AERODYNAMIC THEORY OF THE OSCILLATING WING-AILERON OF FINITE SPAN

Thesis by

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This thesis consists of two parts as listed below:

Part A. Aerodynamic Theory of the Oscillating Wing Aileron of Finite Span;

Part B. Flutter Report No. 5.

(Due to the classification of this report, its inclusion as Part B of this thesis cannot be allowed until the lifting of such restrictions).

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PREFACE

Shortly after arriving at Pasadena in June, 1941, Dr. M. A. Biot presented the author with the problem which is given in reference 1. For shortness of form, reference 1 is simply referred to as Report 5. Dr. Biot had already done some work on this problem and had conceived the idea of the vortex approximations which are set forth in section I-1 part II of Report 5. Under the guidance of Dr. Biot this problem was started, and after it was well along he suggested that it be used for the author's thesis. This was brought to the attention of Dr. Theodore von Kármán who approved the suggestion provided that some part would be solitary work so that the requirements of a thesis would be satisfied. That part of the problem which was to be considered as solitary work was not settled until September, 1942, at which time Report 5 was nearly completed.

It is to be observed that Report 5 considers only the simple wing and does not treat the wing-aileron combination. It was then agreed upon by Drs. von Kármán and Biot that the treatment of this phase of the subject would satisfy the above mentioned requirements, and thus the following pages have been written. It is to be pointed out that this thesis can be considered as a supplement or a sequel to Report 5 and in reading it is necessary to have a copy of this report at hand.

Acknowledgment has been made to the authors concerned within the body of the text except where reference is made to Report 5, in which due credit is given. At this point, however, the author wishes to express his gratitude and thanks for the information and help given him by the staff members of the California Institute of Technology.

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Minneapolis, Minnesota.

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Summary of Symbols

The following groups of symbols are not included in this list: the symbols introduced in Report 5, certain other symbols of a temporary nature which are self-understood, and more or less standard symbols such as are used for the Bessel functions, the logarithmic base e, the imaginary unit i, etc.

Roman, Upper Case

A_β instantaneous value of the total circulation about an oscillating wing-aileron combination introduced in section I-4.

$B_{2\beta}$ defines an integral concerning the downwash along the chord, introduced in section I-4, corresponds to B_2 of Report 5.

$B^o_{2\beta}$ defines an integral which is a component part of $B_{2\beta}$, introduced in section I-4, derived in section I-6.

$B_{3\beta}$ defines an integral concerning the downwash along the chord, introduced in section I-4, corresponds to B_3 of Report 5.

$B^o_{3\beta}$ defines an integral which is a component part of $B_{3\beta}$, introduced in section I-4, derived in section I-6.

C a function of reduced velocity, defined by Theodorsen in reference 2, introduced in section I-9, same as $\bar{P}$ used by Lombard.

E_o, E_1, E_2
define integrals of the vorticity along the aileron chord, introduced in section II-2 as expressions (2.1), (2.2), (2.3).

$E'_o, E''_o, E'''_o, E^o_o, E'_1, \cdots, E'''_2, E^o_2$
define integrals which are component parts of E_o, E_1, and E_2 introduced in section II-3, as expressions (3.9), (3.10), etc.

K_R the real part of $\bar{S}_R$ introduced in II-13.

L the wing lift per unit of span, positive upward.

$L_h, L_{\dot{h}}, L_\alpha, L_\beta$
coefficients in the wing lift expression, used only in section I-8.

M the wing moment per unit of span about the mid-point of the chord, positive when stalling.

M_a the wing moment per unit of span about the axis of rotation, positive when stalling, introduced in section I-9, Theodorsen's notation.

M_e identical to M_α introduced in section I-10, Lombard's notation.

M_β the moment of the aileron per unit of span about its hinge, positive when stalling, introduced in section II-2 as expression (2.4)

$M_h, M_{\dot h}, M_\alpha, M_\beta$
coefficients in the wing moment expression, used only in section I-8.

N_{AR} imaginary part of $\bar{S}_{AR}$ introduced in section II-13.

$\bar{P}$ a function of reduced velocity, defined by Lombard in reference 4, introduced in section I-10.

Q_2 symbolizes an integral, introduced in section II-5 as expression (5.4).

$R_1, R_2, \cdots, R_{14}$
primarily nondimensional functions of aileron chord, Lombard's notation, see appendix B.

$\bar{S}_{AR}$ a function of reduced velocity, aspect ratio, and aileron chord, defined by expression (9.9) section II-9

$S_0(k,\theta_0), S_1(k,\theta_0), \cdots, S_5(k,\theta_0)$
functions of aspect ratio and aileron chord, $k = \frac{4}{AR}$ or $\frac{1}{3AR}$, introduced in section II-4.

$T_1, T_2, \cdots, T_{14}$
primarily nondimensional functions of aileron chord, Theodorsen's notation, see appendix A.

Roman, Lower Case

a ratio of distance between midpoint of the chord and aileron hinge divided by semichord, Theodorsen's Notation.

a_0, a_1, a_2
functions of aspect ratio, repeated here from Report 5, part II, page 35. In this thesis see section II-6.

a_n coefficients of a Fourier series, used in appendix C only.

c chord, same as in Report 5.

e_0, e_1, e_2
auxiliary notation used to assemble aileron hinge moment expression introduced in section II-8 as expressions (8.7), (8.8), and (8.9).

f used here to designate a function of aspect ratio, reduced velocity, and aileron chord, introduced in section II-9, as expression (9.4)

f_0, f_1, f_2
functions of same variables as f above, introduced in section II-8 as expressions (8.2), (8.4), and (8.6), component parts of

$f_{0T}, f_{1T}, f_{2T}, f_{0S}, \cdots, f_{2L}$
component parts of f_0, f_1, and f_2, introduced in sections II-4, II-5, and II-6.

h instantaneous vertical displacement of the midpoint of the chord due to wing oscillations, positive downward, same as in Report 5.

h_a instantaneous vertical displacement of the point on the chord designated as the axis of rotation due to wing oscillations, positive downward, introduced in section I-9.

k used in appendix C to designate $\frac{4}{A\!\!R}$ and $\frac{1}{3A\!\!R}$.

m_L mass of an air cylinder per unit of span with the chord as diameter, introduced in section II-10.

w downwash velocity, positive downward.

w_0 velocity of relative downwash from leading edge to aileron hinge, due to wing oscillations, introduced in section I-5.

w_1 velocity of relative downwash along aileron chord, due to wing oscillations, introduced in section I-5.

$w_{\bar{\beta}}$ used to symbolize the velocity of relative downwash due to wing oscillations, includes both w_0 and w_1, introduced in section II-3.

$w_{\bar{\beta}}'$, $w_{\bar{\beta}}''$, $w_{\bar{\beta}}'''$
downwash due to wake trailing, shed, tip and bound vorticity respectively for wing-aileron combination, introduced in section II-3.

x designates distance measured along the chord, origin at midpoint, positive toward trailing edge.

x_0 distance from midpoint of chord to aileron hinge.

x_1 distance from midpoint of the chord to the center of lift, used in section I-9 only.

y_e instantaneous vertical displacement of the point on chord designated as the axis of rotation, due to wing oscillations, positive upward, introduced in section I-10, Lombard's notation.

Greek, Upper and Lower Case

β angle of aileron deflection relative to its neutral position, positive when it increases the lift. See figure 5.1 section I-5.

Γ_β^0 circulation due to the oscillations of a wing-aileron combination, introduced in section I-4.

Γ_β', Γ_β'', Γ_β'''
circulation due to the wake trailing, shed, tip and bound vorticity respectively of a wing-aileron combination.

ε the ratio of the distance between the forward quarter chord point and the axis of rotation divided by the wing chord. See figure 10.1 section I-10, Lombard's notation.

θ used as a variable of integration, related to x by expression (3.2) section II-3.

θ_0 a term which designates the position of the aileron hinge. See expression (5.6) section I-5.

τ used as a variable of integration, related to x by expression (5.5) section I-5. This term does not appear in final expressions for wing lift, wing moment, and aileron hinge moment. See line below.

τ ratio of aileron chord divided by total chord, used only in sections I-10 and appendix B. Appears in final expressions only. See definition of τ given above.

Chapter I

WING LIFT AND WING MOMENT

I-1 A Preliminary Remark

As is stated in the preface this dissertation is a supplement or a sequel to reference 1 which was written by Dr. M. A. Biot and the author. For shortness of form reference 1 is referred to as Report 5. The basic theory used herein is described in Report 5 and many formulae are taken bodily from this report. In reading this dissertation it is therefore essential that a copy of Report 5 is at hand.

I-2 Purpose

A formula for the lift of an oscillating wing and one for the wing moment are derived in Report 5. The moment formula is taken about a lateral axis through the midpoint of the chord. The formulae of Report 5, however, assume a wing without an aileron, consequently, the effects of a wing with an aileron having oscillatory motion are unknown in so far as Report 5 is concerned.

It is the purpose of this thesis to derive the formulae for an oscillating wing of finite span, which wing has an aileron. These formulae are, the wing lift, the wing moment and the moment of the aileron about its hinge.

I-3 Summary

All the fundamental considerations as set forth in Report 5 also apply here. The summary of Report 5 (page 2 part I) written by Dr. Biot is hereby thought of as a part of this summary. This is especially true of the wing lift and wing moment formulae developed herein, since the introduction of an aileron produces no new functions of the reduced velocity and aspect ratio. The particular functions referred to in Report 5 are $\overline{P}_{AR}$ and $\overline{Q}_{AR}$. These functions occur unchanged in the formulae for the wing lift and the wing moment for an oscillating wing with an aileron. From this point of view the contents contained herein are purely a sequel to Report 5.

The moment of the aileron about its hinge, as derived here, forms a more or less separate derivation, although it is dependent on some of the results of Report 5. The same vortex pattern and the same approximations as used in Report 5 are also used here. Further, there are no new approximations introduced but a new function designated as $\overline{S}_{AR}$ arises from the derivation. This new function $\overline{S}_{AR}$ is similar to $\overline{P}_{AR}$ and $\overline{Q}_{AR}$ given in Report 5 in that it is a function of reduced velocity $\frac{U}{\omega c}$ and aspect ratio, but different in that $\overline{S}_{AR}$ is also a function of the ratio of aileron chord to wing chord or as given here a nondimensional parameter which gives the location of the aileron hinge. When the aspect ratio is infinite then $\overline{P}_{AR} = \overline{Q}_{AR} = \overline{S}_{AR} = C$ where C is the function described by Theodorsen in reference 2. The numerical values of $\overline{S}_{AR}$ have not as yet been calculated.

In order to integrate certain expressions pertaining to $\overline{S}_{AR}$ Fourier series and termwise integration are used. The coefficients of this series are the modified Bessel functions, the variable being proportional to the reciprocal of the aspect ratio, while the trignometric terms depend on the above mentioned nondimensional parameter which locates the aileron hinge. All other integrals are taken from Report 5 with the exception of one for which

no integration was found. Fortunately, this latter integral cancels from the finished formulae.

The results are given in four forms. The first form is as derived, assuming that the vertical displacement of the wing is measured at the midpoint of the chord. In order that the formulae can be easily compared to the results of Report 5 they are given in a second form which is called here the coefficient form. The third form, called here the Theodorsen form, is given so that direct comparison can be made with Theodorsen's work in reference 2. The fourth form is given because it perhaps lends itself better to the numerical calculations of wing flutter analysis; it is called here the Lombard form. If a comparison is made it will be found that this fourth form is identical to that used by Lómbard in reference 4.

I-4 Introduction

As mentioned in the summary the fundamental considerations set forth in Report 5 apply here. The same vortex pattern will be assumed except that in the determination of the circulation the effect of the aileron will be included.

On page 53 of Report 5 the expression for the total circulation is given by means of equation (6.1) which is

$$A = \Gamma^{\circ} + \Gamma' + \Gamma'' + \Gamma'''$$

where Γ', Γ'', Γ''' are the component parts of the circulation about the wing respectively due to the wake trailing vortices, the shed vorticity, and the tip trailing and bound vortices, while Γ° is due to the wing oscillations. Since the only item of change is the addition of an aileron the only term which changes directly is Γ°. This means that Γ° must be derived so as to include the effects of the aileron. The terms Γ', Γ'' and Γ''' to be sure include A as a common factor and do change but only as a consequence of the change of the factor A.

Following Theodorsen's notation given in reference 2, the Greek letter β is chosen to designate aileron deflection, hence β used as subscript will indicate that the terms taken from Report 5 must be adjusted so as to include the effect of the aileron. Following this scheme expression (6.1) of Report 5 can now be written as

$$A_{\beta} = \Gamma_{\beta}^{\circ} + \Gamma_{\beta}' + \Gamma_{\beta}'' + \Gamma_{\beta}'''$$

where A_{β} designates the total circulation about an oscillating wing with aileron. It is evident that Γ_{β}° must be derived. For a wing with aileron equation (6.3) page 53 of Report 5 now takes the following form:

$$A_{\beta} = \frac{\Gamma_{\beta}^{\circ}}{F - Q_{0} - Q_{1}} \tag{4.1}$$

where now the effect of the aileron is included. The terms F, Q_0, and Q_1 have here the same meaning as in Report 5 and likewise the numerical values of Report 5 can be used here.

Before the wing lift and wing moment formulae for the case of a wing with an aileron can be determined two more terms of Report 5 must be modified, viz., B_2 and B_3. If the subscript β is properly attached to expression (2.23) which is given on page 70 of Report 5, it becomes for the wing with aileron

$$B_{2\beta} = B^0_{2\beta} + A_\beta c\left[\frac{1}{2} + \frac{1}{i\lambda c} + \frac{Q_1}{i\lambda c} + F_1\right] \tag{4.2}$$

Here $B^0_{2\beta}$ must be calculated for a wing with aileron but F_1 can be taken from Report 5. If the same operation is applied to expression (2.36) given on page 76 of Report 5, the expression becomes

$$B_{3\beta} = B^0_{3\beta} + A_\beta c^2\left[\frac{1}{8} + \frac{1}{i\lambda c} - \frac{2}{\lambda^2 c^2} + \frac{Q_0}{2i\lambda c} - \frac{2\,Q_1}{\lambda^2 c^2} + F_0\right] \tag{4.3}$$

where $B^0_{3\beta}$ must be derived for a wing with aileron. The values for F_0 however can be taken directly from Report 5.

The equations of section III-3 of Report 5 can now be put in form so that they indicate the inclusion of an aileron. From Report 5 page 78 the lift expression (3.4) is given as

$$L = \rho U A + \frac{\rho}{2} i\omega c A - \rho i\omega B_2$$

If A is replaced by A_β and B_2 by $B_{2\beta}$ the expression becomes

$$L = \rho U A_\beta + \frac{\rho}{2} i\omega c A_\beta - \rho i\omega B_{2\beta} \tag{4.4}$$

where this expression designates symbolically the lift of an oscillating wing of finite span with aileron. It is to be pointed out that the subscript β is omitted from the lift symbol L. This should not cause any confusion, since if $\beta = \dot{\beta} = \ddot{\beta} = 0$ expression (4.4) reduces to expression (3.4) page 78 of Report 5. If expression (3.5) given on page 78 of Report 5 is treated in the same manner as the lift expression it becomes

$$M = -\rho U B_{2\beta} - \frac{\rho c^2}{16} i\omega A_\beta + \frac{\rho}{2} i\omega B_{3\beta} \tag{4.5}$$

which gives the moment of a wing with aileron about a lateral axis through the midpoint of the chord positive when it tends to stall the wing. Here again the subscript β has been omitted from the symbol M for the same reason as in the case of expression (4.4). A symbol M_β will be employed however, to designate the aileron hinge moment.

Before introducing $\overline{P}_{AR}$ and $\overline{Q}_{AR}$ in expressions (4.4) and (4.5) it is advisable to eliminate λ from expressions (4.2) and (4.3) by means of the relation that

$$\omega = \lambda U \tag{4.6}$$

which relation is found as equation (6.3) on page 39 of Report 5. Performing this operation on expression (4.2) it becomes

$$B_{2\beta} = B^{\circ}_{2\beta} + A_\beta c \left[\frac{1}{2} + \frac{U}{i\omega c} + \frac{U Q_1}{i\omega c} + F_1 \right] \tag{4.7}$$

and likewise expression (4.3) can be written as

$$B_{3\beta} = B^{\circ}_{3\beta} + A_\beta c^2 \left[\frac{1}{8} + \frac{U}{i\omega c} + \frac{2U}{(i\omega c)^2} + \frac{U Q_0}{2 i\omega c} + \frac{2 U^2 Q_1}{(i\omega c)^2} + F_0 \right] \tag{4.8}$$

Substituting (4.1) and (4.7) in expression (4.4) it becomes on algebraic reduction as follows:

$$L = -\rho i\omega B^{\circ}_{2\beta} + \rho U \Gamma^{\circ}_{\beta} \frac{Q_1 + \frac{i\omega c}{U} F_1}{Q_0 + Q_1 - F} \tag{4.9}$$

On page 79 of Report 5, $\overline{P}_{AR}$ is defined by expression (3.6) which expression is

$$\overline{P}_{AR} = \frac{Q_1 + \frac{i\omega c}{U} F_1}{Q_0 + Q_1 - F} \tag{4.10}$$

Thus expression (4.9) can be written as

$$L = \rho i\omega B_{2\beta}^{\circ} + \rho U \Gamma_{\beta}^{\circ} \overline{P_{R}} \tag{4.11}$$

In like manner expression (4.5) can be brought into the following form by means of substituting expressions (4.1) (4.7) and (4.8) and reducing algebraically; thus

$$M = -\rho U B_{2\beta}^{\circ} + \frac{\rho}{2} i\omega B_{3\beta}^{\circ} - \frac{\rho}{4} U c \Gamma_{\beta}^{\circ} + \frac{\rho}{4} U c \Gamma_{\beta}^{\circ} \frac{Q_1 - F + 4F_1 - \frac{2i\omega c}{U} F_0}{Q_0 + Q_1 - F} \tag{4.12}$$

On page 80 of Report 5, $\overline{Q_{R}}$ is defined by means of expression (3.8) which expression is

$$\overline{Q_{R}} = \frac{Q_1 - F + 4F_1 - \frac{2i\omega c}{U} F_0}{Q_0 + Q_1 - F} \tag{4.13}$$

Substituting this in expression (4.12) it becomes

$$M = -\rho U B_{2\beta}^{\circ} + \frac{\rho}{2} i\omega B_{3\beta}^{\circ} - \frac{\rho}{4} U c \Gamma_{\beta}^{\circ} + \frac{\rho}{4} U c \Gamma_{\beta}^{\circ} \overline{Q_{R}} \tag{4.14}$$

From the appearance of expressions (4.11) and (4.14) it is evident that the wing lift and the wing moment formulae willbe completed as soon as Γ_{β}°, $B_{2\beta}^{\circ}$, and $B_{3\beta}^{\circ}$ are derived.

The expression for the moment of the aileron about its hinge however, requires considerably more development. Since Report 5 considers the entire wing, i. e. from leading edge to trailing edge, the results of Report 5 do not lend themselves directly to this phase of the problem. The solution of the aileron hinge moment must be introduced by considering the air pressure acting on the aileron. Since the development of the aileron hinge moment is rather lengthy no formulae will be introduced here. Therefore, in the following pages the wing lift and the wing moment equations will be developed first. The aileron hinge moment will be taken up last starting with the pressure equation which is given as equation (1.3) page 55 of Report 5.

I-5 Wing-Aileron Oscillations Replaced by Equivalent Downwash

In Report 5 section I-6 pages 36-41 the expression for the relative downwash is derived from the kinematics of the wing's oscillatory motion. This is worked out for a wing without an aileron. Here it is essential to include the effect of the aileron. Since the solution of this derivation is linear it is only necessary to derive the effect of the aileron alone, then the expression for the wing-aileron combination is obtained by the process of superposition.

Figure 5.1 has been drawn similar in some respects to figure 6.2 page 39 of Report 5 but in addition figure 5.1 possesses an aileron. This figure shows the coördinate system and depicts the angle β i. e., the angle of aileron deflection. The point E designates the leading edge, F the trailing edge, and G the aileron hinge. The origin is at the midpoint of the chord and is designated as point O. The X-axis is considered as the line EG produced. The abscissa of the aileron hinge is taken as X_0. The point P with abscissa X is an arbitrary point on the aileron.

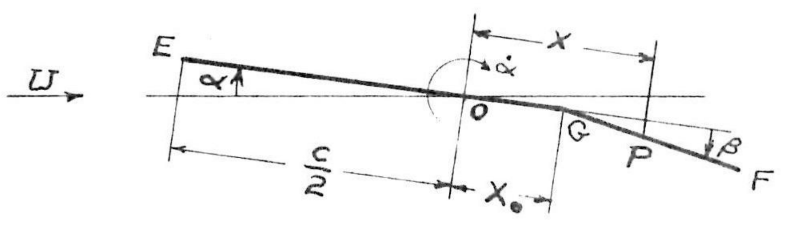

Fig. 5.1 Wing-Aileron Combination.

From figure 5.1 it is apparent that expression (6.6) given on page 40 of Report 5 is applicable to that part of the wing designated by the line EG hence it can be said that the downwash is

$$w_5 = -\dot{h} - U\alpha - i\omega\alpha X , \qquad for \; -\frac{c}{2} \leq X \leq X_0 \tag{5.1}$$

where $\dot{h}$ designates the vertical velocity of the point O, positive when downward. From the information given on page 40 of Report 5 it follows that the angular velocity

$$\dot{\alpha} = i\omega\alpha$$

and making use of this relation, expression (5.1) becomes

$$w_0 = -\dot{h} - U\alpha - \dot{\alpha}x \,, \qquad \text{for } -\frac{c}{2} \leq x \leq x_0 \tag{5.2}$$

The latter expression being the same as (5.1) is however, in a simpler form.

From figure 5.1 it is apparent that the angle of aileron deflection β is measured relative to the line EG produced, which as drawn has an angle of attack α, consequently the angle of attack of the aileron GF is $\alpha+\beta$. The angles α and β are taken so small that with good approximation the $\sin(\alpha+\beta) = \alpha+\beta$ and the $\cos(\alpha+\beta) = 1$. Since the methods of superposition are to be used the angle α will be set equal to zero. Under the imposed conditions the undisturbed air stream U, has an upward component normal to the aileron of magnitude $U\beta$. Since an upward velocity is here considered as a negative downwash the contribution of this component to the downwash is $-U\beta$.

As in Report 5 the angular velocity $\dot{\alpha}$ is considered positive when it is in the direction of increasing α. In like manner let $\dot{\beta}$ designate the angular velocity of the aileron GF, positive with increasing β as indicated in figure 5.1. Since the methods of superposition are used here take $\dot{\alpha} = 0$. Under this condition the linear velocity of the point P due to the angular velocity $\dot{\beta}$ is $(x-x_0)\dot{\beta}$. The angular velocity $\dot{\beta}$ causes the point P of figure 5.1 to move downward, hence the motion of the undisturbed air relative to point P is upward. The contribution of this motion to downwash is evidently $-(x-x_0)\dot{\beta}$.

The total effect produced on the downwash by aileron when $\dot{h} = \alpha = \dot{\alpha} = 0$ is

$$-U\beta - (x-x_0)\dot{\beta}$$

The downwash equivalent to the aileron GF when $\dot{h}$, α, and $\dot{\alpha}$ are not zero is obtained by superposing the above expression on equation (5.2), that is

$$w_1 = w_0 - U\beta - (x-x_0)\dot{\beta} \,, \qquad \text{for} \quad x_0 \leq x \leq \frac{c}{2} \tag{5.3}$$

or

$$w_1 = -\dot{h} - U\alpha - \dot{\alpha}x - U\beta - (x - x_0)\dot{\beta}$$

$$\text{for} \quad x_0 \leq x \leq \frac{c}{2} \tag{5.4}$$

where w_1 = the downwash velocity equivalent to the aileron.

For the convenience of subsequent integration a new variable τ will be introduced, defined by the following relation

$$x = \frac{c}{2}\cos\tau \tag{5.5}$$

In order to designate the hinge G (figure 5.1) a constant Θ_0 is so chosen that

$$x_0 = \frac{c}{2}\cos\Theta_0 \tag{5.6}$$

The particular choice of the Greek letters τ and Θ are consistent with Report 5 except that no constant Θ_0 is introduced in that report. Relation (5.5) is introduced on page 44 of Report 5 as equation (1.3) and Θ becomes one of the variables of integration on page 58 of Report 5. However, as will be seen the present use of τ will in no way confuse its use as designating the ratio of aileron chord to wing chord.

Making use of relations (5.5) and (5.6) expressions (5.2) and (5.4) can be put in the following forms:

$$w_0 = -\dot{h} - U\alpha - \frac{c\dot{\alpha}}{2}\cos\tau$$

$$\text{for} \quad \Theta_0 \leq \tau \leq \pi \tag{5.7}$$

and

$$w_1 = -\dot{h} - U\alpha - \frac{c\dot{\alpha}}{2}\cos\tau - U\beta - \frac{c\dot{\beta}}{2}(\cos\tau - \cos\Theta_0)$$

$$\text{for} \quad 0 \leq \tau \leq \Theta_0 \tag{5.8}$$

I-6 Derivations of Γ_β°, $B_{2\beta}$, $B_{3\beta}$

From page 43 of Report 5 Munk's integral for the determination of the circulation is given as expression (1.2) and is here reproduced as expression (6.1); thus

$$\Gamma = -c\int_0^\pi w(1+\cos\tau)\,d\tau \tag{6.1}$$

where c = the length of the chord. To evaluate this integral, however, the downwash w must be a known function of τ. The expression for Γ_β° can now be obtained by substituting relations (5.7) and (5.8) in Munk's integral. From the limits of integration it is clear however, that w_1 must be integrated from 0 to θ_0, and w_0 from θ_0 to π. Following this scheme Γ_β° becomes

$$\Gamma_\beta^\circ = -c\int_0^{\theta_0}\left[-\dot h - U\alpha - \frac{c\dot\alpha}{2}\cos\tau - U\beta - \frac{c\dot\beta}{2}(\cos\tau - \cos\theta_0)\right](1+\cos\tau)\,d\tau$$
$$-c\int_{\theta_0}^{\pi}\left[-\dot h - U\alpha - \frac{c\dot\alpha}{2}\cos\tau\right](1+\cos\tau)\,d\tau$$

This integral can be written as

$$\Gamma_\beta^\circ = c\int_0^{\pi}\left[\dot h + U\alpha + \frac{c\dot\alpha}{2}\cos\tau\right](1+\cos\tau)\,d\tau$$
$$+c\int_0^{\theta_0}\left[U\beta + \frac{c\dot\beta}{2}(\cos\tau - \cos\theta_0)\right](1+\cos\tau)\,d\tau$$

and on integration it becomes

$$\Gamma_\beta^\circ = \pi c\left(\dot h + U\alpha + \frac{c\dot\alpha}{4}\right) + Uc\beta(\theta_0 + \sin\theta_0)$$
$$+\frac{c^2\dot\beta}{2}\left(\frac{\theta_0}{2} - \theta_0\cos\theta_0 + \sin\theta_0 - \frac{1}{4}\sin 2\theta_0\right) \tag{6.2}$$

It is very convenient to express Γ_β° in terms of the Theodorsen's constants (see reference 2). These constants are given in appendix A in terms of the notation used here. Introducing these constants expression (6.2) becomes

$$\Gamma_\beta^{\circ} = \pi c\left(\dot h + U\alpha + \frac{c\dot\alpha}{4}\right) + Uc\beta T_{10} + \frac{c^2\dot\beta}{4} T_{11} \tag{6.3}$$

To derive $B_{2\beta}^{\circ}$ integral (1.7) page 58 of Report 5 must be used. This integral is

$$B_2 = \frac{c^2}{2}\int_0^{\pi} w \sin^2\tau \, d\tau \tag{6.4}$$

This integral presents the same problem in regards to the interval of integration as Γ_β° it being necessary to integrate from 0 to θ_0 and then from θ_0 to π. Substituting expressions (5.7) and (5.8) integral (6.4) becomes

$$B_{2\beta}^{\circ} = \frac{c^2}{2}\int_0^{\theta_0}\left[-\dot h - U\alpha - \frac{c\dot\alpha}{2}\cos\tau - U\beta - \frac{c\dot\beta}{2}(\cos\tau - \cos\theta_0)\right]\sin^2\tau\, d\tau$$
$$+ \frac{c^2}{2}\int_{\theta_0}^{\pi}\left[-\dot h - U\alpha - \frac{c\dot\alpha}{2}\cos\tau\right]\sin^2\tau\, d\tau$$

which can be written as

$$B_{2\beta}^{\circ} = -\frac{c^2}{2}\int_0^{\pi}\left[\dot h + U\alpha + \frac{c\dot\alpha}{2}\cos\tau\right]\sin^2\tau\, d\tau$$
$$- \frac{c^2}{2}\int_0^{\theta_0}\left[U\beta + \frac{c\dot\beta}{2}(\cos\tau - \cos\theta_0)\right]\sin^2\tau\, d\tau$$

Integrating and rearranging $B_{2\beta}^{\circ}$ takes the following form:

$$B_{2\beta}^{\circ} = -\frac{c^2}{4}\Big[\pi(\dot h + U\alpha) + U\beta(\theta_0 - \sin\theta_0\cos\theta_0)$$
$$+ \frac{c\dot\beta}{2}\left(\frac{2}{3}\sin^3\theta_0 - \theta_0\cos\theta_0 + \sin\theta_0\cos^2\theta_0\right)\Big] \tag{6.5}$$

Making use of the Theodorsen constants as given in appendix A expression (6.5) becomes

$$B_{2\beta}^{\circ} = -\frac{c^2}{4}\left[\pi(\dot h + U\alpha) - U\beta T_4 - \frac{c\dot\beta}{2} T_1\right] \tag{6.6}$$

The last integral of this section is $B^{\circ}_{3\beta}$. This is given as integral (1.12) on page 61 of Report 5 as follows:

$$B_3 = \frac{c^3}{4}\int_0^{\pi} w \sin^2\tau \cos\tau \, d\tau \tag{6.7}$$

Substituting expressions (5.7) and (5.8) integral (6.7) becomes

$$B^{\circ}_{3\beta} = \frac{c^3}{4}\int_0^{\theta_0}\left[-\dot{h} - U\alpha - \frac{c\dot{\alpha}}{2}\cos\tau - U\beta - \frac{c\dot{\beta}}{2}(\cos\tau - \cos\theta_0)\right]\sin^2\tau\cos\tau\, d\tau$$

$$+ \frac{c^3}{4}\int_{\theta_0}^{\pi}\left[-\dot{h} - U\alpha - \frac{c\dot{\alpha}}{2}\cos\tau\right]\sin^2\tau\cos\tau\, d\tau$$

or

$$B^{\circ}_{3\beta} = -\frac{c^3}{4}\int_0^{\pi}\left[\dot{h} + U\alpha + \frac{c\dot{\alpha}}{2}\cos\tau\right]\sin^2\tau\cos\tau\, d\tau$$

$$- \frac{c^3}{4}\int_0^{\theta_0}\left[U\beta + \frac{c\dot{\beta}}{2}(\cos\tau - \cos\theta_0)\right]\sin^2\tau\cos\tau\, d\tau$$

On integrating, the above expression is brought to the following form:

$$B^{\circ}_{3\beta} = -\frac{c^2}{4}\left[\frac{\pi}{16}c^2\dot{\alpha} + \frac{1}{3}Uc\beta\sin^3\theta_0\right.$$

$$\left. + \frac{c^2\dot{\beta}}{2}\left(\frac{\theta_0}{8} - \frac{1}{32}\sin 4\theta_0 - \frac{1}{3}\cos\theta_0\sin^3\theta_0\right)\right] \tag{6.8}$$

Again, making use of Theodorsen's constants it can be shown that

$$B^{\circ}_{3\beta} = -\frac{c^2}{4}\left[\frac{\pi}{16}c^2\dot{\alpha} + Uc\beta(T_8 - T_1) - \frac{c^2\dot{\beta}}{2}(T_7 + T_1\cos\theta_0)\right] \tag{6.9}$$

I-7 Wing Lift and Wing Moment

The formulae for the wing lift and the wing moment can now be put in their final form. The formula first to be assembled is the wing lift, expression (4.11). This expression requires the substitution of $B^{\circ}_{2\beta}$ expression (6.6), and Γ°_{β} expression (6.3). Making these substitutions the formula for wing lift becomes

$$L = -\rho i\omega\left\{-\frac{c^2}{4}\left[\pi(\dot{h}+U\alpha) - U\beta T_4 - \frac{c\dot{\beta}}{2}T_1\right]\right\}$$
$$+\rho U \overline{P}_R\left\{\pi c\left(\dot{h}+U\alpha+\frac{c\dot{\alpha}}{4}\right) + Uc\beta T_{10} + \frac{c^2\dot{\beta}}{4}T_{11}\right\}$$

or

$$L = \frac{\pi}{4}\rho c^2 i\omega\left[\dot{h}+U\alpha - \frac{U\beta}{\pi}T_4 - \frac{c\dot{\beta}}{2\pi}T_1\right]$$
$$+\pi\rho c U \overline{P}_R\left[\dot{h}+U\alpha+\frac{c\dot{\alpha}}{4} + \frac{U\beta}{\pi}T_{10} + \frac{c\dot{\beta}}{4\pi}T_{11}\right] \tag{7.1}$$

which is one form of the expression for the wing lift per unit of span positive when upward. Expression (7.1) is not however, in a suitable form for application.

