B_1 \sinh^2 m\alpha(h_1-Y)\right]\sin 2\alpha x \tag{19}$$

The total velocity potential Φ is

$$\Phi(x,Y) = Ux - \frac{U\pi}{m\alpha \sinh(m\alpha h_1)}\cos\alpha x \cosh m\alpha(h_1-Y) + \left[(a'_2+A_1Y)\sinh 2m\alpha(h_1-Y) + B_1\sinh^2 m\alpha(h_1-Y)\right]\sin 2\alpha x. \tag{20}$$

c) <u>A Wave-Shaped Wall in an Open Tunnel</u>



Let the height of the free-jet where there is no pressure difference be $= h_2$.

The boundary condition (A ii) in this case is

$$\phi_{1Y}(x, h_2) = 0 \tag{A_4 ii}$$

The solution for ϕ_1 is

$$\phi_1(x,y) = -\frac{U\pi}{m\alpha \cosh(m\alpha h_2)} \cos\alpha x \, Sh\, m\alpha(h_2 - y). \tag{21}$$

Substituting for the derivatives of ϕ_1 in (B i and ii)

$$m^2\phi_{2xx} + \phi_{2yy} = \frac{M_\infty^2 \pi^2}{m^2 \cosh^2 m\alpha h_2} \alpha U \left[\frac{\gamma+1}{4} M_\infty^2 \cosh 2m\alpha(h_2-y) - \left(\frac{\gamma+3}{4} M_\infty^2 + 1\right)\right] \sin 2\alpha x \qquad \text{(i)}$$

$$\phi_{2y}(x,0) = \frac{\pi^2(1+m^2)}{2m \cosh m\alpha h_2} U \, Sh(m\alpha h_2) \sin 2\alpha x. \qquad \text{(ii)} \qquad (B_4)$$

$$\phi_{2y}(\infty) = 0. \qquad \text{(iii)}$$

A particular solution of (B_4 i) is

$$\phi_2(x,y) = \left[A_2 y \, Sh\, 2m\alpha(h_2-y) + B_2 Sh^2 m\alpha(h_2-y)\right] \sin 2\alpha x. \tag{22}$$

where

$$A_2 = -\frac{\gamma+1}{16} \frac{M_\infty^4 \pi^2}{m^3 \cosh^2(m\alpha h_2)} U$$

$$B_2 = +\left(\frac{\gamma-3}{4} M_\infty^2 + 1\right) \frac{M_\infty^2 \pi^2}{2m^4 \alpha \cosh^2 m\alpha h_2} U$$

The complete solution of (B_4 i) to satisfy (B_4 iii) is written as

$$\phi_2(x,y) = \sum \left(a_n'' \sin n\alpha x + b_n'' \cos n\alpha x\right) Sh\, 2m\alpha(h_2 - y)$$

$$+ \left[A_2 y \, Sh\, 2m\alpha(h_2-y) + B_2 Sh^2 m\alpha(h_2-y)\right] \sin 2\alpha x \tag{23}$$

Satisfying the boundary condition (B_4 ii)

$$a_2'' = -U\frac{M_\infty^2\pi^2}{4m^4\alpha}\left(1+\frac{3\gamma-5}{8}M_\infty^2\right)\frac{\tanh 2m\alpha h_2}{\cosh^2 m\alpha h_2} - \frac{U\pi^2(1+m^2)}{4m^2\alpha}\frac{\tanh m\alpha h_2}{\cosh 2m\alpha h_2} \quad (24)$$

$$\Phi_2(x,y) = \left[\left(a_2''+Ay\right)\operatorname{sh} 2m\alpha(h_2-y)+B_2\operatorname{sh}^2 m\alpha(h_2-y)\right]\sin 2\alpha x . \quad (25)$$

The total velocity potential Φ is

$$\Phi(x,y) = Ux - \tau\frac{U\pi}{m\alpha\cosh(m\alpha h_2)}\cos\alpha x\,\operatorname{sh} 2m\alpha(h_2-y)$$

$$+\left[\left(a_2''+A_2y\right)\operatorname{sh} 2m\alpha(h_2-y)+B_2\operatorname{sh}^2 m\alpha(h_2-y)\right]\sin 2\alpha x . \quad (26)$$

IV. THE SOLUTION FOR A BODY OF ARBITRARY SHAPE

An attempt can be made to solve the approximate problem for the body of arbitrary shape by extending the reasoning of II by means of Fourier Analysis. The first approximation is thus easily obtained for the free stream case (and in fact could be found even for a closed or open channel (Ref. 7)). However the second approximation is harder to obtain even for bodies of very simple shape.

Thus problem (A)

$$
\left.
\begin{aligned}
&m^2 \phi_{1xx} + \phi_{1yy} = 0 && \text{(i)}\\
&\phi_{1y}(x,0) = U f'(x) && \text{(ii)}\\
&\phi_{1y}(\infty) = 0 && \text{(iii)}
\end{aligned}
\right\} \quad \text{(A)}
$$

has the solution $\phi_1(x,y)$ which can be represented almost everywhere by an integral

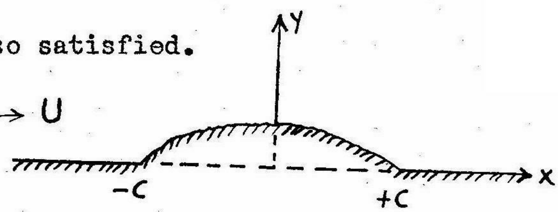
$$\phi_1(x,y) = -\frac{U}{\pi m}\int_0^{\infty}\frac{d\lambda}{\lambda}\int_{-\infty}^{\infty} e^{-m\lambda y} f'(t)\cos\lambda(t-x)\,dt \tag{28}$$

(28) satifies (A i, iii) and since

$$\phi_{1y}(x,0) = +\frac{U}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{+\infty} f'(t)\cos\lambda(t-x)\,dt = \frac{U}{2}\left\{f'(x-\epsilon)+f'(x+\epsilon)\right\} \tag{29}$$

by Fourier's integral theorem, (A iii) is also satisfied.

