I. NOTE ON THE EQUATIONS OF TRIPLE AND QUADRUPLE CORRELATION FUNCTIONS IN ISOTROPIC TURBULENCE

II. ON THE TURBULENT MIXING OF TWO FLUIDS OF DIFFERENT DENSITIES

III. ON THE POSSIBILITY OF KEEPING THE ELECTRONS INSIDE THE DIMENSION OF NUCLEUS AND THE QUANTUM MECHANICAL THEORY OF NEUTRON

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Ning Hu

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CONTENTS

Part I

Note on the Equations of Triple and Quadruple Correlation Functions in Isotropic Turbulence.

I. Introduction and summary.

Recently Kármán and Howarth (1938) have successfully developed a theory of isotropic turbulence by investigating the velocity correlation functions at two points in the field of flow. From the equations of motion they established a partial differential equation connecting the double and triple correlation functions, $f(r,t)$ and $h(r,t)$, which can be solved by using some additional physical assumptions. As suggested by the authors, the indeterminateness of two unknown functions from one equation may be due to the fact that this equation has not exhausted the information obtainable from the equations of motion; so it will be worthwhile to investigate whether this indeterminateness can be reduced by establishing the dynamical equations for the correlation functions of higher orders. This investigation has now been carried out in the present note and it is found that when we come to consider the quadruple correlation functions, the number of kinematically independent unknown functions is increased by ten, while only three dynamical equations are obtained. Consequently the consideration of quadruple correlation functions gives no aid to the solution of the problem. It seems that this situation will continue when we push our consideration to correlation functions of still higher orders. This would mean that the correlation functions cannot be determined from their equations without introducing some additional assumptions.

II. The kinematics of the quadruple correlation functions.

The notation used in the present note is the same as that used in Kármán and Howarths' paper, so no detailed explanation of the notation will be given now. Although Robertson (1940) has recently given a more mathmatically compact presentation of the same theory, the procedure in the present note still follows that of Kármán and Howarth for the sake of physical clearness.

We have two types of quadruple correlation tensors, namely:

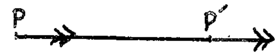
$$(1) \quad (\overline{u^2})^2 Q_{ijkl} = \overline{u_i u_j u'_k u'_l},$$

$$(i, j, k, l = 1, 2, 3)$$

$$(2) \quad (\overline{u^2})^2 P_{ijkl} = \overline{u_i u_j u_k u'_l},$$

where u_i and u'_i are respectively the velocity components at the point P and P'. Consider a particular coordinate system whose x_1- axis is along $\overrightarrow{PP'}$. From the condition of isotropy, all properties of these correlation functions must be symmetrical about $\overrightarrow{PP'}$, so the components of the above tensors obtained by interchanging the indices 2 and 3 must be the same functions of the relative coordinates of the points P and P', $r = x'_1 - x_1$. Furthermore, since the condition of isotropy requires that the above tensors must remain unchanged by a reflection of coordinate axis, the components of which the indices 2 and 3 occurs an odd number of times must vanish. Consequently the only non-vanishing components of the above two tensors are

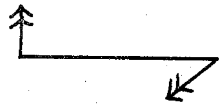
$$Q_1 = Q_{1111},$$



$Q_2 = Q_{2222} = Q_{3333}$,

$Q_3 = Q_{2211} = Q_{3311} = Q_{1122} = Q_{1133}$,

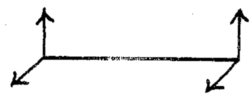
$Q_4 = Q_{2233} = Q_{3322}$,



$Q_5 = Q_{1212} = Q_{1313} = Q_{2121} = Q_{3131} = Q_{1221} = Q_{1331}$

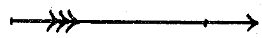
$= Q_{2112} = Q_{3113}$,

$Q_6 = Q_{2323} = Q_{3232} = Q_{2332} = Q_{3223}$;



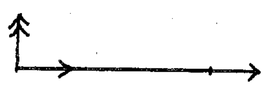
and

$P_1 = P_{1111}$,



$P_2 = P_{2222} = P_{3333}$,

$P_3 = P_{2211} = P_{3311} = P_{2121} = P_{3131} = P_{1221} = P_{1331}$,



$P_4 = P_{2112} = P_{3113} = P_{1212} = P_{1313} = P_{1122} = P_{1133}$,

$$P_5 = P_{2233} = P_{3322} = P_{2323} = P_{3232} = P_{3223} = P_{2332} .$$

It can easily be seen from the form of the above functions that they must all be even functions of the variable $r = x_1' - x_1$, as the indices 1 also occurs an even number of times in each one of the above functions so that the reflection of x_1-axis cannot change their signs.

In the case of double and triple correlation tensors, the condition of isotropy does not impose any relation among their non-vanishing components. It will be shown that this is not so for the quadruple correlation tensors. For, rotating the coordinate system about the x_1-axis through an arbitrary angle θ, we have

$$Q_1^* = Q_1 ,$$

$$Q_2^* = Q_2(\cos^4\theta + \sin^4\theta) + 2Q_4\cos^2\theta\sin^2\theta + 4Q_6\cos^2\theta\sin^2\theta ,$$

$$Q_3^* = Q_3(\cos^2\theta + \sin^2\theta) = Q_3 ,$$

$$Q_4^* = Q_4(\sin^4\theta + \cos^4\theta) + (2Q_2 - 4Q_6)\cos^2\theta\sin^2\theta ,$$

$$Q_5^* = Q_5(\sin^2\theta + \cos^2\theta) = Q_5 ,$$

$$Q_6^* = 2(Q_2 - Q_4)\sin^2\theta\cos^2\theta + 2Q_6(\cos^4\theta + \sin^4\theta) ;$$

and

$$P_1^* = P_1 ,$$

$$P_2^* = P_2(\cos^4\theta + \sin^4\theta) + 6P_5\cos^2\theta\sin^2\theta ,$$

$$(6) \qquad P_3^* = P_3(\cos^2\theta + \sin^2\theta) = P_3 ,$$

$$P_4^* = P_4(\cos^2\theta + \sin^2\theta) = P_4 ,$$

$$P_5^* = (P_2 - 2P_5)\cos^2\theta\sin^2\theta + P_5(\cos^4\theta + \sin^4\theta) ,$$

where the stars denote the quantities in the new coordinate system. Now, due to the condition of isotropy, the components of tensors (1) and (2) must remain unchanged ~~same~~ under this rotation of coordinates, so we must have $Q_i^* = Q_i$ and $P_i^* = P_i$. This can be true only if

$$(7) \qquad Q_6 = \frac{1}{2}(Q_2 - Q_4) ,$$

$$(8) \qquad P_5 = \frac{1}{3}P_2 .$$

Consequently only five of Q's and four of P's are independent.

We shall now write down the transformation equations for the quadruple correlation tensors Q_{ijkl} and P_{ijkl} under general rotation of coordinates. Let $a^s_{\ i}$ be the cosines of the angle between the new coordinates x_i and the coordinates x_s^* used in the foregoing. By the relation (7) and (8), using the relation $\sum_{i=1}^{3} a^s_{\ i}a^t_{\ i} = \delta_{st}$ and combining terms, we can reduce the transformation equation into the form containing $a^1_{\ 1}$, $a^1_{\ 2}$, and $a^1_{\ 3}$ only. Substituting $\xi_i = a^1_{\ i}/r$, we have (dropping out the stars)

$$(9) \qquad Q_{ijk\ell} = (Q_1 + Q_4 + 2Q_6 - 2Q_3 - 4Q_5)\frac{1}{r^4}\cdot \xi_i\xi_j\xi_k\xi_\ell$$

$$+ (Q_3 - Q_4)\frac{1}{r^2}(\xi_i\xi_j\delta_{k\ell} + \xi_k\xi_\ell\delta_{ij})$$

$$+ (Q_5 - Q_6)\frac{1}{r^2}(\xi_i\xi_k\delta_{j\ell} + \xi_i\xi_\ell\delta_{kj} + \xi_j\xi_k\delta_{i\ell} + \xi_j\xi_\ell\delta_{ik})$$

$$+ Q_4\delta_{ij}\delta_{k\ell} + Q_6(\delta_{i\ell}\delta_{kj} + \delta_{j\ell}\delta_{ik}),$$

and

$$(10) \qquad P_{ijk\ell} = (P_1 + P_2 - 3P_3 - 3P_4)\frac{1}{r^4}\xi_i\xi_j\xi_k\xi_\ell$$

$$+ (P_4 - \frac{1}{3}P_2)\frac{1}{r^2}(\xi_i\xi_k\delta_{j\ell} + \xi_j\xi_k\delta_{i\ell} + \xi_i\xi_j\delta_{k\ell})$$

$$+ (P_3 - \frac{1}{3}P_2)\frac{1}{r^2}(\xi_k\xi_\ell\delta_{ij} + \xi_j\xi_\ell\delta_{ik} + \xi_i\xi_\ell\delta_{jk})$$

$$+ P_2(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell}).$$

Now we shall proceed to investigate what further kinematic relations are provided by the equation of continuity. For double and triple correlation tensors, this relation is the vanishing of their divergences. This same relation can be obtained for P_{ijkl}, as

$$(11) \qquad (\overline{u^2})^2\frac{\partial P_{ijkl}}{\partial \xi_\ell} = (\overline{u^2})^2\frac{\partial}{\partial x'_\ell}P_{ijk\ell} = \overline{u_i u_j u_k \frac{\partial}{\partial x'_\ell}u'_\ell}$$

But no such relation can be obtained for $Q_{ijk\ell}$ since

$$\frac{\partial}{\partial x_j}\,\overline{u_i u_j u'_k u'_\ell} \;=\; -\frac{\partial}{\partial x'_k}\,\overline{u_i u_j u'_k u'_\ell} \;\neq\; 0.$$

At first sight it may be thought that the relation imposed by the equation of continuity on the components of $Q_{ijk\ell}$ might be in a form other than the vanishing of the divergence. But physically we can see no reason why this relation should exist, as in constructing $Q_{ijk\ell}$ the quantities considered at both points represent the transport of momentum, which should have no direct connection with the equation of continuity.

Applying (11) to (10), we have

$$(12)\qquad \frac{\partial}{\partial \xi_\ell} P_{ijk\ell} = \left\{ \frac{1}{r^3}\,(P'_1 - 3P'_3) + \frac{2}{r^4}\,(P_1 + 2P_2 - 3P_3 - 6P_4)\right\}\xi_i\xi_j\xi_k$$

$$+ \left\{ \frac{2}{r^2}(P_4 - \frac{1}{3}P_2) + \frac{1}{r}\,P'_3 + \frac{2}{r^2}\,(P_3 - \frac{1}{3}P_2)\right\}\cdot$$

$$(\delta_{ij}\xi_k + \delta_{ik}\xi_j + \delta_{jk}\xi_i)$$

$$= 0.$$

Since ξ_i is an arbitrary vector, the coefficients of $\xi_i\xi_j\xi_k$ and ξ_i must vanish separately. Hence we have

$$\frac{1}{r^3}\,(P'_1 - 3P'_3) - \frac{2}{r^4}\,(P_1 + 2P_2 - 3P_3 - 6P_4) = 0,$$

$$\frac{2}{r^2}\,(P_4 - \frac{2}{3}P_2) + \frac{1}{r}\,P'_3 + \frac{2}{r^2}\,P_3 = 0\,;$$

or

$$(13)\qquad \begin{aligned} P_4 &= \frac{1}{3} P_1 + \frac{r}{6} P_1' , \\ P_2 &= \frac{1}{2} P_1 + \frac{r}{4} P_1' + \frac{3}{4} r P_3' + \frac{3}{2} P_3 \end{aligned}$$

Therefore the final form for $P_{ijk\ell}$ reads

$$(14)\qquad \begin{aligned} P_{ijk\ell} &= \frac{1}{4r^4} (2P_1 - rP_1' - 6P_3' + 3rP_3') \xi_i \xi_j \xi_k \xi_\ell \\ &+ \frac{1}{r^2} \left(\frac{1}{6} P_1 + \frac{r}{12} P_1' - \frac{1}{2} P_3 - \frac{r}{4} P_3'\right) (\delta_{\ell i} \xi_j \xi_k + \delta_{\ell j} \xi_i \xi_k + \delta_{\ell k} \xi_i \xi_j) \\ &+ \frac{1}{r^2} \left(-\frac{1}{6} P_1 - \frac{r}{12} P_1' + \frac{1}{2} P_3 - \frac{r}{4} P_3'\right) (\delta_{ij} \xi_k \xi_\ell + \delta_{jk} \xi_i \xi_\ell + \delta_{ik} \xi_j \xi_\ell) \\ &+ \left(\frac{1}{6} P_1 + \frac{r}{12} P_1' + \frac{1}{2} P_3 + \frac{r}{4} P_3'\right) (\delta_{ij} \delta_{k\ell} + \delta_{ik} \delta_{j\ell} + \delta_{jk} \delta_{i\ell}) . \end{aligned}$$

III. Dynamical equations connecting the triple and quadruple correlation functions.

Let u_i, p and u'_i, p' be the velocities and pressures at the two points $P(x_1, x_2, x_3)$ and $P'(x'_1, x'_2, x'_3)$ respectively. The equation of motion at the point P can be written

$$(15) \qquad \frac{\partial}{\partial t} u_i + u_j \frac{\partial}{\partial x_j} u_i = -\frac{1}{\rho} \frac{\partial}{\partial x_i} p + \nu \nabla^2 u_i .$$

Let us multiply the above equation by u_k and u_ℓ' so that

$$(16) \qquad u'_\ell u_k \frac{\partial}{\partial t} u_i + u'_\ell u_k u_j \frac{\partial}{\partial x_j} u_i = -\frac{1}{\rho} u'_\ell u'_k \frac{\partial}{\partial x_i} p + \nu u'_\ell u_k \nabla^2 u_i ,$$

Interchanging the indices i and k, we obtain

$$(17) \qquad u'_\ell u_i \frac{\partial}{\partial t} u_k + u'_\ell u_i u_j \frac{\partial}{\partial x_j} u_k = -\frac{1}{\rho} u'_\ell u_i \frac{\partial}{\partial x_k} p + \nu u'_\ell u_i \nabla^2 u_k .$$

Similarly multiplying the equation of motion at the point P' by $u_i u_k$, we have

$$(18) \qquad u_i u_k \frac{\partial}{\partial t} u'_\ell + u_i u_k u'_j \frac{\partial}{\partial x'_j} u'_\ell = -\frac{1}{\rho} u_i u_k \frac{\partial}{\partial x'_\ell} p' + \nu u_i u_k \nabla^2 u'_\ell .$$

Add (16),(17) and (18) and note that

$$(19) \qquad u'_\ell u_k u_j \frac{\partial}{\partial x_j} u_i + u'_\ell u_i u_j \frac{\partial}{\partial x_j} u_k = \frac{\partial}{\partial x_j} (u_i u_j u_k u'_\ell)$$

$$= -\frac{\partial}{\partial \xi_j} (u_i u_j u_k u'_\ell) ,$$

$$(20)\qquad u_i u_k u'_j \frac{\partial}{\partial x'_j} u'_\ell = \frac{\partial}{\partial x'_j}(u_i u_k u'_j u'_\ell) = \frac{\partial}{\partial \xi_j}(u_i u_k u'_j u'_\ell),$$

then we have

$$(21)\qquad \frac{\partial}{\partial t}(u_i u_k u'_\ell) + \frac{\partial}{\partial \xi_j}(u_i u_k u'_j u'_\ell) - \frac{\partial}{\partial \xi_j}(u_i u_j u_k u'_\ell)$$

$$= -\frac{1}{\rho}\left(u'_\ell u_k \frac{\partial p}{\partial x_i} + u'_\ell u_i \frac{\partial p}{\partial x_k} + u_i u_k \frac{\partial p'}{\partial x'_\ell}\right) + \nu \nabla^2 u_i u_k u'_\ell .$$

