Rigidity of Three Measure Classes on the Ideal Boundary of Manifolds with Negative Curvature

Dissertation presented by

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Dedicated to Renxiong Tanchu for his pure friendship.

The woods are lovely, dark and deep,

But I have promises to keep,

And miles to go before I sleep.

And miles to go before I sleep.

—— Robert Frost, *Stopping by Woods on a Snowy Evening, New Hampshire* [Fr]

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Yet, unlike Odysseus, who had a home for his destiny, there was nothing similar for Virgil's hero, though he had Italy. And ——

That Italy you think so near, with ports
You think to enter, ignorant as you are,
Lies far, past far lands, by untraveled ways.

—— Virgil, Aeneid

Abstract

On the ideal boundary, $\partial\widetilde{M}$, of the universal covering $\widetilde{M}$ of a negatively curved closed Riemannian manifold M, there exist three natural measure classes: the harmonic measure class $\{v_x\}_{x\in\widetilde{M}}$, the Lebesgue measure class $\{m_x\}_{x\in\widetilde{M}}$, the Bowen-Margulis measure class $\{u_x\}_{x\in\widetilde{M}}$.

A famous conjecture (by A. Katok, F. Ledrappier, D. Sullivan) states that the coincidence of any two of these three measure classes implies that M is locally symmetric. We prove a weaker version of Sullivan's conjecture: the horospheres in $\widetilde{M}$ have constant mean curvature if and only if $m_x = v_x$ for all $x \in \widetilde{M}$.

In investigating these rigidity problems, we come across a class of integral formulas involving Laplacian Δ^u along the unstable foliation of the geodesic flow. One of which is $\int_{SM} (\Delta^u\varphi + < \nabla^u \log g, \nabla^u\varphi >)dm = 0$. Using these formulas, many rigidity problems are discussed, including (i) a simple proof of Hamenstädt's lemma 5.3 which avoids her use of stochastic process; (ii) two functional descriptions of those manifolds which have horospheres with constant mean curvature: the horospheres in $\widetilde{M}$ have constant mean curvature if and only if $\int_{SM} \Delta^u\varphi dm = 0$ for all φ in $C^2_u(SM)$ or $\int_{SM} \Delta^{su}\varphi dm = 0$ for all φ in $C^2_{su}(SM)$.

Finally, we study ergodic properties of Anosov foliations and their applications to manifolds of negative curvature. We obtain an integral formula for topological entropy in terms of Ricci and scalar curvature. We also show that the function $c(x)$ in Margulis's asymptotic formula $c(x) = \lim_{R\to\infty} e^{-hR}S(x,R)$ is almost always nonconstant. In dimension 2, $c(x)$ is a constant function if and only if the manifold has constant negative curvature. Generally, if the Ledrappier-Patterson-Sullivan measure is flip invariant, then $c(x)$ is constant.

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I. Introduction

1. Historical remarks

(0.1) Rigidity of lattices

Mostow remarked in his pioneering work [Mo] that the chronology of rigidity begins with the theorem of A. Selberg (1960) that a discrete co-compact subgroup Γ of $SL(n, R)$ cannot be continuously deformed except trivially, that is, by inner automorphisms of $SL(n, \mathbb{R})$, if $n > 2$. At about the same time, E. Calabi and E. Vesentini (1961) proved the rigidity of complex structure under infinitisimal deformations of compact quotients of bounded symmetric domains, and later Calabi proved the metric analogue for compact hyperbolic n-space forms for $n > 2$. There upon A. Weil (1962) generalized Selberg's and Calabi's results to semi-simple groups having no compact or 3 dimensional simple factors. And then, Mostow proved his famous rigidity theorem ([Mo], 1973).

Mostow Rigidity Theorem. *Let Y be a compact locally symmetric space of non-positive sectional curvature. Then the fundamental group determined Y uniquely up to an isometry and a choice of normalizing constants, provided that Y has no closed one or two dimensional geodesic subspaces which are direct factors locally.*

The development of Mostow's work is culminated in Margulis's superrigidity theorem ([M2], see also [Z]) about the rigidity of lattices in semisimple group.

Gromov generalized Mostow's result to manifolds of nonpositive curvature ([GSB]). Finally, Ballmann, Brin, Burns, Eberlein and Spatzier ([BBE] [BBS]) carried out a program of the classification of nonpositively curved manifolds. They defined the important notion of rank:

$$\text{rank}(M) = \max\{k \mid \text{each geodesic of } M \text{ is contained in a k-flat}\}$$

and proved the following theorem.

Theorem ([B] [BS]). *Suppose that rank(M) ≥ 2, the sectional curvature has a lower bound $-a^2$ and M has finite volume. If the universal covering of M is irreducible, then M is a locally symmetric space of noncompact type.*

(0.2) Rigidity of the geodesic flows with smooth Anosov splitting

At about the same time as Mostow's early work on rigidity (the early 1960's), the Russian mathematicians started their systematic study of the geodesic flow g^t on a closed manifold of negative curvature. Dynamical ideas in the study of manifolds with negative curvature go back to M. Morse, G. A. Hedlund and E. Hopf ([He], [Ho]). In the 1960's Anosov and Sinai proved that the geodesic flow of a closed manifold of negative curvature is an Anosov flow and it is ergodic ([An1] [AnS]). In [An1], Anosov claimed that the Anosov splitting is "obviously not smooth in general ([An1, p.12])." He proved, however, that it is always Hölder continuous. More recent developments can be found in Boris Hasselblatt's thesis ([Ha]). Gromov has asked if the Anosov splitting is C^2 then M should be locally symmetric ([S3]). The first result concerning this question was proven by Kanai.

Theorem [Ka]. *Let M be a closed C^∞ Riemannian manifold of dimension greater than 2. Assume that the sectional curvature k of M satisfies $-\frac{9}{4} < k \leq -1$ and that the Anosov splitting is of class C^∞. Then the geodesic flow g^t of M is C^∞-isomorphic to the geodesic flow of a closed Riemannian manifold of constant negative curvature.*

Katok and Feres improved Kanai's result by showing that one can replace the pinching condition $-\frac{a}{4} < k \leq -1$ by $k < 0$ in the odd dimensional case and by $-4 < k \leq -1$ in the general case ([F]).

Y. Benoist, P. Foulon and F. Labourie recently proved the following theorem.

Theorem ([BFL]). *Let M be a C^∞ closed Riemannian manifold of negative curvature. If the Anosov splitting of the geodesic flow g^t of M is of C^∞ class, then*

g^t is C^∞-isomorphic to the geodesic flow of a locally symmetric space of negative curvature.

2. Rigidity of three measure classes.

(1.1) Notations

In the last section we reviewed the main results concerning the rigidity of lattices in manifolds of nonpositive curvature and the rigidity of the geodesic flow on manifolds of negative curvature with smooth Anosov splitting. Now we come to another kind of rigidity, but before going into the details, let us fix some notation.

Throughout this section, we denote by M a closed Riemannian manifold of negative curvature. Let $\widetilde{M}$ be the universal covering of M. We also denote

- SM (resp. $S\widetilde{M}$) the unit tangent bundle of M (resp. $\widetilde{M}$).
- g^t the geodesic flow on SM or $S\widetilde{M}$.
- $\pi : S\widetilde{M} \to \widetilde{M}$ the canonical projection.
- $\partial\widetilde{M}$ the ideal boundary of $\widetilde{M}$.
- $v(t) = \pi(g^t v)$, is the geodesics with initial velocity $\dot{v}(0) = v$.
- $v(\infty)$ is the point at $\partial\widetilde{M}$ determined by the geodesic $v(t)$.
- $P : S\widetilde{M} \to \partial\widetilde{M}$ is the projection $P(v) = v(\infty)$ and $P_x : S_x\widetilde{M} \to \partial\widetilde{M}$ is the restriction of P to $S_x\widetilde{M}$.
- μ is the Bowen-Margulis measure.
- ν is the harmonic measure (see [L1]).
- m is the Liouville measure on SM or $S\widetilde{M}$, normalized so that $m(SM) = 1$. We also denote by $dm(x)$ the Riemannian volume on M or $\widetilde{M}$, normalized such that $m(M) = 1$.
- (x, ξ) the vector v in S_xM such that $v(\infty) = \xi$.
- ρ_v the Busemann function at $v(-\infty)$ such that $\rho_v(v(0)) = 0$.

We also denote by W^{su}, W^u, W^{ss}, W^s the strong unstable, unstable, strong stable, stable foliations of the geodesic flow. The canonical projection $\pi : S\widetilde{M} \to \widetilde{M}$ maps $W^u(v)$ or $W^s(v)$ diffeomorphically onto $\widetilde{M}$. Thus the Riemannian metric $< \, , \, >$ on $\widetilde{M}$ lifts to a Riemannian metric g^i on $W^i(v)$ which induces a Lebesgue measure $m^i (i = su, \; u, \; ss, \; s)$.

(1.2) Three measure classes on $\partial\widetilde{M}$

There are three natural, and probably, most interesting measure classes on $\partial\widetilde{M}$, which arise either from the geometry on $\widetilde{M}$ or from the dynamics of the geodesic flow.

(i) **The Lebesgue measure class**: Consider the Lebesgue measure m_x on $S_x\widetilde{M}$ induced by the Riemannian metric. By the absolute continuity of the stable foliation of the geodesic flow, m_x and m_y are equivalent for each $x, y \in \widetilde{M}$. Thus they define a measure class of $\partial\widetilde{M}$, which is called the Lebesgue measure class.

(ii) **The Bowen-Margulis measure class**: In his paper [L1], Ledrappier constructed a family of probability measures $\{\mu_x\}_{x\in\widetilde{M}}$ on $\partial\widetilde{M}$ which are transversals (see [L1] for definition) of the Bowen-Margulis measure μ. He proved that there is a continous function F on $\widetilde{M}$ such that

$$\frac{d\mu_y}{d\mu_x}(\xi) = e^{-h\rho_{(x,-\xi)}(y)}\frac{F(y)}{F(x)}$$

for all $x, y \in \widetilde{M}$. Here $\rho_{(x,-\xi)}(y)$ is the Busemann function at ξ such that

$\rho_{(x,-\xi)}(x) = 0.$

(iii) **The harmonic measure class**: In the early 1980's, Anderson, Schoen [AS], and Sullivan [S1] solved the Dirichlet problem for $\partial\widetilde{M}$. Thus for each $x \in \widetilde{M}$, there exists a harmonic measure ν_x on $\partial\widetilde{M}$, such that for any continuous function f on $\partial\widetilde{M}$.

$$F(x) \stackrel{\text{def}}{=} \int_{S\widetilde{M}} f(\xi)d\nu_x(\xi)$$

gives a harmonic function on $\widetilde{M}$ whose restriction to $\partial\widetilde{M}$ is f. They proved that for each $x, y \in \widetilde{M}$, ν_x and ν_y are equivalent and

$$\frac{d\nu_y}{d\nu_x}(\xi) = k(x, y, \xi)$$

where $k(x, y, \xi)$ is a minimal positive harmonic function of y called the Poisson kernel.

For each $x \in \widetilde{M}$, almost all of the Brownian motions starting from x converge to a point in $\partial\widetilde{M}$ ([Pr]). It is well-known that the harmonic measure ν_x is exactly the hitting probability at $\partial\widetilde{M}$ of Brownian motions starting from x.

Corresponding to these three measure classes on $\partial\widetilde{M}$, there are three invariant ergodic probability measures on $S\widetilde{M}$ of the geodesic flow:

(i) The Liouville measure m, which is the unique equilibrium state of the function $\frac{d}{dt}|_{t=0} \det(dg^t|_{TW^u(v)})$.

(ii) The Bowen-Margulis measure μ, which maximizes the metric entropy. It is the unique equilibrium state of the zero function.

(iii) The harmonic measure ν, which is constructed by Ledrappier ([L3]) as the unique equilibrium state of the function

$$\tau(v) \stackrel{\text{def}}{=} \frac{d}{dt}|_{t=0} \log k(v(0), v(t), v(\infty)).$$

(1.3) Statement of results

Katok proved that if $\dim M = 2$, then the equivalence of any two of these three measure classes implies that M is a surface with constant negative curvature (See [K1], [K2]. See also [L2] for another independent proof.) In view of these facts, it makes sense to make the following conjecture:

Conjecture. *Let M be a closed Riemannian manifold of negative curvature. Then the equivalence of any two of the three measure classes (the harmonic, the Bowen-Margulis, the Liouville) implies that M is a locally symmetric space.*

A related famous conjection was made by Katok in 1982:

Katok's entropy rigidity conjecture. *The topological entropy of the geodesic flow equal to the metric entropy h_m, if and only if M is locally symmetric.*

At about the same time, Sullivan made his conjecture:

The Sullivan conjecture. *The harmonic measure class coincides with the Liouville measure class if and only if M is locally symmetric.*

In 1989, Ledrappier proved the following result

Theorem 1([L1]). *Let M be a closed Riemannian manifold of negative curvature. Then the horospheres in $\widetilde{M}$ have constant mean curvature if and only if the Bowen-Margolis measure μ_x and the harmonic measure ν_x coincide at each point $x \in \widetilde{M}$.*

In chapter II, we prove a weak version of the Sullivan conjecture.

Theorem 2([Y1]). *Let M be a closed Riemannian manifold of negative curvature. Then the horospheres in $\widetilde{M}$ have constant mean curvature if and only if the Lebesgue measure m_x and the harmonic measure ν_x coincide at each point $x \in \widetilde{M}$.*

As a corollary, we prove

Corollary ([Y1]). *With the above notations, if $\widetilde{M}$ is harmonic at one point, then the horospheres in $\widetilde{M}$ have constant mean curvature.*

Hamenstädt claimed she proved the following results in [H1] and [H2]:

Theorem 3([H2], 1989). *With the above notations, if the Bowen-Margulis measure class coincides with the harmonic measure class, then the horospheres in $\widetilde{M}$ have constant mean curvature.*

Theorem 4([H1], 1990). *With the above notations, if the Bowen-Margulis measure class coincides with the Liouville measure class, then the horospheres in $\widetilde{M}$ have constant mean curvature.*

Serious gaps have been found in both of her proofs (in [H2] by the author, in [H1] by Gilles Courtois). In both of her proofs, she relies heavily upon stochastic processes to obtain two crucial integral formulas. We give simple geometrical proofs of both lemmas. Although her proofs of both lemmas are correct, our results are in their most general form. One of these is stated in the following theorem (See [Y2] for more details concerning the notations).

Theorem 5([Y2]). *Let M be a closed Riemannian manifold of negative curvature, then for any function φ of class C^2_u on SM, we have*

$$\int_{SM} (\Delta^u \varphi + < \nabla^u \varphi, \nabla^u \log g >) dm = 0.$$

Using this formula, we obtain a functional description of those manifolds M which has horospheres with constant mean curvature:

Theorem 6([Y2]). *A compact manifold M with negative curvature has horospheres with constant mean curvature if and only if*

$$\int_{SM} \Delta^u \varphi dm = 0 \text{ for all } C^2_u \text{ functions } \varphi \text{ on } SM.$$

Finally, we study the ergodic properties of Anosov foliations. We prove that the strong stable and weak stable foliations of the geodesic flow on M are uniquely ergodic. If one denotes by w^{ss} the unique harmonic measure of the W^{ss}-foliation, by R and Ric the scalar and Ricci curvature of M, by R^H the scalar curvature of the horospheres, then we obtain the following

Theorem 7([Y3]). $h^2 = \int_{SM} (R^H(v) - R(\pi(v)) + Ric(v)) dw^{ss}(v)$

Using A. Connes' Gauss-Bonnet theorem for foliation we have

Corollary. *If* $\dim M = 3$, *then* $h^2 = \int_{SM} (Ric(v) - R(\pi(v)))dw^{ss}(v)$

With the help of various descriptions of the Bowen-Ledrappier-Margulis-Patterson-Sullivan measure μ_x at infinity, we study the Margulis's asymptotic formula $\lim_{R\to\infty} e^{-hR}S(x,R) = c(x)$ for the volume of geodesic spheres. We show that

Theorem 8 ([Y3]). *For a manifold M of negative curvature, if $c(x)$ is a constant function, then*

i) *For each x in $\widetilde{M}$, $h = \int_{\partial\widetilde{M}} trU(x,\xi)d\mu_x(\xi)$;*

ii) *If* $\dim M = 2$, *then M has constant negative curvature.*

The organization of this work is as follows. Since chapter II, chapter III and chapter IV use different methods, they are relatively independent. (They also reflect successive stages in the author's efforts.) In chapter II, we discuss the potential theory and Brownian motion on the universal covering $\widetilde{M}$. These are used in the proof of theorem 2 and its corollary.

Chapter III is dedicated to the integral formulas. We prove theorem 5 and another class of integral formulas. Many rigidity problems are discussed, including the proof of theorem 6.

In chapter IV, we study Lucy Garnett's ergodic theory for foliations, and various descriptions of the unique harmonic measure w^{ss}. These are used in the proof of theorem 7 and theorem 8.

II. Contribution to Sullivan's Conjecture

0. Introduction.

Throughout this chapter, we consider a $n-dim.$ closed Riemannian manifold M of negative sectional curvature. Let X be its universal covering. For each $x \in X$ the harmonic measure ν_x is the hitting probability measure of Brownian paths starting at x converging to the ideal boundary ∂X. It is well-known that ν_x and ν_y are equivalent for any x, $y \in X$ ([AS]). For any v in the unit tangent bundle SX, let $v(t)$ be the geodesic with initial velocity $\dot{v}(0) = v$ and let $v(\infty)$ be the point at ∂X determined by the geodesic $v(t)$. We denote by π the canonical projection from SM to M or from SX to X. For any $x \in X$, the restriction P_x of the canonical projection $P : SX \to \partial X, P(v) = v(\infty)$ is a homeomorphism from S_xX to ∂X which transports the canonical Lebesgue measure m_x on S_xX induced by the Riemannian metric to a measure on ∂X which we still denote by m_x. The absolute continuity of the stable foliation of the geodesic flow g^t on SM implies that m_x and m_y are equivalent for any two points $x, y \in X$. Thus we get two measure classes $\{\nu_x\}_{x\in X}$ and $\{m_x\}_{x\in X}$ which are called the harmonic class and the Lebesgue (or geodesic) class. Sullivan suggested that they are singular in general (see [S]). On the other hand, it is easy to see that if M is locally symmetric, then $m_x = \nu_x$ for all $x \in X$. The purpose of this paper is to prove

Theorem A. *If the harmonic measure and the Lebesgue measure coincide at each point $x \in X$: $\nu_x = m_x$, then the horospheres in X have constant mean curvature.*

If $dimM = 2$ or 3, this implies that M has constant curvature ([H3], [K2]). Ledrappier has a result parallel to [K2]. He constructed a family of Bowen-Margulis measures $\{\mu_x\}_{x\in X}$ (see section 2 of this paper) on ∂X and proved the following

Theorem. *If the harmonic measure and the Bowen-Margulis measure coincide at each point $x \in X$: $\nu_x = \mu_x$, then the horospheres in X have constant mean* curvature.