If the airfoil lies between



$$
\begin{aligned}
f'(x) &= \psi(x) \quad \text{for } |x| < c\\
&= 0 \quad \text{for } |x| > c
\end{aligned} \tag{30}
$$

and the order of integration in (28) may be changed, the potential

$$\phi_1(x,y) = \frac{U}{\pi m}\int_{-c}^{+c}\frac{(t-x)\psi(t)}{(t-x)^2+m^2y^2}\,dt \tag{31}$$

Thus, to obtain the second approximation, problem (B) must be solved.

$$\left.\begin{aligned} m^2\phi_{2xx} + \phi_{2yy} &= F[\phi_1(x,y)] && \text{(i)} \\ \phi_{2y} &= G[\phi_1(x,y)] && \text{(ii)} \\ \phi_{2y}(\infty) &= 0\,. && \text{(iii)} \end{aligned}\right\} \quad \text{(B)}$$

After the first approximation ϕ_1 is found, the right hand sides of (B i, ii) are known functions of x, y. Thus the problem is that of solutions of Poisson's equation with certain boundary conditions. By putting

$$\phi_2(x,y) = k(x,y) + g(x,y) \tag{32}$$

where

$$\left.\begin{aligned} m^2 g_{xx} + g_{yy} &= 0 \\ \text{and} \quad g_y(x,0) &= G[\phi_1(x)] \\ g_y(\infty) &= 0\,, \end{aligned}\right\} \tag{33}$$

the problem is reduced to finding a solution to Poisson's equation where y derivatives vanish on the boundary

$$m^2 K_{xx} + K_{yy} = F\left[\phi(x,y)\right]$$

$$K_y(x,0) = K_y(\infty) = 0$$

This may be done by constructing the Greens' function. (33) of course, can be solved just as (A).

In any practical case however, even for the simplest airfoils, the function F is so complicated that this integration procedure has not been carried out explicitily.

V. DISCUSSION OF METHOD AND RESULTS

By the method of power series the solutions of the non-linear equation (B) under certain boundary conditions was approximated by the successive solution of a set of potential problems. As long as the flow is everywhere subsonic, equation (B) is elliptic and the boundary conditions (i and ii), of the elliptic type, are probably sufficient. However, there is some doubt, as expressed by Guderley (Ref. 8) and others, whether the problem is properly stated if there is a local supersonic zone. Guderley concludes that potential flow must break down, or that the series approximation (4) diverges. It should be noted here that integrations of this type are very important, for one never knows by virtue of the non-linearity, whether a local supersonic zone will form. However, from another point of view, experimental results (Ref. 9) do show smooth shock-free transonic flow for a certain range of Mach numbers close to 1. It may be that the exact question of break-down of flow, due to viscosity effects etc., has to be investigated near $M = 1$. In addition a supersonic zone imbedded in a subsonic flow can behave in an elliptic way, for, a small disturbance in it, can affect the entire flow field. Thus it is still felt that solutions may be approximated in this way for local Mach numbers close enough to 1.

Since only the simplified problems were solved completely, the results shown in the curves should be taken as only quantitative for airfoils. Figure 1 shows the distribution of the local Mach number as calculated by the values of the first and second order potentials over two wavy walls of maximum thickness $\epsilon = 0.1$ and $\epsilon = 0.2$ in a free-stream of Mach number 0.8. The distribution is symmetrical over the body as calculated by the first and second order approximations. The first approximation gives a high value of M at the leading and trailing edges and lower value at the center, than the more accurate second order values. The difference between the two values at the maximum Mach number is about 3% for $\epsilon = 0.1$ and 10% for $\epsilon = 0.2$. Also, the difference increases for higher values of M_∞. This is in qualitative agreement with the experimental results at GALCIT transonic tunnel for a 12% circular arc airfoil where the distribution for $M_\infty = 0.8$ is less at the leading and trailing edges and more at the center than the calculated values by the small perturbation theory, (Ref. 7). The velocity distribution is known for two bodies of thickness $\epsilon = 0.1$ and $\epsilon = 0.2$ at $M_\infty = 0.8$. There is no supersonic zone for $\epsilon = 0.1$. As the thickness increases, a supersonic region is developed and there is a great variation between the two approximations. Of course the values obtained for $\epsilon = 0.1$ are closer to the actual conditions of the flow than for the body $\epsilon = 0.2$. The tunnel wall interference on the local velocity distribution over the body in a 10" tunnel as determined by second order approximation is shown in Figure 1. For a closed tunnel, the effect of the wall boundary is to increase the velocity over the body. The first order approximation indicates maximum influence at the center of the body. The second approximation gives maximum values at $\frac{x}{c} = 0.25$ & 0.75. It is min at the center $\frac{x}{c} = 0.5$

and at the end $\frac{x}{c} = 0$ & $1 \cdot 0$. For open tunnel, the effect is just the reverse; i.e., the velocity over the body decreases. For a tunnel of the height chosen the effect is apparently very small. However due to the peculiar problem which was solved, that of an infinite wall, the free stream Mach number is not defined.