Averaging over time and introducing

$$(\overline{u^2})^{3/2}\, T_{ik\ell} = \overline{u_i u_k u'_\ell}\,,$$

$$(\overline{u^2})^{2}\, Q_{ikj\ell} = \overline{u_i u_k u'_j u'_\ell}\,,$$

$$(\overline{u^2})^{2}\, P_{ikj\ell} = \overline{u_i u_k u_j u'_\ell}\,,$$

and

$$(\overline{u^2})^{2}\, S_{ik\ell} = \overline{u_k u'_\ell \partial p/\partial x_i} + \overline{u_i u'_\ell \partial p/\partial x_k} + \overline{u_i u_k \partial p'/\partial x'_\ell}\,,$$

we have

$$(22)\qquad \frac{\partial}{\partial t}\left\{(\overline{u^2})^{3/2}\, T_{ik\ell}\right\} + (\overline{u^2})^{2}\frac{\partial}{\partial \xi_j}(Q_{ikj\ell} - P_{ikj\ell})$$

$$= -\frac{1}{\rho}(\overline{u^2})^{2}\, S_{ik\ell} + \nu(\overline{u^2})^{3/2}\nabla^2 T_{ik\ell} .$$

The triple correlation tensor $T_{ik\ell}$ has already been investigated thoroughly in Kármán and Howarths' paper. It was found to be giesn by the following expression:

$$(23)\quad \cancel{(u^2)}^{3/2}\, T_{ikl} = \left(-\frac{h}{r^3} + \frac{h'}{r^2}\right)\xi_i\xi_k\xi_l + \frac{h}{r}\delta_{ik}\xi_l - \left(\frac{h}{r} + \frac{h'}{r}\right)(\delta_{kl}\xi_i + \delta_{li}\xi_k).$$

The tensor S_{ikl}, being isotropic and symmetrical with respect to the indices i and k, must be of the following form (cf. Robertson's paper)

$$(2.4)\quad S_{ikl} = \frac{1}{r^3}S_1\cdot\xi_i\xi_k\xi_l + \frac{1}{r}S_2\,\delta_{ik}\xi_l + \frac{1}{r}S_3\,(\delta_{il}\xi_k + \delta_{kl}\xi_i).$$

If we substitute (23),(24),(9) and (14) into the equation (22), we shall obtain some relations connecting the functions h, Q_i, P_i and S_i. It will be convenient to evaluate $\nabla^2 T_{ikl}$, $\partial Q_{ikjl}/\partial\xi_j$ and $\partial P_{ikjl}/\partial\xi_j$ first. Since the analysis is very involved, we shall write down their resulting expressions only:

$$(25)\quad \nabla^2 T_{ikl} = \left(\frac{h'''}{r^2} + \frac{3h''}{r^3} - \frac{12h'}{r^4} + \frac{12h}{r^5}\right)\xi_i\xi_k\xi_l$$

$$+\left(\frac{h''}{r} + \frac{4h'}{r^2} - \frac{4h}{r^3}\right)\delta_{ik}\xi_l$$

$$+\left(-\frac{h'''}{2} - \frac{3h''}{r}\right)(\delta_{il}\xi_k + \delta_{kl}\xi_i),$$

$$(26)\quad \frac{\partial}{\partial\xi_j}Q_{ikjl} = \left\{\frac{1}{r^3}(Q_1' - Q_3' - 2Q_5') + \frac{1}{r^4}(2Q_1 + 4Q_4 + 8Q_6 - 6Q_3 - 12Q_5)\right\}\xi_i\xi_k\xi_l$$

$$+\left\{\frac{1}{r}Q_3' + \frac{1}{r^2}(2Q_3 - 2Q_4 + 2Q_5 - 2Q_6)\right\}\delta_{ik}\xi_l$$

$$+\left\{\frac{1}{r}Q_5' + \frac{1}{r^2}(3Q_5 - 3Q_6 + Q_3 - Q_4)\right\}(\delta_{il}\xi_k + \delta_{kl}\xi_i),$$

$$(27)\quad \frac{\partial}{\partial \xi_j} P_{ikjl} = \left\{ \frac{1}{r^3}\left(-\frac{1}{3}P_1' - \frac{1}{3}rP_1'' + 2P_3'\right) + \frac{1}{r^4}\left(\frac{4}{3}P_1 - 4P_3\right) \right\} \xi_i \xi_k \xi_l$$

$$+ \left\{ \frac{1}{r^2}\left(\frac{1}{3}P_1 - P_3\right) + \frac{1}{r}\left(\frac{2}{3}P_1' - P_3'\right) + \frac{1}{6}P_1'' \right\} (\delta_{il}\xi_k + \delta_{kl}\xi_i)$$

$$+ \frac{4}{r^2}\left(-\frac{1}{6}P_1 - \frac{r}{12}P_1' + \frac{1}{2}P_3\right)\delta_{ik}\xi_l \, .$$

where the primes denote the differentiation with respect to r.

Substituting (25), (26) and (27) into (22), we obtain an equation containing the terms multiplied by $\xi_i\xi_k\xi_l$, $\delta_{ik}\xi_l$ and $\delta_{il}\xi_k + \delta_{kl}\xi_i$ respectively. Since the resulting equation must ~~satisfy~~ be satisfied by an arbitrary vector ξ_i, the coefficients of the above three factors must be zero separately. Then we have

$$(28)\quad \frac{\partial}{\partial t}\left\{(\overline{u^2})^{3/2} h\right\} + r\frac{\partial}{\partial t}\left\{(\overline{u^2})^{3/2}\frac{\partial h}{\partial r}\right\} + (\overline{u^2})^2\left\{ \frac{2}{r}(Q_1 + 2Q_4 + 4Q_6 - 6Q_5\right.$$

$$\left. - 3Q_3 - \frac{2}{3}P_1 + 2P_3) + \frac{\partial}{\partial r}(Q_1 - Q_3 - 2Q_5) + \frac{r}{3}\frac{\partial^2}{\partial r^2}P_1 \right\}$$

$$= -\frac{1}{\rho}(\overline{u^2})^2 S_1 + \nu(\overline{u^2})^{3/2}\left(r\frac{\partial^3}{\partial r^3}h + 3\frac{\partial^2}{\partial r^2}h - \frac{12}{r}\frac{\partial}{\partial r}h + \frac{12}{r^2}h \right),$$

$$(29)\quad \frac{\partial}{\partial t}\left\{(\overline{u^2})^{3/2} h\right\} + (\overline{u^2})^2\left\{ \frac{2}{r}\left(Q_3 + Q_4 + Q_5 - Q_6 + \frac{1}{3}P_1 - P_3\right)\right.$$

$$\left. + \frac{\partial}{\partial r}\left(Q_3 + \frac{1}{3}P_1\right) \right\}$$

$$= -\frac{1}{\rho}(\overline{u^2})^2 S_2 + \nu(\overline{u^2})^{3/2}\left(\frac{\partial^2}{\partial r^2}h + \frac{4}{r}\frac{\partial}{\partial r}h - \frac{4}{r}h \right),$$

$$(30)\qquad \frac{\partial}{\partial t}\left\{(\overline{u^2})^{3/2} h\right\} + \frac{r}{2}\frac{\partial}{\partial t}\left\{(\overline{u^2})^{3/2}\frac{\partial h}{\partial r}\right\} - (\overline{u^2})^2\left\{\frac{1}{r}(3Q_5 - 3Q_6 + Q_3 - Q_4 - \frac{1}{3}P_1 + P_3) + \frac{\partial}{\partial r}(Q_5 - \frac{2}{3}P_1 + P_3) - \frac{r}{6}\frac{\partial^2}{\partial r^2}P_1\right\}$$

$$= + \frac{1}{\rho}(\overline{u^2})^2 S_3 + \nu(\overline{u^2})^{3/2}\left(\frac{1}{2}\frac{\partial^3}{\partial r^3}h + \frac{3}{r}\frac{\partial^2}{\partial r^2}h\right).$$

It is seen that the consideration of quadruple correlation functions increases the number of unknown functions by ten, namely, Q_1, Q_3, Q_4, Q_5, Q_6, P_1, P_3, S_1, S_2 and S_3, while only three equations are obtained. Hence the consideration of the quadruple correlation functions not only gives no help to the solution of Kármán and Howarths' equation, but also requires additional physical assumptions in itself in order to solve the Q's and P's even after the double and triple correlation functions have been known.

In conclusion, the author wishes to express his hearty thanks to Professor Theodore von Kármán for suggesting the problem, and for the invaluable help.

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Part II

On The Turbulent Mixing of Two Fluids of Different Densities

I. Introduction and summary.

Tollmien's calculation on the turbulent jets and half jets provides an excellent check of Prandtl's momentum transport theory with the experiment. In the present paper we extend Tollmien's theory to the case in which the mixing fluids are of different densities. Both the change in the distribution in mean velocity and the change of the angular divergence of the jets are investigated. Quite a number of experimental observations of turbulent jets of this kind have been published in literature on account of their importance in connection with the atomization of a spray of fuel in the internal combustion engines. It is found that the predicted density distribution across the jet agrees with the experiment very well.

In the following treatment we shall confine ourselves to the immiscible fluids from which an emulsion is formed by turbulent diffusion. This corresponds to the actual case of a spray of liquid fuel in air.

II. The general theory.

Let us consider two immiscible and incompressible fluids 1 and 2 of densities ρ_1 and ρ_2 respectively, and let v_1 and v_2 be the partial volumes of these two fluids per unit volume at any point x_i $(i = 1, 2, 3)$ so that the actual density ρ at this point is

$$(1) \qquad \rho = v_1 \rho_1 + v_2 \rho_2$$

with

$$(2) \qquad v_1 + v_2 = 1 .$$

From the condition of immiscibility, the change of v_1 and v_2 can only be due to the convection of the fluid:

$$(3) \qquad \frac{\partial}{\partial t} v_1 + \sum_j \frac{\partial}{\partial x_j} (u_j v_1) = 0 ,$$

$$(4) \qquad \frac{\partial}{\partial t} v_2 + \sum_j \frac{\partial}{\partial x_j} (u_j v_2) = 0 ,$$

where u_i is the velocity at the point x_i , $j = 1, 2, 3$. Adding (3) and (4) and using (2), we obtain

$$(5) \qquad \sum_j \frac{\partial}{\partial x_j} u_j = 0 .$$

Using (1), we also have

$$(6) \qquad \frac{\partial}{\partial t} \rho + \sum_j \frac{\partial}{\partial x_j} (u_j \rho) = 0.$$

(5) shows that the mixture of two incompressible fluids is also incompressible, and (6) shows that if we follow the moving element of the fluid, we should find that the density of this element is unchanged. The first result is the direct consequence of the condition of incompressibility, while the second is the direct consequence of immiscibility. Since the validity of (5) and (6) is very obvious, we may use them equally well as our starting point and consider (1)-(4) as the conclusions.

The equation of motion for the mixture can be written into the following form

$$(7) \qquad \frac{\partial}{\partial t}(\rho u_i) + \sum_j \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i$$

Following the same procedure as followed by Reynolds, we write

$$(8) \qquad u_i = U_i + u'_i , \quad \rho = \rho_o + \rho' , \quad p = p_o + p' ,$$

where U_i, ρ_o and p_o are the average values of u_i, ρ and p respectively over time, and u'_i, ρ' and p' may be called the fluctuation of velocity, density and pressure respectively. Thus averaging (5) (6) and (7) over time, we obtain respectively

$$(9) \qquad \frac{\partial}{\partial t}(\rho_o U_i) + \frac{\partial}{\partial t}(\overline{\rho' u'_i}) + \sum_j \frac{\partial}{\partial x_j}\Big(\rho_o U_i U_j + \rho_o \overline{u'_i u'_j} + U_i \overline{\rho' u'_j} + U_j \overline{\rho' u'_i} + \overline{\rho' u'_i u'_j}\Big) = -\frac{\partial}{\partial x_i} p_o , \quad + \nu \nabla^2 U_i$$

? ! neglected in turbulent flow

$$(10) \qquad \sum_j \frac{\partial}{\partial x_j} U_j = 0 ,$$

$$(11)\qquad \frac{\partial \rho_0}{\partial t} + \sum_j \frac{\partial}{\partial x_j}\left(\rho_0 U_j + \overline{\rho' u'_j} \right) = 0 .$$

where the bars denote as usual the averages over time. These are the fundamental equations of the present problem.

When the mean flow is two dimensional and steady, write $U_1 = U$, $U_2 = V$, $U_3 = 0$, $u'_1 = u'$, $u'_2 = v'$, $u'_3 = w'$, $x_1 = x$, $x_2 = y$ and $x_3 = z$, we have

$$(12)\qquad \frac{\partial}{\partial x}\left(\rho_0 U^2 + \rho_0 \overline{u'^2} + 2U\overline{\rho' u'} + \overline{\rho' u'^2} \right)$$

$$+ \frac{\partial}{\partial y}\left(\rho_0 UV + \rho_0 \overline{u'v'} + U\overline{\rho' v'} + V\overline{\rho' u'} + \overline{\rho' u'v'} \right) = - \frac{\partial}{\partial x} p_o ,$$

$$(13)\qquad \frac{\partial}{\partial x}\left(\rho_0 UV + \rho_0 \overline{u'v'} + V\overline{\rho' u'} + U\overline{\rho' v'} + \overline{\rho' u'v'} \right) +$$

$$+ \frac{\partial}{\partial y}\left(\rho_0 V^2 + \rho_0 \overline{v'^2} + 2V\overline{\rho' v'} + \overline{\rho' v'^2} \right) = - \frac{\partial}{\partial y} p_o ,$$

$$(14)\qquad \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 ,$$

$$(15)\qquad \frac{\partial}{\partial x}\left(\rho_0 U + \overline{\rho' u'} \right) + \frac{\partial}{\partial y}\left(\rho_0 V + \overline{\rho' v'} \right) = 0.$$

(12) and (13) are the first and second components of (9), the third component of (9) being identically zero on both sides.

In Tollmien's solution the viscosity terms and $\partial p_o/\partial x$ have been neglected. The terms $\rho_0 \overline{u'^2}$ and $\rho_0 \overline{v'^2}$ are also neglected against the term $\rho_0 \overline{u'v'}$, since according to the mechanism of momentum transport theory only $\rho_0 \overline{u'v'}$ is responsible for the transfer of momentum.

