Superconformal Chern-Simons Theories and Their String Theory Duals

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Para mi abuela.

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Abstract

In this thesis, we consider two aspects of the conjectured gauge theory/string theory correspondence between three-dimensional maximal supersymmetric conformal field theories, which describe the world-volume theory of multiple M2-branes in flat space, and M-theory on $AdS_4 \times S^7$.

First we study three classes of $\mathcal{N} = 6, 8$ superconformal Chern-Simons theories that are related to the gauge theory side of the correspondence: the Bagger-Lambert (BL) theories based on 3-algebras, the Lorentzian signature 3-algebra theories, and the Aharony-Bergman-Jafferis-Maldacena (ABJM) theories. We verify the superconformal symmetry of the BL theory, prove that it is parity conserving and conjecture the (by now proven) uniqueness of its $SO(4)$ realization. We then consider the Lorentzian signature 3-algebra theories and show that although the ghosts can be removed to ensure unitarity by gauging certain global symmetries, the resulting theories spontaneously break the conformal symmetry and reduce to maximally supersymmetric three-dimensional Yang-Mills theories. After this, we recast the ABJM theory in a form for which the $SU(4)$ R-symmetry of the action is manifest; then we use this form to verify in complete detail the $OSp(6|4)$ superconformal symmetry of the theory and to express the scalar potential as a sum of squares.

Next, we study the one-loop correction to the energy of a point-particle and circular string solutions to type IIA string theory on $AdS_4 \times CP^3$. We compute the spectrum of fluctuations for each of these solutions using two techniques, known as the algebraic curve approach and the world-sheet approach. We propose a new prescription for computing the one-loop corrections that gives well-defined results and agrees with the predictions of the all-loop Bethe ansatz for our point-particle and circular string solutions as well as for previous folded-spinning string solutions.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

String theory was initially introduced in an attempt to describe hadrons and their strong interactions. As this theory was developed, and with the formulation of Quantum Chromodynamics (QCD), string theory was ruled out as a theory of hadrons but became a very promising candidate for a quantum theory of gravity.

The idea of string theory as a dual description of QCD was still desired, but it was unclear how to construct it. On one hand, there are fundamental differences between the theories, while in gauge theories local fields are fundamental objects, in string theory, gauge fields are derived as low energy excitations of fundamental open strings and therefore nonfundamental. On the other hand, the idea of a possible duality was supported by the fact discovered by 't Hooft that in the large N limit the Feynman diagrams of the perturbative expansion of the $SU(N)$ gauge field theory organize themselves in terms of a genus expansion of two-dimensional Riemann surfaces that resemble the perturbative expansion of an interacting string theory.

The first concrete example of the gauge theory/string theory duality was proposed by Maldacena in reference [1]. By considering stacks of D3-branes, Maldacena found that in certain limits the gauge theory on the world-volume of the branes describes the same physics as the string theory in the near-horizon geometry created by the branes. The precise Maldacena's conjecture was that type IIB super string theory on the $AdS_5 \times S^5$ curved background is dual to $\mathcal{N} = 4$ four-dimensional super Yang-Mills theory with gauge group $SU(N)$ ($\mathcal{N} = 4$ SYM) [2, 3, 4].

A basic check of the duality is that the symmetries match. In each case the complete symmetry is given by the superalgebra $PSU(2, 2|4)$. In the string theory this supergroup is the isometry group of the $AdS_5 \times S^5$ background, while in the gauge theory side it corresponds to the superconformal symmetry group of the theory.

An important part of the duality is how the parameters of both theories are related. The $\mathcal{N} = 4$ SYM is parametrized by the rank N of the gauge group and the coupling constant g_{YM}, or equivalent the 't Hooft parameter $\lambda = g_{YM}^2 N$. The string theory, on the other hand, depends on the string coupling constant g_s, and the effective string tension R^2/λ' where R is the common radius of the

AdS_5 and S^5 geometries. Maldacena proposed that the precise correspondence is given by

$$g_s = \frac{4\pi\lambda}{N}, \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}.$$

From this relations we can see that when the string theory geometry is weakly curved ($\sqrt{\lambda} \gg 1$), and the low energy effective field theory description of $AdS_5 \times S^5$ in terms of type IIB supergravity is justified, the dual gauge theory is strongly coupled. Conversely, when the gauge theory is weakly coupled ($\lambda \ll 1$), and we have control of the perturbative regime, the dual string theory geometry is strongly curved. Therefore, Maldacena's conjecture is of the strong/weak type. Unfortunately, it is not known how to fully access the strong coupling regime in either theory, or even how to rigorously quantize string theory on a curved background. We can simplify both theories by considering the 't Hooft limit ($N \gg 1$ at fixed λ). In this limit the quantum corrections in the string theory side (given by string loops) are suppressed, and in the gauge theory side only planar diagrams become relevant. But even in this limit the strong/weak correspondence is still present.

To check the duality we also need to understand the relation between the excitations of the two theories. The correspondence identifies the string energy eigenstates $|\mathcal{O}_Q>$ with the gauge invariant local operators $\mathcal{O}_Q$ of the gauge theory, where Q denote the set of conserved charges, and both states and operators transform under the same representation of the superconformal group. However, the conjecture goes beyond the kinematics and claims the full dynamical agreement of both theories. Specifically, the duality requires that the spectrum of scaling dimensions Δ in the conformal gauge theory should coincide with the spectrum of energies E of the string states. This is

$$\langle \mathcal{O}_A(x)\, \mathcal{O}_B(y) \rangle \approx \frac{\delta_{AB}}{|x-y|^{2\Delta_A}} \Longleftrightarrow \mathcal{H}_{String}\, |\mathcal{O}_A\rangle = E_A\, |\mathcal{O}_A\rangle\,, \tag{1.1}$$

with

$$\Delta_A\left(\lambda, \frac{1}{N}, Q\right) = E_A\left(\frac{R^2}{\alpha'}, g_s, Q\right). \tag{1.2}$$

In the special sector of BPS states/operators the energies/dimensions are protected by supersymmetry, i.e., do not depend on λ, and therefore the relation equation (1.2) can be confirmed. However, checking the duality beyond the BPS sector remains a challenge because to the fundamental problem of the strong/weak duality remained.

However, a remarkable development was introduced by Berenstein, Maldacena, and Nastase (BMN) [5] in 2002. Because the energy and scaling dimension are not only functions of λ but also of the conserved charges Q, BMN proposed that for a special subset of states/operators parameterized by large quantum numbers, the string sigma model corrections may be suppressed in the limit in which these quantum numbers became large. The important point from the string perspective is that such a limit can make the semiclassical computations of the string energy also quantum exact.

In this BMN limit, the gauge theory operators of interest will have a large number of constituent fields, and this make the calculation of the anomalous dimensions complicated. This technical problem was solved using the interpretation of the anomalous dimension matrix as an integrable spin-chain Hamiltonian [6, 7]. This allowed to calculate the one loop anomalous dimension by applying the Bethe ansatz techniques [8].

These seminal ideas trigged a tremendous and rapid progress in exploring the $AdS_5 \times S^5$ gauge theory/string theory duality, a complete review collection of all this progress can be found in [9].

Based on the $AdS_5 \times S^5$ gauge theory/string theory duality, it is possible to conjecture other AdS/CFT dualities. For example, one can consider stacks of N M2-branes or M5-branes instead of D3-branes. In this case, the corresponding world-volume theories are three and six-dimensional superconformal field theories (SCFT), while the dual M-theory is the product of an anti–de Sitter spacetime and a sphere. Specifically, the M2-brane duality conjecture that M-theory on $AdS_4 \times S^7$ (with N units of flux threading the sphere) is dual to a three-dimensional SCFT. The supergroup for this case is $OSp(8|4)$. The M5-brane duality conjecture that M-theory on $AdS_7 \times S^4$ is dual to a six-dimensional SCFT, and the supergroup is $OSp(6,2|4)$.

One of the reasons that make the M-brane dualities more challenging than the D3-brane duality is that the M-theory background does not contain a dilaton field, and therefore there is no weak-coupling limit. Also, it is not obvious that a classical action describing the conformal field theory that is dual to the M-theory solution needs to exist. For example, in the case of the M2 duality, we can consider the maximally supersymmetric $SU(N)$ Yang-Mills theory that describes the world-volume theory on a collection of N coincident D2-branes as a weak coupling "description" in the UV of the desired SCFT. To be specific, this three-dimensional $SU(N)$ Yang-Mills theory while maximally supersymmetric it is not conformal, i.e., it has a dimensionfull coupling. But if we flow to the infrared of this gauge theory, the coupling becomes infinite and one reaches the conformally invariant fixed point of the theory. Although, there is no guarantee that this fixed point has a dual Lagrangian description. The M5 case is even worst since there is not even a weak coupling "description" in the ultraviolet of the required SCFT because the theory is six-dimensional.

For these reasons the M-brane dualities have been explored in much less detail than the D3-brane case. However, few years ago Bagger and Lambert [10, 11] and Gustavsson [12] introduced new ideas that triggered a revolution in our understanding of the M2-brane duality.

Following the suggestions by J. H. Schwarz [13] that the desired three-dimensional SCFT should be a Chern-Simons theory, Bagger and Lambert [10, 11] constructed for the first time a Lagrangian description of a class of three-dimensional $\mathcal{N} = 8$ superconformal Chern-Simons of theories. The construction is based on the use of an interesting new type of algebra called 3-algebra. Although their theory was not the desired dual to M-theory on $AdS_4 \times S^7$, it introduced important new ideas in this subject.

Soon after this, Aharony, Bergman, Jafferis and Maldacena (ABJM) [14] obtained the correct construction. By considering only 3/4 maximal supersymmetry, ABJM found a three-dimensional $\mathcal{N}=6$ superconformal Chern-Simons theory with gauge group $U(N)_k \times U(N)_{-k}$ where the subscripts are the level of the Chern-Simons terms. ABJM conjecture that this gauge theory is dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$, with N units of flux. For $k > 2$ this M-theory is 3/4 maximal as the proposed gauge theory, however for $k = 1, 2$ the M-theory is maximal supersymmetric. In the gauge theory side the enhancement of the supersymmetry for $k = 1, 2$ is a nontrivial property of this quantum theory [15]. The ABJM theory has two parameters, the rank of the gauge group N and the Chern-Simons level k. In the large N limit with N/k fixed the effective 't Hooft coupling of the planar diagrams is $\lambda = N/k$. In the limit $N^{1/5} \gg k \gg N$ the dual theory reduces to type IIA string theory on $AdS_4 \times CP^3$. In this limit the string coupling constant and the CP^3 radius of the geometry are related to the ABJM theory by

$$g_s = \frac{\lambda^{5/4}}{N}, \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}.$$

From this relations we can see that this duality is again of the strong/weak type.

In this thesis, we study few aspects of this M2-brane duality between three-dimensional supersymmetric conformal Chern-Simons field theories and M-theory on $AdS_4 \times S^7$.

First, in chapter 2, we study the Bagger-Lambert theories based on 3-algebras. We verify the $OSp(6|4)$ superconformal symmetry of the BL theory, and prove that it is parity conserving. And after describing several unsuccessful attempts to construct theories of this type for other gauge groups and representations, we conjecture the (by now proven) uniqueness of its $SO(4)$ realization. In chapter 3, we study a realization of the BL theory based on a Lorentzian signature 3-algebra. Here, we show that although the ghost degrees of freedom can be removed to ensure unitarity by gauging certain global symmetries, the resulting theories spontaneously break the conformal symmetry and reduce to maximally supersymmetric three-dimensional Yang-Mills theories. In chapter 4, we recast the ABJM theory in a form for which the $SU(4)$ R-symmetry of the action is manifest. Then we use this form to verify in complete detail the $OSp(6|4)$ superconformal symmetry of the theory and to express the scalar potential as a sum of squares.

After this, in chapter 5, we study the one-loop correction to the energy of a point-particle and circular string solutions (with support in CP^3) to type IIA string theory on $AdS_4 \times CP^3$. We compute the spectrum of fluctuations for each of these solutions using two techniques, known as the algebraic curve approach and the world-sheet approach. We proposed a new prescription for computing the one-loop corrections that gives well-defined results and agrees with the predictions of the all-loop Bethe ansatz for our point-particle and circular string solutions with support in CP^3 as well as for previous folded-spinning string solutions with support in AdS_4.

Chapter 2

$\mathcal{N} = 8$ Superconformal Chern-Simons Theories

Following earlier studies of coincident M2-brane systems [16], Bagger and Lambert (BL) [10, 11] have constructed an explicit action for a new maximally supersymmetric superconformal Chern-Simons theory in three dimensions. The motivation for their work, like that in [13], was to construct the superconformal theories that are dual to $AdS_4 \times S^7$ solutions of M-theory. Such theories, which are associated to coincident M2-branes, should be maximally supersymmetric, which in three-dimensions means that they have $\mathcal{N} = 8$ supersymmetry. More precisely, the superconformal symmetry group should be $OSp(8|4)$, which is also the symmetry of the M-theory solution. It is not obvious that a classical action describing the conformal field theory that is dual to the M-theory solution needs to exist. In fact, there are good reasons to be skeptical: These field theories can be defined as the infrared conformal fixed points of nonconformal $SU(N)$ $\mathcal{N} = 8$ Yang-Mills theories, but there is no guarantee that any of these fixed points has a dual Lagrangian description.

J. H. Schwarz in [13] attempted to construct three-dimensional theories with $OSp(8|4)$ superconformal symmetry and $SU(N)$ gauge symmetry using scalar and spinor matter fields in the adjoint representation of the gauge group. These would be analogous to $\mathcal{N} = 4$ $SU(N)$ gauge theory in four dimensions, with one crucial difference, the F^2 gauge field kinetic term has the wrong dimension for a conformal theory in three-dimensions. Also, it would give propagating degrees of freedom, which are not desired. To address both of these issues, [13] proposed using a Chern-Simons term for the gauge fields instead of an F^2 term. The conclusion reached in [13] was that such an action, with $\mathcal{N} = 8$ supersymmetry, does not exist. This was consistent with the widely held belief (at the time) that supersymmetric Chern-Simons theories in three-dimensions only exist for $\mathcal{N} \leq 3$.[1]

The work of Bagger and Lambert [10] presents an explicit action and supersymmetry transformations for an $\mathcal{N} = 8$ Chern-Simons theory in three-dimensions evading the $\mathcal{N} \leq 3$ bound mentioned above. Their construction can be described in terms of an interesting new type of algebra, which

[1]Theories of this type with $\mathcal{N} = 2$ supersymmetry were first constructed by Ivanov [17] and by Gates and Nishino [18]. For a recent discussion see [19].

we call a BL algebra.[2] It involves a totally antisymmetric triple bracket analog of the Lie bracket[3]

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d.$$

There should also be a symmetric invertible metric h^{ab} that can be used to raise and lower indices. The structure constants f^{abcd} defined in this way are required to have total antisymmetry. Furthermore, this tensor is also required to satisfy a quadratic constraint, analogous to the Jacobi identity, which BL call the "fundamental equation."

An important question, of course, is whether BL algebras have any nontrivial realizations. BL settle this question by noting that a solution is provided by a set of four generators T^a that transform as a four-vector of an $SO(4)$ gauge group. In this example $f^{abcd} = \varepsilon^{abcd}$ and $h^{ab} = \delta^{ab}$. After reviewing the free theory in section 2.1, we review the BL $SO(4)$ theory in section 2.2 making a couple of new observations in the process. The first is an explicit verification that the action is invariant under the conformal supersymmetries as well as the Poincaré supersymmetries. Taken together, these generate the entire $OSp(8|4)$ symmetry. The second is a careful demonstration in section 2.3 of a fact noted in [11], namely that the theory is parity conserving. This feature, which is essential for a dual to the M-theory solution, involves combining a spatial reflection with an $SO(4) = SU(2) \times SU(2)$ reflection. The latter reflection can be interpreted as interchanging the two $SU(2)$ factors.

We also explore whether there exist BL theories for other choices of gauge groups and matter representations. Motivated by the $SO(4)$ example, section 2.4 considers parity-conserving theories with gauge group $G \times G$ and matter fields belonging to a representation (R, R), where R is some representation of G. Two classes of such examples that have been examined carefully are based on $G = SO(n)$ and $G = USp(2n)$ with R chosen to be the fundamental representation in each case. The first of these two classes is described in detail. The free theory (appropriate for a single M2-brane) appears in this classification as $G = SO(1)$, and the $SO(4)$ theory appears as $G = USp(2)$. An invariant totally antisymmetric fourth-rank tensor f^{abcd}, where a, b, c, d label components of the representation (R, R), can be constructed. However, it turns out that the fundamental equation is satisfied only for the free theory, the $SO(4)$ theory, and the $G = SO(2)$ case. The $SO(2)$ case does not give a new theory, however, for reasons that are explained later in this chapter.

BL suggested that there may be other theories with $OSp(8|4)$ superconformal symmetry based on nonassociative algebras. Following up on this suggestion, section 2.4 attempts to utilize the algebra of octonions in this manner. This leads to a seven-dimensional BL-type algebra. However, once again it turns out that the fundamental identity is not satisfied. Thus, this approach also does

[2]Gustavsson, studying the same problem in [12], was independently led to formulate conditions that are equivalent to BL algebras. The equivalence is described in [11].

[3]Such brackets, regarded as generalizations of Poisson brackets, were considered by Nambu in 1973 [20]. For a recent discussion of Nambu brackets see [21].

not lead to other consistent field theories with $OSp(8|4)$ superconformal symmetry. Based on these studies, we conjecture that the $SO(4)$ BL-theory is the only nontrivial three-dimensional Lagrangian theory with $OSp(8|4)$ superconformal symmetry, at least if one assumes irreducibility and a finite number of fields.

It is a curious coincidence that three-dimensional gravity with a negative cosmological constant can be formulated as a twisted Chern-Simons theory based on the gauge group $SO(2,2)$. Aside from the noncompact form of the gauge group, this is identical to the Chern-Simons term that is picked out by the BL-theory. This is discussed in section 2.5.

2.1 The Free Theory

Let us start with the well-known free $\mathcal{N} = 8$ superconformal theory. It contains no gauge fields, so it is not a Chern-Simons theory. The action is

$$S = \frac{1}{2} \int \left(-\partial^\mu \phi^I \partial_\mu \phi^I + i\overline{\psi}^A \gamma^\mu \partial_\mu \psi^A \right) d^3x. \tag{2.1}$$

This theory has $OSp(8|4)$ superconformal symmetry. The R-symmetry is $Spin(8)$ and the conformal symmetry is $Sp(4) = Spin(3,2)$. The index I labels components of the fundamental 8_v representation of $Spin(8)$ and the index A labels components of the spinor 8_s representation. In particular, ψ^A denotes 8 two-component Majorana spinors. The Poincaré and conformal supersymmetries belong to the other spinor representation, 8_c, whose components are labeled by dotted indices $\dot{A}$, etc.

The three inequivalent eight-dimensional representations of $Spin(8)$ can couple to form a singlet. The invariant tensor (or Clebsch-Gordan coefficients) describing this is denoted $\Gamma^I_{A\dot{A}}$, since it can be interpreted as eight matrices satisfying a Dirac algebra. We also use the transpose matrix, which is written $\Gamma^I_{\dot{A}A}$ without adding an extra symbol indicating that it is the transpose. These matrices have appeared many times before in superstring theory.

Our conventions for the three-dimensional and $Spin(8)$ Dirac Algebras are summarized in appendix A.1 and A.2. Note that in our conventions γ^μ are 2×2 matrices and Γ^I are 8×8 matrices. They act on different vector spaces and therefore they trivially commute with one another. BL use a somewhat different formalism in which γ^μ and Γ^I are 11 anticommuting 32×32 matrices. We find this formalism somewhat confusing, since the three-dimensional theories in question cannot be obtained by dimensional reduction of a higher-dimensional theory (in contrast to $\mathcal{N} = 4$ super Yang-Mills theory).

The action (2.1) is invariant under the supersymmetry transformations

$$\delta\phi^I = i\bar{\varepsilon}^{\dot{A}}\Gamma^I_{\dot{A}A}\psi^A = i\bar{\varepsilon}\Gamma^I\psi = i\overline{\psi}\Gamma^I\varepsilon, \tag{2.2}$$

$$\delta\psi = -\gamma\cdot\partial\phi^I\Gamma^I\varepsilon. \tag{2.3}$$

One can deduce the conserved supercurrent by the Noether method, which involves varying the action while allowing ε to have arbitrary x dependence. This gives

$$\delta S = -i\int \partial_\mu\bar{\varepsilon}\Gamma^I\gamma\cdot\partial\phi^I\gamma^\mu\psi d^3x.$$

Thus the conserved supercurrent is $i\Gamma^I\gamma\cdot\partial\phi^I\gamma^\mu\psi$. The conservation of this current is easy to verify using the equations of motion.

Let us now explore the superconformal symmetry. As a first try, let us consider taking $\varepsilon^{\dot{A}}(x) = \gamma\cdot x\eta^{\dot{A}}$, since this has the correct dimensions. Using $\partial_\mu\varepsilon(x) = \gamma_\mu\eta$ and $\gamma^\mu\gamma^\rho\gamma_\mu = -\gamma^\rho$, this gives

$$\delta S = i\int \overline{\psi}\gamma\cdot\partial\phi^I\Gamma^I\eta d^3x.$$

This can be canceled by including an additional variation of the form $\delta\psi \sim \Gamma^I\phi^I\eta$. Thus the superconformal symmetry is given by

$$\delta\phi^I = i\overline{\psi}\Gamma^I\gamma\cdot x\eta, \tag{2.4}$$

$$\delta\psi = -\gamma\cdot\partial\phi^I\Gamma^I\gamma\cdot x\eta - \phi^I\Gamma^I\eta. \tag{2.5}$$

One can deduce the various bosonic $OSp(8|4)$ symmetry transformations by commuting ε and η transformations. Of these only the conformal transformation, obtained as the commutator of two η transformations, is not a manifest symmetry of the action. It is often true that scale invariance implies conformal symmetry. However, this is not a general theorem, so it is a good idea to check conformal symmetry explicitly as we have done.

2.2 The SO(4) Theory

The $SO(4)$ gauge theory contains scalar fields ϕ^I_a and Majorana spinor fields ψ^A_a each of which transforms as four-vectors of the gauge group ($a = 1,2,3,4$). In addition there are $SO(4)$ gauge fields A^{ab}_μ with field strengths $F^{ab}_{\mu\nu}$. The field content of the $SO(4)$ theory is summarized in table 2.1 and the index notation in table 2.2. Since four-vector indices are raised and lowered with a Kronecker delta, we do not distinguish superscripts and subscripts.

Table 2.1. Field content of the BL $SO(4)$ theory

Field	Description	Units
ϕ_a^I	Scalar Field	1/2
$\psi_{a\sigma}^A$	Spinor Field	1
$A_{\mu ab}$	Gauge Field	1
$Q_\sigma^{\dot{A}}$	SUSY generator	1/2
$\epsilon_\sigma^{\dot{A}}$	SUSY parameter	−1/2
$S_\sigma^{\dot{A}}$	Super conformal generator	1/2
$\eta_\sigma^{\dot{A}}$	Super conformal parameter	−1/2

The action is a sum of a matter term and a Chern-Simons term:

$$S_k = k\,(S_{\rm m} + S_{\rm CS})\,.$$

We choose normalizations such that the level-k action S_k is k times the level-one action S_1. Then k, which is a positive integer, is the only arbitrary parameter. Perturbation theory is an expansion in $1/k$. So the theory is weakly coupled and can be analyzed in perturbation theory when k is large. The goal here is to construct and describe the classical action.

The required level-one Chern-Simons action is given by

$$S_{\rm CS} = \alpha \int \tilde{\omega}_3,$$

where the "twisted" Chern-Simons form $\tilde{\omega}_3$ is constructed so that

$$d\tilde{\omega}_3 = \frac{1}{2}\epsilon_{abcd} F_{ab} \wedge F_{cd}.$$

This implies that

$$\tilde{\omega}_3 = \frac{1}{2}\epsilon_{abcd} A_{ab} \wedge (dA_{cd} + \frac{2}{3} A_{ce} \wedge A_{ed}).$$

When $SO(4)$ is viewed as $SU(2) \times SU(2)$, this is the difference of the Chern-Simons terms for the two $SU(2)$ factors. The coefficient α is chosen so that these Chern-Simons terms have standard

Table 2.2. Index notation for the BL $SO(4)$ theory

Index	Values	Group	Representation
I	1, 2,. . ., 8	Spin(8)	$8_{\rm v}$
A	1, 2,. . ., 8	Spin(8)	$8_{\rm s}$
$\dot{A}$	1, 2,. . ., 8	Spin(8)	$8_{\rm c}$
a	1, 2, 3, 4	SO(4)	4 − dim
σ	1, 2	SO(1,2)	Spinor
μ	1, 2, 3	SO(1,2)	Vector

level-one normalization. Varying the gauge field by an amount δA, one has (up to a total derivative)

$$\delta\tilde{\omega}_3 = \epsilon_{abcd}\delta A_{ab} \wedge F_{cd},$$

or

$$\delta S_{\rm CS} = \frac{\alpha}{2}\int \epsilon_{abcd}\,\epsilon^{\mu\nu\rho}\delta A_\mu^{ab} F_{\nu\rho}^{cd} d^3x.$$

The $SO(4)$ matter action is a sum of kinetic and interaction terms

$$S_{\rm m} = S_{\rm kin} + S_{\rm int},$$

where

$$S_{\rm kin} = \int d^3x \left(-\frac{1}{2}(D_\mu\phi^I)_a(D^\mu\phi^I)_a + \frac{i}{2}\overline{\psi}_a\gamma^\mu(D_\mu\psi)_a\right),$$

and

$$S_{\rm int} = \int d^3x \left(ic\,\epsilon_{abcd}\overline{\psi}_a\Gamma^{IJ}\psi_b\phi_c^I\phi_d^J - \frac{4}{3}c^2\sum(\epsilon_{abcd}\phi_b^I\phi_c^J\phi_d^K)^2\right).$$

The supersymmetry transformations that leave the action invariant are

$$\delta\phi_a^I = i\overline{\varepsilon}\Gamma^I\psi_a, \tag{2.6a}$$

$$\delta\psi_a = -\gamma^\mu(D_\mu\phi^I)_a\Gamma^I\varepsilon + \frac{2c}{3}\epsilon_{abcd}\Gamma^{IJK}\varepsilon\phi_b^I\phi_c^J\phi_d^K, \tag{2.6b}$$

$$\delta A_{\mu ab} = 4ic\,\epsilon_{abcd}\,\overline{\psi}_c\gamma_\mu\Gamma^I\phi_d^I\varepsilon, \tag{2.6c}$$

for the identification

$$c = \frac{1}{16\alpha}.$$

The formulas agree with BL for $c = 3$, which corresponds to $\alpha = 1/48$. Any apparent minus-sign discrepancies are due to the different treatment of the Dirac matrices discussed earlier.

The conformal supersymmetries also hold. They can be analyzed in the same way that was discussed for the free theory. The result, as before, is to replace ε by $\gamma \cdot x\eta$ and to add a term $-\phi_a^I\Gamma^I\eta$ to $\delta\psi_a$. We have verified the Poincaré and the conformal supersymmetries of this theory in complete detail. Thus this theory has $OSp(8|4)$ superconformal symmetry and $SO(4)$ gauge symmetry. It also has parity invariance, which we explain in the next section.

2.3 Parity Conservation

The relative minus sign between the two $SU(2)$ contributions to the Chern-Simons term has an interesting consequence. Normally, Chern-Simons theories are parity violating. In this case, however, one can define the parity transformation to be a spatial reflection together with interchange of the two $SU(2)$ gauge groups. Then one concludes that the Chern-Simons term is parity conserving.[4]

To conclude that the entire theory is parity conserving, there is one other term that needs to be analyzed. It is the one that has the structure

$$\epsilon_{abcd}\bar{\psi}_a\Gamma^{IJ}\psi_b\phi_c^I\phi_d^J.$$

The interchange of the two $SU(2)$ groups gives one minus sign (due to the epsilon symbol), so invariance will only work if a spinor bilinear of the form $\bar{\psi}_1\psi_2 = \psi_1^\dagger\gamma^0\psi_2$ is a pseudoscalar in three-dimensions. So we must decide whether this is true. Certainly, in four dimensions such a structure is usually considered to be a scalar. The R-symmetry labels are irrelevant to this discussion.

Let us review the parity analysis of spinor bilinears in four dimensions. The usual story is that the parity transform (associated to spatial inversion $\vec{x} \to -\vec{x}$) of a spinor is given by $\psi \to \gamma^0\psi$. There are two points to be made about this. First, spatial inversion is a reflection in four dimensions. This differs from the case in three-dimensional spacetime, where spatial inversion is a rotation, rather than a reflection. Therefore, it is more convenient for generalization to the three-dimensional case to consider a formula for the transformation of a spinor under reflection of only one of the spatial coordinates (x^i, say). Under this reflection, the formula in four dimensions is $\psi \to i\gamma^i\gamma_5\psi$. For this choice reflecting all three coordinates gives the previous rule $\psi \to \gamma^0\psi$ (up to an ambiguous and irrelevant sign). With this rule, one can easily show that $\bar{\psi}_1\psi_2$ is a scalar and $\bar{\psi}_1\gamma_5\psi_2$ is a pseudoscalar, as usual.

The second point is that the Dirac algebra for four-dimensional spacetime has an automorphism $\gamma^\mu \to i\gamma^\mu\gamma_5$. In other words,

$$\{i\gamma^\mu\gamma_5, i\gamma^\nu\gamma_5\} = \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

This automorphism squares to $\gamma^\mu \to -\gamma^\mu$, which is also an automorphism. The kinetic term, which involves $\overline{\psi}\gamma\cdot\partial\psi$, is invariant under this automorphism, since $i\gamma^0\gamma_5 i\gamma^\mu\gamma_5 = \gamma^0\gamma^\mu$. In view of this automorphism, it is equally sensible to define a reflection by the rule $\psi \to \gamma^i\psi$. However, if one makes this choice, then one discovers that $\bar{\psi}_1\psi_2$ is a pseudoscalar and $i\bar{\psi}_1\gamma_5\psi_2$ is a scalar. This makes sense, since they (and their negatives) are interchanged by the automorphism.

In the case of three-dimensions, there is no analog of γ_5, and so the automorphism discussed above has no analog. As a result, the only sensible rule for a reflection is $\psi \to \gamma^i\psi$. Then one is forced to conclude (independent of any conventions) that $\bar{\psi}_1\psi_2$ is a pseudoscalar. This is what we saw is required for the $SO(4)$ super Chern-Simons theory to be parity conserving.

[4]This was pointed out to us by A. Kapustin before the appearance of [11]. This way of implementing parity conservation, including the odd parity of a spinor bilinear, was understood already in [22].

2.4 The Search for Generalizations

Possible generalizations of the $SO(4)$ theory are suggested by the fact that $SO(4) = SU(2)\times SU(2) = USp(2) \times USp(2)$ and that a four-vector field ϕ^a can be reexpressed as a bifundamental field $\phi^{\alpha\alpha'}$.

An infinite class of candidate theories with the same type of structure is based on the gauge group $SO(n) \times SO(n)$ with matter fields $\phi^{\alpha\alpha'}$ assigned to the bifundamental representation $(\mathbf{n}, \mathbf{n})$. In this case one takes the gauge field to be

$$A_{\alpha\alpha'\beta\beta'} = \delta_{\alpha\beta} A'_{\alpha'\beta'} + \delta_{\alpha'\beta'} A_{\alpha\beta},$$

where $A_{\alpha\beta} = -A_{\beta\alpha}$ and $A'_{\alpha'\beta'} = -A'_{\beta'\alpha'}$ are $SO(n)$ gauge fields. The $n = 1$ case is the free theory with 8 scalars and 8 spinors and no gauge fields, which was discussed in section 2.1.

The BL structure constants vanish for $n = 1$, and for $n > 1$ they are given by

$$\begin{aligned} f^{\alpha\alpha'\beta\beta'\gamma\gamma'\delta\delta'} = \frac{1}{2(n-1)}\Big(&- \delta^{\alpha\beta}\delta^{\gamma\delta}\delta^{\alpha'\delta'}\delta^{\beta'\gamma'} + \delta^{\alpha\beta}\delta^{\gamma\delta}\delta^{\alpha'\gamma'}\delta^{\beta'\delta'} - \delta^{\alpha\gamma}\delta^{\delta\beta}\delta^{\alpha'\beta'}\delta^{\gamma'\delta'} \\ &+ \delta^{\alpha\gamma}\delta^{\delta\beta}\delta^{\alpha'\delta'}\delta^{\gamma'\beta'} - \delta^{\alpha\delta}\delta^{\beta\gamma}\delta^{\alpha'\gamma'}\delta^{\delta'\beta'} + \delta^{\alpha\delta}\delta^{\beta\gamma}\delta^{\alpha'\beta'}\delta^{\delta'\gamma'}\Big). \end{aligned} \tag{2.7}$$

For this choice one finds that the dual gauge field is

$$\tilde{A}^{\alpha\alpha'\beta\beta'} = f^{\alpha\alpha'\beta\beta'\gamma\gamma'\delta\delta'} A_{\gamma\gamma'\delta\delta'} = \delta^{\alpha\beta} A'^{\alpha'\beta'} - \delta^{\alpha'\beta'} A^{\alpha\beta}.$$

Therefore the twisted Chern-Simons term again is proportional to the difference of the individual Chern-Simons terms, as required by parity conservation. However, the BL fundamental equation is not satisfied for $n > 2$, and there are a number of inconsistencies in the supersymmetry algebra. This leaves the $n = 2$ case as the only remaining candidate for a new theory. This theory (if it exists) has the same matter content as the BL-theory, but fewer gauge fields. Even though the BL algebra is okay in this case, the elimination of four gauge fields gives a violation of another requirement. Specifically, the antisymmetric tensor f_{abcd} is not $SO(2) \times SO(2)$ adjoint valued in a pair of indices. This is an essential requirement, because the formula for the supersymmetry variation of the gauge field has the form

$$\delta A_{\mu ab} = 4ic\, f_{abcd}\, \overline{\psi}_c \gamma_\mu \Gamma^I \phi^I_d \varepsilon.$$

This equation does not make sense when the right-hand side introduces unwanted degrees of freedom that do not belong to the adjoint representation. This problem arises for all cases with $n > 1$ including the $n = 2$ case in particular. One could try to remove the nonadjoint pieces of the right-hand side, but that leads to other inconsistencies.

A completely analogous analysis exists for candidate theories based on the gauge group $USp(2n)\times$

$USp(2n)$ with matter fields belonging to the bifundamental representation. For the choice $n = 1$ this is the $SO(4)$ theory of section 2.2. Again, one can construct a totally antisymmetric tensor f^{abcd} for all n. However, this does give any new theories, because the BL fundamental equation is not satisfied for $n > 1$.

Let us now describe another attempt to construct new examples. BL describe a systematic way to obtain totally antisymmetric triple brackets based on nonassociative algebras. However, the examples they discuss all involve adjoining "a fixed Hermitian matrix G" that does not seem to be compatible with a conventional Lie algebra interpretation. Here we explore dispensing with such an auxiliary matrix and applying their procedure to the most familiar nonassociative algebra we know, namely the algebra of octonions. The question to be addressed is then whether this gives a new superconformal theory with the gauge group G_2 and with the matter fields belonging to the seven-dimensional representation.

