BOUNDS OF FIXED POINT RATIOS OF PERMUTATION REPRESENTATIONS OF $GL_n(q)$ AND GROUPS OF GENUS ZERO

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Abstract

If G is a transitive subgroup of the symmetric group $Sym(\Omega)$, where Ω is a finite set of order m; and G satisfies the following conditions: $G=\langle S\rangle$, $S=\{g_1,\ldots,g_r\}\subseteq G^{\#}$, $g_1\cdots g_r=1$, and $\sum_{i=1}^{r} c(g_i)=(r-2)m+2$, where $c(g_i)$ is the number of cycles of g_i on Ω, then G is called a group of genus zero. These conditions correspond to the existance of an m-sheeted branched covering of the Riemann surface of genus zero with r branch points. The fixed point ratio of an element g in G is defined as $f(g)/|\Omega|$, where $f(g)$ is the number of fixed points of g on Ω. In this thesis we assume that G satisfies $L_n(q)\leq G\leq PGL_n(q)$ and G is represented primitively on Ω. The primitive permutation representations of G are determined by the maximal subgroups of G. We obtain upper bounds for fixed point ratios of the semisimple and unipotent elements of G. The bounds are expressed as rational functions which depend on n, q, the rational canonical forms of the elements, and the maximal subgroups. Then those bounds are used to prove the following:

Theorem: If G is a group of genus zero, then one of the following holds: (a) $q=2$ and $n\leq 32$, (b) $q=3$ and $n\leq 12$, (c) $q=4$ and $n\leq 11$, (d) $5\leq q\leq 13$ and $n\leq 8$, (e) $16\leq q\leq 83$ and $n\leq 4$, (f) $89\leq q\leq 343$ and $n=2$.

Thus for those G satisfying $L_n(q)\leq G\leq PGL_n(q)$, this theorem confirms the J. Thompson's conjecture which states that except for $\mathbb{Z}_p$, A_k with $k\geq 5$, there are only finitely many finite simple groups which are composition factors of groups of genus zero.

Contents

Introduction

Let $f:X\to Y$ be a branched covering of compact Riemann surfaces. That is f is a continuous surjective map preserving the analytic structures on X and Y. It turns out that for each $x\in X$, there exists a neighborhood N of x such that $f:N\to f(N)$ is conformally equivalent to the map $z\mapsto z^{e(x)}$ for some positive integer $e(x)$ in terms of local coordinates. The integer $e(x)$ is called the ramification index at x. There is a positive integer m such that at any $y\in Y$, $\sum_{x\in f^{-1}(y)} e(x)=m$. Moreover there is a finite set $B=\{b_1,\dots,b_r\}=\{b\in Y:|f^{-1}(b)|<m\}$ of branch points such that $f:X-f^{-1}(B)\to Y-B$ is an m-sheeted topological covering. Thus from elementary algebraic topology, we have that f induces an injection $f_*:\pi_1(X-f^{-1}(B))\to\pi_1(Y-B)$ of the fundamental groups with $|\pi_1(Y-B):\pi_1(X-f^{-1}(B))|=m$. We call f an *m-sheeted branched covering* with branch points B. The genus g of X and the genus h of Y are related by the Riemann-Hurwitz formula: $\sum_{x\in X}\{e(x)-1\}=2\{g+m(1-h)-1\}$.

We say that two branched coverings $f_i:X_i\to Y$ are equivalent if there exists an isomorphism of Riemann surfaces $\alpha:X_1\to X_2$ such that $f_2\circ\alpha=f_1$. There is a natural bijection β between the set of equivalence classes of m-sheeted branched coverings of Y with branch points in $B=\{b_1,\dots,b_r\}\subset Y$ and the set of equivalence classes of transitive permutation representations of $\pi_1(Y-B,b_0)$ of degree m, where b_0 is a base point.

Suppose that Y is of genus zero; that is Y is conformally equivalent to the Riemann sphere $\mathbb{P}^1=\mathbb{C}\cup\{\infty\}$. Let y_i be the path class in $\pi_1(\mathbb{P}^1-B,b_0)$ around b_i. Then $\pi_1(\mathbb{P}^1-B,b_0)$ is the free group on $\{y_i:1\le i\le r\}$ modulo the relation $y_1\cdots y_r=1$. If $\sigma:\pi_1(\mathbb{P}^1-B,b_0)\to S_m$ is the permutation representation supplied by the bijection β above, then $\sigma(y_i)$ has cycle lengths $e(x)$, where $x\in f^{-1}(b_i)$. Thus the Riemann-Hurwitz formula gives $\sum_{i=1}^{r} Ind(y_i)=2(g+m-1)$, where $Ind(y_i)=m-c(y_i)$ and $c(y_i)$ is the number of cycles of $\sigma(y_i)$.

We say that a transitive subgroup G of $Sym(\Omega)$ is a *group of genus* g if there exists $S=\{g_1,\ldots,g_r\}\subseteq G^{\#}$ with $G=\langle S\rangle$, $g_1\cdots g_r=1$, and $\sum_{i=1}^{r} c(g_i)=(r-2)m-2g+2$. If G is a group of genus g with $|\Omega|=m$ and $|S|=r$, we call the triple (G,Ω,S) a *genus* g *system* of degree m and size r. The relation between branched coverings of $\mathbb{P}^1$ of genus g and groups of genus g is exhibited in the following

Riemann's Existence Theorem: There is a natural bijection between:

(a) Equivalence classes of m-sheeted branched covering of $\mathbb{P}^1$ of genus g with branch points $B=\{b_1,\ldots,b_r\}$.

(b) $Sym(\Omega)$-conjugacy classes of genus g systems (G,Ω,S) of degree m with $|S|=r$.

For a proof of this theorem, see [Fr2].

The field of meromorphic functions $\mathbb{C}(Y)$ is by definition the set of all branched coverings $\phi:Y\to\mathbb{P}^1$. Thus the covering $f:X\to Y$ induces an injection $f^*:\mathbb{C}(Y)\to\mathbb{C}(X)$ by $\phi\mapsto\phi\circ f$, i.e., $\mathbb{C}(X)$ is an extension field of $\mathbb{C}(Y)$. Moreover the Galois group $Gal(\mathbb{K}/\mathbb{C}(Y))$ is the image of $\pi_1(Y-B)$ in S_m under the bijection β above, where $\mathbb{K}$ is the Galois closure of $\mathbb{C}(X)$ over $\mathbb{C}(Y)$. This remark shows that the study of groups of genus g is useful in the *inverse Galois problem*; i.e., the problem of showing that each finite group is a Galois group over $\mathbb{Q}$.

It is well known that for each G which is either $\mathbb{Z}_p$, p a prime, or A_m with $m\geq 5$, there is a cover $\psi:\mathbb{P}^1\to\mathbb{P}^1$ such that G is a group of genus zero corresponding to ψ under the bijection in Riemann's existence theorem. Let $E(g)$ be the set of all composition factors other than $\mathbb{Z}_p$ and A_m of all groups of genus g. It was conjectured by J. Thompson that $E(g)$ is a finite set for each $g\geq 0$ (cf [GT]). Since for K a composition factor of a group of genus zero, K is also a composition factor of a group of genus g for any g, $E(0)$ plays a special role in the study of $E(g)$. From now on, we assume $g=0$. In this case, Thompson has observed that the conjecture reduces to the case that G is primitive, i.e., if K is a composition factor of a group of genus zero, then K is also a composition factor of a primitive group of genus zero, see Corollary 2.4 in [GT]. In this case, the general structure of G is described in the following theorem about maximal subgroups by M. Aschbacher and L. Scott (cf [AS]):

Theorem: Suppose G is a finite group and H is a maximal subgroup of G such that $\bigcap_{g\in G} H^g=1$. Let Q be a minimal normal subgroup of G, and L be a minimal normal subgroup of Q. Let $\Delta=\{L_1,\ldots,L_t\}$ be the set of G-conjugates of L. Then $G=HQ$ and precisely one of the following holds:

(a) L is of prime order p.

(b) $F^*(G)=Q$ and $H\cap Q=H_1\times\cdots\times H_t$, where $H_i=H\cap L_i\neq 1$, $1\leq i\leq t$.

(c) $F^*(G)=Q\times R$ with $Q\simeq R$ and $H\cap Q=1$.

(d) $F^*(G)=Q$, $H\cap L=1$, $H\cap Q\neq 1$.

(e) $F^*(G)=Q$, $H\cap Q=1$, and $|L|\neq p$.

There have been some results on groups of genus zero concerning the above cases (a), (c), (d) and (e); see [GT] and [Ne] for case (a), and [Sh], [As4], [GT] for cases (c), (d), and (e) respectively. The case (b) still remains open. These results essentially reduce the problem of groups of genus zero down to the case that G is almost simple.

In this thesis, we consider a subcase of (b), that is $Q=PSL_n(q)$. More precisely, we assume that $\overline{G}$ is a group such that $L_n(q)\leq\overline{G}\leq PGL_n(q)$, where for $n=2$, $q\geq 4$, and $\overline{G}$ has a faithful primitive permutation representation on $\overline{\Omega}$. This is equivalent to requiring that for $\alpha\in\overline{\Omega}$, $\overline{H}=\overline{G}_\alpha$ is a maximal subgroup of $\overline{G}$ with $L_n(q)\nleq\overline{H}$. In that event the permutation representation of $\overline{G}$ on $\overline{\Omega}$ is equivalent to its representation by right multiplication as a subgroup of $Sym(\overline{G}/\overline{H})$. Our result is the following:

Theorem A: *If $\overline{G}$ is a group of genus zero, then one of the following holds:*

(a) $q=2$ and $n \leq 32$.

(b) $q=3$ and $n \leq 12$.

(c) $q=4$ and $n \leq 11$.

(d) $5 \leq q \leq 13$ and $n \leq 8$.

(e) $16 \leq q \leq 83$ and $n \leq 4$.

(f) $89 \leq q \leq 343$ and $n=2$.

This theorem confirms that Thompson's conjecture is true in the case that $\overline{G}$ satisfies the condition $L_n(q) \leq \overline{G} \leq PGL_n(q)$. Let G, H be the preimages of $\overline{G}$, $\overline{H}$ in $GL_n(q)$ rspectively in the rest of this introduction. The approach used in this thesis is roughly the following: we distinguish the representation of $\overline{G}$ on $\overline{\Omega}$ according to whether H is:

(a) reducible.

(b) irreducible and contains a transvection.

(c) irreducible and contains no transvection.

In case (a), the representation is equivalent to the representation on Λ_d, where Λ_d is the set of all d dimensional subspaces of the n dimensional vector space V over $GF(q)$.

Case (b) splits into two subcases:

(b_1) V is a nontrivial direct sum of a set of subspaces stabilized by $\overline{H}$.

(b_2) $\overline{H}$ is primitive on V, i.e., V is not a nontrivial direct sum of subspaces stabilized by $\overline{H}$.

In the first case, the representation is equivalent to the representation on $\Lambda_{k,l}$, where $kl=n$ and

$$\Lambda_{k,l}=\{\{V_1, V_2, \ldots, V_l\} : V=V_1 \oplus V_2 \oplus \cdots \oplus V_l,\ dim(V_j)=k\ \forall j\}.$$

In the second case, a list of maximal subgroups is extracted from W. Kantor's paper [Ka].

In case (c), we appeal to the paper by M. Aschbacher [As2] on maximal subgroups of the classical groups and a theorem of M. Liebeck bounding the order of almost simple subgroups of the classical groups.

In each case, we obtain bounds for the fixed point ratio of unipotent and semisimple elements of $\overline{G}$. For a permutation representation of $\overline{G}$ on $\overline{\Omega}$, the fixed point ratio $\mathcal{N}(\overline{g})$ of $\overline{g}\in\overline{G}$ is defined as $f(\overline{g})/|\overline{\Omega}|$, where $f(\overline{g})$ is the number of fixed points of $\overline{g}$ on $\overline{\Omega}$. The fixed point ratio comes into play in this problem as follows: Define $\mathcal{U}(\overline{g})=c(\overline{g})/|\overline{\Omega}|$. Then the Riemann-Hurwitz formula gives the bound $\sum_{i=1}^{r}\mathcal{U}(\overline{g}_i)>r-2$ when $(\overline{G},\overline{\Omega},\overline{S})$ is a genus zero system and $\overline{S}=(\overline{g}_1,\dots,\overline{g}_r)$. Further $\mathcal{U}(\overline{g})$ can be expressed in terms of fixed point ratios of $\overline{g}^i$ (see (1.1)), so this bound translates to a bound involving fixed point ratios.

The concept of fixed point ratio is of some interest in contexts other than this problem. For example, there is a conjecture which states that if G is a finite almost simple group represented primitively and faithfully on X with $\mathcal{N}(g)\geq\frac{1}{2}$ for some nonidentity element g of G, then G is isomorphic to S_m, A_m, or $Sp_{2m}(2)$ for some m, (or a slight modification obtained by adding a few more examples). A positive solution to this conjecture can be used to study the subgroup structure of A_m. Moreover, P. Kleidman has solved a conjecture of Wielandt using the notion of fixed point ratio. Thus the following result giving bounds for fixed point ratios for $\overline{G}$ between $L_n(q)$ and $PGL_n(q)$ is of independent interest:

Theorem B: Suppose $L_n(q)\leq\overline{G}\leq PGL_n(q)$ and $\overline{G}$ is represented primitively by right multiplication on $\overline{\Omega}=\overline{G}/\overline{H}$. Let $\overline{g}\in\overline{G}$ and assume that $\overline{g}$ is of prime order. Then one of (a), (b), or (c) holds:

(a) $\overline{H}$ is the stabilizer of a d-dimensional subspace and $\mathcal{N}(\overline{g})\leq\frac{1}{q^d}+\frac{1}{q^{n-d}}$.

(b) One of the following holds:

(1) $\overline{H}$ is the stabilizer of a direct sum decomposition $V=V_1\oplus V_2\oplus\cdots\oplus V_l$, with $dim(V_j)=k$ $\forall j$ and $l\geq 2$. Also $\mathcal{N}(\overline{g})\leq\frac{4}{q^{n-k}}$ if $q\geq 3$ and $\mathcal{N}(\overline{g})\leq\frac{56}{2^{n-k}}$ if $q=2$.

(2) $\overline{H}=\overline{G}\cap PGL_n(q_1)$, where $q=q_1^r$, r a prime; and $\mathcal{N}(\overline{g})\leq(n,q-1)\left(\frac{q_1}{q_1-1}\right)^{n-1}\left(\frac{q_1}{q}\right)^n$ if $\overline{g}$ is unipotent, $\mathcal{N}(\overline{g})\leq(n,q-1)\cdot\frac{q-1}{q}\left(\frac{q_1}{q_1-1}\right)^{n+1}\left(\frac{q_1}{q}\right)^{2n-3}$ if $\overline{g}$ is semisimple.

(3) $\overline{H}=\overline{G}\cap PGU_n(q_1)$, with $q=q_1^2$, $n\geq 3$; and $\mathcal{N}(\overline{g})\leq\left(\frac{5}{2}\right)\cdot\frac{(n,q_1^2-1)}{q_1^{2(n-2)}}$ if $\overline{g}$ is unipotent, $\mathcal{N}(\overline{g})\leq\frac{(n,q_1^2-1)}{(q_1-1)^{2n-3}}$ if $\overline{g}$ is semisimple.

(4) $\overline{H}=\overline{G}\cap\overline{\Delta}$, where $\overline{\Delta}=\Delta(\mathrm{V},\mathrm{f}_0)/\mathrm{Z}$, $Z=Z(GL_n(q))$, f_0 a symplectic form on an $n=2l$ dimensional vector space V over $GF(q)$ with $l\geq 2$, $\Delta(V,f_0)$ the group of all similarities; and $\mathcal{N}(\overline{g})\leq\frac{(2l,q-1)}{(q-1)^l q^{l-2}}$ if $\overline{g}$ is unipotent; $\mathcal{N}(\overline{g})\leq\frac{(2l,q-1)}{(q-1)^{l-1}q^{l-1}}$ if $\overline{g}$ is semisimple and $l\geq 3$; and $\mathcal{N}(\overline{g})\leq\frac{(4,q-1)(q+1)(q^2+2)}{q^2(q^3-1)}$ if $\overline{g}$ is semisimple and $l=2$.

(5) $\overline{G}=L_3(4)$ and $\overline{H}=A_6$.

(6) $\overline{G}=L_2(9)$ and $\overline{H}=A_5$.

(c) The preimage H does not contain a transvection. For $\overline{g}$ unipotent, let ν be the number of Jordan blocks. Then $\mathcal{N}(\overline{g})\leq\frac{1}{q^{n-\nu}}$, if the smallest dimension of Jordan blocks of g is 1; and $\mathcal{N}(\overline{g})\leq\frac{1}{q^{n-\nu-1}}$, if all blocks have dimension at least 2; and if g is a transvection, then $\mathcal{N}(\overline{g})=0$. For $\overline{g}$ semisimple, let c be the degree of $min(g)$. Then either $\mathcal{N}(\overline{g})\leq\frac{1}{q^{n-\mu}}$, where μ is the smallest dimension of homogeneous components of g; or one of the following holds:

(1) $H\simeq GL_{\frac{n}{r}}(q^r)\cap G$, where r is a prime dividing n. H acts on V', where V' is V considered as $\frac{n}{r}$ dimensional vector space over the field $GF(q^r)$. $\mathcal{N}(\overline{g})\leq\frac{(n,q-1)}{q^{(n-\frac{n}{c}-1)(n-\frac{n}{r})}(q-1)^{n-\frac{n}{r}}}$ if $r\nmid c$; $\mathcal{N}(\overline{g})\leq\frac{(n,q-1)\binom{m+r-1}{r-1}}{q^{(n-\frac{n}{c}-1)(n-\frac{n}{r})-n}(q-1)^{2n-\frac{n}{r}}}$ if $r|c$, where $m=\frac{n}{c}$. If $n=2$, then $r=c=2$, and $\mathcal{N}(\overline{g})\leq\frac{4}{q(q-1)}$.

(2) $H\simeq GL(V_1)*GL(V_2)\cap G$, where V_1, V_2 are l, m dimensional vector spaces over $GF(q)$ respectively; and $n=lm$, $l\neq m$, $l\neq 1$, $m\neq 1$, $V=V_1\otimes V_2$. Moreover there is a homomorphism $\pi: GL(V_1)\times GL(V_2)\rightarrow GL(V)$ by $(v_1\otimes v_2)(g_1,g_2)\pi=v_1g_1\otimes v_2g_2$, and $g=(g_1,g_2)\pi$. $\mathcal{N}(\overline{g})\leq(n,q-1)\Big/q^{(l^2-2)(m^2-1)(1-\frac{1}{c})-1-(n+l)}(q-1)^{n+l}$ if $c|l$ but $c\nmid m$; $\mathcal{N}(\overline{g})\leq(n,q-1)\Big/q^{(m^2-2)(l^2-1)(1-\frac{1}{c})-1-(n+m)}(q-1)^{n+m}$ if $c|m$ but $c\nmid l$; $\mathcal{N}(\overline{g})\leq\Big\{(n,q-1)\Big/$

$q^{(l^2-2)(m^2-1)(1-\frac{1}{c})-1-(n+l)}(q-1)^{n+l}\Big\}+\Big\{(n,q-1)\Big/q^{(m^2-2)(l^2-1)(1-\frac{1}{c})-1-(n+m)}(q-1)^{n+m}\Big\}$ if $c|l$ and $c|m$.

(3) $H=N_G(R)$, where $n=r^m$ is a power of prime $r\neq p$ and R is an r-group of symplectic type such that $|R{:}Z(R)|=r^{2m}$. Also R is of exponent r if r is odd and of exponent 4 if $r=2$. Moreover $|Z(R)|>2$, $q=p^e$, where $e=|p|$ in the group of units U of $\mathbb{Z}_{r^k}$ with $r^k=|Z(R)|$, and e is required to be odd. $\frac{H}{Z}\simeq C_{Aut(R)}(Z(R))\simeq E_{r^{2m}}\cdot Sp_{2m}(r)$. If $r=2$, then $q=p$. $\mathcal{N}(\overline{g})\leq (n,q-1)n^{2m+3}\Big/\Big\{q^{n^2(1-\frac{1}{c})-n}\ (q-1)^{n-1}\Big\}$.

(4) $H=DS_m\cap G$, where $D\simeq GL(V_1)*GL(V_2)*\cdots*GL(V_m)$, each V_i is a k-dimensional vector space over $GF(q)$, $V=V_1\otimes V_2\otimes\cdots\otimes V_m$, $n=k^m$, $m>1$, DS_m is the semidirect product of D by S_m, and there exists a homomorphism $\pi{:}GL(V_1)wrS_m\rightarrow GL(V)$ by $(v_1\otimes v_2\otimes\cdots\otimes v_m)(x,(g_1,\ldots,g_m))\pi=v_{1x^{-1}}g_1\otimes v_{2x^{-1}}g_2\otimes\cdots\otimes v_{mx^{-1}}g_m$, $x\in S_m$, and $g=(x,(g_1,\ldots,g_m))\pi$. $\mathcal{N}(\overline{g})\leq(n,q-1)\Big/\Big\{(m-1)!\,q^{(k^{2m}-k^2)(1-\frac{1}{c})-(k^m+k)}(q-1)^{k^m+k}\Big\}$ or $\mathcal{N}(\overline{g})\leq(n,q-1)m!\Big/\Big\{q^{k^{2m}(1-\frac{1}{c})-k^2(m-\frac{1}{c})-(k^m+k)}(q-1)^{k^m+k}\Big\}$.

(5) $H=O^+_{2m}(q)GF(q)^{\#}\cap G$, p odd, $n=2m$, c even, and $\mathcal{N}(\overline{g})\leq(n,q-1)(q^{\frac{n}{2}}-1)\Big/\Big\{q^{\frac{1}{2}n^2(1-\frac{2}{c})}(q-1)^{n-1}\Big\}$.

(6) $\mathcal{N}(\overline{g})\leq\dfrac{(n,q-1)}{q^{(n-1)(n-\frac{n}{c}-3)-3}(q-1)^{n-\frac{n}{c}}}$.

For some special cases in this theorem (for example, when $n=2$ or 3 in the case (b)(2)), there are better bounds contained in the body of this thesis. Also note that in some cases, the bounds are not useful, e.g., in case (b)(1) when $q=2$ and $2^{n-k}\leq 56$. For the cases (b)(5) and (b)(6), the bounds can be calculated easily from the character tables. Since all these three cases fall into that finite number of exceptions for groups of genus zero, we don't need their bounds in the process of proving Theorem A.

To apply these bounds for the purpose of reducing the genus zero problem to a finite number of exceptions, a certain amount of calculation is involved. Define $\mathcal{M}(\overline{g})=max\{\mathcal{N}(\overline{x}) : \overline{x}\in\langle\overline{g}\rangle^{\#}\}$. If $\mathcal{M}(\overline{g})\leq\frac{1}{85}$ for all $\overline{g}\in\overline{S}$, then $(\overline{G},\overline{\Omega},\overline{S})$ is not a genus zero system (cf. (2.5)). The first step of the reduction is done by applying the threshold $\frac{1}{85}$. This enables us to eliminate immediately those $\overline{G}$'s with q sufficiently large.

However for small q's, there are elements $\overline{g}$ with $\mathcal{N}(\overline{g})>\frac{1}{85}$. This is where most of the case analyses occurs, especially in the case that $\overline{H}$ is the stablizer of a point, i.e., a one dimensional subspace. In these cases, although the calculations are rather messy, they are entirely elementary. For a generating set $\overline{S}=(\overline{g}_1,\ldots,\overline{g}_r)$, we define the type of $\overline{S}$ as $(|\overline{g}_1|,\ldots,|\overline{g}_r|)$. In the bulk of the case analyses, to show that for a certain type of a generating set $\overline{S}$ the triple $(\overline{G},\overline{\Omega},\overline{S})$ cannot be a *GZS* (i.e., *genus zero system*), we usually show that $\sum_{i=1}^{r}\mathcal{U}(\overline{g}_i)>r-2$ cannot hold. But for some 'small' cases (e.g., $\overline{S}$ is of type (2,3,8), (2,4,5), etc.), a closer look is needed; and usually the analysis is done via elementary number theoretic means (e.g., congruence), to show that the equality $\sum_{i=1}^{r}\mathcal{U}(\overline{g}_i)=(r-2)+\frac{2}{|\Omega|}$ actually cannot hold for n sufficiently large. In some rough sense, the amount of calculation involved in each individual case depends on how big the fixed point ratio can get.

For example, when $\overline{H}$ is the stablizer of a point, for $q=2$, the fixed point ratio of a transvection is $\frac{1}{2}-\frac{1}{2(2^n-1)}$; and for $q=3$, the fixed point ratio of a pseudo-reflection is $\frac{1}{3}+\frac{4}{3(3^n-1)}$. A big portion of the calculation in this thesis is done for such cases. It seems to me that a certain amount of calculation is unavoidable, unless we can better untilize the two conditions, $\langle\overline{g}_1,\ldots,\overline{g}_r\rangle=\overline{G}$ and $\overline{g}_1\cdots\overline{g}_r=\overline{1}$, to exclude more possibilities for small q's and not rely so heavily upon $\sum_{i=1}^{r}c(\overline{g}_i)=(r-2)|\Omega|+2$ to get contradictions.

Now some comments about the notation. In the following chapters, the symbols listed below always have the meanings explained here unless stated otherwise. $\mathbb{F}=GF(q)$ is the field of $q=p^e$ elements of characteristic p. V is the n dimensional vector space over $GF(q)$. $\overline{G}$ satisfies $L_n(q)\leq\overline{G}\leq PGL_n(q)$ and the bar notation indicates that $\overline{G}$, $\overline{g}$ are the images of G, g under the projective map $P{:}GL(V)\rightarrow PGL(V)$. Z is the center of $GL(V)$. $Sym(\Omega)$ is the symmetric group on the set Ω. Most group theoretical notations used in the following chapters are fairly standard. $\lceil x \rceil$ denotes the least integer greater than or equal to x, and $\lfloor x \rfloor$ is the largest integer less than or equal to x. Other notations will be defined in the body of this thesis when they are needed.

Chapter I
Preliminary Results and Miscellaneous Facts

Section 1. Lemmas on Permutation Representations.

Let $\pi: G \to Sym(\Omega)$ be a permutation representation. For $g \in G$, denote $f(g) = |Fix(g\pi)|$ and $c(g) = c(g\pi)$, where $Fix(g\pi)$ is the set of fixed points of g on Ω and $c(g\pi)$ is the number of cycles of $g\pi$. Define $\mathcal{U}(g) = c(g)/|\Omega|$, $\mathcal{N}(g) = f(g)/|\Omega|$, and $\mathcal{M}(g) = max\{\mathcal{N}(x) : x\pi \in \langle g\pi \rangle^{\#}\}$. The material in this section is well known, see for example [As4]. For completeness and easy reference, we include some of the proofs.

(1.1) Suppose we have a permutation representation $\pi: G \to Sym(\Omega)$, not necessarily faithful. Let $g \in G$. Then $c(g) = \frac{1}{|g|} \sum_{d \| g|} \phi(\frac{|g|}{d}) f(g^d)$, where ϕ is the Euler's ϕ-function.

Proof. Denote $\alpha = |g|$ and $\beta = |g\pi|$. So $\alpha = \beta\gamma$ for some γ. Let's count the sum s of the numbers of fixed points $f(g^i)$ as i ranging from 1 to α in two different ways. Let $d|\alpha$ and k be such that $(\frac{\alpha}{d}, k) = 1$. Then as $(\beta, dk) = (\beta, d)$, we have $\langle (g\pi)^{dk} \rangle = \langle (g\pi)^d \rangle$; which gives $f(g^{dk}) = f(g^d)$. Hence $s = \sum_{d|\alpha} \phi(\frac{\alpha}{d}) f(g^d)$. On the other hand, each point in a t-cycle of $g\pi$ appears as the fixed point in $(g\pi)^i$ exactly when $i = tm$, where $1 \le m \le \frac{\alpha}{t}$. In this way, we collect α fixed points for each cycle. Hence $s = \alpha c(g)$. So we have the conclusion.

Remark. In particular, (1.1) says that no matter which preimage g of $g\pi$ we choose, the expression $\frac{1}{|g|} \sum_{d \| g|} \phi(\frac{|g|}{d}) f(g^d)$ always gives us the same thing, that is $c(g)$, although we might have $|g| \neq |g'|$ for a different preimage g' of $g\pi$. Hence, when we calculate $\mathcal{U}(g\pi)$, we can choose any preimage g.

(1.2) Suppose $\pi: G \to Sym(\Omega)$ is a permutation representation. Then

(a) $c(g) \leq c(g^i)$, $\mathcal{U}(g) \leq \mathcal{U}(g^i)$, $f(g) \leq f(g^i)$, $\mathcal{N}(g) \leq \mathcal{N}(g^i)$, $\mathcal{N}(g) \leq \mathcal{M}(g)$.

(b) If π is faithful, then $\mathcal{U}(g) \leq \frac{1}{|g|}(1+\mathcal{M}(g)(|g|-1))$.

(c) If the triple (G,Ω,S) is a GZS, then $\sum_{i=1}^{r} \mathcal{U}(g_i) > r-2$.

(1.3) Suppose a finite group G has a transitive permutation representation on Ω and H is the stabilizer of a point ω_0 in Ω. Then $\mathcal{N}(g) = \frac{|g^G \cap H|}{|g^G|}$.

Proof. By counting the set $X = \{(g^x, \omega) : x \in G,\ \omega \in \Omega,\ \omega g^x = \omega\}$ in two different ways, we have $f(g)|g^G| = |X| = |\Omega||g^G \cap H|$.

Section 2. Lemmas on Groups of Genus Zero.

In this section, let (G,Ω,S) be a GZS. For $S = \{g_1, \cdots, g_r\}$, define the type of S to be $(|g_1|, \cdots, |g_r|)$. For the proofs of (2.1), (2.2), and (2.3), see [As4].

(2.1) Let $S = \{g_1, \cdots, g_r\} \subseteq G^{\#}$ with S of type $(m_1, \cdots m_r)$. Then for each $\sigma \in Sym(\{1, \cdots, r\})$, there exists $T = \{h_1, \cdots, h_r\} \subseteq G^{\#}$ such that (G,Ω,T) is a GZS of type $(m_{1\sigma}, \cdots, m_{r\sigma})$ and $g_{i\sigma} \in h_i^G$.

(2.2) Assume S is of type (k,l,m) with $k \leq l \leq m$ and $1/k + 1/l + 1/m \geq 1$. Then either G is solvable or $(k,l,m) = (2,3,5)$ and $G \simeq A_5$.

(2.3) If S is of type (2,2,2,2), then G is solvable.

(2.4) (a) If $\mathcal{M}(g) \leq \frac{1}{6}$ or if $\mathcal{U}(g) \leq \frac{2}{3}$ $\forall g \in S$, then $r = |S| \leq 5$.

(b) If $\mathcal{M}(g) \leq \frac{1}{10}$ or if $\mathcal{U}(g) \leq \frac{3}{5}$ $\forall g \in S$, then $|S| \leq 4$.

(c) Suppose G is not solvable. If $\mathcal{M}(g) \leq \frac{1}{24}$ or if $\mathcal{U}(g) \leq \frac{1}{2}$ $\forall g \in S$, then $|S| = 3$.

Proof. Let $g \in S$. Since $\mathcal{U}(g) \leq \frac{1}{|g|} + \mathcal{M}(g) \leq \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, we have $r-2 < \sum_{i=1}^{r} \mathcal{U}(g_i) \leq \frac{2r}{3}$. So $r < 6$, and thus $r \le 5$. For part (b), we have that $\mathcal{U}(g) \le \frac{1}{2}+\frac{1}{10}=\frac{3}{5}$, and then $r-2<\sum_{i=1}^{r}\mathcal{U}(g_i)\le\frac{3r}{5}$ implies $r<5$, so $r\le 5$. For part (c), if $\mathcal{U}(g)\le\frac{1}{2}$ $\forall g\in S$, then we have that $r-2<\sum_{i=1}^{r}\mathcal{U}(g_i)\le\frac{r}{2}$, which implies $r<4$, so $r\le 3$. Then as G is not solvable, $r\ne 2$. So $|S|=3$. If $\mathcal{M}(g)\le\frac{1}{24}$, then by (b), $|S|\le 4$. If $|S|=4$, then S is not of type (2,2,2,2). So $\mathcal{U}(g)\le\frac{1}{2}+\frac{1}{24}=\frac{13}{24}$ if $|g|=2$; and $\mathcal{U}(g)\le\frac{1}{3}+\frac{1}{24}=\frac{3}{8}$ if $|g|\ge 3$. Thus $\sum_{i=1}^{4}\mathcal{U}(g_i)\le 3\cdot\frac{13}{24}+\frac{3}{8}=2$, a contradiction. So $|S|=3$ again.

(2.5) Suppose G is neither solvable nor $G\simeq A_5$. Then there is at least one $g\in S$ such that $\mathcal{M}(g)>\frac{1}{85}$.

Proof. Suppose there is no such g in S. Then by (2.4)(b), $|S|\le 4$. Since (2.3) says that S is not of type (2,2,2,2), if $|S|=4$, we have that $\sum_{i=1}^{4}\mathcal{U}(g_i)\le\frac{3}{2}+\frac{1}{3}+\frac{4}{85}=\frac{959}{510}<2=|S|-2$, a contradiction. So $|S|=3$, and (2.2) says that $1/|g_1|+1/|g_2|+1/|g_3|<1$. Then it is easy to calculate that $1/|g_1|+1/|g_2|+1/|g_3|$ achieves the maximum when S is of type (2,3,7), i.e., we have $1/|g_1|+1/|g_2|+1/|g_3|\le 1/2+1/3+1/7=\frac{41}{42}$. Under our assumption, $\sum_{i=1}^{3}\mathcal{U}(g_i)\le\sum_{i=1}^{3}\left\{\frac{1}{|g_i|}+(1-\frac{1}{|g_i|})\mathcal{M}(g_i)\right\}\le\sum_{i=1}^{3}\left\{\frac{1}{|g_i|}+(1-\frac{1}{|g_i|})\frac{1}{85}\right\}\le(1-\frac{1}{85})\sum_{i=1}^{3}\frac{1}{|g_i|}+\frac{3}{85}\le(1-\frac{1}{85})\frac{41}{42}+\frac{3}{85}=1$, which is a contradiction. So at least one $g\in S$ is such that $\mathcal{M}(g)>\frac{1}{85}$.

Remark. This crucial fraction $\frac{1}{85}$ appears in [GT] too. Since we assume G is neither solvable nor $G\simeq A_5$, it follows from (2.2) and (2.3) by purely elementary means as we saw in (2.5). This threshold $\frac{1}{85}$ plays an important role in eliminating those $\overline{G}$'s with n and q sufficiently large, as we will see later.

(2.6) Suppose G is neither solvable nor $G\simeq A_5$. Also suppose that S is of type (k,l,m) with $k\le l\le m$ and for all $g\in G$ of prime order, $\mathcal{M}(g)\le\lambda$. Then

(a) $1>1/k+1/l+1/m>(1-3\lambda)/(1-\lambda)$.

(b) $k<3(1-\lambda)/(1-3\lambda)$.

(c) If $\mathcal{M}(g_1) \leq a \leq \lambda$, then $l < 2(1-\lambda) \Big/ \Big\{(1-\frac{1}{k})(1-a) - 2\lambda\Big\}$.

(d) If $\mathcal{M}(g_1) \leq a \leq \lambda$ and $\mathcal{M}(g_2) \leq b \leq \lambda$, then $m < (1-\lambda) \Big/ \Big\{(1-\frac{1}{k})(1-a) + (1-\frac{1}{l})(1-b) - (1+\lambda)\Big\}$ provided that the right hand side is positive.

Proof. Since G is neither solvable nor $G = A_5$, $1/k + 1/l + 1/m < 1$ by (2.2). Suppose $\mathcal{M}(g_1) \leq a$, $\mathcal{M}(g_2) \leq b$, and $\mathcal{M}(g_3) \leq c$. Then by (1.2)(b), $1 = r-2 < \sum \mathcal{U}(g_i) \leq \frac{1}{k}(1+(k-1)a) + \frac{1}{l}(1+(l-1)b) + \frac{1}{m}(1-(m-1)c) = a+b+c+(1-a)/k+(1-b)/l+(1-c)/m$; call this inequality $(*)$. If we let $a=b=c=\lambda$ in $(*)$, then $1-3\lambda < (1-\lambda)(\frac{1}{k}+\frac{1}{l}+\frac{1}{m}) \leq 3(1-\lambda)/k$, which implies (b) and the latter part of (a). If we substitute $b=c=\lambda$ in $(*)$, then $(1-\text{a})(1-\frac{1}{k}) < 2\lambda + (1-\lambda)(\frac{1}{l}+\frac{1}{m}) \leq 2\lambda + 2(1-\lambda)/l$. Since $(1-\text{a})(1-\frac{1}{k}) - 2\lambda > 0$, we have (c). Similarly, by letting $c=\lambda$ in $(*)$, we can show (d).

Section 3. Elementary Properties Related to $GL_n(q)$.

In this section, we collect some elementary properties about the permutation representations of $GL_n(q)$, or $PGL_n(q)$; about unipotent and semisimple elements of $GL_n(q)$; and about the elements which generates $GL_n(q)$. These properties will be used in the subsequent sections.

Let $\overline{G}$ and $\overline{\Omega}$ be as in the introduction. Let $\overline{H}$ be the stabilizer of a point of $\overline{\Omega}$ in $\overline{G}$. Suppose G is a subgroup of $GL_n(q)$ such that the image GP under the projective map P is such that $GP = \overline{G}$. Denote the restriction of P to G by P_1. Let $H = \overline{H} P_1^{-1}$, the preimage of $\overline{H}$ in G. So $Z(G) = ker(P_1) = Z \cap G$, $Z(G) \leq H$, and H is maximal in G. Since the representation of $\overline{G}$ is faithful, H does not contain $SL(V)$.

(3.1) Let G and H be as above, and G be represented on $\Omega = G/H$ by the right multiplication, then $\frac{f(\overline{g})}{|\overline{\Omega}|} = \frac{f(g)}{|\Omega|}$.

Proof. Denote the representation of G on Ω by π, and the representation of $\overline{G}$ on $\overline{\Omega} = \overline{G}/\overline{H}$ by

$\overline{\pi}$. We have the composition $\alpha = P_1\overline{\pi} : G \to Sym(\overline{\Omega})$. The map $\beta : \Omega \to \overline{\Omega}$ defined by $Hx \mapsto \overline{H}\overline{x}$ is an equivalence between π and α. So in particular, $g\pi$ and $g\alpha$ have the same number of fixed points. Thus $\frac{f(g)}{|\Omega|} = \frac{f(g\alpha)}{|\overline{\Omega}|} = \frac{f(\overline{g})}{|\overline{\Omega}|}$.

Remark. Suppose that $\overline{G}$ is a group of genus of zero. Thus $\langle \overline{g}_1, \ldots, \overline{g}_r \rangle = \overline{G}$ and $\overline{g}_1 \cdots \overline{g}_r = \overline{1}$. Let g_i be a preimage of $\overline{g}_i$ in $GL_n(q)$. So we have $g_1 \cdots g_r = \lambda \in Z$. Since $\overline{g_r\lambda^{-1}} = \overline{g}_r$, we can choose g_i's so that $g_1 \cdots g_r = 1$. Let $G_1 = \langle g_1, \ldots, g_r \rangle$. Since $SL_n(q) \leq G_1 Z$, we have $SL_n(q) = O^{p'}(SL_n(q)) \leq O^{p'}(G_1 Z) \leq G_1 \leq GL_n(q)$. In particular, we have that $\langle g_1, \ldots, g_r \rangle$ is absolutely irreducible. For a group of genus zero $\overline{G}$, when we utilize the generator condition $\langle \overline{g}_1, \ldots, \overline{g}_r \rangle = \overline{G}$ and $\overline{g}_1 \cdots \overline{g}_r = \overline{1}$, we usually work inside G_1. In particular, we will use the absolute irreducibility of $\langle g_1, \ldots, g_r \rangle$ quite often. When we utilize the condition $\sum_{i=1}^{r} \mathcal{U}(\overline{g}_i) = (r-2) + \frac{2}{|\Omega|}$, we usually work inside $\overline{G}$; or sometimes it is convenient to think inside G in view of (1.1) and (3.1), especially when it is given the matrix representation of the preimage g_i. Here G is such that $GP = \overline{G}$; and particularly, we often take G to the preimage of $\overline{G}$ in $GL_n(q)$. Finally, note that we can also choose the preimage g_i's so that $g_1 \cdots g_r = \omega$, where ω is a generator for $GF(q)^{\#}$. If this is the case, then $Z \leq \langle g_1, \ldots, g_r \rangle$, and thus $\langle g_1, \ldots, g_r \rangle$ is the preimage of $\overline{G}$ in $GL_n(q)$.

(3.2) (a) Suppose $L_n(q) \leq \overline{G} = \langle \overline{g}_1, \ldots, \overline{g}_r \rangle$. Then $\sum_{i=1}^{r} dim[V, g_i] \geq n$, where each g_i is a preimage of $\overline{g}_i$.

(b) Suppose that $L_n(q) \leq \overline{G} = \langle \overline{g}_1, \overline{g}_2, \overline{g}_3 \rangle \leq PGL_n(q)$ with $\overline{g}_1\overline{g}_2\overline{g}_3 = 1$ and $|\overline{g}_1| = 2$. Then $L_n(q) \leq \langle \overline{g}_2^2, \overline{g}_3 \rangle$.

Proof. For (b), as $\overline{g}_2\overline{g}_3\overline{g}_2^{-1} = \overline{g}_1^{-1}\overline{g}_2^{-1} = (\overline{g}_2\overline{g}_1)^{-1} = (\overline{g}_2^2\overline{g}_3)^{-1} \in \langle \overline{g}_2^2, \overline{g}_3 \rangle = A$, $\overline{g}_2 \in N_{\overline{G}}(A)$. Also $\overline{g}_3 \in N_{\overline{G}}(A)$. Hence $A \trianglelefteq \overline{G} = \langle \overline{g}_2, \overline{g}_3 \rangle$. So $L_n(q) \leq A$ as we assume that when $n=2$, $q \neq 2$ and 3.

(3.3) Assume $G = \langle g_1, \ldots, g_r \rangle \leq GL_n(q)$.

(a) Suppose G is absolutely irreducible. For each $1 \leq i \leq r-1$, suppose E_i is an eigenspace for

g_i corresponding to the eigenvalue λ_i, where λ_i is in the splitting field of $min(g_i)$. Denote $\nu_i=dim\{E_i\}$. Also suppose E is an eigenspace of g_r^t, $1\leq t\leq|g_r|$, corresponding to λ in the splitting field of $min(g_r)$. Denote $\mu=dim\{E\}$. Then $\mu\leq t\{(r-1)n-\sum_{i=1}^{r-1}\nu_i\}$.

(b) Suppose G is irreducible. Let $1\leq t\leq|g_r|$. Denote $\mu=dim\{C_V(g_r^t)\}$, $\nu_i=dim\{C_V(g_i)\}$. Then $\mu\leq t\{(r-1)n-\sum_{i=1}^{r-1}\nu_i\}$.

(c) In particular, suppose G is absolutely irreducible, $r=2$, and $\nu_1=dim\{C_V(g_1)\}$. Then $n\leq|g_2|(n-\nu_1)$. So if g_1 is a transvection, then $n\leq|g_2|$.

Proof. We prove (b). The proof for (a) is similar. Let $W_i=C_V(g_i)\cap C_V(g_i^{g_r})\cap\cdots\cap C_V(g_i^{g_r^{t-1}})$. We have that for each i with $1\leq i\leq r-1$, g_r acts on $U_i=C_V(g_r^t)\cap W_i$. Hence g_r acts on $U=U_1\cap\cdots\cap U_{r-1}$. Also each of $g_1,\ldots, g_{r-1}$ centralizes U. Then $G=\langle g_1,\ldots,g_r\rangle$ acts on U. Since G is irreducible, $U=0$. But as $U=C_V(g_r^t)\cap W_1\cap\cdots\cap W_{r-1}$, and $dim(W_i)\geq t\nu_i-(t-1)n$, we have $0=dim(U)\geq\mu+\sum_{i=1}^{r-1}dim(W_i)-(r-1)n\geq\mu+t\sum_{i=1}^{r-1}\nu_i-t(r-1)n$.

(3.4) (a) For $g\in GL_n(q)$ with $|g|=p^e$, let d be the largest dimension of all Jordan blocks of g. Then $p^{e-1}+1\leq d\leq p^e$.

(b) Let $f=x^t-a\in GF(q)[x]$ with $(t,p)=1$ and $|a|=b$. Let α be the smallest positive integer such that $(tb)|(q^\alpha-1)$. Then $GF(q^\alpha)$ is the splitting field of f over $GF(q)$.

(c) Let $g\in GL_n(q)$ and $\overline{g}$ be the image in $PGL_n(q)$ such that $|\overline{g}|=t$ with $(t,p)=1$ and $g^t=a\in GF(q)^{\#}$. Let b and α have the same meaning as in (b). Then g has a simple submodule of dimension α.

Proof. (c) follows from (b).

(3.5) Denote $\mathcal{N}(g,G/H)$ the fixed point ratio of g on G/H. Assume that $H\leq G=GL(V)$, and $ZSL(V)\leq G_1\leq G$, $H_1=G_1\cap H$. Then $\mathcal{N}(g,G_1/H_1)\leq(n,q-1)\mathcal{N}(g,G/H)$.

Proof. We have that $\mathcal{N}(g,G_1/H_1)=\frac{|g^{G_1}\cap H_1|}{|g^{G_1}|}\leq\frac{|C_{G_1}(g)||g^{G_1}\cap H|}{|G_1|}\leq(n,q-1)\frac{|C_G(g)||g^G\cap H|}{|G|}=$

$(n,q-1)\mathcal{N}(g,G/H)$.

Now a few words about a notation which will be used later. For $g\in GL_n(q)$, denote by $a^\alpha b^\beta\cdots$ the type of g. Here the notation means that if $|g|=p^e$, then the Jordan decomposition of g has α blocks of dimension a, β blocks of dimension b with $b\neq a$, etc. If $(|g|,p)=1$, the notation means that g has a homogeneous component which is the direct sum of α simple module of dimension a, etc. If there are two or more non-isomorphic simple submodule of dimension a, we use subscripts $a_1^{\alpha_1}a_2^{\alpha_2}\cdots$ to distinguish different homogeneous components, where $a_1=a_2=a$, etc.

Section 4. Some Combinatorial Results on Finite Dimensional Vector Spaces over $GF(q)$.

Let $\Lambda_d=\{W\colon W\leq V$ and $dim_{GF(q)}(W)=d\}$. We also denote the order of Λ_d by $\left[{n \atop d} \right]_q$.

(4.1) $\left[{n \atop d} \right]_q=\frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-d+1}-1)}{(q^d-1)(q^{d-1}-1)\cdots(q-1)}$, and $\left[{n \atop d} \right]_q=\left[{n \atop n-d} \right]_q$.

Proof. This is well known.

Remark. In the following, we always use the convention that $\left[{n \atop d} \right]_q=0$ if $d<0$ or $d>n$.

(4.2) Let W be a fixed d-dimensional subspace of V. Then the number of subspaces U such that $W\oplus U=V$ is $q^{d(n-d)}$; the number of m-dimensional subspaces U such that $W\leq U\leq V$ is $\left[{n-d \atop m-d} \right]_q$; and the number of m-dimensional subspaces U such that $W+U=V$ is $\left[{d \atop m+d-n} \right]_q q^{(n-m)(n-d)}$.

Proof. This can be shown by a counting argument.

(4.3) $\sum_\mu q^{(d-\mu)(\nu-\mu)}\left[{n-\nu \atop d-\mu} \right]_q\left[{\nu \atop \mu} \right]_q=\left[{n \atop d} \right]_q=\sum_\mu q^{\mu(n-d-\nu+\mu)}\left[{n-\nu \atop d-\mu} \right]_q\left[{\nu \atop \mu} \right]_q$, for any $0\leq\nu\leq n$.

Proof. Choose a fixed $U\leq V$ with $dim\{U\}=\nu$. Let $W\leq V$ be such that $dim\{W\}=d$ and

$dim\{W\cap U\}=\mu$. So $dim\{W+U\}=d-\mu+\nu$, $W=(W\cap U)\oplus X$ with $dim\{X\}=d-\mu$, and $W+U=U\oplus X$. For each fixed W the number of X is $q^{(d-\mu)\mu}$. For each fixed subspace Y with $U\leq Y$ and $dim\{Y\}=d-\mu+\nu$, the number of X such that $Y=U\oplus X$ is $q^{(d-\mu)\nu}$. So for each such Y, the number of W with $W+U=Y$ is $q^{(d-\mu)(\nu-\mu)}\left[{\nu \atop \mu}\right]_q$. Also the number of such Y is $\left[{n-\nu \atop d-\mu}\right]_q$. Hence $q^{(d-\mu)(\nu-\mu)}\left[{n-\nu \atop d-\mu}\right]_q\left[{\nu \atop \mu}\right]_q$ is the number of W with $dim\{W\cap U\}=\mu$; and thus we have the conclusion. Exchange the role of μ with $d-\mu$, ν with $n-\nu$, we have the second equality.

Remark. $\left[{n \atop d}\right]_q$ corresponds to the binomial coefficient $\binom{n}{d}$, and the formular in (4.3) corresponds to the identity $\sum_{\mu}\binom{n-\nu}{d-\mu}\binom{\nu}{\mu}=\binom{n}{d}$ in binomial coefficients.

(4.4) Suppose $n=kl$. Let $A_{k,l}=\{\{V_1,V_2,\ldots,V_l\}\ :\ V=V_1\oplus V_2\oplus\cdots\oplus V_l,\ dim(V_j)=k\ \forall j\}$. Then $|A_{k,l}|=\frac{1}{l!}q^{\frac{1}{2}n(n-k)}\left[{n \atop k}\right]_q\left[{n-k \atop k}\right]_q\cdots\left[{2k \atop k}\right]_q\left[{k \atop k}\right]_q$.

Proof. It suffices to show that the number of ordered l-tuples $(V_1,V_2,\ldots,V_l)$ such that $V=V_1\oplus V_2\oplus\cdots\oplus V_l$, $dim(V_j)=k\ \forall j$, is $q^{\frac{1}{2}n(n-k)}\left[{n \atop k}\right]_q\left[{n-k \atop k}\right]_q\cdots\left[{2k \atop k}\right]_q\left[{k \atop k}\right]_q$. The factorial $l!$ in the denominator is due to that $\{V_1,V_2,\ldots,V_l\}$ is considered as a set, hence is not ordered. We induct on l. It is true for $l=1$. The number of chioces for V_1 is $\left[{n \atop k}\right]_q$, and for each fixed V_1, the number of U such that $V=V_1\oplus U$ is $q^{k(n-k)}$. Hence by induction, the number of ordered l-tuples $(V_1,V_2,\ldots,V_l)$ here is $q^{\frac{1}{2}(n-k)(n-2k)+k(n-k)}\left[{n \atop k}\right]_q\left[{n-k \atop k}\right]_q\cdots\left[{2k \atop k}\right]_q\left[{k \atop k}\right]_q=q^{\frac{1}{2}n(n-k)}\left[{n \atop k}\right]_q\left[{n-k \atop k}\right]_q\cdots\left[{2k \atop k}\right]_q\left[{k \atop k}\right]_q$.

Section 5. Some Estimates.

The following inequalities are used later to estimate bounds of fixed point ratios.

