Trigonometric Expansions of Doubly Periodic Functions of the Second Kind in Any Number of Variables

Thesis by

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Summary

A function $F(x_1, x_2, \cdots, x_n)$ involving $\frac{\vartheta_1^{(1)} \vartheta_1(x_1 + x_2 + \cdots + x_n)}{\vartheta_1(x_1) \vartheta_1(x_2) \cdots \vartheta_1(x_n)}$ is defined; F is shown to be analytic in a cylindrical region (T_n) defined by $|\mathrm{Im}\, x_i| < \mathrm{Im}\, \pi\tau$ for $i = 1, 2, \cdots, n$. Another function $\bar{F}(x_1, x_2, \cdots, x_n)$ involving $\frac{\vartheta_1^{(1)} \vartheta_1(x_1 + x_2 + \cdots + x_n)}{\vartheta_1(x_1) \vartheta_1(x_2) \cdots \vartheta_1(x_n)} \cot(x_1 + x_2 + \cdots + x_n)$ is defined; $\bar{F}$, too, is shown to be analytic in (T_n).

Then F can be expressed in terms of $\bar{F}(x_1, \cdots, x_{n-1})$, $F(x_1, \cdots, x_{n-j})$ and $F(x_n, x_1 + \cdots + x_{n-j})$ for $j = 1, 2, \cdots, n-2$ and $\bar{F}(x_1, x_2, \cdots, x_n)$ can be expressed in terms of $F(x_1, x_2, \cdots, x_{n-j})$ and $\bar{F}(x_n, x_1 + x_2 + \cdots + x_{n-j})$. From these representations, formulae for the q-coefficients of the Fourier expansions of F and $\bar{F}$ are obtained.

The Fourier expansion of $F(x, y, z)$ is paraphrased to obtain a theorem concerning the number of ways of representing an integer N in the form $N = k^2 + k(l+m+n) + mn$

I

Introduction

The expansion

$$\frac{\vartheta_1^{(1)}\,\vartheta_1(x+y)}{\vartheta_1(x)\,\vartheta_1(y)} = \cot x + \cot y + 4\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2rs}\sin 2(rx+sy)$$

and the expansions obtained by increasing one or both variables by the half periods $\frac{\pi}{2}$, $\frac{\pi\tau}{2}$ and $\frac{\pi}{2}+\frac{\pi\tau}{2}$ have been the source of a great many number-theoretic results under application of E. T. Bell's method of paraphrase. (1)* These number-theoretic results give relationships between the number of ways of representing an integer N by certain quadratic forms in 2 or 3 variables which take only integral values.

In a thesis presented to the University of Nebraska in 1940, entitled "Algebraic Proofs of Certain Arithmetical Paraphrases," I proved by the purely algebraic methods of Uspensky (2)-(3), a set of eight identities which W. A. Dwyer obtained by paraphrasing certain theta-function identities which he had established by the methods of analysis. (4) A part of the algebraic proof consisted of the following result: Let (N, λ, β) denote given integers and let (l, m, n) denote a set of integers such that

1.1 $\quad N = \lambda^2 l^2 + \beta^2 mn$

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* Underlined numbers in parentheses refer to section entitled References at the end of the thesis.

(ℓ, m, n) is called an integral solution of (1.1), and we consider (ℓ, m, n) as a vector of three components which can be transformed by a matrix.

Let T denote the set of integral solutions of (1.1) such that

i) $\ell \gtrless 0$

ii) $m \geq n \geq 0$

iii) $2\lambda\ell - \beta(m-n) \geq 0$

iv) $2\lambda\ell - \beta(m-n) \equiv 0 \mod (2\beta)$

v) $\beta(m+n) \equiv 0 \mod (2\lambda)$

Then there is a certain 3x3 matrix A, which is identical with its own inverse, such that when all of the integral solutions of the set T are transformed by A, the set T is reproduced in some order.

Upon generalizing A to an n x n matrix, A_n, identical with its inverse, I was able to establish the following result:

Let $[y] \equiv (y_1, y_2, \cdots, y_n)$ denote an integral solution of

1.2 $\quad N = \frac{1}{2}[y]\, Q_n\, [y]'$

where

$$Q_n = \beta^2 \begin{bmatrix} (n-1)(n-2)\frac{\lambda^2}{\beta^2} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 1 \\ 0 & 1 & 0 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 1 & \cdot & \cdot & 0 \end{bmatrix}$$

and (λ, β) are integers.

Let T_n denote the set of integral solutions of (1.2) such that

i) $y_1 \gtreqless 0$

ii) $y_2 \geq y_3 \geq \cdots \geq y_n \geq 0$

iii) $(n-1)\lambda y_1 - \beta \sum_{r=2}^{n} y_r + \beta(n-1) y_n \geq 0$

iv) $(n-1)\lambda y_1 - \beta \sum_{r=2}^{n} y_r \equiv 0 \mod((n-1)\beta)$

v) $\beta \sum_{r=2}^{n} y_r \equiv 0 \mod(\lambda(n-1))$

Then when the integral solutions of the set T_n are transformed by A_n, the set T_n is reproduced in some order.

I have been unsuccessful in attempting to find that generalization of Dwyer's identities which would presumably contain the identity associated with the generalized matrix A_n. E. T. Bell pointed out (1) that number-theoretic results involving the representation of an integer N as a quadratic form in $(n+1)$ variables which take integral values would be associated with the expansion of $\frac{\vartheta(x_1+x_2+\cdots+x_n)}{\vartheta(x_1)\vartheta(x_2)\cdots\vartheta(x_n)}$. Following his suggestion and under his supervision, I have been able to write down the q-coefficients in the Fourier expansion of $\frac{(\vartheta_1^{(1)})^{n-1}\vartheta_1(x_1+x_2+\cdots+x_n)}{\vartheta_1(x_1)\vartheta_1(x_2)\cdots\vartheta_1(x_n)}$ in terms of the q-coefficients in the Fourier expansion of $\frac{(\vartheta_1^{(1)})^{n-1-j}\vartheta_1(x_1+x_2+\cdots+x_{n-j})}{\vartheta_1(x_1)\vartheta_1(x_2)\cdots\vartheta_1(x_{n-j})}$ and $\frac{(\vartheta_1^{(1)})^{n-1-j}\vartheta_1(x_1+x_2+\cdots+x_{n-1})}{\vartheta_1(x_1)\vartheta_1(x_2)\cdots\vartheta_1(x_{n-1})}\cot(x_1+\cdots+x_{n-1})$ for $j = 1, 2, \cdots, n-2$.

The primary purpose of this thesis has been to obtain this expansion with a view towards studying the formation of the coefficients as the number of variables is increased. In this way it is anticipated that some insight may be gained into the appropriate extension of Dwyer's identities.

Incidentally, as a check on the computations for the case of three variables, and as a matter of interest in itself, we write, following Gage, (5)

$$\frac{(\vartheta_1^{(1)})^2\,\vartheta_1(x_1+x_2+x_3)}{\vartheta_1(x_1)\,\vartheta_1(x_2)\,\vartheta_1(x_3)} = \frac{\vartheta_1^{(1)}\,\vartheta_1(x_1+x_2)}{\vartheta_1(x_1)\,\vartheta_1(x_2)}\;\frac{\vartheta_1^{(1)}\,\vartheta_1(x_1+x_2+x_3)}{\vartheta_1(x_1+x_2)\,\vartheta_1(x_3)}\;;$$

paraphrasing the results of the expansions obtained by the two methods, we obtain number-theoretic results involving the representation of N in terms of a quadratic form in four variables.

Functions of Several Complex Variables

Since the theory of functions of several complex variables is perhaps not as widely known as that of a single complex variable, we merely state here for the convenience of the reader, the definitions and theorems which are required to justify the analysis which follows. The definitions and theorems are taken from Osgood's Lehrbuch der Funktionentheorie, Volume II. (6)

Let S denote the space of n complex variables $z_1, \cdots, z_n$ where each z_k ranges over its own complex plane Z_k. A set of complex numbers $(z_1, z_2, \cdots, z_n)$ where z_k is a point of Z_k denotes a point of S. If T_k is a region of Z_k, then the totality of points $(z_1, z_2, \cdots, z_n)$ where z_k is a point of T_k is called a cylindrical region of S and this cylindrical region is denoted by $(T) = (T_1, T_2, \cdots, T_n)$.

Let $F(z_1, z_2, \cdots, z_n)$ be a function uniquely defined at each point of a cylindrical region (T). F is said to be analytic in $(T) = (T_1, T_2, \cdots, T_n)$ if at every point $(z_1^\circ, z_2^\circ, \cdots, z_n^\circ)$ of (T), F has a multiple power series expansion.

$$F(z_1, z_2, \cdots, z_n) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1 k_2 \cdots k_n} (z_1 - z_1^\circ)^{k_1} (z_2 - z_2^\circ)^{k_2} \cdots (z_n - z_n^\circ)^{k_n}$$

the expansion being valid in the cylindrical region $(C_1, C_2, \cdots, C_n)$ where C_k denotes the interior of the circle $|z_k - z_k^\circ| < R_k$ and C_k lies within T_k for $k = 1, 2, \cdots, n$. It is a consequence of this definition that F is continuous in $(z_1, z_2, \cdots, z_n)$ together in the cylindrical region (T) if it is analytic in (T). F is said to be analytic at a point of (T) if F is analytic in some neighborhood of the point. If $(a_1, a_2, \cdots, a_n)$ is a point of (T) at which F is analytic, then $F(a_1, a_2, \cdots, z_k, \cdots, a_n)$ considered as a function of z_k alone, is analytic (regular) in the sense of the theory of functions of a single complex variable.

<u>Hartog's</u> <u>Theorem</u>. Let $F(z_1, z_2, \cdots, z_n)$ be uniquely defined at each point of a cylindrical region $(T) = (T_1, T_2, \cdots, T_n)$ If for every point P of (T) it is true that when all the variables except one, say z_k, are held fixed at P, F is an analytic function of z_k in T_k for $k = 1, 2, \cdots, n$, then F is analytic in (T).

<u>Laurent's</u> <u>Theorem</u>. Let $(T) = (T_1, T_2, \cdots, T_n)$ be a cylindrical region where T_k is the annulus $\rho_k < |z_k - a_k| < P_k$ about a_k as center for $k = 1, 2, \cdots, n$. If $F(z_1, z_2, \cdots, z_n)$ is analytic in (T), then F has a Laurent expansion about the point $(a_1, a_2, \cdots, a_n)$, namely,

$$F(z_1, z_2, \cdots, z_n) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} c_{k_1 k_2 \cdots k_n} (z_1 - a_1)^{k_1} (z_2 - a_2)^{k_2} \cdots (z_n - a_n)^{k_n}$$

where

$$c_{k_1 k_2 \cdots k_n} = \frac{1}{(2\pi i)^n} \int_{\Gamma_1} \int_{\Gamma_2} \cdots \int_{\Gamma_n} \frac{F(t_1, t_2, \cdots, t_n)\, dt_1\, dt_2 \cdots dt_n}{(t_1 - a_1)^{k_1+1} (t_2 - a_2)^{k_2+1} \cdots (t_n - a_n)^{k_n+1}}$$

and Γ_k is a closed path, described in the positive sense, and lying within the annulus T_k.

Proof for the case of 2 complex variables.

Let (z_1, z_2) be a point of the cylindrical region $(T) = (T_1, T_2)$ where T_1 is the annulus $\rho_1 < |z_1 - a_1| < P_1$ and T_2 is the annulus $\rho_2 < |z_2 - a_2| < P_2$. Then $F(z_1, z_2)$ considered as a function of z_1 alone is analytic in T_1 for every fixed value of z_2 in T_2.

By Cauchy's Theorem

$$F(z_1, z_2) = \frac{1}{2\pi i}\int_{\Gamma_1'} \frac{F(t_1, z_2)}{(t_1 - z_1)}\,dt_1 - \frac{1}{2\pi i}\int_{\Gamma_1''} \frac{F(t_1, z_2)}{(t_1 - z_1)}\,dt_1$$

where Γ_1' and Γ_1'' are circles with centers at a_1 and radii r_1', and r_1'', respectively, with $\rho_1 < r_1'' < r_1' < P_1$ and Γ_1' and Γ_1'' are described in the positive sense.

Also,

$$F(t_1, z_2) = \frac{1}{2\pi i}\int_{\Gamma_2'} \frac{F(t_1, t_2)}{(t_2 - z_2)}\,dt_2 - \frac{1}{2\pi i}\int_{\Gamma_2''} \frac{F(t_1, t_2)}{(t_2 - z_2)}\,dt_2$$

where Γ_2' and Γ_2'' are circles with centers at a_2 and radii r_2' and r_2'' respectively, with $\rho_2 < r_2'' < r_2' < P_2$ and Γ_2' and Γ_2'' are described in the positive sense.

Hence

$$F(z_1, z_2) = \frac{1}{(2\pi i)^2}\left\{ \int_{\Gamma_1'}\int_{\Gamma_2'} \frac{F(t_1, t_2)\,dt_2\,dt_1}{(t_1 - z_1)(t_2 - z_2)} - \int_{\Gamma_1'}\int_{\Gamma_2''} \frac{F(t_1, t_2)\,dt_2\,dt_1}{(t_1 - z_1)(t_2 - z_2)} \right.$$

$$\left. - \int_{\Gamma_1''}\int_{\Gamma_2'} \frac{F(t_1, t_2)\,dt_2\,dt_1}{(t_1 - z_1)(t_2 - z_2)} + \int_{\Gamma_1''}\int_{\Gamma_2''} \frac{F(t_1, t_2)\,dt_2\,dt_1}{(t_1 - z_1)(t_2 - z_2)} \right\}$$

Just as in the case of Laurent's Theorem for a function of a single complex variable, we write

$$\frac{1}{t_i - z_i} = \frac{1}{(t_i - a_i)}\,\frac{1}{1 - \frac{(z_i - a_i)}{(t_i - a_i)}} = \sum_{n=0}^{\infty} \frac{(z_i - a_i)^n}{(t_i - a_i)^{n+1}}$$

in the integrals in which Γ_i' is a contour, $i = 1, 2$, and

$$\frac{1}{t_i - z_i} = -\frac{1}{(z_i - a_i)}\,\frac{1}{1 - \frac{(t_i - a_i)}{(z_i - a_i)}} = -\sum_{n=0}^{\infty} \frac{(t_i - a_i)^n}{(z_i - a_i)^{n+1}}$$

in the integrals in which Γ_i'' is a contour, $i = 1, 2$, and note that each of the series is absolutely convergent and

uniformly convergent with respect to the current variable t_i involved. Moreover, $|F(t_1, t_2)| \leq M$ where M is a constant independent of t_i for t_i on Γ_i' or Γ_i'', $i = 1, 2$.
Consequently, the order of summations and integrations can be interchanged and we can write

$$F(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (z_1 - a_1)^n (z_2 - a_2)^m \frac{1}{(2\pi i)^2} \int_{\Gamma_1'} \int_{\Gamma_2'} \frac{F(t_1, t_2)\, dt_2\, dt_1}{(t_1 - a_1)^{n+1} (t_2 - a_2)^{m+1}}$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (z_1 - a_1)^n (z_2 - a_2)^{-m-1} \frac{1}{(2\pi i)^2} \int_{\Gamma_1'} \int_{\Gamma_2''} \frac{F(t_1, t_2)(t_2 - a_2)^m\, dt_2\, dt_1}{(t_1 - a_1)^{n+1}}$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (z_1 - a_1)^{-n-1} (z_2 - a_2)^m \frac{1}{(2\pi i)^2} \int_{\Gamma_1''} \int_{\Gamma_2'} \frac{F(t_1, t_2)(t_1 - a_1)^n\, dt_2\, dt_1}{(t_2 - a_2)^{m+1}}$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (z_1 - a_1)^{-n-1} (z_2 - a_2)^{-m-1} \frac{1}{(2\pi i)^2} \int_{\Gamma_1''} \int_{\Gamma_2''} F(t_1, t_2)(t_1 - a_1)^n (t_2 - a_2)^m\, dt_2\, dt_1$$

Now the contours Γ_i' and Γ_i'' may be replaced by the same contour Γ_i where Γ_i is a circle of center a_i and radius R_i with $r_i'' \leq R_i \leq r_i'$. Then, defining

$$C_{k_1 k_2} = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{F(t_1, t_2)\, dt_2\, dt_1}{(t_1 - a_1)^{k_1+1} (t_2 - a_2)^{k_2+1}}$$

we have

$$F(z_1, z_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} C_{k_1 k_2} (z_1 - a_1)^{k_1} (z_2 - a_2)^{k_2}$$

The proof for the case of n variables is the obvious generalization of the one just given.

We shall be concerned with functions $F(z_1, z_2, \cdots, z_n)$ such that

$$F(z_1, z_2, \cdots, z_k + \pi, \cdots, z_n) = F(z_1, z_2, \cdots, z_n) \quad \text{for} \quad k = 1, 2, \cdots, n$$

and which are analytic in the cylindrical region $(T_n) = (T_1, T_2, \cdots, T_n)$ where T_k is defined by $|\mathfrak{Im}\, z_k| < \mathfrak{Im}\, \pi\tau \quad k = 1, 2, \cdots, n$;

τ is a complex constant with positive imaginary part appearing in the definition of

$$\vartheta_1(z, q) = \vartheta_1(z) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} \sin(2n+1)z \quad ; \quad q = e^{i\pi\tau}$$

With the generalization of Laurent's Theorem at our disposal, we can introduce the generalized Fourier expansion of $F(z_1, z_2, \cdots, z_n)$. For by setting $W_k = e^{2iz_k}$, $k = 1, 2, \cdots, n$ in $F(z_1, z_2, \cdots, z_n)$, F becomes a function $\bar{F}(W_1, W_2, \cdots, W_n)$ which is analytic in a region $(\bar{T}_n) = (\bar{T}_1, \bar{T}_2, \cdots, \bar{T}_n)$ where $\bar{T}_k$ is an annulus in the W_k plane. Hence, $\bar{F}$ has a Laurent expansion in $(\bar{T}_n)$ and this expansion in (T_n) becomes the Fourier expansion

$$F(z_1, z_2, \cdots, z_n) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} C_{k_1 k_2 \cdots k_n} e^{2i(k_1 z_1 + k_2 z_2 + \cdots + k_n z_n)}$$

where

$$C_{k_1 k_2 \cdots k_n} = \frac{1}{\pi^n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(z_1, z_2, \cdots, z_n) e^{-2i(k_1 z_1 + k_2 z_2 + \cdots + k_n z_n)} dz_1 \cdots dz_n.$$

III

The Function $\vartheta_1(z)$

The function $\vartheta_1(z)$ is a doubly periodic function of the second kind, and we enumerate here for the convenience of the reader those properties which it will be necessary to know in the work which follows. (7)

Define τ to be a complex number with positive imaginary part and define $q = e^{i\pi\tau}$

Then we define

3.1. $\vartheta_1(z) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \sin(2n+1)z$

$\vartheta_1(z)$ is an integral function of z. It can be shown that

3.2. $\vartheta_1(z) = 2Gq^{1/4}\sin z \prod_{n=1}^{\infty}(1-q^{2n}e^{2iz})\prod_{n=1}^{\infty}(1-q^{2n}e^{-2iz})$

where

$$G = \prod_{n=1}^{\infty}(1-q^{2n}) \quad ,$$

and also that

$$\vartheta_1^{(1)}(0) = \vartheta_1^{(1)} = 2G^3q^{1/4}$$

From (3.1.) it follows that

3.3. $\vartheta_1(z+\pi) = -\vartheta_1(z)$

and from (3.2.)