Let us denote the imaginary octonions by e_a with $a = 1, 2, \ldots, 7$. These have the nonassociative multiplication table

$$e_a e_b = t_{abc} e_c - \delta_{ab}.$$

The totally antisymmetric tensor t_{abc} has the following nonvanishing components

$$t_{124} = t_{235} = t_{346} = t_{457} = t_{561} = t_{672} = t_{713} = 1.$$

Note that these are related by cyclic permutation of the indices $(a, b, c) \to (a+1, b+1, c+1)$. It is well known that t_{abc} can be regarded as an invariant tensor describing the totally antisymmetric coupling of three seven-dimensional representations of the Lie group G_2.

Let T_{ab} denote a generator of an $SO(7)$ rotation in the ab plane. The $SO(7)$ Lie algebra is

$$[T_{ab}, T_{cd}] = T_{ad}\delta_{bc} - T_{bd}\delta_{ac} - T_{ac}\delta_{bd} + T_{bc}\delta_{ad}.$$

The generators of G_2 can be described as a 14-dimensional subalgebra of this Lie algebra. A possible choice of basis is given by

$$X_1 = T_{24} - T_{56} \quad \text{and} \quad Y_1 = T_{24} - T_{37},$$

and cyclic permutations of the indices. This gives 14 generators X_A consisting of X_a and $X_{a+7} = Y_a$. By representing the generators T_{ab} by seven-dimensional matrices in the usual way, one can represent the G_2 generators by antisymmetrical seven-dimensional matrices. These can then be used in the usual way to express G_2 gauge fields as seven-dimensional matrices A_{ab}.

The group G_2 is a subgroup of $SO(7)$ in which the $\mathbf{7}$ of $SO(7)$ corresponds to the $\mathbf{7}$ of G_2. Thus, the seven-index epsilon symbol, which is an invariant tensor of $SO(7)$, is also an invariant tensor of G_2. It can be used to derive an antisymmetric fourth-rank tensor of G_2:

$$f_{abcd} = \frac{1}{6}\epsilon_{abcdefg}t_{efg}.$$

This tensor has the following nonzero components

$$f_{7356} = f_{1467} = f_{2571} = f_{3612} = f_{4723} = f_{5134} = f_{6245} = 1.$$

These are also related by cyclic permutations. This tensor is the same (up to normalization) as the one given by the construction based on associators that was proposed by BL.

If one defines

$$[abc, def] = \sum_x f_{abcx} f_{defx},$$

the BL fundamental equation takes the form

$$[abw, xyz] - [abx, yzw] + [aby, zwx] - [abz, wxy] = 0.$$

Note that the left-hand side has antisymmetry in the pair (a, b) and total antisymmetry in the four indices (w, x, y, z). One can verify explicitly that these relations are *not* satisfied by the tensor f_{abcd} given above.[5] Thus, the tensor f_{abcd} does not define a seven-dimensional BL algebra, and we do not obtain a new theory for the gauge group G_2.

2.5 Relation to anti–de Sitter Gravity?

Pure three-dimensional gravity with a negative cosmological constant can be formulated as a twisted Chern-Simons theory based on the gauge group $SO(2, 2)$ [23, 24, 25]. The BL-theory, on the other hand, requires a twisted Chern-Simons term for the gauge group $SO(4)$. Aside from the signature, these are exactly the same! What should one make of this coincidence?[6]

The BL-theory was motivated by the desire to construct conformal field theories dual to gravity in four-dimensional anti–de Sitter space. So the notion that it might be possible to interpret it as a gravity theory in three-dimensional anti–de Sitter space is certainly bizarre. The BL-theory can be modified easily to the gauge group $SO(2, 2)$, though this introduces some disturbing minus signs into half of the kinetic terms of the scalar and spinor fields. If one makes this change anyway, the Chern-Simons term is exactly that for gravity. However, there is a serious problem with a gravitational interpretation in addition to the problem of the negative kinetic terms: a gravity theory should

[5] BL did not claim that it necessarily would satisfy the fundamental equation.

[6] This section was motivated by a question raised by Aaron Bergman at a seminar given by JHS.

have diffeomorphism symmetry. The Chern-Simons term has this symmetry, but the matter terms in the Lagrangian contain the three-dimensional Lorentz metric to contract indices, so they are not diffeomorphism invariant. Thus, we believe that there is no sensible interpretation of the BL-theory as a three-dimensional gravity theory. Nonetheless, it is striking that its Chern-Simons term is so closely related to the one that arises in the Chern-Simons description of three-dimensional gravity with a negative cosmological constant.

The $SO(2,2)$ Chern-Simons formulation of three-dimensional gravity in anti–de Sitter space has supergravity generalizations, which can be formulated as Chern-Simons theories for the supergroups [23]

$$OSp(p|2) \times OSp(q|2).$$

The pure gravity case corresponds to $p = q = 0$. The existence of these supergravity theories, together with the bizarre coincidence noted above, suggests trying to generalize the BL-theory to the corresponding supergroup extensions of $SO(4)$. This idea encounters problems with spin and statistics, since the odd generators of this supergroup are not spacetime spinors.

Chapter 3

Ghost-Free Superconformal Action for Multiple M2-Branes

In the last chapter we studied the classical Lagrangian theories in three-dimensions with $OSp(8|4)$ superconformal symmetry discovered by Bagger and Lambert [16, 10, 11], as well as Gustavsson [12, 26]. The general rules for constructing such actions are based on a 3-algebra, which is characterized by structure constants $f^{ABC}{}_D$ and a metric h_{AB}. The initial assumption was that the metric should be positive definite. This led to the discovery of the BL $SO(4)$ theory with $SO(4)$ gauge symmetry [10]. In the last chapter and in [27] we verified the full superconformal symmetry of this theory and conjecture it uniqueness, i.e, that there are no other such theories, at least if one assumes a finite number of fields. This conjecture was subsequently proved in [28, 29]. Also a proposal for the physical interpretation of the BL $SO(4)$ theory in terms of M2-branes in M-theory at an M-fold singularity has been given in [30, 31].

However, these developments left unresolved the question whether it is possible to give a Lagrangian description of the conformal field theory associated with coincident M2-branes in flat 11-dimensional spacetime. That theory is known to correspond to the IR fixed point of $\mathcal{N} = 8$ super Yang-Mills theory. The question is whether there is a dual formulation of this fixed point theory. The only apparent way of evading the uniqueness theorem is to consider 3-algebras with an indefinite signature metric. This possibility was examined by three different groups [32, 33, 34], who proposed a new class of theories based on a 3-algebra with Lorentzian signature. The generators of the 3-algebra are the generators of an arbitrary semisimple Lie algebra plus two additional null generators $T^{\pm}$. The theory based on the 3-algebra associated to the gauge group $SU(N)$ or $U(N)$ looks like a good candidate for the theory of N coincident M2-branes, except for the fact that it contains unwanted negative norm states in the physical spectrum. This makes the theory nonunitary even though these states do not contribute to loops. Subsequent papers discussing the interpretation and application of Lorentzian 3-algebras include [35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. In particular, [46] proved that the Lorentzian 3-algebras considered in [32, 33, 34] are the only indecomposable Lorentzian 3-algebras (aside from the obvious $SO(3,1)$ variant of the Bagger-Lambert theory).

In this chapter we propose modifying the construction in [32, 33, 34] by gauging certain global symmetries.[1] We claim that this eliminates the unwanted ghost degrees of freedom while preserving all of the other symmetries. In section 3.1 we describe the BL-theory for general Lie algebras. In section 3.2 we explain the basic idea of our construction in a simplified model. And in section 3.3 we apply the same procedure to the theory of interest.

3.1 Lorentzian Metric BL-Theory for General Lie Algebras

In this section we describe the BL-theory for general Lie algebras based on a family of 3-algebras with Lorentzian metric proposed in [32, 33, 34]; we will follow the notation of [33]. The Lagrangian of a BL-theory is completely specified once a 3-algebra with a metric is given. The structure constants of the 3-algebra ${f^{ABC}}_D$ must satisfy the fundamental identity and $f^{ABCE} = {f^{ABC}}_D h^{DE}$, where h^{DE} is the 3-algebra metric, must be totally antisymmetric. In [33], the 3-algebra is constructed from an ordinary Lie algebra $\mathfrak{g}$ by adding two generators to $\mathfrak{g}$ called T^+ and T^- so that the 3-algebra has dimension $dim\,(\mathfrak{g})+2$. Its structure constants are given in terms of the $\mathfrak{g}$-structure constants ${f^{ab}}_c$ as

$$ {f^{+ab}}_c = {f^{ab}}_c, \tag{3.1} $$

with all other nonzero components of ${f^{ABC}}_D$ related by permuting, raising, or lowering indices. The generators of $\mathfrak{g}$ satisfy

$$ \begin{aligned} \left[T^a, T^b\right] &= {f^{ab}}_c T^c, \\ \mathrm{Tr}\left(T^a T^b\right) &= \delta^{ab}. \end{aligned} \tag{3.2} $$

The invariant metric of the 3-algebra is given by

$$ h^{+-} = -1, \qquad h^{++} = 0, \qquad h^{--} = 0, \qquad h^{ab} = \delta^{ab}. \tag{3.3} $$

The field content of the theory is summarized in the following table:

[1] After this work had been completed, Hirosi Ooguri informed us that Masahito Yamazaki is also considering this possibility.

Field	3d World-Volume	SO(8)	$\mathfrak{g}$	Dimension
$X_{\pm}^{I}$	Scalar	8_{v}	Singlet	1/2
X^{I}	Scalar	8_{v}	Adjoint	1/2
$\Psi_{\pm}$	Spinor	8_{s}	Singlet	1
Ψ	Spinor	8_{s}	Adjoint	1
$\mathcal{A}_{\mu}$	Gauge field	1	Adjoint	1
$\mathcal{B}_{\mu}$	Gauge field	1	Adjoint	1

With the choice of structure constants and 3-algebra metric given above, the BL-theory reduces to the following Lagrangian,

$$\begin{aligned}\mathcal{L} = & -\frac{1}{2}\mathrm{Tr}\left(D_{\mu}X^{I}D^{\mu}X^{I}\right) + D_{\mu}X_{+}^{I}D^{\mu}X_{-}^{I} + \frac{i}{2}\mathrm{Tr}\left(\bar{\Psi}\Gamma^{\mu}D_{\mu}\Psi\right) - \frac{i}{2}\bar{\Psi}_{+}\Gamma^{\mu}D_{\mu}\Psi_{-} - \frac{i}{2}\bar{\Psi}_{-}\Gamma^{\mu}D_{\mu}\Psi_{+} \\ & + \epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(\mathcal{B}_{\lambda}\left(\partial_{\mu}\mathcal{A}_{\nu} - [\mathcal{A}_{\mu},\mathcal{A}_{\nu}]\right)\right) - \frac{1}{12}\mathrm{Tr}\left(X_{+}^{I}\left[X^{J},X^{K}\right] + X_{+}^{J}\left[X^{K},X^{I}\right] + X_{+}^{K}\left[X^{I},X^{J}\right]\right)^{2} \\ & + \frac{i}{2}\mathrm{Tr}\left(\bar{\Psi}\Gamma_{IJ}X_{+}^{I}\left[X^{J},\Psi\right]\right) + \frac{i}{4}\mathrm{Tr}\left(\bar{\Psi}\Gamma_{IJ}\left[X^{I},X^{J}\right]\Psi_{+}\right) - \frac{i}{4}\mathrm{Tr}\left(\bar{\Psi}_{+}\Gamma_{IJ}\left[X^{I},X^{J}\right]\Psi\right), \end{aligned} \tag{3.4}$$

where $I = 1, ..., 8$ are the transverse coordinates and $X_{\pm}^{I} = \frac{1}{\sqrt{2}}\left(X_{0}^{I} \pm X_{1}^{I}\right)$. The covariant derivatives are defined as

$$D_{\mu}X^{I} = \partial_{\mu}X^{I} - 2\left[\mathcal{A}_{\mu},X^{I}\right] - \mathcal{B}_{\mu}X_{+}^{I}, \tag{3.5a}$$

$$D_{\mu}X_{-}^{I} = \partial_{\mu}X_{-}^{I} - \mathrm{Tr}\left(\mathcal{B}_{\mu}X^{I}\right), \tag{3.5b}$$

$$D_{\mu}X_{+}^{I} = \partial_{\mu}X_{+}^{I}, \tag{3.5c}$$

and similarly for the fermions. The gauge transformations are

$$\delta X^{I} = 2\left[\Lambda,X^{I}\right] + MX_{+}^{I}, \tag{3.6a}$$

$$\delta X_{-}^{I} = \mathrm{Tr}\left(MX^{I}\right), \tag{3.6b}$$

$$\delta X_{+}^{I} = 0, \tag{3.6c}$$

$$\delta\Psi = 2\left[\Lambda,\Psi\right] + M\Psi_{+}, \tag{3.6d}$$

$$\delta\Psi_{-} = \mathrm{Tr}\left(M\Psi\right), \tag{3.6e}$$

$$\delta\Psi_{+} = 0, \tag{3.6f}$$

$$\delta\mathcal{A}_{\mu} = \partial_{\mu}\Lambda + 2\left[\Lambda,\mathcal{A}_{\mu}\right], \tag{3.6g}$$

$$\delta\mathcal{B}_{\mu} = \partial_{\mu}M + 2\left[M,\mathcal{A}_{\mu}\right] + 2\left[\Lambda,\mathcal{B}_{\mu}\right], \tag{3.6h}$$

where Λ and M are infinitesimal matrices in the adjoint of $\mathfrak{g}$. The matrix Λ generates the $\mathcal{G}$ gauge transformations while M generates the noncompact subgroup transformations.

Finally, the $\mathcal{N}=8$ supersymmetry transformations (consistent with scale invariance) are

$$\begin{aligned}
\delta\mathcal{A}_\mu &= \frac{i}{2}\bar{\varepsilon}\Gamma_\mu\Gamma_I\left(X_+^I\Psi - X^I\Psi_+\right), && (3.7a)\\
\delta\mathcal{B}_\mu &= i\bar{\varepsilon}\Gamma_\mu\Gamma_I\left[X^I,\Psi\right], && (3.7b)\\
\delta X_\pm^I &= i\bar{\varepsilon}\Gamma^I\Psi_\pm, && (3.7c)\\
\delta X^I &= i\bar{\varepsilon}\Gamma^I\Psi, && (3.7d)\\
\delta\Psi_+ &= \partial_\mu X_+^I\Gamma^\mu\Gamma^I\varepsilon, && (3.7e)\\
\delta\Psi_- &= D_\mu X_-^I\Gamma^\mu\Gamma^I\varepsilon - \frac{1}{3}\mathrm{Tr}\left(X^IX^JX^K\right)\Gamma_{IJK}\epsilon, && (3.7f)\\
\delta\Psi &= D_\mu X^I\Gamma^\mu\Gamma^I\varepsilon - \frac{1}{2}X_+^I\left[X^J,X^K\right]\Gamma_{IJK}\epsilon. && (3.7g)
\end{aligned}$$

Note that this theory has a noncompact gauge group whose Lie algebra is a semidirect sum of any ordinary Lie algebra $\mathfrak{g}$ of a compact Lie group $\mathcal{G}$, and $\dim(\mathfrak{g})$ abelian generators. The gauge field $\mathcal{A}_\mu$ is associated with the compact part, while the gauge field $\mathcal{B}_\mu$ is associated with the noncompact part. Like all BL theories, it has $\mathcal{N}=8$ supersymmetry, scale invariance, conformal invariance, and $SO(8)$ R-symmetry. These combine to give the supergroup $OSp(8|4)$. The theory also has parity invariance. At the same time, it does not admit any tunable coupling constant, since any coupling constant can be absorbed in field redefinitions. Furthermore $\mathcal{G}$ can be chosen to be any compact Lie group. These are special features that are not shared by the $SO(4)$ BL-theory described in chapter 2, which is based on a 3-algebra with a positive-definite metric.

3.2 The Basic Idea

After integrating out certain auxiliary fields, the Lorentzian metric BL-theory described in section 3.1 contains terms of the form

$$S \sim \int d^3x\left(-\phi_+^{-2}\mathrm{Tr}(F^2) + \partial^\mu\phi_+\partial_\mu\phi_-\right).$$

This has manifest scale invariance if $\phi_\pm$ have dimension 1/2. This theory has a ghost degree of freedom, which (ignoring the first term) is reminiscent of the one contained in the covariant gauge-fixed string world-sheet theory prior to imposing the Virasoro constraints. In the present case, there are no Virasoro constraints, so the theory needs to be modified if we wish to make sense of it.

An important clue is that this theory has a global symmetry given by a constant shift of the field ϕ_-. Our proposal is to modify this theory by gauging this symmetry through the inclusion of a dimension 3/2 Stückelberg field C_μ

$$S \sim \int d^3x \left(-\phi_+^{-2}\mathrm{Tr}(F^2) + \partial^\mu\phi_+(\partial_\mu\phi_- - C_\mu)\right).$$

The gauge symmetry is simply given by

$$\delta\phi_- = \Lambda \quad \text{and} \quad \delta C_\mu = \partial_\mu\Lambda.$$

Classically, this theory is conformally invariant. (In the case of the M2-brane theory in the next section the conformal symmetry is expected to survive in the quantum theory.) This theory can be gauge fixed by setting $\phi_- = 0$. Integrating out C_μ gives a delta functional imposing the constraint $\partial_\mu\phi_+ = 0$. Thus, ϕ_+ is a constant, which is determined by a boundary condition. Calling the constant g_{YM}, we are left with pure Yang-Mills theory,

$$S \sim -g_{\mathrm{YM}}^{-2}\int d^3x \mathrm{Tr}(F^2).$$

The Yang-Mills theory is not conformally invariant, of course, since g_{YM} is dimensionful. However, this construction shows that it arises from spontaneous breaking of the conformal symmetry.

3.3 Modifying the Lorentzian Metric BL-Theory

Despite the numerous properties that make the Lorentzian metric BL-theory described in section 3.1 a promising candidate for describing multiple M2-branes in flat space, it has one very troubling feature. To see this, consider the fields X_-^I and Ψ_-. Note that the full dependence on these fields is given by

$$\mathcal{L}_- = -i\bar{\Psi}_+\Gamma^\mu\partial_\mu\Psi_- + \partial^\mu X_+^I\partial_\mu X_-^I. \tag{3.8}$$

As it stands, these terms describe propagating ghost degrees of freedom, which makes the theory unsatisfactory, since it is not unitary. At this point, it is useful to observe that the action has the following global shift symmetries (pointed out in [33]):

$$\delta X_-^I = \Lambda^I \quad \text{and} \quad \delta\Psi_- = \eta.$$

Also note that Ψ_- and X_-^I do not appear in any of the gauge or supersymmetry transformations of the other fields. We will show that it is possible to eliminate the ghosts from the theory, while preserving all of its desirable properties, by promoting these global shift symmetries to local symmetries.

To gauge the global shift symmetries described above we introduce two new gauge fields: a vector field C^I_μ in the vector representation of $SO(8)$, and a 32-component Majorana-Weyl spinor χ satisfying $\Gamma^{012}\chi = -\chi$. These appear in two new terms that we add to the Lagrangian:

$$\mathcal{L}_{\text{new}} = \bar{\Psi}_+\chi - \partial^\mu X^I_+ C^I_\mu. \tag{3.9}$$

Note that C^I_μ must have dimension 3/2 and χ must have dimension 2 to preserve scale invariance. The new local shift symmetries are

$$\delta X^I_- = \Lambda^I, \quad \delta C^I_\mu = \partial_\mu \Lambda^I, \tag{3.10}$$

and

$$\delta\Psi_- = \eta, \quad \delta\chi = i\Gamma^\mu\partial_\mu\eta. \tag{3.11}$$

There is one additional local symmetry of equation (3.9), which is relatively trivial, namely

$$\delta C^I_\mu = \partial^\rho \tilde{\Lambda}^I_{\mu\rho}, \quad \text{where} \quad \tilde{\Lambda}^I_{\mu\rho} = -\tilde{\Lambda}^I_{\rho\mu}. \tag{3.12}$$

C^I_μ and χ are invariant under the original gauge symmetries.

Now let us consider the supersymmetry of the modified theory. The supersymmetry transformations of all the old fields are unchanged. In particular,

$$\delta X^I_+ = i\bar{\varepsilon}\Gamma^I\Psi_+,$$

and

$$\delta\Psi_+ = \Gamma^\mu\partial_\mu X^I_+\Gamma^I\varepsilon.$$

The supersymmetries of the new gauge fields must be defined in such a way that $\mathcal{L}_{\text{new}}$ is invariant. We will find that the resulting supersymmetry algebra closes on shell when one takes account of the new gauge symmetries. Under supersymmetry

$$\delta C^I_\mu = \bar{\varepsilon}\Gamma^I\Gamma_\mu\chi,$$

and

$$\delta\chi = i\Gamma^I\varepsilon\,\partial^\mu C^I_\mu.$$

Using these four transformation rules, it is easy to see that both $\mathcal{L}_{\text{new}}$ and the equations of motion are supersymmetric.

We will now check the closure of all the algebras. The fact that the supersymmetry variations of C^I_μ and χ are not invariant under the new gauge transformations implies that the supersymmetry transformations do not commute with these gauge transformations. Specifically, one finds that

$$[\delta(\Lambda), \delta(\varepsilon)] = \delta(\eta), \quad \text{where} \quad \eta = \Gamma^\mu \Gamma^I \partial_\mu \Lambda^I \varepsilon,$$

and

$$[\delta(\eta), \delta(\varepsilon)] = \delta(\Lambda) + \delta(\tilde{\Lambda}) \quad \text{where} \quad \Lambda^I = i\bar{\varepsilon}\Gamma^I \eta \quad \text{and} \quad \tilde{\Lambda}^I_{\mu\rho} = i\bar{\varepsilon}\Gamma^I \Gamma_{\mu\rho}\eta.$$

The supersymmetry algebra is slightly affected, as well. Specifically, we find that

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]C^I_\mu = \delta(\xi)C^I_\mu + \delta(\tilde{\Lambda})C^I_\mu,$$

where $\xi^\rho = 2i\bar{\varepsilon}_1\Gamma^\rho\varepsilon_2$, as usual, and $\tilde{\Lambda}^I_{\mu\rho} = \xi_\mu C^I_\rho - \xi_\rho C^I_\mu$. Similarly, for χ we find that

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]\chi = \delta(\xi)\chi + \delta(\eta)\chi,$$

where $\eta = \left(-\bar{\epsilon}_1\Gamma^\mu\epsilon_2\Gamma_\mu + \frac{1}{4}\bar{\epsilon}_1\Gamma^{LM}\epsilon_2\Gamma_{LM}\right)\chi$. One also finds that requiring the on-shell closure of the commutator $[\delta(\varepsilon_1), \delta(\varepsilon_2)]\Psi_-$ gives the expected equation of motion for Ψ_- after noting that the commutator receives a contribution from $\delta(\eta)\Psi_-$. In summary, we have verified that the supersymmetries close on shell into translations, the old gauge transformations, and the new gauge transformations given by Eqs (3.10)–(3.12).

3.4 Discussion

After modifying the theory by introducing the new gauge fields C_μ and χ, it still has scale invariance, $\mathcal{N} = 8$ supersymmetry, no coupling constant, and can accommodate any Lie group in its gauge group, which are all desirable properties for describing multiple M2-branes in flat space. In addition, we can use the new gauge symmetries to make the gauge choices

$$X^I_- = \Psi_- = 0.$$

This removes the kinetic terms for the ghosts and changes the supersymmetry transformations for C_μ and χ by induced gauge transformations, i.e., $\delta C^I_\mu = \bar{\epsilon}\Gamma^I\Gamma_\mu\chi + \partial_\mu\Lambda^I$ and $\delta\chi = i\Gamma^I\epsilon\partial^\mu C^I_\mu + i\Gamma^\mu\partial_\mu\eta$ for appropriate choices of Λ^I and η. Furthermore, the equations of motion that come from varying the new fields are

$$\partial_\mu X^I_+ = 0, \quad \Psi_+ = 0.$$

The first equation implies that X^I_+ is a constant. Any nonzero choice spontaneously breaks conformal symmetry and breaks the R-symmetry to an unbroken $SO(7)$ subgroup. On the other hand, the choice $X_+^I = 0$ gives a free theory.

We can use the $SO(8)$ R-symmetry to choose the nonzero component of X_+^I to be in the 8 direction, $X_+^I = v\delta^{I8}$. Also, the noncompact gauge fields, $\mathcal{B}$, which appear quadratically can be integrated out. This leaves a maximally supersymmetric three-dimensional Yang-Mills theory with $SO(7)$ R-symmetry:

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4v^2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) - \frac{1}{2}\mathrm{Tr}\left(D'_\mu X^i D'_\mu X^i\right) + \frac{i}{2}\mathrm{Tr}\left(\bar{\Psi}\Gamma^\mu D'_\mu \Psi\right) \\ & + \frac{i}{2}\mathrm{Tr}\left(\bar{\Psi}\Gamma_{8i}\left[X^i, \Psi\right]\right) - \frac{v^2}{4}\mathrm{Tr}\left(\left[X^i, X^j\right]\right)^2, \end{aligned} \tag{3.13}$$

where the index $i = 1, ..., 7$, and D'_μ and $F_{\mu\nu}$ depend only the massless gauge field $\mathcal{A}$ associated with the maximally compact subgroup of the original gauge group. Note that this is an exact result—not just the leading term in a large-v expansion. This is a supersymmetric generalization of the toy model described in section 3.2.

To summarize, in this chapter we have proposed a modification of the Bagger-Lambert theory that removes the ghosts when the 3-algebra has a Lorentzian signature metric, thus ensuring unitarity. Such theories evade the no-go theorem, which states that there is essentially only one nontrivial 3-algebra with positive-definite metric. Our modification of the Lorentzian 3-algebra theories in [32, 33, 34] breaks the conformal symmetry spontaneously and reduces them to maximally supersymmetric three-dimensional Yang-Mills theories.[2] This result is somewhat disappointing inasmuch as it means that we are no closer to the original goal of understanding the $v \to \infty$ IR fixed point theory that describes coincident M2-branes in 11 noncompact dimensions.

As things stand, it appears that the BL $SO(4)$ theory is the only genuinely new maximally supersymmetric superconformal theory.

[2]Reference [34] observed that if one chooses X_+^I to be constant and Ψ_+ to be zero, then the theory reduces to $\mathcal{N} = 8$ SYM. However, they did not deduce these choices from an action principle.

Chapter 4

Studies of the ABJM Theory in a Formulation with Manifest SU(4) R-Symmetry

As we saw in the previous chapters the BL theory was conjectured [27] and proved [28, 29] to be the unique three-dimensional superconformal field theory with maximal supersymmetry. The generalizations based on Lorentzian 3-algebras [32, 33, 34] turned out to be equivalent to the original super Yang-Mills theories once the ghosts were eliminated [47, 48, 49]. At that point, it looked like the only possibility left to explore was whether there are other 3-algebras (whose metric is neither positive-definite not Lorentzian) that open new possibilities. However, Aharony, Bergman, Jafferis, and Maldacena (ABJM) in reference [14] showed that a better way to open new possibilities was to consider theories with reduced supersymmetry.

In this chapter we examine the class of three-dimensional superconformal field theories discovered by ABJM [14]. These theories are superconformal Chern-Simons gauge theories with $\mathcal{N}=6$ supersymmetry. When the gauge group is chosen to be $U(N)\times U(N)$ and the Chern-Simons level is k, these theories are conjectured to be dual to M-theory on $AdS_4\times S^7/\mathbb{Z}_k$ with N units of flux. More precisely, this is the appropriate dual description for $N^{1/5} >> k$. In the opposite limit, $N^{1/5} << k << N$, a dual description in terms of type IIA string theory on $AdS_4\times \mathbb{C}P^3$ is more appropriate. A large-N expansion for fixed 't Hooft parameter $\lambda = N/k$ can be defined. These developments raise the hope that this duality can be analyzed in the same level of detail as has been done for the duality between $\mathcal{N}=4$ super Yang-Mills theory with a $U(N)$ gauge group in four dimensions and type IIB superstring theory on $AdS_5\times S^5$ with N units of flux.

After the ABJM paper appears, quite a few papers examined several of its properties as well as possible generalizations. Among the first are [50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63]. New superconformal Chern-Simons theories with $\mathcal{N}=5$ supersymmetry have been constructed in [59]. Some of these $\mathcal{N}=5$ theories should be dual to the D_{k+2} orbifolds described in [60].

In reference [61], Bagger and Lambert show that the ABJM theories correspond to a class of 3-algebras in which the bracket $[T^a, T^b, T^c]$ is no longer antisymmetric in all three indices. The actions and supersymmetry transformations that are derived in [59, 61] appear to be equivalent to the actions and supersymmetry transformations that are obtained in this chapter (without reference to 3-algebras). Also, a large class of superconformal Chern-Simons theories with $\mathcal{N} = 4$ supersymmetry was constructed by Gaiotto and Witten [64]. This was generalized to include twisted hypermultiplets in [59, 43]. This generalization includes the Bagger-Lambert theory as a special case. Moreover, all the ABJM theories turn out to be special cases of the generalized Gaiotto-Witten theories in which the supersymmetry is enhanced to $\mathcal{N} = 6$.

The purpose of this chapter is to recast the ABJM theory in a form for which the $SU(4)$ R-symmetry of the action and the supersymmetry transformations is manifest and to use this form to study some of its properties. The existence of such formulas is a consequence of what was found in [14]. We also verify the conformal supersymmetry of the action, which is not a logical consequence of previous results. Since this symmetry is a necessary requirement for the validity of the proposed duality, its verification can be viewed as an important and nontrivial test of the duality. We also recast the potential, which is sixth order in the scalar fields, in a new form.[1] This new form should be useful for studying the moduli space of supersymmetric vacua of the theory, as well as the vacuum structure of various deformations of the ABJM theory. Although we discuss the gauge group $U(N) \times U(N)$, all of our analysis also holds for the straightforward generalization to $U(M) \times U(N)$.

Some of our results are new and others confirm results that have been obtained previously. The ABJM theories were formulated in [14] using auxiliary fields associated with $\mathcal{N} = 2$ superfields. In this formulation only an $SU(2) \times SU(2)$ subgroup of the $SU(4)$ R-symmetry is manifest, though the full $SU(4)$ symmetry has been deduced. In addition, [14] deduced a manifestly $SU(4)$ invariant form of the scalar field potential, which is sixth order in the scalar fields. The quartic interaction terms that have two scalar and two spinor fields were also recast in an $SU(4)$ covariant form in [50]. Our results are in agreement with both of these.

4.1 The $\mathbf{U(1) \times U(1)}$ Theory

The field content of ABJM theories consists of scalars, spinors, and gauge fields. The $U(1) \times U(1)$ theory has fewer indices to keep track of, and it is quite a bit simpler, than the full $U(N) \times U(N)$ theory; so it is a good place to start.

There are four complex scalars X_A and their adjoints X^A. (We choose not to use adjoint or complex conjugation symbols to keep the notation from becoming too cumbersome.) A lower index

[1] A similar formula also appears in [61].

labels the **4** representation of the global $SU(4)$ R-symmetry and an upper index labels the complex-conjugate $\bar{\mathbf{4}}$ representation.

Similarly, the Fermi fields are Ψ^A and Ψ_A. These are also two-component spinors, though that index is not displayed. As usual, the notation $\bar{\Psi}^A$ or $\bar{\Psi}_A$ implies transposing the spinor index and right multiplication by γ^0. Note, however, that for our definition there is no additional complex conjugation, so in all cases a lower index indicates a **4** and an upper index indicates a $\bar{\mathbf{4}}$. With these conventions various identities that hold for Majorana spinors can be used for these spinors, as well, even though they are complex (Dirac), for example, $\bar{\Psi}^A \Psi_B = \bar{\Psi}_B \Psi^A$. The 2×2 Dirac matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The index $\mu = 0, 1, 2$ is a 3-dimensional Lorentz index, and the signature is $(-, +, +)$. It is convenient to use a Majorana representation, which implies that γ^μ is real. We also choose a representation for which $\gamma^{\mu\nu\lambda} = \varepsilon^{\mu\nu\lambda}$. In particular, this means that $\gamma^0\gamma^1\gamma^2 = 1$. For example, one could choose $\gamma^0 = i\sigma^2$, $\gamma^1 = \sigma^1$, and $\gamma^2 = \sigma^3$.

The $U(1)$ gauge fields are denoted A_μ and $\hat{A}_\mu$. The fields X_A and Ψ^A have $U(1)$ charges $(+, -)$, while their adjoints have charges $(-, +)$. Thus, for example,

$$D_\mu X_A = \partial_\mu X_A + i(A_\mu - \hat{A}_\mu) X_A,$$

and

$$D_\mu X^A = \partial_\mu X^A - i(A_\mu - \hat{A}_\mu) X^A.$$

We choose to normalize fields so that the level-k Lagrangian is k times the level-1 Lagrangian. With this convention, the $N = 1$ action is

$$S = \frac{k}{2\pi} \int d^3x \left(-D^\mu X^A D_\mu X_A + i\bar{\Psi}_A \gamma^\mu D_\mu \Psi^A + \frac{1}{2}\varepsilon^{\mu\nu\lambda}(A_\mu \partial_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda) \right).$$

The claim is that this action describes an $\mathcal{N} = 6$ superconformal theory with $OSp(6|4)$ superconformal symmetry. The R-symmetry is $Spin(6) = SU(4)$ and the conformal symmetry is $Sp(4) = Spin(3, 2)$. The supercharges transform as the **6** representation of $SU(4)$. Both the Poincaré and conformal supercharges are 6-vectors. Each accounts for 12 of the 24 fermionic generators of the superconformal algebra.

The antisymmetric product of two **4**s gives a **6**. The invariant tensor (or Clebsch-Gordan coefficients) describing this is denoted $\Gamma^I_{AB} = -\Gamma^I_{BA}$, since these can be interpreted as six matrices satisfying a Clifford algebra. More precisely, if one also defines $\tilde{\Gamma}^I = (\Gamma^I)^\dagger$, or in components

$$\tilde{\Gamma}^{IAB} = \frac{1}{2}\varepsilon^{ABCD}\Gamma^I_{CD} = -\left(\Gamma^I_{AB}\right)^*,$$

then[2]

$$\Gamma^I\tilde{\Gamma}^J + \Gamma^J\tilde{\Gamma}^I = 2\delta^{IJ}. \tag{4.1}$$

Note that γ^μ are 2×2 matrices and Γ^I are 4×4 matrices. They act on different vector spaces, and therefore they trivially commute with one another.

The supersymmetry transformations of the matter fields are

$$\delta X_A = i\Gamma^I_{AB}\bar{\Psi}^B\varepsilon^I, \tag{4.2}$$

$$\delta\Psi_A = \Gamma^I_{AB}\gamma^\mu\varepsilon^I D_\mu X^B, \tag{4.3}$$

and their adjoints, which are

$$\delta X^A = -i\tilde{\Gamma}^{IAB}\bar{\Psi}_B\varepsilon^I, \tag{4.4}$$

$$\delta\Psi^A = -\tilde{\Gamma}^{IAB}\gamma^\mu\varepsilon^I D_\mu X_B. \tag{4.5}$$

For the gauge fields we have

$$\delta A_\mu = \delta\hat{A}_\mu = -\Gamma^I_{AB}\bar{\Psi}^A\gamma_\mu\varepsilon^I X^B - \tilde{\Gamma}^{IAB}\bar{\Psi}_A\gamma_\mu\varepsilon^I X_B. \tag{4.6}$$

The verification that these leave the action invariant is given in the appendix C.