As a by-product of our machinery, we also prove a corollary of purely geometric flavour:

Corollary 2.2. *If the universal covering X of a compact negatively curved manifold is harmonic, then the horospheres in X have constant mean curvature.*

1. The Potential theory

Let $\triangle$ be the Laplace-Beltrami operator on X. By Anderson and Sullivan ([A], [S]), for any continuous function f on ∂X, there exists unique a function F on $X \cup \partial X$ such that

i) $\triangle F = 0$ on X

ii) $\lim_{x \to \xi} F(x) = f(\xi)$ for any $\xi \in \partial X$.

We denote by $C(\partial X)$ the space of continuous functions on ∂X. For each $x \in X$, the map

$$C(\partial X) \longrightarrow R : f \longmapsto F(x)$$

defines a positive linear functional on $C(\partial X)$. By the Riesz representation theorem, it defines a unique probability measure on ∂X. This is exactly the harmonic measure ν_x determined by the hitting probability of Brownian motion (see introduction) such that

$$F(x) = \int_{\partial X} f(\xi) d\nu_x(\xi)$$

According to [AS], given any $\xi \in \partial X$, there corresponds a unique function $k(x, y, \xi)$ such that

i) $k(x, x, \xi) = 1$

ii) $\triangle_y k(x, y, \xi) = 0$

iii) $\lim_{y \to \eta} k(x, y, \xi) = 0$ for any $\eta \in \partial X - \{\xi\}$ and for all $x, y \in X$, for $\nu_x - a.e. \xi \in \partial X$, we have

$$\frac{d\nu_y}{d\nu_x}(\xi) = k(x, y, \xi)$$

By [AS], X possesses a gloabally defined Green function $G(x,y)$, i.e.,

i) $G(x,y) = G_x(y)$ is harmonic on $X - \{x\}$.

ii) $G(x,y) \geq 0$

iii) $G_x(y) \sim \rho^{2-n}$ if $n > 2$, and $\sim \ln\frac{1}{\rho}$ if $n = 2$

there are constants $\sigma > 0, \delta > 0$ such that

$$\frac{1}{\sigma}e^{-\frac{1}{\delta}\rho(x,y)} \leq G(x,y) \leq \sigma e^{-\delta\rho(x,y)} \cdots\cdots\cdots\cdots (1.1)$$

for all $x \in X, y \in X - B(x,1)$. In particular, G_x extends continuously to $(X - \{x\}) \cup \partial X$ with zero boundary value on ∂X.

The Poisson kernel $k(x,y,\xi)$ is also given by

$$k(x,y,\xi) = \lim_{z\to\xi}\frac{G(x,z)}{G(y,z)}$$

We define the exponential growth rate of the Green function at any point ξ at infinity by

$$G(\xi) = \limsup_{T\to\infty} -\frac{1}{T}lnG(v(0),v(T))$$

where $v \in SX$ is any vectorsatisfies $v(\infty) = \xi$. It is easy to see that $G(\xi)$ does not depend on the choice of v. By (1.1)

$$\delta \leq G(\xi) \leq \frac{1}{\delta}$$

Lemma 1.1 For any $\xi \in \partial X$ and $v \in SX$ such that $v(\infty) = \xi$,

$$G(\xi) = \limsup_{R\to\infty}\frac{1}{R}lnk(v(0),v(R),v(\infty)) = \limsup_{R\to\infty}\frac{1}{R}lnk(v(R),v(0),v(-\infty))$$

Proof. According to [H1], there are numbers $C_1 > 0, C_2 > 0$ such that for all $v \in SX$ and $t \geq 1$, we have

$$C_1^{-1} \leq k(v(0),v(T),v(\infty))k(v(0),v(T),v(-\infty)) \leq C_1$$

$$C_2^{-1} \leq k(v(0),v(T),v(-\infty))G(v(0),v(T))^{-1} \leq C_2$$

which shows the lemma.

Remark: By the Harnack inequality at infinity (corollary 5.2 of [AS]), for any $\xi \in \partial X$ and any two positive harmonic functions u, v defined on a neighborhood U of ξ in $X \cup \partial X$ such that $u|_{U \cap \partial X} = 0 = v|_{U \cap \partial X}$, we have

$$\limsup_{T \to \infty} -\frac{1}{T} ln u(w(T)) = \limsup_{T \to \infty} -\frac{1}{T} ln v(w(T)) = G(\xi)$$

for any $w \in S(U - \partial X)$ such that $w(\infty) = \xi$. Thus $G(\xi)$ is nothing but the exponential growth rate of all positive harmonic functions vanishing on a neighborhood of ξ in ∂X. $G(\xi)$ reflects the asymptotic property of X. Yet, since X is the universal covering of a compact manifold M, one also expects $G(\xi)$ to provide information about the geometry of M.

The next proposition shows that the limit of $-\frac{1}{T} ln G(v(0), v(T))$ as T tends to infinity exists for a total probability set (defined below) on SX.

We first explain the meaning of total probability. According to the classical ergodic theory, there exists a decomposition

$$\partial X = (\cup_\alpha X_\alpha) \cup X_0$$

such that

i) For each α, there exists a unique ergodic measure μ_α of the geodesic flow g^t on SM such that for any $\xi \in X_\alpha$ and $v \in SM, v(\infty) = \xi$, we have
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(g^t v) dt = \int_{SM} f d\mu_\alpha$$
for any continuous function f on SM and

ii) Consider the set $M_0 = \{v \in SM | v(\infty) \in X_0\}$. For any g_t- invariant measure μ on SM, we have $\mu(M_0) = 0$. ($U_\alpha X_\alpha$ is called the total probability set)

Proposition 1.1. *For any α and $v \in SX$ such that $v(\infty) \in X_\alpha$,*

$$G(v(\infty)) = lim_{T\to\infty} - \frac{1}{T} lnG(v(0), v(T))$$

exists and $G(v(\infty)) \geq h_{\mu_\alpha}$, the metric entropy of the geodesic flow with respect to μ_α

Proof. Consider the following function on SX

$$\tau(v) = \frac{d}{dt}|_{t=0} lnk(v(0), v(t), v(\infty))$$

It is easy to see that $\tau(v)$ is invariant under the fundamental group of M , so it can be projected into a function on SM which we still denote by τ. By [AS], τ is Holder continuous and there exists unique equilibrium state ν, which we call once again the harmonic measure. The topological pressure of τ is zero by a result in [L2] . According to the variational principle, we have for any α

$$h_{\mu_\alpha} - \int \tau d\mu_\alpha \leq h_\nu - \int \tau d\nu = 0$$

Given any $v \in SX$ such that $v(\infty) \in X_\alpha$, by the Birkhoff ergodic theorem and lemma 1.1,

$$\begin{aligned}
&\lim_{T\to\infty} -\frac{1}{T} lnG(v(0), v(T)) \\
&= \lim_{T\to\infty} \frac{1}{T} lnk(v(0), v(T), v(\infty)) \\
&= \lim_{T\to\infty} \int_0^T \tau(g^t v)dt \\
&= \int \tau d\mu_\alpha \geq h_{\mu_\alpha}
\end{aligned}$$

The equality implies $h_{\mu_\alpha} = \int \tau d\mu_\alpha$. By the uniqueness of equilibrium state, $\mu_\alpha = \nu$.

2. Construction of three invariant measures and a corollary about harmonic manifold.

In the last section we introduced the harmonic measure ν as the unique equilibrium state of the function τ. We need to study the construction of ν and the other two natural invariant measures as well for later consideration.

Let $W^{ss}, W^{s}, W^{su}, W^{u}$ be respectively the strong stable, stable, strong unstable, unstable foliation of the geodesic flow. Fix a point $o \in X$, for any $v \in SX$, let ω_o be the measure on $W^{su}(v)$ which satisfies $\omega_o = \nu_o \circ P$, where $P : W^{su}(v) \to \partial X$ is the canonical projection. We then obtain a measure ν^{su} on $W^{su}(v)$ by

$$\frac{d\nu^{su}}{d\omega^o}(W) = k(o, w(0), w(\infty))$$

It is easy to see that ν^{su} does not depend on the choice of the base point o in X and transforms under the geodesic flow via

$$\frac{d\nu^{su}}{d\nu^{su} \circ g^{-t}}(g^t v) = k(v(0), v(t), v(\infty))$$

One can similarly obtain a family of measures ν^s on the leaves of W^s satisfying

$$\frac{d\nu^{s}}{d\nu^{s} \circ g^{-t}}(g^t v) = k(v(0), v(t), v(\infty))^{-1}.$$

By [H1], there exists a constant $\sigma > 0$ such that $\nu = \sigma d\nu^{su} \times d\nu^s$.

The second natural invariant measure is the Liouville measure m. We also denote by the same symbol m the normalized Riemannian volume on M. And we denote by m_x the normalized Lebesgue measure on S_xM. Then

$$\int_{SM} dm(v) = \int_M dm(x) \int_{S_xM} dm_x(v)$$

It is well-known that m is the equilibrium state of the Holder function

$$trU(v) = \frac{d}{dt}|_{t=0} ln det(dg^t|_{E^u})$$

or

$$trU(-v) = -\frac{d}{dt}|_{t=0} ln det(dg^t|_{E^s})$$

where $U(v)$ is the second fundamental form of the horosphere $\pi(W^{su}(v))$. And the metric on $E^u = TW^{su}(v)$ is the lift via $\pi : W^{su}(v) \to \pi(W^{su}(v))$ of the Riemannian metric on $\pi(W^{su}(v))$ induced by the Riemannian metric on M.

The third natural invariant measure, the Bowen-Margulis measure μ, is the unique equilibrium state of the zero function which realizes the topological entropy. By Margulis's construction ([M]), there is a family of conditional measures μ^i on the leaves of $W^i (i = ss, s, su, u)$ such that $d\mu^u = d\mu^{su} \times dt$ is invariant under the pseudo group of holonomy maps with respect to the strong stable foliation. Another characteristic property is its uniform expansion under the geodesic flow: $\mu^{su} \circ g^{-t} = e^{ht}\mu^{su}$. Moreover, μ^{ss} (resp. μ^s) are the images of the measures μ^{su} (resp. μ^u) under the flip map $v \mapsto -v$ and

$$d\mu = d\mu^{su} \times d\mu^s = d\mu^u \times d\mu^{ss}$$

In [L2], Ledrappier constructed a family of probability measures $\{\mu_x\}_{x \in X}$ on ∂X (or $S_x X$ via the projection P) which are transversals of μ such that there is a continuous function F on X satisfying

$$\frac{d\mu_y}{d\mu_x}(\xi) = e^{-h\rho_{(x,\xi)}(y)} \frac{F(y)}{F(x)}$$

where $\rho_{(x,\xi)}$ is the Busemann function at ξ with zero value at x. The equivalence class of these measures is called the Bowen-Margulis measure class.

Proposition 2.1. Let M be a closed negatively curvatured manifold . The following properties are equivalent:

(i) $G(\xi) \equiv h$

(ii) $G(\xi)$ is continuous at some point $\xi \in \partial X$

(iii) The Bowen-Margulis measure μ and the harmonic measure ν coincide

(iv) (Conjecture) The horospheres in X have constant mean curvature.

Proof. (i) $\Rightarrow (ii)$ obvious

(ii) $\Rightarrow (iii)$. Recall that all of the three measure classes are positive on open sets on ∂X. One can find sequences $\xi_k^m \to \xi_0, \xi_k^\mu \to \xi_0, \xi_k^\nu \to \xi_0$ such that for all $v \in SX, v(\infty) = \xi_k^i (i = m, \mu, \nu,)$, $g^t v$ reproduces the measures m, μ, ν as t tends to infinity. By proposition 1.1, we have

$$G(\xi_k^m) = \int \tau dm$$

$$G(\xi_k^\mu) = \int \tau d\mu$$

$$G(\xi_k^\nu) = \int \tau d\nu$$

for all k. Thus by the continuity at ξ_0, we have $\int \tau dm = \int \tau d\mu = \int \tau d\nu$ which implies $\mu = \nu$

$(iii) \Rightarrow (i)$. If the harmonic measure ν coincides with the Bowen-Margulis measure μ, then by the variational principal, there exists a continuous function f on SM such that $\tau(v) - h \equiv \frac{d}{dt}|_{t=0} f(g^t v)$ and our conclusion follows immediately.

Note: The above conditions should be equivalent to the condition that the horospheres in X have constant mean curvature. In fact, this is exactly the result of [H2]. But there is a gap in the proof there. So whether $\mu = \nu$ implies constant mean curvature still remains open.

Recall that a Riemannian manifold N is called harmonic at the point $n \in N$ if the function θ defined by

$$\theta(m) = \frac{\sigma_{exp_n^* g}}{\sigma_{g_n}}(exp_n^{-1} m)$$

(i.e., the quotient of the canonical measure of the Riemannian metric $exp_n^* g$ on $T_n N$ (pull back of g by the map exp_n) by the Lebesgue measure of the Eucilidean structure g_n on $T_n N$, depends only on the distance $d(n, m)$. N is called harmonic if it is harmonic for each $n \in N$. It is a well-known conjecture that all harmonic manifolds are locally symmetric. This has been confirmed when $dimN \leq 5$ or when the manifold is compact and simply connected. Using the dynamics of the geodesic flow, we can obtain the following corollary

Corollary 2.2. *If the universal covering X of a compact negatively curved manifold M is harmonic, then the horospheres of X have constant mean curvature.*

Note: It is easy to see that X is harmonic at $x_0 \in X$ if and only if the Green function $G(x_0, y)$ depends only on $d(x_0, y)$. The corollary would follow easily from proposition 2.1 if the coincidence of the harmonic and the Bowen-Margulis measure implies constant mean curvature of the horospheres.

Proof. The harmonicity of X implies that the Green function $G(x, y)$ depends only on $d(x, y)$ (By a result of Cheeger and Yau ([CY]): the heat kernel $P(x, y, t)$ of a harmonic manifold depends only on $d(x, y)$. On the other hand, we know that $G(x, y) = \int_0^\infty P(x, y, t)dt$. Since

$$k(x, y, \xi) = \lim_{z \to \xi} \frac{G(x, z)}{G(y, z)}$$

it follows that $\tau(v) = \frac{d}{dt}|_{t=0} lnk(v(0), v(t), v(\infty))$ is a constant function on SX. Since the harmonic and the Bowen-Margulis measure coincide, by the variational principle, there exists a function f on SM such that

$$\frac{d}{dt}|_{t=0} f(g^t v) = \tau(v) - h$$

The boundedness of f on SM implies that $\tau(v) - h$ must be zero. So the corollary follows from proposition 2 and theorem 1 of [L3].

The following conjecture seems a step closer than the entropy rigidity conjecture:

Conjecture. *The universal covering of a compact negatively curved manifold is harmonic if and only if its horospheres have constant mean curvature.*

3. Continuity of the Radon-Nikodym Derivative.

From now on we suppose the harmonic measure class and the Liouville measure class are equivalent, that is, $\nu = m$.

We use the notations of the above sections. For $x \in X$, $v \in SX$, there is a unique $\psi_x(v) \in S_xX$ such that $\psi_x(v)(\infty) = v(\infty)$. The map $\psi_x : v \to \psi_x(v)$ is continuous and for every $w \in SX$ its restriction to $W^{su}(w)$ is a homomorphism onto $S_xX - \{\psi_x(-w)\}$ which is absolutely continuous with respect to the Lebesgue measure. Its Jacobian with respect to m^{su} (induced by the Riemannian metric) on the horosphere $\pi(W^{su})$ and m_x on S_xX is given in [H1]:

Lemma 3.1. *For each x in X and v in S_xX we have*

$$\frac{dm^{su} \circ \psi_x^{-1}}{dm_x(v)} = det(U(v) + U(-v))^{-1}$$

We denote by $\rho(v)$ the Radon-Nikodym derivative of ν_x with respect to m_x:

$$\rho(v) = \frac{d\nu_x}{dm_x}(v)$$

whenever it exists and let $\rho(v) = \infty$ otherwise. Since ν_x and m_x are equivalent, $0 < \rho(v) < \infty$ for $m-$ a.e. $v \in SX$.

Lemma 3.2

$$\rho(v) = e^{f(v)}$$

where

$$\frac{d}{dt}|_{t=0} f(g^t v) = \tau(v) - trU(-v)$$

and f is of class C_s^∞

Note: We say that a function f on SM is of class C_s^j for some $j \in [o, \infty]$ if f restricted to every stable manifold is of class C^j and if the jets of order up to j of these restrictions are continuous on SM.

Proof. By definition

$$\rho(v) = \frac{d\nu_x}{dm_x}(v)$$

$$= \frac{d\nu_x}{d\nu^{su}\circ\psi_x^{-1}}(v)\cdot\frac{d\nu^{su}\circ\psi_x^{-1}}{dm^{su}\circ\psi_x^{-1}}(v)\cdot\frac{dm^{su}\circ\psi_x^{-1}}{dm_x}(v)$$

$$= k(v(0),v(0),v(\infty))\cdot\frac{d\nu^{su}}{dm^{su}}(v)\cdot det(U(v)+U(-v))^{-1}$$

$$= \frac{1}{det(U(v)+U(-v))}\cdot\frac{d\nu^{su}}{dm^{su}}(v)$$

By the same technique as in [H2], it is easy to see that the Radon-Nikodym derivative

$$\frac{d\nu^{su}}{dm^{su}}(v)$$

exists everywhere and it is continuous.Thus $\rho(v)$ exists everywhere and it is continuous. Moreover

$$\rho(g^t v) = \frac{1}{det(U(g^t v)+U(-g^t v))}\cdot\frac{d\nu^{su}}{dm^{su}}(g^t v)$$

$$= \frac{1}{det(U(g^t v)+U(-g^t v))}\cdot\frac{d\nu^{su}}{d\nu^{su}\circ g^{-t}}(g^t v)\cdot\frac{d\nu^{su}}{dm^{su}}(v)\cdot\frac{dm^{su}\circ g^{-t}}{dm^{su}}(g^t v)$$

$$= \frac{det(U(v)+U(-v))}{det(U(g^t v)+U(-g^t v))}\cdot\frac{k(v(0),v(t),v(\infty))}{det(dg^t|_{E^u})}\cdot\rho(v)$$

Let $Y_u(t)$ ($Y_s(t)$) be the fundamental unstable (stable) matrix solution of the Jacobi equation $\ddot{Y}(t)+R(t)Y(t)=o$ with respect to a parallel orthonormal basis $\{e_i(t)\}_{i=1}^{n-1}$ along the geodesic $v(t)$ such that $Y_u(0)=Y_s(0)=I$. By the Sturm-Liouville formula

$$\dot{Y}_u^*(t)Y_s(t)-Y_u^*(t)\dot{Y}_s(t)\equiv\dot{Y}_u^*(0)Y_s(0)-Y_u^*(0)\dot{Y}_s(0)=U(v)+U(-v)$$

but the left hand side is equal to

$$= Y_u^*(t)[(\dot{Y}_uY_u^{-1})^*-(\dot{Y}_sY_s^{-1})]Y_s(t)$$

$$= Y_u^*(t)[U(g^t v)+U(-g^t v)]Y_s(t)$$

Hence we have

$$det(U(g^t v)+U(-g^t v))det(dg^t|_{E^u})det(dg^t|_{E^s}) = det(U(v)+U(-v))$$

and

$$\rho(g^t v) = k(v(0), v(t), v(\infty)) det(dg^t|_{E^s})\rho(v)$$

By the variational principle, there exists a function f on SM such that

$$\frac{d}{dt}|_{t=0} f(g^t v) = \tau(v) - trU(-v)$$

So we have $\rho(g^t v) = e^{f(g^t v) - f(v)}\rho(v)$ which means that ρe^{-f} is g^t–invariant. By the ergodicity of the geodesic flow and the continuity of ρe^{-f}, we have

$$\rho(v) \equiv e^{f(v)}$$

after normalizing f by a constant.

