Appendix A

Residual Phase Error in an OPLL with Nonuniform FM Response

In this appendix, we calculate the effect of the SCL FM response of the form

$$F_{FM}(f) = \frac{1}{b}\left(\frac{b-\sqrt{jf/f_c}}{1+\sqrt{jf/f_c}}\right), \tag{A.1}$$

where f_c denotes the corner frequency of the thermal response and depends on the device material and structure, and $b = K_{th}/K_{el} - 1$ denotes the relative strength of the thermal and electronic responses, on the minimum residual phase error in a Type I SCL-OPLL. For typical SCLs, $b > 0$, and f_c lies in the range of 0.1–10 MHz. For example, the fit to the experimental data in figure 2.9 was obtained with $b = 1.64$ and $f_c = 1.8$ MHz.

The open loop transfer function of a Type I loop with the nonuniform SCL FM response is therefore given by

$$G_{op}(\bar{f}) = \frac{\bar{K_F}}{jb\bar{f}}\left(\frac{b-\sqrt{j\bar{f}}}{1+\sqrt{j\bar{f}}}\right), \tag{A.2}$$

where the loop gain K_F and the frequency are normalized according to

$$\begin{aligned} \bar{f} &\doteq f/f_c, \\ \bar{K_F} &\doteq K_F/f_c. \end{aligned} \tag{A.3}$$

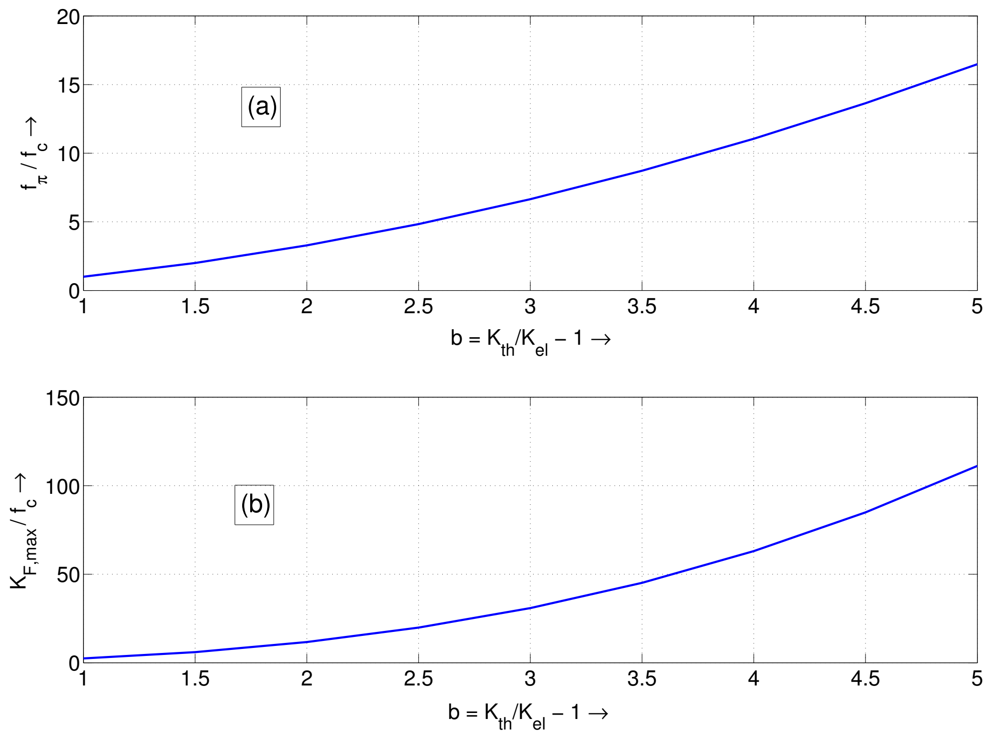

Figure A.1. Variation of (a) the normalized π-crossover frequency and (b) the normalized maximum gain as a function of the parameter b in equation (A.1).