Note that the covariant derivatives only involve A_-, where

$$A_\pm = A \pm \hat{A}.$$

Therefore, let us rewrite the Chern-Simons terms using [65]

$$\int (A\wedge dA - \hat{A}\wedge d\hat{A}) = \int A_+\wedge dA_- = \int A_-\wedge dA_+.$$

Since this is the only appearance of A_+ in the action, it can be integrated out to give the delta functional constraint

$$F_- = dA_- = 0.$$

The A_- equation of motion, on the other hand, just identifies F_+ with the dual of the charge current. Since the kinetic terms are defined with a flat connection A_-, this is just a free theory when the topology is trivial, which is the case for $k=1$. Then this theory has $\mathcal{N}=8$ superconformal symmetry.

ABJM proposes to treat F_+ as an independent variable and to add a Lagrange multiplier term

[2]An explicit realization in terms of Pauli matrices is given by $\Gamma^1 = i\sigma_2\otimes 1$, $\Gamma^2 = \sigma_2\otimes\sigma_1$, $\Gamma^3 = \sigma_2\otimes\sigma_3$, $\Gamma^4 = 1\otimes\sigma_2$, $\Gamma^5 = i\sigma_1\otimes\sigma_2$, $\Gamma^6 = i\sigma_3\otimes\sigma_2$.

to ensure that F_+ is a curl

$$S_\tau = \frac{1}{4\pi}\int \tau\varepsilon^{\mu\nu\lambda}\partial_\mu F_{+\nu\lambda}d^3x.$$

Then the quantization condition on F_+ requires that τ has period 2π. They then explain that after gauge fixing $\tau = 0$ one is left with a residual $\mathbb{Z}_k$ gauge symmetry under which $X^A \to \exp(2\pi i/k)X^A$ and similarly for Ψ_A. Thus one is left with a sigma model on $\mathbb{C}^4/\mathbb{Z}_k$. This breaks the supersymmetry from $\mathcal{N} = 8$ to $\mathcal{N} = 6$ for $k > 2$. The reason for this is that the 8-component $Spin(8)$ supercharge decomposes with respect to the $SU(4) \times U(1)$ subgroup as $6_0 + 1_2 + 1_{-2}$. Because of their $U(1)$ charges, the singlets transform under a $\mathbb{Z}_k$ transformation as $Q \to \exp(\pm 4\pi i/k)Q$. Therefore two of the supersymmetries are broken for $k > 2$.

This analysis of the $U(1)$ factors continues to apply in the $U(N) \times U(N)$ theories with $N > 1$. The Bagger-Lambert theory corresponds to the gauge group $SU(2) \times SU(2)$. Since it has no $U(1)$ factors, no discrete Z_k gauge symmetry arises, and this theory has $\mathcal{N} = 8$ superconformal symmetry for all values of k. So, it is different from the $U(2) \times U(2)$ ABJM theory, and its interpretation in terms of branes or geometry (see [66, 31]) must also be different.

4.2 The U(N) × U(N) Theory

The field content of the $U(N) \times U(N)$ ABJM theory consists of four $N \times N$ matrices of complex scalars $(X_A)^a{}_{\hat{a}}$ and their adjoints $(X^A)^{\hat{a}}{}_a$. These transform as $(\bar{\mathbf{N}}, \mathbf{N})$ and $(\mathbf{N}, \bar{\mathbf{N}})$ representations of the gauge group, respectively. Similarly, the spinor fields are matrices $(\Psi^A)^a{}_{\hat{a}}$ and their adjoints $(\Psi_A)^{\hat{a}}{}_a$. The $U(N)$ gauge fields are hermitian matrices $A^a{}_b$ and $\hat{A}^{\hat{a}}{}_{\hat{b}}$. In matrix notation, the covariant derivatives are

$$D_\mu X_A = \partial_\mu X_A + i(A_\mu X_A - X_A\hat{A}_\mu),$$

and

$$D_\mu X^A = \partial_\mu X^A + i(\hat{A}_\mu X^A - X^A A_\mu),$$

with similar formulas for the spinors. Infinitesimal gauge transformations are given by

$$\delta A_\mu = D_\mu\Lambda = \partial_\mu\Lambda + i[A_\mu, \Lambda], \tag{4.7a}$$

$$\delta \hat{A}_\mu = D_\mu\hat{\Lambda} = \partial_\mu\hat{\Lambda} + i[\hat{A}_\mu, \hat{\Lambda}], \tag{4.7b}$$

$$\delta X_A = -i\Lambda X_A + iX_A\hat{\Lambda}, \tag{4.7c}$$

and so forth.

The action consists of terms that are straightforward generalizations of those of the $U(1) \times U(1)$ theory, as well as new interaction terms that vanish for $N = 1$. The kinetic and Chern-Simons terms are

$$S_{\rm kin} = \frac{k}{2\pi}\int d^3x\,{\rm tr}\left(-D^\mu X^A D_\mu X_A + i\bar{\Psi}_A\gamma^\mu D_\mu\Psi^A\right),$$

and

$$S_{\rm CS} = \frac{k}{2\pi}\int d^3x\,\varepsilon^{\mu\nu\lambda}{\rm tr}\Big(\frac{1}{2}A_\mu\partial_\nu A_\lambda + \frac{i}{3}A_\mu A_\nu A_\lambda - \frac{1}{2}\hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\Big).$$

Additional interaction terms of the schematic form $X^2\Psi^2$ and X^6 remain to be determined. These terms are not required to deduce the equations of motion of the gauge fields, which are

$$J^\mu = \frac{1}{2}\varepsilon^{\mu\nu\lambda}F_{\nu\lambda} \quad \text{and} \quad \hat{J}^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\lambda}\hat{F}_{\nu\lambda},$$

where

$$J^\mu = iX_A D^\mu X^A - iD^\mu X_A X^A - \bar{\Psi}^A\gamma^\mu\Psi_A,$$

and

$$\hat{J}^\mu = iX^A D^\mu X_A - iD^\mu X^A X_A - \bar{\Psi}_A\gamma^\mu\Psi^A.$$

Note that in the special case of $U(1)\times U(1)$ one has $J^\mu = -\hat{J}^\mu$, and hence the equations of motion imply $F_{\mu\nu} = \hat{F}_{\mu\nu}$.

In matrix notation, the supersymmetry transformations of the matter fields are

$$\delta X_A = i\Gamma^I_{AB}\bar{\varepsilon}^I\Psi^B,$$

and

$$\delta\bar{\Psi}_A = -\Gamma^I_{AB}\bar{\varepsilon}^I\gamma^\mu D_\mu X^B + \delta_3\bar{\Psi}_A,$$

or equivalently

$$\delta\Psi_A = \Gamma^I_{AB}\gamma^\mu\varepsilon^I D_\mu X^B + \delta_3\Psi_A,$$

and their adjoints, which are

$$\delta X^A = -i\tilde{\Gamma}^{IAB}\bar{\Psi}_B\varepsilon^I,$$

and

$$\delta\Psi^A = -\tilde{\Gamma}^{IAB}\gamma^\mu\varepsilon^I D_\mu X_B + \delta_3\Psi^A,$$

or equivalently

$$\delta\bar{\Psi}^A = \tilde{\Gamma}^{IAB}\bar{\varepsilon}^I\gamma^\mu D_\mu X_B + \delta_3\bar{\Psi}^A.$$

The terms denoted δ_3 are cubic in X and are given below. The supersymmetry transformations of the gauge fields are

$$\delta A_\mu = \Gamma^I_{AB}\bar{\varepsilon}^I\gamma_\mu\Psi^A X^B - \tilde{\Gamma}^{IAB}X_B\bar{\Psi}_A\gamma_\mu\varepsilon^I,$$

$$\delta\hat{A}_\mu = \Gamma^I_{AB} X^B \bar{\varepsilon}^I \gamma_\mu \Psi^A - \tilde{\Gamma}^{IAB} \bar{\Psi}_A \gamma_\mu \varepsilon^I X_B.$$

Note that $\delta A_\mu \neq \delta\hat{A}_\mu$ for $N > 1$. They are matrices in different spaces.

In the appendix C we show that supersymmetry requires the choice

$$\delta_3 \Psi^A = N^{IA} \varepsilon^I \quad \text{and} \quad \delta_3 \Psi_A = N^I_A \varepsilon^I, \tag{4.8}$$

where

$$N^{IA} = \tilde{\Gamma}^{IAB}(X_C X^C X_B - X_B X^C X_C) - 2\tilde{\Gamma}^{IBC} X_B X^A X_C, \tag{4.9}$$

and

$$N^I_A = (N^{IA})^\dagger = \Gamma^I_{AB}(X^C X_C X^B - X^B X_C X^C) - 2\Gamma^I_{BC} X^B X_A X^C. \tag{4.10}$$

Note that these expressions vanish when the matrices X^A (and their adjoints X_A) are diagonal.

All the possible structures for the $\Psi^2 X^2$ terms are

$$L_{4\mathrm{a}} = i\varepsilon^{ABCD}\mathrm{tr}(\bar{\Psi}_A X_B \Psi_C X_D) - i\varepsilon_{ABCD}\mathrm{tr}(\bar{\Psi}^A X^B \Psi^C X^D), \tag{4.11a}$$

$$L_{4\mathrm{b}} = i\mathrm{tr}(\bar{\Psi}^A \Psi_A X_B X^B) - i\mathrm{tr}(\bar{\Psi}_A \Psi^A X^B X_B), \tag{4.11b}$$

$$L_{4\mathrm{c}} = 2i\mathrm{tr}(\bar{\Psi}_A \Psi^B X^A X_B) - 2i\mathrm{tr}(\bar{\Psi}^B \Psi_A X_B X^A). \tag{4.11c}$$

The coefficients are chosen so that $L_4 = L_{4\mathrm{a}} + L_{4\mathrm{b}} + L_{4\mathrm{c}}$ is the correct result required by supersymmetry, as is demonstrated in the appendix C.

The Lagrangian also contains a term $L_6 = -V$ that is sixth order in the scalar fields. The scalar potential V is expected to be nonnegative and to vanish for a supersymmetric vacuum. An $SU(4)$ covariant formula for V in terms of the fields X^A and X_A has been given in [14, 50]

$$\begin{aligned} V = -\frac{1}{3}\mathrm{tr}\Big[& X^A X_A X^B X_B X^C X_C + X_A X^A X_B X^B X_C X^C \\ & + 4 X_A X^B X_C X^A X_B X^C - 6 X^A X_B X^B X_A X^C X_C \Big], \end{aligned} \tag{4.12}$$

a result that we confirm in the appendix C.

This formula for V is not expressed as a sum of squares, which makes it inconvenient for determining the extrema. For a supersymmetric vacuum, $\delta\Psi^A = \delta\Psi_A = 0$. In particular, for a solution in which the scalar fields X^A and X_A are constant, and the gauge fields vanish, the variations $\delta_3\Psi^A$ and $\delta_3\Psi_A$ should vanish. This implies that $N^{IA} = 0$ and $N^I_A = (N^{IA})^\dagger = 0$. The way to ensure these requirements, as well as manifest $SU(4)$ symmetry, is for the potential to take the form

$$V = \frac{1}{6}\mathrm{tr}(N^{IA} N^I_A). \tag{4.13}$$

The definitions of N^{IA} and N^I_A are given in equations (4.9) and (4.10). It is straightforward to verify the equivalence of equations (4.12) and (4.13) for this choice of the coefficient by using the key identity

$$\Gamma^I_{AB}\tilde{\Gamma}^{ICD} = -2\delta^{CD}_{AB}.$$

The indicated relationship between the potential and $\delta_3\Psi$ in equation (4.13) should be quite general in theories of this type. As has already been noted, N^{IA} and N^I_A vanish when the scalar fields are diagonal matrices. To get the expected moduli space, these should be the only choices for which they vanish (modulo gauge transformations).

4.3 Conclusion

The study of ABJM theories has become a hot topic. The technology that has been developed in the study of the duality between four-dimensional superconformal gauge theories and AdS_5 vacua of type IIB superstring theory can now be adapted to a new setting. It should now be possible to study the duality between three-dimensional superconformal Chern-Simon theories and AdS_4 vacua of type IIA superstring theory and M-theory. A great deal should be learned in the process, and there may even be applications to other areas of physics.

The contributions of this chapter to this subject are modest: We have verified the Poincaré supersymmetries of the ABJM theory in a formalism with manifest $SU(4)$ symmetry. The action that we obtained agrees with results given in [14, 50, 61]. We have also verified by explicit calculation that this action has the conformal supersymmetries that are required by the proposed duality. Since this is not implied by any previous calculations, it is an important (and nontrivial) test of the duality. Taken together with the Poincaré supersymmetries, this implies the full $OSp(6|4)$ superconformal symmetry of the action. We have also recast the sextic potential as a sum of squares in equation (4.13), a form that should prove useful in future studies.

Chapter 5

One-Loop Corrections to Type IIA String Theory on $\mathbf{AdS_4 \times CP^3}$

As we saw in the previous chapter, ABJM discovered a new example of the AdS/CFT correspondence that relates type IIA string theory on $AdS_4 \times CP^3$ to a three-dimensional $\mathcal{N} = 6$ Chern-Simons theory [14].

Since then, much of the analysis that was done to test the AdS_5/CFT_4 duality has been repeated for the AdS_4/CFT_3 duality. For example, various sectors of the planar Chern-Simons theory were shown to be integrable up to four loops in perturbation theory, i.e., it was shown that the dilatation operator in these sectors corresponds to a spin-chain Hamiltonian that can be diagonalized by solving Bethe equations [55, 67, 68, 69]. Moreover, the classical string theory dual to the planar gauge theory was also shown to be integrable, i.e., the equations of motion for the string theory sigma model were recast as a flatness condition for a certain one-form known as the Lax connection [70, 71, 72, 73]. It should be noted that classical integrability has only been demonstrated in the subsector of the $AdS_4 \times CP^3$ superspace described by the $OSp(6|4)/(U(3) \times SO(3,1))$ supercoset, and that κ-symmetry in the coset sigma model breaks down for string solutions that move purely in AdS_4 [72]. Demonstrating integrability in the full superspace requires more general methods [74]. The pure spinor string theory on $AdS_4 \times CP^3$ was studied in [75, 76]. An important consequence of the Lax connection is that any classical solution to the sigma model equations of motion can be mapped into a multisheeted Riemann surface known as an algebraic curve [77, 78, 79]. The AdS_4/CFT_3 algebraic curve was constructed in [80]. Following these developments, a set of all-loop Bethe equations, which interpolate between the gauge theory Bethe equations at weak coupling and the string theory algebraic curve at strong coupling, were proposed in [81]. The all-loop Bethe ansatz is a powerful tool for testing the AdS/CFT correspondence.

While the AdS_4/CFT_3 duality shares certain features with the AdS_5/CFT_4 duality, it also exhibits several new features. For example, when one looks at quantum excitations to the string theory sigma model in the Penrose limit of type IIA string theory on $AdS_4 \times CP^3$, one finds that half of the excitations are twice as massive as the other half [52, 58, 57]. The latter are subsequently referred to as "light" and the former are referred to as "heavy." This is in contrast to what was found when looking at the Penrose limit of type IIB string theory on $AdS_5 \times S^5$, where all the excitations have the same mass [5]. Various properties of the heavy and light modes were studied in [82, 83, 84]. Furthermore, the AdS_4/CFT_3 magnon dispersion relation was found to be $\epsilon = \frac{1}{2}\sqrt{1+8h(\lambda)\sin^2\frac{p}{2}}$ where $h(\lambda) = \lambda$ for $\lambda \gg 1$ and $h(\lambda) = 2\lambda^2$ for $\lambda \ll 1$. This is in contrast to the magnon dispersion relation for AdS_5/CFT_4, where $h(\lambda) = \frac{\sqrt{\lambda}}{4\pi}$ for all values of λ. One possible reason why the AdS_4/CFT_3 magnon dispersion receives corrections at strong coupling is that the theory only has 3/4 maximal supersymmetry. Another consequence of the less than maximal supersymmetry is that the radius of $AdS_4 \times CP^3$ varies as a function of λ, although this only becomes relevant at two loops in the sigma model [85].

Perhaps the most puzzling new feature of the AdS_4/CFT_3 correspondence arises when computing the one-loop correction to the energy of classical solutions to type IIA string theory in $AdS_4 \times CP^3$. Note that the one-loop corrections we are describing correspond to quantum corrections to the world-sheet theory and α' corrections to the classical string theory. In particular, several groups found a disagreement with the all-loop Bethe ansatz after computing the one-loop correction to the energy of the folded spinning string in $AdS_4 \times CP^3$. In computing the one-loop correction, these groups used the same prescription for adding up fluctuation frequencies that was used in $AdS_5 \times S^5$ [86, 87, 88]. The authors of [89] subsequently proposed an alternative summation prescription that achieves agreement with the all-loop Bethe ansatz by treating the frequencies of heavy and light modes on unequal footing. This prescription is not applicable to type IIB string theory on $AdS_5 \times S^5$ because there is no distinction between heavy and light frequencies in this theory. Hence, the prescription proposed in [89] is special to the AdS_4/CFT_3 correspondence. Reference [90] pointed out that the discrepancy can also be resolved if one takes $\sqrt{h(\lambda)} = \sqrt{\lambda} + a_1 + \mathcal{O}\left(1/\sqrt{\lambda}\right)$ with $a_1 \neq 0$ when doing world-sheet calculations. Although the algebraic curve calculation in [91] found that this correction should be zero, the authors in [90] argue that different values of a_1 can be consistent because a_1 may be scheme dependent.

In this chapter we extend the study of one-loop corrections in $AdS_4 \times CP^3$ by computing one-loop corrections for solutions with nontrivial support in CP^3 and trivial support in AdS_4, notably a rotating point-particle and a circular string with two equal angular momenta in CP^3, which we refer to as the spinning string. The latter solution is the AdS_4/CFT_3 analogue of the $SU(2)$ circular string that was discovered in [92] and studied extensively in the AdS_5/CFT_4 correspondence [93, 94, 95]. The point-particle and spinning string solutions are especially interesting to study in the AdS_4/CFT_3 context because they avoid the κ-symmetry issues described above (since they have trivial support in AdS_4). Various string solutions with support in CP^3 were also constructed in [96, 97], however one-loop corrections were not considered in those papers.

In order to compute the one-loop correction to the energy of a classical solution, we must first compute the spectrum of fluctuations about the solution. This can be computed by expanding the Green-Schwarz (GS) action to quadratic order in the fluctuations and finding the normal modes of the resulting equations of motion. We refer to this method as the world-sheet (WS) approach. Alternatively, the spectrum can be computed from the algebraic curve corresponding to this solution using semiclassical techniques. We refer to this as the algebraic curve (AC) approach. This approach was developed for type IIB string theory in $AdS_5 \times S^5$ in [98] and then adapted to type IIA string theory in $AdS_4 \times CP^3$ in [80]. In this chapter, we compute the spectrum of fluctuations about the point-particle and spinning string using both approaches and find that the algebraic curve frequencies agree with the world-sheet frequencies up to constant shifts and shifts in mode number.

Although the algebraic curve and world-sheet spectra look very similar, they have very different properties. In particular, the algebraic curve spectrum gives a divergent one-loop correction if we use the same prescription for adding up the frequencies that was used in $AdS_5 \times S^5$. Since the point-particle is a BPS solution we expect that its one-loop correction should vanish. Furthermore, since the spinning string solution becomes near-BPS in a certain limit, we expect its one-loop correction to be nonzero but finite. Hence the algebraic curve does not give one-loop corrections that are compatible with supersymmetry if one uses the standard summation prescription.

We propose a new summation prescription that gives a vanishing one-loop correction for the point-particle and a finite one-loop correction for the spinning string when used with both the algebraic curve spectrum and the world-sheet spectrum. This prescription has certain similarities to the one that was proposed by Gromov and Mikhaylov in [89], however our motivation for introducing it is somewhat different. Whereas they proposed a new summation prescription in order to get a one-loop correction to the energy of the folded spinning string that agrees with the all-loop Bethe ansatz, we find that a new summation prescription is required for a much more basic reason: consistency of the algebraic curve with supersymmetry. In principle, we obtain three predictions for the one-loop correction to the spinning-string energy; one coming from the algebraic curve and two coming from the world-sheet (since the world-sheet spectrum gives finite results using both the old and new summation prescriptions). However, if we expand in the large-$\mathcal{J}$ limit (where $\mathcal{J} = \frac{J}{\sqrt{2\pi^2\lambda}}$ and J is the spin) and evaluate the sums at each order of $\mathcal{J}$ using ζ-function regularization, we find that all three predictions are the same (up to so-called nonanalytic and exponentially suppressed terms, which are subdominant). In this way we get a single prediction for the one-loop correction to the spinning string energy. Furthermore, we show that this result is consistent with the predictions of the Bethe ansatz.

The structure of this chapter is as follows. In section 5.1, we review the world-sheet approach, the algebraic curve approach, and summation prescriptions. It should be noted that our versions of the world-sheet and algebraic curve formalisms have some new features. In particular, we recast the quadratic GS action in the $AdS_4 \times CP^3$ supergravity background in a way that removes half of the fermionic degrees of freedom explicitly and we reformulate the algebraic curve approach using off-shell techniques that make calculations much more efficient. In section 5.2.1, we present the classical solution for a point-particle rotating in CP^3 and describe the gauge theory operator dual to this solution. In the rest of section 5.2 we summarize the fluctuation frequencies for the point-particle solution and compute the one-loop correction using the standard summation prescription used in $AdS_5 \times S^5$ as well as our new summation prescription.[1] In section 5.3.1, we present the classical solution for a spinning string with two equal angular momenta in CP^3 and propose the gauge theory operator dual to this solution. In the rest of section 5.3 we summarize the fluctuation frequencies for the spinning-string solution, analyze various properties of the one-loop correction to its energy, and make a prediction for the anomalous dimension of its dual gauge theory operator.[2] In section 5.4, we use the Bethe ansatz to compute the leading two contributions to the anomalous dimension of operator dual to the spinning and verify that they agree with the prediction we obtain using string theory. section 5.5 presents our conclusions. Appendix D reviews some basic properties of the dual gauge theory. Appendices E and A.3 review the geometry of $AdS_4 \times CP^3$ as well as our Dirac matrix conventions.

5.1 Review of Formalism

5.1.1 World-Sheet Formalism

The world-sheet approach for computing the spectrum of fluctuations about a classical solution in $AdS_5 \times S^5$ was developed in [99]. In this section we review how to compute the spectrum of fluctuations around a classical solution to type IIA string theory in a supergravity background which consists of the following string frame metric, dilaton, and Ramond-Ramond forms [14]:

$$\begin{aligned} ds^2 &= G_{MN}dx^M dx^N = R^2\left(\frac{1}{4}ds^2_{AdS_4} + ds^2_{CP^3}\right), && (5.1a)\\ e^{\phi} &= \frac{R}{k}, && (5.1b)\\ F_4 &= \frac{3}{8}kR^2 Vol_{AdS_4}, && (5.1c)\\ F_2 &= k\mathbf{J}, && (5.1d)\end{aligned}$$

where R^2 is the radius of curvature in string units, $\mathbf{J}$ is the Kähler form on CP^3, and k is an integer corresponding to the level of the dual Chern-Simons theory. Note that the AdS_4 space has radius

[1] Although the spectrum of the point-particle was already computed using the algebraic curve in [80], we present it again using more efficient techniques.

[2] The authors in [96] made similar conjectures for the gauge theory operators dual to the point-particle and spinning string, however the classical solutions considered in that paper have different charges than the ones constructed in this chapter.

$R/2$ while the CP^3 space has radius R. The metric for a unit AdS_4 space given by

$$ds^2_{AdS_4} = -\cosh^2\rho \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \tag{5.2}$$

and the metric for a unit CP^3 space is given by

$$\begin{aligned} ds^2_{CP^3} = &\mathrm{d}\xi^2 + \cos^2\xi\sin^2\xi\left(\mathrm{d}\psi + \frac{1}{2}\cos\theta_1\mathrm{d}\varphi_1 - \frac{1}{2}\cos\theta_2\mathrm{d}\varphi_2\right)^2 \\ &+ \frac{1}{4}\cos^2\xi\left(\mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\varphi_1^2\right) + \frac{1}{4}\sin^2\xi\left(\mathrm{d}\theta_2^2 + \sin^2\theta_2\mathrm{d}\varphi_2^2\right), \end{aligned} \tag{5.3}$$

where $0 \leq \xi < \pi/2$, $0 \leq \psi < 2\pi$, $0 \leq \theta_i \leq \pi$, and $0 \leq \varphi_i < 2\pi$. More details about the geometry of $AdS_4 \times CP^3$ are given in appendix E.

Using the metric in equation (5.1), the bosonic part of the string Lagrangian in conformal gauge is given by

$$\mathcal{L}_{bose} = \frac{1}{4\pi}\eta^{ab}G_{MN}\partial_a X^M \partial_b X^N, \tag{5.4}$$

where $a, b = \tau, \sigma$ are world-sheet indices, $\eta^{ab} = diag\,[-1, 1]$, and we have set $\alpha' = 1$. Because AdS_4 has two Killing vectors and CP^3 has three Killing vectors, any solution to the bosonic equations of motion has at least five conserved charges. In particular, the two AdS_4 charges are given by

$$E = \sqrt{\lambda/2}\int_0^{2\pi} d\sigma \cosh^2\rho\dot{t}, \tag{5.5}$$

$$S = \sqrt{\lambda/2}\int_0^{2\pi} d\sigma \sinh^2\rho\sin^2\theta\dot{\phi}, \tag{5.6}$$

and the three CP^3 charges are given by

$$J_\psi = 2\sqrt{2\lambda}\int_0^{2\pi} d\sigma\cos^2\xi\sin^2\xi\left(\dot{\psi} + \frac{1}{2}\cos\theta_1\dot{\phi}_1 - \frac{1}{2}\cos\theta_2\dot{\phi}_2\right), \tag{5.7a}$$

$$\begin{aligned} J_{\phi_1} = &\sqrt{\lambda/2}\int_0^{2\pi} d\sigma\cos^2\xi\sin^2\theta_1\dot{\phi}_1 \\ &+\sqrt{2\lambda}\int_0^{2\pi} d\sigma\cos^2\xi\sin^2\xi\left(\dot{\psi} + \frac{1}{2}\cos\theta_1\dot{\phi}_1 - \frac{1}{2}\cos\theta_2\dot{\phi}_2\right)\cos\theta_1, \end{aligned} \tag{5.7b}$$

$$\begin{aligned} J_{\phi_2} = &\sqrt{\lambda/2}\int_0^{2\pi} d\sigma\sin^2\xi\sin^2\theta_2\dot{\phi}_2 \\ &+\sqrt{2\lambda}\int_0^{2\pi} d\sigma\cos^2\xi\sin^2\xi\left(\dot{\psi} + \frac{1}{2}\cos\theta_1\dot{\phi}_1 - \frac{1}{2}\cos\theta_2\dot{\phi}_2\right)\cos\theta_2, \end{aligned} \tag{5.7c}$$

where E is the energy and S, J_ψ, J_{ϕ_1}, and J_{ϕ_2} are angular momenta.

A solution to the bosonic equations of motion is said to be a classical solution if it also satisfies the Virasoro constraints

$$G_{MN}\left(\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N\right) = 0, \;\; G_{MN}\partial_\tau X^M \partial_\sigma X^N = 0. \tag{5.8}$$

Note that these are the only constraints that relate motion in AdS_4 to motion in CP^3.

The spectrum of bosonic fluctuations around a classical solution can be computed by expanding the bosonic Lagrangian in equation (5.4) to quadratic order in the fluctuations and finding the normal modes of the resulting equations of motion. In the examples we consider, we find that two of the bosonic modes are massless and the other eight are massive. While the eight massive modes correspond to the physical transverse degrees of freedom, the two massless modes can be discarded. One way to see that the massless modes can be discarded is by expanding the Virasoro constraints to linear order in the fluctuations [99].

To compute the spectrum of fermionic fluctuations, we only need the quadratic part of the fermionic GS action for type IIA string theory. This action describes two 10-dimensional Majorana-Weyl spinors of opposite chirality that can be combined into a single non-chiral Majorana spinor Θ. The quadratic GS action for type IIA string theory in a general background can be found in [100]. For the supergravity background in equation (5.1), the quadratic Lagrangian for the fermions is given by

$$\mathcal{L}_{Fermi} = \bar{\Theta}\left(\eta^{ab} - \epsilon^{ab}\Gamma_{11}\right) e_a \left[(\partial_b + \frac{1}{4}\omega_b) + \frac{1}{8} e^{\phi}\left(-\Gamma_{11}\Gamma \cdot F_2 + \Gamma \cdot F_4\right) e_b \right] \Theta, \tag{5.9}$$

where $\bar{\Theta} = \Theta^\dagger \Gamma^0$, $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$, $e_a = \partial_a X^M e_M^A \Gamma_A$, $\omega_a = \partial_a X^M \omega_M^{AB} \Gamma_{AB}$, and $\Gamma \cdot F_{(n)} = \frac{1}{n!}\Gamma^{N_1 \dots N_n} F_{N_1 \dots N_n}$. Note that M is a base-space index while $A, B = 0, ..., 9$ are tangent-space indices. Explicit formulas for e_M^A, ω_M^{AB}, $\Gamma \cdot F_2$, and $\Gamma \cdot F_4$ are provided in appendix E. Explicit formulas for the Dirac matrices are provided in appendix A.3.

We will now recast the fermionic Lagrangian in equation (5.9) in form that allows us to compute the fermionic fluctuation frequencies in a straightforward way. First we note that after rearranging terms, equation (5.9) can be written as

$$\frac{\mathcal{L}_{Fermi}}{2K} = -\bar{\Theta}_+ \Gamma_0 \left[\partial_\tau - \Gamma_{11}\partial_\sigma + \frac{1}{4}\left(\omega_\tau - \Gamma_{11}\omega_\sigma\right) \right] \Theta - 2K \bar{\Theta}_+ \Gamma_0 \Gamma \cdot F \Gamma_0 \Theta_+, \tag{5.10}$$

where we define $K = \partial_\tau X^M e_M^0$, $\Theta_+ = P_+ \Theta$, and

$$P_+ = -\frac{1}{2K}\Gamma_0\left(e_\tau + e_\sigma \Gamma_{11}\right), \tag{5.11}$$

$$\Gamma \cdot F = \frac{1}{8} e^{\phi}\left(-\Gamma_{11}\Gamma \cdot F_2 + \Gamma \cdot F_4\right). \tag{5.12}$$

Note that $P_+ = P_+^\dagger$ and if the classical solution satisfies

$$\partial_\sigma X^M e_M^0 = 0, \tag{5.13}$$

then P_+ is a projection operator, i.e., $P_+^2 = P_+$. In addition, if the classical solution satisfies

$$P_+ \left[P_+, \omega_\tau - \Gamma_{11}\omega_\sigma\right] = 0, \tag{5.14}$$

then the fermionic Lagrangian simplifies to

$$\frac{\mathcal{L}_{Fermi}}{2K} = -\bar{\Theta}_+ \Gamma_0 \left[\partial_\tau - \Gamma_{11}\partial_\sigma + \frac{1}{4}\left(\omega_\tau - \Gamma_{11}\omega_\sigma\right) + 2K\left(\Gamma \cdot F\Gamma_0\right)\right]\Theta_+. \tag{5.15}$$

Finally, if we consider the Fourier mode $\Theta(\sigma, \tau) = \tilde{\Theta}\exp\left(-i\omega\tau + in\sigma\right)$, where $\tilde{\Theta}$ is a constant spinor, then the equations of motion for the fermionic fluctuations are given by

$$\left\{P_+\left[i\omega + in\Gamma_{11} - \frac{1}{4}\left(\omega_\tau - \Gamma_{11}\omega_\sigma\right) - 2K\left(\Gamma \cdot F\Gamma_0\right)\right]P_+\right\}\tilde{\Theta} = 0. \tag{5.16}$$

One can choose a basis where P_+ has the form $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (where each element in the 2×2 matrix corresponds to a 16×16 matrix). In this basis, the matrix on the left-hand side of equation (5.16) will have the form $\begin{pmatrix} \mathbf{A} & 0 \\ 0 & 0 \end{pmatrix}$. The fermionic frequencies are determined by taking the determinant of $\mathbf{A}$ and finding its roots.

Only half of the fermionic components appear in the Lagrangian in equation (5.15). Hence, a natural choice for fixing kappa-symmetry is to set the other components to zero by imposing the gauge condition $\Theta = \Theta_+$. This gives the desired number of fermionic degrees of freedom. In particular, before imposing the Majorana condition, Θ has 32 complex degrees of freedom. When the classical solution satisfies equations (5.13) and (5.14), the quadratic GS action can be recast in terms of projection operators that remove half of Θ's components, leaving 16 complex degrees of freedom. After solving the fermionic equations of motion, one then finds that only half the solutions have positive energy, leaving eight complex degrees of freedom. Finally, after imposing the Majorana condition we should be left with eight real degrees of freedom, which matches the number of transverse bosonic degrees of freedom. Explicit calculations of the fermionic frequencies for the classical solutions studied in this chapter are described in sections 5.2.2 and 5.3.2.

5.1.2 Algebraic Curve Formalism

The procedure for computing the spectrum of excitations about a classical string solution using the AdS_4/CFT_3 algebraic curve was first presented in [80]. In this section, we reformulate this procedure in terms of an off-shell formalism similar to the one that was developed for the AdS_5/CFT_4 algebraic curve in [101]. The off-shell formalism makes things much more efficient. First we describe how to construct the classical algebraic curve. Then we describe how to semiclassically quantize the curve and obtain the spectrum of excitations.

5.1.2.1 Classical Algebraic Curve

For type IIA string theory in $AdS_4 \times CP^3$, any classical solution can be encoded in a 10-sheeted Riemann surface whose branches, called quasi-momenta, are denoted by

$$\{q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}\}\,.$$

This algebraic curve corresponds to the fundamental representation of $OSp(6|4)$, which is ten-dimensional. Furthermore, the quasi-momenta are not all independent. In particular

$$(q_1(x), q_2(x), q_3(x), q_4(x), q_5(x)) = -\left(q_{10}(x), q_9(x), q_8(x), q_7(x), q_6(x)\right), \tag{5.17}$$

where x is a complex number called the spectral parameter. To compute the quasi-momenta, it is useful to parameterize AdS_4 and CP^3 using the following embedding coordinates

$$\begin{aligned} n_1^2 + n_2^2 - n_3^2 - n_4^2 - n_5^2 &= 1, \\ \sum_{I=1}^{4} \left|z^I\right|^2 &= 1, \qquad z^I \sim e^{i\lambda} z^I, \end{aligned}$$

where $\lambda \in \mathbb{R}$. A classical solution in the global coordinates of equations (5.2) and (5.3) can be converted to embedding coordinates using equations (E.2) and (E.7) provided in appendix E. One can then compute the following connection:

$$j_a(\tau,\sigma) = 2\begin{pmatrix} n_i \partial_a n_j - n_j \partial_a n_i & 0 \\ 0 & z_I^\dagger D_a z^J - z^J D_a z_I^\dagger \end{pmatrix}, \tag{5.18}$$

where $a \in \{\tau, \sigma\}$, $D_a = \partial_a + iA_a$, and $A_a = i\sum_{I=1}^{4} z_I^\dagger \partial_a z^I$ [80]. This connection is a 9×9 matrix and transforms under the bosonic part of the supergroup $OSp(6|4)$, notably $SU(4) \times SO(3,2) \sim O(6) \times Sp(4)$. A key property is that it is flat, which allows us to construct the following monodromy matrix:

$$\Lambda(x) = P \exp \frac{1}{x^2-1} \int_0^{2\pi} d\sigma \left[j_\sigma(\tau,\sigma) + x j_\tau(\tau,\sigma) \right], \tag{5.19}$$

where P is the path-ordering symbol and the integral is over a loop of constant world-sheet time τ. It can be shown that the eigenvalues of $\Lambda(x)$ are independent of τ.