(5.1) $(q-1)^{\beta}(q+\beta)q^{\frac{1}{2}\beta(2\alpha-\beta-1)-1}\leq(q^{\alpha}-1)(q^{\alpha-1}-1)\cdots(q^{\alpha-\beta+1}-1)\leq q^{\frac{1}{2}\beta(2\alpha-\beta+1)}$.

Proof. As $(q^{\alpha}-1)(q^{\alpha-1}-1)\cdots(q^{\alpha-\beta+1}-1)=(q-1)^{\beta}(q^{\alpha-1}+q^{\alpha-2}+\cdots+q+1)(q^{\alpha-2}+q^{\alpha-3}+\cdots+q+1)\cdots(q^{\alpha-\beta}+q^{\alpha-\beta-1}+\cdots+q+1)=(q-1)^{\beta}\{q^{(\alpha-1)+(\alpha-2)+\cdots+(\alpha-\beta)}+\beta q^{(\alpha-1)+(\alpha-2)+\cdots+(\alpha-\beta)-1}+$

$\cdots\}\geq(q-1)^{\beta}(q+\beta)q^{\frac{1}{2}\beta(2\alpha-\beta-1)-1}$, the first inequality holds. The second one is clear.

(5.2) $q^{\beta(\alpha-\beta)}\leq\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_q\leq\frac{q}{q+\beta}\cdot\left(\frac{q}{q-1}\right)^{\beta}\cdot q^{\beta(\alpha-\beta)}\leq q^{\beta(\alpha-\beta+1)}$, and $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_q\sim q^{\beta(\alpha-\beta)}$ as $q\to+\infty$, where $f(q)\sim g(q)$ means $f(q)$ and $g(q)$ are asymptotically equal as $q\to+\infty$.

Proof. As $\frac{q^{\alpha-i}-1}{q^{\beta-i}-1}\geq q^{\alpha-\beta}$ $\forall 0\leq i\leq\beta-1$; by (4.1), we have $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_q\geq q^{\beta(\alpha-\beta)}$. For the second inequality, we use (5.1). So $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_q\leq\frac{q^{\frac{1}{2}\beta(2\alpha-\beta+1)}}{(q-1)^{\beta}(q+\beta)q^{\frac{1}{2}\beta(\beta-1)-1}}$, but $\frac{1}{2}\beta(2\alpha-\beta+1)-\frac{1}{2}\beta(\beta-1)=\beta(\alpha-\beta)+\beta$.

(5.3) $\left[\begin{smallmatrix}n-\alpha\\ d-\beta\end{smallmatrix}\right]_q\Big/\left[\begin{smallmatrix}n\\ d\end{smallmatrix}\right]_q\leq\frac{1}{q^{(\alpha-\beta)d+\beta(n-\alpha-d+\beta)}}$.

Proof. We have $\left[\begin{smallmatrix}n-\alpha\\ d-\beta\end{smallmatrix}\right]_q\Big/\left[\begin{smallmatrix}n\\ d\end{smallmatrix}\right]_q=\frac{(q^{n-\alpha}-1)\cdots(q^{n-\alpha-d+\beta+1}-1)}{(q^{n}-1)\cdots(q^{n-d+\beta+1}-1)}\cdot\frac{(q^{d}-1)\cdots(q^{d-\beta+1}-1)}{(q^{n-d+\beta}-1)\cdots(q^{n-d+1}-1)}$, and $\frac{q^{i}-1}{q^{j}-1}\leq\frac{1}{q^{j-i}}$ for $j\geq i$, also $\alpha(d-\beta)+\beta(n-2d+\beta)=(\alpha-\beta)d+\beta(n-d-\alpha+\beta)$.

(5.4) (a) $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_q\left[\begin{smallmatrix}n-\alpha\\ d-\beta\end{smallmatrix}\right]_q\Big/\left[\begin{smallmatrix}n\\ d\end{smallmatrix}\right]_q\leq\left(\frac{q}{q-1}\right)^{\beta}\frac{1}{q^{\alpha(d-\beta)+\beta(n-\alpha-2d+2\beta)}}$.

(b) Let $\alpha+\gamma=n$, $\beta+\delta=d$. Then $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_q\left[\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right]_q\Big/\left[\begin{smallmatrix}n\\ d\end{smallmatrix}\right]_q\leq\left(\frac{q}{q-1}\right)^{d}\cdot\frac{q^2}{(q+\beta)(q+\delta)}\cdot\frac{1}{q^{f}}$, and $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_q\left[\begin{smallmatrix}\gamma\\ \delta\end{smallmatrix}\right]_q\Big/\left[\begin{smallmatrix}n\\ d\end{smallmatrix}\right]_q\sim\frac{1}{q^{f}}$, where $f=d(n-d)-\beta(\alpha-\beta)-\delta(\gamma-\delta)$.

Proof. We only prove (a), for (b) is similar. So we have $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_q\left[\begin{smallmatrix}n-\alpha\\ d-\beta\end{smallmatrix}\right]_q\Big/\left[\begin{smallmatrix}n\\ d\end{smallmatrix}\right]_q=xy\left[\begin{smallmatrix}d\\ d-\beta\end{smallmatrix}\right]_q$, where $x=\frac{(q^{\alpha}-1)\cdots(q^{\alpha-\beta+1}-1)}{(q^{n}-1)\cdots(q^{n-\beta+1}-1)}\leq\frac{1}{q^{\beta(n-\alpha)}}$, $y=\frac{(q^{n-\alpha}-1)\cdots(q^{n-\alpha-d+\beta+1}-1)}{(q^{n-\beta}-1)\cdots(q^{n-d+1}-1)}\leq\frac{1}{q^{(d-\beta)(\alpha-\beta)}}$, and $\left[\begin{smallmatrix}d\\ d-\beta\end{smallmatrix}\right]_q=\left[\begin{smallmatrix}d\\ \beta\end{smallmatrix}\right]_q\leq\left(\frac{q}{q-1}\right)^{\beta}q^{\beta(d-\beta)}$. Since $\beta(n-\alpha)+(d-\beta)(\alpha-\beta)-\beta(d-\beta)=\alpha(d-\beta)+\beta(n-\alpha-2d+2\beta)$, we have the conclusion.

(5.5) Suppose $\alpha\geq\beta$. Then

(a) $\sum_{\mu}\left[\begin{smallmatrix}\alpha\\ d-\mu\end{smallmatrix}\right]_q\left[\begin{smallmatrix}\beta\\ \mu\end{smallmatrix}\right]_q\leq\sum_{\mu}\left[\begin{smallmatrix}\alpha+1\\ d-\mu\end{smallmatrix}\right]_q\left[\begin{smallmatrix}\beta-1\\ \mu\end{smallmatrix}\right]_q$.

(b) In particular, $\sum_{\mu}\begin{bmatrix} n-\nu \\ d-\mu \end{bmatrix}_q\begin{bmatrix} \nu \\ \mu \end{bmatrix}_q \leq \begin{bmatrix} n-1 \\ d \end{bmatrix}_q+\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q$ for any ν with $1\leq\nu\leq n-1$.

Proof. Denote $x=\sum_{\mu}\begin{bmatrix} \alpha \\ d-\mu \end{bmatrix}_q\begin{bmatrix} \beta \\ \mu \end{bmatrix}_q$ and $y=\sum_{\mu}\begin{bmatrix} \alpha+1 \\ d-\mu \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ \mu \end{bmatrix}_q$. By (4.3), we have $\begin{bmatrix} \beta-1 \\ \mu \end{bmatrix}_q+\begin{bmatrix} \beta-1 \\ \mu-1 \end{bmatrix}_q q^{\beta-\mu}=\begin{bmatrix} \beta \\ \mu \end{bmatrix}_q=\begin{bmatrix} \beta-1 \\ \mu \end{bmatrix}_q q^{\mu}+\begin{bmatrix} \beta-1 \\ \mu-1 \end{bmatrix}_q$, which gives $x=\sum_{d-\mu\geq\mu}\begin{bmatrix} \alpha \\ d-\mu \end{bmatrix}_q \left\{\begin{bmatrix} \beta-1 \\ \mu \end{bmatrix}_q q^{\mu}+\begin{bmatrix} \beta-1 \\ \mu-1 \end{bmatrix}_q\right\}+\sum_{d-\mu<\mu}\begin{bmatrix} \alpha \\ d-\mu \end{bmatrix}_q\left\{\begin{bmatrix} \beta-1 \\ \mu \end{bmatrix}_q+\begin{bmatrix} \beta-1 \\ \mu-1 \end{bmatrix}_q q^{\beta-\mu}\right\}$, and $y=\sum_{d-\mu\geq\mu}\left\{\begin{bmatrix} \alpha \\ d-\mu \end{bmatrix}_q q^{d-\mu}+\begin{bmatrix} \alpha \\ d-\mu-1 \end{bmatrix}_q\right\}\begin{bmatrix} \beta-1 \\ \mu \end{bmatrix}_q+\sum_{d-\mu<\mu}\left\{\begin{bmatrix} \alpha \\ d-\mu \end{bmatrix}_q+\begin{bmatrix} \alpha \\ d-\mu-1 \end{bmatrix}_q q^{\alpha-d+\mu+1}\right\}$ $\begin{bmatrix} \beta-1 \\ \mu \end{bmatrix}_q=\sum_{d-\mu\geq\mu}\begin{bmatrix} \alpha \\ d-\mu \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ \mu \end{bmatrix}_q q^{d-\mu}+\sum_{d-\mu+1\geq\mu-1}\begin{bmatrix} \alpha \\ d-\mu \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ \mu-1 \end{bmatrix}_q+\sum_{d-\mu<\mu}\begin{bmatrix} \alpha \\ d-\mu \end{bmatrix}_q$ $\begin{bmatrix} \beta-1 \\ \mu \end{bmatrix}_q+\sum_{d-\mu+1<\mu-1}\begin{bmatrix} \alpha \\ d-\mu \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ \mu-1 \end{bmatrix}_q q^{\alpha-d+\mu}$. For $d-\mu<\mu$, we have $\alpha-d+\mu>\beta-\mu$ as $\alpha\geq\beta$. Hence if $d=2c-1$, then $y-x\geq\begin{bmatrix} \alpha \\ c \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ c-1 \end{bmatrix}_q(q^c-q^{c-1})+\begin{bmatrix} \alpha \\ c-1 \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ c-1 \end{bmatrix}_q-$ $\begin{bmatrix} \alpha \\ c-1 \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ c-1 \end{bmatrix}_q q^{\beta-c}=\begin{bmatrix} \alpha \\ c-1 \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ c-1 \end{bmatrix}_q\left\{\frac{q^{\alpha-c+1}-1}{q^c-1}q^{c-1}(q-1)-(q^{\beta-c}-1)\right\}$. But $(q^{\alpha-c+1}-1)q^{c-1}(q-1)-(q^{\beta-c}-1)(q^c-1)=\{q^{\alpha}(q-1)-q^{\beta}\}+(q^{\beta-c}+q^{c-1}-1)\geq0$, as $\alpha\geq\beta$, so we have $y\geq x$. If d is even, $d=2c$, then $y-x\geq\begin{bmatrix} \alpha \\ c+1 \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ c-1 \end{bmatrix}_q(q^{c+1}-q^{c-1})+\begin{bmatrix} \alpha \\ c-1 \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ c \end{bmatrix}_q$ $-\begin{bmatrix} \alpha \\ c-1 \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ c \end{bmatrix}_q q^{\beta-(c+1)}=\begin{bmatrix} \alpha \\ c-1 \end{bmatrix}_q\begin{bmatrix} \beta-1 \\ c \end{bmatrix}_q\left\{\frac{(q^{\alpha-c+1}-1)(q^{\alpha-c}-1)}{(q^{c+1}-1)(q^{\beta-c}-1)}q^{c-1}(q^2-1)-(q^{\beta-c-1}-1)\right\}$. Similarly to previous case, as $\alpha\geq\beta$, we have that the expression in the curly bracket is ≥0, which gives $y\geq x$. So in any case, we have $y\geq x$.

Remark. Suppose $g,h\in GL(V)$ such that g has two eigenspaces A, B in V of dimension α, β respectively with $V=A\oplus B$, and h has two eigenspaces C, D in V of dimension γ, δ respectively with $V=C\oplus D$. Without loss of generality, we can assume that $\gamma-\delta\geq\alpha-\beta\geq0$. Then in view of (6.2), (5.5) is equivalent to the assertion that h fixes at least as many d dimensional subspaces in V as g does. In particular, pseudo-reflections fix at least as many subspaces as any g of the above type does. Later we will see that only elements in the center Z fix more d-dimensional subspaces than pseudo-reflections, see the remark after (6.6).

(5.6) Denote $\lceil x \rceil$ the least integer $\geq x$. Then $n - \lceil \frac{n}{r} \rceil \geq \lceil \frac{(r-1)n}{r} \rceil - 1$, where $n \geq r \geq 1$.

Proof. This is easy to verify.

Chapter II
Maximal Parabolic Subgroups

In this chapter, we assume that H is the stabilizer of a d-dimensional subspace and that $n \geq 2d$; as H is the stabilizer of an $(n-d)$-dimensional subspace in the dual representation, there is no loss of generality in this assumption. Let $f(g, \Lambda_d)$ be the number of W in Λ_d fixed by g. Note that $f(g, \Lambda_d) = f(\lambda g, \Lambda_d)$ for any $\lambda \in GF(q)^{\#}$.

This chapter is devoted to a proof of the following result:

Proposition 1. *$\overline{G}$ is not a group of genus zero unless one of the following holds:*

(a) $q=2$ and $n \leq 32$.

(b) $q=3$ and $n \leq 12$.

(c) $4 \leq q \leq 13$ and $n \leq 8$.

(d) $16 \leq q \leq 83$ and $n \leq 3$.

(e) $89 \leq q \leq 167$ and $n=2$.

Section 6. Bounds for the General Cases and the Initial Reduction.

(6.1) $\mathcal{N}(g) = f(g, \Lambda_d)/|\Lambda_d|$.

Proof. G is transitive on Λ_d, as $SL(V)$ is and $SL(V) \subseteq G$. Since $H = Stab_G(W)$ for some W in Λ_d, the representation of G on $\Omega = G/H$ and on Λ_d are equivalent.

Suppose $g \in GL_n(q)$ with $(|g|, char(GF(q))) = 1$. So g is semisimple. Thus $min(g)$ has no multiple roots. Let $f = min(g) = f_1 f_2 \cdots f_\alpha$, where each f_μ is irreducible in $GF(q)[x]$. So all f_1, f_2, $\ldots$, f_α are distinct. By the elementary divisor theorem, $V = V_1 \oplus V_2 \oplus \cdots \oplus V_\alpha$, $V_\mu = V_{\mu 1} \oplus V_{\mu 2} \oplus \cdots \oplus V_{\mu d_\mu}$, $1 \leq \mu \leq \alpha$, where each $V_{\mu\nu}$ is an irreducible $GF(q)\langle g \rangle$-module and V_μ

is the homogeneous component corresponding to the irreducible factor f_μ. Denote $c_\mu = deg(f_\mu)$. So $dim_{\mathbf{F}}(V_{\mu\nu}) = c_\mu$, $dim_{\mathbf{F}}(V_\mu) = d_\mu c_\mu$, and $n = \sum_{\mu=1}^{\alpha} d_\mu c_\mu$.

(6.2) $f(g, \Lambda_d) = \sum_{s_1c_1+s_2c_2+\cdots+s_\alpha c_\alpha = d} \begin{bmatrix} d_1 \\ s_1 \end{bmatrix}_{q^{c_1}} \begin{bmatrix} d_2 \\ s_2 \end{bmatrix}_{q^{c_2}} \cdots \begin{bmatrix} d_\alpha \\ s_\alpha \end{bmatrix}_{q^{c_\alpha}}$. In particular, if $min(g)$ splits in $GF(q)[x]$, $f(g, \Lambda_d) = \sum_{s_1+s_2+\cdots+s_\alpha = d} \begin{bmatrix} d_1 \\ s_1 \end{bmatrix}_q \begin{bmatrix} d_2 \\ s_2 \end{bmatrix}_q \cdots \begin{bmatrix} d_\alpha \\ s_\alpha \end{bmatrix}_q$.

Proof. Suppose $W \leq V$ with $Wg = W$. As $Wf(g) = 0$, $min(g|_W) | f$. Hence $min(g|_W) = f_{\eta_1} f_{\eta_2} \cdots f_{\eta_\beta}$, $1 \leq \beta \leq \alpha$. By the elementary divisor theorem, $W = W_{\eta_1} \oplus W_{\eta_2} \oplus \cdots \oplus W_{\eta_\beta}$, $W_{\eta_\tau} = W_{\eta_\tau 1} \oplus W_{\eta_\tau 2} \oplus \cdots \oplus W_{\eta_\tau s_{\eta_\tau}}$, where each $W_{\eta_\tau k}$ is a g-cyclic subspace; thus $dim_{GF(q)}(W_{\eta_\tau k}) = deg(f_{\eta_\tau}) = c_{\eta_\tau}$ $\forall k$. We have $W_{\eta_\tau} \leq V_{\eta_\tau}$. Also if $W_{\mu 1}$, $W_{\mu 2}, \ldots, W_{\mu s_\mu}$ are g-cyclic subspaces in V_μ with $W_{\mu 1} \oplus W_{\mu 2} \oplus \cdots \oplus W_{\mu s_\mu} = W_\mu \leq V_\mu$, then $W_\mu g = W_\mu$. Thus the number of W_μ in V_μ such that $dim_{GF(q)}(W_\mu) = s_\mu c_\mu$ and $W_\mu g = W_\mu$ is same as the number of W_μ in V_μ such that $W_\mu = W_{\mu 1} \oplus W_{\mu 2} \oplus \cdots \oplus W_{\mu s_\mu}$, where each $W_{\mu k}$ is a g-cyclic subspace in V_μ. To find the number of such W_μ, we count the number of pairs $((W_{\mu 1}, W_{\mu 2}, \ldots, W_{\mu s_\mu}), W_\mu)$, where each $W_{\mu k}$ is a g-cyclic subspace in V_μ, and $W_\mu = W_{\mu 1} \oplus W_{\mu 2} \oplus \cdots \oplus W_{\mu s_\mu}$. The number of choices for $W_{\mu 1}$ is $(q^{c_\mu d_\mu} - 1)/(q^{c_\mu} - 1)$, as each non-zero vector in V_μ generates a c_μ-dimensional g-cyclic subspace and every such subspace has $q^{c_\mu} - 1$ non-zero vectors. Similarly, the number of choices for $W_{\mu k}$ is $(q^{c_\mu d_\mu} - q^{c_\mu(k-1)})/(q^{c_\mu} - 1)$. Hence the number of choices for the ordered s_μ-tuple $(W_{\mu 1}, W_{\mu 2}, \ldots, W_{\mu s_\mu})$ is $x = (q^{c_\mu d_\mu} - 1)(q^{c_\mu d_\mu} - q^{c_\mu})(q^{c_\mu d_\mu} - q^{2c_\mu}) \cdots (q^{c_\mu d_\mu} - q^{c_\mu(s_\mu - 1)}) / (q^{c_\mu} - 1)^{s_\mu}$. Once we have chosen $W_{\mu 1}$, $W_{\mu 2}, \ldots, W_{\mu s_\mu}$, W_μ is uniquely determined. Hence the number of pairs $((W_{\mu 1}, W_{\mu 2}, \ldots, W_{\mu s_\mu}), W_\mu)$ is x. On the other hand, for each W_μ, the number of $(W_{\mu 1}, W_{\mu 2}, \ldots, W_{\mu s_\mu})$ is $y = (q^{c_\mu s_\mu} - 1)(q^{c_\mu s_\mu} - q^{c_\mu})(q^{c_\mu s_\mu} - q^{2c_\mu}) \cdots (q^{c_\mu s_\mu} - q^{c_\mu(s_\mu - 1)}) / (q^{c_\mu} - 1)^{s_\mu}$. So the number of W_μ such that $W_\mu \leq V_\mu$, $dim_{GF(q)}(W_\mu) = c_\mu s_\mu$, $W_\mu g = W_\mu$ is equal to $\frac{x}{y} = (q^{c_\mu d_\mu} - 1)(q^{c_\mu d_\mu} - q^{c_\mu})(q^{c_\mu d_\mu} - q^{2c_\mu}) \cdots (q^{c_\mu d_\mu} - q^{c_\mu(s_\mu - 1)}) / (q^{c_\mu s_\mu} - 1)(q^{c_\mu s_\mu} - q^{c_\mu})(q^{c_\mu s_\mu} - q^{2c_\mu}) \cdots (q^{c_\mu s_\mu} - q^{c_\mu(s_\mu - 1)}) = \begin{bmatrix} d_\mu \\ s_\mu \end{bmatrix}_{q^{c_\mu}}$. Therefore the number of d-dimensional subspaces fixed by g is $\sum_{s_1c_1+s_2c_2+\cdots+s_\alpha c_\alpha = d} \begin{bmatrix} d_1 \\ s_1 \end{bmatrix}_{q^{c_1}} \begin{bmatrix} d_2 \\ s_2 \end{bmatrix}_{q^{c_2}} \cdots \begin{bmatrix} d_\alpha \\ s_\alpha \end{bmatrix}_{q^{c_\alpha}}$.

(6.3) Suppose g is semisimple and $V=V_1\oplus V_2\oplus\cdots\oplus V_\alpha$ with V_μ a homogeneous component of g for each μ, $1\leq\mu\leq\alpha$. Let β be such that $1\leq\beta\leq\alpha$ and $\nu=dim\{V_1\oplus V_2\oplus\cdots\oplus V_\beta\}$. Then $f(g,A_d)\leq\sum_\mu\left[\begin{smallmatrix} n-\nu \\ d-\mu\end{smallmatrix}\right]_q\left[\begin{smallmatrix} \nu \\ \mu\end{smallmatrix}\right]_q$.

Proof. This follows from the argument of (6.2).

(6.4) Let A, B, C be subspaces of V satisfying $Ag=A$, $A=B\oplus C$, g acts as a scalar $\lambda\in GF(q)^{\#}$ on C. Then either there exists at most one subspace W such that $Wg=W$, $W\cap A=B$, $W+A=V$, or g has the eigenvalue λ on $\widetilde{V}=V/A$. In particular, if A is the eigenspace of λ on V, and U is such that $A\leq U\leq V$, then there is at most one W such that $Wg=W$ and $A\oplus W=U$.

Proof. We may assume $A\neq V$ and λ is not an eigenvalue of g on $\widetilde{V}$. Thus g is not unipotent. So replacing g by its semisimple part, we may assume g is semisimple. Now W is the sum of B with the homogeneous components of g distinct from the eigenspace E_λ for λ.

(6.5) Let g be any element of $GL_n(q)$ and ν be the dimension of an eigenspace E of g corresponding to an eigenvalue $\lambda\in GF(q)^{\#}$. Then $f(g,A_d)\leq\sum_\mu\left[\begin{smallmatrix} n-\nu \\ d-\mu\end{smallmatrix}\right]_q\left[\begin{smallmatrix} \nu \\ \mu\end{smallmatrix}\right]_q$.

Proof. Let E be the eigenspace above, and $E\leq U\leq V$ with $dim\{U\}=d+\nu-\mu$. The number of such U's is $\left[\begin{smallmatrix} n-\nu \\ d-\mu\end{smallmatrix}\right]_q$. Fix such a U and suppose there is at least one W such that $Wg=W$, $dim\{W\}=d$, and $W+E=U$. Let D be such that $E\leq D\leq U$ and $\widetilde{D}=D/E$ is the eigenspace corresponding to λ for g on $\widetilde{U}=U/E$. Then $\lambda^{-1}g$ acts on D either as an element of order p or as the identity. Since $W+E=U$, we have $dim\{W\cap D\}=\mu+\alpha$, where $\alpha=dim\{\widetilde{D}\}$. Also as $(W\cap D)+E=D$, $W\cap D$ has a surjective projection on each non-trivial Jordan block of $\lambda^{-1}g$ on D. The number of subspaces S of D of dimension $\mu+\alpha$ with $Sg=S$ and $S+E=D$ is at most $q^{\beta(\nu-\mu)}\left[\begin{smallmatrix} \nu-\beta \\ \mu-\beta\end{smallmatrix}\right]_q$, where β is the number of non-trivial Jordan blocks of $\lambda^{-1}g$ on D. Since for each fixed S, the number of W with $Wg=W$, $W+D=U$, and $W\cap D=S$ is at most one by (6.4), and also $q^{\beta(\nu-\mu)}\left[\begin{smallmatrix} \nu-\beta \\ \mu-\beta\end{smallmatrix}\right]_q\leq\left[\begin{smallmatrix} \nu \\ \mu\end{smallmatrix}\right]_q$, we have the conclusion.

(6.6) Let $g \in GL_n(q)$ with g not in Z. Then $\mathcal{N}(g) \leq \frac{1}{q^d} + \frac{1}{q^{n-d}} \leq \frac{2}{q^d}$.

Proof. For g unipotent, we have $1 \leq dim\{C_V(g)\} = \nu \leq n-1$. For g semisimple and $min(g)$ not irreducible, let ν be the dimension of a homogeneous component of g; and thus $1 \leq \nu \leq n-1$. In these two cases, by (6.5) and (6.3), we have $f(g, A_d) \leq \sum_{\mu} \left[{n-\nu \atop d-\mu} \right]_q \left[{\nu \atop \mu} \right]_q$. Then by (5.5), $\mathcal{N}(g) \leq \left\{ \left[{n-1 \atop d} \right]_q + \left[{n-1 \atop d-1} \right]_q \right\} \Big/ \left[{n \atop d} \right]_q \leq \frac{1}{q^d} + \frac{1}{q^{n-d}} \leq \frac{2}{q^d}$. Now suppose g is semisimple and $f = min(g)$ is irreducible in $GF(q)[x]$. Let $c = deg(f)$. Then $f(g, A_d) = 0$ if $c \nmid d$; and $f(g, A_d) = \left[{\lambda \atop s} \right]_{q^c}$ otherwise, where $\lambda = \frac{n}{c}$, $s = \frac{d}{c}$, and $c \geq 2$. So $\mathcal{N}(g) = \left[{\lambda \atop s} \right]_{q^c} \Big/ \left[{c\lambda \atop cs} \right]_q \leq q^{cs(\lambda-s+1)} \Big/ q^{cs(c\lambda-cs)} = \frac{1}{q^{cs\{(c-1)(\lambda-s)-1\}}} \leq \frac{1}{q^{cs}} = \frac{1}{q^d}$, unless $c=2$ and $\lambda=2$, $s=1$. But in this case, we have directly that $\mathcal{N}(g) = \left[{2 \atop 1} \right]_{q^2} \Big/ \left[{4 \atop 2} \right]_q = \frac{1}{q^2+q+1} \leq \frac{1}{q^2} = \frac{1}{q^d}$. So the bound $\frac{1}{q^d} + \frac{1}{q^{n-d}}$ works for any unipotent or semisimple g. Hence it works for any g not in Z.

Remark. Actually, among all the elements $g \in GL_n(q)$ with g not in Z, the pseudo-reflections, i.e., semisimple elements with two eigenspaces in V of dimension 1 and $n-1$, fix the most number of d-dimensional subspaces, because in the notation of (6.6), $\left[{\lambda \atop s} \right]_{q^c} \leq \left[{n-1 \atop d} \right]_q$.

(6.7) $\overline{G}$ is not a group of genus zero unless one of the following holds: (a) $q=2$ and $d \leq 7$; in addition if $d=7$ then $n \leq 14$. (b) $q=3$ and $d \leq 4$. (c) $q=4$, 5, and $d \leq 3$. (d) $7 \leq q \leq 13$ and $d \leq 2$. (e) $16 \leq q \leq 167$ and $d=1$.

Proof. This follows from (6.6), because $\frac{1}{q^d} + \frac{1}{q^{n-d}} \leq \frac{1}{85}$ except for those cases listed above.

Remark. This initial reduction still leaves an infinite number of possibilities open. To reduce down to a finite number of exceptions in the case of maximal parabolics, we carry out the analyses according to $q=2$, $3 \leq q \leq 13$, $16 \leq q \leq 167$ in the following sections. We already have an exact formula for $f(g, A_d)$ in (6.2) when g is semisimple. For unipotent g, the next few lemmas give better bounds than the overall bound $\frac{1}{q^d} + \frac{1}{q^{n-d}}$, and are used in later sections.

In the following, assume $|g|=p^e$ for some e. Thus $min(g)=(x-1)^m$ for some m. So each Jordan block of g has the form $\begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}$. Let $\{v_1,\ldots,v_j,v_{j+1},\ldots,v_n\}$ be the basis of V corresponding to the Jordan canonical form of g, such that $\{v_1,\ldots,v_j\}$ is the basis for a Jordan block of g of dimension j. So $v_1g=v_1+v_2$, $v_2g=v_2+v_3,\ldots,$ $v_{j-1}g=v_{j-1}+v_j$, $v_jg=v_j$. Denote by π_i the canonical projection of V onto $\langle v_i\rangle$.

(6.8) (a) Let s_k be the number of d-dimensional subspaces W such that $Wg=W$, and $W\pi_1=W\pi_2=\cdots=W\pi_{k-1}=0$, $W\pi_k\neq 0$, $1\leq k\leq j$. Then $s_k\leq q^{n-d-k+1}\cdot\xi$, where ξ is the number of $(d-j+k-1)$-dimensional subspaces W_1 of $\langle v_{j+1},\ldots,v_n\rangle$ such that $W_1^g=W_1$. In particular, $s_k\leq q^{n-d-k+1}\begin{bmatrix} n-j \\ d-j+k-1 \end{bmatrix}_q$, with equality if $v_ig=v_i$ $\forall i>j$.

(b) The number $f(g,\Lambda_d)$ of d-dimensional subspaces fixed by g is at most $\sum_{k=1}^{l-1} q^{n-d-k+1}\begin{bmatrix} n-j \\ d-j+k-1 \end{bmatrix}_q+\begin{bmatrix} n-l+1 \\ d \end{bmatrix}_q$ for any l with $j\geq l$; and it is equal to $\sum_{k=1}^{j-1} q^{n-d-k+1}\begin{bmatrix} n-j \\ d-j+k-1 \end{bmatrix}_q+\begin{bmatrix} n-j+1 \\ d \end{bmatrix}_q$ if $v_ig=v_i$ $\forall i>j$. Here if $l=1$, then the summation is to be understood being empty by convention.

(c) Suppose g has two Jordan blocks of dimension j_1, j_2. Then for any l_1, l_2 with $l_1\leq j_1$, $l_2\leq j_2$, we have $f(g,\Lambda_d)\leq \sum_{k_1=1}^{l_1-1} q^{n-d-k_1+1}\Big\{\sum_{k_2=1}^{l_2-1} q^{n-d-k_1-k_2+2}\begin{bmatrix} n-j_1-j_2 \\ d-j_1-j_2+k_1+k_2-2 \end{bmatrix}_q+$ $\begin{bmatrix} n-j_1-l_2+1 \\ d-j_1+k_1-1 \end{bmatrix}_q\Big\}+\sum_{k_2=1}^{l_2-1} q^{n-d-l_1-k_2+2}\begin{bmatrix} n-l_1-j_2+1 \\ d-j_2+k_2-1 \end{bmatrix}_q+\begin{bmatrix} n-l_1-l_2+2 \\ d \end{bmatrix}_q$.

(d) Suppose g is of type $2^\alpha 1^{n-2\alpha}$. Then $f(g,\Lambda_d)=\sum_i q^{i(n-d-\alpha+i)}\begin{bmatrix} \alpha \\ i \end{bmatrix}_q\begin{bmatrix} n-\alpha-i \\ d-2i \end{bmatrix}_q$.

Proof. Suppose that W satisfy those conditions given in (a). Then W contains a vector of the form v_k+x+u with $x\in\langle v_{k+1},\ldots,v_j\rangle$, $u\in U=\langle v_{j+1},v_{j+2},\ldots,v_n\rangle$. Since $(v_k+x+u)(g-1)^i=v_{k+i}+x(g-1)^i+u(g-1)^i$ with $x(g-1)^i\in\langle v_{k+1+i},\ldots,v_j\rangle$, W contains a vector of the form $w=v_k+u$ with $u\in U$. For a fixed $w=v_k+u$, as $w\in W$ and $Wg=W$, $w_i=v_{k+i}+u(g-1)^i=w(g-1)^i\in W$, $0\leq i\leq j-k$, where $(g-1)^0=1$. Thus $\langle w_0,w_1,\ldots,w_{j-k}\rangle\leq W$ and clearly $dim_{\mathbb{F}}\langle w_0,w_1,\ldots,w_{j-k}\rangle=$

$j-k+1$. As $W\pi_1=W\pi_2=\cdots=W\pi_{k-1}=0$, $W+U=\langle v_k,v_{k+1},\ldots,v_n\rangle$, which implies $dim_{\mathbb{F}}(W\cap U)=d+(n-j)-(n-k+1)=d-j+k-1$. So $W=\langle w_0,w_1,\ldots,w_{j-k}\rangle\oplus(W\cap U)$, and $(W\cap U)^g=(W\cap U)$. Now we count the number of pairs (w,X), where $w=v_k+u$ with $u\in U$, and $X\leq U$, $X^g=X$, with $dim\{X\}=d-j+k-1$. Since the number of such w is q^{n-j}, the number of pairs (w,X) is $q^{n-j}\cdot\xi$. For each fixed W satisfying the conditions in (a), the number of such (w,X)'s with $w\in W$, and $X=W\cap U$ is $q^{d-j+k-1}$, as for $w'=v_k+u'$ with $u'\in U$, $u'-u=w'-w\in W\cap U$. So the number s_k of W's satisfying the conditions in (a) is at most $q^{n-j}\cdot\xi/q^{d-j+k-1}=q^{n-d-k+1}\cdot\xi$. Also it is easy to see that in particular, $s_k\leq q^{n-d-k+1}\begin{bmatrix} n-j \\ d-j+k-1\end{bmatrix}_q$, with equality if $v_ig=v_i$ $\forall i>j$. Finally, (b) follows from (a); and if we apply (a) twice, we have (c). For part (d), we have that the number of W with $Wg=W$ and $dim\{[W,g]\}=i$ is equal to $q^{i(n-d-\alpha+i)}\begin{bmatrix} \alpha \\ i\end{bmatrix}_q\begin{bmatrix} n-\alpha-i \\ d-2i\end{bmatrix}_q$.

(6.9) (a) Let j be the dimension of one of the Jordan blocks of g and suppose l is such that $j\geq l$. Then $\mathcal{N}(g)\leq\sum_{k=1}^{l-1}\frac{1}{q^{(j-k)(n-d-k+1)+(k-1)d}}+\frac{1}{q^{(l-1)d}}$. In particular, if j is the dimension of any non-trivial Jordan block of g, then $\mathcal{N}(g)\leq\frac{1}{q^{(j-1)(n-d)}}+\frac{1}{q^d}\leq\frac{2}{q^d}$; and if $j\geq3$, then $\mathcal{N}(g)\leq\frac{1}{q^{(j-1)(n-d)}}+\frac{1}{q^{(j-2)(n-d-1)+d}}+\frac{1}{q^{2d}}$.

(b) If g is a transvection, then $\mathcal{N}(g)=\frac{q^d-1}{q^n-1}\cdot\frac{q^{n-1}-q^{n-d}}{q^{n-1}-1}+\frac{q^{n-d}-1}{q^n-1}$. If g is of type $2^\alpha1^{n-2\alpha}$ and g is not a transvection, i.e., $\alpha\geq2$, then $\mathcal{N}(g)\leq\frac{1}{q^{n-d}}+\frac{1}{q^{n-1}}+\frac{1}{q^{2d}}$.

(c) Supoose g has at least t non-trivial Jordan blocks. Then $\mathcal{N}(g)\leq\frac{t}{q^{n-d}}+\frac{1}{q^{td}}$.

(d) Suppose g has two Jordan blocks of dimension j_1, j_2; and $j_1\geq l_1$, $j_2\geq l_2$. Then

$$\mathcal{N}(g)\leq\sum_{k_1=1}^{l_1-1}\sum_{k_2=1}^{l_2-1}\frac{1}{q^{(j_1+j_2-k_1-k_2)(n-d-k_1-k_2+2)+(k_1-1)d+(k_2-1)(d-1)}}+$$

$$\sum_{k_1=1}^{l_1-1}\frac{1}{q^{(j_1-k_1)(n-d-k_1-l_2+2)+(k_1-1)d+(l_2-1)(d-1)}}+$$

$$\sum_{k_2=1}^{l_2-1}\frac{1}{q^{(j_2-k_2)(n-d-l_1-k_2+2)+(l_1+k_2-2)d}}+\frac{1}{q^{(l_1+l_2-2)d}}.$$

(e) Suppose g is of type $2^{\alpha}1^{n-2\alpha}$. Then $\mathcal{N}(g)\leq \sum_{i=0}^{min\{\alpha,[\frac{d}{2}]\}} \left(\frac{q}{q-1}\right)^{i}\cdot\frac{1}{q^{i(n-2d+2i)+\alpha(d-2i)}}$.

Proof. By (5.3), $q^{n-d-k+1}\begin{bmatrix} n-j \\ d-j+k-1 \end{bmatrix}_q\Big/\begin{bmatrix} n \\ d \end{bmatrix}_q\leq\frac{q^{n-d-k+1}}{q^{(j-k+1)(n-d-k+1)+(k-1)d}}=\frac{1}{q^{(j-k)(n-d-k+1)+(k-1)d}}$, and $\begin{bmatrix} n-l+1 \\ d \end{bmatrix}_q\Big/\begin{bmatrix} n \\ d \end{bmatrix}_q\leq\frac{1}{q^{(l-1)d}}$; thus by (6.8)(b) we have that $\mathcal{N}(g)\leq \left\{\sum_{k=1}^{l-1} q^{n-d-k+1}\begin{bmatrix} n-j \\ d-j+k-1 \end{bmatrix}_q+\begin{bmatrix} n-l+1 \\ d \end{bmatrix}_q\right\}\Big/\begin{bmatrix} n \\ d \end{bmatrix}_q\leq\sum_{k=1}^{l-1}\frac{1}{q^{(j-k)(n-d-k+1)+(k-1)d}}+\frac{1}{q^{(l-1)d}}$; i.e., the first part of (a) holds. Since $j\geq 2$ always, by letting $l=2$ or letting $l=3$ if $j\geq 3$ in the first part of (a), and noting that $(j-1)(n-d)\geq n-d\geq d$, we get the rest of (a). If g is a transvection, then $\mathcal{N}(g)=\left\{\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q-\begin{bmatrix} n-2 \\ d-1 \end{bmatrix}_q+\begin{bmatrix} n-1 \\ d \end{bmatrix}_q\right\}\Big/\begin{bmatrix} n \\ d \end{bmatrix}_q=\frac{q^d-1}{q^n-1}\left(1-\frac{q^{n-d}-1}{q^{n-1}-1}\right)+\frac{q^{n-d}-1}{q^n-1}$, hence the first part of (b) holds. If g is of type $2^{\alpha}1^{n-2\alpha}$ with $\alpha\geq 2$, let $\{v_1,v_2\}$, $\{v_3,v_4\}$ be the bases for two Jordan blocks of dimension 2 respectively. The number of W with $Wg=W$, $W\pi_1\neq 0$ is less than or equal to $q^{n-d}\begin{bmatrix} n-2 \\ d-2 \end{bmatrix}_q$. The number of W with $Wg=W$, $W\pi_1=0$, $W\pi_3\neq 0$ is less than or equal to $q^{n-1-d}\begin{bmatrix} n-3 \\ d-2 \end{bmatrix}_q$. The number of W with $Wg=W$, $W\pi_1=0$, $W\pi_3=0$ is less than or equal to $\begin{bmatrix} n-2 \\ d \end{bmatrix}_q$. Thus $\mathcal{N}(g)\leq\left\{q^{n-d}\begin{bmatrix} n-2 \\ d-2 \end{bmatrix}_q+q^{n-1-d}\begin{bmatrix} n-3 \\ d-2 \end{bmatrix}_q+\begin{bmatrix} n-2 \\ d \end{bmatrix}_q\right\}\Big/\begin{bmatrix} n \\ d \end{bmatrix}_q\leq \frac{1}{q^{n-d}}+\frac{1}{q^{n-1}}+\frac{1}{q^{2d}}$ by (5.3), i.e., the second part of (b) holds. For part (c), let $v_1^{(s)},\ldots,v_{j_s}^{(s)}$ be the Jordan canonical basis for s-th block of g, $1\leq s\leq t$, and let $\pi_1^{(s)}$ be the canonical projection of V onto $\langle v_1^{(s)}\rangle$. Then the number of W such that $Wg=W$, and $W\pi_1^{(s)}\neq 0$ is less than or equal to $q^{n-d}\begin{bmatrix} n-j_s \\ d-j_s \end{bmatrix}_q$; and the number of W such that $Wg=W$, and $W\pi_1^{(s)}=0$ $\forall 1\leq s\leq t$ is less than or equal to $\begin{bmatrix} n-t \\ d \end{bmatrix}_q$. Hence $\mathcal{N}(g)\leq\left\{\sum_{s=1}^{t} q^{n-d}\begin{bmatrix} n-j_s \\ d-j_s \end{bmatrix}_q+\begin{bmatrix} n-t \\ d \end{bmatrix}_q\right\}\Big/\begin{bmatrix} n \\ d \end{bmatrix}_q\leq\sum_{s=1}^{t}\frac{1}{q^{(j_s-1)(n-d)}}+\frac{1}{q^{td}}$ $\leq\frac{t}{q^{n-d}}+\frac{1}{q^{td}}$. For part (d), we have for example by (5.3), $\begin{bmatrix} n-j_1-j_2 \\ d-j_1-j_2+k_1+k_2-2 \end{bmatrix}_q\Big/\begin{bmatrix} n \\ d \end{bmatrix}_q\leq \frac{1}{q^{(j_1+j_2-k_1-k_2+2)(n-d-k_1-k_2+2)+(k_1+k_2-2)d}}$. Since $(j_1+j_2-k_1-k_2+2)(n-d-k_1-k_2+2)+(k_1+k_2-2)d-(n-d-k_1+1)-(n-d-k_1-k_2+2)=(j_1+j_2-k_1-k_2)(n-d-k_1-k_2+2)+(k_1-1)d+(k_2-1)(d-1)$, we have $q^{n-d-k_1+1}q^{n-d-k_1-k_2+2}\begin{bmatrix} n-j_1-j_2 \\ d-j_1-j_2+k_1+k_2-2 \end{bmatrix}_q\Big/\begin{bmatrix} n \\ d \end{bmatrix}_q\leq \frac{1}{q^{(j_1+j_2-k_1-k_2)(n-d-k_1-k_2+2)+(k_1-1)d+(k_2-1)(d-1)}}$. The estimations for other terms are similar, so by (6.8)(c), we have (d). For part (e), we have $\begin{bmatrix} \alpha \\ i \end{bmatrix}_q\begin{bmatrix} n-\alpha-i \\ d-2i \end{bmatrix}_q\Big/\begin{bmatrix} n \\ d \end{bmatrix}_q=$
$\frac{(q^{\alpha}-1)\cdots(q^{\alpha-i+1}-1)}{(q^{n}-1)\cdots(q^{n-i+1}-1)}\cdot\frac{(q^{n-\alpha-i}-1)\cdots(q^{n-\alpha-d+i+1}-1)}{(q^{n-i}-1)\cdots(q^{n-d+i+1}-1)}\cdot\frac{(q^{d}-1)\cdots(q^{d-i+1}-1)}{(q^{n-d+i}-1)\cdots(q^{n-d+1}-1)}$.

$\frac{(q^{d-i}-1)\cdots(q^{i+1}-1)}{(q^{d-2i}-1)\cdots(q-1)}\leq\frac{1}{q^{i(n-\alpha)+\alpha(d-2i)+i(n-2d+i)}}\cdot\frac{(q^{d-i}-1)\cdots(q^{i+1}-1)}{(q^{d-2i}-1)\cdots(q-1)}$. But $\frac{(q^{d-i}-1)\cdots(q^{i+1}-1)}{(q^{d-2i}-1)\cdots(q-1)}=\left[{d-i \atop d-2i}\right]_q=\left[{d-i \atop i}\right]_q\leq\left(\frac{q}{q-1}\right)^i\cdot q^{i(d-2i)}$ by (5.2), and as $i(n-\alpha)+\alpha(d-2i)+i(n-2d+i)-i(d-2i)-i(n-d-\alpha+i)=\alpha(d-2i)+i(n-2d+2i)$ thus $q^{i(n-d-\alpha+i)}\left[{\alpha \atop i}\right]_q\left[{n-\alpha-i \atop d-2i}\right]_q\Big/\left[{n \atop d}\right]_q\leq\left(\frac{q}{q-1}\right)^i\cdot\frac{1}{q^{i(n-2d+2i)+\alpha(d-2i)}}$.

(6.10) Suppose $|g|=p^e$. If g has only one Jordan block, then $f(g,A_d)=1$. If g has exactly two Jordan blocks of dimension ν_1, ν_2 respectively, then $f(g,A_d)=\sum_{\mu_1+\mu_2=d,\ 0\leq\mu_i\leq\nu_i} q^{min(\mu_1,\nu_2-\mu_2)}$.

Proof. Let V_1, V_2 be subspaces corresponding to two blocks. Suppose $Wg=W$. Let $\mu_2=dim\{W\cap V_2\}$. So $dim\{W+V_2\}=\nu_2+\mu_1$, where $\mu_1=d-\mu_2$. Let U be the unique subspace of V_1 with $Ug=U$ and $dim\{U\}=\mu_1$. Let X, Y be the Jordan canonical bases of $W\cap V_2$ and V_2 respectively. So $X\subseteq Y$. Since $W+V_2=U\oplus V_2$, we can complete X to a basis X' of W so that $X'-X=\{u_i+v_i\colon\ 1\leq i\leq\mu_1\}$, where $T=\{u_i\colon\ 1\leq i\leq\mu_1\}$ is the Jordan canonical basis of U and $v_i\in V_2$, $\forall\ 1\leq i\leq\mu_1$. Then $M'=M_{X'\cup Y}(g|_{W+V_2})=\begin{bmatrix} A & 0 & B \\ 0 & C & D \\ 0 & 0 & E \end{bmatrix}$. Also $M=M_{T\cup Y}(g|_{W+V_2})=\begin{bmatrix} A & 0 & 0 \\ 0 & C & D \\ 0 & 0 & E \end{bmatrix}$ and there exists $N=\begin{bmatrix} I & Q & R \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ such that $M'N=NM$. This implies that $AQ=QC$ and $AR+B=QD+RE$. Since A and C are Jordan blocks of dimension μ_1, $\nu_2-\mu_2$ respectively, the number of Q is $q^{min(\mu_1,\nu_2-\mu_2)}$. For each fixed choice of Q and R, we can choose $B=QD+RE-AR$. Hence the number of d-dimensional W with $Wg=W$ and $\mu_2=dim\{W\cap V_2\}$ is equal to the number of Q, which is $q^{min(\mu_1,\nu_2-\mu_2)}$. So we have the conclusion.

(6.11) Suppose $|g|=p^e$. If g has only one Jordan block, then $\mathcal{N}(g)\leq\frac{1}{q^{d(n-d)}}$. If g has exactly two Jordan blocks, then $\mathcal{N}(g)\leq\frac{1}{q^{d(n-d-1)}}$.

Proof. If g has only one Jordan block, then $\mathcal{N}(g)=1\Big/\left[{n \atop d}\right]_q\leq\frac{1}{q^{d(n-d)}}$. Suppose g has two Jordan blocks of dimension ν_1, ν_2. Without loss of generality, we can assume that $\nu_2\geq\nu_1$. So $d\leq\nu_2$ and thus $min(\mu_1,\nu_2-\mu_2)=min(d-\mu_2,\nu_2-\mu_2)=d-\mu_2$. Then by (6.10) we have that

$$\mathcal{N}(g)\leq(1+q+\cdots+q^d)\Big/\left[\begin{smallmatrix} n \\ d \end{smallmatrix}\right]_q=\frac{(q^{d+1}-1)(q^d-1)\cdots(q^2-1)}{(q^n-1)\cdots(q^{n-d+1}-1)}\leq\frac{1}{q^{d(n-d-1)}}.$$

(6.12) Suppose $d=1$. Let g be any element in $GL_n(q)$, and $\lambda_1,\ldots,\lambda_r$ be all the eigenvalues of g with $\lambda_i\in GF(q)$, and $V_1,\ldots,V_r$ be the corresponding eigenspaces. Denote $\nu_i=dim(V_i)$. Also let $\nu=max\{\nu_i:1\leq i\leq r\}$. Denote $\overline{g}$ the image of g in $PGL_n(q)$. Then we have

(a) $\mathcal{N}(g)=\dfrac{q^{\nu_1}+\cdots+q^{\nu_r}-r}{q^n-1}$; in particular if g is unipotent, then $\mathcal{N}(g)=\dfrac{q^{\nu}-1}{q^n-1}<\dfrac{1}{q^{n-\nu}}$, where $\nu=dim\{C_V(g)\}$. Also if $|\overline{g}|=s$, then $\mathcal{N}(g)\leq\frac{s}{q^{n-\nu}}$; in particular, if each $\nu_i\leq 1$, then $\mathcal{N}(g)\leq\frac{s}{q^{n-1}}$.

(b) $\mathcal{N}(g)\leq\frac{1}{q^{\nu}}+\frac{1}{q^{n-\nu}}$; in particular for any $\overline{g}\in PGL_n(q)^{\#}$, $\mathcal{N}(\overline{g})\leq\frac{1}{q}+\frac{1}{q^{n-1}}$.

(c) If g has an eigenspace of dimension $n-1$, then either λg is a transvection for some $\lambda\in GF(q)^{\#}$ or g is a pseudo-reflection, and $\mathcal{N}(g)\leq\frac{1}{q}$, $\frac{1}{q}+\frac{1}{q^{n-1}}$ respectively. If $g\notin Z$ and for any $\lambda\in GF(q)^{\#}$, λg is neither a transvection nor a pseudo-reflection, then $\mathcal{N}(g)\leq\frac{1}{q^2}+\frac{1}{q^{n-2}}$.

Proof. (a) follows from $\mathcal{N}(g)=\left[\begin{smallmatrix} \nu_1 \\ 1 \end{smallmatrix}\right]_q+\cdots+\left[\begin{smallmatrix} \nu_r \\ 1 \end{smallmatrix}\right]_q\Big/\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]_q$ and $\left[\begin{smallmatrix} \lambda \\ 1 \end{smallmatrix}\right]_q=\frac{q^{\lambda}-1}{q-1}$. (b) holds as $f(g,A_1)\leq\left[\begin{smallmatrix} n-\nu \\ 1 \end{smallmatrix}\right]_q+\left[\begin{smallmatrix} \nu \\ 1 \end{smallmatrix}\right]_q$.

Section 7. The Case $q=2$.

In this section, we always assume $q=2$. So $GL_n(2)=SL_n(2)=PGL_n(2)=L_n(2)=G=\overline{G}$, where G and $\overline{G}$ have the same meanings as in Sec. 3. The conclusion of this section is the following:

Proposition: *If G is a group of genus zero, then n<33.*

Proof. This follows from (7.31), (7.53) and (7.54) in this section.

(7.1) (a) Suppose $d=1$. Then for any $g\in G$, $\mathcal{N}(g)=\frac{2^{\nu}-1}{2^n-1}\leq\frac{1}{2^{n-\nu}}$, where $\nu=dim\{C_V(g)\}$.

(b) $f(g,A_2)=\left[\begin{smallmatrix} \nu \\ 2 \end{smallmatrix}\right]_2+2^{\nu-1}\left[\begin{smallmatrix} \alpha-\nu \\ 1 \end{smallmatrix}\right]_2+\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]_{2^2}$, where $\nu=dim\{C_V(g)\}$, $\alpha=dim\{C_V(g^2)\}$, and $x=\frac{1}{2}(dim\{C_V(g^3)\}-dim\{C_V(g)\})$.