Following the same reasoning, we may neglect $\overline{\rho' u'}$ against $\overline{\rho' v'}$ also. The other terms $\overline{\rho' u'^2}$, $\overline{\rho' v'^2}$, $\overline{\rho' u' v'}$ can also be neglected if we consider the case in which the fluctuations in velocity and density are small compared with the mean velocity and density of the flow. By momentum transport theory we may write

$$(16) \qquad \rho_0 \overline{u'v'} = - k_1 \rho_0 \ell^2 \left|\frac{\partial U}{\partial y}\right| \frac{\partial U}{\partial y}$$

$$(17) \qquad \overline{\rho' v'} = - k_2 \ell^2 \left|\frac{\partial U}{\partial y}\right| \frac{\partial \rho_0}{\partial y}$$

Equation (13) may be reserved for the determination of p_0; the other three equations (12), (14) and (15) then become

$$(18) \qquad \frac{\partial}{\partial x}(\rho_0 U^2) + \frac{\partial}{\partial y}\left(\rho_0 UV - k_1 \rho_0 \ell^2 \left|\frac{\partial U}{\partial y}\right| \frac{\partial U}{\partial y} - k_2 \ell^2 \left|\frac{\partial U}{\partial y}\right| \frac{\partial \rho_0}{\partial y} U\right) = 0,$$

$$(19) \qquad \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0,$$

$$(20) \qquad \frac{\partial}{\partial x}(\rho_0 U) + \frac{\partial}{\partial y}\left(\rho_0 V - k_2 \ell^2 \left|\frac{\partial U}{\partial y}\right| \frac{\partial \rho_0}{\partial y}\right) = 0.$$

In the following sections we shall apply the general theory to a half jet and a full jet in succession.

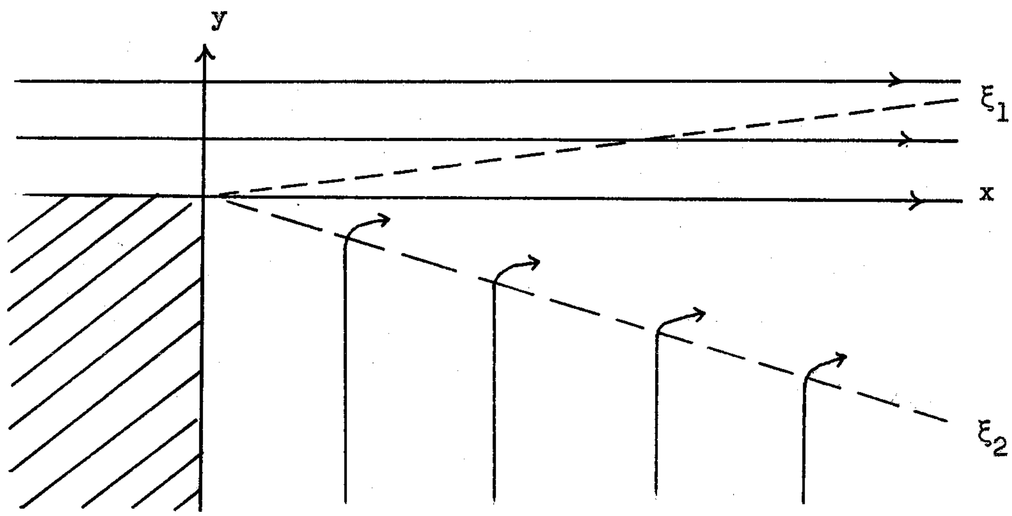


Fig. 1.

III. The half jet.

Fig. 1 shows the device for producing the half jet considered in Tollmien's paper. Let the fluid originally at rest to the right of the obstacle be of density ρ_2, and the moving fluid above the obstacle be of density ρ_1. Let U_o be the undisturbed velocity of the fluid 1 (density ρ_1) above the jet. Assuming the boundaries of the jet be straight lines and using (19), as in Tollmien's solution, we may put

$$\frac{U}{U_o} = F'(\eta), \quad \frac{V}{U_o} = -F(\eta) + \eta F'(\eta), \quad \rho_o = \rho_o(\eta), \tag{21}$$
$$\ell = cx, \qquad \eta = \frac{y}{x}.$$

Then (18) and (20) become respectively

$$(22) \quad -\eta\frac{d}{d\eta}(\rho_o F'^2) + \frac{d}{d\eta}\left\{\rho_o F'(-F+\eta F') - k_1 c^2 \rho_o F''^2 - k_2 c^2 \rho_o' F' F''\right\} = 0,$$

$$(23) \quad -\eta\frac{d}{d\eta}(\rho_o F') + \frac{d}{d\eta}\left\{\rho_o(-F+\eta F') - k_2 c^2 \rho_o' F''\right\} = 0.$$

Carrying out the differentiation, we have

$$(24) \quad -\frac{d}{d\eta}(\rho_o F')F = k_1 c^2 \frac{d}{d\eta}(\rho_o F''^2) + k_2 c^2 \frac{d}{d\eta}(\rho_o' F' F''),$$

$$(25) \quad -\rho_o' F = k_2 c^2 \frac{d}{d\eta}(\rho_o' F'').$$

We can now show that if we follow the momentum transport theory strictly, we should have $k_1 = k_2$. For according to that theory, we ~~we~~ must have in the first step

$$-\overline{u'v'} = \overline{v'\ell}\,\frac{dU}{dy},$$

$$-\overline{\rho'v'} = \overline{v'\ell}\,\frac{d\rho_0}{dy}.$$

By Prandtl's assumption

$$\overline{v'\ell} = k\ell^2\frac{dU}{dy}$$

so we have $k = k_1 = k_2$. It will be noted that this strict interpretation of momentum transport theory has resulted a serious difficulty. For, following the above reasoning, we conclude that the correlation of the temperature fluctuation T' with v' must also be given by the formula

$$\overline{T'v'} = -\overline{v'\ell}\,\frac{dT}{dy} = -k\ell^2\left|\frac{dU}{dy}\right|\frac{dT}{dy},$$

where T is the mean temperature. But this would require that the distribution of mean temperature and mean velocity must be the same, while it has been shown experimentally by Fage and Falkner (1932) that they are quite different from each other in turbulent wakes. However a recent measurement by Thiele (1942) shows that for the ← No! axial symmetrical jet the velocity and temperature distributions are nearly the same (No measurement of the temperature distribution

---

An earlier experiment by Ruden shows, however, a lesser agreement between these two distributions (Die Naturwissenschaft 21(1933) 375).

for two dimensional jet seems to have been made). This seems to show that the momentum transport theory may account for the phenomena more satisfactotily in jets than in wakes. Again, unless the condition $k_1 = k_2$ is used, the solution of (24) and (25) is very diffecult. Therefore we shall be content with the solution under the condition $k_1 = k_2 = k$.

The equations(24) and (25) then become

$$(26) \qquad -\frac{d}{d\eta}(\rho_0 F')F = kc^2 \frac{d}{d\eta}\left\{\frac{d}{d\eta}(\rho_0 F')F''\right\}$$

$$(27) \qquad -\rho_0' F = kc^2 \frac{d}{d\eta}(\rho_0' F'').$$

If we put

$$(28) \qquad \varphi = \frac{d}{d\eta}(\rho_0 F')F''$$

and

$$(29) \qquad \psi = \rho_0' F'',$$

(26) and (27) become respectively

$$(30) \qquad -\varphi\frac{F}{F''} = kc^2 \varphi' ,$$

$$(31) \qquad -\psi\frac{F}{F''} = kc^2 \psi' .$$

It is seen that the above two equations have exactly the same form. Dividing (30) by (31), we have

$$\frac{\varphi'}{\varphi} = \frac{\psi'}{\psi}$$

i.e., $\varphi = A\,\psi$

where A is the constant of integration. Substituting (28) and (29), we have

$$\rho_0' F'' = A \frac{d}{d\eta}(\rho_0 F') F'' .$$

Since $F'' \neq 0$ every where inside the jet ($F'' \approx \partial U/\partial y$), we may cancel out the common factor F'' on both sides of the above equation. Integrate the resulting equation once, then we have

$$\rho_0 = \frac{B}{1 - AF'} , \tag{32}$$

where B is the integration constant. A and B can be determined immediately from the following boundary conditions

$$\begin{aligned} \rho_0 &= \rho_1 , \quad F' = 1 \quad \text{at} \quad \eta = \eta_1 \\ \rho_0 &= \rho_2 , \quad F' = 0 \quad \text{at} \quad \eta = \eta_2 \end{aligned} \tag{33}$$

which give $B = \rho_2$ and $A = (\rho_1 - \rho_2)/\rho_1$. Then (32) becomes

$$\rho_0 = \frac{\rho_2}{1 - \frac{\rho_1 - \rho_2}{\rho_1} F'} = \frac{\rho_2}{1 - \frac{\rho_1 - \rho_2}{\rho_1}\frac{U}{U_0}}$$

or

$$\frac{\rho_0 - \rho_2}{\rho_1 - \rho_2} = \frac{\rho_2}{\rho_1}\frac{U}{U_0}\,\frac{1}{1 - \frac{\rho_1 - \rho_2}{\rho_1}\frac{U}{U_0}} .$$

We see that when $\rho_1 - \rho_2$ is small, the distribution of $\rho_0 - \rho_2$ is the same as that of U. Substituting $\rho_0 = \rho_2/(1 - AF')$ into (31), we have

$$(35) \qquad F + 2kc^2F''' = \frac{kc^2AF''}{1 - AF'}, \quad A = \frac{\rho_1 - \rho_2}{\rho_1}.$$

If we put $A = 0$, i.e., $\rho_1 = \rho_2$, the above equation reduces to

$$(36) \qquad F + 2kc^2F''' = 0$$

which is just Tollmien's equation for the case of the homogeneous fluid.

Since our original equations (18) and (20) are expected to hold only when ρ' and therefore $\rho_1 - \rho_2$ is small, we may solve the equation (35) by the method of perturbations. We can write (35) in the following form

$$F + 2kc^2F''' = kc^2AF''^2\{1 + AF' + A^2F'^2 + \ \ldots\ldots\}.$$

If all terms involving A^2 are neglected, then the above equation becomes

$$F + 2kc^2F''' = kc^2AF''^2.$$

This equation can be reduced to a more simple form by putting

$$(37) \qquad \eta = (2kc^2)^{1/3}\xi, \qquad F(\eta) = (2kc^2)^{1/3}G(\xi).$$

Then we have

$$(38) \qquad G + G''' = \frac{1}{2} A G''^2 = \bar{A} G''^2,$$

and Tollmien's equation (36) becomes

$$(39) \qquad G + G''' = 0.$$

Before proceeding further, let us write down the boundary conditions of the present problem, which are exactly the same as in Tollmien's case, namely

$$G(\xi_1) = \xi_1$$

$$G'(\xi_1) = 1, \quad \text{i.e.,} \quad U/U_o = 1 \quad \text{at } \xi = \xi_1,$$

$$G''(\xi_1) = 0, \quad " \quad \partial U/\partial y = 0 \text{ at } \xi = \xi_1,$$

$$G'(\xi_2) = 0, \quad " \quad U = 0 \text{ at } \xi = \xi_2,$$

$$G''(\xi_2) = 0, \quad " \quad \partial U/\partial y = 0 \text{ at } \xi = \xi_2,$$

where $\xi_1 = (2kc^2)^{-\frac{1}{3}} y_1/x$, $\xi_2 = (2kc^2)^{-\frac{1}{3}} y_2/x$, y_1 and y_2 being respectively the upper and lower boundaries of the jet. It is convenient to introduce a new variable $\bar{\xi} = \xi - \xi_1$ so that the above five boundary conditions become

$$(40) \qquad \begin{aligned} &G(0) = \xi_1, \; G'(0) = 1, \; G''(0) = 0, \\ &G'(\bar{\xi}_2) = 0, \quad G''(\bar{\xi}_2) = 0. \end{aligned}$$

Now we put

$$(41) \qquad G(\bar{\xi}) = G_o(\bar{\xi}) + \bar{A}\, G_1(\bar{\xi}),$$

$$(42) \qquad \xi_1 = \xi_{1o} + \bar{A}\, \xi_{11},$$

$$(43) \qquad \bar{\xi}_2 = \bar{\xi}_{2o} + \bar{A}\, \bar{\xi}_{21}.$$

The first equation expresses the perturbed form of $G(\bar{\xi})$, the latter two equations express the shift of the boundary of the jet when $\bar{A}$ changes. Substituting (41) into (38) and equating the terms not involving $\bar{A}$, and the coefficient of $\bar{A}$ in the resulting equation to zero separately, we obtain respectively

$$(44) \qquad G_o(\bar{\xi}) + G_o'''(\bar{\xi}) = 0,$$

$$(45) \qquad G_1(\bar{\xi}) + G_1(\bar{\xi}) = \bar{A}\;\; G_o''(\bar{\xi})^{\,2}.$$

We note that (44) is just the Tollmien's equation (39). So $G_o(\bar{\xi})$ is given by Tollmien's solution

$$(46) \qquad G_o(\bar{\xi}) \doteq d_1 e^{-\bar{\xi}} + d_2 e^{\bar{\xi}/2} \cos\frac{\sqrt{3}}{2}\bar{\xi} + d_3 e^{\bar{\xi}/2} \sin\frac{\sqrt{3}}{2}\bar{\xi}\,.$$

Substituting (46) into (45), and solving for $G_1(\bar{\xi})$, we have

$$(47) \qquad G_1(\bar{\xi}) = C_1\, e^{-\bar{\xi}} + C_2\, e^{\bar{\xi}/2} \cos\frac{\sqrt{3}}{2}\bar{\xi} + C_3\, e^{\bar{\xi}/2} \sin\frac{\sqrt{3}}{2}\bar{\xi} + f(\bar{\xi})$$

$$f(\bar{\xi}) = B_1 e^{-2\bar{\xi}} + B_2 e^{\bar{\xi}} + B_3 e^{\bar{\xi}} \cos\sqrt{3}\bar{\xi} + B_4 e^{-\bar{\xi}/2} \cos\frac{\sqrt{3}}{2}\bar{\xi}$$

$$+ B_5 e^{-\bar{\xi}/2} \sin\frac{\sqrt{3}}{2}\bar{\xi} + B_6 e^{\bar{\xi}} \sin\sqrt{3}\,\bar{\xi}\,,$$

where C_1, C_2 and C_3 are arbitrary constants and

$$
(48)\qquad
\begin{aligned}
B_1 &= -\frac{1}{7} d_1^{\,2},\\
B_2 &= \frac{1}{4}\ (-\frac{1}{2} d_2 + \frac{\sqrt{3}}{2} d_3)^2 + (-\frac{\sqrt{3}}{2} d_2 - \frac{1}{2} d_3)^2,\\
B_3 &= -\frac{1}{14}\ (-\frac{1}{2} d_2 + \frac{\sqrt{3}}{2} d_3)^2 - (-\frac{\sqrt{3}}{2} d_2 - \frac{1}{2} d_3)^2,\\
B_4 &= d_1\,(-\frac{1}{2} d_2 + \frac{\sqrt{3}}{2} d_3),\\
B_5 &= d_1\,(-\frac{\sqrt{3}}{2} d_2 - \frac{1}{2} d_3),\\
B_6 &= -\frac{1}{7}(-\frac{1}{2} d_2 + \frac{\sqrt{3}}{2} d_3)\,(-\frac{\sqrt{3}}{2} d_2 - \frac{1}{2} d_3).
\end{aligned}
$$