Figure (2) indicates the variation of the maximum local Mach number with free-stream values for two bodies of maximum thickness $\epsilon = 0 \cdot 1$ and $\epsilon = 0 \cdot 2$. The difference between the first and second approximations, for low free-stream Mach numbers is negligible. For higher values of M_∞ there is a rapid increase in M due to ϕ_2 . In general, the linearized small perturbation theory gives a lower local Mach number and there is considerable error in neglecting the second perturbation potential for calculations at high subsonic Mach numbers. It is also evident the error increases as the thickness of the body increases. It can be seen from the form of the solutions that $\frac{\tau}{1 - M_\infty^2}$ is the important parameter.

Figures (3) and (4) show the variation of M with Y for $M_\infty = 0 \cdot 8$ and $0 \cdot 9$. They indicate the considerable error in calculating M close to the body by the first approximation. The error decreases for increasing distances along the y-axis until the two values reach the free-stream Mach number at large distances along the y-axis. The maximum height of the supersonic zone for $\epsilon = 0 \cdot 2$ and $M_\infty = 0 \cdot 9$ is $0 \cdot 5$ by the first approximation and about $1 \cdot 1$ by the second approximation.

Another approach, which is restricted to transonic flow, and should be the more accurate then, is the solution of an approximate non-linear equation. By assuming (Ref. 8)

$$u = a^* + \phi_x$$
$$v = \phi_y \tag{35}$$

where a^* sound velcotiy = flow velocity at $M = 1$, For thin bodies equation (3) is approximated by

$$\frac{\gamma + 1}{a^*} \phi_x \phi_{xx} - \phi_{yy} = 0 \tag{36}$$

The main advantage of (36) compared with (3) is its simplicity but the same fundamental difficulties of non-linearity and proper boundary conditions remain.

Thus, summarizing, the method of power series makes it possible to obtain approximations to some simple compressible flow problems up to the second order. Qualititively, it is shown that the first order theory, (Prandtl's rule) tends to underestimate the compressibility effects. A further investigation of the general problem, the proper boundary conditions, and the transonic flow equation seems desirable.

EXAMPLES

For the purpose of computations, a wavy wall with the following dimensions was chosen

$$\epsilon = 0.1 \quad \text{and} \quad 0.2$$

$$\lambda = 2C = 6.0$$

$$\alpha = \frac{2\pi}{\lambda} = 1.05 \; ; \quad \tau = \frac{\epsilon\alpha}{\pi}$$

$$M_\infty = 0.8$$

Figure 1. $M^2_{(\text{max on the body})}$ vs $\frac{X}{C}$

From (8)

$$M^2(x,0) = M_\infty^2\left[1 + \left(1 + \frac{\gamma-1}{2}M_\infty^2\right)\left\{\frac{2}{U}\tau\,\phi_{1x}(x,0) + \frac{2\tau^2}{U}\left[\phi_{2x}(x,0) + f(x)\,\phi_{1xy}(x,0)\right]\right.\right.$$
$$\left.\left. + \frac{\tau^2}{U^2}\left[\phi_{1y}^2(x,0) + \phi_{1x}^2(x,0)\left\{1 + 2(\gamma-1)M_\infty^2\right\}\right]\right\}\right].$$

$$\phi_1(x,y) = -\frac{U\pi}{m\alpha}\cos\alpha x\, e^{-m\alpha y}. \tag{10}$$

$$\phi_2(x,y) = -\frac{U\pi^2}{4m^2\alpha}\left[1 + m^2 + \frac{\gamma+1}{4}\frac{M_\infty^4}{m^2} + \frac{\gamma+1}{2}\frac{M_\infty^4}{m}\alpha y\right]\sin 2\alpha x\, e^{-2m\alpha y}. \tag{13}$$

Substituting for the derivatives of ϕ_1 and ϕ_2 in (8)

$$M^2(x,0) = M_\infty^2\left[1 + \left(1 + \frac{\gamma-1}{2}M_\infty^2\right)\left\{\frac{2\epsilon\alpha}{m}\sin\alpha x - \frac{\epsilon^2\alpha^2}{m^2}\left[1+m^2+\frac{\gamma+1}{4}\frac{M_\infty^4}{m^2}\right]\cos 2\alpha x\right.\right.$$

$$\left.\left. - 2\epsilon^2\alpha^2\sin^2\alpha x + \epsilon^2\alpha^2\cos^2\alpha x + \frac{\epsilon^2\alpha^2}{m^2}\sin^2\alpha x\left[1+2(\gamma-1)M_\infty^2\right]\right\}\right]$$

For $M_\infty = 0.8$ and $\epsilon = 0.2$, $\gamma = 1.405$,

$$M^2(x,0) = .64\left[1 + 1.128\left\{.7\sin\alpha x - .25\cos 2\alpha x + .053\sin^2\alpha x + 0.044\right\}\right].$$

At $\frac{x}{c} = 0.5$:

$$M^2_{(max\ on\ the\ body)} = .64\left[1 + 1.128\left\{.7 + .25 + .053 + .044\right\}\right] = 1.40.$$

By the first approximation

$$M^2(x,0) = M_\infty^2\left[1 + \left(1 + \frac{\gamma-1}{2}M_\infty^2\right)\left\{\frac{2\epsilon\alpha}{m}\sin\alpha x\right\}\right]$$

$$M^2_{max\,(1.5,0)} = .64\left[1 + 1.128\times.7\right] = 1.05.$$

The Computations are similarly carried out for different values of $\frac{x}{c}$.