Expression (7.1) can be put in a more suitable form as follows: From page 5 part I of Report 5 the following relations are obtained:

$$h = \bar{h}e^{i\omega t}$$
$$\alpha = \bar{\alpha}e^{i\omega t}$$

A third expression concerning the angle β can be written as

$$\beta = \bar{\beta}e^{i\omega t}$$

where $\bar{\beta}$ is the complex amplitude.

Differentiating the above three expressions gives

$$\dot{h} = i\omega \bar{h} e^{i\omega t}$$

$$\dot{\alpha} = i\omega \bar{\alpha} e^{i\omega t}$$

$$\dot{\beta} = i\omega \bar{\beta} e^{i\omega t}$$

which by means of the first three expressions can be put in the following form:

$$\begin{aligned} \dot{h} &= i\omega h \\ \dot{\alpha} &= i\omega \alpha \\ \dot{\beta} &= i\omega \beta \end{aligned} \tag{7.2}$$

If the same operation is applied again the above three relations become

$$\begin{aligned} \ddot{h} &= i\omega \dot{h} = -\omega^2 h \\ \ddot{\alpha} &= i\omega \dot{\alpha} = -\omega^2 \alpha \\ \ddot{\beta} &= i\omega \dot{\beta} = -\omega^2 \beta \end{aligned} \tag{7.3}$$

The first bracketed quantity of (7.1) has $i\omega$ as a factor. If each term of the brackets is multiplied by this factor and relations (7.2) and (7.3) are applied, expression (7.1) becomes

$$L = \frac{\pi}{4}\rho c^2 \left[\ddot{h} + U\dot{\alpha} - \frac{U\dot{\beta}}{\pi} T_4 - \frac{c\ddot{\beta}}{2\pi} T_1 \right] + \pi \rho c U \bar{P}_{AR} \left[\dot{h} + U\alpha + \frac{c\dot{\alpha}}{4} + \frac{U\beta}{\pi} T_{10} + \frac{c\dot{\beta}}{4\pi} T_{11} \right] \tag{7.4}$$

This form is suitable for application, but for betterment and comparison three others will be introduced. Before introducing these other forms the wing moment equation will be assembled.

The wing moment is given by expression (4.14). To assemble expression (4.14), the expressions for Γ_β^0, $B_{2\beta}^0$ and $B_{3\beta}^0$ are needed. These are given by expressions (6.3), (6.6), and (6.9). Substituting, M becomes

$$M = -\rho U\left\{-\frac{c^2}{4}\left[\pi(\dot{h}+U\alpha) - U\beta T_4 - \frac{c\dot{\beta}}{2}T_1\right]\right\}$$

$$+\frac{\rho}{2}i\omega\left\{-\frac{c^2}{4}\left[\frac{\pi c^2}{16}\dot{\alpha} + Uc\beta(T_8 - T_1) - \frac{c^2\dot{\beta}}{2}(T_7 + T_1\cos\theta_0)\right]\right\}$$

$$-\frac{\rho}{4}Uc\left\{\pi c(\dot{h} + U\alpha + \frac{c\dot{\alpha}}{4}) + U\beta T_{10} + \frac{c^2\dot{\beta}}{4}T_{11}\right\}$$

$$+\frac{\rho}{4}Uc\overline{Q}_{R}\left\{\pi c(\dot{h} + U\alpha + \frac{c\dot{\alpha}}{4}) + Uc\beta T_{10} + \frac{c^2\dot{\beta}}{4}T_{11}\right\}$$

Before applying the operations contained in relations (7.2) and (7.3), an intermediate form of this expression will be written as follows:

$$M = -\frac{\pi}{4}\rho c^2\left\{U\left[\frac{c\dot{\alpha}}{4} + \frac{U\beta}{\pi}(T_4 + T_{10}) + \frac{c\dot{\beta}}{2\pi}(T_1 + \tfrac{1}{2}T_{11})\right]\right.$$

$$+i\omega\left[\frac{c^2}{32}\dot{\alpha} + \frac{Uc}{2\pi}\beta(T_8 - T_1) - \frac{c^2\dot{\beta}}{4\pi}(T_7 + T_1\cos\theta_0)\right]$$

$$\left.-U\overline{Q}_{R}\left[\dot{h} + U\alpha + \frac{c\dot{\alpha}}{4} + \frac{U\beta}{\pi}T_{10} + \frac{c\dot{\beta}}{4\pi}T_{11}\right]\right\}$$

(7.5)

Applying the operations expressed by relations (7.2) and (7.3) the above expression becomes

$$M = -\frac{\pi}{4}\rho c^2\left\{\frac{Uc}{4}\dot{\alpha} + \frac{c^2}{32}\ddot{\alpha} + \frac{U^2\beta}{\pi}(T_4 + T_{10})\right.$$

$$+\frac{Uc}{2\pi}\dot{\beta}(T_8 + \tfrac{1}{2}T_{11}) - \frac{c^2\ddot{\beta}}{4\pi}(T_7 + T_1\cos\theta_0)$$

$$\left.-U\overline{Q}_{R}\left[\dot{h} + U\alpha + \frac{c\dot{\alpha}}{4} + \frac{U\beta}{\pi}T_{10} + \frac{c\dot{\beta}}{4\pi}T_{11}\right]\right\}$$

(7.6)

which gives the wing moment per unit of span, positive when stalling and about a lateral axis through the midpoint of the chord.

I-8 Wing Lift and Wing Moment, Coefficient Form

Expressions (3.7) and (3.9) given in section III-3 of Report 5 are in the so-called coefficient form. This form is explained in section 3 of Report 5, in particular, page 6 part I. These forms as shown on page 6 of the above reference are

$$L = L_h h + L_\alpha \alpha$$

$$M = M_h h + M_\alpha \alpha$$

If the coefficient of β is added the above will appear as

$$L = L_h h + L_\alpha \alpha + L_\beta \beta$$

$$M = M_h h + M_\alpha \alpha + M_\beta \beta$$

The symbol M_β is here used to designate a coefficient, and in Chapter II it is used to designate the aileron hinge moment. No confusion should arise for the use of M_β as a coefficient is limited to this section only. In Report 5 however, the above two expressions are slightly modified as shown below;

$$L = L_{\dot h} \dot h + L_\alpha \alpha + L_\beta \beta$$

$$M = M_{\dot h} \dot h + M_\alpha \alpha + M_\beta \beta$$

where, by applying the first of the relations (7.2) it is seen that

$$L_{\dot h} = \frac{1}{i\omega} L_h \text{ and } M_{\dot h} = \frac{1}{i\omega} M_h .$$

Applying relation (7.2) with the exception of the first, to expression (7.1) it beomces

$$L = \frac{\pi}{4} \rho c^2 i\omega \left[\dot h + U\alpha - \frac{U\beta}{\pi} T_4 - \frac{i\omega c}{2\pi} \beta T_1 \right]$$
$$+ \pi \rho c U \bar{P}_R \left[\dot h + U\alpha + \frac{i\omega c}{4} \alpha + \frac{U\beta}{\pi} T_{10} + \frac{i\omega c}{4\pi} T_{11} \right]$$

On collecting terms the expression becomes in coefficient form the following:

$$L=\pi\rho cU\left[\frac{i\omega c}{4U}+\bar{P}_{R}\right]\dot{h}+\pi\rho cU^{2}\left[\frac{i\omega c}{4U}+\left(1+\frac{i\omega c}{4U}\right)\bar{P}_{R}\right]\alpha$$

$$+\rho cU^{2}\left[\frac{1}{8}\left(\frac{\omega c}{U}\right)^{2}T_{1}-\frac{i\omega c}{4U}T_{4}+\left(T_{10}+\frac{i\omega c}{4U}T_{11}\right)\bar{P}_{R}\right]\beta \tag{8.1}$$

If this is compared to expression (3.7) page 79 of Report 5 it will be observed that the coefficients of $\dot{h}$ and α are identical.

Applying the same procedure to expression (7.5) as is done to the wing lift expression, the wing moment can be changed into the coefficient form. This involves the substitution of $\dot{\alpha}=i\omega\alpha$ and $\dot{\beta}=i\omega\beta$. Substituting in expression (7.5) it becomes

$$M=-\frac{\pi}{4}\rho c^{2}\Big\{U\Big[\frac{i\omega c}{4}\alpha+\frac{U\beta}{\pi}(T_{4}+T_{10})+\frac{i\omega c}{2\pi}\beta\left(T_{1}+\tfrac{1}{2}T_{11}\right)\Big]$$

$$+i\omega\left[\frac{i\omega c^{2}}{32}\alpha+\frac{Uc\beta}{2\pi}(T_{8}-T_{1})-\frac{i\omega c^{2}}{4\pi}\beta(T_{7}+T_{1}\cos\theta_{0})\right]$$

$$-U\bar{Q}_{R}\left[\dot{h}+U\alpha+\frac{i\omega c}{4}\alpha+\frac{U\beta}{\pi}T_{10}+\frac{i\omega c}{4\pi}\beta T_{11}\right]\Big\}$$

On rearranging terms the expression can be written as

$$M=\frac{\pi}{4}\rho c^{2}U\bar{Q}_{R}\dot{h}+\frac{\pi}{4}\rho c^{2}U^{2}\left[\frac{\omega^{2}c^{2}}{32U^{2}}-\frac{i\omega c}{4U}+\left(1+\frac{i\omega c}{4U}\right)\bar{Q}_{R}\right]\alpha$$

$$-\frac{\rho c^{2}}{4}U^{2}\Big[\frac{\omega^{2}c^{2}}{4U}(T_{7}+T_{1}\cos\theta_{0})+\frac{i\omega c}{2U}\left(T_{8}+\tfrac{1}{2}T_{11}\right)$$

$$-\left(T_{10}+\frac{i\omega c}{4U}T_{11}\right)\bar{Q}_{R}+T_{4}+T_{10}\Big]\beta \tag{8.2}$$

Comparing this with expression (3.9) page 80 of Report 5 it is observed that the first two coefficients coincide.

I-9 Wing Lift and Wing Moment, Theodorsen Form

As the above subtitle implies the Theodorsen forms as they are called herein are the forms of the wing lift and the wing moment expressions as Theodorsen presented them in the N. A. C. A., T. R. No. 496, see reference 2. For this purpose figure 2 of Theodorsen's report has been reproduced as figure 9.1 consistent with the notation used here.

In figure 9.1 the point A is the axis of rotation. It is located a distance $\frac{ac}{2}$ aft of the midpoint O of the chord. The quantity a is here used in the same sense as it is by Theodorsen and is to be sure a pure number but may be either positive or negative. The only conflict between Theodorsen's notation and the notation as used here is the lower case c. Here the lower case c is used to designate the length of the chord, while Theodorsen uses it as a nondimensional quantity to designate the distance OG shown on figure 9.1. See also appendix A. The balance of the notation is the same as shown in figure 5.1.

Here $\dot{h}$ is assumed to be the vertical component of the velocity (positive downward) of the point O, figure 9.1. In order to distinguish between the vertical velocity of point O and that of point A let $\dot{h}_a$ designate the vertical velocity of point A which, according to Theodorsen, is considered positive when downward.

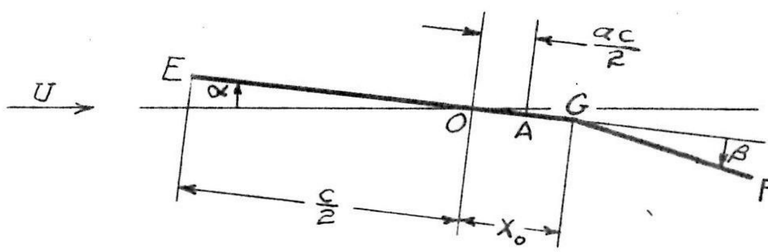

Fig. 9.1 - Point A, the Axis of Rotation.

Due to the oscillatory motion of the wing and the meaning of the axis of rotation it follows that the point A is moving on a straight vertical line. This is not true for the point O because its motion is the vector sum of $\dot{h}_a$ and $\frac{ac}{2}\dot{\alpha}$. The path of point O due to

$\frac{ac}{2}\dot{\alpha}$ is on a circular arc described about point A as a center, and therefore, the resultant path of point O is not a vertical straight line. Since here the amplitude of α is assumed to be small, it follows with good approximation that the point O can be considered as moving on a straight vertical line; hence

$$\dot{h} = \dot{h}_a - \frac{ac}{2}\dot{\alpha} \tag{9.1}$$

and from the above argument it also follows that

$$\ddot{h} = \ddot{h}_a - \frac{ac}{2}\ddot{\alpha} \tag{9.2}$$

The lift, expression (7.4) can now be put in Theodorsen's form by substituting expressions (9.1) and (9.2) as shown below;

$$L = \frac{\pi}{4}\rho c^2\left[\ddot{h}_a - \frac{ac}{2}\ddot{\alpha} + U\dot{\alpha} - \frac{U\dot{\beta}}{\pi}T_4 - \frac{c\ddot{\beta}}{2\pi}T_1\right]$$
$$+\pi\rho c U\overline{P_R}\left[\dot{h}_a - \frac{ac}{2}\dot{\alpha} + U\alpha + \frac{c\dot{\alpha}}{4} + \frac{U\beta}{\pi}T_{10} + \frac{c\dot{\beta}}{4\pi}T_{11}\right]$$

which can be written as

$$L = \frac{\pi}{4}\rho c^2\left[\ddot{h}_a + U\dot{\alpha} - \frac{ac}{2}\ddot{\alpha} - \frac{U\dot{\beta}}{\pi}T_4 - \frac{c\ddot{\beta}}{2\pi}T_1\right]$$
$$+\pi\rho c U\overline{P_R}\left[\dot{h}_a + U\alpha + \frac{c}{2}\left(\frac{1}{2} - a\right)\dot{\alpha} + \frac{U\beta}{\pi}T_{10} + \frac{c\dot{\beta}}{4\pi}T_{11}\right] \tag{9.3}$$

If the notation of this expression is completely translated into that of Theodorsen, it will be found that this expression is identical to expression (XVIII) of N. A. C. A., T. R. No. 496, except that $\overline{P_R}$ replaces Theodorsen's C . Attention is also called to the fact that Theodorsen takes the lift as positive downward.

Expression (7.6) gives the moment about the midpoint of the chord which is considered positive if it is a stalling moment. Theodorsen takes the moment about the point A of figure 9.1 but in like manner he considers it also positive when stalling. Let M_a designate the moment about the point A. Figure 9.2 shows the lift vector L, at a distance X_1, from the midpoint O of the chord EF. The point E is the leading edge. The axis of rotation is shown as point A on this figure. From figure 9.2 it follows that

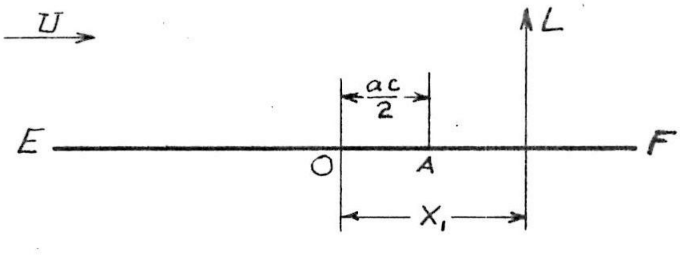

Fig. 9.2 - Showing Lift Vector L.

$$M = -x_1 L$$

and

$$M_a = -\left(x_1 - \frac{ac}{2}\right) L$$

$$= -x_1 L + \frac{ac}{2} L$$

from which it follows that

$$M_a = M + \frac{ac}{2} L \tag{9.4}$$

Before making use of expression (9.4), relation (9.1) will be substituted in expression (7.6) as shown below:

$$M = -\frac{\pi}{4}\rho c^2 \Big\{ \frac{Uc}{4}\dot{\alpha} + \frac{c^2}{32}\ddot{\alpha} + \frac{U^2}{\pi}\beta(T_4 + T_{10}) + \frac{Uc}{2\pi}\dot{\beta}\left(T_8 + \frac{1}{2}T_{11}\right) - \frac{c^2}{4\pi}\ddot{\beta}\left(T_7 + T_1 \cos\theta_0\right)$$
$$- U\bar{Q}_R \Big[\dot{h}_a + U\alpha + \frac{c}{2}\left(\frac{1}{2} - a\right)\dot{\alpha} + \frac{U\beta}{\pi}T_{10} + \frac{c\dot{\beta}}{4\pi}T_{11} \Big] \Big\} \tag{9.5}$$

Substituting expressions (9.3) and (9.5) in expression (9.4) the wing moment about the axis of rotation is obtained as follows:

$$M_a = -\frac{\pi}{4}\rho c^2\Big\{\frac{Uc}{4}\dot{\alpha} + \frac{c^2}{32}\ddot{\alpha} + \frac{U^2}{\pi}\beta(T_4+T_{10})$$

$$+\frac{Uc}{2\pi}\dot{\beta}\left(T_8+\frac{1}{2}T_{11}\right) - \frac{c^2\ddot{\beta}}{4\pi}(T_7+T_1\cos\theta_0)$$

$$-U\bar{Q}_R\Big[\dot{h}_a + U\alpha + \frac{c}{2}\left(\frac{1}{2}-a\right)\dot{\alpha} + \frac{U\beta}{\pi}T_{10} + \frac{c\dot{\beta}}{4\pi}T_{11}\Big]\Big\}$$

$$+\frac{ac}{2}\Big\{\frac{\pi}{4}\rho c^2\Big[\ddot{h}_a + U\dot{\alpha} - \frac{ac}{2}\ddot{\alpha} - \frac{U\dot{\beta}}{\pi}T_4 - \frac{c\ddot{\beta}}{2\pi}T_1\Big]$$

$$+\pi\rho c U\bar{P}_R\Big[\dot{h}_a + U\alpha + \frac{c}{2}\left(\frac{1}{2}-a\right)\dot{\alpha}$$

$$+\frac{U\beta}{\pi}T_{10} + \frac{c\dot{\beta}}{4\pi}T_{11}\Big]\Big\}$$

The above expression can be put in the following form;

$$M_a = -\frac{\pi}{4}\rho c^2\Big\{-\frac{ac}{2}\ddot{h}_a + \frac{Uc}{2}\left(\frac{1}{2}-a\right)\dot{\alpha} + \frac{c^2}{4}\left(\frac{1}{8}+a^2\right)\ddot{\alpha}$$

$$+\frac{U^2}{\pi}(T_4+T_{10})\beta + \frac{Uc}{2\pi}\left(T_8 + aT_4 + \frac{1}{2}T_{11}\right)\dot{\beta}$$

$$-\frac{c^2}{4\pi}\Big[T_7 + (\cos\theta_0 - a)T_1\Big]\ddot{\beta}\Big\}$$

$$+\frac{\pi}{4}\rho c^2 U\left(2a\bar{P}_R + \bar{Q}_R\right)\Big[\dot{h}_a + U\alpha + \frac{c}{2}\left(\frac{1}{2}-\alpha\right)\dot{\alpha}$$

$$+\frac{U\beta}{\pi}T_{10} + \frac{c\dot{\beta}}{4\pi}T_{11}\Big]$$

(9.8)

The above coefficient of $\dot{\beta}$ can be altered through an identity which exists between three of the Theodorsen constants. This identity is as follows;

$$T_8 = T_1 - T_8 - T_4 \cos\theta_0 \tag{9.9}$$

This identity can be quickly proven by the substitution of the constants as given in appendix A. If this identity is substituted as written above for T_8 in expression (9.8) the coefficient of $\dot{\beta}$ becomes

$$\frac{Uc}{2\pi}\left(T_8 + aT_4 + \tfrac{1}{2}T_{11}\right) = \frac{Uc}{2\pi}\left[T_1 - T_8 - (\cos\theta_0 - a)T_4 + \tfrac{1}{2}T_{11}\right]$$

If the coefficient of $\dot{\beta}$ is changed as indicated and expression (9.8) is translated into Theodorsen's notation, it will be found identical with expression (XX) of reference 2 provided the factor $(2a\overline{P}_{AR} + \overline{Q}_{AR})$, in expression (9.8) is replaced by $2(a + \frac{1}{2})C$. Attention is called to the fact that $\overline{P}_{AR} = \overline{Q}_{AR} = C$ when $AR = \infty$ (see Report 5 page 2 part I) from which it follows that

$$2a\overline{P}_{AR} + \overline{Q}_{AR} = 2\left(a + \tfrac{1}{2}\right)C \qquad \text{when} \qquad AR = \infty .$$

I-10 Wing Lift and Wing Moment, Lombard Form

The wing lift and the wing moment expression will now be put in what is called here the Lombard form. To do this it is necessary to introduce some of the Lombard notation. The complete explanation of Lombard's notation is found in appendix II of reference 4. The notation of Report 5 will also be retained in this section, however, it is necessary to adopt some new notation and this will be taken from Lombard's work.

Here, as well as in Report 5, the symbol τ has been used as one of the variables of integration. It will now be used to designate the ratio of aileron chord divided by wing chord. This should cause no confusion since the aforesaid variable of integration does not occur in the final formulae. In section I-5, x_0 is defined as the abscissa of the aileron hinge, see figure 5.1. From this it follows that

$$\tau = \frac{\frac{c}{2} - x_0}{c}$$

This relation can be put in terms of $\cos\theta_0$ if expression (5.6) is employed, thus

$$\tau = \frac{1}{2}\left(1 - \cos\theta_0\right) \tag{10.1}$$

or

$$\cos\theta_0 = 1 - 2\tau \tag{10.2}$$

The next variable to be introduced is ε ; this is similar to Theodorsen's a in that it locates the axis of rotation. The relation between a and ε can best be derived by means of a figure, and for this purpose figure 10.1 is shown. In this figure EF represents the wing chord where the point E is the leading edge and F the trailing edge. As in figure 9.1 the point A represents the axis of rotation. From figure 10.1 it is apparent that

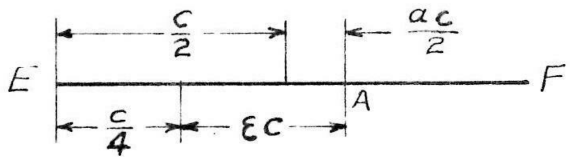

Fig. 10.1 - Lombard's ε, in relation to Theodorsen's a.

$$\varepsilon c - \frac{ac}{2} = \frac{c}{4}$$

and from this it follows that

$$\varepsilon = \frac{1}{2}\left(a + \frac{1}{2}\right) \tag{10.3}$$

or

$$a = 2\left(\varepsilon - \frac{1}{4}\right) \tag{10.4}$$

In order to make the results appear more nearly like those of Lombard, the following additional notation will be used. This notation is defined below; thus

$$M_e = M_\alpha$$

$$m_L = \frac{\pi}{4}\rho c^2$$

and

$$y_e = -h_a$$

It is to be pointed out that the barred notation is not used here as it is used by Lombard, see reference 4, e. g. Lombard writes $\bar{y}_e$.

By substitution of the above notation and the R's given in appendix B, expression (9.3) divided by m_L appears as follows;

$$\frac{L}{m_L} = -\left[\ddot{y}_e - U\dot{\alpha} + c\left(\varepsilon - \frac{1}{4}\right)\ddot{\alpha} - R_3 U\dot{\beta} - R_4 c\ddot{\beta}\right]$$
$$-4\frac{U}{c}\bar{F}_{R}\left[\dot{y}_e - U\alpha - c\left(\frac{1}{2} - \varepsilon\right)\dot{\alpha} - \frac{R_1}{4}U\beta - \frac{R_2}{4}c\dot{\beta}\right] \tag{10.5}$$

The above expression is in reality the Theodorsen form in Lombard's notation. Lombard expresses the formula for wing lift in the coefficient form.

In order to put expression (10.5) in coefficient form, relations (7.2) and (7.3) must be applied. Applying these relations and collecting the terms in such a manner as to form the coefficients of y_e, αC, and βC expression (10.5) becomes

$$\frac{L}{m_L} = -\left[4i\omega\frac{U}{C}\overline{P}_{AR} - \omega^2\right]y_e$$
$$-\left[\left(\frac{1}{4}-\varepsilon\right)\omega^2 - 4\frac{U^2}{C^2}\overline{P}_{AR} - 4i\omega\frac{U}{C}\overline{P}_{AR}\left(\frac{1}{2}-\varepsilon\right) - i\omega\frac{U}{C}\right]\alpha C$$
$$-\left[R_4\omega^2 - R_1\frac{U^2}{C^2}\overline{P}_{AR} - i\omega\frac{U}{C}\left(R_2\overline{P}_{AR} + R_3\right)\right]\beta C \tag{10.6}$$

The minus signs before the bracketed quantities in the above expression have been introduced in order to make expression (10.6) directly comparable to Lombard's work. In the arrangement of the dynamical equations as given by Lombard the aerodynamic coefficients appear with opposite signs, which is the usual thing in problems of harmonic motion.

If the terms due to the elastic constants of the wing and those of the wing mass are omitted in expressions (5.04), (5.05), and (5.06) page 119 of Lombard's Thesis (reference 4), then with a slight change of notation, these expressions coincide precisely with the three bracketed quantities of formula (10.6) except that $\overline{P}_{AR}$ replaces the $\overline{P}$ in Lombard's expressions.*

---

* Lombard credits the symbol $\overline{P}$ to R. Kassner and H. Fingado, see page 56 of Lombard's Thesis, reference 4. It is identical to the C' of Theodorsen.

To put the wing moment formula into the Lombard form, the notations of this section and that of appendix B is substituted in expression (9.8), section I-9. Attention is called to the fact that expression (9.8) is the wing moment about the axis of rotation positive when stalling; it is designated as M_e in this section. Performing the above mentioned substitutions the expression for the wing moment divided by cm_L can be written as

$$
\begin{aligned}
\frac{M_e}{cm_L} = -\Big[&(\varepsilon - \tfrac{1}{4})\ddot{y}_e + U(\tfrac{1}{2} - \varepsilon)\dot{\alpha} + c(\varepsilon^2 - \tfrac{\varepsilon}{2} + \tfrac{3}{32})\ddot{\alpha} \\
&+ R_5 \frac{U^2}{c}\beta + U(R_6 - R_3\varepsilon)\dot{\beta} + (R_7 - R_4\varepsilon)\ddot{\beta} \Big] \\
&- 4\frac{U}{c}\left(\varepsilon \bar{P}_{AR} + \frac{\bar{Q}_{AR} - \bar{P}_{AR}}{4}\right)\Big[\dot{y}_e - U\alpha - c(\tfrac{1}{2} - \varepsilon)\dot{\alpha} \\
&\qquad - \frac{R_1}{4}U\beta - \frac{c}{4}R_2\dot{\beta}\Big]
\end{aligned}
\tag{10.7}
$$

which again might be called the Theodorsen form in Lombard's notation.

The next step in the procedure is to apply relations (7.2) and (7.3) of section I-7. Applying these relations expression (10.7) becomes

$$\frac{M_e}{c m_L} = -\left[\left(\frac{1}{4}-\varepsilon\right)\omega^2 + 4i\omega\frac{U}{c}\left(\varepsilon\overline{P}_{AR} + \frac{\overline{Q}_{AR}-\overline{P}_{AR}}{4}\right)\right] y_e$$

$$-\left[-\left(\varepsilon^2 - \frac{\varepsilon}{2} + \frac{3}{32}\right)\omega^2 + i\omega\frac{U}{c}\left(\frac{1}{2}-\varepsilon\right)\right.$$

$$\left. + 4i\omega\frac{U}{c}\left(\varepsilon\overline{P}_{AR} + \frac{\overline{Q}_{AR}-\overline{P}_{AR}}{4}\right)\left(\varepsilon - \frac{1}{2}\right) - 4\frac{U^2}{c^2}\left(\varepsilon\overline{P}_{AR} + \frac{\overline{Q}_{AR}-\overline{P}_{AR}}{4}\right)\right]\alpha c$$

$$-\left\{-(R_7 - R_4\varepsilon)\omega^2 + \frac{U^2}{c^2}\left[R_5 - R_1\left(\varepsilon\overline{P}_{AR} + \frac{\overline{Q}_{AR}-\overline{P}_{AR}}{4}\right)\right]\right.$$

$$\left. + i\omega\frac{U}{c}\left[R_6 - R_3\varepsilon - R_2\left(\varepsilon\overline{P}_{AR} + \frac{\overline{Q}_{AR}-\overline{P}_{AR}}{4}\right)\right]\right\}\beta c \tag{10.8}$$

The above expression is in a form so that it can be compared with expressions (5.07), (5.08), and (5.09) page 119 of Lombard's Thesis (reference 4). Here it will be noticed that $\varepsilon\overline{P}_{AR} + \frac{\overline{Q}_{AR}-\overline{P}_{AR}}{4}$ replaces $\overline{P}\varepsilon$ in the expressions as given by Lombard. On page 2 part I of Report 5, it is pointed out that for a wing of infinite aspect ratio $\overline{P}_{AR} = \overline{Q}_{AR} = C$. The symbol C used by Theodorsen is identical to the $\overline{P}$ used by Lombard, hence it can be said that for a wing of infinite aspect ratio $\overline{P}_{AR} = \overline{Q}_{AR} = \overline{P}$ from which it follows that $\varepsilon\overline{P}_{AR} + \frac{\overline{Q}_{AR}-\overline{P}_{AR}}{4} = \overline{P}\varepsilon$ when the aspect ratio $AR = \infty$. Under this condition the two expressions become identical.