Next we substitute (41),(42) and (43) into the five boundary conditions (40). With the help of Taylor's expansion

$$\psi(\bar{\xi}_2) = \psi(\bar{\xi}_{2o} + \bar{A}\,\bar{\xi}_{21}) = \psi(\bar{\xi}_{2o}) + \bar{A}\,\bar{\xi}_{21}\,\psi'(\bar{\xi}_{2o}),$$

we get for the terms not involving $\bar{A}$:

$$
(49)\qquad
\begin{aligned}
&G_o(0) = \xi_{1o},\quad G_o'(0) = 1,\quad G_o''(0) = 0,\\
&G_o'(\bar{\xi}_{2o}) = 0,\quad G_o''(\bar{\xi}_{2o}) = 0,
\end{aligned}
$$

and for the coefficients of $\bar{A}$:

$$
(50)\qquad
\begin{aligned}
&G_1(0) = \xi_{11},\quad G_1'(0) = 0,\quad G_1''(0) = 0,\\
&\bar{\xi}_{21} G_o''(\bar{\xi}_{2o}) + G_1'(\bar{\xi}_{2o}) = 0,\\
&\bar{\xi}_{21} G_o'''(\bar{\xi}_{2o}) + G_1''(\bar{\xi}_{2o}) = 0.
\end{aligned}
$$

The first three equations of (50) can also be written in the following form by substituting (47) and (48):

$$C_1 + C_2 + B_1 + B_2 + B_3 + B_4 = \xi_{11},$$

$$(51) \quad -C_1 + \frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 - 2B_1 + B_2 + B_3 + \sqrt{3}B_6 - \frac{1}{2}B_4 + \frac{\sqrt{3}}{2}B_5 = 0,$$

$$C_1 - \frac{1}{2}C_2 + \frac{\sqrt{3}}{2}C_3 + 4B_1 + B_2 - 2B_3 + 2\sqrt{3}B_6 - \frac{1}{2}B_4 - \frac{\sqrt{3}}{2}B_5 = 0.$$

It is seen that the boundary conditions (49) are just those given in Tollmien's solution. Thus Tollmien's results

$$(52) \quad \begin{aligned} &d_1 = -0.0062, \quad d_2 = 0.987, \quad d_3 = 0.577, \\ &\bar{\xi}_{2o} = -3.02, \quad \xi_{1o} = 0.981, \quad \xi_{2o} = -2.04 \end{aligned}$$

can be used here. These values may be substituted into (48) and (51) to solve C_1, C_2 and C_3 as functions of ξ_{11}. With the help of the fifth equation of (49), the last two equations of (50) can be written

$$(53) \quad \begin{aligned} &G_1'(\bar{\xi}_{2o}) = G_1'(-3.02) = 0, \\ &\bar{\xi}_{21} = -G_1''(\bar{\xi}_{2o})/G_o'''(\bar{\xi}_{2o}) = -G_1''(-3.02)/G_o'''(-3.02). \end{aligned}$$

Since C_1, C_2 and C_3 have been determined as functions of ξ_{11}, we may use the first equation of (53) to determine ξ_{11}. $\bar{\xi}_{21}$ is given directly by the second equation of (53). Carrying out the calculation, we obtain finally

$$\xi_{11} = 0.139, \qquad \bar{\xi}_{21} = 0.212.$$

Our final solution can be written ~~inte~~ the following form

$$(54) \quad \frac{U}{U_o} = G'(\bar{\xi}) = G_o'(\bar{\xi}) + \bar{A}\, G_1'(\bar{\xi}),$$

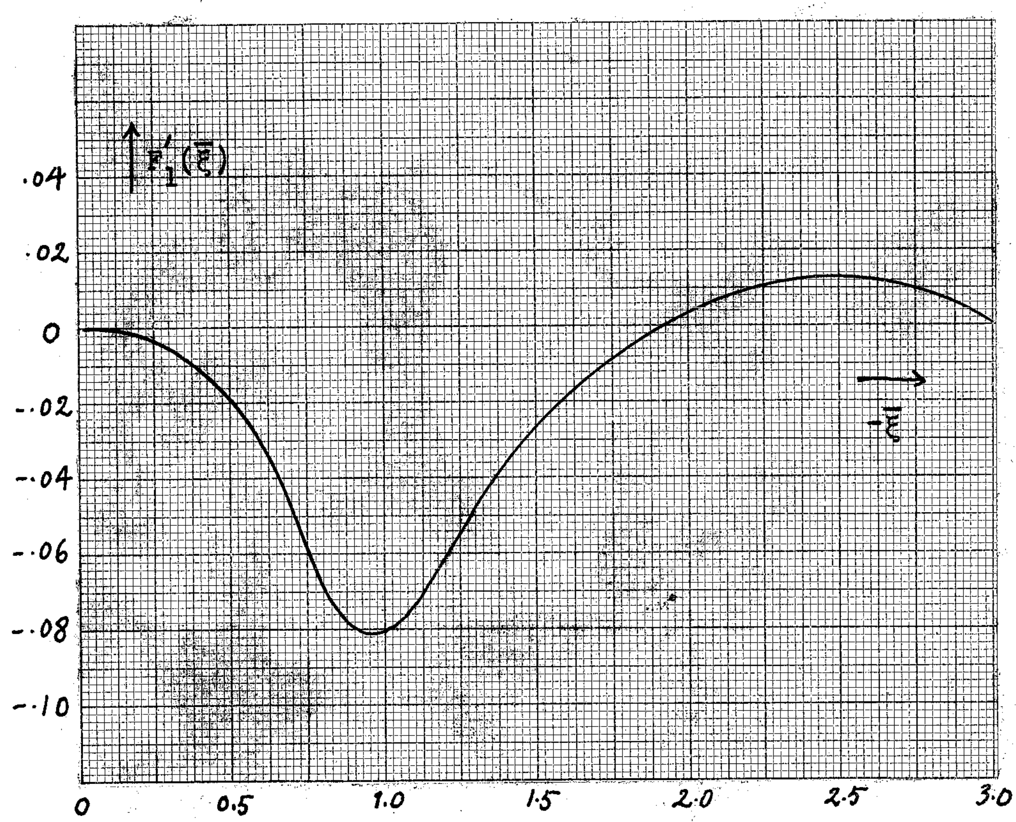
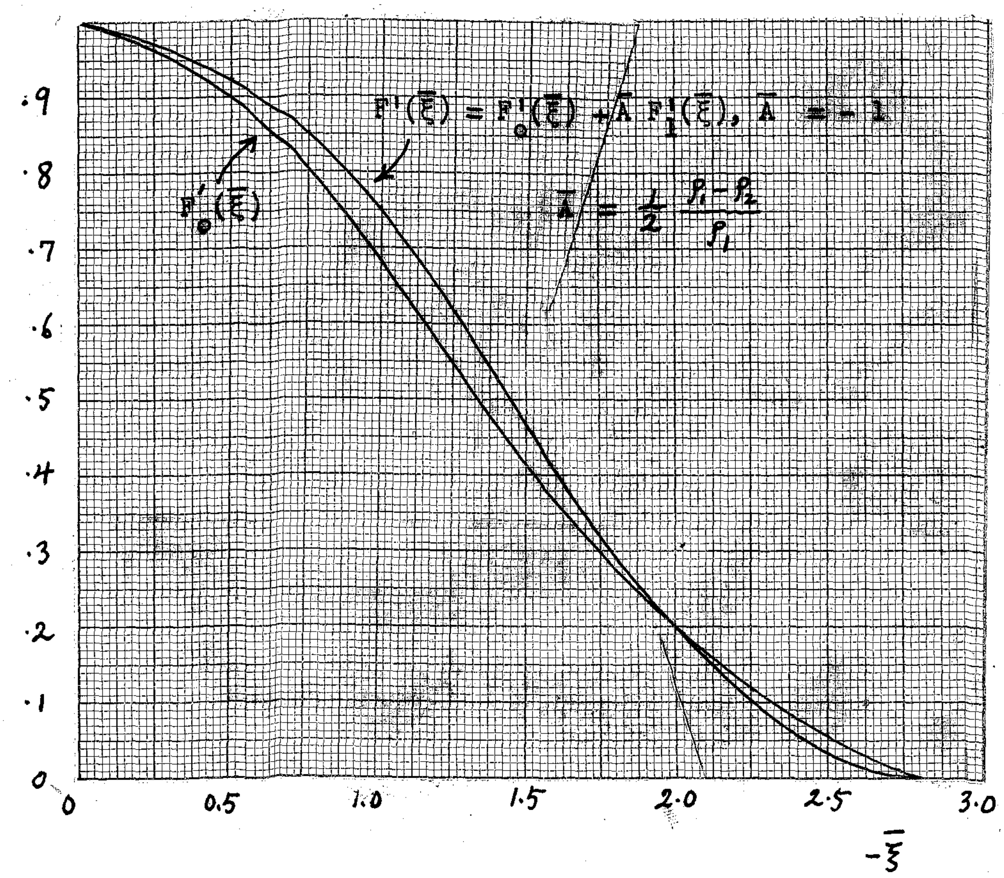


Fig.2.



$F'(\bar{\xi}) = F'_0(\bar{\xi}) + \bar{A}\,F'_1(\bar{\xi}),\ \bar{A} = -1$
$\bar{A} = \frac{1}{2}\,\frac{\rho_1 - \rho_2}{\rho_1}$
$F'_0(\bar{\xi})$
·9
·8
·7
·6
·5
·4
·3
·2
·1
0
0
0.5
1.0
1.5
2.0
2.5
3.0
$-\bar{\xi}$

Fig. 3.

$$G_0'(\bar{\xi}) = 0.0062e^{-\bar{\xi}} + 0.0062e^{\bar{\xi}/2}\cos\sqrt{\tfrac{3}{2}}\,\bar{\xi} - 1.143e^{\bar{\xi}/2}\sin\sqrt{\tfrac{3}{2}}\,\bar{\xi}$$

$$\begin{aligned} G_1'(\bar{\xi}) = &- 0.0031e^{-\bar{\xi}} - 0.425e^{\bar{\xi}/2}\cos\sqrt{\tfrac{3}{2}}\,\bar{\xi} + 0.0824e^{\bar{\xi}/2}\sin\sqrt{\tfrac{3}{2}}\,\bar{\xi} \\ &- 0.000011e^{-\bar{\xi}} + 0.327e^{\bar{\xi}} + 0.0951e^{\bar{\xi}}\cos\sqrt{3}\,\bar{\xi} \\ &- 0.161e^{\bar{\xi}}\sin\sqrt{3}\,\bar{\xi} + 0.00612e^{-\bar{\xi}/2}\cos\sqrt{\tfrac{3}{2}}\,\bar{\xi} \\ &- 0.00358e^{-\bar{\xi}/2}\sin\sqrt{\tfrac{3}{2}}\,\bar{\xi} \end{aligned}$$

(55) $$\xi_1 = 0.981 + 0.140\,\bar{A}$$

(56) $$\xi_2 = \bar{\xi}_2 + \xi_1 = - 2.04 - 0.054\,\bar{A}$$

The distributions $G_1'(\bar{\xi})$ and $G'(\bar{\xi})$ are plotted respectively in Fig. 2 and 3. It is seen that even with $\bar{A} \approx 1$, the deviation of the mean velocity distribution from Tollmien's solution is exceedingly small. Further discussions will be given in Section V.

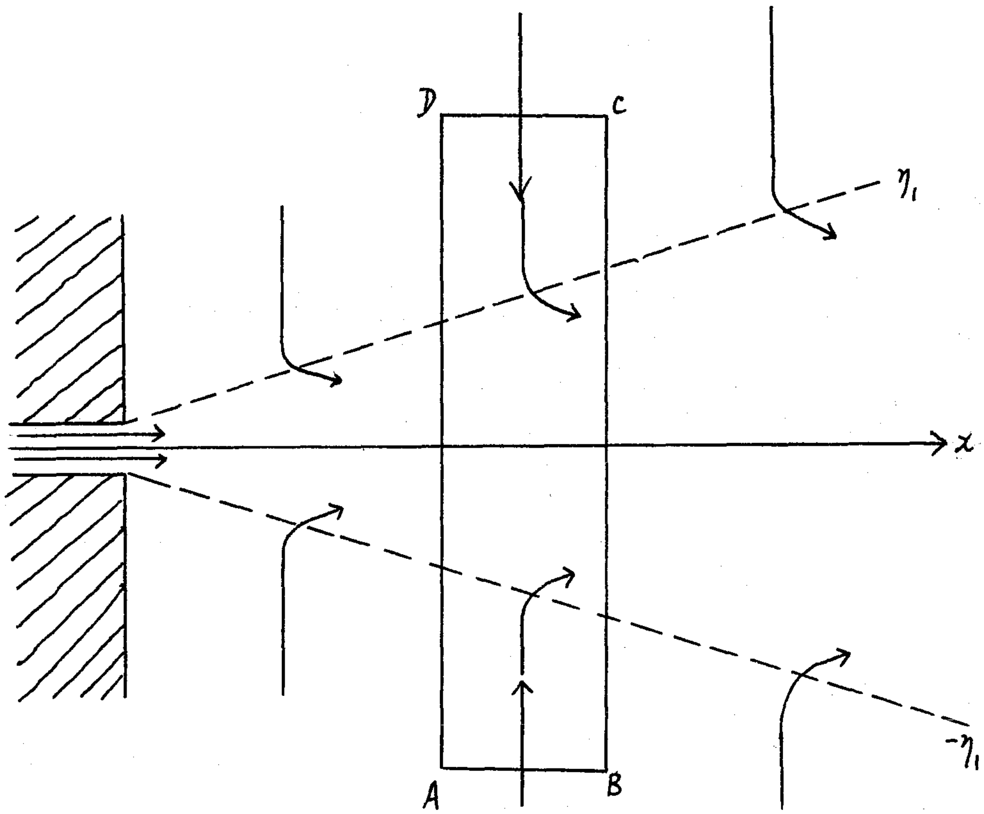


Fig. 4.

IV. The full jet.

Fig. 4 shows a two dimensional symmetrical jet from a narrow slit of a fluid of density ρ_1 into a rest fluid of density ρ_2. Since the total momentum across the different cross-sections of the jet must be constant, we have

$$(57) \qquad \int_{-\infty}^{+\infty} \rho_0 U^2 \, dy = \text{Const.}.$$

Assume the boundaries of the jet be straight lines and introduce the variable $\eta = y/x$ as in the last section, then (57) becomes

$$(58) \qquad \int_{-\infty}^{+\infty} \rho_0 U^2 \, d\eta \, x = \text{Const.}.$$

Since ρ_0 must be intermediate between ρ_1 and ρ_2, it cannot increase or decrease without limit when x is increased to infinity, so we must have

$$U \propto 1/\sqrt{x}$$

in order that the integral (58) should be independent of x. We can now deduce another relation from (5) and (6). Let us imagine four plane surfaces ABCD perpendicular to the plane of flow (Fig. 4). Since the mean flow is steady there should be no accumulation of mass inside the domain ABCD. Hence

$$\int_{AD} \rho_0 U dy - \int_{BC} \rho_0 U dy - \int_{AB} \rho_2 V dx + \int_{DC} \rho_2 V dx = 0$$

Furthermore, since the fluid is incompressible, we have

$$\int_{AD} U dy - \int_{BC} U dy - \int_{DC} V dx + \int_{AB} V dx = 0.$$

Multiply the second equation by - and add tb the first, we get

(59) $$\int_{AD} (\rho_0 - \rho_2) U dy - \int_{BC} (\rho_0 - \rho_2) U dy = 0.$$

Since U vanishes outside the jet and since the length AB = CD can be arbitrary, the above relation can be put into the following form

(60) $$\int_{-\infty}^{+\infty} (\rho_0 - \rho_2) U \, dy = \text{Const.},$$

or

$$\int_{-\infty}^{+\infty} (\rho_0 - \rho_2) U \, d\eta \, x = \text{Const.}.$$

Now $U \propto 1/\sqrt{x}$, we have therefore $\rho_0 - \rho_2 \propto 1/\sqrt{x}$ also. This means that ρ_0 will be very near to ρ_2 when x is very large. This is just what should be expected since at large distances from the slit nearly all the fluid inside the jet is the fluid of density ρ_2 that has been sucked in from both sides of the jet.