With a Straight-Wall Boundary:-

$$\bar{\phi}_1(x,y) = -\frac{U\pi}{m\alpha\, Sh(m\alpha h_1)}\cos\alpha x\, Cosh.\, m\alpha(h_1-y)$$

$$\bar{\phi}_2(x,y) = \left[(a_2 + A_1 y)\, Sh\, 2m\alpha(h_1-y) + B_1\, Sh^2 m\alpha(h_1-y)\right]\sin 2\alpha x.$$

$$M^2_{Wall} - M^2_{F.S} = M_\infty^2\left[\left(1 + \frac{\gamma-1}{2}M_\infty^2\right)\left\{\frac{2}{U}\tau\left(\bar{\phi}_{1x}(x,0) - \phi_{1x}(x,0)\right) + \frac{2\tau^2}{U}\left[\bar{\phi}_{2x}(x,0) - \phi_{2x}(x,0) + f(x)\left\{\bar{\phi}_{1xy}(x,0) - \phi_{1xy}(x,0)\right\}\right]\right.\right.$$

$$\left.\left. + \frac{\tau^2}{U^2}\left[\bar{\phi}^2_{1y}(x,0) - \phi^2_{1y}(x,0) + \left\{\bar{\phi}^2_{1x}(x,0) - \phi^2_{1x}(x,0)\right\}\left\{1 + 2(\gamma-1)M_\infty^2\right\}\right]\right\}\right].$$

Substituting for the derivatives of $\bar{\phi}_1, \phi_1, \bar{\phi}_2$ & ϕ_2 for $M_\infty = 0.8$

$$\left[M^2_{Wall} - M^2_{F.S}\right]_{max} = .64\left[.003\sin\alpha x + .00298\cos 2\alpha x + .00295\sin^2\alpha x - .00034\right]$$

Maximum difference = .0021 by second approximation,
at $\frac{x}{c} = 0.5$

= .0019 by first approximation.

The values for various values of $\frac{x}{c}$ are plotted in Figure 1.

Figure 2 — M^2_{max} vs M_∞. (at $\frac{x}{c} = 0.5$).

$$M^2_{1\,max} = M^2_\infty\left[1 + \left(1 + \frac{\gamma-1}{2}M^2_\infty\right)\frac{2\epsilon\alpha}{\sqrt{1-M^2_\infty}}\right].$$

$$M^2_{2\,max} = M^2_\infty\left[1 + \left(1 + \frac{\gamma-1}{2}M^2_\infty\right)\left\{\frac{2\epsilon\alpha}{\sqrt{1-M^2_\infty}} + \frac{\epsilon^2\alpha^2}{1-M^2_\infty}\left[2 - M^2_\infty + \frac{\gamma+1}{4}\frac{M^4_\infty}{1-M^2_\infty}\right]\right.\right.$$

$$\left.\left. - 2\epsilon^2\alpha^2 + \left\{1 + 2(\gamma-1)M^2_\infty\right\}\frac{\epsilon^2\alpha^2}{1-M^2_\infty}\right\}\right].$$

Substituting for $\epsilon = 0.2$,

$M_\infty = 0.7$,

$$M^2_{1\,max} = 0.82;$$

$$M^2_{2\,max} = 0.91;$$

Figure 3

M^2 vs $\frac{y}{c}$; $M_\infty = 0.8$.

$$M^2_2(1.5, y) = .64\left[1 + 1.128\left\{.7e^{-m\alpha y} + (.25 + .053 + .044)e^{-2m\alpha y}\right\}\right]$$

For $\frac{y}{c} = \frac{1}{3}$; $M^2_1 = 0.910$;

$M^2_2 = 1.02$;

Figure 4

M^2 vs $\frac{y}{c}$ for $M_\infty = 0\cdot 9$

At $\frac{x}{c} = 0\cdot 5$;

$$M_1^2 = \cdot 81\left[1 + 1\cdot 162\left\{\cdot 964\, e^{-m\alpha y}\right\}\right] = 1\cdot 385$$

$$M_2^2 = \cdot 81\left[1 + 1\cdot 12\, e^{-m\alpha y} + (1\cdot 317 + \cdot 512\, y)\, e^{-2m\alpha y}\right] = 1\cdot 975.$$

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APPENDIX

(i) Expression for Mach Number

$$M^2(x,y) = \frac{W^2}{a^2} = \frac{\Phi_x^2 + \Phi_y^2}{a^2}$$

Substituting for the derivatives of Φ and for a^2 from (4) and (2)

$$M^2(x,y) = \frac{\left[U^2 + 2U\tau\phi_{1x} + \tau^2\phi_{1x}^2 + \tau^2\phi_{1y}^2 + 2U\tau^2\phi_{2x} + \cdots\right]}{a_\infty^2 - \frac{\gamma-1}{2}\left[2U\tau\phi_{1x} + \tau^2\phi_{1x}^2 + \tau^2\phi_{1y}^2 + 2U\tau^2\phi_{2x} + \cdots\right]}$$