The C_s^∞-regularity of f comes from the smooth-Livsic theorem (lemma 2.2 of [LMM]) and the fact that both $trU(-v)$ and $\tau(v)$ are of C_s^∞–class.

4. Brownian Motion on X and Sullivan's Conjecture.

Let $\Omega = C(R_+, M)$ be the space of continuous paths in M, and $\{P_x, x \in M\}$ the family of probability measures on Ω which describe the Brownian motion on M. Let $\widetilde{\Omega} = C(R_+, X)$ be the space of continuous paths in X and $\Pi : X \to M$ the covering map. For every $x \in X$ and all $\omega \in \Omega$ such that $\omega(0) = \Pi(x)$, there is a unique path $\tilde{\omega} \in \widetilde{\Omega}$ such that

$$\Pi\tilde{\omega}(t) = \omega(t)$$

for all $t \geq 0$.

We denote by $(r(\omega, t), \theta(\omega, t))$ the polar coordinates about x of the path $\tilde{\omega}(t)$. Note that for all $x \in X$ and $P_{\Pi x}$–a.e. $\omega \in \Omega$ we have $\omega(0) = \Pi x$. So (r, θ) is defined $P_{\Pi x}$–a.e.

Let $P(t, x, y)$ be the heat kernel on X, i.e., the fundamental solution of

$$\frac{\partial u}{\partial t} = \triangle u$$

The density of the distribution of the Brownian paths at time t under $P_{\Pi x}$ with respect to the Riemannian volume is $P(t,x,y)$.

By [Pr], [L1], [Kai1], for all X in X and $P_{\Pi x}$-a.e. $\omega \in \Omega$, $\theta(\omega,t)$ converges as t tends to ∞ towards some point $\theta(\omega,\infty)$ in ∂X and

$$\lim_{t\to\infty} \frac{1}{t} d(\tilde{\omega}(t),(r(\omega,t),\theta(\omega,\infty))) = 0$$

We list two asymptotic invariants of the Brownian motion:

1) α , the growth rate of distance along the Brownian paths. It was proved in [Kai2] that for every x in X and $P_{\Pi x}$-a.e. $\omega \in \Omega$, $\lim_{t\to\infty} \frac{1}{t} r(\omega,t) = \alpha$ and

$$\alpha = \int_M dm(x) \int_{\partial X} trU(-(x,\xi))d\nu_x(\xi)$$

where (x,ξ) is the vector v in S_xM such that $v(\infty) = (x,\xi)$

2) β,the exponential decay rate of heat kernel along the Brownian paths. For every x in X and $P_{\Pi x}$-a.e. $\omega \in \Omega$ we have

$$\lim_{t\to\infty} \frac{1}{t} lnP(t,x,\tilde{\omega}(t)) = -\beta$$

By [L2], [Kai2]

$$\beta = h_\nu \alpha = \int_M dm(x) \int_{S_xM} || \nabla lnk(v(0),\cdot,v(\infty))||^2 d\nu_x(v)$$

Now we are ready to prove the following

Theorem 4.1. The horospheres of the universal covering X of a compact negatively curvatured manifold M have constant mean curvature if and only if the harmonic measure ν_x coincides at each point with the Lebesgue measure m_x.

Proof.

1). If the horospheres in X have constant mean curvature h, then $\tau(v) = trU(-v) = h$ for all v by the result of [L3] . It follows from lemma 3.2 that $\nu_x = m_x$ for all x.

2). Recall that the Liouville measure m is given by

$$\int_{SM} dm(v) = \int_M dm(x) \int_{S_x M} dm_x(v)$$

If ν_x and m_x coincide at each point, then we have $trU(-v) \equiv \tau(v)$. By lemma 3.2 and

$$\alpha = \int_{SM} trU(-v)dm(v) = \int_{SM} \tau(v)dm(v) = h_\nu$$

$$\beta = \int_{SM} || \bigtriangledown lnk||^2 dm$$

By the Schwartz inequality,

$$\beta \geq (\int_{SM} || \bigtriangledown lnk||dm)^2$$

with equality if and only if $|| \bigtriangledown lnk|| \equiv C$

But $\beta = h_\nu \alpha = (\int_{SM} \tau(v)dm)^2$ and

$$(\int_{SM} || \bigtriangledown lnk||dm)^2 \geq (\int_{SM} \frac{d}{dt} lnkdm)^2 = (\int_{SM} \tau(v)dm)^2$$

So we do have equality. So $|| \bigtriangledown lnk|| \equiv h_\nu$ and

$$trU(-v) = \tau(v) \leq || \bigtriangledown lnk|| = h_\nu$$

By the variational principle

$$h - \int \tau(v)d\mu \leq h_\nu - \int \tau(v)d\nu = 0$$

So $h \leq \int \tau(v)d\mu \leq h_\nu$. Thus the Bowen-Margulis measure and the harmonic measure coincide.

By proposition 2 of [L3], there exists a function f on SM such that for all x, y in X and all ξ in ∂X (we denote the lift of f to SX by the same notation)

$$k(x, y, \xi) = e^{-h\rho_{x,\xi}(y) + f(y,\xi) - f(x,\xi)}$$

If we denote by Y the geodesic spray then $|| \nabla \ln k||^2 = ||hY + \nabla f||^2 = h^2 + || \nabla f||^2 + 2h\frac{d}{dt}|_{t=0} f(g^t v)$ thus

$$\beta = \int || \nabla \ln k||^2 dm = h^2 + \int || \nabla f||^2 dm = h^2$$

and $\int || \nabla f||^2 dm = 0$ which implies that f is a constant function and $\tau(v) \equiv trU(v) \equiv h$.

Added in proof: I received an oral communication from Dr. Hamenstadt that she has corrected her proof in [H2]. So all the properties in proposition 2.1 are equivalent to constant mean curvature of horospheres. Also the result of corollary 2.2 can be improved by requiring only that X is harmonic at one point.

III. Integral Formulas for the Laplacian along the Unstable Foliation and Applications to Rigidity Problems For Manifolds of Negative Curvature

1. Introduction. Statement of results.

Let X be a C^1-vector field on a complete Riemannian manifold N. Let dv be the volume element of N, and $\{\phi_t\}_{t\in\mathbb{R}}$ be the induced flow of X on N. Fix any compact set K in N, and set

$$V(t) = \int_{\phi_t(K)} dv$$

then standard calculation shows that

$$\dot{V}(0) = \int_K div\ X\ dv$$

Thus the divergence of X measures the infinitisimal distortion of volume by the flow generated by X

Divergence Theorem. *If X is a C^1 vector field on N with compact support, then*

$$\int_N div\ X\ dv = 0$$

Remember that $div(h \cdot \nabla f) = h\Delta f + < \nabla h, \nabla f >$ ($\nabla f \stackrel{\text{def}}{=} \text{grad} f$ and $\Delta f \stackrel{\text{def}}{=} div(\nabla f)$), we also have

Greens's Formula. *Let $h \in C^1$, $f \in C^2$ be functions on N such that $h\nabla f$ has compact support. Then*

$$\int_N (h\Delta f + < \nabla h, \nabla f >) dv = 0$$

These formulas are of basic importance in many different areas of Riemannian geometry. In studying rigidity problems for manifolds of negative curvature, we come across a class of integral formulas involving the Laplacian or the divergence along leaves of foliation. These formulas play a cruical role in many rigidity problems concerning manifolds of negative curvature, besides being of interest in their own right.

Throughout this chapter we consider a compact n-dimensional Riemannian manifold M, with negative sectional curvatures. We denote by

- $\widetilde{M}$ the universal cover of M.
- SM (resp. $S\widetilde{M}$) the unit tangent bundle of M (resp. $\widetilde{M}$).
- g^t the geodesic flow on SM or $S\widetilde{M}$, and X is the geodesic spray generating g^t.
- $\pi : S\widetilde{M} \to \widetilde{M}$ the canonical projection.
- $\partial\widetilde{M}$ the ideal boundary of $\widetilde{M}$.
- $v(t) = \pi(g^t v)$,
- $v(\infty)$ is the point at $\partial\widetilde{M}$ determined by the geodesic $v(t)$.
- $P : S\widetilde{M} \to \partial\widetilde{M}$ is the projection $P(v) = v(\infty)$ and $P_x : S_x\widetilde{M} \to \partial\widetilde{M}$ is the restriction of P to $S_x\widetilde{M}$.
- μ is the Bowen-Margulis measure.
- ν is the harmonic measure (see [L1]).
- m is the Liouville measure on SM or $S\widetilde{M}$, normalized so that $m(SM) = 1$. We also denote by $dm(x)$ the Riemannian volume on M or $\widetilde{M}$, normalized such that $m(M) = 1$.
- (x, ξ) the vector v in S_xM such that $v(\infty) = \xi$.
- ρ_v the Busemann function at $v(-\infty)$ such that $\rho_v(v(0)) = 0$.
- SD the unit tangent bundle over any set D in $\widetilde{M}$.

We also denote by W^{su}, W^u, W^{ss}, W^s the strong unstable, unstable, strong stable, stable foliations of the geodesic flow. The canonical projection $\pi : S\widetilde{M} \to \widetilde{M}$

maps $W^u(v)$ or $W^s(v)$ diffeomorphically onto $\widetilde{M}$. Thus the Riemannian metric $<\ ,\ >$ on $\widetilde{M}$ lifts to a Riemannian metric g^i on $W^i(v)$ which induces a Lebesgue measure $m^i (i = su,\ u,\ ss,\ s)$.

We say a function φ on SM (or $S\widetilde{M}$) is of class C_u^j (resp. C_s^j) for some integer $j \in [0, \infty]$, if φ restricted to every unstable (resp. stable) leaf is of class C^j and if the j-jets of these restrictions are continuous on SM (or $S\widetilde{M}$).

Let Δ, ∇, div denote the Laplacian, the gradient, and the divergence operator on $\widetilde{M}$. The metric g^u (resp. g^i, $i = s, su, ss$) induces on each unstable leaf W^u (resp. each W^i leaf, $i = s, su, ss$) a Laplacian, gradient and divergence operator which we denote by the symbols Δ^u, ∇^u, div^u (resp. Δ^i, ∇^i, div^i). Whenever we specify a function φ to be of Class C_u^2 (or C_i^2), we denote by $\Delta^u\varphi$ (resp. $\Delta^i\varphi$) the Laplacian of the restriction of φ to the leaves of W^u (resp. W^i) with respect to g^u (resp. g^i). We also denote by $\nabla^u\varphi$ (resp. $\nabla^i\varphi$) the gradient of φ along the W^u(resp. W^i) leaves whenever we specify φ to be of class C_u^1 (or C_i^1).

The geodesic flow g^t acting on SM is an Anosov flow which preserves the Liouville measure m. For any measurable partition ξ on SM subordinate to the W^u-foliation, i.e., for $m - a.e.\ v \in SM$

(1) $\xi(v) \subset W^u(v)$ and

(2) $\xi(v)$ contains a neighborhood of v open in the submanifold topology of $W^u(v)$

we have a canonical system of conditional measures $\{m_v^\xi\}_{v \in SM}$ associated with ξ, such that for any measurable set A in SM, we have $m(A) = \int m_v^\xi(A) dm(v)$.

Let ρ be the density of m_v^ξ with respect to m^u. We prove that there is a C_u^∞ function g defined globally on $S\widetilde{M}$ such that $\nabla \log g$ is invariant under the group $\Gamma = \pi_1(M)$ of deck transformations and g coincides with ρ locally up to a scalar multiplication.

Theorem 1.

(i) *For any function* φ *of class* C_u^2 *on* SM *we have* $\int\limits_{SM}(\Delta^u\varphi + < \nabla^u\varphi, \nabla^u \log g >)dm = 0$.

(ii) *For any* C_u^2 *function* φ *on* $S\widetilde{M}$ *with compact support*
$\int\limits_{S\widetilde{M}}(\Delta^u\varphi + < \nabla^u\varphi, \nabla^u \log g >)dm = 0$.

Formula (ii) includes Lemma 5.3 of [H1] as a special case. See section 7 for more details.

The second class of our integral formulas is related to another foliation of $S\widetilde{M}$: $S\widetilde{M} = \{S_x\widetilde{M}\}_{x\in\widetilde{M}}$. Let $\{\sigma_x\}_{x\in\widetilde{M}}$ be a family of finite Borel measures on $S\widetilde{M} = \{S_x\widetilde{M}\}$. Via the canonical projection $P_x : S_x\widetilde{M} \to \partial\widetilde{M}$, $\{\sigma_x\}_{x\in\widetilde{M}}$ can be viewed as a family of measures on $\partial\widetilde{M}$ and one can talk about their Radon-Nikodym derivatives with respect to one another.

Theorem 2. *If for any* x *in* $\widetilde{M}$, *the Radon-Nikodym derivative*

$$\frac{d\sigma_y}{d\sigma_x}(\xi) \stackrel{\text{def}}{=} \sigma(x, y, \xi)$$

is a C_s^2 *function of* $(y, \xi) \in S\widetilde{M}$, *then*

(i) *For any* C_s^2 *function* φ *on* $S\widetilde{M}$ *of compact support,*

$$\int\limits_{S\widetilde{M}} [\Delta^s\varphi + \varphi(\|\nabla^s \log\sigma\|^2 + \Delta^s \log\sigma) + 2 < \nabla^s \log\sigma, \nabla^s\varphi >]dm(x)d\sigma_x = 0$$

(ii) *If in addition,* $\{\sigma_x\}_{x\in\widetilde{M}}$ *is invariant under the fundamental group* Γ *of* M, *then for any* C_s^2 *function* φ *on* SM

$$\int\limits_{SM} [\Delta^s\varphi + \varphi(\|\nabla^s \log\sigma\|^2 + \Delta^s \log\sigma) + 2 < \nabla^s \log\sigma, \nabla^s\varphi >]dm(x)d\sigma_x = 0$$

This formula is more general than the ones that appeared in [L1] and [H2].

Using these formulas, we give a functional description of compact manifolds of negative curvature whose horospheres have constant mean curvature.

Theorem 3. *A compact manifold M with negative curvature has horospheres with constant mean curvature if and only if*

(i) $\int_{SM} \Delta^u \varphi dm = 0$ *for all C_u^2 function φ on SM, or*

(ii) $\int_{S\widetilde{M}} \Delta^u \varphi dm = 0$ *for all C_u^2 function φ on $S\widetilde{M}$ with compact support.*

We also consider the horospherical foliation W^{su} and obtain the following theorem:

Theorem 1′. *Suppose M is a compact manifold of negative curvature, then*

(i) *For any function φ of class C_{su}^2 on SM we have*

$$\int_{SM} (\Delta^{su}\varphi + < \nabla^{su}\varphi, \nabla^{su}\log g >) dm = 0$$

(ii) *For any function φ of class C_{su}^2 on $S\widetilde{M}$ with compact support*

$$\int_{S\widetilde{M}} (\Delta^{su}\varphi + < \nabla^{su}\varphi, \nabla^{su}\log g >) dm = 0$$

Using this theorem, we give another functional description of compact manifolds of negative curvature whose horospheres have constant mean curvature.

Theorem 3′. *A compact manifold M with negative curvature has horospheres with constant mean curvature if and only if*

(i) $\int_{SM} \Delta^{su}\varphi dm = 0$ *for all C_{su}^2 functions φ on SM, or*

(ii) $\int_{S\widetilde{M}} \Delta^{su}\varphi dm = 0$ *for all C_{su}^2 functions φ on $S\widetilde{M}$ with compact support.*

There are three natural invariant measures of g^t on SM: the harmonic measure ν, the Bowen-Margulis measure μ, the Liouville measure m. A general conjecture is that the coincidence of any two of these three measures implies that M is locally symmetric. This has been confirmed in dimension 2 by Katok ([K2]) and Ledrappier ([L2]). The earliest conjecture is given by Katok as the entropy rigidity conjecture.

Katok's entropy rigidity conjecture. *The topological entropy h of the geodesic flow equals to the metric entropy h_m, if and only if M is locally symmetric.*

He first proved this to be true if $\dim M = 2$ ([K2]) by using conformal geometry. Hamenstadt ([H1]) proved the conjecture in dimension 3 by using the following theorem:

Theorem 4 ([H1]). *If $h = h_m$ then the horospheres in M have constant mean curvature.*

A major part of her paper consists of the proof of Lemma 5.3 which relies heavily upon stochastic processes. Our approach provides a shorter geometric proof of her Lemma. Although a serious gap is found at the end of her paper, we feel that our proof is sufficiently different from hers and the Lemma has a special flavour to merit an independent treatment.

Another famous conjecture is the Sullivan's conjecture.

Sullivan's conjecture. *The harmonic measure ν coincides with the Liouville measure m if and only if M is locally symmetric.*

This conjecture has been proved in dimension 2 by Katok ([K2]) and Ledrappier ([L2]). A weaker version in higher dimension is discussed in [Y1]. We give two integral formulas under the assumption that $\nu = m$ which might be useful for further investigation.

We also consider the case when the harmonic measure ν and the Bowen-Margulis measure μ coincide. We provide a simple proof of an important lemma in

[H2].