Proof. (a) follows from (6.12)(a). For part (b), suppose $W \leq V$ with $dim\{W\}=2$ and $Wg=W$. We have that the number of such W's on which g acts as the identity is $\left[{\nu \atop 2} \right]_2$, the number of W's on which g acts as an element of order 2 is $2^{\nu-1}\left[{\alpha-\nu \atop 1} \right]_2$, the number of W's on which g acts as an element of order 3 is $\left[{x \atop 1} \right]_{2^2}$. Since g acts on W as an element of order at most 3, we have (b).

In the following, sometimes we denote $\mathcal{N}(g)$ by $\mathcal{N}(g,d)$ to indicate a particular value of d involved. Similar for $\mathcal{U}(g,d)$.

(7.2) Let $g \in G$, and $\nu = dim\{C_V(g)\}$. Let $|g|=2^e m$ with m odd. Define $d(m)=min\{k>0 : 2^k \equiv 1 (mod\ m)\}$ if $m>1$ and $d(1)=0$, $\alpha=2^{e-1}$ if $e>0$ and $\alpha=0$ if $e=0$. Then $\nu \leq n-d(m)-\alpha$. Also $n \geq d(m)+\alpha+1$ if $e>0$ and $n \geq d(m)$ if $e=0$.

Proof. Let $x=g^m$ and $y=g^{2^e}$. Then the minimum dimension of a faithful $GF(2)\langle g \rangle$-module is achieved when $V=[V,y] \oplus C_V(y)$ with $dim([V,y])=d(m)$ and $C_V(y)$ a Jordan block for x. By (3.4)(a), $dim(C_V(y))=2^{e-1}+1$ in our minimum case. Hence we have the conclusion.

In order to prove the proposition stated in the beginning of this section, in view of (6.7), we always assume in the rest of this section that $n \geq 33$ and $d \leq 6$.

(7.3) Suppose $|g|=2^e$ with $e \geq 1$. Then $\mathcal{N}(g) \leq \frac{1}{2^7}$ unless one of the following holds:

(a) $e=3$ and $d=1$, and g is one of the following types: $7^1 1^{n-7}$, $6^1 2^1 1^{n-8}$, $6^1 1^{n-6}$, $5^1 3^1 1^{n-8}$, $5^1 2^2 1^{n-9}$, $5^1 2^1 1^{n-7}$, $5^1 1^{n-5}$.

(b) $e=2$ and $d=2$, and g is one of the following types: $3^1 2^1 1^{n-5}$, $3^1 1^{n-3}$, $4^1 1^{n-4}$, and also $\mathcal{N}(g) \leq \frac{1}{2^{32}}+\frac{1}{2^6}$, $\frac{1}{2^{32}}+\frac{1}{2^4}$, $\frac{1}{2^{33}}+\frac{1}{2^6}$ respectively; or $e=2$ and $d=3$, and g is of type $3^1 1^{n-3}$, and $\mathcal{N}(g) \leq \frac{1}{2^{31}}+\frac{1}{2^6}$.

(c) $e=2$ and $d=1$, and g is one of the following types: $4^2 1^{n-8}$; $4^1 3^1 2^\alpha 1^{n-7-2\alpha}$ for $\alpha=0$ or

1; $3^2 2^\alpha 1^{n-6-2\alpha}$ for $\alpha=0$, 1, or 2; $4^1 2^\alpha 1^{n-4-2\alpha}$ for $0\leq\alpha\leq 3$; $3^1 2^\alpha 1^{n-3-2\alpha}$ for $0\leq\alpha\leq 4$.

(d) $e=1$ and $d=4$, 5, or 6, and g is of type $2^1 1^{n-2}$, and $\mathcal{N}(g)\leq\frac{1}{2^{n-d}}+\frac{1}{2^d}$; or $e=1$ and $d=3$, and g is of type $2^1 1^{n-2}$ or $2^2 1^{n-4}$, and $\mathcal{N}(g)\leq\frac{1}{2^{30}}+\frac{1}{2^3}$, $\frac{1}{2^{28}}+\frac{1}{2^6}$ respectively; or $e=1$ and $d=2$, and g is of type $2^1 1^{n-2}$, $2^2 1^{n-4}$, or $2^3 1^{n-6}$, and $\mathcal{N}(g)\leq\frac{1}{2^{31}}+\frac{1}{2^2}$, $\frac{1}{2^{30}}+\frac{1}{2^4}$, $\frac{1}{2^{29}}+\frac{1}{2^6}$ respectively.

(e) $e=1$ and $d=1$, and g is of type $2^\alpha 1^{n-2\alpha}$ with $\alpha\leq 6$.

Proof. If $e\geq 4$, then g has a block of dimension $j\geq 9$. Hence by (6.9)(a), $\mathcal{N}(g)\leq \sum_{k=1}^{8}\frac{1}{2^{(9-k)(n-d-k+1)+(k-1)d}}+\frac{1}{2^{8d}}\leq\frac{1}{2^{17}}+\frac{1}{2^{8d}}\leq\frac{1}{2^7}$, as $(9-k)(n-d-k+1)+(k-1)d\geq n-d-k+1 \geq 20$.

If $e=3$ and g has two blocks of dimension j_1, j_2 both ≥ 5, then by (6.9)(d), $\mathcal{N}(g)\leq\sum_{k_1=1}^{4}\sum_{k_2=1}^{4}\frac{1}{2^{(10-k_1-k_2)(n-d-k_1-k_2+2)+(k_1-1)d+(k_2-1)(d-1)}}+\sum_{k_1=1}^{4}\frac{1}{2^{(5-k_1)(n-d-k_1-3)+(k_1-1)d+4d-4}}$ $+\sum_{k_2=1}^{4}\frac{1}{2^{(5-k_2)(n-d-k_2-3)+(3+k_2)d}}+\frac{1}{2^{8d}}\leq\frac{1}{2^{38}}+\frac{1}{2^{18}}+\frac{1}{2^{25}}+\frac{1}{2^{8d}}\leq\frac{1}{2^7}$, as for example $(5-k_2)(n-d-k_2-3)+(3+k_2)d\geq(n-d-k_2-3)+(3+k_2)=n-d\geq 27$. So g has only one block of dimension $j\geq 5$. If $d\geq 2$, then by (5.9)(a), $\mathcal{N}(g)\leq\sum_{k=1}^{4}\frac{1}{2^{(5-k)(n-d-k+1)+(k-1)d}}+\frac{1}{2^{4d}}\leq\frac{1}{2^{22}}+\frac{1}{2^8}\leq\frac{1}{2^7}$, as $(5-k)(n-d-k+1)+(k-1)d\geq n-d-k+1\geq 24$. So $d=1$. Then as $\mathcal{N}(g)=\frac{2^\nu-1}{2^n-1}$, where $\nu=dim\{C_V(g)\}$, we have $\mathcal{N}(g)\leq\frac{1}{2^7}$ if $\nu\leq n-7$. Thus g is one of the following types: $7^1 1^{n-7}$, $6^1 2^1 1^{n-8}$, $6^1 1^{n-6}$, $5^1 3^1 1^{n-8}$, $5^1 2^2 1^{n-9}$, $5^1 2^1 1^{n-7}$, $5^1 1^{n-5}$.

Suppose $e=2$. If $d\geq 2$ and g has two blocks of dimension j_1, j_2 both ≥ 3, then by (6.9)(d), $\mathcal{N}(g)\leq\sum_{k_1=1}^{2}\sum_{k_2=1}^{2}\frac{1}{2^{(6-k_1-k_2)(n-d-k_1-k_2+2)+(k_1-1)d+(k_2-1)(d-1)}}+$ $\sum_{k_1=1}^{2}\frac{1}{2^{(3-k_1)(n-d-k_1-1)+(k_1-1)d+2d-2}}+\sum_{k_2=1}^{2}\frac{1}{2^{(3-k_2)(n-d-k_2-1)+(1+k_2)d}}+\frac{1}{2^{4d}}\leq\frac{4}{2^{50}}+\frac{2}{2^{24}}+\frac{2}{2^{28}}+\frac{1}{2^8}\leq$ $\frac{1}{2^7}$, as for example $(6-k_1-k_2)(n-d-k_1-k_2+2)+(k_1-1)d+(k_2-1)(d-1)\geq 2(n-d-k_1-k_2+2)$ ≥ 50. Also if g has ≥ 4 non-trivial blocks, then by (6.9)(c), $\mathcal{N}(g)\leq\frac{4}{2^{n-d}}+\frac{1}{2^{4d}}\leq\frac{1}{2^7}$, as $n-d-2\geq 25$. Hence if $d\geq 2$, then g is one of the following types: $a^1 2^\alpha 1^{n-a-2\alpha}$, where $a=3$ or 4, and $0\leq\alpha\leq 2$. Suppose $a=3$ first. Then by (6.9)(a), $\mathcal{N}(g)\leq\frac{1}{2^{2(n-d)}}+\frac{1}{2^{n-1}}+\frac{1}{2^{2d}}\leq\frac{1}{2^7}$ if $d\geq 4$. If

$d=3$ and $\alpha\geq 1$, then (6.9)(d) gives that $\mathcal{N}(g)\leq\frac{1}{2^7}$. So $\alpha=0$, i.e., for $d=3$, g is of type $3^1 1^{n-3}$, in which case $\mathcal{N}(g)\leq\frac{1}{2^{2(n-3)}}+\frac{1}{2^{n-1}}+\frac{1}{2^6}\leq\frac{1}{2^{31}}+\frac{1}{2^6}$. If $d=2$, then by (7.1)(b), $\mathcal{N}(g)=\left\{\left[{n-2-\alpha \atop 2}\right]_2+2^{n-3-\alpha}\left[{\alpha+1 \atop 1}\right]_2\right\}/\left[{n \atop 2}\right]_2\leq\frac{1}{2^{2(\alpha+2)}}+\frac{1}{2^{n-1}}$, which is $\leq\frac{1}{2^7}$ if $\alpha=2$, and $\leq\frac{1}{2^{32}}+\frac{1}{2^6}$, $\frac{1}{2^{32}}+\frac{1}{2^4}$ for $\alpha=1$, or 0 respectively. Now suppose that $a=4$. Then by (6.9)(a) $\mathcal{N}(g)\leq\frac{1}{2^{3(n-d)}}+\frac{1}{2^{2n-d-2}}+\frac{1}{2^{n+d-2}}+\frac{1}{2^{3d}}\leq\frac{1}{2^7}$ if $d\geq 3$. If $d=2$, then by (7.1)(b), $\mathcal{N}(g)=\left\{\left[{n-3-\alpha \atop 2}\right]_2+2^{n-4-\alpha}\left[{\alpha+1 \atop 1}\right]_2\right\}/\left[{n \atop 2}\right]_2\leq\frac{1}{2^{2(\alpha+3)}}+\frac{1}{2^n}$, which is $\leq\frac{1}{2^7}$ if $\alpha=1$, or 2, and $\leq\frac{1}{2^{33}}+\frac{1}{2^6}$ for $\alpha=0$. If $d=1$, then as before, $\nu\leq n-7$ implies $\mathcal{N}(g)\leq\frac{1}{2^7}$. So g is one of the following types: $4^2 1^{n-8}$; $4^1 3^1 2^\alpha 1^{n-7-2\alpha}$ for $\alpha=0$ or 1; $3^2 2^\alpha 1^{n-6-2\alpha}$ for $\alpha=0$, 1, or 2; $4^1 2^\alpha 1^{n-4-2\alpha}$ for $0\leq\alpha\leq 3$; $3^1 2^\alpha 1^{n-3-2\alpha}$ for $0\leq\alpha\leq 4$.

Suppose $e=1$. So g is of type $2^\alpha 1^{n-2\alpha}$. If $d\geq 2$ and $\alpha\geq 4$, then as before by (6.9)(c), $\mathcal{N}(g)\leq\frac{4}{2^{n-d}}+\frac{1}{2^{4d}}\leq\frac{1}{2^7}$. So $\alpha\leq 3$. If $d=1$, then $\alpha\leq 6$.

(7.4) Suppose g is semisimple. Then $\mathcal{N}(g)\leq\frac{1}{2^7}$ unless one of the following holds:

(a) $|g|=3$ and $d\leq 3$; g is of type $2^\alpha 1^{n-2\alpha}$ with $\alpha\leq 3$ if $d=1$; with $\alpha=1$, and $\mathcal{N}(g)\leq\frac{1}{2^{62}}+\frac{1}{2^4}$, $\frac{1}{2^{32}}+\frac{1}{2^6}$ for $d=2$, 3 respectively.

(b) $|g|=5$ and $d=1$, and g is of type $4^1 1^{n-4}$.

(c) $|g|=7$, $d\leq 2$, g is of type $3_1^{\alpha_1} 3_2^{\alpha_2} 1^{n-3(\alpha_1+\alpha_2)}$, where $\alpha_1+\alpha_2\leq 2$ if $d=1$; and $\alpha_1+\alpha_2=1$, $\mathcal{N}(g)\leq\frac{1}{2^6}$ if $d=2$.

(d) $|g|=9$, $d=1$, and g is of type $6^1 1^{n-6}$.

(e) $|g|=15$, $d=1$, and g is of type $4^1 1^{n-4}$ or $4^1 2^1 1^{n-6}$.

(f) $|g|=21$, $d=1$, and g is of type $6^1 1^{n-6}$.

(g) $|g|=31$, $d=1$, and g is of type $5^1 1^{n-5}$.

(h) $|g|=63$, $d=1$, and g is of type $6^1 1^{n-6}$.

Proof. Let a be the dimension of a simple submodule of g with $a \geq 2$, and αa be the dimension of the cooresponding homogeneous component. Suppose $a > d$. Then $\mathcal{N}(g) \leq \genfrac{[}{]}{0pt}{}{n-\alpha a}{d}_2 \Big/ \genfrac{[}{]}{0pt}{}{n}{d}_2 \leq \frac{1}{2^{d\alpha a}}$. In paticular, if $|g| \neq 3,5,7,9,15,21,31,63$, then as g has a simple submodule of dimension $a \geq 7 > d$, we have $\mathcal{N}(g) \leq \frac{1}{2^{7d}} \leq \frac{1}{2^7}$.

Suppose a is such that $2a > d \geq a$. Then $\mathcal{N}(g) \leq \left\{ \genfrac{[}{]}{0pt}{}{\alpha}{1}_{2^a} \genfrac{[}{]}{0pt}{}{n-\alpha a}{d-a}_2 + \genfrac{[}{]}{0pt}{}{n-\alpha a}{d}_2 \right\} \Big/ \genfrac{[}{]}{0pt}{}{n}{d}_2$. Since by (5.3), $\genfrac{[}{]}{0pt}{}{n-\alpha a}{d-a}_2 \Big/ \genfrac{[}{]}{0pt}{}{n}{d}_2 \leq \frac{1}{2^x}$ with $x = (\alpha-1)ad + a(n-d-\alpha a+a) = (\alpha-1)a(d-a) + a(n-d) \geq a(n-d)$ and $\genfrac{[}{]}{0pt}{}{\alpha}{1}_{2^a} \leq 2^{\alpha a}$, we have $\mathcal{N}(g) \leq \frac{1}{2^y} + \frac{1}{2^{d\alpha a}}$, where $y = a(n-d-\alpha)$. Suppose $|g| = 5,9,15,21,31$, or 63. Then g has a simple submodule of dimension $a \geq 4$. Thus $2a > d$, and $y = a(n-d-\alpha) \geq 4(n-\frac{n}{2}-\frac{n}{4}) = n$. So we have that $\mathcal{N}(g) \leq max\{\frac{1}{2^{d\alpha a}}, \frac{1}{2^n} + \frac{1}{2^{d\alpha a}}\} \leq \frac{1}{2^7}$ if $d \geq 2$. If $d=1$, then those statements follow from (7.1)(a) easily.

Suppose $|g|=3$. So g is of type $2^\alpha 1^{n-2\alpha}$. If $d=1$ and $\alpha \geq 4$, then $\mathcal{N}(g) \leq \frac{2^{n-8}-1}{2^n-1} \leq \frac{1}{2^8}$. If $d=2$ and $\alpha \geq 2$, then $\mathcal{N}(g,2) = \left\{ \genfrac{[}{]}{0pt}{}{\alpha}{1}_{2^2} + \genfrac{[}{]}{0pt}{}{n-2\alpha}{2}_2 \right\} \Big/ \genfrac{[}{]}{0pt}{}{n}{2}_2 \leq \frac{1}{2^{n-2}} + \frac{1}{2^{4\alpha}} \leq \frac{1}{2^7}$; and for $\alpha=1$, $\mathcal{N}(g,2) = \left\{ 1 + \genfrac{[}{]}{0pt}{}{n-2}{2}_2 \right\} \Big/ \genfrac{[}{]}{0pt}{}{n}{2}_2 \leq \frac{1}{2^{2(n-2)}} + \frac{1}{2^4} \leq \frac{1}{2^{62}} + \frac{1}{2^4}$. Similarly, if $d=3$ and $\alpha \geq 2$, then $\mathcal{N}(g,3) \leq \frac{1}{2^{n-6+2\alpha}} + \frac{1}{2^{6\alpha}} \leq \frac{1}{2^7}$; and for $\alpha=1$, $\mathcal{N}(g,3) = \left\{ \genfrac{[}{]}{0pt}{}{n-2}{1}_2 + \genfrac{[}{]}{0pt}{}{n-2}{3}_2 \right\} \Big/ \genfrac{[}{]}{0pt}{}{n}{3}_2 \leq \frac{1}{2^{n-1}} + \frac{1}{2^6} \leq \frac{1}{2^{32}} + \frac{1}{2^6}$. Suppose $d \geq 4$. We have $\mathcal{N}(g,d) \leq \sum_{i=0}^{min\{\alpha,[\frac{d}{2}]\}} \genfrac{[}{]}{0pt}{}{\alpha}{i}_{2^2} \genfrac{[}{]}{0pt}{}{n-2\alpha}{d-2i}_2 \Big/ \genfrac{[}{]}{0pt}{}{n}{2}_2$ and for $i \geq 1$, $\genfrac{[}{]}{0pt}{}{\alpha}{i}_{2^2} \genfrac{[}{]}{0pt}{}{n-2\alpha}{d-2i}_2 \Big/ \genfrac{[}{]}{0pt}{}{n}{2}_2 \leq \left(\frac{2^2}{2^2-1}\right)^i 2^{2i(\alpha-i)} \cdot 2^{d-2i+(d-2i)(n-d-2\alpha+2i)} \Big/ 2^{d(n-d)} \leq \frac{1}{2^{17}}$, because $d(n-d) - 2i(\alpha-i) - (d-2i)(n-d-2\alpha+2i) - d = 2i(n-d-\alpha+i) + 2(\alpha-i)(d-2i) - d \geq 2i(n-d-\alpha+i) - d \geq 2(\frac{n}{2}-d+i) - d = n-3d+2i \geq 17$. For $i=0$, $\genfrac{[}{]}{0pt}{}{n-2\alpha}{d}_2 \Big/ \genfrac{[}{]}{0pt}{}{n}{2}_2 \leq \frac{1}{2^{2d\alpha}} \leq \frac{1}{2^8}$. Hence $\mathcal{N}(g,d) \leq \frac{3}{2^{17}} + \frac{1}{2^8} \leq \frac{1}{2^7}$ for $d \geq 4$.

Suppose $|g|=7$. So g is of type $3_1^{\alpha_1} 3_2^{\alpha_2} 1^{n-3(\alpha_1+\alpha_2)}$. If $d=1$ or 2, then $\mathcal{N}(g,d) \leq \genfrac{[}{]}{0pt}{}{n-3(\alpha_1+\alpha_2)}{d}_2 \Big/ \genfrac{[}{]}{0pt}{}{n}{d}_2 \leq \frac{1}{2^{3d(\alpha_1+\alpha_2)}}$, which is $\leq \frac{1}{2^7}$ if $\alpha_1+\alpha_2 \geq 3$ and $d=1$; or $\alpha_1+\alpha_2 \geq 2$ and $d=2$. For $d=2$ and $\alpha_1+\alpha_2=1$, we have that $\mathcal{N}(g,2) = \genfrac{[}{]}{0pt}{}{n-3}{2}_2 \Big/ \genfrac{[}{]}{0pt}{}{n}{2}_2 \leq \frac{1}{2^6}$. If $d=3$, 4, or 5, then $\mathcal{N}(g,d) = \left\{ \genfrac{[}{]}{0pt}{}{\alpha_1}{1}_{2^3} \genfrac{[}{]}{0pt}{}{n-3(\alpha_1+\alpha_2)}{d-3}_2 + \genfrac{[}{]}{0pt}{}{\alpha_2}{1}_{2^3} \genfrac{[}{]}{0pt}{}{n-3(\alpha_1+\alpha_2)}{d-3}_2 + \genfrac{[}{]}{0pt}{}{n-3(\alpha_1+\alpha_2)}{d}_2 \right\} \Big/$

$\begin{bmatrix} n \\ d \end{bmatrix}_2 \leq \frac{2}{2^{2n-12}} + \frac{1}{2^{3(\alpha_1+\alpha_2)d}} \leq \frac{1}{2^7}$. If $d=6$, then $\mathcal{N}(g,6) = \Big\{ \begin{bmatrix} \alpha_1 \\ 1 \end{bmatrix}_{2^3} \begin{bmatrix} \alpha_2 \\ 1 \end{bmatrix}_{2^3} + \begin{bmatrix} \alpha_1 \\ 1 \end{bmatrix}_{2^3} \begin{bmatrix} n-3(\alpha_1+\alpha_2) \\ 3 \end{bmatrix}_2 + \begin{bmatrix} \alpha_2 \\ 1 \end{bmatrix}_{2^3} \begin{bmatrix} n-3(\alpha_1+\alpha_2) \\ 3 \end{bmatrix}_2 + \begin{bmatrix} n-3(\alpha_1+\alpha_2) \\ 6 \end{bmatrix}_2 \Big\} \Big/ \begin{bmatrix} n \\ 6 \end{bmatrix}_2 \leq \frac{1}{2^{4n-26}} + \frac{2}{2^{2n-15}} + \frac{1}{2^{18(\alpha_1+\alpha_2)}} \leq \frac{1}{2^7}$.

(7.5) Suppose $|g| = 2^e s$ with $(2,s)=1$, $e \geq 1$ and $s \geq 3$. Then $\mathcal{N}(g) \leq \frac{1}{2^7}$ unless one of the following holds:

(a) $|g|=6$, $d \leq 3$, and $\mathcal{N}(g) \leq \frac{1}{2^6} + \frac{1}{2^{31}}$, $\frac{1}{2^6} + \frac{1}{2^{32}}$ for $d=2$, 3 respectively.

(b) $|g| = 10$, 12, 14, 20, 24, 28, 30, 42, 60, 62; and $d=1$.

Proof. By (7.3) and (7.4), $e \leq 3$ and $s = 3,5,7,9,15,21,31$, or 63. Suppose $e=1$. For $|g|=6$, as $\mathcal{N}(g) \leq \mathcal{N}(g^2)$, by (7.4)(a), $d \leq 3$. Since we have that $\nu = dim\{C_V(g)\} \leq n-3$, for $d=1$, $\mathcal{N}(g) \leq \frac{1}{2^3}$. If $d=2$, then by (7.4)(a), g^2 is of type $2^1 1^{n-2}$, because otherwise $\mathcal{N}(g) \leq \mathcal{N}(g^2) \leq \frac{1}{2^7}$. So $dim\{C_V(g^2)\} = n-2$. Also $x = \frac{1}{2}(dim\{C_V(g^3)\} - \nu) \leq \frac{1}{2}(n-1)$. Thus by (7.1)(b), $\mathcal{N}(g,2) \leq \Big\{ \begin{bmatrix} n-3 \\ 2 \end{bmatrix}_2 + 2^{\nu-1} \begin{bmatrix} n-2-\nu \\ 1 \end{bmatrix}_2 + \begin{bmatrix} x \\ 2 \end{bmatrix}_{2^2} \Big\} \Big/ \begin{bmatrix} n \\ 2 \end{bmatrix}_2 \leq \frac{1}{2^6} + \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}} \leq \frac{1}{2^6} + \frac{1}{2^{31}}$. If $d=3$, then by (6.4)(a), $\mathcal{N}(g) \leq \mathcal{N}(g^2) \leq \frac{1}{2^6} + \frac{1}{2^{32}}$. For $|g| = 10$, 18, 30, 42, 62, 126, by (7.4) we have $d=1$; and by (7.2) for $|g|=18$ or 126, we have $\mathcal{N}(g) \leq \frac{1}{2^7}$. For $|g|=14$, we have $d=1$ or 2. If $d=2$, then by (7.4)(c), g^2 is of type $3^1 1^{n-3}$. So by (7.1)(b), $\mathcal{N}(g,2) \leq \Big\{ \begin{bmatrix} n-4 \\ 2 \end{bmatrix}_2 + 2^{\nu-1} \begin{bmatrix} n-3-\nu \\ 1 \end{bmatrix}_2 \Big\} \Big/ \begin{bmatrix} n \\ 2 \end{bmatrix}_2 \leq \frac{1}{2^8} + \frac{1}{2^{n-1}} \leq \frac{1}{2^7}$.

Suppose $e=2$. For $|g|=12$, as $\mathcal{N}(g) \leq \mathcal{N}(g^4)$, by (7.4)(a), $d \leq 3$. Since we have that $\nu = dim\{C_V(g)\} \leq n-4$, for $d=1$, $\mathcal{N}(g) \leq \frac{1}{2^4}$. If $d=2$, then by (7.4)(a), g^4 is of type $2^1 1^{n-2}$, because otherwise $\mathcal{N}(g) \leq \mathcal{N}(g^2) \leq \frac{1}{2^7}$. So $dim\{C_V(g^2)\} = n-2$. Also by (7.3)(b), g^3 is of type $3^1 2^1 1^{n-5}$, $3^1 1^{n-3}$, or $4^1 1^{n-4}$. So $x = \frac{1}{2}(dim\{C_V(g^3)\} - \nu) \leq \frac{1}{2}(n-2)$. Thus by (7.1)(b), $\mathcal{N}(g,2) \leq \Big\{ \begin{bmatrix} n-4 \\ 2 \end{bmatrix}_2 + 2^{\nu-1} \begin{bmatrix} n-2-\nu \\ 1 \end{bmatrix}_2 + \begin{bmatrix} x \\ 2 \end{bmatrix}_{2^2} \Big\} \Big/ \begin{bmatrix} n \\ 2 \end{bmatrix}_2 \leq \frac{1}{2^8} + \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}} \leq \frac{1}{2^7}$. Similarly, if $d=3$, then $\mathcal{N}(g) \leq \frac{1}{2^7}$. For $|g| = 20$, 36, 60, 84, 124, 252, by (7.4) we have $d=1$; and by (7.2) for $|g| = 36$, 84, 124 or 252, we have $\mathcal{N}(g) \leq \frac{1}{2^7}$. For $|g|=28$, we have $d=1$ or 2. If $d=2$, then by

(7.4)(c), g^4 is of type $3^1 1^{n-3}$. So by (7.1)(b), $\mathcal{N}(g,2) \leq \left\{ \begin{bmatrix} n-5 \\ 2 \end{bmatrix}_2 + 2^{\nu-1} \begin{bmatrix} n-4-\nu \\ 1 \end{bmatrix}_2 \right\} \Big/ \begin{bmatrix} n \\ 2 \end{bmatrix}_2 \leq \frac{1}{2^{10}} + \frac{1}{2^{n-1}} \leq \frac{1}{2^7}$.

Suppose $e=3$. As $|g^s|=8$, and $\mathcal{N}(g) \leq \mathcal{N}(g^s)$, by (7.3)(a), $d=1$. By (7.2), for $|g|=8s$ with $s=5,7,9,15,21,31,63$, we have $\mathcal{N}(g) \leq \frac{1}{2^7}$. So it left only $|g|=24$.

(7.6) If $d \geq 2$, then $\mathcal{U}(g) \leq \frac{27}{16} \cdot \frac{1}{|g|} + \frac{1}{2^7}$.

Proof. Suppose $d \geq 2$. Then by (7.4), (7.5), and (7.6), we have $\mathcal{N}(g) \leq \frac{1}{2^7}$ except possibly when $|g|=2,3,4,6,7$; and in which cases, $\mathcal{N}(g) \leq \frac{1}{2^2} + \frac{1}{2^{31}}$, $\frac{1}{2^4} + \frac{1}{2^{62}}$, $\frac{1}{2^4} + \frac{1}{2^{32}}$, $\frac{1}{2^6} + \frac{1}{2^{31}}$, $\frac{1}{2^6}$ respectively. So $\mathcal{U}(g) \leq \frac{1}{|g|}\{1 + \phi(2)(\frac{1}{2^2} + \frac{1}{2^{31}}) + \phi(3)(\frac{1}{2^4} + \frac{1}{2^{62}}) + \phi(4)(\frac{1}{2^4} + \frac{1}{2^{32}}) + \phi(6)(\frac{1}{2^6} + \frac{1}{2^{31}}) + \phi(7) \cdot \frac{1}{2^6} + \sum_{d||g|, d>4, d \neq 6,7} \phi(d) \mathcal{N}(g^{\frac{|g|}{d}})\} \leq \frac{1}{|g|}(\frac{13}{8} + \frac{1}{2^{27}}) + \frac{1}{|g|} \cdot \frac{1}{2^7} \sum_{d||g|, d>4, d \neq 6,7} \phi(d) \leq \frac{27}{16} \cdot \frac{1}{|g|} + \frac{1}{2^7}$.

(7.7) Assume $d=1$. We have the following upper bounds for $\mathcal{U}(g)$:

(a) Suppose $|g|$ is odd:

(i) $(|g|,21)=1$, $\mathcal{U}(g) \leq \frac{1}{|g|} + \frac{1}{16}$.

(ii) $3||g|$ but $7 \nmid |g|$, or $3 \nmid |g|$ but $7||g|$, $\mathcal{U}(g) \leq \frac{21}{16} \cdot \frac{1}{|g|} + \frac{1}{16}$.

(iii) $|g| \equiv 0 (mod\ 21)$, $\mathcal{U}(g) \leq \frac{27}{16} \cdot \frac{1}{|g|} + \frac{1}{16}$.

(b) $|g|=2^e$ and $e \geq 3$, $\mathcal{U}(g) \leq \frac{63}{32} \cdot \frac{1}{|g|} + \frac{1}{2^8}$.

(c) $2||g|$ but $|g| \neq 2^e$:

(i) $4 \nmid |g|$, $3 \nmid |g|$ and $7 \nmid |g|$, $\mathcal{U}(g) \leq \frac{11}{8} \cdot \frac{1}{|g|} + \frac{1}{16}$.

(ii) $4||g|$, $3 \nmid |g|$ and $7 \nmid |g|$, $\mathcal{U}(g) \leq \frac{7}{4} \cdot \frac{1}{|g|} + \frac{1}{16}$.

(iii) $4 \nmid |g|$, $3||g|$ and $7 \nmid |g|$, $\mathcal{U}(g) \leq \frac{15}{8} \cdot \frac{1}{|g|} + \frac{1}{16}$.

(iv) $4 \nmid |g|$, $3 \nmid |g|$ and $7||g|$, $\mathcal{U}(g) \leq \frac{7}{4} \cdot \frac{1}{|g|} + \frac{1}{16}$.

(v) $4||g|$, $3||g|$ and $7\not||g|$, $\mathcal{U}(g)\leq\frac{9}{4}\cdot\frac{1}{|g|}+\frac{1}{16}$.

(vi) $4\not||g|$, $3||g|$ and $7||g|$, $\mathcal{U}(g)\leq\frac{9}{4}\cdot\frac{1}{|g|}+\frac{1}{16}$.

(vii) $4||g|$, $3\not||g|$ and $7||g|$, $\mathcal{U}(g)\leq\frac{17}{8}\cdot\frac{1}{|g|}+\frac{1}{16}$.

(viii) $4||g|$, $3||g|$ and $7||g|$, $\mathcal{U}(g)\leq\frac{21}{8}\cdot\frac{1}{|g|}+\frac{1}{16}$.

In any case, we always have $\mathcal{U}(g)\leq\frac{21}{8}\cdot\frac{1}{|g|}+\frac{1}{16}$.

Proof. For example, we show part (viii) in (c). So $|g|\equiv 0(mod\ 84)$. We have by (7.2) that

$$\mathcal{U}(g)=\frac{1}{|g|}\{1+\phi(2)\mathcal{N}(g^{\frac{|g|}{2}})+\phi(3)\mathcal{N}(g^{\frac{|g|}{3}})+\phi(4)\mathcal{N}(g^{\frac{|g|}{4}})+\phi(6)\mathcal{N}(g^{\frac{|g|}{6}})+\phi(7)\mathcal{N}(g^{\frac{|g|}{7}})+$$

$$\sum_{d||g|,d>4,d\neq 6,7}\phi(d)\mathcal{N}(g^{\frac{|g|}{d}})\}\leq\frac{1}{|g|}(1+\tfrac{1}{2}+2\cdot\tfrac{1}{4}+2\cdot\tfrac{1}{4}+2\cdot\tfrac{1}{8}+6\cdot\tfrac{1}{8})+\frac{|g|-14}{|g|}\cdot\frac{1}{16}=\frac{21}{8}\cdot\frac{1}{|g|}+\frac{1}{16}.$$

(7.8) For $1\leq d\leq 6$, we have $\mathcal{U}(g)\leq\frac{21}{8}\cdot\frac{1}{|g|}+\frac{1}{16}$.

Proof. This combines (7.6) and the last statement in (7.7).

In the following, unless we explicitly change the definition, ν_i denotes $dim\{C_V(g_i)\}$ for $g_i\in S$. In particular, ν_i is the number of Jordan blocks of g_i when g_i is unipotent. Also when we write $\mathcal{U}(g)\leq b$, it means that it holds for any d with $1\leq d\leq 6$ unless it is mentioned explicitely what specific value of d is under consideration.

(7.9) (a) Suppose $|g|=2$. If $d=1$, then $\mathcal{U}(g)\leq\frac{3}{4}$, $\frac{5}{8}$ for transvection and non-transvection respectively. If $d\geq 2$, then $\mathcal{U}(g)\leq\frac{5}{8}+\frac{1}{2^{32}}$, $\frac{17}{32}+\frac{1}{2^{31}}$ for transvection and non-transvection respectively. If $d\geq 2$ and $dim([V,g])>3$, then $\mathcal{U}(g)\leq\frac{129}{256}$.

(b) Suppose $d\geq 2$. Then for $|g|=3$, 4, and 5, $\mathcal{U}(g)\leq\frac{3}{8}+\frac{1}{3\cdot 2^{61}}$, $\frac{11}{32}+\frac{1}{2^{32}}$, $\frac{33}{160}$ respectively. If $|g|\geq 6$, then $\mathcal{U}(g)\leq\frac{37}{128}$. If $|g|=3$ and g is not of type $2^1 1^{n-2}$, then $\mathcal{U}(g)\leq\frac{65}{192}$. If $|g|=4$ and g is not of types: $3^1 2^1 1^{n-5}$, $3^1 1^{n-3}$, $4^1 1^{n-4}$, then $\mathcal{U}(g)\leq\frac{81}{256}+\frac{1}{2^{33}}$.

(c) For $|g|\geq 3$, $\mathcal{U}(g)\leq\frac{1}{2}$. For $|g|\geq 42$, $\mathcal{U}(g)\leq\frac{1}{8}$.

(d) Suppose $d=1$. If $|g|=5$ or $|g|\geq 7$, then $\mathcal{U}(g)\leq\frac{1}{4}$.

(e) Also we have bounds of $\mathcal{U}(g)$ for g of certain orders:

$\|g\|$	$\mathcal{U}(g)$	$\|g\|$	$\mathcal{U}(g)$
41	57/656	40	17/160
39	5/52	38	15/152
37	53/592	36	1/24
35	1/10	34	7/68
33	9/88	32	111/2048
31	47/496	30	17/160
29	45/464	27	1/9
26	3/26	25	41/400
23	39/368	22	389/5632
19	35/304	17	33/272
16	13/128	13	316/4096
11	47/512		

Proof. The first part of (a) follows from (7.2) immediately. If $d\geq 2$, then by (7.3)(d), $\mathcal{N}(g)\leq\frac{1}{2^{31}}+\frac{1}{2^2}$, $\frac{1}{2^{30}}+\frac{1}{2^4}$ respectively. Thus we have the second part of (a). The third part of (a) also follows from (7.3)(d). Part (b) follows from (7.4)(a), (7.3)(b) and (d), (7.4)(b) and (7.6). For example, if $|g|=3$ and g is not of type $2^1 1^{n-2}$, then by (7.4)(a), $\mathcal{U}(g)\leq\frac{1}{3}(1+\frac{2}{2^7})=\frac{65}{192}$. If $|g|=4$ and g is not of types: $3^1 2^1 1^{n-5}$, $3^1 1^{n-3}$, $4^1 1^{n-4}$, then by (7.3)(b), $\mathcal{N}(g)\leq\frac{1}{2^7}$; this together with (7.3)(d) gives $\mathcal{U}(g)\leq\frac{1}{4}\{1+(\frac{1}{2^2}+\frac{1}{2^{31}})+\frac{2}{2^7}\}=\frac{81}{256}+\frac{1}{2^{33}}$. For part (c), by (7.8), if $|g|\geq 6$, then $\mathcal{U}(g)\leq\frac{21}{8}\cdot\frac{1}{6}+\frac{1}{16}=\frac{1}{2}$. If $|g|=5$, then by (7.4)(b), $\mathcal{U}(g)\leq\frac{1}{5}(1+\frac{4}{2^4})=\frac{1}{4}$. If $|g|=4$, then by (7.3)(b) and (c), $\mathcal{U}(g)\leq\frac{1}{4}(1+\frac{1}{2}+\frac{2}{2^2})=\frac{1}{2}$. If $|g|=3$, then by (7.4)(a), $\mathcal{U}(g)\leq\frac{1}{3}(1+\frac{2}{2^2})=\frac{1}{2}$. So the first part of (c) holds. Similarly, the second part of (c) follows directely from (7.8). For part (d), suppose $d=1$. If $|g|\geq 14$, then $\mathcal{U}(g)\leq\frac{21}{8}\cdot\frac{1}{14}+\frac{1}{16}=\frac{1}{4}$. For $|g|=13$, 11, 10, 9, 8, or 7, by (7.7) we have $\mathcal{U}(g)\leq\frac{29}{208}$, $\frac{27}{176}$, $\frac{1}{5}$, $\frac{5}{24}$, $\frac{1}{4}$, $\frac{1}{4}$ respectively. For

$|g|=12$, we have $\mathcal{U}(g)\leq\frac{1}{12}(1+\frac{1}{2}+\frac{2}{2^2}+\frac{2}{2^2}+\frac{2}{2^3}+\frac{4}{2^4})=\frac{1}{4}$; and for $|g|=5$, $\mathcal{U}(g)\leq\frac{1}{5}(1+\frac{4}{2^4})=\frac{1}{4}$ by (7.2). Since $\max\{\frac{29}{208},\frac{27}{176},\frac{1}{5},\frac{5}{24},\frac{1}{4}\}=\frac{1}{4}$, (d) holds. For part (e), except for $|g|=36$, 32, 22, 16, 13, and 11, by plugging in the formulars in (7.6) and (7.7) directly, we imediately have those bounds. For $|g|=32$, since g has at least one block of dimension ≥ 17, $dim\{C_V(g^2)\}\leq n-15$, $dim\{C_V(g^4)\}\leq n-13$, $dim\{C_V(g^4)\}\leq n-9$, so we have $\mathcal{U}(g,1)\leq \frac{1}{32}(1+\frac{1}{2}+\frac{2}{2^9}+\frac{4}{2^{13}}+\frac{8}{2^{15}}+\frac{16}{2^{16}})=\frac{1541}{2^{15}}<\frac{111}{2048}$. Also for $d\geq 2$, by (7.3), $\mathcal{U}(g,d)\leq \frac{1}{32}(1+\frac{1}{2}+\frac{2+4+8+16}{2^7})=\frac{111}{2048}$. It is similar for $|g|=16$. For $|g|=13$, since the smallest positive integer t such that $13|2^t-1$ is 12, g has at least one non trivial simple submodule of dimension 12. As $d\leq 6$, $\mathcal{N}(g,d)\leq\left[\begin{smallmatrix} n-12 \\ d\end{smallmatrix}\right]_2\Big/\left[\begin{smallmatrix} n \\ d\end{smallmatrix}\right]_2\leq\frac{1}{2^{12d}}\leq\frac{1}{2^{12}}$. Thus $\mathcal{U}(g)\leq\frac{1}{13}(1+\frac{12}{2^{12}})=\frac{316}{4096}$. Similar for $|g|=11$. For $|g|=36$, g^4 has at least one simple submodule of dimension 6, and thus g^{12} has at least 3 simple submodules each of dimension 2. So $\mathcal{N}(g)$, $\mathcal{N}(g^2)$, $\mathcal{N}(g^3)$, $\mathcal{N}(g^4)$, $\mathcal{N}(g^6)$, $\mathcal{N}(g^{12})$ are all $\leq\frac{1}{2^6}$ by (7.2) for $d=1$ and by (7.4)(a) for $d\geq 2$. Thus $\mathcal{U}(g)\leq \frac{1}{36}(1+\frac{1}{2}+\frac{2}{2^2}+\frac{2+2+6+4+6+12}{2^6})=\frac{1}{24}$. For $|g|=22$, since g^2 has at least one simple submodule of dimension 10, $\mathcal{N}(g)$, $\mathcal{N}(g^2)$ both $\leq\frac{1}{2^{10}}$ for any d with $1\leq d\leq 6$. Thus $\mathcal{U}(g)\leq \frac{1}{22}(1+\frac{1}{2}+\frac{10}{2^{10}}+\frac{10}{2^{10}})=\frac{389}{5632}$.

(7.10) $|S|\leq 5$.

Proof. Let $r=|S|$. By (7.9)(a) and (b), for all $g\in G^{\#}$, $\mathcal{U}(g)\leq\frac{3}{4}$. Thus $r-2<\sum_{g\in S}\mathcal{U}(g)\leq\frac{3}{4}r$, which gives $r\leq 7$.

Let α be the number of transvections in S. Then we have $r-2<\sum_{g\in S}\mathcal{U}(g)\leq\frac{3}{4}\alpha+(r-\alpha)\frac{5}{8}$ which gives $\alpha>3r-16$. Further as $n>r-1$, G is not generated by $r-1$ transvections, so $\alpha\leq r-2$. Hence $r\leq 6$.

Suppose $r=6$. Then as $3r-16<\alpha\leq r-2$, we have $\alpha=3$ or 4. Let β be the number of involutions in S which are non transvections. Since $r-2<\sum_{g\in S}\mathcal{U}(g)\leq\frac{3}{4}\alpha+\frac{5}{8}\beta+\frac{1}{2}(r-\alpha-\beta)$, we have $4r-16<2\alpha+\beta$, which implies that either $\alpha=4$ and $\beta>0$, or $\alpha=\beta=3$.

Suppose $\alpha=3$ and assume without loss of generality that g_1, g_2, g_3 are transvections and g_4, g_5, g_6 are non transvection involutions. Then we have $4<\sum_{g\in S}\mathcal{U}(g)\leq\frac{9}{4}+\sum_{i=4}^{6}\frac{1}{2}\{1+\mathcal{N}(g_i)\}$ $=\frac{15}{4}+\sum_{i=4}^{6}\mathcal{N}(g_i)$, which implies at least one i, say $i=4$, is such that $\mathcal{N}(g_4)>\frac{1}{12}$. But this forces $d=1$ and $n-\nu_4=2$ by (7.3). Then we have $\sum_{i=1}^{5}dim\{[V,g_i]\}\leq 5+\frac{n}{2}<n$, contradicting $G=\langle g_1,\ldots,g_5\rangle$.

So $\alpha=4$ and $\beta\geq 1$. Say g_5 is an involution. Then we have $\sum_{i=1}^{5}dim\{[V,g_i]\}\leq 4+\frac{n}{2}<n$, contradicting $G=\langle g_1,\ldots,g_5\rangle$.

So $|S|\leq 5$.

(7.11) $|S|\leq 4$.

Proof. Suppose $|S|=r=5$. Assume first that $d\geq 2$. As there are at most 3 transvections in S, we have by (7.9)(a) and (b) the contradiction $\sum\mathcal{U}(g_i)\leq 3\cdot(\frac{5}{8}+\frac{1}{2^{32}})+2\cdot(\frac{17}{32}+\frac{1}{2^{31}})<3$. So $d=1$.

Let α be the number of transvections in S, and β be the number of non-transvection involutions in S. First suppose $\alpha+\beta=5$, i.e., all elements in S are involutions. Since $q=2$ and $d=1$, we have $|\Omega|=|G/H|=2^n-1$. The condition $\sum_{i=1}^{5}c(g_i)=(r-2)|\Omega|+2$ thus implies $\frac{5}{2}+\frac{1}{2}\sum_{i=1}^{5}\frac{2^{\nu_i}-1}{2^n-1}=3+\frac{2}{2^n-1}$, and this gives $\sum_{i=1}^{5}2^{\nu_i}=2^n+8$. Without loss of generality, assume $\nu_1\geq\cdots\geq\nu_5$. Then $2^{\nu_5}|8$, which gives $\nu_5\leq 3$. But $\nu_5\geq\lceil\frac{n}{2}\rceil$. So as we assume $n\geq 33$, not all elements in S are involutions.

Similar as in (7.10), since $4r-16<2\alpha+\beta$ and $\alpha+\beta\neq 5$, we have $\alpha+\beta=3$ or 4, and $\alpha\geq 1$. Also clearly $\alpha\leq 3$.

Suppose $\alpha=3$, say g_1, g_2, g_3 are transvections. Then by (3.3), $|g_4|$ and $|g_5|$ are both $\geq\frac{n}{3}\geq 11$, which implies that $\mathcal{U}(g_i)\leq\frac{21}{8}\cdot\frac{1}{11}+\frac{1}{16}=\frac{53}{176}$ for $i=4$ and 5. Then the contradiction $\sum\mathcal{U}(g_i)\leq 3\cdot\frac{3}{4}+2\cdot\frac{53}{176}<3$ shows that $\alpha=1$ or 2.

Suppose $\alpha=2$, say g_1, g_2 are transvections and g_3 is an involution. Then $\nu_i\leq\frac{n}{2}+2$ for

$i=4$ and 5. Thus for $i=4$ and 5, if $|g_i|=3$, then $\mathcal{U}(g_i)\le\frac{1}{3}(1+\frac{2}{2^{\frac{n}{2}-2}})\le\frac{1}{3}+\frac{1}{3\cdot2^{13}}$. Similarly, for $|g_i|=4$, 5, or 6, we have $\mathcal{U}(g_i)\le\frac{3}{8}+\frac{1}{2^{15}}$, $\frac{1}{5}+\frac{1}{5\cdot2^{12}}$, $\frac{1}{3}+\frac{1}{3\cdot2^{14}}$ respectively. If $|g_i|\ge7$, then $\mathcal{U}(g_i)\le\frac{21}{8}\cdot\frac{1}{7}+\frac{1}{16}=\frac{7}{16}$. Since $max\{\frac{1}{3}+\frac{1}{3\cdot2^{13}}, \frac{3}{8}+\frac{1}{2^{15}}, \frac{1}{5}+\frac{1}{5\cdot2^{12}}, \frac{1}{3}+\frac{1}{3\cdot2^{14}}, \frac{7}{16}\}=\frac{7}{16}$, we have that if $\alpha+\beta=3$, then $\sum\mathcal{U}(g_i)\le2\cdot\frac{3}{4}+\frac{5}{8}+2\cdot\frac{7}{16}=3$. If $\alpha+\beta=4$, say g_4 is an involution, then $\nu_3+\nu_4\le n+2$. Since ν_3 and ν_4 are both $\ge\lceil\frac{n}{2}\rceil$, we have ν_3 and ν_4 are both $\le n-\lceil\frac{n}{2}\rceil+2\le n-4$, which gives that $\mathcal{U}(g_3)$ and $\mathcal{U}(g_4)$ are both $\le\frac{1}{2}(1+\frac{1}{2^4})=\frac{17}{32}$. Thus $\sum\mathcal{U}(g_i)\le2\cdot\frac{3}{4}+2\cdot\frac{17}{32}+\frac{7}{16}=3$.

Hence $\alpha=1$. If $\beta=2$, then $\sum\mathcal{U}(g_i)\le\frac{3}{4}+2\cdot\frac{5}{8}+2\cdot\frac{1}{2}<3$. So $\beta=3$. Say g_1 is a transvection and g_2, g_3, g_4 are non-transvection involutions. Then there is at most one g_i in $\{g_2, g_3, g_4\}$ with $\nu_i=n-2$. This implies $\sum\mathcal{U}(g_i)\le\frac{3}{4}+\frac{5}{8}+2\cdot\frac{9}{16}+\frac{1}{2}=3$.

Therefore $r\le4$.

(7.12) If S is of type $(2,2,2,k)$, then $d=1$ and $3\le k\le6$.

Proof. By (2.3), not all g_i's in S are involutions, i.e., $k\ge3$. As $G=\langle g_1, \ldots, g_3\rangle$, we have $\nu_1+\nu_2+\nu_3\le2n$. Also as $n\ge33$, there is at most one element among $\{g_1,g_2,g_3\}$ of types $2^{\alpha}1^{n-2\alpha}$ with $1\le\alpha\le8$. Suppose $d\ge2$ first. If there is a transvection, say g_1, then $\mathcal{U}(g_4)\le\frac{1}{3}(1+\frac{2}{2^7})=\frac{65}{192}$ for $|g_4|=3$ as g_4 is not of type 2^11^{n-2}. Hence by (7.9)(b), $\mathcal{U}(g_4)\le max\{\frac{65}{192}, \frac{11}{32}+\frac{1}{2^{32}}, \frac{33}{160}, \frac{37}{128}\}=\frac{11}{32}+\frac{1}{2^{32}}$, which gives the contradiction $\sum\mathcal{U}(g_i)\le(\frac{5}{8}+\frac{1}{2^{32}})+2\cdot\frac{129}{256}+(\frac{11}{32}+\frac{1}{2^{32}})<2$. If there is no transvection in S, then we have by (7.9)(a) and (b) the contradiction $\sum\mathcal{U}(g_i)\le(\frac{17}{32}+\frac{1}{2^{31}})+2\cdot\frac{129}{256}+(\frac{3}{8}+\frac{1}{3\cdot2^{61}})<2$. So $d=1$.

Since $n-1\ge\nu_i\ge\frac{n}{2}$ $\forall1\le i\le3$, $2^{\nu_1}+2^{\nu_2}+2^{\nu_3}$ achieves the maximum when $\{\nu_1, \nu_2, \nu_3\}=\{n-1, \frac{n}{2}+1, \frac{n}{2}\}$; and thus $\sum_{i=1}^{3}\mathcal{U}(g_i)\le\frac{3}{2}+\frac{1}{2}(\frac{1}{2}+\frac{1}{2^{\frac{n}{2}-1}}+\frac{1}{2^{\frac{n}{2}}})\le\frac{451}{256}$. For $k\ge15$, $\mathcal{U}(g_4)\le\frac{21}{8}\cdot\frac{1}{15}+\frac{1}{16}=\frac{19}{80}$ by (7.7). If $k=14$, $\mathcal{U}(g_4)\le\frac{7}{4}\cdot\frac{1}{14}+\frac{1}{16}=\frac{3}{16}$ by (7.7)(c)(iv). Similarly, for $k=13$ and 11, we have respectively $\mathcal{U}(g_4)\le\frac{29}{208}, \frac{27}{176}$. Since $max\{\frac{19}{80}, \frac{3}{16}, \frac{29}{208}, \frac{27}{176}\}=\frac{19}{80}$, we have $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq\frac{451}{256}+\frac{19}{80}<2$, a contradiction. So $k\leq 10$ or $k=12$.