AILERON HINGE MOMENT

II-1 The Pressure Expression

As pointed out in the introduction, section I-4, the derivation of the moment expression for the aileron about its hinge starts with the pressure expression given as (1.3) page 55 of Report 5. It will be noticed that this expression is given on the second page of chapter III of Report 5 and that this chapter starts with Euler's equation for fluid motion, consequently, it can be rightfully said that the aileron hinge moment starts with first principles. Attention is called to the fact that Euler's equation in Report 5 was linearized, hence this approximation is inherent with the above mentioned pressure expression.

The pressure expression referred to is as follows:

$$p_2(x) - p_1(x) = -\rho U\left[u_2(x) - u_1(x)\right] - \rho\frac{\partial}{\partial t}\int_{-\varepsilon-\frac{c}{2}}^{x}\left[u_2(x) - u_1(x)\right]dx \tag{1.1}$$

This expression gives the difference in pressure between upper and lower surfaces of the airfoil at the point x as shown on figure 1.1. In this figure

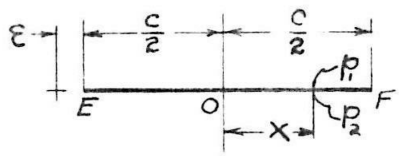

Fig. 1.1 - Flat Plate Wing

EF represents the chord where the point E designates the leading edge. As the figure indicates the origin O is at the midpoint of the chord. The dimension ε establishes a point slightly upstream from the leading edge, the reason for which is given on page 55 of Report 5. Expression (1.1) is written in functional notation to emphasize the terms dependent on x.

As explained on page 56 of Report 5, the strength of the circulation γ, per unit of length can be written in functional notation as

$$\gamma = -\left[u_2(x) - u_1(x)\right] \tag{1.2}$$

where $u_1(x)$ and $u_2(x)$ are the perturbation velocities at the point x (figure 1.1) on the upper and lower surfaces respectively. Substituting expression (1.2) in (1.1) and dropping the functional notation it becomes

$$p_2 - p_1 = \rho U \gamma + \rho \frac{\partial}{\partial t} \int_{-t-\frac{c}{2}}^{x} \gamma \, dx$$

At this point the t in the lower limit can be set equal to zero. It is true that γ tends to infinity as the leading edge of a plane airfoil is approached, however, the integral exists. As a matter of fact the integral $\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma \, dx$ determines the circulation about a wing of infinite span. Following the procedure mentioned above the pressure equation becomes

$$p_2 - p_1 = \rho U \gamma + \rho \frac{\partial}{\partial t} \int_{-\frac{c}{2}}^{x} \gamma \, dx \tag{1.3}$$

II-2 Aileron Hinge Moment, Fundamental Form

Before starting the derivation of the aileron hinge moment, three terms symbolized by E_0, E_1 and E_2 will be introduced. These represent integrals which will appear in the subsequent development and are defined as follows:

$$E_0 = \int_{x_0}^{\frac{c}{2}} \gamma \, dx \tag{2.1}$$

$$E_1 = \int_{x_0}^{\frac{c}{2}} \gamma x \, dx \tag{2.2}$$

$$E_2 = \int_{x_0}^{\frac{c}{2}} \gamma x^2 \, dx \tag{2.3}$$

In the above, γ is the strength of the circulation per unit of length, x_0 is the coordinate of the hinge, $\frac{c}{2}$ is the point at the trailing edge of the airfoil, and the integration is taken along the aileron chord GF as shown in figure 2.1.

In figure 2.1 the wing-aileron combination is schematically represented by the straight line EGF. The point E represents the leading edge, the point G the aileron hinge, and the point F the trailing edge, hence the line segment GF is the aileron chord. If dx is an element of aileron chord, the lift per unit of span is $(p_2 - p_1)\,dx$ and the moment of this element about the point G is $(p_2 - p_1)(x - x_0)\,dx$.

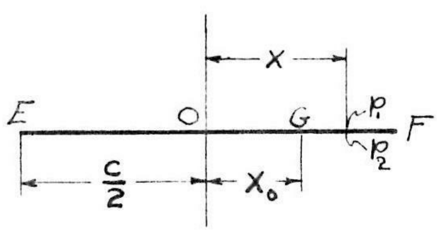

Fig. 2.1 - Wing-Aileron.

From this it follows that the aileron hinge moment per unit of span is

$$M_\beta = -\int_{x_0}^{\frac{c}{2}} (p_2 - p_1)(x - x_0)\,dx \tag{2.4}$$

where the minus sign is introduced such that stalling moments are positive.

Substituting expression (1.3) in (2.4) it becomes

$$M_\beta = -\int_{x_0}^{\frac{c}{2}} \left[\rho U \gamma + \rho \frac{\partial}{\partial t} \int_{-\frac{c}{2}}^{x} \gamma dx \right] (x - x_0)\, dx$$

which can be arranged as

$$M_\beta = -\rho U \int_{x_0}^{\frac{c}{2}} \gamma x dx + \rho U x_0 \int_{x_0}^{\frac{c}{2}} \gamma dx$$

$$- \rho \frac{\partial}{\partial t} \left\{ \int_{x_0}^{\frac{c}{2}} \left[\int_{-\frac{c}{2}}^{x} \gamma dx \right] x dx - x_0 \int_{x_0}^{\frac{c}{2}} \left[\int_{-\frac{c}{2}}^{x} \gamma dx \right] dx \right\}$$

Use can now be made of integrals (2.1) and (2.2); thus M_β becomes

$$M_\beta = -\rho U (E_1 - x_0 E_0)$$

$$- \frac{\partial}{\partial t} \left\{ \int_{x_0}^{\frac{c}{2}} \left[\int_{-\frac{c}{2}}^{x} \gamma dx \right] x dx - x_0 \int_{x_0}^{\frac{c}{2}} \left[\int_{-\frac{c}{2}}^{x} \gamma dx \right] dx \right\} \qquad (2.5)$$

The two latter integrals of (2.5) can be brought into a better form by means of an integration by parts. Before carrying out this work a function $F(x)$ is defined as follows; thus let

$$F(x) = \int_{-\frac{c}{2}}^{x} \gamma dx \qquad (2.6)$$

The first of the two latter integrals can now be written as

$$\int_{x_0}^{\frac{c}{2}} \left[\int_{-\frac{c}{2}}^{x} \gamma dx \right] x dx = \int_{x_0}^{\frac{c}{2}} F(x)\, x\, dx$$

In applying the parts formula $\int u dv = uv - \int v du$ let $u = F(x)$ and $dv = x dx$ then $du = F'(x) dx$ and $v = \frac{x^2}{2}$. From (2.6) it follows that $du = \gamma dx$ hence

$$\int_{x_0}^{\frac{c}{2}} \left[\int_{-\frac{c}{2}}^{x} \gamma dx \right] x dx = \frac{x^2}{2} F(x) \Big]_{x_0}^{\frac{c}{2}} - \int_{x_0}^{\frac{c}{2}} \frac{x^2}{2} \gamma dx$$

$$= \frac{c^2}{8} F\left(\frac{c}{2}\right) - \frac{x_0}{2} F(x_0) - \frac{1}{2} \int_{x_0}^{\frac{c}{2}} \gamma x^2 dx$$

The terms $F(\frac{c}{2})$ and $F(X_o)$ must be interpreted, the latter term of the above being E_2 as defined by expression (2.3).

The term $F(\frac{c}{2})$ by expression (2.6) is

$$F(\tfrac{c}{2}) = \int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma \, dx$$

which is the total circulation about the wing, here symbolized by A_β , see section I-4, hence

$$F(\tfrac{c}{2}) = A_\beta$$

The term $F(x_o)$ in like manner is

$$F(X_o) = \int_{-\frac{c}{2}}^{X_o} \gamma \, dx$$

With reference to the coordinate system shown in figure 2.1 it is evident that

$$F(X_o) = \int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma \, dx - \int_{X_o}^{\frac{c}{2}} \gamma \, dx$$

and by the above discussion and expression (2.1) this can be written as

$$F(X_o) = A_\beta - E_o$$

The first of the two latter integrals from expression (2.5) now becomes

$$\int_{X_o}^{\frac{c}{2}} \left[\int_{-\frac{c}{2}}^{X} \gamma \, dx \right] x \, dx = \frac{c^2}{8} A_\beta - \frac{X_o^2}{2}(A_\beta - E_o) - \frac{1}{2} E_2$$

In a similar way the last integral of expression (2.5) can be put in the following form

$$\int_{X_o}^{\frac{c}{2}} \left[\int_{-\frac{c}{2}}^{X} \gamma \, dx \right] dx = \frac{c}{2} A_\beta - X_o(A_\beta - E_o) - E_1$$

The above procedure is similar to that given on pages 57 and 58 of Report 5.

Substituting the last two results of the above paragraph expression (2.5) becomes

$$M_\beta = -\rho U(E_1 - x_0 E_0)$$

$$-\rho \frac{\partial}{\partial t}\left\{ \frac{c^2}{8} A_\beta - \frac{x_0^2}{2}(A_\beta - E_0) - \frac{1}{2} E_2 \right.$$

$$\left. - x_0 \left[\frac{c}{2} A_\beta - x_0(A_\beta - E_0) - E_1 \right] \right\}$$

which expression can be arranged as follows:

$$M_\beta = -\rho U(E_1 - x_0 E_0)$$

$$-\frac{\rho}{2} \frac{\partial}{\partial t}\left\{ \left(\frac{c}{2} - x_0\right)^2 A_\beta - (x_0^2 E_0 - 2 x_0 E_1 + E_2) \right\} \tag{2.7}$$

Making use of the operational methods as set forth in section I-7 the above expression can be written as

$$M_\beta = \rho U(x_0 E_0 - E_1)$$

$$-\frac{\rho}{2} i\omega \left\{ \left(\frac{c}{2} - x_0\right)^2 A_\beta - (x_0^2 E_0 - 2 x_0 E_1 + E_2) \right\}$$

Substituting relation (5.6) of section I-5 the above expression becomes

$$M_\beta = \rho U\left(\frac{c}{2}\cos\theta_0 E_0 - E_1\right) - \frac{\rho}{8} i\omega c^2 (1 - \cos\theta_0)^2 A_\beta$$

$$+ \frac{\rho}{2} i\omega c \left(\frac{c}{4}\cos^2\theta_0 E_0 - \cos\theta_0 E_1 + \frac{E_2}{c} \right) \tag{2.8}$$

Expression (2.8) appears to be in the most convenient form for the work as presented here.

II-3 Integrable Forms of E_0, E_1, E_2

The terms E_0, E_1, and E_2 are analogons to the terms Γ, B_2, and B_3 of Report 5. It is now necessary to evaluate these terms. In order to evaluate the terms E_0, E_1, and E_2 they must be put in more definite form. From page 100 of Report 5 expression (1D) gives γ as

$$\gamma = \frac{2}{\pi \sin\theta} \int_0^{\pi} \frac{w \sin^2\tau}{\cos\theta - \cos\tau}\, d\tau + \frac{2\Gamma}{\pi c \sin\theta} \tag{3.1}$$

In this expression w, the downwash is considered as a function of τ, and τ is related to x through relation (5.5) of section I-5. In the same way the variable θ of expression (3.1) is related to the x of the terms E_0, E_1, and E_2, see expressions (2.1), (2.2) and (2.3). This relation is

$$x = \frac{c}{2}\cos\theta \tag{3.2}$$

Substituting (3.1) and making use of relation (3.2), integrals (2.1), (2.2) and (2.3) can be put in the following form

$$E_0 = \frac{c}{\pi} \int_0^{\theta_0} \int_0^{\pi} \frac{w \sin^2\tau}{\cos\theta - \cos\tau}\, d\tau\, d\theta + \frac{\Gamma \theta_0}{\pi} \tag{3.3}$$

$$E_1 = \frac{c^2}{2\pi} \int_0^{\theta_0} \int_0^{\pi} \frac{w \sin^2\tau}{\cos\theta - \cos\tau} \cos\theta\, d\tau\, d\theta + \frac{c\Gamma}{2\pi} \sin\theta_0 \tag{3.4}$$

$$E_2 = \frac{c^3}{4\pi} \int_0^{\theta_0} \int_0^{\pi} \frac{w \sin^2\tau}{\cos\theta - \cos\tau} \cos^2\theta\, d\tau\, d\theta + \frac{c^2\Gamma}{4\pi}\left(\frac{\theta_0}{2} + \frac{1}{4}\sin 2\theta_0\right) \tag{3.5}$$

In the above expressions one integration has been performed, viz., that part which embraces the term $\frac{2\Gamma}{2c\sin\theta}$ of expression (3.1). To complete the evaluation the function of τ which determines the downwash w must be given.

In Report 5 the downwash w is determined by a piecemeal process where the resultant downwash in the notation of that report is given as

$$w = w' + w'' + w''' + w_0$$

see page 44 of Report 5. The formulae for w', w'', and w''' are the same as given in Report 5 except that the A in these formulae is here replaced by A_β, see expression (4.1) section I-4. The w_0 must be modified for that part of the wing which is aileron, otherwise it is the same.

Let the symbol w_β be defined such that

$$w_\beta = w_1 \qquad \text{for} \qquad 0 \leq \tau \leq \theta_0$$

and

$$w_\beta = w_0 \qquad \text{for} \qquad \theta_0 \leq \tau \leq \pi$$

Both w_0 and w_1 are given here in section I-5 as expressions (5.7) and (5.8) respectively. The resultant downwash as used here can now be written as

$$w = w_\beta' + w_\beta'' + w_\beta''' + w_\beta \tag{3.6}$$

where the subscript β has been attached to the terms w', w'', and w''' to indicate that A_β is used in their respective formulae in place of the A of Report 5.

Attention is called to the fact that the Γ present in expressions (3.3), (3.4) and (3.5) is the total circulation about the wing and is here equal to A_β. From this and expression (3.6), E_0 can be written as follows;

$$E_0 = \frac{c}{\pi}\int_0^{\theta_0}\int_0^{\pi}\left(w_\beta' + w_\beta'' + w_\beta''' + w_\beta\right)\frac{\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta + \frac{A_\beta\theta_0}{\pi} \tag{3.7}$$

In section I-4 the term A_β is given as

$$A_\beta = \Gamma_\beta' + \Gamma_\beta'' + \Gamma_\beta''' + \Gamma_\beta^0$$

Substituting the above relation in expression (3.7) it becomes

$$E_o = \frac{c}{\pi}\int_0^{\theta_o}\int_0^{\pi}(w_\beta' + w_\beta'' + w_\beta''' + w_\beta)\frac{\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta$$

$$+ (\Gamma_\beta' + \Gamma_\beta'' + \Gamma_\beta''' + \Gamma_\beta^o)\frac{\theta_o}{\pi} \qquad (3.8)$$

Let four symbols E_o', E_o'', E_o''', and E_o^o be defined as follows;

$$E_o' = \frac{c}{\pi}\int_0^{\theta_o}\int_0^{\pi} w_\beta' \frac{\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta + \frac{\Gamma_\beta'\theta_o}{\pi} \qquad (3.9)$$

$$E_o'' = \frac{c}{\pi}\int_0^{\theta_o}\int_0^{\pi} w_\beta'' \frac{\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta + \frac{\Gamma_\beta''\theta_o}{\pi} \qquad (3.10)$$

$$E_o''' = \frac{c}{\pi}\int_0^{\theta_o}\int_0^{\pi} w_\beta''' \frac{\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta + \frac{\Gamma_\beta'''\theta_o}{\pi} \qquad (3.11)$$

$$E_o^o = \frac{c}{\pi}\int_0^{\theta_o}\int_0^{\pi} w_\beta \frac{\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta + \frac{\Gamma_\beta^o\theta_o}{\pi} \qquad (3.12)$$

Comparing these four expressions with (3.8) it follows that

$$E_o = E_o' + E_o'' + E_o''' + E_o^o \qquad (3.13)$$

and following similar logic with E_1 and E_2 it can be shown that

$$E_1 = E_1' + E_1'' + E_1''' + E_1^o \qquad (3.14)$$

and that

$$E_2 = E_2' + E_2'' + E_2''' + E_2^o \qquad (3.15)$$

where expressions similar to (3.9), (3.10), (3.11) and (3.12) exist defining the component parts of (3.14) and (3.15). These eight expressions will not be given here as in the case of E_o but will be taken up as each one of these parts is evaluated. As in Report 5, it can be said that E_o' is that part of E_o due to the trailing vortices, that E_o'' is that part of E_o due to the shed vorticity, etc.

II-4 Derivation of E_0', E_1', E_2'

In the evaluation of the twelve terms given in expressions (3.13), (3.14) and (3.15) it is, from the nature of the integrals, more logical to derive these terms in sequence, E_0', E_1', E_2', $E_0''E_1''E_2''$etc. The first term to be considered is E_0'. The formula for w_β' which represents the downwash induced on the wing by the trailing vortices, can be written from the expression given at the top of page 45 Report 5 as

$$w_\beta' = \frac{A_\beta e^{-\frac{c}{r_1}}}{2\pi r_1(2+i\lambda r_1)}\left[\frac{4+i\lambda r_1}{2+i\lambda r_1}e^{\frac{c}{r_1}\cos\tau} + \frac{c}{r_1}\left(e^{\frac{c}{r_1}\cos\tau} - e^{\frac{c}{r_1}\cos\tau}\cos\tau\right)\right] \tag{4.1}$$

It is here to be noticed that A_β has replaced the A in the aforesaid reference, otherwise the two expressions are identical.

Substituting (4.1) in expression (3.9) it becomes

$$E_0' = \frac{c}{\pi}\int_0^{\theta_0}\int_0^{\pi}\left\{\frac{A_\beta e^{-\frac{c}{r_1}}}{2\pi r_1(2+i\lambda r_1)}\left[\frac{4+i\lambda r_1}{2+i\lambda r_1}e^{\frac{c}{r_1}\cos\tau} + \frac{c}{r_1}\left(e^{\frac{c}{r_1}\cos\tau} - e^{\frac{c}{r_1}\cos\tau}\cos\tau\right)\right]\right\}\frac{\sin^2\tau}{\cos\theta-\cos\tau}\,d\tau\,d\theta + \frac{\Gamma_\beta'\theta_0}{\pi}$$

which can be arranged as follows:

$$E_0' = \frac{c A_\beta e^{-\frac{c}{r_1}}}{2\pi^2 r_1(2+i\lambda r_1)}\left\{\left[\frac{4+i\lambda r_1}{2+i\lambda r_1} + \frac{c}{r_1}\right]\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\frac{\sin^2\tau}{\cos\theta-\cos\tau}\,d\tau\,d\theta - \frac{c}{r_1}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\frac{\cos\tau\sin^2\tau}{\cos\theta-\cos\tau}\,d\tau\,d\theta\right\} + \frac{\Gamma_\beta\theta_0}{\pi} \tag{4.2}$$

In the above there are two distinct integrals which are symbolized as $S_0(\frac{c}{r_1}, \theta_0)$ and $S_1(\frac{c}{r_1}, \theta_0)$ thus

$$S_0(\tfrac{c}{r_1}, \theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\frac{\sin^2\tau}{\cos\theta-\cos\tau}\,d\tau\,d\theta \tag{4.3}$$

and

$$S_1(\tfrac{c}{r_1}, \theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\frac{\cos\tau\sin^2\tau}{\cos\theta-\cos\tau}\,d\tau\,d\theta \tag{4.4}$$

These symbols are written with functional notation to emphasize the fact that both $S_0(\frac{c}{r_1},\theta_0)$ and $S_1(\frac{c}{r_1},\theta_0)$ are functions of $\frac{c}{r_1}$ and θ_0.

With this new notation expression (4.2) can now be written as

$$E_0' = \frac{c A_\beta e^{-\frac{c}{r_1}}}{2\pi r_1(2+i\lambda r_1)}\left\{\left[\frac{4+i\lambda r_1}{2+i\lambda r_1}+\frac{c}{r_1}\right]S_0(\tfrac{c}{r_1},\theta_0) - \frac{c}{r_1}S_1(\tfrac{c}{r_1},\theta_0)\right\} + \frac{\Gamma_\beta'\theta_0}{\pi} \tag{4.5}$$

The integrals which $S_0(\frac{c}{r_1},\theta_0)$ and $S_1(\frac{c}{r_1},\theta_0)$ represent are worked out in appendix C. The values of these two expressions are given below:

$$S_0(\tfrac{c}{r_1},\theta_0) = \frac{2r_1}{c}\sigma(\tfrac{c}{r_1},\theta_0) \tag{4.6}$$

and

$$S_1(\tfrac{c}{r_1},\theta_0) = \frac{r_1}{c}\left[I_0(\tfrac{c}{r_1})\sin\theta_0 - 2\left(\frac{r_1}{c}-\cos\theta_0\right)\sigma(\tfrac{c}{r_1},\theta_0)\right] \tag{4.7}$$

where $I_0(\frac{c}{r_1})$ is the modified Bessel function of zero order. The symbol $\sigma(\frac{c}{r_1},\theta_0)$ represents the following Fourier series;

$$\sigma(\tfrac{c}{r_1},\theta_0) = \sum_{n=1}^{\infty} I_n(\tfrac{c}{r_1})\sin n\theta_0 \tag{4.8}$$

This series is the result of the integration described in Appendix C, and as shown, is also a function of $\frac{c}{r_1}$ and θ_0.

To finish the above formulae substitute for r_1 the expression,

$$r_1 = \frac{c\,AR}{4} \tag{4.9}$$

which relation is expression (2.2) part II - page 13 of Report 5. Thus expression (4.5) becomes

$$E_0' = \frac{A_\beta e^{-\frac{4}{AR}}}{\pi AR(1+\frac{i\lambda c}{8}AR)}\left\{\left[\frac{1}{1+\frac{i\lambda c}{8}AR}+\frac{4}{AR}+1\right]S_0(\tfrac{4}{AR},\theta_0) - \frac{4}{AR}S_1(\tfrac{4}{AR},\theta_0)\right\} + \frac{\Gamma_\beta'\theta_0}{\pi} \tag{4.10}$$

and expressions (4.6) (4.7) and (4.8) become

$$S_0\left(\frac{4}{AR}, \theta_0\right) = \frac{AR}{2}\,\sigma\left(\frac{4}{AR}, \theta_0\right) \tag{4.11}$$

$$S_1\left(\frac{4}{AR}, \theta_0\right) = \frac{AR}{4}\left[I_0\left(\frac{4}{AR}\right)\sin\theta_0 - 2\left(\frac{AR}{4} - \cos\theta_0\right)\sigma\left(\frac{4}{AR}, \theta_0\right)\right. \tag{4.12}$$

$$\sigma\left(\frac{4}{AR}, \theta_0\right) = \sum_{n=1}^{\infty} I_n\left(\frac{4}{AR}\right)\sin n\theta_0 \tag{4.13}$$

Following the same scheme as exemplified in expression (2.6) page 46 of Report 5 a symbol f_{OT} is chosen such that

$$f_{OT} = \frac{e^{-\frac{4}{AR}}}{AR\left(1 + \frac{i\lambda c}{8}AR\right)}\left\{\left[\frac{1}{1+\frac{i\lambda c}{8}AR} + \frac{4}{AR} + 1\right] S_0\left(\frac{4}{AR}, \theta_0\right) - \frac{4}{AR} S_1\left(\frac{4}{AR}, \theta_0\right)\right\} \tag{4.14}$$

The subscript O of the symbol f_{OT} is chosen so as to designate it as part of the E_0 group and the T associates it with the trailing vortices. By means of this E_0' can be written as

$$E_0' = \frac{A_\beta f_{OT}}{\pi} + \frac{\Gamma_\beta'\theta_0}{\pi} \tag{4.15}$$

It is to be pointed out that f_{OT} is a function of AR, θ_0 and λc, which means that it is a function of aspect ration, aileron chord, and the reduced velocity.

As outlined at the beginning of this section, the E's would be evaluated in the order E_0', E_1', E_2', etc. The next term in this sequence is E_1'. This term is defined by substituting w_β' (expression (4.1)) for w and Γ_β' for Γ in expression (3.4); thus

$$E_1' = \frac{c^2}{2\pi}\int_0^{\theta_0}\int_0^{\pi} \frac{A_\beta e^{-\frac{c}{r_1}}}{2\pi r_1(2+i\lambda r_1)}\left[\frac{4+i\lambda r_1}{2+i\lambda r_1} e^{\frac{c}{r_1}\cos\tau} + \frac{c}{r_1}\left(e^{\frac{c}{r_1}\cos\tau} - e^{\frac{c}{r_1}\cos\tau}\cos\tau\right)\right]\frac{\sin^2\tau\cos\theta}{\cos\theta-\cos\tau}\,d\tau\, d\theta + \frac{c\Gamma_\beta'}{2\pi}\sin\theta_0 \tag{4.16}$$

This expression can be arranged as follows;

$$E_1' = \frac{c^2 A_B e^{-\frac{c}{r_1}}}{4\pi^2 r_1(2+i\lambda r_1)} \left\{ \left[\frac{4+i\lambda r_1}{2+i\lambda r_1} + \frac{c}{r_1}\right] \int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau} \frac{\sin^2\tau\cos\theta}{\cos\theta-\cos\tau}\, d\tau\, d\theta \right.$$

$$\left. -\frac{c}{r_1}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\cos\tau\,\frac{\sin^2\tau\cos\theta}{\cos\theta-\cos\tau}\, d\tau\, d\theta \right\} + \frac{c\,\Gamma_B'}{2\pi}\sin\theta_0 \qquad (4.17)$$

As was done for expression (4.2) here also the two integrals will be symbolized by $S_2(\frac{c}{r_1},\theta_0)$ and $S_3(\frac{c}{r_1},\theta_0)$; thus

$$S_2\left(\frac{c}{r_1},\theta_0\right) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\frac{\sin^2\tau\cos\theta}{\cos\theta-\cos\tau}\, d\tau\, d\theta \qquad (4.18)$$

$$S_3\left(\frac{c}{r_1},\theta_0\right) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\cos\tau\,\frac{\sin^2\tau\cos\theta}{\cos\theta-\cos\tau}\, d\tau\, d\theta \qquad (4.19)$$

Substituting these two expressions E_1' becomes

$$E_1' = \frac{c^2 A_B e^{-\frac{c}{r_1}}}{4\pi r_1(2+i\lambda r_1)} \left\{ \left[\frac{4+i\lambda r_1}{2+i\lambda r_1} + \frac{c}{r_1}\right] S_2\left(\frac{c}{r_1},\theta_0\right) - \frac{c}{r_1} S_3\left(\frac{c}{r_1},\theta_0\right) \right\} + \frac{c\,\Gamma_B'}{2\pi}\sin\theta_0 \qquad (4.20)$$

Expressions (4.18) and (4.19) as worked out in appendix C are given below in terms of aspect ratio AR, rather than $\frac{c}{r_1}$; thus

$$S_2\left(\frac{4}{AR},\theta_0\right) = \frac{AR}{4}\left[I_1\left(\frac{4}{AR}\right)\theta_0 + I_0\left(\frac{4}{AR}\right)\sin\theta_0 - 2\left(\frac{AR}{4}-\cos\theta_0\right)\sigma\left(\frac{4}{AR},\theta_0\right)\right] \qquad (4.21)$$

$$S_3\left(\frac{4}{AR},\theta_0\right) = \frac{AR}{4}\left[1+\frac{AR^2}{4} - AR\cos\theta_0 + \cos 2\theta_0\right]\sigma\left(\frac{4}{AR},\theta_0\right) + \frac{AR}{4} I_2\left(\frac{4}{AR}\right)\theta_0$$

$$+\frac{AR}{8}\left[-\left(\frac{AR^2}{2}+1\right)I_1\left(\frac{4}{AR}\right) + 2AR\, I_2\left(\frac{4}{AR}\right) + 3 I_3\left(\frac{4}{AR}\right)\right]\sin\theta_0$$

$$+\frac{1}{8}\left[\left(AR^2+1\right)I_1\left(\frac{4}{AR}\right) - AR\left(\frac{AR^2}{2}+1\right)I_2\left(\frac{4}{AR}\right) + AR^2 I_3\left(\frac{4}{AR}\right)\right.$$

$$\left. - AR\, I_4\left(\frac{4}{AR}\right) - I_5\left(\frac{4}{AR}\right)\right]\sin 2\theta_0 \qquad (4.22)$$

Substituting relation (4.9) in expression (4.20) it becomes

$$E_1' = \frac{c\,A_\beta\, e^{-\frac{4}{AR}}}{2\pi AR\left(1+\frac{i\lambda c}{8}AR\right)}\left\{\left[\frac{1}{1+\frac{i\lambda c}{8}AR}+\frac{4}{AR}+1\right]S_2\left(\frac{4}{AR},\theta_0\right) - \frac{4}{AR}S_3\left(\frac{4}{AR},\theta_0\right)\right\} + \frac{c\,\Gamma_\beta'}{2\pi}\sin\theta_0 \tag{4.23}$$

Let f_{1T} designate the following expression:

$$f_{1T} = \frac{e^{-\frac{4}{AR}}}{AR\left(1+\frac{i\lambda c}{8}AR\right)}\left\{\left[\frac{1}{1+\frac{i\lambda c}{8}AR}+\frac{4}{AR}+1\right]S_2\left(\frac{4}{AR},\theta_0\right) - \frac{4}{AR}S_3\left(\frac{4}{AR},\theta_0\right)\right\} \tag{4.24}$$

Here the subscript 1 of f_{1T} designates its relation to E_1 while the T as before, associates it with the trailing vortices. The term E_1' can now be written as

$$E_1' = \frac{A_\beta c f_{1T}}{2\pi} + \frac{c\,\Gamma_\beta'}{2\pi}\sin\theta_0 \tag{4.25}$$

It is to be noted that f_{1T} differs from f_{0T} in that $S_2\left(\frac{4}{AR},\theta_0\right)$ and $S_3\left(\frac{4}{AR},\theta_0\right)$ replace $S_0\left(\frac{4}{AR},\theta_0\right)$ and $S_1\left(\frac{4}{AR},\theta_0\right)$ respectively.