2. Laplacian along the unstable foliation. Proof of theorem 1.

For the sake of convenience, we take a Markov parition $P = \{P_i\}_{i=1}^{k}$ of the geodesic flow, such that each P_i is a u-parallelepiped and ∂P has zero Liouville measure (Markov paritions for transitive Anosov flow were constructed by Ratner. See [Ra] for more details). We then define two new partitions ξ and η by

$$\xi(v) \stackrel{\text{def}}{=} P(v) \cap W^{u}_{\text{loc}}(v)$$

(which is the connected piece of $W^u(v)$ containing v)

$$\eta(v) \stackrel{\text{def}}{=} P(v) \cap W^{su}_{\text{loc}}(v)$$

(which is the connected piece of $W^{su}(v)$ containing v)

It is easy to see that ξ (resp. η) is a measurable partition subordinate to the W^u-foliation (resp. W^{su}-foliation) and for all $v \in intP_i$, $g^t(\eta(v)) = \eta(g^t v)$ for t small enough (See [LY] for more details). We denote by $\{m_v^\xi\}_{v\in SM}$ and $\{m_v^\eta\}_{v\in SM}$ the canonical systems of conditional measures associated with ξ, η. Let $\rho(v)$ be the Radon-Nikodym derivative of m_v^η with respect to the induced Riemannian measure m_v^{su}:

$$\rho(v) \stackrel{\text{def}}{=} \frac{dm_v^\eta}{dm_v^{su}}(v)$$

Since $\xi < \eta$, according to the classical measure theory (See [Ro]), $\{m_w^\eta\}_{w\in\xi(v)}$ also constitute the canonical family of conditional measures with respect to the partition $\{\eta(w)\}_{w\in\xi(v)}$ of $\xi(v)$. By proposition 11.1 of [KS] or Proposition 4.1 of [LS], we have

$$\rho(v) = \frac{dm_v^\xi}{dm_v^u}(v).$$

We lift everything up to $\widetilde{SM}$ and denote the corresponding objects by the same symbols. We then get partitions P, ξ, η and a function ρ on $\widetilde{SM}$.

Remark. The properties of Markov partition we are actually using are

(i) For each $v \in int(P_i)$, $g^t(\eta(v)) = \eta(g^t v)$ for t small enough. This is used in proving proposition 2.2.

(ii) $int(P_i)$ is a nice open set which facilitates our computation in Theorem 1 and Theorem 3.

Lemma 2.1. *ρ is of class C_u^∞ locally.*

Proof. According to lemma 2.5 in [LMM], there is a local parametrization of the W^u-foliation $\Lambda_u : U \times V \to SM$ (where $U \subset \mathbb{R}^{n+1}$ and $V \subset \mathbb{R}^n$ are open sets), such that

(i) Λ_u is a homeomorphism from $U \times V$ to an open set in $int(P(v))$ containing v.

(ii) For each $y \in V$, $\Lambda_{u,y} : U \to SM$ given by $\Lambda_{u,y}(x) = \Lambda_u(x,y)$ is a C^∞ immersion whose image is an open subset of a leaf of the W^u-foliation and, moreover, for any α, $\partial^\alpha \Lambda / \partial x^\alpha$ is continuous on $U \times V$.

(iii) $\Lambda_u^*(dm) = \lambda_u(x,y)dxdy$, $\lambda_u \in C_u^\infty(U \times V)$.

By proposition 11.1 of [KS], we have

$$\Lambda_{u,y}^*(dm_v^\xi) = \frac{\lambda_u(x,y)}{\int\limits_U \lambda_u(x,y)dx} dx$$

But since m_v^u is a smooth measure on $W^u(v)$ (See the note following Proposition 2.2 and corollary 2.3 for more details), by the property of Λ_u, we have

$$\Lambda_{u,y}^*(dm_v^u) = \beta_u(x,y)dx, \text{ for some function } \beta_u \in C_u^\infty(U \times V)$$

Thus $\Lambda_u^*(\rho) = \frac{\Lambda^* u,y(dm_v^\xi)}{\Lambda^* u,y(dm_v^u)} = \frac{1}{\int\limits_U \lambda_u(x,y)dx} \frac{\lambda_u(x,y)}{\beta_u(x,y)}$ and it is a function in $C_u^\infty(U \times V)$, which implies, by the remark following Definition 2.2 in [LMM], that ρ is of class C_u^∞ locally.

Proposition 2.2. *There exists a globally defined function g of class C^∞_u on $\widetilde{SM}$ such that g coincides with ρ on each $\xi(v)$ up to a scalar multiplication, and $\nabla^u \log g$ is invariant under the deck transformations of M.*

Proof. Compare our proof with that of [LS]. Fix v_0 in $\widetilde{SM}$. As $P(v_0)$ is a u-parallelepiped, $g^{-t}(\eta)(v) = \eta(v)$ for all $v \in int\, P(v_0)$ and t small enough, and we can write for any Borel subset K in the interior of $P(v_0)$

$$m_v^{g^{-t}\eta}(K) = \frac{m_v^\eta(K \cap (g^{-t}\eta)(v)}{m_v^\eta((g^{-t}\eta)(v))} = \int_{\eta(v)\cap K} \rho(w)dm_v^{su}(w).$$

Secondly by the invariance of m we also have

$$\begin{aligned} m_v^{g^{-t}\eta}(K) &= m_{g^t v}^\eta(g^t K) \\ &= \int_{\eta(g^t v)\cap g^t K} \rho(w)dm_{g^t v}^{su}(w) \\ &= \int_{g^t(g^{-t}(\eta)(v)\cap K)} \rho(w)dm_{g^t v}^{su}(w) \\ &= \int_{g^{-t}(\eta)(v)\cap K} \rho(g^t w)dm_{g^t v}^{su}(g^t w) \\ &= \int_{\eta(v)\cap K} \rho(g^t w)J_t^u(w)dm_v^{su}(w) \end{aligned}$$

where $J_t^u(w) \stackrel{\text{def}}{=} \det(dg^t|_{TW^{su}(w)})$, with respect to the metric g^{su}.

Consequently we get for m almost every $w \in \rho(v_0)$,

$$\rho(w) = \rho(g^t w)J_t^u(w).$$

Now we can define g on $\widetilde{SM}$ as follows:

(i) For all $t \in \mathbb{R}$, $g(g^t v_0) \stackrel{\text{def}}{=} J_t^u(v_0)^{-1}$. In particular, $g(v_0) = 1$;

(ii) For all $v \in W^{su}(v_0)$, $g(v) \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{J_t^u(g^{-t}v_0)}{J_t^u(g^{-t}v)}$;

(iii) For all $v \in W^{ss}(v_0)$, $g(v) \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{J_t^s(v)}{J_t^s(v_0)}$;

(iv) For each $w \in \widetilde{SM}$, by the transversality of the two foliations W^u and W^{ss}, there exists a unique $v \in W^u(w) \cap W^{ss}(v_0)$, and a number

$t_0 \in \mathbb{R}$, such that $g^{t_0}v \in W^{su}(w)$. We define $g(g^{t_0}v) = J^u_{t_0}(v)^{-1}g(v)$ and

$$g(w) = g(g^{t_0}v)\lim_{t\to\infty}\frac{J^u_t(g^{t_0-t}v)}{J^u_t(g^{-t}w)}$$

It is easy to see from this construction that

(1) g coincides with ρ on each $\xi(v)$ up to a scalar multiple $c(v)$;

(2) g is of class C^∞_u since ρ is of that class locally;

(3) $\nabla^u \log g$ is invariant under the deck transformations and thus it can be projected to a vector field on SM.

Note: Identify $T_vTM = T_{\pi(v)}M \oplus T_{\pi(v)}M$ via the map $\xi \to (d\pi(\xi), K(\xi))$, where $K : TTM \to TM$ is the Riemannian connection. The splitting induces a canonical metric on TTM given by

$$<< \xi, \eta >> \stackrel{\text{def}}{=} < d\pi(\xi),\ d\pi(\eta) > + < K(\xi),\ K(\eta) >,$$

where $<,>$ denotes the metric on TM. In the same way, we obtain a canonical metric on TSM. Let $\widetilde{g}^i$ be the metric on each $W^i (i = ss, s, su, u)$ leaf induced by this canonical metric. And let $\widetilde{m}^i$ be the corresponding volume on each W^i-leaf.

Now let $\{e_i(t)\}$ $(i = 1, \cdots, n)$ be the system of vector fields obtained by parallel displacement of an orthonomal system at $v(0)$, where $e_n(t) = \dot{v}(0)$. Let $Y_i(t)$ be the Jacobi field along $v(t)$ satisfying

$$Y_i(0) = e_i(0), \qquad \dot{Y}_i(0) = U_v(e_i(0))$$

where U_v is the second fundamental form at $v(0)$ of the horosphere $\rho_v^{-1}(0)$. Then

$$\{(Y_i(0),\ \dot{Y}_i(0))\}_{i=1}^{n-1}$$

gives a basis of $T_vW^{su}(v)$ and

$$dg^t|_{T_vW^{su}(v)}(Y_i(0),\ \dot{Y}_i(0)) = (Y_i(t),\ U_{g^tv}Y_i(t)).$$

Thus the determinant of $dg^t|_{T_v W^{su}(v)}$ with respect to the canonical metric $\widetilde{g}^{su}$ is given by

$$(2.1) \qquad \widetilde{J}^u_t(v) = \frac{\det(I + U^2_{g^t v})}{\det(I + U^2_v)} J^u_t(v) \quad \text{where } I \text{ is the identity operator.}$$

And it is also easy to see that

$$d\widetilde{m}^{su}(v) = \det(I + U^2_v) dm^{su}(v).$$

Let us define

$$\widetilde{\rho}(v) = \frac{dm^\eta_v}{d\widetilde{m}^{su}_v}(v)$$

then $\widetilde{\rho}(v) = \det(I + U^2_v)\rho(v)$. By Lemma 3.2 of [H1], U_v is an operator of class C^∞_u. Thus $\widetilde{\rho}$ and ρ have the same regularity.

Actually, if we consider the "partially hyperbolic" diffeomorphism g^1 on SM, by theorem 3 of [PS], it is easy to see that

$$\begin{aligned} \frac{\widetilde{\rho}(w)}{\widetilde{\rho}(v)} &= \lim_{n\to\infty} \frac{\widetilde{J}^u_n(g^{-n}v)}{\widetilde{J}^u_n(g^{-n}w)} = \lim_{t\to\infty} \frac{\widetilde{J}^u_t(g^{-t}v)}{\widetilde{J}^u_t(g^{-t}w)} \\ &= \frac{\rho(w)}{\rho(v)} \frac{\det(I + U^2_w)}{\det(I + U^2_v)} \end{aligned}$$

Since the Liouville measure m is the unique equilibrium state of the function

$$\varphi^u(v) \stackrel{\text{def}}{=} \frac{d}{dt}|_{t=0} \log\det \widetilde{J}^u_t(v)$$

And we have by (2.1)

$$\varphi^u(v) - trU_v = \frac{d}{dt}|_{t=0} \log\det(I + U^2_{g^t v}).$$

Here $\det(I + U^2_v)$ is a well-defined function on SM. According to the variational principle, we have the following

Corollary 2.3. *The Liouville measure m is the unique equilibrium state of the function $tr\,U$.*

Recall the definitions of Δ^u, ∇^u, div^u. We need the following lemma:

Lemma 2.4. *Let φ be any function of class C_u^J, then $\Delta^u\varphi$ is a function of class C_u^{J-2} and $\nabla^u\varphi$ is a vector field of class C_u^{J-1}. Let X be any vector field of class C_u^J, then $div^u(X)$ is a function of class C_u^{J-1}.*

Proof. We consider only $\Delta^u\varphi$ (the other cases are similar). Let $x : U \to M$ be a local chart on M (where $U \subset \mathbb{R}^n$ is a open set). We denote the corresponding vector fields $\frac{\partial}{\partial x^i}$ on M by ∂^i, $1 \leq i \leq n$. The Christoffel symbols Γ_{ij}^k are determined by

$$D_{\partial^j}\partial^i = \Sigma\Gamma_{ij}^k\partial_k$$

For the Riemannian metric, define

$$g_{ik} = < \partial_i, \partial_k >, \; G = (g_{ik}),$$

$$g = \det G, \quad G^{-1} = (g^{jk})$$

Then the Christoffel symbols Γ_{ij}^k and the functions g, g^{jk}, lift to local smooth functions on SM which we denote by the same symbols. Remember that the projection $\pi : S\widetilde{M} \to \widetilde{M}$ maps $W^u(v)$ diffeomorphically onto $\widetilde{M}$. One can lift the vector fields ∂^i via π to each W^u-leaf locally to get local vector fields on $S\widetilde{M}$ which we denote by the same symbols. By the argument in Lemma 2.1, it is easy to see that the ∂^i's are local vector fields on $S\widetilde{M}$ of class C_u^∞. We have for each C_u^J function φ on SM

$$\Delta^u\varphi = \frac{1}{\sqrt{g}}\sum_{j,k}\partial_j(g^{jk}\sqrt{g}\partial_k\varphi)$$

which implies that $\Delta^u\varphi$ is of class C_u^{J-2}.

Now we are ready to prove

Theorem 1.

(i) *For any C_u^2 function φ on SM we have*

$$\int_{SM}(\Delta^u\varphi + < \nabla^u\varphi, \nabla^u \log g >)dm = 0$$

(ii) *For any C_u^2 function φ on $S\widetilde{M}$ with compact support,*

$$\int_{S\widetilde{M}}(\Delta^u\varphi + < \nabla^u\varphi, \nabla^u \log g >)dm = 0.$$

Proof. (i) By definition, Δ^u and ∇^u are the lifts of the Laplacian and gradient on $\widetilde{M}$ to $W^u(v)$ via the projection π. In particular, $\nabla^u\varphi$ can be viewed as a C^1_u vector field on SM. Denote by φ_t the flow on SM induced by this vector field. Note that the W^u leaves are φ_t-invariant.

As in the proof of the divergence theorem, take any compact subset K in SM such that $K \subset \operatorname{int}(P_i)$ for some u-parallelepiped P_i of our Markov partition. If t is small enough, then $\varphi_t K$ remains in $\operatorname{int}(P_i)$. Using the partition ξ and the family of conditional measures $\{m_v^\xi\}$, we have the following computation:

$$
\begin{aligned}
V(t) &\stackrel{\text{def}}{=} m(\varphi_t K) \\
&= \int_{P_i} dm(v) \int_{\xi(v)\cap\varphi_t K} dm_v^\xi(w) \\
&= \int_{P_i} dm(v) \int_{\xi(v)\cap\varphi_t K} c(v) g(w) dm_v^u(w) \\
&= \int_{P_i} dm(v) \int_{\xi(v)\cap K} c(v) g(\varphi_t w) \det(d\varphi_t)(w) dm_v^u(w)
\end{aligned}
$$

where $c(v)$ is the constant on $\xi(v)$ such that $\rho(w) = c(v)g(w)$ for all $w \in \xi(v)$.

Thus

$$
\begin{aligned}
\dot{V}(0) &= \int_{P_i} dm(v) \int_{\xi(v)\cap K} c(v)[< \nabla^u g, \nabla^u \varphi > + g \operatorname{div}(\nabla^u\varphi)] dm_v^u(w) \\
&= \int_{P_i} dm(v) \int_{\xi(v)\cap K} [< \nabla^u \log g, \nabla^u\varphi > + \Delta^u\varphi] dm_v^\xi(w) \\
&= \int_K [< \nabla^u \log g, \nabla^u\varphi > + \Delta^u\varphi] dm
\end{aligned}
$$

which has nothing to do with the Markov partition. Obviously this is true for any compact set K in SM. Take $K = SM$, we have $V(t) \equiv 1$ and $\dot{V}(0) = 0$ and (i) follows.

The proof of (ii) is similar.

3. Measure classes on the ideal boundary. Proof of Theorem 2.

Recall that the canonical projection $P : S\widetilde{M} \to \partial\widetilde{M}$ restricted to each $S_x\widetilde{M}$ is a homeomorphism $P_x : S_x\widetilde{M} \to \partial\widetilde{M}$. Thus each family of finite Borel measures $\{\sigma_x\}_{x\in\widetilde{M}}$ on $\{S_x\widetilde{M}\}_{x\in\widetilde{M}}$ can be viewed as a family of measures on $\partial\widetilde{M}$ via P_x and vice versa. From now on we will use the notation σ_x for measures on both $S_x\widetilde{M}$ or $\partial\widetilde{M}$. We say that $\{\sigma_x\}_{x\in\widetilde{M}}$ defines a measure class on $\partial\widetilde{M}$ if and only if they are absolutely continuous with respect to each other when viewed as measures on $\partial\widetilde{M}$. Then we can talk about their Radon-Nikodym derivatives

$$\sigma(x,y,\xi) \overset{\text{def}}{=} \frac{d\sigma_y}{d\sigma_x}(\xi).$$

Also we have a measure $d\sigma$ on $S\widetilde{M}$ (or SM, if $\{\sigma_x\}_{x\in\widetilde{M}}$ is invariant under $\Gamma = \pi_1(M)$) defined by

$$\int_{S\widetilde{M}} \cdot\, d\sigma = \int_{\widetilde{M}} dm(x) \int_{S_x\widetilde{M}} \cdot\, d\sigma_x \quad \left(\text{or } \int_{SM} \cdot\, d\sigma = \int_{M} dm(x) \int_{S_xM} \cdot\, d\sigma_x\right).$$

In other words, we consider $d\sigma_x$, $x \in \widetilde{M}$ as conditional measures and the Riemannian volume as the factor measure for $d\sigma$.

Theorem 2. *If for any $x \in \widetilde{M}$, $\sigma(x,y,\xi)$ is a C_s^2 function of $(y,\xi) \in S\widetilde{M}$, then*

(i) *For any C_s^2 function φ on $S\widetilde{M}$ with compact support,*

$$\int_{S\widetilde{M}} [\Delta^s\varphi + \varphi(\|\Delta^s \log\sigma + \|\nabla^s \log\sigma\|^2) + 2 < \nabla^s \log\sigma, \nabla^s\varphi >]d\sigma = 0$$

(ii) *If in addition, $\{\sigma_x\}_{x\in\widetilde{M}}$ is invariant under Γ, then for any C_s^2 function φ on SM,*

$$\int_{SM} [\Delta^s\varphi + \varphi(\|\nabla^s \log\sigma\|^2 + \Delta^s \log\sigma) + 2 < \nabla^s \log\sigma, \nabla^s\varphi >]d\sigma = 0$$

Proof.