Now write

(61) $$\frac{U}{U_0} = \frac{F'(\eta)}{\sqrt{x}}, \quad \frac{V}{U_0} = \frac{1}{2\sqrt{x}}\{2\eta F'(\eta) - F(\eta)\}, \rho_0 = \rho_1 + \frac{f(\eta)}{\sqrt{x}},$$

$$\ell = cx, \quad \eta = \frac{y}{x}$$

and substitute these into (18), we find that when terms involving higher powers of 1/x are neglected, the resulting equation reduces to the same equation as that in the case of the homogeneous fluid, namely

$$F'^2 + F F'' = 2kc^2 \frac{d}{d\eta} (F'')^2,$$

or by integration

$$(62) \qquad F F' = 2kc^2 F''^2.$$

Introducing a change of variable $\xi = (2kc^2)^{-1/3}\eta$ as in Tollmien's treatment, we have, as in the last section:

$$(63) \qquad G(\xi)\, G'(\xi) = G''(\xi)^2$$

This means that neither the velocity distribution nor the boundaryies of the jet, ξ_1, is influenced by the density of the spraying fluid at large distances from the slit. Tollmien's solution for the homogeneous case is therefore entirely applicable to the present case so far as the first approximation is concerned.

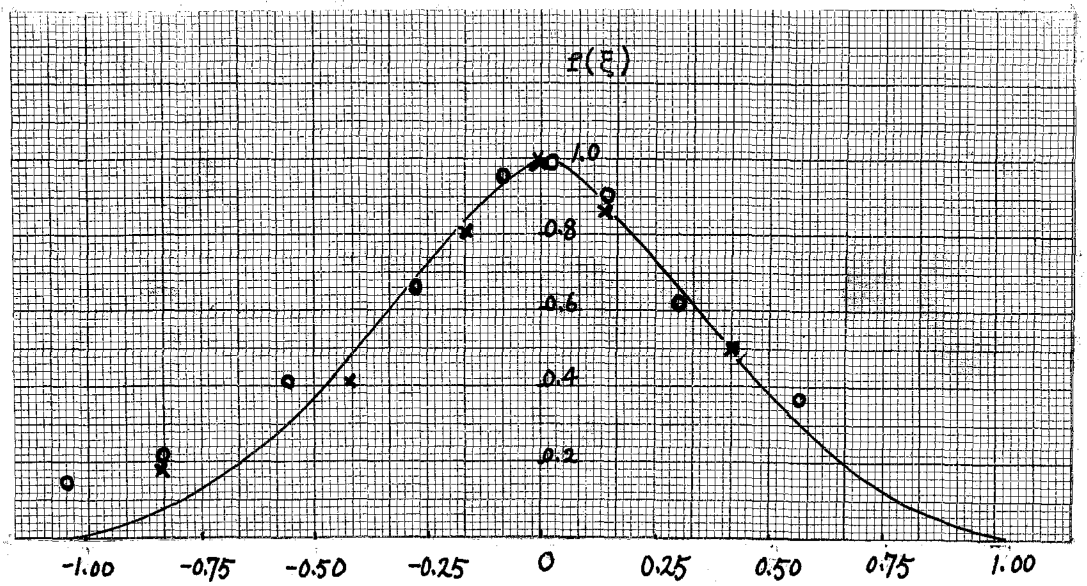
Next we shall proceed to consider the density distribution $f(\eta)$. Substituting (61) into (20) and neglecting terms of higher powers in $1/x$, we obtain

$$\frac{d}{d\eta} (F f) = 2kc_2^2 \frac{d}{d\eta} (f'F'').$$

Integrating once, we have

$$(64) \qquad F f = 2kc_2^2 f'F''.$$

Divide (62) by (64) and integrate, we obtain



$\xi = \eta / \eta_1$

$\eta_1 \backsimeq$ boundary.

Fig. 5

$$(65) \qquad f = C\,(F')^{k_1/k_2} \; ,$$

where C is the constant of integration. If as in the last section we put $k_1 = k_2$, then the distribution of $\rho_0 - \rho_2$ will be the same as that of mean velocity.

It can be shown that the same conclusions hold for the axial symmetrical jet, namely, (i) that the velocity distribution at large distances from the orifice is not influenced by the density of the spraying fluid and (ii) that the distribution of density is also given by (65). The analysis leading to the above conclusions is essentially similar to that we have for the plane symmetrical jet, so it is omitted here.

From (1) and (2) we obatin

$$v_1 = \frac{\rho_0 - \rho_2}{\rho_1 - \rho_2} = \frac{1}{\rho_1 - \rho_2}\frac{1}{\sqrt{x}}\, f(\eta).$$

Experimentally the density distribution is measured by placing a small cup in the jet and determining v_1 at any point from the sample collected in the cup. Fig. 5 shows an experimental result of the density distribution observed by Lee. It will be noted that since Lee's experiment was originally intended for fuel-injection studies, the injection of the spraying fluid was not continuous (with 750 separate sprays per minute), so the jet was actually not steady. Nevertheless the experimental result checks with our theory quite satisfactorily.

V. Discussion of the results.

It has been found by Tollmien that the boundary of the jet corresponds to a definite value of ξ_1. From the following relation for the full jet

$$y_1 = (2k_1c^2)^{1/3}\xi_1 x,$$

we see that the divergence of Tollmien's jet can be known when k_1c^2 is fixed. In the transport theories, however, the value of k_1c^2 can only be determined experimentally. Therefore the divergence of Tollmien's jet cannot be predicted in the theory. Now from experimental observations we know that when the density of the rest fluid, ρ_2, is different from that of the spraying fluid, the divergence of the jet increases with the increase of ρ_2. Thus the spray of water into the air will be wider when the pressure of the air is increased. According to the present theory, this may be due to two causes: (i) ξ_1 increases with the increase of ρ_2, or (ii) k_1c^2 or ℓ/x decreases when ρ_2 is increased. The second cause can easily be visualized by the fact that the diffusing masses of the spraying fluid will encounter more retardation and therefore have a shorter mixing length when the density of the surrounding fluid (ρ_2) is increased. In the present theory we can only take into account the first cause, since, as we have mentioned before, the value of k_1c^2 itself cannot be predicted theoretically in the transport theories.

From our foregoing results we know that ξ_1 (and also ξ_2) changes only slightly for the half jet and remains entirely unchanged at large distances for the full jet. Therefore the observed change of divergence of the jet with different ρ_2 must largely be due to the change of the mixing length.

Our foregoing treatment for a full jet has been confined to large distances from the slit or orifice. At small distances, since the higher powers of 1/x can no more be neglected, the resulting equations will contain x as well as η and consequently the similarity in the different sections of the flow is no more possible. It will be noted that the usual procedure of treating the flows in jets and wakes fails whenever the similarity in different cross-sections of the flow is impossible. For instance, no theoretical treatment has been achieved for the wakes at small distances from the obstacle.

In our case, the situation at small distances from the orifice or slit is even more complicated by the fact that the atomization of the spraying fluid must be influenced greatly by the surface tension of these two immiscible fluids.

In conclusion, the author wishes to express his hearty thanks to Professor Theodore von Kármán for suggesting the problem and the the helpful suggestions and discusions.

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Part III.

On the Possibility of Keeping the Electrons Inside the Dimension of Nucleus and the Quantum Mechanical Theory of Neutron

ABSTRACT

In the present paper an attempt is given to treat the β-activity along the lines of Gamow's theory of α-activity without using Fermi's conception of likening the process to the emission of light photons by the charged particle. It is found that if we consider the proton as a sphere with constant high potential inside, then the electron can be kept inside the proton for any length of time depending on the energy of the electron and the height of the potential chosen. The present theory explains satisfactorily the β-activity of the free neutron and gives the right order of magnitude of the nuclear force in contrast to the too small value obtained in Fermi's theory. The latter result means that the magnitude of the nuclear force can be explained by the β-exchange force alone without the introduction of mesotron. At the present stage the theory cannot explain satisfactorily the different types of β-active nuclei, and therefore no detailed quantitative study of the mean lifes of the β-transmutations is given. The present model of proton can also be used to explain the binding of mesotron with the heavy particles. This provides a new way of treating the exchange interactions between the heavy particles in the mesotron theory.

I. Introduction

From Heisenberg's principle of uncertainty and from actually solving the wave equation of the electron in Gamow's model of the nucleus, it is concluded that the electron cannot possibly exist inside the nucleus.* On the other hand, β-electrons are found to be emitted directly from the radioactive nuclei. To meet this difficulty, Fermi† assumed that the β-electrons do not already exist inside the nucleus, but are created instantaneously during the transformation of the neutron into proton inside the nucleus in just the same way as the light photons are created and annihilated by the charged particles.

Fermi assigned to the heavy nuclear particles a new kind of charge "g" analogous to the electromagnetic charge "e" of the charged particles. In this way he bestowed upon the heavy particles a new power of creation and annihilation of electrons. This enables us to speak of the emitted electrons from heavy particles just as light is given off from a light source. This conception has been extended later to the emission and absorption of mesotrons.

The significance of this new charge "g" can be looked at in two different ways. First, we may consider it simply as a coupling constant behind which is hidden an unknown mechanism of binding the electron in the heavy particle, so that the electron is not really created and annihilatedaas the terms would imply. This situation can best be understood by considering the case of α-disintegration. If we did not

* For a detailed discussion of this point see Bethe and Bacher, Review of Modern Physics, 8 (1936) 82, § 38

† E. Fermi, Zeit. f. Phys. 88 (1934) 161.

know Gamow's model of the nucleus, we might have explained the α-disintegration by assigning to the α-active nucleus an "α-particle" charge analogous to the terminology "meson" charge and "β-particle" charge "g" in Fermi's theory, and developing a formal theory of α-disintegration in close analogy to the Fermi's theory. This "α-particle" charge then represents nothing fundamental and ultimate. The second way of looking is that the charge "g" really represents a fundamental and ultimate interaction constant similar to the electromagnetic charge "e". This conclusion is inevitable if no satisfactory mechanism of binding can be found. It invalidates our old conception that all mass particles cannot be created and annihilated.

It will be noted that this concept of creation and annihilation of electron during the β-disintegration is of a different nature as that of the creation and annihilation of electron pairs by the light wave. In the latter case, the electron is considered as merely jumping from the negative energy state to a positive energy state and leaving a positive electron hole. So although experimentally we observe the creation and annihilation of electron pairs, we deny it theoretically by saying that it only involves a jumping of states as a necessary consequence of Dirac theory of electron. Here in the case of β-disintegration, however, the creation and annihilation of electron cannot be considered as a jumping of states as in the case of the pair production. Consequently the theory of β-disintegration is the only occasion in which the concept of actual creation and annihilation of electrons is needed.

It seems worthwhile to investigate whether this conception of creation and annihilation of electron can also be avoided in the theory of β-disintegration. As we have mentioned before this attempt can only be achieved by finding out a possible mechanism of binding along the lines of Gamow's theory of α-disintegration. We see that even the process of pair production must also be considered as actual creation and annihilation of mass particles, the process of creation and annihilation of β-particles is of an entirely different type (for instance, the charge "g" plays no role in the process of pair production), and so the simplification in our conception will be very great if this attempt is successful. At any rate, success in this investigation is also of considerable interest in itself because it leads to a new explanation of the β-disintegration. This investigation is the aim of the present paper.

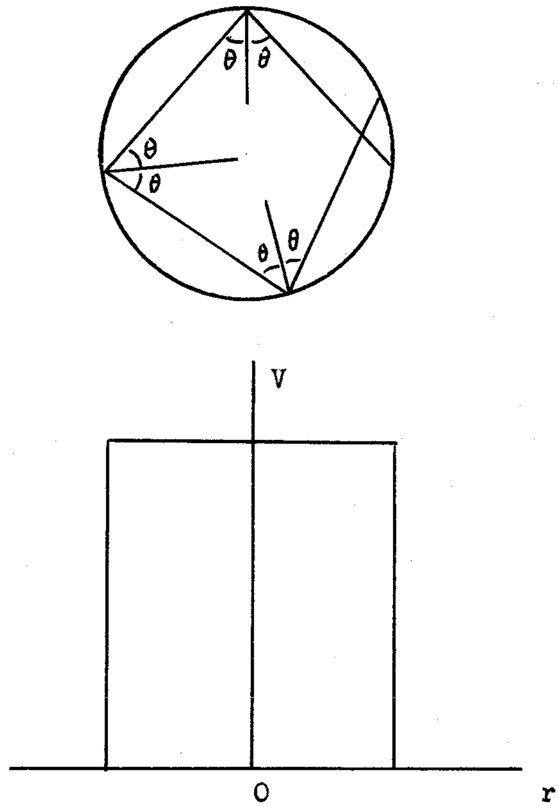

Fig. 1.

II. A semi-classical consideration.

Before we proceed to obtain the rigorous solution, let us consider a semi-wave semi-classical aspect of the problem which is origianlly the idea that started the present investigation. We shall consider the proton as a sphere with extremely high potential inside, and assume that the drop of potential at the boundary is so abrupt that the gradient of potential both outside and inside the sphere is negligible in comparison with it. This assumption is essentially that the proton exerts a strong attractive force on the electron at the boundary surface of the sphere. A schematic representation of this sphere is given in Fig. 1. The potential is equal to a positive constant value inside the radius r_1 and vanishes for $r > r_1$ (the Coulumb potential outside being neglected as extremely weak in comparison with the potential inside the sphere. This point can be verified later). Since the electron has a negative charge, it will gain an extremely high kinetic energy when it falls inside the sphere. Let us consider an electron inside the sphere with an angular momentum about the center of the sphere as shown in the figure. From the principle of relativity we know that the velocity of the electron is equal to the group velocity of the wave and that the wave velocity of the wave is decreased when the group velocity is increased, so the electron will have a smaller wave velocity inside the sphere. Classically we may consider the electron as a narrow pencil of waves striking at the inner surface of the sphere at an angle of incidence θ (θ is different from zero when the angular momentum of the electron about the center is not zero). Since the wave velocity of the electron is smaller inside the sphere we shall have total reflection as in the case of light wave when θ is sufficiently large. It can be seen from the figure that if total reflection occurs, the electron will be totally reflected again and again at the same angle of incidence θ inside the surface of the sphere and consequently there is no way for the electron to escape. Such a sphere with an electron inside has a total charge zero and so we may consider it as a neutron except that the resulting spin is different owing to the fact that the role of neutrino has not been considered. According to Fermi, the neutrino is assumed to satisfy the same Dirac wave equation for the electron except that the mass and charge are put equal to zero. Hence the binding of the neutrino can be treated in the same way as the electrn (the electrostatic potential inside the sphere must be replaced by some other kind of potential since the neutrino has no charge).