The term E_2' is defined by substituting expression (4.1) for w in expression (3.4) and Γ_β' for Γ; thus

$$E_2' = \frac{c^3}{4\pi}\int_0^{\theta_0}\int_0^{\pi}\left\{\frac{A_\beta e^{-\frac{c}{r_1}}}{2\pi r_1(2+i\lambda r_1)}\left[\frac{4+i\lambda r_1}{2+i\lambda r_1}e^{\frac{c}{r_1}\cos\tau} + \frac{c}{r_1}\left(e^{\frac{c}{r_1}\cos\tau} - e^{\frac{c}{r_1}\cos\tau}\cos\tau\right)\right]\right\}\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}\,d\tau\, d\theta + \frac{c^2\Gamma_\beta'}{4\pi}\left(\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0\right) \tag{4.26}$$

which can be written as

$$E_2' = \frac{c^3 A_\beta e^{-\frac{c}{r_1}}}{8\pi^2 r_1(2+i\lambda r_1)}\left\{\left[\frac{4+i\lambda r_1}{2+i\lambda r_1}+\frac{c}{r_1}\right]\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}\,d\tau\, d\theta - \frac{c}{r_1}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\cos\tau\,\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}\,d\tau\, d\theta\right\} + \frac{c^2\Gamma_\beta'}{4\pi}\left(\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0\right) \tag{4.27}$$

This expression involves two more integrals which are defined as

$$S_4\left(\frac{c}{r_1}, \theta_0\right) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau} \frac{\sin^2\tau\cos^2\theta}{\cos\theta - \cos\tau}\, d\tau\, d\theta \tag{4.28}$$

$$S_5\left(\frac{c}{r_1}, \theta_0\right) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau} \cos\tau \frac{\sin^2\tau\cos^2\theta}{\cos\theta - \cos\tau}\, d\tau\, d\theta \tag{4.29}$$

Substituting the above two symbols in expression (4.27) it becomes

$$E_2' = \frac{c^3 A_B e^{-\frac{c}{r_1}}}{8\pi r_1(2 + i\lambda r_1)} \left\{\left[\frac{4 + i\lambda r_1}{2 + i\lambda r_1} + \frac{c}{r_1}\right] S_4\left(\frac{c}{r_1}, \theta_0\right) - \frac{c}{r_1} S_5\left(\frac{c}{r_1}, \theta_0\right)\right\} + \frac{c^2 \Gamma_B'}{4\pi}\left(\frac{\theta_0}{2} + \frac{1}{4}\sin 2\theta_0\right) \tag{4.30}$$

The integrals (4.28) and (4.29) are worked out in Appendix C and are shown below in their AR form;

$$\begin{aligned} S_4\left(\tfrac{4}{AR}, \theta_0\right) &= \frac{AR}{4}\left[1 + \frac{AR}{4} - AR\cos\theta_0 + \cos 2\theta_0\right]\sigma\left(\tfrac{4}{AR}, \theta_0\right) + \frac{AR}{4} I_2\left(\tfrac{4}{AR}\right)\theta_0 \\ &+ \frac{AR}{8}\left[-\left(\frac{AR^2}{2} - 1\right) I_1\left(\tfrac{4}{AR}\right) + 2AR\, I_2\left(\tfrac{4}{AR}\right) + 3I_3\left(\tfrac{4}{AR}\right)\right]\sin\theta_0 \\ &+ \frac{1}{8}\left[(AR^2 + 1) I_1\left(\tfrac{4}{AR}\right) - AR\left(\frac{AR^2}{2} + 1\right) I_2\left(\tfrac{4}{AR}\right) + AR^2 I_3\left(\tfrac{4}{AR}\right) \right. \\ &\left. - AR\, I_4\left(\tfrac{4}{AR}\right) - I_5\left(\tfrac{4}{AR}\right)\right]\sin 2\theta_0 \end{aligned} \tag{4.31}$$

If expression (4.22) is subtracted from expression (4.31) it will be found that $S_4\left(\frac{4}{AR}, \theta_0\right)$ can be written as follows;

$$S_4\left(\tfrac{4}{AR}, \theta_0\right) = S_3\left(\tfrac{4}{AR}, \theta_0\right) + \frac{AR}{4} I_1\left(\tfrac{4}{AR}\right)\sin\theta_0 \tag{4.32}$$

In like manner it can also be shown that

$$S_2\left(\tfrac{4}{AR}, \theta_0\right) = S_1\left(\tfrac{4}{AR}, \theta_0\right) + \frac{AR}{4} I_1\left(\tfrac{4}{AR}\right)\theta_0 \tag{4.33}$$

The above two relations can also be obtained if the two integrands defining these expressions are first subtracted and then integrated, see appendix C.

The last expression of this group is $S_5\left(\frac{4}{AR}, \theta_0\right)$ which is given below:

$$S_5\left(\tfrac{4}{AR},\theta_0\right) = \frac{1}{8}\left[I_0\left(\tfrac{4}{AR}\right) - I_4\left(\tfrac{4}{AR}\right)\right]\theta_0 + \frac{1}{8}\Big[AR\cos 3\theta_0 - \frac{3AR^2}{2}\cos 2\theta_0$$

$$+ 3AR\left(1+\frac{AR^2}{2}\right)\cos\theta_0 - \frac{3AR^2}{2}\left(1+\frac{AR^2}{4}\right)\Big]\sigma\left(\tfrac{4}{AR},\theta_0\right)$$

$$+ \frac{1}{8}\Big[\left(4+\frac{3AR^2}{2}+\frac{3AR^4}{8}\right)I_1\left(\tfrac{4}{AR}\right) - \frac{3AR}{2}\left(1+AR^2\right)I_2\left(\tfrac{4}{AR}\right)$$

$$-3\left(1-\frac{AR^2}{4}\right)I_3\left(\tfrac{4}{AR}\right) - I_5\left(\tfrac{4}{AR}\right)\Big]\sin\theta_0$$

$$+ \frac{1}{8}\Big[-\frac{3AR^3}{4}I_1\left(\tfrac{4}{AR}\right) + \left(\frac{1}{2}+AR^2+\frac{3AR^4}{8}\right)I_2\left(\tfrac{4}{AR}\right)$$

$$-\frac{AR}{2}\left(1+\frac{3AR^2}{2}\right)I_3\left(\tfrac{4}{AR}\right) + \frac{3AR^2}{4}I_4\left(\tfrac{4}{AR}\right) - \frac{AR}{2}I_5\left(\tfrac{4}{AR}\right) - \frac{1}{2}I_6\left(\tfrac{4}{AR}\right)\Big]\sin 2\theta_0$$

$$+ \frac{1}{8}\Big[\left(\frac{1}{3}+\frac{3AR^2}{4}\right)I_1\left(\tfrac{4}{AR}\right) - \frac{AR}{2}\left(1+\frac{3AR^2}{2}\right)I_2\left(\tfrac{4}{AR}\right)$$

$$+AR^2\left(1+\frac{3AR^2}{8}\right)I_3\left(\tfrac{4}{AR}\right) - \frac{AR}{2}\left(1+\frac{3AR^2}{4}\right)I_4\left(\tfrac{4}{AR}\right)$$

$$+\frac{3AR^2}{4}I_5\left(\tfrac{4}{AR}\right) - \frac{AR}{2}I_6\left(\tfrac{4}{AR}\right) - \frac{1}{3}I_7\left(\tfrac{4}{AR}\right)\Big]\sin 3\theta_0 \qquad (4.34)$$

Expression (4.30) in terms of aspect ratio is as follows;

$$E_2' = \frac{c^2 A_\beta e^{-\frac{4}{AR}}}{4\pi AR\left(1+\frac{i\lambda c}{8}AR\right)}\left\{\left[\frac{1}{1+\frac{i\lambda c}{8}AR} + \frac{4}{AR} + 1\right]S_4\left(\tfrac{4}{AR},\theta_0\right) - \frac{4}{AR}S_5\left(\tfrac{4}{AR},\theta_0\right)\right\} + \frac{c^2\Gamma_\beta'}{4\pi}\left(\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0\right) \qquad (4.35)$$

A symbol f_{2T} can now be defined as shown below:

$$f_{2T} = \frac{e^{-\frac{4}{AR}}}{AR\left(1+\frac{i\lambda c}{8}AR\right)}\left\{\left[\frac{1}{1+\frac{i\lambda c}{8}AR} + \frac{4}{AR} + 1\right]S_4\left(\tfrac{4}{AR},\theta_0\right) - \frac{4}{AR}S_5\left(\tfrac{4}{AR},\theta_0\right)\right\} \qquad (4.36)$$

The term E_2' can now be written as

$$E_2' = \frac{c^2 A_\beta}{4\pi} f_{2T} + \frac{c^2\Gamma_\beta'}{4\pi}\left(\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0\right) \qquad (4.37)$$

II-5 Derivation of E_0'', E_1'', E_2''

The second group of terms to be derived is E_0'', E_1'', and E_2''. This group arises from the shed vorticity. To obtain the downwash for this group take the expression given at the top of page 47 of Report 5 and replace A by A_β ; thus

$$w_\beta'' = \frac{i\lambda A_\beta}{\pi c}\int_0^\infty \frac{e^{-i\lambda\xi}\,d\xi}{\frac{2\xi}{c}+1-\cos\tau} - \frac{1.1358\,i\lambda A_\beta}{\pi(1+6i\lambda\mu_1)}\,e^{-\frac{c}{12\mu_1}(1-\cos\tau)}$$

As in appendix C page 95 of Report 5 the variable ξ will be changed to the nondimensional variable ς through the relation that

$$\xi = \frac{c}{2}(\varsigma-1)$$

Making this substitution, w_β'' becomes

$$w_\beta'' = \frac{i\lambda A_\beta}{2\pi}\,e^{\frac{i\lambda c}{2}}\int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{\varsigma-\cos\tau}\,d\varsigma - \frac{1.1358\,i\lambda A_\beta}{\pi(1+i\lambda\mu_1)}\,e^{-\frac{c}{12\mu_1}(1-\cos\tau)} \qquad (5.1)$$

Substituting the above in expression (3.10), E_0'' takes on the following form;

$$E_0'' = \frac{c}{\pi}\int_0^{\theta_0}\int_0^\pi \left\{ \frac{i\lambda A_\beta}{2\pi}\,e^{\frac{i\lambda c}{2}}\int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{\varsigma-\cos\tau}\,d\varsigma - \frac{1.1358\,i\lambda A_\beta}{\pi(1+6i\lambda\mu_1)}\,e^{-\frac{c}{12\mu_1}(1-\cos\tau)} \right\} \frac{\sin^2\tau}{\cos\theta-\cos\tau}\,d\tau\,d\theta + \frac{\Gamma_\beta''\theta_0}{\pi}$$

Rearranging the above expression it becomes

$$E_0'' = \frac{i\lambda c}{2\pi^2}A_\beta\,e^{\frac{i\lambda c}{2}}\int_0^{\theta_0}\int_0^\pi\int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\varsigma\,d\tau\,d\theta - \frac{1.1358\,i\lambda c}{\pi^2(1+6i\lambda\mu_1)}A_\beta\,e^{-\frac{c}{12\mu_1}}\int_0^{\theta_0}\int_0^\pi e^{\frac{c}{12\mu_1}\cos\tau}\,\frac{\sin^2\tau}{\cos\theta-\cos\tau}\,d\tau\,d\theta + \frac{\Gamma_\beta''\theta_0}{\pi} \qquad (5.2)$$

The first integral of expression (5.2) is worked out in appendix D. The value of this integral is

$$\int_0^{\theta_0}\int_0^{\pi}\int_1^{\infty} \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\varsigma\,d\tau\,d\theta$$

$$= \frac{2\pi}{i\lambda c}e^{-\frac{i\lambda c}{2}}(\pi-\theta_0) - \frac{2\pi}{i\lambda c}e^{-\frac{i\lambda c}{2}}Q_2\sin\theta_0 \tag{5.3}$$

where Q_2 is defined in appendix D as

$$Q_2 = e^{\frac{i\lambda c}{2}}\int_1^{\infty}\frac{e^{-\frac{i\lambda c}{2}\varsigma}}{(\varsigma-\cos\theta_0)\sqrt{\varsigma^2-1}}\,d\varsigma \tag{5.4}$$

No integration was found for the above integral, however, it fortunately cancels from the aileron hinge moment equation.

The second integral of expression (5.2) is seen to be the same as integral (4.3) if $\frac{c}{\pi_1}$ is replaced by $\frac{c}{12\pi_1}$. Following the scheme of integral (4.3) let

$$S_0\left(\tfrac{c}{12\pi_1},\theta_0\right) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{12\pi_1}\cos\tau}\frac{\sin^2\tau}{\cos\theta-\cos\tau}\,d\tau\,d\theta \tag{5.5}$$

and from appendix C it follows that

$$S_0\left(\tfrac{c}{12\pi_1},\theta_0\right) = \frac{24\pi_1}{c}\,\sigma\left(\tfrac{c}{12\pi_1},\theta_0\right) \tag{5.6}$$

where in this case

$$\sigma\left(\tfrac{c}{12\pi_1},\theta_0\right) = \sum_{n=1}^{\infty} I_n\left(\tfrac{c}{12\pi_1}\right)\sin n\theta_0$$

In order to write the above expressions in terms of aspect ratio substitute relation (4.9); thus

$$S_0\left(\tfrac{1}{3AR},\theta_0\right) = 6AR\,\sigma\left(\tfrac{1}{3AR},\theta_0\right) \tag{5.7}$$

and

$$\sigma\left(\tfrac{1}{3AR},\theta_0\right) = \sum_{n=1}^{\infty} I_n\left(\tfrac{1}{3AR}\right)\sin n\theta_0 \tag{5.8}$$

where $I_n\left(\frac{1}{3AR}\right)$ is the modified Bessel function of order n.

Substituting expressions (5.3) and (5.5) in expression (5.2), it becomes

$$E_0'' = \frac{i\lambda c}{2\pi^2} A_\beta e^{\frac{i\lambda c}{2}} \left[\frac{2\pi}{i\lambda c} e^{-\frac{i\lambda c}{2}} (\pi - \theta_0) - \frac{2\pi}{i\lambda c} e^{-\frac{i\lambda c}{2}} Q_2 \sin\theta_0 \right]$$

$$- \frac{1.1358\, i\lambda c}{\pi^2(1+6i\lambda M_1)} A_\beta e^{-\frac{c}{12M_1}} \left[\pi S_0\left(\tfrac{1}{3AR}, \theta_0\right) \right] + \frac{\Gamma_\beta'' \theta_0}{\pi}$$

Making use of relation (4.9) again the above expression can be brought to the following form;

$$E_0'' = \frac{A_\beta}{\pi} \left[\pi - \theta_0 - Q_2 \sin\theta_0 \right]$$

$$- \frac{1.1358\, i\lambda c}{\pi\left(1 + \frac{3i\lambda c}{2} AR\right)} A_\beta e^{-\frac{1}{3AR}} S_0\left(\tfrac{1}{3AR}, \theta_0\right) + \frac{\Gamma_\beta'' \theta_0}{\pi} \tag{5.9}$$

As in section II-4 let f_{os} be defined as follows

$$f_{os} = - \frac{1.1358\, i\lambda c}{1 + \frac{3i\lambda c}{2} AR} e^{-\frac{1}{3AR}} S_0\left(\tfrac{1}{3AR}, \theta_0\right) \tag{5.10}$$

The letter s , in the subscript indicates its association with the shed vorticity. The term E_0'' can now be written as

$$E_0'' = \frac{A_\beta}{\pi} \left[\pi - \theta_0 - Q_2 \sin\theta_0 \right] + \frac{A_\beta f_{os}}{\pi} + \frac{\Gamma_\beta'' \theta_0}{\pi} \tag{5.11}$$

The next term of the group is E_1''. This term is defined when ω_β'' (expression (5.1)) is substituted in expression (3.4) of section II-3. Performing these operations, E_1'' becomes

$$E_1'' = \frac{c^2}{2\pi} \int_0^{\theta_0} \int_0^{\pi} \left\{ \frac{i\lambda A_\beta}{2\pi} e^{\frac{i\lambda c}{2}} \int_1^{\infty} \frac{e^{-\frac{i\lambda c}{2}\zeta}}{\zeta - \cos\tau} d\zeta \right.$$

$$\left. - \frac{1.1358\, i\lambda A_\beta}{\pi(1+6i\lambda M_1)} e^{-\frac{1}{12M_1}(1-\cos\tau)} \right\} \frac{\sin^2\tau \cos\theta}{\cos\theta - \cos\tau} d\tau\, d\theta$$

$$+ \frac{c\Gamma_\beta''}{2\pi} \sin\theta_0$$

which can be put in the following form:

$$E_1'' = \frac{i\lambda c^2}{4\pi^2} A_\beta e^{\frac{i\lambda c}{2}} \int_0^{\theta_0}\int_0^{\pi}\int_1^{\infty} \frac{e^{-\frac{i\lambda c}{2}\varsigma} \sin^2\tau \cos\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)} d\varsigma\, d\tau\, d\theta$$

$$-\frac{1.1358\, i\lambda c^2}{2\pi^2(1+6i\lambda\eta_1)} A_\beta e^{-\frac{c}{12\eta_1}} \int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{12\eta_1}\cos\tau} \frac{\sin^2\tau\cos\theta}{\cos\theta-\cos\tau} d\tau\, d\theta$$

$$+\frac{c\Gamma_\beta''}{2\pi}\sin\theta_0 \qquad (5.12)$$

Like expression (5.2), expression (5.12) involves two integrals, the first of which is treated in appendix D. In this appendix the following result is given;

$$\int_0^{\theta_0}\int_0^{\pi}\int_1^{\infty} \frac{e^{-\frac{i\lambda c}{2}\varsigma} \sin^2\tau\cos\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)} d\varsigma\, d\tau\, d\theta$$

$$= \frac{2\pi}{i\lambda c} e^{-\frac{i\lambda c}{2}} \left[\pi - \sin\theta_0 + \frac{2}{i\lambda c}(\pi-\theta_0)\right]$$

$$-\frac{\pi}{i\lambda c} e^{-\frac{i\lambda c}{2}} \left[\sin 2\theta_0 + \frac{4}{i\lambda c}\sin\theta_0\right] Q_2$$

$$-\frac{4\pi}{(i\lambda c)^2} e^{-\frac{i\lambda c}{2}} \sin\theta_0\, Q_0 + \frac{4\pi}{(i\lambda c)^2} e^{-\frac{i\lambda c}{2}} \theta_0 \left[1+Q_1\right] \qquad (5.13)$$

The terms Q_0 and Q_1 which appear in the above integral are defined on page 49 of Report 5 as expressions (3.3) and (3.4) which are here repeated as follows:

$$Q_0 = \frac{\pi}{4} i\lambda c\, e^{\frac{i\lambda c}{2}}\, i\, H_0^{(2)}\left(\frac{\lambda c}{2}\right) \qquad (5.14)$$

$$Q_1 = \frac{\pi}{4} i\lambda c\, e^{\frac{i\lambda c}{2}}\, H_1^{(2)}\left(\frac{\lambda c}{2}\right) \qquad (5.15)$$

where $H_0^{(2)}(\frac{\lambda c}{2})$ and $H_1^{(2)}(\frac{\lambda c}{2})$ are the so-called Hankel functions. The term Q_2 is defined by expression (5.4).

The second integral of (5.12) can be made the same as the integral (4.18) if $\frac{c}{\eta_1}$ is replaced by $\frac{c}{12\eta_1}$; from this it follows that

$$S_2\left(\frac{c}{12\eta_1}, \theta_0\right) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{12\eta_1}\cos\tau} \frac{\sin^2\tau\cos\theta}{\cos\tau-\cos\theta} \qquad (5.16)$$

This integral is worked out in appendix C and $S_2(\frac{c}{12\eta_1}, \theta_0)$ is given below in terms of aspect ratio; thus

$$S_2\left(\tfrac{1}{3\mathit{AR}},\theta_0\right) = 3\mathit{AR}\Big[I_1\left(\tfrac{1}{3\mathit{AR}}\right)\theta_0 + I_0\left(\tfrac{1}{3\mathit{AR}}\right)\sin\theta_0$$
$$- 2\left(3\mathit{AR} - \cos\theta_0\right)\sigma\left(\tfrac{1}{3\mathit{AR}},\theta_0\right)\Big] \tag{5.17}$$

Substituting expression (5.13) and replacing the second integral by $\pi S_2\left(\frac{1}{3\mathit{AR}},\theta_0\right)$ expression (5.12) becomes

$$E_1'' = \frac{i\lambda c^2}{4\pi^2} A_\beta e^{\frac{i\lambda c}{2}} \Big\{ \frac{2\pi}{i\lambda c} e^{-\frac{i\lambda c}{2}} \left[\pi - \sin\theta_0 + \frac{2}{i\lambda c}(\pi-\theta_0)\right]$$
$$- \frac{\pi}{i\lambda c} e^{-\frac{i\lambda c}{2}} \left[\sin 2\theta_0 + \frac{4}{i\lambda c}\sin\theta_0\right] Q_2$$
$$+ \frac{4\pi}{(i\lambda c)^2} e^{-\frac{i\lambda c}{2}} \sin\theta_0\, Q_0 + \frac{4\pi}{(i\lambda c)^2} e^{-\frac{i\lambda c}{2}} \theta_0 \left[1 + Q_1\right]\Big\}$$
$$- \frac{1.1358\, i\lambda c^2}{2\pi^2(1+6i\lambda r_1)} A_\beta e^{-\frac{c}{12 r_1}} \left[\pi S_2\left(\tfrac{1}{3\mathit{AR}},\theta_0\right)\right]$$
$$+ \frac{c\,\Gamma_\beta''}{2\pi}\sin\theta_0$$

Substituting for $r_1 = \frac{\mathit{AR}\,c}{4}$, the above expression reduces to

$$E_1'' = \frac{cA_\beta}{2\pi}\Big[\pi - \sin\theta_0 + \frac{2\pi}{i\lambda c} + \frac{2\sin\theta_0}{i\lambda c} Q_0 + \frac{2\theta_0}{i\lambda c} Q_1$$
$$- \left(\tfrac{1}{2}\sin 2\theta_0 + \frac{2}{i\lambda c}\sin\theta_0\right) Q_2\Big]$$
$$- \frac{1.1358\, i\lambda c^2}{2\pi\left(1 + \frac{3i\lambda c}{2}\mathit{AR}\right)} A_\beta e^{-\frac{1}{3\mathit{AR}}} S_2\left(\tfrac{1}{3\mathit{AR}},\theta_0\right)$$
$$+ \frac{c\,\Gamma_\beta''}{2\pi}\sin\theta_0 \tag{5.18}$$

Like expression (5.10) a f_{1S} is defined such that

$$f_{1S} = - \frac{1.1358\, i\lambda c}{1 + \frac{3i\lambda c}{2}\mathit{AR}} e^{-\frac{1}{3\mathit{AR}}} S_2\left(\tfrac{1}{3\mathit{AR}}\,\theta_0\right) \tag{5.19}$$

With this, expression (5.18) can be written as follows;

$$E_1'' = \frac{cA_\beta}{2\pi}\left[\pi\left(1+\frac{2}{i\lambda c}\right) + \sin\theta_0 + \frac{2\sin\theta_0}{i\lambda c}Q_0 + \frac{2\theta_0}{i\lambda c}Q_1 - \left(\sin\theta_0\cos\theta_0 + \frac{2}{i\lambda c}\sin\theta_0\right)Q_2\right] + \frac{cA_\beta}{2\pi}f_{1s} \mp \frac{c\Gamma_\beta''}{2\pi}\sin\theta_0 \tag{5.20}$$

The last term in this group is E_2'' which is derived in the same manner as E_0'' and E_1''. To define this term substitute expression (5.1) in expression (3.5) of section II-3; thus

$$E_2'' = \frac{c^3}{4\pi}\int_0^{\theta_0}\int_0^{\pi}\left\{\frac{i\lambda A_\beta}{2\pi}e^{\frac{i\lambda c}{2}}\int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{\varsigma-\cos\tau}d\varsigma - \frac{1.1358\, i\lambda A_\beta}{\pi(1+6i\lambda\pi_1)}e^{-\frac{c}{12\pi_1}(1-\cos\tau)}\right\}\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}d\tau d\theta + \frac{c^2\Gamma_\beta''}{4\pi}\left(\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0\right)$$

Writing the above in the same form as was done in the preceeding work it becomes

$$E_2'' = \frac{i\lambda c^3}{8\pi^2}A_\beta e^{\frac{i\lambda c}{2}}\int_0^{\theta_0}\int_0^{\pi}\int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos^2\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}d\tau d\theta - \frac{1.1358\, i\lambda c^3A_\beta}{4\pi^2(1+6i\lambda\pi_1)}e^{-\frac{c}{12\pi_1}}\int_0^{\theta_0}\int_0^{\pi}e^{\frac{c}{12\pi_1}\cos\tau}\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}d\tau d\theta + \frac{c^2\Gamma_\beta''}{4\pi}\left(\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0\right) \tag{5.21}$$

The first integral of the above is treated in appendix D and the second in appendix C. The result for the first integral is

$$\int_0^{\theta_0}\int_0^{\pi}\int_1^{\infty} \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos^2\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\varsigma\,d\tau\,d\theta$$

$$\begin{aligned}=\frac{2\pi}{i\lambda c}e^{-\frac{i\lambda c}{2}}\Big\{&\pi-\frac{\theta_0}{2}-\frac{1}{4}\sin 2\theta_0+\frac{4\pi}{i\lambda c}\left(1+\frac{2}{i\lambda c}\right)\\&+\frac{2}{i\lambda c}\left[\frac{1}{2}\sin 2\theta_0+\frac{4\sin\theta_0}{i\lambda c}+\theta_0\right]Q_0\\&+\frac{4}{i\lambda c}\left[\frac{2\theta_0}{i\lambda c}+\sin\theta_0\right]Q_1\\&-\sin\theta_0\left[\cos^2\theta_0+\frac{4}{i\lambda c}\left(\cos\theta_0+\frac{2}{i\lambda c}\right)\right]Q_2\Big\}\end{aligned} \qquad (5.22)$$

where Q_0 , Q_1 , and Q_2 are defined by expressions (5.14), (5.15) and (5.4) respectively.

If in expression (4.28) $\frac{c}{\pi_1}$ is replaced by $\frac{c}{12\pi_1}$ it becomes

$$S_4\left(\frac{c}{12\pi_1},\theta_0\right)=\frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\frac{c}{12\pi_1}\cos\tau}\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}\,d\tau\,d\theta \qquad (5.23)$$

where on comparison it is seen that the above integral is the same as the second integral of expression (5.21). As worked out in appendix C expression (5.23) in terms of aspect ratio is as follows;

$$\begin{aligned}S_4\left(\frac{1}{3AR},\theta_0\right)=&\,3AR\left[1+36AR^2-12AR\cos\theta_0+\cos 2\theta_0\right]\sigma\left(\frac{1}{3AR},\theta_0\right)\\&+3AR\,I_2\left(\frac{1}{3AR}\right)\theta_0\\&+\frac{3AR}{2}\left[-(72AR^2-1)I_1\left(\frac{1}{3AR}\right)+24AR\,I_2\left(\frac{1}{3AR}\right)+3AR\,I_3\left(\frac{1}{3AR}\right)\right]\sin\theta_0\\&+\frac{1}{8}\Big[(144AR^2+1)I_1\left(\frac{1}{3AR}\right)-12AR(72AR^2+1)I_2\left(\frac{1}{3AR}\right)\\&\qquad+144AR^2I_3\left(\frac{1}{3AR}\right)-12AR\,I_4\left(\frac{1}{3AR}\right)-I_5\left(\frac{1}{3AR}\right)\Big]\sin 2\theta_0\end{aligned} \qquad (5.24)$$

Substituting expressions (5.22), (5.23) and (4.9) in expression (5.21) it reduces to

$$E_2'' = \frac{c^2 A_\beta}{4\pi}\left\{\pi - \frac{\theta_0}{2} - \frac{1}{4}\sin 2\theta_0 + \frac{4\pi}{i\lambda c}\left(1+\frac{2}{i\lambda c}\right)\right.$$

$$+ \frac{2}{i\lambda c}\left[\frac{1}{2}\sin 2\theta_0 + \frac{4\sin\theta_0}{i\lambda c} + \theta_0\right] Q_0$$

$$+ \frac{4}{i\lambda c}\left[\frac{2\theta_0}{i\lambda c} + \sin\theta_0\right] Q_1$$

$$\left. + \sin\theta_0\left[\cos^2\theta_0 + \frac{4}{i\lambda c}\left(\cos\theta_0 + \frac{2}{i\lambda c}\right)\right] Q_2 \right\}$$

$$- \frac{1.1358\, i\lambda c^3}{4\pi\left(1+\frac{3i\lambda c}{2}AR\right)} A_\beta\, e^{-\frac{1}{3AR}} S_4\left(\frac{1}{3AR}, \theta_0\right)$$

$$+ \frac{c^2 \Gamma_\beta''}{4\pi}\left(\frac{\theta_0}{2} + \frac{1}{4}\sin 2\theta_0\right) \qquad (5.25)$$

A symbol f_{2S} will now be defined for the aspect ratio term of E_2'' as follows:

$$f_{2S} = -\frac{1.1358\, i\lambda c}{1+\frac{3i\lambda c}{2}AR}\, e^{-\frac{1}{3AR}} S_4\left(\frac{1}{3AR}, \theta_0\right) \qquad (5.26)$$

Making use of expression (5.26) E_2'' can be put in its final form;

$$E_2'' = \frac{c^2 A_\beta}{4\pi}\left\{\pi - \frac{\theta_0}{2} - \frac{1}{4}\sin 2\theta_0 + \frac{4\pi}{i\lambda c}\left(1+\frac{2}{i\lambda c}\right)\right.$$

$$+ \frac{2}{i\lambda c}\left[\frac{1}{2}\sin 2\theta_0 + \frac{4\sin\theta_0}{i\lambda c} + \theta_0\right] Q_0$$

$$+ \frac{4}{i\lambda c}\left[\frac{2\theta_0}{i\lambda c} + \sin\theta_0\right] Q_1$$

$$\left. - \sin\theta_0\left[\cos^2\theta_0 + \frac{4}{i\lambda c}\left(\cos\theta_0 + \frac{2}{i\lambda c}\right)\right] Q_2 \right\}$$

$$+ \frac{c^2 A_\beta}{4\pi} f_{2S} + \frac{c^2 \Gamma_\beta''}{4\pi}\left(\frac{\theta_0}{2} + \frac{1}{4}\sin 2\theta_0\right) \qquad (5.27)$$

II-6 Derivations of E_0''', E_1''', E_2'''

The group of terms E_0''', E_1''', and E_2''' arise from the bound and tip trailing vortices discussed on pages 7-10 part II of Report 5. The effects of the tip trailing and the bound vortices on the downwash are summed in a single expression given on page 32 part II of Report 5. It is expression (5.5). A table of constants to be used in connection with this expression is given on page 35 part II of Report 5. This same table can be used here. The only item which is changed in the above mentioned expression is to replace A by A_β. Expression (4.1) given on page 51 of Report 5 is more convenient than the above mentioned expression and is here introduced with the subscript β ; thus

$$w_\beta''' = \frac{A_\beta}{2\pi c}\left[a_0 + \frac{a_1}{2} + \frac{a_2}{4} - \frac{1}{2}(a_1 + a_2)\cos\tau + \frac{a_2}{4}\cos^2\tau\right] \tag{6.1}$$

where a_0 , a_1 , and a_2 are given in Table I page 35 part II of Report 5.