If there is no transvection among g_1, g_2, g_3, then $n-2\geq\nu_i\geq\frac{n}{2}$ $\forall 1\leq i\leq 3$. These three conditions plus the condition $\nu_1+\nu_2+\nu_3\leq 2n$ implies that $max\{2^{\nu_1}+2^{\nu_2}+2^{\nu_3}\}=2^{n-2}+2^{\frac{n}{2}+2}+2^{\frac{n}{2}}$. Hence $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{3}{2}+\frac{1}{2}(\frac{1}{4}+\frac{1}{2^{\frac{n}{2}-2}}+\frac{1}{2^{\frac{n}{2}}})\leq\frac{421}{256}$.

Suppose $k\geq 9$, by (7.7), $\mathcal{U}(g_4)\leq\frac{21}{8}\cdot\frac{1}{9}+\frac{1}{16}=\frac{17}{48}$. Also by (7.7)(b), for $k=8$, $\mathcal{U}(g_4)\leq\frac{1}{4}$. So if there is no transvection among g_1, g_2, g_3, then $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq\frac{421}{256}+\frac{17}{48}<2$, a contradiction. Thus for $k\geq 8$, one of g_1, g_2, g_3 is a transvection. Then this implies that $\nu_4\leq\frac{n}{2}+1$, and thus $\mathcal{N}(g_4)\leq\frac{1}{2^{\frac{n}{2}-1}}\leq\frac{1}{2^6}$. So for $k=12$, we have $\mathcal{U}(g_4)=\frac{1}{12}\{1+\mathcal{N}(g_4^6)+2\mathcal{N}(g_4^4)+2\mathcal{N}(g_4^3)+2\mathcal{N}(g_4^2)+4\mathcal{N}(g_4)\}\leq\frac{1}{12}(1+\frac{1}{2}+\frac{2}{4}+\frac{2}{4}+\frac{2}{8}+\frac{4}{2^6})=\frac{15}{64}$. For $k=9$, as g_4^3 has at least 3 simple submodules of dimension 2, we have $\mathcal{U}(g_4)\leq\frac{1}{9}(1+\frac{2}{2^6}+\frac{6}{2^6})=\frac{1}{8}$. For $k=10$, as $|g_4^2|=5$, we have that $\mathcal{N}(g_4^2)\leq\frac{1}{2^4}$, and thus $\mathcal{U}(g_4)=\frac{1}{12}\{1+\mathcal{N}(g_4^5)+4\mathcal{N}(g_4^2)+4\mathcal{N}(g_4)\}\leq\frac{1}{10}(1+\frac{1}{2}+\frac{4}{16}+\frac{4}{2^6})=\frac{29}{160}$. For $k=8$, as g_4^2 is not of type $3^1 1^{n-3}$, $dim\{C_V(g_4^2)\}\leq n-3$. So $\mathcal{N}(g_4^2)\leq\frac{1}{2^3}$, and thus $\mathcal{U}(g_4)=\frac{1}{8}\{1+\mathcal{N}(g_4^4)+2\mathcal{N}(g_4^2)+4\mathcal{N}(g_4)\}\leq\frac{1}{8}(1+\frac{1}{2}+\frac{2}{8}+\frac{4}{2^6})=\frac{29}{128}$. Hence for $k=8$, 9, 10, and 12, $\mathcal{U}(g_4)\leq max\{\frac{29}{128},\frac{1}{8},\frac{29}{160},\frac{15}{64}\}=\frac{15}{64}$, which gives $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq\frac{451}{256}+\frac{15}{64}=\frac{511}{256}<2$, a contradiction. So $k\leq 7$.

Suppose $k=7$. If one of g_1, g_2, g_3 is a transvection, then $\nu_4\leq\frac{n}{2}+1$, and thus $\mathcal{N}(g_4)\leq\frac{1}{2^{\frac{n}{2}-1}}\leq\frac{1}{2^6}$. Then $\mathcal{U}(g_4)\leq\frac{1}{7}(1+\frac{6}{2^6})=\frac{5}{32}$, and we have $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq\frac{451}{256}+\frac{5}{32}=\frac{491}{256}<2$, a contradiction. So no transvection in S. Since we always have $\mathcal{U}(g)\leq\frac{1}{7}(1+\frac{6}{8})=\frac{1}{4}$ for $|g|=7$, in this case, we have $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq\frac{421}{256}+\frac{1}{4}=\frac{485}{256}<2$, a contradiction again. So $k\leq 6$.

(7.13) S is not of type (2,2,2,6) for $d=1$.

Proof. Denote $\alpha=dim\{C_V(g_4^3)\}$ and $\beta=dim\{C_V(g_4^2)\}$. Since $|\Omega|=2^n-1$ and $\sum_{i=1}^{4}\mathcal{U}(g_i)=2+\frac{2}{|\Omega|}=2+\frac{2}{2^n-1}$, we have $\frac{3}{2}+\frac{1}{2}\sum_{i=1}^{3}\frac{2^{\nu_i}-1}{2^n-1}+\frac{1}{6}+\frac{1}{6}\cdot\frac{2^{\alpha}-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\beta}-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\nu_4}-1}{2^n-1}=2+\frac{2}{2^n-1}$. This identity can be transformed to $2^{n+1}-2^{\alpha}-3(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})=2^{\beta+1}+2^{\nu_4+1}-24$.

Suppose $\nu_4 \geq 3$. As $\beta \geq \nu_4$, 2^4 divides $2^{n+1}-2^{\alpha}-3(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})-2^{\beta+1}-2^{\nu_4+1}$; which implies $2^4|24$, a contradiction. So if $n\geq 8$, then $\nu_4=0$, 1, or 2. Suppose $\nu_4=0$. Then $2^{\beta+1}+2^{\nu_4+1}-24 =2(2^{\beta}-11)$. So for $n\geq 6$, we have $4|(2^{\beta}-11)$. But $2^{\beta}-11$ is even only when $\beta=0$ and in which case $11-2^{\beta}=10$. Thus for $n\geq 6$, $\nu_4\neq 0$. Suppose $\nu_4=1$. Then $2^{\beta+1}+2^{\nu_4+1}-24= 4(2^{\beta-1}-5)$. So for $n\geq 10$, we have $8|(2^{\beta-1}-5)$. But $2^{\beta-1}-5$ is even only when $\beta=1$ and in which case $5-2^{\beta-1}=4$. Thus for $n\geq 10$, $\nu_4\neq 1$. So if we assume $n\geq 10$, we must have $\nu_4=2$. Then $2^{\beta+1}+2^{\nu_4+1}-24=2^{\beta+1}-16$. Thus for $n\geq 10$, $2^5|(2^{\beta+1}-16)$. Then $\beta\geq 4$ and $\beta=2$ give contradictions. So for $n\geq 10$, we must have $\nu_4=2$ and $\beta=3$, in which case $2^{\beta+1}+2^{\nu_4+1}-24=0$. That is for $n\geq 10$, we have $2^{n+1}-2^{\alpha}-3(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})=0$. Now if g_1 and g_4^3 are both transvections, then $\nu_1=n-1=\alpha$. Thus $2^{n+1}-2^{\alpha}-3(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})=-3(2^{\nu_2}+2^{\nu_3})<0$ and this contradiction shows that g_1 and g_4^3 cannot both be transvections. If g_1 is a transvection but g_4^3 is not, then $\nu_1=n-1$ and $\alpha\leq n-2$. Thus $2^{n-2}\leq 2^{n+1}-2^{\alpha}-3\cdot 2^{\nu_1}=3(2^{\nu_2}+2^{\nu_3})$. On the other hand, since $\nu_1+\nu_2+\nu_3\leq 2n$, we have $\nu_2+\nu_3\leq n+1$. This together with $\nu_i\geq\frac{n}{2}$ gives $\{\nu_2, \nu_3\}=\{\frac{n}{2}, \frac{n}{2}+1\}$ if n even, and $\{\frac{n+1}{2}, \frac{n+1}{2}\}$ if n odd. So either $2^{n-2}\leq 3(2^{\frac{n}{2}}+2^{\frac{n}{2}+1})$ or $2^{n-2}\leq 3\cdot 2^{\frac{n+1}{2}+1}$; which implies either $2^{\frac{n}{2}-2}\leq 9$ or $2^{\frac{n-7}{2}}\leq 3$. So if $n\geq 11$, we have a contradiction. If no g_i, $1\leq i\leq 3$, is a transvection, then $\nu_i\leq n-2$, and we always have $\alpha\leq n-1$. Thus $2^{n-1}\leq\frac{1}{3}(2^{n+1}-2^{\alpha})=2^{\nu_1}+2^{\nu_2}+2^{\nu_3}$. Then at least one, say $\nu_1=n-2$. So $2^{n-2}\leq 2^{\nu_2}+2^{\nu_3}$. Since $\nu_1+\nu_2+\nu_3\leq 2n$, we have $\nu_2+\nu_3\leq n+2$. This together with $n-2\geq\nu_i\geq\frac{n}{2}$ gives $2^{\nu_2}+2^{\nu_3}\leq 2^{\frac{n}{2}}+2^{\frac{n}{2}+2}$ if $n\geq 8$. Then we have $2^{\frac{n}{2}-2}\leq 5$, which is a contradiction if $n\geq 10$. So in conclusion, we have that if $n\geq 11$, then S is not of type (2,2,2,6).

(7.14) S is not of type (2,2,2,5) for $d=1$.

Proof. We have $\frac{3}{2}+\frac{1}{2}\sum_{i=1}^{3}\frac{2^{\nu_i}-1}{2^n-1}+\frac{1}{5}+\frac{4}{5}\cdot\frac{2^{\nu_4}-1}{2^n-1}=2+\frac{2}{2^n-1}$. This identity can be transformed to $3\cdot 2^n-5(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})=8(2^{\nu_4}-5)$. If $n\geq 12$, then $\nu_i\geq 6$ $\forall 1\leq i\leq 3$. Hence 2^6 divides the left hand side of the equation. Then $8|(2^{\nu_4}-5)$. But $2^{\nu_4}-5$ is even only when $\nu_4=0$ and in which case $5-2^{\nu_4}=4$. So we have a contradiction. Thus if $n\geq 12$, then S is not of type (2,2,2,5).

(7.15) S is not of type (2,2,2,4) for $d=1$.

Proof. Denote $\alpha=dim\{C_V(g_4^2)\}$. Similar to the above, this time we have $\frac{3}{2}+\frac{1}{2}\sum_{i=1}^{3}\frac{2^{\nu_i}-1}{2^n-1}+\frac{1}{4}+\frac{1}{4}\cdot\frac{2^\alpha-1}{2^n-1}+\frac{2}{4}\cdot\frac{2^{\nu_4}-1}{2^n-1}=2+\frac{2}{2^n-1}$. This identity can be transformed to $2^n-2(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})-2^\alpha=16(2^{\nu_4-3}-1)$. If $n\geq 10$, then α and $\nu_i\geq 5$ $\forall 1\leq i\leq 3$. Hence 2^5 divides the left hand side of the equation. Then $2|(2^{\nu_4-3}-1)$. This implies $\nu_4=3$. But if $n\geq 13$, then $\nu_4\geq 4$. So for $n\geq 13$, S is not of type (2,2,2,4).

(7.16) S is not of type (2,2,2,3) for $d=1$.

Proof. We have $\frac{3}{2}+\frac{1}{2}\sum_{i=1}^{3}\frac{2^{\nu_i}-1}{2^n-1}+\frac{1}{3}+\frac{2}{3}\cdot\frac{2^{\nu_4}-1}{2^n-1}=2+\frac{2}{2^n-1}$. This identity can be transformed to $2^n-3(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})=8(2^{\nu_4-1}-3)$. If $n\geq 10$, then $\nu_i\geq 5$ $\forall 1\leq i\leq 3$. Hence 2^5 divides the left hand side of the equation. Then $4|(2^{\nu_4-1}-3)$. But 2^{ν_4-1} is even only when $\nu_4=1$ and in which case $3-2^{\nu_4-1}=2$. So we have a contradiction. So for $n\geq 10$, S is not of type (2,2,2,3).

(7.17) Suppose $|S|=4$. Then there are at most 2 involutions in S.

Proof. This combines (7.12) to (7.16).

(7.18) Suppose $|S|=4$ and S has 2 involutions. Then $d=1$ and S is one of the following types: (2,2,4,6), (2,2,4,4), (2,2,3,4), (2,2,3,3), or (2,2,3,6).

Proof. Suppose without loss of generality that g_1 and g_2 are involutions. Suppose $d\geq 2$ first. If g_1 and g_2 are both of types $2^\alpha 1^{n-2\alpha}$ with $\alpha\leq 3$, then for $|g_i|=3$, $i=3$ or 4, g_i is not of type $2^1 1^{n-2}$. So as in (7.12), $\mathfrak{U}(g_i)\leq\frac{11}{32}+\frac{1}{2^{32}}$, which gives the contradiction $\sum\mathfrak{U}(g_i)\leq 2\cdot(\frac{5}{8}+\frac{1}{2^{32}})+2\cdot(\frac{11}{32}+\frac{1}{2^{32}})<2$. If there is only one among g_1, g_2 of type: $2^\alpha 1^{n-2\alpha}$ with $\alpha\leq 3$, then we have the contradiction $\sum\mathfrak{U}(g_i)\leq(\frac{5}{8}+\frac{1}{2^{32}})+\frac{129}{256}+2\cdot(\frac{3}{8}+\frac{1}{3\cdot 2^{61}})<2$. So $d=1$.

Suppose for $i=3$ and 4, $|g_i|\neq 3$, 4, and 6. Then by (7.9)(d), we have the contradiction $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq 2\cdot\frac{3}{4}+2\cdot\frac{1}{4}=2$. Hence one of g_3 and g_4 has order equal to 3, 4, or 6.

First suppose that both g_1 and g_2 are transvections. Then by (3.3), $|g_i|\geq\lceil\frac{n}{2}\rceil\geq 17$ for $i=3$ and 4. Thus (7.9)(d) implies $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq 2\cdot\frac{3}{4}+2\cdot\frac{1}{4}=2$. So there is at most one transvection among g_1, g_2.

Without loss of generality, suppose that g_1 is a transvection. Suppose $|g_3|=6$. As $\nu_1+\nu_2+\nu_3\leq 2n$, $\nu_1=n-1$, $\nu_2\geq\frac{n}{2}$, we have $\nu_3\leq\frac{n}{2}+1$. So $\mathcal{N}(g_3)\leq\frac{1}{2^{\frac{n}{2}-1}}\leq\frac{1}{2^4}$. If g_3^3 is a transvection, then $dim\{C_V(g_1)\cap C_V(g_2)\cap C_V(g_3^3)\}\geq\frac{n}{2}-2$, which implies $dim\{C_V(g_3^2)\}\leq\frac{n}{2}+2$ and in this case $\mathcal{N}(g_3^2)\leq\frac{1}{2^{\frac{n}{2}-2}}\leq\frac{1}{2^3}$. If g_3^3 is not a transvection, then $\mathcal{N}(g_3^3)\leq\frac{1}{4}$. So whether g_3 is a transvection or not, we always have $\mathcal{N}(g_3^3)+2\mathcal{N}(g_3^2)\leq max\{\frac{1}{2}+\frac{2}{8},\ \frac{1}{4}+\frac{2}{4}\}=\frac{3}{4}$. Hence $\mathcal{U}(g_3)\leq\frac{1}{6}(1+\frac{3}{4}+\frac{2}{2^4})=\frac{5}{16}$. As $max\{\frac{5}{16},\ \frac{1}{4}\}=\frac{5}{16}$, we have that if $|g_3|$ and $|g_4|$ are both ≥ 5, then $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq\frac{3}{4}+\frac{5}{8}+2\cdot\frac{5}{16}=2$, a contradiction. So one of the elements in g_3, g_4 is of order 3 or 4.

Suppose that $|g_3|=3$ or 4. Since $\nu_1=n-1$, we have $\nu_2+\nu_3\leq n+1$. As $\frac{n}{2}\leq\nu_2\leq n-2$, $0\leq\nu_3\leq n-2$, we have that $2^{\nu_2}+2^{\nu_3}\leq 2^{n-2}+2^3$ and $\frac{1}{2}\cdot 2^{\nu_2}+\frac{2}{3}\cdot 2^{\nu_3}\leq\frac{1}{2}\cdot 2^{n-2}+\frac{2}{3}\cdot 2^3$. This implies $\frac{1}{2}\{\mathcal{N}(g_2)+\mathcal{N}(g_3)\}\leq\frac{1}{8}+\frac{1}{2^{n-2}}$ for $|g_3|=4$; and $\frac{1}{2}\mathcal{N}(g_2)+\frac{2}{3}\mathcal{N}(g_3)\}\leq\frac{1}{8}+\frac{1}{3\cdot 2^{n-4}}$ for $|g_3|=3$. So $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{3}{4}+\frac{1}{2}+\frac{1}{4}(1+\frac{1}{2})+\frac{1}{2}\mathcal{N}(g_2)+\frac{2}{4}\mathcal{N}(g_3)\leq\frac{13}{8}+\frac{1}{2}(\frac{1}{4}+\frac{1}{2^{n-3}})=\frac{7}{4}+\frac{1}{2^{n-2}}$ for $|g_3|=4$; and $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{3}{4}+\frac{1}{2}+\frac{1}{3}+\frac{1}{2}\mathcal{N}(g_2)+\frac{2}{3}\mathcal{N}(g_3)\leq\frac{19}{12}+\frac{1}{8}+\frac{1}{3\cdot 2^{n-4}}=\frac{41}{24}+\frac{1}{3\cdot 2^{n-4}}$ for $|g_3|=3$. So in either cases, we have $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{7}{4}+\frac{1}{3\cdot 2^{n-4}}$. For $|g_4|\geq 15$, we have $\mathcal{U}(g_4)\leq\frac{21}{8}\cdot\frac{1}{15}+\frac{1}{16}=\frac{19}{80}$. If $|g_4|=14$, 13, 11, 10, or 9, then $\mathcal{U}(g_4)\leq\frac{3}{16}$, $\frac{29}{208}$, $\frac{27}{176}$, $\frac{1}{5}$, $\frac{5}{24}$ respectively. As $max\{\frac{19}{80},\ \frac{3}{16},\ \frac{29}{208},\ \frac{27}{176},\ \frac{1}{5},\ \frac{5}{24}\}=\frac{19}{80}$, we have $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq\frac{7}{4}+\frac{1}{3\cdot 2^{n-4}}+\frac{19}{80}<2$, a contradiction. So $|g_4|\leq 8$ or $|g_4|=12$. Since $\nu_1+\nu_2+\nu_4\leq 2n$ and $\frac{n}{2}\leq\nu_2$, we have $\nu_4\leq\frac{n}{2}+1$. Thus $\mathcal{N}(g_4)\leq\frac{1}{2^{\frac{n}{2}-1}}$. So for $|g_4|=12$, $\mathcal{U}(g_4)\leq\frac{1}{12}(1+\frac{1}{2}+\frac{2}{4}+\frac{2}{4}+\frac{2}{8}+\frac{4}{2^{\frac{n}{2}-1}})\leq\frac{11}{48}+\frac{1}{2^{\frac{n}{2}-1}}$. Similraly, $\mathcal{U}(g_4)\leq\xi+\frac{1}{2^{\frac{n}{2}-1}}$, where $\xi=\frac{7}{32}$, $\frac{1}{7}$, $\frac{1}{5}$, for $|g_4|=8$, 7, 5 respectively. For example, $\xi=\frac{7}{32}$ for $|g_4|=8$ is because g_4^2 is not of type

$3^1 1^{n-3}$. Thus $\mathcal{N}(g_4^2)\le\frac{1}{8}$ and $\mathcal{U}(g_4)\le\frac{1}{8}(1+\frac{1}{2}+\frac{2}{8}+\frac{4}{2^{\frac{n}{2}-1}})\le\frac{7}{32}+\frac{1}{2^{\frac{n}{2}-1}}$. As $max\{\frac{11}{48},\frac{7}{32},\frac{1}{7},\frac{1}{5}\}=\frac{11}{48}$, we have $\sum_{i=1}^{4}\mathcal{U}(g_i)\le\frac{7}{4}+\frac{1}{3\cdot 2^{n-4}}+\frac{11}{48}+\frac{1}{2^{\frac{n}{2}-1}}=\frac{95}{48}+\frac{1}{3\cdot 2^{n-4}}+\frac{1}{2^{\frac{n}{2}-1}}\le\frac{95}{48}+\frac{1}{3\cdot 2^{29}}+\frac{1}{2^{15}}<2$. So $|g_4|\le 4$ or $|g_4|=6$, i.e., S is one of the following types: (2,2,4,6), (2,2,4,4), (2,2,3,4), (2,2,3,3), or (2,2,3,6).

Now suppose none of g_1, g_2 is a transvection. So $\mathcal{U}(g_1)$ and $\mathcal{U}(g_2)$ both $\le\frac{5}{8}$. If one of g_3, g_4 is not of order 3, 4, and 6, say g_3, then $\mathcal{U}(g_3)\le\frac{1}{4}$ by (7.9)(d). Hence we have the contradiction $\sum_{i=1}^{4}\mathcal{U}(g_i)\le 2\cdot\frac{5}{8}+\frac{1}{4}+\frac{1}{2}=2$. So again S is one of the following types: (2,2,4,6), (2,2,4,4), (2,2,3,4), (2,2,3,3), or (2,2,3,6).

(7.19) S is not of type (2,2,4,6) for $d=1$.

Proof. Suppose S is of type (2,2,4,6). Denote $\alpha=dim\{C_V(g_3^2)\}$, $\beta=dim\{C_V(g_4^3)\}$, $\gamma=dim\{C_V(g_4^2)\}$. We have $1+\frac{1}{2}\sum_{i=1}^{2}\frac{2^{\nu_i}-1}{2^n-1}+\frac{1}{4}+\frac{1}{4}\cdot\frac{2^{\alpha}-1}{2^n-1}+\frac{2}{4}\cdot\frac{2^{\nu_3}-1}{2^n-1}+\frac{1}{6}+\frac{1}{6}\cdot\frac{2^{\beta}-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\gamma}-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\nu_4}-1}{2^n-1}=2+\frac{2}{2^n-1}$. This identity can be transformed to $7\cdot 2^n-3\cdot 2^{\alpha}-2^{\beta+1}-6(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})=2^{\gamma+2}+2^{\nu_4+2}-48$. Suppose $\nu_4\ge 3$. We have $\nu_3\ge 4$ as $|g_3|=4$, and also α, β, ν_1, ν_2 all $\ge\frac{n}{2}\ge 16$. As $\beta\ge\nu_4$, 2^5 divides $7\cdot 2^n-3\cdot 2^{\alpha}-2^{\beta+1}-6(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})-2^{\gamma+2}-2^{\nu_4+2}$; which implies $2^5|48$, a contradiction. So $\nu_4=0$, 1, or 2. Suppose $\nu_4=0$. Then $2^{\gamma+2}+2^{\nu_4+2}-48=4(2^{\gamma}-11)$. We have $4|(2^{\gamma}-11)$. But $2^{\gamma}-11$ is even only when $\gamma=0$ and in which case $11-2^{\gamma}=10$. Thus $\nu_4\ne 0$. Suppose $\nu_4=1$. Then $2^{\gamma+2}+2^{\nu_4+2}-48=8(2^{\gamma-1}-5)$. As $\nu_3\ge 5$, α, β, ν_1, ν_2 all ≥ 8, we have $8|(2^{\gamma-1}-5)$. But $2^{\gamma-1}-5$ is even only when $\gamma=1$ and in which case $5-2^{\gamma-1}=4$. Thus $\nu_4\ne 1$. So we must have $\nu_4=2$. Then $2^{\gamma+2}+2^{\nu_4+2}-48=2^{\gamma+2}-32$. Thus $2^6|(2^{\gamma+2}-32)$. Then $\gamma\ge 4$ and $\gamma=2$ give contradictions. So we must have $\nu_4=2$ and $\gamma=3$, in which case $2^{\gamma+2}+2^{\nu_4+2}-48=0$. That is we have $7\cdot 2^n-3\cdot 2^{\alpha}-2^{\beta+1}-6(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})=0$. Since $\nu_1+\nu_2+\nu_3\le 2n$, $\frac{n}{2}\le\nu_i\le n-1$ for $i=1$, 2, $\frac{n}{4}\le\nu_3\le n-2$, we have $2^{\nu_1}+2^{\nu_2}+2^{\nu_3}\le 2^{n-1}+2^{\frac{3n}{4}+1}+2^{\frac{n}{4}}$. So $3\cdot 2^{\alpha}+2^{\beta+1}+6(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})\le$

$3\cdot2^{n-1}+2^n+3\cdot2^n+3(2^{\frac{3n}{4}+2}+2^{\frac{n}{4}+1})=5\cdot2^n+2^{n-1}+3(2^{\frac{3n}{4}+2}+2^{\frac{n}{4}+1})$. But when $n\geq16$, $\frac{3n}{4}+2\leq n-2$; and thus $2^{\frac{3n}{4}+2}+2^{\frac{n}{4}+1}<2^{n-1}$. Then $3\cdot2^\alpha+2^{\beta+1}+6(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})<5\cdot2^n+2^{n-1}+3\cdot2^{n-1}=7\cdot2^n$, and this is a contradiction. Therefore, S is not of type (2,2,4,6).

(7.20) S is not of type (2,2,4,4) for $d=1$.

Proof. Suppose S is of type (2,2,4,4). Denote $\alpha=dim\{C_V(g_3^2)\}$, $\beta=dim\{C_V(g_4^2)\}$. We have $1+\frac{1}{2}\sum_{i=1}^{2}\frac{2^{\nu_i}-1}{2^n-1}+\frac{1}{4}+\frac{1}{4}\cdot\frac{2^\alpha-1}{2^n-1}+\frac{2}{4}\cdot\frac{2^{\nu_3}-1}{2^n-1}+\frac{1}{4}+\frac{1}{4}\cdot\frac{2^\beta-1}{2^n-1}+$
$\frac{2}{4}\cdot\frac{2^{\nu_4}-1}{2^n-1}=2+\frac{2}{2^n-1}$. This identity can be transformed to $2^{\alpha-1}+2^{\beta-1}+2^{\nu_1}+2^{\nu_2}+2^{\nu_3}+2^{\nu_4}=2^n+8$. We have ν_3 and ν_4 both ≥4, and also α, β, ν_1, ν_2 all $\geq\frac{n}{2}\geq16$. Thus $2^4|(2^n+8)$. This is a contradiction. So S is not of type (2,2,4,4).

(7.21) S is not of type (2,2,3,4) for $d=1$.

Proof. Suppose S is of type (2,2,3,4). Denote $\alpha=dim\{C_V(g_4^2)\}$. We have $1+\frac{1}{2}\sum_{i=1}^{2}\frac{2^{\nu_i}-1}{2^n-1}+\frac{1}{3}+\frac{2}{3}\cdot\frac{2^{\nu_3}-1}{2^n-1}+\frac{1}{4}+\frac{1}{4}\cdot\frac{2^\alpha-1}{2^n-1}+\frac{2}{4}\cdot\frac{2^{\nu_4}-1}{2^n-1}=2+\frac{2}{2^n-1}$. This identity can be transformed to $3\cdot2^\alpha+6(2^{\nu_1}+2^{\nu_2}+2^{\nu_4})-5\cdot2^n=16(3-2^{\nu_3-1})$. We have $\nu_4\geq5$, and also α, ν_1, ν_2 all $\geq\frac{n}{2}\geq16$. Thus $2^2|(3-2^{\nu_3-1})$. But $3-2^{\nu_3-1}$ is even only when $\nu_3=1$ and in which case $3-2^{\nu_3-1}=2$. This contradiction shows that S is not of type (2,2,3,4).

(7.22) S is not of type (2,2,3,3) for $d=1$.

Proof. Suppose S is of type (2,2,3,3). We have $1+\frac{1}{2}\sum_{i=1}^{2}\frac{2^{\nu_i}-1}{2^n-1}+\frac{2}{3}+\frac{2}{3}\cdot\sum_{i=3}^{4}\frac{2^{\nu_i}-1}{2^n-1}=2+\frac{2}{2^n-1}$. This identity can be transformed to $3(2^{\nu_1}+2^{\nu_2})-2^{n+1}=24-4(2^{\nu_3}+2^{\nu_4})$. We have both ν_3, $\nu_4\geq2$, then we have the contradiction $2^4|24$. So one of ν_3, ν_4, say $\nu_3\leq1$. If $\nu_3=0$, then $24-4(2^{\nu_3}+2^{\nu_4})=4(5-2^{\nu_4})$. Then $n\geq10$ implies that $2^3|(5-2^{\nu_4})$. But $5-2^{\nu_4}$ is even only when $\nu_4=0$ and in which case $5-2^{\nu_4}=4$. So we must have $\nu_3=1$. Then $24-4(2^{\nu_3}+2^{\nu_4})=4(4-2^{\nu_4})$. Thus we have $2^3|(4-2^{\nu_4})$, which forces $\nu_4=2$. But $\nu_3=1$ and

$\nu_4=2$ cannot occur at the same time, because for $|g|=3$, every non trivial simple submodule is of dimension 2, and thus $\nu_3=1$ implies n is odd, while $\nu_4=2$ implies n is even. So S is not of type (2,2,3,3).

(7.23) S is not of type (2,2,3,6) for $d=1$.

Proof. Suppose S is of type (2,2,3,6). Denote $\alpha=dim\{C_V(g_4^3)\}$, $\beta=dim\{C_V(g_4^2)\}$. We have $1+\frac{1}{2}\sum_{i=1}^{2}\frac{2^{\nu_i}-1}{2^n-1}+\frac{1}{3}+\frac{2}{3}\cdot\frac{2^{\nu_3}-1}{2^n-1}+\frac{1}{6}+\frac{1}{6}\cdot\frac{2^{\alpha}-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\beta}-1}{2^n-1}+$ $\frac{2}{6}\cdot\frac{2^{\nu_4}-1}{2^n-1}=2+\frac{2}{2^n-1}$. This identity can be transformed to $3(2^{\nu_1}+2^{\nu_2})+4\cdot 2^{\nu_3}+2\cdot 2^{\nu_4}+2^{\alpha}+2^{\beta+1}=3\cdot 2^n+24$. If $\nu_1=n-1=\nu_2$, then $3(2^{\nu_1}+2^{\nu_2})+2^{\alpha}-3\cdot 2^n=2^{\alpha}\geq 32$ for $n\geq 10$ gives a contradiction. So say without loss of generality that $\frac{n}{2}\leq\nu_1\leq n-1$, $\frac{n}{2}\leq\nu_2\leq n-2$. Also we have $0\leq\nu_3\leq n-2$, $\nu_1+\nu_2+\nu_3\leq 2n$. Thus similar as before, we have $3(2^{\nu_1}+2^{\nu_2})+4\cdot 2^{\nu_3}\leq 3(2^{n-1}+2^{n-2})+4\cdot 2^3=9\cdot 2^{n-2}+32$. Suppose $\nu_4\geq 4$. As $\beta\geq\nu_4$, if $n\geq 10$, then $2^5|(24-4\cdot 2^{\nu_3})$, which cannot be satisfied. So $\nu_4\leq 3$. Then as $\alpha+\beta\leq n+\nu_4\leq n+3$, we have $2^{\alpha}+2^{\beta+1}\leq 2^{n-1}+2^5$. Hence $3(2^{\nu_1}+2^{\nu_2})+4\cdot 2^{\nu_3}+2\cdot 2^{\nu_4}+2^{\alpha}+2^{\beta+1}\leq 9\cdot 2^{n-2}+32+16+2^{n-1}+2^5$ $=11\cdot 2^{n-2}+80<3\cdot 2^n+24$ if $n\geq 8$. Therefore S is not of type (2,2,3,6).

(7.24) Suppose $|S|=4$. Then S has at most 1 involution.

Proof. This combines (7.16) to (7.22).

(7.25) Suppose $|S|=4$ and S has 1 involution. Then $d=1$ and S is one of the following types: (2,3,3,3), (2,3,3,4), (2,3,4,4), (2,4,4,4).

Proof. Suppose without loss of generality that g_1 is an involution. Consider $d\geq 2$ first. Then by (7.9), $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq(\frac{5}{8}+\frac{1}{2^{32}})+3\cdot(\frac{3}{8}+\frac{1}{3\cdot 2^{61}})<2$, a contradiction. So $d=1$.

If there is at least one element among g_2, g_3, g_4, say g_4, of order $=5$ or ≥ 7, then $\mathcal{U}(g_4)\leq\frac{1}{4}$ by (7.9)(d). Then by (7.9)(c), $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq\frac{3}{4}+2\cdot\frac{1}{2}+\frac{1}{4}=2$, a contradiction. So for $i=2$, 3, and 4, $|g_i|=3$, 4, or 6. Since for $|g|=6$, $\mathcal{U}(g)\leq\frac{3}{8}$, there is at most one element among g_2, g_3, g_4 of order 6.

Suppose S is of type (2,3,3,6), or (2,3,4,6), or (2,4,4,6). If g_1 is not a transvection, then $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq\frac{5}{8}+2\cdot\frac{1}{2}+\frac{3}{8}=2$, a contradiction. So g_1 is a transvection. Then as both g_2 and g_3 have eigenspaces of dimension $\geq\frac{n}{4}$, both ν_2 and $\nu_3\leq\frac{3n}{4}+1$, which implies both $\mathcal{N}(g_2)$ and $\mathcal{N}(g_3)\leq\frac{1}{2^{\frac{n}{4}-1}}\leq\frac{1}{2^7}$. So we have $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq\frac{3}{4}+\frac{3}{8}+x=\frac{9}{8}+x$, where $x=2\cdot\frac{1}{3}(1+\frac{2}{2^7})$, $\frac{1}{3}(1+\frac{2}{2^7})+\frac{1}{4}(1+\frac{1}{2}+\frac{2}{2^7})$, $2\cdot\frac{1}{4}(1+\frac{1}{2}+\frac{2}{2^7})$ for S of types (2,3,3,6), (2,3,4,6), (2,4,4,6) respectively. In any case, $x\leq\frac{7}{8}$, which gives the contradiction $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq2$. So no element in S is of order 6. Thus S is one of the following types: (2,3,3,3), (2,3,3,4), (2,3,4,4), (2,4,4,4).

(7.26) S is not of type (2,3,3,3) for $d=1$.

Proof. Suppose S is of type (2,3,3,3). We have $\frac{1}{2}+\frac{1}{2}\cdot\frac{2^{\nu_1}-1}{2^n-1}+1+\frac{2}{3}\cdot\sum_{i=2}^{4}\frac{2^{\nu_i}-1}{2^n-1}=2+\frac{2}{2^n-1}$. This identity can be transformed to $3\cdot2^{\nu_1}+4(2^{\nu_2}+2^{\nu_3}+2^{\nu_4})=3\cdot2^n+24$. If all ν_2, ν_3, $\nu_4\geq2$, then we have the contradiction $2^4|24$. So one of ν_3, ν_4, say $\nu_4\leq1$. First suppose $\nu_4=0$. In this case, as $3\cdot2^{\nu_1}+4(2^{\nu_2}+2^{\nu_3})=3\cdot2^n+20$, one of ν_2, ν_3, say $\nu_3=0$. Now suppose $\nu_4=1$. In this case as $3\cdot2^{\nu_1}+4(2^{\nu_2}+2^{\nu_3})=3\cdot2^n+16$, we have that one of ν_2, ν_3, say $\nu_3\leq2$. So in any case we have that $3\cdot2^{\nu_1}+4(2^{\nu_2}+2^{\nu_3}+2^{\nu_4})\leq3\cdot2^{n-1}+4(2^{n-2}+2+2^2)=5\cdot2^{n-1}+24<3\cdot2^n+24$. So S is not of type (2,3,3,3).

(7.27) S is not of type (2,3,3,4) for $d=1$.

Proof. Suppose S is of type (2,3,3,4). Denote $\alpha=dim\{C_V(g_4^2)\}$. We have $\frac{1}{2}+\frac{1}{2}\cdot\frac{2^{\nu_1}-1}{2^n-1}+\frac{2}{3}+\frac{2}{3}\cdot\sum_{i=2}^{3}\frac{2^{\nu_i}-1}{2^n-1}+\frac{1}{4}+\frac{1}{4}\cdot\frac{2^{\alpha}-1}{2^n-1}+\frac{2}{4}\cdot\frac{2^{\nu_4}-1}{2^n-1}=2+\frac{2}{2^n-1}$. This identity can be transformed to $6(2^{\nu_1}+2^{\nu_4})+8(2^{\nu_2}+2^{\nu_3})+3\cdot2^{\alpha}=7\cdot2^n+48$. We have $\nu_4\geq5$. Suppose both ν_2, $\nu_3\geq2$, then we have the contradiction $2^5|48$. So one of ν_2, ν_3, say $\nu_3\leq1$. First suppose $\nu_3=0$. In this case, as $6(2^{\nu_1}+2^{\nu_4})+8\cdot2^{\nu_2}+3\cdot2^{\alpha}=7\cdot2^n+40$, we must also have $\nu_2=0$. Now suppose $\nu_3=1$. In this case as $6(2^{\nu_1}+2^{\nu_4})+8\cdot 2^{\nu_2}+3\cdot 2^{\alpha}=7\cdot 2^n+32$, if $\nu_2\geq 3$, then we have the contradiction $2^6|32$. So $\nu_2\leq 2$. Thus in any case, we have that $6(2^{\nu_1}+2^{\nu_4})+8(2^{\nu_2}+2^{\nu_3})+3\cdot 2^{\alpha}\leq 6(2^{n-1}+2^{n-2})+8(2^2+2)+3\cdot 2^{n-1}=6\cdot 2^n+48<7\cdot 2^n+48$. So S is not of type (2,3,3,4).

(7.28) S is not of type (2,3,4,4) for $d=1$.

Proof. Suppose S is of type (2,3,4,4). We have $\frac{1}{2}+\frac{1}{2}\cdot\frac{2^{\nu_1}-1}{2^n-1}+\frac{1}{3}+\frac{2}{3}\cdot\frac{2^{\nu_2}-1}{2^n-1}+\frac{2}{4}+\frac{1}{4}\cdot\frac{2^{\alpha}-1}{2^n-1}+\frac{1}{4}\cdot\frac{2^{\beta}-1}{2^n-1}+\frac{2}{4}\cdot\sum_{i=3}^{4}\frac{2^{\nu_i}-1}{2^n-1}=2+\frac{2}{2^n-1}$, where $\alpha=dim\{C_V(g_3^2)\}$, $\beta=dim\{C_V(g_4^2)\}$. This identity can be transformed to $6(2^{\nu_1}+2^{\nu_3}+2^{\nu_4})+3(2^{\alpha}+2^{\beta})-8\cdot 2^n=2^4(3-2^{\nu_2-1})$. Since both ν_3 and $\nu_4\geq 5$, and as 2^6 divides the left hand side of the equation, we have $2^2|(3-2^{\nu_2-1})$. But $3-2^{\nu_2-1}$ is even only when $\nu_2=1$ and in which case $3-2^{\nu_2-1}=2$. So S is not of type (2,3,4,4).

(7.29) S is not of type (2,4,4,4) for $d=1$.

Proof. Suppose S is of type (2,4,4,4). We have $\frac{1}{2}+\frac{1}{2}\cdot\frac{2^{\nu_1}-1}{2^n-1}+\frac{3}{4}+\frac{1}{4}\cdot\frac{2^{\alpha}-1}{2^n-1}+\frac{1}{4}\cdot\frac{2^{\beta}-1}{2^n-1}+\frac{1}{4}\cdot\frac{2^{\gamma}-1}{2^n-1}+\frac{2}{4}\cdot\sum_{i=2}^{4}\frac{2^{\nu_i}-1}{2^n-1}=2+\frac{2}{2^n-1}$, where $\alpha=dim\{C_V(g_2^2)\}$, $\beta=dim\{C_V(g_3^2)\}$, and $\gamma=dim\{C_V(g_4^2)\}$. This identity can be transformed to $2(2^{\nu_1}+2^{\nu_2}+2^{\nu_3}+2^{\nu_4})+2^{\alpha}+2^{\beta}+2^{\gamma}=3\cdot 2^n+16$. We have $\nu_i\geq 4$ $\forall i$ and also α, β, γ are all ≥ 5. Then we have the contradiction $2^5|16$. So S is not of type (2,4,4,4).

(7.30) Suppose $|S|=4$. Then there is no involution in S.

Proof. This is the combination of (7.24) to (7.29).

(7.31) $|S|=3$.

Proof. By (7.9)(c), for any g with $|g|\geq 3$, $\mathfrak{U}(g)\leq\frac{1}{2}$. Thus we have $\sum_{i=1}^{4}\mathfrak{U}(g_i)\leq 4\cdot\frac{1}{2}=2$, a contradiction. Therefore $|S|=3$.

(7.32) Suppose $d=1$ and S is of type $(2,l,m)$. Then we have the following bounds for $\xi=\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)$, where g is either g_2 or g_3.

$\lvert g\rvert$	ξ	$\lvert g\rvert$	ξ
28	47/112	24	41/96
21	183/448	20	69/160
18	71/192	15	5/12
14	57/128	12	233/512
10	521/1280	9	25/64
8	245/512	7	45/112
6	6179/12288	5	579/1280
4	321/512	3	299/512

Proof. In the following, denote $\nu_1=dim\{C_V(g_1)\}$ and $\nu_2=dim\{C_V(g)\}$.

Let $\lvert g\rvert=28$. Then $\mathcal{U}(g)\leq\frac{17}{8}\cdot\frac{1}{28}+\frac{1}{16}=\frac{31}{224}$ by (7.7). So if $\nu_1\leq n-3$, then $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^3}+\frac{31}{224}=\frac{47}{112}$. By (3.3), $\nu_1=n-1$ is impossible, as we assume $n\geq 33$. If $\nu_1=n-2$, then by (7.1)(a), $\mathcal{N}(g^t)\leq\frac{1}{2^{n-2t}}$, because $dim\{C_V(g^t)\}\leq t(n-\nu_1)=2t$. So $\mathcal{U}(g)\leq\frac{1}{28}(1+\frac{1}{2^{n-28}}+\frac{2}{2^{n-14}}+\frac{6}{2^{n-8}}+\frac{6}{2^{n-4}}+\frac{12}{2^{n-2}})\leq\frac{1}{28}(1+\frac{1}{2^{n-28}}+\frac{1}{2^{n-18}})\leq\frac{97}{1792}$. Thus $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^2}+\frac{97}{1792}=\frac{657}{1792}<\frac{47}{112}$.

Let $\lvert g\rvert=24$. Then $\mathcal{U}(g)\leq\frac{1}{24}(1+\frac{1}{2}+\frac{2}{8}+\frac{4}{16}+\frac{2}{4}+\frac{2}{8}+\frac{4}{16}+\frac{8}{16})=\frac{7}{48}$. So if $\nu_1\leq n-3$, then $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^3}+\frac{7}{48}=\frac{41}{96}$. Similar to the previous case, $\nu_1=n-1$ is impossible. If $\nu_1=n-2$, then $\mathcal{N}(g^t)\leq\frac{1}{2^{n-2t}}$. So $\mathcal{U}(g)\leq\frac{1}{24}(1+\frac{1}{2^{n-24}}+\frac{2}{2^{n-12}}+\frac{4}{2^{n-6}}+\frac{2}{2^{n-16}}+\frac{2}{2^{n-8}}+\frac{4}{2^{n-4}}+\frac{8}{2^{n-2}})\leq\frac{1}{24}(1+\frac{1}{2^{n-25}})\leq\frac{257}{6144}$. Thus $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^2}+\frac{257}{6144}=\frac{2177}{6144}<\frac{41}{96}$.

Let $\lvert g\rvert=21$. Then $\mathcal{U}(g)\leq\frac{27}{16}\cdot\frac{1}{21}+\frac{1}{16}=\frac{1}{7}$ by (7.7). So if $\nu_1\leq n-4$, then $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^4}+\frac{1}{7}=\frac{183}{448}$. Since g^7 has an eigenspace E with $dim\{E\}\geq\frac{n}{3}$, and by (3.3), $dim\{E\}\leq 7(n-\nu_1)$, we have that $\nu_1=n-1$ is impossible. If $n-2\geq\nu_1\geq n-3$, then similar to the previous case, $\mathcal{N}(g^t)\leq\frac{1}{2^{n-3t}}$. So $\mathcal{U}(g)\leq\frac{1}{21}(1+\frac{2}{2^{n-21}}+\frac{6}{2^{n-9}}+\frac{12}{2^{n-3}})\leq\frac{1}{21}(1+\frac{1}{2^{n-23}})\leq\frac{1025}{21504}$.

Thus $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^2}+\frac{1025}{21504}=\frac{7745}{21504}<\frac{183}{448}$.

Let $|g|=20$. Then $\mathcal{U}(g)\leq\frac{7}{4}\cdot\frac{1}{20}+\frac{1}{16}=\frac{3}{20}$. So if $\nu_1\leq n-3$, then $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^3}+\frac{3}{20}=\frac{69}{160}$. Similar to the previous case, $\nu_1=n-1$ is impossible. If $\nu_1=n-2$, then $\mathcal{N}(g^t)\leq\frac{1}{2^{n-2t}}$. So $\mathcal{U}(g)\leq\frac{1}{20}(1+\frac{1}{2^{n-20}}+\frac{2}{2^{n-10}}+\frac{4}{2^{n-8}}+\frac{4}{2^{n-4}}+\frac{8}{2^{n-2}})\leq\frac{1}{20}(1+\frac{1}{2^{n-21}})\leq\frac{205}{4096}$. Thus $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^2}+\frac{205}{4096}=\frac{1485}{4096}<\frac{69}{160}$.

Let $|g|=18$. As g^2 has at least one simple module of dimension 6, g^6 has at least 3 simple modules of dimension 2, which implies $\mathcal{N}(g)$, $\mathcal{N}(g^2)$, $\mathcal{N}(g^3)$ and $\mathcal{N}(g^6)\leq\frac{1}{2^6}$. Thus $\mathcal{U}(g)\leq\frac{1}{18}(1+\frac{1}{2}+\frac{2}{2^6}+\frac{2}{2^6}+\frac{6}{2^6}+\frac{6}{2^6})=\frac{5}{48}$. So if $\nu_1\leq n-4$, then $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^4}+\frac{5}{48}=\frac{71}{192}$. We have that $\nu_1=n-1$ is impossible. If $n-2\geq\nu_1\geq n-3$, then similar to the previous case, $\mathcal{N}(g^t)\leq\frac{1}{2^{n-3t}}$. So $\mathcal{U}(g)\leq\frac{1}{18}(1+\frac{1}{2^{n-27}}+\frac{2}{2^{n-18}}+\frac{2}{2^{n-9}}+\frac{6}{2^{n-6}}+\frac{6}{2^{n-3}})\leq\frac{1}{18}(1+\frac{1}{2^{n-28}})\leq\frac{11}{192}$. Thus $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^2}+\frac{11}{192}=\frac{71}{192}$.

Let $|g|=15$. Since g^5 has an eigenspace E with $dim\{E\}\geq\frac{n}{3}$, and by (3.3), $dim\{E\}\leq 5(n-\nu_1)$, we have that $\nu_1\geq n-2$ is impossible. So $\nu_1\leq n-3$. Let 2^a1^b, 4^c1^d be the types of g^5, g^3 respectively. As $G=\langle g_1,g\rangle=\langle g_1,g^3,g^5\rangle$, we have $\nu_1+b+d\leq 2n$. Then $b+d\leq\frac{3n}{2}$. This together with $0\leq b\leq n-2$, $0\leq d\leq n-4$, gives that $\frac{2}{15}\cdot 2^b+\frac{4}{15}\cdot 2^d\leq\frac{2}{15}\cdot 2^{n-2}+\frac{4}{15}\cdot 2^{\frac{n}{2}+2}$. So $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^3}+\frac{1}{15}(1+\frac{2}{2^2}+\frac{4}{2^{\frac{n}{2}-2}}+\frac{8}{2^4})=\frac{199}{480}+\frac{1}{15\cdot 2^{\frac{n}{2}-4}}\leq\frac{5}{12}$.

Let $|g|=14$. Then $\mathcal{U}(g)\leq\frac{7}{4}\cdot\frac{1}{14}+\frac{1}{16}=\frac{3}{16}$. So if $\nu_1\leq n-5$, then $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^5}+\frac{3}{16}=\frac{57}{128}$. Similarly, if $\nu_1\geq n-2$, then $\frac{n}{2}\leq dim\{C_V(g^7)\}\leq 7(n-\nu_1)\leq 14$, contradicting to $n\geq 33$. If $n-3\geq\nu_1\geq n-4$, then $\mathcal{N}(g^t)\leq\frac{1}{2^{n-4t}}$. So $\mathcal{U}(g)\leq\frac{1}{14}(1+\frac{1}{2^{n-28}}+\frac{6}{2^{n-8}}+\frac{6}{2^{n-4}})\leq\frac{1}{14}(1+\frac{1}{2^{n-29}})\leq\frac{17}{224}$. Thus $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^3}+\frac{17}{224}=\frac{5}{14}<\frac{57}{128}$.

Let $|g|=12$. Suppose $\nu_1\geq n-2$. Then by (3.3), $\frac{n}{2}\leq\mu=dim\{C_V(g^6)\}\leq 6(n-\nu_1)\leq 12$, contradicts to the assumption $n\geq 33$. So $\nu_1\leq n-3$. Denote $\alpha=dim\{C_V(g^3)\}$ and let 2^a1^b be the type of g^4. As $G=\langle g_1,g\rangle=\langle g_1,g^3,g^4\rangle$, we have $\nu_1+\alpha+b\leq 2n$. If $b\geq n-4$, then $\nu_1+\alpha\leq n+4$. Since $\frac{n}{2}\leq\nu_1\leq n-3$, $\frac{n}{4}\leq\alpha\leq n-2$, we have $\frac{1}{4}\cdot 2^{\nu_1}+\frac{6}{12}\cdot 2^{\alpha}\leq\frac{1}{4}\cdot 2^{\frac{3n}{4}+4}+\frac{6}{12}\cdot 2^{\frac{n}{4}}$. Since $\mathcal{N}(g)\leq\mathcal{N}(g^3)$, we have $\frac{1}{4}\mathcal{N}(g_1)+\frac{1}{12}\{2\mathcal{N}(g^3)+4\mathcal{N}(g)\}\leq\frac{1}{4}\cdot\frac{1}{2^{\frac{n}{4}-4}}+\frac{6}{12}\cdot\frac{1}{2^{\frac{3n}{4}}}$, which implies that $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq$ $\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^4}+\frac{1}{12}(1+\frac{1}{2}+\frac{2}{2^2}+\frac{2}{2^3})+\frac{6}{12}\cdot\frac{1}{2^{24}}=\frac{29}{64}+\frac{1}{2^{25}}<\frac{233}{512}$. Now suppose that $b\leq n-6$. Thus $\mathcal{N}(g)\leq\mathcal{N}(g^2)\leq\mathcal{N}(g^4)\leq\frac{1}{2^6}$. Since g^4 has an eigenvalue of multiplicity ≥ 11, $\nu_1+\alpha\leq 2n-11$. This together with $\frac{n}{2}\leq\nu_1\leq n-3$, $\frac{n}{4}\leq\alpha\leq n-2$ gives $\frac{1}{4}\cdot 2^{\nu_1}+\frac{2}{12}\cdot 2^{\alpha}\leq\frac{1}{4}\cdot 2^{n-9}+\frac{2}{12}\cdot 2^{n-2}$. Hence $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^9}+\frac{1}{12}(1+\frac{1}{2}+\frac{2}{2^6}+\frac{2}{2^2}+\frac{2}{2^6}+\frac{4}{2^6})=\frac{2627}{6144}<\frac{233}{512}$.