The π-crossover frequency f_π (frequency where the phase of $G_{op}(f)$ goes to π) and the maximum gain (gain at which $|G_{op}(f_\pi)| = 1$ can now be calculated. Setting $\angle G_{op}(f_\pi) = -\pi$ in equation (A.2), we obtain

$$\bar{f}_\pi = 2\left(\frac{b-1+\sqrt{b^2+6b+1}}{4}\right)^2. \tag{A.4}$$

Next, setting $|G_{op}(f_\pi)| = 1$, we have

$$\bar{K}_{F,max} = b\bar{f}_\pi\sqrt{\frac{1+\bar{f}_\pi+\sqrt{2\bar{f}_\pi}}{b^2+\bar{f}_\pi-b\sqrt{2\bar{f}_\pi}}}. \tag{A.5}$$

The behavior of the normalized π-crossover frequency and the normalized maximum gain as a function of b are shown in figure A.1. $\bar{f}_\pi$ and $\bar{K}_{F,max}$ increase monotonically with b, and larger values of b and f_c therefore lead to higher loop bandwidths.

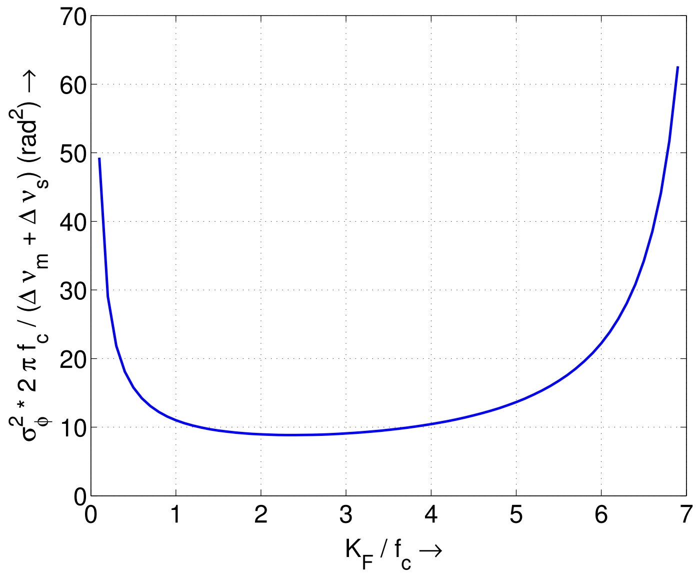

Figure A.2. Variation of the integral $\mathcal{I}(b, \bar{K_F})$ in equation (A.6) as a function of $\bar{K_F}$, for $b = 1.64$.

We now calculate the variance of the residual phase error by using equation (A.2) in equation (2.18), to obtain

$$\begin{aligned}\sigma_\phi^2 &= \frac{\Delta\nu_m + \Delta\nu_s}{2\pi f_c} \int_{-\infty}^{\infty} d\bar{f} \frac{b^2|1+\sqrt{j\bar{f}}|^2}{|K_F(b-\sqrt{j\bar{f}}) + jb\bar{f}(1+\sqrt{j\bar{f}})|^2} \\ &= \frac{\Delta\nu_m + \Delta\nu_s}{2\pi f_c} \mathcal{I}(b, \bar{K_F}). \end{aligned} \tag{A.6}$$

To understand the behavior of the variance of the phase error, we first note that σ_ϕ^2 scales inversely with f_c as expected, since the loop bandwidth increases with f_c. Further, the behavior of the integral $\mathcal{I}(b, \bar{K_F})$ for a given value of b, chosen to be $b = 1.64$ to match the experimental result of figure 2.9, is shown in figure A.2. For this value of b, the maximum stable gain is $\bar{K}_{F,max} = 7.36$. At low gains, the loop has little effect, leading to a high phase error. As the gain approaches the maximum possible value, the phase error again increases since the loop begins to go unstable. Therefore, there is an optimum value of the gain—$\bar{K}_{F,opt} \approx 2.4$ in this case—for which the variance of the phase error is minimized. The ratio of the optimum gain to the maximum stable loop gain lies between 0.25 and 0.35.