(i) Define a function Φ on $\widetilde{M}$ by

$$\Phi(y) = \int\limits_{S_y\widetilde{M}} \varphi(y,\xi)d\sigma_y(\xi)$$

Fix a point x, $\Phi(y) = \int\limits_{S_y\widetilde{M}} \varphi(y,\xi)\sigma(x,y,\xi)d\sigma_x(\xi)$

And $\Delta\Phi(y) = \int\limits_{S_x\widetilde{M}} [(\Delta^s\varphi)\sigma + \varphi\Delta^s\sigma + 2 < \nabla^s\varphi, \nabla^s\sigma >]d\sigma_x(\xi)$

But $\sigma(x,x,\xi) = 1$ and $\sigma^{-1}\Delta^s\sigma = \Delta^s \log\sigma + \nabla^s\|\log\sigma\|^2$

By Green's formula, we have $\int\limits_{\widetilde{M}} \Delta\Phi dm = 0$ and (i) follows.

The proof of (ii) is the same.

There are three natural measure classes on $\partial\widetilde{M}$.

(i) **The harmonic measure class**: Let ν_x be the hitting probability at $\partial\widetilde{M}$ of Brownian motions on $\widetilde{M}$ starting from x. We know that for any two points x, y in $\widetilde{M}$, ν_x and ν_y are equivalent and

$$\frac{d\nu_y}{d\nu_x}(\xi) = K(x,y,\xi)$$

is a minimal positive harmonic function of y, which is called the Poisson kernel (see [AS]).

(ii) **The Lebesgue measure class**: Consider the Lebesgue measure m_x on $S_x\widetilde{M}$ induced by the Riemannian metric. By the absolute continuity of the stable foliation of the geodesic flow, m_x and m_y are equivalent for each $x, y \in \widetilde{M}$ and they define a measure class of $\partial\widetilde{M}$, which is called the Lebesgue measure class.

(iii) **The Bowen-Margulis measure class**: In his paper [L1], Ledrappier constructed a family of probability measures $\{\mu_x\}_{x\in\widetilde{M}}$ on $\partial\widetilde{M}$

which are transversals (see [L1] for definition) of the Bowen-Margulis measure μ. He proved that there is a continous function F on $\widetilde{M}$ such that

$$\frac{d\mu_y}{d\mu_x}(\xi) = e^{-h\rho_{(x,-\xi)}(y)}\frac{F(y)}{F(x)}$$

for all $x, y \in \widetilde{M}$. Here $\rho_{(x,-\xi)}(y)$ is the Busemann function at ξ such that

$\rho_{(x,-\xi)}(x) = 0.$

Note that for any $x \in \widetilde{M}$, $K(x, y, \xi)$ can be thought of as a C_s^∞ function of $(y, \xi) \in S\widetilde{M}$, and we have $\Delta^s \log K + \|\nabla^s \log K\|^2 = K^{-1}\Delta^s K = 0$. Applying theorem 2 to $\sigma_x = \nu_x$, we have

Corollary 3.1.

(i) *For any function φ of class C_s^2 on SM, we have*

$$\int_{SM} (\Delta^s\varphi + 2 < \nabla^s \log K, \nabla^s\varphi >) dm(x) d\nu_x(\xi) = 0$$

(ii) *For any C_s^2 function φ on $S\widetilde{M}$ with compact support*

$$\int_{S\widetilde{M}} (\Delta^s\varphi + 2 < \nabla^s\varphi, \nabla^s \log K >) dm(x) d\nu_x(\xi) = 0$$

4. A functional description of constant mean curvature.

Proof of theorem 3.

We continue to use the assumptions and notations of the above sections. We denote by $U(v)$ the second fundamental form at $v(0)$ of the horosphere $\rho_v^{-1}(0)$. We know that $tr\,U(v)$ is the mean curvature of the horosphere at $v(0)$ and $tr\,U(v) = \frac{d}{dt}|_{t=0} \log J_t^u(v)$. We denote by $\tau(v)$ the function $\frac{d}{dt}|_{t=0} \log K(v(0), v(t), v(\infty))$.

There are several equivalent descriptions of negatively curved compact manifolds whose horospheres have constant mean curvature:

(i) $tr\, U \equiv h$;

(ii) ([Y1]) $tr\, U(-v) \equiv \tau(v)$;

(iii) ([L1]) $K(x, y, \xi) \equiv e^{-h\rho_{x,-\xi}(y)}$;

(iv) ([L1]) $\mu_x \equiv \nu_x$;

(v) ([Y1]) $m_x \equiv \nu_x$;

(vi) ([L1]) $\lambda_1 = -\frac{h^2}{4}$, where λ_1 is the top of the spectrum of the Laplacian Δ in $L^2(\widetilde{M})$;

We give another description:

Theorem 3. *The horospheres in $\widetilde{M}$ have constant mean curvature if and only if*

(i) *For any C^2_u function φ on SM, $\int_{SM} \Delta^u \varphi dm = 0$. or*

(ii) *For any C^2_u function φ on $S\widetilde{M}$ with compact support, $\int_{S\widetilde{M}} \Delta^u \varphi dm = 0$.*

Proof.

(i) "$\Longrightarrow$". If $tr\; U \equiv h$, then by [L1], $K(x, y, \xi) = e^{-h\rho_{x,-\xi}(y)}$. By [Y], $m_x \equiv \nu_x$. Applying Corollary 3.1, note that $\nabla^u \log K = hX$ (here X is the geodesic spray), and $dm = dm(x) \cdot d\nu_x$, we have $\int_{SM} [\Delta^u \varphi + 2h\, X(\varphi)] dm = 0$. So

$$\int_{SM} \Delta^u \varphi dm = 0.$$

since $\int_{SM} X(\varphi) dm = 0$.

"$\Longleftarrow$" If $\int\limits_{SM} \Delta^u \varphi dm = 0$ for all C_u^2 function φ on SM. According to theorem 1, we have for all C_u^2 function φ

$$\int\limits_{SM} < \nabla^u \varphi, \nabla^u \log g > dm = 0.$$

For each P_i of our Markov partition $P = \{P_i\}_{i=1}^K$, consider all the C_u^2-functions φ with compact support in $int(P_i)$. We have

$$\begin{aligned}
0 &= \int\limits_{SM} < \nabla^u \varphi, \nabla^u \log g > dm \\
&= \int\limits_{P_i} dm(v) \int\limits_{\xi(v)} < \nabla^u \varphi, \nabla^u \rho > \rho^{-1} dm_v^{\xi} \\
&= \int\limits_{P_i} dm(v) \int\limits_{\xi(v)} < \nabla^u \varphi, \nabla^u \rho > dm_v^u \\
&= \int\limits_{P_i} dm(v) \int\limits_{\xi(v)} - \varphi \Delta^u \rho \, dm_v^u (\text{by the Green's formula})
\end{aligned}$$

Since φ is arbitrary, it is easy to see that $\Delta^u \rho \equiv 0$ m-a.e. on $int(P_i)$. By the continuity of $\Delta^u g$, we have $\Delta^u g \equiv 0$ on $\widetilde{SM}$. This means that for each $\xi \in \partial\widetilde{M}$, the function

$$G_\xi(y) \stackrel{\text{def}}{=} g(y, \xi)$$

is a harmonic function of y.

On the other hand, we know that $g(g^t v) = J_t^u(v)^{-1} g(v)$. Thus $\lim\limits_{y \to \eta} G_\xi(y) = 0$ for any $\eta \in \partial\widetilde{M}$, $\eta \neq -\xi$. Therefore $G_\xi(\cdot)$ is a kernel function. By the minimality of the Poisson kernel, as well as the uniqueness of kernel function (Corollary 5.3 of [AS]), we have

$$G_\xi(y) \equiv c\, K(x, y, -\xi)$$

for some constant c. Thus

$$\tau(-v) = \frac{d}{dt}|_{t=0} \log K(v(0)\ v(-t),\ -\xi) = \frac{d}{dt}|_{t=0} \log J_t^u(v) = trU\,(v)$$

And it follows from [Y] that $tr\, U \equiv h$.

(ii) The proof of (ii) is similar.

5. Another functional description of constant mean curvature

Proof of theorem 1′ and theorem 3′.

Recall that for any fixed point ξ in $\partial\widetilde{M}$, the horospheres at ξ constitute a codimension-1 foliation on $\widetilde{M}$. If we denote by Δ^h the Laplacian on each horosphere with respect to the induced Riemannian metric, then we have the following relation:

Lemma 5.1. *For any C^2–function φ on $\widetilde{M}$ we have*

$$\Delta\varphi = \Delta^h\varphi + \ddot{\varphi} - tr\; U\dot{\varphi}$$

Here the derivative $\dot{\varphi}$ is taken along the gradient flow of the Busemann function at ξ.

Proof. Let N be the gradient vector field of the Busemann function at ξ. Let $(e_i)_{1\leq i\leq n-1}$ be an orthonormal basis of T_xH (here H is the horosphere at ξ passing through x), and $(E_1, \cdots, E_{n-1}, N)$ be a field of orthonormal frames in a neighborhood of x in $\widetilde{M}$, whose value at x is $(e_1, \cdots, e_{n-1}, N_x)$. For any point of H where this frame field is defined, we have

$$\Delta\varphi = trD\, d\varphi = \sum_{i=1}^{n-1}(E_i d\varphi(E_i) - d\varphi(D_{E_i}E_i)) + D\, d\varphi(N,N)$$

Since $D_N N = 0$, we have $D\, d\varphi(N,N) = \ddot{\varphi}$. Therefore, we get

$$\Delta\varphi = tr\, D\, d\varphi = \sum_{i=1}^{n-1}(E_i d\varphi(E_i) - d\varphi(D_{E_i}E_i)) + \ddot{\varphi}$$

Yet $D_X Y = D^h_X Y - < U\; X, Y > N$ for any two vector fields X, Y on $\widetilde{M}$ (here D^h is the canonical submanifold connection and U is the second fundamental form of H). So we get

$$\Delta\varphi = \Delta^h\varphi - \dot{\varphi} tr\; U + \ddot{\varphi}$$

Recall that via the cannonical projection $\pi : S\widetilde{M} \to \widetilde{M}$, the induced Riemannian metric on $H(\pi(v))$ lifts to a Riemannian metric g^{su} on $W^{su}(v)$, which induces a Laplacian operator Δ^{su} (resp. gradient ∇^{su}, divergence div^{su}).

Theorem 1′.

(i) *For any C^2_{su} function φ on SM.*

$$\int_{SM} (\Delta^{su}\varphi + < \nabla^{su}\varphi, \nabla^{su}\log g >) dm = 0$$

(ii) *For any C^2_{su} function φ on $S\widetilde{M}$ with compact support,*

$$\int_{S\widetilde{M}} (\Delta^{su}\varphi + < \nabla^{su}\varphi, \nabla^{su}\log g >) dm = 0.$$

Proof.

(i) By Lemma 5.1, $\Delta^u\varphi + < \nabla^u\varphi, \nabla^u \log g >$

$= \Delta^{su}\varphi + < \nabla^{su}\varphi, \nabla^{su}(\log g) > + \ddot{\varphi} + \dot{\varphi} tr\, U + \dot{\varphi}(\log g)^{\cdot}$

$= \Delta^{su}\varphi + < \nabla^{su}\varphi, \nabla^{su}(\log g) > + \ddot{\varphi}$

since $(\log\, g)^{\cdot} = -tr\, U$. And the theorem follows by the invariance of the measure m. (Note that here we need φ to be of class C^2_u in order to use Lemma 5.1. Actually, one can prove theorem 1′ using the same method as in the proof of theorem 1. So it is enough to require φ belonging to C^2_{su} class.)

(ii) The proof of (ii) is similar.

Theorem 3′. *A compact manifold M with negative curvature has constant mean curvatured horospheres if and only if*

(i) $\int_{SM} \Delta^{su}\varphi dm = 0$ *for all C^2_{su} functions φ on SM, or*

(ii) $\int_{SM} \Delta^{su}\varphi dm = 0$ *for all C^2_{su} functions φ on $S\widetilde{M}$ with compact support.*

Proof.

(i)"$\Longrightarrow$": If the horospheres in $\widetilde{M}$ have constant mean curvature, then g $\equiv$constant along W^{su} leave and $\int_{SM} \Delta^{su}\varphi dm = 0$ follows from theorem $1'$.

"$\Longleftarrow$": If $\int_{SM} \Delta^{su}\varphi dm = 0$ for all C^2_{su} functions φ on SM, then

$$\int_{SM} < \nabla^{su}\varphi, \nabla^{su}\log g > dm = 0.$$

Using the partition η in section 2 and the same technique as in the proof of theorem 3, we get

$$\Delta^{su} g \equiv 0$$

on each W^{su}–leaf. It is well-known that the horospheres have subexponential growth (see for example [HIH]). Thus the entropy of BM(H) (Brownian motion on H) is zero by theorem 6 in [Kai3]. According to theorem 2 there, H has no nonconstant bounded harmonic function. So the function g must be constant along the W^{su}–leaves. It follows that $tr\, U = \frac{d}{dt}|_{t=0} \log g(g^t v)$ is also constant on SM.

6. If the harmonic measure coincides with the Bowen-Margulis measure.

In this case, since ν is the equilibrium state of $\tau(v)$ and μ is the equilibrium state of zero function, by the variational principle, there exists a function f on SM such that

$$\dot{f}(v) = \tau(v) - h$$

By the smooth Livsic theory ([LMM]), f is of C^∞_s class because so is $\tau(v)$. On the other hand, according to Proposition 2 of [L1], we have

$$\log K(x, y, \xi) = -h\rho_{x,-\xi}(y) + f(y, \xi) - f(x, \xi)$$

Applying corollary 3.1, we have

Lemma 6.1. *If $\nu = u$, then*

(i) *For every function φ of class C^2_s on SM,*

$$\int\limits_{SM} [\Delta^s \varphi + 2 < \nabla^s \varphi, \nabla^s f > + 2h\dot{\varphi}] dm(x) d\nu_x = 0$$

(ii) *For every C^2_s function φ on $S\widetilde{M}$ with compact support*

$$\int\limits_{S\widetilde{M}} [\Delta^s \varphi + 2 < \nabla^s \varphi, \nabla^s f > + 2h\dot{\varphi}] dm(x) d\nu_x = 0$$

Note that our proof of (i) is much simpler than that of [H2] where a generalized kind of Brownian motion was employed.

7. If the harmonic measure coincides with the Liouville measure.

The famous Sullivan conjecture states that this can only happen when M is locally symmetric. Note that in this case, by Lemma 3.2 of [Y1], there exists a C^∞_s function f on SM such that for $m_x - a.e. \nu \in S_x M$

$$\frac{d\nu_x}{dm_x}(v) = e^{f(v)}$$

Moreover, $\dot{f}(v) = \tau(v) - tr\, U(-v)$.

Via the flip map $v \mapsto -v$, we can think of ξ as a partition of SM into stable pieces and we also get a function g globally defined on $S\widetilde{M}$ such that

(1) g is of C^∞_s class

(2) $g(w) = \frac{dm^\xi_v}{dm^s_w}(v)$ on $\xi(v)$ up to a scalar multiplication

(3) $g(g^t v) = g(v) J^s_t(v)^{-1}$

Therefore, for each $w \in W^{ss}(v) \cap \xi(v)$, we have

$$\begin{aligned} \frac{g(w)}{g(v)} &= \lim_{t \to \infty} \frac{J^s_t(w)}{J^s_t(v)} = \lim_{t \to \infty} \frac{e^{f(g^t w) - f(w)} K(w(0), w(t), w(\infty))^{-1}}{e^{f(g^t v) - f(v)} K(v(0), v(t), v(\infty))^{-1}} \\ &= e^{f^{(v)} - f^{(w)}} K(v(0), w(0), v(\infty)) \end{aligned}$$

Applying theorem 1, we have

Lemma 7.1. *If $\nu = m$, then*

(i) *For any C_s^2 function φ on SM*

$$\int_{SM} (\Delta^s\varphi + < \nabla^s \log K - \nabla^s f, \nabla^s\varphi >) dm = 0$$

(ii) *For any C_s^2 function φ on $S\widetilde{M}$ with compact support*

$$\int_{S\widetilde{M}} (\Delta^s\varphi + < \nabla^s \log K - \nabla^s f, \nabla^s\varphi >) dm = 0$$

Applying Corollary 3.1 to φe^{-f}, we have

Lemma 7.2. *If $\nu = m$, then*

(i) *For every C_s^2 function φ on SM,*

$$\int_{SM} [\Delta^s\varphi + 2 < \nabla^s \log K - \nabla^s f, \nabla^s\varphi >$$
$$+\varphi(\|\nabla^s f\|^2 - \Delta^s f - 2 < \nabla^s \log K, \nabla^s f >)] dm = 0$$

(ii) *For every C_s^2 function φ on $S\widetilde{M}$ with compact support*

$$\int_{S\widetilde{M}} [\Delta^s\varphi + 2 < \nabla^s \log K - \nabla^s f, \nabla^s\varphi >$$
$$+\varphi(\|\nabla^s f\|^2 - \Delta^s f - 2 < \nabla^s \log K, \nabla^s f >)] dm = 0$$

Applying lemma 7.1 to $\varphi = f$ and lemma 7.2 to $\varphi \equiv 1$, we obtain

Corollary 7.3. *: If $v = m$, then* $\int_{SM} \|\nabla^s f\|^2 dm = \int_{SM} \Delta^s f \; dm$, $\int_{sm} < \nabla^s f, \; \nabla^s \log K > dm = 0$

Remark. By corollary 7.3, it is easy to see that the Sullivan conjecture would follow if one can prove that $\int_{SM} \Delta^s f \, dm = 0$.

8. If the Liouville measure and the Bowen-Margulis measure coincide.

In this case, since m is the equilibrium state of $trU(v)$, there exists a function f on SM such that

$$\dot{f}(v) = h - tr\, U(v)$$

By the smooth Livsic theorem, F is of class C_u^∞ because $tr\, U(v)$ is C_u^∞.