Let $|g|=10$. Suppose $\nu_1\geq n-3$. Then by (3.3), $\frac{n}{2}\leq\mu=dim\{C_V(g^5)\}\leq 5(n-\nu_1)\leq 15$, contradicts to the assumption $n\geq 33$. So $\nu_1\leq n-4$. Denote $\alpha=dim\{C_V(g^5)\}$ and let 4^a1^b be the type of g^2. As $G=\langle g_1,g\rangle=\langle g_1,g^5,g^2\rangle$, we have $\nu_1+\alpha+b\leq 2n$. If $b=n-4$, then $\nu_1+\alpha\leq n+4$. Since $\frac{n}{2}\leq\nu_1\leq n-4$, $\frac{n}{2}\leq\alpha\leq n-1$, we have $\frac{1}{4}\cdot 2^{\nu_1}+\frac{5}{10}\cdot 2^{\alpha}\leq\frac{1}{4}\cdot 2^{\frac{n}{2}}+\frac{5}{10}\cdot 2^{\frac{n}{2}+4}$. Since $\mathcal{N}(g)\leq\mathcal{N}(g^5)$, we have $\frac{1}{4}\mathcal{N}(g_1)+\frac{1}{10}\{\mathcal{N}(g^5)+4\mathcal{N}(g)\}\leq\frac{1}{4}\cdot\frac{1}{2^{\frac{n}{2}}}+\frac{5}{10}\cdot\frac{1}{2^{\frac{n}{2}-4}}$, which implies that we have $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^{16}}+\frac{1}{10}(1+\frac{5}{2^{12}}+\frac{4}{2^4})=\frac{3}{8}+\frac{33}{2^{18}}<\frac{193}{512}<\frac{521}{1280}$. Now suppose that $b\leq n-8$. Thus $\mathcal{N}(g)\leq\mathcal{N}(g^2)\leq\frac{1}{2^8}$. Since g^2 has an eigenvalue of multiplicity ≥ 7, $\nu_1+\alpha\leq 2n-7$. This together with $\frac{n}{2}\leq\nu_1\leq n-4$, $\frac{n}{2}\leq\alpha\leq n-1$ gives $\frac{1}{4}\cdot 2^{\nu_1}+\frac{1}{10}\cdot 2^{\alpha}\leq\frac{1}{4}\cdot 2^{n-6}+\frac{1}{10}\cdot 2^{n-1}$. Hence $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^6}+\frac{1}{10}(1+\frac{1}{2}+\frac{4}{2^8}+\frac{4}{2^8})=\frac{521}{1280}$.

Let $|g|=9$. Suppose $\nu_1\geq n-3$. Since g^3 has an eigenspace of dimension $\mu\geq\frac{n}{3}$, and by (3.3), $\mu\leq 3(n-\nu_1)\leq 9$, which contradicts to $n\geq 33$. So $\nu_1\leq n-4$. Let $6^a3^b1^c$ be the type of g. Since $a\geq 1$, g^3 has at least 3 simple modules of dimension 2, so $\mathcal{N}(g^3)\leq\frac{1}{2^6}$. Hence $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^4}+\frac{1}{9}(1+\frac{2}{2^6}+\frac{6}{2^6})=\frac{25}{64}$.

Let $|g|=8$. Suppose $\nu_1\geq n-4$. Then by (3.3), $\frac{n}{2}\leq\mu=dim\{C_V(g^4)\}\leq 4(n-\nu_1)\leq 16$, contradicts to the assumption $n\geq 33$. So $\nu_1\leq n-5$. Since $\nu_1+\nu_2\leq n$, $\frac{n}{2}\leq\nu_1$, we have $\nu_2\leq\frac{n}{2}$. Hence $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^5}+\frac{1}{8}(1+\frac{1}{2}+\frac{2}{2^3}+\frac{4}{2^{\frac{n}{2}}})=\frac{61}{128}+\frac{1}{2^{\frac{n}{2}+1}}\leq\frac{245}{512}$.

Let $|g|=7$. Let $3_1^{a_1}3_2^{a_2}1^b$ be the type of g. Since g has an eigenspace of dimension ≥ 5, $\nu_1\leq n-5$. Also as $\nu_1+b\leq n$, $\frac{n}{2}\leq\nu_1\leq n-5$, $0\leq b\leq n-3$, we have $\frac{1}{4}\cdot 2^{\nu_1}+\frac{6}{7}\cdot 2^b\leq\frac{1}{4}\cdot 2^{n-5}+\frac{6}{7}\cdot 2^5$. So we have $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\le\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^5}+\frac{1}{7}(1+\frac{6}{2^{n-5}})=\frac{359}{896}+\frac{3}{7\cdot 2^{n-6}}<\frac{45}{112}$.

Let $|g|=6$. Suppose $\nu_1\ge n-5$. Then by (3.3), $\frac{n}{2}\le dim\{C_V(g^3)\}\le 3(n-\nu_1)\le 15$, contradicts to the assumption $n\ge 33$. So $\nu_1\le n-6$. Denote $\alpha=dim\{C_V(g^3)\}$ and let 2^a1^b be the type of g^2. As $G=\langle g_1,g\rangle=\langle g_1,g^3,g^2\rangle$, we have $\nu_1+\alpha+b\le 2n$. If $b\ge n-6$, then $\nu_1+\alpha\le n+6$. Since $\frac{n}{2}\le\nu_1\le n-6$, $\frac{n}{2}\le\alpha\le n-1$, we have $\frac{1}{4}\cdot 2^{\nu_1}+\frac{3}{6}\cdot 2^{\alpha}\le\frac{1}{4}\cdot 2^{\frac{n}{2}}+\frac{3}{6}\cdot 2^{\frac{n}{2}+6}$. Since $\mathcal{N}(g)\le\mathcal{N}(g^3)$, we have $\frac{1}{4}\mathcal{N}(g_1)+\frac{1}{6}\{\mathcal{N}(g^3)+2\mathcal{N}(g)\}\le\frac{1}{4}\cdot\frac{1}{2^{\frac{n}{2}}}+\frac{3}{6}\cdot\frac{1}{2^{\frac{n}{2}-6}}$, which implies that we have $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\le\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^{16}}+\frac{1}{6}(1+\frac{3}{2^{10}}+\frac{2}{2^2})=\frac{1}{2}+\frac{129}{2^{18}}<\frac{6179}{12288}$. Now suppose that $b\le n-8$. Thus $\mathcal{N}(g)\le\mathcal{N}(g^2)\le\frac{1}{2^8}$. Since g^2 has an eigenvalue of multiplicity ≥ 11, $\nu_1+\alpha\le 2n-11$. This together with $\frac{n}{2}\le\nu_1\le n-6$, $\frac{n}{2}\le\alpha\le n-1$ gives $\frac{1}{4}\cdot 2^{\nu_1}+\frac{1}{6}\cdot 2^{\alpha}\le\frac{1}{4}\cdot 2^{n-10}+\frac{1}{6}\cdot 2^{n-1}$. Hence $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\le\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^{10}}+\frac{1}{6}(1+\frac{1}{2}+\frac{2}{2^8}+\frac{2}{2^8})=\frac{6179}{12288}$.

Let $|g|=5$. Since g has an eigenspace of dimension ≥ 7, $\nu_1\le n-7$. Let 4^a1^b be the type of g. Then $\nu_1+b\le n$. As $\frac{n}{2}\le\nu_1\le n-7$, $0\le b\le n-4$, we have $\frac{1}{4}\cdot 2^{\nu_1}+\frac{4}{5}\cdot 2^b\le\frac{1}{4}\cdot 2^{n-7}+\frac{4}{5}\cdot 2^7$. So we have $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\le\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^7}+\frac{1}{5}(1+\frac{4}{2^{n-7}})=\frac{1157}{2560}+\frac{1}{5\cdot 2^{n-9}}<\frac{579}{1280}$.

Let $|g|=4$. Since $\nu_1+\nu_2\le n$, $\frac{n}{2}\le\nu_1\le n-1$ and $\frac{n}{4}\le\nu_2\le n-2$, we have $\frac{1}{4}\cdot 2^{\nu_1}+\frac{2}{4}\cdot 2^{\nu_2}\le\frac{1}{4}\cdot 2^{\frac{3n}{4}}+\frac{2}{4}\cdot 2^{\frac{n}{4}}$, and this implies that $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\le\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^{\frac{n}{4}}}+\frac{1}{4}(1+\frac{1}{2}+\frac{2}{2^{\frac{3n}{4}}})<\frac{5}{8}+\frac{1}{2^{10}}+\frac{1}{2^{25}}<\frac{321}{512}$.

Let $|g|=3$. Since g has an eigenspace of dimension ≥ 11, $\nu_1\le n-11$. Let 2^a1^b be the type of g. Then $\nu_1+b\le n$. As $\frac{n}{2}\le\nu_1\le n-11$, $0\le b\le n-2$, we have $\frac{1}{4}\cdot 2^{\nu_1}+\frac{2}{3}\cdot 2^b\le\frac{1}{4}\cdot 2^{n-11}+\frac{2}{3}\cdot 2^{11}$. So we have $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g)\le\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^{11}}+\frac{1}{3}(1+\frac{2}{2^{n-11}})\le\frac{7}{12}+\frac{1}{2^{13}}+\frac{1}{3\cdot 2^{21}}<\frac{299}{512}$.

(7.33) Suppose $|S|=3$ with $|g_1|=2$. Then one of g_2, g_3 is of order 3 or 4 if $d\ge 2$; and one of g_2, g_3 is of order 3, 4, or 6 if $d=1$.

Proof. Suppose $d\ge 2$ first. If g_1 is one of the types: $2^{\alpha}1^{n-2\alpha}$ with $\alpha\le 3$, then by (3.3)(c), both $|g_2|$ and $|g_3|\ge\frac{n}{3}\ge 11$, which implies that both $\mathcal{U}(g_2)$ and $\mathcal{U}(g_3)\le\frac{27}{16}\cdot\frac{1}{11}+\frac{1}{2^7}=\frac{227}{1408}$ by (7.8).

Then we have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\le(\frac{5}{8}+\frac{1}{2^{32}})+2\cdot\frac{227}{1408}<1$. Hence g_1 is not one of the types: $2^\alpha 1^{n-2\alpha}$ where $\alpha\le 3$, which implies that $\mathcal{U}(g_1)\le\frac{129}{256}$ by (7.9)(a). For $|g|\ge 8$, (7.8) implies that $\mathcal{U}(g)\le\frac{27}{16}\cdot\frac{1}{8}+\frac{1}{2^7}=\frac{7}{32}$. For $|g_i|=7$, where $i=2$ or 3, g_i cannot be of type $3^1 1^{n-3}$. Thus by (7.4)(c), we have that $\mathcal{U}(g_i)\le\frac{1}{7}(1+\frac{6}{2^7})<\frac{7}{32}$. By (7.9)(b), for $|g_i|=5$, $\mathcal{U}(g_i)\le\frac{33}{160}<\frac{7}{32}$. For $|g_i|=6$, as $G=\langle g_1,g_i\rangle=\langle g_1,g_i^3,g_i^2\rangle$, we cannot have both g_i^2 of type $2^1 1^{n-2}$ and g_i^3 of type $2^\alpha 1^{n-2\alpha}$ with $\alpha\le 3$. So either $\mathcal{N}(g_i^2)$ and $\mathcal{N}(g_i)$ both $\le\frac{1}{2^7}$, or $\mathcal{N}(g_i^3)$ and $\mathcal{N}(g_i)$ both $\le\frac{1}{2^7}$. Then either $\mathcal{U}(g_i)\le\frac{1}{6}\{1+(\frac{1}{2^2}+\frac{1}{2^{31}})+\frac{2}{2^7}+\frac{2}{2^7}\}=\frac{41}{192}+\frac{1}{3\cdot 2^{32}}<\frac{7}{32}$ or $\mathcal{U}(g_i)\le\frac{1}{6}\{1+\frac{1}{2^7}+2(\frac{1}{2^4}+\frac{1}{2^{62}})+\frac{2}{2^7}\}=\frac{49}{256}+\frac{1}{3\cdot 2^{62}}<\frac{7}{32}$. Thus if no element among g_2, g_3 is of order 3 and 4, then $\sum_{i=1}^{3}\mathcal{U}(g_i)\le\frac{129}{256}+2\cdot\frac{7}{32}<1$, a contradiction. So one of g_2, g_3 is of order 3 or 4 if $d\ge 2$.

Now consider $d=1$. Suppose neither $|g_2|$ nor $|g_3|$ is of order 3, 4, and 6. Let $g_i=g_2$ or g_3. If $|g_i|\ge 42$, or g_i is one of the elements on the table in (7.9), then $\mathcal{U}(g_i)\le\frac{1}{8}$, which gives $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_i)\le\frac{1}{2}$. If g_i is one of the elements on the table in (6.32), as we assume $|g_i|\ne 3$, 4, and 6, we still have $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_i)\le\frac{1}{2}$. Then $\sum_{i=1}^{3}\mathcal{U}(g_i)\le 1$, a contradiction.

(7.34) Suppose $d\ge 2$ and S is of type $(2,l,m)$ with $l=3$ or 4. Then $d=2$, $l=3$ and $m=8$.

Proof. Since g_2 has an eigenspace of dimension $\ge\frac{n}{4}\ge 8$, we have $\mathcal{U}(g_1)\le\frac{129}{256}$. Also if $|g_2|=3$, then g_2 is not of type $2^1 1^{n-2}$, so by (7.9)(b), $\mathcal{U}(g_2)\le\frac{65}{192}$. If $|g_2|=4$, then g_2 is not of types: $3^1 2^1 1^{n-5}$, $3^1 1^{n-3}$, $4^1 1^{n-4}$; and thus by (7.9)(b), $\mathcal{U}(g_2)\le\frac{81}{256}+\frac{1}{2^{33}}<\frac{65}{192}$; i.e., in either cases, $\mathcal{U}(g_2)\le\frac{65}{192}$. For $|g_3|\ge 12$, by (7.8), $\mathcal{U}(g_3)\le\frac{27}{16}\cdot\frac{1}{12}+\frac{1}{2^7}=\frac{19}{128}$. For $|g_3|=11$, the table in (7.9)(e) gives $\mathcal{U}(g_3)\le\frac{47}{512}$. For $|g_3|=10$, we have by (7.3)(d), (7.4)(b), and (7.5) that $\mathcal{U}(g_3)\le\frac{1}{10}\{1+(\frac{1}{2^2}+\frac{1}{2^{31}})+\frac{4}{2^7}+\frac{4}{2^7}\}=\frac{21}{160}+\frac{1}{5\cdot 2^{32}}$. For $|g_3|=9$, as g_3^3 has at least 3 simple modules each of dimension 2, we have $\mathcal{U}(g_3)\le\frac{1}{9}(1+\frac{2}{2^7}+\frac{6}{2^7})=\frac{17}{144}$. For $|g_3|=7$, as g_3 is not of type $3^1 1^{n-3}$, we have by (7.4)(c) that $\mathcal{U}(g_3)\le\frac{1}{7}(1+\frac{6}{2^7})=\frac{67}{448}$. Since $max\{\frac{19}{128},\frac{47}{512},\frac{21}{160}+\frac{1}{5\cdot 2^{32}},\frac{17}{144},\frac{67}{448}\}=\frac{67}{448}$, if $|g_3|\ge 9$ or $|g_3|=7$, then we have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\le\frac{129}{256}+\frac{65}{192}+\frac{67}{448}<1$. So $|g_3|=8$.

Suppose S is of type (2,4,8). By (3.2), as $G=\langle g_2^2,\ g_3\rangle=\langle g_2,\ g_3^2\rangle$, we have that g_2^2 is not of types $2^{\alpha}1^{n-2\alpha}$ with $\alpha\leq 4$, and g_3^2 is not of types: $3^12^11^{n-5}$, 3^11^{n-3}, 4^11^{n-4}. Thus by (7.3), $\mathcal{N}(g_2)$, $\mathcal{N}(g_2^2)$, $\mathcal{N}(g_3)$, $\mathcal{N}(g_3^2)$ are all $\leq\frac{1}{2^7}$. This gives that $\mathcal{U}(g_2)\leq\frac{1}{4}(1+\frac{1}{2^7}+\frac{2}{2^7})=\frac{131}{512}$, and $\mathcal{U}(g_3)\leq\frac{1}{8}\{1+(\frac{1}{2^2}+\frac{1}{2^{31}})+\frac{2}{2^7}+\frac{4}{2^7}\}=\frac{83}{512}+\frac{1}{2^{34}}$. Then we have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{129}{256}+\frac{131}{512}+\frac{83}{512}+\frac{1}{2^{34}}<1$. So S is not of type (2,4,8).

Suppose $d\geq 3$ and S is of type (2,3,8). By (3.2)(b), $G=\langle g_2,\ g_3^2\rangle$. As g_2 has an eigenspace of dimension ≥ 8, g_3^2 is not one of the following types: $3^12^11^{n-5}$, 3^11^{n-3}, 4^11^{n-4}. Thus by (7.3)(b), $\mathcal{N}(g_3^2)\leq\frac{1}{2^7}$. Hence by (7.3)(d), we have $\mathcal{U}(g_3)\leq\frac{1}{8}\{1+(\frac{1}{2^3}+\frac{1}{2^{30}})+\frac{2}{2^7}+\frac{4}{2^7})=\frac{75}{512}+\frac{1}{2^{33}}$. Then we have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{129}{256}+\frac{65}{192}+\frac{75}{512}+\frac{1}{2^{33}}<1$.

Hence $d=2$ and S is of type (2,3,8).

(7.35) Suppose $d=1$ and S is of type $(2,6,m)$. Then $m=4$ or 6.

Proof. Suppose $\nu_1\geq n-5$. Then by (3.3), $\frac{n}{2}\leq dim\{C_V(g_2^3)\}\leq 3(n-\nu_1)\leq 15$, contradicts to the assumption $n\geq 33$. So $\nu_1\leq n-6$. Denote $\alpha=dim\{C_V(g_2^3)\}$ and let 2^a1^b be the type of g_2^2. As $G=\langle g_1,g_2\rangle=\langle g_1,g_2^3,g_2^2\rangle$, we have $\nu_1+\alpha+b\leq 2n$. If $b\geq n-6$, then $\nu_1+\alpha\leq n+6$. Since $\frac{n}{2}\leq\nu_1\leq n-6$, $\frac{n}{2}\leq\alpha\leq n-1$, we have $\frac{1}{2}\cdot 2^{\nu_1}+\frac{3}{6}\cdot 2^{\alpha}\leq\frac{1}{2}\cdot 2^{\frac{n}{2}+6}+\frac{3}{6}\cdot 2^{\frac{n}{2}}$. Since $\mathcal{N}(g_2)\leq\mathcal{N}(g_2^3)$, we have $\frac{1}{2}\mathcal{N}(g_1)+\frac{1}{6}\{\mathcal{N}(g^3)+2\mathcal{N}(g_2)\}\leq\frac{1}{2}\cdot\frac{1}{2^{\frac{n}{2}-6}}+\frac{3}{6}\cdot\frac{1}{2^{\frac{n}{2}}}$, which implies that we have $\mathcal{U}(g_1)+\mathcal{U}(g_2)\leq\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2^{10}}+\frac{1}{6}(1+\frac{3}{2^{16}}+\frac{2}{2^2})=\frac{3}{4}+\frac{65}{2^{17}}<\frac{4627}{6144}$. Now suppose that $b\leq n-8$. Thus $\mathcal{N}(g_2)\leq\mathcal{N}(g_2^2)\leq\frac{1}{2^8}$. Since g_2^2 has an eigenvalue of multiplicity ≥ 11, $\nu_1+\alpha\leq 2n-11$. This together with $\frac{n}{2}\leq\nu_1\leq n-6$, $\frac{n}{2}\leq\alpha\leq n-1$ gives $\frac{1}{2}\cdot 2^{\nu_1}+\frac{1}{6}\cdot 2^{\alpha}\leq\frac{1}{2}\cdot 2^{n-10}+\frac{1}{6}\cdot 2^{n-1}$. Hence $\mathcal{U}(g_1)+\mathcal{U}(g_2)\leq\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2^{10}}+\frac{1}{6}(1+\frac{1}{2}+\frac{2}{2^8}+\frac{2}{2^8})=\frac{4627}{6144}$. So in any case, we have $\mathcal{U}(g_1)+\mathcal{U}(g_2)\leq\frac{4627}{6144}$. Suppose $|g_3|\geq 15$. Then by (7.8), $\mathcal{U}(g_3)\leq\frac{21}{8}\cdot\frac{1}{15}+\frac{1}{16}=\frac{19}{80}$. By the table in (7.9), for $|g_3|=13$ or 11, $\mathcal{U}(g_3)\leq\frac{316}{4096}$, $\frac{47}{512}$ respectively. As $max\{\frac{19}{80},\ \frac{316}{4096},\ \frac{47}{512}\}=\frac{19}{80}$, we have that for $|g_3|\geq 15$ or $|g_3|=13$ or 11, $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{4627}{6144}+\frac{19}{80}<1$. This contradiction shows that $|g_3|\leq 10$ or $|g_3|=14$ or 12. But by the table in (7.32), for $|g_3|=5$, 7, 8, 9, 10, 12, 14, we have

$\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_3)\leq max\{\frac{579}{1280}, \frac{45}{112}, \frac{245}{512}, \frac{25}{64}, \frac{521}{1280}, \frac{233}{512}, \frac{57}{128}\}=\frac{245}{512}$. Also by (7.32), $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_2)\leq\frac{6179}{12288}$. Thus $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{6179}{12288}+\frac{245}{512}<1$. This contradiction shows that $|g_3|=4$ or 6.

(7.36) Suppose $d=1$ and S is of type $(2,4,m)$. Then $m=6$.

Proof. Since $\nu_1+\nu_2\leq n$, $\frac{n}{2}\leq\nu_1\leq n-1$ and $\frac{n}{4}\leq\nu_2\leq n-2$, we have $\frac{1}{2}\cdot2^{\nu_1}+\frac{2}{4}\cdot2^{\nu_2}\leq\frac{1}{2}\cdot2^{\frac{3n}{4}}+\frac{2}{4}\cdot2^{\frac{n}{4}}$. Hence $\mathcal{U}(g_1)+\mathcal{U}(g)\leq\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2^{\frac{n}{4}}}+\frac{1}{4}(1+\frac{1}{2}+\frac{2}{2^{\frac{3n}{4}}})<\frac{7}{8}+\frac{1}{2^{10}}+\frac{1}{2^{25}}<\frac{7}{8}+\frac{5}{2^{12}}$. For $|g_3|\geq43$, by (7.7), $\mathcal{U}(g_3)\leq\frac{21}{8}\cdot\frac{1}{43}+\frac{1}{16}=\frac{85}{688}$. For $|g_3|=42$, by (7.7)(c)(vi), $\mathcal{U}(g_3)\leq\frac{9}{4}\cdot\frac{1}{42}+\frac{1}{16}=\frac{13}{112}<\frac{85}{688}$. From the table in (7.9), if $41\geq|g_3|\geq11$ with $|g_3|\neq12$, 14, 15, 18, 20, 21, 24, 28, then $\mathcal{U}(g_3)\leq max\{\frac{57}{656}, \frac{17}{160}, \frac{5}{52}, \frac{15}{152}, \frac{53}{592}, \frac{1}{24}, \frac{1}{10}, \frac{7}{68}, \frac{9}{88}, \frac{1541}{2^{15}}, \frac{47}{496}, \frac{17}{160}, \frac{45}{464}, \frac{1}{9}, \frac{3}{26}, \frac{41}{400}, \frac{39}{368}, \frac{389}{5632}, \frac{35}{304}, \frac{33}{272}, \frac{13}{128}, \frac{316}{4096}, \frac{47}{512}\}<\frac{85}{688}$. So if $|g_3|\geq11$ with $|g_3|\neq12$, 14, 15, 18, 20, 21, 24, 28, then $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{7}{8}+\frac{5}{2^{12}}+\frac{85}{688}<1$. This contradiction shows that $|g_3|=28$, 24, 21, 20, 18, 15, 14, 12, or $|g_3|\leq10$.

By (3.2), $G=\langle g_2^2, g_3\rangle$. If g_2^2 is a transvection, i.e., $\alpha=dim\{C_V(g_2^2)\}=n-1$, then by (3.3)(c), $|g_3|\geq n\geq33$. So for $|g_3|=28$, 24, 21, 20, or 18, g_2^2 is not a transvection. Then similar to the calculation in (7.32), we have that $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_2)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^{\frac{n}{4}}}+\frac{1}{4}(1+\frac{1}{2^2}+\frac{2}{2^{\frac{3n}{4}}})<\frac{9}{16}+\frac{1}{2^{10}}+\frac{1}{2^{25}}<\frac{289}{512}$. Also from (7.32), we have $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_3)\leq max\{\frac{47}{112}, \frac{41}{96}, \frac{183}{448}, \frac{69}{160}, \frac{71}{192}\}=\frac{69}{160}$. This implies that $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{289}{512}+\frac{69}{160}<1$, and thus $|g_3|\neq28$, 24, 21, 20, or 18.

Similarly, for $|g_3|=15$, 14, 12, 10, 9, 7 or 5, by (3.3)(c), $dim\{C_V(g_2^2)\}\leq n-3$. Hence we have $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_2)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^{\frac{n}{4}}}+\frac{1}{4}(1+\frac{1}{2^3}+\frac{2}{2^{\frac{3n}{4}}})<\frac{17}{32}+\frac{1}{2^{10}}+\frac{1}{2^{25}}<\frac{273}{512}$. Also from (7.32), we have $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_3)\leq max\{\frac{5}{12}, \frac{57}{128}, \frac{233}{512}, \frac{521}{1280}, \frac{25}{64}, \frac{45}{112}, \frac{579}{1280}\}=\frac{233}{512}$. This implies that $\sum_{i=1}^{3}\mathcal{U}(g_i)$ $\leq\frac{273}{512}+\frac{233}{512}<1$, and thus $|g_3|\neq15$, 14, 12, 10, 9, 7 or 5.

For $|g_3|=8$, by (3.3)(c), $dim\{C_V(g_2^2)\}\leq n-5$. Hence we have $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_2)\leq\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{2^{\frac{n}{4}}}+\frac{1}{4}(1+\frac{1}{2^5}+\frac{2}{2^{\frac{3n}{4}}})<\frac{65}{128}+\frac{1}{2^{10}}+\frac{1}{2^{25}}<\frac{261}{512}$. Also from (7.32), We have $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_3)\leq\frac{245}{512}$. This implies that $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{261}{512}+\frac{245}{512}<1$, and thus $|g_3|\neq 8$.

Therefore $|g_3|=6$.

(7.37) Suppose $d=1$ and S is of type $(2,3,m)$. Then $m=12$ or 8.

Proof. Since g_2 has an eigenspace of dimension ≥ 11, $\nu_1\leq n-11$. Let 2^a1^b be the type of g_2. Then $\nu_1+b\leq n$. As $\frac{n}{2}\leq\nu_1\leq n-11$, $0\leq b\leq n-2$, we have $\frac{1}{2}\cdot 2^{\nu_1}+\frac{2}{3}\cdot 2^b\leq\frac{1}{2}\cdot 2^{n-11}+\frac{2}{3}\cdot 2^{11}$. So we have $\mathcal{U}(g_1)+\mathcal{U}(g_2)\leq\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2^{11}}+\frac{1}{3}(1+\frac{2}{2^{n-11}})\leq\frac{5}{6}+\frac{1}{2^{12}}+\frac{1}{3\cdot 2^{21}}<\frac{5123}{6144}$. For $|g_3|\geq 26$, by (7.7), $\mathcal{U}(g_3)\leq\frac{21}{8}\cdot\frac{1}{26}+\frac{1}{16}=\frac{17}{104}$. From the table in (7.9), if with $|g_3|=25$, 23, 22, 19, 17, 16, 13, or 11, then $\mathcal{U}(g_3)\leq max\{\frac{41}{400},\frac{39}{368},\frac{389}{5632},\frac{35}{304},\frac{33}{272},\frac{13}{128},\frac{316}{4096},\frac{47}{512}\}<\frac{17}{104}$. So if $|g_3|\geq 11$ with $|g_3|\neq 12$, 14, 15, 18, 20, 21, 24, then $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{5123}{6144}+\frac{17}{104}<1$. This contradiction shows that $|g_3|=24$, 21, 20, 18, 15, 14, 12, or $|g_3|\leq 10$.

For $|g_3|=21$, 18, 10, 9, or 7, by the table in (7.32), $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_3)\leq max\{\frac{183}{448},\frac{71}{192},\frac{521}{1280},\frac{25}{64},\frac{45}{112}\}=\frac{183}{448}$. Also by (7.32), $\frac{1}{2}\mathcal{U}(g_1)+\mathcal{U}(g_2)\leq\frac{299}{512}$. This implies that $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{299}{512}+\frac{183}{448}<1$, and thus $|g_3|=24$, 20, 15, 14, 12, or 8.

For $|g_3|=24$, or 20, we saw in the proof of (7.32) that $\mathcal{U}(g_3)\leq\frac{7}{48},\frac{3}{20}$ respectively. For $|g_3|=15$, by (7.7), we have $\mathcal{U}(g_3)\leq\frac{21}{16}\cdot\frac{1}{15}+\frac{1}{16}=\frac{3}{20}$. Since $max\{\frac{7}{48},\frac{3}{20}\}=\frac{3}{20}$, we have $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{5123}{6144}+\frac{3}{20}<1$. This shows that $|g_3|=14$, 12, or 8.

For $|g_3|=14$, as $G=\langle g_2, g_3^2\rangle$ and g_2 has an eigenspace of dimension $\alpha\geq\frac{n}{3}$, we have that $dim\{C_V(g_3^2)\}\leq n-\alpha\leq\frac{2n}{3}$. Hence $\mathcal{N}(g_3)$ and $\mathcal{N}(g_3^2)$ both $\leq\frac{1}{2^{\frac{n}{3}}}$, which gives that $\mathcal{U}(g_3)\leq\frac{1}{14}(1+\frac{1}{2}+\frac{6}{2^{\frac{n}{3}}}+\frac{6}{2^{\frac{n}{3}}})<\frac{3}{28}+\frac{3}{7\cdot 2^{10}}=\frac{771}{7168}$. Then $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{5123}{6144}+\frac{771}{7168}<1$. So $|g_3|\neq 14$.

Therefore $|g_3|=12$ or 8.

(7.38) Suppose $d=2$. Then S is not of type (2,3,8).

Proof. Since $d=2$, $|\Omega|=\left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]_2=\frac{1}{3}(2^n-1)(2^{n-1}-1)$. Suppose that S is of type (2,3,8). Denote $\alpha=dim\{C_V(g_3^4)\}$, $\beta=dim\{C_V(g_3^2)\}$. Then by (7.1)(b), we have that $\frac{1}{2}+\frac{1}{2}\cdot\frac{(2^{\nu_1}-1)(2^{\nu_1-1}-1)}{(2^n-1)(2^{n-1}-1)}$ $+\frac{1}{2}\cdot\frac{3\cdot 2^{\nu_1-1}(2^{n-\nu_1}-1)}{(2^n-1)(2^{n-1}-1)}+\frac{1}{3}+\frac{2}{3}\cdot\frac{(2^{\nu_2}-1)(2^{\nu_2-1}-1)}{(2^n-1)(2^{n-1}-1)}+\frac{2}{3}\cdot\frac{2^{n-\nu_2}-1}{(2^n-1)(2^{n-1}-1)}+\frac{1}{8}+$ $\frac{1}{8}\cdot\frac{(2^\alpha-1)(2^{\alpha-1}-1)}{(2^n-1)(2^{n-1}-1)}+\frac{1}{8}\cdot\frac{3\cdot 2^{\alpha-1}(2^{n-\alpha}-1)}{(2^n-1)(2^{n-1}-1)}+\frac{2}{8}\cdot\frac{(2^\beta-1)(2^{\beta-1}-1)}{(2^n-1)(2^{n-1}-1)}+\frac{2}{8}\cdot\frac{3\cdot 2^{\beta-1}(2^{\alpha-\beta}-1)}{(2^n-1)(2^{n-1}-1)}+$ $\frac{4}{8}\cdot\frac{(2^{\nu_3}-1)(2^{\nu_3-1}-1)}{(2^n-1)(2^{n-1}-1)}+\frac{4}{8}\cdot\frac{3\cdot 2^{\nu_3-1}(2^{\beta-\nu_3}-1)}{(2^n-1)(2^{n-1}-1)}=1+\frac{6}{(2^n-1)(2^{n-1}-1)}$. This identity can be transformed into $3\cdot 2^{2\nu_1+1}-9\cdot 2^{\nu_1+2}+2^{2\nu_2+3}-3\cdot 2^{\nu_2+3}+2^{n-\nu_2+4}+3\cdot 2^{2\alpha-1}+45\cdot 2^{n-1}+3\cdot 2^{2\beta}+$ $3\cdot 2^{2\nu_3+1}-9\cdot 2^{\nu_3+2}=2^{2n-1}-2^n-2^{n-1}-26$. Since $\nu_1\geq\frac{n}{2}$, $\nu_3\geq\frac{n}{8}$, $\alpha\geq\frac{n}{2}$, $\beta\geq\frac{n}{4}$, $\nu_2\geq 0$, and $n\geq 33$ as we have assumed, 4 divides the left hand side. But $4\nmid 26$. So S is not of type (2,3,8).

(7.39) Suppose $d=1$. Then S is not of type (2,4,6).

Proof. Suppose that S is of type (2,4,6). Denote $\alpha=dim\{C_V(g_2^2)\}$, $\beta=dim\{C_V(g_3^3)\}$, $\gamma=dim\{C_V(g_3^2)\}$. Then $\frac{1}{2}+\frac{1}{2}\cdot\frac{2^{\nu_1}-1}{2^n-1}+\frac{1}{4}+\frac{1}{4}\cdot\frac{2^\alpha-1}{2^n-1}+\frac{2}{4}\cdot\frac{2^{\nu_2}-1}{2^n-1}+\frac{1}{6}+\frac{1}{6}\cdot\frac{2^\beta-1}{2^n-1}+$ $\frac{2}{6}\cdot\frac{2^\gamma-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\nu_3}-1}{2^n-1}=1+\frac{2}{2^n-1}$. This identity can be transformed to $2^n-3\cdot 2^\alpha-2^{\beta+1}-$ $6(2^{\nu_1}+2^{\nu_2})=2^{\gamma+2}+2^{\nu_3+2}-48$. Since $\gamma\geq\nu_3$, and ν_1, α, β all $\geq\frac{n}{2}\geq 16$, $\nu_2\geq\frac{n}{4}\geq 8$, if $\nu_3\geq 3$, then $2^5|48$, a contradiction. So $\nu_3\leq 2$. If $\nu_3=0$, then $2^{\gamma+2}+2^{\nu_3+2}-48=4(2^\gamma-11)$. But 2^4 divides the left hand side, which implies that $4|(2^\gamma-11)$, a contradiction. If $\nu_3=1$, then $2^{\gamma+2}+2^{\nu_3+2}-48=8(2^{\gamma-1}-5)$. As 2^6 divides the left hand side, $8|(2^{\gamma-1}-5)$, a contradiction. If $\nu_3=2$, then $2^{\gamma+2}+2^{\nu_3+2}-48=2^5(2^{\gamma-3}-1)$, which implies that $\gamma=3$. Thus n is odd and $2^n-3\cdot 2^\alpha-2^{\beta+1}-3\cdot 2^{\nu_1+1}=3\cdot 2^{\nu_2+1}$. Then ν_1, α, β all $\geq\frac{n+1}{2}$. As $\nu_1+\nu_2\leq n$, we have $\nu_2\leq\frac{n-1}{2}$. Since $2^{\frac{n+1}{2}}$ divides the left hand side, $\frac{n+1}{2}\leq\nu_2+1$. So $\nu_2=\frac{n-1}{2}$. Then $\alpha=\nu_2+1$. Thus $2^n-2^{\beta+1}-3\cdot 2^{\nu_1+1}=3\cdot 2^{\frac{n+1}{2}+1}$. Then either $\beta=\frac{n+1}{2}$ or $\nu_1=\frac{n+1}{2}$. If $\beta=\frac{n+1}{2}$, then

$2^n-3\cdot2^{\nu_1+1}=2^{\frac{n+1}{2}+3}$. Hence $\nu_1=\frac{n+1}{2}+2$ and $2^n=2^{\frac{n+1}{2}+5}$. This forces $n=11$. As we assume that $n\geq33$, this is a contradiction. If $\nu_1=\frac{n+1}{2}$, then $2^n-2^{\beta+1}=3\cdot2^{\frac{n+1}{2}+2}$. So $\beta=\frac{n+1}{2}+1$, and thus $2^n=2^{\frac{n+1}{2}+4}$. This forces $n=9$, a contradiction again. So S is not of type (2,4,6).

(7.40) Suppose $d=1$. Then S is not of type (2,6,6).

Proof. Suppose that S is of type (2,6,6). Denote $\alpha_i=dim\{C_V(g_i^3)\}$, $\beta_i=dim\{C_V(g_i^2)\}$, $i=2$, 3. Then $\frac{1}{2}+\frac{1}{2}\cdot\frac{2^{\nu_1}-1}{2^n-1}+\sum_{i=2}^{3}\left\{\frac{1}{6}+\frac{1}{6}\cdot\frac{2^{\alpha_i}-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\beta_i}-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\nu_i}-1}{2^n-1}\right\}=1+\frac{2}{2^n-1}$. This identity can be transformed to $2^n+24=3\cdot2^{\nu_1}+2(2^{\nu_2}+2^{\nu_3})+2^{\alpha_2}+2^{\alpha_3}+2^{\beta_2+1}+2^{\beta_3+1}$. As $G=\langle g_1,g_2\rangle$, by (3.3)(c), $n\leq|g_2|(n-\nu_1)=6(n-\nu_1)$, thus $\nu_1\leq n-6$. Also as $\nu_1+\nu_i\leq n$, $\nu_1\geq\frac{n}{2}$, we have $\nu_i\leq\frac{n}{2}$, $i=2$, 3. By (3.2), $G=\langle g_2^2,\ g_3\rangle$ and thus by (3.3)(c), $n\leq|g_3|(n-\beta_2)=6(n-\beta_2)$, which gives $\beta_2\leq n-6$. Similarly, $\beta_3\leq n-6$. Denote $\xi=3\cdot2^{\nu_1}+2(2^{\nu_2}+2^{\nu_3})+2^{\beta_2+1}+2^{\beta_3+1}$. Then $3\cdot2^{\frac{n}{2}}<\xi\leq3\cdot2^{n-6}+2\cdot2^{\frac{n}{2}+1}+2\cdot2^{n-5}\leq3\cdot2^{n-6}+2^{n-6}+4\cdot2^{n-6}=2^{n-3}$, as $\frac{n}{2}+2\leq n-6$. If $\alpha_2=n-1=\alpha_3$, then $3\cdot2^{\nu_1}+2(2^{\nu_2}+2^{\nu_3})+2^{\alpha_2}+2^{\alpha_3}+2^{\beta_2+1}+2^{\beta_3+1}=2^n+\xi>2^n+3\cdot2^{\frac{n}{2}}>2^n+24$, a contradiction. If one of α_2, α_3 is $\leq n-2$, then $2^{\alpha_2}+2^{\alpha_3}+\xi\leq2^{n-1}+\ 2^{n-2}+2^{n-3}<2^n+24$, a contradiction again. So S is not of type (2,6,6).

(7.41) Suppose $d=1$. Then S is not of type (2,3,8).

Proof. Suppose that S is of type (2,3,8). Denote $\alpha=dim\{C_V(g_3^4)\}$, $\beta=dim\{C_V(g_3^2)\}$. Then $\frac{1}{2}+\frac{1}{2}\cdot\frac{2^{\nu_1}-1}{2^n-1}+\frac{1}{3}+\frac{2}{3}\cdot\frac{2^{\nu_2}-1}{2^n-1}+\frac{1}{8}+\frac{1}{8}\cdot\frac{2^{\alpha}-1}{2^n-1}+\frac{2}{8}\cdot\frac{2^{\beta}-1}{2^n-1}+$
$\frac{4}{8}\cdot\frac{2^{\nu_3}-1}{2^n-1}=1+\frac{2}{2^n-1}$. This identity can be transformed to $12(2^{\nu_1}+2^{\nu_3})+3\cdot2^{\alpha}+3\cdot2^{\beta+1}-2^n=2^5(3-2^{\nu_2-1})$. We have ν_1, α, both $\geq\frac{n}{2}\geq16$, $\beta\geq\frac{n}{4}\geq8$, and as $n\geq33$, $\nu_3\geq5$. So 2^7 divides the left hand side of the equation, and thus $2^2|(3-2^{\nu_2-1})$, a contradiction. So S is not of type (2,3,8).

(7.42) Suppose $d=1$. Then S is not of type (2,3,12).

Proof. Suppose that S is of type (2,3,12). Denote $\alpha=dim\{C_V(g_3^6)\}$, $\beta=dim\{C_V(g_3^4)\}$, $\gamma=dim\{C_V(g_3^3)\}$, $\delta=dim\{C_V(g_3^2)\}$. Then $\frac{1}{2}+\frac{1}{2}\cdot\frac{2^{\nu_1}-1}{2^n-1}+\frac{1}{3}+\frac{2}{3}\cdot\frac{2^{\nu_2}-1}{2^n-1}+\frac{1}{12}+\frac{1}{12}\cdot\frac{2^{\alpha}-1}{2^n-1}+\frac{2}{12}\cdot\frac{2^{\beta}-1}{2^n-1}+\frac{2}{12}\cdot\frac{2^{\gamma}-1}{2^n-1}+\frac{2}{12}\cdot\frac{2^{\delta}-1}{2^n-1}+\frac{4}{12}\cdot\frac{2^{\nu_3}-1}{2^n-1}=1+\frac{2}{2^n-1}$. This identity can be transformed to $2^n+48=6\cdot 2^{\nu_1}+8\cdot 2^{\nu_2}+4\cdot 2^{\nu_3}+2^{\alpha}+2^{\beta+1}+2^{\gamma+1}+2^{\delta+1}$. As before, since $\nu_1+\nu_i\le n$, $\nu_1\ge\frac{n}{2}$, we have $\nu_i\le\frac{n}{2}$, $i=2$, 3. Since g_2 has an eigenspace of dimension ≥ 11, and $G=\langle g_2,\ g_3^2\rangle$, we have $\nu_1\le n-11$ and $\delta\le n-11$. Denote $x=6\cdot 2^{\nu_1}+8\cdot 2^{\nu_2}+4\cdot 2^{\nu_3}+2^{\delta+1}$ and $y=2^{\alpha}+2^{\beta+1}+2^{\gamma+1}$. So we have $x+y=2^n+48$. Also $3\cdot 2^{\frac{n}{2}}<x\le 6\cdot 2^{n-11}+8\cdot 2^{\frac{n}{2}}+4\cdot 2^{\frac{n}{2}}+2^{n-10}\le 2^{n-8}+8\cdot 2^{n-16}+4\cdot 2^{n-16}<2^{n-7}$, as $\frac{n}{2}\le n-16$. If at least two from $\{\alpha,\ \beta+1,\ \gamma+1\}$ are equal to $n-1$ or as sets $\{\alpha,\ \beta+1,\ \gamma+1\}=\{n-1,\ n-2,\ n-2\}$, then $y\ge 2^n$, which implies $x+y>2^n+3\cdot 2^{\frac{n}{2}}>2^n+48$, a contradiction. Otherwise, $y\le 2^{n-1}+2^{n-2}+2^{n-3}$, which implies that $x+y\le 2^{n-1}+2^{n-2}+2^{n-3}+2^{n-7}<2^n+48$, a contradiction again. So S is not of type (2,3,12).

(7.43) If $|S|=3$, then there is no involution in S.

Proof. This is the combination of (7.33) to (7.42).

(7.44) Suppose $|S|=3$ and the smallest order of elements in S is 3. Then $d=1$ and S is one of the following types: (3,3,6), (3,4,6), (3,3,4), or (3,4,4).

Proof. Suppose without loss of generality that $|g_1|=3$. Let $2^a1^{\nu_1}$ be the type of g_1. Since g_1 has an eigenspace of dimension ≥ 11, we always have $\nu_i\le n-11$ for $i=2$ and 3. If $\nu_1=n-2$, then by (3.3), $n\le|g_i|(n-\nu_1)=2|g_i|$ for $i=2$, 3. Hence $|g_i|\ge 17$, as we assume that $n\ge 33$. Then for $i=2$, 3, by (7.8) we have $\mathcal{U}(g_i)\le\frac{21}{8}\cdot\frac{1}{17}+\frac{1}{16}=\frac{59}{272}$. Hence $\sum_{i=1}^{3}\mathcal{U}(g_i)\le\frac{1}{2}+2\cdot\frac{59}{272}<1$, a contradiction. So $\nu_1\le n-4$.

Consider the case that $d\ge 2$ first. So by (7.9)(b), $\mathcal{U}(g_i)\le\frac{65}{192}$ for any g_i with $|g_i|=3$. If $|g_i|=4$, where $i=2$ or 3, then as g_1 has an eigenspace of dimension ≥ 11, g_i is not of types: $3^12^11^{n-5}$, 3^11^{n-3}, 4^11^{n-4}. Hence by (7.9)(b), $\mathcal{U}(g_i)\le\frac{81}{256}+\frac{1}{2^{33}}$. Thus again by (7.9)(b), for

$|g_i|\geq 4$, where $i=2$ or 3, we have $\mathcal{U}(g_i)\leq max\{\frac{81}{256}+\frac{1}{2^{33}}, \frac{33}{160}, \frac{37}{128}\}=\frac{81}{256}+\frac{1}{2^{33}}<\frac{65}{192}$. Since S is not of type (3,3,3), we have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq 2\cdot\frac{65}{192}+(\frac{81}{256}+\frac{1}{2^{33}})<1$. So $d=1$.

Since $\nu_1\leq n-4$, we have $\mathcal{U}(g_1)\leq\frac{1}{3}(1+\frac{2}{2^4})=\frac{3}{8}$. For $|g|\geq 11$, we have $\mathcal{U}(g)\leq\frac{21}{8}\cdot\frac{1}{11}+\frac{1}{16}=\frac{53}{176}$. For $|g|=10$, 9, 7, or 5, we have by (7.7), $\mathcal{U}(g)\leq\frac{1}{5}$, $\frac{5}{24}$, $\frac{1}{4}$, $\frac{21}{80}$ respectively. For $|g|=8$, $\mathcal{U}(g)\leq\frac{1}{8}(1+\frac{1}{2}+\frac{2}{2^3}+\frac{4}{2^4})=\frac{1}{4}$. Since $max\{\frac{53}{176}, \frac{1}{5}, \frac{5}{24}, \frac{1}{4}, \frac{21}{80}, \frac{1}{4}\}=\frac{53}{176}$, thus if no element in $S-\{g_1\}$ is of order 3, 4, and 6, then $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{3}{8}+2\cdot\frac{53}{176}<1$, a contradiction. So one of g_2, g_3 is of order 3, 4, or 6.

Without loss of generality, suppose $|g_2|=6$. Since g_1 has an eigenspace of dimension ≥ 11, $\nu_2\leq n-11$. As $G=\langle g_1, g_2^3, g_2^2\rangle$, we have $\alpha+\beta\leq 2n-11$, where $\alpha=dim\{C_V(g_2^3)\}$, $\beta=dim\{C_V(g_3^2)\}$. Also $\frac{n}{2}\leq\alpha\leq n-1$, $0\leq\beta\leq n-2$, we have $2^\alpha+2\cdot 2^\beta\leq 2^{n-1}+2\cdot 2^{n-10}$. Hence $\mathcal{U}(g_2)\leq\frac{1}{6}(1+\frac{1}{2}+\frac{2}{2^{10}}+\frac{2}{2^{11}})=\frac{513}{2048}$. So if $|g_3|\geq 5$, then we have $\mathcal{U}(g_3)\leq max\{\frac{53}{176}, \frac{513}{2048}\}=\frac{53}{176}$, which gives the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{3}{8}+\frac{513}{2048}+\frac{53}{176}<1$. Therefore if one of g_2, g_3 is of order 6, then without loss of generality, S is of type (3,3,6) or (3,4,6).

Suppose $|g_2|=4$. As before, since $\nu_2\leq n-11$, we have $\mathcal{U}(g_2)\leq\frac{1}{4}(1+\frac{1}{2}+\frac{2}{2^{11}})=\frac{1537}{4096}$. By (7.8), for $|g_3|\geq 15$, $\mathcal{U}(g_3)\leq\frac{21}{8}\cdot\frac{1}{15}+\frac{1}{16}=\frac{19}{80}$. For $|g_3|=14$, 13, 11, 10, or 9, by (7.8), $\mathcal{U}(g_3)\leq\frac{3}{16}$, $\frac{29}{208}$, $\frac{27}{176}$, $\frac{1}{5}$, $\frac{5}{24}$ respectively. Suppose $|g_3|=12$. Denote $\alpha=dim\{C_V(g_3^4)\}$, $\beta=dim\{C_V(g_3^3)\}$. Since $G=\langle g_1, g_3^4, g_3^3\rangle$, we have $\alpha+\beta\leq 2n-11$. Also $0\leq\alpha\leq n-2$, $\frac{n}{4}\leq\beta\leq n-2$, we have $2^\alpha+2^\beta\leq 2^{n-2}+2^{n-9}$. Thus $\mathcal{U}(g_3)\leq\frac{1}{12}(1+\frac{1}{2}+\frac{2}{2^2}+\frac{2}{2^9}+\frac{2}{2^3}+\frac{4}{2^{11}})=\frac{385}{2048}$. Suppose $|g_3|=8$. As $\nu_3\leq n-11$, $\mathcal{U}(g_3)\leq\frac{1}{8}(1+\frac{1}{2}+\frac{2}{2^3}+\frac{4}{2^{11}})=\frac{897}{4096}$. Suppose $|g_3|=7$. As $\nu_3\leq n-11$, $\mathcal{U}(g_3)\leq\frac{1}{7}(1+\frac{6}{2^{11}})=\frac{1027}{7168}$. Suppose $|g_3|=5$. As $\nu_3\leq n-11$, $\mathcal{U}(g_3)\leq\frac{1}{5}(1+\frac{4}{2^{11}})=\frac{513}{2560}$. So for $|g_3|\geq 7$ or $|g_3|=5$, we have $\mathcal{U}(g_3)\leq max\{\frac{19}{80}, \frac{3}{16}, \frac{29}{208}, \frac{27}{176}, \frac{1}{5}, \frac{5}{24}, \frac{385}{2048}, \frac{897}{4096}, \frac{1027}{7168}, \frac{513}{2560}\}=\frac{19}{80}$. So if $|g_3|\neq 3$, 4, and 6, then we have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{3}{8}+\frac{1537}{4096}+\frac{19}{80}<1$. So without loss of generality, if one of the element in g_2, g_3 is of order 4, then S is of type (3,3,4), (3,4,4), or (3,4,6).

Suppose $|g_2|=3$. Since g_1, g_2 both have an eigenspace of dimension ≥ 11, $\nu_1\leq n-11$ and

$\nu_2 \leq n-11$. So we have $\mathcal{U}(g_i) \leq \frac{1}{3}(1+\frac{2}{2^{11}}) = \frac{1025}{3072}$ for $i=1$ and 2. By (7.8), for $|g_3| \geq 10$, $\mathcal{U}(g_3) \leq \frac{21}{8} \cdot \frac{1}{10} + \frac{1}{16} = \frac{13}{40}$. As we have seen already, for $|g_3| = 9$, 8, 7, or 5, we have by (7.7), $\mathcal{U}(g) \leq \frac{25}{144}$, $\frac{1}{4}$, $\frac{1}{4}$, $\frac{21}{80}$ respectively. As $max\{\frac{13}{40}, \frac{5}{24}, \frac{1}{4}, \frac{21}{80}\} = \frac{13}{40}$, if $|g_3| \neq 4$ and 6, then we have the contradiction $\sum_{i=1}^{3} \mathcal{U}(g_i) \leq 2 \cdot \frac{1025}{3072} + \frac{13}{40} < 1$. So without loss of generality, if one of the element in g_2, g_3 is of order 3, then S is of type (3,3,4) or (3,3,6).