The first term E_0''' of this group is defined by expression (3.11) of section II-3, which on substitution of expression (6.1) becomes

$$E_0''' = \frac{c}{\pi}\int_0^{\theta_0}\int_0^{\pi}\left\{\frac{A_\beta}{2\pi c}\left[a_0 + \frac{a_1}{2} + \frac{a_2}{4} - \frac{1}{2}(a_1 + a_2)\cos\tau + \frac{a_2}{4}\cos^2\tau\right]\right\}\frac{\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta + \frac{\Gamma_\beta'''\theta_0}{\pi}$$

This expression is now put in the following form;

$$\begin{aligned} E_0''' = \frac{A_\beta}{2\pi c}\Bigg\{\left(a_0 + \frac{a_1}{2} + \frac{a_2}{4}\right)&\int_0^{\theta_0}\int_0^{\pi}\frac{\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta \\ &- \frac{1}{2}(a_1 + a_2)\int_0^{\theta_0}\int_0^{\pi}\frac{\cos\tau\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta \\ &+ \frac{a^2}{4}\int_0^{\theta_0}\int_0^{\pi}\frac{\cos^2\tau\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta\Bigg\} + \frac{\Gamma_\beta'''\theta_0}{\pi} \end{aligned} \tag{6.2}$$

Expression (6.2) involves a double integration, the first of which can be integrated by means of the following formula;

$$\int_0^{\pi} \frac{\cos n\tau}{\cos\tau - \cos\theta}\, d\tau = \pi \frac{\sin n\theta}{\sin\theta} \tag{6.3}$$

where $n = 0, 1, 2, 3, \cdots$. This integral is not discussed here but is well known in the subject of thin airfoil theory and its solution represents a Cauchy principal value. The derivation of this formula is given by von Kármán and Burgers on pages 173 and 174 of Aerodynamic Theory, volume II, reference 3. The second part of this integration being elementary in nature will not be discussed and the final value of E_o''' will be given at once; thus

$$E_o''' = \frac{A_\beta}{96\pi}\Big[(48a_o + 24a_1 + 15a_2)\sin\theta_o - 6(a_1 + a_2)\sin 2\theta_o + a_2 \sin 3\theta_o\Big] + \frac{\Gamma_\beta'''\theta_o}{\pi} \tag{6.4}$$

A symbol f_{oc} will be defined such that

$$f_{oc} = \frac{1}{96}\Big[(48a_o + 24a_1 + 15a_2)\sin\theta_o - 6(a_1 + a_2)\sin 2\theta_o + a_2 \sin 3\theta_o\Big] + \frac{\Gamma_\beta'''\theta_o}{\pi} \tag{6.5}$$

from which it follows that

$$E_o''' = \frac{A_\beta}{\pi} f_{oc} + \frac{\Gamma_\beta'''\theta_o}{\pi} \tag{6.7}$$

The second term of this group is E_1'''. It is defined by substituting expression (6.1) and replacing Γ by Γ_β''' in expression (3.4) of section II-4; thus

$$E_1''' = \frac{c^2}{2\pi}\int_0^{\theta_o}\int_0^{\pi}\left\{\frac{A_\beta}{2\pi c}\Big[a_o + \frac{a_1}{2} + \frac{a_2}{4} - \frac{1}{2}(a_1 + a_2)\cos\tau + \frac{a_2}{4}\cos^2\tau\Big]\right\}\frac{\sin^2\tau\cos\theta}{\cos\theta - \cos\tau}\, d\tau\, d\theta + \frac{c\,\Gamma_\beta'''}{2\pi}\sin\theta_o \tag{6.8}$$

The technique of the integration of expression (6.8) being the same as that used on expression (6.2), the intervening steps will be omitted and the result written down directly;

$$E_1''' = \frac{cA_\beta}{384\pi}\Big[(48a_0+24a_1+15a_2)\theta_0 - 12(a_1+a_2)\sin\theta_0$$

$$+(24a_0+12a_1+9a_2)\sin 2\theta_0 - 4(a_1+a_2)\sin 3\theta_0$$

$$+\frac{3}{4}a_2\sin 4\theta_0\Big] + \frac{c\Gamma_\beta'''}{2\pi}\sin\theta_0 \tag{6.9}$$

A symbol f_{1c} will now be established as follows;

$$f_{1c} = \frac{1}{192}\Big[(48a_0+24a_1+15a_2) - 12(a_1+a_2)\sin\theta_0$$

$$+(24a_0+12a_1+9a_2)\sin 2\theta_0 - 4(a_1+a_2)\sin 3\theta_0$$

$$+\frac{3}{4}a_2\sin 4\theta_0\Big] \tag{6.10}$$

Using this expression, E_1''' can be put in the following form;

$$E_1''' = \frac{cA_\beta}{2\pi}f_{1c} + \frac{c\Gamma_\beta'''}{2\pi}\sin\theta_0 \tag{6.11}$$

The last term of this group is E_2 and defining it by substitution expression (6.1) for w and Γ_β''' for Γ in expression (3.5) it becomes

$$E_2''' = \frac{c^3}{4\pi}\int_0^{\theta_0}\int_0^{\pi}\Big\{\frac{A_\beta}{2\pi c}\Big[a_0 + \frac{a_1}{2} + \frac{a_2}{4} - \frac{1}{2}(a_1+a_2)\cos\tau$$

$$+\frac{a_2}{4}\cos^2\tau\Big]\Big\}\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}\,d\tau\,d\theta$$

$$+\frac{c^2\Gamma_\beta'''}{4\pi}\left(\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0\right) \tag{6.12}$$

Here again, owing to the similarity of the integration, the outline of this procedure will be omitted and the result given directly as shown below:

$$E_2''' = \frac{C^2 A_\beta}{1536\pi}\Big[(144a_0 + 72a_1 + 48a_2)\sin\theta_0 + (16a_0 + 8a_1 + 7a_2)\sin 3\theta_0$$

$$+ \frac{3}{5}a_2 \sin 5\theta_0 - 3(a_1 + a_2)(4\theta_0 + 4\sin 2\theta_0 + \sin 4\theta_0)\Big]$$

$$+ \frac{C^2 \Gamma_\beta'''}{4\pi}\left(\frac{\theta_0}{2} + \frac{1}{4}\sin 2\theta_0\right) \tag{6.13}$$

A term f_{2c} is defined as

$$f_{2c} = \frac{1}{384}\Big[(144a_0 + 72a_1 + 48a_2)\sin\theta_0 + (16a_0 + 8a_1 + 7a_2)\sin 3\theta_0$$

$$+ \frac{3}{5}a_2 \sin 5\theta_0 - 3(a_1 + a_2)(4\theta_0 + 4\sin 2\theta_0 + \sin 4\theta_0)\Big] \tag{6.14}$$

The term E_2''' can now be written as

$$E_2''' = \frac{C^2 A_\beta}{4\pi} f_{2c} + \frac{C^2 \Gamma_\beta'''}{4\pi}\left(\frac{\theta_0}{2} + \frac{1}{4}\sin 2\theta_0\right) \tag{6.15}$$

II-7 Derivation of E_0°, E_1°, E_2°

The last group of terms are E_0°, E_1°, and E_2°. This group arises from the motion of the wing-aileron combination. The downwash equivalent to this motion is given in section I-5. The expressions are

$$w_1 = -\dot{h} - U\alpha - \frac{c\dot{\alpha}}{2}\cos\tau - U\beta - \frac{c\dot{\beta}}{2}(\cos\tau - \cos\theta_0) \qquad (7.1)$$

for $0 \leqslant \tau \leqslant \theta_0$, and

$$w_0 = -\dot{h} - U\alpha - \frac{c\dot{\alpha}}{2}\cos\tau \qquad (7.2)$$

for $\theta_0 \leqslant \tau \leqslant \pi$.

Substituting these two expressions in expression (3.12) and integrating in their respective intervals as defined above E_0° becomes

$$E_0^\circ = \frac{c}{\pi}\int_0^{\theta_0}\left\{\int_0^{\theta_0}\left[-\dot{h} - U\alpha - \frac{c\dot{\alpha}}{2}\cos\tau - U\beta - \frac{c\dot{\beta}}{2}(\cos\tau - \cos\theta_0)\right]\frac{\sin^2\tau}{\cos\theta - \cos\tau}d\tau + \int_{\theta_0}^{\pi}\left[-\dot{h} - U\alpha - \frac{c\dot{\alpha}}{2}\cos\tau\right]\frac{\sin^2\tau}{\cos\theta - \cos\tau}d\tau\right\}d\theta + \frac{\Gamma_\beta^\circ\theta_0}{\pi}$$

The above integration can be arranged as follows;

$$E_0^\circ = -\frac{c}{\pi}\left\{\int_0^{\theta_0}\int_0^{\pi}\left[\dot{h} + U\alpha + \frac{c\dot{\alpha}}{2}\cos\tau\right]\frac{\sin^2\tau}{\cos\theta - \cos\tau}d\tau d\theta + \int_0^{\theta_0}\int_0^{\theta_0}\left[U\beta + \frac{c\dot{\beta}}{2}(\cos\tau - \cos\theta_0)\right]\frac{\sin^2\tau}{\cos\theta - \cos\tau}d\tau d\theta\right\} + \frac{\Gamma_\beta^\circ\theta_0}{\pi} \qquad (7.3)$$

The first integration of the first integral of expression (7.3) is

$$\int_0^{\pi}\left[\dot{h} + U\alpha + \frac{c\dot{\alpha}}{2}\cos\tau\right]\frac{\sin^2\tau}{\cos\theta - \cos\tau}d\tau$$

which can be handled by the integral formula (6.3) section II-6. After applying this formula the second integration becomes elementary.

It will be noticed that the second integral of expression (7.3), i.e.

$$\int_0^{\theta_0}\int_0^{\theta_0}\left[U\beta + \frac{c\dot{\beta}}{2}(\cos\tau - \cos\theta_0)\right]\frac{\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta$$

is symmetrical, which means that τ and θ can be interchanged without changing the value of the integral. To evaluate this integral the same technique can be applied here as was applied by von Kármán and Burgers in Volume II of "Aerodynamic Theory" section 11 pages 53-56, (reference 3). The integral can also be integrated in the ordinary way, however, as it turns out it is a Couchy principal value. If the above mentioned process of integration is carried out the value of E_0° becomes

$$E_0^\circ = -c\Big[(\dot{h}+U\alpha)\sin\theta_0 + \frac{c\dot{\alpha}}{8}\sin 2\theta_0 + \frac{U\beta\theta_0}{\pi}\sin\theta_0 - \frac{c\dot{\beta}}{8\pi}(\theta_0\sin 2\theta_0 + \cos 2\theta_0 - 1)\Big] + \frac{\Gamma_\beta^\circ\theta_0}{\pi} \tag{7.4}$$

The definition of E_1° involves expressions (7.1) and (7.2) as well as expression (3.4) of section II-3, however, Γ must be replaced by Γ_β°.

Following this procedure E_1° becomes

$$E_1^\circ = \frac{c^2}{2\pi}\int_0^{\theta_0}\Big\{\int_0^{\theta_0}\Big[-\dot{h} - U\alpha - \frac{c\dot{\alpha}}{2}\cos\tau - U\beta - \frac{c\dot{\beta}}{2}(\cos\tau - \cos\theta_0)\Big]\frac{\sin^2\tau\cos\theta}{\cos\theta - \cos\tau}\,d\tau$$
$$+ \int_0^{\pi}\Big[-\dot{h} - U\alpha - \frac{c\dot{\alpha}}{2}\cos\tau\Big]\frac{\sin^2\tau\cos\theta}{\cos\theta - \cos\tau}\,d\tau\Big\}\,d\theta + \frac{c\Gamma_\beta^\circ}{2\pi}\sin\theta_0$$

which can be arranged as

$$E_1^\circ = -\frac{c^2}{2\pi}\Big\{\int_0^{\theta_0}\int_0^{\pi}\Big[\dot{h} + U\alpha + \frac{c\dot{\alpha}}{2}\cos\tau\Big]\frac{\sin^2\tau\cos\theta}{\cos\theta - \cos\tau}\,d\tau\,d\theta$$
$$+ \int_0^{\theta_0}\int_0^{\theta_0}\Big[U\beta + \frac{c\dot{\beta}}{2}(\cos\tau - \cos\theta_0)\Big]\frac{\sin^2\tau\cos\theta}{\cos\theta - \cos\tau}\,d\tau\,d\theta\Big\} \tag{7.5}$$

The outline of the integration is the same here as in the case of expression

(7.3) hence the result will be given directly; thus

$$E_1^\circ = -\frac{c^2}{4}\Big[(\dot{h}+U\alpha)\left(\theta_0+\tfrac{1}{2}\sin 2\theta_0\right)+\frac{c\dot{\alpha}}{4}\left(\sin\theta_0+\tfrac{1}{3}\sin 3\theta_0\right)$$
$$+\frac{U\beta}{\pi}\left(\theta_0^2+\sin^2\theta_0\right)+\frac{c\dot{\beta}}{2\pi}\left(\theta_0\sin\theta_0-\theta_0^2\cos\theta_0\right)\Big]+\frac{c\,\Gamma_\beta^\circ}{2\pi}\sin\theta_0 \tag{7.6}$$

The last term of this group is E_2°. This is defined by substituting expressions (7.1) and (7.2) for w and replacing Γ by Γ_β° in expression (3.5); thus

$$E_2^\circ = \frac{c^3}{4\pi}\int_0^{\theta_0}\Big\{\int_0^{\theta_0}\Big[-\dot{h}-U\alpha-\frac{c\dot{\alpha}}{2}\cos\tau$$
$$-U\beta-\frac{c\dot{\beta}}{2}\left(\cos\tau-\cos\theta_0\right)\Big]\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}\,d\tau$$
$$+\int_{\theta_0}^{\pi}\Big[-\dot{h}-U\alpha-\frac{c\dot{\alpha}}{2}\cos\tau\Big]\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}\,d\tau\Big\}d\theta$$
$$+\frac{c^2\Gamma_\beta^\circ}{4\pi}\left(\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0\right)$$

which can be written as

$$E_2^\circ = -\frac{c^3}{4\pi}\Big\{\int_0^{\theta_0}\int_0^{\pi}\Big[\dot{h}+U\alpha+\frac{c\dot{\alpha}}{2}\cos\tau\Big]\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}\,d\tau\,d\theta$$
$$+\int_0^{\theta_0}\int_0^{\theta_0}\Big[U\beta+\frac{c\dot{\beta}}{2}\left(\cos\tau-\cos\theta_0\right)\Big]\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}\,d\tau\,d\theta\Big\}$$
$$+\frac{c^2\Gamma_\beta^\circ}{4\pi}\left(\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0\right) \tag{7.7}$$

Here also, the integration procedure being the same as in the preceeding, the result will be given directly; thus

$$E_2^\circ = -\frac{c^3}{8}\Big[(\dot{h}+U\alpha)\left(2\sin\theta_0-\tfrac{2}{3}\sin^3\theta_0\right)$$
$$+\frac{c\dot{\alpha}}{8}\left(\theta_0+\sin 2\theta_0-\tfrac{1}{4}\sin 4\theta_0\right)+\frac{2U\beta}{\pi}\theta_0\sin\theta_0$$
$$+\frac{c\dot{\beta}}{2\pi}\left(\sin^2\theta_0+\frac{\theta_0^2}{4}-\frac{3\theta_0}{4}\sin 2\theta_0+\tfrac{1}{16}\sin^2 2\theta_0\right)\Big]$$
$$+\frac{c^2\Gamma_\beta^\circ}{4\pi}\left(\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0\right) \tag{7.8}$$

II-8 Assembly of E_0 , E_1 , E_2

Expression (3.13) section II-3 indicates that E_0 is the sum of expressions (4.15), (5.11), (6.7), and (7.4). In assembling E_0 the last term E_5^0 will be written first; thus

$$E_0 = -C\left[(\dot{h}+U\alpha)\sin\theta_0 + \frac{C\dot{\alpha}}{8}\sin 2\theta_0 + \frac{U\beta}{\pi}\theta_0\sin\theta_0 - \frac{C\dot{\beta}}{8\pi}\left(\theta_0\sin 2\theta_0 + \cos 2\theta_0 - 1\right)\right] + \frac{\Gamma_\beta^{\,0}\theta_0}{\pi}$$

$$+ \frac{A_\beta f_{oT}}{\pi} + \frac{\Gamma_\beta^{\,\prime}\theta_0}{\pi}$$

$$+ \frac{A_\beta}{\pi}\left[\pi - \theta_0 - Q_2\sin\theta_0\right] + \frac{A_\beta f_{os}}{\pi} + \frac{\Gamma_\beta^{\,\prime\prime}\theta_0}{\pi}$$

$$+ \frac{A_\beta f_{oc}}{\pi} + \frac{\Gamma_\beta^{\,\prime\prime\prime}\theta_0}{\pi}$$

which can be written as

$$E_0 = -C\left[(\dot{h}+U\alpha)\sin\theta_0 + \frac{C\dot{\alpha}}{8}\sin 2\theta_0 + \frac{U\beta}{\pi}\theta_0\sin\theta_0 - \frac{C\dot{\beta}}{8\pi}\left(\theta_0\sin 2\theta_0 + \cos 2\theta_0 - 1\right)\right]$$

$$+ \frac{A_\beta}{\pi}\left[\pi - \theta_0 - Q_2\sin\theta_0 + f_{oT} + f_{os} + f_{oc}\right]$$

$$+ \left(\Gamma_\beta^{\,0} + \Gamma_\beta^{\,\prime} + \Gamma_\beta^{\,\prime\prime} + \Gamma_\beta^{\,\prime\prime\prime}\right)\frac{\theta_0}{\pi}$$

In section I-4 under the discussion of circulation it was pointed out that

$$A_\beta = \Gamma_\beta^{\,0} + \Gamma_\beta^{\,\prime} + \Gamma_\beta^{\,\prime\prime} + \Gamma_\beta^{\,\prime\prime\prime}$$

where A_β is the total circulation about the wing. If this is substituted in the expression for E_0 it will be noticed that the last term will cancel the θ_0 term in the second bracketed quantity and that E_0 becomes

$$E_0 = -c\left[(\dot{h} + U\alpha)\sin\theta_0 + \frac{c\dot{\alpha}}{8}\sin 2\theta_0 + \frac{U\beta}{\pi}\theta_0\sin\theta_0 - \frac{c\dot{\beta}}{8\pi}\left(\theta_0\sin 2\theta_0 + \cos 2\theta_0 - 1\right)\right] + \frac{A_\beta}{\pi}\left[\pi - Q_2\sin\theta_0 + f_0\right] \tag{8.1}$$

where

$$f_0 = f_{0T} + f_{0S} + f_{0C} \tag{8.2}$$

From expression (3.14) it is apparent that E_1 is the sum of expressions (4.25), (5.20), (6.11) and (7.6). The procedure for determining E_1 being similar to E_0, the result will be written as follows:

$$E_1 = -\frac{c^2}{4}\left[(\dot{h} + U\alpha)\left(\theta_0 + \tfrac{1}{2}\sin 2\theta_0\right) + \frac{c\dot{\alpha}}{4}\left(\sin\theta_0 + \tfrac{1}{3}\sin 3\theta_0\right) + \frac{U\beta}{\pi}\left(\theta_0^2 + \sin^2\theta_0\right) + \frac{c\dot{\beta}}{2\pi}\left(\theta_0\sin\theta_0 - \theta_0^2\cos\theta_0\right)\right] + \frac{cA_\beta}{2\pi}\left[\pi\left(1 + \frac{2}{i\lambda c}\right) + \frac{2\sin\theta_0}{i\lambda c}Q_0 + \frac{2\theta_0}{i\lambda c}Q_1 - \sin\theta_0\left(\cos\theta_0 + \frac{2}{i\lambda c}\right)Q_2 + f_1\right] \tag{8.3}$$

where

$$f_1 = f_{1T} + f_{1S} + f_{1C} \tag{8.4}$$

Expression (3.15) designates E_2 as a sum of expression (4.37), (5.27), (6.14) and (7.8). This result is also given without comment as follows;

$$E_2 = -\frac{c^3}{8}\Big[(\dot{h} + U\alpha)(2\sin\theta_0 - \tfrac{2}{3}\sin^3\theta_0)$$

$$+ \frac{c\dot{\alpha}}{8}(\theta_0 + \sin 2\theta_0 + \tfrac{1}{4}\sin 4\theta_0) + \frac{2U\beta}{\pi}\theta_0\sin\theta_0$$

$$+ \frac{c\dot{\beta}}{2\pi}\Big(\sin^2\theta_0 + \frac{\theta_0^2}{4} - \frac{3\theta_0}{4}\sin 2\theta_0 + \frac{1}{16}\sin^2 2\theta_0\Big)\Big]$$

$$+ \frac{c^2 A_\beta}{4\pi}\Big\{\pi + \frac{4\pi}{i\lambda c}\Big(1 + \frac{2}{i\lambda c}\Big)$$

$$+ \frac{2}{i\lambda c}\Big[\frac{1}{2}\sin 2\theta_0 + \frac{4\sin\theta_0}{i\lambda c} + \theta_0\Big] Q_0$$

$$+ \frac{4}{i\lambda c}\Big[\frac{2\theta_0}{i\lambda c} + \sin\theta_0\Big] Q_1$$

$$- \sin\theta_0\Big[\cos^2\theta_0 + \frac{4}{i\lambda c}\Big(\cos\theta_0 + \frac{2}{i\lambda c}\Big)\Big] Q_2 + f_2\Big\} \qquad (8.5)$$

where

$$f_2 = f_{2T} + f_{2S} + f_{2C} \qquad (8.6)$$

Owing to the fact that the expressions which make up the aileron hinge moment are bulky, the following symbols will be defined as a temporary expedient; thus let

$$e_0 = E_0^\circ - \frac{\Gamma_\beta^\circ\theta_0}{\pi} \qquad (8.7)$$

$$e_1 = E_1^\circ - \frac{c\Gamma_\beta^\circ}{2\pi}\sin\theta_0 \qquad (8.8)$$

$$e_2 = E_2^\circ - \frac{c^2\Gamma_\beta^\circ}{4\pi}\Big(\frac{\theta_0}{2} + \frac{1}{4}\sin 2\theta_0\Big) \qquad (8.9)$$

From this and expressions (8.1), (8.3), and (8.5) it follows that

$$E_0 = e_0 + \frac{A_\beta}{\pi}\left[\pi - \sin\theta_0 \, Q_2 + f_0\right] \tag{8.10}$$

$$E_1 = e_1 + \frac{cA_\beta}{2\pi}\left[\pi\left(1+\frac{2}{i\lambda c}\right) + \frac{2\sin\theta_0}{i\lambda c}Q_0 + \frac{2\theta_0}{i\lambda c}Q_1 - \sin\theta_0\left(\cos\theta_0 + \frac{2}{i\lambda c}\right)Q_2 + f_1\right] \tag{8.11}$$

$$E_2 = e_2 + \frac{c^2A_\beta}{4\pi}\left\{\pi\left[1+\frac{4}{i\lambda c}\left(1+\frac{2}{i\lambda c}\right)\right] + \frac{2}{i\lambda c}\left[\sin\theta_0\cos\theta_0 + \frac{4\sin\theta_0}{i\lambda c} + \theta_0\right]Q_0 + \frac{2}{i\lambda c}\left[\frac{4\theta_0}{i\lambda c} + 2\sin\theta_0\right]Q_1 - \sin\theta_0\left[\cos^2\theta_0 + \frac{4}{i\lambda c}\left(\cos\theta_0 + \frac{2}{i\lambda c}\right)\right]Q_2 + f_2\right\} \tag{8.12}$$

Also from expressions (7.4), (7.6) and (7.8) it follows that

$$e_0 = -c\left[(\dot{h} + U\alpha)\sin\theta_0 + \frac{c\dot{\alpha}}{8}\sin 2\theta_0 + \frac{U\beta}{\pi}\theta_0\sin\theta_0 - \frac{c\dot{\beta}}{8\pi}\left(\theta_0\sin 2\theta_0 + \cos 2\theta_0 - 1\right)\right] \tag{8.13}$$

$$e_1 = -\frac{c^2}{4}\left[(\dot{h}+U\alpha)\left(\theta_0 + \frac{1}{2}\sin 2\theta_0\right) + \frac{c\dot{\alpha}}{4}\left(\sin\theta_0 + \frac{1}{3}\sin 3\theta_0\right) + \frac{U\beta}{\pi}\left(\theta_0^2 + \sin^2\theta_0\right) + \frac{c\dot{\beta}}{2\pi}\left(\theta_0\sin\theta_0 - \theta_0^2\cos\theta_0\right)\right] \tag{8.14}$$

$$e_2 = -\frac{c^3}{8}\left[(\dot{h}+U\alpha)\left(2\sin\theta_0 - \frac{2}{3}\sin^3\theta_0\right) + \frac{c\dot{\alpha}}{8}\left(\theta_0 + \sin 2\theta_0 + \frac{1}{4}\sin 4\theta_0\right) + \frac{2U\beta}{\pi}\theta_0\sin\theta_0 + \frac{c\dot{\beta}}{2\pi}\left(\sin^2\theta_0 + \frac{\theta_0^2}{4} - \frac{3}{4}\theta_0\sin 2\theta_0 + \frac{1}{16}\sin^2 2\theta_0\right)\right] \tag{8.15}$$

II-9 Aileron Hinge Moment

Before assembling the aileron hinge moment forumla it will be found convenient to replace ω in expression (2.8) by its equivalent λU, and write expression (2.8) as follows;

$$M_\beta = \rho U\left(\frac{c}{2}\cos\theta_0 E_0 - E_1\right)$$
$$+ \frac{\rho U}{2} i\lambda c\left[\frac{c}{4}\cos^2\theta_0 E_0 - \cos\theta_0 E_1 + \frac{E_2}{c} - \frac{cA_\beta}{4}(1-\cos\theta_0)^2\right] \tag{9.1}$$

The fact that $\omega = \lambda U$ is set forth on pages 38 and 39 of Report 5 and is here given as expression (4.6) of section I-4.

From the above it appears that there are two quantities to construct with the $E's$. The first is given in the parenthesis and the second in the brackets. The quantity first to be treated is the parenthesis and substituting expressions (8.10) and (8.11) it becomes

$$\frac{c}{2}\cos\theta_0 E_0 - E_1 = \frac{c}{2}\cos\theta_0 e_0 - e_1$$
$$+ \frac{cA_\beta}{2\pi}\left[\pi\left(\cos\theta_0 - 1 - \frac{2}{i\lambda c}\right) - \frac{2\sin\theta_0}{i\lambda c}Q_0\right.$$
$$\left. - \frac{2\theta_0}{i\lambda c}Q_1 + \frac{2\sin\theta_0}{i\lambda c}Q_2 + f_0\cos\theta_0 - f_1\right] \tag{9.2}$$

The second expression of (9.1) requires the substitution of expressions (8.10), (8.11) and (8.12); performing this substitution it reduces to

$$\frac{c}{4}\cos^2\theta_0 E_0 - \cos\theta_0 E_1 + \frac{E_2}{c} - \frac{cA_\beta}{4}(1-\cos\theta_0)^2$$
$$= \frac{c}{4}\cos^2\theta_0 e_0 - \cos\theta_0 e_1 + \frac{e_2}{c}$$
$$+ \frac{cA_\beta}{4\pi}\frac{2}{i\lambda c}\left\{2\pi\left[1 + \frac{2}{i\lambda c} - \cos\theta_0\right]\right.$$
$$+ \left[\frac{4\sin\theta_0}{i\lambda c} + \theta_0 - \sin\theta_0\cos\theta_0\right]Q_0$$
$$+ 2\left[\sin\theta_0 + \frac{2\theta_0}{i\lambda c} - \theta_0\cos\theta_0\right]Q_1 - \frac{4\sin\theta_0}{i\lambda c}Q_2$$
$$\left. + \frac{i\lambda c}{2}\left[f_0\cos^2\theta_0 - 2f_1\cos\theta_0 + f_2\right]\right\} \tag{9.3}$$

Substituting expressions (9.2) and (9.3) in expression (9.1), M_β becomes

$$M_\beta = \rho U \left\{ \frac{c}{2}\cos\theta_0 e_0 - e_1 + \frac{i\lambda c}{2}\left[\frac{c}{4}\cos^2\theta_0 e_0 - \cos\theta_0 e_1 + \frac{e_2}{c}\right]\right\}$$
$$+ \frac{\rho U c}{4\pi} A_\beta \left\{\left[\theta_0 - \sin\theta_0\cos\theta_0\right]Q_0 + 2\left[\sin\theta_0 - \theta_0\cos\theta_0\right]Q_1 + 2\left(f_0\cos\theta_0 - f_1\right) + \frac{i\lambda c}{2}\left(f_0\cos^2\theta_0 - 2f_1\cos\theta_0 + f_2\right)\right\}$$

This can be put in a more condensed form by making use of the Theodorsen constants and by the introduction of a new symbol f. From appendix A it appears that

$$T_4 = -(\theta_0 - \sin\theta_0\cos\theta_0)$$

and

$$T_{12} - T_4 = 2(\sin\theta_0 - \theta_0\cos\theta_0)$$

For the symbol f let

$$f = 2(f_0\cos\theta_0 - f_1) + \frac{i\lambda c}{2}\left(f_0\cos^2\theta_0 - 2f_1\cos\theta_0 + f_2\right) \tag{9.4}$$

From this it follows that the above expression for M_β can be written as

$$M_\beta = \rho U\left(\frac{c}{2}\cos\theta_0 e_0 - e_1\right) + \frac{\rho}{2} i\omega c\left(\frac{c}{4}\cos^2\theta_0 e_0 - \cos\theta_0 e_1 + \frac{e_2}{c}\right) + \frac{\rho U c}{4\pi}A_\beta\left[-T_4 Q_0 + (T_{12} - T_4)Q_1 + f\right] \tag{9.5}$$

Before substituting expressions (8.13), (8.14), and (8.15) for e_0, e_1, and e_2 respectively, it is advisable to change the trigonometric functions in these expressions to functions of a single angle, in order that the Theodorsen constants of appendix A may be quickly applied. If these changed expressions are substituted and the factor $i\omega$ in expression (9.5) treated by expressions (7.2) and (7.3) of section I-7, expression (9.5) becomes

$$M_\beta = \frac{\rho c^2}{4}\Big[\frac{c\ddot{h}}{2}(\theta_0\cos\theta_0 - \sin\theta_0 + \tfrac{1}{3}\sin^3\theta_0)$$

$$+ U(\dot{h} + U\alpha)(\theta_0 - \sin\theta_0\cos\theta_0)$$

$$+ \frac{Uc\dot{\alpha}}{2}(\theta_0\cos\theta_0 - \tfrac{1}{3}\sin^3\theta_0 - \sin\theta_0\cos^2\theta_0)$$

$$+ \frac{c^2\ddot{\alpha}}{4}(\tfrac{1}{8}\sin\theta_0\cos\theta_0 + \tfrac{1}{12}\sin^3\theta_0\cos\theta_0 - \tfrac{\theta_0}{8})$$

$$+ \frac{U^2\beta}{\pi}(\theta_0^2 + \sin^2\theta_0 - 2\theta_0\sin\theta_0\cos\theta_0)$$

$$+ \frac{c^2\ddot{\beta}}{4\pi}\Big(\tfrac{1}{2}\cos^4\theta_0 - \tfrac{1}{8}\sin^2\theta_0\cos^2\theta_0 - \tfrac{1}{2} - \tfrac{\theta_0^2}{8} - \theta_0^2\cos^2\theta_0$$

$$+ \tfrac{\theta_0}{2}\sin\theta_0\cos^3\theta_0 + \tfrac{7}{4}\theta_0\sin\theta_0\cos\theta_0\Big)\Big]$$

$$+ \frac{\rho Uc}{4\pi}A_\beta\big[-Q_0T_4 + Q_1(T_{12} - T_4) + f\big] \tag{9.6}$$