Recall that for each $w \in W^{su}(v)$,

$$\begin{aligned} \frac{g(w)}{g(v)} &= \lim_{t\to\infty} \frac{J_t^u(g^{-t}v)}{J_t^u(g^{-t}w)} \\ &= \lim_{t\to\infty} \frac{e^{ht-f(v)+f(g^{-t}v)}}{e^{ht-f(w)+f(g^{-t}w)}} \\ &= e^{f(v)-f(w)} \end{aligned}$$

Thus $\nabla^u \log g(v) \equiv \nabla^u f(v) + hX$ (recall that X is the geodesic spray), and by theorem 1, we have

Lemma 8.1. *If $m = u$, then*

(i) *For each C_u^2 function φ on SM,* $\int_{SM} (\Delta^u\varphi + < \nabla^u f, \nabla^u\varphi >)dm = 0$

(ii) *For each C_u^2 function φ on $\widetilde{SM}$ with compact support, we have*

$$\int_{\widetilde{SM}} [\Delta^u\varphi + < \nabla^u f, \nabla^u\varphi > - hX(\varphi)]dm = 0$$

Applying Lemma 8.1 to $\varphi = f$, we obtain

Corollary 8.2. *If $m = u$, then* $\int \|\nabla^u f\|^2 dm = -\int \Delta^u f\, dm$

IV. Brownian Motion on Anosov Foliations, Integral Formula and Rigidity

0. Introduction.

We generalize Lucy Garnett's ergodic theory for C^3 foliations to foliations $\mathcal{F}$ of class $C^3_{\mathcal{F}}$. We apply it to study the ergodic properties of Anosov foliations. We prove in section 3 the following

Theorem 3.1. *The horocycle foliations (W^{su} or W^{ss}) of a C^3-transitive Anosov system with leafwise Riemannian metric of class C^3_i ($i = su, ss$) are uniquely ergodic (i.e., they have precisely one harmonic measure).*

Then we generalize the integral formulas in [Y2] to Anosov foliations and apply them to obtain the following rigidity result:

Theorem 4.2. *For a Anosov system with its unique harmonic measure w^{ss}, the following properties are equivalent:*

1° w^{ss} *is a invariant measure of the Anosov system.*

2° J^{ss}_t *is constant along W^{ss}-leaves.*

We apply the above theory to the geodesic flow on a compact Riemannian manifold M of negative curvature. We give an explicit description of the harmonic measure w^{ss} as the weak limit of the normalized spherical measure of geodesic balls. This settles a problem raised by Katok. We also give two formulas for topological entropy.

Theorem 6.2. *Let R be the scalar curvature of M and R^H the scalar curvature of the horospheres, let Ric be the Ricci curvature of M and $\mathrm{tr}\,U$ be the mean curvature of the horospheres*

1° $h = \int_{SM} \mathrm{tr}\, U dw^{ss}$

2° $h^2 = \int_{SM} (R^H(v) - R(\pi(v)) + Ric(v)) dw^{ss}$

Using A. Connes' Gauss-Bonnet theorem for foliation we get

Corollary 6.3. *For a 3-dimensional closed Riemannian manifold of negative curvature,*

$$h^2 = \int_{SM} (Ric(v) - R(\pi(v))dw^{ss}(v)$$

In section 7, we study the Margulis's asymptotic formula

$$\lim_{R\to\infty} \frac{1}{e^{hR}} S(x,R) = c(x)$$

for the volume of geodesic spheres. We show that

Theorem 7.2. *For any compact manifold M of negative curvature, if $c(x)$ is a constant function, then for each x in $\widetilde{M}$,*

$$h = \int_{\partial\widetilde{M}} trU(x,\xi)d\mu_x(\xi)$$

where μ_x is the Bowen-Ledrappier-Margulis-Patterson-Sullivan measure at infinity.

This implies particularly that

Theorem 7.3. *If* dim $M = 2$, *then $c(x)$ is a constant function if and only if M has constant negative curvature.*

1. Ergodic properties of foliations.

In this section we review the main results in [Ga] and generalize them to foliations $\mathcal{F}$ of class $C^3_{\mathcal{F}}$.

Let $\mathcal{F}$ be any foliation on a compact manifold M equipped with a Riemannian metric on its tangent bundle. We assume that both $\mathcal{F}$ and the Riemannian metric on its tangent bundle are of class C^3. Each leaf L of the foliation inherits a C^3 Riemannian structure making it into a connected C^3 Riemannian manifold. The induced geometries on the leaves are uniformly bounded because M is compact. Thus each leaf L is complete for diffusion (i.e., the integral of the heat kernel over the whole space equals one).

We have a Laplace-Beltrami operator Δ^L on each leaf L. The measure on the leaf L induced by the Riemannian metric is denoted by dx. Let $P_t(x,y)$ be the heat kernel of the operator Δ^L. There is a one parameter semigroup of operators D_t corresponding to the diffusion of heat in the leaf directions:

$$D_t f(x) = \int_L f(y) P_t(x,y) dy$$

where f is a global function $f : M \to R$. If m is a measure on M, the measure diffused along the leaves of the foliation $D(t)m$ is defined by

$$\int_M f \, d(D(t)m) = \int_M D_t f \, dm$$

The set of probability measures on a compact finite-dimensional foliated manifold M is a nonempty convex set. The leaf diffusion operator $D(t)$ is a continuous affine mapping and any fixed point will be diffusion invariant for the time t. The Markov-Kakutani fixed point theorem insures that a fixed point exists for all times.

Definition.

(i) *A probability measure on M is said to be diffusion invariant if the integral of f with respect to that measure equals the integral of $D_t f$ with respect to the measure for any continuous function f.*

(ii) *A diffusion invariant measure is said to be ergodic if the manifold M cannot be split into two disjoint measurable leaf saturated sets with intermediate measure.*

(iii) *A probability measure m on M is harmonic if $\int_M \Delta^L f \, dm = 0$, where f is any bounded measurable function on M which is smooth in the leaf direction and Δ^L denotes the Laplacian in the leaf direction.*

Let E be any flow box of the foliation $\mathcal{F}$. The quotient of such an E by the $\mathcal{F}$-leaves is called the quotient transversal $I = I(E)$. If $P : E \to I$ is the projection, then by the classical measure theory, any measure m on E may be disintegrated uniquely into the projected measure ν on the transversal I and a system of measures $\sigma(s)$ on the leaf slices $p^{-1}(s) = E(s)$ for each s in I. These measures satisfy the following conditions:

(i) $\sigma(s)$ is a probability measure on $E(s)$.

(ii) If s is a measurable subset of I, then $\nu(s) = m(^{-1}(s))$.

(iii) If f is m-integrable and $\text{supp}(f) \subset E$, then

$$\int f(x)dm(x) = \int f(y)d\sigma(s)(y)\, d\nu(s).$$

The following theorem is due to Lucy Garnett:

Theorem 1 ([Ga]). *Let M be a compact foliated manifold with C^3-foliation $\mathcal{F}$ and a C^3-Riemannian metric on the tangent bundle of $\mathcal{F}$. Let m be any probability measure on M, then the following conditions are equivalent:*

(i) *m is diffusion invariant, i.e., $D(t)m = m$ for all t;*

(ii) *m is harmonic, i.e., $\int_M \Delta^L \varphi dm = 0$ for any bounded measurable function on M which is smooth in the leaf direction.*

(iii) *For any flow box E of the foliation $\mathcal{F}$ and ν almost all s (see the above construction), $\sigma(s)$ is a harmonic function times the Riemannian measure restricted to $E(s)$.*

Recall that a holonomy invariant measure of the foliation $\mathcal{F}$ is a family of measures defined on each transversal of the foliation $\mathcal{F}$, which is invariant under all the canonical homeomorphisms of the holonomy pseudogroup (see [Pl]). Given any transverse invariant measure, a global measure may be formed by locally integrating the Riemannian leaf measures with respect to the transverse invariant measure. Such a measure is called completely invariant. Obviously, any such measure disintegrates locally to a constant function times the Riemannian leaf measure and thus, by Theorem 1, is a harmonic measure for the foliation $\mathcal{F}$.

If L is a leaf of $\mathcal{F}$, let $x \in L$ and let $B(x,R)$ denote the ball in L of radius R around x under the leaf Riemannian metric. Define the growth function of $\mathcal{F}$ at x by $G_x(R) = \text{vol } B(x,R)$ where vol denotes the Riemannian volume on the leaf L. L is said to have exponential growth if

$$\varliminf_{R\to\infty} \frac{1}{R} \log G_x(R) > 0$$

and non-exponential growth otherwise.

Every foliation admits a nontrivial harmonic measure. But the following theorem tells us that there are many foliations which have no holomomy invariant measure at all.

Theorem 2 ([Pl]). *For a codimension one foliation $\mathcal{F}$ of class C^1 of a compact manifold M the following are equivalent.*

(i) *$\mathcal{F}$ has a leaf with non-exponential growth.*

(ii) *$\mathcal{F}$ has a leaf with polynomial growth.*

(iii) *$\mathcal{F}$ has a nontrivial holonomy invariant measure.*

for arbitrary codimension, we have

Theorem 3 ([Pl]). *Let $\mathcal{F}$ be a foliation of class C^1 of a compact manifold M. If L is a leaf of $\mathcal{F}$ having non-exponential growth, then there exists a nontrivial holonomy invariant for $\mathcal{F}$ which is finite on compact sets and which has support contained in the closure of L.*

As a direct corollary of Yosida's ergodic theorem for Markov processes (see [Yo]), one has the following:

The foliation ergodic theorem ([Ga]). *Let M be a harmonic probability measure. For any m-integrable function f, there exists an m-integrable function $\widetilde{f}$ which is constant along leave and satisfies*

(i) $\widetilde{f}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T D_t f(x) dt$ *for m almost all x*

(ii) $\int \widetilde{f}(x)dm = \int f(x)dm$

(iii) *If m is ergodic then* $\widetilde{f} = \int f(x)dm(x)$.

Let us denote by $\{w_t\}$ the set of Brownian paths lying on the leaves of the foliation $\mathcal{F}$ (induced by the Riemannian metric on each leaf). One has another interpretation of the foliation ergodic theorem.

The leaf path ergodic theorem ([Ga]). *Let m be any harmonic probability measure. For any m-integrable function f on M, the limit*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(w_t)dt$$

exists for m-a.e. x and almost any path w (in the sense of Wiener measure) starting at x and lying on the leaf on x. This limit is constant on leaves and equals the leaf diffused time average of f.

And finally, we have the Kryloff-Bogoluboff theory of harmonic measures.

Theorem 4. *There is a leaf saturated measurable set R in M having the following properties:*

(i) *For any $x \in R$ the diffused Dirac measure $\widetilde{\delta}_x$ exists, is ergodic and contains x in its support. $\widetilde{\delta}_x$ is defined by $\int f\, d\widetilde{\delta}_x = \widetilde{f}(x)$ for any continuous function $f : M \to R$.*

(ii) *Any two points on the same leaf in R have same diffused dirac measures.*

(iii) *R has full probability (i.e., $u(R) = 1$ for any harmonic probability measure)*

If one checks carefully all the steps of [Ga] (particularly the proof of Facts 1–4 on pp. 289–292), one easily sees that

Proposition 1. *All the results in* [Ga] *are true for foliations with C^3-leaves and C^3 Riemannian metric on each leaf whose 3-jets depend continuously on the points in M.*

Let us be more specific about the regularity requirements for the foliations and for the leafwise Riemannian metrics. We say that a foliation $\mathcal{F}$ has C^k leaves and C^k Riemannian metric on each leaf whose k-jets depend continuously on the points in M if for each point in M there is a local parametrization of the $\mathcal{F}$ foliation $\varphi : U \times V \to M$ (where $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^c$ are open sets, $d = \dim \mathcal{F}$ and $c = \operatorname{codim}\mathcal{F}$), such that

(i) φ is a homeomorphism from $U \times V$ to an open set in M

(ii) For each $y \in V$, $\varphi_y : U \to M$ given by $\varphi_y(x) = \varphi(x, y)$ is a C^k immersion whose image is an open subset of a leaf of the W^u-foliation and, moreover, for any $1 \le \alpha \le k$, $\frac{\partial^\alpha \varphi}{\partial x^\alpha}$ is continuous on $U \times V$.

(iii) For each $y \in V$, the pull-back of the Riemannian metric $g^{\mathcal{F}}$ on the leaf $\varphi_y(U)$ is a Riemannian metric on U satisfying

$$\varphi_y^*(g) = g_{ij} dx^i \wedge dx^j, \quad 0 \le i,\ j \le d$$

such that for each $1 \le \alpha \le k$, $\frac{\partial^\alpha g_{ij}}{\partial x^\alpha}$ is continuous on $Y \times V$.

We will call such a foliation of class $C^k_{\mathcal{F}}$. Such a Riemannian metric on leaves are also called to be of class $C^k_{\mathcal{F}}$. We will say that a function $\psi : M \to R$ is of class $C^j_{\mathcal{F}}$, $0 \le j \le k$, if $\psi \circ \varphi : U \times V \to R$ has derivatives of orders $1 \le \alpha \le j$ with respect to the arguments in U and that they are continuous.

2. Entropy properties of foliations.

We review the main results in [Kai3]. Our setting is a $C^3_{\mathcal{F}}$ foliation $\mathcal{F}$ on a compact manifold M and leafwise Riemannian metrics of class $C^3_{\mathcal{F}}$. Let $P_t(x, y)$ be the heat kernel of the leaf L_x. For any harmonic ergodic measure m, the densities φ of the conditional measures m on the $\mathcal{F}$-leaves are uniquely determined up to a scalar multiplication. Then we have a biased random motion (the φ-process) corresponding to the second order operator $\Delta^L + 2\nabla^L \log\varphi$ (∇^L is the gradient operator on L). The transition probability densities for this φ-process are

$$P_t'(x,y) = P_t(x,y)\varphi(y)/\varphi(x)$$

Theorem 2.1. *(Kaimanovich, [Kai3]) For m-a.e. x in M the following limits exist (not depending on x)*

$$h(\mathcal{F},m) = -\lim_{T\to\infty}\frac{1}{t}\int P_t(x,y)\log P_t(x,y)dy$$
$$h'(\mathcal{F},m) = -\lim_{T\to\infty}\frac{1}{t}\int P_t'(x,y)\log P_t'(x,y)dy$$

And $h(\mathcal{F},m)$ is called the entropy of Brownian motion on the foliation $\mathcal{F}$ with respect to m.

We list the following facts in [Kai3] (Recall that a Riemannian manifold M is called Liouvillian if there are no nonconstant bounded harmonic functions on M):

1° $h(\mathcal{F},m) = h'(\mathcal{F},m) + \int_M \|\nabla^L \log\varphi\|^2 dm$

2° $h(\mathcal{F},m) = 0$ (or $h'(\mathcal{F},m) = 0$) if and only if m-a.e. leaves are Liouvillian.

3° If $h(\mathcal{F},m) > 0$ (or $h'(\mathcal{F},m) > 0$), then for m-a.e. leaves of the foliation, the space of bounded harmonic functions is infinite-dimensional.

4° Every harmonic ergodic measure with almost all Liouvillian leaves is completely invariant.

5° If m-a.e. leaves of the foliation have subexponential growth, then m-a.e. leaves are Liouvillian.

Note that if the foliation $\mathcal{F}$ is trivial (i.e., the manifold M itself), then for all $x \in M$, the following limit exists:

$$h(M) = -\lim_{T\to\infty}\frac{1}{t}\int_M P_t(x,y)\log P_t(x,y)dm(y)$$

And

1° M is Liouvillian if and only if $h(M) = 0$.

2° If M has subexponential growth, then M is Liouvillian.

3. Harmonic measures for Anosov foliations.

In this section we consider a transitive Anosov flow g_t on a closed manifold M (or a transitive Anosov diffeomorphism on M). We denote by W^{su} (resp. W^{ss}) the strong unstable (resp. strong stable) foliation of the Anosov flow g_t or the Anosov diffeomorphism f. These foliations are also known as horospheric foliations. For Anosov flows, we also get the weak stable foliations W^s (resp. weak unstable foliations W^u):

$$W^s(x) = \bigcup_{t\in\mathbb{R}} W^{ss}(g_t x) \text{ (resp. } W^u = \bigcup_{t\in\mathbb{R}} W^{su}(g_t x))$$

All these Anosov foliations are always Hölder continuous but may fail to be C^1, even if the flow (or the diffeomorphism) itself is C^∞. But it is well-known that if the Anosov flow g^t (or the Anosov diffeomorphism f) is C^K ($K \geq 2$ is any integer or ∞), then each W^i-leaf ($i = s, u, ss, su$) is a C^k immersed manifold (see, e.g., [PS1]). Moreover, the four foliations have C^k leaves whose k-jet are continuously depending on the point. As is remarked by R. De La Llave, J. M. Marco and R. Moriyon in their fundamental work ([LMM], pp. 578), their regularity results for the Livsic cohomology equation can also be stated in terms of functions and flows or diffeomorphisms of class C^K by using the Sobolev's embedding theorem.

Let us consider any Riemannian metric g^i defined on the W^i-foliation ($i =$ s,u,ss,su) of class C^3_i. (Note that for any C^3-Riemannian metric on M, the induced Riemannian metrics g^i ($i = ss, su, s, u$) are of class C^3_i.) By proposition 1.1, each of these foliations has nontrivial harmonic measures.

Theorem 3.1. *The horocycle foliations (W^{su} or W^{ss}) of a C^3-transitive Anosov system with leafwise Riemannian metric of class C^3_i ($i = su, ss$) are uniquely ergodic* *in the sense that there is precisely one harmonic probability measure.*

Proof. We consider only the W^{su} foliation (the W^{ss} foliation can be treated similarly). According to a result of D. Sullivan and R. Williams ([SW]), the leaves of any strong Anosov foliations have polynomial growth. By Fact 4° of section 2, any W^{su}-harmonic ergodic measure m is completely invariant. Now theorem 3.1 follows from Bowen and Marcus's result that the horocycle foliations have unique holonomy invariant measure ([BM]).

We denote this measure by w^{su}. It has a local description as the product of Lebesgue measure m^{su} on W^{su} and Bowen-Margulis measure μ^{ss} on W^{ss} (in the diffeomorphism situation) or μ^{s} on W^{s} (in the flow case):

$$dw^{su} = m^{su} \times \mu^{ss} \quad (\text{ or } m^{su} \times \mu^{s} \text{ in the flow case}).$$

Note that the unique harmonic measure w^{ss} of the W^{ss}-foliation has a similar description.

$$d\,w^{ss} = m^{ss} \times \mu^{su} \ (\text{ or } m^{ss} \times \mu^{u} \text{ in the flow case}).$$

Recall that a flow φ_t on a compact metric space X is uniquely ergodic if any only if the following sequence

$$f_T(x) \triangleq \frac{1}{T} \int_0^T f(\varphi_t(x))dt$$

converges uniformly for any continuous function f on X:

$$f_T(x) \to \int f(x)d\mu(x) \ \ (T \to \infty)$$

where μ is the unique ergodic measure. We have an analogous result for the W^{ss} (or W^{su}) foliation.

Theorem 3.2. *For any continuous function f on M,*

$$\frac{1}{m^{ss}B_x^{ss}(R)} \int_{B_x^{ss}(R)} f(y)\, dm^{ss}(y) \longrightarrow \int f(x)dw^{ss}(x)$$

uniformly on M, where $B_x^{ss}(R)$ is the ball on $W^{ss}(x)$ under the g^{ss} metric.