(7.45) Suppose $d=1$. Then S is not of type (3,3,4).

Proof. Suppose that S is of type (3,3,4). Denote $\alpha = dim\{C_V(g_3^2)\}$. Then $\sum_{i=1}^{2} \left\{\frac{1}{3} + \frac{2}{3} \cdot \frac{2^{\nu_i}-1}{2^n-1}\right\} + \frac{1}{4} + \frac{1}{4} \cdot \frac{2^{\alpha}-1}{2^n-1} + \frac{2}{4} \cdot \frac{2^{\nu_3}-1}{2^n-1} = 1 + \frac{2}{2^n-1}$. This identity can be transformed to $8(2^{\nu_1}+2^{\nu_2}) + 3 \cdot 2^{\alpha} + 3 \cdot 2^{\nu_3+1} = 2^n + 48$. As seen before, $\nu_i \leq n-11$, for $i=1$, 2, and 3. Also as $\frac{n}{4} \leq \nu_3$, we have $3 \cdot 2^{\frac{n}{4}+1} < 8(2^{\nu_1}+2^{\nu_2}) + 3 \cdot 2^{\nu_3+1} \leq 3 \cdot 2^{n-8}$. So if $\alpha \leq n-2$, then $8(2^{\nu_1}+2^{\nu_2}) + 3 \cdot 2^{\alpha} + 3 \cdot 2^{\nu_3+1} \leq 2^{n-1} + 2^{n-2} + 2^{n-7} + 2^{n-8} < 2^n + 48$, a contradiction. If $\alpha = n-1$, then $8(2^{\nu_1}+2^{\nu_2}) + 3 \cdot 2^{\alpha} + 3 \cdot 2^{\nu_3+1} > 2^n + 3 \cdot 2^{\frac{n}{4}+1} > 2^n + 48$, a contradiction again. So S is not of type (3,3,4).

(7.46) Suppose $d=1$. Then S is not of type (3,3,6).

Proof. Suppose that S is of type (3,3,6). Denote $\alpha = dim\{C_V(g_3^3)\}$, $\beta = dim\{C_V(g_3^2)\}$. Then $\sum_{i=1}^{2} \left\{\frac{1}{3} + \frac{2}{3} \cdot \frac{2^{\nu_i}-1}{2^n-1}\right\} + \frac{1}{6} + \frac{1}{6} \cdot \frac{2^{\alpha}-1}{2^n-1} + \frac{2}{6} \cdot \frac{2^{\beta}-1}{2^n-1} + \frac{2}{6} \cdot \frac{2^{\nu_3}-1}{2^n-1} = 1 + \frac{2}{2^n-1}$. This identity can be transformed to $4(2^{\nu_1}+2^{\nu_2}) + 2^{\alpha} + 2^{\beta+1} + 2^{\nu_3+1} = 2^n + 24$. As seen before, $\nu_i \leq n-11$, for $i=1$, 2, and 3. We have $4(2^{\nu_1}+2^{\nu_2}) + 2^{\nu_3+1} \leq 2^{n-8} + 2^{n-10}$. So if both or one of α, $\beta+1$ is $\leq n-2$, then $4(2^{\nu_1}+2^{\nu_2}) + 2^{\alpha} + 2^{\beta+1} + 2^{\nu_3+1} \leq 2^{n-1} + 2^{n-2} + 2^{n-8} + 2^{n-10} < 2^n + 24$, a contradiction. Hence $\alpha = n-1$ and $\beta = n-2$. But on the other hand, as $G = \langle g_1, g_3^3, g_3^2 \rangle$, we have $\alpha + \beta \leq 2n-11$. Thus this contradiction shows that S is not of type (3,3,6).

(7.47) Suppose $d=1$. Then S is not of type (3,4,4).

Proof. Suppose that S is of type (3,4,4). Denote $\alpha_i = dim\{C_V(g_i^2)\}$ for $i=2$, 3. Then

$\frac{1}{3}+\frac{2}{3}\cdot\frac{2^{\nu_1}-1}{2^n-1}+\sum_{i=2}^{3}\left\{\frac{1}{4}+\frac{1}{4}\cdot\frac{2^{\alpha_i}-1}{2^n-1}+\frac{2}{4}\cdot\frac{2^{\nu_i}-1}{2^n-1}\right\}=1+\frac{2}{2^n-1}$. This identity can be transformed to $2^{\nu_1+3}+6(2^{\nu_2}+2^{\nu_3})+3(2^{\alpha_2}+2^{\alpha_3})=2^{n+1}+48$. As seen before, $\nu_i\leq n-11$, for $i=2$ and 3. We also have $\frac{n}{4}\leq\nu_2$ and $\nu_1+\nu_2\leq n$, which implies $\nu_1\leq\frac{3n}{4}\leq n-8$ as $n\geq 33$. So $2^{\nu_1+3}+6(2^{\nu_2}+2^{\nu_3})\leq 2^{n-5}+3\cdot 2^{n-9}$. If one of α_1, α_2 is equal to $n-1$ and the other is $\geq n-2$, then $3(2^{\alpha_2}+2^{\alpha_3})\geq 3(2^{n-1}+2^{n-2})=2^{n+1}+2^{n-2}>2^{n+1}+48$, a contradiction. Hence either both α_1 and α_2 are $\leq n-2$, or one of α_1, α_2 is equal to $n-1$ and the other is $\leq n-3$. In both cases, we have $3(2^{\alpha_2}+2^{\alpha_3})\leq 3(2^{n-1}+2^{n-3})=2^n+2^{n-1}+2^{n-2}+2^{n-3}$. But this implies that $2^{\nu_1+3}+6(2^{\nu_2}+2^{\nu_3})+3(2^{\alpha_2}+2^{\alpha_3})\leq 2^n+2^{n-1}+2^{n-2}+2^{n-3}+2^{n-5}+3\cdot 2^{n-9}<2^{n+1}+48$, a contradiction again. So S is not of type (3,4,4).

(7.48) Suppose $d=1$. Then S is not of type (3,4,6).

Proof. Suppose that S is of type (3,4,6). Denote $\alpha=dim\{C_V(g_2^2)\}$, $\beta=dim\{C_V(g_3^3)\}$, $\gamma=dim\{C_V(g_3^2)\}$. Then $\frac{1}{3}+\frac{2}{3}\cdot\frac{2^{\nu_1}-1}{2^n-1}+\frac{1}{4}+\frac{1}{4}\cdot\frac{2^{\alpha}-1}{2^n-1}+\frac{2}{4}\cdot\frac{2^{\nu_2}-1}{2^n-1}+\frac{1}{6}+\frac{1}{6}\cdot\frac{2^{\beta}-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\gamma}-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\nu_3}-1}{2^n-1}=1+\frac{2}{2^n-1}$. This identity can be transformed to $2^{\nu_1+3}+3\cdot 2^{\nu_2+1}+2^{\nu_3+2}+3\cdot 2^{\alpha}+2^{\beta+1}+2^{\gamma+2}=3\cdot 2^n+48$. Similar as before, $\nu_i\leq n-11$, for $i=2$ and 3. Also $\nu_1\leq n-8$. So $2^{\nu_1+3}+3\cdot 2^{\nu_2+1}+2^{\nu_3+2}\leq 2^{n-5}+2^{n-8}+2^{n-10}$. If one of $\beta+1$, $\gamma+2$ is $\leq n-2$, then $3\cdot 2^{\alpha}+2^{\beta+1}+2^{\gamma+2}\leq 3\cdot 2^{n-1}+2^n+2^{n-2}=2^{n+1}+2^{n-1}+2^{n-2}$, which gives $2^{\nu_1+3}+3\cdot 2^{\nu_2+1}+2^{\nu_3+2}+3\cdot 2^{\alpha}+2^{\beta+1}+2^{\gamma+2}\leq 2^{n+1}+2^{n-1}+2^{n-2}+2^{n-5}+2^{n-8}+2^{n-10}<2^{n+1}+2^n<3\cdot 2^n+48$, a contradiction. Hence both $\beta+1$, $\gamma+2$ are $\geq n-1$, which implies $\beta+\gamma\geq 2n-5$. But on the other hand, as $G=\langle g_1, g_3^3, g_3^2\rangle$, and g_1 has an eigenspace on dimension ≥ 11, we have $\beta+\gamma\leq 2n-11$. This contradiction shows that S is not of type (3,4,6).

(7.49) If $|S|=3$, then any element in S has order at least 4.

Proof. This is the combination of (7.43) to (7.48).

(7.50) Suppose $|S|=3$ and the smallest order of elements in S is 4. Then $d=1$ and S is of type (4,4,4).

Proof. Consider the case that $d\geq 2$ first. Suppose $g_i\in S$ is such that $|g_i|=4$. If $\nu_i\geq n-3$, then by (3.3), $n\leq |g_j|(n-\nu_i)\leq 3|g_j|$ for $j\neq i$. Hence $|g_j|\geq 11$, as we assume that $n\geq 33$. Then by (7.6) we have $\mathcal{U}(g_j)\leq\frac{27}{16}\cdot\frac{1}{11}+\frac{1}{2^7}=\frac{227}{1408}$. Hence $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq(\frac{11}{32}+\frac{1}{2^{32}})+2\cdot\frac{227}{1408}<1$, a contradiction. So if $|g_i|=4$, then g_i is not of types: $3^1 2^1 1^{n-5}$, $3^1 1^{n-3}$, $4^1 1^{n-4}$. Thus by (7.9)(b), we have then $\mathcal{U}(g)\leq\frac{81}{256}+\frac{1}{2^{33}}$ for any $g\in S$. But then we again have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq 3\cdot(\frac{81}{256}+\frac{1}{2^{33}})<1$. So $d=1$.

Without loss of generality, assume $|g_1|=4$. Then as $\nu_1\geq\frac{n}{4}$, and $n\geq 33$, we always have $\nu_i\leq\frac{3n}{4}$, so $\nu_i\leq n-9$ for $i=2$ and 3. Also $\mathcal{U}(g_1)\leq\frac{1}{4}(1+\frac{1}{2}+\frac{2}{2^2})=\frac{1}{2}$. For $g\in\{g_2,\ g_3\}$ with $|g|\geq 14$, $\mathcal{U}(g)\leq\frac{21}{8}\cdot\frac{1}{14}+\frac{1}{16}=\frac{1}{4}$. As in the proof for (7.20), we have that for $|g|=13$, 12, 11, 10, 9, 8, or 7, $\mathcal{U}(g)\leq\frac{29}{208},\ \frac{1}{4},\ \frac{27}{176},\ \frac{1}{5},\ \frac{5}{24},\ \frac{1}{4},\ \frac{1}{4}$ respectively. For $g\in\{g_2,\ g_3\}$ with $|g|=6$. Denote $\alpha=dim\{C_V(g^3)\}$, $\beta=dim\{C_V(g^2)\}$. Since $G=\langle g_1,\ g^3,\ g^2\rangle$, we have $\alpha+\beta\leq 2n-\nu_1\leq\frac{7n}{4}$. So $2^\alpha+2\cdot 2^\beta\leq 2^{n-1}+2\cdot 2^{\frac{3n}{4}+1}$. Hence $\mathcal{U}(g)\leq\frac{1}{6}(1+\frac{1}{2}+\frac{2}{2^{\frac{n}{4}-1}}+\frac{2}{2^9})\leq\frac{389}{2048}$. For $g\in\{g_2,\ g_3\}$ with $|g|=5$. Then $\mathcal{U}(g)\leq\frac{1}{5}(1+\frac{4}{2^9})=\frac{129}{640}$. Suppose $|g_2|$ and $|g_3|$ are both ≥ 5. Since $max\{\frac{1}{4},\ \frac{29}{208},\ \frac{1}{4},\ \frac{27}{176},\ \frac{1}{5},\ \frac{5}{24},\ \frac{1}{4},\ \frac{1}{4},\ \frac{389}{2048},\ \frac{129}{640}\}=\frac{1}{4}$, we have $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{1}{2}+2\cdot\frac{1}{4}=1$, a contradiction. So at least one of the element among g_2, g_3 is of order 4.

Suppose without loss of generality that $|g_2|=4$. Then both $\mathcal{U}(g_1)$ and $\mathcal{U}(g_2)\leq\frac{1}{4}(1+\frac{1}{2}+\frac{2}{2^9})=\frac{193}{512}$. For $|g_3|\geq 15$, $\mathcal{U}(g_3)\leq\frac{21}{8}\cdot\frac{1}{15}+\frac{1}{16}=\frac{19}{80}$. For $|g_3|=14$, $\mathcal{U}(g_3)\leq\frac{3}{16}$ by (7.7). For $|g_3|=12$, as $\nu_3\leq n-9$, we have $\mathcal{U}(g_3)\leq\frac{1}{12}(1+\frac{1}{2}+\frac{2}{2^2}+\frac{2}{2^2}+\frac{2}{2^3}+\frac{4}{2^9})=\frac{353}{1536}$. For $|g_3|=8$, as $\nu_3\leq n-9$, we have $\mathcal{U}(g_3)\leq\frac{1}{8}(1+\frac{1}{2}+\frac{2}{2^3}+\frac{4}{2^9})=\frac{225}{1024}$. For $|g_3|=7$, as $\nu_3\leq n-9$, we have $\mathcal{U}(g_3)\leq\frac{1}{7}(1+\frac{6}{2^9})=\frac{37}{256}$. So if $|g_3|\neq 4$, then $\mathcal{U}(g_3)\leq max\{\frac{19}{80},\ \frac{3}{16},\ \frac{29}{208},\ \frac{353}{1536},\ \frac{27}{176},\ \frac{1}{5},\ \frac{5}{24},\ \frac{225}{1024},\ \frac{37}{256},\ \frac{389}{2048},\ \frac{129}{640}\}=\frac{19}{80}$, which gives the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq 2\cdot\frac{193}{512}+\frac{19}{80}<1$. Therefore S is of type (4,4,4).

(7.51) Suppose $d=1$. Then S is not of type (4,4,4).

Proof. Suppose that S is of type (4,4,4). Denote $\alpha_i = dim\{C_V(g_i^2)\}$ for $1 \leq i \leq 3$. Then $\sum_{i=1}^{3}\left\{\frac{1}{4}+\frac{1}{4}\cdot\frac{2^{\alpha_i}-1}{2^n-1}+\frac{2}{4}\cdot\frac{2^{\nu_i}-1}{2^n-1}\right\}=1+\frac{2}{2^n-1}$. This identity can be transformed to $\sum_{i=1}^{3}(2^{\nu_i+1}+2^{\alpha_i})=2^n+16$. Since $\alpha_i \geq \frac{n}{2}$, $\nu_i \geq \frac{n}{4}$, $\forall\ i$, and as we assume that $n \geq 33$, 2^{10} divides the left hand side, which gives the contradiction $2^{10}|16$. Thus S is not of type (4,4,4).

(7.52) Suppose $|S|=3$. Then any element in S has order at least 5.

Proof. This follows from (7.49) to (7.51).

(7.53) Suppose $|S|=3$. Then $d=1$ and S is of type (6,6,6).

Proof. Suppose S is not of type (6,6,6). Then by (7.52), at least one element in S is of order 5 or ≥ 7. For $|g|=5$ or $|g| \geq 7$, we have by (7.9)(d) $\mathfrak{U}(g) \leq \frac{1}{4}$. For $|g|=6$, $\mathfrak{U}(g) \leq \frac{1}{6}(1+\frac{1}{2}+\frac{2}{2^2}+\frac{2}{2^3})=\frac{3}{8}$. Hence $\sum_{i=1}^{3}\mathfrak{U}(g_i) \leq \frac{1}{4}+2\cdot\frac{3}{8}=1$, a contradiction. So S is of type (6,6,6). Suppose $d \geq 2$. Then by (7.9)(b), $\sum_{i=1}^{3}\mathfrak{U}(g_i) \leq 3\cdot\frac{37}{128}<1$, a contradiction. So $d=1$.

(7.54) Suppose $d=1$. Then S is not of type (6,6,6).

Proof. Suppose that S is of type (6,6,6). Denote $\alpha_i = dim\{C_V(g_i^3)\}$, $\beta_i = dim\{C_V(g_i^2)\}$ for $1 \leq i \leq 3$. Then $\sum_{i=1}^{3}\left\{\frac{1}{6}+\frac{1}{6}\cdot\frac{2^{\alpha_i}-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\beta_i}-1}{2^n-1}+\frac{2}{6}\cdot\frac{2^{\nu_i}-1}{2^n-1}\right\}=1+\frac{2}{2^n-1}$. This identity can be transformed to $2(2^{\nu_1}+2^{\nu_2}+2^{\nu_3})+(2^{\alpha_1}+2^{\alpha_2}+2^{\alpha_3})+2(2^{\beta_1}+2^{\beta_2}+2^{\beta_3})=3\cdot 2^n+24$. Suppose $\nu_i \geq n-3$. Then by (3.3)(c), $n \leq 6(n-\nu_i) \leq 18$ which contradicting to our assumption $n \geq 33$. So $\nu_i \leq n-4$, $\forall i$. Let x be the number terms among α_i, β_i+1, $1 \leq i \leq 3$, which are equal to $n-1$. Then $3\cdot 2^n+24 \leq 3\cdot 2^{n-3}+x\cdot 2^{n-1}+(6-x)2^{n-2}=3\cdot 2^{n-1}+3\cdot 2^{n-3}+x\cdot 2^{n-2}$, which implies $4\cdot 2^{n-2}+2^{n-3}+24 \leq x\cdot 2^{n-2}$. Hence $x \geq 5$. Then there exists an i such that $\alpha_i=n-1$ and $\beta_i=n-2$. But as $C_V(g_i^3)\cap C_V(g_i^2)=C_V(g_i)$, this implies $\nu_i \geq \alpha_i+\beta_i-n=n-3$, which we have seen is impossible. So S is not of type (6,6,6).

Section 8. The Cases: $3 \leq q \leq 13$.

The result of this section is the following:

Proposition: *Suppose $\overline{G}$ is a group of genus zero. Then (a) $n \leq 12$, if $q=3$. (b) $n \leq 8$, if $4 \leq q \leq 13$.*

Proof. This follows from (8.30) to (8.35).

(8.1) Suppose $q \geq 3$, $n \geq 9$, and $2 \leq d \leq 4$. Then $\mathcal{N}(g) \leq \frac{1}{q^5}$, unless g has an eigenspace E for some $\lambda \in GF(q)^{\#}$ and with $dim\{E\} \geq n-2$.

Proof. Let ν be the maximal dimension of eigenspaces of g. Suppose $\nu \leq n-3$. If also $3 \leq \nu$, or if g is semisimple and V has a direct summand of dimension ν consisting of homogeneous components and with $3 \leq \nu \leq n-3$, then by (6.5), (6.3) and (5.5), $\mathcal{N}(g) \leq \left\{ \left[{n-3 \atop d} \right]_q + \left[{3 \atop 1} \right]_q \left[{n-3 \atop d-1} \right]_q + \left[{3 \atop 2} \right]_q \left[{n-3 \atop d-2} \right]_q + \left[{n-3 \atop d-3} \right]_q \right\} \Big/ \left[{n \atop d} \right]_q \leq \frac{1}{q^{3d}} + \frac{1}{q^{n+d-5}} + \frac{1}{q^{2n-d-6}} + \frac{1}{q^{3(n-d)}} \leq \frac{1}{q^6} + \frac{1}{q^6} + \frac{1}{q^8} + \frac{1}{q^{15}} \leq \frac{1}{q^5}$. So $\nu=0$, 1, or 2. First consider that g is unipotent, then $\nu \neq 0$; and by (6.11), as $d(n-d)$ and $d(n-d-1)$ are both ≥ 5, we have $\mathcal{N}(g) \leq \frac{1}{q^5}$. Now consider that g is semisimple. Suppose $min(g)$ is not irreducible. Then g has a homogeneous component of dimension $n-2$ or $n-1$ and g is of type $a^{\alpha}2^1$, $a^{\alpha}1^2$, $a^{\alpha}1_1^1 1_2^1$, or $a^{\alpha}1^1$, with $a \geq 2$. As $n \geq 9$, we have $\alpha \geq 4$. Suppose g is of type $a^{\alpha}1^1$. If $d \not\equiv 0$ and 1 modulo a, then $f(g, \Lambda_d)=0$. If $d=a\beta$, then $\mathcal{N}(g) = \left[{\alpha \atop \beta} \right]_{q^a} \Big/ \left[{a\alpha+1 \atop a\beta} \right]_q \leq \frac{1}{q^{a\beta(a-1)(\alpha-\beta)}}$. Since $(a,\beta)=(2,1)$, $(3,1)$, $(2,2)$, or $(4,1)$, we have $a\beta(a-1)(\alpha-\beta) \geq 5$ always. So $\mathcal{N}(g) \leq \frac{1}{q^5}$. If $d=a\beta+1$, then $\mathcal{N}(g) = \left[{\alpha \atop \beta} \right]_{q^a} \Big/ \left[{a\alpha+1 \atop a\beta+1} \right]_q \leq \frac{1}{q^{a\beta(a-1)(\alpha-\beta)+a(\alpha-2\beta)}}$. Since $(a,\beta)=(2,1)$, or $(3,1)$, we have $a\beta(a-1)(\alpha-\beta)+a(\alpha-2\beta) \geq 5$ always. So $\mathcal{N}(g) \leq \frac{1}{q^5}$. It is similar for g of type $a^{\alpha}2^1$, $a^{\alpha}1^2$, $a^{\alpha}1_1^1 1_2^1$. Now suppose that $min(g)$ is irreducible. So $n=a\alpha$ with $a \geq 2$ and either $f(g,\Lambda_d)=0$, or $d=a\beta$ and thus $\mathcal{N}(g) = \left[{\alpha \atop \beta} \right]_{q^a} \Big/ \left[{a\alpha \atop a\beta} \right]_q \leq \frac{1}{q^{a\beta\{(a-1)(\alpha-\beta)-\}}}$. Since for $(a,\beta)=(2,1)$, $(3,1)$, $(2,2)$, or $(4,1)$, we have respectively that $a\beta\{(a-1)(\alpha-\beta)-1\} \geq 6$, 9, 8, or 20. So $\mathcal{N}(g) \leq \frac{1}{q^5}$. Therefore the claim is true for g unipotent or semisimple. Now suppose g is neither unipotent nor semisimple. So $|g|=p^e s$ with $(p,s)=1$, and $e \geq 1$, $s>1$. Let ν_1, ν_2 be the maximal dimensions of eigenspaces of g^s and

g^{p^e} respectively. If both ν_1, ν_2 are $\geq n-2$, then $\nu \geq n-4 \geq 5$ and as we have seen above, $\mathcal{N}(g) \leq \frac{1}{q^5}$. If one of ν_1, ν_2 is $\leq n-3$, say for example ν_1, then $\mathcal{N}(g) \leq \mathcal{N}(g^s) \leq \frac{1}{q^5}$. Therefore $\mathcal{N}(g) \leq \frac{1}{q^5}$, unless $\nu \geq n-2$.

(8.2) Suppose $q \geq 3$, $n \geq 9$, and $2 \leq d \leq 4$. Then $\mathcal{N}(g) \leq \frac{2}{q^4}$, unless g has an eigenspace of dimension E corresponding to some $\lambda \in GF(q)^{\#}$ and with $dim\{E\} \geq n-1$.

Proof. Let ν be the maximal dimension of eigenspaces of g. Suppose $\nu \leq n-2$. If $2 \leq \nu$, or if g is semisimple and V has a direct summand of dimension ν consisting of homogeneous components and with $2 \leq \nu \leq n-2$, then by (5.5), (5.3) and (5.5), $\mathcal{N}(g) \leq \left\{ \left[{n-2 \atop d} \right]_q + \left[{2 \atop 1} \right]_q \left[{n-2 \atop d-1} \right]_q + \left[{n-2 \atop d-2} \right]_q \right\} / \left[{n \atop d} \right]_q \leq \frac{1}{q^{2d}} + \frac{q+1}{q^{n-1}} + \frac{1}{q^{2(n-d)}} \leq \frac{1}{q^4} + \frac{1}{q^7} + \frac{1}{q^8} + \frac{1}{q^{10}} \leq \frac{2}{q^4}$. So $\nu=0$, or 1. First consider that g is unipotent, then $\nu \neq 0$; and by (6.11), as $d(n-d) \geq 5$, we have $\mathcal{N}(g) \leq \frac{2}{q^4}$. Now consider that g is semisimple. Suppose $min(g)$ is not irreducible. Then g is of type $a^{\alpha}1^1$, with $a \geq 2$. Exactly the same as in (8.1), for this case and the case that $min(g)$ irreducible, we have $\mathcal{N}(g) \leq \frac{1}{q^5} \leq \frac{2}{q^4}$. That is the claim is true for g unipotent or semisimple. The case that g is neither unipotent nor semisimple is also exactly the same as in (8.1). Therefore $\mathcal{N}(g) \leq \frac{2}{q^4}$, unless $\nu \geq n-1$.

(8.3) If g acts as a scalar $\lambda \in GF(q)^{\#}$ on a subspace W of V, and α is the order of g on V/W, then the order of g on V divides $l.c.m.(\alpha, |\lambda|)$ unless g has the eigenvalue λ on V/W. If g acts on W and g acts as a scalar $\lambda \in GF(q)^{\#}$ on V/W, and α is the order of g on W, then the order of g on V divides $l.c.m.(\alpha, |\lambda|)$ unless g has the eigenvalue λ on W. In particular, if g has an eigenspace of dimension $n-1$ coresponding to $\lambda \in GF(q)^{\#}$, then either $\lambda^{-1}g$ is a transvection or g is a pseudo-reflection; and thus $|g|\,|\,p(q-1)$.

Proof. Suppose g acts as a scalar $\lambda \in GF(q)^{\#}$ on a subspace W of V. Then there exists a basis X of V so that $M = M_X(g) = \begin{bmatrix} A & B \\ 0 & \lambda I \end{bmatrix}$. Denote $\beta = l.c.m.(\alpha, |\lambda|)$. We have that $M^i = \begin{bmatrix} A^i & CB \\ 0 & \lambda^i I \end{bmatrix}$, where $C = A^{i-1} + \lambda A^{i-2} + \cdots + \lambda^{i-2}A + \lambda^{i-1}I$, $\forall \ i \geq 1$. Since $(A-\lambda I)(A^{\beta-1} + \lambda A^{\beta-2} + \cdots + \lambda^{\beta-2}A + \lambda^{\beta-1}I) = A^{\beta} - \lambda^{\beta}I = 0$, if λ is not an eigenvalue of g on V/W, then $A - \lambda I$ is invertible, and thus $A^{\beta-1}+\lambda A^{\beta-2}+\cdots+\lambda^{\beta-2}A+\lambda^{\beta-1}I=0$, which implies that $M^{\beta}=I$. So the order of g on V divides $l.c.m.(\alpha, |\lambda|)$. The second part is similar. The last statement is evident from the first part.

(8.4) Let S be the set of all possible orders of the elements in $GL_2(q)$.

(a) Let A, B be the sets of all divisors of $q-1$ and q^2-1 respectively. Let $C=\{s : s=px$ with $x\in A\}$. Then $S=B\cup C$.

(b) Let $g\in GL_n(q)$ with $n\geq 3$. Denote ν the largest dimension of the eigenspaces of g. Let T be the set of all possible orders of those g's with $\nu\geq n-2$. Then $T=S$ if p is odd. If $p=2$, then $T=S\cup R$, where R is the set of all divisors of $4(q-1)$; and in this case, $T\neq S$.

(c) Let $\overline{g}\in PGL_n(q)$ with $n\geq 3$ with a preimage $g\in GL_n(q)$. Suppose g has an eigenspace of dimension $\nu\geq n-2$. Let $\overline{T}$ be the set of all possible orders of such $\overline{g}$'s. Then $\overline{T}=S$ for p odd; and $\overline{T}=S\cup\{4\}$ for $p=2$. Also the set of all possible order of $|\overline{g}|$ with $\nu\geq n-1$ is $\{p\}\cup A$, where A is the same set as in (a).

Proof. Part (a) is clear. Let E be the eigenspace of g corresponding to $\lambda\in GF(q)^{\#}$ with $dim\{E\}=\nu\geq n-2$. If $\nu=n-1$, then by (8.3), either $\lambda^{-1}g$ is a transvection or g is a pseudo-reflection. So $|g|\in S$. Suppose $\nu=n-2$. If g has no eigenvalue λ on V/E, then by (8.3), $|g|$ divides $\alpha\beta$, where $\alpha\in S$ and $\beta=\frac{|\lambda|}{(\alpha,|\lambda|)}$. Since β divides $q-1$, β is the order of some element in $Z(GL_2(q))$. Thus we have $\alpha\beta\in$S, which gives $|g|\in S$. If g has the eigenvalue λ on V/E, then there exist D with $E<D<V$, $\lambda^{-1}g$ acts as a transvection on D, and g acts as a scalar δ on V/D. If $\delta\neq\lambda$, then by (8.3), $|g|$ divides $\alpha\gamma$, where α is the order of g on D and $\gamma=\frac{|\delta|}{(\alpha,|\delta|)}$. Again as $\alpha\in S$ and $\gamma|(q-1)$, we have $|g|\in S$. If $\delta=\lambda$, then there exists basis X so that g has the matrix representation $M=M_X(g)=\begin{bmatrix} A & B \\ 0 & \lambda I_{n-2}\end{bmatrix}$, where $A=\begin{bmatrix}\lambda & b\\ 0 & \lambda\end{bmatrix}$. We have $A^i=\begin{bmatrix}\lambda^i & i\lambda^{i-1}b\\ 0 & \lambda^i\end{bmatrix}$ $\forall$ i. So $|A||(q-1)p$. Also $M^s=\begin{bmatrix}A^s & CB\\ 0 & \lambda^s I\end{bmatrix}$, where $C=A^{s-1}+\lambda A^{s-2}+\cdots+\lambda^{s-2}A+\lambda^{s-1}I=$ $\begin{bmatrix} s\lambda^{s-1} & \lambda^{s-2}bx \\ 0 & s\lambda^{s-1}\end{bmatrix}$ with $x=\sum_{i=0}^{s-1} i=\frac{1}{2}(s-1)s$. Now for p odd, take $s=(q-1)p\in S$. Then as s

and $\frac{1}{2}(s-1)s$ are both $\equiv 0(mod\ p)$, we have $C=0$, which gives $|g||s$. So $|g|\in S$. That is for p odd, we always have $|g|\in S$. If $p=2$, take $s=4(q-1)$. Then $\frac{1}{2}(s-1)s=2(s-1)(q-1)\equiv 0(mod\ 2)$, which gives $C=0$. Hence $|g||4(q-1)$. So $T\subseteq S\cup R$. On the other hand, since $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has order 4, it is clear that $R\subseteq T$. Hence $T=S\cup R$. Since $4\notin S$, we have $S\neq T$. So (b) holds. For part (c), clearly we have $\overline{T}\subseteq T$. To see that $S\subseteq\overline{T}$, note that since $n\geq 3$, for g such that $M_X(g)=\begin{bmatrix} A & 0 \\ 0 & I_{n-2} \end{bmatrix}$ we have $|\overline{g}|=|g|$. For p odd, by (b), as $T=S$, we have $\overline{T}=S$. For $p=2$, from the proof of (b), we can see that $|g|\in S$, which gives $|\overline{g}|\in S$, unless $M=M_X(g)=\begin{bmatrix} A & B \\ 0 & \lambda I_{n-2} \end{bmatrix}$, with $A=\begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix}$. Also we can see from above that in this case $M^4=\lambda^4 I$, which implies $|\overline{g}||4$. Thus it is clear that $\overline{T}=S\cup\{4\}$.

(8.5) Let $1\neq g\in GL_n(q)$, where $n\geq 3$. Suppose g has an eigenspace of dimension $\nu\geq n-2$. If $q=3$, then $|g|\in\{2, 3, 4, 6, 8\}$. If q=4, then $|g|\in\{2, 3, 4, 5, 6, 12, 15\}$. If $q=5$, then $|g|\in\{2, 3, 4, 5, 6, 8, 10, 12, 20, 24\}$. If $q=7$, then $|g|\in\{2, 3, 4, 6, 7, 8, 12, 14, 16, 21, 24, 42, 48\}$.

Proof. Using the notation in (8.4), for example, when $q=4$, $S=\{1, 2, 3, 5, 6, 15\}$. Since the divisors of $4(q-1)=12$ are R$=\{1, 2, 3, 4, 6, 12\}$. Hence $T=S\cup R=\{1, 2, 3, 4, 5, 6, 12, 15\}$.

(8.6) Let $\overline{1}\neq\overline{g}\in PGL_n(q)$, where $n\geq 3$. Suppose a preimage g of $\overline{g}$ has an eigenspace of dimension $\nu\geq n-2$. If $q=3$, then $|\overline{g}|\in\{2, 3, 4, 6, 8\}$. If $q=4$, then $|\overline{g}|\in\{2, 3, 4, 5, 6, 15\}$. If $q=5$, then $|\overline{g}|\in\{2, 3, 4, 5, 6, 8, 10, 12, 20, 24\}$. If $q=7$, then $|\overline{g}|\in\{2, 3, 4, 6, 7, 8, 12, 14, 16, 21, 24, 42, 48\}$.

Proof. Using the notation in (8.4), for example, when $q=4$, $S=\{1, 2, 3, 5, 6, 15\}$. Hence $\overline{T}=S\cup\{4\}=\{1, 2, 3, 4, 5, 6, 15\}$.

(8.7) Let $\nu=\nu(g)$ denote the maximal dimension of eigenspaces of g. Suppose for all $g\neq 1$, $\mathcal{N}(g)\leq a$; for all g with $\nu=n-2$, $\mathcal{N}(g)\leq b\leq a$; and for all g with $\nu\leq n-3$, $\mathcal{N}(g)\leq c\leq b$. Then $\mathfrak{U}(g)\leq\{1+(p+q-3)a+\{q(q-1)+(p-1)(q-2)+x\}b\}\frac{1}{|\overline{g}|}+c$, where $x=0$ if p odd, and $x=2$ if

$p=2$.

Proof. We have $\mathcal{U}(g)=\mathcal{U}(\overline{g})=\frac{1}{|\overline{g}|}\sum_{d||\overline{g}|}\phi(d)\mathcal{N}(\overline{g}^{\frac{|\overline{g}|}{d}})$. We break the summation into parts according to whether $d=1$, $1\neq d\in\{p\}\cup A=D$, $d\in\overline{T}-D$, $d\notin\overline{T}$, where $\overline{T}$ has the same meaning as in (8.4)(c), and A is the set of all divisors of $q-1$. So if $d\in\overline{T}-D$, then $\nu(g^{\frac{|g|}{d}})\leq n-2$; which gives $\mathcal{N}(\overline{g}^{\frac{|\overline{g}|}{d}})\leq b$. If $d\notin\overline{T}$, then $\nu(g^{\frac{|g|}{d}})\leq n-3$; which gives $\mathcal{N}(\overline{g}^{\frac{|\overline{g}|}{d}})\leq c$. Since $\sum_{1\neq d\in D}\phi(d)\leq (p-1)+(q-1)-1=p+q-3$, the part of the summation over d with $1\neq d\in D$ is $\leq(p+q-3)a$. Since $\sum_{d\in\overline{T}-D}\phi(d)=(q^2-1)+(p-1)(q-1)+x-(p+q-2)=q(q-1)+(p-1)(q-2)+x$, where $x=0$ if p odd; and $x=\phi(4)=2$ if $p=2$, we have that the part of the summation over $d\in\overline{T}-D$ is $\leq\{q(q-1)+(p-1)(q-2)+x\}b$, where $x=0$, 2 for p odd, even respectively. The rest is clear.

(8.8) Suppose $n\geq 9$, $d\geq 2$ and $q=3$. Then $\mathcal{U}(g)\leq\frac{1117}{729}\cdot\frac{1}{|\overline{g}|}+\frac{1}{3^5}$.

Proof. In any case, we have $\mathcal{N}(g)\leq\frac{1}{q^d}+\frac{1}{q^{n-d}}\leq\frac{1}{q^2}+\frac{1}{q^7}$. Also $\mathcal{N}(g)\leq\frac{2}{q^4}$ except when $|\overline{g}|=1,2,3$; and $\mathcal{N}(g)\leq\frac{1}{q^5}$ except when $|\overline{g}|=1,2,3,4,6,8$. Hence $\mathcal{U}(g)\leq\frac{1}{|\overline{g}|}\{1+(\phi(2)+\phi(3))\cdot(\frac{1}{3^2}+\frac{1}{3^7})+(\phi(4)+\phi(6)+\phi(8))\cdot\frac{2}{3^4}+\frac{1}{3^5}\cdot\sum_{d||g|,d>4,d\neq 6,8}\phi(d)\}\leq\frac{1}{|\overline{g}|}(1+\frac{244}{3^6}+\frac{16}{3^4})+\frac{1}{3^5}=\frac{1117}{729}\cdot\frac{1}{|\overline{g}|}+\frac{1}{3^5}$.

(8.9) Suppose $q\geq 4$, $n\geq 9$ and $d\geq 2$. Then $\mathcal{U}(\overline{g})\leq\frac{225}{128}\cdot\frac{1}{|\overline{g}|}+\frac{1}{4^5}$.

Proof. Using the notation in (8.7). Since we have $a\leq\frac{1}{q^d}+\frac{1}{q^{n-d}}$ and $(p+q-3)\leq 2q$, and thus $(p+q-3)a\leq\frac{2}{q^{d-1}}+\frac{2}{q^{n-d-1}}\leq\frac{2}{4}+\frac{2}{4^4}$. Similarly, as $\{q(q-1)+(p-1)(q-2)+x\}\leq 2q^2$ and $b\leq\frac{2}{q^4}$ for $d\geq 2$; we have $\{q(q-1)+(p-1)(q-2)+x\}b\leq\frac{4}{q^2}\leq\frac{4}{4^2}$. Also $c=\frac{1}{4^5}$ for $d\geq 2$. Hence $\mathcal{U}(\overline{g})\leq(1+\frac{2}{4}+\frac{2}{256}+\frac{4}{16})\frac{1}{|\overline{g}|}+\frac{1}{4^5}=\frac{225}{128}\cdot\frac{1}{|\overline{g}|}+\frac{1}{4^5}$.

(8.10) Suppose $n\geq 9$, $d=1$. If $q=3$, then $\mathcal{U}(\overline{g})\leq\frac{460}{243}\cdot\frac{1}{|\overline{g}|}+(\frac{1}{3^3}+\frac{1}{3^6})$. If $q=4$, then we have both $\mathcal{U}(\overline{g})\leq(\frac{7}{4}+\frac{3}{4^8})\frac{1}{|\overline{g}|}+(\frac{1}{4^2}+\frac{1}{4^7})$ and $\mathcal{U}(\overline{g})\leq\frac{196691}{65536}\cdot\frac{1}{|\overline{g}|}+(\frac{1}{4^3}+\frac{1}{4^6})$. If $q=5$, then $\mathcal{U}(\overline{g})\leq$

$(\frac{12}{5}+\frac{7}{5^8})\frac{1}{|\bar{g}|}+(\frac{1}{5^2}+\frac{1}{5^7})$. If $3\le q\le 13$, then $\mathcal{U}(\bar{g})\le\frac{37}{13}\cdot\frac{1}{|\bar{g}|}+(\frac{1}{q^2}+\frac{1}{q^7})$.

Proof. For example, we consider the case that $q=3$. In any case, we have $\mathcal{N}(g)\le\frac{1}{q^\nu}+\frac{1}{q^{n-\nu}}\le\frac{1}{q^2}+\frac{1}{q^{n-2}}$ except when $|\bar{g}|=1,2,3$; and $\mathcal{N}(g)\le\frac{1}{q^3}+\frac{1}{q^{n-3}}$ except when $|\bar{g}|=1,2,3,4,6,8$. Hence $\mathcal{U}(\bar{g})\le\frac{1}{|\bar{g}|}\{1+(\phi(2)+\phi(3))\cdot(\frac{1}{3}+\frac{1}{3^8})+(\phi(4)+\phi(6)+\phi(8))\cdot(\frac{1}{3^2}+\frac{1}{3^7})+(\frac{1}{3^3}+\frac{1}{3^6})\cdot\sum_{d\,||\,g|,d>4,d\ne 6,8}\phi(d)\}$ $\le\frac{1}{|\bar{g}|}\{1+(1+\frac{1}{3^7})+8(\frac{1}{3^2}+\frac{1}{3^7})\}+(\frac{1}{3^3}+\frac{1}{3^6})=\frac{460}{243}\cdot\frac{1}{|\bar{g}|}+(\frac{1}{3^3}+\frac{1}{3^6})$. It is similar for $q=4$. Suppose $3\le q\le 13$. Since the preimages of $\bar{g}^{\frac{|\bar{g}|}{d}}$ have an eigenspce of dimension $=n-1$ iff $1\ne d\in\{p\}\cup A$, where A is the set of all divisors of $q-1$. Thus similar to (8.9), we have $\mathcal{U}(\bar{g})\le\{1+(p+q-3)(\frac{1}{q}+\frac{1}{q^{n-1}})\}\frac{1}{|\bar{g}|}+(\frac{1}{q^2}+\frac{1}{q^{n-2}})$. As for $3\le q\le 13$, $(p+q-3)(\frac{1}{q}+\frac{1}{q^{n-1}})\le\frac{24}{13}$, we have the last conclusion.

(8.11) Denote $\nu(\bar{g})$ the maximal dimension of eigenspaces of the preimages of $\bar{g}$. Suppose $q\ge 3$, $n\ge 9$.

(a) Suppose $d\ge 2$. Then $\mathcal{U}(\bar{g})\le\frac{2431}{4374}$, $\frac{2675}{6561}$, $\frac{47}{162}+\frac{1}{4\cdot 3^7}$, $\frac{1}{5}+\frac{2}{5\cdot 4^3}$, $\frac{56}{243}+\frac{1}{2\cdot 3^7}$, $\frac{55}{343}+\frac{6}{7^8}$, $\frac{17}{108}+\frac{1}{8\cdot 3^7}$, $\frac{11}{81}+\frac{2}{3^9}+\frac{2}{3^6}$, $\frac{3}{25}+\frac{4}{5^5}+\frac{1}{2\cdot 5^7}$ for $|\bar{g}|=2, 3, 4, 5, 6, 7, 8, 9, 10$ respectively.

(b) Suppose $d\ge 2$ and $\nu(\bar{g})\le n-2$. Then $\mathcal{U}(\bar{g})\le\frac{83}{162}$, $\frac{85}{243}$, $\frac{5}{18}+\frac{1}{3^4}+\frac{1}{4\cdot 3^7}$, $\frac{1}{5}+\frac{2}{5\cdot 4^3}$ for $|\bar{g}|=2, 3, 4, 5$ respectively.

(c) Suppose $d\ge 2$ and $\nu(\bar{g})\le n-3$. Then $\mathcal{U}(\bar{g})\le\frac{1}{2}+\frac{1}{2\cdot 3^5}$, $\frac{1}{3}+\frac{2}{3^6}$ for $|\bar{g}|=2, 3$ respectively.

(d) Suppose $d=1$. For $|\bar{g}|=2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$ we have respectively $\mathcal{U}(\bar{g})\le\frac{8749}{13122}$, $\frac{10937}{19683}$, $\frac{2}{5}+\frac{3}{4\cdot 5^8}$, $\frac{9}{25}+\frac{4}{5^9}$, $\frac{10}{27}+\frac{1}{2\cdot 3^7}$, $\frac{13}{49}+\frac{6}{7^9}$, $\frac{1}{4}+\frac{13}{8\cdot 3^8}$, $\frac{17}{81}+\frac{2}{3^{10}}+\frac{2}{3^7}$, $\frac{27}{125}+\frac{1}{2\cdot 5^7}$, $\frac{21}{121}+\frac{10}{11^9}$, $\frac{2}{13}+\frac{11}{12\cdot 13^8}$, $\frac{25}{169}+\frac{12}{13^9}$, $\frac{52}{343}+\frac{1}{2\cdot 7^7}$. For $|\bar{g}|=2$, we have $\mathcal{U}(\bar{g})\le\frac{5}{8}+\frac{1}{2\cdot 4^8}$, $\frac{3}{5}+\frac{1}{2\cdot 5^8}$, $\frac{4}{7}+\frac{1}{2\cdot 7^8}$ for $q\ge 4$, 5, 7 respectively. For $|\bar{g}|=3$, we have $\mathcal{U}(\bar{g})\le\frac{1}{2}+\frac{2}{3\cdot 4^8}$, $\frac{7}{15}+\frac{2}{3\cdot 5^8}$ for $q\ge 4$, 5 respectively.

(e) Suppose $d=1$ and $\nu(\bar{g})\le n-2$. Then $\mathcal{U}(\bar{g})\le\frac{5}{9}+\frac{1}{2\cdot 3^7}$, $\frac{11}{27}+\frac{2}{3^8}$, $\frac{7}{18}+\frac{7}{4\cdot 3^8}$, $\frac{1}{4}+\frac{1}{5\cdot 4^6}$ for $|\bar{g}|=2, 3, 4, 5$ respectively. For $|\bar{g}|=2$, we have $\mathcal{U}(\bar{g})\le\frac{17}{32}+\frac{1}{2\cdot 4^7}$, $\frac{13}{25}+\frac{1}{2\cdot 5^7}$, $\frac{25}{49}+\frac{1}{2\cdot 7^7}$ for $q\ge 4$, 5, 7 respectively. For $|\bar{g}|=3$, we have $\mathcal{U}(\bar{g})\leq\frac{3}{8}+\frac{2}{3\cdot 4^7}$, $\frac{9}{25}+\frac{2}{3\cdot 5^7}$ for $q\geq 4$, 5 respectively. For $|\bar{g}|=4$, we have $\mathcal{U}(\bar{g})\leq\frac{11}{32}+\frac{9}{4^9}$ if $q\geq 4$.

(f) Suppose $d=1$ and $\nu(\bar{g})\leq n-3$. Then $\mathcal{U}(\bar{g})\leq\frac{14}{27}+\frac{1}{2\cdot 3^6}$, $\frac{29}{81}+\frac{2}{3^7}$, $\frac{41}{128}+\frac{33}{4^9}$, $\frac{31}{135}+\frac{4}{5\cdot 3^6}$, $\frac{25}{81}+\frac{23}{2\cdot 3^9}$, $\frac{11}{63}+\frac{2}{7\cdot 3^5}$, $\frac{7}{36}+\frac{43}{8\cdot 3^8}$, for $|\bar{g}|=2$, 3, 4, 5, 6, 7, 8 respectively. For $|\bar{g}|=2$, we have $\mathcal{U}(\bar{g})\leq\frac{65}{128}+\frac{1}{2\cdot 4^6}$, $\frac{63}{125}+\frac{1}{2\cdot 5^6}$ for $q\geq 4$, 5 respectively. For $|\bar{g}|=3$, we have $\mathcal{U}(\bar{g})\leq\frac{11}{32}+\frac{2}{3\cdot 4^6}$, $\frac{127}{375}+\frac{2}{3\cdot 5^6}$ for $q\geq 4$, 5 respectively.

Proof. We only show several examples, since for the cases which are not shown here the calculations are similar. Suppose $d\geq 2$ first. If $|\bar{g}|=2$, then $\mathcal{U}(\bar{g})\leq\frac{1}{2}\{1+(\frac{1}{q^d}+\frac{1}{q^{n-d}})\}\leq\frac{1}{2}\{1+(\frac{1}{3^2}+\frac{1}{3^{n-2}})\}\leq\frac{2431}{4374}$. Suppose $|\bar{g}|=4$. If $\nu(\bar{g})\leq n-2$, then $\mathcal{U}(\bar{g})\leq\frac{1}{4}(1+(\frac{1}{3^2}+\frac{1}{3^7})+2\cdot\frac{2}{3^4}))=\frac{47}{162}+\frac{1}{4\cdot 3^7}$. If $\nu(\mathrm{g})=n-1$, then $4|(q-1)$. So $q\geq 5$ and thus $\mathcal{U}(\bar{g})\leq\frac{1}{4}(1+3\cdot(\frac{1}{5^2}+\frac{1}{5^7}))=\frac{7}{25}+\frac{3}{4\cdot 5^7}\leq\frac{47}{162}+\frac{1}{4\cdot 3^7}$.

Now suppose $d=1$ in the following. Suppose $|\bar{g}|=4$. If $\nu(\bar{g})=n-1$, then $4|(q-1)$. So $q\geq 5$ and thus $\mathcal{U}(\bar{g})\leq\frac{1}{4}(1+3\cdot(\frac{1}{5}+\frac{1}{5^8}))=\frac{2}{5}+\frac{3}{4\cdot 5^8}$. Otherwise, we have that $\mathcal{U}(\bar{g})\leq\frac{1}{4}(1+(\frac{1}{3}+\frac{1}{3^8})+2(\frac{1}{3^2}+\frac{1}{3^7}))\leq\frac{2}{5}+\frac{3}{4\cdot 5^8}$. Suppose $|\bar{g}|=5$. If $\nu(\bar{g})=n-1$, then as either $p=5$ or $5|(q-1)$, we have $q\geq 5$. Thus $\mathcal{U}(\bar{g})\leq\frac{1}{5}(1+4\cdot(\frac{1}{5}+\frac{1}{5^8}))=\frac{9}{25}+\frac{4}{5^9}$. If $\nu(\bar{g})=n-2$, then by (8.6), $q\geq 4$; and thus $\mathcal{U}(\bar{g})\leq\frac{1}{5}(1+4\cdot(\frac{1}{4^2}+\frac{1}{4^7}))=\frac{1}{4}+\frac{1}{5\cdot 4^6}\leq\frac{9}{25}+\frac{4}{5^9}$. Otherwise, we have that $\mathcal{U}(\bar{g})\leq\frac{1}{5}(1+4\cdot(\frac{1}{3^3}+\frac{1}{3^6}))\leq\frac{9}{25}+\frac{4}{5^9}$. Suppose $|\bar{g}|=6$. If $\nu(\bar{g})\leq n-2$, then $\mathcal{U}(\bar{g})\leq\frac{1}{6}(1+3\cdot(\frac{1}{3}+\frac{1}{3^8})+2\cdot(\frac{1}{3^2}+\frac{1}{3^7}))=\frac{10}{27}+\frac{1}{2\cdot 3^7}$. If $\nu(\bar{g})=n-1$, then as $6|(q-1)$, we have $q\geq 7$. Thus $\mathcal{U}(\bar{g})\leq\frac{1}{6}(1+5\cdot(\frac{1}{7}+\frac{1}{7^8}))=\frac{2}{7}+\frac{5}{6\cdot 7^8}\leq\frac{10}{27}+\frac{1}{2\cdot 3^7}$. Suppose $|\bar{g}|=5$ and $\nu(\bar{g})\leq n-2$. So $\mathcal{N}(g)\leq\frac{1}{4^2}+\frac{1}{4^7}$ if $q\geq 4$. By (8.6), for $q=3$, $\nu(\bar{g})\leq n-3$, which gives $\mathcal{N}(g)\leq\frac{1}{3^3}+\frac{1}{3^6}\leq\frac{1}{4^2}+\frac{1}{4^7}$. Thus $\mathcal{U}(\bar{g})\leq\frac{1}{5}(1+4\cdot(\frac{1}{4^2}+\frac{1}{4^7}))=\frac{1}{4}+\frac{1}{5\cdot 4^6}$. Suppose $|\bar{g}|=4$ and $\nu(\bar{g})\leq n-3$. If $q=3$, then $\nu(\bar{g}^2)\leq n-2$. So $\mathcal{N}(\bar{g}^2)+2\mathcal{N}(\bar{g})\leq(\frac{1}{3^2}+\frac{1}{3^7})+2(\frac{1}{3^3}+\frac{1}{3^6})\leq(\frac{1}{4}+\frac{1}{4^8})+2(\frac{1}{4^3}+\frac{1}{4^6})$, which gives that in any case we have $\mathcal{U}(\bar{g})\leq\frac{1}{4}(1+(\frac{1}{4}+\frac{1}{4^8})+2(\frac{1}{4^3}+\frac{1}{4^6}))=\frac{41}{128}+\frac{33}{4^9}$.