The quasi-momenta are related to the eigenvalues of the monodromy matrix. In particular, if we diagonalize the monodromy matrix we will find that the eigenvalues of the AdS_4 part are in general given by

$$\left\{ e^{i\hat{p}_1(x)}, e^{i\hat{p}_2(x)}, e^{i\hat{p}_3(x)}, e^{i\hat{p}_4(x)}, 1 \right\}, \tag{5.20}$$

where $\hat{p}_1(x) + \hat{p}_4(x) = \hat{p}_2(x) + \hat{p}_3(x) = 0$, while the eigenvalues from the CP^3 part are given by

$$\left\{ e^{i\tilde{p}_1(x)}, e^{i\tilde{p}_2(x)}, e^{i\tilde{p}_3(x)}, e^{i\tilde{p}_4(x)} \right\}, \tag{5.21}$$

where $\sum_{i=1}^4 \tilde{p}_i(x) = 0$. The classical quasi-momenta are then defined as

$$(q_1, q_2, q_3, q_4, q_5) = \left(\frac{\hat{p}_1 + \hat{p}_2}{2}, \frac{\hat{p}_1 - \hat{p}_2}{2}, \tilde{p}_1 + \tilde{p}_2, \tilde{p}_1 + \tilde{p}_3, \tilde{p}_1 + \tilde{p}_4 \right), \tag{5.22}$$

where we have suppressed the x-dependence. From this formula, we see that $q_1(x)$ and $q_2(x)$ correspond to the AdS_4 part of the algebraic curve, while $q_3(x)$, $q_4(x)$, and $q_5(x)$ correspond to the CP^3 part of the algebraic curve.

5.1.2.2 Semiclassical Quantization

The algebraic curve will generically have cuts connecting several pairs of sheets. These cuts encode the classical physics. To perform semiclassical quantization, we add poles to the algebraic curve which correspond to quantum fluctuations. Each pole connects two sheets. In particular the bosonic fluctuations connect two AdS sheets or two CP^3 sheets and the fermionic fluctuations connect an AdS sheet to a CP^3 sheet. See figure (5.1) for a depiction of the fluctuations. In total there are eight bosonic and eight fermionic fluctuations, and they are labeled by the pairs of sheets that their poles connect. The labels are referred to as polarizations and are summarized in table 5.1.

Notice that every fluctuation can be labeled by two equivalent polarizations because every pole connects two equivalent pairs of sheets as a consequence of equation (5.17). Fluctuations connecting sheet 5 or 6 to any other sheet are defined to be light. Notice that there are eight light excitations. All the others are defined to be heavy excitations. The physical significance of this terminology will become clear later on. When we compute the spectrum of fluctuations about the point particle in section 5.2 for example, we will find that the heavy excitations are twice as massive as the light excitations.

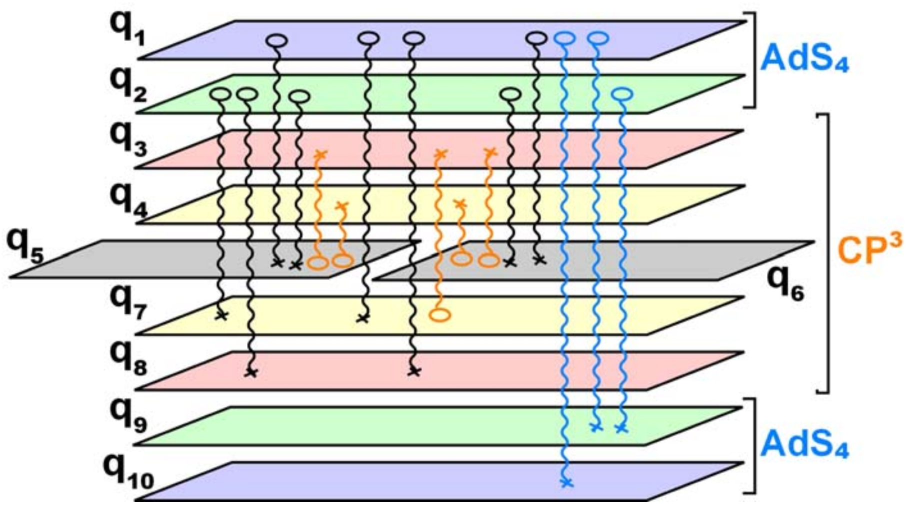

Figure 5.1. Depiction of the fluctuations of the $AdS_4 \times CP^3$ algebraic curve. Each fluctuation corresponds to a pole that connects two sheets.

Table 5.1. Labels for the fluctuations (**heavy**, **light**) of the $AdS_4 \times CP^3$ algebraic curve

	Polarizations $(\mathbf{i}, \mathbf{j})$
AdS	$(\mathbf{1}, \mathbf{10}/\mathbf{1}, \mathbf{10}); (\mathbf{2}, \mathbf{9}/\mathbf{2}, \mathbf{9}); (\mathbf{1}, \mathbf{9}/\mathbf{2}, \mathbf{10})$
Fermions	$(\mathbf{1}, \mathbf{7}/\mathbf{4}, \mathbf{10}); (\mathbf{1}, \mathbf{8}/\mathbf{3}, \mathbf{10}); (\mathbf{2}, \mathbf{7}/\mathbf{4}, \mathbf{9}); (\mathbf{2}, \mathbf{8}/\mathbf{3}, \mathbf{9})$ $(\mathbf{1}, \mathbf{5}/\mathbf{6}, \mathbf{10}); (\mathbf{1}, \mathbf{6}/\mathbf{5}, \mathbf{10}); (\mathbf{2}, \mathbf{5}/\mathbf{6}, \mathbf{9}); (\mathbf{2}, \mathbf{6}/\mathbf{5}, \mathbf{9})$
CP^3	$(\mathbf{3}, \mathbf{7}/\mathbf{4}, \mathbf{8})$ $(\mathbf{3}, \mathbf{5}/\mathbf{6}, \mathbf{8}); (\mathbf{3}, \mathbf{6}/\mathbf{5}, \mathbf{8}); (\mathbf{4}, \mathbf{5}/\mathbf{6}, \mathbf{7}); (\mathbf{4}, \mathbf{6}/\mathbf{5}, \mathbf{7})$

When adding poles, we must take into account the level-matching condition

$$\sum_{n=-\infty}^{\infty} n \sum_{ij} N_n^{ij} = 0, \tag{5.23}$$

where N_n^{ij} is the number of excitations with polarization ij and mode number n. Furthermore, the locations of the poles are not arbitrary; they are determined by the following equation:

$$q_i\left(x_n^{ij}\right) - q_j\left(x_n^{ij}\right) = 2\pi n, \tag{5.24}$$

where x_n^{ij} is the location of a pole corresponding to a fluctuation with polarization ij and mode number n.

In addition to adding poles to the algebraic curve, we must also add fluctuations to the classical quasi-momenta. These fluctuations will depend on the spectral parameter x as well as the locations of the poles, which we will denote by the collective coordinate y. The functional form of the fluctuations is determined by some general constraints:

- They are not all independent:

$$\begin{pmatrix} \delta q_1(x,y) \\ \delta q_2(x,y) \\ \delta q_3(x,y) \\ \delta q_4(x,y) \\ \delta q_5(x,y) \end{pmatrix} = - \begin{pmatrix} \delta q_{10}(x,y) \\ \delta q_9(x,y) \\ \delta q_8(x,y) \\ \delta q_7(x,y) \\ \delta q_6(x,y) \end{pmatrix}.$$

- They have poles near the points $x = \pm 1$ and the residues of these poles are synchronized as follows:

$$\lim_{x \to \pm 1} \left(\delta q_1(x,y), \delta q_2(x,y), \delta q_3(x,y), \delta q_4(x,y), \delta q_5(x,y) \right) \propto \frac{1}{x \pm 1} \left(1,1,1,1,0\right). \tag{5.25}$$

- There is an inversion symmetry:

$$\begin{pmatrix} \delta q_1(1/x,y) \\ \delta q_2(1/x,y) \\ \delta q_3(1/x,y) \\ \delta q_4(1/x,y) \\ \delta q_5(1/x,y) \end{pmatrix} = \begin{pmatrix} -\delta q_2(x,y) \\ -\delta q_1(x,y) \\ -\delta q_4(x,y) \\ -\delta q_3(x,y) \\ \delta q_5(x,y) \end{pmatrix}. \tag{5.26}$$

- The fluctuations have the following large-x behavior:

$$\lim_{x \to \infty} \begin{pmatrix} \delta q_1(x,y) \\ \delta q_2(x,y) \\ \delta q_3(x,y) \\ \delta q_4(x,y) \\ \delta q_5(x,y) \end{pmatrix} \simeq \frac{1}{2gx} \begin{pmatrix} \Delta(y) + N_{19} + 2N_{1\,10} + N_{15} + N_{16} + N_{17} + N_{18} \\ \Delta(y) + 2N_{29} + N_{2\,10} + N_{25} + N_{26} + N_{27} + N_{28} \\ -N_{35} - N_{36} - N_{37} - N_{39} - N_{3\,10} \\ -N_{45} - N_{46} - N_{48} - N_{49} - N_{4\,10} \\ N_{35} + N_{45} - N_{57} - N_{58} - N_{15} - N_{25} + N_{59} + N_{5\,10} \end{pmatrix}, \tag{5.27}$$

where $g = \sqrt{\lambda/8}$, $N_{ij} = \sum_{n=-\infty}^{\infty} N_n^{ij}$, and $\Delta(y)$ is called the anomalous part of the energy shift. Whereas the N_n^{ij} are inputs of the calculation, $\Delta(y)$ will be determined in the process of determining the fluctuations of the quasi-momenta. The factor of two that appears in front of $N_{1\,10}$ and N_{29} is a consequence of the symmetry in equation (5.17). The coefficients of the other terms on the right-hand side of equation (5.27) can be determined using arguments similar to those in [98].

- Finally, when the spectral parameter approaches the location of one of the poles, the fluctua-

Table 5.2. Relations between heavy and light off-shell frequencies

Heavy	**Light**	
$\Omega_{29} =$	$2\Omega_{25}$	**AdS**
$\Omega_{1\ 10} =$	$2\Omega_{15}$	
$\Omega_{19} =$	$\Omega_{15} + \Omega_{25}$	
$\Omega_{27} =$	$\Omega_{25} + \Omega_{45}$	**Fermions**
$\Omega_{17} =$	$\Omega_{15} + \Omega_{45}$	
$\Omega_{28} =$	$\Omega_{25} + \Omega_{35}$	
$\Omega_{18} =$	$\Omega_{15} + \Omega_{35}$	
$\Omega_{37} =$	$\Omega_{35} + \Omega_{45}$	$\mathbb{CP}^3$

tions have the following form:

$$\lim_{x \to x_n^{ij}} \delta q_k \propto \frac{\alpha(x_n^{ij}) N_n^{ij}}{x - x_n^{ij}}, \quad \alpha(x) = \frac{1}{2g} \frac{x^2}{x^2 - 1}, \tag{5.28}$$

where the proportionality constants can be read off from the coefficient of N_{ij} in the kth row of equation (5.27).

After computing the anomalous part of the energy shift, the fluctuation frequency is given by

$$\Omega(y) = \Delta(y) + \sum_{AdS_4} N^{ij} + \frac{1}{2} \sum_{ferm} N^{ij}. \tag{5.29}$$

It is useful to consider the fluctuation frequency without fixing the value of y. In this case, the fluctuation frequency is said to be off-shell.

Using arguments similar to those in [101], we find all the relations among the off-shell frequencies. First, all the light off-shell frequencies are related by

$$\Omega_{i6}(y) = \Omega_{i5}(y), \tag{5.30}$$

where $i = 1, 2, 3, 4$.

Second, all the heavy off-shell frequencies can be written as the sum of two light off-shell frequencies as summarized in table 5.2.

Finally, any off-shell frequency Ω_{ij} is related to its mirror off-shell frequency $\Omega_{\overline{ij}}$ by

$$\Omega_{ij}(y) = -\Omega_{\overline{ij}}(1/y) + \Omega_{\overline{ij}}(0) + C,$$

where $C = 1, 1/2$, or 0 for AdS, Fermionic, or CP^3 polarizations respectively. The mirror polarization $\overline{(i,j)}$ of the polarization (i,j) can be readily found using equation (5.26), e.g., $\overline{(1,10)} = (2,9)$, $\overline{(2,5)} = (1,5)$, $\overline{(4,5)} = (3,5)$, $\overline{(3,7)} = (3,7)$, etc. Using these relations, only two of the eight light off-shell frequencies are independent. For example,

$$\Omega_{35}(y) = -\Omega_{45}(1/y) + \Omega_{45}(0), \tag{5.31a}$$

$$\Omega_{25}(y) = -\Omega_{15}(1/y) + \Omega_{15}(0) + 1/2. \tag{5.31b}$$

In conclusion, if we compute the off-shell frequencies Ω_{15} and Ω_{45}, then we can determine all the other off-shell frequencies automatically from the relations in equations (5.30), (5.31) and table 5.2.

The on-shell frequencies are then obtained by evaluating the off-shell frequencies at the location of the poles which are determined by solving equation (5.24), i.e., $\omega_n^{ij} = \Omega_{ij}\left(x_n^{ij}\right)$. It will be convenient to organize them into the following linear combinations:

$$\omega_L(n) = \omega_n^{35} + \omega_n^{36} + \omega_n^{45} + \omega_n^{46} - \omega_n^{15} - \omega_n^{16} - \omega_n^{25} - \omega_n^{26}, \tag{5.32}$$

$$\omega_H(n) = \omega_n^{19} + \omega_n^{29} + \omega_n^{1\,10} + \omega_n^{37} - \omega_n^{17} - \omega_n^{18} - \omega_n^{27} - \omega_n^{28}, \tag{5.33}$$

where L stands for light and H stands for heavy. It should be noted that heavy and light frequencies are not as well-defined in the world-sheet approach. In general, the only way to identify heavy and light frequencies in the world-sheet approach is by comparing the world-sheet spectrum to the algebraic curve spectrum, i.e., a world-sheet frequency is said to be heavy/light if the corresponding algebraic curve frequency is heavy/light.

5.1.3 Summation Prescriptions

Given the spectrum of fluctuations about a classical string solution, we compute the one-loop correction to the string energy by adding up the spectrum. The standard formula is

$$\delta E_{1-loop,old} = \lim_{N\to\infty} \frac{1}{2\kappa} \sum_{n=-N}^{N} \left(\sum_{i=1}^{8} \omega_{n,i}^{B} - \sum_{i=1}^{8} \omega_{n,i}^{F} \right), \tag{5.34}$$

where κ is proportional to the classical energy (the exact formula is given in sections 3 and 4), B/F stands for bosonic/fermionic, n is the mode number, and i is some label. For example, if we are dealing with frequencies computed from the algebraic curve, then they will be labeled by a pair of integers called a polarization, as explained in section 2.2. Although this formula works well for string solutions in $AdS_5 \times S^5$, it gives a one-loop correction which disagrees with the all-loop Bethe ansatz when applied to the folded-spinning string in $AdS_4 \times CP^3$. In [89] Gromov and Mikhaylov subsequently proposed the following formula for computing one-loop corrections in $AdS_4 \times CP^3$:

$$\delta E_{1-loop,\mathrm{GM}} = \lim_{N\to\infty} \frac{1}{2\kappa} \sum_{n=-N}^{N} K_n, \quad K_n = \begin{cases} \omega^H(n) + \omega^L(n/2) & n \in even \\ \omega^H(n) & n \in odd \end{cases}, \tag{5.35}$$

where ω_n^L/ω_n^H are referred to as heavy/light frequencies and are defined in equations (5.32) and (5.33). For later convenience, we note that equation (5.34) can be written in terms of heavy and light frequencies as follows:

$$\delta E_{1-loop,old} = \lim_{N\to\infty} \frac{1}{2\kappa} \sum_{-N}^{N} \left(\omega_L(n) + \omega_H(n)\right). \tag{5.36}$$

In the large-κ limit, equation (5.35) can be approximated as the following integral:

$$\delta E_{1-loop} \approx \lim_{N\to\infty} \frac{1}{2\kappa} \int_{-N}^{N} \left(\omega_H(n) + \frac{1}{2}\omega_L(n/2)\right) dn. \tag{5.37}$$

In [89] it was shown that equation (5.37) gives a one-loop correction which agrees with the all-loop Bethe ansatz when applied to the spectrum of the folded spinning string.

In this chapter we propose a new summation prescription:

$$\delta E_{1-loop,new} = \lim_{N\to\infty} \frac{1}{2\kappa} \sum_{-N}^{N} \left(2\omega_H(2n) + \omega_L(n)\right). \tag{5.38}$$

This sum can be motivated physically using the observation in [89] that heavy modes with mode number $2n$ can be thought of as bound states of two light modes with mode number n. This suggests that only heavy modes with even mode number should contribute to the one-loop correction. The formula for the one-loop correction should therefore have the form

$$\delta E_{1-loop,new} = \lim_{N\to\infty} \frac{1}{2\kappa} \sum_{-N}^{N} \left(A\omega_H(2n) + B\omega_L(n)\right).$$

The coefficients A and B can then be fixed uniquely by requiring that the integral approximation to this formula reduces to equation (5.37) in the large-κ limit, ensuring that this summation prescription gives a one-loop correction to the folded spinning string energy which agrees with the all-loop Bethe ansatz. One then finds that $A = 2$ and $B = 1$.

One virtue of the new summation prescription in equation (5.38) compared to the one in equation (5.35) is that it gives more well-defined results for one-loop corrections. For example, consider the case where $\omega_L(n) = -2\omega_H(n) = C$, where C is some constant (the AC frequencies for the point-particle will have this form). In this case, equation (5.35) does not have a well-defined $N \to \infty$ limit; in particular the sum alternates between $\pm C/(4\kappa)$ depending on whether N is even or odd. On the other hand, equation (5.38) vanishes for all N.

5.2 Point-Particle

5.2.1 Classical Solution and Dual Operator

In terms of the coordinates of equations (5.2) and (5.3), the solution for a point-particle rotating with angular momentum J in CP^3 is given by

$$t = \kappa\tau,\ \ \rho = 0,\ \ \xi = \pi/4,\ \ \theta_1 = \theta_2 = \pi/2,\ \ \psi = \mathcal{J}\tau,\ \ \phi_1 = \phi_2 = 0, \tag{5.39}$$

where $\mathcal{J} = \frac{J}{4\pi g}$ and $g = \sqrt{\lambda/8}$. This version of the solution will be useful for doing calculations in the world-sheet formalism. Alternatively, we can write this solution in embedding coordinates by plugging equation (5.39) into equations (E.2) and (E.7):

$$n_1 = \cos\kappa\tau,\ \ n_2 = \sin\kappa\tau,\ \ n_3 = n_4 = n_5 = 0,\ \ z^1 = z^2 = z_3^\dagger = z_4^\dagger = \frac{1}{2}e^{i\mathcal{J}\tau/2}. \tag{5.40}$$

This version of the solution will be useful for doing calculations in the algebraic curve formalism. The energy and angular momenta of the particle can be read off from equations (5.5)–(5.7): $E = 4\pi g\kappa$, $S = 0$, $J_\psi = J$, $J_{\phi_1} = J_{\phi_2} = 0$. Furthermore, the Virasoro constraints in equation (5.8) give $\kappa = \mathcal{J}$, or equivalently $E = J$. Note that this is a BPS condition. We therefore expect that the dimension of the dual gauge theory operator should be protected by supersymmetry.

The gauge theory operator dual to the point-particle rotating in CP^3 should have the form

$$\mathcal{O} = \mathrm{tr}\left[\left(Z^1 Z_3^\dagger\right)^J\right].$$

This can be understood heuristically by associating the scalars Z^1,Z^2, Z^3, Z^4 with the embedding coordinates z^1, z^2, z^3, z^4 and noting that

$$\frac{1}{2}\begin{pmatrix} e^{i\mathcal{J}\tau/2} \\ e^{i\mathcal{J}\tau/2} \\ e^{-i\mathcal{J}\tau/2} \\ e^{-i\mathcal{J}\tau/2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\mathcal{J}\tau/2} \\ 0 \\ e^{-i\mathcal{J}\tau/2} \\ 0 \end{pmatrix}.$$

Since the transformation on the right-hand side is an $SU(4)$ transformation, the solution in equation (5.40) is equivalent to $z^1 = z_3^\dagger = \frac{1}{\sqrt{2}}e^{i\mathcal{J}\tau/2}$, $z^2 = z^4 = 0$. Furthermore, the engineering dimension of this operator is J, which matches the energy of the point-particle solution, and the two-loop dilatation operator in equation (D.1) vanishes when applied to this operator, which is consistent with our expectation that the anomalous dimension of the operator should vanish.

5.2.2 Point-Particle Spectrum from the World-Sheet

Bosonic Spectrum

To compute the spectrum of bosonic fluctuations about the point-particle, first we add fluctuations to the classical solution in equation (5.39):

$$t = \kappa\tau + \delta t(\tau,\sigma), \;\; \eta_i = \delta\eta_i(\tau,\sigma), \;\; \xi = \pi/4 + \delta\xi(\tau,\sigma),$$

$$\theta_j = \pi/2 + \delta\theta_j(\tau,\sigma), \;\; \psi = \kappa\tau + \delta\psi(\tau,\sigma), \;\; \phi_j = \delta\phi_j(\tau,\sigma),$$

where $i = 1,2,3$ and $j = 1,2$. Expanding the bosonic Lagrangian in equation (5.4) to quadratic order gives

$$\begin{aligned} 4\pi\mathcal{L}_{bos} \;\; = \;\; & -\frac{1}{4}\left(\partial\delta t\right)^2 + \frac{1}{4}\left(\partial\delta\psi\right)^2 + \sum_{i=1}^{3}\left[\left(\partial\delta\eta_i\right)^2 + \kappa^2\delta\eta_i^2\right] + \left(\partial\delta\xi\right)^2 + \kappa^2\delta\xi^2 \\ & + \frac{1}{8}\left(\partial\delta\theta_1\right)^2 + \frac{1}{8}\left(\partial\delta\theta_2\right)^2 + \frac{1}{8}\left(\partial\delta\phi_1\right)^2 + \frac{1}{8}\left(\partial\delta\phi_2\right)^2 + \frac{1}{4}\kappa\delta\theta_1\delta\dot{\phi}_1 - \frac{1}{4}\kappa\delta\theta_2\delta\dot{\phi}_2, \end{aligned}$$

where $\left(\partial f\right)^2 = -\left(\partial_\tau f\right)^2 + \left(\partial_\sigma f\right)^2$. We immediately see that the fluctuations δt and $\delta\psi$ are massless, while $\delta\eta_i$ and $\delta\xi$ have mass κ. If we consider Fourier modes of the form $f(\tau,\sigma) = \tilde{f}e^{i(\omega\tau+n\sigma)}$, then the equations of motion for the remaining fields reduce to

$$\begin{pmatrix} \omega^2-n^2 & -i\omega\kappa & 0 & 0 \\ i\omega\kappa & \omega^2-n^2 & 0 & 0 \\ 0 & 0 & \omega^2-n^2 & i\omega\kappa \\ 0 & 0 & -i\omega\kappa & \omega^2-n^2 \end{pmatrix} \begin{pmatrix} \delta\tilde{\theta}_1 \\ \delta\tilde{\phi}_1 \\ \delta\tilde{\theta}_2 \\ \delta\tilde{\phi}_2 \end{pmatrix} = 0.$$

The dispersion relations for the normal modes of this system are obtained by taking the determinant of the matrix on the left-hand side, setting it to zero, and solving for ω. The positive solutions are

$$\frac{\omega}{\kappa} = \sqrt{\frac{1}{4} + \frac{n^2}{\kappa^2}} \pm \frac{1}{2}. \tag{5.41}$$

Each of these solutions has multiplicity two, giving a total of four positive solutions.

In summary, we find that there are eight massive modes and two massless modes. Three of the massive modes come from AdS_4. Their dispersion relations are given by

$$\frac{\omega}{\kappa} = \sqrt{1 + \frac{n^2}{\kappa^2}}.$$

The remaining five massive modes come from CP^3. One of them has the dispersion relation in the equation above and the other four have the dispersion relations in equation (5.41). The two massless modes are longitudinal and can be discarded. This can be seen by expanding the Virasoro constraints in equation (5.8) to linear order in the perturbations. Doing so gives

$$\partial_\tau \left(\delta t - \delta\psi\right) = \partial_\sigma \left(\delta t - \delta\psi\right) = 0.$$

Noting that $\left(\partial\delta t\right)^2 - \left(\partial\delta\psi\right)^2 = \partial\left(\delta t - \delta\psi\right)\partial\left(\delta t + \delta\psi\right)$, we see that the equation above implies that all terms in the action involving δt and $\delta\psi$ vanish.

Fermionic Spectrum

In order to compute the spectrum of fermionic fluctuations about the point-particle solution given by equation (5.39), we only need to know the pullback of the vielbein and the spin connection in the background of this classical solution. These are given by

$$e_\tau = \frac{R}{2}\mathcal{J}\left(-\Gamma^0 + \Gamma^4\right), \qquad e_\sigma = 0, \tag{5.42}$$

and

$$\omega_\tau = \mathcal{J}\left(\Gamma^{89} - \Gamma^{67}\right), \qquad \omega_\sigma = 0. \tag{5.43}$$

Plugging these expressions into equation (5.11) gives

$$P_+ = \frac{1}{2}\left(1 + \Gamma^0\Gamma^4\right), \tag{5.44}$$

where we used $K = \partial_\tau X^\mu e^0_\mu = (R/2)\mathcal{J}$. It is straightforward to check that equations (5.13), (5.14) are satisfied for the point-particle solution. Therefore, by plugging equations (E.10), (5.43), (5.44) into equation (5.16) and using the Dirac matrices in appendix E, we obtain an explicit form of the equation of motion for the fermionic fluctuations. The frequencies are then determined using the procedure described at the end of section 2.1. In particular, the positive fermionic frequencies are given by

$$\begin{aligned}
\omega_1 &= \sqrt{\kappa^2 + n^2}, \\
\omega_2 &= \frac{1}{2}\sqrt{\kappa^2 + 4n^2} + \frac{\kappa}{2}, \\
\omega_3 &= \frac{1}{2}\sqrt{\kappa^2 + 4n^2} - \frac{\kappa}{2},
\end{aligned}$$

where ω_2 and ω_3 have multiplicity two while ω_1 has multiplicity four, for a total of 8 fermionic frequencies.

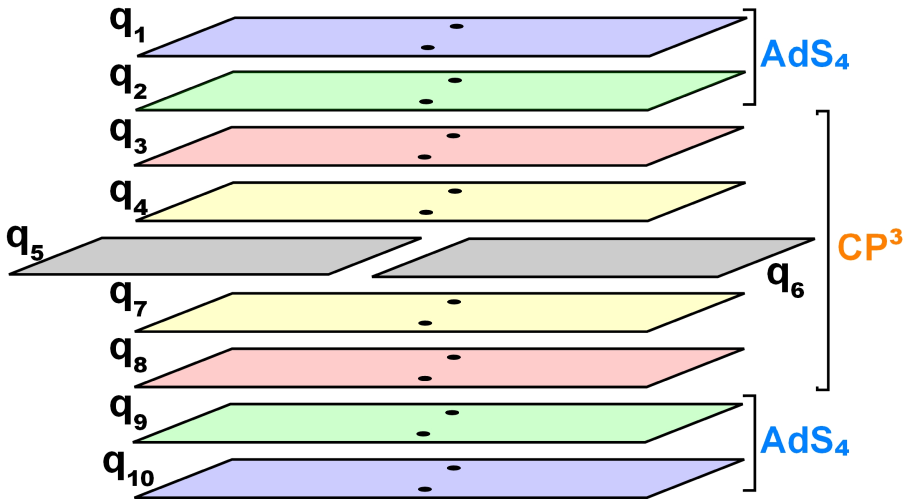

Figure 5.2. Classical algebraic curve for the point-particle rotating in CP^3.

5.2.3 Point-Particle Algebraic Curve

Classical Quasi-momenta

In this section, we compute the algebraic curve for the classical solution given in equation (5.40). First we plug this solution into equation (5.18):

$$(j_\tau)_{AdS_4} = 2\kappa \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (j_\tau)_{CP^3} = i\mathcal{J} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad j_\sigma = 0.$$

Note that this connection is independent of σ, so it is trivial to compute the monodromy matrix in equation (5.19) since path ordering is not an issue. Diagonalizing the monodromy matrix and comparing the eigenvalues to equations (5.20) and (5.21) then gives $\hat{p}_1 = -\hat{p}_4 = \frac{4\pi\kappa x}{x^2-1}$, $\tilde{p}_1 = -\tilde{p}_4 = \frac{2\pi\mathcal{J}x}{x^2-1}$, and $\hat{p}_2 = \hat{p}_3 = \tilde{p}_2 = \tilde{p}_3 = 0$. Recalling that $\kappa = \mathcal{J}$ and plugging these results into equation (5.22), we find that the classical quasi-momenta are

$$q_1 = q_2 = q_3 = q_4 = \frac{2\pi\mathcal{J}x}{x^2-1}, \quad q_5 = 0. \tag{5.45}$$

The algebraic curve corresponding to these quasi-momenta is depicted in figure (5.2). Note that all sheets except those corresponding to q_5 and q_6 have poles at $x = \pm 1$.

Off-shell Frequencies

Recall from equations (5.30,5.31) and table 5.2 that if we know the off-shell frequencies $\Omega_{15}(y)$ and $\Omega_{45}(y)$, then all the others are determined. Let us begin by computing $\Omega_{15}(y)$. Suppose we have two fluctuations between q_1 and q_5. To satisfy level-matching, let us take one of these fluctuations to have mode number $+n$ and the other to have mode number $-n$. Each fluctuation corresponds to adding a pole to the classical algebraic curve. The locations of the poles are determined by solving equation (5.24). We will denote the pole locations by $x^{15}_{\pm n}$. We then make the following ansatz for the fluctuations:

$$\begin{aligned}
\delta q_1(x,y) &= \sum_{\pm} \frac{\alpha\left(x_n^{15}\right)}{x - x_n^{15}}, \quad \delta q_2(x,y) = -\delta q_1(1/x, y), \\
\delta q_5(x,y) &= -\sum_{\pm} \frac{\alpha\left(x\right)}{x - x_n^{15}} - \sum_{\pm} \frac{\alpha\left(1/x\right)}{1/x - x_n^{15}},
\end{aligned}$$

where $\alpha(x)$ is defined in equation (5.28), $\pm$ stands for the sum over the positive and negative mode number, and y is a collective coordinate for the positions of the two poles $x^{15}_{\pm n}$. We have not made an ansatz for δq_3 and δq_4 because they are not needed to compute $\Omega_{15}(y)$. Notice that this ansatz satisfies the inversion symmetry in equation (5.26) and has pole structure in agreement with equation (5.28). In the large-x limit, the fluctuations reduce to

$$\begin{aligned}
\lim_{x\to\infty} \delta q_1(x,y) &\sim \frac{1}{x}\sum_{\pm} \alpha\left(x_n^{15}\right), \\
\lim_{x\to\infty} \delta q_2(x,y) &\sim \frac{1}{2gx}\sum_{\pm} \frac{1}{\left(x_n^{15}\right)^2 - 1}, \\
\lim_{x\to\infty} \delta q_5(x,y) &\sim -\frac{1}{gx},
\end{aligned}$$

where we neglect $\mathcal{O}\left(x^{-2}\right)$ terms. Comparing these expressions to equation (5.27) implies that the anomalous energy shift is given by

$$\Delta(y) = \sum_{\pm} \frac{1}{\left(x_n^{15}\right)^2 - 1}.$$

The off-shell fluctuation frequency is then obtained by plugging this into equation (5.29) and recalling that the (1,5) fluctuation is fermionic:

$$\Omega_{15}(y) = \Delta(y) + \frac{1}{2}N^{15} = \Delta(y) + 1 = \sum_{\pm} \frac{1}{2}\frac{\left(x_n^{15}\right)^2 + 1}{\left(x_n^{15}\right)^2 - 1}.$$

This implies that the off-shell frequency for a single fluctuation between q_1 and q_5 is given by

$$\Omega_{15}(y) = \frac{1}{2}\frac{y^2+1}{y^2-1}.$$

Now let us compute $\Omega_{45}(y)$. Once again, let us suppose that we have two fluctuations between q_4 and q_5 which have opposite mode numbers $\pm n$. We make the following ansatz for the fluctuations:

$$\begin{aligned}
\delta q_1(x,y) &= \frac{\alpha_+(y)}{x+1} + \frac{\alpha_-(y)}{x-1}, \;\; \delta q_2(x,y) = -\delta q_1(1/x,y), \\
\delta q_4(x,y) &= -\sum_{\pm} \frac{\alpha\,(x)}{x - x_n^{45}}, \;\; \delta q_3(x,y) = -\delta q_4(1/x,y), \\
\delta q_5(x,y) &= \sum_{\pm} \frac{\alpha\,(x)}{x - x_n^{45}} + \sum_{\pm} \frac{\alpha\,(1/x)}{1/x - x_n^{45}},
\end{aligned}$$

where $\alpha_\pm(y)$ are some functions to be determined. Note that this ansatz satisfies the inversion symmetry in equation (5.26) and has pole structure in agreement with equation (5.28). Taking the large-x limit gives

$$\begin{aligned}
\lim_{x\to\infty} \delta q_1(x,y) &\sim \frac{\alpha_+(y)+\alpha_-(y)}{x}, \;\; \lim_{x\to\infty} \delta q_2(x,y) \sim \alpha_-(y) - \alpha_+(y) + \frac{\alpha_+(y)+\alpha_-(y)}{x}, \\
\lim_{x\to\infty} \delta q_3(x,y) &\sim 0, \;\; \lim_{x\to\infty} \delta q_4(x,y) \sim -\frac{1}{gx}, \;\; \lim_{x\to\infty} \delta q_5(x,y) \sim \frac{1}{gx}.
\end{aligned}$$

Comparing these limits with equation (5.27) implies that

$$\alpha_+(y) = \alpha_-(y) = \frac{\Delta(y)}{4g}. \tag{5.46}$$

Furthermore, the residues of the poles at $x = \pm 1$ must be synchronized according to equation (5.25). For example, if we equate the residues of δq_1 and δq_4 near $x = +1$ we find that

$$\lim_{x\to+1} \delta q_1(x,y) \sim \frac{\alpha_-(y)}{x-1} = \lim_{x\to+1} \delta q_4 \sim \left(\frac{1}{4g}\sum_{\pm}\frac{1}{x_n^{45}-1}\right)\frac{1}{x-1} \to \alpha_-(y) = \frac{1}{4g}\sum_{\pm}\frac{1}{x_n^{45}-1}.$$

Combining this with the equation (5.46) implies that

$$\Delta(y) = \sum_{\pm} \frac{1}{x_n^{45}-1}. \tag{5.47}$$

At this point it is useful to recall that x_n^{45} is a root of the following equation (which comes from plugging equation (5.45) into equation (5.24)):

$$\frac{2\pi\mathcal{J}x_n^{45}}{\left(x_n^{45}\right)^2-1} = 2\pi n.$$

Table 5.3. Off-shell frequencies for fluctuations about the point-particle solution

	$\mathbf{\Omega(y)}$	**Polarizations**
AdS	$\frac{y^2+1}{y^2-1}$	$(\mathbf{1},\mathbf{10});(\mathbf{2},\mathbf{9});(\mathbf{1},\mathbf{9})$
Fermions	$\frac{y^2+3}{2(y^2-1)}$ $\frac{y^2+1}{2(y^2-1)}$	$(\mathbf{1},\mathbf{7});(\mathbf{1},\mathbf{8});(\mathbf{2},\mathbf{7});(\mathbf{2},\mathbf{8})$ $(\mathbf{1},\mathbf{5});(\mathbf{1},\mathbf{6});(\mathbf{2},\mathbf{5});(\mathbf{2},\mathbf{6})$
$\mathbf{CP^3}$	$\frac{2}{y^2-1}$ $\frac{1}{y^2-1}$	$(\mathbf{3},\mathbf{7})$ $(\mathbf{3},\mathbf{5});(\mathbf{3},\mathbf{6});(\mathbf{4},\mathbf{5});(\mathbf{4},\mathbf{6})$

Note that this equation has two roots. The convention that we will follow is to assign the pole to the root with larger magnitude. Hence, if $n<0$ then $x_n^{45}=\frac{\mathcal{J}}{n}-\sqrt{1+\frac{\mathcal{J}^2}{n^2}}$ and if $n>0$ then $x_n^{45}=\frac{\mathcal{J}}{n}+\sqrt{1+\frac{\mathcal{J}^2}{n^2}}$. The point to take away from this discussion is that

$$x_{+n}^{45}=-x_{-n}^{45}.$$

Using this fact, equation (5.47) can be written as follows:

$$\Delta(y)=\frac{1}{x_{+n}^{45}-1}-\frac{1}{x_{+n}^{45}+1}=\frac{2}{\left(x_{+n}^{45}\right)^2-1}=\sum_{\pm}\frac{1}{\left(x_n^{45}\right)^2-1}.$$

The off-shell fluctuation frequency is then obtained by plugging this into equation (5.29) and recalling that the $(4,5)$ fluctuation is a CP^3 fluctuation:

$$\Omega_{45}(y)=\Delta(y)=\sum_{\pm}\frac{1}{\left(x_n^{45}\right)^2-1}.$$

It follows that the off-shell frequency for a single fluctuation between q_4 and q_5 is given by

$$\Omega_{45}(y)=\frac{1}{y^2-1}.$$

The remaining off-shell frequencies are now easily computed from equations (5.30,5.31) and table 5.2. We summarize the off-shell frequencies in table 5.3.