3.4. $\vartheta_1(z \pm n\pi\tau) = (-1)^n q^{-n^2} e^{\mp 2inz}\vartheta_1(z)$

3.5. $\lim_{z\to 0}\frac{z}{\vartheta_1(z)} = \frac{1}{\vartheta_1^{(1)}}$

For

$$\lim_{z\to 0}\frac{z}{\vartheta_1(z)} = \lim_{z\to 0}\frac{z}{z\vartheta_1^{(1)} + \frac{1}{6}z^3\vartheta_1^{(3)} + \cdots} = \frac{1}{\vartheta_1^{(1)}}$$

We call attention to the fact that $\vartheta_1(n\pi + m\pi\tau) = 0$ for any integral values of n and m, positive, negative, or zero; hence $\frac{1}{\vartheta_1(z)}$ has poles at $z = n\pi + m\pi\tau$ and these are simple poles in view of (3.3.), (3.4.), and (3.5.).

The Function $F(x, y)$

Define:

$$F(x,y) = \frac{\vartheta_1^{(1)}\,\vartheta_1(x+y)}{\vartheta_1(x)\,\vartheta_1(y)} - \cot x - \cot y \qquad \text{for} \quad x \neq 0 \quad y \neq 0$$

$$= \frac{\vartheta_1^{(1)}(y)}{\vartheta_1(y)} - \cot y \qquad \text{for} \quad x = 0 \quad y \neq 0$$

$$= \frac{\vartheta_1^{(1)}(x)}{\vartheta_1(x)} - \cot x \qquad \text{for} \quad x \neq 0 \quad y = 0$$

$$= 0 \qquad \text{for} \quad x = 0 \quad y = 0$$

Then

$$F(x+\pi, y) = F(x, y+\pi) = F(x,y) = -F(-x,-y)$$

By Hartog's Theorem, $F(x,y)$ is analytic in $(T)=(T_x, T_y)$ where T_x is the region $|\mathfrak{Im}\, x| < \mathfrak{Im}\,\pi\tau$ and T_y is the region $|\mathfrak{Im}\, y| < \mathfrak{Im}\,\pi\tau$. For $\vartheta_1(z)$ is an integral function of z with simple zeros at the points $z = n\pi + m\pi\tau$ where n and m are integers, positive, negative, or zero. Hence, for fixed x in T_x, $F(x,y)$ has at most a simple pole at $y = n\pi$ in T_y; by the periodicity of F, it suffices to analyze $F(x,y)$ at $y=0$.

The residue at $y = 0$ for $x \neq 0$ is by (3.5)

$$\lim_{y\to 0} y\, F(x,y) = \lim_{y \to 0} y\left[\frac{\vartheta_1^{(1)}\,\vartheta_1(x+y)}{\vartheta_1(x)\,\vartheta_1(y)} - \cot x - \cot y\right] = 0$$

and the residue at $y=0$ for $x=0$ is

$$\lim_{y\to 0} y\, F(0,y) = \lim_{y\to 0} y\left[\frac{\vartheta_1^{(1)}(y)}{\vartheta_1(y)} - \cot y\right] = 0$$

Hence, the apparent singularity at $y=0$ is removable and since we have defined $F(x,0) = \lim_{y\to 0} F(x,y)$, $F(x,y)$ is analytic in y for fixed x in T_x and y in T_y. By symmetry in x and y, $F(x,y)$ is analytic in x for fixed y in T_y and x in T_x. Accordingly, the hypotheses of Hartog's

Theorem are satisfied and $F(x,y)$ is analytic in x and y in (T).

It follows from our remarks on Fourier's expansion that

$$F(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{nm}\, e^{2i(nx+my)}$$

where

$$c_{nm} = \frac{1}{\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(x,y)\, e^{-2i(nx+my)}\, dx\, dy$$

Now $F(-x,-y) = -F(x,y)$;

hence,

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{nm}\left[\cos 2(nx+my) - i \sin 2(nx+my)\right]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{nm}\left[\cos 2(nx+my) + i \sin 2(nx+my)\right]$$

and

$$F(x,y) = i \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{nm} \sin 2(nx+my)$$

By substituting $-x$ for x and $-y$ for y in

$$c_{-n,-m} = \frac{1}{\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(x,y)\, e^{2i(nx+my)}\, dx\, dy$$

we have

$$c_{-n,-m} = -\frac{1}{\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(x,y)\, e^{-2i(nx+my)}\, dx\, dy$$

$$= -c_{nm}$$

Hence,

$$F(x,y) = 2i\left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[c_{nm} \sin 2(nx+my) + c_{n,-m} \sin 2(nx-my) \right] \right.$$

$$\left. + \sum_{n=1}^{\infty} \left[c_{n,0} \sin nx + c_{0,n} \sin ny \right] \right.$$

Consider now

$$c_{nm} = \frac{1}{\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(x,y) e^{-2i(nx+my)} \, dx \, dy$$

for $n \geq 1$ and define

$$G(y) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(x,y) e^{-2inx} \, dx$$

To evaluate $G(y)$, we consider the integral

$\int_{\Gamma_N} F(x,y) e^{-2inx} \, dx$ where Γ_N is the parallelogram with vertices at $-\frac{\pi}{2}$, $-\frac{\pi}{2} - \frac{2N+1}{2}\pi\tau$, $\frac{\pi}{2} - \frac{2N+1}{2}\pi\tau$, $\frac{\pi}{2}$, described in the positive sense. By the periodicity of $F(x,y)e^{-2inx}$

$$\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2} - \frac{2N+1}{2}\pi\tau} F(x,y) e^{-2inx} \, dx + \int_{\frac{\pi}{2} - \frac{2N+1}{2}\pi\tau}^{\frac{\pi}{2}} F(x,y) e^{-2inx} \, dx = 0$$

If we can show that

$$\lim_{N \to \infty} \int_{-\frac{\pi}{2} - \frac{2N+1}{2}\pi\tau}^{\frac{\pi}{2} - \frac{2N+1}{2}\pi\tau} F(x,y) e^{-2inx} \, dx = 0$$

then by Cauchy's Theorem we shall have that

$$G(y) = -2i \sum_{j=1}^{\infty} R_j$$

where R_j is the residue of $F(x,y)e^{-2inx}$ at $x = -j\pi\tau$.

We define

$$I_n = \int_{-\frac{\pi}{2}-\frac{2N+1}{2}\pi\tau}^{\frac{\pi}{2}-\frac{2N+1}{2}\pi\tau} F(x,y)e^{-2inx}\,dx$$

and make the substitution $x = x' - N\pi\tau$

Then

$$I_n = q^{2nN}\int_{-\frac{\pi}{2}-\frac{\pi\tau}{2}}^{\frac{\pi}{2}-\frac{\pi\tau}{2}} F(x'-N\pi\tau, y)e^{-2inx'}\,dx'$$

Since we are concerned only with $F(x,y)$ for x on the line from $-\frac{\pi}{2}-\frac{2N+1}{2}\pi\tau$ to $\frac{\pi}{2}-\frac{2N+1}{2}\pi\tau$ and y on the line from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we have $x \neq 0$ and accordingly,

$$F(x'-N\pi\tau, y) = \frac{\vartheta_1^{(1)}\vartheta_1(x'+y-N\pi\tau)}{\vartheta_1(x'-N\pi\tau)\vartheta_1(y)} - \cot(x'-N\pi\tau) - \cot y$$

and

$$F(x'-N\pi\tau, 0) = \frac{\vartheta_1^{(1)}(x'-N\pi\tau)}{\vartheta_1(x'-N\pi\tau)} - \cot(x'-N\pi\tau)$$

$$= \lim_{y\to 0} F(x'-N\pi\tau, y)$$

We write

$$F(x'-N\pi\tau, y) = e^{2iNy}\left[\frac{\vartheta_1^{(1)}\vartheta_1(x'+y)}{\vartheta_1(x')\vartheta_1(y)} - \cot y\right] - \cot(x'-N\pi\tau)$$

$$+\cot y\left[e^{2iNy} - 1\right]$$

$\frac{\vartheta_1^{(1)}\vartheta_1(x'+y)}{\vartheta_1(x')\vartheta_1(y)} - \cot y$ is analytic for the values of x' and y under consideration, the apparent pole at $y=0$ being removable; hence we can write.

$$\left|\frac{\vartheta_1^{(1)}\vartheta_1(x'+y)}{\vartheta_1(x')\vartheta_1(y)} - \cot y\right| \leq M$$

where M is a constant independent of N.

Also,

$$\left|\cot y\left[e^{2iNy}-1\right]\right| = O(N)$$

and

$$\left|\cot(x'-N\pi\tau)\right| \leq 2$$

for N sufficiently large and the values of x' and y under consideration. Hence,

$$|I_n| = |q^{2nN}|\,O(N)$$

and therefore

$$\lim_{N\to\infty} I_n = 0$$

Then, as noted above

$$G(y) = -2i\sum_{j=1}^{\infty} R_j$$

where

$$\begin{aligned} R_j &= \lim_{x\to -j\pi\tau}(x+j\pi\tau)\,F(x,y)\,e^{-2inx} \\ &= \lim_{x\to -j\pi\tau}(x+j\pi\tau)\,\frac{\vartheta_1'\,\vartheta_1(x+y)}{\vartheta_1(x)\,\vartheta_1(y)}\,e^{-2inx} \\ &= q^{2nj}\,e^{2ijy} \end{aligned}$$

Then

$$\begin{aligned} c_{nm} &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} G(y)\,e^{-2imy}\,dy \\ &= \frac{-2i}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{j=1}^{\infty} q^{2nj}\,e^{2i(j-m)y}\,dy \end{aligned}$$

The series being uniformly convergent, the term-by-term integration shows

$$c_{nm} = \left\{\begin{matrix} 0 & \text{if } m \leq 0 \\ -2i\,q^{2nm} & \text{if } m \geq 1 \end{matrix}\right\} \quad \text{if } n \geq 1$$

By symmetry in x and y, we have

$$c_{0m} = c_{m0} = 0$$

Accordingly,

$$F(x,y) = 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q^{2nm} \sin 2(nx+my)$$

We note that

$$\lim_{\substack{y\to 0 \\ x\neq 0}} F(x,y) = F(x,0) = 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin 2nx = \frac{\vartheta_1^{(1)}(x)}{\vartheta_1(x)} - \cot x$$

$$\lim_{\substack{x\to 0 \\ y\neq 0}} F(x,y) = F(0,y) = 4 \sum_{m=1}^{\infty} \frac{q^{2m}}{1-q^{2m}} \sin 2my = \frac{\vartheta_1^{(1)}(y)}{\vartheta_1(y)} - \cot y$$

$$\lim_{x\to 0} \lim_{y\to 0} F(x,y) = F(0,0) = 0$$

The expansion for $\frac{\vartheta_1^{(1)}(x)}{\vartheta_1(x)} - \cot x$ may be obtained directly by the infinite parallelogram method used in evaluating $G(y)$.

The Function $\bar{F}(x,y)$

We define:

$$\bar{F}(x,y) = \frac{\vartheta_1^{(1)}\vartheta_1(x+y)}{\vartheta_1(x)\vartheta_1(y)}\cot(x+y) - \cot x \cot y \qquad \text{for } x \neq 0,\ y \neq 0,\ x+y \neq 0$$

$$= \frac{\vartheta_1^{(1)}(y)}{\vartheta_1(y)}\cot y - \csc^2 y \qquad \text{for } x = 0 \quad y \neq 0$$

$$= \frac{\vartheta_1^{(1)}(x)}{\vartheta_1(x)}\cot x - \csc^2 x \qquad \text{for } x \neq 0 \quad y = 0$$

$$= \frac{-(\vartheta_1^{(1)})^2}{\vartheta_1^2(x)} + \cot^2 x \qquad \text{for } x+y = 0 \quad x \neq 0$$

$$= \frac{1}{3}\frac{\vartheta_1^{(3)}}{\vartheta_1^{(1)}} - \frac{2}{3} \qquad \text{for } x = y = 0$$

Then $\bar{F}(x+\pi, y) = \bar{F}(x, y+\pi) = \bar{F}(x,y)$ and $\bar{F}(x,y)$ is analytic in $(T) = (T_x, T_y)$ where T_x, T_y are the strips $|\Im x| < \Im \pi\tau$, $|\Im y| < \Im \pi\tau$, respectively. The argument is essentially the same as in the case of $\frac{\vartheta_1^{(1)}\vartheta_1(x+y)}{\vartheta_1(x)\vartheta_1(y)} - \cot x - \cot y$; we should perhaps remark that the apparent poles of order two at $x = y = 0$ are removable as are all the apparent simple poles.

In view of the periodicity and the parity of $\bar{F}(x,y)$, namely, $\bar{F}(-x,-y) = \bar{F}(x,y)$, the Fourier expansion is given by

$$\bar{F}(x,y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_{nm} \cos 2(nx+my)$$

where

$$c_{nm} = \frac{1}{\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{F}(x,y)e^{-2i(nx+my)}\,dx\,dy$$

Moreover, $c_{-n,-m} = c_{nm}$, and hence,

$$\bar{F}(x,y) = c_{00} + 2\sum_{n=1}^{\infty}[c_{n,0}\cos 2nx + c_{0,n}\cos 2ny]$$
$$+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[c_{nm}\cos 2(nx+my) + c_{n,-m}\cos 2(nx-my)]$$

The calculation of c_{nm} for $n \geq 1$ is essentially the same as that for c_{nm} in the case of the function $F(x,y)$ defined in the preceding chapter.

We define, for $n \geq 1$

5.1 $$\bar{f}_n(y) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{F}(x,y)e^{-2inx}\,dx$$

and evaluate $\bar{f}_n(y)$ by the infinite parallelogram method. The result is

$$\bar{f}_n(y) = -2i\sum_{j=1}^{\infty} R_j$$

where

$$R_j = \lim_{x\to -j\pi\tau}(x+j\pi\tau)\bar{F}(x,y)e^{-2inx} = q^{2nj}e^{2ijy}\cot(y-j\pi\tau)$$

Now

$$\cot(y-j\pi\tau) = i\left[1+2\sum_{r=1}^{\infty}q^{2rj}e^{-2iry}\right]$$

hence

5.2 $$\bar{f}_n(y) = 2\sum_{r=1}^{\infty}q^{2nr}e^{2iry} + 4\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}q^{2r(n+s)}e^{2iy(r-s)}$$

Then

$$c_{nm} = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\bar{f}_n(y)e^{-2imy}\,dy$$

The series are absolutely and uniformly convergent over the range of integration and integrating term-by-term, we find

$$\underset{n\geq 1}{c_{nm}} = \begin{cases} 2q^{2nm} & \text{if } m \geq 1 \\ 0 & \text{if } m \leq 0 \end{cases} + \begin{cases} 4\sum_{j=m+1}^{\infty} q^{2j(n-m+j)} & \text{if } m \geq 0 \\ 4\sum_{j=1}^{\infty} q^{2j(n-m+j)} & \text{if } m \leq 0 \end{cases}$$

The last result may be written

$$c_{nm} = 4\sum_{j=1}^{\infty} q^{2j^2+2j(n-m)} \quad \text{if } n \geq 1 \text{ and } m \leq 0$$

$$c_{nm} = 2q^{2nm} + 4\sum_{j=1}^{\infty} q^{2j^2+2j(m+n)+2mn} \quad \text{if } n \geq 1 \text{ and } m \geq 1.$$

By symmetry, we find

$$c_{nm} = c_{mn}$$

and in particular

$$c_{on} = c_{no}$$

Hence

$$c_{on} = 4\sum_{j=1}^{\infty} q^{2j^2+2jn} \quad \text{for } n \geq 1$$

It remains to calculate c_{oo}. To do this, we write

$$\frac{\vartheta_1^{(1)}\vartheta_1(x+y)}{\vartheta_1(x)\,\vartheta_1(y)}\cot(x+y) - \cot x \cot y = -1 + 4\cot(x+y)\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2rs}\sin 2(rx+sy)$$

since

$$\frac{\vartheta_1^{(1)}\vartheta_1(x+y)}{\vartheta_1(x)\,\vartheta_1(y)} = \cot x + \cot y + 4\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2rs}\sin 2(rx+sy)$$

Hence,

$$c_{oo} = -1 + \frac{4}{\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cot(x+y)\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2rs}\sin 2(rx+sy)\,dx\,dy$$

Define

$$K(x,y) = \cot(x+y)\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2rs}\sin 2(rx+sy)$$

Then $K(x,y)$ is analytic in x and y in $(T) = (T_x, T_y)$ as already defined. For $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, the contour $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ may be deformed into the other three sides of the parallelogram with vertices $-\frac{\pi}{2}$, $-\frac{\pi}{2}-\frac{\pi\tau}{2}$, $\frac{\pi}{2}-\frac{\pi\tau}{2}$, $\frac{\pi}{2}$ without encountering any singularities of $K(x,y)$. By periodicity

$$\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}-\frac{\pi\tau}{2}} K(x,y)\,dx + \int_{\frac{\pi}{2}-\frac{\pi\tau}{2}}^{\frac{\pi}{2}} K(x,y)\,dx = 0$$

Hence

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K(x,y)\,dx = \int_{-\frac{\pi}{2}-\frac{\pi\tau}{2}}^{\frac{\pi}{2}-\frac{\pi\tau}{2}} K(x,y)\,dx$$

and then by the transformation

$$x = x' - \frac{\pi\tau}{2}$$

we find

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K(x,y)\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K(x'-\tfrac{\pi\tau}{2}, y)\,dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cot(x'+y-\tfrac{\pi\tau}{2}) \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2rs} \sin 2\left[r(x'-\tfrac{\pi\tau}{2}) + sy\right] dx'$$

Now

$$\cot(x'+y-\tfrac{\pi\tau}{2}) = i\left[1 + 2\sum_{k=1}^{\infty} q^{k} e^{-2ik(x'+y)}\right]$$

Therefore,

$$C_{oo} = -1 + 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K(x, y)\, dx\, dy$$

$$= -1 + 4i \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} q^{2rs} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2\left[r\left(x' - \frac{\pi\tau}{2}\right) + sy\right] dx'\, dy$$

$$+ 8i \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} q^{2rs+k} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 2\left[r\left(x' - \frac{\pi\tau}{2}\right) + sy\right] e^{-2ik(x'+y)}\, dx'\, dy$$

$$= -1 + 4 \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} q^{2rs+k-r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2i(r-k)x'}\, e^{2i(s-k)y}\, dx'\, dy$$

$$= -1 + 4 \sum_{k=1}^{\infty} q^{2k^2}$$

Finally then we have shown that

5.3 $$\bar{F}(x, y) = -1 + 4 \sum_{k=1}^{\infty} q^{2k^2}$$

$$+ 8 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} q^{2k^2+2kn} \left[\cos 2nx + \cos 2ny\right]$$

$$+ 8 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} q^{2k^2+2k(n+m)} \cos 2(nx - my)$$

$$+ 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[q^{2nm} + 2 \sum_{k=1}^{\infty} q^{2k^2+2k(m+n)+2mn}\right] \cos 2(nx + my)$$

This result may also be obtained by multiplying $\cot(x+y)$ into $\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} q^{2rs} \sin 2(rx + sy)$ and writing $\cot(x+y) \sin 2(rx + sy)$ in terms of cosines.