Factoring out a_∞^2 and using the binomial expansion for the denominator,

$$M^2(x,y) = M_\infty^2\left[1 + \frac{\tau}{U}2\phi_{1x} + \frac{\tau^2}{U^2}\left\{\phi_{1x}^2 + 2U\phi_{2x} + \phi_{1y}^2\right\}\right]\left[1 + \frac{\gamma-1}{2a_\infty^2}\left\{2U\tau\phi_{1x} + \right.\right.$$

$$\left.\left.\tau^2\left(\phi_{1x}^2 + \phi_{1y}^2 + 2U\phi_{2x}\right)\right\} + \frac{(\gamma-1)^2}{a_\infty^4}\tau^2U^2\phi_{1x}^2\right]$$

$$M^2(x,y) = M_\infty^2\left[1 + \left(1 + \frac{\gamma-1}{2}M_\infty^2\right)\left\{\frac{2}{U}\tau\phi_{1x} + \frac{2}{U}\tau^2\phi_{2x} + \frac{\tau^2}{U^2}\left[\phi_{1y}^2 + \phi_{1x}^2\left\{1 + 2(\gamma-1)M_\infty^2\right\}\right]\right\}\right] \quad (7)$$

The local Mach number on the body is obtained as

$$M^2(x,0) = M_\infty^2\left[1 + \left(1 + \frac{\gamma-1}{2}M_\infty^2\right)\left\{\frac{2}{U}\tau\phi_{1x}(x,0) + \frac{2\tau^2}{U}\left[\phi_{2x}(x,0) + f(x)\,\phi_{1xy}(x,0)\right]\right.\right.$$

$$\left.\left. + \frac{\tau^2}{U^2}\left[\phi_{1y}^2(x,0) + \phi_{1x}^2(x,0)\left\{1 + 2(\gamma-1)M_\infty^2\right\}\right]\right\}\right]. \quad (8)$$

(ii) Experssion for the Pressure Coefficient :—

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2} = \left(\frac{p}{p_\infty} - 1\right)\frac{2}{\gamma M_\infty^2} \quad (9)$$

$$\frac{p}{p_\infty} = \left[\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}M_\infty^2}\right]^{\frac{\gamma}{\gamma-1}} \tag{10}$$

Substituting for M^2 from equation (7) in (10)

$$p = p_\infty - \frac{2}{U}\tau\,\phi_{1x}q_\infty - \frac{2}{U}\tau^2\phi_{2x}q_\infty - \frac{\tau^2}{U^2}q_\infty\left[\phi_{1y}^2 + \phi_{1x}^2\left\{1+2(\gamma-1)M_\infty^2\right\}\right].$$

Substituting in (9)

$$C_p(x,y) = -\frac{2\tau}{U}\phi_{1x} - \frac{2\tau^2}{U}\phi_{2x} - \frac{\tau^2}{U^2}\left[\phi_{1y}^2 + \phi_{1x}^2\left\{1+2(\gamma-1)M_\infty^2\right\}\right]. \tag{11}$$

Expanding the derivatives of ϕ by Taylor Series and substituting for

$$y = \tau f(x),$$

$$C_p(x,0) = -\frac{2\tau}{U}\left[\phi_{1x}(x,0) + \tau f(x)\phi_{1xy}(x,0) + \cdots\right] - \frac{2\tau^2}{U}\left[\phi_{2x}(x,0) + \cdots\right]$$

$$- \frac{\tau^2}{U^2}\left[\phi_{1y}^2(x,0) + \cdots\right] - \frac{\tau^2}{U^2}\left[1+2(\gamma-1)M_\infty^2\right]\left[\phi_{1x}^2(x,0) + \cdots\right].$$

The contributions to $C_p(x,0)$ from the perturbation potentials by the first and second order approximations are:

$$C_p(x,0) = C_{p(1)} + C_{p(2)} \qquad \text{Where} \qquad C_{p(1)} = -\frac{2\tau\phi_{1x}(x,0)}{U} \quad \&$$

$$C_{p(2)} = -\frac{2\tau^2}{U}\left[\phi_{2x}(x,0) + f(x)\phi_{1xy}(x,0)\right] - \frac{\tau^2}{U^2}\left[\phi_{1y}^2(x,0) + \phi_{1x}^2\left\{1+2(\gamma-1)M_\infty^2\right\}\right]$$

$$C_p(x,0) = -\frac{2\tau\phi_{1x}(x,0)}{U} - \frac{2\tau^2}{U}\left[\phi_{2x}(x,0) + f(x)\phi_{1xy}(x,0)\right] - \frac{\tau^2}{U^2}\left[\phi_{1y}^2(x,0) + \phi_{1x}^2\left\{1+2(\gamma-1)M_\infty^2\right\}\right]. \tag{12}$$

Fig 1.