On-shell Frequencies

To compute the on-shell frequencies, we must compute the locations of the poles by solving equation (5.24). Recall that fluctuations that connect q_5 or q_6 to any other sheets are referred to as light, and all the others are referred to as heavy. A little thought shows that for light fluctuations, equation (5.24) reduces to

$$\frac{\mathcal{J}x_n}{x_n^2-1}=n,$$

Table 5.4. Spectrum of fluctuations about the point-particle solution computed using the world-sheet (WS) and algebraic curve (AC) formalisms ($\omega_n = \sqrt{n^2+\kappa^2}$)

	WS	**AC**	**Polarizations**
AdS	ω_n	ω_n	$(\mathbf{1},\mathbf{10}); (\mathbf{2},\mathbf{9}); (\mathbf{1},\mathbf{9})$
Fermions	$\omega_n \pm \frac{\kappa}{2}$ $\frac{1}{2}\omega_{2n}$	$\omega_n - \frac{\kappa}{2}$ $\frac{1}{2}\omega_{2n}$	$(\mathbf{1},\mathbf{7}); (\mathbf{1},\mathbf{8}); (\mathbf{2},\mathbf{7}); (\mathbf{2},\mathbf{8})$ $(\mathbf{1},\mathbf{5}); (\mathbf{1},\mathbf{6}); (\mathbf{2},\mathbf{5}); (\mathbf{2},\mathbf{6})$
$\mathbf{CP^3}$	ω_n $\frac{1}{2}\omega_{2n} \pm \frac{\kappa}{2}$	$\omega_n - \kappa$ $\frac{1}{2}\omega_{2n} - \frac{\kappa}{2}$	$(\mathbf{3},\mathbf{7})$ $(\mathbf{3},\mathbf{5}); (\mathbf{3},\mathbf{6}); (\mathbf{4},\mathbf{5}); (\mathbf{4},\mathbf{6})$

and for heavy fluctuations it reduces to

$$\frac{\mathcal{J} x_n}{x_n^2 - 1} = \frac{n}{2}.$$

Each of these equations admits two solutions. We will assign the location of the pole to the solution with greater magnitude. Assuming $n > 0$, the location of the pole for light excitations is then given by

$$x_n = \frac{\mathcal{J}}{2n} + \sqrt{\frac{\mathcal{J}^2}{4n^2} + 1},$$

and the location of the pole for heavy excitations is given by

$$x_n = \frac{\mathcal{J}}{n} + \sqrt{\frac{\mathcal{J}^2}{n^2} + 1}.$$

Plugging these solutions into the off-shell frequencies in table 5.3 readily gives the on-shell algebraic curve frequencies in table 5.4.

5.2.4 Excitation Spectrum

We summarize the spectrum of fluctuations obtained with the algebraic curve and the world-sheet in table 5.4, the polarizations (**heavy**/**light**) indicate which pairs of sheets are connected by a fluctuation in the AC formalism, and $\pm$ indicates that half of the frequencies have a $+$ and the other half have a $-$. The algebraic curve frequencies have been rescaled by a factor of κ in order to compare them to the world-sheet frequencies. The derivations of these frequencies are described in sections 5.2.2 and 5.2.3. Note that the fluctuations in this table are labeled by polarizations. Although this notation was only defined for the algebraic curve formalism, we find that the world-sheet frequencies match the algebraic curve frequencies up to constant shifts, so it is convenient to label the world-sheet frequencies with polarizations as well. Also note that both sets of frequencies agree with the spectrum of fluctuations that were found in the Penrose limit (up to constant shifts) [52, 58, 57].

While the constant shifts in the world-sheet spectrum occur with opposite signs and can be removed by gauge transformations, this is not the case for the algebraic curve frequencies. In fact, the constant shifts in the algebraic curve frequencies have physical significance, which can be seen by taking the mode number $n = 0$. In this limit, the AdS frequencies reduce to κ, the CP^3 frequencies reduce to 0, and the Fermi frequencies reduce to $\kappa/2$. In this sense, the $n = 0$ algebraic curve frequencies have "flat-space" behavior. This property was also observed for algebraic curve frequencies computed about solutions in $AdS_5 \times S^5$ [98]. On the other hand, the world-sheet frequencies do not have this property. In the next subsection, we will see that the constant shifts in the algebraic curve spectrum have important implications for the one-loop correction to the classical energy.

5.2.5 One-Loop Correction to Energy

Using equations (5.32) and (5.33) we see that ω_H and ω_L are constants for both the world-sheet and algebraic curve spectra. In particular, for the world-sheet spectrum we find that $\omega_H(n) = \omega_L(n) = 0$. As a result, both the standard summation prescription in equation (5.36) and the new summation prescription in equation (5.38) give a vanishing one-loop correction to the energy. On the other hand, for the algebraic curve we find that $\omega_H(n) = \kappa$ and $\omega_L(n) = -2\kappa$. For these values of ω_H and ω_L, the new summation prescription gives a vanishing one-loop correction but the standard summation prescription gives a linear divergence:

$$\delta E_{1-loop,old} = \lim_{N\to\infty} -(N + 1/2).$$

Thus we find that both summation prescriptions are consistent with supersymmetry if we use the spectrum computed from the world-sheet, but only the new summation is consistent with supersymmetry if we use the spectrum computed from the algebraic curve.

5.3 Spinning String

5.3.1 Classical Solution and Dual Operator

In the global coordinates of equations (5.2) and (5.3), the solution for a circular spinning string with two equal nonzero spins in CP^3 is

$$t = \kappa\tau, \;\; \rho = 0, \;\; \xi = \pi/4, \;\; \theta_1 = \theta_2 = \pi/2, \;\; \psi = m\sigma, \;\; \phi_1 = \phi_2 = 2\mathcal{J}\tau, \tag{5.48}$$

where $\mathcal{J} = J/4\pi g$ and m is the winding number. Using equations (E.2) and (E.7), we can also write this solution in embedding coordinates (which are useful for doing algebraic curve calculations):

$$n_1 = \cos\kappa\tau, \;\; n_2 = \sin\kappa\tau, \;\; n_3 = n_4 = n_5 = 0, \;\; z^1 = z_4^\dagger = \frac{1}{2}e^{i(\mathcal{J}\tau + m\sigma/2)}, \;\; z^3 = z_2^\dagger = \frac{1}{2}e^{i(\mathcal{J}\tau - m\sigma/2)}. \tag{5.49}$$

Equations (5.5-5.7) imply that $E = 4\pi g\kappa$, $S = 0$, $J_\psi = 0$, and $J_{\phi_1} = J_{\phi_2} = J$. Furthermore, the Virasoro constraints in equation (5.8) give $\kappa = \sqrt{m^2 + 4\mathcal{J}^2}$, or equivalently $E = 2J\sqrt{1 + \frac{\pi^2 m^2 \lambda}{2J^2}}$. In the limit $\mathcal{J} \gg m$, this reduces to the BPS condition $E = 2J$, so we expect that the dual operator should have engineering dimension $2J$ and a finite but nonzero anomalous dimension. Furthermore, the dispersion relation has a BMN expansion in the parameter λ/J^2, which allows us to make a prediction for anomalous dimension of the dual operator. Expanding the dispersion relation to first order in the BMN parameter gives

$$E = 2J + \frac{\pi^2 m^2 \lambda}{2J} + \mathcal{O}\left(\lambda^2/J^3\right). \tag{5.50}$$

To extrapolate this formula to the gauge theory, we must make the replacement $\lambda \to 2\lambda^2$. One way to understand this replacement is by comparing the magnon dispersion relation at strong and weak 't Hooft coupling, as explained in the introduction. We therefore get the following prediction for the anomalous dimension of the dual gauge theory operator

$$\mathbf{\Delta} - 2J = \frac{\pi^2 \lambda^2 m^2}{J} + \mathcal{O}\left(\lambda^2/J^2\right). \tag{5.51}$$

The higher-order terms in the expansion of the classical string energy in equation (5.50) correspond to $\mathcal{O}\left(\lambda^4/J^3\right)$ corrections to the anomalous dimension, but the one-loop correction to the energy provides $\mathcal{O}\left(\lambda^2/J^2\right)$ corrections to the anomalous dimension (see equation (5.77)).

The dual gauge theory operator should have the form

$$\mathcal{O} = \operatorname{tr}\left[\left(Z^1 Z_2^\dagger\right)^J \left(Z^3 Z_4^\dagger\right)^J + \ldots\right], \tag{5.52}$$

where the dots stand for permutations of $\left(Z^1 Z_2^\dagger\right)$ and $\left(Z^3 Z_4^\dagger\right)$. Note that the engineering dimension of the operator is $2J$, as expected. When we apply the two loop dilatation operator in equation (D.1) to the operator in equation (5.52), it reduces to

$$\mathbf{\Delta} - 2J = \lambda^2 \sum_{i=1}^{2J} \left(1 - P_{2i-1,2i+1} + 1 - P_{2i,2i+2}\right). \tag{5.53}$$

This is the Hamiltonian for two identical Heisenberg spin chains; one located on the even sites and the other on the odd sites. If one thinks of Z^1 and $Z_2^\dagger$ as being up spins and Z^3 and $Z_4^\dagger$ as being down spins, then each spin chain has J up spins and J down spins. In section 5.4, we use the Bethe ansatz to show that the anomalous dimension is indeed given by equation (5.51).

5.3.2 Spinning String Spectrum from the World-Sheet

Bosonic Spectrum

In this section we calculate the spectrum of bosonic fluctuations about the circular spinning string in $AdS_4 \times CP^3$. Let us begin by adding fluctuations to the solution in equation (5.48):

$$\begin{aligned} t &= \kappa\tau + \delta t(\tau,\sigma), \;\; \eta_i = \delta\eta_i(\tau,\sigma), \;\; \xi = \pi/4 + \delta\xi(\tau,\sigma), \\ \theta_j &= \pi/2 + \delta\theta_j(\tau,\sigma), \;\; \psi = m\sigma + \delta\psi(\tau,\sigma), \;\; \phi_j = 2\mathcal{J}\tau + \delta\phi_j(\tau,\sigma), \end{aligned}$$

where $i = 1,2,3$, $j = 1,2$, and $\kappa = \sqrt{4\mathcal{J}^2 + m^2}$. Expanding the bosonic Lagrangian in equation (5.4) to quadratic order in the fluctuations gives

$$\begin{aligned} 4\pi\mathcal{L}_{bos} &= m^2/2 - \frac{1}{4}\left(\partial\delta t\right)^2 + \sum_{i=1}^{3}\left[\left(\partial\delta\eta_i\right)^2 + \kappa^2\delta\eta_i^2\right] \\ &\quad +\partial\left(\delta\bar{\psi}\right)^2 + \partial\left(\delta\bar{\xi}\right)^2 - m^2\delta\bar{\xi}^2 + \partial\left(\delta\theta_+\right)^2 + 4\mathcal{J}^2\left(\delta\theta_+\right)^2 + \partial\left(\delta\theta_-\right)^2 \\ &\quad +\partial\left(\delta\phi_+\right)^2 + \partial\left(\delta\phi_-\right)^2 + 4\mathcal{J}\left(\delta\theta_-\partial_\tau\delta\bar{\psi} + \delta\bar{\xi}\partial_\tau\delta\bar{\phi}_-\right) \\ &\quad -2m\left(\delta\theta_-\partial_\sigma\delta\phi_+ + \delta\theta_+\partial_\sigma\delta\phi_-\right), \end{aligned}$$

where $\delta\bar{\psi} = \sqrt{2}\delta\psi$, $\delta\bar{\xi} = 2\sqrt{2}\delta\xi$, $\delta\theta_\pm = \frac{1}{\sqrt{2}}\left(\delta\theta_1 \pm \delta\theta_2\right)$, $\delta\phi_\pm = \frac{1}{\sqrt{2}}\left(\delta\phi_1 \pm \delta\phi_2\right)$, and $\left(\partial f\right)^2 = -\left(\partial_\tau f\right)^2 + \left(\partial_\sigma f\right)^2$. Note that the AdS_4 fluctuations are the same as those of the point-particle. In particular, we see that δt is massless and $\delta\eta_i$ have mass κ. If we consider Fourier modes of the form $f(\tau,\sigma) = \tilde{f}e^{i(\omega\tau + n\sigma)}$, then the equations of motion for the CP^3 fluctuations reduce to

$$\begin{pmatrix} \omega^2 - n^2 + m^2 & 2i\mathcal{J}\omega & 0 & 0 & 0 & 0 \\ -2i\mathcal{J}\omega & \omega^2 - n^2 & -imn & 0 & 0 & 0 \\ 0 & imn & \omega^2 - n^2 - 4\mathcal{J}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 - n^2 & 0 & -2i\mathcal{J}\omega \\ 0 & 0 & 0 & 0 & \omega^2 - n^2 & -imn \\ 0 & 0 & 0 & 2i\mathcal{J}\omega & imn & \omega^2 - n^2 \end{pmatrix} \begin{pmatrix} \delta\tilde{\bar{\xi}} \\ \delta\tilde{\phi}_- \\ \delta\tilde{\theta}_+ \\ \delta\tilde{\bar{\psi}} \\ \delta\tilde{\phi}_+ \\ \delta\tilde{\theta}_- \end{pmatrix} = 0. \tag{5.54}$$

The fluctuations $\left(\delta\tilde{\bar{\xi}}, \delta\tilde{\phi}_-, \delta\tilde{\theta}_+\right)$ and $\left(\delta\tilde{\bar{\psi}}, \delta\tilde{\phi}_+, \delta\tilde{\theta}_-\right)$ are decoupled because the matrix in equation (5.54) is block diagonal. The frequencies are determined by taking the determinant of the matrix and finding its roots. The equation we must solve is

$$\left(n^2-\omega^2\right)\left(4\mathcal{J}^2-m^2+n^2-\omega^2\right)\left(n^4-m^2n^2-\left(4\mathcal{J}^2+2n^2\right)\omega^2+\omega^4\right)^2=0.$$

This polynomial has 12 roots, which come in opposite signs. Of the six positive roots, three correspond to the fluctuations $\left(\delta\tilde{\tilde{\xi}},\delta\widetilde{\phi}_-,\delta\widetilde{\theta}_+\right)$:

$$\omega=\sqrt{4\mathcal{J}^2+n^2-m^2},\ \ \sqrt{2\mathcal{J}^2+n^2\pm\sqrt{4\mathcal{J}^4+n^2\kappa^2}},$$

and three correspond to the fluctuations $\left(\delta\tilde{\tilde{\psi}},\delta\widetilde{\phi}_+,\delta\widetilde{\theta}_-\right)$:

$$\omega=|n|\,,\ \ \sqrt{2\mathcal{J}^2+n^2\pm\sqrt{4\mathcal{J}^4+n^2\kappa^2}}.$$

Note that the solution $\omega=|n|$ corresponds to a massless mode, which can be discarded along with the other massless mode δt. The remaining eight modes are massive and correspond to the transverse degrees of freedom.

Fermionic Spectrum

In this section we compute the spectrum of fermionic fluctuations about the spinning string solution in equation (5.48). The pullback of the vielbein and the spin connection in the background of this classical solution are given by

$$e_\tau=R\left(-\frac{\kappa}{2}\Gamma^0+\frac{\mathcal{J}}{\sqrt{2}}\left(\Gamma^6+\Gamma^8\right)\right),\quad e_\sigma=\frac{Rm\Gamma^4}{2},\tag{5.55}$$

and

$$\omega_\tau=\sqrt{2}\mathcal{J}\left(\Gamma^{74}+\Gamma^{85}+\Gamma^{49}+\Gamma^{56}\right),\quad \omega_\sigma=m\left(\Gamma^{89}+\Gamma^{76}\right).\tag{5.56}$$

Plugging these expressions into equation (5.11) then gives

$$P_+=\frac{1}{2}\left(1+\frac{\sqrt{2}\mathcal{J}}{\kappa}\Gamma^0\left(\Gamma^6+\Gamma^8\right)+\frac{m}{\kappa}\Gamma^0\Gamma^4\Gamma_{11}\right),\tag{5.57}$$

where we used $K=\partial_\tau X^\mu e^0_\mu=(R/2)\kappa$. It is straightforward to check that equations (5.13,5.14) are satisfied for the spinning string solution. Therefore, by plugging equations (E.10,5.56,5.57) into equation (5.16) and using the Dirac matrices in appendix E, we obtain an explicit form of the equation of motion for the fermionic fluctuations. The frequencies are then determined using the procedure described at the end of section 2.1. In this way, the positive fermionic frequencies are given by

$$\begin{aligned}
\omega_1 &= \sqrt{4\mathcal{J}^2 + n^2} + \frac{\kappa}{2}, \\
\omega_2 &= \sqrt{4\mathcal{J}^2 + n^2} - \frac{\kappa}{2}, \\
\omega_3 &= \frac{1}{2}\sqrt{\kappa^2 + 4n^2},
\end{aligned}$$

where ω_1 and ω_2 have multiplicity two while ω_3 has multiplicity four, for a total of 8 fermionic frequencies. In obtaining these expressions, equation 5.61 is useful.

5.3.3 Spinning String Algebraic Curve

Classical Quasi-momenta

Since the spinning string has the same motion in AdS_4 as the point particle, the AdS_4 quasi-momenta have the same structure and are given by

$$q_1(x) = q_2(x) = \frac{2\pi\kappa x}{x^2 - 1},$$

where $\kappa = \sqrt{4\mathcal{J}^2 + m^2}$ for the spinning string. Therefore, we just have to find the CP^3 quasi-momenta. For the classical solution in equation (5.49), one finds that the connection in equation (5.18) is given by

$$\begin{aligned}
(j_0)_{CP^3} &= i\mathcal{J}\begin{pmatrix} 1 & e^{-im\sigma} & 0 & 0 \\ e^{im\sigma} & 1 & 0 & 0 \\ 0 & 0 & -1 & -e^{-im\sigma} \\ 0 & 0 & -e^{im\sigma} & -1 \end{pmatrix}, \\
(j_1)_{CP^3} &= im\begin{pmatrix} 1 & 0 & e^{-2i\mathcal{J}\tau} & 0 \\ 0 & -1 & 0 & -e^{-2i\mathcal{J}\tau} \\ e^{2i\mathcal{J}\tau} & 0 & 1 & 0 \\ 0 & -e^{2i\mathcal{J}\tau} & 0 & -1 \end{pmatrix}.
\end{aligned}$$

Using equation (5.19), the CP^3 part of the monodromy matrix is given by

$$\Lambda(x) = P\exp\frac{1}{x^2-1}\int_0^{2\pi} d\sigma J(\sigma, x),$$

where

$$J(\sigma,x)=\begin{pmatrix} i\left(\mathcal{J}x+m/2\right) & i\mathcal{J}xe^{-im\sigma} & im/2 & 0 \\ i\mathcal{J}xe^{im\sigma} & i\left(\mathcal{J}x-m/2\right) & 0 & -im/2 \\ im/2 & 0 & -i\left(\mathcal{J}x-m/2\right) & -i\mathcal{J}xe^{-im\sigma} \\ 0 & -im/2 & -i\mathcal{J}xe^{im\sigma} & -i\left(\mathcal{J}x+m/2\right) \end{pmatrix}. \tag{5.58}$$

and we set $\tau=0$ since the eigenvalues of $\Lambda(x)$ are independent of τ. At this point, it is useful to observe that under a gauge transformation of the form $J(\sigma,x)\to g^{-1}(\sigma)J(\sigma,x)g(\sigma)-g^{-1}(\sigma)\partial_\sigma g(\sigma)$, the monodromy matrix transforms as $\Lambda(x)\to g^{-1}(0)\Lambda\left(x\right)g(2\pi)$. If $g(0)=\pm g(2\pi)$, then the eigenvalues of $\Lambda(x)$ are gauge invariant up to a sign. Furthermore, if we can choose $g(\sigma)$ such that the σ-dependence of $J(\sigma,x)$ is removed, then the monodromy matrix would be trivial to evaluate since path-ordering would not be an issue. This can be accomplished using the gauge transformation $g\left(\sigma\right)=diag(e^{-im\sigma/2},e^{im\sigma/2},e^{-im\sigma/2},e^{im\sigma/2})$ [102]. Under this transformation, $J(\sigma,x)$ becomes

$$J(\sigma,x)\to J(0,x)+i\,\frac{m}{2}\;diag[1,-1,1,-1]. \tag{5.59}$$

When m is odd, $g(0)=-g(2\pi)$ so we must supplement this gauge transformation with $\Lambda(x)\to -\Lambda(x)$.

Diagonalizing $\Lambda(x)$ and comparing to equation (5.21) gives

$$\begin{aligned} \tilde{p}_1 &= \tfrac{2\pi x}{x^2-1}\left[K(x)+K(1/x)\right]-\pi m, \\ \tilde{p}_2 &= \tfrac{2\pi x}{x^2-1}\left[K(x)-K(1/x)\right]-\pi m, \\ \tilde{p}_3 &= -\tilde{p}_2, \qquad \tilde{p}_4=-\tilde{p}_1, \end{aligned} \tag{5.60}$$

where $K(x)=\sqrt{\mathcal{J}^2+m^2x^2/4}$. In deriving equation (5.60), we made use of the following identity:

$$\sqrt{A\pm\sqrt{B}}=\frac{1}{2}\left(\sqrt{2A+2\sqrt{A^2-B}}\pm\sqrt{2A-2\sqrt{A^2-B}}\right). \tag{5.61}$$

Furthermore, we subtracted πm from $\tilde{p}_1$ and $\tilde{p}_2$ and added πm to $\tilde{p}_3$ and $\tilde{p}_4$ so that the quasi-momenta are $\mathcal{O}(1/x)$ in the large-x limit. This also implements the transformation $\Lambda(x)\to-\Lambda(x)$ when m is odd. The quasi-momenta $q_3(x)$, $q_4(x)$, and $q_5(x)$ are then given by plugging equation (5.60) into equation (5.22)

$$\begin{aligned} q_3(x) &= \tfrac{4\pi x}{x^2-1}K(x)-2\pi m, \\ q_4(x) &= -q_3(1/x)-2\pi m=\tfrac{4\pi x}{x^2-1}K(1/x), \\ q_5(x) &= 0. \end{aligned} \tag{5.62}$$

From these quasi-momenta, we see that the spinning string algebraic curve has a cut between q_3

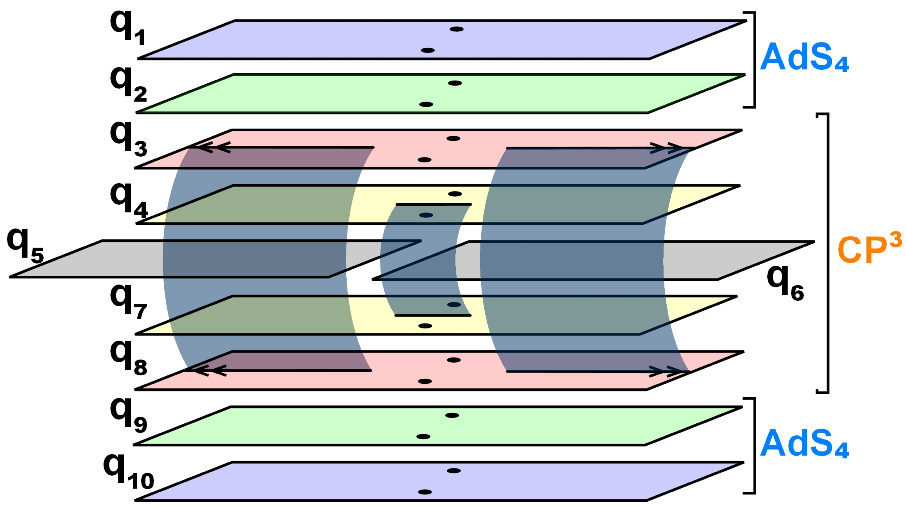

Figure 5.3. Classical algebraic curve for the circular spinning string in CP^3.

and q_8 and between q_4 and q_7 (by inversion symmetry). The classical algebraic curve is depicted in figure (5.3).

Off-shell Frequencies

Since q_1 and q_5 have the same structure as they did for the point-particle solution, a little thought shows that $\Omega_{15}(y)$ should be the same as we found for the point-particle. In particular,

$$\Omega_{15}(y) = \frac{1}{2}\frac{y^2+1}{y^2-1}.$$

From equations (5.30,5.31) and table 5.2, it follows that the only off-shell frequency we need to compute is $\Omega_{45}(x)$.

Let us suppose that we have two fluctuations between q_4 and q_5; one with mode number $+n$ and the other with mode number $-n$. These fluctuations correspond to adding poles to the classical algebraic curve. The locations of the poles will be denoted $x^{45}_{\pm n}$. Looking at equation (5.62), we see that q_4 is proportional to a square root coming from $K(1/x)$. We therefore expect that $\delta q_4(x)$ should be proportional to $\partial_x K(1/x) \propto 1/K(1/x)$ and make the following ansatz for the fluctuations:

$$\begin{aligned}
\delta q_1(x,y) &= \frac{\alpha_+(y)}{x+1} + \frac{\alpha_-(y)}{x-1}, \;\; \delta q_2(x,y) = -\delta q_1(1/x,y), \\
\delta q_5(x,y) &= \sum_{\pm} \frac{\alpha(x)}{x - x^{45}_n} + \sum_{\pm} \frac{\alpha(1/x)}{1/x - x^{45}_n}, \\
\delta q_4(x,y) &= h(x,y)/K(1/x), \;\; \delta q_3(x,y) = -\delta q_4(1/x,y),
\end{aligned}$$

where $\sum_{\pm}$ stands for the sum over positive and negative mode number, y is a collective coordinate for $x^{45}_{\pm n}$, $\alpha(x)$ is defined in equation (5.28), and $\alpha_{\pm}(y)$ are some functions to be determined. Note that this ansatz is consistent with the inversion symmetry in equation (5.26). We also make the following ansatz for $h(x,y)$:

$$h(x,y) = \frac{\alpha_+(y)K(1)}{x+1} + \frac{\alpha_-(y)K(1)}{x-1} - \sum_{\pm} \frac{\alpha(x_n^{45})K(1/x_n^{45})}{x - x_n^{45}}.$$

For this choice of $h(x,y)$, the residue of δq_4 at $x = x_{\pm n}^{45}$ agrees with equation (5.28) and the residues of all the fluctuations are synchronized $x = \pm 1$ according to equation (5.25). To compute the anomalous energy shift, we must look at the large-x behavior of the fluctuations and compare it to equation (5.27). At large x, δq_2 and δq_4 are given by

$$\begin{aligned}
\lim_{x\to\infty} \delta q_2(x,y) &\sim \alpha_-(y) - \alpha_+(y) + \frac{1}{x}\left(\alpha_+(y) + \alpha_-(y)\right), \\
\lim_{x\to\infty} \delta q_4(x,y) &\sim \frac{1}{\mathcal{J} x}\left[K(1)\left(\alpha_+(y) + \alpha_-(y)\right) - \sum_{\pm} \alpha(x_n^{45})K(1/x_n^{45})\right].
\end{aligned}$$

where we neglect terms of $\mathcal{O}(x^{-2})$. Comparing the asymptotic forms of δq_2 and δq_4 with equation (5.27) gives

$$\alpha_+(y) = \alpha_-(y) = \frac{\Delta(y)}{4g} = \frac{1}{\kappa}\left[-\frac{\mathcal{J}}{g} + \sum_{\pm} \alpha\left(x_n^{45}\right) K\left(1/x_n^{45}\right)\right],$$

where $\kappa = 2K(1)$. Recalling that the (4,5) fluctuation is a CP^3 fluctuation, equation (5.29) implies that

$$\Omega_{45}(y) = \frac{1}{K(1)}\left[\sum_{\pm} \frac{\left(x_n^{45}\right)^2 K(1/x_n^{45})}{(x_n^{45})^2 - 1}\right] - \frac{2\mathcal{J}}{K(1)}.$$

This implies that for a single fluctuation

$$\Omega_{45}(y) = \frac{2}{\kappa}\frac{y^2 K(1/y)}{y^2 - 1} - \frac{2\mathcal{J}}{\kappa}.$$

Now it is trivial to write down all the other off-shell frequencies using the relations in equations (5.30,5.31) and table 5.2. The off-shell frequencies are summarized in table 5.5.

On-Shell Frequencies

The structure of this section is as follows: for each row of table 5.5, we find the solutions of the equation in the second column, plug the solution with greatest magnitude into the off-shell frequency in the first column, and simplify the resulting expression to obtain the on-shell algebraic curve frequency in the corresponding row in table 5.6. We use the following notation for on-shell frequencies:

$$\omega_n = \Omega\left(x_n\right).$$

Table 5.5. Off-shell frequencies for the fluctuations about the spinning string solution

$\mathbf{\Omega(y)}$	**Pole Location**	**Polarizations**
$\frac{y^2+1}{y^2-1}$	$2\kappa x_n = n\left(x_n^2-1\right)$	$(\mathbf{1},\mathbf{10});(\mathbf{2},\mathbf{9})$ $(\mathbf{1},\mathbf{9})$
$\frac{1}{2}\frac{y^2+1}{y^2-1}+\frac{2}{\kappa}\left(\frac{y^2K(1/y)}{y^2-1}-\mathcal{J}\right)$ $\frac{1}{2}\frac{y^2+1}{y^2-1}+\frac{2}{\kappa}\frac{K(y)}{y^2-1}$ $\frac{1}{2}\frac{y^2+1}{y^2-1}$	$x_n\left(\kappa+2K(1/x_n)\right)=n\left(x_n^2-1\right)$ $x_n\left(\kappa+2K(x_n)\right)=(n+m)\left(x_n^2-1\right)$ $\kappa x_n = n\left(x_n^2-1\right)$	$(\mathbf{1},\mathbf{7});(\mathbf{2},\mathbf{7})$ $(\mathbf{1},\mathbf{8});(\mathbf{2},\mathbf{8})$ $(\mathbf{1},\mathbf{5});(\mathbf{1},\mathbf{6});$ $(\mathbf{2},\mathbf{5});(\mathbf{2},\mathbf{6})$
$\frac{2}{\kappa}\left[\frac{y}{y^2-1}\left(K(y)/y+yK(1/y)\right)-\mathcal{J}\right]$ $\frac{2}{\kappa}\frac{K(y)}{y^2-1}$ $\frac{2}{\kappa}\left(\frac{y^2K(1/y)}{y^2-1}-\mathcal{J}\right)$	$2x_n\left(K(x_n)+K(1/x_n)\right)=(n+m)\left(x_n^2-1\right)$ $2x_nK(x_n)=(n+m)\left(x_n^2-1\right)$ $2x_nK(1/x_n)=n\left(x_n^2-1\right)$	$(\mathbf{3},\mathbf{7})$ $(\mathbf{3},\mathbf{5});(\mathbf{3},\mathbf{6})$ $(\mathbf{4},\mathbf{5});(\mathbf{4},\mathbf{6})$

- **(1,9); (2,9); (1,10)**

  The equation for the pole location implies that $\frac{1}{x_n^2-1}=\frac{n}{2\kappa x_n}$. Plugging this into the formula for the off-shell frequency implies that

$$\omega_n = \frac{n}{2\kappa}\left(x_n + 1/x_n\right). \tag{5.63}$$

  Solving for the pole location gives

$$x_n = \frac{1}{n}\left(\kappa \pm \sqrt{\kappa^2+n^2}\right).$$

  Choosing solution with larger magnitude and plugging it into equation (5.63) then leads to

$$\omega_n = \sqrt{1+\frac{n^2}{\kappa^2}}.$$

- **(1,7); (2,7)**

  The equation for the pole location implies that $\frac{x_nK(1/x_n)}{x_n^2-1}=\frac{n}{2}-\frac{\kappa x_n}{2(x_n^2-1)}$. Plugging this into the off-shell frequency and doing a little algebra gives

$$\omega_n = \frac{n}{\kappa}x_n - \frac{2\mathcal{J}}{\kappa} - 1/2. \tag{5.64}$$

  Solving for the pole location gives

$$x_n = \frac{1}{n}\left(\kappa \pm \sqrt{4\mathcal{J}^2+n^2}\right).$$

Taking the solution with larger magnitude and plugging it onto equation (5.64) then gives

$$\frac{1}{\kappa}\left(\sqrt{4\mathcal{J}^2+n^2}-2\mathcal{J}\right)+\frac{1}{2}.$$

- **(1,8); (2,8)**

  From the equation for the pole location we find $\frac{K(x_n)}{x_n^2-1}=\frac{1}{2x_n}(n+m)-\frac{\kappa}{2(x_n^2-1)}$. Plugging this into the off-shell frequency and doing a little algebra gives

$$\omega_n=\frac{n+m}{\kappa x_n}+1/2. \tag{5.65}$$

  The solutions to the equation for the pole location are

$$x_n=\frac{(m+n)}{n(2m+n)}\left(\kappa\pm\sqrt{4\mathcal{J}^2+(m+n)^2}\right).$$

  Taking the solution with larger magnitude and plugging it into equation (5.65) gives

$$\omega_n=\frac{n(2m+n)}{\kappa}\frac{1}{\kappa+\sqrt{4\mathcal{J}^2+(m+n)^2}}+1/2.$$

  Finally, multiplying the numerator and denominator in first term by $\kappa-\sqrt{4\mathcal{J}^2+(m+n)^2}$ and doing a little more algebra gives

$$\omega_n=\frac{1}{\kappa}\sqrt{4\mathcal{J}^2+(m+n)^2}-\frac{1}{2}.$$

- **(1,5); (1,6); (2,5); (2,6)**

  This is very similar to the calculation for 19, 29, 1 10, so we omit it.