The Function $F(x,y,z)$

Define:

$$F(x,y,z) = \frac{(\vartheta_1^{(1)})^2 \vartheta_1(x+y+z)}{\vartheta_1(x)\,\vartheta_1(y)\,\vartheta_1(z)} + \cot x \cot y + \cot x \cot z + \cot y \cot z$$

$$- \frac{\vartheta_1^{(1)} \vartheta_1(x+y)}{\vartheta_1(x)\,\vartheta_1(y)} \cot z - \frac{\vartheta_1^{(1)} \vartheta_1(x+z)}{\vartheta_1(x)\,\vartheta_1(z)} \cot y - \frac{\vartheta_1^{(1)} \vartheta_1(y+z)}{\vartheta_1(y)\,\vartheta_1(z)} \cot x$$

for $x \neq 0 \quad y \neq 0 \quad z \neq 0$

$$= \frac{\vartheta_1^{(1)} \vartheta_1^{(1)}(y+z)}{\vartheta_1(y)\,\vartheta_1(z)} - \frac{\vartheta_1^{(1)}(y)}{\vartheta_1(y)} \cot z - \frac{\vartheta_1^{(1)}(z)}{\vartheta_1(z)} \cot y + \cot y \cot z$$

for $x = 0 \quad y \neq 0 \quad z \neq 0$

$$= \frac{\vartheta_1^{(1)} \vartheta_1^{(1)}(x+z)}{\vartheta_1(x)\,\vartheta_1(z)} - \frac{\vartheta_1^{(1)}(x)}{\vartheta_1(x)} \cot z - \frac{\vartheta_1^{(1)}(z)}{\vartheta_1(z)} \cot x + \cot x \cot z$$

for $x \neq 0 \quad y = 0 \quad z \neq 0$

$$= \frac{\vartheta_1^{(1)} \vartheta_1(x+y)}{\vartheta_1(x)\,\vartheta_1(y)} - \frac{\vartheta_1^{(1)}(x)}{\vartheta_1(x)} \cot y - \frac{\vartheta_1^{(1)}(y)}{\vartheta_1(y)} \cot x + \cot x \cot y$$

for $x \neq 0 \quad y \neq 0 \quad z = 0$

$$= \frac{\vartheta_1^{(2)}(z)}{\vartheta_1(z)} \quad \text{for} \quad x = y = 0 \quad z \neq 0$$

$$= \frac{\vartheta_1^{(2)}(y)}{\vartheta_1(y)} \quad \text{for} \quad x = z = 0 \quad y \neq 0$$

$$= \frac{\vartheta_1^{(2)}(x)}{\vartheta_1(x)} \quad \text{for} \quad y = z = 0 \quad x \neq 0$$

$$= \frac{\vartheta_1^{(3)}}{\vartheta_1^{(1)}} = -1 + 24 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} \quad \text{for} \quad x = y = z = 0$$

Then

$$F(x+\pi, y, z) = F(x, y+\pi, z) = F(x, y, z+\pi) = F(x, y, z) = F(-x, -y, -z)$$

By Hartog's Theorem, $F(x, y, z)$ is analytic in $(T) = (T_x, T_y, T_z)$ where T_α $(\alpha = x, y, z)$ is defined by $|\mathrm{Im}\,\alpha| < \mathrm{Im}\,\pi\tau$. The argument as before is that for fixed y, z, $F(x, y, z)$ has at most a simple pole at $x = 0$ and the residue at $x = 0$ is zero, hence $F(x, y, z)$ is analytic in x in T_x since by definition $\lim\limits_{x \to 0} F(x, y, z) = F(0, y, z)$. By symmetry in x, y, z, $F(x, y, z)$ is analytic in x, y, z in (T).

Having in mind to define a similar function of n variables involving $\dfrac{(\vartheta_1^{(1)})^{n-1}\,\vartheta_1(x_1 + x_2 + \cdots + x_n)}{\vartheta_1(x_1)\,\vartheta_1(x_2)\cdots\vartheta_1(x_n)}$, it is clear that we shall need a more compact notation. We introduce here a portion of the notation to be used later, partly to familiarize the reader with it, but mostly to reduce the writing, already grown cumbersome. In this direction, we begin by abbreviating the definition of $F(x, y, z)$ to that for $x \neq 0$, $y \neq 0$, $z \neq 0$, the remainder of the definition of F being merely the appropriate limits as one or more of the variables involved go to zero.

Now define:

$$[x\,y\,z] = \frac{(\vartheta_1^{(1)})^2\,\vartheta_1(x+y+z)}{\vartheta_1(x)\,\vartheta_1(y)\,\vartheta_1(z)} \qquad [x\,y] = \frac{\vartheta_1^{(1)}\,\vartheta_1(x+y)}{\vartheta_1(x)\,\vartheta_1(y)} \qquad [x] \equiv 1$$

$$(x) = \cot x \qquad (x\,y) = \cot x \cot y \qquad (x+y) = \cot(x+y)$$

$$S(x,y) = 4\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2rs} \sin 2(rx + sy) = [x\,y] - (x) - (y)$$

$$\bar{S}(x,y) = -1 + (x+y)\,S(x,y) = [x\,y](x+y) - (x\,y)$$

The use of the parentheses to denote the cotangent will not be ambiguous since we shall avoid their use in any other connection except: after a functional symbol F , ϑ_1 etc.; in the designation of a cylindrical region $(T_n) = (T_1, T_2, \cdots, T_n)$; in $(\vartheta_1^{(1)})^n$, $(-1)^n$.

We shall use $\sum$ without subscripts to denote the symmetric function on x, y, z of the argument which follows $\sum$; for example

$$\sum [xy](z) = [xy](z) + [xz](y) + [yz](x)$$

In this notation we abbreviate the statement

$$\frac{(\vartheta_1^{(1)})^2 \vartheta_1(x+y+z)}{\vartheta_1(x)\vartheta_1(y)\vartheta_1(z)} = \frac{\vartheta_1^{(1)} \vartheta_1(y+z)}{\vartheta_1(y)\vartheta_1(z)} \frac{\vartheta_1^{(1)} \vartheta_1(x+y+z)}{\vartheta_1(x)\vartheta_1(y+z)}$$

to

$$[xyz] = [yz][x, y+z]$$

Then

$$\begin{aligned} F(x,y,z) &= [xyz] - \sum [xy](z) + \sum [x](yz) \\ &= [yz]\{(x) + (y+z) + S(x,y+z)\} - [yz](x) \\ &\quad - \{(x) + (y) + S(x,y)\}(z) - \{(x) + (z) + S(x,z)\}(y) \\ &\quad + (xy) + (xz) + (yz) \\ &= [yz](y+z) - (yz) \\ &\quad + \{[yz] - (y) - (z)\} S(x,y+z) \\ &\quad + (y)\{S(x,y+z) - S(x,z)\} \\ &\quad + (z)\{S(x,y+z) - S(x,y)\} \end{aligned}$$

We note that

$$[yz](y+z)-(yz) = \frac{\vartheta_1^{(1)}\vartheta_1(y+z)}{\vartheta_1(y)\,\vartheta_1(z)}\cot(y+z) - \cot y \cot z = \bar{S}(y,z)$$

is just the function $\bar{F}(y,z)$ defined in the preceding chapter and

$$[yz] - (y) - (z) = \frac{\vartheta_1^{(1)}\vartheta_1(y+z)}{\vartheta_1(y)\,\vartheta_1(z)} - \cot y - \cot z = S(y,z)$$

is just the function $F(y,z)$ defined in Chapter IV; we will ordinarily denote the Fourier expansions of F and $\bar{F}$ by S and $\bar{S}$, respectively. Thus,

$$F(x,y,z) = \bar{F}(y,z) + F(y,z)\,S(x,y+z)$$
$$+(y)\{S(x,y+z) - S(x,z)\} + (z)\{S(x,y+z) - S(x,y)\}$$

Now $\bar{F}(y,z)$, $F(y,z)$, $S(x,y+z)$, $S(x,y)$, $S(x,z)$ are all analytic for real x, y, z ; so are $(y)\{S(x,y+z)-S(x,z)\}$ and $(z)\{S(x,y+z)-S(x,y)\}$, since they have at most a simple pole at $y=0$ $z=0$ respectively and the residue there is zero, e.g.,

$$\lim_{y\to 0} y\,(y)\{S(x,y+z) - S(x,z)\} = 0$$

Hence, we have written $F(x,y,z)$ as the sum of four functions, all analytic for real x, y, z and this form will be useful in determining the Fourier expansion of $F(x,y,z)$ in (T), namely,

$$F(x,y,z) = \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} C_{lmn} \cos 2(lx+my+nz)$$

where

$$C_{lmn} = \frac{1}{\pi^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(x,y,z)\,e^{-2i(lx+my+nz)}\,dx\,dy\,dz$$

From the definition of $c_{\ell mn}$ and the symmetry of $F(x,y,z)$ in x,y,z, we have

$$c_{\ell mn} = c_{\ell nm} = c_{m\ell n} = c_{mn\ell} = c_{n\ell m} = c_{nm\ell}$$

and from the parity of $F(x,y,z)$ we have

$$c_{\ell mn} = c_{-\ell,-m,-n};$$

hence

$$F(x,y,z) = c_{000} + 2\sum_{\ell=1}^{\infty} c_{\ell 00}\left[\cos 2\ell x + \cos 2\ell y + \cos 2\ell z\right]$$

$$+2\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} c_{\ell m0}\left[\cos 2(\ell x+my) + \cos 2(\ell x+mz) + \cos 2(\ell y+mz)\right]$$

$$+2\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} c_{\ell,-m,0}\left[\cos 2(\ell x-my) + \cos 2(\ell x-mz) + \cos 2(\ell y-mz)\right]$$

$$+2\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[c_{\ell mn}\cos 2(\ell x+my+nz) + c_{\ell,m,-n}\cos 2(\ell x+my-nz)\right.$$

$$\left.+ c_{\ell,-m,n}\cos 2(\ell x-my+nz) + c_{\ell,-m,-n}\cos 2(\ell x-my-nz)\right]$$

Hence we need only compute $c_{\ell mn}$ for $\ell \geq 1$ and c_{000}. To this end, we consider

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(x,y,z)e^{-2i\ell x}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{F}(y,z)e^{-2i\ell x}dx$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(y,z)\,S(x,y+z)e^{-2i\ell x}dx$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (y)\left\{S(x,y+z) - S(x,z)\right\}e^{-2i\ell x}dx$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (z)\left\{S(x,y+z) - S(x,y)\right\}e^{-2i\ell x}dx$$

The first integral is zero if $l \neq 0$; the last three are zero if $l = 0$ as is evident from term-by-term integration of $S(x, y+z)$, $S(x,y)$ and $S(x, z)$; the series are all absolutely and uniformly convergent for $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, $\alpha = x, y, z$, so that the term-by-term integration is valid. Thus,

$$\underset{l \geq 1}{C_{lmn}} = \frac{1}{\pi^3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(y,z)\, S(x,y+z) e^{-2i(lx+my+nz)}\, dx\, dy\, dz$$

$$+ \frac{1}{\pi^3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (y) \left\{ S(x,y+z) - S(x,z) \right\} e^{-2i(lx+my+nz)}\, dx\, dy\, dz$$

$$+ \frac{1}{\pi^3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (z) \left\{ S(x,y+z) - S(x,y) \right\} e^{-2i(lx+my+nz)}\, dx\, dy\, dz$$

and

$$C_{omn} = \frac{1}{\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{F}(y,z) e^{-2i(my+nz)}\, dy\, dz$$

In particular,

$$C_{ooo} = -1 + 4 \sum_{k=1}^{\infty} q^{2k^2}$$

To carry out the evaluation of $c_{\ell m n}$ for $\ell \geq 1$ we define

6.1 $$f_\ell(u) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S(x,u)\, e^{-2i\ell x}\, dx$$

$$= \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} q^{2rs} \sin 2(rx+su)\, e^{-2i\ell x}\, dx$$

$$= \frac{-2i}{\pi} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} q^{2rs} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ e^{2i(r-\ell)x + 2isu} - e^{-2i(r+\ell)x - 2isu} \right\} dx$$

$$= -2i \sum_{s=1}^{\infty} q^{2\ell s} e^{2isu}$$

$$= \frac{-2i\, q^{2\ell} e^{2iu}}{1 - q^{2\ell} e^{2iu}}$$

Then

$$\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (y) \left\{ S(x, y+z) - S(x,z) \right\} e^{-2i\ell x}\, dx$$

$$= (y) \left\{ f_\ell(y+z) - f_\ell(z) \right\}$$

$$= -2i \sum_{s=1}^{\infty} q^{2\ell s} e^{2isz}\, (y) \left\{ e^{2isy} - 1 \right\}$$

Hence, for $\ell \geq 1$,

$$c_{\ell m n} = \frac{-2i}{\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(y,z) \sum_{s=1}^{\infty} q^{2\ell s} e^{2is(y+z)} e^{-2i(my+nz)} \, dy \, dz$$

$$- \frac{2i}{\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{s=1}^{\infty} q^{2\ell s} (y) \left\{ e^{2isy} - 1 \right\} e^{-2imy} e^{2i(s-n)z} \, dy \, dz$$

$$- \frac{2i}{\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{s=1}^{\infty} q^{2\ell s} (z) \left\{ e^{2isz} - 1 \right\} e^{-2inz} e^{2i(s-m)y} \, dy \, dz$$

Now define

$$C_{rs} = \frac{1}{\pi^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(y,z) \, e^{-2i(ry+sz)} \, dy \, dz$$

and

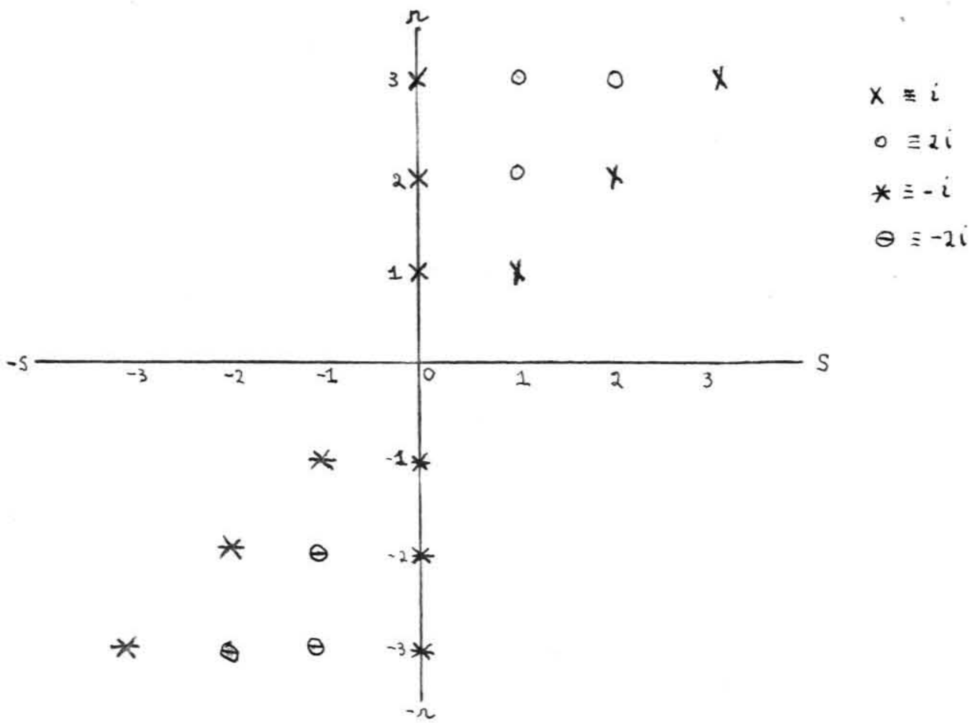
$$K_{rs} = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (y) \left\{ e^{2iry} - 1 \right\} e^{-2isy} \, dy$$

Then

6.2 $$c_{rs} = \begin{cases} -2i \, q^{2rs} & \text{if } r \geq 1 \text{ and } s \geq 1 \\ 2i \, q^{2rs} & \text{if } r \leq -1 \text{ and } s \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

6.3 $$K_{rs} = \begin{cases} i \quad \text{if } s = 0 \text{ and } r \geq 1 & -i \quad \text{if } s = 0 \text{ and } r \leq -1 \\ 2i \quad \text{if } r-1 \geq s \geq 1 & -2i \quad \text{if } r+1 \leq s \leq -1 \\ i \quad \text{if } r = s \geq 1 & -i \quad \text{if } r = s \leq -1 \\ 0 \quad \text{otherwise} & \end{cases}$$

The definition of K_{rs} is visualized quickly by the diagram below.