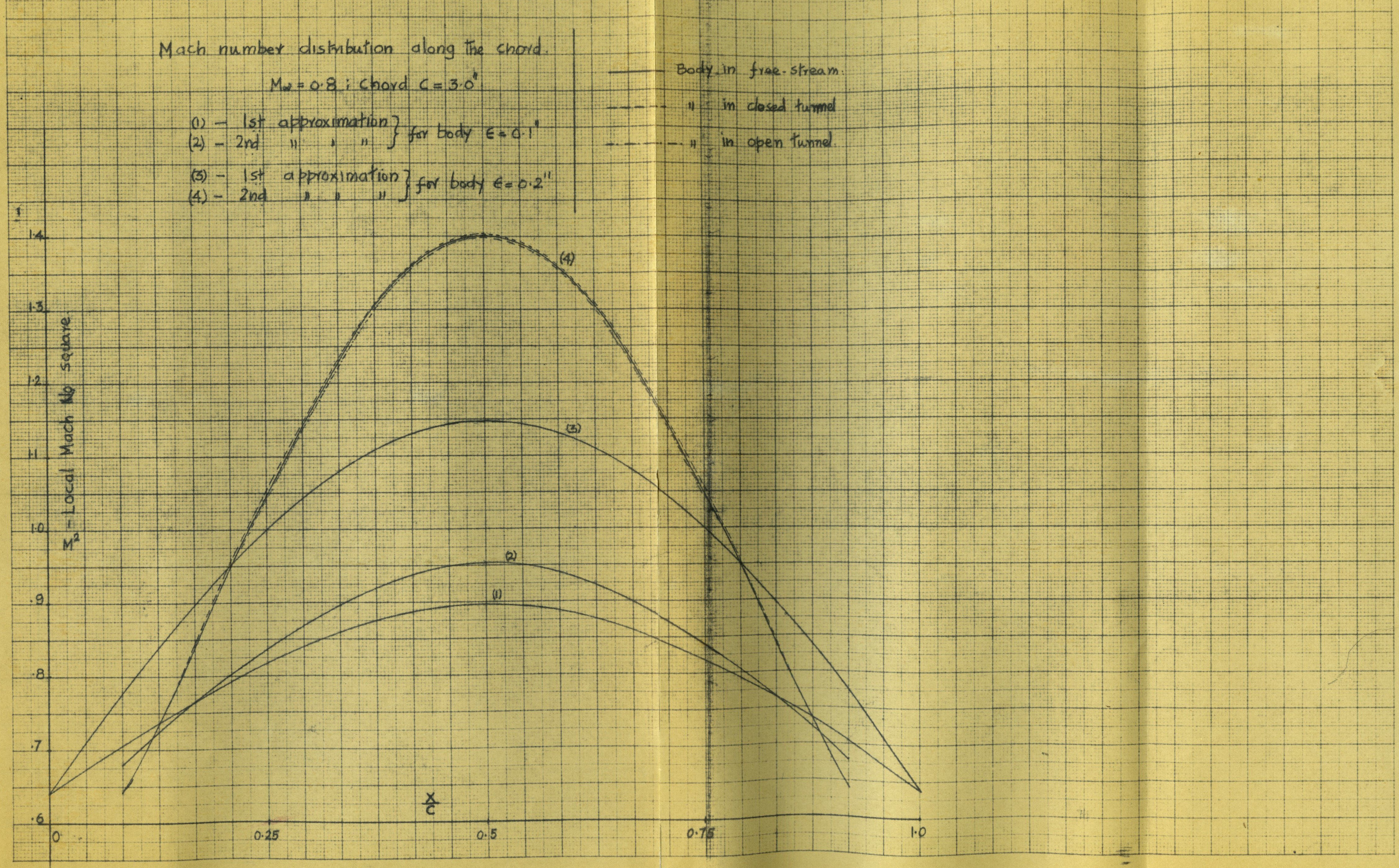
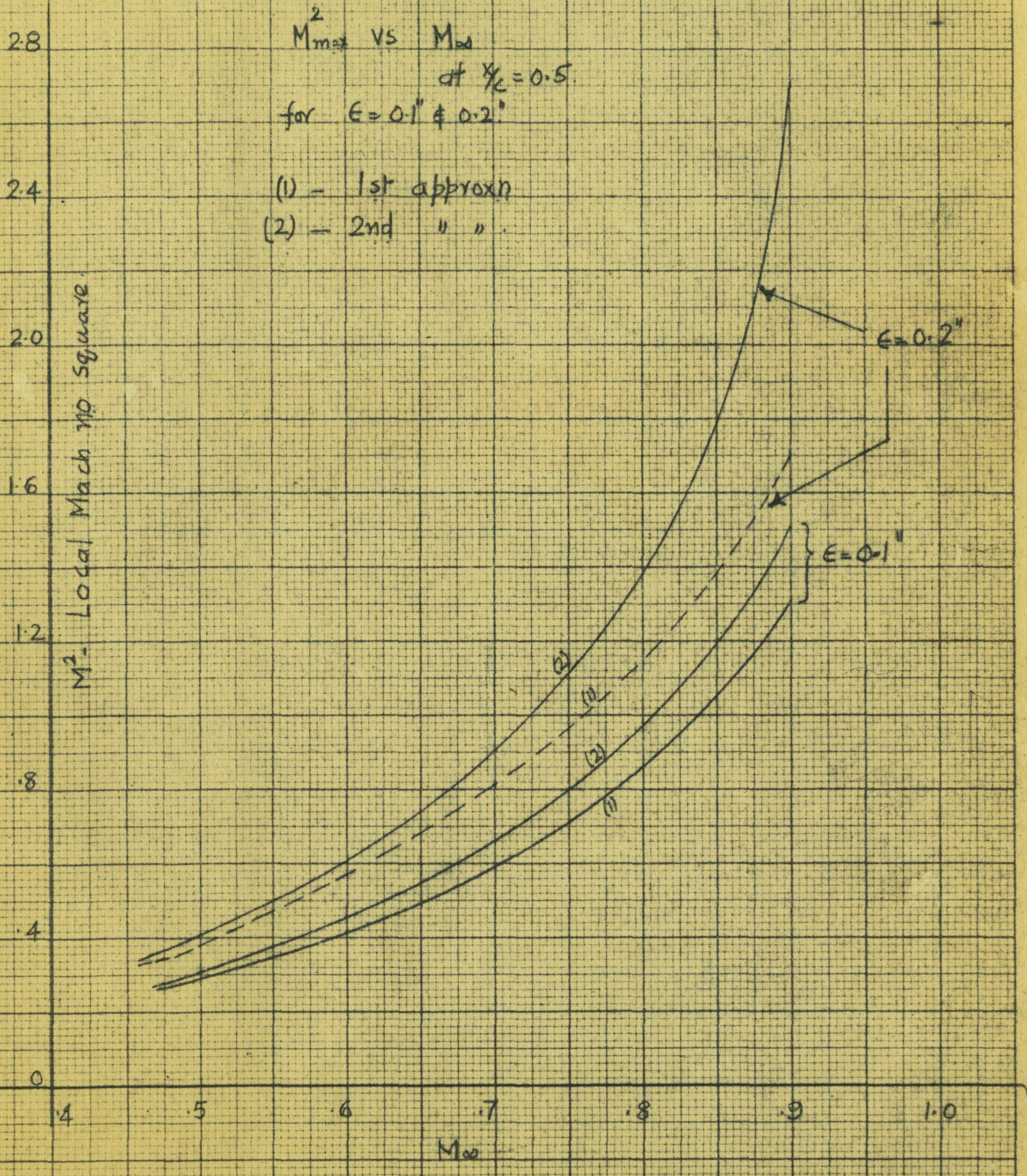