Proof. By the arguments of Sullivan ([S4]) and Plante ([Pl]), for any foliation $\mathcal{F}$ with subexponential growth, the normalized measures

$$\frac{1}{m^{\mathcal{F}} B_x^{\mathcal{F}}(R)} \int_{B_x^{\mathcal{F}}(R)} \bullet \, dm^{\mathcal{F}}(y)$$

is weakly coverging to some harmonic measure of the foliation. Our theorem follows from the fact that

(i) W^{ss} is a foliation with polynomial growth.

(ii) The harmonic measure on W^{ss} is unique.

As is remarked in [BM], the weak stable (or weak unstable) foliation W^s (or W^u) for an Anosov flow has no completely invariant measure. However, they have at least one nontrivial harmonic measure.

Conjecture 1. *The weak stable (or unstable) Anosov foliations are uniquely ergodic in the sense that they have unique harmonic measure.*

This is true in the special case of geodesic flows on manifolds with negative curvature (see section 5).

4. Integral formula and rigidity.

We continue to use the assumptions and notations of section 3. For a C^3 Anosov flow g_t or an Anosov diffeomorphism f with leafwise Riemannian metrics g^i on W^i of class C_i^3 ($i = ss, su$), we define

$$\varphi^u(x) = -\frac{d}{dt}|_{t=0} \log J_t^{su}(x) \qquad (J_t^{su}(x) \triangleq \det \; dg^t|_{W^{su}(x)})$$

$$\varphi^s(x) = \frac{d}{dt}|_{t=0} \log J_t^{ss}(x) \qquad (J_t^{ss}(x) \triangleq \det \; dg^t|_{W^{ss}(x)})$$

for flow. For diffeomorphism, we define

$$\varphi^u(x) = -\log J_1^{su}(x) \qquad (J_t^{su}(x) \triangleq \det \; df^t|_{W^{su}(x)})$$

$$\varphi^s(x) = \log J_1^{ss}(x) \qquad (J_t^{ss}(x) \triangleq \det \; df^t|_{W^{ss}(x)})$$

It is well known that φ^u (resp. φ^s) is Hölder continuous and it has a unique equilibrium state m^+ (resp. m^-) which is an invariant ergodic measure of the Anosov system. It is also uniquely determined by the fact that it disintegrates into absolutely continuous measures along the W^{su} (resp. W^{ss}) leaves.

If we denote by ρ^{su} (resp. ρ^{ss}) the local density of conditional measures of m^+ (resp. m^-) with respect to the Riemannian volume m^{su} (resp. m^{ss}), then ρ^{su} is (resp. ρ^{ss}) of class C^3_{su} (resp. C^3_{ss}) (see [LMM], [LY], [Y2]), then $\nabla^{su}\log\rho^{su}$ (resp. $\nabla^{ss}\log\rho^{ss}$) is a continuous vector field on M of class C^2_{su} (resp. C^2_{ss}).

Theorem 1 in [Y2] can be generalized to an arbitrary Anosov system.

Theorem 4.1.

(i) *For any C^2_{su} function φ on M, we have*

$$\int_M (\Delta^{su}\varphi + < \nabla^{su}\varphi, \nabla^{su}\log\rho^{su} >)dm^+ = 0$$

(ii) *For any C^2_{ss} function φ on M, we have*

$$\int_M (\Delta^{ss}\varphi + < \nabla^{ss}\varphi, \nabla^{ss}\log\rho^{ss} >)dm^- = 0$$

And we have the following rigidity results.

Theorem 4.2. *For an Anosov system (flow or diffeomorphism), the following properties are equivalent:*

(a) *The measure m^+ (resp. m^-) and the measure w^{su} (resp. w^{ss}) coincide.*

(b) *w^{su} (resp. w^{ss}) is an invariant measure of the Anosov system.*

(c) *$J^{su}_t(x)$ (resp. $J^{ss}_t(x)$) is constant along W^{su} (resp. W^{ss}) leaves.*

Proof. (a) $\Longrightarrow$ (b) obvious

(b) $\Longrightarrow$ (a): Since w^{su} is an invariant measure of the Anosov system and, moreover, it is absolutely continuous along the W^{su} foliation. By the uniqueness of the Bowen-Sinai-Margulis measure, we have $w^{su} = m^+$.

(a) $\Longrightarrow$ (c). Since w^{su} is harmonic and m^+ coincides with w^{su}, we have by theorem 4.1

$$\int < \nabla^{su}\varphi,\ \nabla^{su}\log\rho^{su} > dm^+ = 0$$

for all function φ of class C^2_{su}. Consider all those φ with compact support in a local W^{su}-flow box P. We have

$$\begin{aligned}
0 &= \int_P < \nabla^{su}\varphi,\ \nabla^{su}\log P^{su} > dm^+ \\
&= \int_P dm^+(x) \int_{W^{su}_{\mathrm{loc}}(x)\cap P} < \nabla^{su}\varphi,\ \nabla^{su}\log\rho^{su} > \rho^{su}(y)dm^{su}(y) \\
&= \int_P dm^+(x) \int_{W^{su}_{\mathrm{loc}}(x)\cap P} \varphi\Delta^{su}\rho^{su}dm^{su}(y)
\end{aligned}$$

By the arbitrariness of φ, $\Delta^{su}\rho^{su} = 0$. Yet according to [LY]

$$\frac{\rho^{su}(y)}{\rho^{su}(x)} = \prod_{i=1}^{\infty} \frac{J_1^{su}(f^{-i}x)}{J_1^{su}(f^{-i}y)}$$

Thus ρ^{su} is a bounded harmonic function along each W^{su}-leaf. It must be constant along each W^{su}-leaf.

(c)$\Longrightarrow$(b) obvious by the description of Bowen-Margulis measure.

5. Applications to manifolds of negative curvature.

Let M be a closed C^∞ Riemannian manifold of negative curvature. Let $\widetilde{M}$ be its universal covering. The geodesic flow g^t on the unit tangent bundle SM is Anosov. We denote

- $\pi : S\widetilde{M} \to \widetilde{M}$ the canonical projection.
- $\partial\widetilde{M}$ the ideal boundary of $\widetilde{M}$.
- $v(t) = \pi(g^t v)$ is the geodesic in $\widetilde{M}$ with initial velocity v.
- $P : S\widetilde{M} \to \partial\widetilde{M}$ is the projection $P(v) = v(\infty) \triangleq \lim_{T\to\infty} v(t) \in \partial\widetilde{M}$.

- $P_x : S_x\widetilde{M} \to \partial\widetilde{M}$ is the restriction of P to $S_x\widetilde{M}$.
- (x, ξ) the vector v in S_xM such that $v(\infty) = \xi$.
- P_v the Busemann function at $v(\infty)$ such that $\rho_v(v(0)) = 0$.
- H_v the horosphere at $v(\infty) \in \partial\widetilde{M}$ passing through $v(0) \in \widetilde{M}$.
- μ, ν, m is the Bowen-Margulis, the harmonic, the Liouville measure of g^t.

The canonical projection $\pi : S\widetilde{M} \to \widetilde{M}$ maps $W^i(v)$ $(i = s, u)$ diffeomorphically onto $\widetilde{M}$. Thus the Riemannian metric on $\widetilde{M}$ lifts to a Riemannian metric g^i on $W^i(v)$ which induces a Riemannian volume $m^i (i = s, u)$. π also maps $W^i(v)$ $(i = su, ss)$ diffeomorphically to horospheres on $\widetilde{M}$. The induced Riemannian metrics on horospheres lift to Riemannian metrics g^{su} or g^{ss} on $W^{su}(v)$ or $W^{ss}(v)$ which induces Riemannian volumes m^{su} or m^{ss}.

Note that all the foliations W^i and the metrics g^i are of class C_i^∞, $i = s, u, ss, su$ (see for example [Y2]). Thus all the results in section 1–4 apply here.

We sum up as

Theorem 5.1. *The W^{su} foliation has a unique harmonic measure w^{su} and locally, $dw^{su} = dm^{su} \times d\mu^s$. Moreover, the following properties are equivalent:*

(a) $w^{su} = m$

(b) $w^{su} = u$

(c) $w^{su} = \nu$

(d) w^{su} *is g^t-invariant*

(e) *Horospheres in $\widetilde{M}$ have constant mean curvature.*

A similar result can be stated for the W^{ss} foliation and the measure w^{ss}.

The purpose of this section is to prove that the weak stable or unstable foliations are also uniquely ergodic. But first let us recall some basic facts.

Let $\Omega = C(R_+, M)$ be the space of continuous paths in M, and $\{P_x, x \in M\}$

the family of probability measures on Ω which describe the Brownian motion on M. Let $\widetilde{\Omega} = C(R_+, \widetilde{M})$ be the space of continuous paths in $\widetilde{M}$ and $\Pi : \widetilde{M} \to M$ the covering map. For each $x \in \widetilde{M}$ and $w \in \Omega$ such that $w(0) = \Pi(x)$, there is a unique path $\widetilde{w} \in \widetilde{\Omega}$ such that $\Pi(\widetilde{w}(t)) = w(t)$ for all $t \geq 0$. We denote by $(r(w,t), \theta(w,t))$ the polar coordinate about x of the path $\widetilde{w}(t)$.

1° For all x in $\widetilde{M}$ and $P_{\pi x}$-a.e. $w \in \Omega$, $\theta(w,t) \overset{t\to\infty}{\longrightarrow} \theta(w,\infty) \in \partial\widetilde{M}$ ([Pr]). We denote by ν_x the hitting probability measure on $\partial\widetilde{M}$ of Brownian motion starting at x, and

$$\frac{d\nu_y}{d\nu_x}(\xi) = k(x,y,\xi)$$

for all $x, y \in \widetilde{M}$ and almost all $\xi \in \widetilde{M}$. $k(x,y,\xi)$ is called the Poisson kernel.

2° ([L1]) For all $x \in \widetilde{M}$ and $P_{\pi x}$-a.e. $w \in \Omega$,

$$\lim_{T\to\infty} \frac{1}{t} d(\widetilde{w}(t),\ (\gamma(w,t),\ \theta(w,\infty))) = 0$$

3° ([Kai2]) For all $x \in \widetilde{M}$ and $P_{\pi x}$-a.e. $w \in \Omega$,

$$\lim_{T\to\infty} \frac{1}{t} \gamma(w,t) = \alpha$$

4° ([Kai2]) $\alpha = \int\limits_M dm(x) \int\limits_{\partial\widetilde{M}} \mathrm{tr}U(x,\xi)\, d\nu_x(\xi)$
where dm is the Riemannian volume on M and $U(x,\xi)$ is the second fundamental form at x of the horosphere $H(x,\xi)$.
$\mathrm{tr}U(x,\xi) = -\frac{d}{dt}|_{t=0} \log J_t^{ss}(x,\xi)$ is the mean curvature of $H(x,\xi)$ at x.

5° ([Kai2]) For every $x \in \widetilde{M}$ and $P_{\pi x}$-a.e. $w \in \Omega$

$$\lim_{T\to\infty} \frac{1}{t} \log P(t, x, \widetilde{w}(t)) = -\beta$$

6° ([Kai2]) $\beta = \int\limits_M dm(x) \int\limits_{S_x M} \|\nabla \log k\|^2 d\nu_x(v) = h_\nu \alpha$ (h_ν is the metric entropy of ν).

Note the measure $dm(x) \times d\nu_x(\xi)$ appeared in 4°, 6°. We prove this is the only harmonic measure of the W^s foliation.

Theorem 5.2. *The weak stable foliation of the geodesic flow has a unique harmonic measure w^s, which is described by $\int_{SM} \cdot\, dw^s = \int_M dm(x) \int_{S_xM} \cdot\, d\nu_x(v)$.*

Our proof is inspired by Garnett's proof of a special case where M is a surface of constant curvature -1.

Proof. Let us consider a W^s-flow box of the form

$$E = \{(x,\xi) | x \in B,\ \xi \in U\}$$

where B is a ball in M centered at a point x_0 and U is an open set in $\partial\widetilde{M}$. For any continuous function f with compact support in E, we have

$$\begin{aligned}\int_E f\, dw^s &= \int_B dm(x) \int_{\partial\widetilde{M}} f\, d\nu_x(\xi)\\ &= \int_B\int_U f(x,\xi)\ k(x_0,x,\xi) dm(x)\, d\nu_{x_0}(\xi)\\ &= \int_U d\nu_{x_0}(\xi) \int_B f(x,\xi)\ k(x_0,x,\xi) dm(x)\end{aligned}$$

which means that dw^s disintegrates locally into the harmonic function $k(x_0,x,\xi)$ times Riemannian volume of the W^s leaf. Thus w^s is a harmonic measure of the W^s foliation.

Given any other ergodic harmonic measure σ and any continuous function f on SM, by the leaf path ergodic theorem, we have for σ-a.e. leaf $W^s(x_0,\xi)$, for all point $(y,\xi) \in W^s(x_0,\xi)$ and P_y almost any path w starting at y.

$$\int_{SM} f\, d\sigma = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\widetilde{w}(t),\xi) dt$$

Given any other w^s-typical leaf $W^s(x_0, \eta)$, then for P_y almost any path w starting at y

$$\int_{SM} f\, dw^s = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\widetilde{w}(t), \eta) dt$$

Consider a typical path w starting at y such that $\widetilde{w}(t) \to e \in \partial\widetilde{M}$, $e \neq \xi$, $e \neq \eta$. By comparison with manifolds of constant negative curvature, it is easy to see that

$$d_{w(t)}((\widetilde{w}(t), \xi),\ (\widetilde{w}(t), \eta)) \to 0 \quad (t \to \infty)$$

where $d_{\widetilde{w}(t)}$ is the induced Riemannian metric on $S_{\widetilde{w}(t)}\widetilde{M}$. We get

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T [f(\widetilde{w}(t), \xi) - f(\widetilde{w}(t), \eta)] dt = 0$$

and $\int_{SM} f\, d\sigma = \int_{SM} f\, d\,w^s$, which proves the uniqueness of harmonic measure of the weak stable foliation.

Note.

i) For any function φ on SM of class C^2_s, we have

$$\begin{aligned} 0 &= \int_M dm(x) \Delta(\int_{S_x M} \varphi d\nu_x) \\ &= \int_M dm(x) \Delta(\int_{S_{x_0} M} \varphi k(x_0, x, \xi) d\nu_{x_0}(\xi) \\ &= \int_{SM} (\Delta^s \varphi + 2 < \nabla^s \varphi, \nabla^s \log k >) dm(x) d\nu_x \end{aligned}$$

ii) We also have

$$\begin{aligned} 0 &= \int_M dm(x) \mathrm{div}(\int_{S_x M} \nabla^s \varphi d\nu_x) \\ &= \int_M dm(x) \mathrm{div}(\int_{S_{x_0} M} \nabla^s \varphi \cdot k(x_0, x, \xi) d\nu_{x_0}(\xi)) \\ &= \int_{SM} (\Delta^s \varphi + < \nabla^s \varphi, \nabla^s \log k >) dm(x) d\nu_x \end{aligned}$$

Combining i), ii), we get $\int_{SM} \Delta^s \varphi dm(x) d\nu_x = 0$. This gives another proof of the fact that $\int_{SM} \cdot\, dm(x) d\nu_x$ is a harmonic measure of the W^s-foliation.

Theorem 5.3. *Let M be a compact manifold of negative curvature. The following properties are equivalent:*

(a) $w^s = m$

(b) $w^s = u$

(c) $w^s = v$

(d) w^s *is* g^t*-invariant*

(e) *Horospheres in* $\widetilde{M}$ *have constant mean curvature.*

Proof. See [Y1].

A similar result can be stated for the unique harmonic measure w^u of the weak stable foliation.

6. Integral formulas for topological entropy.

We continue to use the symbols and notations of section 5. Let w^{ss} be the unique harmonic measure of the strong stable foliation W^{ss} of the geodesic flow. By theorem 3.2, w^{ss} is a limit of the average measure on balls $B_x^{ss}(R)$ in $W^{ss}(x)$:

$$\frac{1}{m^{ss}B_x^{ss}(R)} \int_{B_x^{ss}(R)} \bullet\, dm^{ss}(y) \longrightarrow w^{ss} \quad (\text{as } R \to \infty)$$

On the other hand, we know that horospheres in $\widetilde{M}$ can be approximated by geodesic spheres, $H(x,\xi) = \lim_{T\to\infty} S_{v(t)}(t)$, where $v(t)$ is the geodesic in $\widetilde{M}$ satisfying $v(0) = x$ and $v(\infty) = \xi$. Thus the harmonic measure is the weak limit of the averaged measures on geodesic spheres.

To be more specific, let φ be a continuous function on SM and x any point on $\widetilde{M}$. We define a function φ_x on $\widetilde{M}$ by $\varphi_x(y) = \varphi(v(y))$ where $v(y) \in S\widetilde{M}$ is the unique vector such that $v(y)(0) = y$ and $v(y)(t) = x$.

For any $\epsilon > 0$, according to theorem 3.2, there exists $R_1 > 0$ such that

$$\left| \frac{1}{m^{ss}B_v^{ss}(R)} \int_{B_v^{ss}(R)} \varphi(w) dm^{ss}(w) - \int_{SM} \varphi\, dw^{ss} \right| < \epsilon$$

for all $R \geq R_1$ and $v \in SM$. According to the estimates in [H4], there exists $R_2 > R_1$ such that for all $R \geq R_2$ and $y \in S_x(R)$ ($S_x(R)$ is the geodesic sphere in $\widetilde{M}$)

$$\left| \frac{1}{\mathrm{vol} D(y, R_1)} \int_{D(y,R_1)} \varphi_x(z) dz - \frac{1}{m^{ss} B^{ss}_{v(y)}} \int_{B^{ss}_{v(y)}(R_1)} \varphi_x(z) dz \right| < \epsilon$$

where $D(y, R_1)$ is the ball in $S_x(R)$ and dz is the volume element of the induced Riemannian metric on $S_x(R)$.