Then in terms of C_{rs} and K_{rs} we may write for $\ell \geq 1$

$$C_{\ell m n} = -2i\left\{ \sum_{s=1}^{\infty} q^{2\ell s}\left[K_{sm}\,\delta_s^n + K_{sn}\,\delta_s^m + C_{(m-s)\,(n-s)}\right]\right\}$$

where

$$\delta_j^i = \left\{ \begin{array}{ll} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{array} \right\}$$

6.4 $$F(x,y,z) = -1 + 4\sum_{k=1}^{\infty} q^{2k^2}$$

$$+2\sum_{l=1}^{\infty}\left\{4\sum_{k=1}^{\infty} q^{2k^2+2lk}\right\}\left\{\cos 2lx + \cos 2ly + \cos 2lz\right\}$$

$$+2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left\{2q^{2lm} + 4\sum_{k=1}^{\infty} q^{2k^2+2k(l+m)+2lm}\right\}\left\{\cos 2(lx+my) + \cos 2(lx+mz) + \cos 2(ly+mz)\right\}$$

$$+2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left\{4\sum_{k=1}^{\infty} q^{2k^2+2k(l+m)}\right\}\left\{\cos 2(lx-my) + \cos 2(lx-mz) + \cos 2(ly-mz)\right\}$$

$$+2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{4\sum_{k=1}^{\infty} q^{2k^2+2k(l+m+n)+2mn}\right\}\left\{\cos 2(lx-my-nz) + \cos 2(mx-ly-nz) + \cos 2(nx-my-lz)\right\}$$

$$+2\sum_{l=1}^{\infty}\left\{4q^{2l} + 4\sum_{k=1}^{\infty} q^{2k^2+2kl+2l}\right\}\cos 2(lx+y+z)$$

$$+2\sum_{l=1}^{\infty}\sum_{m=2}^{\infty}\left\{4q^{2lm} - 4\sum_{k=1}^{m-1} q^{2k^2+2k(l-2m)+2m^2} + 4\sum_{k=1}^{\infty} q^{2k^2+2kl+2lm}\right\}\cos 2(lx+my+mz)$$

$$+2\sum_{l=1}^{\infty}\sum_{m=2}^{\infty}\left\{4q^{2lm} + 4\sum_{k=1}^{\infty} q^{2k^2+2k(l+m-1)+2lm}\right\}\left\{\cos 2(lx+my+z) + \cos 2(lx+y+mz)\right\}$$

$$+2\sum_{l=1}^{\infty}\sum_{\substack{m=3 \\ m>n}}^{\infty}\sum_{n=2}^{\infty}\left\{4q^{2lm} - 4\sum_{k=1}^{n-1} q^{2k^2+2k(l-m-n)+2mn} + 4\sum_{k=1}^{\infty} q^{2k^2+2k(l+m-n)+2lm}\right\}\left\{\cos 2(lx+my+nz) + \cos 2(lx+ny+mz)\right\}$$

The Functions $F(x_1, x_2, \cdots, x_n)$ and $\bar{F}(x_1, x_2, \cdots, x_n)$

Because of the length of this chapter, we present a brief outline of the developments which it contains.

1. Notation.

The functions F and $\bar{F}$ to be considered are symmetric in each pair of the variables $x_1, x_2, \cdots, x_n$; accordingly, we introduce a symmetric function notation slightly more general than that ordinarily used in the treatment of rational symmetric functions. But before doing this it is necessary to introduce a notation which eliminates needless writing of "cot",

$$\frac{(\vartheta_1^{(1)})^{n-1}\,\vartheta_1(x_1 + x_2 + \cdots + x_n)}{\vartheta_1(x_1)\,\vartheta_1(x_2)\cdots\vartheta_1(x_n)}$$

and

$$S(x_n, x_1 + x_2 + \cdots + x_{n-1})$$

2. The Definition of $F(x_1, x_2, \cdots, x_n)$

The definition of F is necessarily lengthy due to the fact that it has apparent singularities whenever $x_j = 0$, $j = 1, 2, \cdots, n$. We give the definition of F at all these apparent singularities and apply Hartog's Theorem to show that it is analytic in a certain cylindrical region (T_n).

3. The Definition of $\bar{F}(x_1, x_2, \cdots, x_n)$

The definition of $\bar{F}$ is slightly more complicated than that of F since $\bar{F}$ has apparent singularities at the points $x_1 + x_2 + \cdots + x_{n-j} = 0$ for $j = 1, 2, \cdots, n-1$ We define $\bar{F}$ so that it too is analytic in (T_n).

4. Another Representation of $F(x_1, x_2, \cdots, x_n)$

   We express $F(x_1, x_2, \cdots, x_n)$ in terms of $\bar{F}(x_1, x_2, \cdots, x_{n-1})$ and $F(x_1, x_2, \cdots, x_{n-j})$ for $j = 1, 2, \cdots, n-2$ and $n \geq 3$.

5. Another Representation of $\bar{F}(x_1, x_2, \cdots, x_n)$

   We express $\bar{F}(x_1, x_2, \cdots, x_n)$ in terms of $F(x_1, x_2, \cdots, x_{n-j})$ and $\bar{F}(x_n, x_1 + x_2 + \cdots + x_{n-j})$ for $j = 1, 2, \cdots, n-2$ and $n \geq 3$.

6. The Fourier Coefficients of $F(x_1, x_2, \cdots, x_n)$

   We show how to compute the Fourier coefficients of $F(x_1, x_2, \cdots, x_n)$ using the representation given in section 4.

7. The Fourier Coefficients of $\bar{F}(x_1, x_2, \cdots, x_n)$

   We show how to compute the Fourier coefficients of $\bar{F}(x_1, x_2, \cdots, x_n)$ using the representation given in section 5.

It is hoped that this outline will guide the reader through the extensive calculations which must be made in this chapter.

1. Notation

We present here, in its generality, the notation which we introduced in part in the preceding chapter.

Define:

7.1.1 $$[x_1 x_2 \cdots x_n] = \frac{\left(\vartheta_1^{(1)}\right)^{n-1} \vartheta_1(x_1+x_2+\cdots+x_n)}{\vartheta_1(x_1)\,\vartheta_1(x_2)\cdots\vartheta_1(x_n)} \qquad n = 2, 3, \cdots$$

7.1.2 $$[x_j] \equiv 1 \qquad j = 1, 2, \cdots, n$$

$$\vartheta_1^{(j)}(x) = \frac{d^j \vartheta_1(x)}{dx^j}$$

$$\vartheta_1^{(j)} = \vartheta_1^{(j)}(0)$$

7.1.3 $$[x_1 x_2 \cdots x_n]^{(j)} = \frac{\left(\vartheta_1^{(1)}\right)^{n-1} \vartheta_1^{(j)}(x_1+x_2+\cdots+x_n)}{\vartheta_1(x_1)\,\vartheta_1(x_2)\cdots\vartheta_1(x_n)}$$

7.1.4 $$c_j = \frac{\vartheta_1^{(j)}}{\vartheta_1^{(1)}}$$

Note that

$$c_1 = 1$$

$$c_{2r} = 0 \qquad r = 1, 2, \cdots$$

7.1.5 $$(x_1 x_2 \cdots x_n) = \cot x_1 \cot x_2 \cdots \cot x_n$$

7.1.6 $$(x_1 + x_2 + \cdots + x_n) = \cot(x_1 + x_2 + \cdots + x_n)$$

7.1.7 $$(x)^{(l)} = \frac{d^l \cot x}{dx^l}$$

7.1.8 $$(x_1 + x_2 + \cdots + x_n)^{(l)} = \left.\frac{d^l \cot u}{du^l}\right]_{u = x_1+x_2+\cdots+x_n}$$

Define:

7.1.9 $$f(u) = \begin{cases} \cot u \, \vartheta_1(u) & u \neq 0 \\ \vartheta_1^{(1)} & u = 0 \end{cases}$$

Then $f(u)$ is analytic at $u = 0$ and $f(-u) = f(u)$

Define:

7.1.10 $$f^{(k)}(0) = \lim_{u \to 0} \frac{d^k \cot u \, \vartheta_1(u)}{du^k} = \lim_{u \to 0} f^{(k)}(u)$$

7.1.11 $$\gamma_k = \frac{f^{(k)}(0)}{\vartheta_1^{(1)}}$$

Note that

7.1.12 $$\lim_{u \to 0} \left\{ \frac{f^{(k)}(u)}{\vartheta_1(u)} - \gamma_k \cot u \right\} = \gamma_{k+1}$$

For if k is odd, $f^{(k)}(0) = 0$ and

$f^{(k)}(u) = u f^{(k+1)}(0) + O(u^3)$; hence by (3.5),

$$\lim_{u \to 0} \frac{f^{(k)}(u)}{\vartheta_1(u)} = \frac{f^{(k+1)}(0)}{\vartheta_1^{(1)}} = \gamma_{k+1} \;;$$

if k is even, $f^{(k)}(u) = f^{(k)}(0) + O(u^2)$; hence

$$\lim_{u \to 0} \left\{ \frac{f^{(k)}(u)}{\vartheta_1(u)} - \gamma_k \cot u \right\} = 0 = \gamma_{k+1}$$

Define:

7.1.13 $$S(x_1, x_2) = 4 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} q^{2rs} \sin 2(r x_1 + s x_2) = [x_1 x_2] - (x_1) - (x_2)$$

7.1.14 $$\bar{S}(x_1, x_2) = -1 + (x_1 + x_2) S(x_1, x_2)$$

Note that

$$S(x_1, x_2) = S(x_2, x_1) \quad ,$$

$$\bar{S}(x_1, x_2) = \bar{S}(x_2, x_1),$$

and

7.1.15 $\quad \bar{S}(x_1, x_2) = [x_1 x_2](x_1 + x_2) - (x_1 x_2)$

We shall have to deal with certain functions which are symmetric in each pair of the variables $x_1, x_2, \cdot\cdot, x_n$; such functions will be termed symmetric functions or symmetric functions on $x_1, x_2, \cdot\cdot, x_n$. Following the usual notation of rational symmetric functions, we write

$$F(x_1, x_2, \cdot\cdot, x_n) = \sum f(x_1, x_2, \cdot\cdot, x_n)$$

to denote a symmetric function F on $x_1, x_2, \cdot\cdot, x_n$ and we call $f(x_1, x_2, \cdot\cdot, x_n)$ the argument of $\sum$. We shall say that $\sum$ is written on $x_1, x_2, \cdot\cdot, x_n$ with respect to the argument f . Hence, the symmetric function on x_1, x_2, x_3 of argument $[x_1 x_2](x_3)$ is:

$$\sum [x_1 x_2](x_3) = [x_1 x_2](x_3) + [x_1 x_3](x_2) + [x_2 x_3](x_1)$$

When a function is symmetric on $x_{n-j}, x_{n-j+1}, \cdot\cdot, x_n$ but not all of these variables appear in the argument of $\sum$, we will write $\sum\limits_{n-j}^{n}$ in place of the usual $\sum$. Thus,

$$\sum_{3}^{4}(x_3) = (x_3) + (x_4)$$

We shall need to write sums of such symmetric functions, e.g.,

$$\sum_{j=1}^{n-1} (-1)^j \sum [x_1 \cdot\cdot x_{n-j}](x_{n-j+1} \cdot\cdot x_n);$$

hence, if $n = 3$, we write,

$$\sum_{j=1}^{2} (-1)^j \sum [x_1 \cdots x_{3-j}](x_{4-j} \cdots x_3) = -\{[x_1 x_2](x_3) + [x_1 x_3](x_2) + [x_2 x_3](x_1)\}$$
$$+ \{(x_2 x_3) + (x_1 x_3) + (x_1 x_2)\}$$

where we have taken note of (7.1.2).

If a function involving $x_1, x_2, \cdots, x_n$ is not symmetric on all of the variables, those on which it is not symmetric are to be written to the left of a bar in the argument of $\sum$; thus,

$$\sum [x_4 | x_1 x_2](x_4 | + x_1 + x_2)(x_3) = [x_1 x_2 x_4](x_1 + x_2 + x_4)(x_3)$$
$$+ [x_1 x_3 x_4](x_1 + x_3 + x_4)(x_2)$$
$$+ [x_2 x_3 x_4](x_2 + x_3 + x_4)(x_1)$$

We may regard $\sum$ as being written on x_1, x_2, x_3 or as being written on x_1, x_2, x_3, x_4 with x_4 held fixed.

Using the bar notation, we can write

7.1.16 $$\sum_{k=1}^{n-1} (-1)^k \sum [x_1 \cdots x_{n-k}](x_{n-k+1} \cdots x_n)$$

$$= -[x_1 \cdots x_{n-1}](x_n) + \sum_{k=2}^{n-1} (-1)^k \sum [x_1 \cdots x_{n-k}](x_n | x_{n-k+1} \cdots x_{n-1})$$
$$+ \sum_{k=1}^{n-2} (-1)^k \sum [x_n | x_1 \cdots x_{n-k-1}](x_{n-k} \cdots x_{n-1}) + (-1)^{n-1}(x_1 \cdots x_{n-1}) ;$$

the first line of the right hand side results from writing the terms which involve $\cot x_n$ and the second line from writing terms which involve x_n in a function denoted by brackets.

The argument of $\sum\limits_{n-j}^{n-1}$ will usually be an expression enclosed in numbered brackets and consisting of several terms, e.g.,

$$\sum_{n-2}^{n-1} \Big[_1\, S(x_n, x_1 + \cdots + x_{n-3} | + x_{n-2}) - S(x_n, x_1 + \cdots + x_{n-1}) \Big]_1$$

In such cases, the rule for forming the symmetric function is the following:

<u>Rule</u> <u>A</u>: Add the results of applying to all the terms in the numbered brackets those permutations which are required to write the symmetric function which has as its argument the first term in the numbered brackets.

Thus,

$$\sum_{n-2}^{n-1} \Big[_1\, S(x_n, x_1 + \cdots + x_{n-3} | + x_{n-2}) - S(x_n, x_1 + \cdots + x_{n-1}) \Big]_1$$

$$= S(x_n, x_1 + \cdots + x_{n-3} + x_{n-2}) - S(x_n, x_1 + \cdots + x_{n-1})$$

$$+ S(x_n, x_1 + \cdots + x_{n-3} + x_{n-1}) - S(x_n, x_1 + \cdots + x_{n-1})$$

We shall have to write complicated expressions such as symmetric functions on $x_1, x_2, \cdots, x_n$ with respect to an argument which itself involves symmetric functions on only some of these variables. We refine the notation in an obvious way which may best be understood from the following example. We write:

7.1.17
$$
\begin{aligned}
&\sum {\Big[}_1 [x_1 \cdots x_{n-4}](x_{n-3}\, x_{n-2}\, x_{n-1}) \sum_{n-3}^{n-1} {\Big[}_2\, S(x_n\|, x_1 + \cdots + x_{n-4}| + x_{n-3}) \\
&\qquad - \sum_{n-2}^{n-1} {\Big[}_3\, S(x_n|||, x_1 + \cdots + x_{n-4}\| + x_{n-3}| + x_{n-2}) {\Big]}_3\, {\Big]}_2\, {\Big]}_1 \\
&= \sum {\Big[}_1 [x_1 \cdots x_{n-4}](x_{n-3}\, x_{n-2}\, x_{n-1}) \sum_{n-3}^{n-1} {\Big[}_2\, S(x_n\|, x_1 + \cdots + x_{n-4}| + x_{n-3}) \\
&\qquad - S(x_n\|, x_1 + \cdots + x_{n-4}| + x_{n-3} + x_{n-2}) \\
&\qquad - S(x_n\|, x_1 + \cdots + x_{n-4}| + x_{n-3} + x_{n-1}) {\Big]}_2 {\Big]}_1 \\
&= \sum {\Big[}_1 [x_1 \cdots x_{n-4}](x_{n-3}\, x_{n-2}\, x_{n-1}) {\Big[}_2 \quad {\Big]}_2 {\Big]}_1
\end{aligned}
$$

where

$$
\begin{aligned}
{\Big[}_2 \quad {\Big]}_2 \equiv {\Big[}_2\, & S(x_n|, x_1 + \cdots + x_{n-4} + x_{n-3}) - S(x_n|, x_1 + \cdots + x_{n-4} + x_{n-3} + x_{n-2}) \\
& - S(x_n|, x_1 + \cdots + x_{n-4} + x_{n-3} + x_{n-1}) \\
& + S(x_n|, x_1 + \cdots + x_{n-4} + x_{n-2}) - S(x_n|, x_1 + \cdots + x_{n-4} + x_{n-2} + x_{n-3}) \\
& - S(x_n|, x_1 + \cdots + x_{n-4} + x_{n-2} + x_{n-1}) \\
& + S(x_n|, x_1 + \cdots + x_{n-4} + x_{n-1}) - S(x_n|, x_1 + \cdots + x_{n-4} + x_{n-1} + x_{n-2}) \\
& - S(x_n|, x_1 + \cdots + x_{n-4} + x_{n-1} + x_{n-3}) {\Big]}_2
\end{aligned}
$$

The following important points are to be noted:

1) The double bar following x_n indicates that it remains fixed with respect to Σ and $\sum_{n-3}^{n-1}$ and the triple bar, that it remains fixed with respect to Σ, $\sum_{n-3}^{n-1}$, and $\sum_{n-2}^{n-1}$; a bar is removed as $\sum_{n-2}^{n-1}$ and $\sum_{n-3}^{n-1}$ are expanded; similarly, for the double bar following $(x_1 + \cdots + x_{n-4})$

2) $\sum_{n-3}^{n-1}$ must be written instead of Σ since x_{n-1} does not appear in the first term of the argument of $\sum_{n-3}^{n-1}$; similarly for $\sum_{n-2}^{n-1}$.

3) While there would be little chance of ambiguity between the functions denoted by brackets and terms included in numbered brackets if the numbers were not used, it is felt that the numbered brackets facilitate the reading of formulas which require several lines and clarify to which terms Σ and $\sum_{n-j}^{n-1}$ apply, sufficiently to justify their use. The brackets should be expanded according to the decreasing order of their indices, as illustrated in the example.

4) Note the application of Rule A.

It is clear from the example just given that in some instances we are not essentially concerned with the fact that x_n appears as an argument of S, nor that the first $n-j$ variables, $x_1, x_2, \cdots, x_{n-j}$ appear in the second argument of S. When this is the case, we shall use an abbreviated form of the notation as shown in the following example. Unfortunately, (see explanation below), we have had to delay the introduction of these abbreviations to this point in order to explain the iterated symmetric function notation.

7.1.18 $$S(x_n|, x_1 + x_2 + \cdots + x_{n-k}) \equiv S(n-k)$$

$$S(x_n||, x_1 + x_2 + \cdots + x_{n-j} | + x_{n-j+1} + \cdots + x_{n-k}) \equiv S(n-j \,|\, n-k)$$

$$S(x_n|||, x_1 + x_2 + \cdots + x_{n-j} || + x_{n-j+1} | + x_{n-j+2} + \cdots + x_{n-k}) \equiv S(n-j \,||\, n-j+1 \,|\, n-k)$$

$$[x_1 \cdots x_{n-j}](x_{n-j+1} \cdots x_{n-k}) \equiv T(n-j;\, n-k)$$

Then (7.1.17) becomes

$$\sum \Big[_1 T(n-4;\, n-1) \sum_{n-3}^{n-1} \Big[_2 S(n-4 \,|\, n-3) - \sum_{n-2}^{n-1} \Big[_3 S(n-4 \,||\, n-3 \,|\, n-2) \Big]_3 \Big]_2 \Big]_1$$

(The reader will note that we cannot write the expanded form of this expression in the notation just given without considerable elaboration, which is the reason for delaying it to this point.)