- **(3,7)**

  From the equation for the pole location, we have $\frac{2x_n}{x_n^2-1}=\frac{n+m}{K(x_n)+K(1/x_n)}$. Plugging this into the off-shell frequency gives

$$\omega_n=\frac{1}{\kappa}\left[(n+m)\,\beta-2\mathcal{J}\right],\quad \beta=\frac{\frac{1}{x_n}K(x_n)+x_nK(1/x_n)}{K(x_n)+K(1/x_n)}. \tag{5.66}$$

  Let us focus on the term β. Multiplying the numerator and denominator by $K(x_n)-K(1/x_n)$ gives

$$\beta=\frac{\frac{1}{x_n}K(x_n)^2-x_nK(1/x_n)^2+(x_n-1/x_n)K(x_n)K(1/x_n)}{m^2(x_n-1/x_n)(x_n+1/x_n)/4}. \tag{5.67}$$

By squaring the equation for the pole location, we find that

$$K(x_n)K(1/x_n) = \frac{1}{2}\left[\frac{1}{4}(n+m)^2(x_n - 1/x_n)^2 - K(x_n)^2 - K(1/x_n)^2\right].$$

Plugging this into equation (5.67) and doing some algebra gives

$$\beta = \frac{\frac{1}{2}m^2\left[3(x_n - 1/x_n) + 1/x_n^3 - x_n^3\right] + 8\mathcal{J}^2(1/x_n - x_n) + \frac{1}{2}(n+m)^2(x_n - 1/x_n)^3}{m^2(x_n - 1/x_n)(x_n + 1/x_n)}.$$

Noting that $x^3 - 1/x^3 = \left(x^2 + 1/x^2 + 1\right)(x - 1/x)$ and doing some more algebra then gives

$$\beta = \frac{-8\mathcal{J}^2 + n\left(\frac{n}{2} + m\right)(x_n - 1/x_n)^2}{m^2(x_n + 1/x_n)}.$$

Noting that $(x - 1/x)^2 = (x + 1/x)^2 - 4$ finally gives

$$\beta = \frac{n\,(n/2 + m)}{m^2}(x + 1/x) - \frac{4n\,(n/2 + m) + 8\mathcal{J}^2}{m^2\,(x + 1/x)}.$$

Combining this with equation (5.66), we find

$$\omega_n = \frac{1}{\kappa}\left[\frac{n+m}{m^2}\left[\begin{array}{c} n(m + n/2)(x_n + 1/x_n) \\ -\left(8\mathcal{J}^2 + 2n(2m+n)\right)(x_n + 1/x_n)^{-1} \end{array}\right] - 2\mathcal{J}\right]. \tag{5.68}$$

The solutions for the pole location are

$$x_n = \pm\frac{1}{n(2m+n)}\left[\begin{array}{c} 8\mathcal{J}^2(m+n)^2 + n(2m+n)\left(2m^2 + n(2m+n)\right) \\ \pm 4|m+n|\sqrt{\left(4\mathcal{J}^2 + n(2m+n)\right)\left(\mathcal{J}^2(m+n)^2 + m^2 n(2m+n)/4\right)} \end{array}\right]^{1/2}.$$

Taking the solution with + sign out front, we see that for either choice of sign inside square root we have

$$x_n + 1/x_n = \frac{2(m+n)}{n(2m+n)}\sqrt{4\mathcal{J}^2 + n(2m+n)}.$$

Plugging this into equation (5.68) and doing a little more algebra finally gives

$$\omega_n = \frac{1}{\kappa}\left(\sqrt{4\mathcal{J}^2 - m^2 + (m+n)^2} - 2\mathcal{J}\right).$$

- **(3,5); (3,6)**

  The equation for the pole location implies that $\frac{K(x_n)}{x_n^2 - 1} = (n+m)\frac{1}{2x_n}$. Using this in the formula for the off-shell frequency leads to

$$\omega_n = \frac{1}{\kappa}(n+m)\frac{1}{x_n}. \tag{5.69}$$

Solving the equation for the pole location gives

$$x_n = \pm \frac{m+n}{\sqrt{2\mathcal{J}^2 + (m+n)^2 \pm \sqrt{4\mathcal{J}^4 + 4\kappa^2 (m+n)^2}}}.$$

The solution with larger magnitude is the one with relative $-$ sign in denominator. Taking the solution with greater magnitude and $+$ sign out front and plugging this into equation (5.69) gives

$$\omega_n = \frac{1}{\kappa}\sqrt{2\mathcal{J}^2 + (m+n)^2 - \sqrt{4\mathcal{J}^4 + (m+n)^2 \kappa^2}}.$$

- **(4,5); (4,6)**

  Plugging the equation for the pole location into the off-shell frequency gives

$$\omega_n = \frac{n}{\kappa} x_n - \frac{2\mathcal{J}}{\kappa}. \tag{5.70}$$

  The solutions for the pole location are

$$x_n = \pm \frac{1}{n}\sqrt{2\mathcal{J}^2 + n^2 \pm \sqrt{4\mathcal{J}^4 + n^2\kappa^2}}.$$

  If we choose the solution with greater magnitude and plug it into equation (5.70), we have

$$\frac{1}{\kappa}\left(\sqrt{2\mathcal{J}^2 + n^2 + \sqrt{4\mathcal{J}^4 + n^2\kappa^2}} - 2\mathcal{J}\right).$$

5.3.4 Excitation Spectrum

We summarize the spectrum of fluctuations about the spinning string in table 5.6, the notation for the frequencies is given in table 5.7 and the polarizations (**heavy**/**light**) indicate which pairs of sheets are connected by a fluctuation in the AC formalism. The algebraic curve frequencies have been rescaled by a factor of κ in order to compare them to the world-sheet frequencies. The derivations are presented in sections 5.3.2 and 5.3.3. We find that the algebraic curve spectrum matches the world-sheet spectrum up to constant shifts and shifts in mode number. Furthermore, if we set the winding number $m = 0$ and take $\mathcal{J} \to \mathcal{J}/2$, we find that all the frequencies in table 5.6 reduce to the corresponding frequencies in table 5.4, which is expected since setting the winding number to zero reduces the string to a point-particle.[3] This is an important check of our results for the spinning string. On the other hand, if we set the mode number $n = 0$, we find that the algebraic curve frequencies once again have flat-space behavior, i.e., the AdS frequencies reduce to κ, the CP^3 frequencies reduce to 0, and the Fermi frequencies reduce to $\kappa/2$.

[3] In showing that the (3,5), (3,6), (4,5), and (4,6) frequencies in table 5.6 reduce to those in table 5.4, the identity in equation (5.61) is useful.

Table 5.6. Spectrum of fluctuations about the spinning string solution computed using the world-sheet (WS) and algebraic curve (AC) formalisms

	WS	**AC**	**Polarizations**
AdS	ω_n^A	ω_n^A	$(\mathbf{1},\mathbf{10});(\mathbf{2},\mathbf{9});(\mathbf{1},\mathbf{9})$
Fermions	$\omega_n^F+\frac{\kappa}{2}$ $\omega_n^F-\frac{\kappa}{2}$ $\omega_{2n}^A/2$	$\omega_n^F+\frac{\kappa}{2}-2\mathcal{J}$ $\omega_{m+n}^F-\frac{\kappa}{2}$ $\omega_{2n}^A/2$	$(\mathbf{1},\mathbf{7});(\mathbf{2},\mathbf{7})$ $(\mathbf{1},\mathbf{8});(\mathbf{2},\mathbf{8})$ $(\mathbf{1},\mathbf{5});(\mathbf{1},\mathbf{6});(\mathbf{2},\mathbf{5});(\mathbf{2},\mathbf{6})$
CP^3	ω_n^C $\omega_n^{C_-}$ $\omega_n^{C_+}$	$\omega_{m+n}^C-2\mathcal{J}$ $\omega_{m+n}^{C_-}$ $\omega_n^{C_+}-2\mathcal{J}$	$(\mathbf{3},\mathbf{7})$ $(\mathbf{3},\mathbf{5});(\mathbf{3},\mathbf{6})$ $(\mathbf{4},\mathbf{5});(\mathbf{4},\mathbf{6})$

Table 5.7. Notation for spinning string frequencies

eigenmodes	**notation**
$\sqrt{2\mathcal{J}^2+n^2\pm\sqrt{4\mathcal{J}^4+n^2\kappa^2}}$ $\sqrt{4\mathcal{J}^2+n^2-m^2}$	$\omega_n^{C_\pm}$ ω_n^C
$\sqrt{n^2+\kappa^2}$	ω_n^A
$\sqrt{4\mathcal{J}^2+n^2}$	ω_n^F

Finally, we would like to point out that both the algebraic curve and world-sheet spectra have instabilities when $|m|\geq 2$. For example if we set $m=2$, then the algebraic curve frequencies labeled by (3,5) and (3,6) become imaginary for $n=-3$ and $n=-1$ and the corresponding world-sheet frequencies become imaginary for $n=\pm 1$.[4]

5.3.5 One-Loop Correction to the Energy

For the spinning string, $\omega_H(n)$ and $\omega_L(n)$ defined in equations (5.32) and (5.33) are nontrivial:

$$\begin{aligned}
\omega_H^{WS}(n) &= 3\omega_n^A+\omega_n^C-4\omega_n^F,\\
\omega_L^{WS}(n) &= 2\omega_n^{C_+}+2\omega_n^{C_-}-2\omega_{2n}^A,\\
\omega_H^{AC}(n) &= 3\omega_n^A+\omega_{n+m}^C-2\omega_n^F-2\omega_{n+m}^F+2\mathcal{J},\\
\omega_L^{AC}(n) &= 2\omega_n^{C_+}+2\omega_{n+m}^{C_-}-2\omega_{2n}^A-4\mathcal{J},
\end{aligned}$$

where WS stands for world-sheet, AC stands for algebraic curve, and we used the notation in table 5.7.

[4] We would like to thank Victor Mikhaylov for showing us his unpublished notes on the spinning string algebraic curve [102]. In these notes, he also derives the algebraic curve for the spinning string and uses it to compute the fluctuation frequencies, however the asymptotics that he imposes on the algebraic curve are different from the asymptotics we use in equation (5.27). The differences occur in the signs of several terms on the right-hand side of equation (5.27). As a result, we obtain frequencies with different constant shifts.

To compute the one-loop correction, we must evaluate an infinite sum of the form

$$\delta E_{1-loop} = \sum_{n=-\infty}^{\infty} \Omega\left(\mathcal{J}, n, m\right). \tag{5.71}$$

Note that the frequency Ω in this equation should not be confused with the off-shell frequencies defined in section 5.1. Since we have two summation prescriptions (the old one in equation (5.36) and the new one in equation (5.38)) and two sets of frequencies (world-sheet and algebraic curve) there are four choices for $\Omega\left(\mathcal{J}, n, m\right)$:

$$\begin{aligned}
\Omega_{old,WS} &= \frac{1}{2\kappa}\left(\omega_H^{WS}(n) + \omega_L^{WS}(n)\right), && (5.72a)\\
\Omega_{new,WS} &= \frac{1}{2\kappa}\left(2\omega_H^{WS}(2n) + \omega_L^{WS}(n)\right), && (5.72b)\\
\Omega_{old,AC} &= \frac{1}{2\kappa}\left(\omega_H^{AC}(n) + \omega_L^{AC}(n)\right), && (5.72c)\\
\Omega_{new,AC} &= \frac{1}{2\kappa}\left(2\omega_H^{AC}(2n) + \omega_L^{AC}(n)\right), && (5.72d)
\end{aligned}$$

where *old*/*new* refers to the summation prescription.

To gain further insight, let us look at the summands in equation (5.72) in two limits: the large-n limit and the large-$\mathcal{J}$ limit. By looking at the large-n limit, we will learn about the convergence properties of the one-loop corrections, and by looking at the large-$\mathcal{J}$ limit and evaluating the sums over n using ζ-function regularization, we will be able to compute the $\mathcal{J}^{-2n}$ contributions to the one-loop corrections. These are referred to as the analytic terms. In general there can also be terms proportional to $\mathcal{J}^{-2n+1}$, which are referred to as the non-analytic terms, and exponentially suppressed terms, i.e., terms that scale like $e^{-\mathcal{J}}$. These terms are subdominant compared to the analytic terms in the large-$\mathcal{J}$ limit.

Large-n Limit

Note that in all four cases $\Omega\left(\mathcal{J}, -n, m\right) = \Omega\left(\mathcal{J}, n, -m\right)$, so the one-loop correction in equation (5.71) can be written as

$$\delta E_{1-loop} = \Omega\left(\mathcal{J}, 0, m\right) + \sum_{n=1}^{\infty}\left(\Omega\left(\mathcal{J}, n, m\right) + \Omega\left(\mathcal{J}, n, -m\right)\right). \tag{5.73}$$

The large-n limit of $\Omega\left(\mathcal{J}, n, m\right) + \Omega\left(\mathcal{J}, n, -m\right)$ for the four choices of $\Omega\left(\mathcal{J}, n, m\right)$ is summarized in table 5.8.

From this table we see that all one-loop corrections are free of quadratic and logarithmic divergences because terms of order n and order $1/n$ cancel out in the large-n limit. At the same time, we find a linear divergence when we apply the old summation prescription to the algebraic curve

Table 5.8. Large-n limit of $\Omega(\mathcal{J},n,m)+\Omega(\mathcal{J},n,-m)$ for the old summation (where $\Omega(\mathcal{J},n,m)=\omega_H(n)+\omega_L(n)$) and the new summation (where $\Omega(\mathcal{J},n,m)=2\omega_H(2n)+\omega_L(n)$) applied to the world-sheet (WS) spectrum and algebraic curve (AC) spectrum

	WS	**AC**
Old Sum	$-\frac{m^2\left(5m^2/4+3\mathcal{J}^2\right)}{2\kappa n^3}+\mathcal{O}\left(n^{-5}\right)$	$-\frac{2\mathcal{J}}{\kappa}-\frac{m^2\left(11m^2/4+5\mathcal{J}^2\right)}{2\kappa n^3}+\mathcal{O}\left(n^{-4}\right)$
New Sum	$-\frac{m^4}{4\kappa n^3}+\mathcal{O}\left(n^{-5}\right)$	$\frac{m^2\left(\mathcal{J}^2-5m^2/4\right)}{2\kappa n^3}+\mathcal{O}\left(n^{-4}\right)$

spectrum since the summand has a constant term. In all other cases however, the summands are at most $\mathcal{O}(n^{-3})$, which suggests that the one-loop corrections are convergent. Hence we find that both summation prescriptions give finite one-loop corrections when applied to the world-sheet spectrum, but only the new summation prescription gives a finite result when applied to the algebraic curve spectrum. This is the same thing we found for the point-particle. The new feature of the spinning string is that the one-loop correction is nonzero and therefore provides a nontrivial prediction to be compared with the dual gauge theory.

Large-$\mathcal{J}$ Limit

In the previous section we found that when $\Omega(\mathcal{J},n,m)=\Omega_{old,AC}(\mathcal{J},n,m)$, the one-loop correction is divergent but for the other three cases in equation (5.72), it is convergent. This means we have three possible predictions for the one-loop correction, however by expanding the summands in the large-$\mathcal{J}$ limit and evaluating the sums over n at each order of $\mathcal{J}$ using ζ-function regularization, we find that all three cases give the same result. The technique of ζ-function regularization is convenient for computing the analytic terms in the large-$\mathcal{J}$ expansion of one-loop corrections but does not capture nonanalytic and exponentially suppressed terms [103]. We now describe this procedure in more detail.

If we expand in the summand in the large-$\mathcal{J}$ limit, only even powers of $\mathcal{J}$ appear:

$$\sum_{n=-\infty}^{\infty}\Omega(\mathcal{J},n,m)=\sum_{k=1}^{\infty}\mathcal{J}^{-2k}\sum_{n=-\infty}^{\infty}\Omega_k(n,m). \tag{5.74}$$

For each power of $\mathcal{J}$, the sum over n can be written as follows

$$\sum_{n=-\infty}^{\infty}\Omega_k(n,m)=\Omega_k(0,m)+\sum_{n=1}^{\infty}\left(\Omega_k(n,m)+\Omega_k(n,-m)\right). \tag{5.75}$$

If we expand $\Omega_k(n,m)$ in the limit $n\to\infty$, we find that it splits into two pieces:

$$\Omega_k(n,m)=\sum_{j=-1}^{2k}c_{k,j}(m)n^j+\tilde{\Omega}_k(n,m),$$

where $\tilde{\Omega}_k(n,m)$ is $\mathcal{O}\left(n^{-2}\right)$. We will refer to $\tilde{\Omega}_k(n,m)$ as the finite piece because it converges when summed over n, and $\sum_{j=-1}^{2k} c_{k,j}(m)n^j$ as the divergent piece because it diverges when summed over n. Furthermore, we find that $\tilde{\Omega}_k(n,m) = \tilde{\Omega}_k(n,-m)$ and $c_{k,j}(m) \propto m^{2k-j}$. Hence, the odd powers of n cancel out of the divergent piece when we add $\Omega_k(n,m)$ to $\Omega_k(n,-m)$ and we get

$$\Omega_k(n,m) + \Omega_k(n,-m) = 2\left[\sum_{j=0}^{k} c_{k,2j}(m)n^{2j} + \tilde{\Omega}_k(n,m)\right].$$

Noting that $\zeta(0) = -1/2$ and $\zeta(2j) = 0$ for $j > 0$, we see that only the constant term in the divergent piece contributes if we evaluate the sum over n using ζ-function regularization:

$$\sum_{n=1}^{\infty}\left(\Omega_k(n,m) + \Omega_k(n,-m)\right) \rightarrow -c_{k,0} + 2\sum_{n=1}^{\infty}\tilde{\Omega}_k(n,m).$$

Combining this with equations (5.74) and (5.75) then gives

$$\delta E_{1-loop} = \sum_{k=1}^{\infty}\mathcal{J}^{-2k}\left[\Omega_k(0,m) - c_{k,0} + 2\sum_{n=1}^{\infty}\tilde{\Omega}_k(n,m)\right].$$

Using the procedure described above, we obtain a single prediction for the one-loop correction to the energy of the spinning string:

$$\begin{aligned}\delta E_{1-loop} \quad &= \quad \frac{1}{2\mathcal{J}^2}\left[m^2/4 + \sum_{n=1}^{\infty}\left(n\left(\sqrt{n^2-m^2}-n\right) + m^2/2\right)\right] \\ &\quad -\frac{1}{8\mathcal{J}^4}\left[3m^4/16 + \sum_{n=1}^{\infty}\left(\begin{array}{c} 3m^4/8 - n^4 \\ +n\sqrt{n^2-m^2}\left(m^2/2+n^2\right)\end{array}\right)\right] + \mathcal{O}\left(\frac{1}{\mathcal{J}^6}\right).\end{aligned} \tag{5.76}$$

In showing that equation (5.71) gives this prediction when $\Omega(\mathcal{J},n,m) = \Omega_{new,AC}(\mathcal{J},n,m)$, it is convenient to shift the index of summation as follows: $\Omega_{new,AC}(\mathcal{J},n,m) \rightarrow \Omega_{new,AC}(\mathcal{J},n-m,m)$. Since the sum is convergent, this shift does not change its value. Recalling that $\mathcal{J} = J/\sqrt{2\pi^2\lambda}$ and making the replacement $\lambda \rightarrow 2\lambda^2$ in equation (5.76) then gives a prediction for the $1/J$ correction to the anomalous dimension of the gauge theory operator in equation (5.52):

$$\Delta - 2J = \left(\frac{\pi^2\lambda^2 m^2}{J} + ...\right) + \frac{1}{J}\left(\frac{2a\pi^2\lambda^2}{J} + ...\right), \tag{5.77}$$

where

$$a \quad = \quad m^2/4 + \sum_{n=1}^{\infty}\left(n\left(\sqrt{n^2-m^2}-n\right) + m^2/2\right).$$

Note that the first term in equation (5.77) came from expanding the classical dispersion relation for the spinning string to first order in the BMN parameter λ/J^2 and then making the replacement $\lambda \to 2\lambda^2$.

5.4 Comparison with Bethe Ansatz

In this section we verify equation (5.77) from the gauge theory side by computing the leading two contributions to the anomalous dimension of the operator dual to the spinning string in $AdS_4 \times CP^3$. First let us consider the operator dual to the $SU(2)$ spinning string in $AdS_5 \times S^5$ which has the form

$$\mathcal{O} = \mathrm{tr}\left[Z^J W^J + \text{permutations}\right], \tag{5.78}$$

where Z and W are complex scalar fields in $\mathcal{N} = 4$ SYM. In this sector, the one-loop planar dilatation operator corresponds to the Hamiltonian of a Heisenberg spin chain of length $2J$ [7]:

$$\mathbf{\Delta} - 2J = \frac{\lambda}{8\pi^2} \sum_{i=1}^{2J} \left(1 - P_{i,i+1}\right). \tag{5.79}$$

The dilatation operator can be diagonalized by solving a set of Bethe ansatz equations [8]:

$$\begin{aligned}
\left(\frac{u_j + i/2}{u_j - i/2}\right)^{2J} &= \prod_{k \neq j}^{J} \frac{u_j - u_k + i}{u_j - u_k - i}, && \text{(5.80a)}\\
\prod_{j=1}^{J} \left(\frac{u_j + i/2}{u_j - i/2}\right) &= 1 \Longrightarrow \sum_{j=1}^{J} \ln\left(\frac{u_j + i/2}{u_j - i/2}\right) = -2\pi m i, && \text{(5.80b)}\\
\Delta - 2J &= \frac{\lambda}{8\pi^2} \sum_{j=1}^{J} \frac{1}{u_j^2 + 1/4}, && \text{(5.80c)}
\end{aligned}$$

where m is an integer which is introduced after taking the log of both sides of equation (5.80b). In the large-J limit, the Bethe equations simplify and can be solved using the methods described in [78, 104, 95]. In particular, [95] found that the anomalous dimension is given by

$$\begin{aligned}
\Delta - 2J &= \left(\frac{\lambda m^2}{4J} + \ldots\right) + \frac{1}{J}\left(\frac{a\lambda}{8J} + \ldots\right),\\
a &= m^2 + \sum_{n=1}^{\infty} \left(n\sqrt{n^2 - 4m^2} - n^2 + 2m^2\right).
\end{aligned} \tag{5.81}$$

Now let us turn to the operator in equation (5.52). In this case, the two-loop planar dilatation operator is given by equation (5.53). As explained in section 5.3.1, this corresponds to the Hamiltonian for two identical Heisenberg spin chains of length $2J$ which are only coupled by a momentum constraint. With this in mind, the Bethe equations are

$$\left(\frac{u_j+i/2}{u_j-i/2}\right)^{2J} = \prod_{k\neq j}^{J}\frac{u_j-u_k+i}{u_j-u_k-i}, \tag{5.82a}$$

$$\left(\prod_{j=1}^{J}\left(\frac{u_j+i/2}{u_j-i/2}\right)\right)^{2} = 1 \Longrightarrow \sum_{j=1}^{J}\ln\left(\frac{u_j+i/2}{u_j-i/2}\right) = -\pi m i, \tag{5.82b}$$

$$\Delta - 2J = 2\lambda^2\sum_{j=1}^{J}\frac{1}{u_j^2+1/4}. \tag{5.82c}$$

Comparing both sets of Bethe equations we see that equation (5.80) can be mapped into equation (5.82) by making the following relabeling:

$$m \to m/2, \qquad \lambda \to 16\pi^2\lambda^2.$$

Making these substitutions in equation (5.81) gives equation (5.77), which we obtained using string theory.

5.5 Conclusions

In this chapter, we studied various methods for computing one-loop corrections to the energies of classical solutions to type IIA string theory in $AdS_4 \times CP^3$. Previous studies which computed the one-loop correction to the folded spinning string in AdS_4 found that agreement with the all-loop AdS_4/CFT_3 Bethe ansatz is not achieved using the standard summation prescription that was used for type IIB string theory in $AdS_5 \times S^5$. Rather, a new summation prescription seems to be required, which distinguishes between so-called light modes and heavy modes. We extended this investigation by analyzing the one-loop correction to the energy of a point-particle and a circular spinning string, both of which are located at the spatial origin of AdS_4 and have nontrivial support in CP^3. The spinning string considered in this chapter has two equal nonzero spins in CP^3 and is the analogue of the $SU(2)$ spinning string in $AdS_5 \times S^5$. The point-particle and spinning string are important examples to analyze because they have trivial support in AdS_4 and therefore avoid the κ-symmetry issues that arise for solutions which purely have support in AdS_4, such as the folded spinning string.

We used two techniques to compute the spectrum of fluctuations about these solutions. One technique, called the world-sheet approach, involves expanding the GS action to quadratic order in the fluctuations and computing the normal modes of the resulting action. The other technique, called the algebraic curve approach, involves computing the algebraic curve for the classical solutions and then carrying out semiclassical quantization. For the point-particle, we found that the world-sheet and algebraic curve fluctuation frequencies match the spectrum of fluctuations obtained in the

Penrose limit up to constant shifts. Furthermore, for the spinning string we found that the algebraic curve spectrum matches the world-sheet spectrum up to constant shifts and shifts in mode number. In particular, the AC and WS frequencies for the spinning string both reduce to the corresponding point-particle frequencies when the winding number is set to zero and become unstable when the winding number $|m| \geq 2$. This is familiar from the $SU(2)$ spinning string in $AdS_5 \times S^5$ [92, 4], which has instabilities for $|m| \geq 1$.

Although the algebraic curve spectrum looks very similar to the world-sheet spectrum, it exhibits some important differences. For example, we find that the algebraic curve frequencies have flat-space behavior when the mode number $n = 0$. This was also found for algebraic curve frequencies in $AdS_5 \times S^5$. More importantly, if we compute one-loop corrections by adding up the algebraic curve frequencies using the standard summation prescription that was used in $AdS_5 \times S^5$, then we get a linear divergence. This is inconsistent with supersymmetry because we expect the one-loop correction to vanish for the point-particle and to be nonzero but finite for the spinning string. We propose a new summation prescription in equation (5.38) which gives precisely these results when applied both to the algebraic curve spectrum and the world-sheet spectrum. This summation prescription has certain similarities to the one that was proposed in [89]. In particular, it also gives a one-loop correction to the folded spinning string which agrees with the all-loop Bethe ansatz. At the same time, it has some important differences which are described in section 2.3. For example, we find that our summation prescription generally gives more well-defined results for one-loop corrections.

In principle we can get three predictions for the one-loop correction to the spinning string (one coming from the algebraic curve and two coming from the world-sheet, because the world-sheet gives finite results using both the old summation prescription in equation (5.36) and the new summation prescription in equation (5.38)), but by expanding the one-loop corrections in the large-$\mathcal{J}$ limit (where $\mathcal{J} = J/\sqrt{2\lambda}\pi$ and J is the spin) and evaluating the sum at each order in $\mathcal{J}$ using ζ-function regularization, we find that all three cases actually give the same result. This is very nontrivial considering that our new summation prescription looks very different than the old one. Furthermore, we show that this result agrees with the predictions of the Bethe ansatz. Thus, while the old summation prescription only seems to work when applied to the world-sheet frequencies of solutions with trivial support in AdS_4, our summation prescription works more generally. Fully understanding why the old summation prescription breaks down for solutions with nontrivial support in AdS_4 warrants further study.

It would be useful to confirm our results using methods more rigorous than ζ-function regularization. This can be done using the contour integral techniques developed in [105], which can also be used to compute $1/\mathcal{J}^{2n+1}$ and exponentially suppressed terms in the large-$\mathcal{J}$ expansion of the one-loop corrections. It would also be interesting to evaluate the one-loop correction to the spinning string energy in a way that does not rely on summation prescriptions. The basic idea would be to identify the one-loop correction with a normal ordering constant which can be then determined by demanding that the quantum generators of certain symmetries preserved by the classical solution have the right algebra. Something along these lines was done for the type IIB superstring in plane-wave background in [106]. Ultimately, fully understanding how to compute one-loop corrections to type IIA string theory in $AdS_4 \times CP^3$ may lead to important tests of the AdS_4/CFT_3 correspondence.

Appendix A

Dirac Algebras

A.1 Three-dimensional Dirac Algebra

The three-dimensional Dirac algebra is defined by

$$\{\gamma_\mu, \gamma_v\} = 2g_{\mu\nu}.$$

We chose the following representation

$$\gamma_0 = i\sigma_2, \qquad \gamma_1 = \sigma_1, \qquad \gamma_2 = \sigma_3,$$

where the σ's are the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A.1}$$

The following relations are useful

$$\begin{aligned}
[\gamma_\mu, \gamma_v] &= 2\epsilon_{\mu\nu\rho}\gamma^\rho, \\
\gamma^\mu\gamma_\nu\gamma_\mu &= -\gamma_\nu, \\
\gamma^\mu\gamma_\mu &= 3, \\
(\gamma \cdot D)(\gamma \cdot D) &= D^2 + \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}, \overline{\Psi}_1\Psi_2 = \overline{\Psi}_2\Psi_1, \\
\overline{\Psi}_1\Gamma^{IJ}\Psi_2 &= -\overline{\Psi}_2\Gamma^{IJ}\Psi_1, \\
\overline{\Psi}_1\Gamma^{IJKL}\Psi_2 &= \overline{\Psi}_2\Gamma^{IJKL}\Psi_1, \\
\overline{\Psi}_1\gamma^{\mu_1\cdots\mu_m}\Psi_2 &= (-1)^m\,\overline{\Psi}_2\gamma^{\mu_m\cdots\mu_1}\Psi_1.
\end{aligned}$$

A.2 Spin(8) Dirac Algebra

The $Spin(8)$ Dirac algebra is defined by

$$\left\{\gamma^I, \gamma^J\right\} = 2\delta^{IJ}.$$

We chose the following representation (from reference [107])

$$\gamma^I = \begin{pmatrix} 0 & \Gamma^I_{A\dot{A}} \\ \left(\Gamma^I\right)^T_{\dot{A}A} & 0 \end{pmatrix}.$$

The Dirac algebra is satisfy if $\dot{B}$

$$\Gamma^I_{A\dot{A}} \left(\Gamma^J\right)^T_{\dot{A}B} + \Gamma^J_{A\dot{A}} \left(\Gamma^I\right)^T_{\dot{A}B} = 2\delta^{IJ}\delta_{AB}.$$

A specific set of matrices $\Gamma^I_{A\dot{A}}$ that satisfy these equations, expressed as direct products of 2×2 blocks, are

$$\begin{aligned}
\Gamma^1 &= i\sigma_2 \times i\sigma_2 \times i\sigma_2, & \Gamma^2 &= I \times \sigma_1 \times i\sigma_2, \\
\Gamma^3 &= I \times \sigma_3 \times i\sigma_2, & \Gamma^4 &= \sigma_1 \times i\sigma_2 \times I, \\
\Gamma^5 &= \sigma_3 \times i\sigma_2 \times I, & \Gamma^6 &= i\sigma_2 \times I \times \sigma_1, \\
\Gamma^7 &= i\sigma_2 \times I \times \sigma_3, & \Gamma^8 &= I \times I \times I.
\end{aligned}$$

Some useful identities

$$\begin{aligned}
\left[\Gamma^{IJ}, \Gamma^{LM}\right] &= 2\delta^{IM}\Gamma^{JL} - 2\delta^{IL}\Gamma^{JM} + 2\delta^{JL}\Gamma^{IM} - 2\delta^{JM}\Gamma^{IL}, \\
\Gamma^{IJ} &= \Gamma^I \left(\Gamma^J\right)^T - \delta^{IJ}, \\
\Gamma^J\Gamma^{LMNO} \left(\Gamma^J\right)^T &= 0, \\
\Gamma^J\Gamma^{IK} \left(\Gamma^J\right)^T &= 4\Gamma^{IK}, \\
\Gamma^{IJ}\Gamma^{KLM} &= \Gamma^{IJKLM} + 6\Gamma^{NOP}\delta^{[IJ]}_{NQ}\delta^{[KLM]}_{QOP} + 6\Gamma^N\delta^{[KLM]}_{[JI]N}, \\
\Gamma^{IJ}\Gamma^J &= 7\Gamma^I, \\
\Gamma^{IJ}\Gamma^{LM}\Gamma^J &= 3\Gamma^{LM}\Gamma^I - 8\Gamma^L\delta^{IM} + 8\Gamma^M\delta^{IL}, \\
\Gamma^{IJ}\Gamma^{LMNO}\Gamma^J &= -\Gamma^{LMNO}\Gamma^I, \\
\Gamma^I\Gamma^{LMN} &= \Gamma^{ILMN} + 3\Gamma^{M'N'}\delta^{[LMN]}_{IM'N'}.
\end{aligned}$$

A.3 Ten-dimensional Dirac Algebra

We use the following representation of the 10d Dirac matrices ($\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$):

$$\begin{aligned}
&\Gamma^0 = i\gamma^0 \otimes \mathrm{I} \otimes \mathrm{I} \otimes \mathrm{I}, && \Gamma^1 = i\gamma^1 \otimes \mathrm{I} \otimes \mathrm{I} \otimes \mathrm{I} && \Gamma^2 = i\gamma^2 \otimes \mathrm{I} \otimes \mathrm{I} \otimes \mathrm{I}, \\
&\Gamma^3 = i\gamma^3 \otimes \mathrm{I} \otimes \mathrm{I} \otimes \mathrm{I}, && \Gamma^4 = \gamma^5 \otimes \sigma_2 \otimes \mathrm{I} \otimes \sigma_1, && \Gamma^5 = \gamma^5 \otimes \sigma_2 \otimes \mathrm{I} \otimes \sigma_3, \\
&\Gamma^6 = \gamma^5 \otimes \sigma_1 \otimes \sigma_2 \otimes \mathrm{I}, && \Gamma^7 = \gamma^5 \otimes \sigma_3 \otimes \sigma_2 \otimes \mathrm{I}, && \Gamma^8 = \gamma^5 \otimes \mathrm{I} \otimes \sigma_1 \otimes \sigma_2, \\
&\Gamma^9 = \gamma^5 \otimes \mathrm{I} \otimes \sigma_3 \otimes \sigma_2,
\end{aligned}$$

where I is the 2×2 identity matrix, the $\gamma's$ are 4d Dirac matrices given by

$$\begin{aligned}
&\gamma^0 = \sigma_1 \otimes \mathrm{I}, && \gamma^1 = i\sigma_2 \otimes \sigma_1, \\
&\gamma^2 = i\sigma_2 \otimes \sigma_2, && \gamma^3 = i\sigma_2 \otimes \sigma_3, \\
&\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3,
\end{aligned}$$

and the Pauli matrices are given by equation (A.1).