The above expression lends itself to the use of the Theodorsen constants (see appendix A). In some cases, however, the constants must be transformed by means of the trigonometric identities. Making use of these constants expression (9.6) becomes

$$M_\beta = \frac{\rho c^2}{4}\Big[\frac{c\ddot{h}}{2}T_1 - U(\dot{h}+U\alpha)T_4 + \frac{Uc\dot{\alpha}}{2}T_8$$

$$+ \frac{c^2\ddot{\alpha}}{4}(T_7 + T_1\cos\theta_0) - \frac{U^2\beta}{\pi}T_5 + \frac{c^2\ddot{\beta}}{4\pi}T_3\Big]$$

$$+ \frac{\rho Uc}{4\pi}A_\beta\big[-Q_0T_4 + Q_1(T_{12} - T_4) + f\big] \tag{9.7}$$

The next step in this development is to insert the value of A_β . In order to minimize the work the latter part of expression (9.7) will be treated alone. This part will be combined later with the first part thus completing the formula for the aileron hinge moment. Substituting expression (4.1) of section I-4 the above mentioned latter part of expression (9.7) becomes

$$A_\beta\left[-Q_0 T_4 + Q_1(T_{12}-T_4)+f\right]$$

$$= \frac{\Gamma_\beta^{\,0}}{F-Q_0-Q_1}\left[-Q_0 T_4 + Q_1(T_{12}-T_4)+f\right]$$

which can be put in the following form;

$$A_\beta\left[-Q_0 T_4 + Q_1(T_{12}-T_4)+f\right]$$

$$= \Gamma_\beta^{\,0}\left[T_4 - T_{12}\,\frac{Q_1 - \frac{F\,T_4}{T_{12}} + \frac{f}{T_{12}}}{Q_0+Q_1-F}\right] \tag{9.8}$$

At this point let a quantity $\overline{S}_{AR}$ be defined such that

$$\overline{S}_{AR} = \frac{Q_1 - \frac{F\,T_4}{T_{12}} + \frac{f}{T_{12}}}{Q_0+Q_1-F} \tag{9.9}$$

provided $0<\theta_0\leq\pi$. The reason for the interval being open at the lower endpoint is due to the fact that $\frac{T_4}{T_{12}}\to\infty$ as $\theta_0\to 0$. The quantity F is independent of θ_0 and therefore remains constant. A rough investigation was made of the term $\frac{f}{T_{12}}$ which indicated that this too tends to infinity as $\theta_0\to 0$. For $\theta_0=\pi$ however, $T_{12}=\pi$, $T_4=-\pi$ and f is finite, hence no difficulty is to be expected at the upper end point. In the expression for M_β, it is seen that $\overline{S}_{R}$ is multiplied by T_{12} and from the rough investigation it appears that $T_{12}\overline{S}_{AR}\to 0$ as $\theta_0\to 0$. The reason for defining $\overline{S}_{AR}$ as shown in expression (9.9) is to make the hinge moment formula as it appears here comparable to the existing formulae for infinite aspect ratio. If $\overline{S}_{AR}$ is written in terms of Lombard's notation (see appendix B) it becomes

$$\overline{S}_{AR} = \frac{Q_1 + \frac{F R_3}{R_8} + \frac{f}{\pi R_8}}{Q_0+Q_1-F} \tag{9.10}$$

provided $0<\tau\leqslant 1$. Here τ designates the ratio of aileron chord divided by wing chord, see expression (10.2) section I-10. Since $\overline{S}_{AR}$ is complex, notation is here given for its real and imaginary parts; thus

$$\overline{S}_{R} = K_{R} + i N_{R} \tag{9.11}$$

Substituting expression (9.9) and expression (6.3) of section I-6 for $\Gamma_S^{\,0}$, in expression (9.8) it becomes

$$A_\beta \left[-Q_0 T_4 + Q_1 (T_{12} - T_4) + f \right]$$
$$= \left[\pi c \left(\dot{h} + U\alpha + \frac{c\dot{\alpha}}{4} \right) + Uc\beta T_{10} + \frac{c^2 \dot{\beta}}{4} T_{11} \right] \left[T_4 - T_{12} \overline{S}_R \right] \tag{9.12}$$

Placing (9.10) in expression (9.7) and combining terms, it becomes

$$M_\beta = \frac{\rho c^2}{4} \left[\frac{c\ddot{h}}{2} T_1 + \frac{Uc\dot{\alpha}}{4} (T_4 + 2T_8) + \frac{c^2 \ddot{\alpha}}{4} (T_7 + T_1 \cos\theta_0) \right.$$
$$+ \frac{U^2 \beta}{\pi} (T_4 T_{10} - T_5) + \frac{U^2 c \dot{\beta}}{4\pi} T_4 T_{11} + \frac{c^2 \ddot{\beta}}{4\pi} T_3$$
$$\left. - U T_{12} \overline{S}_R \left(\dot{h} + U\alpha + \frac{c\dot{\alpha}}{4} + \frac{U\beta}{\pi} T_{10} + \frac{c\dot{\beta}}{4\pi} T_{11} \right) \right] \tag{9.13}$$

which is the complete formula for the aileron hinge moment.

II-10 Aileron Hinge Moment, Coefficient Form

Although expression (9.13) is the finished formula, it might be desirable to put this expression in coefficient form as is done with the wing lift and wing moment formulae in section I-8. In order to put formula (9.13) in coefficient form it is necessary only to apply identities (7.2) and (7.3) of section I-7 and then collect the terms which form the coefficients of $\dot{h}$, α and β (see section I-8). Applying identities (7.2) and (7.3) to expression (9.13) it becomes

$$M_\beta = \frac{\rho C^2}{4}\Big[\frac{i\omega c}{2}\dot{h}T_1 + \frac{i\omega c}{4}U\alpha(T_4+2T_8) - \frac{\omega^2 c^2}{4}\alpha(T_7+T_1\cos\theta_0)$$
$$+\frac{U^2\beta}{\pi}(T_4T_{10}-T_5) + \frac{i\omega c}{4\pi}U\beta T_4T_{11} - \frac{\omega^2 c^2}{4\pi}\beta T_3$$
$$- U T_{12}\bar{S}_R\Big(\dot{h}+U\alpha+\frac{i\omega c}{4}\alpha + \frac{U\beta}{\pi}T_{10} + \frac{i\omega c}{4\pi}\beta T_{11}\Big)\Big] \tag{10.1}$$

The terms of the above expression will now be collected and arranged as coefficients of $\dot{h}$, α, and β; thus

$$M_\beta = \frac{\rho C^2}{4}\Big\{U\Big[\frac{i\omega c}{2U}T_1 - \bar{S}_R T_{12}\Big]\dot{h}$$
$$+U^2\Big[\frac{i\omega c}{4U}(T_4+2T_8) - \frac{\omega^2 c^2}{4U^2}(T_7+T_1\cos\theta_0)$$
$$-\Big(1+\frac{i\omega c}{4U}\Big)\bar{S}_R T_{12}\Big]\alpha$$
$$+\frac{U^2}{\pi}\Big[T_4T_{10} - T_5 + \frac{i\omega c}{4U}T_4T_{11} - \frac{\omega^2 c^2}{4U^2}T_3$$
$$-\Big(T_{10}+\frac{i\omega c}{4U}T_{11}\Big)\bar{S}_R T_{12}\Big]\beta\Big\} \tag{10.2}$$

The above result is now in the form used in Report 5, however, it is obvious that no comparison can be made with the formulae of Report 5.

II-11 Aileron Hinge Moment, Theodorsen Form

The Theodorsen form of the aileron hinge moment equation is practically the same as expression (9.13) except that $\dot{h}$ and $\ddot{h}$ must be replaced by $\dot{h}_a$ and $\ddot{h}_a$ by means of relations (9.1) and (9.2) of section I-9. If this is done expression (9.13) becomes

$$M_\beta = \frac{\rho C^2}{4}\Big\{\frac{C}{2}\left(\ddot{h}_a - \frac{aC}{2}\ddot{\alpha}\right)T_1 + \frac{UC\dot{\alpha}}{4}(T_4 + 2T_8)$$
$$+ \frac{C^2\ddot{\alpha}}{4}(T_7 + T_1\cos\theta_0) + \frac{U^2\beta}{\pi}(T_4T_{10} - T_5)$$
$$+ \frac{UC\dot{\beta}}{4\pi}T_4T_{11} + \frac{C^2\ddot{\beta}}{4\pi}T_3$$
$$- U\,T_{12}\,\overline{S}_{R}\left(\dot{h}_a - \frac{aC}{2}\dot{\alpha} + U\alpha + \frac{C\dot{\alpha}}{4} + \frac{U\beta}{\pi}T_{10} + \frac{C\dot{\beta}}{4\pi}T_{11}\right)\Big\}$$

which on collecting terms takes the following form;

$$M_\beta = \frac{\rho C^2}{4}\Big\{\frac{C}{2}\ddot{h}_aT_1 + \frac{UC\dot{\alpha}}{4}(T_4 + 2T_8)$$
$$+ \frac{C^2}{4}\ddot{\alpha}\left[T_7 + (\cos\theta_0 - a)T_1\right] + \frac{U^2\beta}{\pi}(T_4T_{10} - T_5)$$
$$+ \frac{UC\dot{\beta}}{4\pi}T_4T_{11} + \frac{C^2\ddot{\beta}}{4\pi}T_3$$
$$- U\,T_{12}\,\overline{S}_{R}\left[\dot{h}_a + U\alpha + \frac{C\dot{\alpha}}{2}\left(\frac{1}{2} - a\right) + \frac{U\beta}{\pi}T_{10} + \frac{C\dot{\beta}}{4\pi}T_{11}\right]\Big\} \qquad (11.1)$$

If the above formula is compared with expression (XIX) of Theodorsen's work (reference 2) it will be found that the coefficient of $\frac{UC\dot{\alpha}}{4}$ does not appear the same as the corresponding term in Theodorsen's expression. This coefficient as given in expression (11.1) can be brought into the form as given in Theodorsen's work if use is made of the following identity existing between the Theodorsen constants. This identity is

$$T_8 = 2T_9 + T_1 - aT_4 \qquad (11.2)$$

From this it follows that

$$T_4 + 2T_8 = T_4 + 2\left[2T_9 + T_1 - aT_4\right]$$

$$= 2\left[2T_9 + T_1 - T_4\left(a - \tfrac{1}{2}\right)\right] \tag{11.3}$$

Also from appendix A it follows that

$$-2T_{13} = T_7 + (\cos\theta_0 - a)T_1 \tag{11.4}$$

which is the coefficient of $\frac{c^2\ddot{\alpha}}{4}$. Substituting expressions (11.3) and (11.4) in formula (11.1) it becomes

$$M_\beta = \frac{\rho c^2}{4}\left\{\frac{c}{2}\ddot{h}_a T_1 + \frac{Uc\dot{\alpha}}{2}\left[2T_9 + T_1 - T_4\left(a - \tfrac{1}{2}\right)\right] - \frac{c^2\ddot{\alpha}}{2}T_{13} + \frac{U^2\beta}{\pi}(T_4T_{10} - T_5) + \frac{Uc\dot{\beta}}{4\pi}T_4T_{11} + \frac{c^2\ddot{\beta}}{4\pi}T_3 - UT_{12}\bar{S}_R\left[\dot{h}_a + U\alpha + \frac{c\dot{\alpha}}{2}\left(\tfrac{1}{2} - a\right) + \frac{U\beta}{\pi}T_{10} + \frac{c\dot{\beta}}{4\pi}T_{11}\right]\right\} \tag{11.5}$$

which, aside from a difference of notation and a slightly different arrangement, is identical to formula (XIX) of reference 2 if $\bar{S}_R$ is replaced by Theodorsen's C.

II-12 Aileron Hinge Moment, Lombard Form

To put the aileron hinge moment formula in the Lombard form, the terms defined in section I-10 and the R's given in appendix B are substituted in expression (11.1). Performing these substitutions the aileron hinge moment divided by cm_L becomes

$$\frac{M_\beta}{cm_L} = R_4 \ddot{y}_e - R_9 U\dot{\alpha} - c(R_7 - R_4\varepsilon)\ddot{\alpha}$$
$$- R_{10}\frac{U^2}{c}\beta - R_{11}U\dot{\beta} - R_{12}c\ddot{\beta}$$
$$- R_8 \frac{U}{c}\bar{S}_{AR}\Big[-\dot{y}_e + U\alpha + c\left(\tfrac{1}{2}-\varepsilon\right)\dot{\alpha}$$
$$+ \frac{R_1}{4}U\beta + \frac{R_2}{4}c\dot{\beta}\Big] \qquad (12.1)$$

Attention is also called to the fact that $y_e = -h_a$ and that M_β is the moment per unit of span.

To put expression (12.1) in its final form, apply relations (7.2) and (7.3) of section I-7 and arrange in coefficient form as shown below;

$$\frac{M_\beta}{cm_L} = -\left[R_4\omega^2 - i\omega\frac{U}{c}R_8\bar{S}_{AR}\right]y_e$$
$$- \left\{-(R_7 - R_4\varepsilon)\omega^2 + i\omega\frac{U}{c}\left[R_9 + R_8\bar{S}_{AR}\left(\tfrac{1}{2}-\varepsilon\right)\right] + \frac{U^2}{c^2}R_8\bar{S}_{AR}\right\}\alpha c$$
$$- \left[-R_{12}\omega^2 + i\omega\frac{U}{c}(R_{11} + R_{14}\bar{S}_{AR}) + \frac{U^2}{c^2}(R_{10} + R_{13}\bar{S}_{AR})\right]\beta c \qquad (12.2)$$

The above expression is in a form which is comparable with Lombard's work, see expressions (5:10), (5:11), and (5:12), page 119, reference 4.

II-13 Conclusion

The principal theme of this thesis is to develop aerodynamic formulae for an oscillating wing-aileron combination of finite span. This is done in the preceeding pages. Hence, the principal theme of this conclusion is the effect of span or, better say, the effect of aspect ratio on the aerodynamic formulae of an oscillating wing-aileron combination.

From the nature of this problem it appears advisable to split the conclusion into two parts, one concerning the wing lift and wing moment, the other concerning the aileron hinge moment. The derivations of the formulae for wing lift and wing moment are in reality an extension of the work which is given in Report 5, while the derivation of the aileron hinge moment is a problem which starts from first principles.

Part I

In chapter I, the wing lift and wing moment formulae as developed in section I-8 are comparable to the corresponding formulae of Report 5. On comparing expressions (8.1) and (8.2) of section I-8 it will be observed that they are respectively identical with expressions (3.7) page 79 and (3.9) page 80 of Report 5, except that the two former expressions have terms due to the presence of the aileron, i. e. the coefficients of β. Further, if the coefficients of β in expressions (8.1) and (8.2) of section I-8 are examined it will be observed that the only terms affected by aspect ratio are $\overline{P}_{AR}$ and $\overline{Q}_{AR}$. From section I-9 it follows that expression (XVIII) for the wing lift as given by Theodorsen in reference 2 can be made to apply to a wing of finite span if Theodorsen's C is replaced by the $\overline{P}_{AR}$ of Report 5. In like manner it also follows from section I-9 that the wing moment expression, i. e. expression XX of reference 2, can be made to apply to a wing of finite span if the factor $2(a+\frac{1}{2})C$ in Theodorsen's expression is replaced by the factor $(2a\overline{P}_{AR} + \overline{Q}_{AR})$ where $\overline{P}_{AR}$ and $\overline{Q}_{AR}$ are taken from Report 5. A similar comparison is pointed out in section I-10 concerning Lombard's work in reference 4.

From the above paragraph it can be concluded that there are no basic differences between the formulae for finite aspect ratio and those for infinite aspect ratio. Hence, to study the effect of aspect ratio it is necessary only to study its effect on $\overline{P_{AR}}$ and $\overline{Q_{AR}}$. This has been done in Report 5 and the effect of the aspect ratio on $\overline{P_{AR}}$ and $\overline{Q_{AR}}$ is shown in figure 3 part I of that report. It can therefore be said that the conclusions given in section 7 part I of Report 5, which were written by Dr. M. A. Biot, can be taken over as a unit insofar as the wing lift and wing moment formulae of chapter I are concerned.

Part II

The aileron hinge moment formula as developed here has been compared with Theodorsen's formula in section II-11 and with Lombard's formula in section II-12. From these comparisons it can be concluded that there are no basic differences. For example, it is necessary only to replace the C in formula XIX of reference 2 by $\overline{S_{AR}}$ in order to obtain the aileron hinge moment formula for a wing of finite aspect ratio. It was also pointed out that $\overline{S_{AR}} \rightarrow C = \overline{P}$ as $AR \rightarrow \infty$, in this case the formulae become identical. Hence, to study the effect of aspect ratio on the aileron hinge moment means the study of $\overline{S_{AR}}$ in comparison with C or $\overline{P}$.

Attention is called to the fact that $\overline{S_{AR}}$ is a function of reduced velocity $\frac{U}{\omega c}$, aspect ratio, and aileron chord or, more precisely, τ whereas C or $\overline{P}$ are functions of reduced velocity only. In addition, the quantity $\overline{S_{AR}}$ like C or $\overline{P}$ is complex. Consequently, since the numerical work is somewhat lengthy, the author was excused from evaluating $\overline{S_{AR}}$ insofar as this thesis is concerned. It is therefore deemed as inadvisable to attempt any concrete conclusion about the variations $\overline{S_{AR}}$ with aspect ratio, without the numerical values.

References

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2. Theodorsen, Theodore; General Theory of Aerodynamic Instability and the Mechanism of Flutter; T. R. No. 496, N. A. C. A. 1940. Herein referred to as N. A. C. A. - T. R. - 496.

3. von Kármán, Theodore and Burgers, J. M.; Aerodynamic Theory; Volume II; W. F. Durand, Editor-in-chief.

4. Lombard, Albert E. Jr.; An Investigation of the Conditions for the Occurrence of Flutter in Aircraft and the Development of Criteria for the Prediction and Elimination of such Flutter, (a doctor's thesis); California Institute of Technology; Pasadena; 1939.

5. Hardy, G. H.; A Course of Pure Mathematics; seventh edition; Cambridge at the University Press, 1938; in particular, Chapter VIII, section 203.

Appendix A

Theodorsen Constants

The definitions of the following symbols are taken from N. A. C. A. TR-496, reference 2, however, the lower case c, used by Theodorsen has been replaced by $\cos\theta_0$, likewise the $\sqrt{1-c^2}$ has been replaced by $\sin\theta_0$, and $\cos^{-1}C$ by θ_0.

$$T_1 = -\frac{1}{2}\sin\theta_0(2+\cos^2\theta_0) + \theta_0\cos\theta_0$$

$$T_2 = \cos\theta_0(1-\cos^2\theta_0) - \theta_0\sin\theta_0(1+\cos^2\theta_0) + \theta_0^2\cos\theta_0$$

$$T_3 = -\left(\frac{1}{8}+\cos^2\theta_0\right)\theta_0^2 + \frac{1}{4}\theta_0\cos\theta_0\sin\theta_0(7+2\cos^2\theta_0) - \frac{1}{8}(1-\cos^2\theta_0)(5\cos^2\theta_0+4)$$

$$T_4 = -\theta_0 + \cos\theta_0\sin\theta_0$$

$$T_5 = -(1-\cos^2\theta_0) - \theta_0^2 + 2\theta_0\cos\theta_0\sin\theta_0$$

$$T_6 = T_2$$

$$T_7 = -\left(\frac{1}{8}+\cos^2\theta_0\right)\theta_0 + \frac{1}{8}\cos\theta_0\sin\theta_0(7+2\cos^2\theta_0)$$

$$T_8 = -\frac{1}{3}\sin\theta_0(2\cos^2\theta_0+1) + \theta_0\cos\theta_0$$

$$T_9 = \frac{1}{2}\left[\frac{1}{3}\sin^3\theta_0 + aT_4\right] = \frac{1}{2}(-p + aT_4) \qquad \text{where} \quad p = -\frac{1}{3}\sin^3\theta_0$$

$$T_{10} = \sin\theta_0 + \theta_0$$

$$T_{11} = \theta_0(1-2\cos\theta_0) + \sin\theta_0(2-\cos\theta_0)$$

$$T_{12} = \sin\theta_0(2+\cos\theta_0) - \theta_0(2\cos\theta_0+1)$$

$$T_{13} = \frac{1}{2}\left[-T_7 - (\cos\theta_0 - a)T_1\right]$$

$$T_{14} = \frac{1}{16} + \frac{a}{2}\cos\theta_0$$

Identities:

$$2T_8 = T_1 - T_4\cos\theta_0$$ see section I-9

$$T_8 = 2T_9 + T_1 - aT_4$$ see section II-11

Appendix B

Lombard Constants

Below is given a table of the Lombard-R's in terms of the Theodorsen T's, see appendix A. This table is taken from the nomenclature, appendix II of Lombard's Thesis, reference 4.

$$R_1 = \frac{4T_{10}}{\pi}$$

$$R_2 = \frac{T_{11}}{\pi}$$

$$R_3 = -\frac{T_4}{\pi}$$

$$R_4 = -\frac{T_1}{\pi}$$

$$R_5 = \frac{1}{\pi}(T_4 + T_{10})$$

$$R_6 = \frac{1}{4\pi}\left[2T_1 - 2T_8 + T_{11} + 4T_4(\tau - \tfrac{3}{4})\right] = \frac{1}{4\pi}(2T_8 - T_4 + T_{11})$$

$$R_7 = \frac{1}{4\pi}\left[-T_7 + 2T_1(\tau - \tfrac{3}{4})\right]$$

$$R_8 = \frac{T_{12}}{\pi}$$

$$R_9 = \frac{1}{4\pi}\left[2p - 2T_1 - T_4\right] = -\frac{1}{4\pi}(T_4 + 2T_8)$$

$$R_{10} = \frac{1}{\pi^2}(T_5 - T_4 T_{10})$$

$$R_{11} = -\frac{T_4 T_{11}}{4\pi^2}$$

$$R_{12} = -\frac{T_3}{4\pi^2}$$

$$R_7 - R_4 \varepsilon = \frac{T_{13}}{2\pi}$$

$$R_{13} = \frac{T_{10} T_{12}}{\pi^2} = \frac{R_1 R_8}{4}$$

$$R_{14} = \frac{T_{11} T_{12}}{4\pi^2} = \frac{R_2 R_8}{4}$$

Note: R_{13} and R_{14} were taken from a report on wing flutter written by Dr. M. A. Biot. The second equivalents for R_6 and R_9 are not given in Lombard's Thesis.

Appendix C

Aspect Ratio Integrals

The integrals of this appendix are called aspect ratio because they are the results of the aspect ratio terms in the functions E_0 , E_1 , and E_2 as given in chapter II, however, these integrals are also functions of the aileron chord. For the purpose of working out these integrals a symbol k will be defined such that

$$k = \frac{c}{R_1} \tag{C 1}$$

or

$$k = \frac{c}{12R_1} \tag{C 2}$$

By this means integrals (4.3) and (5.5) of chapter II appear the same, and one derivation suffices for both.

Integral (4.3) now becomes

$$S_0(k,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{k\cos\tau}\,\frac{\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta \tag{C 3}$$

In order to integrate, the first step is to write $e^{k\cos\tau}$ in a Fourier half range cosine series; thus

$$e^{k\cos\tau} = a_0 + a_1\cos\tau + a_2\cos 2\tau + \cdots + a_n\cos n\tau + \cdots \tag{c 4}$$

Multiply the above expression by $d\tau$ and integrate from 0 to π as shown below:

$$\int_0^{\pi} e^{k\cos\tau}\,d\tau = \int_0^{\pi} a_0\,d\tau + \int_0^{\pi} a_1\cos\tau\,d\tau + \int_0^{\pi} a_2\cos 2\tau\,d\tau + \cdots$$

$$+ \int_0^{\pi} a_n\cos n\tau\,d\tau + \cdots$$

All of the above integrals of the right member reduce to zero except the first, and from it the following result is obtained:

$$\pi a_0 = \int_0^{\pi} e^{k\cos\tau}\,d\tau \tag{C 5}$$

Integral (B 1) given on page 85 of Report 5 becomes after substitution of (C 1) the following;

$$\int_0^{\pi} e^{k\cos\tau}\cos n\tau\, d\tau = \pi I_n(k) \qquad \text{(C 6)}$$

where $I_n(k)$ is the modified Bessel function. If n takes on the value zero the above becomes

$$\int_0^{\pi} e^{k\cos\tau}\, d\tau = \pi I_0(k) \qquad \text{(C 7)}$$

Substituting this result in (C 5) the value of a_0 becomes

$$a_0 = I_0(k) \qquad \text{(C 8)}$$

To obtain the balance of the coefficients, multiply expression (C 4) by $\cos n\tau\, d\tau$ and integrate from 0 to π. Following this procedure a_n becomes

$$a_n = 2 I_n(k) \qquad \text{(C 9)}$$

for $n = 1, 2, 3, \cdots$. Making use of (C 8) and (C 9) the series (C 4) can be written as

$$e^{k\cos\tau} = I_0(k) + 2\sum_{n=1}^{\infty} I_n(k)\cos n\tau \qquad \text{(C 10)}$$

Before substituting series (C 10) in (C 3) the integral will be slightly modified in form; thus

$$S_0(k,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{k\cos\tau}\left\{\cos\tau + \cos\theta - \frac{\sin^2\theta}{\cos\tau - \cos\theta}\right\} d\tau\, d\theta$$

With the exception of the fraction the above can be integrated by means of (C 6) and becomes

$$S_0(k,\theta_0) = \int_0^{\theta_0}\left[I_1(k) + I_0(k)\cos\theta\right] d\theta - \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{k\cos\tau}\frac{\sin^2\theta}{\cos\tau - \cos\theta}\, d\tau\, d\theta \qquad \text{(C 11)}$$

Substituting (C 10) and (C 11) it becomes

$$S_0(k,\theta_0) = \int_0^{\theta_0}\left[I_1(k) + I_0(k)\cos\theta\right] d\theta - \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi}\left\{I_0(k) + 2\sum^{\infty} I_n(k)\cos n\tau\right\}\frac{\sin^2\theta}{\cos\tau - \cos\theta}\, d\tau\, d\theta$$

Assuming that the above satisfies the requirements such that it can be written as

$$S_o(k,\theta_o) = \int_0^{\theta_o} \left[I_1(k) + I_o(k)\cos\theta \right] d\theta - \frac{1}{\pi} I_o(k) \int_0^{\theta_o} \int_0^{\pi} \frac{\sin^2\theta}{\cos\tau - \cos\theta}\, d\tau\, d\theta$$

$$- \frac{2}{\pi} \sum_{n=1}^{\infty} I_n(k) \int_0^{\theta_o} \int_0^{\pi} \frac{\sin^2\theta \cos n\tau}{\cos\tau - \cos\theta}\, d\tau\, d\theta \qquad \text{(C 12)}$$

It is apparent that in expression (C 12) the following integration is involved; thus

$$\int_0^{\pi} \frac{\cos n\tau}{\cos\tau - \cos\theta}\, d\tau = \pi \frac{\sin n\theta}{\sin\theta} \qquad \text{(C 13)}$$

where $n = 0, 1, 2, \cdots$. This is integral (2 D) given on page 101 of Report 5 except that the symbols have been changed to suit this problem. The result is a Cauchy principal value, see also pages 173 and 174 of reference 3. Using (C 13) expression (C 12) can be written as

$$S_o(k,\theta) = \int_0^{\theta_o} \left[I_1(k) + I_o(k)\cos\theta \right] d\theta$$

$$-2 \sum_{n=1}^{\infty} I_n(k) \int_0^{\theta_o} \sin\theta \sin n\theta\, d\theta \qquad \text{(C 14)}$$

From the trigonometric identities the integrand in the above integral can be transformed as follows;

$$\sin\theta \sin n\theta = \frac{1}{2} \left[\cos(n-1)\theta - \cos(n+1)\theta \right] \qquad \text{(C 15)}$$

Substituting (C 15) in (C 14) and integrating it becomes

$$S_o(k,\theta_o) = I_1(k)\,\theta_o + I_o(k) \sin\theta_o$$

$$- \sum_{n=1}^{\infty} I_n(k) \int_0^{\theta_o} \left[\cos(n-1)\theta - \cos(n+1)\theta \right] d\theta$$

In order to avoid the difficulty apparent in the remaining integral when $n = 1$ the series will be separated and the above expression written as

$$S_0(k,\theta_0) = I_1(k)\theta_0 + I_0(k)\sin\theta_0 - I_1(k)\int_0^{\theta_0} d\theta$$
$$-\sum_{n=2}^{\infty} I_n(k)\int_0^{\theta_0}\cos(n-1)\theta\, d\theta$$
$$+\sum_{n=1}^{\infty} I_n(k)\int_0^{\theta_0}\cos(n+1)\theta\, d\theta$$

which becomes on integration

$$S_0(k,\theta_0) = I_0(k)\sin\theta_0 - \sum_{n=2}^{\infty} I_n(k)\frac{\sin(n-1)\theta_0}{n-1}$$
$$+\sum_{n=1}^{\infty} I_n(k)\frac{\sin(n+1)\theta_0}{n+1} \tag{C 16}$$

Before changing the form of (C 16) the limit $S_0(k,\theta_0)$ as the aspect ratio tends to infinity will be evaluated. From expression (C 1) and (C 2), and from the fact that $r_1 = \frac{c\mathcal{R}}{4}$ which is given in Report 5, it is evident that

$$k = \frac{4}{\mathcal{R}} \tag{C 17}$$

or

$$k = \frac{1}{3\mathcal{R}} \tag{C 18}$$

hence as $\mathcal{R} \to \infty$ it follows for either of the above two equivalents that $k \to 0$ From the properties of the Bessel functions it follows that

$$\lim_{k\to 0} I_n(k) = 0$$

for $n = 1, 2, 3, \cdots$ and for $n = 0$

$$\lim_{k\to 0} I_0(k) = 1$$

hence if $\theta_0 \neq 0$ it follows from expression (C 16) that

$$\text{limit}\, S_0(k,\theta_0) = \sin\theta_0 \tag{C 19}$$

The procedure of rearranging expression (C 16) will now be given in detail. In the first series in this expression let $m = n-1$ and in the second let $m = n+1$ then

$$S_0(k,\theta_0) = I_0(k)\sin\theta_0 - \sum_{m=1}^{\infty} I_{m+1}(k)\frac{\sin m\theta_0}{m}$$
$$+\sum_{m=2}^{\infty} I_{m-1}(k)\frac{\sin m\theta_0}{m}$$

By including the first term of the right member in the second series the above can be written as

$$S_0(k,\theta_0) = \sum_{m=1}^{\infty}\left[I_{m-1}(k) - I_{m+1}(k)\right]\frac{\sin m\theta_0}{m} \qquad \text{(C 20)}$$

From the properties of the Bessel functions the following recurrence formula exists; thus

$$I_{n-1}(k) - I_{n+1}(k) = \frac{2n}{k} I_n(k) \qquad \text{(C 21)}$$

Substituting this in expression (C 20) and at the same time writing n for m it becomes

$$S_0(k,\theta_0) = \frac{2}{k}\sum_{n=1}^{\infty} I_n(k)\sin n\theta_0 \qquad \text{(C 22)}$$

At this point a symbol $\sigma(k,\theta_0)$ will be defined such that

$$\sigma(k,\theta_0) = \sum_{n=1}^{\infty} I_n(k)\sin n\theta_0 \qquad \text{(C 23)}$$

Expression (C 22) can now be written in its final form as

$$S_0(k,\theta_0) = \frac{2}{k}\sigma(k,\theta_0) \qquad \text{(C 24)}$$

This expression is given in section II-4 as expression (4.6) where $k = \frac{c}{n_1}$ and in section II-5 as expression (5.6) where $k = \frac{c}{12 n_1}$.