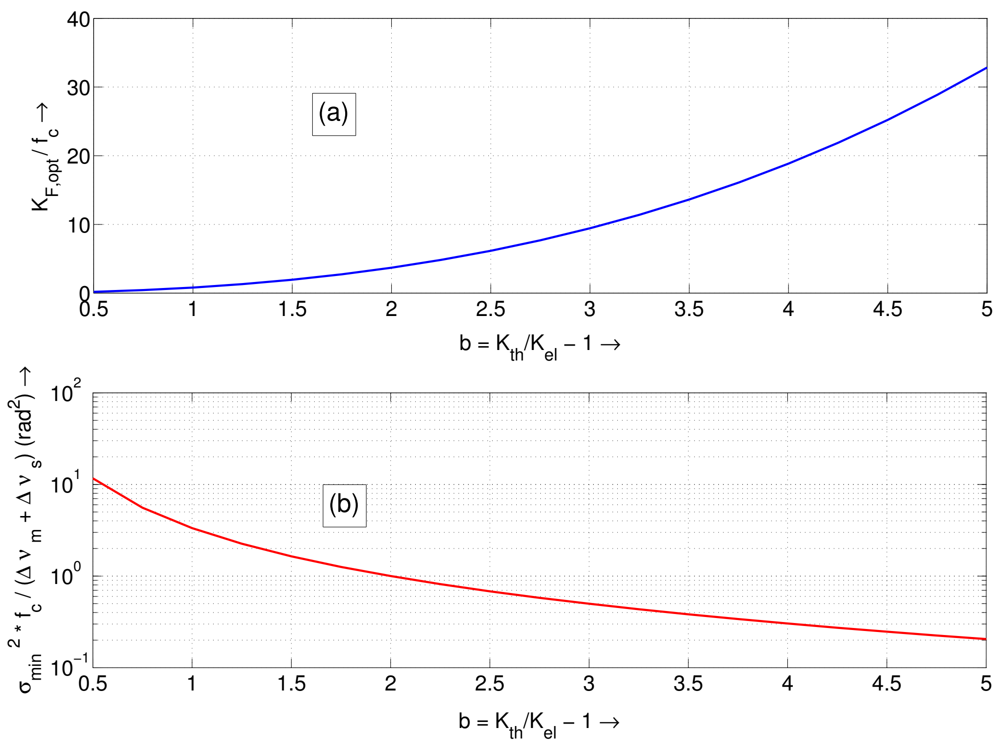

Figure A.3. Variation of the (a) normalized optimum gain $\bar{K}_{F,opt} = K_{F,opt}/f_c$ and (b) the normalized minimum residual phase error $\sigma^2_{min} f_c/(\Delta\nu_m + \Delta\nu_s)$ as a function of the parameter b, for a first-order OPLL with a SCL with nonuniform FM response.

The value of $\bar{K}_{F,opt}$ is a function of b, as is the value of the minimum phase error σ^2_{min}. As b increases, the loop bandwidth is higher, leading to a larger value of $\bar{K}_{F,opt}$ and a smaller value of σ^2_{min}. The values of the (normalized) optimum gain and minimum residual phase error vs. the parameter b are plotted in figure A.3. As a concrete example, consider the experimentally measured FM response of figure 2.9, for which $b = 1.64$ and $f_c = 1.8$ MHz. For these values, we obtain

$$\frac{\sigma^2_{min}}{\Delta\nu_m + \Delta\nu_s} = 8 \times 10^{-7}\ \mathrm{rad}^2/\mathrm{Hz}. \tag{A.7}$$