Finally, we define the 10d chirality operator as

$$\Gamma_{11} = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\Gamma^6\Gamma^7\Gamma^8\Gamma^9.$$

Appendix B

Poincaré and Conformal Supersymmetries of the BL SO(4) Theory

In this appendix we verified in complete detail the Poincaré and conformal supersymmetries of the BL $SO(4)$ theory.

The BL $SO(4)$ Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\left(D_\mu \phi^I\right)_a \left(D^\mu \phi^I\right)_a + \frac{i}{2}\overline{\psi}^A_a \gamma^\mu \left(D_\mu \psi^A\right)_a \\ &+ ic_1 \epsilon^{abcd}\overline{\psi}^A_a \left(\Gamma^{IJ}\right)_{AB} \psi^B_b \phi^I_c \phi^J_d - c_2 \epsilon^{abcd}\epsilon^{efgd}\phi^I_a \phi^J_b \phi^K_c \phi^K_g \phi^I_e \phi^J_f \\ &+ c_3 \epsilon^{\mu\nu\lambda}\epsilon^{abcd}\left(A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}A_{\mu ab}A_{\nu cg}A_{\lambda gd}\right),\end{aligned}$$

where the c_i, $i = 1, 2, 3, 4$ will be fixed by supersymmetry. The field content and the index notation are given in tables 2.1 and 2.2. The supersymmetry transformations are

$$\delta\phi^I_a = i\overline{\psi}^A_a \Gamma^I_{A\dot{A}}\epsilon^{\dot{A}} = i\overline{\epsilon}^{\dot{A}}\Gamma^I_{\dot{A}A}\psi^{Aa}, \tag{B.1}$$

$$\delta\psi_{Aa} = -\gamma_\mu \left(D_\mu \phi^I\right)_a \Gamma^I_{A\dot{A}}\epsilon^{\dot{A}} + c_4 \epsilon^{abcd}\left(\Gamma^{IJK}\right)_{A\dot{A}}\epsilon^{\dot{A}}\phi^I_b \phi^J_c \phi^K_d - \phi^I_a \Gamma^I_{A\dot{A}}\eta^{\dot{A}}, \tag{B.2}$$

$$\delta A^\mu_{ab} = i\epsilon^{abcd}\overline{\psi}^A_c \gamma_\mu \Gamma^I_{A\dot{A}}\epsilon^{\dot{A}}\phi^I_d, \tag{B.3}$$

and

$$\epsilon^{\dot{A}}(x) = \epsilon^{\dot{A}}_0 + \gamma^\mu x_\mu \eta^{\dot{A}},$$

where $\epsilon^{\dot{A}}_0$ is a constant 8_c-spinor that correspond to the Poincaré supersymmetry parameter while $\eta^{\dot{A}}$ correspond to the superconformal parameter.

Variations

Now, we will take the supersymmetry variations of the Lagrangian by separating it in five terms: A, B, C, D and the Chern-Simons term.

- Term A$=-\frac{1}{2}\left(D_\mu\phi^I\right)_a\left(D^\mu\phi^I\right)_a$

  First, we can write A up to boundary terms in the following way

$$\mathrm{A}=-\frac{1}{2}\left(D_\mu\phi^I\right)_a\left(D^\mu\phi^I\right)_a=\frac{1}{2}\left(D^\mu D_\mu\phi^I\right)_a\phi^I_a.$$

  Then variation is

$$\begin{aligned}\delta(\mathrm{A})&=\frac{1}{2}\delta\left(\left(D^\mu D_\mu\phi^I\right)_a\phi^I_a\right)\\&=D^\mu D_\mu\phi^I_a\delta\phi^I_a+\frac{1}{2}\delta A^\mu_{ab}\left(D_\mu\phi^I\right)_b\phi^I_a+\frac{1}{2}D_\mu\left(\delta A^\mu_{ab}\phi^I_b\right)\phi^I_a\\&=D^\mu D_\mu\phi^I_a\delta\phi^I_a+\delta A^\mu_{ab}\left(D_\mu\phi^I\right)_b\phi^I_a\\&=iD^\mu D_\mu\phi^I_a\left(\overline{\psi}_a\Gamma^I\epsilon\right)+i\epsilon^{abcd}\left(\overline{\psi}_c\gamma_\mu\Gamma^J\epsilon\right)\phi^J_d\left(D_\mu\phi^I\right)_b\phi^I_a.\end{aligned}$$

- Term B$=\frac{i}{2}\overline{\psi}^A_a\gamma^\mu\left(D_\mu\psi^A\right)_a$

$$\begin{aligned}\delta(\mathrm{B})&=\delta\left(\frac{i}{2}\overline{\psi}_a\gamma^\mu\left(D_\mu\psi\right)_a\right)\\&=i\overline{\psi}_a\gamma^\mu\partial_\mu\delta\psi_a+iA_{\mu ab}\overline{\psi}_a\gamma^\mu\delta\psi_b+\frac{i}{2}\delta A_{\mu ab}\overline{\psi}_a\gamma^\mu\psi_b\\&=i\overline{\psi}_a\gamma^\mu D_\mu\delta\psi_a+\frac{i}{2}\delta A_{\mu ab}\overline{\psi}_a\gamma^\mu\psi_b\\&=-i\left(\overline{\psi}_a\gamma^\mu\gamma_v\Gamma^I\epsilon\right)\left(D_\mu D_v\phi^I\right)_a+3ic_4\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\Gamma^{IJK}\epsilon\right)D_\mu\phi^I_b\phi^J_c\phi^K_d\\&\quad-i\left(\overline{\psi}_a\gamma^\mu\Gamma^I\eta\right)D_\mu\phi^I_a-i\left(\overline{\psi}_a\gamma^\mu\gamma_v\Gamma^ID_\mu\epsilon\right)\left(D_v\phi^I\right)_a\\&\quad+ic_4\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\Gamma^{IJK}D_\mu\epsilon\right)\phi^I_b\phi^J_c\phi^K_d+\frac{i}{2}\delta A_{\mu ab}\overline{\psi}_a\gamma^\mu\psi_b\\&\quad-2ic_4\epsilon^{abcd}\left(\overline{\psi}_e\gamma^\mu\Gamma^{IJK}\epsilon\right)A^\mu_{ae}\phi^I_b\phi^J_c\phi^K_d\\&=-i\left(\overline{\psi}_a\gamma^\mu\gamma_v\Gamma^I\epsilon\right)\left(D_\mu D_v\phi^I\right)_a+3ic_4\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\Gamma^{IJK}\epsilon\right)D_\mu\phi^I_b\phi^J_c\phi^K_d\\&\quad+3ic_4\epsilon^{abcd}\left(\overline{\psi}_a\Gamma^{IJK}\eta\right)\phi^I_b\phi^J_c\phi^K_d-2ic_4\epsilon^{becd}\left(\overline{\psi}_a\gamma^\mu\Gamma^{IJK}\epsilon\right)A^\mu_{ab}\phi^I_e\phi^J_c\phi^K_d\\&\quad-\frac{1}{2}\epsilon^{abcd}\left(\overline{\psi}_c\gamma_\mu\Gamma^I\epsilon\right)\left(\overline{\psi}_a\gamma^\mu\psi_b\right)\phi^I_d\\&=-i\left(\overline{\psi}_a\Gamma^I\epsilon\right)\left(D_\mu D^\mu\phi^I\right)_a-i\frac{1}{2}\left(\overline{\psi}_a\gamma^\mu\gamma^v\Gamma^I\epsilon\right)F_{\mu v ab}\phi^I_b\\&\quad+3ic_4\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\Gamma^{IJK}\epsilon\right)\left(D_\mu\phi^I\right)_b\phi^J_c\phi^K_d\\&\quad+3ic_4\epsilon^{abcd}\left(\overline{\psi}_a\Gamma^{IJK}\eta\right)\phi^I_b\phi^J_c\phi^K_d-2ic_4\epsilon^{becd}\left(\overline{\psi}_a\gamma^\mu\Gamma^{IJK}\epsilon\right)A^\mu_{ab}\phi^I_e\phi^J_c\phi^K_d\\&\quad-\frac{1}{2}\epsilon^{abcd}\left(\overline{\psi}_c\gamma_\mu\Gamma^I\epsilon\right)\left(\overline{\psi}_a\gamma^\mu\psi_b\right)\phi^I_d.\end{aligned}$$

- Term C$=ic_1\epsilon^{abcd}\overline{\psi}^A_a\left(\Gamma^{IJ}\right)_{AB}\psi^B_b\phi^I_c\phi^J_d$

$$\begin{aligned}
\delta(\mathrm{C}) &= 2ic_1\epsilon^{abcd}\overline{\psi}^A_a\left(\Gamma^{IJ}\right)_{AB}\delta\psi^B_b\phi^I_c\phi^J_d + 2ic_1\epsilon^{abcd}\overline{\psi}^A_a\left(\Gamma^{IJ}\right)_{AB}\psi^B_b\delta\phi^I_c\phi^J_d, \\
&= 2ic_1\epsilon^{abcd}\overline{\psi}^A_a\left(\Gamma^{IJ}\right)_{AB}[-\gamma_\mu\left(D_\mu\phi^K\right)_b\Gamma^K\epsilon + c_4\epsilon^{befg}\Gamma^{LMN}\epsilon\phi^L_e\phi^M_f\phi^N_g \\
&\quad - \phi^K_b\Gamma^K\eta]\phi^I_c\phi^J_d - 2c_1\epsilon^{abcd}\overline{\psi}_a\Gamma^{IJ}\psi_b\overline{\psi}_c\Gamma^I\epsilon\phi^J_d, \\
&= -2ic_1\epsilon^{abcd}\left(\overline{\psi}_a\Gamma^{IJ}\gamma_\mu\Gamma^K\epsilon\right)\left(D_\mu\phi^K\right)_b\phi^I_c\phi^J_d \\
&\quad + 2ic_1c_4\epsilon^{abcd}\epsilon^{befg}\left(\overline{\psi}_a\Gamma^{IJ}\Gamma^{KLM}\epsilon\right)\phi^I_c\phi^J_d\phi^K_e\phi^L_f\phi^M_g \\
&\quad - 2ic_1\epsilon^{abcd}\left(\overline{\psi}_a\Gamma^{IJ}\Gamma^K\eta\right)\phi^K_b\phi^I_c\phi^J_d \\
&\quad - 2c_1\epsilon^{abcd}\left(\overline{\psi}_a\Gamma^{IJ}\psi_b\right)\left(\overline{\psi}_c\Gamma^I\epsilon\right)\phi^J_d.
\end{aligned}$$

- Term D$= -c_2\epsilon^{abcd}\epsilon^{efgd}\phi^I_a\phi^J_b\phi^K_c\phi^I_e\phi^J_f\phi^K_g$

$$\begin{aligned}
\delta(D) &= -3c_2\epsilon^{abcd}\epsilon^{efgd}\delta\left(\phi^I_a\phi^I_e\right)\phi^J_b\phi^J_f\phi^K_c\phi^K_g \\
&= -3ic_2\epsilon^{abcd}\epsilon^{efgd}\left(\bar{\psi}_a\phi^I_e + \bar{\psi}_e\phi^I_a\right)\Gamma^I\epsilon\phi^J_b\phi^J_f\phi^K_c\phi^K_g.
\end{aligned}$$

After relabeling dummy indices we get

$$\delta(D) = -6ic_2\epsilon^{abcd}\epsilon^{befg}\left(\bar{\psi}_a\Gamma^K\epsilon\right)\phi^I_c\phi^J_d\phi^K_c\phi^J_f\phi^I_g.$$

- Chern-Simons term$=c_3\epsilon^{\mu\nu\lambda}\epsilon^{abcd}\left(A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}A_{\mu ab}A_{\nu cg}A_{\lambda gd}\right)$

$$\begin{aligned}
\delta(\mathrm{CS}) &= 2c_3\epsilon^{\mu\nu\lambda}\epsilon^{abcd}\left(\delta A_{\mu ab}\partial_\nu A_{\lambda cd} + \delta A_{\mu ab}A_{\nu cg}A_{\lambda gd}\right) \\
&= c_3\epsilon^{\mu\nu\lambda}\epsilon^{abcd}F_{\nu\lambda}\delta A_{\mu ab} \\
&= ic_3\epsilon^{\mu\nu\lambda}\epsilon^{abcd}\epsilon^{abef}\left(\overline{\psi}_e\gamma_\mu\Gamma^I\epsilon\right)\phi^I_f F_{\nu\lambda} \\
&= ic_3 4\epsilon^{\mu\nu\lambda}\left(\delta^{cd}_{ef}\right)\left(\overline{\psi}_e\gamma_\mu\Gamma^I\epsilon\right)\phi^I_f F_{\nu\lambda cd} \\
&= ic_3 4\epsilon^{\mu\nu\lambda}\left(\overline{\psi}_c\gamma_\mu\Gamma^I\epsilon\right)\phi^I_d F_{\nu\lambda cd},
\end{aligned}$$

where we used the fact that

$$\epsilon^{\mu\nu\lambda}\widetilde{A}_{\mu cd}\delta A_{\nu cg}A_{\lambda gd} = \epsilon^{\mu\nu\lambda}\delta\widetilde{A}_{\mu cd}A_{\nu cg}A_{\lambda gd}.$$

Cancellations

We can classify the terms we obtain from the supersymmetries variations in five different types as shown in table B.1. Now we will show how these different types of contributions cancel by choosing the right c_i's parameters.

Table B.1. Classification of the supersymmetry variations of the BL $SO(4)$ Lagrangian

	$DD\phi\psi$	$F\phi\psi$	$\psi\,(D\phi)\,\phi^2$	$\psi\phi^3\eta$	$\psi^3\phi$	$\phi^5\psi$
A	$\times$	0	$\times$	0	0	0
B	$\times$	$\times$	$\times$	$\times$	$\times$	0
C	0	0	$\times$	$\times$	$\times$	$\times$
D	0	0	0	0	0	$\times$
CS	0	$\times$	0	0	0	0

- Term $DD\phi\psi$

  This cancelation is trivial

$$iD^{\mu}D_{\mu}\phi_a^I\left(\overline{\psi}_a\Gamma^I\epsilon\right)-i\left(D_{\mu}D^{\mu}\phi^I\right)_a\left(\overline{\psi}_a\Gamma^I\epsilon\right)=0.$$

- Term $F\phi\psi$

  We first simplify the contribution from the B term as

$$-i\frac{1}{2}\left(\overline{\psi}_a\gamma^{\mu}\gamma^{v}\Gamma^I\epsilon\right)F_{\mu vab}\phi_b^I=-i\frac{1}{2}\epsilon^{\mu v\rho}\left(\overline{\psi}_a\gamma_{\rho}\Gamma^I\epsilon\right)\phi_b^IF_{\mu vab},$$

  then for the cancelation we need

$$-i\kappa c_3 4\epsilon^{\mu v\rho}\left(\overline{\psi}_a\gamma_{\rho}\Gamma^I\epsilon\right)\phi_b^IF_{\mu vab}-i\frac{1}{2}\epsilon^{\mu v\rho}\left(\overline{\psi}_a\gamma_{\rho}\Gamma^I\epsilon\right)\phi_b^IF_{\mu vab}=0,$$

  therefore this implies that

$$c_3=\frac{1}{8}.$$

- Term $\psi\phi^3\eta$

  First, let us simplify both contributions in the following way

$$\begin{aligned}3ic_4\epsilon^{abcd}\left(\overline{\psi}_a\Gamma^{IJK}\eta\right)\phi_b^I\phi_c^J\phi_d^K&=3ic_4\epsilon^{abcd}\left(\overline{\psi}_a\Gamma^I\Gamma^J\Gamma^K\eta\right)\phi_b^I\phi_c^J\phi_d^K,\\-2ic_1\epsilon^{abcd}\left(\overline{\psi}_a\Gamma^{IJ}\Gamma^K\eta\right)\phi_b^K\phi_c^I\phi_d^J&=-2ic_1\epsilon^{abcd}\left(\overline{\psi}_a\Gamma^I\Gamma^J\Gamma^K\eta\right)\phi_b^I\phi_c^J\phi_d^K,\end{aligned}$$

  then we need

$$c_4=\frac{2}{3}c_1.$$

- Term $\psi\,(D\phi)\,\phi^2$

  We want

$$i\epsilon^{abcd}\left(\overline{\psi}_a\gamma_{\mu}\Gamma^J\epsilon\right)\left(D_{\mu}\phi^I\right)_b\phi_d^I\phi_c^J+3ic_4\epsilon^{abcd}\left(\overline{\psi}_a\gamma^{\mu}\Gamma^{IJK}\epsilon\right)\left(D_{\mu}\phi^I\right)_b\phi_c^J\phi_d^K$$

$$-2ic_1\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\Gamma^{IJ}\Gamma^K\epsilon\right)\left(D_\mu\phi^K\right)_b\phi_c^I\phi_d^J=0,$$

expanding the second and third contributions we get

$$\begin{aligned}&i\epsilon^{abcd}\left(\overline{\psi}_a\gamma_\mu\Gamma^J\epsilon\right)\phi_b^J\left(D_\mu\phi^I\right)_d\phi_c^I\\&\quad+ic_4\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\left(\Gamma_K\Gamma_I^T\Gamma_J+\Gamma_I\Gamma_J^T\Gamma_K-\Gamma_I\Gamma_K^T\Gamma_J\right)\epsilon\right)\left(D_\mu\phi^K\right)_b\phi_c^I\phi_d^J\\&\qquad-2ic_1\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\Gamma^I\Gamma_J^T\Gamma_K\epsilon\right)\left(D_\mu\phi^K\right)_b\phi_c^I\phi_d^J=0,\end{aligned}$$

and using the following simplification of the second term

$$\begin{aligned}&\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\left(2\delta^{IK}\Gamma^J+3\Gamma_I\Gamma_J^T\Gamma_K-4\Gamma_I\delta^{KJ}\right)\epsilon\right)\left(D_\mu\phi^K\right)_b\phi_c^I\phi_d^J=\\&\quad3\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\Gamma_I\Gamma_J^T\Gamma_K\epsilon\right)\left(D_\mu\phi^K\right)_b\phi_c^I\phi_d^J-6\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\Gamma^J\epsilon\right)\left(D_\mu\phi^I\right)_d\phi_c^I\phi_b^J,\end{aligned}$$

we find that the coefficients are

$$\begin{aligned}c_1&=\frac{1}{4},\\c_4&=\frac{1}{6}.\end{aligned}$$

- Term $\psi^3\phi$

  The required cancellation is

$$\begin{aligned}0&=-\frac{1}{2}\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\psi_b\right)\left(\overline{\psi}_c\gamma_\mu\Gamma^I\epsilon\right)\phi_d^I+2c_1\epsilon^{abcd}\left(\overline{\psi}_a\Gamma^{IJ}\psi_b\right)\left(\overline{\psi}_c\Gamma^J\epsilon\right)\phi_d^I\\&=-\frac{1}{2}\mathrm{T1}+2c_1\mathrm{T2}.\end{aligned}$$

  Then, we have two structures

$$\mathrm{T1}=\epsilon^{abcd}\left(\overline{\psi}_a\gamma^\mu\psi_b\right)\left(\overline{\psi}_c\gamma_\mu\Gamma^I\epsilon\right)\phi_d^I,$$

  and

$$\mathrm{T2}=\epsilon^{abcd}\left(\overline{\psi}_a\Gamma^{IJ}\psi_b\right)\left(\overline{\psi}_c\Gamma^J\epsilon\right)\phi_d^I,$$

  and we want to prove they are equivalent.

  We decompose the first structure in the following way

$$\mathrm{T1}=\epsilon^{abcd}\left(\overline{\psi}_a\gamma_b^\mu\psi_b\right)\left(\overline{\psi}_c\gamma_\mu\Gamma^I\epsilon\right)\phi_d^I=\frac{1}{3}\epsilon^{abcd}\begin{pmatrix}\overline{\psi}_a\gamma^\mu\left(\psi_b\overline{\psi}_c-\psi_c\overline{\psi}_b\right)\gamma_\mu\Gamma^I\epsilon\\+\left(\overline{\psi}_a\gamma^\mu\psi_b\right)\left(\overline{\psi}_c\gamma_\mu\Gamma^I\epsilon\right)\end{pmatrix}\phi_d^I.$$

Using the following Fierz transformation

$$\psi_b\overline{\psi}_c - \psi_c\overline{\psi}_b = \frac{1}{8}\gamma^\mu\overline{\psi}_b\gamma_\mu\psi_c - \frac{1}{16}\Gamma^{IJ}\overline{\psi}_b\Gamma^{IJ}\psi_c + \frac{1}{384}\Gamma^{IJKL}\gamma^\mu\overline{\psi}_b\Gamma^{IJKL}\gamma_\mu\psi_c,$$

we get

$$\begin{aligned}
\mathrm{T1} &= \frac{1}{3}\epsilon^{abcd}\begin{pmatrix} -\frac{1}{8}\left(\psi_a\Gamma^I\gamma^\nu\epsilon\right)\left(\overline{\psi}_b\gamma_\nu\psi_c\right) - \frac{3}{16}\left(\overline{\psi}_a\Gamma^{LM}\Gamma^I\epsilon\right)\left(\overline{\psi}_b\Gamma^{LM}\psi_c\right) \\ -\frac{1}{384}\left(\overline{\psi}_a\Gamma^{LMNO}\Gamma^I\gamma^\nu\epsilon\right)\left(\overline{\psi}_b\Gamma^{LMNO}\gamma_\nu\psi_c\right) \\ +\left(\overline{\psi}_a\Gamma^I\gamma_\mu\epsilon\right)\left(\overline{\psi}_b\gamma^\mu\psi_c\right)\end{pmatrix}\phi^I_d, \\
&= \frac{1}{3}\epsilon^{abcd}\begin{pmatrix} \frac{7}{8}\left(\psi_a\Gamma^I\gamma^\nu\epsilon\right)\left(\overline{\psi}_b\gamma_\nu\psi_c\right) - \frac{3}{16}\left(\overline{\psi}_a\Gamma^{LM}\Gamma^I\epsilon\right)\left(\overline{\psi}_b\Gamma^{LM}\psi_c\right) \\ -\frac{1}{384}\left(\overline{\psi}_a\Gamma^{LMNO}\Gamma^I\gamma^\nu\epsilon\right)\left(\overline{\psi}_b\Gamma^{LMNO}\gamma_\nu\psi_c\right)\end{pmatrix}\phi^I_d.
\end{aligned}$$

For the second structure we get

$$\mathrm{T2} = \epsilon^{abcd}\left(\overline{\psi}_a\Gamma^{IJ}\psi_b\right)\left(\overline{\psi}_c\Gamma^J\epsilon\right)\phi^I_d = \frac{1}{3}\left[\epsilon^{abcd}\begin{pmatrix}\overline{\psi}_a\Gamma^{IJ}\left(\psi_b\overline{\psi}_c - \psi_c\overline{\psi}_b\right)\Gamma^J\epsilon \\ +\left(\overline{\psi}_a\Gamma^{IJ}\psi_b\right)\left(\overline{\psi}_c\Gamma^J\epsilon\right)\end{pmatrix}\phi^I_d\right].$$

Using the Fierz transformation and the following identities

$$\begin{aligned}
\Gamma^{IJ}\Gamma^J &= 7\Gamma^I, \\
\Gamma^{IJ}\Gamma^{LM}\Gamma^J &= 3\Gamma^{LM}\Gamma^I - 8\Gamma^L\delta^{IM} + 8\Gamma^M\delta^{IL}, \\
\Gamma^{IJ}\Gamma^{LMNO}\Gamma^J &= -\Gamma^{LMNO}\Gamma^I,
\end{aligned}$$

we get

$$\begin{aligned}
\mathrm{T2} &= \frac{1}{3}\left[\epsilon^{abcd}\begin{pmatrix} \frac{7}{8}\left(\overline{\psi}_a\Gamma^I\gamma^\nu\epsilon\right)\left(\bar{\psi}_b\gamma_\nu\psi_c\right) - \left(\overline{\psi}_a\Gamma^J\epsilon\right)\left(\overline{\psi}_b\Gamma^{IJ}\psi_c\right) \\ -\frac{1}{384}\left(\overline{\psi}_a\Gamma^{LMNO}\Gamma^I\gamma^\nu\epsilon\right)\left(\overline{\psi}_b\Gamma^{LMNO}\gamma_\nu\psi_c\right) \\ -\frac{3}{16}\left(\overline{\psi}_a\Gamma^{LM}\Gamma^I\epsilon\right)\left(\overline{\psi}_b\Gamma^{LM}\psi_c\right) + \left(\overline{\psi}_c\Gamma^J\epsilon\right)\left(\overline{\psi}_a\Gamma^{IJ}\psi_b\right)\end{pmatrix}\phi^I_d\right] \\
&= \frac{1}{3}\epsilon^{abcd}\begin{pmatrix} \frac{7}{8}\left(\overline{\psi}_a\Gamma^I\gamma^\nu\epsilon\right)\left(\bar{\psi}_b\gamma_\nu\psi_c\right) - \frac{3}{16}\left(\overline{\psi}_a\Gamma^{LM}\Gamma^I\epsilon\right)\left(\overline{\psi}_b\Gamma^{LM}\psi_c\right) \\ -\frac{1}{384}\left(\overline{\psi}_a\Gamma^{LMNO}\Gamma^I\gamma^\nu\epsilon\right)\left(\overline{\psi}_b\Gamma^{LMNO}\gamma_\nu\psi_c\right)\end{pmatrix}\phi^I_d.
\end{aligned}$$

Therefore $\mathrm{T1} = \mathrm{T2}$, and

$$c_1 = \frac{1}{4}.$$

- Term $\phi^5\psi$

  We want

$$2ic_1c_4\epsilon^{abcd}\epsilon^{befg}\overline{\psi}^a\Gamma^{IJ}\Gamma^{KLM}\epsilon\phi^I_c\phi^J_d\phi^K_e\phi^L_f\phi^M_g - 6ic_2\epsilon^{abcd}\epsilon^{befg}\overline{\psi}^a\Gamma^K\epsilon\phi^I_c\phi^J_d\phi^K_e\phi^J_f\phi^I_g = 0,$$

then we would like to show that

$$3c_2\epsilon^{abcd}\epsilon^{befg}\Gamma^K\phi_c^I\phi_d^J\phi_e^K\phi_f^J\phi_g^I = c_1c_4\epsilon^{abcd}\epsilon^{befg}\Gamma^{IJ}\Gamma^{KLM}\phi_c^I\phi_d^J\phi_e^K\phi_f^L\phi_g^M.$$

The right-hand side can be simplified by noting that $\Gamma^{IJ}\Gamma^{KLM}$ can be written as

$$\Gamma^{IJ}\Gamma^{KLM} = \Gamma^{IJKLM} + 6\Gamma^{NOP}\delta_{NQ}^{[IJ]}\delta_{QOP}^{[KLM]} + 6\Gamma^N\delta_{[JI]N}^{[KLM]}.$$

and $\epsilon^{bacd}\epsilon^{befg} = 6\delta_{efg}^{[acd]}$ then

$$\begin{aligned}
&\epsilon^{abcd}\epsilon^{befg}\Gamma^{IJ}\Gamma^{KLM}\phi_c^I\phi_d^J\phi_e^K\phi_f^L\phi_g^M \\
&\qquad\qquad = -6\bar{\psi}^a\left(6\Gamma^{NOP}\delta_{NQ}^{[IJ]}\delta_{QOP}^{[KLM]} + 6\Gamma^N\delta_{[JI]N}^{[KLM]}\right)\epsilon\phi_c^I\phi_d^J\phi_a^K\phi_c^L\phi_d^M \\
&\qquad\qquad = -36\overline{\psi}^a\Gamma^{M'}\delta_{[ML]M'}^{[KIJ]}\phi_c^I\phi_d^J\phi_a^K\phi_c^L\phi_d^M \\
&\qquad\qquad = -36\overline{\psi}^a\Gamma^{M'}\delta_{MLM'}^{[KIJ]}\phi_c^I\phi_d^J\phi_a^K\phi_c^L\phi_d^M \\
&\qquad\qquad = 36\psi^a\Gamma^{M'}\delta_{LMM'}^{IJK}\delta_{efg}^{[acd]}\phi_e^K\phi_f^J\phi_g^I\phi_c^L\phi_d^M \\
&\qquad\qquad = 36\psi^a\delta_{efg}^{[acd]}\Gamma^K\phi_e^K\phi_f^J\phi_g^I\phi_c^I\phi_d^J,
\end{aligned}$$

where the term $\Gamma^{NOP}\delta_{NQ}^{[IJ]}\delta_{QOP}^{[KLM]}$ in the second line cancel since only its symmetric part in $I \longleftrightarrow L$ and $J \longleftrightarrow M$ contribute but this part is clearly zero if we rewrite this term as

$$\begin{aligned}
\Gamma^{NOP}\delta_{NQ}^{[IJ]}\delta_{QOP}^{[KLM]} &= \left(2\Gamma^{ILM}\delta_J^K + 2\Gamma^{IKL}\delta_J^M\right) + 2\left(\Gamma^{IMK}\delta_J^L - \Gamma^{LJK}\delta_M^I\right) \\
&\quad - \left(2\Gamma^{JLM}\delta_I^K + 2\Gamma^{JMK}\delta_I^L\right).
\end{aligned}$$

Now, the left-hand side can be written like

$$3c_2\epsilon^{abcd}\epsilon^{befg}\Gamma^K\phi_c^I\phi_d^J\phi_e^K\phi_f^J\phi_g^I = 18c_2\delta_{efg}^{[acd]}\Gamma^K\phi_c^I\phi_d^J\phi_e^K\phi_f^J\phi_g^I.$$

Finally, for the cancellation we need

$$c_2 = 2c_1c_4.$$

- Parameters of the Lagrangian

Finally the constraints on the parameters of the Lagrangian that come from supersymmetry are

$$\begin{aligned}
c_1 &= \frac{1}{4}, \\
c_2 &= \frac{1}{12},
\end{aligned}$$

$$c_3 = \frac{1}{8},$$
$$c_4 = \frac{1}{6}.$$

Note that they agree exactly with those of Bagger and Lambert up to minus signs.

Appendix C

Verification of Superconformal Symmetry

C.1 The U(1) × U(1) Theory

Let us check the supersymmetry of the $U(1) \times U(1)$ theory. We only analyze half of the terms, since the other half are just their adjoints. Omitting the factor of $k/2\pi$, the variation of the Lagrangian contains (dropping total derivatives)

$$\Delta_1 = -D^\mu X^A D_\mu \delta X_A = iD^2 X^A \bar{\varepsilon}^I \Gamma^I_{AB} \Psi^B,$$

and

$$\begin{aligned}\Delta_2 &= i\delta\bar{\Psi}_A \gamma \cdot D\Psi^A = -i\Gamma^I_{AB}\bar{\varepsilon}^I \gamma \cdot DX^B \gamma \cdot D\Psi^A \\ &= i\Gamma^I_{AB}\bar{\varepsilon}^I D^2 X^B \Psi^A - \frac{1}{2}\Gamma^I_{AB}\bar{\varepsilon}^I \gamma^{\rho\mu}(F_{\rho\mu} - \hat{F}_{\rho\mu}) X^B \Psi^A.\end{aligned}$$

Note that the gauge fields only appear in the covariant derivatives in the combination $A - \hat{A}$, which has a vanishing supersymmetry variation. The variation of the Chern-Simons term, using the first term in equation (4.6), contributes

$$\Delta_3 = \frac{1}{2}\varepsilon^{\mu\nu\lambda}\bar{\varepsilon}^I \gamma_\mu \Psi^A \Gamma^I_{AB} X^B (F_{\nu\lambda} - \hat{F}_{\nu\lambda}).$$

Using $\varepsilon^{\mu\nu\lambda}\gamma_\mu = \gamma^{\nu\lambda}$, we see that $\Delta_1 + \Delta_2 + \Delta_3 = 0$. The other half of the terms in the variation of the action, which are the adjoints of the ones considered here, cancel in the same way. The conserved supersymmetry current can be computed by the standard Noether procedure. This gives (aside from an arbitrary normalization)

$$Q^I_\mu = \Gamma^I_{AB}\gamma \cdot DX^A \gamma_\mu \Psi^B - \tilde{\Gamma}^{IAB}\gamma \cdot DX_A \gamma_\mu \Psi_B.$$

One can check this result by computing the divergence. This vanishes as a consequence of the equations of motion $\gamma \cdot D\Psi^B = 0$, $D \cdot DX^A = 0$, and $F_{\mu\nu} - \hat{F}_{\mu\nu} = 0$.

Let us now explore the conformal supersymmetry, with an infinitesimal spinor parameter η^I, using the method explained in [27]. As a first try, consider replacing ε^I by $\gamma \cdot x\eta^I$ in the preceding equations, since this has the correct dimensions. Using $\partial_\mu \varepsilon(x) = \gamma_\mu \eta$ and $\gamma^\mu \gamma^\rho \gamma_\mu = -\gamma^\rho$, this gives a variation of the action that almost cancels, except for a couple of terms. These remaining terms can be canceled by including an additional variation of the spinor fields. It has the form

$$\delta' \Psi^A = -\tilde{\Gamma}^{IAB} \eta^I X_B \quad \text{and} \quad \delta' \Psi_A = \Gamma^I_{AB} \eta^I X^B.$$

Correspondingly, the conserved superconformal current is

$$S^I_\mu = \gamma \cdot x\, Q^I_\mu + \Gamma^I_{AB} X^A \gamma_\mu \Psi^B - \tilde{\Gamma}^{IAB} X_A \gamma_\mu \Psi_B.$$

As a check, one can compute the divergence using the conservation of Q^I_μ and the spinor field equation of motion

$$\partial^\mu S^I_\mu = \gamma^\mu Q^I_\mu + \Gamma^I_{AB} \gamma \cdot DX^A \Psi^B - \tilde{\Gamma}^{IAB} \gamma \cdot DX_A \Psi_B = 0.$$

The various bosonic $OSp(6|4)$ symmetry transformations are obtained by commuting ε and η transformations. Of these only the conformal transformation, obtained as the commutator of two η transformations, is not a manifest symmetry of the action. It is often true that scale invariance implies conformal symmetry. However, this is not a general theorem, so it is a good idea to check the conformal symmetry (or the conformal supersymmetry) explicitly.

C.2 The $\mathbf{U(N) \times U(N)}$ Theory

Let us now examine the supersymmetry of the $U(N) \times U(N)$ theory. Some of the terms are simple generalizations of those examined in the $N = 1$ case and will not be described here. Rather, we focus on those that only arise for $N > 1$. We will first determine the quartic $\Psi^2 X^2$ term (called L_4) in the action by requiring that the variation of its X fields cancels the terms that arise from varying the gauge fields in the spinor kinetic term. Since these terms are cubic in Ψ, various Fierz identities are required. The second step is to determine the variation $\delta_3 \Psi$ by requiring that this variation of the spinor kinetic term cancels against the lowest-order variation of the Ψ fields in L_4 and the variation of the gauge fields in the scalar kinetic term. The third and final step is to determine L_6 by arranging that its variation cancels against the $\delta_3 \Psi$ variation of L_4. After this has been completed, we verify the conformal supersymmetry.