Next in order of derivation is $S_1(\frac{c}{n_1},\theta_0)$ which is first defined by the integral (4.4) given in section II-4. Making use of expression (C 1) this integral can be written as

$$S_1(k,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{k\cos\tau}\frac{\cos\tau\sin^2\tau}{\cos\theta - \cos\tau}\,d\tau\,d\theta \qquad \text{(C 25)}$$

To bring this into a form easier to handle, the fraction in the integrand will be changed as follows;

$$\begin{aligned}\frac{\cos\tau\sin^2\tau}{\cos\theta - \cos\tau} &= \frac{\cos^3\tau - \cos\tau}{\cos\theta - \cos\tau} \\ &= \cos^2\tau + \cos\theta\cos\tau - \sin^2\theta - \frac{\sin^2\theta\cos\theta}{\cos\tau - \cos\theta}\end{aligned} \qquad \text{(C 26)}$$

Substituting this in (C 25) it becomes

$$S_1(\kappa,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\kappa\cos\tau}\Big\{\cos^2\tau + \cos\theta\cos\tau$$

$$- \sin^2\theta - \frac{\sin^2\theta\cos\theta}{\cos\tau-\cos\theta}\Big\}d\tau\, d\theta \qquad (C\ 27)$$

The first three terms of the integrand can be integrated by means of (C 6) as shown below;

$$S_1(\kappa,\theta_0) = \int_0^{\theta_0}\Big\{\frac{1}{2}\big[I_0(\kappa) + I_2(\kappa)\big] + I_1(\kappa)\cos\theta - I_0(\kappa)\sin^2\theta\Big\}d\theta$$

$$-\frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\kappa\cos\tau}\frac{\sin^2\theta\cos\theta}{\cos\tau-\cos\theta}\,d\tau\, d\theta \qquad (C\ 28)$$

Substituting expression (C 10) and rearranging it becomes

$$S_1(\kappa,\theta_0) = \int_0^{\theta_0}\Big[\frac{1}{2}I_2(\kappa) + I_1(\kappa)\cos\theta + \frac{1}{2}I_0(\kappa)\cos 2\theta\Big]d\theta$$

$$-\frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi}\Big\{I_0(\kappa) + 2\sum_{n=1}^{\infty} I_n(\kappa)\cos n\theta\Big\}\frac{\sin^2\theta\cos\theta}{\cos\tau-\cos\theta}\,d\tau\, d\theta \qquad (C\ 29)$$

which as in the case of $S_0(\kappa,\theta_0)$ can be written as

$$S_1(\kappa,\theta_0) = \int_0^{\theta_0}\Big[\frac{1}{2}I_2(\kappa) + I_1(\kappa)\cos\theta + \frac{1}{2}I_0(\kappa)\cos 2\theta\Big]d\theta$$

$$-2\sum_{n=1}^{\infty} I_n(\kappa)\int_0^{\theta_0}\sin n\theta\sin\theta\cos\theta\, d\theta \qquad (C\ 30)$$

where use of integral (C 13) has been made at this point. By means of the trigonometric identities

$$\sin n\theta\sin\theta\cos\theta = \frac{1}{4}\big[\cos(n-2)\theta - \cos(n+2)\theta\big]$$

substituting this and integrating the first three terms (C 30) becomes

$$S_1(\kappa,\theta_0) = \frac{1}{2}I_2(\kappa)\theta_0 + I_1(\kappa)\sin\theta_0 + \frac{1}{4}I_0(\kappa)\sin 2\theta_0$$

$$-\frac{1}{2}\sum_{n=1}^{\infty} I_n(\kappa)\int_0^{\theta_0}\big[\cos(n-2)\theta - \cos(n+2)\theta\big]d\theta$$

In this case the term which gives trouble is obtained for $n=2$, hence the above is written as

$$S_1(\kappa,\theta_0) = \tfrac{1}{2} I_2(\kappa)\,\theta_0 + I_1(\kappa)\sin\theta_0 + \tfrac{1}{4} I_0(\kappa)\sin 2\theta_0$$

$$- \tfrac{1}{2} I_1(\kappa)\int_0^{\theta_0} \cos\theta \, d\theta - \tfrac{1}{2} I_2(\kappa)\int_0^{\theta_0} d\theta$$

$$- \tfrac{1}{2}\sum_{n=3}^{\infty} I_n(\kappa)\int_0^{\theta_0} \cos(n-2)\theta \, d\theta$$

$$+ \tfrac{1}{2}\sum_{n=1}^{\infty} I_n(\kappa)\int_0^{\theta_0} \cos(n+2)\theta \, d\theta$$

which on integration becomes

$$S_1(\kappa,\theta_0) = \tfrac{1}{2} I_1(\kappa)\sin\theta_0 + \tfrac{1}{4} I_0(\kappa)\sin 2\theta_0$$

$$- \tfrac{1}{2}\sum_{n=3}^{\infty} I_n(\kappa)\,\frac{\sin(n-2)\theta_0}{n-2}$$

$$+ \tfrac{1}{2}\sum_{n=1}^{\infty} I_n(\kappa)\,\frac{\sin(n+2)\theta_0}{n+2} \qquad \text{(C 31)}$$

From expression (C 31) it is also quite evident that

$$\lim_{\kappa\to 0} S_1(\kappa,\theta_0) = \tfrac{1}{4}\sin 2\theta_0 \qquad \text{(C 32)}$$

The procedure to be used in transforming (C 31) into a form more suitable for numerical calculation is the same as that used in the case of $S_0(\kappa,\theta_0)$ but here the work is somewhat more involved. To begin, let $m = n-1$ in the first series of (C 31) and let $m = n+2$ in the second, then

$$S_1(\kappa,\theta_0) = \tfrac{1}{2} I_1(\kappa)\sin\theta_0 + \tfrac{1}{4} I_0(\kappa)\sin 2\theta_0$$

$$- \tfrac{1}{2}\sum_{m=1}^{\infty} I_{m+2}(\kappa)\,\frac{\sin m\theta_0}{m}$$

$$+ \tfrac{1}{2}\sum_{m=3}^{\infty} I_{m-2}(\kappa)\,\frac{\sin m\theta_0}{m}$$

which can be written as

$$S_1(\kappa,\theta_0) = \tfrac{1}{2} I_1(\kappa)\sin\theta_0 - \tfrac{1}{2} I_3(\kappa)\sin\theta_0$$

$$+ \tfrac{1}{2}\sum_{m=2}^{\infty} \left[I_{m-2}(\kappa) - I_{m+2}(\kappa) \right] \frac{\sin m\theta_0}{m} \qquad \text{(C 33)}$$

In the recurrence formula (C 21) first let $n = m-1$ and then let $n = m+1$, which given the following two formulae:

$$I_{m-2}(k) - I_m(k) = \frac{2(m-1)}{k} I_{m-1}(k)$$

and

$$I_m(k) - I_{m+2}(k) = \frac{2(m+1)}{k} I_{m+1}(k)$$

Adding the above two expressions gives

$$I_{m-2}(k) - I_{m+2}(k) = \frac{2m}{k}\Big[I_{m-1}(k) + I_{m+1}(k)\Big] - \frac{2}{k}\Big[I_{m-1}(k) - I_{m+1}(k)\Big]$$

Applying the recurrence formula (C 21) to the latter brackets, the expression can be written as

$$I_{m-2}(k) - I_{m+2}(k) = \frac{2m}{k}\Big[I_{m-1}(k) + I_{m+1}(k) - \frac{2}{k} I_m(k)\Big] \tag{C 34}$$

Substituting the above in (C 33) and separating the series it can be brought to the following form;

$$\begin{aligned} S_1(k,\theta_0) = {} & \frac{1}{2} I_1(k)\sin\theta_0 - I_3(k)\sin\theta_0 \\ & + \frac{1}{k}\sum_{m=2}^{\infty} I_{m-1}(k)\sin m\theta_0 + \frac{1}{k}\sum_{m=2}^{\infty} I_{m+1}(k)\sin m\theta_0 \\ & - \frac{2}{k^2}\sum_{m=2}^{\infty} I_m(k)\sin m\theta_0 \end{aligned} \tag{C 35}$$

Now consider each of the above series separately. In the first of the above series let $n = m-1$, then

$$\begin{aligned} \sum_{m=2}^{\infty} I_{m-1}(k)\sin m\theta_0 &= \sum_{n=1}^{\infty} I_n(k)\sin(n+1)\theta_0 \\ &= \sum_{n=1}^{\infty} I_n(k)\Big[\sin n\theta_0\cos\theta_0 + \cos n\theta_0 \sin\theta_0\Big] \end{aligned}$$

In the second series let $n = m+1$, then

$$\sum_{m=2}^{\infty} I_{m+1}(\kappa)\sin m\theta_0 = \sum_{n=3}^{\infty} I_n(\kappa)\sin(n-1)\theta_0$$
$$= \sum_{n=1}^{\infty} I_n(\kappa)\left[\sin n\theta_0\cos\theta_0 - \cos n\theta_0\sin\theta_0\right] - I_2(\kappa)\sin\theta_0$$

Adding the above two expressions gives

$$\sum_{m=2}^{\infty} I_{m-1}(\kappa)\sin m\theta_0 + \sum_{m=2}^{\infty} I_{m+1}(\kappa)\sin m\theta_0$$
$$= 2\cos\theta_0\sum_{n=1}^{\infty} I_n(\kappa)\sin n\theta_0 - I_2(\kappa)\sin\theta_0$$
$$= 2\cos\theta_0\,\sigma(\kappa,\theta_0) - I_1(\kappa) - I_2(\kappa)\sin\theta_0$$

where expression (C 23) has been applied. Combining the first two terms of (C 35) by means of the recurrence formula and substituting the above expression (C 35) becomes

$$S_1(\kappa,\theta_0) = \frac{2}{\kappa}I_2(\kappa)\sin\theta_0 + \frac{1}{\kappa}\left[2\cos\theta_0\,\sigma(\kappa,\theta_0) - I_2(\kappa)\sin\theta_0\right] - \frac{2}{\kappa^2}\sum_{m=2}^{\infty} I_m(\kappa)\sin m\theta_0$$

which can be written as

$$S_1(\kappa,\theta_0) = \frac{1}{\kappa}I_2(\kappa)\sin\theta_0 + \frac{2}{\kappa}\sigma(\kappa,\theta_0)\cos\theta_0 - \frac{2}{\kappa^2}\left[\sum_{m=1}^{\infty} I_m(\kappa)\sin m\theta_0 - I_1(\kappa)\sin\theta_0\right]$$

Again applying expression (C 23) to the above expression and eliminating $I_2(\kappa)$ by means of the recurrence formula the above expression becomes

$$S_1(\kappa,\theta_0) = \frac{1}{\kappa}\left[I_0(\kappa)\sin\theta_0 - 2\left(\frac{1}{\kappa} - \cos\theta_0\right)\sigma(\kappa,\theta_0)\right] \qquad \text{(C 36)}$$

which is identical to expression (4.21) of section II-4 if κ is set equal to $\frac{4}{AR}$.

For the next integral substitute $\kappa = \frac{c}{R_1}$ in integral (4.18) section II-4 as shown below;

$$S_2(\kappa,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\kappa\cos\tau}\frac{\sin^2\tau\cos\theta}{\cos\theta - \cos\tau}\,d\tau\,d\theta \qquad \text{(C 37)}$$

Comparing the above integrand with the integrand of (C 3) it is seen that the two are the same except that (C 37) has an additional factor $\cos\theta$, in the numerator. Hence, multiplying the integrands of (C 14) by $\cos\theta$, the function $S_2(K,\theta_0)$ can be written at once as

$$S_2(K,\theta_0) = \int_0^{\theta_0} \left[I_1(K)\cos\theta + I_0(K)\cos^2\theta \right] d\theta$$

$$- 2\sum_{n=1}^{\infty} I_n(K) \int_0^{\theta_0} \sin\theta\cos\theta\sin n\theta \, d\theta \qquad \text{(C 38)}$$

To bring the second integrand into form substitute the following trigonometric identity;

$$\sin\theta\cos\theta\sin n\theta = \frac{1}{4}\left[\cos(n-2)\theta - \cos(n+2)\theta\right] \qquad \text{(C 39)}$$

Substituting (C 39) and integrating the first expression (C 38) becomes

$$S_2(K,\theta_0) = I_1(K)\sin\theta_0 + I_0(K)\left(\frac{\theta_0}{2} + \frac{1}{4}\sin 2\theta_0\right)$$

$$- \frac{1}{2}\sum_{n=1}^{\infty} I_n(K) \int_0^{\theta_0} \left[\cos(n-2)\theta - \cos(n+2)\theta\right] d\theta \qquad \text{(C 40)}$$

Before integrating the series, they will be separated and the one containing the $\cos(n-2)\theta$ terms will be expanded for $n=1$ and $n=2$; thus integrating (C 40) becomes

$$S_2(K,\theta_0) = I_1(K)\sin\theta_0 + I_0(K)\left(\frac{\theta_0}{2} + \frac{1}{4}\sin 2\theta_0\right)$$

$$- \frac{1}{2} I_1(K)\sin\theta_0 - \frac{1}{2} I_2(K)\theta_0$$

$$- \frac{1}{2}\sum_{n=3}^{\infty} I_n(K) \frac{\sin(n-2)\theta_0}{n-2}$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty} I_n(K) \frac{\sin(n+2)\theta_0}{n+2} \qquad \text{(C 41)}$$

At this point the $\lim_{K\to 0} S_2(K,\theta_0)$ can be seen at once to be

$$\lim_{K\to 0} S_2(K,\theta_0) = \frac{\theta_0}{2} + \frac{1}{4}\sin 2\theta_0 \qquad \text{(C 42)}$$

The procedure used in reducing (C 41) to its final form is much the same as that used in reducing (C 31) and will not be given in detail. If, however, the procedure is carried out, it will be found that the result becomes

$$S_2(\kappa,\theta_0) = \frac{1}{\kappa}\left[I_1(\kappa)\,\theta_0 + I_0(\kappa)\sin\theta_0 - 2\left(\frac{1}{\kappa} - \cos\theta_0\right)\sigma(\kappa,\theta_0)\right] \tag{C 43}$$

Before leaving $S_2(\kappa,\theta_0)$ it might be well to show the direct derivation of formula (4.33) of section II-4 which in κ notation can be written as

$$S_2(\kappa,\theta_0) = S_1(\kappa,\theta_0) + \frac{1}{\kappa} I_1(\kappa)\,\theta_0$$

To derive this directly subtract integral (C 25) from (C 37); the resulting expression is

$$S_2(\kappa,\theta_0) - S_1(\kappa,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\kappa\cos\tau}\left[\frac{\sin^2\tau\cos\theta}{\cos\theta-\cos\tau} - \frac{\cos\tau\sin^2\tau}{\cos\theta-\cos\tau}\right]d\tau\,d\theta$$

This result reduces to

$$S_2(\kappa,\theta_0) - S_1(\kappa,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\kappa\cos\tau}\sin^2\tau\,d\tau\,d\theta$$

which integrates at once as

$$S_2(\kappa,\theta_0) - S_1(\kappa,\theta_0) = \left[I_0(\kappa) - I_2(\kappa)\right]\frac{\theta_0}{2}$$

The above can be reduced by means of the recurrence formula to

$$S_2(\kappa,\theta_0) - S_1(\kappa,\theta_0) = \frac{1}{\kappa} I_1(\kappa)\,\theta_0 \tag{C 44}$$

From section II-4 expression (4.19) becomes on substitution of $\kappa = \frac{C}{\pi}$ the following;

$$S_3(\kappa,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{\kappa\cos\tau}\cos\tau\,\frac{\sin^2\tau\cos\theta}{\cos\theta-\cos\tau}\,d\tau\,d\theta \tag{C 45}$$

This integral is similar to (C 25) for if the integrand of (C 25) be multiplied by $\cos\theta$ the integral becomes (C 45). Multiplying the integrands of (C 30) by $\cos\theta$ the first integral of (C 45) is obtained; thus

$$S_3(\kappa,\theta_0) = \int_0^{\theta_0}\left[\frac{1}{2}I_2(\kappa)\cos\theta + I_1(\kappa)\cos^2\theta + \frac{1}{2}I_0(\kappa)\cos 2\theta\cos\theta\right]d\theta$$
$$- 2\sum_{n=1}^{\infty} I_n(\kappa)\int_0^{\theta_0}\sin n\theta\,\sin\theta\cos^2\theta\,d\theta$$

Since the above can be used in the derivation of $S_5(\kappa,\theta_0)$ it is well to rewrite this expression as

$$S_3(k,\theta_0) = \int_0^{\theta_0} \left[\tfrac{1}{2} I_2(k)\cos\theta + \tfrac{1}{2} I_1(k)(1+\cos 2\theta) + \tfrac{1}{4} I_0(k)(\cos\theta + \cos 3\theta) \right] d\theta - \sum_{n=1}^{\infty} I_n(k) \int_0^{\theta_0} \sin n\theta \sin 2\theta \cos\theta \, d\theta \tag{C 46}$$

The trigonometric identity which must be substituted for the second integrand in (C 46) is shown below;

$$\sin n\theta \sin 2\theta \cos\theta = \tfrac{1}{4}\left[\cos(n-1)\theta + \cos(n-3)\theta - \cos(n+3)\theta - \cos(n+1)\theta\right] \tag{C 47}$$

From this expression it is apparent that after substitution, each term of the resulting series will be made up of four terms. From the procedure which follows it is advisable to write the result with four separate series. As in the preceeding cases it is necessary before integrating to separate from the series containing the $\cos(n-1)\theta$, the term for which $n=1$ and in addition the series containing the $\cos(n-3)\theta$, the term for which $n=3$. In the latter series however, the terms for which $n=1$ and $n=2$ will also be separated. Performing this substitution and integrating (C 46) becomes

$$\begin{aligned} S_3(k,\theta_0) = {} & \tfrac{1}{2} I_2(k)\sin\theta_0 + \tfrac{1}{2} I_1(k)(\theta_0 + \tfrac{1}{2}\sin 2\theta_0) + \tfrac{1}{4} I_0(k)(\sin\theta_0 + \tfrac{1}{3}\sin 3\theta_0) \\ & - \tfrac{1}{4} I_1(k)\theta_0 - \tfrac{1}{4}\sum_{n=2}^{\infty} I_n(k)\frac{\sin(n-1)\theta_0}{n-1} \\ & - \tfrac{1}{8} I_1(k)\sin 2\theta_0 - \tfrac{1}{4} I_2(k)\sin\theta_0 - \tfrac{1}{4} I_3(k)\theta_0 \\ & - \tfrac{1}{4}\sum_{n=4}^{\infty} I_n(k)\frac{\sin(n-3)\theta_0}{n-3} \\ & + \tfrac{1}{4}\sum_{n=1}^{\infty} I_n(k)\frac{\sin(n+3)\theta_0}{n+3} + \tfrac{1}{4}\sum_{n=1}^{\infty} I_n(k)\frac{\sin(n+1)\theta_0}{n+1} \end{aligned} \tag{C 48}$$

which can be written as

$$S_3(k,\theta_0) = \frac{1}{4}\Big[I_0(k)(\sin\theta_0 + \tfrac{1}{3}\sin 3\theta_0) + I_1(k)(\theta_0 + \tfrac{1}{2}\sin 2\theta_0)$$

$$+ I_2(k)\sin\theta_0 - I_3(k)\theta_0$$

$$+\sum_{n=1}^{\infty} I_n(k)\frac{\sin(n+1)\theta_0}{n+1} - \sum_{n=2}^{\infty} I_n(k)\frac{\sin(n-1)\theta_0}{n-1}$$

$$+\sum_{n=1}^{\infty} I_n(k)\frac{\sin(n+3)\theta_0}{n+3} - \sum_{n=4}^{\infty} I_n(k)\frac{\sin(n-3)\theta_0}{n-3} \qquad \text{(C 49)}$$

From the above expression it is apparent that

$$\lim_{k \to 0} S_3(k,\theta_0) = \frac{1}{4}(\sin\theta_0 + \tfrac{1}{3}\sin 3\theta_0) \qquad \text{(C 50)}$$

Following a procedure similar to that used in $S_1(k,\theta_0)$, the series of (C 49) can be written in terms of $\sigma(k,\theta_0)$. The manipulation of the terms is quite long and only the final result will be given here, which is

$$S_3(k,\theta_0) = \frac{1}{k}\left[1 + \frac{4}{k^2} - \frac{4}{k}\cos\theta_0 + \cos 2\theta_0\right]\sigma(k,\theta_0) + \frac{1}{k}I_2(k)\theta_0$$

$$+\left[-\frac{1}{2k}\left(\frac{8}{k^2}+1\right)I_1(k) + \frac{4}{k^2}I_2(k) + \frac{3}{2k}I_3(k)\right]\sin\theta_0$$

$$+\left[\left(\frac{2}{k^2}+\frac{1}{8}\right)I_1(k) - \frac{1}{2k}\left(\frac{8}{k^2}+1\right)I_2(k) + \frac{2}{k^2}I_3(k)\right.$$

$$\left. - \frac{1}{2k}I_4(k) - \frac{1}{8}I_5(k)\right]\sin 2\theta_0$$

(C 51)

The above expression is not as yet in its best form for numerical calculations, since the Bessel functions should be reduced to orders of zero and one. This form, however, is much better for numerical calculation than (C 49) since here it is only necessary to compute one series i. e., $\sigma(k,\theta_0)$ and this must be computed for $S_1(k,\theta_0)$.

Expression (4.28) section II-4 defines $S_4(\frac{L}{h},\theta_0)$ which in terms of the k notation becomes

$$S_4(k,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{k\cos t}\,\frac{\sin^2 t}{\cos\theta}\,\frac{\cos^2\theta}{\cos t}\,dt\,d\theta \qquad \text{(C 52)}$$

This integral can be obtained by multiplying the integrand of (C 47) by $\cos\theta$.

The first integral is therefore derived from expression (C 38) by introducing the factor $\cos\theta$ as follows;

$$S_4(k,\theta_0) = \int_0^{\theta_0} \left[I_1(k)\cos^2\theta + I_0(k)\cos^3\theta \right] d\theta$$

$$- 2\sum_{n=1}^{\infty} I_n(k) \int_0^{\theta_0} \sin\theta \cos^2\theta \sin n\theta \, d\theta$$

which can be written as

$$S_4(k,\theta_0) = \int_0^{\theta_0} \left[\tfrac{1}{2} I_1(k)(1+\cos 2\theta) + \tfrac{1}{4} I_0(k)(3\cos\theta + \cos 3\theta) \right] d\theta$$

$$- \sum_{n=1}^{\infty} I_n(k) \int_0^{\theta_0} \sin n\theta \sin 2\theta \cos\theta \, d\theta \qquad \text{(C 53)}$$

If the above expression is compared with (C 46) it will be noticed that the two integrals involving the series are the same. With the aid of (C 48) the above expression can be integrated at once; thus

$$S_4(k,\theta_0) = \tfrac{1}{2} I_1(k)(\theta_0 + \tfrac{1}{2}\sin 2\theta_0) + \tfrac{1}{4} I_0(k)(3\sin\theta_0 + \tfrac{1}{3}\sin 3\theta_0)$$

$$- \tfrac{1}{4} I_1(k)\theta_0 - \tfrac{1}{4} \sum_{n=2}^{\infty} I_n(k) \frac{\sin(n-1)\theta_0}{n-1}$$

$$- \tfrac{1}{8} I_1(k)\sin 2\theta_0 - \tfrac{1}{4} I_2(k)\sin\theta_0 - \tfrac{1}{4} I_3(k)\theta_0$$

$$- \tfrac{1}{4} \sum_{n=4}^{\infty} I_n(k) \frac{\sin(n-3)\theta_0}{n-3}$$

$$+ \tfrac{1}{4} \sum_{n=1}^{\infty} I_n(k) \frac{\sin(n+3)\theta_0}{n+3} + \tfrac{1}{4} \sum_{n=1}^{\infty} I_n(k) \frac{\sin(n+1)\theta_0}{n+1}$$

which can be written as

$$S_4(k,\theta_0) = \tfrac{1}{4} \Big[I_0(k)(3\sin\theta_0 + \tfrac{1}{3}\sin 3\theta_0) + I_1(k)(\theta_0 + \tfrac{1}{2}\sin 2\theta_0)$$

$$- I_2(k)\sin\theta_0 - I_3(k)\theta_0$$

$$+ \sum_{n=1}^{\infty} I_n(k) \frac{\sin(n+1)\theta_0}{n+1} - \sum_{n=2}^{\infty} I_n(k) \frac{\sin(n-1)\theta_0}{n-1}$$

$$+ \sum_{n=1}^{\infty} I_n(k) \frac{\sin(n+3)\theta_0}{n+3} - \sum_{n=4}^{\infty} I_n(k) \frac{\sin(n-3)\theta_0}{n-3}$$

(C 54)

From the above it can be seen that the

$$\text{limit } S_4(k,\theta_0) = \frac{1}{4}\left(3\sin\theta_0 + \frac{1}{3}\sin 3\theta_0\right) \tag{C 55}$$

Manipulating the series of (C 54) by methods used heretofor and by using the recurrence formula it can be brought to the following form;

$$\begin{aligned} S_4(k,\theta_0) = {} & \frac{1}{k}\left[1+\frac{4}{k^2}-\frac{4}{k}\cos\theta_0+\cos 2\theta_0\right]\sigma(k,\theta_0) + \frac{1}{k}I_2(k)\,\theta_0 \\ & +\left[-\frac{1}{2k}\left(\frac{8}{k^2}-1\right)I_1(k)+\frac{4}{k^2}I_2(k)+\frac{3}{2k}I_3(k)\right]\sin\theta_0 \\ & +\left[\left(\frac{2}{k^2}+\frac{1}{8}\right)I_1(k)-\frac{1}{2k}\left(\frac{8}{k^2}+1\right)I_2(k)+\frac{2}{k^2}I_3(k)\right. \\ & \left. -\frac{1}{2k}I_4(k)-\frac{1}{8}I_5(k)\right]\sin 2\theta_0 \end{aligned} \tag{C 56}$$

The derivation of expression (4.32) section II-4 will now be shown which in the k notation can be written as

$$S_4(k,\theta_0) - S_3(k,\theta_0) = \frac{1}{k}I_1(k)\sin\theta_0 \tag{C 57}$$

Making use of the expressions (C 45) and (C 52) the above difference can be written as

$$S_4(k,\theta_0) - S_3(k,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{k\cos\tau}\left[\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau} - \cos\tau\frac{\sin^2\tau\cos\theta}{\cos\theta-\cos\tau}\right]d\tau\, d\theta$$

which reduces to

$$S_4(k,\theta_0) - S_3(k,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{k\cos\tau}\sin^2\tau\cos\theta\, d\tau\, d\theta \tag{C 58}$$

The first integration of (C 58) is made by use of (C 6) which becomes

$$S_4(k,\theta_0) - S_3(k,\theta_0) = \frac{1}{2}\left[I_0(k) - I_2(k)\right]\int_0^{\theta_0}\cos\theta\, d\theta$$

Since $I_0(k) - I_1(k) = \frac{2}{k}I_1(k)$ expression (C 57) follows at once.

The last expression to be derived is $S_5(k,\theta_0)$. This is defined by integral (4.29) which in k notation becomes

$$S_5(K,\theta_0) = \frac{1}{\pi}\int_0^{\theta_0}\int_0^{\pi} e^{K\cos\tau}\cos\tau\,\frac{\sin^2\tau\cos^2\theta}{\cos\theta-\cos\tau}\,d\tau\,d\theta \qquad \text{(C 59)}$$

This integral can be obtained if the integrand of (C 45) is multiplied by $\cos\theta$ and from this it follows that the first integration can be obtained by multiplying the integrand of (C 46) by $\cos\theta$. This is done as shown below;

$$S_5(K,\theta_0) = \int_0^{\theta_0}\Big[\tfrac{1}{2}I_2(K)\cos^2\theta + \tfrac{1}{2}I_1(K)(\cos\theta + \cos 2\theta\cos\theta)$$

$$+ \tfrac{1}{4}I_0(K)(\cos^2\theta + \cos 3\theta\cos\theta)\Big]\,d\theta$$

$$-\sum_{n=1}^{\infty} I_n(K)\int_0^{\theta_0}\sin n\theta\sin 2\theta\cos^2\theta\,d\theta \qquad \text{(C 60)}$$

which can be arranged as

$$S_5(K,\theta_0) = \frac{1}{4}\int_0^{\theta_0}\Big[\tfrac{1}{2}I_0(K)(1+2\cos 2\theta+\cos 4\theta)$$

$$+ I_1(K)(3\cos\theta + \cos 3\theta) + I_2(K)(1+\cos 2\theta)\Big]\,d\theta$$

$$-\frac{1}{2}\sum_{n=1}^{\infty} I_n(K)\int_0^{\theta_0}\sin n\theta\sin 2\theta\,(1+\cos 2\theta)\,d\theta \qquad \text{(C 61)}$$

Integrating the first integral of (C 61) and rearranging the second, it becomes

$$S_5(K,\theta_0) = \frac{1}{4}\Big[I_0(K)(\theta_0 + \sin 2\theta_0 + \tfrac{1}{4}\sin 4\theta_0)$$

$$+ I_1(K)(3\sin\theta_0 + \tfrac{1}{3}\sin 3\theta_0) + I_2(K)(\theta_0 + \tfrac{1}{2}\sin 2\theta_0)\Big]$$

$$-\frac{1}{2}\sum_{n=1}^{\infty} I_n(K)\int_0^{\theta_0}\sin n\theta\sin 2\theta\,d\theta$$

$$-\frac{1}{4}\sum_{n=1}^{\infty} I_n(K)\int_0^{\theta_0}\sin n\theta\sin 4\theta\,d\theta \qquad \text{(C 62)}$$

The trigonometric identities to be applied in this case are

$$\sin n\theta\sin 2\theta = \frac{1}{2}\Big[\cos(n-2)\theta - \cos(n+2)\theta\Big]$$

and

$$\sin n\theta\sin 4\theta = \frac{1}{2}\Big[\cos(n-4)\theta - \cos(n+4)\theta\Big]$$

Applying these trigonometric identities, and separating the proper terms, expression (C 62) can be written as

$$S_5(\kappa,\theta_o) = \frac{1}{8}\Big\{\left[I_0(\kappa) - I_4(\kappa)\right]\theta_o + \left[4I_1(\kappa) - I_3(\kappa)\right]\sin\theta_o$$

$$+\left[I_0(\kappa) + \frac{1}{2}I_2(\kappa)\right]\sin 2\theta_o + \frac{1}{3}I_1(\kappa)\sin 3\theta_o + \frac{1}{4}I_0(\kappa)\sin 4\theta_o\Big\}$$

$$-\frac{1}{4}\sum_{n=3}^{\infty} I_n(\kappa)\frac{\sin(n-2)\theta_o}{n-2} + \frac{1}{4}\sum_{n=1}^{\infty} I_n(\kappa)\frac{\sin(n+2)\theta_o}{n+2}$$

$$-\frac{1}{8}\sum_{n=5}^{\infty} I_n(\kappa)\frac{\sin(n-4)\theta_o}{n-4} + \frac{1}{8}\sum_{n=1}^{\infty} I_n(\kappa)\frac{\sin(n+4)\theta_o}{n+4} \tag{C 63}$$

The value of $S_5(\kappa,\theta_o)$ for $\kappa=0$ will be taken from (C 63); thus

$$\lim_{\kappa\to 0} S_5(\kappa,\theta_o) = \frac{1}{8}\left(\theta_o + \sin 2\theta_o + \frac{1}{4}\sin 4\theta_o\right) \tag{C 64}$$

The function will now be given in its final form as shown below;

$$S_5(\kappa,\theta_o) = \frac{1}{8}\left[I_0(\kappa) - I_4(\kappa)\right]\theta_o$$

$$+\frac{1}{2}\left[\frac{1}{\kappa}\cos 3\theta_o - \frac{6}{\kappa^2}\cos 2\theta_o + \left(\frac{3}{\kappa}+\frac{24}{\kappa^3}\right)\cos\theta_o - \left(\frac{6}{\kappa^2}+\frac{24}{\kappa^4}\right)\right]\sigma(\kappa,\theta_o)$$

$$+\frac{1}{8}\Big[\left(4+\frac{24}{\kappa^2}+\frac{96}{\kappa^4}\right)I_1(\kappa) - \left(\frac{6}{\kappa}+\frac{96}{\kappa^3}\right)I_2(\kappa)$$

$$-\left(3-\frac{12}{\kappa^2}\right)I_3(\kappa) - I_5(\kappa)\Big]\sin\theta_o$$

$$+\frac{1}{8}\Big[-\frac{48}{\kappa^3}I_1(\kappa) + \left(\frac{1}{2}+\frac{16}{\kappa^2}+\frac{96}{\kappa^4}\right)I_2(\kappa) - \left(\frac{2}{\kappa}+\frac{48}{\kappa^3}\right)I_3(\kappa)$$

$$+\frac{12}{\kappa^2}I_4(\kappa) - \frac{2}{\kappa}I_5(\kappa) - \frac{1}{2}I_6(\kappa)\Big]\sin 2\theta_o$$

$$+\frac{1}{8}\Big[\left(\frac{1}{3}+\frac{12}{\kappa^2}\right)I_1(\kappa) - \left(\frac{2}{\kappa}+\frac{48}{\kappa^3}\right)I_2(\kappa) + \left(\frac{16}{\kappa^2}+\frac{96}{\kappa^4}\right)I_3(\kappa)$$

$$-\left(\frac{2}{\kappa}+\frac{48}{\kappa^3}\right)I_4(\kappa) + \frac{12}{\kappa^2}I_5(\kappa) - \frac{2}{\kappa}I_6(\kappa) - \frac{1}{3}I_7(\kappa)\Big]\sin 3\theta_o \tag{C 65}$$

Although the higher order Bessel functions in (C 65) can be reduced, they must be computed for $\sigma(\kappa,\theta_o)$. It was therefore thought best to leave expression (C 65) as is, for it may be necessary to compute the Bessel functions up to and including $I_7(\kappa)$ in order to obtain a satisfactory value for $\sigma(\kappa,\theta_o)$.