2. The Function $F(x_1, x_2, \cdots, x_n)$

Using the definitions (7.1.1 - 7.1.5), we define

7.2.1a $$F(x_1, x_2, \cdots, x_n) = [x_1 \cdots x_n] + \sum_{k=1}^{n-1} (-1)^k \sum [x_1 \cdots x_{n-k}](x_{n-k+1} \cdots x_n)$$

$$x_j \neq 0 \quad j = 1, 2, \cdots, n; \quad n \geq 1$$

7.2.1b $$F(x_1, x_2, \cdots, x_{n-1}, 0) = [x_1 \cdots x_{n-1}]^{(1)} + \sum_{k=1}^{n-2} (-1)^k \sum [x_1 \cdots x_{n-k-1}]^{(1)} (x_{n-k} \cdots x_{n-1})$$

$$+ (-1)^{n-1} C_1 (x_1 \cdots x_{n-1})$$

$$x_j \neq 0 \quad j = 1, 2, \cdots, n-1$$

7.2.1c $$F(x_1, x_2, \cdots, x_{n-2}, 0, 0) = [x_1 \cdots x_{n-2}]^{(2)}$$

$$+ \sum_{k=1}^{n-3} (-1)^k \sum [x_1 \cdots x_{n-k-2}]^{(2)} (x_{n-k-1} \cdots x_{n-2})$$

$$x_j \neq 0 \quad j = 1, 2, \cdots, n-2$$

7.2.1d $$F(x_1, x_2, \cdots, x_{n-\ell}, 0, \cdots, 0) = [x_1 \cdots x_{n-\ell}]^{(\ell)}$$

$$+ \sum_{k=1}^{n-\ell-1} (-1)^k \sum [x_1 \cdots x_{n-k-\ell}]^{(\ell)} (x_{n-k-\ell+1} \cdots x_{n-\ell})$$

$$+ (-1)^{n-\ell} C_\ell (x_1 \cdots x_{n-\ell})$$

$$x_j \neq 0 \quad j = 1, 2, \cdots, n-\ell; \quad \ell = 1, 2, \cdots, n-2$$

7.2.1e $F(x_1, 0, \cdots, 0) = [x_1]^{(n-1)} - C_{n-1}(x_1)$; $x_1 \neq 0$

7.2.1f $F(0, 0, \cdots, 0) = C_n$

With the assertion that F is symmetric in each pair of the variables $x_1, x_2, \cdots, x_n$, we have defined F completely.

We show that F is analytic in the cylindrical region $(T_n) = (T_1, T_2, \cdots, T_n)$ where T_k is the strip $|\mathrm{Im}\, x_k| < \mathrm{Im}\, \pi\tau$ for $k = 1, 2, \cdots, n$. We fix all of the variables except one, say x_1, and show that

7.2.2 $$\lim_{x_1 \to 0} F(x_1, x_2, \cdots, x_n) = F(0, x_2, \cdots, x_n)$$

thus the apparent singularity at $x_1 = 0$ is removable and F as defined is analytic in x_1; by the symmetry of F on $x_1, x_2, \cdots, x_n$ and Hartog's Theorem, F is analytic in $x_1, x_2, \cdots, x_n$ in (T_n).

In order to establish (7.2.2), we observe that

7.2.3 $$\lim_{x_1 \to 0} \left\{ [x_1 x_2 \cdots x_j] - [x_2 \cdots x_j](x_1) \right\} = [x_2 \cdots x_j]^{(1)} \qquad j = 2, 3, \cdots$$

7.2.4 $$\lim_{x_1 \to 0} \left\{ [x_1 x_2 \cdots x_j]^{(k)} - [x_2 \cdots x_j]^{(k)}(x_1) \right\} = [x_2 \cdots x_j]^{(k+1)} \qquad k = 1, 2, \cdots$$

7.2.5 $$\lim_{x_1 \to 0} \left\{ [x_1]^{(k)} - c_k(x_1) \right\} = c_{k+1}$$

the last three results are established easily by writing the Taylor's expansion about $x_1 = 0$ of numerator and denominator in the functions denoted by brackets.

For $x_j \neq 0$, $j = 1, 2, \cdots, n$, we have by (7.2.1a) and (7.1.16)

$$\lim_{x_1 \to 0} F(x_1, \cdots, x_n) = \lim_{x_1 \to 0} \Big[_1 [x_1 x_2 \cdots x_n] - [x_2 \cdots x_n](x_1)$$
$$+ \sum_{k=1}^{n-2} (-1)^k \sum [x_1 | x_2 \cdots x_{n-k}](x_{n-k+1} \cdots x_n)$$
$$+ \sum_{k=2}^{n-1} (-1)^k \sum [x_2 \cdots x_{n-k+1}](x_1 | x_{n-k+2} \cdots x_n)$$
$$+ (-1)^{n-1} (x_2 \cdots x_n) \Big]_1$$

Then by (7.2.3) and the introduction of $c_1 = 1$

$$\lim_{x_1 \to 0} F(x_1, \cdots, x_n) = [x_2 \cdots x_n]^{(1)} + \sum_{k=1}^{n-2} (-1)^k \sum [x_2 \cdots x_{n-k}]^{(1)} (x_{n-k+1} \cdots x_n)$$
$$+ (-1)^{n-1} c_1 (x_2 \cdots x_n)$$

Hence, by (7.2.1b) and the symmetry of F in x_1 and x_n,

$$\lim_{x_1\to 0} F(x_1, x_2, \cdots, x_n) = F(0, x_2, \cdots, x_n)$$

For $x_j \neq 0$, $j = 2, 3, \cdots, n-1$, we have by (7.2.1b)

$$\begin{aligned}\lim_{x_1\to 0} F(x_1, x_2, \cdots, x_{n-1}, 0) = \lim_{x_1\to 0} \Big[_1 [x_1 \cdots x_{n-1}]^{(1)} - [x_2 \cdots x_{n-1}]^{(1)}(x_1) \\ + \sum_{k=1}^{n-3} (-1)^k \sum [x_1 | x_2 \cdots x_{n-k-1}]^{(1)} (x_{n-k} \cdots x_{n-1}) \\ + \sum_{k=2}^{n-2} (-1)^k \sum [x_2 \cdots x_{n-k}]^{(1)} (x_1 | x_{n-k+1} \cdots x_{n-1}) \\ + (-1)^{n-2} \Big[_2 [x_1]^{(1)} - c_1(x_1)\Big]_2 (x_2 \cdots x_{n-1}) \Big]_1\end{aligned}$$

By (7.2.4) and (7.2.5)

$$\lim_{x_1\to 0} F(x_1, x_2, \cdots, x_{n-1}, 0) = [x_2 \cdots x_{n-1}]^{(2)} + \sum_{k=1}^{n-3} (-1)^k \sum [x_2 \cdots x_{n-1-k}]^{(2)} (x_{n-k} \cdots x_{n-1})$$

Hence, by (7.2.1c) and the symmetry of F in x_1 and x_{n-1}

$$\lim_{x_1\to 0} F(x_1, x_2, \cdots, x_{n-1}, 0) = F(0, x_2, \cdots, x_{n-1}, 0)$$

In the same way, using (7.2.1d) and induction,

$$\lim_{x_1\to 0} F(x_1, x_2, \cdots, x_{n-\ell}, 0, \cdots, 0) = F(0, x_2, \cdots, x_{n-\ell}, 0, \cdots, 0)$$

and using (7.2.1e) and (7.2.1f)

$$\lim_{x_1\to 0} F(x_1, 0, \cdots, 0) = F(0, 0, \cdots, 0)$$

This concludes the proof of (7.2.2) and as was asserted above, establishes the result that F is analytic in (T_n).

3. The Function $\bar{F}(x_1, x_2, \cdots, x_n)$

Using (7.1.1 - 7.1.11) and the binomial coefficients $\binom{k}{l}$, we define:

7.3.1a $$\bar{F}(x_1, x_2, \cdots, x_n) = [x_1 x_2 \cdots x_n](x_1 + x_2 + \cdots + x_n)$$

$$+ \sum_{j=1}^{n-1} (-1)^j \sum [x_1 \cdots x_{n-j}](x_1 + \cdots + x_{n-j})(x_{n-j+1} \cdots x_n)$$

$$+ (-1)^n (x_1 \cdots x_n)$$

$$\text{for } \begin{cases} x_i \neq 0 & i = 1, 2, \cdots, n \\ x_1 + \cdots + x_{n-j} \neq 0 & j = 0, 1, \cdots, n-2 \end{cases}$$

7.3.1b $$\bar{F}(x_1, x_2, \cdots, x_{n-k}, 0, \cdots, 0) = \sum_{l=0}^{k} \binom{k}{l} [x_1 \cdots x_{n-k}]^{(k-l)} (x_1 + \cdots + x_{n-k})^{(l)}$$

$$+ \sum_{j=1}^{n-1-k} (-1)^j \sum \Big[_1 \sum_{l=0}^{k} \binom{k}{l} [x_1 \cdots x_{n-k-j}]^{(k-l)} (x_1 + \cdots + x_{n-k-j})^{(l)} (x_{n-k-j+1} \cdots x_{n-k}) \Big]_1$$

$$+ (-1)^{n-k} \gamma_k (x_1 \cdots x_{n-k})$$

$$\text{for } \begin{cases} x_i \neq 0 & i = 1, 2, \cdots, n-k \\ x_1 + \cdots + x_{n-k-j} \neq 0 & j = 0, 1, \cdots, n-k-2 \\ k = 1, 2, \cdots, n-2 \end{cases}$$

7.3.1c $$\bar{F}(x_1, 0, \cdots, 0) = \sum_{l=0}^{n-1} \binom{n-1}{l} [x_1]^{(n-1-l)} (x_1)^{(l)} - \gamma_{n-1}(x_1)$$

$$\text{for } x_1 \neq 0$$

7.3.1d $$\bar{F}(0, 0, \cdots, 0) = \gamma_n$$

Using (7.1.9), we can write

7.3.1a' $$\bar{F}(x_1, \cdots, x_n) = \frac{\left(\vartheta_1^{(1)}\right)^{n-1}}{\prod_{i=1}^{n} \vartheta_1(x_i)} f(x_1 + \cdots + x_n)$$

$$+ \sum_{j=1}^{n-1} (-1)^j \sum \left[\frac{\left(\vartheta_1^{(1)}\right)^{n-j-1}}{\prod_{i=1}^{n-j} \vartheta_1(x_i)} f(x_1 + \cdots + x_{n-j}) \prod_{i=n-j+1}^{n} \cot x_i \right]_1$$

$$+ (-1)^n \prod_{i=1}^{n} \cot x_i$$

7.3.1b' $$\bar{F}(x_1, x_2, \cdots, x_{n-k}, 0, \cdots, 0) = \frac{\left(\vartheta_1^{(1)}\right)^{n-k-1}}{\prod_{i=1}^{n-k} \vartheta_1(x_i)} f^{(k)}(x_1 + \cdots + x_{n-k})$$

$$+ \sum_{j=1}^{n-1-k} (-1)^j \sum \left[\frac{\left(\vartheta_1^{(1)}\right)^{n-k-j-1}}{\prod_{i=1}^{n-k-j} \vartheta_1(x_i)} f^{(k)}(x_1 + \cdots + x_{n-k-j}) \prod_{i=n-k-j+1}^{n-k} \cot x_i \right]_1$$

$$+ (-1)^{n-k} \gamma_k \prod_{i=1}^{n-k} \cot x_i$$

7.3.1c' $$\bar{F}(x_1, 0, 0, \cdots, 0) = \frac{f^{(n-1)}(x_1)}{\vartheta_1(x_1)} - \gamma_{n-1} \cot x_1$$

7.3.1d' $$\bar{F}(0, 0, \cdots, 0) = \gamma_n$$

Thus, when we use the definition (7.3.1a - 7.3.1d), there are apparent singularities of $\bar{F}$ at $x_1 + x_2 + \cdots + x_{n-j} = 0$ for $j = 0, 1, \cdots, n-2$; they are seen to be removable when we use the definition (7.3.1a' - 7.3.1d') since $f(u)$ is analytic at $u = 0$

Finally, with the assertion that $\bar{F}$ is symmetric in each pair of the variables $x_1, x_2, \cdots, x_n$, the function $\bar{F}$ is defined completely.

$\bar{F}$ is analytic in $x_1, x_2, \cdots, x_n$ in the cylindrical region (T_n) defined in section 2 of this chapter. The demonstration is essentially that of the analyticity of $F(x_1, x_2, \cdots, x_n)$. We give only the details showing that

7.3.2 $$\lim_{x_{n-k}\to 0} \bar{F}(x_1, x_2, \cdots, x_{n-k}, 0, \cdots, 0) = \bar{F}(x_1, x_2, \cdots, x_{n-k-1}, 0, \cdots, 0)$$

for $x_i \neq 0$, $i = 1, 2, \cdots, n-k-1$; $k = 1, 2, \cdots, n-2$; the procedure for the case $k = 0$ is exactly the same; the result

$$\lim_{x_1 \to 0} \bar{F}(x_1, 0, \cdots, 0) = \bar{F}(0, 0, \cdots, 0)$$

is given by (7.1.12).

For compactness in writing the verification of (7.3.2), we abbreviate the notation in the spirit of (7.1.18) as follows:

$$(x_1 + x_2 + \cdots + x_{n-k-j})^{(l)} \equiv (+\, n-k-j)^{(l)}$$

$$(x_{n-k} | + x_1 + \cdots + x_{n-k-j})^{(l)} \equiv (x_{n-k} | + n-k-j)^{(l)}$$

$$[x_1 \cdots x_{n-k-j}]^{(k-l)} \equiv [n-k-j]^{(k-l)}$$

$$[x_{n-k} | x_1 \cdots x_{n-k-j-1}]^{(k-l)} \equiv [x_{n-k} | n-k-j-1]^{(k-l)}$$

We will not abbreviate the expressions like

$$(x_{n-k-j} \cdots x_{n-k-1})$$

In this notation then, (7.3.1b) becomes:

$$F(x_1,\cdots,x_{n-k},0,\cdots,0) = \sum_{\ell=0}^{k}\binom{k}{\ell}[n-k]^{(k-\ell)}(+n-k)^{(\ell)}$$
$$+\sum_{j=1}^{n-1-k}(-1)^j\sum\Big[_1\sum_{\ell=0}^{k}\binom{k}{\ell}[n-k-j]^{(k-\ell)}(+n-k-j)^{(\ell)}(x_{n-k-j+1}\cdots x_{n-k})\Big]_1$$
$$+(-1)^{n-k}\gamma_k(x_1\cdots x_{n-k})$$

The method of establishing (7.3.2) is to write first the terms which involve x_{n-k} as argument of a function denoted by brackets and to add terms as necessary in order to use the results (7.2.3 - 7.2.4). Then the terms which were added are subtracted and grouped with those which involve x_{n-k} as argument of the cotangent. Thus,

$$F(x_1,\cdots,x_{n-k},0,\cdots,0) = \sum_{\ell=0}^{k}\binom{k}{\ell}(+n-k)^{(\ell)}\Big[_1[n-k]^{(k-\ell)}-[n-k-1]^{(k-\ell)}(x_{n-k})\Big]_1$$
$$+\sum_{j=1}^{n-2-k}(-1)^j\sum\Big[_1\sum_{\ell=0}^{k}\binom{k}{\ell}(x_{n-k}|+n-k-j-1)^{(\ell)}\Big[_2[x_{n-k}|n-k-j-1]^{(k-\ell)}$$
$$-[n-k-j-1]^{(k-\ell)}(x_{n-k}|)\Big]_2(x_{n-k-j}\cdots x_{n-k-1})\Big]_1$$

$$+(-1)^{n-1-k}\sum_{\ell=0}^{k}\binom{k}{\ell}[x_{n-k}]^{(k-\ell)}(x_{n-k})^{(\ell)}(x_1\cdots x_{n-k-1})+(-1)^{n-k}\gamma_k(x_1\cdots x_{n-k})$$

$$+\sum_{\ell=0}^{k}\binom{k}{\ell}[n-k-1]^{(k-\ell)}\Big[_1(+n-k)^{(\ell)}(x_{n-k})-(+n-k-1)^{(\ell)}(x_{n-k})\Big]_1$$

$$+\sum_{j=1}^{n-2-k}(-1)^j\sum\Big[_1\sum_{\ell=0}^{k}\binom{k}{\ell}[n-k-j-1]^{(k-\ell)}(x_{n-k}|+n-k-j-1)^{(\ell)}(x_{n-k}|x_{n-k-j}\cdots x_{n-k-1})\Big]_1$$
$$+\sum_{j=2}^{n-1-k}(-1)^j\sum\Big[_1\sum_{\ell=0}^{k}\binom{k}{\ell}[n-k-j]^{(k-\ell)}(+n-k-j)^{(\ell)}(x_{n-k}|x_{n-k-j+1}\cdots x_{n-k-1})\Big]_1$$

By (7.2.4)

$$\lim_{x_{n-k}\to 0}\Big[_1[x_{n-k}|n-k-j-1]^{(k-\ell)}-[n-k-j-1]^{(k-\ell)}(x_{n-k}|)\Big]_1=[n-k-j-1]^{(k-\ell+1)}$$

By (7.1.12)

$$\lim_{x_{n-k}\to 0} \left[{}_1 \sum_{l=0}^{k} \binom{k}{l} [x_{n-k}]^{(k-l)} (x_{n-k})^{(l)} - \gamma_k(x_{n-k})\right]_1 = \gamma_{k+1}$$

Note that

$$\lim_{x_{n-k}\to 0} \left[{}_1 (x_{n-k}|+n-k-j-1)^{(l)}(x_{n-k}|) - (+n-k-j-1)^{(l)}(x_{n-k}|)\right]_1 = (+n-k-j-1)^{(l+1)}$$

Hence,

$$\lim_{x_{n-k}\to 0} \bar{F}(x_1, \cdots, x_{n-k}, 0, \cdots, 0) = \sum_{l=0}^{k} \binom{k}{l} (+n-k-1)^{(l)} [n-k-1]^{(k-l+1)}$$

$$+ \sum_{l=0}^{k} \binom{k}{l} [n-k-1]^{(k-l)} (+n-k-1)^{(l+1)}$$

$$+ \sum_{j=1}^{n-2-k} (-1)^j \sum \left[{}_1 \sum_{l=0}^{k} \binom{k}{l} (+n-k-j-1)^{(l)} [n-k-j-1]^{(k-l+1)} (x_{n-k-j} \cdots x_{n-k-1})\right]_1$$

$$+ \sum_{j=1}^{n-2-k} (-1)^j \sum \left[{}_1 \sum_{l=0}^{k} \binom{k}{l} [n-k-j-1]^{(k-l)} (+n-k-j-1)^{(l+1)} (x_{n-k-j} \cdots x_{n-k-1})\right]_1$$

$$+ (-1)^{n-1-k} \gamma_{k+1}(x_1 \cdots x_{n-k-1})$$

$$\lim_{x_{n-k}\to 0} \bar{F}(x_1, \cdots, x_{n-k}, 0, \cdots, 0) = \sum_{l=0}^{k+1} \binom{k+1}{l} (+n-k-1)^{(l)} [n-k-1]^{(k+1-l)}$$

$$+ \sum_{j=1}^{n-2-k} (-1)^j \sum \left[{}_1 \sum_{l=0}^{k+1} \binom{k+1}{l} (+n-k-j-1)^{(l)} [n-k-j-1]^{(k+1-l)} (x_{n-k-j} \cdots x_{n-k-1})\right]_1$$

$$+ (-1)^{n-1-k} \gamma_{k+1}(x_1 \cdots x_{n-k-1})$$

$$\lim_{x_{n-k}\to 0} \bar{F}(x_1, \cdots, x_{n-k}, 0, \cdots, 0) = \bar{F}(x_1, \cdots, x_{n-k-1}, 0, \cdots 0)$$

This concludes the proof of (7.3.2), showing that $\bar{F}$ is analytic in x_1. By the symmetry of $\bar{F}$ and Hartog's Theorem, $\bar{F}$ is analytic in $x_1, x_2, \cdots, x_n$ in (T_n).