Appendix B

Four-Wave Mixing in a Multisegment Nonlinear Waveguide

In this appendix, we derive a general expression for the power generated by four-wave mixing in the multisegment nonlinear waveguide shown in figure B.1 for arbitrary values of the phase mismatch in each segment.[1] This is important in order to understand practical implementations of the dispersion compensation technique described in section 6.1.1.3, where it is often difficult to precisely control the value of the dispersion parameter D_c. Let the waveguide consist of N segments, labeled $k = 1, 2, \ldots, N$. The length of segment k is given by L_k, and let the propagation constants of the chirped, reference and output fields in this segment be denoted by $\beta_{ch,k}$, $\beta_{R,k}$ and $\beta_{out,k}$ respectively. The phase mismatch in this segment is therefore given by $\Delta\beta_k = 2\beta_{ch,k} - \beta_{R,k} - \beta_{out,k}$, and related to the value of the dispersion parameter in the segment by equation (6.13). For the sake of notational simplicity, we assume that the loss, refractive index, nonlinear susceptibility and the effective mode area of the different segments are equal. Our goal is to calculate the output field and optical power generated by FWM at $z = \sum_{k=1}^{N} L_k$. We ignore splice losses in this calculation.

We begin by describing a separate frame of reference for the kth segment, denoted by the position variable $z_k = \left(z - \sum_{i=1}^{k-1} L_i\right) \in [0, L_k]$. Similar to equation (6.1), the

[1] A similar calculation has been performed by Inoue [135] for the case of a chain of fiber amplifiers with different dispersions.

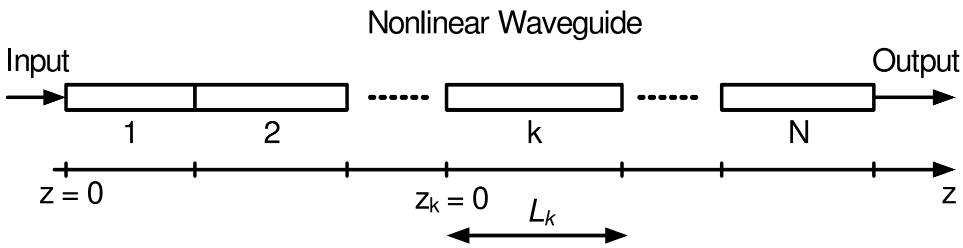

Figure B.1. A multisegment nonlinear waveguide for four-wave mixing.

electric field in this segment is described by its slowly varying complex amplitude as

$$E(z_k, t) = \frac{1}{2} A_k(z_k) \exp\left(j(\omega t - \beta_k z_k)\right) + \text{c.c.} \tag{B.1}$$

The continuity of the electric field at $z_k = 0$ requires that

$$A_k(z_k = 0) = A_{k-1}(z_{k-1} = L_{k-1}) \exp(-j\beta_{k-1} L_{k-1}). \tag{B.2}$$

It is to be understood henceforth that the argument of the function A_k is the variable z_k. We consider the FWM process where the chirped wave of frequency ω_{ch} and the reference wave of frequency ω_R generate an output wave with a frequency $\omega_{out} = 2\omega_{ch} - \omega_R$, and assume that the chirped and reference waves are undepleted by the FWM process. For the chirped wave, we have

$$\begin{aligned} A_{ch,k}(0) &= A_{ch,k-1}(L_{k-1}) \exp(-j\beta_{ch,k-1} L_{k-1}) \\ &= A_{ch,k-1}(0) \exp\left[-(\alpha/2 + j\beta_{ch,k-1}) L_{k-1}\right] \\ &= A_{ch,1}(0) \exp\left(-\frac{\alpha}{2} \sum_{i=1}^{k-1} L_i - j \sum_{i=1}^{k-1} \beta_{ch,i} L_i\right). \end{aligned} \tag{B.3}$$

Similarly, the reference wave at $z_k = 0$ is given by

$$A_{R,k}(0) = A_{R,1}(0) \exp\left(-\frac{\alpha}{2} \sum_{i=1}^{k-1} L_i - j \sum_{i=1}^{k-1} \beta_{R,i} L_i\right). \tag{B.4}$$