In the following, unless mentioned otherwise, we denote $\nu_i=\nu(\bar{g}_i)$ the maximal dimension of eigenspaces of the preimages of $\bar{g}_i$. Also we assume in the rest of this section that for $q=3$, $n\geq 13$, and for $4\leq q\leq 13$, $n\geq 9$.

(8.12) $|S|\leq 4$.

Proof. Suppose $d\geq 2$ first. Then $\mathcal{U}(\bar{g})\leq max\{\frac{2431}{4374}, \frac{2675}{6561}, \frac{1117}{729}\cdot\frac{1}{4}+\frac{1}{3^5}, \frac{225}{128}\cdot\frac{1}{4}+\frac{1}{4^5}\}=\frac{2431}{4374}\leq\frac{3}{5}$, which implies that $|S|\leq 4$ by (2.4)(b).

Now suppose $d=1$. For $|\bar{g}|\geq 7$, by (8.10), we have $\mathcal{U}(\bar{g})\leq max\{\frac{460}{243}\cdot\frac{1}{7}+(\frac{1}{3^3}+\frac{1}{3^6})$, $\frac{196691}{65536}\cdot\frac{1}{7}+(\frac{1}{4^3}+\frac{1}{4^6})$, $\frac{37}{13}\cdot\frac{1}{7}+(\frac{1}{5^2}+\frac{1}{5^7})\}=\frac{37}{13}\cdot\frac{1}{7}+(\frac{1}{5^2}+\frac{1}{5^7})$. Hence for $|\bar{g}|\geq 3$, we have $\mathcal{U}(\bar{g})\leq max\{\frac{10937}{19683}, \frac{2}{5}+\frac{3}{4\cdot 5^8}, \frac{9}{25}+\frac{4}{5^9}, \frac{10}{27}+\frac{1}{2\cdot 3^7}, \frac{37}{13}\cdot\frac{1}{7}+(\frac{1}{5^2}+\frac{1}{5^7})\}=\frac{10937}{19683}<\frac{8749}{13122}$; that is for any $\bar{g}\neq\bar{1}$, we have $\mathcal{U}(\bar{g})\leq\frac{8749}{13122}$; which gives $r-2<\sum_{i=1}^{r}\mathcal{U}(\bar{g}_i)\leq r\cdot\frac{8749}{13122}$. This implies that $|S|=r\leq 6$.

Suppose $|S|=6$. If not all elements in S are involutions, then $\sum_{i=1}^{6}\mathcal{U}(\bar{g}_i)\leq 5\cdot\frac{8749}{13122}+\frac{10937}{19683}\leq 4$, a contradiction. So all elements in S are involutions. Since $n\geq 9$, not all $\nu(\bar{g}_i)=n-1$. Then $\sum_{i=1}^{6}\mathcal{U}(\bar{g}_i)\leq 5\cdot\frac{8749}{13122}+(\frac{5}{9}+\frac{1}{2\cdot 3^7})\leq 4$. Hence $|S|\leq 5$.

Suppose $|S|=5$. Let α be the number of involutions in S. Then we have $3<\sum_{i=1}^{5}\mathcal{U}(\bar{g}_i)\leq \alpha\cdot\frac{8749}{13122}+(5-\alpha)\frac{10937}{19683}$, which implies that $\alpha\geq 2$.

Suppose $\alpha=5$. There are at most 3 involutions in S which have eigenspaces of dimension $n-1$. So if $q\geq 4$, then $\sum_{i=1}^{5}\mathcal{U}(\bar{g}_i)\leq 3(\frac{5}{8}+\frac{1}{2\cdot 4^8})+2(\frac{17}{32}+\frac{1}{2\cdot 4^7})\leq 3$, a contradiction. So $q=3$. Suppose $n\geq 13$. There are at most two $\bar{g}_i$'s such that $\nu(\bar{g}_i)=n-1$, because any preimage of $\bar{g}_i$ is either of order 2 or 4, hence has an eigenspace of dimension ≥ 4. Suppose that $\nu(\bar{g}_i)=n-1$, say for $i=1$ and 2. Then at least one of $\bar{g}_i$'s, $i=3,4,5$, is such that $\nu(\bar{g}_i)\leq n-3$. In this case, we have $\sum_{i=1}^{5}\mathcal{U}(\bar{g}_i)\leq 2\cdot\frac{8749}{13122}+2(\frac{5}{9}+\frac{1}{2\cdot 3^7})+(\frac{14}{27}+\frac{1}{2\cdot 3^6})\leq 3$. If there is at most one $\bar{g}_i$ such that $\nu(\bar{g}_i)=n-1$, then $\sum_{i=1}^{5}\mathcal{U}(\bar{g}_i)\leq\frac{8749}{13122}+4(\frac{5}{9}+\frac{1}{2\cdot 3^7})\leq 3$, again a contradiction. Thus $\alpha\leq 4$.

Suppose $\alpha=4$. Say $|\bar{g}_i|=2$ for $1\leq i\leq 4$. From the reasoning in the previous paragraph, we see that $\sum_{i=1}^{4}\mathcal{U}(\bar{g}_i)\leq max\{3(\frac{5}{8}+\frac{1}{2\cdot 4^8})+(\frac{17}{32}+\frac{1}{2\cdot 4^7})$, $2\cdot\frac{8749}{13122}+(\frac{5}{9}+\frac{1}{2\cdot 3^7})+(\frac{14}{27}+\frac{1}{2\cdot 3^6})$, $\frac{8749}{13122}+3(\frac{5}{9}+\frac{1}{2\cdot 3^7})\}=2\cdot\frac{8749}{13122}+(\frac{5}{9}+\frac{1}{2\cdot 3^7})+(\frac{14}{27}+\frac{1}{2\cdot 3^6})$. Since $\mathcal{U}(\bar{g}_5)\leq\frac{10937}{19683}$, $\sum_{i=1}^{5}\mathcal{U}(\bar{g}_i)\leq 3$, a contradiction. So $\alpha \leq 3$.

Suppose $\alpha=3$. Say $|\bar{g}_i|=2$ for $1\leq i\leq 3$. If one of these involutions is such that $\nu(\bar{g}_i)\leq n-3$, then $\sum_{i=1}^{5}\mathcal{U}(\bar{g}_i)\leq 2\cdot\frac{8749}{13122}+(\frac{14}{27}+\frac{1}{2\cdot 3^6})+2\cdot\frac{10937}{19683}\leq 3$. So $\nu(\bar{g}_i)\geq n-2\ \forall\ 1\leq i\leq 3$. Then $\nu(\bar{g}_i)\leq n-3$ for $i=4$ and 5. So for $i=4$ and 5, we have $\mathcal{U}(\bar{g}_i)\leq max\{\frac{11}{27}+\frac{2}{3^8}, \frac{2}{5}+\frac{3}{4\cdot 5^8}, \frac{9}{25}+\frac{4}{5^9}, \frac{10}{27}+\frac{1}{2\cdot 3^7}, \frac{37}{13}\cdot\frac{1}{7}+(\frac{1}{5^2}+\frac{1}{5^7})\}=\frac{37}{13}\cdot\frac{1}{7}+(\frac{1}{5^2}+\frac{1}{5^7})$. Hence we have the contradiction $\sum_{i=1}^{5}\mathcal{U}(\bar{g}_i)\leq 3\cdot\frac{8749}{13122}+2\cdot\{\frac{37}{13}\cdot\frac{1}{7}+(\frac{1}{5^2}+\frac{1}{5^7})\}\leq 3$. So $\alpha=2$.

Say $|\bar{g}_i|=2$ for $1\leq i\leq 2$. If $\nu(\bar{g}_i)=n-1$ for both $i=1$ and 2, then at least one of $\bar{g}_i$'s, $i=3,4,5$, is such that $\nu(\bar{g}_i)\leq n-3$. In this case, we have $\sum_{i=1}^{5}\mathcal{U}(\bar{g}_i)\leq 2\cdot\frac{8749}{13122}+2\cdot\frac{10937}{19683}+\{\frac{37}{13}\cdot\frac{1}{7}+(\frac{1}{5^2}+\frac{1}{5^7})\}\leq 3$. If one of $\nu(\bar{g}_i)\leq n-2$, $i=1$, 2, then $\sum_{i=1}^{5}\mathcal{U}(\bar{g}_i)\leq\frac{8749}{13122}+(\frac{5}{9}+\frac{1}{2\cdot 3^7})+3\cdot\frac{10937}{19683}\leq 3$.

Therefore $|S|\leq 4$.

(8.13) Suppose $|S|=4$. Then $d=1$ and S is of the type: (2,2,2,3) with $q=3$ or 4.

Proof. Suppose $d\geq 2$ first. For $|\bar{g}|\geq 3$, we have $\mathcal{U}(\bar{g})\leq max\{\frac{2675}{6561}, \frac{47}{162}+\frac{1}{4\cdot 3^7}, \frac{1117}{729}\cdot\frac{1}{5}+\frac{1}{3^5}, \frac{225}{128}\cdot\frac{1}{5}+\frac{1}{4^5}\}=\frac{2675}{6561}$. If there are at most 2 involutions in S, then $\sum_{i=1}^{4}\mathcal{U}(\bar{g}_i)\leq 2\cdot\frac{2431}{4374}+2\cdot\frac{2675}{6561}\leq 2$. So S has 3 involutions. Say $|\bar{g}_i|=2$ for $1\leq i\leq 3$. If there is at most one $\bar{g}_i$, $1\leq i\leq 3$, with $\nu(\bar{g}_i)=n-1$, then $\sum_{i=1}^{4}\mathcal{U}(\bar{g}_i)\leq\frac{2431}{4374}+2\cdot\frac{83}{162}+\frac{2675}{6561}\leq 2$. So there are two $\bar{g}_i$'s, $1\leq i\leq 3$, with $\nu(\bar{g}_i)=n-1$. Then $\nu(\bar{g}_4)\leq n-2$, which implies that $\mathcal{U}(\bar{g}_4)\leq max\{\frac{85}{243}, \frac{47}{162}+\frac{1}{4\cdot 3^7}, \frac{1117}{729}\cdot\frac{1}{5}+\frac{1}{3^5}, \frac{225}{128}\cdot\frac{1}{5}+\frac{1}{4^5}\}=\frac{225}{128}\cdot\frac{1}{5}+\frac{1}{4^5}$. Then $\sum_{i=1}^{4}\mathcal{U}(\bar{g}_i)\leq 2\cdot\frac{2431}{4374}+\frac{83}{162}+(\frac{225}{128}\cdot\frac{1}{5}+\frac{1}{4^5})\leq 2$. So $d=1$.

Let α be the number of involutions in S. Suppose $\alpha=3$ first. Say $|\bar{g}_i|=2$ for $1\leq i\leq 3$. Since $g_i^2\in Z$, $1\leq i\leq 3$, each g_i has exactly two eigenspaces of dimensions of dimension ν_i and $n-\nu_i$ when p is odd; and $\nu_i=dim\{C_V(g_i)\}\geq\lceil\frac{n}{2}\rceil$ when $p=2$. So $\mathcal{N}(\bar{g}_1)+\mathcal{N}(\bar{g}_2)+\mathcal{N}(\bar{g}_3)\leq (q^{\nu_1}+\cdots+q^{\nu_3}+q^{n-\nu_1}+\cdots+q^{n-\nu_3}-6)/(q^n-1)$. Since $\nu_1+\nu_2+\nu_3\leq 2n$, $\lceil\frac{n}{2}\rceil\leq\nu_i\leq n-1$, we have $q^{\nu_1}+\cdots+q^{\nu_3}\leq q^{n-1}+q^{\frac{n}{2}+1}+q^{\frac{n}{2}}$. Also $q^{n-\nu_1}+\cdots+q^{n-\nu_3}\leq 3q^{\frac{n}{2}}$. Thus $\mathcal{N}(\bar{g}_1)+\mathcal{N}(\bar{g}_2)+\mathcal{N}(\bar{g}_3)\leq$

$\frac{1}{q}+\frac{1}{q^{\frac{n}{2}-1}}+\frac{4}{q^{\frac{n}{2}}}$. This gives $\mathcal{U}(\overline{g}_1)+\mathcal{U}(\overline{g}_2)+\mathcal{U}(\overline{g}_3)\leq\frac{3}{2}+\frac{1}{2}(\frac{1}{q}+\frac{1}{q^{\frac{n}{2}-1}}+\frac{4}{q^{\frac{n}{2}}})$, which is $\leq\frac{5}{3}+\frac{7}{2\cdot 3^6}$ if $q=3$ and $n\geq 13$; is $\leq\frac{13}{8}+\frac{1}{4^4}$ if $q=4$; is $\leq\frac{8}{5}+\frac{9}{2\cdot 5^5}$ if $q\geq 5$. If $q=3$ and $|\overline{g}_4|\geq 7$, then $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq(\frac{5}{3}+\frac{7}{2\cdot 3^6})+(\frac{460}{243}\cdot\frac{1}{7}+(\frac{1}{3^3}+\frac{1}{3^6}))\leq 2$. So in this case $3\leq|\overline{g}_4|\leq 6$. If $q=4$ and $|\overline{g}_4|\geq 5$, then by (8.10), $\mathcal{U}(\overline{g}_4)\leq max\{\frac{9}{25}+\frac{4}{5^9},\ (\frac{7}{4}+\frac{3}{4^8})\frac{1}{6}+(\frac{1}{4^2}+\frac{1}{4^7})\}$, which implies $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq 2$. So for $q=4$, $3\leq|\overline{g}_4|\leq 4$. If $q\geq 5$ and $|\overline{g}_4|\geq 7$, then $\mathcal{U}(\overline{g}_4)\leq max\{\frac{13}{49}+\frac{6}{7^9},\ \frac{37}{13}\cdot\frac{1}{8}+(\frac{1}{5^2}+\frac{1}{5^7})\}$, which implies that $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq 2$. So for $q\geq 5$, $3\leq|\overline{g}_4|\leq 6$. Since $\nu_i\geq\lceil\frac{n}{2}\rceil\ \forall\ 1\leq i\leq 3$, if some ν_i, say $\nu_1\geq n-4$ when $q=3$, $\nu_1\geq n-2$ when $q\geq 4$, then $\nu_4\leq n-3$; which implies that for $4\leq|\overline{g}_4|\leq 6$ we have $\mathcal{U}(\overline{g}_4)\leq max\{\frac{41}{128}+\frac{33}{4^9},\ \frac{31}{135}+\frac{4}{5\cdot 3^6},\ \frac{25}{81}+\frac{23}{2\cdot 3^9}\}=\frac{41}{128}+\frac{33}{4^9}$. Also $max\{\frac{5}{3}+\frac{7}{2\cdot 3^6},\ \frac{13}{8}+\frac{1}{4^4},\ \frac{8}{5}+\frac{9}{2\cdot 5^5}\}=\frac{5}{3}+\frac{7}{2\cdot 3^6}$. This gives $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq(\frac{5}{3}+\frac{7}{2\cdot 3^6})+(\frac{41}{128}+\frac{33}{4^9})\leq 2$; and in this case we also have for $q\geq 5$, and $|\overline{g}_4|=3$, $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq(\frac{8}{5}+\frac{9}{2\cdot 5^5})+(\frac{127}{375}+\frac{2}{3\cdot 5^6})\leq 2$. If $\nu_i\leq n-5$ when $q=3$, $\nu_i\leq n-3$ when $q\geq 4$, $\forall\ 1\leq i\leq 3$; then $\mathcal{U}(\overline{g}_1)+\mathcal{U}(\overline{g}_2)+\mathcal{U}(\overline{g}_3)\leq 3\cdot\frac{1}{2}(1+max\{\frac{1}{4^3}+\frac{1}{4^6},\ \frac{1}{3^5}+\frac{1}{3^8}\})\leq 3\cdot\frac{1}{2}(1+(\frac{1}{4^3}+\frac{1}{4^6}))$. Since for $4\leq|\overline{g}_4|\leq 6$ we have $\mathcal{U}(\overline{g}_4)\leq max\{\frac{2}{5}+\frac{3}{4\cdot 5^8},\ \frac{9}{25}+\frac{4}{5^9},\ \frac{10}{27}+\frac{1}{2\cdot 3^7}\}=\frac{2}{5}+\frac{3}{4\cdot 5^8}$, $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq 2$; and in this case we also have for $q\geq 5$, $|\overline{g}_4|=3$, $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq 3\cdot\frac{1}{2}(1+(\frac{1}{5^3}+\frac{1}{5^6}))+(\frac{7}{15}+\frac{2}{3\cdot 5^8})\leq 2$. Hence $|\overline{g}_4|=3$ and $q=3$ or 4.

Suppose $\alpha=2$. Say $|\overline{g}_i|=2$ for $1\leq i\leq 2$. If one of ν_1, ν_2, say $\nu_1\geq n-4$ when $q=3$, $\nu_1\geq n-2$ when $q\geq 4$, then ν_3 and $\nu_4\leq n-3$; which implies that $\mathcal{U}(\overline{g}_i)\leq\frac{41}{128}+\frac{33}{4^9}$ if $|\overline{g}_i|\geq 4$, $i=3$, 4. Suppose $|\overline{g}_3|\geq 3$ and $|\overline{g}_4|\geq 4$ if $q\geq 4$; and $|\overline{g}_3|$ and $|\overline{g}_4|$ both ≥ 4 if $q=3$. Then $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq 2\cdot\frac{8749}{13122}+(\frac{11}{32}+\frac{2}{3\cdot 4^6})+(\frac{41}{128}+\frac{33}{4^9})\leq 2$. Hence if $q\geq 4$, then $|\overline{g}_3|=|\overline{g}_4|=3$; if $q=3$, then $|\overline{g}_3|=3$. Consider $q\geq 4$ first. Since $\nu_3\geq\lceil\frac{n}{3}\rceil\geq 3$, ν_1 and ν_2 cannot be both $\geq n-1$. Thus $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq(\frac{5}{8}+\frac{1}{2\cdot 4^8})+(\frac{17}{32}+\frac{1}{2\cdot 4^7})+2\cdot(\frac{11}{32}+\frac{2}{3\cdot 4^6})\leq 2$. Now consider $q=3$. Since $\nu_3\geq\lceil\frac{n}{3}\rceil\geq 5$, $\nu_1+\nu_2\leq 2n-5$, which implies that $\mathcal{N}(\overline{g}_1)+\mathcal{N}(\overline{g}_2)\leq(\frac{1}{3}+\frac{1}{3^{12}})+(\frac{1}{3^4}+\frac{1}{3^9})$. Thus $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq 1+\frac{1}{2}\cdot\{(\frac{1}{3}+\frac{1}{3^{12}})+(\frac{1}{3^4}+\frac{1}{3^9})\}+2\cdot(\frac{29}{81}+\frac{2}{3^7})\leq 2$. So both ν_1 and ν_2 are $\leq n-5$ when $q=3$; $\leq n-3$ when $q\geq 4$. Also we have $\nu_3+\nu_4\leq 2n-5$, which implies that $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq 2\cdot(\frac{14}{27}+\frac{1}{2\cdot 3^6})+\frac{10937}{19683}+(\frac{29}{81}+\frac{2}{3^7})\leq 2$. So $\alpha\neq 2$.

Suppose $\alpha=1$. Say $|\overline{g}_1|=2$. If $|\overline{g}_i|\geq 4\ \forall\ 2\leq i\leq 4$, then $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq\frac{8749}{13122}+3\cdot(\frac{2}{5}+\frac{3}{4\cdot 5^8})\leq$

2. So say $|\overline{g}_2|=3$. Suppose both ν_1 and ν_2 are $\geq n-3$. Then ν_3 and ν_4 are both $\leq n-3$. Then $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq\frac{8749}{13122}+\frac{10937}{19683}+2\cdot(\frac{29}{81}+\frac{2}{3^7})\leq 2$. So one of ν_1, ν_2 is $\leq n-4$, which implies $\mathcal{U}(\overline{g}_1)+\mathcal{U}(\overline{g}_2)\leq max\{\frac{8749}{13122}+(\frac{29}{81}+\frac{2}{3^7}),\ (\frac{14}{27}+\frac{1}{2\cdot 3^6})+\frac{10937}{19683}\}=(\frac{14}{27}+\frac{1}{2\cdot 3^6})+\frac{10937}{19683}$. If $|\overline{g}_i|\geq 4$ for $i=3$ and 4, then $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq(\frac{14}{27}+\frac{1}{2\cdot 3^6})+\frac{10937}{19683}+2\cdot(\frac{2}{5}+\frac{3}{4\cdot 5^8})\leq 2$. So say $|\overline{g}_3|=3$. Since $\nu_2+\nu_3\leq 2n-5$, $\mathcal{U}(\overline{g}_2)+\mathcal{U}(\overline{g}_3)\leq(\frac{29}{81}+\frac{2}{3^7})+\frac{10937}{19683}$. Also similar as above, $\mathcal{U}(\overline{g}_1)+\mathcal{U}(\overline{g}_2)$ and $\mathcal{U}(\overline{g}_1)+\mathcal{U}(\overline{g}_3)$ are both $\leq(\frac{14}{27}+\frac{1}{2\cdot 3^6})+\frac{10937}{19683}$. Thus if $|\overline{g}_4|\geq 4$, then $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq(\frac{14}{27}+\frac{1}{2\cdot 3^6})+\frac{10937}{19683}+\frac{1}{2}\{(\frac{29}{81}+\frac{2}{3^7})+\frac{10937}{19683}\}+(\frac{2}{5}+\frac{3}{4\cdot 5^8})\leq 2$. So $|\overline{g}_4|=3$. Since $\nu_2+\nu_3+\nu_4\leq 2n$, $\lceil\frac{n}{3}\rceil\leq\nu_i\leq n-1$, we have $q^{\nu_2}+\cdots+q^{\nu_4}\leq q^{n-1}+q^{\frac{2n}{3}+1}+q^{\frac{n}{3}}$. Also $q^{n-\nu_2}+\cdots+q^{n-\nu_4}\leq 3q^{\frac{2n}{3}}$. Thus $\mathcal{N}(\overline{g}_2)+\mathcal{N}(\overline{g}_3)+\mathcal{N}(\overline{g}_4)\leq\frac{1}{q}+\frac{1}{q^{\frac{n}{3}-1}}+\frac{1}{q^{\frac{2n}{3}}}+\frac{3}{q^{\frac{n}{3}}}$. This gives $\mathcal{U}(\overline{g}_2)+\mathcal{U}(\overline{g}_3)+\mathcal{U}(\overline{g}_4)\leq 1+\frac{2}{3}(\frac{1}{q}+\frac{1}{q^{\frac{n}{3}-1}}+\frac{1}{q^{\frac{2n}{3}}}+\frac{3}{q^{\frac{n}{3}}})$, which is $\leq 1+\frac{22}{81}+\frac{2}{3^9}$ if $q=3$ and $n\geq 13$; and is $\leq 1+\frac{1}{3}+\frac{2}{3\cdot 4^6}$ if $q\geq 4$ and $n\geq 9$. Thus we have respectively $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq\frac{8749}{13122}+1+\frac{22}{81}+\frac{2}{3^9}$; $(\frac{5}{8}+\frac{1}{2\cdot 4^8})+1+\frac{1}{3}+\frac{2}{3\cdot 4^6}$, both of them ≤ 2. So $\alpha\neq 1$.

Suppose $\alpha=0$. If there are at most two elements in S with $|\overline{g}_i|=3$, then $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq 2\cdot\frac{10937}{19683}+2(\frac{2}{5}+\frac{3}{4\cdot 5^8})\leq 2$. So say $|\overline{g}_i|=3\ \forall\ 1\leq i\leq 3$. Then the same as in the previous paragraph, $\mathcal{U}(\overline{g}_1)+\mathcal{U}(\overline{g}_2)+\mathcal{U}(\overline{g}_3)\leq max\{1+\frac{22}{81}+\frac{2}{3^9},\ 1+\frac{1}{3}+\frac{2}{3\cdot 4^6}\}=1+\frac{1}{3}+\frac{2}{3\cdot 4^6}$, which implies $\sum_{i=1}^{4}\mathcal{U}(\overline{g}_i)\leq 1+\frac{1}{3}+\frac{2}{3\cdot 4^6}+\frac{10937}{19683}\leq 2$. So $\alpha\neq 0$.

Therefore S is of the type: (2,2,2,3) with $q=3$ or 4.

(8.14) Suppose $d=1$. Then S is not of type (2,2,2,3) when $q=3$ and $q=4$.

Proof. Suppose $q=3$. Pick a fixed preimage g_i for each $\overline{g}_i$. Let ν_i, μ_i be the dimensions of eigenspaces of g_i corrsponding to the eigenvalues 1 and -1 respectively. For $1\leq i\leq 3$, the preimages of $\overline{g}_i$ has exactly two eigenvalues and either they are $\{1,-1\}$ or $\{\omega^2,\omega^6\}$, where ω is such that $GF(3^2)^{\#}=\langle\omega\rangle$. If they are $\{\omega^2,\omega^6\}$, then $\nu_i=0=\mu_i$. Otherwise, $\nu_i+\mu_i=n$, $\nu_i\neq 0$, $\mu_i\neq 0$; and in this case we can pick g_i so that $\nu_i\geq\mu_i\ \forall\ 1\leq i\leq 3$. Also either $\nu_4=0$ or $\mu_4=0$, and we can pick g_4 so that $\mu_4=0$. Since $|\bar{\Omega}|=\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]_3=\frac{1}{2}(3^n-1)$, we have $\frac{3}{2}+\frac{1}{2}\cdot\sum_{i=1}^{3}\frac{3^{\nu_i}+3^{\mu_i}-2}{3^n-1}+\frac{1}{3}+\frac{2}{3}\cdot\frac{3^{\nu_4}-1}{3^n-1}=2+\frac{4}{3^n-1}$. This identity can be transformed into $3(3^{\nu_1}+3^{\nu_2}+3^{\nu_3}+3^{\mu_1}+3^{\mu_2}+3^{\mu_3})+4\cdot 3^{\nu_4}=3^n+5\cdot 3^2$. If all ν_i, μ_i are 0 for $1\le i\le 3$, then $4\cdot 3^{\nu_4}=3^n+3^3$. But if $\nu_4\le n-2$, then $4\cdot 3^{\nu_4}<3^n$; and if $\nu_4=n-1$, then $4\cdot 3^{\nu_4}=3^n+3^{n-1}>3^n+3^3$. So not all ν_i, μ_i are 0 for $1\le i\le 3$. If ν_i, μ_i are 0 for, say i=1 and 2, then $3(3^{\nu_3}+3^{\mu_3})+4\cdot 3^{\nu_4}=3^n+11\cdot 3$. This is a contradiction as $\nu_4\ge\lceil\frac{n}{3}\rceil\ge 3$ and $\nu_3\ge\mu_3\ge 1$. If only say $\nu_1=\mu_1=0$, then $3(3^{\nu_2}+3^{\nu_3}+3^{\mu_2}+3^{\mu_3})+4\cdot 3^{\nu_4}=3^n+13\cdot 3$. Again, this is a contradiction as 3^2 divides the left hand side but not the right hand side. So none of ν_i, μ_i is 0 for $1\le i\le 3$. Then $\nu_i\ge\lceil\frac{n}{2}\rceil\ge 2$ $\forall$ $1\le i\le 3$. If we also have $\mu_i\ge 2$ $\forall$ $1\le i\le 3$, then 3^3 divides the left hand side, which is a contradiction. So for some i, $\mu_i=1$. Then the left hand side is $\ge 3^n+4\cdot 3^{\nu_4}>3^n+5\cdot 3^2$, a contradiction again. So S is not of type (2,2,2,3) when $q=3$.

Suppose $q=4$. We have $\nu_i=dim\{C_V(g_i)\}\ge\lceil\frac{n}{2}\rceil$, $\forall$ $1\le i\le 3$. The preimages of $\bar{g}_4$ has three eigenvalues. They are $\{1,\omega^{21},\omega^{42}\}=GF(4)^{\#}$, or $\{\omega^7,\omega^{28},\omega^{49}\}$, or $\{\omega^{14},\omega^{56},\omega^{35}\}$, where ω is such that $GF(4^3)^{\#}=\langle\omega\rangle$. In the last two cases, we have $\mathcal{N}(\bar{g}_4)=0$. For the first case, $\mathcal{N}(\bar{g}_4)=\frac{4^\alpha+4^\beta+4^\gamma-3}{4^n-1}$, where α, β, γ are the dimensions for the eigenspaces corresponding to 1, ω^{21}, ω^{42} respectively. Also $|\bar{\Omega}|=\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]_4=\frac{1}{3}(4^n-1)$. Thus if $\mathcal{N}(\bar{g}_4)\ne 0$, then we have $\frac{3}{2}+\frac{1}{2}\cdot\sum_{i=1}^{3}\frac{4^{\nu_i}-1}{4^n-1}+\frac{1}{3}+\frac{2}{3}\cdot\frac{4^\alpha+4^\beta+4^\gamma-3}{4^n-1}=2+\frac{6}{4^n-1}$. This identity can be transformed into $3(4^{\nu_1}+4^{\nu_2}+4^{\nu_3})+4(4^\alpha+4^\beta+4^\gamma)=4^n+7\cdot 2^3$. So if all α, β, γ are ≥ 1, then we have a contradiction. So exactly one of them, say $\gamma=0$. Then $3(4^{\nu_1}+4^{\nu_2}+4^{\nu_3})+4(4^\alpha+4^\beta)=4^n+13\cdot 2^2$, which gives a contradiction again as α and β are both ≥ 1 and $\nu_i\ge\lceil\frac{n}{2}\rceil\ge 2$, $\forall$ $1\le i\le 3$. Thus $\mathcal{N}(\bar{g}_4)=0$. Similarly, this time we have $3(4^{\nu_1}+4^{\nu_2}+4^{\nu_3})=4^n+11\cdot 2^2$. This supplies a contradiction too as all $\nu_i\ge 2$. So S is not of type (2,2,2,3) when $q=4$.

(8.15) $|S|=3$.

Proof. This follows from (8.12) through (8.14).

(8.16) Suppose S has an involution, say $|\overline{g}_1|=2$. Then $d=1$ and S is one of the following types: (2,6,6) with $q=3$; (2,4,5) or (2,4,6) with $q=3$, 4, or 5; (2,3,8), (2,3,9), (2,3,12) with $q=3$ or 4.

Proof. Suppose $d\geq 2$ first. If $\nu(\overline{g}_1)=n-1$, then by (3.3) $|\overline{g}_2|$ and $|\overline{g}_3|$ are $\geq n\geq 9$. Then for $i=2$ and 3, $\mathcal{U}(\overline{g}_i)\leq max\{\frac{1117}{729}\cdot\frac{1}{9}+\frac{1}{3^5},\ \frac{225}{128}\cdot\frac{1}{9}+\frac{1}{4^5}\}=\frac{225}{128}\cdot\frac{1}{9}+\frac{1}{4^5}$, which implies $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq\frac{2431}{4374}+2\cdot(\frac{225}{128}\cdot\frac{1}{9}+\frac{1}{4^5})\leq 1$. So $\nu(\overline{g}_1)\leq n-2$. Suppose $|\overline{g}_i|\geq 5$ for $i=2$ and 3. Then $\mathcal{U}(\overline{g}_i)\leq max\{\frac{1}{5}+\frac{2}{5\cdot 4^3},\ \frac{56}{243}+\frac{1}{2\cdot 3^7},\ \frac{55}{343}+\frac{6}{7^8},\ \frac{1117}{729}\cdot\frac{1}{8}+\frac{1}{3^5},\ \frac{225}{128}\cdot\frac{1}{8}+\frac{1}{4^5}\}=\frac{56}{243}+\frac{1}{2\cdot 3^7}$, which implies that $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq\frac{83}{162}+2(\frac{56}{243}+\frac{1}{2\cdot 3^7})\leq 1$. So say $|\overline{g}_2|=3$ or 4. Then both $\nu(\overline{g}_1)$ and $\nu(\overline{g}_2)$ are $\leq n-3$. Since for $|\overline{g}_3|\geq 7$, we have $\mathcal{U}(\overline{g}_i)\leq max\{\frac{55}{343}+\frac{6}{7^8},\ \frac{17}{108}+\frac{1}{8\cdot 3^7},\ \frac{11}{81}+\frac{2}{3^9}+\frac{2}{3^6},\ \frac{3}{25}+\frac{4}{5^5}+\frac{1}{2\cdot 5^7},\ \frac{1117}{729}\cdot\frac{1}{11}+\frac{1}{3^5},\ \frac{225}{128}\cdot\frac{1}{11}+\frac{1}{4^5}\}=\frac{55}{343}+\frac{6}{7^8}$, and $\mathcal{U}(\overline{g}_2)\leq max\{(\frac{1}{3}+\frac{2}{3^6}),\ \frac{5}{18}+\frac{1}{3^4}+\frac{1}{4\cdot 3^7}\}=\frac{1}{3}+\frac{2}{3^6}$, we have $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq(\frac{1}{2}+\frac{1}{2\cdot 3^5})+(\frac{1}{3}+\frac{2}{3^6})+(\frac{55}{343}+\frac{6}{7^8})\leq 1$. For $|\overline{g}_2|=4$ and $|\overline{g}_3|=5$, we have $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq(\frac{1}{2}+\frac{1}{2\cdot 3^5})+(\frac{5}{18}+\frac{1}{3^4}+\frac{1}{4\cdot 3^7})+(\frac{1}{5}+\frac{2}{5\cdot 4^3})\leq 1$. For $|\overline{g}_2|=4$ and $|\overline{g}_3|=6$, as $SL_n(q)\leq\langle g_2,g_3^2\rangle$, we have $\nu(g_3^2)\leq n-2$. Hence $\mathcal{U}(\overline{g}_3)\leq\frac{1}{6}\{1+(\frac{1}{3^2}+\frac{1}{3^7})+4\cdot\frac{2}{3^4}\}=\frac{49}{243}+\frac{1}{2\cdot 3^8}$, which implies that $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq(\frac{1}{2}+\frac{1}{2\cdot 3^5})+(\frac{5}{18}+\frac{1}{3^4}+\frac{1}{4\cdot 3^7})+(\frac{49}{243}+\frac{1}{2\cdot 3^8})\leq 1$. Hence $d=1$.

Suppose $\nu(\overline{g}_1)=n-1$ first. So $|\overline{g}_2|$ and $|\overline{g}_3|$ are $\geq n$ and each eigenspace of them is of dimension 1. Suppose $q=3$. If $|\overline{g}_2|=13$, then $\mathcal{U}(\overline{g}_2)\leq\frac{1}{13}+\frac{12}{3^{12}}$. If $|\overline{g}_2|=14$, as g_2 is semisimple and each eigenspace of g_2 is of dimension 1, each eigenspace of g_2^2 is of dimension ≤ 2, and each eigenspace of g_2^7 is of dimension ≤ 7. Hence $\mathcal{U}(\overline{g}_2)\leq\frac{1}{14}(1+\frac{2}{3^{n-7}}+6\cdot\frac{7}{3^{n-2}}+6\cdot\frac{14}{3^{n-1}})\leq\frac{1}{14}+\frac{1}{7\cdot 3^6}+\frac{1}{3^9}$. For $|\overline{g}_2|\geq 15$, we have by (8.10), $\mathcal{U}(\overline{g}_2)\leq\frac{460}{243}\cdot\frac{1}{15}+\frac{1}{27}+\frac{1}{3^6}$. Similar for $\mathcal{U}(\overline{g}_3)$. Since $max\{\frac{1}{13}+\frac{12}{3^{12}},\ \frac{1}{14}+\frac{1}{7\cdot 3^6}+\frac{1}{3^9},\ \frac{460}{243}\cdot\frac{1}{15}+\frac{1}{27}+\frac{1}{3^6}\}=\frac{460}{243}\cdot\frac{1}{15}+\frac{1}{27}+\frac{1}{3^6}$, we have $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq\frac{8749}{13122}+2\cdot(\frac{460}{243}\cdot\frac{1}{15}+\frac{1}{27}+\frac{1}{3^6})\leq 1$. So for $q=3$, $\nu(\overline{g}_1)\leq n-2$. Similarly for $q\geq 4$, if $|\overline{g}_2|=9$, 10, 11, 12, 13, 14, 15 then $\mathcal{U}(\overline{g}_2)\leq\frac{1}{9}+\frac{50}{3\cdot 4^8},\ \frac{3}{20}+\frac{57}{10\cdot 4^8},\ \frac{1}{11}+\frac{10}{4^8},\ \frac{1}{6}+\frac{67}{12\cdot 4^8},\ \frac{1}{13}+\frac{12}{4^8},\ \frac{13}{112}+\frac{109}{14\cdot 4^8},\ \frac{17}{120}+\frac{11}{5\cdot 4^7}$ respectively. For $|\overline{g}_2|\geq 16$, we have by (8.10), $\mathcal{U}(\overline{g}_2)\leq(\frac{7}{4}+\frac{3}{4^8})\frac{1}{16}+(\frac{1}{4^2}+\frac{1}{4^7}),\ (\frac{12}{5}+\frac{7}{5^8})\frac{1}{16}+(\frac{1}{5^2}+\frac{1}{5^7}),\ \frac{37}{13}\cdot\frac{1}{16}+(\frac{1}{7^2}+\frac{1}{7^7})$ for $q=4$, 5, $q\geq 7$ respectively. These bounds implies that $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq(\frac{5}{8}+\frac{1}{2\cdot 4^8})+2\{(\frac{7}{4}+\frac{3}{4^8})\frac{1}{16}+(\frac{1}{4^2}+\frac{1}{4^7})\}$, $(\frac{3}{5}+\frac{1}{2\cdot 5^8})+2\{(\frac{12}{5}+\frac{7}{5^8})\frac{1}{16}+(\frac{1}{5^2}+\frac{1}{5^7})\}$, $(\frac{4}{7}+\frac{1}{2\cdot 7^8})+2\{\frac{37}{13}\cdot\frac{1}{16}+(\frac{1}{7^2}+\frac{1}{7^7})\}$ for $q=4$, 5, $q\geq 7$

respectively; each of them is ≤ 1. So $\nu(\overline{g}_1)\le n-2$.

Since $\overline{g}_1$ has an eigenspace of dimension $\ge\frac{n}{2}$, every eigenspace of $\overline{g}_2$ is of dimension $\le\frac{n}{2}\le n-3$. So by (8.11), for $7\le|\overline{g}_2|\le 14$, we have $\mathcal{U}(\overline{g}_2)\le max\{\frac{11}{63}+\frac{2}{7\cdot 3^5}, \frac{7}{36}+\frac{43}{8\cdot 3^8}, \frac{17}{81}+\frac{2}{3^{10}}+\frac{2}{3^7}, \frac{27}{125}+\frac{1}{2\cdot 5^7}, \frac{21}{121}+\frac{10}{11^9}, \frac{2}{13}+\frac{11}{12\cdot 13^8}, \frac{25}{169}+\frac{12}{13^9}, \frac{52}{343}+\frac{1}{2\cdot 7^7}\}=\frac{27}{125}+\frac{1}{2\cdot 5^7}$. By (8.10), for $|\overline{g}_2|\ge 15$, we have that $\mathcal{U}(\overline{g}_2)\le max\{\frac{460}{243}\cdot\frac{1}{15}+(\frac{1}{27}+\frac{1}{3^6}), (\frac{7}{4}+\frac{3}{4^8})\frac{1}{15}+(\frac{1}{4^2}+\frac{1}{4^7}), (\frac{12}{5}+\frac{7}{5^8})\frac{1}{15}+(\frac{1}{5^2}+\frac{1}{5^7}), \frac{37}{13}\cdot\frac{1}{15}+(\frac{1}{7^2}+\frac{1}{7^7})\}=\frac{37}{13}\cdot\frac{1}{15}+(\frac{1}{7^2}+\frac{1}{7^7})$. So if $|\overline{g}_2|\ge 7$, then $\mathcal{U}(\overline{g}_2)\le max\{\frac{27}{125}+\frac{1}{2\cdot 5^7}, \frac{37}{13}\cdot\frac{1}{15}+(\frac{1}{7^2}+\frac{1}{7^7})\}=\frac{27}{125}+\frac{1}{2\cdot 5^7}$. Similar for $\mathcal{U}(\overline{g}_3)$. Also by (8.11), we have that $\mathcal{U}(\overline{g}_1)\le$ $max\{\frac{5}{9}+\frac{1}{2\cdot 3^7}, \frac{17}{32}+\frac{1}{2\cdot 4^7}, \frac{13}{25}+\frac{1}{2\cdot 5^7}, \frac{25}{49}+\frac{1}{2\cdot 7^7}\}=\frac{5}{9}+\frac{1}{2\cdot 3^7}$. Hence if both $|\overline{g}_2|$ and $|\overline{g}_3|$ are ≥ 7, then $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\le(\frac{5}{9}+\frac{1}{2\cdot 3^7})+2\{\frac{27}{125}+\frac{1}{2\cdot 5^7}\}\le 1$. So one of $|\overline{g}_2|$ and $|\overline{g}_3|$ is ≤ 6. Say $3\le|\overline{g}_2|\le 6$. We also suppose that $|\overline{g}_2|\le|\overline{g}_3|$ in the following.

Suppose $|\overline{g}_2|=6$. Let α, β be the maximal dimension of eigenspaces of $\overline{g}_2^3$, $\overline{g}_2^2$ respectively. Then as $\widetilde{G}=\langle\overline{g}_1,\overline{g}_2^3, \overline{g}_2^2\rangle$, we have $\nu_1+\alpha+\beta\le 2n$, $\lceil\frac{n}{2}\rceil\le\nu_1\le n-2$, $\lceil\frac{n}{2}\rceil\le\alpha\le n-1$, $\lceil\frac{n}{3}\rceil\le\beta\le n-1$. So $\frac{1}{2}\mathcal{N}(\overline{g}_1)+\frac{1}{6}\mathcal{N}(\overline{g}_2^3)+\frac{2}{6}\mathcal{N}(\overline{g}_2^2)\le\frac{1}{q^n-1}\{\frac{1}{2}(q^{\nu_1}+q^{n-\nu_1}-2)+\frac{1}{6}(q^{\alpha}+q^{n-\alpha}-2)+\frac{2}{6}(q^{\beta}+q^{n-\beta}-2)\}\le\frac{1}{q^n-1}\{\frac{1}{2}(q^{n-\lceil\frac{n}{2}\rceil+1}+q^{\lceil\frac{n}{2}\rceil-1}-2)+\frac{1}{6}(q^{n-\lceil\frac{n}{2}\rceil}+q^{\lceil\frac{n}{2}\rceil}-2)+\frac{2}{6}(q^{n-1}+q-2)\}\le$

$\frac{1}{2}(\frac{1}{q^{\lceil\frac{n}{2}\rceil-1}}+\frac{1}{q^{n-\lceil\frac{n}{2}\rceil+1}})+\frac{1}{6}(\frac{1}{q^{\lceil\frac{n}{2}\rceil}}+\frac{1}{q^{n-\lceil\frac{n}{2}\rceil}})+\frac{2}{6}(\frac{1}{q}+\frac{1}{q^{n-1}})\le$

$\frac{4}{3}\cdot\frac{1}{q^{\lceil\frac{n}{2}\rceil-1}}+\frac{1}{3}(\frac{1}{q}+\frac{1}{q^{n-1}})$. Since every eigenspace of $\overline{g}_2$ is of dimension $\le[\frac{n}{2}]$, we have

$\mathcal{U}(\overline{g}_1)+\mathcal{U}(\overline{g}_2)\le\frac{1}{2}+\frac{1}{6}+\frac{4}{3}\cdot\frac{1}{q^{\lceil\frac{n}{2}\rceil-1}}+\frac{1}{3}(\frac{1}{q}+\frac{1}{q^{n-1}})+\frac{2}{6}\cdot\frac{6}{q^{\lceil\frac{n}{2}\rceil}}$, which is less than or equal to

$\frac{7}{9}+\frac{2}{3^6}+\frac{1}{3^{13}}, \frac{3}{4}+\frac{22}{3\cdot 4^5}+\frac{1}{3\cdot 4^8}, \frac{11}{15}+\frac{26}{3\cdot 5^5}+\frac{1}{3\cdot 5^8}, \frac{5}{7}+\frac{34}{3\cdot 7^5}+\frac{1}{3\cdot 7^8}$ for $q=3$, 4, 5, and $q\ge 7$ respectively. Now for $|\overline{g}_3|\ge 7$ and $q=3$, we have $\mathcal{U}(\overline{g}_3)\le max\{\frac{11}{63}+\frac{2}{7\cdot 3^5}, \frac{7}{36}+\frac{43}{8\cdot 3^8}, \frac{17}{81}+\frac{2}{3^{10}}+\frac{2}{3^7}, \frac{27}{125}+\frac{1}{2\cdot 5^7}, \frac{460}{243}\cdot\frac{1}{11}+(\frac{1}{27}+\frac{1}{3^6})\}=\frac{27}{125}+\frac{1}{2\cdot 5^7}$. Similarly, for $|\overline{g}_3|\ge 7$, we have $\mathcal{U}(\overline{g}_3)\le(\frac{7}{4}+\frac{3}{4^8})\frac{1}{11}+(\frac{1}{4^2}+\frac{1}{4^7})$, $(\frac{12}{5}+\frac{7}{5^8})\frac{1}{11}+(\frac{1}{5^2}+\frac{1}{5^7})$, $\frac{37}{13}\cdot\frac{1}{11}+(\frac{1}{7^2}+\frac{1}{7^7})$ for $q=4$, 5, and $q\ge 7$ respectively. In any case, we have $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\le 1$. Hence $|\overline{g}_2|=|\overline{g}_3|=6$. Suppose S is of type (2,6,6). Let γ, δ be the maximal dimension of eigenspaces of $\overline{g}_3^3$, $\overline{g}_3^2$ respectively. Since $PSL_n(q)\le\langle\overline{g}_2, \overline{g}_3^2\rangle=\langle\overline{g}_2^3, \overline{g}_2^2, \overline{g}_3^2\rangle$, we have $\alpha+\beta+\gamma\le 2n$. Also as $\lceil\frac{n}{2}\rceil\le\alpha\le n-1$, $\lceil\frac{n}{3}\rceil\le\beta\le n-1$, $\lceil\frac{n}{2}\rceil\le\gamma\le n-1$, we have $\mathcal{N}(\bar{g}_2^3)+\mathcal{N}(\bar{g}_2^2)+\mathcal{N}(\bar{g}_3^2)\le\frac{1}{q^n-1}\{(q^\alpha+q^{n-\alpha}-2)+(q^\beta+q^{n-\beta}-2)+(q^\gamma+q^{n-\gamma}-2)\}\le\frac{1}{q^n-1}\{(q^{n-1}+q-2)+(q^{n-\lceil\frac{n}{3}\rceil+1}+q^{\lceil\frac{n}{3}\rceil-1}-2)+(q^{\lceil\frac{n}{3}\rceil}+q^{n-\lceil\frac{n}{3}\rceil}-2)\}\le(\frac{1}{q}+\frac{1}{q^{n-1}})+(\frac{1}{q^{\lceil\frac{n}{3}\rceil-1}}+\frac{1}{q^{n-\lceil\frac{n}{3}\rceil+1}})+(\frac{1}{q^{\lceil\frac{n}{3}\rceil}}+\frac{1}{q^{n-\lceil\frac{n}{3}\rceil}})\le(\frac{1}{q}+\frac{1}{q^{n-1}})+2(\frac{1}{q^{\lceil\frac{n}{3}\rceil-1}}+\frac{1}{q^{n-\lceil\frac{n}{3}\rceil}})$. Similarly for $\mathcal{N}(\bar{g}_3^3)+\mathcal{N}(\bar{g}_3^2)+\mathcal{N}(\bar{g}_2^2)$. So $\frac{1}{6}\mathcal{N}(\bar{g}_2^3)+\frac{2}{6}\mathcal{N}(\bar{g}_2^2)+\frac{1}{6}\mathcal{N}(\bar{g}_3^3)+\frac{2}{6}\mathcal{N}(\bar{g}_3^2)=\frac{1}{6}\{\mathcal{N}(\bar{g}_2^3)+\mathcal{N}(\bar{g}_2^2)+\mathcal{N}(\bar{g}_3^2)\}+\frac{1}{6}\{\mathcal{N}(\bar{g}_3^3)+\mathcal{N}(\bar{g}_3^2)+\mathcal{N}(\bar{g}_2^2)\}\le\frac{1}{3}(\frac{1}{q}+\frac{1}{q^{n-1}})+\frac{2}{3}(\frac{1}{q^{\lceil\frac{n}{3}\rceil-1}}+\frac{1}{q^{n-\lceil\frac{n}{3}\rceil}})$. This implies that $\mathcal{U}(\bar{g}_2)+\mathcal{U}(\bar{g}_3)\le\frac{1}{3}+\frac{1}{3}(\frac{1}{q}+\frac{1}{q^{n-1}})+\frac{2}{3}(\frac{1}{q^{\lceil\frac{n}{3}\rceil-1}}+\frac{1}{q^{n-\lceil\frac{n}{3}\rceil}})+\frac{2}{3}\cdot\frac{6}{q^{\lceil\frac{n}{2}\rceil}}$, which is less than or equal to $\frac{5}{12}+\frac{2}{3\cdot 4^2}+\frac{14}{3\cdot 4^5}+\frac{1}{3\cdot 4^8}$, $\frac{2}{5}+\frac{2}{3\cdot 5^2}+\frac{14}{3\cdot 5^4}+\frac{1}{3\cdot 5^8}$, $\frac{8}{21}+\frac{2}{3\cdot 7^2}+\frac{14}{3\cdot 7^4}+\frac{1}{3\cdot 7^8}$ for $q=4$, 5, and $q\ge 7$ respectively. This implies that $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\le(\frac{3}{4}+\frac{22}{3\cdot 4^5}+\frac{1}{3\cdot 4^8})+\frac{1}{2}(\frac{5}{12}+\frac{2}{3\cdot 4^2}+\frac{14}{3\cdot 4^5}+\frac{1}{3\cdot 4^8})$, $(\frac{11}{15}+\frac{26}{3\cdot 5^5}+\frac{1}{3\cdot 5^8})+\frac{1}{2}(\frac{2}{5}+\frac{2}{3\cdot 5^2}+\frac{14}{3\cdot 5^4}+\frac{1}{3\cdot 5^8})$, $(\frac{5}{7}+\frac{34}{3\cdot 7^5}+\frac{1}{3\cdot 7^8})+\frac{1}{2}(\frac{8}{21}+\frac{2}{3\cdot 7^2}+\frac{14}{3\cdot 7^4}+\frac{1}{3\cdot 7^8})$ for $q=4$, 5, and $q\ge 7$ respectively. Since each of them is ≤ 1, we have that $q=3$ and S is of type (2,6,6).