4. Another Representation of $F(X_1, X_2, \cdots, X_n)$

We proceed now to express $F(X_1, X_2, \cdots, X_n)$ in terms of $\bar{F}(X_1, X_2, \cdots, X_{n-1})$ and $F(X_1, X_2, \cdots, X_{n-j})$ for $j = 1, 2, \cdots, n-2$ and $n \geq 3$. We abbreviate the definition of F to the statement (7.2.1a).

We note that

$$[X_1 \cdots X_n] = [X_1 \cdots X_{n-1}][X_1 + X_2 + \cdots + X_{n-1}, X_n]$$

Using (7.1.13), this becomes

7.4.1 $$[X_1 \cdots X_n] = [X_1 \cdots X_{n-1}]\Big[_1 (X_1 + \cdots + X_{n-1}) + (X_n) + S(X_n, X_1 + \cdots + X_{n-1})\Big]_1$$

Then, using (7.1.16) and (7.4.1)

$$F(X_1, X_2, \cdots, X_n) = [X_1 \cdots X_n] + \sum_{k=1}^{n-1} (-1)^k \sum [X_1 \cdots X_{n-k}](X_{n-k+1} \cdots X_n)$$

$$=$$

$$[X_1 \cdots X_{n-1}](X_1 + \cdots + X_{n-1}) + [X_1 \cdots X_{n-1}]\, S(X_n, X_1 + \cdots + X_{n-1})$$

$$+ [X_1 \cdots X_{n-1}](X_n) - [X_1 \cdots X_{n-1}](X_n)$$

$$+ \sum_{k=2}^{n-1} (-1)^k \sum [X_1 \cdots X_{n-k}](X_n | X_{n-k+1} \cdots X_{n-1})$$

$$+ \sum_{k=1}^{n-2} (-1)^k \sum [X_n | X_1 \cdots X_{n-k-1}](X_{n-k} \cdots X_{n-1})$$

$$+ (-1)^{n-1} [X_n](X_1 \cdots X_{n-1})$$

Using (7.4.1) again on $[x_n | x_1 \cdots x_{n-k-1}]$ and introducing the abbreviations of (7.1.18) we have

$$F(x_1, \cdots, x_n) = [x_1 \cdots x_{n-1}](x_1 + \cdots + x_{n-1}) + [x_1 \cdots x_{n-1}]\, S(n-1)$$

$$+ \sum_{k=2}^{n-1} (-1)^k \sum T(n-k\,;\, n-1)\,(x_n|)$$

$$+ \sum_{k=1}^{n-2} (-1)^k \sum \Big[_1\, T(n-k-1; n-1)\, \Big[_2\, (x_n|)$$

$$+ (x_1 + \cdots + x_{n-k-1}) + S(n-k-1) \Big]_2 \Big]_1$$

$$+ (-1)^{n-1} (x_1 \cdots x_{n-1})$$

Recalling (7.3.1a), the terms not involving x_n are exactly $\bar{F}(x_1, x_2, \cdots, x_{n-1})$; the terms involving (x_n) vanish; hence

7.4.2 $$F(x_1, \cdots, x_n) = \bar{F}(x_1, \cdots, x_{n-1}) + [x_1 \cdots x_{n-1}]\, S(n-1) + \sum_{k=1}^{n-2} (-1)^k \sum T(n-k-1; n-1)\, S(n-k-1)$$

By adding and subtracting

7.4.3 $$A_1 = \Big[_1 \sum_{k=1}^{n-2} (-1)^k \sum T(n-1-k\,;\, n-1) \Big]_1\, S(n-1)$$

to $[x_1 \cdots x_{n-1}]\, S(n-1)$, the latter expression becomes $F(x_1, \cdots, x_{n-1})\, S(n-1) - A_1$; hence using the abbreviations

7.4.4 $F(x_1, \cdots, x_n) = F(n)$

7.4.5 $\bar{F}(x_1, \cdots, x_{n-1}) = \bar{F}(n-1)$

we can write (7.4.2) in the form

7.4.6 $$F(n) = \bar{F}(n-1) + F(n-1)\, S(n-1) - A_1 + \sum_{k=1}^{n-2} (-1)^k \sum T(n-k-1; n-1)\, S(n-k-1)$$

We define

7.4.7 $G_1 = F(n) - \bar{F}(n-1) - F(n-1)\,S(n-1)$

Recalling Rule A (Page 37) we have from (7.4.6) and (7.4.3),

$$G_1 = \sum_{k=1}^{n-2} (-1)^k \sum \Big[_1\, T(n-k-1;\, n-1)\,\big[_2\, S(n-k-1) - S(n-1)\big]_2\,\Big]_1$$

Then

7.4.8 $$G_1 = (-1) \sum T(n-2;\, n-1)\,\big[_1\, S(n-2) - S(n-1)\big]_1$$

$$+ \sum_{k=2}^{n-2} (-1)^k \sum \Big[_1\, T(n-k-1;\, n-1)\,\big[_2\, S(n-k-1) - S(n-1)\big]_2\,\Big]_1$$

By adding and subtracting

7.4.9 $$A_2 = -\sum \Big[_1\, \Big[_2 \sum_{k=1}^{n-3} (-1)^k \sum T(n-2-k;\, n-2)\Big]_2 (x_{n-1})\,\big[_2\, S(n-2) - S(n-1)\big]_2\,\Big]_1$$

the expression

$$-\sum \Big[_1\, T(n-2;\, n-1)\,\big[_2\, S(n-2) - S(n-1)\big]_2\,\Big]_1$$

becomes

$$-\sum \Big[_1\, F(n-2)\,(x_{n-1})\,\big[_2\, S(n-2) - S(n-1)\big]_2\,\Big]_1 - A_2\ ;$$

hence, using (7.4.4), we obtain for (7.4.8)

7.4.10 $$G_1 = -\sum \Big[_1\, F(n-2)\,(x_{n-1})\,\big[_2\, S(n-2) - S(n-1)\big]_2\,\Big]_1 - A_2$$

$$+ \sum_{k=2}^{n-2} (-1)^k \sum \Big[_1\, T(n-k-1;\, n-1)\,\big[_2\, S(n-k-1) - S(n-1)\big]_2\,\Big]_1$$

Now consider

$$-A_2 = \sum \Big[_1 \Big[_2 \sum_{k=1}^{n-3} (-1)^k \sum T(n-2-k; n-2) \Big]_2 (x_{n-1}) \Big[_2 S(n-2) - S(n-1) \Big]_2 \Big]_1 .$$

$-A_2$ is symmetric on $x_1, x_2, \cdots, x_{n-1}$; by Rule A, we need only consider the coefficient of $T(n-2-k; n-1)$ for $k = 1, 2, \cdots, n-3$. In particular, for $k = 1$, the only terms involving $T(n-3 ; n-1)$ are those obtained by permuting x_{n-2} and x_{n-1} in $-A_2$; therefore, we have

$$(-1)\, T(n-3; n-1) \sum_{n-2}^{n-1} \Big[_1 S(n-3 \mid n-2) - S(n-1) \Big]_1$$

Likewise, for $k = 2$,

$$(-1)^2\, T(n-4; n-1) \sum_{n-3}^{n-1} \Big[_1 S(n-4 \mid n-2) - S(n-1) \Big]_1$$

Generally, then

$$(-1)^k\, T(n-2-k; n-1) \sum_{n-1-k}^{n-1} \Big[_1 S(n-2-k \mid n-2) - S(n-1) \Big]_1$$

Hence,

$$7.4.11 \quad -A_2 = \sum_{k=1}^{n-3} (-1)^k \sum \Big[_1 T(n-2-k; n-1) \sum_{n-1-k}^{n-1} \Big[_2 S(n-2-k \mid n-2) - S(n-1) \Big]_2 \Big]_1$$

Now define

$$7.4.12 \quad G_2 = G_1 + \sum \Big[_1 F(n-2)(x_{n-1}) \Big[_2 S(n-2) - S(n-1) \Big]_2 \Big]_1$$

Then, from (7.4.10) and (7.4.11), we have

7.4.13 $$G_2 = \sum_{k=2}^{n-2} (-1)^k \sum \Big[_1 T(n-1-k\,;\, n-1) \Big[_2 S(n-1-k) - S(n-1)\Big]_2 \Big]_1$$

$$+ \sum_{k=1}^{n-3} (-1)^k \sum \Big[_1 T(n-2-k\,;\, n-1) \sum_{n-1-k}^{n-1} \Big[_2 S(n-2-k\,|\,n-2) - S(n-1)\Big]_2 \Big]_1$$

The next step is to write

$$G_2 = (-1)^2 \sum \Big[_1 T(n-3\,;\, n-1) \Big[_2 S(n-3) - \sum_{n-2}^{n-1} \Big[_3 S(n-3\,|\,n-2)\Big]_3 + S(n-1)\Big]_2 \Big]_1$$

$$+ \sum_{k=3}^{n-2} (-1)^k \sum \Big[_1 T(n-1-k\,;\, n-1) \Big[_2 S(n-1-k) - S(n-1)\Big]_2 \Big]_1$$

$$+ \sum_{k=2}^{n-3} (-1)^k \sum \Big[_1 T(n-2-k\,;\, n-1) \sum_{n-1-k}^{n-1} \Big[_2 S(n-2-k\,|\,n-2) - S(n-1)\Big]_2 \Big]_1$$

where the first line results from adding the terms corresponding to $k = 2$ and $k = 1$ respectively in the second and third lines. Again, we add and subtract:

$$A_3 = \sum \Big[_1 \Big[_2 \sum_{k=1}^{n-4} (-1)^k \sum T(n-3-k\,;\, n-3)\Big]_2 (x_{n-2}\, x_{n-1}) \Big[_2 S(n-3) - \sum_{n-2}^{n-1} \Big[_3 S(n-3\,|\,n-2)\Big]_3 + S(n-1)\Big]_2 \Big]_1$$

$$= \sum_{k=1}^{n-4} (-1)^k \sum \Big[_1 T(n-3-k\,;\, n-1) \sum_{n-2-k}^{n-1} \Big[_2 S(n-3-k\,|\,n-3) - \sum_{n-2}^{n-1} \Big[_3 S(n-3-k\,\|\,n-3\,|\,n-2)\Big]_3 + S(n-1)\Big]_2 \Big]_1$$

Then,

7.4.14 $$G_2 = (-1)^2 \sum \Big[_1\, F(n-3)(X_{n-2}X_{n-1}) \Big[_2\, S(n-3) - \sum_{n-2}^{n-1} \Big[_3\, S(n-3|n-2)\Big]_3 + S(n-1)\Big]_2 \Big]_1$$

$$+ \sum_{k=3}^{n-2} (-1)^k \sum \Big[_1\, T(n-1-k;n-1) \Big[_2\, S(n-1-k) - S(n-1)\Big]_2 \Big]_1$$

$$+ \sum_{k=2}^{n-3} (-1)^k \sum \Big[_1\, T(n-2-k;n-1) \sum_{n-1-k}^{n-1} \Big[_2\, S(n-2-k|n-2) - S(n-1)\Big]_2 \Big]_1$$

$$- \sum_{k=1}^{n-4} (-1)^k \sum \Big[_1\, T(n-3-k;n-1) \sum_{n-2-k}^{n-1} \Big[_2\, S(n-3-k|n-3) - \sum_{n-2}^{n-1} \Big[_3\, S(n-3-k||n-3|n-2)\Big]_3 + S(n-1)\Big]_2 \Big]_1$$

Define

7.4.15 $$G_3 = G_2 - (-1)^2 \sum \Big[_1\, F(n-3)(X_{n-2}X_{n-1}) \Big[_2\, S(n-3) - \sum_{n-2}^{n-1} \Big[_3\, S(n-3|n-2)\Big]_3 + S(n-1)\Big]_2 \Big]_1$$

The desired representation of F is now established by induction; to this end we define

7.4.16 $$G_r = G_{r-1} - (-1)^{r-1} \sum \big[_1\, F(n-r)(x_{n-r+1} \cdots x_{n-1}) \big[_2\, S(n-r)$$

$$- \sum_{n-r+1}^{n-1} \big[_2\, S(n-r|n-r+1)\big]_2 + \sum_{n-r+1}^{n-1} \big[_2\, S(n-r|n-r+2)\big]_2$$

$$\cdots + (-1)^r \sum_{n-r+1}^{n-1} \big[_2\, S(n-r|n-2)\big]_2 + (-1)^{r-1}\, S(n-1)\ \big]_1$$

7.4.17 $$G_r = \sum_{k=r}^{n-2} (-1)^k \sum \big[_1\, T(n-1-k\,;n-1) \big[_2\, S(n-1-k) - S(n-1)\big]_2 \big]_1$$

$$+ \sum_{k=r-1}^{n-3} (-1)^k \sum \big[_1\, T(n-2-k\,;n-1) \sum_{n-1-k}^{n-1} \big[_2\, S(n-2-k|n-2) - S(n-1)\big]_2 \big]_1$$

$$- \sum_{k=r-2}^{n-4} (-1)^k \sum \big[_1\, T(n-3-k\,;n-1) \sum_{n-2-k}^{n-1} \big[_2\, S(n-3-k|n-3)$$

$$- \sum_{n-r+1}^{n-1} \big[_3\, S(n-3-k\|n-3|n-2)\big]_3 + S(n-1)\big]_2 \big]_1$$

$$\vdots$$

$$+(-1)^r \sum_{k=1}^{n-1-r} (-1)^k \sum \big[_1\, T(n-r-k\,;n-1) \sum_{n-r+1-k}^{n-1} \big[_2\, S(n-r-k|n-r)$$

$$- \sum_{n-r+1}^{n-1} \big[_3\, S(n-r-k\|n-r|n-r+1)\big]_3$$

$$+ \sum_{n-r+1}^{n-1} \big[_3\, S(n-r-k\|n-r|n-r+2)\big]_3$$

$$\vdots$$

$$+(-1)^r \sum_{n-r+1}^{n-1} \big[_3\, S(n-r-k\|n-r|n-2)\big]_3$$

$$+ (-1)^{r+1}\, S(n-1)\ \big]_2 \big]_1$$

This definition agrees with G_3 when (7.4.14) is substituted in (7.4.15); thus G_3 serves as a basis for the induction.

Now consider the coefficient of $T(n-r-1\,;\,n-1)$ in G_r by setting $k=r,\ k=r-1,\ \cdots,\ k=1$ respectively, in $\sum_{k=r}^{n-2},\ \sum_{k=r-1}^{n-3},\ \cdots,\ \sum_{k=1}^{n-1-r}$ in (7.4.16); denote this coefficient by B_{r+1}.

Then

7.4.18 $$B_{r+1} = (-1)^r \Big[_0 \Big[_1\, S(n-1-r) - S(n-1)\Big]_1$$

$$- \sum_{n-r}^{n-1} \Big[_2\, S(n-1-r\,|\,n-2) - S(n-1)\Big]_2$$

$$- \sum_{n-r}^{n-1} \Big[_3\, S(n-1-r\,|\,n-3) - \sum_{n-r+1}^{n-1} \Big[_{3'}\, S(n-1-r\,||\,n-3\,|\,n-2)\Big]_{3'}$$

$$+ S(n-1)\Big]_3$$

$$\vdots$$

$$- \sum_{n-r}^{n-1} \Big[_r\, S(n-1-r\,|\,n-r) - \sum_{n-r+1}^{n-1} \Big[_{r'}\, S(n-1-r\,||\,n-r\,|\,n-r+1)\Big]_{r'}$$

$$+ \sum_{n-r+1}^{n-1} \Big[_{r''}\, S(n-1-r\,||\,n-r\,|\,n-r+2)\Big]_{r''}$$

$$\vdots$$

$$+(-1)^r \sum_{n-r+1}^{n-1} \Big[_{r^{(r-2)}}\, S(n-1-r\,||\,n-r\,|\,n-2)\Big]_{r^{(r-2)}}$$

$$+(-1)^{r+1}\, S(n-1)\Big]_r\Big]_0$$

The coefficient of $S(n-1)$ in $(-1)^r B_{r+1}$ is

$$(-1) + \binom{r}{r-1} - \binom{r}{r-2} + \cdots + (-1)^r \binom{r}{1} = (-1)^r$$

The coefficient of $S(n-2)$ in $(-1)^r B_{r+1}$ is

$$(-1) + \binom{r-1}{r-2} - \binom{r-1}{r-3} + \cdots + (-1)^{r-1} \binom{r-1}{1} = (-1)^{r-1}$$

The term $S(n-\ell)$ first appears in the bracket $[_\ell \quad]_\ell$ and its coefficient is (-1) ; in the bracket $[_{\ell+1} \quad]_{\ell+1}$ the coefficient is

$$\binom{r}{r-\ell}\binom{\ell}{1} \div \binom{r}{r-\ell+1} = r-\ell+1 = \binom{r-\ell+1}{r-\ell} ;$$

in the bracket $[_{\ell+2} \quad]_{\ell+2}$ the coefficient is

$$-\binom{r}{r-\ell-1}\binom{\ell+1}{2} \div \binom{r}{r-\ell+1} = -\binom{r-\ell+1}{r-\ell-1}$$

The coefficient of $S(n-\ell)$ in $(-1)^r B_{r+1}$ is therefore

$$(-1) + \binom{r-\ell+1}{r-\ell} - \cdots + (-1)^{r-\ell+1} \binom{r-\ell+1}{1} = (-1)^{r-\ell+1}$$