It will be noted that according to the following investigation we find that the total reflection of a wave inside the spherical boundary is of a slightly different nature as the total reflection on a plane boundary as we are familiar with in optics. In the case of total reflection at the plane boundary we find that the wave on the other side of the plane dies down exponentially with the distance so the intensity of the transmitted wave will vanish at great distances. Now in the case of spherical boundary, the intensity of the transmitted wave at great distances does not vanish although it falls to a very small value, which is the smaller the higher the angular momentum of the electron inside the sphere. This point is even an advantage of the present theory because it provides a small probabitity for the electron to leak through the boundary as is required by the experimental fact of β-disintegration.

We shall proceed to solve rigorously the Dirac's equation of the electron in the model of Fig. 1 in the following two sections. detailed physical discussions will be given in Section VI.

III. The rigorous solution.

To proceed rigorously we use the spherical coordinates for the wave functions of the electron. We shall start from the relativistic equation of Dirac for an electron. This equation has been written by Darwin in rectangular coordinates in the following form:

$$
(1)\quad \left\{
\begin{aligned}
\alpha i \psi_1 &+ \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_4 + \frac{\partial}{\partial z}\psi_3 = 0,\\
\alpha i \psi_2 &+ \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_3 - \frac{\partial}{\partial z}\psi_4 = 0,\\
\beta i \psi_3 &+ \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_2 + \frac{\partial}{\partial z}\psi_1 = 0,\\
\beta i \psi_4 &+ \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_1 - \frac{\partial}{\partial z}\psi_2 = 0,
\end{aligned}
\right.
$$

where

$$
(2)\quad \left\{
\begin{aligned}
\alpha &= \frac{2\pi}{h}\left(\frac{W+eV}{c} + mc\right),\\
\beta &= \frac{2\pi}{h}\left(\frac{W+eV}{c} - mc\right).
\end{aligned}
\right.
$$

We look for a solution for the region inside the sphere with angular momentum k ($k \neq 0$). Such a solution has been given by Darwin:

$$
(3)\quad \left\{
\begin{aligned}
\psi_1 &= -i F_k(r) P_{k+1}^{u}, & \psi_2 &= -i F_k(r) P_{k+1}^{u},\\
\psi_3 &= (k+u+1) G_k(r) P_k^{u}, & \psi_4 &= (-k+u) G_k(r) P_k^{u+1}
\end{aligned}
\right.
$$

where P_k^u is the surface harmonic defined by

$$P_k^u = (k+u)!\, \sin^u\theta \left(\frac{d}{d\cos\theta}\right)^{k+u} \frac{1}{2^k \cdot k!} (\cos^2\theta - 1)^k e^{iu\varphi} \tag{4}$$

$$= (k-u)!\, P_k^u(\cos\theta)\, e^{iu\varphi}$$

where $P_k^u(\cos\theta)$ is the associated Legendre polynormal and k and u are integers, k being positive and $-k-1 \leqslant u \leqslant k$. $F(r)$ and $G(r)$ are solutions of the following set of equations

$$\left\{ \begin{array}{l} \alpha F + \dfrac{dG}{dr} - \dfrac{k}{r} G = 0, \\ \\ -\beta G + \dfrac{dF}{dr} + \dfrac{k+2}{r} F = 0. \end{array} \right. \tag{5}$$

Now since V = constant inside the sphere we may consider   and as constant. Eliminating $F(r)$ from the above equations, we have

$$\frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} + \left(\alpha\beta - \frac{k(k+1)}{\alpha r^2}\right) G = 0. \tag{6}$$

Put $G(r) = u(z)/\sqrt{z}$ and $z = ar$ where $a = \sqrt{\alpha\beta} = \frac{2\pi}{h}\sqrt{\frac{(W+2V)^2}{c^2} - m^2c^2}$,
we obtain

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{(k+\frac{1}{2})^2}{z^2}\right) u = 0, \tag{7}$$

which is the Bessel equation of order $k + \frac{1}{2}$ and the two independent solutions of which are given by

$$(8) \quad \begin{cases} u_1 = J_{k+\frac{1}{2}}(z), \\ u_2 = J_{-k-\frac{1}{2}}(z). \end{cases}$$

The second solution of (8) must be excluded since $J_{-k-\frac{1}{2}}(z)$ is not finite at the origin ($z = 0$). Using u_1 we obtain

$$(9) \quad \begin{cases} G_k = \frac{1}{\sqrt{ar}} J_{k+\frac{1}{2}}(ar), \\ F_k = \frac{1}{\sqrt{ar}} J_{k+\frac{3}{2}}(ar). \end{cases}$$

Darwin's solution inside the sphere then becomes

$$(10) \quad \begin{aligned} \psi_1 &= -Ai \frac{a}{\alpha\sqrt{ar}} J_{k+\frac{3}{2}}(ar) P_{k+1}^{u}, \\ \psi_2 &= -Ai \frac{a}{\alpha\sqrt{ar}} J_{k+\frac{3}{2}}(ar) P_{k+1}^{u+1}, \\ \psi_3 &= A(k+u+1) \frac{1}{\sqrt{ar}} J_{k+\frac{1}{2}}(ar) P_k^{u}, \\ \psi_3 &= A(-k+u) \frac{1}{\sqrt{ar}} J_{k+\frac{1}{2}}(ar) P_k^{u+1}. \end{aligned}$$

To obtain the solution outside the sphere where $V = 0$ we notice that the solution u_2 of (8) cannot be excluded since the region considered does not contain the origin. Denoting all quantities referred to the solution outside the sphere by a prime, we have

$$(11) \quad \begin{cases} \psi_1' = -\left[A'J_{k+\frac{3}{2}}(a'r) + B'J_{-k-\frac{3}{2}}(a'r)\right] i \dfrac{a'}{\alpha'\sqrt{a'r}} P_{k+1}^{u}, \\ \psi_2' = -\left[A'J_{k+\frac{3}{2}}(a'r) + B'J_{-k-\frac{3}{2}}(a'r)\right] i \dfrac{a'}{\alpha'\sqrt{a'r}} P_{k+1}^{u+1}, \\ \psi_3' = \left[A'J_{k+\frac{1}{2}}(a'r) + B'J_{-k-\frac{1}{2}}(a'r)\right](k+u+1)\dfrac{1}{\sqrt{a'r}} P_{k}^{u}, \\ \psi_4' = \left[A'J_{k+\frac{1}{2}}(a'r) + B'J_{-k-\frac{1}{2}}(a'r)\right](-k+u)\dfrac{1}{\sqrt{a'r}} P_{k}^{u+1}, \end{cases}$$

where

$$(12) \quad \begin{cases} \alpha' = \dfrac{2\pi}{h}\left(\dfrac{W}{c} + mc\right), \\ \beta' = \dfrac{2\pi}{h}\left(\dfrac{W}{c} - mc\right), \\ a' = \sqrt{\alpha'\beta'} = \dfrac{2\pi}{h}\sqrt{\dfrac{W^2}{c^2} - m^2c^2}, \end{cases}$$

where A' and B' are constants. To this solution we must impose the condition that there should be no wave coming from infinity. Thus the only possible solution is a progressive wave diverging from the sphere. Such a solution is represented by a Hankel function defined by

$$H_{k+\frac{1}{2}}(a'r) = J_{k+\frac{1}{2}}(a'r) + i(-1)^{k+1} J_{-k-\frac{1}{2}}(a'r).$$

So we must replace (9) by

$$G_k' = \frac{1}{\sqrt{a'r}} H_{k+\frac{1}{2}}(a'r),$$

$$F_k' = \frac{1}{\sqrt{a'r}} H_{k+\frac{3}{2}}(a'r).$$

Comparing this with (11), we obtain

$$B' = i(-1)^{k+1} A'$$

Therefore Darwin's solution outside the sphere is given by

$$(13) \quad \begin{cases} \psi_1' = -A' i \dfrac{a'}{\alpha' \sqrt{a'r}} H_{k+\frac{3}{2}}(a'r) P_{k+1}^{u}, \\ \psi_2' = -A' i \dfrac{a'}{\alpha' \sqrt{a'r}} H_{k+\frac{3}{2}}(a'r) P_{k+1}^{u+1}, \\ \psi_3' = A'(k+u+1) \dfrac{1}{\sqrt{a'r}} H_{k+\frac{1}{2}}(a'r) P_k^{u}, \\ \psi_4 = A'(-k+u) \dfrac{1}{\sqrt{a'r}} H_{k+\frac{1}{2}}(a'r) P_k^{u+1}. \end{cases}$$

From the above result we know that the only solution outside the sphere is a divergent progressive wave. On the other hand since (10) represents a stationary wave, so the only solution inside the sphere is a stationary wave. Strictly speaking, when we have a divergent progressive wave outside the sphere the stationary wave inside the sphere should be represented by a stationary wave with damped amplitude by the law of conservation of charge and mass of the electron. However, in our present case the effect of damping can be neglected with sufficient approximation as we shall see later that the amplitude of the wave outside the sphere is extremely small in comparison with the amplitude of the wave inside the sphere.

The condition to be satisfied at the boundary $r = r_1$ is

$$\psi_i = \psi_i' \qquad (i = 1, 2, 3, 4). \tag{14}$$

Substituting (10) and (13) into (14), we notice that the equations for i =1 and 3 are identical with those for i = 2 and 4. Thus we obtain only two equations:

$$\begin{cases} \dfrac{a}{\alpha\sqrt{ar_1}} A\, J_{k+\frac{3}{2}}(ar_1) = \dfrac{a'}{\alpha'\sqrt{a'r_1}} A' H_{k+\frac{3}{2}}(a'r_1), \\[2ex] \dfrac{1}{\sqrt{ar_1}} A\, J_{k+\frac{1}{2}}(ar_1) = \dfrac{1}{\sqrt{a'r_1}} A' H_{k+\frac{1}{2}}(a'r_1). \end{cases} \tag{15}$$

Since above are two homogeneous linear equations for A and A', they can have solution only when the determinant vanishes. Thus we have

$$\frac{a}{\alpha} \frac{J_{k+\frac{3}{2}}(ar_1)}{J_{k+\frac{1}{2}}(ar_1)} = \frac{a'}{\alpha'} \frac{H_{k+\frac{3}{2}}(a'r_1)}{H_{k+\frac{1}{2}}(a'r_1)} . \tag{16}$$

This result shows that a' must be a solution of (16), i.e., W cannot have arbitrary values but must take only descrete values determined by (16).

IV. The charge and current densities.

We shall now proceed to calculate the charge and current densities at any point. According to Dirac these are given by the following expressions:

$$
(17)\quad
\begin{cases}
\rho = e(\psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 + \psi_4^*\psi_4), \\[2mm]
j_x = -ce(\psi_1^*\psi_4 + \psi_2^*\psi_3 + \psi_3^*\psi_2 + \psi_4^*\psi_1), \\[2mm]
j_y = -ce(-i\psi_1^*\psi_4 + i\psi_2^*\psi_3 - i\psi_3^*\psi_2 + i\psi_4^*\psi_1), \\[2mm]
j_z = -ce(\psi_1^*\psi_3 - \psi_2^*\psi_4 + \psi_3^*\psi_1 - \psi_4^*\psi_2).
\end{cases}
$$

Now we are interested in the radial component of the current given by

$$
\begin{aligned}
j_r &= j_x \sin\theta\cos\varphi + j_z \sin\theta\sin\varphi + j_z\cos\theta \\
&= Re\,\{(j_x - i j_y)e^{i\varphi}\sin\theta + j_z\cos\theta\}.
\end{aligned}
$$

Since from (17)

$$j_x - i j_y = -2ce(\psi_2^*\psi_3 + \psi_4^*\psi_1),$$

$$j_z = -2ce\,Re\,\{\psi_1^*\psi_3 - \psi_2^*\psi_4\},$$

we have

$$j_r = Re\{2ce(\psi_2^*\psi_3 + \psi_4^*\psi_1)e^{i\varphi}\sin\theta + 2ce(\psi_1^*\psi_3 - \psi_2^*\psi_4)\cos\theta\}.$$

Since from (15)

$$(18)\quad \begin{cases} \psi_2'^{*}\psi_3' = A'^2 i\,(k+u+1)\dfrac{1}{\alpha' r}H^{*}_{k+\frac{1}{2}}(a'r)H_{k+\frac{3}{2}}(a'r)\,P^{u+1*}_{k+1}P^{u}_{k}, \\ \psi_1'^{*}\psi_3' = A'^2 i\,(k+u+1)\dfrac{1}{\alpha' r}H^{*}_{k+\frac{1}{2}}(a'r)\,H_{k+\frac{3}{2}}(a'r)\,P^{u*}_{k+1}P^{u}_{k}, \\ \psi_4'^{*}\psi_1' = A'^2 i\,(k-u)\dfrac{1}{\alpha' r}H_{k+\frac{1}{2}}(a'r)\,H^{*}_{k+\frac{3}{2}}(a'r)\,P^{u}_{k+1}\,P^{u+1*}_{k}, \\ \psi_2'^{*}\psi_4' = -A'^2 i(k-u)\dfrac{1}{\alpha' r}H^{*}_{k+\frac{1}{2}}(a'r)H_{k+\frac{3}{2}}(a'r)\,P^{u+1*}_{k}P^{u+1}_{k+1}, \end{cases}$$

and

$$(19)\quad \begin{cases} P^{u+1*}_{k+1}P^{u}_{k}e^{i\varphi} = [(k-u)!]^2 P^{u+1}_{k+1}(\cos\theta)\,P^{u}_{k}(\cos\theta), \\ P^{u*}_{k+1}P^{u}_{k} = (k+1-u)!(k-u)!\,P^{u}_{k+1}(\cos\theta)\,P^{u}_{k}(\cos\theta), \\ P^{u}_{k+1}P^{u+1*}_{k}e^{i\varphi} = [(k-1-u)!]^2 P^{u}_{k+1}(\cos\theta)\,P^{u+1}_{k}(\cos\theta), \\ P^{u+1*}_{k}P^{u+1}_{k+1} = (k-u-1)!(k-u)!\,P^{u+1}_{k}(\cos\theta)\,P^{u+1}_{k+1}(\cos\theta), \end{cases}$$

then j_r is given by

$$(20)\qquad j_r = -2ceA'^2 f(r)[h(\theta)+g(\theta)],$$

where

$$f(r) = \frac{(-1)^{k+1}}{\alpha' r}\left[J_{k+\frac{1}{2}}(a'r)J_{-k-\frac{3}{2}}(a'r) - J_{k+\frac{3}{2}}(a'r)\,J_{-k-\frac{1}{2}}(a'r)\right]$$

$$= \frac{2\sin(k+\frac{3}{2})\pi}{\pi\alpha' r}\,\frac{(-1)^{k+1}}{\alpha' r} = \frac{2}{\pi\alpha'^2 r^2},$$

$$h(\theta) = (k+u+1)\left[(k+u)!\,(k-u)!\,P_{k+1}^{u+1}(\cos\theta)P_k^u(\cos\theta)\sin\theta\right.$$

$$\left.-(k+1-u)!\,(k-u)!\,P_{k+1}^{u}(\cos\theta)P_k^u(\cos\theta)\cos\theta\right],$$

and

$$g(\theta) = (k-u)\left[-(k+1-u)!\,(k-u-1)!\,P_{k+1}^{u}(\cos\theta)P_k^{u+1}(\cos\theta)\sin\theta\right.$$

$$\left.+(k-u-1)!\,(k-u)!\,P_k^{u+1}(\cos\theta)P_{k+1}^{u+1}(\cos\theta)\cos\theta\right].$$