In the frame of reference we have set up to describe the kth segment, the equation for the evolution of the output field is identical to equation (6.10):

$$\frac{dA_{out,k}}{dz_k} = -\frac{\alpha}{2}A_{out,k} - jgA^2_{ch,k}(0)A^*_{R,k}(0)e^{-3\alpha z_k/2}e^{-j\Delta\beta_k z_k}, \tag{B.5}$$

where we have defined $g = nc\gamma\epsilon_0 A_{eff}/2$. The solution to this differential equation is

$$A_{out,k}(L_k) = e^{-\alpha L_k/2}\left[A_{out,k}(0) - jgA^2_{ch,k}(0)A^*_{R,k}(0)\left(\frac{1-e^{-(\alpha+j\Delta\beta_k)L_k}}{\alpha+j\Delta\beta_k}\right)\right]. \tag{B.6}$$

Using equations (B.2), (B.3) and (B.4), we obtain

$$\begin{aligned} A_{out,k}(L_k) =& e^{-\alpha L_k/2}\left[A_{out,k-1}(L_{k-1})\exp\left(-j\beta_{out,k-1}L_{k-1}\right) - jg\left(\frac{1-e^{-(\alpha+j\Delta\beta_k)L_k}}{\alpha+j\Delta\beta_k}\right)\right. \\ &\left.\times A^2_{ch,1}(0)A^*_{R,1}(0)\exp\left(-\frac{3\alpha}{2}\sum_{i=1}^{k-1}L_i - j\sum_{i=1}^{k-1}(2\beta_{ch,i}-\beta_{R,i})L_i\right)\right], \end{aligned} \tag{B.7}$$

which can be rewritten as

$$\begin{aligned} A_{out,k}(L_k) = \exp\left(-\frac{\alpha L_k}{2} - j\sum_{i=1}^{k-1}\beta_{out,i}L_i\right)&\left[A_{out,k-1}(L_{k-1})\exp\left(j\sum_{i=1}^{k-2}\beta_{out,i}L_i\right)\right. \\ &\left. -jgA^2_{ch,1}(0)A^*_{R,1}(0)\left(\frac{1-e^{-(\alpha+j\Delta\beta_k)L_k}}{\alpha+j\Delta\beta_k}\right)\exp\left(-\frac{3\alpha}{2}\sum_{i=1}^{k-1}L_i - j\sum_{i=1}^{k-1}\Delta\beta_i L_i\right)\right]. \end{aligned} \tag{B.8}$$

We note that the phase term $\exp\left(-j\sum_{i=1}^{k-1}\beta_{out,i}L_i\right)$ in equation (B.8) has no effect on the power of the output wave, which only depends on the magnitude of $A_{out,k}$ as given by equation (6.14). This term depends on the propagation constants $\beta_{out,i}$ of the output wave, and is difficult to evaluate in general. The physics of the process is mainly determined by the phase-mismatch terms $\Delta\beta_i$. We therefore find it convenient to define a new amplitude

$$\tilde{A}_{out,k}(z_k) = A_{out,k}(z_k)\exp\left(j\sum_{i=1}^{k-1}\beta_{out,i}L_i\right), \tag{B.9}$$

and equation (B.8) can be rewritten in the form

$$\tilde{A}_{out,k}(L_k) = \exp\left(-\frac{\alpha L_k}{2}\right)\Big[\tilde{A}_{out,k-1}(L_{k-1}) - jgA_{ch,1}^2(0)A_{R,1}^*(0) \\ \times \left(\frac{1-e^{-(\alpha+j\Delta\beta_k)L_k}}{\alpha+j\Delta\beta_k}\right)\exp\left(-\frac{3\alpha}{2}\sum_{i=1}^{k-1}L_i - j\sum_{i=1}^{k-1}\Delta\beta_i L_i\right)\Big]. \tag{B.10}$$

Equation (B.10) is the general solution for the output field generated by FWM in a multiple-segment nonlinear waveguide. The values of the phase mismatch in the various segments is related to the frequency chirp by equation (6.13), and the output power is evaluated using equation (6.14):

$$P_{out}\left(z = \sum_{i=1}^{N} L_i\right) = \frac{nc\epsilon_0 A_{eff}}{2}\left|\tilde{A}_{out,N}(L_N)\right|^2 = \frac{g}{\gamma}\left|\tilde{A}_{out,N}(L_N)\right|^2. \tag{B.11}$$