Hence,

7.4.19 $$B_{r+1} = (-1)^r \Big[_1\, S(n-1-r) - \sum_{n-r}^{n-1} \big[_2\, S(n-1-r \mid n-r)\big]_2$$

$$+ \sum_{n-r}^{n-1} \big[_2\, S(n-1-r \mid n-r+1)\big]_2$$

$$\vdots$$

$$+ (-1)^{r-1} \sum_{n-r}^{n-1} \big[_2\, S(n-1-r \mid n-2)\big]_2$$

$$+ (-1)^r S(n-1) \Big]_1$$

Now add and subtract to (7.4.17)

$$A_{r+1} = \sum \Big[_1 \Big[_2 \sum_{k=1}^{n-r-2} (-1)^k \sum T(n-r-1-k; n-r-1) \Big]_2 (x_{n-r} \cdots x_{n-1})\, B_{r+1} \Big]_1$$

$$= (-1)^r \sum_{k=1}^{n-r-2} (-1)^k \sum T(n-r-1-k;\, n-1)\, D_{r+1}$$

where

$$D_{r+1} = \sum_{n-r-k}^{n-1} \Big[_1\, S(n-r-1-k \mid n-r-1) - \sum_{n-r}^{n-1} \Big[_2\, S(n-r-1-k \| n-r-1 \mid n-r) \Big]_2$$

$$+ \sum_{n-r}^{n-1} \Big[_2\, S(n-r-1-k \| n-r-1 \mid n-r+1) \Big]_2$$

$$\vdots$$

$$+ (-1)^{r+1} \sum_{n-r}^{n-1} \Big[_2\, S(n-r-1-k \| n-r-1 \mid n-2) \Big]_2$$

$$+ (-1)^r S(n-1) \Big]_1$$

Now

7.4.20 $\sum T(n-r-1;\, n-1)\, B_{r+1} + A_{r+1} = \sum F(n-r-1)(x_{n-r} \cdots x_{n-1})\, B_{r+1}$

Using an obvious abbreviation, we write (7.4.17) in the form

$$G_r = \sum T(n-r-1;\, n-1)\, B_{r+1} + \sum_{k=r+1}^{n-2} + \sum_{k=r}^{n-3} - \cdots + (-1)^r \sum_{k=2}^{n-r-1}\ ;$$

adding and subtracting A_{r+1} and using (7.4.20), we have

$$G_r = \sum F(n-r-1)(x_{n-r}\cdots x_{n-1})\, B_{r+1} + \sum_{k=r+1}^{n-2} + \cdots + (-1)^r \sum_{k=2}^{n-r-1} - A_{r+1}$$

By (7.4.17)

$$\sum_{k=r+1}^{n-2} + \sum_{k=r}^{n-3} + \cdots + (-1)^r \sum_{k=2}^{n-r-1} - A_{r+1} = G_{r+1}$$

hence

7.4.21 $G_{r+1} = G_r - \sum F(n-r-1)(x_{n-r}\cdots x_{n-1})\, B_{r+1}$;

when we substitute (7.4.19) for B_{r+1} in (7.4.21), this becomes the expression (7.4.16) with r replaced by $r+1$ and the induction is completed.

Hence,

7.4.22
$$F(n) = \bar{F}(n-1) + F(n-1)\, S(n-1) + \sum_{k=1}^{n-2} (+1)^k \sum F(n-1-k)(x_{n-k}\cdots x_{n-1})\, B_{k+1}$$

where

$$F(j) \equiv 1 \qquad j = 1, 2, \cdots, n-1$$

$$B_{k+1} = (-1)^k \Big[_1\, S(n-1-k) - \sum_{n-k}^{n-1} \Big[_2\, S(n-1-k \mid n-k) + \sum_{n-k}^{n-1} \Big[_2\, S(n-1-k \mid n-k+1)\Big]_2$$

$$\cdots + (-1)^{k-1} \sum_{n-k}^{n-1} \Big[_2\, S(n-1-k \mid n-2)\Big]_2 + (-1)^k\, S(n-1)\Big]_1$$

Note that the arguments of S in B_{k+1} are in 1 - 1 correspondence with the exponents of e in the expansion of

7.4.23
$$e^{x_1 + \cdots + x_{n-1-k}} \prod_{j=n-k}^{n-1} \{1 - e^{x_j}\}$$

From this remark it is easy to see that the expressions

$$(x_{n-k} \cdots x_{n-1})\, B_{k+1}$$

are analytic at $x_j = 0$ for $j = n-k, \cdots, n-1$. For they have at most simple poles at those points and the residues there are zero;
for example,

$$\lim_{x_{n-k} \to 0} x_{n-k}(x_{n-k})\,(x_{n-k+1} \cdots x_{n-1})\, B_{k+1} = 0$$

Since we have in mind to compute the Fourier coefficients of F from the form (7.4.22), we shall be interested in this form only for real values of all the variables x_i, $i = 1, 2, \cdots, n$. By our remark that $(x_{n-k} \cdots x_{n-1})\, B_{k+1}$ is analytic at $x_j = 0$ for $j = n-k, \cdots, n-1$ and the fact that $\bar{F}(n-1)$ and the $F(n-1-k)$ are analytic in the cylindrical regions (T_{n-1-k}), we can assert that at least for real values of all the variables x_i, we have written F as a sum of functions which are analytic.

5. Another Representation of $\bar{F}(x_1, x_2, \cdots, x_n)$

We show now that $\bar{F}(x_1, x_2, \cdots, x_n)$ can be expressed in terms of $F(x_1, x_2, \cdots, x_{n-j})$ and $\bar{F}(x_n, x_1 + \cdots + x_{n-j})$ for $j = 1, 2, \cdots, n-2$ and $n \geq 3$. We abbreviate the definition of $\bar{F}$ to the statement (7.3.1a).

Recalling (7.4.1), we write

$$[x_1 \cdots x_n](x_1 + \cdots + x_n) = [x_1 \cdots x_{n-1}]\left[_1 (x_1 + \cdots + x_{n-1}) + (x_n)\right.$$

$$\left. + S(x_n, x_1 + \cdots + x_{n-1})\right]_1 (x_1 + \cdots + x_n)$$

$$= [x_1 \cdots x_{n-1}]\left[_1 (x_1 + \cdots + x_{n-1})(x_n)\right.$$

$$\left. - 1 + S(x_n, x_1 + \cdots + x_{n-1})(x_1 + \cdots + x_n)\right]_1$$

By (7.1.14),

$$7.5.1 \quad [x_1 \cdots x_n](x_1 + \cdots + x_n) = [x_1 \cdots x_{n-1}]\left[_1 (x_1 + \cdots + x_{n-1})(x_n)\right.$$

$$\left. + \bar{S}(x_n, x_1 + \cdots + x_{n-1})\right]_1$$

Now we write (7.3.1a), observing that the term which corresponds to $j = n-1$ can be added to the last term, namely,

$$\bar{F}(x_1, \cdots, x_n) = [x_1 \cdots x_n](x_1 + \cdots + x_n)$$

$$+ \sum_{j=1}^{n-2} (-1)^j \sum [x_1 \cdots x_{n-j}](x_1 + \cdots + x_{n-j})(x_{n-j+1} \cdots x_n)$$

$$+ (-1)^{n-1} \{n-1\} (x_1 \cdots x_n)$$

By (7.1.16)

$$7.5.2 \quad \sum_{j=1}^{n-2} (-1)^j \sum [x_1 \cdots x_{n-j}](x_1 + \cdots + x_{n-j})(x_{n-j+1} \cdots x_n)$$

$$= \quad (-1)[x_1 \cdots x_{n-1}](x_1 + \cdots + x_{n-1})(x_n)$$

$$+ \sum_{j=2}^{n-2} (-1)^j \sum [x_1 \cdots x_{n-j}](x_1 + \cdots + x_{n-j})(x_n | x_{n-j+1} \cdots x_{n-1})$$

$$+ \sum_{j=1}^{n-2} (-1)^j \sum [x_n | x_1 \cdots x_{n-j-1}](x_n | + x_1 + \cdots + x_{n-j-1})(x_{n-j} \cdots x_{n-1})$$

Substituting (7.5.1) and (7.5.2) in the definition of $\bar{F}$ and writing $[x_n | x_1 \cdots x_{n-j-1}](x_n | + x_1 + \cdots + x_{n-j-1})$ in terms of (7.5.1), we have:

$$7.5.3 \quad \bar{F}(x_1, \cdots, x_n) = \Big[_1 [x_1 \cdots x_{n-1}] \bar{S}(x_n, x_1 + \cdots + x_{n-1})$$

$$+ \sum_{j=2}^{n-2} (-1)^j \sum [x_1 \cdots x_{n-j}](x_1 + \cdots + x_{n-j})(x_n | x_{n-j+1} \cdots x_{n-1})$$

$$+ \sum_{j=1}^{n-2} (-1)^j \sum \Big[_2 [x_1 \cdots x_{n-j-1}] \Big[_3 (x_1 + \cdots + x_{n-j-1})(x_n |)$$

$$+ \bar{S}(x_n |, x_1 + \cdots + x_{n-j-1}) \Big]_3 (x_{n-j} \cdots x_{n-1}) \Big]_2$$

$$+ (-1)^{n-1} \{n-1\}(x_1 \cdots x_n) \Big]_1$$

The terms involving $\cot x_n$ cancel and we obtain

7.5.4 $$\bar{F}(x_1, \cdots, x_n) = \Big[_1 [x_1 \cdots x_{n-1}]\ \bar{S}(x_n, x_1 + \cdots + x_{n-1})$$

$$+ \sum_{j=1}^{n-2} (-1)^j \sum [x_1 \cdots x_{n-j-1}](x_{n-j} \cdots x_{n-1})\ \bar{S}(x_n |, x_1 + \cdots + x_{n-j-1})\Big]_1$$

Using the notation of (7.1.18), we write (cf. (7.4.4 - 7.4.5))

$$\bar{S}(x_n |, x_1 + \cdots + x_{n-k}) = \bar{S}(n-k);$$

then (7.5.4) becomes

7.5.5 $$\bar{F}(n) = [x_1 \cdots x_{n-1}]\ \bar{S}(n-1)$$

$$+ \sum_{k=1}^{n-2} (-1)^k \sum T(n-k-1; n-1)\ \bar{S}(n-k-1)$$

Comparing this with (7.4.2), we see that we can write (7.5.5) directly from (7.4.22) in the form

7.5.6 $$\bar{F}(n) = F(n-1)\ \bar{S}(n-1)$$

$$+ \sum_{k=1}^{n-2} (+1)^k \sum F(n-1-k)(x_{n-k} \cdots x_{n-1})\ \bar{B}_{k+1}$$

where, corresponding to (7.4.19),

7.5.7 $$\bar{B}_{k+1} = (-1)^k \Big[_1 \bar{S}(n-1-k) - \sum_{n-k}^{n-1} \Big[_2 \bar{S}(n-1-k \,|\, n-k)\Big]_2$$

$$+ \sum_{n-k}^{n-1} \Big[_2 \bar{S}(n-1-k \,|\, n-k)\Big]_2$$

$$\vdots$$

$$+ (-1)^{k-1} \sum_{n-k}^{n-1} \Big[_2 \bar{S}(n-1-k \,|\, n-2)\Big]_2$$

$$+ (-1)^k\ \bar{S}(n-1)\Big]_1$$

We remark that at least for real values of all the variables $x_i \quad i = 1, 2, \cdot\cdot, n$, we have written $\bar{F}$ as a sum of functions which are analytic. (See the remark at the end of the previous section.)

6. The Fourier Coefficients of $F(x_1, x_2, \cdots, x_n)$

$$F(x_1, x_2, \cdots, x_n) = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} c_{k_1 \cdots k_n} \, e^{2i(k_1 x_1 + \cdots + k_n x_n)}$$

where

7.6.1 $c_{k_1 \cdots k_n}$

$$= \frac{1}{\pi^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(x_1, \cdots, x_n) e^{-2i(k_1 x_1 + \cdots + k_n x_n)} \, dx_1 \cdots dx_n$$

Now $F(-x_1, -x_2, \cdots, -x_n) = (-1)^{n+1} F(x_1, x_2, \cdots, x_n)$

hence $c_{-k_1, -k_2, \cdots, -k_n} = (-1)^{n+1} c_{k_1 k_2 \cdots k_n}$

Then

7.6.2 $F(x_1, x_2, \cdots, x_n)$

$$= \sum_{k_n=1}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{n-1}=-\infty}^{\infty} c_{k_1 \cdots k_n} \left[e^{2i(k_1 x_1 + \cdots + k_n x_n)} + (-1)^{n+1} e^{-2i(k_1 x_1 + \cdots + k_n x_n)} \right]_1$$

$$+ \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{n-1}=-\infty}^{\infty} c_{k_1 \cdots k_{n-1} 0} \left[e^{2i(k_1 x_1 + \cdots + k_{n-1} x_{n-1})} + (-1)^{n+1} e^{-2i(k_1 x_1 + \cdots + k_{n-1} x_{n-1})} \right]_1$$

$$\left[e^{2i(k_1 x_1 + \cdots + k_n x_n)} + (-1)^{n+1} e^{-2i(k_1 x_1 + \cdots + k_n x_n)} \right] = \begin{cases} 2i \sin(k_1 x_1 + \cdots + k_n x_n) \\ \text{if } n \text{ is even} \\ 2 \cos(k_1 x_1 + \cdots + k_n x_n) \\ \text{if } n \text{ is odd.} \end{cases}$$

From (7.6.2) we see that we need only consider $c_{k_1 \cdots k_n}$ for $k_n \geq 1$ and $c_{k_1 \cdots k_{n-1} 0}$.

By the remark at the end of section 4 of this chapter, it is possible to write

7.6.3 $$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(n)\, e^{-2ik_n x_n}\, dx_n$$

$$= \frac{1}{\pi}\, \bar{F}(n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-2i k_n x_n}\, dx_n$$

$$+ \frac{1}{\pi}\, F(n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S(n-1)\, e^{-2i k_n x_n}\, dx_n$$

$$+ \sum_{k=1}^{n-2} (+1)^k \sum \frac{1}{\pi} F(n-1-k) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x_{n-k} \cdots x_{n-1})\, B_{k+1}\, e^{-2ik_n x_n}\, dx_n$$

Now for $k_n \geq 1$ we have from (6.1) and (7.1.18)

7.6.4 $$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S(n-j)\, e^{-2ik_n x_n}\, dx_n = f_{k_n}(x_1 + \cdots + x_{n-j}) \equiv f_{k_n}(n-j)$$

and for $k_n = 0$

7.6.5 $$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S(n-j)\, dx_n = 0$$

Writing $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x_{n-k} \cdots x_{n-1})\, B_{k+1}\, e^{-2ik_n x_n}\, dx_n$ as a sum of integrals and using (7.6.5) we have for $k_n = 0$

7.6.6 $$c_{k_1 \cdots k_{n-1} 0} = \frac{1}{\pi^{n-1}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{F}(n-1)\, e^{-2i(k_1 x_1 + \cdots + k_{n-1} x_{n-1})}\, dx_1 \cdots dx_{n-1}$$

Using (7.6.4) we have for $k_n \geq 1$

7.6.7 $$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(n) e^{-2ik_n x_n}\, dx_n$$

$$= F(n-1) f_{k_n}(n-1) + \sum_{k=1}^{n-2} (-1)^k \sum F(n-1-k)(x_{n-k} \cdots x_{n-1}) \Big[_1 \quad \Big]_1$$

where

$$\Big[_1 \quad \Big]_1 \equiv (-1)^k \Big[_1\, f_{k_n}(n-1-k) - \sum_{n-k}^{n-1} \Big[_2\, f_{k_n}(n-1-k \mid n-k)\Big]_2$$
$$+ \sum_{n-k}^{n-1} \Big[_2\, f_{k_n}(n-1-k \mid n-k+1)\Big]_2$$
$$\vdots$$
$$+ (-1)^{k-1} \sum_{n-k}^{n-1} \Big[_2\, f_{k_n}(n-1-k \mid n-2)\Big]_2$$
$$+ (-1)^k f_{k_n}(n-1) \Big]_1$$

Substituting from (6.1) the q-series for f_{k_n} and taking into account the remark concerning (7.4.23), we have

$$\Big[_1 \quad \Big]_1 = -2i \sum_{s=1}^{\infty} q^{2k_n s} e^{2is(x_1 + \cdots + x_{n-1-k})} \prod_{j=n-k}^{n-1} \{ e^{2isx_j} - 1 \}$$

Thus (7.6.7) becomes

7.6.8 $$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(n) e^{-2ik_n x_n}\, dx_n$$

$$= F(n-1) f_{k_n}(n-1)$$
$$-2i \sum_{k=1}^{n-2} (+1)^k \sum \Big[_1 F(n-1-k) \sum_{s=1}^{\infty} q^{2k_n s} e^{2is(x_1 + \cdots + x_{n-1-k}} \prod_{j=n-k}^{n-1} (x_j) \{ e^{2isx_j} - 1 \} \Big]_1$$

Then, from (7.6.1), (7.6.8), and (6.3) we have for $k_n \geq 1$

$$c_{k_1 \cdots k_n} = -2i \sum_{s=1}^{\infty} q^{2k_n s} \frac{1}{\pi^{n-1}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(n-1) \prod_{j=1}^{n-1} \{ e^{2ix_j(s-k_j)} dx_j \}$$

$$-2i \sum_{s=1}^{\infty} q^{2k_n s} \sum_{k=1}^{n-2} (+1)^k \frac{1}{\pi^{n-1}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum \Big[_1 \ \Big]_1$$

where

$$\Big[_1 \ \Big]_1 = F(n-1-k) \prod_{j=1}^{n-k-1} e^{2ix_j(s-k_j)} dx_j \prod_{\ell=n-k}^{n-1} K_{sk_\ell}$$

We denote the Fourier coefficient of $F(n-j)$ with indices $k_1, k_2, \cdots, k_{n-j}$ by $C(n-j; k_1, k_2, \cdots, k_{n-j})$; of $\bar{F}(n-j)$, by $\bar{C}(n-j; k_1, k_2, \cdots, k_{n-j})$.