Mach number distribution along the chord.
$M_\infty = 0.8$; Chord $c = 3.0''$
(1) – 1st approximation } for body $\epsilon = 0.1''$
(2) – 2nd " " "
(3) – 1st approximation } for body $\epsilon = 0.2''$
(4) – 2nd " " "
Body in free-stream.
" in closed tunnel
" in open tunnel.
M^2 – Local Mach No. Square
$\frac{x}{c}$
0
0.25
0.5
0.75
1.0
·6
·7
·8
·9
1.0
1.1
1.2
1.3
1.4
(1)
(2)
(3)
(4)

Fig 2



M^2_{max} vs M_∞
at $x/c = 0.5$
for $\epsilon = 0.1''$ & $0.2''$
(1) – 1st approxn
(2) – 2nd " "
$\epsilon = 0.2''$
$\epsilon = 0.1''$
(1)
(2)
(1)
(2)
M^2 Local Mach no square
M_∞
2.8
2.4
2.0
1.6
1.2
.8
.4
0
.4
.5
.6
.7
.8
.9
1.0

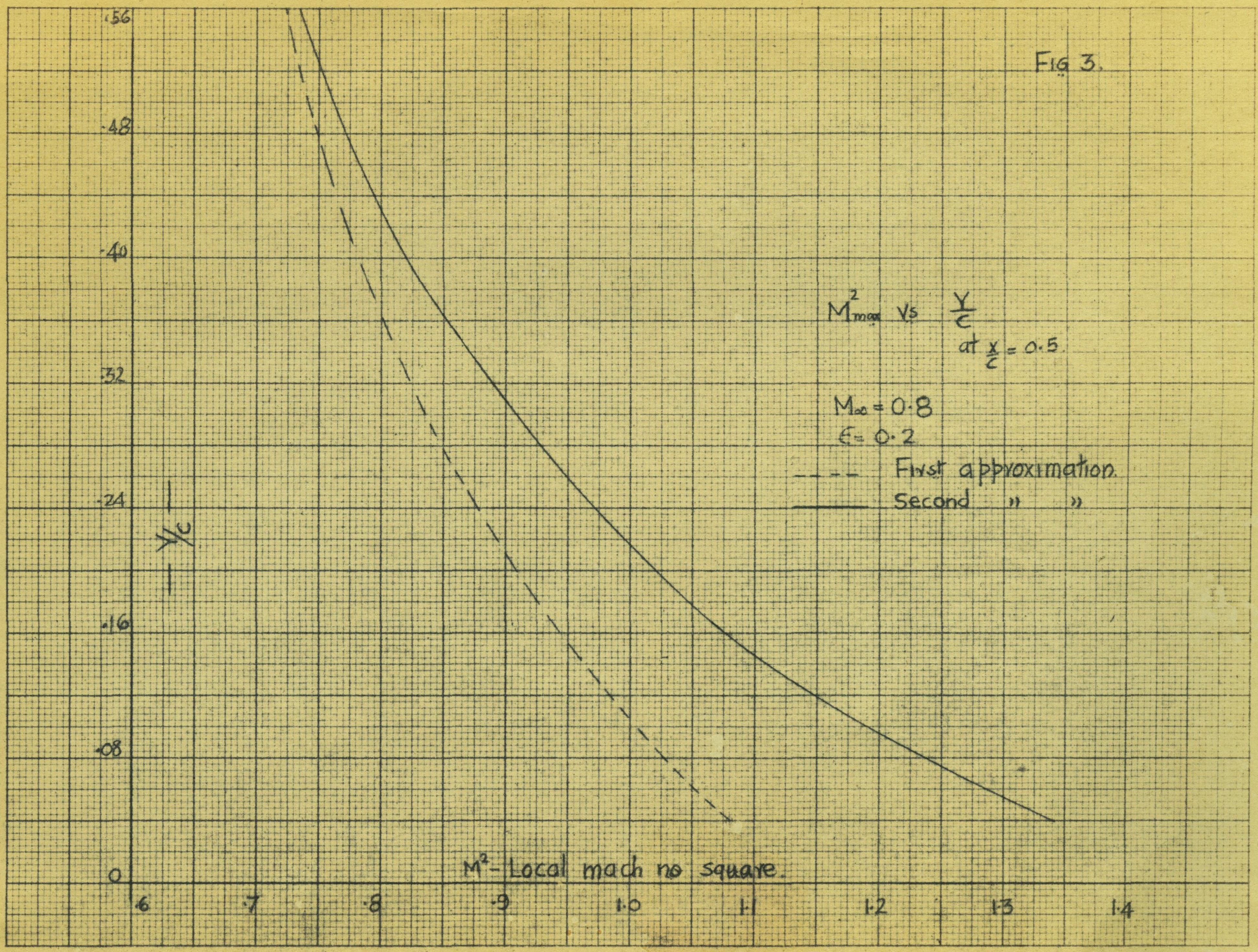
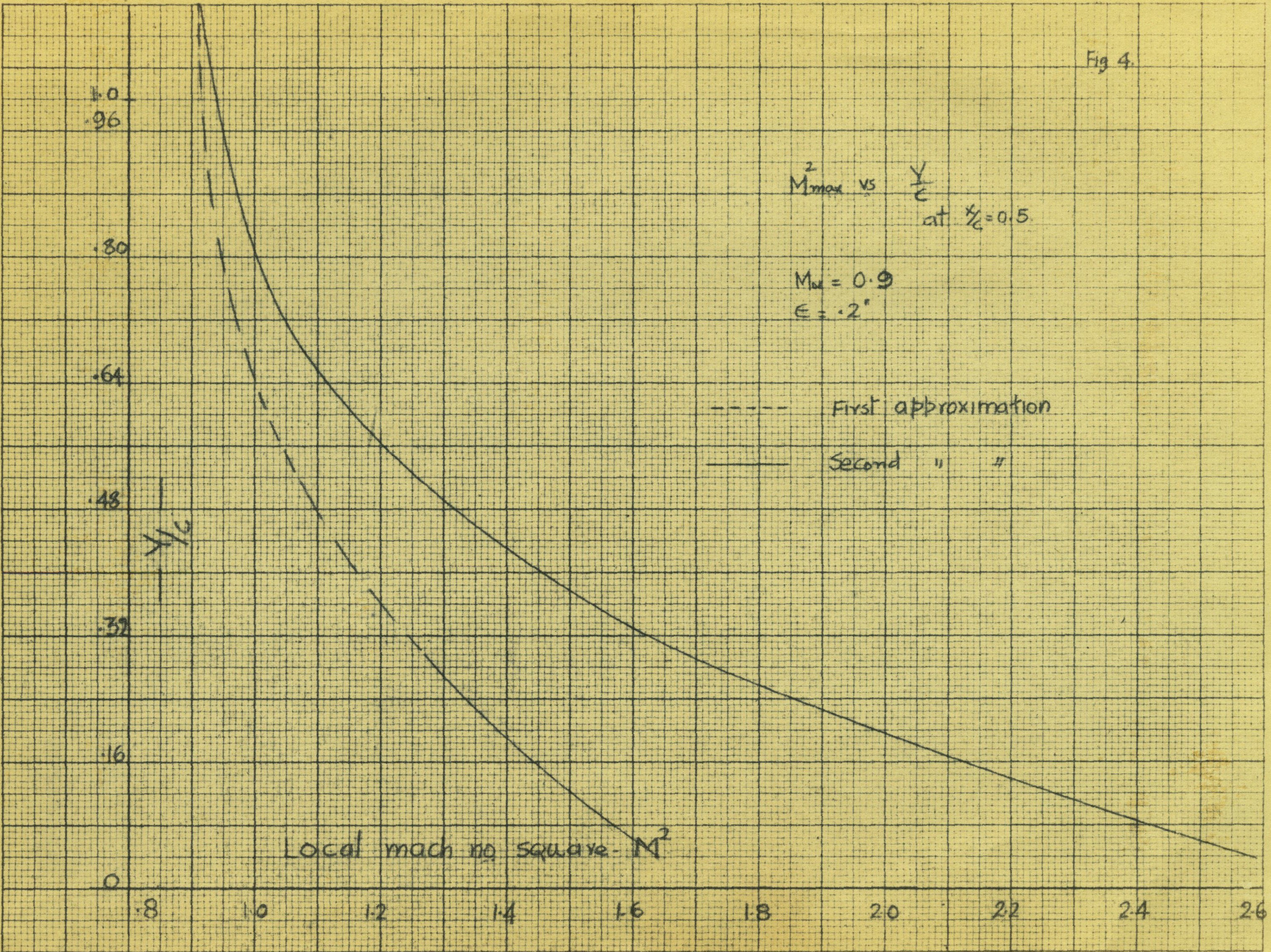


Fig 3.
M^2_{max} vs $\frac{Y}{c}$
at $\frac{x}{c} = 0.5$
$M_\infty = 0.8$
$\epsilon = 0.2$
First approximation.
Second " "
Y/c
M^2 - Local mach no square.
·56
·48
·40
·32
·24
·16
·08
0
·6
·7
·8
·9
1·0
1·1
1·2
1·3
1·4

Fig 4.



M^2_{max} vs $\frac{Y}{c}$
at $x/c = 0.5$
$M_\infty = 0.9$
$\epsilon = \cdot 2''$
----- First approximation
—— Second " "
Y/c
Local mach no square - M^2
1·0
·96
·80
·64
·48
·32
·16
0
·8
1·0
1·2
1·4
1·6
1·8
2·0
2·2
2·4
2·6