The total current J is therefore given by

(22) $$J = \int_0^\pi 2\pi r^2 j_r \sin\theta\, d\theta$$

$$= 8\pi c e A'^2 \frac{1}{\alpha'^2}\int_0^\pi [h(\theta)+g(\theta)]\sin\theta\, d\theta.$$

With the help of the following two well known formulae for Legendre functions,

$$P_k^{u+1}(\cos\theta) = -(k+1+u)\sin\theta P_{k+1}^u(\cos\theta)$$

$$+\cos\theta P_{k+1}^{u+1}(\cos\theta),$$

$$P_k^u(\cos\theta) = \frac{1}{k+u+1}\left[P_{k+1}^{u+1}(\cos\theta)\sin\theta\right.$$

$$\left.+(k-u+1)\cos\theta P_{k+1}^u(\cos\theta)\right],$$

(22) can be reduced to

$$J = \frac{8\pi c e A'^2}{\alpha'^2}\Big\{\int_0^{\pi}(k+u+1)(k+u+2)\,[(k-u)!]^2\,[P_k^u(\cos\theta)]^2\sin\theta\, d\theta \tag{23}$$

$$+\int_0^{\pi}(k-u)^2[(k-u-1)!]^2\,[P_k^{u+1}(\cos\theta)]^2\sin\theta\, d\theta\Big\},$$

$$= \frac{8\pi c e A'^2}{\alpha'^2}\Big\{(k+u+1)(k+u+2)(k-u)!\,(k+u)!\,\frac{2}{2k+1}$$

$$+ (k-u)^2(k-u-1)!\,(k+u+1)!\,\frac{2}{2k+1}\Big\},$$

or finally

$$J = \frac{16\pi c e A'^2}{(2k+1)\alpha'^2}\Big\{(k+u+2)!\,(k-u)! + (k-u)(k-u)!\,(k+u+1)!\Big\}. \tag{24}$$

We next consider the current inside the sphere. We may choose the orientation of the z axis such that u = k, i.e., the component of the angular momentum in the z-direction is just the total angular momentum. Substituting (10) into (17), we have

$$j_x = j\sin\varphi\,, \qquad j_y = -j\cos\varphi, \qquad j_z = 0,$$

(25)

$$j = \frac{2ceA^2}{\alpha r}(2k+1)\, J_{k+\frac{1}{2}}(\alpha r) J_{k+\frac{3}{2}}(\alpha r)(k-u)!\,(k+1-u)!\,P_{k+1}^{u}(\cos\theta)P_k^{u}(\cos\theta).$$

Thus the current inside the sphere flows arround circular path parallel to the xy-plane with the center at the origin. We see immediately that this is in accordance with the current in the rough picture given in Fig. 1.

To calculate the total charge inside the sphere we first form the expressions for $\psi_i^*\psi_i$ as follows

$$
(26)\qquad
\begin{cases}
\psi_1^*\psi_1 = -\dfrac{A^2 a}{\alpha^2 r}\left[J_{k+\frac{3}{2}}(ar)\right]^2 P_{k+1}^{u*}\,P_{k+1}^{u}, \\[2ex]
\psi_2^*\psi_2 = -\dfrac{A^2 a}{\alpha^2 r}\left[J_{k+\frac{3}{2}}(ar)\right]^2 P_{k+1}^{u+1*}\,P_{k+1}^{u+1}, \\[2ex]
\psi_3^*\psi_3 = (k+u+1)^2\dfrac{A^2}{ar}\left[J_{k+\frac{1}{2}}(ar)\right]^2 P_{k}^{u*}\,P_{k}^{u}, \\[2ex]
\psi_4^*\psi_4 = (k-u)^2\dfrac{A^2}{ar}\left[J_{k+\frac{1}{2}}(ar)\right]^2 P_{k}^{u+1*}\,P_{k}^{u+1}.
\end{cases}
$$

Substituting this into the first equation of (17), we have

$$
(27)\qquad \rho = e\sum_{i=1}^{4}\psi_i^*\psi_i = eA^2\Big\{-\frac{a}{\alpha^2 r}\left[J_{k+\frac{3}{2}}(ar)\right]^2\left[P_{k+1}^{u*}P_{k+1}^{u}+P_{k+1}^{u+1*}P_{k+1}^{u+1}\right]
$$
$$
+\frac{1}{ar}\left[J_{k+\frac{1}{2}}(ar)\right]^2\left[(k+u+1)^2P_k^{u*}P_k^{u}+(k-u)\,P_k^{u+1*}P_k^{u+1}\right]\Big\}.
$$

The total charge Q inside the sphere is then given by

$$
(28)\qquad Q = \int_0^{\pi}\int_0^{2\pi}\int_0^{r_1}\rho r^2\sin\theta\,dr\,d\theta\,d\varphi
$$
$$
= 2\pi eA^2\Big\{-\frac{a}{\alpha^2}\int_0^{r_1} r\left[J_{k+\frac{3}{2}}(ar)\right]^2 dr\int_0^{\pi}\left[P_{k+1}^{u*}P_{k+1}^{u}+P_{k+1}^{u+1*}P_{k+1}^{u+1}\right]\sin\theta\,d\theta
$$
$$
+\frac{1}{a}\int_0^{r_1} r\left[J_{k+\frac{1}{2}}(ar)\right]^2 dr\int_0^{\pi}\left[(k+u+1)^2P_k^{u*}P_k^{u}+(k-u)^2P_k^{u+1*}P_k^{u+1}\right]\sin\theta\,d\theta\Big\}.
$$

The final expression for Q is then given by

(29)
$$Q = 4\pi e A^2 \{ [-\int_0^{r_1} \frac{a}{\alpha^2} r [J_{k+\frac{3}{2}}(ar)]^2 dr] (k+u+1)!(k-u)! + [\int_0^{r_1} \frac{1}{\alpha} r [J_{k+\frac{1}{2}}(ar)]^2 dr] (k-u-1)!(k+u)! \}.$$

When Q and J are known the half life τ of the electron being contained inside the sphere can approximately be estimated. From the law of conservation of charge we have

(30)
$$-\frac{dQ}{dt} = J.$$

Let

(31)
$$Q = Q_0 e^{-t/\tau}.$$

Then substituting (31) into (30), we have

$$Q/\tau = J,$$

or

(32)
$$\tau = Q/J.$$

Since a and A' are given by (15) and (16), the numerical value of Q/J can be calculated for any given values of k and u when a' is known. From the experimental result of β-disintegration, we take the greatest value of W as $\leqq 7\ mc^2$ (usually $W \cong 2$ or $3\ mc^2$). Then

(33)
$$a' = \sqrt{\alpha'\beta'} = \frac{2\pi}{h}\sqrt{\frac{W^2}{c^2} - m^2c^2} \lesssim \frac{2\pi}{h} \times 7\,mc = 2\pi \times 7 \times \frac{1}{2.4\times 10^{-10}}$$
$$\cong 18 \times 10^{10}\ cm^{-1}.$$

The radius of the proton can be taken as

$$(34) \qquad r_1 \cong 2.8 \times 10^{-13}\ cm.$$

Consequently $a'r_1 \lesssim 18\times10^{10} \times 2.8\times10^{-13} \cong 5\times10^{-2}$. For such small value of argument, we have

$$(35) \qquad H_{k+\frac{1}{2}}(a'r_1) \cong \frac{1\cdot3\cdot5\cdots(2k+1)}{\sqrt{2\pi}}(a'r_1)^{-k-\frac{1}{2}},$$

$$(36) \qquad H_{k+\frac{3}{2}}(a'r_1) \cong \frac{1\cdot3\cdot5\cdots(2k+3)}{\sqrt{2\pi}}(a'r_1)^{-k-\frac{3}{2}}.$$

Substituting these into (15), we obtain

$$(37) \qquad \frac{a}{\alpha}\frac{J_{k+\frac{3}{2}}(ar_1)}{J_{k+\frac{1}{2}}(ar_1)} = \frac{a'}{\alpha'}\frac{H_{k+\frac{3}{2}}(a'r_1)}{H_{k+\frac{1}{2}}(a'r_1)} \cong \frac{a'}{\alpha'}(2k+3)\frac{1}{\alpha' r_1} \gtrsim (2k+3)\cdot2\cdot10^{3}$$

The numerical values of $J_{k+\frac{3}{2}}(ar_1)$ for k 0 are always less than 0.5. Therefore from (37) $J_{k+\frac{1}{2}}(ar_1)$ must be less than $2.5\times10^{-4}/(2k+3)$, which shows that ar_1 must be very near to one of the zeros of $J_{k+\frac{1}{2}}(ar_1)$. We assume that this is the first zero of $J_{k+\frac{1}{2}}(ar_1)$ beside the origin in order to make the value of a not unreasonably large. For a reason which will be plain later we choose $k = 3$. Then from consulting a table of Bessel functions, we obtain the first zero of $J_{\frac{7}{2}}(ar_1) = 0$ as $ar_1 \cong 7$, this gives immediately

$$(38) \qquad a \cong \frac{7}{2.8\times10^{-13}} \cong 2.5\times10^{13}\ cm^{-1}.$$

It is seen that this approximate value of a is independent of a'. This means that when a' varies the variation of a is very small and confined within the range $|a_1 - a_2|$ given by

$$J_{\frac{7}{2}}(a_1 r_1) = \frac{1}{9} \times 2.5 \times 10^{-4}, \quad J_{\frac{7}{2}}(a_2 r_1) = -\frac{1}{9} \times 2.5 \times 10^{-4}.$$

Substituting (38) and (36) into the first equation of (15), we obtain

(39)
$$A' \cong \frac{\alpha'}{\alpha}\sqrt{\frac{a}{a'}}\,\frac{J_{9/2}(ar_1)}{H_{9/2}(a'r_1)}A \cong \sqrt{\frac{a'}{a}}\,J_{9/2}(ar_1)\frac{\sqrt{2\pi}\,(a'r)^{9/2}}{1\cdot3\cdot5\cdot7\cdot9}A$$

$$\cong 1.6 \times 10^{-3} A \sqrt{\frac{a'}{a}}\,(a'r_1)^{9/2}$$

where we have substituted $J_{9/2}(ar_1) \cong 0.3$. Substitute k = 3 and u = 0 into (24) and (29), we get respectively

(40)
$$J = \frac{16\pi c e A'^2}{7\alpha'^2}[5!\,3! + 3\cdot 3!\,4!]$$

$$\cong 2.2 \times 10^{-2}\,\frac{ceA^2}{(a'r_1)^9}$$

(41)
$$Q \cong 8 \times 10^{-25}\,eA^2$$

Therefore we have finally

(42)
$$\tau \cong \frac{Q}{J} \cong \frac{a'a}{c(a'r_1)^9}\cdot 10^{-22} \cong \text{Const.}\ a'^{-8} \cong \text{Const.}\ W^{-8}$$

Thus τ is proportional to the inverse eighth power of energy W. We can easily see that for other values of k, τ is proportional to (2k + 2)-th power of W. If we put W = 7 mc^2, we obtain

(43) $$\tau \cong 10^3 \text{ sec.}$$

The foregoing result shows that the current J outside the sphere is a very small quantity in comparison with the total charge Q inside the sphere and the mean life estimated is also of the right order of magnitude.

In order to estimate the dependence of J and τ on the angular momentum k we repeat the above calculation for k = 1 and k = 6. The calculated results are based on $W = 7\ mc^2$ and are tabulated with the case k = 3 in the following table. It is seen that τ increases rapidly

Table I.

k	u	A'/A	Q/eA^2	J/eA^2, cm/sec.	τ, sec.	V, MV.
1	0	$1.4\ 10^{-5}$	$5\ 10^{-26}$	10^{-19}	$5\ 10^{-7}$	400 *
3	0	$0.46\ 10^{-9}$	$8\ 10^{-25}$	$7\ 10^{-28}$	10^3	500
6	0	$0.25\ 10^{-17}$	$4\ 10^{-22}$	$5\ 10^{-38}$	10^{16}	800

* The values of v calculated here are not accurate owing to (38) being not accurate. We can easily see that if r_1 is considered as two times larger than the value used in (38), then the value of V estimated will be two times smaller.

with the angular momentum. The last column of the above table gives the order of magnitude of V calculated according to (2) under the condition $W \lesssim 7\ mc^2 \ll eV$. Since V can only have one definite value, it follows that we can have atmost only one possible value of k inside the sphere. ~~Consequently only one electron can be absorbed by the sphere according to Pauli's principle.~~

In Table 1 we have not considered the effect of change of u. Here we notice that since u only represents the component of k on the z-axis, and since the total current and charge cannot depend on the choice of z-axis, J and Q and therefore τ should be independent of u. At first sight this conclusion seems to be inconsistent with our foregoing result that both (24) and (29) contain u. However, since the normalization constant A still remains arbitrary, we expect that u will disappear from our final expressions for (24) and (29) after the normalization has been carried out. We first consider the following two integrals contained in (29)

$$\int_0^{r_1} r[J_{k+\frac{1}{2}}(ar)]^2 dr = \frac{1}{2} r_1^2 \{[J_{k+\frac{1}{2}}(ar_1)]^2 + [J_{k+\frac{3}{2}}(ar_1)]^2\} - (k+\tfrac{1}{2}) r_1 J_{k+\frac{1}{2}}(ar_1) J_{k+\frac{3}{2}}(ar_1),$$

$$\int_0^{r_1} r[J_{k+\frac{3}{2}}(ar)]^2 dr = \frac{1}{2} r_1^2 \{[J_{k+\frac{3}{2}}(ar_1)]^2 + [J_{k+\frac{5}{2}}(ar_1)]^2\} - (k+\tfrac{3}{2}) r_1 J_{k+\frac{3}{2}}(ar_1) J_{k+\frac{5}{2}}(ar_1).$$

Since r_1 is an extremely small quantity, we may neglect the terms involving r_1^2. Now $J_{k+\frac{1}{2}}(ar)$ vanishes, so the first integral is zero. (29) then becomes

$$Q = 4\pi e A^2 \left[-\int_0^{r_1} \frac{a}{\alpha^2} r [J_{k+\frac{3}{2}}(ar)]^2 dr\right] (k+u+1)!\,(k-u)!$$

Dividing this by (24), we obtain

$$\frac{Q}{J} = T \frac{(k+u+1)!\;\;(k-u)!}{(k+u+2)!\,(k-u)! + (k-u)(k-u)!\,(k+u+1)!} = T\frac{1}{(k+u+2)+(k-u)} = T\frac{1}{2k+2},$$

where T is a function independent of u. Thus we see that Q/J is independent of u. Now the process of normalization is essentially to put Q = e, hence e/J and consequently J must be independent of u. This proves that both Q and J given by (24) and (29) are independent of u after carrying out the normalization as we have expected.

It will be noted that our foregoing estimated values of V agree with the Heisenberg's principle of uncertainty. Let p be the momentum of the electron inside the sphere, then by Heisenberg's principle

$$\Delta p \geqq \frac{h}{2\pi r_1} \cong \frac{hc}{2\pi e^2} mc = 137\, mc$$

whence

$$W + eV \gtrsim \Delta p \cdot c \cong 137\, mc^2$$

Since $W \ll eV$, we have

(44) $$eV \gtrsim 137\, mc^2 \quad \text{or} \quad 70\ \text{MEV.}$$

i.e., $V \gtrsim 70$ MV., which is seen to be satisfied by our foregoing results.