Then

$$\underset{k_n \geq 1}{c(n; k_1, k_2, \cdots, k_n)} = -2i \sum_{s=1}^{\infty} q^{2k_n s} \Big[_1 \, c(n-1; k_1 - s, \cdots, k_{n-1} - s)$$

$$+ \sum_{k=1}^{n-2} (+1)^k \sum c(n-1-k; k_1 - s, \cdots, k_{n-k-1} - s) \prod_{\ell=n-k}^{n-1} K_{sk_\ell} \Big]_1$$

where $\sum$ is on $k_1, k_2, \cdots, k_{n-1}$;

$$c(n; k_1, k_2, \cdots, k_{n-1}, 0) = \bar{c}(n-1; k_1, k_2, \cdots, k_{n-1})$$

7. The Fourier Coefficients of $\bar{F}(x_1, x_2, \cdots, x_n)$

Define

7.7.1 $$\bar{c}_{k_1 \cdots k_n} = \frac{1}{\pi^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{F}(n) e^{-2i(k_1 x_1 + \cdots + k_n x_n)} dx_1 \cdots dx_n$$

Then, since

$$\bar{F}(-x_1, -x_2, \cdots, -x_n) = (-1)^n \bar{F}(x_1, x_2, \cdots, x_n)$$

we have (cf. 7.6.2)

7.7.2 $$\bar{F}(x_1, x_2, \cdots, x_n)$$

$$= \sum_{k_n=1}^{\infty} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{n-1}=-\infty}^{\infty} \bar{c}_{k_1 \cdots k_n} \left[{}_1 e^{2i(k_1 x_1 + \cdots + k_n x_n)} + (-1)^n e^{-2i(k_1 x_1 + \cdots + k_n x_n)} \right]_1$$

$$+ \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{n-1}=-\infty}^{\infty} \bar{c}_{k_1 \cdots k_{n-1} 0} \left[{}_1 e^{2i(k_1 x_1 + \cdots + k_{n-1} x_{n-1})} + (-1)^n e^{-2i(k_1 x_1 + \cdots + k_{n-1} x_{n-1})} \right]_1$$

hence, we need only consider $\bar{c}_{k_1 \cdots k_n}$ for $k_n \geq 1$ and $\bar{c}_{k_1 \cdots k_{n-1} 0}$

By the remark at the end of section 5, we can write:

7.7.3 $$\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{F}(n) e^{-2ik_n x_n} dx_n$$

$$= \frac{1}{\pi} F(n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{S}(n-1) e^{-2ik_n x_n} dx_n$$

$$+ \sum_{k=1}^{n-2} (+1)^k \sum F(n-1-k) \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x_{n-k} \cdots x_{n-1}) \bar{B}_{k+1} e^{-2ik_n x_n} dx_n$$

By (5.1) we have for $k_n \geq 1$

7.7.4 $$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{S}(n-1)e^{-2ik_nx_n}\,dx_n = \bar{f}_{k_n}(x_1+\cdots+x_{n-1}) \equiv \bar{f}_{k_n}(n-1)$$

By (5.3) we have for $k_n = 0$

7.7.5 $$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{S}(n-1)\,dx_n = -1 + 4\sum_{s=1}^{\infty} q^{2s^2} + 8\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2s^2+2sr}\cos 2r(x_1+\cdots+x_{n-1})$$

$$= -1 + 4\sum_{s=1}^{\infty}\sum_{r=-\infty}^{\infty} q^{2s^2+2s|r|} e^{2ir(x_1+\cdots+x_{n-1})}$$

$$\equiv \bar{f}_0(n-1)$$

Then, writing $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x_{n-k}\cdots x_{n-1})\,\bar{B}_{k+1}e^{-2ik_nx_n}\,dx_n$ as a sum of integrals,

and using (7.7.4 - 7.7.5), we have (7.7.3) in the form

7.7.6 $$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{F}(n)\,e^{-2ik_nx_n}\,dx_n$$

$$= F(n-1)\,\bar{f}_{k_n}(n-1) + \sum_{k=1}^{n-2}(+1)^k \sum F(n-1-k)(x_{n-k}\cdots x_{n-1})\Big[_1 \quad \Big]_1$$

where

$$\Big[_1 \quad \Big]_1 \equiv (-1)^k\Big[_1\, \bar{f}_{k_n}(n-1-k) - \sum_{n-k}^{n-1}\Big[_2\,\bar{f}_{k_n}(n-1-k\,|\,n-k)\Big]_2 + \sum_{n-k}^{n-1}\Big[_2\,\bar{f}_{k_n}(n-1-k\,|\,n-k+1)\Big]_2$$

$$\cdots + (-1)^{k-1}\sum_{n-k}^{n-1}\Big[_2\,\bar{f}_{k_n}(n-1-k\,|\,n-2)\Big]_2 + (-1)^k\,\bar{f}_{k_n}(n-1)\Big]_1$$

When $k_n \geq 1$ we substitute for $\bar{f}_{k_n}$ in (7.7.6) the q-series given by (5.2); taking into account the remark concerning (7.4.23), we have

7.7.7 $$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{F}(n)\, e^{-2ik_n x_n}\, dx_n$$

$$= F(n-1)\Big[_1\, 2\sum_{r=1}^{\infty} q^{2k_n r} e^{2ir(x_1+\cdots+x_{n-1})} + 4\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2r(k_n+s)} e^{2i(r-s)(x_1+\cdots+x_{n-1})}\Big]_1$$

$$+ \sum_{k=1}^{n-2}(+1)^k \sum F(n-1-k)\Big[_1\, 2\sum_{r=1}^{\infty} q^{2k_n r} e^{2ir(x_1+\cdots+x_{n-1-k})} \prod_{j=n-k}^{n-1}(x_j)\{e^{2irx_j}-1\}$$

$$+ 4\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2r(k_n+s)} e^{2i(r-s)(x_1+\cdots+x_{n-1-k})} \prod_{j=n-k}^{n-1}(x_j)\{e^{2i(r-s)x_j}-1\}$$

When $k_n = 0$, we substitute for $\bar{f}_0(n-1)$ in (7.7.6) the q-series of (7.7.5). Then

7.7.8 $$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{F}(n)\, dx_n$$

$$= F(n-1)\Big[_1 -1 + 4\sum_{s=1}^{\infty}\sum_{r=-\infty}^{\infty} q^{2s^2+2s|r|} e^{2ir(x_1+\cdots+x_{n-1})}\Big]_1$$

$$+ 4\sum_{k=1}^{n-2}(+1)^k \sum_{s=1}^{\infty}\sum_{r=-\infty}^{\infty} q^{2s^2+2s|r|} \sum F(n-1-k)\, e^{2ir(x_1+\cdots+x_{n-k-1})} \prod_{j=n-k}^{n-1}(x_j)\{e^{2irx_j}-1\}$$

Then, using the notation at the end of the last section,

$$\begin{aligned}
\bar{c}(n;k_1,\cdots,k_n) &= 2\sum_{r=1}^{\infty} q^{2k_n r}\, c(n-1;k_1-r,k_2-r,\cdots,k_{n-1}-r) \\
{\scriptstyle k_n\ge 1}\qquad &+ 4\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2r(k_n+s)}\, c(n-1;k_1-r+s,\cdots,k_{n-1}-r+s) \\
&+ 2\sum_{r=1}^{\infty} q^{2k_n r}\sum_{k=1}^{n-2}\sum c(n-1-k;k_1-r,\cdots,k_{n-1}-r)\prod_{l=n-k}^{n-1} K_{r k_l} \\
&+ 4\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} q^{2r(k_n+s)}\sum_{k=1}^{n-2}\sum c(n-1-k;k_1-r+s,\cdots,k_{n-1-k}-r+s)\prod_{l=n-k}^{n-1} K_{(r-s)k_l}
\end{aligned}$$

$$\begin{aligned}
\bar{c}(n;k_1,\cdots,k_{n-1},0) &= -c(n-1;k_1,\cdots,k_{n-1}) \\
&+ 4\sum_{s=1}^{\infty}\sum_{r=-\infty}^{\infty} q^{2s^2+2s|r|}\, c(n-1;k_1-r,\cdots,k_{n-1}-r) \\
&+ 4\sum_{s=1}^{\infty}\sum_{r=-\infty}^{\infty} q^{2s^2+2s|r|}\sum_{k=1}^{n-2}\sum c(n-1-k;k_1-r,\cdots,k_{n-k-1}-r)\prod_{l=n-k}^{n-1} K_{r k_l}
\end{aligned}$$

A Paraphrase Result Obtained From $F(x, y, z)$

We return now to the function $F(x, y, z)$ defined in Chapter VI. Let $S(x, y, z)$ denote the Fourier expansion of $F(x, y, z)$ (see (6.4)), let $S(y, z)$ denote the Fourier expansion of $F(y, z)$, and let $\bar{S}(y, z)$ denote the Fourier expansion of $\bar{F}(y, z)$.

Then we write

$$S(x, y, z) = [xyz] - \sum [xy](z) + \sum [x](yz);$$

hence,

$$[xyz] = S(x, y, z) + S(x, y)(z) + S(x, z)(y) + S(y, z)(x) + (xy) + (xz) + (yz)$$

Again, following Gage (5), we have

$$[xyz] = [yz][x, y+z]$$

$$= \{(y) + (z) + S(y, z)\}\{(x) + (y+z) + S(x, y+z)\}$$

$$= -1 + (y+z)\, S(y, z) + (xy) + (xz) + (yz)$$

$$+ S(x, y+z)(y) + S(x, y+z)(z) + S(y, z)(x)$$

$$+ S(y, z)\, S(x, y+z)$$

Combining the two results,

8.1 $$S(x, y, z) - \bar{S}(y, z) = (y)\{S(x, y+z) - S(x, z)\} + (z)\{S(x, y+z) - S(x, y)\} + S(x, y+z)\, S(y, z)$$

Recall now that the terms in $S(x,y,z)$ which do not involve x are exactly $\bar{S}(y,z)$; we denote by $S^*(x,y,z)$ the result of writing $\{S(x,y,z)-\bar{S}(y,z)\}$ with $2x$, $2y$, $2z$, q^2 replaced by x, y, z, q and terms arranged according to the quadratic form of the exponent of q. Thus,

8.2 $$S^*(x,y,z) = 8\sum_{l=1}^{\infty}\sum_{k=1}^{\infty} q^{k^2+lk}\cos lx$$

$$+4\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} q^{lm}\{\cos(lx+my)+\cos(lx+mz)+2\cos(lx+my+mz)\}$$

$$+8\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} q^{k^2+k(l+m)+lm}\{\cos(lx+my)+\cos(lx+mz)\}$$

$$+8\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} q^{k^2+k(l+m)}\{\cos(lx-my)+\cos(lx-mz)\}$$

$$+8\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} q^{k^2+k(l+m+n)+mn}\{\cos(lx-my-nz)+\cos(mx-ly+nz)$$
$$+\cos(nx+my-lz)\}$$

$$+8\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} q^{k^2+kl+lm}\cos(lx+my+mz)$$

$$-8\sum_{l=1}^{\infty}\sum_{m=2}^{\infty}\sum_{k=1}^{m-1} q^{k^2+k(l-2m)+m^2}\cos(lx+my+mz)$$

$$+8\sum_{l=1}^{\infty}\sum_{m=2}^{\infty}\sum_{n=1}^{m-1} q^{lm}\{\cos(lx+my+nz)+\cos(lx+ny+mz)\}$$

$$+8\sum_{l=1}^{\infty}\sum_{\substack{m=2\\ m>n}}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} q^{k^2+k(l+m-n)+lm}\{\cos(lx+my+nz)+\cos(lx+ny+mz)\}$$

$$-8\sum_{l=1}^{\infty}\sum_{\substack{m=3\\ m>n}}^{\infty}\sum_{n=2}^{\infty}\sum_{k=1}^{n-1} q^{k^2+k(l-m-n)+mn}\{\cos(lx+my+nz)+\cos(lx+ny+mz)\}$$

We exhibit next the result of writing in terms of cosines alone the right hand side of (8.1), again with $2x, 2y, 2z, q^2$ replaced by x, y, z, q ; we denote this result also by $S^*(x, y, z)$.

8.3
$$S^*(x, y, z) = 8 \sum_{d=1}^{\infty} \sum_{n=1}^{\infty} q^{d(2n)} \{\cos(dx + ny + 2nz) + \cos(dx + 2ny + nz)\}$$
$$+ 4 \sum_{d=1}^{\infty} \sum_{\delta=1}^{\infty} q^{d\delta} \{\cos(dx + \delta y) + \cos(dx + \delta z) + 2\cos(dx + \delta y + \delta z)\}$$
$$+ 8 \sum_{d=1}^{\infty} \sum_{n=2}^{\infty} q^{d(2n)} \sum_{r=1}^{n-1} \{\cos[dx + (n+r)y + 2nz] + \cos[dx + (n-r)y + 2nz]$$
$$+ \cos[dx + 2ny + (n+r)z] + \cos[dx + 2ny + (n+r)z]\}$$
$$+ 8 \sum_{d=1}^{\infty} \sum_{\substack{\tau=3 \\ \tau \text{ odd}}}^{\infty} q^{d\tau} \sum_{r=1}^{\frac{\tau-1}{2}} \{\cos[dx + (\tfrac{\tau-1}{2} + r)y + \tau z] + \cos[dx + (\tfrac{\tau+1}{2} - r)y + \tau z]$$
$$+ \cos[dx + \tau y + (\tfrac{\tau-1}{2} + r)z] + \cos[dx + \tau y + (\tfrac{\tau+1}{2} - r)z]\}$$
$$+ 8 \sum_{d_1=1}^{\infty} \sum_{\delta_1=1}^{\infty} \sum_{d_2=1}^{\infty} \sum_{\delta_2=1}^{\infty} q^{d_1\delta_1 + d_2\delta_2} \{\cos[d_2 x + (\delta_2 - d_1)y + (\delta_2 - \delta_1)z]$$
$$- \cos[d_2 x + (\delta_2 + d_1)y + (\delta_2 + \delta_1)z]\}$$

Note that the second line in (8.2) is identical with the second line in (8.3), and when these terms are cancelled, there is a common factor 8 throughout (8.2) and (8.3). We are now ready to paraphrase (1) the results; before doing so, we introduce a new use of $\sum$, namely $\sum_{()}' f(\alpha, \beta, \gamma)$ which denotes summation over all integral solutions of a certain representation of N.

Also, we enumerate all of the representations of N as follows:

1. $N = k^2 + lk \qquad k \geq 1 \quad l \geq 1$

2. $N = k^2 + k(l+m) + lm \qquad m \geq 1$

3. $N = k^2 + k(l+m)$

4. $N = k^2 + k(l+m+n) + mn \qquad n \geq 1$

5. $N = k^2 + kl + lm$

6. $N = k'^2 + k'(l-2m') + m'^2 \qquad 1 \leq k' \leq m'-1 \quad m' \geq 2$
   $= k'^2 + l(m'-k')$

7. $N = lm'$

8. $N = k^2 + k(l+m'-n') + lm' \qquad m' > n' \geq 1$

9. $N = k''^2 + k''(l-m''-n'') + m''n'' \qquad 1 \leq k'' \leq n''-1 \quad 2 \leq n'' < m''$
   $= k''^2 + k''(m''-n'') + l(n''-k'')$

10. $N = d(2\delta) \qquad d \geq 1 \quad \delta \geq 1$

11. $N = d(2\delta') \qquad \delta' \geq 2$

12. $N = d\gamma \qquad \gamma$ odd $\quad \gamma \geq 3$

13. $N = d_1\delta_1 + d_2\delta_2 \qquad d_1 \geq 1 \quad \delta_1 \geq 1 \quad d_2 \geq 1 \quad \delta_2 \geq 1$

Let $f(x, y, z)$ be an arbitrary junction which is symmetric on x, y, z and satisfying

$$f(-x, -y, -z) = f(x, y, z)$$

Then paraphrasing (8.2) and (8.3) we have:

8.4
$$\sum_{(1)} f(l,0,0) + \sum_{(2)} [f(l,m,0) + f(l,0,m)] + \sum_{(3)} [f(l,-m,0) + f(l,0,-m)]$$

$$+ \sum_{(4)} [f(l,-m,-n) + f(m,-l,n) + f(n,m,-l)] + \sum_{(5)} f(l,m,m)$$

$$- \sum_{(6)} f(l,m',m') + \sum_{(7)} \sum_{n'=1}^{m'-1} [f(l,m',n') + f(l,n',m')]$$

$$+ \sum_{(8)} [f(l,m',n') + f(l,n',m')] - \sum_{(9)} [f(l,m'',n'') + f(l,n'',m'')]$$

$$\equiv$$

$$\sum_{(10)} [f(d,\delta,2\delta) + f(d,2\delta,\delta)]$$

$$+ \sum_{(11)} \sum_{r=1}^{\delta'-1} [f(d,\delta'+r,2\delta') + f(d,\delta'-r),2\delta') + f(d,2\delta',\delta'+r) + f(d,2\delta',\delta'-r)]$$

$$+ \sum_{(12)} \sum_{r=1}^{\frac{\tau-1}{2}} [f(d,\tfrac{\tau-1}{2}+r,\tau) + f(d,\tfrac{\tau+1}{2}-r,\tau) + f(d,\tau,\tfrac{\tau-1}{2}+r) + f(d,\tau,\tfrac{\tau+1}{2}-r)]$$

$$+ \sum_{(13)} [f(d_2,\delta_2-d_1,\delta_2-\delta_1) - f(d_2,\delta_2+d_1,\delta_2+\delta_1)]$$

Note that if N is odd, $\sum_{(10)} \equiv \sum_{(11)} \equiv 0$ and $\sum_{(7)} \equiv \sum_{(12)}$; also, if N is a prime and $N \geq 3$, $\sum_{(2)} \equiv 0$

Now define $N(\alpha)$ to be the number of representations of N in the form α, where α is one of the representations 1-13; e.g., $N(4) = 1$, $N(6) = 4$ if $N=5$. Let $f(x,y,z) \equiv 1$ Then we have the

Theorem: If N is a prime and $N \geq 3$, then

$$N(1) + 2N(3) + 3N(4) + N(5) - N(6) + 2N(8) - 2N(9) = 0$$

The integral solutions of 1-13 for $N=5$ are listed below.

1. $(k,l) = (1,4)$
2. No solutions.
3. $(k,l,m) = (1,1,3); (1,2,2); (1,3,1)$
4. $(k,l,m,n) = (1,1,1,1)$
5. $(k,l,m) = (1,2,1); (1,1,3)$
6. $(k,l,m') = (1,1,5); (1,2,3); (1,4,2); (2;1;3)$
7. $(l,m') = (1,5)$
8. $(k,l,m',n') = (1,1,2,1)$
9. $(k'', l, m'', n'') = (1,1,5,4); (1,3,3,2); (1,1,5,3); (1,2,4,2); (1,1,5,2)$
10. No solutions.
11. No solutions.
12. $(d,\gamma) = (1,5)$
13. $(d_1,\delta_1,d_2,\delta_2) = (1,1,1,4); (1,1,4,1); (1,1,2,2); (1,2,1,3); (2,1,1,3); (1,2,3,1); (2,1,3,1)$ $(1,3,1,2); (3,1,1,2); (1,3,2,1); (3,1,2,1)$ $(1,4,1,1); (4,1,1,1); (2,2,1,1)$

Then, as stated by the theorem,

$$1 + 2\cdot 3 + 3\cdot 1 + 2 - 4 + 2\cdot 1 - 2\cdot 5 = 0$$

If (8.4) is written out for $N=5$, it is observed that the identity holds without taking account of the symmetry or parity of f.

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