Determination of $\mathbf{L_4}$

A useful identity involving four two-component Majorana spinors, obtained by a Fierz transformation, is

$$\bar{\psi}_1\gamma_\mu\psi_2\bar{\psi}_3\gamma^\mu\varepsilon = -2\bar{\varepsilon}\psi_1\bar{\psi}_2\psi_3 - \bar{\psi}_1\psi_2\bar{\varepsilon}\psi_3.$$

Juggling the indices this can be recast in the form

$$\bar{\varepsilon}\gamma_\mu\psi_1\bar{\psi}_2\gamma^\mu\psi_3 = -2\bar{\psi}_1\psi_2\bar{\varepsilon}\psi_3 - \bar{\varepsilon}\psi_1\bar{\psi}_2\psi_3.$$

These will be useful for eliminating Dirac matrices from equations that arise later. As written, these relations preserve the 123 sequence of the spinors, which is convenient if they are matrices that are to be multiplied. However, the right-hand sides can be rewritten in other ways without Dirac matrices using the relation

$$\psi_1\bar{\psi}_2\psi_3 + \psi_2\bar{\psi}_3\psi_1 + \psi_3\bar{\psi}_1\psi_2 = 0. \tag{C.1}$$

This equation will also be useful.

Varying the gauge fields in the spinor kinetic term of the $U(N)\times U(N)$ theory (dropping a factor of $k/2\pi$) gives

$$\mathrm{tr}\left(\bar{\Psi}_A\gamma^\mu(-\delta A_\mu\Psi^A + \Psi^A\delta\hat{A}_\mu)\right).$$

Keeping only the terms with two superscripts on spinor fields, since the other terms are just their adjoints, leaves

$$\Gamma^I_{BC}\mathrm{tr}(-\bar{\Psi}^A\gamma^\mu\Psi_A\bar{\Psi}^B\gamma_\mu\varepsilon^I X^C + \bar{\varepsilon}^I\gamma^\mu\Psi^B\bar{\Psi}_A\gamma_\mu\Psi^A X^C).$$

Inserting the identities above, so as to eliminate Dirac matrices while retaining the order of the matrices, which are implicitly multiplied, leaves

$$\Gamma^I_{BC}\mathrm{tr}\Big(2\bar{\varepsilon}^I\Psi^A\bar{\Psi}_A\Psi^B X^C + \bar{\Psi}^A\Psi_A\bar{\varepsilon}^I\Psi^B X^C - 2\bar{\Psi}^B\Psi_A\bar{\varepsilon}^I\Psi^A X^C - \bar{\varepsilon}^I\Psi^B\bar{\Psi}_A\Psi^A X^C\Big)$$
$$= i\mathrm{tr}(\bar{\Psi}^A\Psi_A\delta X_B X^B) - i\mathrm{tr}(\bar{\Psi}_A\Psi^A X^B\delta X_B) + 2\Gamma^I_{BC}\mathrm{tr}(\bar{\varepsilon}^I\Psi^A[\bar{\Psi}_A\Psi^B X^C - X^C\bar{\Psi}^B\Psi_A]).$$

Now consider varying the X fields in the second term in L_{4a}. This gives

$$
\begin{aligned}
-2i\varepsilon_{ABCD}\mathrm{tr}(\bar{\Psi}^A\delta X^B\Psi^C X^D) &= -2\tilde{\Gamma}^{IBE}\varepsilon_{ABCD}\mathrm{tr}(\bar{\Psi}^A\bar{\varepsilon}^I\Psi_E\Psi^C X^D) \\
&= -\varepsilon^{BEFG}\varepsilon_{ABCD}\Gamma^I_{FG}\mathrm{tr}(\bar{\Psi}^A\bar{\varepsilon}^I\Psi_E\Psi^C X^D) \\
&= \delta^{EFG}_{ACD}\Gamma^I_{FG}\mathrm{tr}(\bar{\Psi}^A\bar{\varepsilon}^I\Psi_E\Psi^C X^D) \\
&= -\delta^{EFG}_{ACD}\Gamma^I_{FG}\mathrm{tr}(\bar{\Psi}^A\Psi_E\bar{\varepsilon}^I\Psi^C X^D + \bar{\Psi}^A\varepsilon^I\bar{\Psi}_E\Psi^C X^D) \\
&= -2i\mathrm{tr}(\bar{\Psi}^A\Psi_A\delta X_B X^B) + 2i\mathrm{tr}(\bar{\Psi}_A\Psi^A X^B\delta X_B) \\
&\quad + 2i\mathrm{tr}(\bar{\Psi}^A\Psi_B\delta X_A X^B) - 2i\mathrm{tr}(\bar{\Psi}_A\Psi^B X^A\delta X_B) \\
&\quad - 2\Gamma^I_{BC}\mathrm{tr}(\bar{\varepsilon}^I\Psi^A[\bar{\Psi}_A\Psi^B X^C - X^C\bar{\Psi}^B\Psi_A]),
\end{aligned}
$$

where we have used equation (C.1). Here we have used the definition

$$\delta^{DEF}_{ABC} = 6\delta^{[D}_A\delta^E_B\delta^{F]}_C.$$

These two sets of terms combine to leave

$$-i\mathrm{tr}(\bar{\Psi}^A\Psi_A\delta X_B X^B) + i\mathrm{tr}(\bar{\Psi}_A\Psi^A X^B\delta X_B) + 2i\mathrm{tr}(\bar{\Psi}^B\Psi_A\delta X_B X^A) - 2i\mathrm{tr}(\bar{\Psi}_A\Psi^B X^A\delta X_B).$$

These terms are canceled in turn by varying X_B in $L_{4\mathrm{b}}$ and $L_{4\mathrm{c}}$. Thus, terms of this structure in the supersymmetry transformations cancel for the choice of L_4 given in section 4.2. The adjoint terms cancel in the same way.

Since we now have the complete dependence of the action on spinor fields, we can deduce the spinor field equations of motion. They are

$$
\begin{aligned}
\gamma\cdot D\Psi^A = -2\varepsilon^{ABCD}X_B\Psi_C X_D - X_B X^B\Psi^A + \Psi^A X^B X_B \\
- 2\Psi^B X^A X_B + 2X_B X^A\Psi^B,
\end{aligned}
\tag{C.2}
$$

and its adjoint

$$
\begin{aligned}
\gamma\cdot D\Psi_A = 2\varepsilon_{ABCD}X^B\Psi^C X^D + X^B X_B\Psi_A - \Psi_A X_B X^B \\
+ 2\Psi_B X_A X^B - 2X^B X_A\Psi_B.
\end{aligned}
\tag{C.3}
$$

Determination of $\delta_3\Psi$

Having determined L_4, we are now in a position to determine $\delta_3\Psi$ by computing terms of the schematic structure $\mathrm{tr}(\Psi_A DX_B X^C X_D)$, $\mathrm{tr}(\Psi_A X_B DX^C X_D)$, and $\mathrm{tr}(\Psi_A X_B X^C DX_D)$ that arise from varying the gauge fields in the X kinetic term and varying the spinor fields in L_4. The adjoint terms work the same way. The terms of the indicated structure that arise from varying the gauge fields in the X kinetic term are

$$i\tilde{\Gamma}^{IBC}\mathrm{tr}\big[\bar{\Psi}_B\gamma^\mu\varepsilon^I(X_CX^AD_\mu X_A - D_\mu X_AX^AX_C + X_AD_\mu X^AX_C - X_CD_\mu X^AX_A)\big].$$

The terms of the indicated structure that arise from varying $L_{4\mathrm{a}}$ are

$$\begin{aligned}
-2i\varepsilon^{ABCD}\mathrm{tr}(\delta\bar{\Psi}_DX_A\Psi_BX_C) &= -2i\varepsilon^{ABCD}\Gamma^I_{DE}\mathrm{tr}(\bar{\Psi}_B\gamma^\mu\varepsilon^IX_CD_\mu X^EX_A)\\
&= i\delta^{ABC}_{EFG}\tilde{\Gamma}^{IFG}\mathrm{tr}(\bar{\Psi}_B\gamma^\mu\varepsilon^IX_CD_\mu X^EX_A)\\
&= 2i\tilde{\Gamma}^{IBC}\mathrm{tr}\left(\begin{array}{c}\bar{\Psi}_B\gamma^\mu\varepsilon^IX_CD_\mu X^AX_A\\ +\bar{\Psi}_C\gamma^\mu\varepsilon^IX_AD_\mu X^AX_B + \bar{\Psi}_A\gamma^\mu\varepsilon^IX_BD_\mu X^AX_C\end{array}\right).
\end{aligned}$$

The terms of the indicated structure that arise from varying $L_{4\mathrm{b}}$ are

$$\begin{aligned}
&i\mathrm{tr}(\delta\bar{\Psi}^B\Psi_BX_AX^A) - i\mathrm{tr}(\bar{\Psi}_B\delta\Psi^BX^AX_A)\\
&\qquad = i\tilde{\Gamma}^{IBC}\mathrm{tr}\big[\bar{\Psi}_B\gamma^\mu\varepsilon^I(D_\mu X_CX^AX_A - X_AX^AD_\mu X_C)\big].
\end{aligned} \tag{C.4}$$

The terms of the indicated structure that arise from varying $L_{4\mathrm{c}}$ are

$$\begin{aligned}
&2i\mathrm{tr}(\bar{\Psi}_A\delta\Psi^BX^AX_B) - 2i\mathrm{tr}(\delta\bar{\Psi}^B\Psi_AX_BX^A)\\
&\qquad = 2i\tilde{\Gamma}^{IBC}\mathrm{tr}\big[\bar{\Psi}_A\gamma^\mu\varepsilon^I(X_BX^AD_\mu X_C + D_\mu X_BX^AX_C)\big].
\end{aligned} \tag{C.5}$$

Adding these up, we obtain

$$\begin{aligned}
&2i\tilde{\Gamma}^{IBC}\mathrm{tr}\big[\bar{\Psi}_A\gamma^\mu\varepsilon^ID_\mu(X_BX^AX_C)\big]\\
&\qquad + i\tilde{\Gamma}^{IBC}\mathrm{tr}\Big[\bar{\Psi}_B\gamma^\mu\varepsilon^I\big(D_\mu(X_CX^AX_A) - D_\mu(X_AX^AX_C)\big)\Big].
\end{aligned} \tag{C.6}$$

Thus, this can cancel against a variation of the spinor field in the spinor kinetic term for the choice

$$\delta_3\Psi^A = \tilde{\Gamma}^{IAB}\varepsilon^I(X_CX^CX_B - X_BX^CX_C) - 2\tilde{\Gamma}^{IBC}\varepsilon^IX_BX^AX_C. \tag{C.7}$$

Determination of $\mathbf{V = -L_6}$

The next step is to determine L_6 by requiring that its δX variation cancels against the $\delta_3\Psi$ variation of L_4. A key identity in the analysis is

$$\Gamma^I_{AB}\tilde{\Gamma}^{ICD} = -2\delta^{CD}_{AB}. \tag{C.8}$$

This is verified by showing that the two sides agree when contracted with δ^B_C as well as with $(\tilde{\Gamma}^J\Gamma^K - \tilde{\Gamma}^K\Gamma^J)^B{}_C$. Since these are 16 linearly independent 4×4 matrices, this constitutes a complete proof.

The supersymmetry variation of L_4, keeping all terms containing Ψ^A but not Ψ_A (since the Ψ_A terms work in the same way) is

$$\begin{aligned}\delta L_4 = &-2i\epsilon_{ABCD}\mathrm{tr}\Big(\delta_3\bar{\Psi}^A X^B\Psi^C X^D\Big)\\ &+ i\mathrm{tr}\Big(\delta_3\bar{\Psi}_A\big(X_BX^B\Psi^A - \Psi^AX^BX_B + 2\Psi^BX^AX_B - 2X_BX^A\Psi^B\big)\Big),\end{aligned} \tag{C.9}$$

where, as derived previously,

$$\delta_3\bar{\Psi}^A = \Gamma^I_{HK}\left[\frac{1}{2}\epsilon^{ACHK}\left(X_DX^DX_C - X_CX^DX_D\right) - \epsilon^{FGHK}X_FX^AX_G\right]\bar{\epsilon}^I, \tag{C.10}$$

$$\delta_3\bar{\Psi}_A = \left[-\Gamma^I_{AC}\left(X^CX_DX^D - X^DX_DX^C\right) + 2\Gamma^I_{HK}X^KX_AX^H\right]\bar{\epsilon}^I. \tag{C.11}$$

Expanding δL_4 is straightforward algebra and gives

$$\begin{aligned}\mathrm{tr}\,\big(&3X^A\delta X_AX^BX_BX^CX_C + 3\delta X_AX^AX_BX^BX_CX^C - 2X^A\delta X_BX^BX_AX^CX_C\\ &- 2X^AX_BX^B\delta X_AX^CX_C - 2X^AX_BX^BX_AX^C\delta X_C + 4i\Gamma^I_{HK}\bar{\epsilon}^I\Psi^A\big[X^HX_AX^BX_BX^K\\ &+ X^BX_BX^HX_AX^K + X^HX_BX^KX_AX^B - X^HX_BX^BX_AX^K - X^BX_AX^HX_BX^K\\ &-X^HX_AX^KX_BX^B\big] + 2i\epsilon_{ABCD}\epsilon^{FGHK}\Gamma^I_{HK}\bar{\epsilon}^I\Psi^AX^BX_FX^CX_GX^D\big).\end{aligned} \tag{C.12}$$

The first two lines can be reproduced by varying

$$V_1 = \mathrm{tr}\Big(X^AX_AX^BX_BX^CX_C + X_AX^AX_BX^BX_CX^C - 2X^AX_BX^BX_AX^CX_C\Big). \tag{C.13}$$

The last line cancels the third and fourth lines and contributes additional terms to V_1, as we will now show. For this purpose, the following identity is useful:

$$\begin{aligned}2\epsilon_{ABCD}\epsilon^{FGHK}\Gamma^I_{HK} &= \epsilon_{LBCD}\epsilon^{FGHK}\Gamma^J_{HK}\left(2\delta^{IJ}\delta^L_A\right)\\ &= \epsilon_{LBCD}\epsilon^{FGHK}\Gamma^J_{HK}\Big(\Gamma^I_{AM}\tilde{\Gamma}^{JML} + \Gamma^J_{AM}\tilde{\Gamma}^{IML}\Big)\\ &= 4\delta^{FGM}_{BCD}\Gamma^I_{AM} + 2\left(\delta^{GPQ}_{BCD}\delta^F_A - \delta^{FPQ}_{BCD}\delta^G_A\right)\Gamma^I_{PQ},\end{aligned}$$

where we have used (C.8) to go from the second line to the third line. Plugging this identity into the last line of (C.12) gives

$$\begin{aligned}\mathrm{tr}\Big(&-4\delta^{FGM}_{BCD}\delta X_MX^BX_FX^CX_GX^D\\ &+ 2i\Gamma^I_{HK}\bar{\epsilon}^I\Psi^A\left(\delta^{GHK}_{BCD}\delta^F_A - \delta^{FHK}_{BCD}\delta^G_A\right)X^BX_FX^CX_GX^D\Big).\end{aligned} \tag{C.14}$$

Expanding the first term in (C.14) gives

$$4\mathrm{tr}\left[-X^D\delta X_D X^F X_F X^G X_G - \delta X_B X^B X_C X^C X_D X^D - \delta X_C X^G X_D X^C X_G X^D \right.$$
$$\left. +\delta X_C X^F X_F X^C X_D X^D + \delta X_B X^B X_D X^G X_G X^D + \delta X_D X^G X_C X^C X_G X^D\right],$$

which also comes from varying

$$V_2 = \mathrm{tr}\Big(- \frac{4}{3} X^A X_A X^B X_B X^C X_C - \frac{4}{3} X_A X^A X_B X^B X_C X^C$$
$$- \frac{4}{3} X_A X^B X_C X^A X_B X^C + 4 X^A X_B X^B X_A X^C X_C\Big).$$

Adding this potential to equation (C.13) gives the total potential

$$V = -\frac{1}{3}\mathrm{tr}\Big[X^A X_A X^B X_B X^C X_C + X_A X^A X_B X^B X_C X^C$$
$$+ 4 X_A X^B X_C X^A X_B X^C - 6 X^A X_B X^B X_A X^C X_C\Big].$$

Furthermore, straightforward algebra shows that the second term in equation (C.14) precisely cancels the terms in the third and fourth lines of equation (C.12). So we conclude that the variation of L_4 is completely canceled by varying $-V$. This expression agrees with the potential obtained in [14, 50].

It is also interesting to note that V is proportional to the trace of the absolute square of the X^3 expression that appears in $\delta_3\Psi$. Specifically,

$$V = \frac{1}{6}\mathrm{tr}(N^{IA} N^I_A),$$

which is straightforward to verify using equation (C.8).

Conserved Supersymmetry Current

The conserved supersymmetry current of the $U(N)\times U(N)$ theory, generalizing the expression given earlier for the $U(1)\times U(1)$ theory, is

$$Q^I_\mu = \mathrm{tr}\Big(M^I_A \gamma_\mu \Psi^A\Big) + \mathrm{tr}\Big(M^{IA}\gamma_\mu \Psi_A\Big).$$

Here

$$M^I_A = -\Gamma^I_{AB}\gamma\cdot DX^B + N^I_A,$$

and

$$M^{IA} = \tilde{\Gamma}^{IAB}\gamma\cdot DX_B + N^{IA},$$

are quantities that appear in the supersymmetry variations of the spinor fields $\bar{\Psi}_A$ and $\bar{\Psi}^A$, respectively. The quantity N^I_A and its adjoint N^{IA} were defined in equations (4.9) and (4.10). The verification that this current is conserved as a consequence of the equations of motion is rather tedious. In any case, it would be redundant, since it is equivalent to the verification of the supersymmetry of the action, which we have just carried out.

Conformal Supersymmetry

In the $U(1)\times U(1)$ case, we found that the conformal supersymmetries can be described by replacing ε^I in the Poincaré supersymmetries by $\gamma \cdot x\, \eta^I$ and by adding an additional term to the spinor field transformations

$$\delta' \Psi_A = \Gamma^I_{AB} X^B \eta^I ,$$

and its adjoint. Let us now verify that the same rule continues to work for $N > 1$. Most terms cancel as a consequence of the Poincaré supersymmetry. The remaining ones that need to cancel separately are those that arise from the derivative in $i\bar{\Psi}_A \gamma \cdot D\delta\Psi^A$ acting on the explicit x^μ in the η^I transformation. This gives

$$i\bar{\Psi}_A \Big[\tilde{\Gamma}^{IAB} (\gamma \cdot DX_B + 3X_C X^C X_B - 3X_B X^C X_C) - 6\tilde{\Gamma}^{IBC} X_B X^A X_C \Big] \eta^I .$$

The first term in this expression is canceled by the $\delta'\Psi^A$ variation of the spinor kinetic term. The remaining terms need to cancel against the $\delta'\Psi$ variation of L_4. The relevant terms that arise in this way are

$$2i\varepsilon^{ABCD} \mathrm{tr}(\delta' \bar{\Psi}_A X_B \Psi_C X_D) + i\mathrm{tr}(\delta' \bar{\Psi}^A \Psi_A X_B X^B) - i\mathrm{tr}(\bar{\Psi}_A \delta' \Psi^A X^B X_B) \\ + 2i\mathrm{tr}(\bar{\Psi}_A \delta' \Psi^B X^A X_B) - 2i\mathrm{tr}(\delta' \bar{\Psi}^B \Psi_A X_B X^A). \quad \text{(C.15)}$$

By manipulations similar to those described previously, the first term in this expression can be recast in the form

$$2i\tilde{\Gamma}^{IBC} \mathrm{tr}(\bar{\Psi}_A X_B X^A X_C + \bar{\Psi}_B X_C X^A X_A + \bar{\Psi}_C X_A X^A X_B)\eta^I .$$

Combining this with the other four terms leaves

$$i\bar{\Psi}_A \Big[\tilde{\Gamma}^{IAB} (-3X_C X^C X_B + 3X_B X^C X_C) + 6\tilde{\Gamma}^{IBC} X_B X^A X_C \Big] \eta^I .$$

This provides the desired cancellation, which proves that the theory has conformal supersymmetry.

Taken together with the $\mathcal{N} = 6$ Poincaré supersymmetry, the conformal supersymmetry implies that the theory has the full $OSp(6|4)$ superconformal symmetry. Even though this result is necessary for a dual AdS interpretation, it was not at all obvious that this symmetry would hold. After all, it is not a logical consequence of the other symmetries that have been verified.

Accordingly, the conserved conformal supersymmetry currents in the $U(N) \times U(N)$ theory are given by

$$S^I_\mu = \gamma \cdot x\, Q^I_\mu - \Gamma^I_{AB} \mathrm{tr}\Big(X^B \gamma_\mu \Psi^A\Big) + \tilde{\Gamma}^{IAB} \mathrm{tr}\Big(X_B \gamma_\mu \Psi_A\Big).$$

As a check on our analysis, let us compute the divergence. The DX^B terms cancel leaving

$$\partial^\mu S^I_\mu = \mathrm{tr}\Big(3N^I_A \Psi^A + 3N^{IA}\Psi_A - \Gamma^I_{AB} X^B \gamma \cdot D\Psi^A + \tilde{\Gamma}^{IAB} X_B \gamma \cdot D\Psi_A\Big),$$

where N^I_A and N^{IA} are as before. Using the spinor field equations of motion (C.2) and (C.3) to eliminate $\gamma \cdot D\Psi^A$ and $\gamma \cdot D\Psi_A$, the terms in $\partial^\mu S^I_\mu$ that involve Ψ^A are

$$\begin{aligned} &3\mathrm{tr}\Big(N^I_A \Psi^A\Big) + 2\varepsilon_{ACDE} \tilde{\Gamma}^{IAB} \mathrm{tr}\Big(X_B X^C \Psi^D X^E\Big) \\ &\qquad - \Gamma^I_{AB} \mathrm{tr}\Big(X^B [-X_C X^C \Psi^A + \Psi^A X^C X_C - 2\Psi^C X^A X_C + 2X_C X^A \Psi^C]\Big). \qquad \text{(C.16)} \end{aligned}$$

A short calculation, similar to previous ones, shows that this vanishes.

Appendix D

Review of ABJM

The ABJM theory is a three-dimensional superconformal Chern-Simons gauge theory with $\mathcal{N}=6$ supersymmetry. The bosonic field content consists of four complex scalars Z^1,Z^2, Z^3, Z^4 and their adjoints $Z_1^\dagger$,$Z_2^\dagger$, $Z_3^\dagger$, $Z_4^\dagger$ (which transform in the $(\bar{N},N)$ and $(N,\bar{N})$ representations of the gauge group $U(N)\times U(N)$) as well as two $U(N)$ gauge fields A_μ and $\hat{A}_\mu$. The kinetic and Chern-Simons terms for these fields are

$$\mathcal{L}_{kin} = -\frac{k}{2\pi}\mathrm{tr}\left(D_\mu Z^I D^\mu Z_I^\dagger\right),$$

$$\mathcal{L}_{CS} = \frac{k}{2\pi}\epsilon^{\mu\nu\lambda}\mathrm{tr}\left(\frac{1}{2}A_\mu\partial_\nu A_\lambda + \frac{i}{3}A_\mu A_\nu A_\lambda - \frac{1}{2}\hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right),$$

where $D_\mu Z^I = \partial_\mu Z^I + i\left(A_\mu Z^I - Z^I\hat{A}_\mu\right)$ and k is called the level. For the complete action see [108, 50]. The scalars have mass dimension $1/2$ and transform in the fundamental representation of the R-symmetry group $SU(4)$. Their adjoints transform in the antifundamental representation of $SU(4)$. The theory has a large-N expansion with 't Hooft parameter $\lambda = N/k$. For $k=1,2$, the theory is conjectured to have $\mathcal{N}=8$ supersymmetry. For $k \ll N \ll k^5$, the theory is conjectured to be dual to type IIA string theory on $AdS_4 \times CP^3$.

For operators of the form

$$\mathcal{O} = W^{i_1 \dots i_J}_{k_1 \dots k_J}\,\mathrm{tr}\left(Z^{k_1}Z^\dagger_{i_1}\dots Z^{k_J}Z^\dagger_{i_J}\right),$$

the two-loop dilatation operator is given by

$$\mathbf{\Delta} - J = \frac{\lambda^2}{2}\sum_{i=1}^{2J}\left(2 - 2P_{i,i+2} + P_{i,i+2}T_{i,i+1} + T_{i,i+1}P_{i,i+2}\right), \tag{D.1}$$

where $\lambda = N/k$, P is the permutation operator, and T is the trace operator [55]. Note that the indices are periodic, i.e., $2J+1 \sim 1$ and $2J+2 \sim 2$.

Appendix E

$\mathbf{AdS_4 \times CP^3}$ Geometry

We use $M, N = (\mathbf{0}, \mathbf{1}, \ldots, \mathbf{9})$ to label base-space indices and $A, B = (0, 1, \ldots, 9)$ to label tangent-space indices. We assign the first four indices to AdS_4 and the last six indices to CP^3. In this appendix, we take the AdS_4 and CP^3 spaces to have unit radii. A radius R can be readily incorporated by $ds^2 \to R^2 ds^2$ and $e_M{}^A \to R e_M{}^A$.

E.1 AdS$_4$

The metric for an AdS_4 space with unit radius in global coordinates (t, ρ, θ, ϕ) is given by

$$ds^2_{AdS_4} = -\cosh^2 \rho \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2 \rho \left(\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\phi^2\right),$$

where $-\infty < t < \infty$, $0 \le \rho < \infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$.

The embedding coordinates are defined by

$$n_1^2 + n_2^2 - n_3^2 - n_4^2 - n_5^2 = 1, \tag{E.1}$$

and they are related to the global coordinates by

$$\begin{aligned} n_1 &= \cosh \rho \cos t, \\ n_2 &= \cosh \rho \sin t, \\ n_3 &= \sinh \rho \cos \theta \sin \phi, \\ n_4 &= \sinh \rho \sin \theta \sin \phi, \\ n_5 &= \sinh \rho \cos \phi. \end{aligned} \tag{E.2}$$

Because the global coordinates are not well defined at $\rho = 0$, it is useful to define Cartesian coordinates $(t,\eta_1,\eta_2,\eta_3)=(\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3})$ for which the metric is given by

$$ds^2_{AdS_4}=g^{AdS_4}_{MN}dX^M dX^N=\frac{1}{(1-\eta^2)^2}\left[-\left(1+\eta^2\right)^2 dt^2+4d\vec{\eta}\cdot d\vec{\eta}\right]. \tag{E.3}$$

Note that this metric is only valid for $\eta^2=\vec{\eta}\cdot\vec{\eta}=\eta_1^2+\eta_2^2+\eta_3^2<1$. These coordinates are related to the global coordinates by $\cosh\rho=(1+\eta^2)/(1-\eta^2)$.

The vielbein (defined by $g^{AdS_4}_{MN}=e_M{}^A e_N{}^B\eta_{AB}$ where $\eta_{AB}=diag\,(-1,1,1,1)$) is given by

$$e_M A=\begin{pmatrix} \frac{(1+\eta^2)}{(1-\eta^2)} & 0 & 0 & 0 \\ 0 & \frac{2}{(1-\eta^2)} & 0 & 0 \\ 0 & 0 & \frac{2}{(1-\eta^2)} & 0 \\ 0 & 0 & 0 & \frac{2}{(1-\eta^2)} \end{pmatrix},$$

where $M=(\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3})$ labels the rows and $A=(0,1,2,3)$ labels the columns.

The nonzero components of the spin connection $\left(\omega_M{}^{AB}=-\omega_M{}^{BA}\right)$ are

$$\begin{aligned}
\omega_{\mathbf{0}}{}^{01} &= 2\eta_1/\left(1-\eta^2\right), \quad \omega_{\mathbf{0}}{}^{02}=2\eta_2/\left(1-\eta^2\right), \quad \omega_{\mathbf{0}}{}^{03}=-2\eta_3/\left(1-\eta^2\right),\\
\omega_{\mathbf{1}}{}^{12} &= 2\eta_2/\left(1-\eta^2\right), \quad \omega_{\mathbf{1}}{}^{13}=2\eta_3/\left(1-\eta^2\right),\\
\omega_{\mathbf{2}}{}^{21} &= 2\eta_1/\left(1-\eta^2\right), \quad \omega_{\mathbf{2}}{}^{23}=2\eta_3/\left(1-\eta^2\right),\\
\omega_{\mathbf{3}}{}^{31} &= 2\eta_1/\left(1-\eta^2\right), \quad \omega_{\mathbf{3}}{}^{32}=2\eta_2/\left(1-\eta^2\right).
\end{aligned}$$

E.2 CP³

The metric for a unit radius CP^3 space in global coordinates $(\psi,\xi,\varphi_1,\theta_1,\varphi_2,\theta_2)=(\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8},\mathbf{9})$ is given by

$$\begin{aligned}
ds^2_{CP^3} &= g^{CP^3}_{MN}dX^M dX^N=\mathrm{d}\xi^2+\cos^2\xi\sin^2\xi\left(\mathrm{d}\psi+\frac{1}{2}\cos\theta_1\mathrm{d}\varphi_1-\frac{1}{2}\cos\theta_2\mathrm{d}\varphi_2\right)^2\\
&\quad+\frac{1}{4}\cos^2\xi\left(\mathrm{d}\theta_1^2+\sin^2\theta_1\mathrm{d}\varphi_1^2\right)+\frac{1}{4}\sin^2\xi\left(\mathrm{d}\theta_2^2+\sin^2\theta_2\mathrm{d}\varphi_2^2\right),
\end{aligned} \tag{E.4}$$

where $0\le\xi<\pi/2$, $0\le\psi<2\pi$, $0\le\theta_i\le\pi$, and $0\le\varphi_i<2\pi$ [109, 52, 85]. The CP^3 Kähler form is given by **J**$=dA$ where

$$A=\frac{1}{2}\left(\cos\theta_1\cos^2\xi d\phi_1+\cos\theta_2\sin^2\xi d\phi_2+\cos 2\xi d\psi\right). \tag{E.5}$$

The embedding or homogeneous coordinates ($z^I \in \mathbb{C}$) are defined by

$$\sum_{I=1}^{4} \left|z^I\right|^2 = 1, \qquad z^I \sim e^{i\lambda} z^I, \tag{E.6}$$

where $\lambda \in \mathbb{R}$. The embedding coordinates are related to the global coordinates by

$$\begin{aligned}
z_1 &= \cos\xi \cos\tfrac{\theta_1}{2} \exp\left(i\tfrac{\psi+\varphi_1}{2}\right), \\
z_2 &= \cos\xi \sin\tfrac{\theta_1}{2} \exp\left(i\tfrac{\psi-\varphi_1}{2}\right), \\
z_3 &= \sin\xi \cos\tfrac{\theta_2}{2} \exp\left(i\tfrac{-\psi+\varphi_2}{2}\right), \\
z_4 &= \sin\xi \sin\tfrac{\theta_2}{2} \exp\left(i\tfrac{-\psi-\varphi_2}{2}\right).
\end{aligned} \tag{E.7}$$

Note that the metric in equation (E.4) can be written in terms of embedding coordinates as follows:

$$ds^2_{CP^3} = dz \cdot dz^\dagger - \left(z^\dagger \cdot dz\right)\left(z \cdot dz^\dagger\right),$$

where $z \cdot z^\dagger = \sum_{I=1}^{4} z^I z_I^\dagger$.

The vielbein (defined by $g_{MN}^{CP^3} = e_M{}^A e_N{}^B \delta_{AB}$) is

$$e_M{}^A = \begin{pmatrix}
\cos\xi\sin\xi & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\cos\xi\sin\xi\cos\theta_1/2 & 0 & \cos\xi\sin\theta_1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos\xi/2 & 0 & 0 \\
-\cos\xi\sin\xi\cos\theta_2/2 & 0 & 0 & 0 & \sin\xi\sin\theta_2/2 & 0 \\
0 & 0 & 0 & 0 & 0 & \sin\xi/2
\end{pmatrix},$$

where $M = (\mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9})$ labels the rows and $A = (4, 5, 6, 7, 8, 9)$ labels the columns.

The nonzero components of the spin connection $\left(\omega_M{}^{AB} = -\omega_M{}^{BA}\right)$ are

$$\begin{aligned}
&\omega_{\mathbf{4}}{}^{45} = \cos(2\xi), && \omega_{\mathbf{4}}{}^{76} = \sin^2\xi, && \omega_{\mathbf{4}}{}^{89} = \cos^2\xi, \\
&\omega_{\mathbf{6}}{}^{45} = \cos\theta_1 \cos(2\xi)/2, && \omega_{\mathbf{6}}{}^{74} = \omega_{\mathbf{6}}{}^{56} = \sin\theta_1 \sin\xi/2, && \\
&\omega_{\mathbf{6}}{}^{67} = -\cos\theta_1(\sin^2\xi - 2)/2, && \omega_{\mathbf{6}}{}^{89} = \cos\theta_1 \cos^2\xi/2, && \\
&\omega_{\mathbf{7}}{}^{46} = \omega_{\mathbf{7}}{}^{57} = \sin\xi/2, && && \\
&\omega_{\mathbf{8}}{}^{54} = \cos\theta_2 \cos(2\xi)/2, && \omega_{\mathbf{8}}{}^{49} = \omega_{\mathbf{8}}{}^{85} = \sin\theta_2 \cos\xi/2, && \\
&\omega_{\mathbf{8}}{}^{67} = \cos\theta_2 \sin^2\xi/2, && \omega_{\mathbf{8}}{}^{98} = \cos\theta_2(\cos^2\xi - 2)/2, && \\
&\omega_{\mathbf{9}}{}^{84} = \omega_{\mathbf{9}}{}^{95} = \cos\xi/2. && &&
\end{aligned}$$

E.3 Fluxes

Using the vielbein, we can convert between base-space and tangent-space coordinates. In particular, by writing the four-form field strength in equation (5.1c) in tangent-space coordinates, one finds that

$$F_{ABCD} = \frac{6k}{R^2}\epsilon_{ABCD}, \tag{E.8}$$

where $\epsilon_{0123} = 1$ and all other non-zero components are related by antisymmetry. Furthermore, if one takes the exterior derivative of equation (E.5), plugs this into equation (5.1d), and converts to tangent-space coordinates, one finds that

$$F_{AB} = \frac{2k}{R^2}\epsilon_{AB}, \tag{E.9}$$

where $\epsilon_{45} = \epsilon_{67} = \epsilon_{89} = 1$ and all other non-zero components are related by antisymmetry. Equations (5.1b), (E.8), and (E.9) then imply that

$$\begin{aligned} e^{\phi}\Gamma \cdot F_2 &= \frac{2}{R}\left(\Gamma^{45} + \Gamma^{67} + \Gamma^{89}\right), \\ e^{\phi}\Gamma \cdot F_4 &= \frac{6}{R}\Gamma^{0123}. \end{aligned}$$

Plugging these expressions into equation (5.12) then gives

$$\Gamma \cdot F = \frac{1}{4R}\left[-\Gamma_{11}\left(\Gamma^{45} + \Gamma^{67} + \Gamma^{89}\right) + 3\Gamma^{0123}\right]. \tag{E.10}$$

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