The Coulumbian potential outside the sphere is given by

(45) $$\frac{e}{r_1} \cong \frac{4.8 \times 10^{-10}}{2.8 \times 10^{-13}} \cong 1.7 \times 10^{3}\ \text{e.s.u.} \quad \text{or} \quad 0.51\ \text{MV.}$$

Thus it can be neglected entirely in comparison with the constant potential inside the sphere. This conclusion is in agreement with the conception of Born and Infeld, who considered the field outside the charged particle as extremely weak in comparison with that inside.

V. The exchange force.

Now we shall proceed to find the exchange potential between two spheres considered in the foregoing with one electron exchanging between them.* The whole system may be compared with the deuteron nucleus. Let these two spheres be at distance $\vec{r}$ apart and let $\vec{r}'$ and $\vec{r}''$ be the distance from any point of the field to the centers of these two spheres ($\vec{r}'' - \vec{r}' = \vec{r}$). Write $\psi(\vec{r}')$ for the wave function of the electron when the latter is combined with the first sphere and $\psi(\vec{r}'')$ for that when the electron is combined with the second sphere. Following the usual procedure of evaluating the exchange integral, we use solution (10) and (13)† for $\psi(\vec{r}')$ and $\psi(\vec{r}'')$ corresponding to the unperturbed states of the electron in these two separate spheres. We may express the unperturbed zero order solution as a linear combination of $\psi(\vec{r}')$ and $\psi(\vec{r}'')$, namely

$$\Psi(\vec{r},\vec{r}') = \frac{1}{\sqrt{2+2S}}\left[\psi(\vec{r}') + \psi(\vec{r}'')\right], \quad \vec{r}' = \vec{r}'' - \vec{r}, \tag{46}$$

where the factor $1/\sqrt{2+2S}$ is the constant of normalization and

$$S = \int \psi(\vec{r}')\psi(\vec{r}'')\,d\vec{r}', \qquad \vec{r}' = \vec{r}'' - \vec{r}, \quad 0 \leq S \leq 1.$$

The interaction energy between these two spheres is

$$\int \Psi^{\dagger}(\vec{r},\vec{r}')\,\mathcal{U}\,\Psi(\vec{r},\vec{r}')\,d\vec{r}',$$

---

* The following treatment follows a similar procedure in the treatment of the hydrogen molecule-ion, see L. Pauling: Chemical Review, 5 (1928) 173.

† With the argument $\vec{r}$ in these solutions replaced by $\vec{r}'$ or $\vec{r}''$

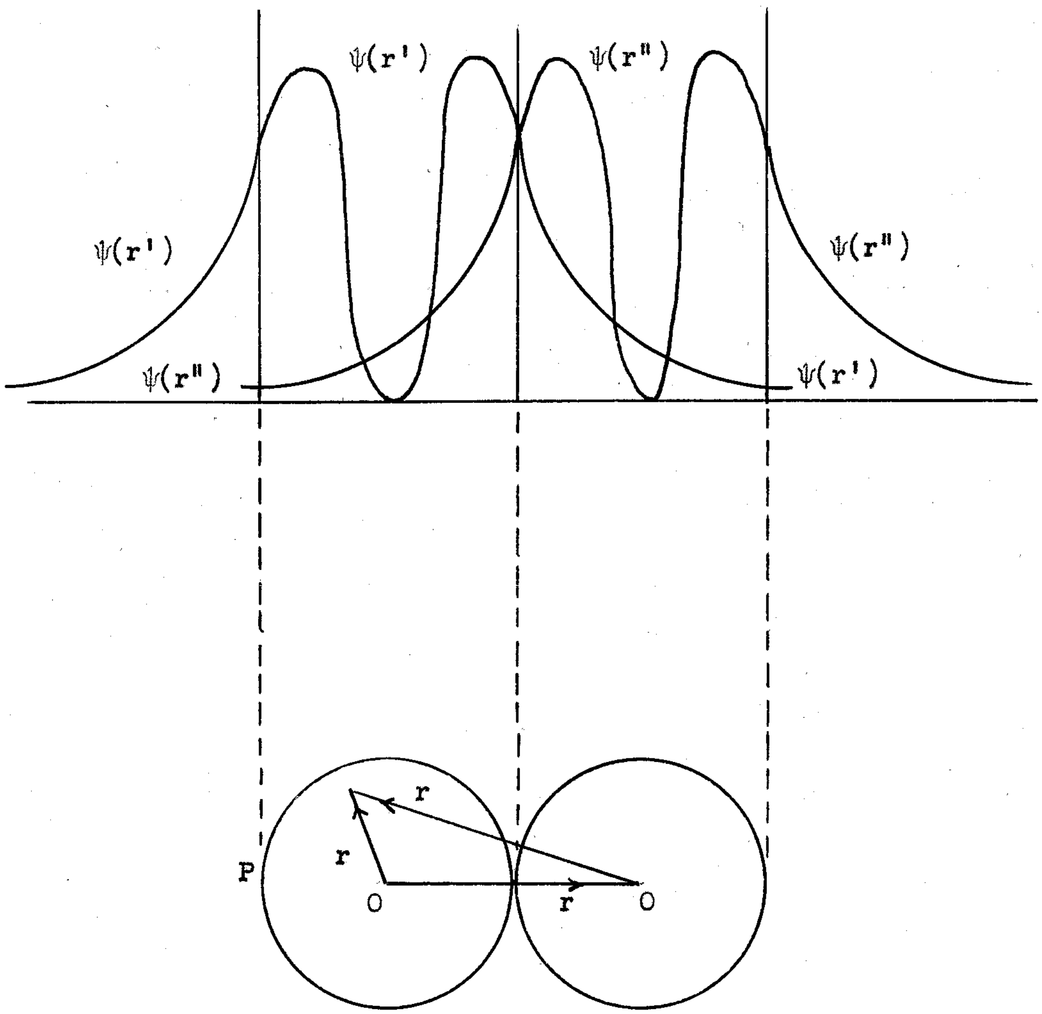

Fig. 2

of which the exchange integral is

$$J(\vec{r}) = \frac{e}{1+S}\int \psi^{+}(\vec{r}')\,\mathcal{U}\,\psi(\vec{r}'')\,d\vec{r}', \quad \vec{r}' = \vec{r}'' - \vec{r}, \tag{47}$$

where $\mathcal{U}$ is the interaction potential between the sphere and the electron: $\mathcal{U} = V$ for $r' < r_1$ and $\mathcal{U} = 0$ for $r' > r_1$. Thus

$$J(\vec{r}) = \frac{e}{1+S} V \int_{r' \leq r_1} \psi^{+}(\vec{r}')\,\psi(\vec{r}'')\,d\vec{r}', \quad \vec{r}' = \vec{r}'' - \vec{r}, \tag{48}$$

We can easily estimate the order of magnitude of $J(\vec{r})$. For $r' < r_1$, $\psi(\vec{r}')$ is given by (10) and $\psi(\vec{r}'')$ by (13), since from $r' < r$, $\vec{r}' = \vec{r}'' - \vec{r}$ and $r \geqslant 2r_1$, we have $r'' > r_1$. As a rough estimation we consider these two spheres in contact with each other, i.e., $r = 2r_1$ (Fig. 2).

From (3.6) we have

$$\psi(\vec{r}'') \propto \frac{1}{(a'r_1)^{k+3/2}},$$

hence

$$[\psi(r'')]_{r''=3r_1} = [\psi(r'')]_{r''=r_1} \frac{(a'r_1)^{k+3/2}}{(3a'r_1)^{k+3/2}} = [\psi(r'')]_{r''=r_1} \left(\frac{1}{3}\right)^{k+3/2}. \tag{49}$$

Since the total charge inside the sphere must be of order e, we have

$$\int_{r' \leq r_1} \psi^{+}(\vec{r}')\,\psi(\vec{r}')\,d\vec{r}' \approx 1. \tag{50}$$

Therefore

$$\psi(r') \approx \frac{1}{v_0^{1/2}}, \qquad r' \leq r_1, \tag{51}$$

where $v_0 = \frac{4}{3}\pi r_1^3 =$ the volume of the sphere. As a rough estimation, we may substitute (51) into (48). Since (49) corresponds to the smallest value of $\psi(r'')$ inside the sphere O' (at point P of Fig. 2), we may also replace $\psi(r'')$ in (48) by the smallest value given by (49). Thus we have

$$(52) \qquad J(r) \gtrsim \frac{1}{1+S} V \left[\psi(r'')\right]_{r''=r_1} \left(\frac{1}{3}\right)^{k+3/2} v_0^{1/2}.$$

Now since $\psi(r'')_{r''=r} = \psi(r')_{r'=r} = 1/v_0$ according to (51), finally we obtain approximately, replacing the factors $1/2+2S$ by $1/\sqrt{2}$ and putting $k = 3$,

$$(53) \qquad J(r) \approx e V \left(\frac{1}{3}\right)^{k+3/2} \approx e \times 500 \times \frac{1}{\sqrt{729}} \approx -16 \text{ MEV}.$$

which is seen to be of the right order of magnitude as the nuclear force between a proton and a neutron. It will be noted that the above estimation should not be taken too seriously since at such small distance of the two spheres ($r = 2r_1$), our method of perturbation might be invalidated by the strong interaction.

VI. Physical discussions.

In this section we shall identify the spherical model treated in the foregoing ~~two~~ sections with the proton and proceed to investigate physically how much this model of the proton represents reality. According to the present theory we may consider the neutron as the proton sphere with an electron contained in it. However, from the requirement of the spin and the continuous β-ray spectra we must assume the co-presence of a neutrino with the electron in the proton. The wave functions for the electron and the neutrino can be treated separately if the condition is satisfied that the interaction between the electron and the neutrino is much smaller than the interaction of both with the proton. According to Fermi, the wave equation for the neutrino is assumed to be the same as the Dirac equation for the electron except that the charge and mass in the equation are put equal to zero. If we further assume that the boundary of the proton sphere exerts a very strong attractive surface force on the neutrino as on the electron (only in the present case the force cannot be represented by a constant electrostatic potential inside the sphere since the neutrino has no charge), then the wave functions for the bounded neutrino will be given by the same solution (10) and (13) for the bounded electron except that m is put equal to zero and eV is replaced by U, the constant potential per neutrino inside the proton sphere, which gives rise to a strong attractive surface force on the neutrino at the boundary of the sphere. Consequently the results in the foregoing **three** sections can equally be applied to the electron and the neutrino. In particular when the interaction between the electron and the neutrino can be neglected in comparison with the interaction of both with the proton, the wave functions for the electron and the neutrino are represented separately by (10), (13) and the modified form of (10), (13) respectively.

It will be noted that for the "permitted" β-disintegration defined in Fermi's theory for which the change of a spin of the heavy particle is zero, the wave functions for the electron and the neutrino must have the same value of k and opposite values of u in order to make the total angular momentum of them vanish. From the further condition that both the electron and the neutrino must have the same mean life, from (42) we have the following condition

$$W_{neutrino} \approx W_{electron}$$

and therefore, from $a_{neutrino} \cong a_{electron}$,

$$eV \cong U.$$

Further discussions will be grouped into the following topics:

(i) The β-activity of the free neutron.

According to the results of the last three sections, the free neutron is β-active (The same conclusion is also reached in Fermi's theory). This is in agreement with the fact that only the proton is found to exist in the free state permanently in the universe.

(ii) The stable nuclei.

In Fermi's theory the neutron is assumed to posses a new kind of charge "g" in analogy to the electromagnetic charge "e" of the charged particles. This new charge "g" is responsible for the β-

activity of the neutron. In fact the value of "g" was determined from the life time of the β -disintegration. It follows from this that the neutron in all states, free or combined, possessing the charge "g" which is intrinsically connected with the β-activity, should also be β -active. When a neutron is coupled with a proton to form a deuteron, the emitted β-particles are absorbed immediately by the proton. This absorption and re-emission of β-particles by these two heavy particles reduces tremendously the probability for the electron and the neutrino to escape from the deuteron and thus renders the latter non-active toward β-disintegration. We repeat that there is no need that the individual neutron in the stable nucleus should also be stable. In fact advantage has been taken of this instability of the neutron in emitting β-particles or mesotrons to account for the nuclear exchange force among the nuclear particles. Now in the present theory we find that the neutron is always β -active. Therefore the situation is exactly the same as in the Fermi's theory and the stable nucleus can be explained in the same way.

(iii) The nuclear exchange force.

In the last section we have evaluated the exchange integral for an electron exchanging between two proton spheres. Since we have neglected the interaction between the electron and the neutrino in comparison with their interactions with the proton, the contribution to the exchange integral from the electron and the neutrino are additive. Hence the total exchange potential must be greater than the value given in (55). This definitely shows that according to the present theory the force caused by the exchange of the β -particles is sufficient to account for the whole magnitude of the nuclear force, of order 10 MEV, in contrast to the estimation in Fermi's theory which shows that the exchange force due to β-particles is 10^{-12} times smaller than the above order.* This means that according to the present theory the nuclear binding can be explained already by β-activity without the introduction of the mesotron. This result is of considerable interest since it was only due to this defect of Fermi's theory that the Yukawa's theory of mesotron was proposed.

(iv) The mesotron theory of nuclear force.

Owing to the failure of Fermi's theory to explain the nuclear force, Yukawa put forth the idea that the nuclear force is due to the exchange of a medium heavy particle laterly called mesotron. The heavy particles are assumed to possessess "meson" charge in analogy to the "g" charge in the case of β-disintegration. In prodeeding to evaluate the exchange force between the heavy particles we meet with the divergence diffaculties which are more severe in nature than those in the radiation theory. Remedy has been introcuced to remove this difficulty first by a cut off procedure and then by developing a strong coupling theory in both of which the heavy particles are essentially assumed to have a finite radius which is just in line with the present theory. Recently Duffin and Kemmer† have successfully formulated the wave equation for the mesotron in the Dirac form. From the view point of the present theory it seems worthwhile to treat the mesotron analogously as we have done for the β-particles in the present paper. It is expected that the final solution should be of the same form as we

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* cf. Bethe and Bacher, loc. cit. p 203.

† N. Kemmer, Proc. Roy. Soc. A. Vol. 173 (1939) 91.

obtained here for the electron and the exchange force can be evaluated easily.

(v) The β-active nuclei

A β-active nucleus will result when one of the neutron is not coupled with other heavy particle which can absorb the β-particles emitted by the neutron. However, here we meet a difficulty of the present theory. From the result of the last section, we know that whenever k is known and $W \lesssim 7\ mc^2 \ll eV$ is used, V is determined completely. Now it is only natural to assume that the proton should have the same definite value of V in all states. If we use $V \approx 500$ MV as given in Table 1, then the only possible value for k is $k = 3$. The value of W is also fixed from (38). This would mean that all β-active nuclei should have the same value of W and k and consequently the same value of τ. This, however, disagrees with the experimental fact that both W and τ are found to be different for different nuclei.

It seems that the above difficulty can be avoided by assuming that either V or r_1 is different when the state of the proton is different in the different neclei. This explaination, however, is very vague and uncertain and calls for further investigation.

Conclusion

No quantitative result has been given in the foregoing discussions owing to the uncertainty of the present theory. The present theory has achieved the task of avoiding the use of the concept of creation and annihilation of electron, explained satisfactorily the β-activity of the neutron, and shown that the exchange of β-particles is sufficient to account for the nuclear force in order of magnitude, but failed in explaining the β-disintegration of the different radioactive nuclei.