Using the results of the preceeding paragraphs the limit of the function

f will now be shown to be zero as the aspect ratio tends to infinity. From section II-9 expression (9.4) is given as

$$f = 2(f_0 \cos\theta_0 - f_1) + \frac{i\lambda c}{2}(f_0 \cos^2\theta_0 - 2 f_1 \cos\theta_0 + f_2) \tag{C 66}$$

It can now be shown that

$$\underset{AR \to \infty}{\text{limit}}\; f = 0 \tag{C 67}$$

From section II-8 expressions (8.2), (8.4), and (8.6) are respectively

$$f_0 = f_{0T} + f_{0S} + f_{0C} \tag{C 68}$$

$$f_1 = f_{1T} + f_{1S} + f_{1C} \tag{C 69}$$

$$f_2 = f_{2T} + f_{2S} + f_{2C} \tag{C 70}$$

It can be shown for the nine terms given in the right hand members of (C 68), (C 69), and (C 70), that

$$\underset{AR \to \infty}{\text{limit}}\; f_{0T} = \underset{AR \to \infty}{\text{limit}}\; f_{0S} = \cdots = \underset{AR \to \infty}{\text{limit}}\; f_{2C} = 0 \tag{C 71}$$

from which it follows that

$$\underset{AR \to \infty}{\text{limit}}\; f_0 = \underset{AR \to \infty}{\text{limit}}\; f_1 = \underset{AR \to \infty}{\text{limit}}\; f_2 = 0 \tag{C 72}$$

and hence the limit (C 67).

To start this proof take the function f_{0T} which is given as expression (4.14) section II-4. Since the limit of a product is equal to the product of the limits it follows that the limit of (4.14) is

$$\underset{AR \to \infty}{\text{limit}}\; f_{0T} = \underset{AR \to \infty}{\text{limit}} \frac{e^{-\frac{4}{AR}}}{AR\left(1 + \frac{i\lambda c}{8} AR\right)} \left\{ \left[\frac{1}{1 + \frac{i\lambda c}{8} AR} + \frac{4}{AR} + 1 \right] \underset{AR \to \infty}{\text{limit}}\; S_0\left(\frac{4}{AR}, \theta_0\right) - \frac{4}{AR} \underset{AR \to \infty}{\text{limit}}\; S_1\left(\frac{4}{AR}, \theta_0\right) \right\}$$

From (C 19) and (C 32) it follows that the above can be written as

$$\underset{AR \to \infty}{\text{limit}}\; f_{0T} = \underset{AR \to \infty}{\text{limit}} \frac{e^{-\frac{4}{AR}}}{AR\left(1 + \frac{i\lambda c}{8} AR\right)} \left\{ \left[\frac{1}{1 + \frac{i\lambda c}{8} AR} + \frac{4}{AR} + 1 \right] \sin\theta_0 - \frac{4}{AR}\left(\frac{1}{4} \sin 2\theta_0\right) \right\} \tag{C 73}$$

from which it follows that

$$\underset{AR \to \infty}{\text{limit}}\; f_{0T} = 0 \tag{C 74}$$

The function f_{os} is given as expression (5.10) section II-5. The limit of this expression can be obtained as shown below;

$$\underset{AR\to\infty}{limit}\; f_{os} = \underset{AR\to\infty}{limit}\left[-\frac{1.1358\, i\lambda c}{1+\frac{3i\lambda c}{2}AR}\, e^{-\frac{1}{3AR}}\right]\underset{AR\to\infty}{limit}\; S_o\left(\frac{1}{3AR}, \theta_o\right) \qquad (C\ 75)$$

Substituting (C 19) this expression becomes

$$\underset{AR\to\infty}{limit}\; f_{os} = \underset{AR\to\infty}{limit}\left[-\frac{1.1358\, i\lambda c}{1+\frac{3i\lambda c}{2}AR}\, e^{-\frac{1}{3AR}}\right]\sin\theta_o = 0 \qquad (C\ 76)$$

The last function of f_o is f_{oc} and it is given as expression (6.5) section II-6. The limit equation is

$$\underset{AR\to\infty}{limit}\; f_{oc} = \frac{1}{96}\underset{AR\to\infty}{limit}\Big[(48a_o + 24a_1 + 15a_2)\sin\theta_o - 6(a_1+a_2)\sin 2\theta_o + a_2\sin 3\theta_o\Big] \qquad (C\ 77)$$

From expressions (5.6), (5.7) and (5.8) of section I-5 part II of Report 5 it follows that

$$\underset{AR\to\infty}{limit}\; a_o = \underset{AR\to\infty}{limit}\; a_1 = \underset{AR\to\infty}{limit}\; a_2 = 0 \qquad (C\ 78)$$

It can now be said that

$$\underset{AR\to\infty}{limit}\; f_{oc} = 0 \qquad (C\ 79)$$

From expressions (C 74), (C 76) and (C 79) it follows that

$$\underset{AR\to\infty}{limit}\; f_o = 0 \qquad (C\ 80)$$

In order to obtain the limits for f_1 and f_2 expressions (4.24), (4.36), (5.19), (5.26), (6.10) and (6.14) from sections II-4, II-5, and II-6 are needed; from appendix C the results given by (C 42), (C 50), (C 55) and (C 64) are also needed. The procedure is the same as that for f_o and the details will not be given here, but if this work is carried out it will be found that expression (C 72) is true and hence the validity of (C 67) follows.

Appendix D

Integrals of Section II-5

In section II-5 the integral (5.3) is given as

$$\int_0^{\theta_0}\int_0^{\pi}\int_1^{\infty} \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\varsigma\,d\tau\,d\theta \tag{D 1}$$

Reversing the order of integration the above expression becomes

$$\int_1^{\infty}\int_0^{\theta_0}\int_0^{\pi} \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\tau\,d\theta\,d\varsigma \tag{D 2}$$

The integrand will be separated in partial fractions as shown below;

$$\frac{\sin^2\tau}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)} = \frac{\varsigma^2-1}{\varsigma-\cos\theta}\cdot\frac{1}{\varsigma-\cos\tau}$$
$$-\frac{\sin^2\theta}{\varsigma-\cos\theta}\cdot\frac{1}{\cos\tau-\cos\theta} - 1$$

Substituting this in (D 2) it becomes

$$\int_1^{\infty}\int_0^{\theta_0}\int_0^{\pi} \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\tau\,d\theta\,d\varsigma$$
$$=\int_1^{\infty}\int_0^{\theta_0} e^{-\frac{i\lambda c}{2}\varsigma}\left\{\int_0^{\pi}\left[\frac{\varsigma^2-1}{\varsigma-\cos\theta}\cdot\frac{1}{\varsigma-\cos\tau}-\frac{\sin^2\theta}{\varsigma-\cos\theta}\cdot\frac{1}{\cos\tau-\cos\theta}-1\right]d\tau\right\}d\theta\,d\varsigma \tag{D 3}$$

Within the brackets of (D 3) are three quantities which form three separate integrals with respect to τ which will be taken up one at a time. The first integral is the type given by B. O. Peirce as No. 300 which is

$$\int\frac{dx}{a+b\cos x} = -\frac{1}{\sqrt{a^2-b^2}}\sin^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right] \tag{D 4}$$

where the constant of integration has been omitted. Applying (D 4) to the first term of integral (D 3) and omitting the fractional coefficient it becomes

$$\int_0^{\pi}\frac{d\tau}{\varsigma-\cos\tau} = -\frac{1}{\sqrt{\varsigma^2-1}}\sin^{-1}\left[\frac{\varsigma\cos\tau-1}{\varsigma-\cos\tau}\right]_0^{\pi} = \frac{\pi}{\sqrt{\varsigma^2-1}} \tag{D 5}$$

The second quantity in the brackets is the type given by (C 13) and therefore

$$\int_0^{\pi}\frac{d\tau}{\cos\tau-\cos\theta} = 0 \tag{D 6}$$

The third quantity is the -1 which on integration yields $-\pi$. Substituting these three results the integral (D 2) takes the following form;

$$\int_1^\infty \int_0^{\theta_0} \int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma} \sin^2\tau}{(\varsigma - \cos\tau)(\cos\theta - \cos\tau)} \, d\tau \, d\theta \, d\varsigma$$

$$= \pi \int_1^\infty \int_0^{\theta_0} e^{-\frac{i\lambda c}{2}\varsigma} \left\{ \frac{\sqrt{\varsigma^2 - 1}}{\varsigma - \cos\theta} - 1 \right\} d\theta \, d\varsigma \qquad \text{(D 7)}$$

The integration with respect to θ in the above integral is given by expression (D 4) and applying this formula (D 7) becomes

$$\int_1^\infty \int_0^{\theta_0} \int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma} \sin^2\tau}{(\varsigma - \cos\tau)(\cos\theta - \cos\tau)} \, d\tau \, d\theta \, d\varsigma$$

$$= -\pi \int_1^\infty e^{-\frac{i\lambda c}{2}\varsigma} \left[\sin^{-1} \frac{\varsigma \cos\theta_0 - 1}{\varsigma - \cos\theta_0} - \frac{\pi}{2} + \theta_0 \right] d\varsigma \qquad \text{(D 8)}$$

For the convergence of this integral as well as those which follow see G. H. Hardy, section 203, reference 5. To bring (D 8) into its final form the parts formula $\int u\,dv = uv - \int v\,du$ is used. In applying the parts formula let

$$u = \sin^{-1} \frac{\varsigma \cos\theta_0 - 1}{\varsigma - \cos\theta_0} - \frac{\pi}{2} + \theta_0$$

and

$$dv = e^{-\frac{i\lambda c}{2}\varsigma} \, d\varsigma$$

From this it follows that

$$du = \frac{\sin\theta_0}{(\varsigma - \cos\theta_0)\sqrt{\varsigma^2 - 1}} \, d\varsigma$$

and

$$v = -\frac{2}{i\lambda c} e^{-\frac{i\lambda c}{2}\varsigma}$$

Substituting the above in the parts formula, integral (D 8) becomes

$$\int_1^\infty \int_0^{\theta_0} \int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma} \sin^2\tau}{(\varsigma - \cos\tau)(\cos\theta - \cos\tau)} \, d\tau \, d\theta \, d\varsigma$$

$$= \frac{2\pi}{i\lambda c} \left[e^{-\frac{i\lambda c}{2}\varsigma} \left(\sin^{-1} \frac{\varsigma \cos\theta_0 - 1}{\varsigma - \cos\theta_0} - \frac{\pi}{2} + \theta_0 \right) \right]_1^\infty$$

$$- \frac{2\pi}{i\lambda c} \sin\theta_0 \int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{(\varsigma - \cos\theta_0)\sqrt{\varsigma^2 - 1}} \, d\varsigma \qquad \text{(D 9)}$$

The bracketed quantity in (D 9) tends to zero as ς tends to infinity, however, at the lower limit, i. e. $\varsigma = 1$ it becomes $-\pi + \theta_0$. No solution was found for the remaining integral of (D 9) so a symbol Q_2 was defined as shown below;

$$Q_2 = e^{\frac{i\lambda c}{2}} \int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{(\varsigma - \cos\theta)\sqrt{\varsigma^2-1}}\, d\varsigma \tag{D 10}$$

This is given as expression (5.4) section II-5. Integral (D 9) can now be written in its final form as

$$\int_1^\infty\int_0^{\theta_0}\int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\, d\tau\, d\theta\, d\varsigma$$

$$= \frac{2\pi}{i\lambda c} e^{-\frac{i\lambda c}{2}} (\pi - \theta_0) - \frac{2\pi}{i\lambda c} e^{-\frac{i\lambda c}{2}} Q_2 \sin\theta_0 \tag{D 11}$$

The above is given as integral (5.3) section II-5.

The next integral of this appendix is given as (5.13) section II-5. On comparing this integral with (D 1) it will be observed that if the integrand of (D 1) is multiplied by $\cos\theta$ it becomes integral (5.13) of section II-5. From this it follows that a first integral can be obtained by multiplying the integrand of (D 7) by $\cos\theta$. Performing this multiplication (D 7) becomes

$$\int_1^\infty\int_0^{\theta_0}\int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\, d\tau\, d\theta\, d\varsigma$$

$$= \pi\int_0^\infty\int_0^{\theta_0} e^{-\frac{i\lambda c}{2}\varsigma}\left\{\frac{\sqrt{\varsigma^2-1}}{\varsigma-\cos\theta}\cos\theta - \cos\theta\right\} d\theta\, d\varsigma \tag{D 12}$$

which can be written as

$$\int_1^\infty\int_0^{\theta_0}\int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\, d\tau\, d\theta\, d\varsigma$$

$$= \pi\int_1^\infty\int_0^{\theta_0} e^{-\frac{i\lambda c}{2}\varsigma}\left\{\frac{\varsigma\sqrt{\varsigma^2-1}}{\varsigma-\cos\theta} - \sqrt{\varsigma^2-1} - \cos\theta\right\} d\theta\, d\varsigma \tag{D 13}$$

Within the brackets of (D 13) are three expressions, insofar as θ is concerned. The first forms an integral of type (D 4) and the remaining two integrals are very simple, hence (D 13) becomes

$$\int_1^\infty\int_0^{\theta_0}\int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\tau\,d\theta\,d\varsigma$$

$$=\pi\int_1^\infty e^{-\frac{i\lambda c}{2}\varsigma}\Big[\frac{\pi\varsigma}{2}-\varsigma\sin^{-1}\frac{\varsigma\cos\theta_0-1}{\varsigma-\cos\theta_0}$$

$$-\theta_0\sqrt{\varsigma^2-1}-\sin\theta_0\Big]d\varsigma$$

(D 14)

The parts formula $\int u\,dv = uv-\int v\,du$ must now be applied, and for this let

$$u=\frac{\pi\varsigma}{2}-\varsigma\sin^{-1}\frac{\varsigma\cos\theta_0-1}{\varsigma-\cos\theta_0}-\theta_0\sqrt{\varsigma^2-1}-\sin\theta_0$$

and

$$dv=e^{-\frac{i\lambda c}{2}\varsigma}\,d\varsigma$$

From which it follows that

$$du=\Big(\frac{\pi}{2}-\frac{\varsigma\sin\theta_0}{(\varsigma-\cos\theta_0)\sqrt{\varsigma^2-1}}-\sin^{-1}\frac{\varsigma\cos\theta_0-1}{\varsigma-\cos\theta_0}-\frac{\theta_0\varsigma}{\sqrt{\varsigma^2-1}}\Big)d\varsigma$$

and

$$v=-\frac{2}{i\lambda c}e^{-\frac{i\lambda c}{2}\varsigma}$$

Substituting the above in (D 14) it becomes

$$\int_1^\infty\int_0^{\theta_0}\int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\tau\,d\theta\,d\varsigma$$

$$=-\frac{2\pi}{i\lambda c}\Big[e^{-\frac{i\lambda c}{2}\varsigma}\Big(\frac{\pi\varsigma}{2}-\varsigma\sin^{-1}\frac{\varsigma\cos\theta_0-1}{\varsigma-\cos\theta_0}-\theta_0\sqrt{\varsigma^2-1}-\sin\theta_0\Big)\Big]_1^\infty$$

$$+\frac{2\pi}{i\lambda c}\int_1^\infty e^{-\frac{i\lambda c}{2}\varsigma}\Big[\frac{\pi}{2}-\frac{\varsigma\sin\theta_0}{(\varsigma-\cos\theta_0)\sqrt{\varsigma^2-1}}-\sin^{-1}\frac{\varsigma\cos\theta_0-1}{\varsigma-\cos\theta_0}-\frac{\theta_0\varsigma}{\sqrt{\varsigma^2-1}}\Big]d\varsigma \qquad \text{(D 15)}$$

In the above expression the lower limit of the uv-term offers no difficulty, the upper limit, however, must be evaluated by means of l'Hospital's rule considering the following expression; thus

$$\operatorname*{limit}_{\varsigma\to\infty}\frac{\frac{\pi}{2}-\sin^{-1}\frac{\varsigma\cos\theta_0-1}{\varsigma-\cos\theta_0}-\theta_0\sqrt{1-\frac{1}{\varsigma^2}}-\frac{\sin\theta_0}{\varsigma}}{\frac{1}{\varsigma}}$$

(D 16)

From which it follows that the limit of the above expression is zero. The remaining integral of (D 15) can also be separated into two convergent integrals thus (D 15) can be written as

$$\int_1^\infty \int_0^{\theta_0} \int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma} \sin^2\tau \cos\theta}{(\varsigma - \cos\tau)(\cos\theta - \cos\tau)} \, d\tau \, d\theta \, d\varsigma$$

$$= \frac{2\pi}{i\lambda c}(\pi - \sin\theta_0)$$

$$- \frac{2\pi \sin\theta_0}{i\lambda c} \int_1^\infty \frac{\varsigma e^{-\frac{i\lambda c}{2}\varsigma}}{(\varsigma - \cos\theta_0)\sqrt{\varsigma^2 - 1}} \, d\varsigma$$

$$- \frac{2\pi}{i\lambda c} \int_1^\infty e^{-\frac{i\lambda c}{2}\varsigma} \left(\sin^{-1} \frac{\varsigma \cos\theta_0 - 1}{\varsigma - \cos\theta_0} + \frac{\theta_0 \varsigma}{\sqrt{\varsigma^2 - 1}} - \frac{\pi}{2} \right) d\varsigma \qquad \text{(D 17)}$$

Operating once with long division on the integrand of the first integral of the right hand member of (D 17) it can be separated into two convergent integrals. If θ_0 is added and subtracted in the integrand of the second, it also can be separated into two convergent integrals. By these means (D 17) can be written in the following form;

$$\int_1^\infty \int_0^{\theta_0} \int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma} \sin^2\tau \cos\theta}{(\varsigma - \cos\tau)(\cos\theta - \cos\tau)} \, d\tau \, d\theta \, d\varsigma$$

$$= \frac{2\pi}{i\lambda c} e^{-\frac{i\lambda c}{2}} (\pi - \sin\theta_0) - \frac{2\pi}{i\lambda c} \sin\theta_0 \int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{\sqrt{\varsigma^2 - 1}} \, d\varsigma$$

$$- \frac{2\pi}{i\lambda c} \sin\theta_0 \cos\theta_0 \int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{(\varsigma^2 - \cos\theta_0)\sqrt{\varsigma^2 - 1}} \, d\varsigma$$

$$- \frac{2\pi}{i\lambda c} \int_1^\infty e^{-\frac{i\lambda c}{2}\varsigma} \left(\sin^{-1} \frac{\varsigma \cos\theta_0 - 1}{\varsigma - \cos\theta_0} - \frac{\pi}{2} + \theta_0 \right) d\varsigma$$

$$- \frac{2\pi}{i\lambda c} \theta_0 \int_1^\infty e^{-\frac{i\lambda c}{2}\varsigma} \left(\frac{\varsigma}{\sqrt{\varsigma^2 - 1}} - 1 \right) d\varsigma \qquad \text{(D 18)}$$

Considering the integrals of the right hand member of the above expression, the first is given on page 96 of Report 5 and is

$$\int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{\sqrt{\varsigma^2 - 1}} \, d\varsigma = -\frac{\pi i}{2} H_0^{(2)}\left(\frac{\lambda c}{2}\right) \qquad \text{(D 19)}$$

where $H_0^{(2)}(\frac{\lambda c}{2})$ is the Hankel function. The second integral is $e^{-\frac{i\lambda c}{2}} Q_2$, see expression (D 10), and the third is integral (D 8). The fourth integral is given on page 98 of Report 5 which is

$$\int_1^{\infty} e^{-\frac{i\lambda c}{2}\varsigma}\left(\frac{\varsigma}{\sqrt{\varsigma^2-1}}-1\right)d\varsigma = -\frac{2}{i\lambda c}e^{-\frac{i\lambda c}{2}} - \frac{\pi}{2}H_1^{(2)}\left(\frac{\lambda c}{2}\right) \tag{D 20}$$

where $H_1^{(2)}\left(\frac{\lambda c}{2}\right)$ is the Hankel function of order one.

Before substituting it is advisable to write (D 19) and (D 20) in terms of Q_0 and Q_1 respectively. This is done by making use of expressions (5.14) and (5.15) of section II-5 which converts (D 19) and (D 20) into the forms shown below;

$$\int_1^{\infty} \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{\sqrt{\varsigma^2-1}}\,d\varsigma = -\frac{2}{i\lambda c}e^{-\frac{i\lambda c}{2}}Q_0 \tag{D 21}$$

$$\int_1^{\infty} e^{-\frac{i\lambda c}{2}\varsigma}\left(\frac{\varsigma}{\sqrt{\varsigma^2-1}}-1\right) = -\frac{2}{i\lambda c}e^{-\frac{i\lambda c}{2}}\left(1+Q_1\right) \tag{D 22}$$

Substituting the above results in (D 18) it becomes

$$\begin{aligned}\int_1^{\infty}\int_0^{\theta_0}\int_0^{\pi} &\frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\tau\,d\theta\,d\varsigma \\ &= \frac{2\pi}{i\lambda c}e^{-\frac{i\lambda c}{2}}\left[\pi-\sin\theta_0+\frac{2}{i\lambda c}(\pi-\theta_0)\right] \\ &\quad -\frac{\pi}{i\lambda c}e^{-\frac{i\lambda c}{2}}\left[\sin 2\theta_0+\frac{4}{i\lambda c}\sin\theta_0\right]Q_2 \\ &\quad +\frac{4\pi\sin\theta_0}{(i\lambda c)^2}e^{-\frac{i\lambda c}{2}}Q_0+\frac{4\pi\theta_0}{(i\lambda c)^2}e^{-\frac{i\lambda c}{2}}\left[1+Q_1\right]\end{aligned} \tag{D 23}$$

which is the form as presented in section II-5, see integral (5.13).

The last integral of this appendix is integral (5.22) of section II-5. On examination of this integral it is seen that the first integration is obtained if the integrands of expression (D 13) are multiplied by $\cos\theta$; thus

$$\begin{aligned}\int_1^{\infty}\int_0^{\theta_0}\int_0^{\pi} &\frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos^2\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\tau\,d\theta\,d\varsigma \\ &= \pi\int_1^{\infty}\int_0^{\theta_0} e^{-\frac{i\lambda c}{2}\varsigma}\left[\frac{\varsigma\sqrt{\varsigma^2-1}}{\varsigma-\cos\theta}\cos\theta-\sqrt{\varsigma^2-1}\cos\theta-\cos^2\theta\right]d\theta\,d\varsigma\end{aligned} \tag{D 24}$$

which can be written as

$$\int_1^\infty\int_0^{\theta_0}\int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos^2\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\tau\,d\theta\,d\varsigma$$

$$= \pi\int_1^\infty\int_0^{\theta_0} e^{-\frac{i\lambda c}{2}\varsigma}\Big[\sqrt{\varsigma^2-1}\,\Big(\frac{\varsigma^2}{\varsigma-\cos\theta}-\cos\theta-\varsigma\Big)$$

$$-\frac{1}{2}-\frac{1}{2}\cos 2\theta\Big]\,d\theta\,d\varsigma \qquad \text{(D 25)}$$

Integral (D 4) applies to the first term in the above parenthesis, and the balance of the integration is evident; hence (D 25) becomes

$$\int_1^\infty\int_0^{\theta_0}\int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos^2\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\tau\,d\theta\,d\varsigma$$

$$= -\pi\int_1^\infty e^{-\frac{i\lambda c}{2}\varsigma}\Big[\varsigma^2\sin^{-1}\frac{\varsigma\cos\theta_0-1}{\varsigma-\cos\theta_0}-\frac{\pi}{2}\varsigma^2+\sqrt{\varsigma^2+1}\,(\sin\theta_0+\theta_0\varsigma)$$

$$-\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0\Big]\,d\varsigma \qquad \text{(D 26)}$$

In the above integral the part in the brackets can be shown by l'Hospital's rule to tend to zero as a limit when ς tends to infinity; hence, according to G. H. Hardy, section 203, reference 5, the integral converges.

To continue the integration the parts formula $\int u\,dv = uv - \int v\,du$ is used, where

$$u = \varsigma^2\sin^{-1}\frac{\varsigma\cos\theta_0-1}{\varsigma-\cos\theta_0}-\frac{\pi}{2}\varsigma^2+\sin\theta_0\sqrt{\varsigma^2-1}$$

$$+\theta_0\varsigma\sqrt{\varsigma^2-1}+\frac{\theta_0}{2}+\frac{1}{4}\sin 2\theta_0$$

and

$$dv = e^{-\frac{i\lambda c}{2}\varsigma}\,d\varsigma$$

The expressions for du and v will not be given here, however, after integrating by parts, (D 26) takes the following form;

$$\int_1^\infty\int_0^{\theta_0}\int_0^\pi \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos^2\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\tau\,d\theta\,d\varsigma = \frac{\pi}{i\lambda c}e^{-\frac{i\lambda c}{2}}\Big[2\pi-\theta_0-\frac{1}{2}\sin 2\theta_0\Big]$$

$$-\frac{2\pi}{i\lambda c}\int_1^\infty e^{-\frac{i\lambda c}{2}\varsigma}\Big[2\varsigma\sin^{-1}\frac{\varsigma\cos\theta_0-1}{\varsigma-\cos\theta_0}+\frac{\varsigma^2\sin\theta_0}{(\varsigma-\cos\theta_0)\sqrt{\varsigma^2-1}}-\pi\varsigma$$

$$+\frac{\varsigma\sin\theta_0}{\sqrt{\varsigma^2-1}}+\theta_0\frac{2\varsigma^2-1}{\sqrt{\varsigma^2-1}}\Big]\,d\varsigma \qquad \text{(D 27)}$$

The details of the following integration will not be given here, however, the procedure will be outlined below. The first step is to take the second term in the brackets of the integral and separate it by means of long division so that it appears as shown below;

$$\frac{\varsigma^2 \sin\theta_0}{(\varsigma-\cos\theta_0)\sqrt{\varsigma^2-1}} = \frac{\varsigma\sin\theta_0}{\sqrt{\varsigma^2-1}} + \frac{\sin\theta_0\cos\theta_0}{\sqrt{\varsigma^2-1}} + \frac{\sin\theta_0\cos^2\theta_0}{(\varsigma-\cos\theta_0)\sqrt{\varsigma^2-1}} \qquad \text{(D 28)}$$

If (D 28) is resubstituted in (D 27) the second fraction can be integrated by (D 21) and the third fraction can be handled by (D 10) which leaves the first fraction unintegrated. Assuming that the above operations have been performed, add to the integrand of the remaining integral the following two zeros;

$$2\theta_0\sqrt{\varsigma^2-1} - 2\theta_0\sqrt{\varsigma^2-1}$$

and

$$2\sin\theta_0 - 2\sin\theta_0$$

The remaining integral can now be separated into three convergent integrals, one of which, say the first, can be recognized through (D 14) to be integral (D 23), except for the common factor π. The second reduces algebraically to integral (D 21), and the third is integral (D 22). If this method is followed, integral (D 27) becomes

$$\int_0^{\theta_0}\int_0^{\pi}\int_1^{\infty} \frac{e^{-\frac{i\lambda c}{2}\varsigma}\sin^2\tau\cos^2\theta}{(\varsigma-\cos\tau)(\cos\theta-\cos\tau)}\,d\varsigma\,d\tau\,d\theta$$

$$\begin{aligned} = \frac{2\pi}{i\lambda c}e^{-\frac{i\lambda c}{2}}\Big\{&\pi - \frac{\theta_0}{2} - \frac{1}{4}\sin 2\theta_0 + \frac{4\pi}{i\lambda c}\left(1+\frac{2}{i\lambda c}\right) \\ &+ \frac{2}{i\lambda c}\left[\frac{1}{2}\sin 2\theta_0 + \frac{4\sin\theta_0}{i\lambda c} + \theta_0\right]Q_0 \\ &+ \frac{4}{i\lambda c}\left[\frac{2\theta_0}{i\lambda c} + \sin\theta_0\right]Q_1 \\ &- \sin\theta_0\left[\cos^2\theta_0 + \frac{4}{i\lambda c}\left(\cos\theta_0 + \frac{2}{i\lambda c}\right)\right]Q_2\Big\} \end{aligned} \qquad \text{(D 29)}$$

This is integral (5.22) of section II-5.