But since $|\varphi(w) - \varphi_x(\pi(w)| \underset{R \to \infty}{\longrightarrow} 0$ uniformly for all $y \in S_x(R)$ and $w \in B^{ss}_{v(y)}(R_1)$, we can assume R_2 so large that

$$\left| \frac{1}{m^{ss} B^{ss}_{v(y)}} \int_{B^{ss}_{v(y)}(R_1)} (\varphi(w) - \varphi_x(\pi(w)) dm^{ss}(w) \right| < \epsilon$$

and we get for all $R \geq R_2$ (we denote the volume of $S_x(R)$ by $S(x, R)$)

$$\left| \frac{1}{S(x,R)} \int_{S_x(R)} \varphi_x(y) dy - \int_{SM} \varphi dw^{ss} \right| < 3\epsilon$$

Proposition 6.1. $\frac{1}{S(x,R)} \int\limits_{S_x(R)} \varphi_x(y) dy \overset{R \to \infty}{\longrightarrow} \int\limits_{SM} \varphi dw^{ss}$ *uniformly on* $\widetilde{M}$ *for any continuous function* φ *on* SM

Now we are ready to prove

Theorem 6.2. *Let* $R(x)$ *be the scalar curvature at* x *in a closed Riemannian manifold* M. *Let* $R^H(v)$ *be the scalar curvature of the horospheres* $H(v)$, *let* $Ric(v)$ *be the Ricci curvature of* v, *then the topological entropy* h *of the geodesic flow satisfies*

1°. $h = \int\limits_{SM} trU(v) dw^{ss}(v)$

2°. $h^2 = \int\limits_{SM} (R^H(v) - R(\pi(v)) + Ric(v)) dw^{ss}(v)$

Proof. Margulis ([M3]) proved that for any closed Riemannian manifold M of negative curvature,

$$\lim_{R\to\infty} \frac{S(x,R)}{e^{hR}} = c(x) \qquad \cdots\cdots(*)$$

for some positive continuous function c on M. Let us calculate the derivatives of the function $G_x(R) = \dfrac{S(x,R)}{e^{hR}} = \dfrac{1}{e^{hR}} \displaystyle\int_{S_x(R)} dy$.

$$G'_x(R) = -h\,G_x(R) + \frac{1}{e^{hR}} \int_{S_x(R)} \mathrm{tr} U_R(y)\,dy$$

where $U_R(y)$ is the second fundamental form of $S_x(R)$ at y and $\mathrm{tr}U_R(y)$ is the mean curvature of $S_x(R)$ at y.

$$G''_x(R) = -h^2 G_x(R) - 2h\,G'_x(R) + \frac{1}{e^{hR}} \int_{S_x(R)} [-\mathrm{tr}\dot{U}_R(y) + (\mathrm{tr}U_R)^2]dy$$

$$G'''_x(R) = -h^3 G_x(R) - 3h^2 G'_x(R) - 3hG''_x(R) + \frac{1}{e^{hR}} \int_{S_x(R)} (\mathrm{tr}\ddot{U}_R - 3\mathrm{tr}\dot{U}_R \mathrm{tr}U_R(y) + (\mathrm{tr}U_R)^3)dy$$

Note that $\mathrm{tr}U_R(y) \to \mathrm{tr}U(v(y))$ $(R \to \infty)$ uniformly. Using Proposition 6.1 and (*), we get

$$\lim_{R\to\infty} G'_x(R) = -h\,c(x) + c(x) \int_{SM} \mathrm{tr}U dw^{ss}$$

$$\lim_{R\to\infty} G''_x(R) = -h^2\,c(x) - 2h \lim_{R\to\infty} G'_x(R) + c(x) \int_{SM} [-\mathrm{tr}\dot{U} + (\mathrm{tr}U)^2]dw^{ss}$$

$$\lim_{R\to\infty} G'''_x(R) = -h^3\,c(x) - 3h^2 \lim_{R\to\infty} G'_x(R) - 3\,h \lim_{R\to\infty} G''_x(R) + c(x) \int_{SM} [\mathrm{tr}\ddot{U} - 3\mathrm{tr}U\mathrm{tr}\dot{U} + (\mathrm{tr}U)^3]dw^{ss}$$

But since $\lim_{R\to\infty} G''_x(R)$ is bounded and $\lim_{R\to\infty} G'''_x(R)$ exists, we must have $\lim_{R\to\infty} G'_x(R) = \lim_{R\to\infty} G''_x(R) = \lim_{R\to\infty} G'''_x(R) = 0$. Thus we get

(i) $h = \int_{SM} \mathrm{tr}\,U\,dw^{ss}$

(ii) $h^2 = \int_{SM} [-\mathrm{tr}\dot{U} + (\mathrm{tr}U)^2]dw^{ss}$

Let us recall some submanifold geometry. If we denote by K^H the Gaussian curvature of $H(v)$ under the induced Riemannian metric, then for any two orthonomal vector X, Y in $T_{\pi(v)}H(v)$, the Gauss equation tells us

$$K^H(X,Y) = K(X,Y) + < U(v)X, X >< U(v)Y, Y > - < U(v)X, Y >< X, U(v)Y >$$

U_v is a positive symmetric operator. Let $e_1, \cdots, e_{n-1}$ be its unit eigenvectors with eigenvalue $\lambda_1, \cdots, \lambda_{n-1}$. Then

$$K^H(e_i, e_j) = K(e_i, e_j) + \lambda_i \lambda_j$$

and

$$\sum_{i,j} K^H(e_i, e_j) = \sum_{i,j} (K(e_i, e_j) + \lambda_i \lambda_j)$$

Thus $R^H(v) = R(\pi(v)) + (\operatorname{tr} U)^2 - \operatorname{tr} U^2 - 2\mathrm{Ric}(v)$.

Remember that U satisfies the Ricatti equation $-\dot{U} + U^2 + S = 0$ where $S(v)X = R(X, v)v$, R is the curvature tensor. So $\operatorname{tr} S(v) = \mathrm{Ric}(v)$ and

$$R^H(v) = R(\pi(v)) + (\operatorname{tr} U)^2 - \operatorname{tr} \dot{U} - \mathrm{Ric}(v)$$

Combining (ii), we get

$$h^2 = \int [R^H(v) - R(\pi(v)) + \mathrm{Ric}(v)] dw^{ss}$$

Remark. By our proof, there is another integral formula for entropy:

$$h^3 = \int [\operatorname{tr}\ddot{U} + 3 \operatorname{tr}\dot{U} \operatorname{tr} U + (\operatorname{tr} U)^3] dw^{ss}$$

Actually, one can get a family of integral formulas for h^n in terms of a polynomial combination of $\operatorname{tr} U$ and its derivatives.

Corollary 6.3. *For a 3-dimensional closed Riemannian manifold of negative curvature*

$$h^2 = \int_{SM} (Ric(v) - R(\pi(v))dw^{ss}(v)$$

Proof. By A. Connes' Gauss-Bonnet theorem ([C]) for 2-dimensional foliation (see also [Gh])

$$\beta_0 - \beta_1 + \beta_2 = (2\pi)^{-1} \int k(x)\, d\mu(x)$$

where μ is a completely invariant measure of the foliation, $k(x)$ denotes the Gaussian curvature function of the leaves and $\beta_0, \beta_1, \beta_2$ are the "average Betti numbers" of the leaves relative to μ. Note that in our case, the W^{ss}-leaves have polynomial growth, each of them is conformally equivalent to the Eucilidean plane and diffeomorphic to the Eucilidean plane. Thus

$$\beta_0 - \beta_1 + \beta_2 = 0$$

Now Corollary 6.3 follows from the fact that in dimension 3, $R^H(v) = 2\,K^H(\pi(v))$ and the formula 2° in theorem 6.2

7. On Margulis's asymptotic formula.

We continue to use the symbols and notations of section 6. In [L1], Ledrappier constructed a family of finite measures $\{\mu_x\}_{x\in\widetilde{M}}$ on the sphere at infinity, satisfying the following property:

$$\frac{d\mu_y}{d\mu_x}(\xi) = e^{-h\rho_{x,\xi}(y)} \quad \cdots \quad \cdots \quad \circledast$$

He calls them the Bowen-Margulis measures, because $P_x^*(\mu_x)$ and $P|^*_{W^{su}(x)}(\mu^{su})$ are in the same measure class. (Recall that $P : S\widetilde{M} \to \partial\widetilde{M}$ is the canonical projection, μ^{su} is the Margulis measure on $W^{su}(x)$.)

To be more specific, let us recall Ledrappier's construction. Take a small open subset A of S_xM and consider the following transversal T of the W^{ss}-foliation

$$T = \bigcup_{-\delta\leq t\leq\delta} g_t A$$

for δ small enough. By the unique ergodicity of the W^{ss}-foliation, one can obtain a measure μ_T on T by sliding along W^{ss} leaves the Margulis measure $d\mu^u = d\mu^{su}dt$ on $W^u(x)$ which satisfies

$$d\mu_T = e^{ht}dt\, d\mu_A$$

for some measure μ_A on A. μ_A is exactly the measure $\mu_x|_A$ (up to a scalar constant). By the unique ergodicity of the W^{ss}-foliation and by the reversibility of Ledrappier's construction, it is easy to see that any family $\{\tau_x\}_{x\in\widetilde{M}}$ of finite measures on $\partial\widetilde{M}$ satisfying ⊛ must coincide with $\{\mu_x\}_{x\in\widetilde{M}}$ (up to a scalar constant). For example:

1). The Patterson-Sullivan measure ([Su2]): Fix *a* point $x \in \widetilde{M}$ and consider the Poincaré series

$$g_s(y,y) = \sum_{\sigma\in\Gamma} e^{-sd(y,\sigma y)}$$

where Γ is the fundamental group of M. It converges for $s > h$ and diverges for $s < h$. Now consider the family of measures

$$\mu_x(s) = \frac{1}{g_s(y,y)}\sum_{\sigma\in\Gamma} e^{-sd(x,\sigma y)}\delta(\sigma y)$$

where $\delta(\sigma y)$ is the unit Dirac mass at σy. Let $\widetilde{\mu}_x$ be a weak limit of the family $\{\mu_x(s)\}$ as $s \to h$, then it is easy to see that

(a) $\widetilde{\mu}_x$ is defined on $\partial\widetilde{M}$

(b) $\frac{d\widetilde{\mu}_y}{d\widetilde{\mu}_x}(\xi) = e^{-h\rho_{x,\xi}(y)}$

By the above remark, they coincide with $\{\mu_x\}_{x\in\widetilde{M}}$ (up to a scalar constant).

2). (Idea comes from a comment by A. Katok.) Via the canonical projection $S_x(R) \to \partial\widetilde{M}$, $y \to v_y(\infty)$ (v_y is the unit normal vector of the geodesic sphere $S_x(R)$ at y), we have a sequence of finite measures defined on $\partial\widetilde{M}$

$$\mu_x(R) = \frac{1}{e^{hR}}\int_{S_x(R)} \bullet dy$$

By the Margulis's asymptotic formula, it is easy to see that any weak limit $\widetilde{\mu}_x$ as $R \to \infty$ satisfies

(a) $\widetilde{\mu}_x$ is a measure on $\partial\widetilde{M}$ with $\widetilde{\mu}_x(\partial\widetilde{M}) = c(x)$

(b) $\frac{d\widetilde{\mu}_y}{d\widetilde{\mu}_x}(\xi) = e^{-h\rho_{x,\xi}(y)}$

Thus they also coincide with $\{\mu_x\}_{x\in\widetilde{M}}$ up to a scalar constant.

Proposition 7.1. *If we denote by $dm(x)$ the normalized Riemannian volume on M, $\widetilde{\mu}_x$ the normalized Patterson-Sullivan measure. Then the unique harmonic measure w^{ss} of the W^{ss}-foliation can be described as*

$$C\int_{SM} \bullet dw^{ss} = \int_{SM} \bullet c(x)\, dm(x)\, d\widetilde{\mu}_x$$

for some constant C ($C = \int_M c(x)\, dm(x)$) (Note: $\int_{SM} \varphi(x,\xi)c(x)dm(x)d\widetilde{\mu}_x(\xi) \stackrel{\text{def}}{=} \int_M c(x) \left(\int_{S_xM} \varphi(x,\xi)d\widetilde{\mu}_x(\xi)\right) dm(x)$)

Proof. Let $\{\mu_x\}_{x\in\widetilde{M}}$ be the family of Ledrappier-Patterson-Sullivan measures. We have

$$\frac{d\mu_y}{d\mu_x}(\xi) = e^{-h\rho_{x,\xi}(y)}$$

i) Let X be the geodesic spray. We have for any function f of class C^1_{ss},

$$\begin{aligned} 0 &= \int_M dm(x)\operatorname{div}|_{y=x}\Big(\int_{S_xM} f\,X\,d\mu_x\Big) \\ &= \int_M dm(x)\operatorname{div}\Big(\int_{S_xM} (f\,X)\,e^{-h\rho_{x,\xi}(y)}d\mu_x\Big) \\ &= \int_{SM} [\dot{f} + (h - \operatorname{tr} U)f]dm(x)d\mu_x \end{aligned}$$

ii) For any function f of class C^2_{ss}, we have

$$0 = \int_M dm(x)\Delta\Big(\int_{S_xM} f d\mu_x\Big)$$
$$= \int_M dm(x)\Delta|_{y=x}\Big(\int_{S_xM} f\, e^{-h\rho_{x,\xi}(y)} d\mu_x\Big)$$
$$= \int_{SM} (\Delta^s f + h\, f(h - \mathrm{tr}U) + 2h\dot{f})dm(x)d\mu_x$$
$$= \int_{SM} [\Delta^{ss} f + (\ddot{f} + (h - \mathrm{tr}U)\dot{f}) + h(\dot{f} + (h - \mathrm{tr}U)f)]dm(x)d\mu_x$$

Combining i), ii), we have $\int_{SM} \Delta^{ss} f\ dm(x)d\mu_x = 0$. By the uniqueness of the W^{ss}-harmonic measure, $dw^{ss} = dm(x)d\mu_x$ (up to normalization). Using proposition 6.1 and Margulis's asymptotic formula, one can see that

$$dw^{ss}(x,\xi) = \frac{c(x)}{\int_M c(x)dm(x)} dm(x)d\widetilde{\mu}_x(\xi)$$

where $\widetilde{\mu}_x$ is the normalized Ledrappier-Patterson-Sullivan measure.

The following theorem implies that for compact manifolds with negative curvature, the function $c(x)$ is almost always not a constant function.

Theorem 7.2. *If $c(x) \equiv C$ then for each $x \in \widetilde{M}$,*

$$h = \int_{\partial\widetilde{M}} trU(x,\xi)d\widetilde{\mu}_x(\xi)$$

Proof. Note that

$$\frac{d\widetilde{\mu}_y}{d\widetilde{\mu}_x}(\xi) = \frac{c(x)}{c(y)} e^{-h\rho_{x,\xi}(y)}$$

If $c(x) \equiv C$, then $\int e^{-h\rho_{x,\xi}(y)} d\widetilde{\mu}_x(\xi) \equiv 1$. Taking the Laplacian on both side yields

$$\int h\,(h - \mathrm{tr}\,U)e^{-h\rho_{x,\xi}(y)} d\widetilde{\mu}_x(\xi) = 0$$

Thus $h = \int_{\partial\widetilde{M}} \mathrm{tr}\,U(x,\xi)d\widetilde{\mu}_x(\xi)$.

Theorem 7.3. *If* dim $M = 2$ *and* $c(x) \equiv C$, *then* M *has constant negative curvature.*

Proof. According to theorem 6.2, $h^2 = \int(-\mathrm{tr}\dot{U} + (\mathrm{tr}U)^2)dw^{ss}$. Using the Ricatti equation $-\dot{U} + U^2 + S = 0$, note that in dimension 2, $\mathrm{tr}\,U^2 = (\mathrm{tr}\,U)^2$, we have

$$h^2 = \int -\mathrm{tr}\,S\,dw^{ss} = -\int K\,dm(x) = -2\pi E$$

where E is the Euler characteristic of M and K is the Gaussian curvature. Theorem 7.3 follows from A. Katok's result ([K1]) that $h^2 = -2\pi E$ if and only if M has constant negative curvature.

The following corollary measures the deviation of metrics from constant negative curvature.

Corollary 7.2. *If* dim $M = 2$, *then*

$$h^2 = \frac{1}{C}\int_M - c(x)\,k(x)\,dm(x)$$

Due to the above facts, it makes sense to have the following conjecture.

Conjecture. *For a compact Riemannian manifold M of negative curvature, $c(x) \equiv C$ if and only if M is locally symmetric.*

Recall that the strong unstable foliation W^{su} also has a unique harmonic measure w^{su}. By the flip map, we get $C\;dw^{su}(x,\xi) = c(x)\,dm(x)\;d\widetilde{\mu}_x(-\xi)$. Thus $w^{su} = w^{ss}$ if and only if $d\widetilde{\mu}_x(-\xi) = d\widetilde{\mu}_x(\xi)$. Ledrappier ([L4]) proves that if dim $M = 2$, then $w^{su} = w^{ss}$ if and only if M has constant curvature. The following result indicates that in multi-dimensional cases, locally symmetric spaces might be the only cases of manifolds of negative curvature for which $w^{ss} = w^{su}$.

Corollary 7.4. *If* $w^{ss} = w^{su}$, *then* $c(x) \equiv C$

Proof. Any C^2 function φ on M can be lifted to a function on SM which we denote by the same symbol. By the proof of our proposition 7.1 or by corollary 1 of [L2], we have

$$
\begin{aligned}
\int\limits_M \Delta\varphi\; c(x)dm &= C \int\limits_{SM} \Delta^s\varphi\; dw^{ss} \\
&= C \int\limits_{SM} (\Delta^{ss}\varphi + \ddot{\varphi} - \mathrm{tr}U\dot{\varphi})dw^{ss} \\
&= C \left[\int\limits_{SM} \Delta^{ss}\varphi\; dw^{ss} + \int\limits_{SM} (\ddot{\varphi} + (h - \mathrm{tr}\, U)\dot{\varphi})dw^{ss} - \int\limits_{SM} h\dot{\varphi}\; dw^{ss} \right] \\
&= -h \int\limits_M c(x)\, dm(x) \int\limits_{\partial\widetilde{M}} \dot{\varphi}(x,\xi)\, d\widetilde{\mu}_x(\xi)
\end{aligned}
$$

Note that $d\widetilde{\mu}_x(\xi) = d\widetilde{\mu}_x(-\xi)$ and $\dot{\varphi}(x,\xi) = -\dot{\varphi}(x,-\xi)$. Thus

$$\int\limits_M \Delta\varphi\; c(x)\; dm(x) = 0$$

for all C^2 function φ on M. It follows that $c(x) \equiv C$.

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The idea of proposition 1 comes from a comment of A. Katok that the normalized measures of the geodesic balls should converge to the Bowen-Margulis measures μ_x constructed by Ledrappier ([L1]). After finishing the first version (section 1–6) of this work, I received a preprint ([L4]) from Professor Ledrappier in which he proves the unique ergodicity for Anosov foliations of geodesic flows, proposition 6.1 as well as formula 1° of our theorem 6.2. Professor Kneiper [Kn] informed me that he also proves formula 2°. Section 7 is inspired by [L4].

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