Suppose $|\bar{g}_2|=5$. Since $\nu_2\le[\frac{n}{2}]$, $\mathcal{U}(\bar{g}_2)\le\frac{1}{5}(1+4\cdot\frac{5}{q^{\lceil\frac{n}{2}\rceil}})\le\frac{1}{5}+\frac{4}{3^5}$. For $|\bar{g}_3|\ge 7$ or $|\bar{g}_2|=5$, we have $\mathcal{U}(\bar{g}_3)\le\frac{27}{125}+\frac{1}{2\cdot 5^7}$, $(\frac{7}{4}+\frac{3}{4^8})\frac{1}{11}+(\frac{1}{4^2}+\frac{1}{4^7})$, $(\frac{12}{5}+\frac{7}{5^8})\frac{1}{11}+(\frac{1}{5^2}+\frac{1}{5^7})$, $\frac{37}{13}\cdot\frac{1}{11}+(\frac{1}{7^2}+\frac{1}{7^7})$ for $q=3$, 4, 5, and $q\ge 7$ respectively. Since we also have $\mathcal{U}(\bar{g}_1)\le max\{\frac{5}{9}+\frac{1}{2\cdot 3^7}$, $\frac{17}{32}+\frac{1}{2\cdot 4^7}$, $\frac{13}{25}+\frac{1}{2\cdot 5^7}$, $\frac{25}{49}+\frac{1}{2\cdot 7^7}$ for $q=3$, 4, 5, and $q\ge 7$ respectively, in any case, we have $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\le 1$. Suppose S is of type (2,5,6). Then as in the previous paragraph, $\mathcal{U}(\bar{g}_1)+\mathcal{U}(\bar{g}_3)\le\frac{7}{9}+\frac{2}{3^6}+\frac{1}{3^{13}}$, which gives $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\le 1$ again.

Suppose $|\bar{g}_2|=4$. Since $\nu_1+\nu_2\le n$, $\lceil\frac{n}{2}\rceil\le\nu_1\le n-2$, $\lceil\frac{n}{4}\rceil\le\nu_2\le n-1$, we have $\frac{1}{2}\mathcal{N}(\bar{g}_1)+\frac{2}{4}\mathcal{N}(\bar{g}_2)\le\frac{1}{2}\cdot\frac{1}{q^n-1}\{(q^{\nu_1}+q^{n-\nu_1}-2)+(q^{\nu_2}+q^{n-\nu_2}-2)\}\le\frac{1}{q^n-1}(q^{n-\lceil\frac{n}{4}\rceil}+q^{\lceil\frac{n}{4}\rceil}-2)\le\frac{1}{q^{\lceil\frac{n}{4}\rceil}}+\frac{1}{q^{n-\lceil\frac{n}{4}\rceil}}$. If $q=4$, then as g_1, g_2 are unipotent, we have $\frac{1}{2}\mathcal{N}(\bar{g}_1)+\frac{2}{4}\mathcal{N}(\bar{g}_2)\le$

$\frac{1}{2}\cdot\frac{1}{4^n-1}\{(4^{\nu_1}-1)+(4^{\nu_2}-1)\}\le\frac{1}{2}\cdot\frac{1}{4^n-1}(4^{n-\lceil\frac{n}{4}\rceil}+4^{\lceil\frac{n}{4}\rceil}-2)\le\frac{1}{2}(\frac{1}{4^{\lceil\frac{n}{4}\rceil}}+\frac{1}{4^{n-\lceil\frac{n}{4}\rceil}})$. Since $\mathcal{N}(\overline{g}_2^2)\le\frac{1}{3^2}+\frac{1}{3^{11}}$, $\frac{1}{4}+\frac{1}{4^8}$, $\frac{1}{5}+\frac{1}{5^8}$, $\frac{1}{9}+\frac{1}{9^8}$ for $q=3$, 4, 5, and $q\ge7$ respectively, we have respectively $\mathcal{U}(\overline{g}_1)+\mathcal{U}(\overline{g}_2)\le\frac{64}{81}+\frac{1}{3^9}+\frac{1}{4\cdot3^{11}}$, $\frac{105}{128}+\frac{3}{4^9}$, $\frac{101}{125}+\frac{1}{4\cdot5^7}$, $\frac{7}{9}+\frac{1}{7^3}+\frac{1}{7^8}+\frac{1}{4\cdot9^8}$. For $q=3$ and $|\overline{g}_3|=9$, as $\nu_3\le[\frac{n}{2}]$, we have $\mathcal{U}(\overline{g}_3)\le\frac{1}{9}(1+2\cdot\frac{1}{3}+6\cdot\frac{1}{3^7})=\frac{5}{27}+\frac{2}{3^8}$. For $q=3$ and $|\overline{g}_3|=10$, as $|g_3^2|=5$ or 10, g_3^2 has a simple submodule of dimension 4, hence $\mathcal{U}(\overline{g}_3)\le\frac{1}{10}\{1+(\frac{1}{3}+\frac{1}{3^{12}})+4(\frac{1}{3^4}+\frac{1}{3^9})+4(\frac{1}{3^6}+\frac{1}{3^7})\}=\frac{56}{405}+\frac{121}{10\cdot3^9}+\frac{1}{10\cdot3^{12}}$. For $q=3$ and $|\overline{g}_3|=11$, we have $\mathcal{U}(\overline{g}_3)\le\frac{1}{11}(1+10\cdot\frac{11}{3^7})=\frac{1}{11}+\frac{10}{3^7}$. So for $q=3$ and $|\overline{g}_3|\ge7$, $\mathcal{U}(\overline{g}_3)\le max\{\frac{11}{63}+\frac{2}{7\cdot3^5}$, $\frac{7}{36}+\frac{43}{8\cdot3^8}$, $\frac{5}{27}+\frac{2}{3^8}$, $\frac{56}{405}+\frac{121}{10\cdot3^9}+\frac{1}{10\cdot3^{12}}$, $\frac{1}{11}+\frac{10}{3^7}$, $\frac{460}{243}\cdot\frac{1}{12}+(\frac{1}{27}+\frac{1}{3^6})\}=\frac{460}{243}\cdot\frac{1}{12}+(\frac{1}{27}+\frac{1}{3^6})$; which implies that $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\le(\frac{64}{81}+\frac{1}{3^9}+\frac{1}{4\cdot3^{11}})+\{\frac{460}{243}\cdot\frac{1}{12}+(\frac{1}{27}+\frac{1}{3^6})\}\le1$. Thus if $q=3$, then $|\overline{g}_3|=5$ or 6. Now consider $q\ge4$. For example if $|\overline{g}_3|=12$, then as $PSL_n(q)\le\langle\overline{g}_2, \overline{g}_3^2\rangle$, the maximal dimension $\nu(\overline{g}_3^2)$ of eigenspaces of $\overline{g}_3^2$ is such that $\nu(\overline{g}_3^2)\le n-\lceil\frac{n}{4}\rceil\le n-3$. Then as $\langle\overline{g}_3\rangle=\langle\overline{g}_3^3,\overline{g}_3^4\rangle$, one of $\nu(\overline{g}_3^3)$, $\nu(\overline{g}_3^4)$ is $\le n-2$. Also as $\nu_3\le[\frac{n}{2}]$, we have $\mathcal{U}(\overline{g}_3)\le\frac{1}{12}\{1+3(\frac{1}{q}+\frac{1}{q^8})+2(\frac{1}{q^2}+\frac{1}{q^7})+2(\frac{1}{q^3}+\frac{1}{q^6})+4(\frac{1}{q^5}+\frac{1}{q^4})\}\le\frac{61}{384}+\frac{5}{3\cdot4^5}+\frac{43}{3\cdot4^9}$. Similarly, for $|\overline{g}_3|=7$, 8, 9, 10, 11, 13, 14, 15, we have respectively that $\mathcal{U}(\overline{g}_3)\le\frac{1}{7}+\frac{30}{7\cdot4^5}$, $\frac{41}{256}+\frac{41}{4^7}+\frac{1}{2\cdot4^9}$, $\frac{1}{6}+\frac{10}{3\cdot4^5}+\frac{2}{9\cdot4^8}$, $\frac{21}{160}+\frac{1}{2\cdot4^4}+\frac{13}{2\cdot4^8}$, $\frac{1}{11}+\frac{50}{11\cdot4^5}$, $\frac{1}{13}+\frac{15}{13\cdot4^4}$, $\frac{5}{56}+\frac{9}{4^5}+\frac{577}{14\cdot4^8}$, $\frac{1}{6}+\frac{2}{3\cdot4^4}+\frac{2}{5\cdot4^8}$. By taking the maximum, for $q\ge4$ and $7\le|\overline{g}_3|\le15$, we have $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\le(\frac{105}{128}+\frac{3}{4^9})+(\frac{1}{6}+\frac{10}{3\cdot4^5}+\frac{2}{9\cdot4^8})\le1$. Also for $|\overline{g}_3|\ge16$, we have $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\le(\frac{105}{128}+\frac{3}{4^9})+\{(\frac{7}{4}+\frac{3}{4^8})\frac{1}{16}+(\frac{1}{4^2}+\frac{1}{4^7})\}$, $(\frac{101}{125}+\frac{1}{4\cdot5^7})+\{(\frac{12}{5}+\frac{7}{5^8})\frac{1}{16}+(\frac{1}{5^2}+\frac{1}{5^7})\}$, $(\frac{7}{9}+\frac{1}{7^3}+\frac{1}{7^8}+\frac{1}{4\cdot9^8})+\{\frac{37}{13}\cdot\frac{1}{16}+(\frac{1}{7^2}+\frac{1}{7^7})\}$ for $q=4$, 5, and $q\ge7$ respectively. Since each of them is ≤1, we also have that for $q\ge4$, $|\overline{g}_3|=5$ or 6. Suppose $q\ge7$. If $|\overline{g}_3|=6$, then as $\langle\overline{g}_3\rangle=\langle\overline{g}_3^3,\overline{g}_3^2\rangle$ and $\nu_3\le[\frac{n}{2}]$, we have $\nu(\overline{g}_3^3)+\nu(\overline{g}_3^2)\le n+[\frac{n}{2}]$. Thus $\mathcal{U}(\overline{g}_3)\le\frac{1}{6}\{1+(\frac{1}{q^2}+\frac{1}{q^7})+2(\frac{1}{q}+\frac{1}{q^8})+2(\frac{1}{q^5}+\frac{1}{q^4})\}\le\frac{32}{147}+\frac{8}{3\cdot7^5}+\frac{3}{2\cdot7^8}$. If $|\overline{g}_3|=5$, then $\mathcal{U}(\overline{g}_3)\le\frac{1}{5}+\frac{32}{5\cdot7^5}$. This implies that if $q\ge7$ and $|\overline{g}_3|=5$ or 6, then $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\le(\frac{7}{9}+\frac{1}{7^3}+\frac{1}{7^8}+\frac{1}{4\cdot9^8})+(\frac{32}{147}+\frac{8}{3\cdot7^5}+\frac{3}{2\cdot7^8})\le1$. So in conclusion, if $|\overline{g}_2|=4$, then $|\overline{g}_3|=5$ or 6 with $q=3$, 4, or 5.

Suppose $|\overline{g}_2|=3$. Since $\nu_1+\nu_2\le n$, $\lceil\frac{n}{2}\rceil\le\nu_1\le n-2$, $\lceil\frac{n}{3}\rceil\le\nu_2\le n-1$, we have

$\frac{1}{2}\mathcal{N}(\bar{g}_1)+\frac{2}{3}\mathcal{N}(\bar{g}_2)\leq\frac{1}{6}\cdot\frac{1}{q^n-1}\{3(q^{\nu_1}+q^{n-\nu_1}-2)+4(q^{\nu_2}+q^{n-\nu_2}-2)\}\leq\frac{1}{6}\cdot\frac{1}{q^n-1}\cdot 7\cdot(q^{n-\lceil\frac{n}{3}\rceil}+q^{\lceil\frac{n}{3}\rceil}-2)\leq$ $\frac{7}{6}(\frac{1}{q^{\lceil\frac{n}{3}\rceil}}+\frac{1}{q^{n-\lceil\frac{n}{3}\rceil}})$. For $q=3$, we have assumed that $n\geq 13$, thus $\mathcal{U}(\bar{g}_1)+\mathcal{U}(\bar{g}_2)\leq\frac{5}{6}+\frac{98}{3^9}$. For $q=4$, 5, or $q\geq 7$, as $n\geq 9$, we have respectively $\mathcal{U}(\bar{g}_1)+\mathcal{U}(\bar{g}_2)\leq\frac{5}{6}+\frac{455}{6\cdot 4^6}$, $\frac{5}{6}+\frac{147}{5^6}$, $\frac{5}{6}+\frac{172}{3\cdot 7^5}$. Similar to above, by applying the formula for the bounds in (8.10), we have $|\bar{g}_3|\leq 15$, 19, 20, 20 for $q=3$, 4, 5, and $q\geq 7$ respectively. For $q\geq 4$ and $|\bar{g}_3|=20$, as $PSL_n(q)\leq\langle\bar{g}_2,\ \bar{g}_3^2\rangle$ and $\lceil\frac{n}{3}\rceil\leq\nu_2$, we have $\nu(\bar{g}_3^2)\leq n-\lceil\frac{n}{3}\rceil\leq n-3$. Also as $\nu_3\leq[\frac{n}{2}]$, we have $\mathcal{U}(\bar{g}_3)\leq\frac{1}{20}\{1+7(\frac{1}{4}+\frac{1}{4^8})+4(\frac{1}{4^3}+\frac{1}{4^6})+8(\frac{1}{4^5}+\frac{1}{4^4})\}=\frac{9}{64}+\frac{41}{5\cdot 4^6}+\frac{7}{5\cdot 4^9}$. Similarly, for $q\geq 4$ and $|\bar{g}_3|=16$, 17, 18, 19, we have respectively $\mathcal{U}(\bar{g}_3)\leq\frac{29}{256}+\frac{41}{4^7}+\frac{1}{4^{10}}$, $\frac{1}{17}+\frac{5}{17\cdot 4^3}$, $\frac{25}{192}+\frac{7}{4^6}+\frac{5}{18\cdot 4^8}$, $\frac{1}{19}+\frac{90}{19\cdot 4^5}$. These bounds all lead to $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\leq 1$. So both for $q\geq 4$ and $q=3$, we have $|\bar{g}_3|\leq 15$. Consider $q\geq 4$ first. For $|\bar{g}_3|=15$, since $\langle\bar{g}_3\rangle=\langle\bar{g}_3^5,\bar{g}_3^3\rangle$ and $\nu_3\leq[\frac{n}{2}]$, we have $\nu(\bar{g}_3^5)+\nu(\bar{g}_3^3)\leq n+[\frac{n}{2}]$. Thus $\mathcal{U}(\bar{g}_3)\leq\frac{1}{15}\{1+2(\frac{1}{q^2}+\frac{1}{q^7})+4(\frac{1}{q}+\frac{1}{q^8})+8(\frac{1}{q^5}+\frac{1}{q^4})\}\leq\frac{17}{120}+\frac{2}{3\cdot 4^4}+\frac{1}{5\cdot 4^7}$. Similarly, for $|\bar{g}_3|=7$, 10, 11, 13, 14, we have respectively $\mathcal{U}(\bar{g}_3)\leq\frac{1}{7}+\frac{30}{7\cdot 4^5}$, $\frac{21}{160}+\frac{21}{10\cdot 4^5}+\frac{1}{10\cdot 4^8}$, $\frac{1}{11}+\frac{50}{11\cdot 4^5}$, $\frac{1}{13}+\frac{15}{13\cdot 4^4}$, $\frac{43}{448}+\frac{63}{7\cdot 4^6}+\frac{1}{14\cdot 4^8}$. So for $q\geq 4$ and $7\leq|\bar{g}_3|\leq 15$ with $|\bar{g}_3|\neq 8$, 9, 12, by taking maximum, we have that $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\leq(\frac{5}{6}+\frac{455}{6\cdot 4^6})+(\frac{1}{7}+\frac{30}{7\cdot 4^5})\leq 1$. For $q\geq 5$ and $|\bar{g}_3|=8$, as $PSL_n(q)\leq\langle\bar{g}_2,\ \bar{g}_3^2\rangle$ and $\lceil\frac{n}{3}\rceil\leq\nu_2$, we have $\nu(\bar{g}_3^2)\leq n-\lceil\frac{n}{3}\rceil\leq n-3$. So $\mathcal{U}(\bar{g}_3)\leq\frac{1}{8}\{1+(\frac{1}{5}+\frac{1}{5^8})+2(\frac{1}{5^3}+\frac{1}{5^6})+4(\frac{1}{5^5}+\frac{1}{5^4})\}=\frac{19}{125}+\frac{61}{4\cdot 5^6}+\frac{1}{8\cdot 5^8}$. This gives $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\leq(\frac{5}{6}+\frac{147}{5^6})+(\frac{19}{125}+\frac{61}{4\cdot 5^6}+\frac{1}{8\cdot 5^8})\leq 1$. For $q\geq 7$ and $|\bar{g}_3|=9$, then $\mathcal{U}(\bar{g}_3)\leq\frac{1}{9}\{1+2(\frac{1}{7}+\frac{1}{7^8})+6(\frac{1}{7^5}+\frac{1}{7^4})\}=\frac{1}{7}+\frac{16}{3\cdot 7^5}+\frac{2}{9\cdot 7^8}$. For $q=5$ and $|\bar{g}_3|=9$, since $\bar{g}_3^3$ has a simple submodule of dimension 2, we have $\mathcal{N}(\bar{g}_3^3)\leq\frac{1}{5^2}+\frac{1}{5^7}$, which gives $\mathcal{U}(\bar{g}_3)\leq\frac{1}{9}\{1+2(\frac{1}{5^2}+\frac{1}{5^7})+6(\frac{1}{5^5}+\frac{1}{5^4})\}=\frac{3}{25}+\frac{4}{5^5}+\frac{2}{9\cdot 5^7}$. So in case $q\geq 5$ and $|\bar{g}_3|=9$, we have $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\leq(\frac{5}{6}+\frac{147}{5^6})+(\frac{1}{7}+\frac{16}{3\cdot 7^5}+\frac{2}{9\cdot 7^8})\leq 1$. For $q\geq 5$ and $|\bar{g}_3|=12$, as $PSL_n(q)\leq\langle\bar{g}_2,\ \bar{g}_3^2\rangle$, $\nu(\bar{g}_3^2)\leq n-\lceil\frac{n}{3}\rceil\leq n-3$. Since $\langle\bar{g}_3\rangle=\langle\bar{g}_3^3,\bar{g}_3^4\rangle$, one of $\nu(\bar{g}_3^3)$, $\nu(\bar{g}_3^4)$ is $\leq n-2$. Also as $\nu_3\leq[\frac{n}{2}]$, we have $\mathcal{U}(\bar{g}_3)\leq\frac{1}{12}\{1+3(\frac{1}{5}+\frac{1}{5^8})+2(\frac{1}{5^2}+\frac{1}{5^7})+2(\frac{1}{5^3}+\frac{1}{5^6})+4(\frac{1}{5^5}+\frac{1}{5^4})\}=\frac{53}{375}+\frac{61}{6\cdot 5^6}+\frac{13}{12\cdot 5^8}$. So it leaves only $q=4$ with $|\bar{g}_3|=8$, 9, 12. Now suppose $q=3$. For $|\bar{g}_3|=15$, since $\bar{g}_3^3$ has a simple submodule of dimension 4, we have $\mathcal{U}(\bar{g}_3)\leq\frac{1}{15}\{1+2(\frac{1}{3}+\frac{1}{3^{12}})+4(\frac{1}{3^4}+\frac{1}{3^9})+8(\frac{1}{3^7}+\frac{1}{3^6})\}=\frac{1}{9}+\frac{1264}{5\cdot 3^{10}}+\frac{1}{5\cdot 3^{13}}$. Similarly, for $|\bar{g}_3|=7$, 10, 11, 13, 14, we have respectively $\mathcal{U}(\bar{g}_3)\leq\frac{1}{7}+\frac{24}{7\cdot 3^7}$,

$\frac{2}{15}+\frac{112}{3^9}+\frac{1}{10\cdot 3^{12}}$, $\frac{1}{11}+\frac{40}{11\cdot 3^{12}}$, $\frac{1}{13}+\frac{48}{13\cdot 3^7}$, $\frac{2}{21}+\frac{40}{7\cdot 3^7}+\frac{1}{14\cdot 3^{12}}$. So for $q=3$ and $7\le|\overline{g}_3|\le 15$ with $|\overline{g}_3|\ne 8$, 9, 12, by taking maximum, we have that $\sum_{i=1}^{3}\mathfrak{U}(\overline{g}_i)\le(\frac{5}{6}+\frac{98}{3^9})+(\frac{1}{7}+\frac{24}{7\cdot 3^7})\le 1$. Hence $|\overline{g}_3|=8$, 9, or 12. In conclusion, if $|\overline{g}_2|=3$, then $|\overline{g}_3|=8$, 9, or 12 with $q=3$ or 4.

(8.17) If $d=1$ and $q=3$, then S is not of type (2,6,6).

Proof. Pick a fixed preimage g_i for each $\overline{g}_i$. Let ν_i, μ_i be the dimensions of eigenspaces of g_i corrsponding to the eigenvalues 1 and -1 respectively, and we can choose g_i so that $\nu_i\ge\mu_i$. We have either $\nu_1=0=\mu_1$; or $\nu_1+\mu_1=n$, $\nu_1\ne 0$, $\mu_1\ne 0$. For $i=2$ and 3, let α_i, β_i be the dimensions of eigenspaces of g_i^3 corrsponding to the eigenvalues 1 and -1 respectively. Since g_i^2, $i=2$, 3, has either 1 or -1 as its eigenvalue, but not both of them, we denote by γ_i the dimension of the eigenspace of g_i^2 with the eigenvalue in $GF(3)$. We have $\alpha_i\ge\nu_i$ and $\beta_i\ge\mu_i$. Also $\alpha_i=\beta_i=0$ iff $g_i^6=-I$, in which case $\nu_i=\mu_i=0$ and γ_i is the dimension of the eigenspace corresponding to -1. If one of ν_i, μ_i is not 0, then γ_i is the dimension of the eigenspace corresponding to 1, and $\gamma_i\ge\nu_i+\mu_i$. Since $|\Omega|=\left[\begin{smallmatrix} n\\ 1\end{smallmatrix}\right]_3=\frac{1}{2}(3^n-1)$, we have $(\frac{1}{2}+2\cdot\frac{1}{6})+\frac{1}{3^n-1}\{\frac{1}{2}(3^{\nu_1}+3^{\mu_1}-2)+\frac{1}{6}\cdot\sum_{i=2}^{3}(3^{\alpha_i}+3^{\beta_i}-2)+\frac{2}{6}\cdot\sum_{i=2}^{3}(3^{\gamma_i}-1)+\frac{2}{6}\cdot\sum_{i=2}^{3}(3^{\nu_i}+3^{\mu_i}-2)\}=1+\frac{4}{3^n-1}$. This identity can be transformed into $3(3^{\nu_1}+3^{\mu_1})+(3^{\alpha_2}+3^{\beta_2}+3^{\alpha_3}+3^{\beta_3})+2(3^{\gamma_2}+3^{\gamma_3})+2(3^{\nu_2}+3^{\mu_2}+3^{\nu_3}+3^{\mu_3})=3^n+5\cdot 3^2$. Suppose $\nu_1=n-1$. Then as $\gamma_i\ge\lceil\frac{n}{3}\rceil\ge 5$ for $i=2$ and 3, we have that the left hand side is greater than $3\cdot 3^{\nu_1}+2\cdot 3^{\gamma_2}>3^n+5\cdot 3^2$. Suppose $\nu_1=n-2$. Then $3(3^{\nu_1}+3^{\mu_1})=3^{n-1}+3^3$, which implies that both γ_2 and γ_3 are $\le n-2$. Also $\nu_1=n-2$ implies that ν_i and μ_i both ≤ 2 for $i=2$ and 3. Since $\langle\overline{g}_i^3,\ \overline{g}_i^2\rangle=\langle\overline{g}_i\rangle$, $i=2$ and 3, we have $\alpha_i+\gamma_i\le n+max\{\nu_i,\mu_i\}\le n+2$ and similarly $\beta_i+\gamma_i\le n+2$. This together with $n-2\ge\gamma_i\ge 5$ gives that both $3^{\alpha_i}+3^{\gamma_i}$ and $3^{\beta_i}+3^{\gamma_i}$ are $\le 3^{n-2}+3^4$. Hence the left hand side is $\le(3^{n-1}+3^3)+4(3^{n-2}+3^4)+4\cdot 3^2<3^n+5\cdot 3^2$. Suppose $\nu_1=n-3$. Then $3(3^{\nu_1}+3^{\mu_1})=3^{n-2}+3^4$. So if one of γ_2, γ_3 is $n-1$, then the other is $\le n-3$. Since ν_2 and μ_2 are both ≤ 3, if one of γ_2, γ_3 is $n-1$, say $\gamma_2=n-1$, then $3^{\alpha_2}+3^{\gamma_2}$ and $3^{\beta_2}+3^{\gamma_2}$ are $\le 3^{n-1}+3^4$. Also as $PSL_n(3)\le\langle\overline{g}_2^2,$

$\overline{g}_3\rangle$, $\gamma_2=n-1$ implies that ν_3 and μ_3 are both ≤ 1, which gives that $\alpha_3+\gamma_3$ and $\beta_3+\gamma_3$ are $\leq n+1$. Then $5\leq\gamma_3\leq n-3$ implies that we have $3^{\alpha_3}+3^{\gamma_3}$ and $3^{\beta_3}+3^{\gamma_3}$ are both $\leq 3^{n-3}+3^4$. So the left hand side is $\leq(3^{n-2}+3^4)+2(3^{n-1}+3^4)+2(3^{n-3}+3^4)+2\cdot 3^3+2\cdot 3<3^n+5\cdot 3^2$. If both γ_2, γ_3 are $\leq n-2$, then both $3^{\alpha_i}+3^{\gamma_i}$ and $3^{\beta_i}+3^{\gamma_i}$ are $\leq 3^{n-2}+3^5$ for $i=2$ and 3. This implies that the left hand side is $\leq(3^{n-2}+3^4)+4(3^{n-2}+3^5)+4\cdot 3^3<3^n+5\cdot 3^2$. Hence we must have $\nu_1\leq n-4$. So $3(3^{\nu_1}+3^{\mu_1})\leq 3^{n-3}+3^5$. It is easy to see that we cannot have both γ_2 and γ_3 equal to $n-1$. So $3^{\gamma_2}+3^{\gamma_3}\leq 3^{n-2}+3^{n-1}$. Suppose one of γ_2, γ_3, say for example $\gamma_2\geq n-2$. Then ν_3 and μ_3 are both ≤ 2, which implies both α_3 and β_3 are $\leq n-3$. So $3^{\alpha_3}+3^{\beta_3}\leq 3^{n-3}+3^3$. Since ν_i and μ_i both $\leq[\frac{n}{2}]\leq n-7$ for $i=2$ and 3, we have $\alpha_2+\gamma_2\leq n+[\frac{n}{2}]$ and $\beta_2+\gamma_2\leq n+[\frac{n}{2}]$. So both α_2 and β_2 are $\leq[\frac{n}{2}]+2\leq n-5$, which gives $3^{\alpha_2}+3^{\beta_2}\leq 2\cdot 3^{n-5}$. Hence in this case the left hand side is $\leq(3^{n-3}+3^5)+2\cdot 3^{n-5}+(3^{n-3}+3^3)+2(3^{n-2}+3^{n-1})+4\cdot 3^{n-7}<3^n+5\cdot 3^2$. Suppose both γ_2 and γ_3 are $\leq n-3$. Since $3^{\alpha_i}+3^{\beta_i}\leq 3^{n-1}+3$ and ν_i, μ_i both $\leq n-7$ for $i=2$ and 3, we have that the left hand side is $\leq(3^{n-3}+3^5)+2(3^{n-1}+3)+4\cdot 3^{n-3}+4\cdot 3^{n-7}<3^n+5\cdot 3^2$. Therefore in conclusion, we have that S is not of type (2,6,6).

(8.18) If $d=1$ and $q=3$, then S is not of type (2,4,5).

Proof. Pick a fixed preimage g_i for each $\overline{g}_i$. Let ν_i, μ_i with $\nu_i\geq\mu_i$ have the same meaning as in (8.17). Let α, β be the dimensions of eigenspaces of g_2^2 corrsponding to the eigenvalues 1 and -1 respectively. We have $\alpha\geq\nu_2+\mu_2$. Since $|\Omega|=\left[{n \atop 1}\right]_3=\frac{1}{2}(3^n-1)$, we have $(\frac{1}{2}+\frac{1}{4}+\frac{1}{5})+\frac{1}{3^n-1}\{\frac{1}{2}(3^{\nu_1}+3^{\mu_1}-2)+\frac{1}{4}(3^{\alpha}+3^{\beta}-2)+\frac{2}{4}(3^{\nu_2}+3^{\mu_2}-2)+\frac{4}{5}(3^{\nu_3}+3^{\mu_3}-2)\}=1+\frac{4}{3^n-1}$. This identity can be transformed into $10(3^{\nu_1}+3^{\mu_1}+3^{\nu_2}+3^{\mu_2})+5(3^{\alpha}+3^{\beta})+16(3^{\nu_3}+3^{\mu_3})=3^n+161$. Clearly, we have $\nu_i\leq n-3$ $\forall$ $1\leq i\leq 3$, and $max\{\alpha,\beta\}\leq n-2$. Suppose $max\{\nu_1,\nu_2\}\geq n-4$. Then ν_3 and μ_3 both ≤ 4; and also $3^{\nu_1}+3^{\mu_1}+3^{\nu_2}+3^{\mu_2}\leq 3^{n-3}+3^4$. Since $3^{\alpha}+3^{\beta}\leq 3^{n-2}+3^2$, we have that the left hand side is $\leq 10(3^{n-3}+3^4)+5(3^{n-2}+3^2)+32\cdot 3^4<3^n+161$. Now suppose $max\{\nu_1,\nu_2\}\leq n-5$. So $3^{\nu_i}+3^{\mu_i}\leq 3^{n-5}+3^5$ for $i=1$ and 2. If one of α, β is $n-2$, then ν_3 and μ_3 both ≤ 2. Hence the left hand side is $\leq 20(3^{n-5}+3^5)+5(3^{n-2}+3^2)+32\cdot 3^2<3^n+161$. If

$max\{\alpha,\beta\}\leq n-3$, then we have that the left hand side is $\leq 20(3^{n-5}+3^5)+5(3^{n-3}+3^3)+16(3^{n-3}+3^3)<3^n+161$. Therefore S is not of type (2,4,5).

(8.19) If $d=1$ and $q=3$, then S is not of type (2,4,6).

Proof. Pick a fixed preimage g_i for each $\overline{g}_i$. Let ν_i, μ_i with $\nu_i\geq\mu_i$ have the same meaning as in (8.17). Let α_λ, β_λ, where $\lambda=1$ or -1, be the dimensions of eigenspaces of g_2^2, g_3^3 corrsponding to the eigenvalues λ respectively. Let γ be the dimension of the eigenspace of g_3^2 with the eigenvalue in $GF(3)$. So $\gamma\geq\lceil\frac{n}{3}\rceil\geq 5$. Then as $PSL_n(3)\leq\langle\overline{g}_2, \overline{g}_3^2\rangle$, we have $\mu_2\leq\nu_2\leq n-5$. Similar as before, we have $(\frac{1}{2}+\frac{1}{4}+\frac{1}{6})+\frac{1}{3^n-1}\{\frac{1}{2}(3^{\nu_1}+3^{\mu_1}-2)+\frac{1}{4}(3^{\alpha_1}+3^{\alpha_{-1}}-2)+\frac{2}{4}(3^{\nu_2}+3^{\mu_2}-2)+\frac{1}{6}(3^{\beta_1}+3^{\beta_{-1}}-2)+\frac{2}{6}(3^\gamma-1)+\frac{2}{6}(3^{\nu_3}+3^{\mu_3}-2)\}=1+\frac{4}{3^n-1}$. This identity can be transformed into $6(3^{\nu_1}+3^{\mu_1}+3^{\nu_2}+3^{\mu_2})+3(3^{\alpha_1}+3^{\alpha_{-1}})+2(3^{\beta_1}+3^{\beta_{-1}})+4\cdot 3^\gamma+4(3^{\nu_3}+3^{\mu_3})=3^n+31\cdot 3$. Denote the left hand side of this identity by x. It is easy to see that $max\{\alpha_1,\alpha_{-1}\}$, ν_1, and γ are all $\leq n-2$. As β_1 and γ both $\geq\nu_3$, we have $\nu_3\leq n-3$. Also if $max\{\beta_1, \beta_{-1}\}=n-1$, then $\gamma\leq n-3$. So $2(3^{\beta_1}+3^{\beta_{-1}})+4\cdot 3^\gamma\leq 2(3^{n-1}+3)+4\cdot 3^{n-3}$. Suppose $\nu_3\geq n-6$. Then α_1, α_{-1}, ν_i, μ_i, $i=1$ and 2, are all ≤ 6. So $x\leq 6\cdot 4\cdot 3^6+3\cdot 2\cdot 3^6+2(3^{n-1}+3)+4\cdot 3^{n-3}+4(3^{n-3}+3^3)<3^n+31\cdot 3$. So $\nu_3\leq n-7$. It is easy to see that $max\{\alpha_1,\alpha_{-1}\}=n-2=\nu_1$ is impossible. Suppose $max\{\alpha_1,\alpha_{-1}\}=n-2$. Then $\nu_3\leq 2$; which implies that $max\{\beta_1, \beta_{-1}\}\leq n-3$ and $2(3^{\beta_1}+3^{\beta_{-1}})\leq 2(3^{n-3}+3^3)$. In this case, if $\gamma=n-2$ and $\nu_1=n-3$, then $6(3^{\nu_1}+3^{\mu_1})+3(3^{\alpha_1}+3^{\alpha_{-1}})+4\cdot 3^\gamma=6(3^{n-3}+3^3)+3(3^{n-2}+3^2)+4\cdot 3^{n-2}=3^n+7\cdot 3^3>3^n+31\cdot 3$. So either both γ and ν_1 are $\leq n-3$; or $\gamma=n-2$ and $\nu_1\leq n-4$. Since $\mu_2\leq\nu_2\leq n-5$, in the first case, we have $x\leq 6(3^{n-3}+3^3)+6(3^{n-5}+3^5)+3(3^{n-2}+3^2)+2(3^{n-3}+3^3)+4\cdot 3^{n-3}+4\cdot 2\cdot 3^2<3^n+31\cdot 3$. In the second case, we have $x\leq 6(3^{n-4}+3^4)+6(3^{n-5}+3^5)+3(3^{n-2}+3^2)+2(3^{n-3}+3^3)+4\cdot 3^{n-2}+4\cdot 2\cdot 3^2<3^n+31\cdot 3$ too. Thus in any case, we have $max\{\alpha_1,\alpha_{-1}\}\leq n-3$. Suppose $\nu_1=n-2$. Then $\nu_2\leq 2$ and $\nu_3\leq 2$. Since $6\cdot 3^{\nu_1}+4\cdot 3^\gamma\geq 6\cdot 3^{n-2}+4\cdot 3^{n-2}>3^n+31\cdot 3$ if $\gamma=n-2$, we must have $\gamma\leq n-3$. Also as before, $\nu_3\leq 2$ and $\gamma\geq 5$ implies $\max\{\beta_1, \beta_{-1}\}\leq n-3$. This together with $\gamma\leq n-3$ implies $2(3^{\beta_1}+3^{\beta_{-1}})+4\cdot 3^\gamma\leq 4(3^{n-3}+3^3)$. Then $x\leq 6(3^{n-2}+3\cdot 3^2)$

$+3(3^{n-3}+3^3)+4(3^{n-3}+3^3)+4\cdot2\cdot3^2<3^n+31\cdot3$. Thus in any case, we have $\nu_1\leq n-3$. Suppose $max\{\beta_1,\ \beta_{-1}\}=n-1$. If $\gamma=n-2$, then $2(3^{\beta_1}+3^{\beta_{-1}})+4\cdot3^{\gamma}=2(3^{n-1}+3)+4\cdot3^{n-2}>3^n+31\cdot3$. So $\gamma\leq n-3$. Since $\gamma\geq5$, we have $\nu_3\geq4$; which implies that ν_1 and $max\{\alpha_1,\alpha_{-1}\}$ are both $\leq n-4$. Then $x\leq6(3^{n-4}+3^4)+6(3^{n-5}+3^5)+3(3^{n-4}+3^4)+2(3^{n-1}+3)+4\cdot3^{n-3}+4(3^{n-7}+3^7)<3^n+31\cdot3$. Thus in any case, we have $max\{\beta_1,\ \beta_{-1}\}\leq n-2$. Also as $max\{\beta_1,\beta_{-1}\}+\gamma\leq n+\nu_3\leq2n-7$, we have $2(3^{\beta_1}+3^{\beta_{-1}})+4\cdot3^{\gamma}\leq4(3^{n-2}+3^{n-5})$. Now as $\nu_1\leq n-3$, $\nu_2\leq n-5$, $max\{\alpha_1,\alpha_{-1}\}\leq n-3$, and $\nu_3\leq n-7$, we have $x\leq6(3^{n-3}+3^3)+6(3^{n-5}+3^5)+3(3^{n-3}+3^3)+4(3^{n-2}+3^{n-5})+4(3^{n-7}+3^7)<3^n+31\cdot3$. Therefore S is not of type (2,4,6).

(8.20) If $d=1$ and $q=3$, then S is not of type (2,3,8).

Proof. Pick a fixed preimage g_i for each $\overline{g}_i$. Let ν_i, μ_i with $\nu_i\geq\mu_i$ have the same meaning as in (8.17). Since $|\overline{g}_2|=3$, we can pick g_2 so that $\mu_2=0$. Thus $\nu_2\geq\lceil\frac{n}{3}\rceil\geq5$. Let α_λ, β_λ, where $\lambda=1$ or -1, be the dimensions of eigenspaces of g_3^4, g_3^2 corrsponding to the eigenvalues λ respectively. We have $(\frac{1}{2}+\frac{1}{3}+\frac{1}{8})+\frac{1}{3^n-1}\{\frac{1}{2}(3^{\nu_1}+3^{\mu_1}-2)+\frac{2}{3}(3^{\nu_2}-1)+\frac{1}{8}(3^{\alpha_1}+3^{\alpha_{-1}}-2)+\frac{2}{8}(3^{\beta_1}+3^{\beta_{-1}}-2)+\frac{4}{8}(3^{\nu_3}+3^{\mu_3}-2)\}=1+\frac{4}{3^n-1}$. This identity can be transformed into $12(3^{\nu_1}+3^{\mu_1}+3^{\nu_3}+3^{\mu_3})+16\cdot3^{\nu_2}+3(3^{\alpha_1}+3^{\alpha_{-1}})+6(3^{\beta_1}+3^{\beta_{-1}})=3^n+59\cdot3$. Denote the left hand side of this identity by x. Since $\nu_2\geq5$, we have ν_1 and ν_3 both $\leq n-5$. Also as $PSL_n(3)\leq\langle\overline{g}_2,\ \overline{g}_3^2\rangle$, we have $\max\{\beta_1,\ \beta_{-1}\}\leq n-5$. This gives $12(3^{\nu_1}+3^{\mu_1}+3^{\nu_3}+3^{\mu_3})+6(3^{\beta_1}+3^{\beta_{-1}})\leq10(3^{n-4}+3^6)$. Also clearly $\nu_2\leq n-3$ and $max\{\alpha_1,\alpha_{-1}\}\leq n-2$. Suppose $\nu_2=n-3$. Then ν_1, ν_3, and $max\{\beta_1,\ \beta_{-1}\}$ are all ≤3. This implies that $x\leq12\cdot4\cdot3^3+16\cdot3^{n-3}+3(3^{n-2}+3^2)+6\cdot2\cdot3^3<3^n+59\cdot3$. So $\nu_2\leq n-4$. This together with $max\{\alpha_1,\alpha_{-1}\}\leq n-2$ implies that $x\leq10(3^{n-4}+3^6)+16\cdot3^{n-4}+3(3^{n-2}+3^2)<3^n+59\cdot3$. Therefore S is not of type (2,3,8).

(8.21) If $d=1$ and $q=3$, then S is not of type (2,3,9).

Proof. Pick a fixed preimage g_i for each $\overline{g}_i$. Let ν_i, μ_i with $\nu_i\geq\mu_i$ have the same meaning as in (8.17). Since $|\overline{g}_2|=3$ and $|\overline{g}_3|=9$, we can pick g_2 and g_3 so that $\mu_2=0$ and $\mu_3=0$. Thus

$\nu_2 \geq \lceil \frac{n}{3} \rceil \geq 5$ and $\nu_3 \geq \lceil \frac{n}{9} \rceil \geq 2$. Let α be the dimension of the eigenspace of g_3^3 corresponding to the eigenvalue 1. So $\alpha \geq \lceil \frac{n}{3} \rceil \geq 5$. We have $(\frac{1}{2}+\frac{1}{3}+\frac{1}{9})+\frac{1}{3^n-1}\{\frac{1}{2}(3^{\nu_1}+3^{\mu_1}-2)+\frac{2}{3}(3^{\nu_2}-1)+\frac{2}{9}(3^{\alpha}-1)+\frac{6}{9}(3^{\nu_3}-1)\}=1+\frac{4}{3^n-1}$. This identity can be transformed into $9(3^{\nu_1}+3^{\mu_1})+12(3^{\nu_2}+3^{\nu_3})+4\cdot 3^{\alpha}=3^n+13\cdot 3^2$. Denote the left hand side of this identity by x. If $\mu_1 \geq 1$, then 3^3 divides x, which is a contradiction. So $\mu_1=0$. Since either ν_1 and μ_1 are both non zero, or both are 0, we have $\nu_1=\mu_1=0$. Then $12(3^{\nu_2}+3^{\nu_3})+4\cdot 3^{\alpha}=3^n+11\cdot 3^2$. But as $\nu_2 \geq 5$, $\nu_3 \geq 2$, and $\alpha \geq 5$, we still have that 3^3 divides the left hand side of the equation, but not the right hand side. So S is not of type (2,3,9).

(8.22) If $d=1$ and $q=3$, then S is not of type (2,3,12).

Proof. Pick a fixed preimage g_i for each $\overline{g}_i$. Let ν_i, μ_i with $\nu_i \geq \mu_i$ have the same meaning as in (8.17). Since $|\overline{g}_2|=3$, we can pick g_2 so that $\mu_2=0$. Thus $\nu_2 \geq \lceil \frac{n}{3} \rceil \geq 5$. Let α_λ, β_λ, γ_λ, where $\lambda=1$ or -1, be the dimensions of eigenspaces of g_3^6, g_3^3, g_3^2 corresponding to the eigenvalues λ respectively. Since $|\overline{g}_3^3|=4$, the eigenvalues of g_3^3 which are not ± 1 are either all in pairs $\{\omega,\omega^3\}$ or all in pairs $\{\omega^2,\omega^6\}$, where ω is such that $GF(3^2)^{\#}=\langle\omega\rangle$. So we have $\beta_1+\beta_{-1}=n-2k$ with $k \geq 1$. Also clearly $\alpha_1 \geq \beta_1+\beta_{-1}$. Let δ be the dimension of the eigenspace of g_3^4 corresponding to the eigenvalue in $GF(3)$. Then $\delta \geq \lceil \frac{n}{3} \rceil \geq 5$. We have $(\frac{1}{2}+\frac{1}{3}+\frac{1}{12})+\frac{1}{3^n-1}\{\frac{1}{2}(3^{\nu_1}+3^{\mu_1}-2)+\frac{2}{3}(3^{\nu_2}-1)+\frac{1}{12}(3^{\alpha_1}+3^{\alpha_{-1}}-2)+\frac{2}{12}(3^{\delta}-1)+\frac{2}{12}(3^{\beta_1}+3^{\beta_{-1}}-2)+\frac{2}{12}(3^{\gamma_1}+3^{\gamma_{-1}}-2)+\frac{4}{12}(3^{\nu_3}+3^{\mu_3}-2)\}=1+\frac{4}{3^n-1}$. This identity can be transformed into $6(3^{\nu_1}+3^{\mu_1})+8\cdot 3^{\nu_2}+(3^{\alpha_1}+3^{\alpha_{-1}})+2\cdot 3^{\delta}+2(3^{\beta_1}+3^{\beta_{-1}})+2(3^{\gamma_1}+3^{\gamma_{-1}})+4(3^{\nu_3}+3^{\mu_3})=3^n+29\cdot 3$. Denote the left hand side of this identity by x. Since $\nu_2 \geq 5$, we have ν_1, ν_3, and $max\{\gamma_1,\gamma_{-1}\}$ are all $\leq n-5$. So $6(3^{\nu_1}+3^{\mu_1})+2(3^{\gamma_1}+3^{\gamma_{-1}})+4(3^{\nu_3}+3^{\mu_3}) \leq 12(3^{n-5}+3^5)=3^{n-3}+3^{n-4}+4\cdot 3^6$. Also clearly $max\{\alpha_1,\alpha_{-1}\}=max\{\beta_1, \beta_{-1}\}=n-1$ and $max\{\alpha_1,\alpha_{-1}\}=\delta=n-1$ are both impossible; and $\nu_2 \leq n-2$. Suppose $\nu_2=n-2$. Then ν_1, ν_3, and $max\{\gamma_1,\gamma_{-1}\}$ are all ≤ 2, which gives that $6(3^{\nu_1}+3^{\mu_1})+2(3^{\gamma_1}+3^{\gamma_{-1}})+4(3^{\nu_3}+3^{\mu_3}) \leq 4\cdot 3^3$. Since $8\cdot 3^{\nu_2}=2\cdot 3^{n-1}+2\cdot 3^{n-2}$, we have $max\{\alpha_1,\alpha_{-1}\}$, δ, and $max\{\beta_1, \beta_{-1}\}$ are all $\leq n-3$. Also as $max\{\alpha_1,\alpha_{-1}\}+\delta \leq$

$n+max\{\gamma_1,\gamma_{-1}\}\leq n+2$, we have $(3^{\alpha_1}+3^{\alpha_{-1}})+2\cdot 3^{\delta}\leq 2(3^{n-3}+3^5)$. Now if $max\{\beta_1,\ \beta_{-1}\}=n-3$, then as $\alpha_1\geq\beta_1+\beta_{-1}=n-2k$ with $k\geq 1$, we must have $k=1$ here, which contradicts to $\alpha_1\leq n-3$. So $max\{\beta_1,\ \beta_{-1}\}\leq n-4$, which gives $2(3^{\beta_1}+3^{\beta_{-1}})\leq 2(3^{n-4}+1)$. Thus $x\leq 4\cdot 3^3+(2\cdot 3^{n-1}+2\cdot 3^{n-2})+2(3^{n-3}+3^5)+2(3^{n-4}+1)<3^n+29\cdot 3$. Hence $\nu_2\leq n-3$. Suppose $\delta=n-1$. If $\nu_2=n-3$, then ν_1, ν_3, and $max\{\gamma_1,\gamma_{-1}\}$ are all ≤ 3. Since $max\{\alpha_1,\alpha_{-1}\}+\delta\leq n+max\{\gamma_1,\gamma_{-1}\}\leq n+3$, we have $\beta_1+\beta_{-1}\leq max\{\alpha_1,\alpha_{-1}\}\leq 4$. Hence $x\leq 24\cdot 3^3+8\cdot 3^{n-3}+2\cdot 3^4+2\cdot 3^{n-1}+4\cdot 3^4<3^n+29\cdot 3$. For $\nu_2\leq n-4$, as $max\{\alpha_1,\alpha_{-1}\}+\delta\leq\ n+max\{\gamma_1,\gamma_{-1}\}\leq 2n-5$, we have $max\{\alpha_1,\alpha_{-1}\}\leq n-4$, which implies that $max\{\beta_1,\ \beta_{-1}\}\leq n-4$ and $(3^{\alpha_1}+3^{\alpha_{-1}})+2\cdot 3^{\delta}\leq 3^{n-4}+3^4+2\cdot 3^{n-1}$. So $x\leq(3^{n-3}+3^{n-4}+4\cdot 3^6)+8\cdot 3^{n-4}+(3^{n-4}+3^4+2\cdot 3^{n-1})+2(3^{n-4}+1)<3^n+29\cdot 3$. Hence $\delta\leq n-2$. This together with $max\{\alpha_1,\alpha_{-1}\}+\delta\leq 2n-5$ implies that $(3^{\alpha_1}+3^{\alpha_{-1}})+2\cdot 3^{\delta}\leq 3^{n-1}+3+2\cdot 3^{n-4}$. Also as $\nu_2\leq n-3$ and $\beta_1+\beta_{-1}\leq n-2$, we have $8\cdot 3^{\nu_2}+2(3^{\beta_1}+3^{\beta_{-1}})\leq 3^{n-1}+3^{n-2}+\ 2\cdot 3^{n-3}+2$. Thus $x\leq(3^{n-3}+3^{n-4}+4\cdot 3^6)+(3^{n-1}+3^{n-2}+2\cdot 3^{n-3}+2)+(3^{n-1}+3+2\cdot 3^{n-4})<3^n+29\cdot 3$. So S is not of type (2,3,12).

(8.23) If $d=1$ and $q=4$, then S is not of type (2,4,5).

Proof. Since for $i=1$, 2, and 3, $(|\overline{g}_i|,q-1)=1$, we can choose a preimage g_i of $\overline{g}_i$ so that $g_i^k=1$, where $k=|\overline{g}_i|$. Then if g_i has an eigenvalue in $GF(4)$, it must be 1; and similar for g_2^2. Let ν_i be the dimension of $C_V(g_i)$ for $i=1$, 2, and 3. Let α be the dimension of $C_V(g_2^2)$. Since $|\overline{\Omega}|=\left[\begin{smallmatrix} n\\ 1\end{smallmatrix}\right]_4=\frac{1}{3}(4^n-1)$, we have $(\frac{1}{2}+\frac{1}{4}+\frac{1}{5})+\frac{1}{4^n-1}\{\frac{1}{2}(4^{\nu_1}-1)+\frac{1}{4}(4^{\alpha}-1)+\frac{2}{4}(4^{\nu_2}-1)+\frac{4}{5}(4^{\nu_3}-1)\}=1+\frac{6}{4^n-1}$. This identity can be transformed into $10(4^{\nu_1}+4^{\nu_2})+5\cdot 4^{\alpha}+16\cdot 4^{\nu_3}=4^n+5\cdot 2^5$. Since $\nu_1\geq\lceil\frac{n}{2}\rceil\geq 5$, $\nu_2\geq\lceil\frac{n}{4}\rceil\geq 3$, and $\alpha\geq\lceil\frac{n}{2}\rceil\geq 5$, if $\nu_3\geq 1$, then 2^6 divides the left hand side of the equation but not the right hand side. So $\nu_3=0$. Then we have $10(4^{\nu_1}+4^{\nu_2})+5\cdot 4^{\alpha}=4^n+9\cdot 2^4$. But this time we still have that 2^5 divides the left hand side of the equation but not the right hand side. So S is not of type (2,4,5).

(8.24) If $d=1$ and $q=4$, then S is not of type (2,4,6).

Proof. For $i=1$ and 2, we choose g_i in the same way as in (8.23), and let ν_i and α have the same meaning as in (8.23). For g_3, let μ_λ be the dimension of the eigenspace corresponding to the eigenvalue $\lambda \in GF(4)^{\#}=\langle\omega\rangle$. Similarly, let γ_λ be the dimension of the eigenspace of g_3^2 corresponding to the eigenvalue λ. Thus $\gamma_{\lambda^2} \geq \mu_\lambda \ \forall \ \lambda \in \langle\omega\rangle$. So if $g_3^6=\omega^2 I$ or ωI, then all μ_λ and all γ_λ are 0. For g_3^3, it can have at most one eigenvalue in $GF(4)^{\#}$; they are 1, ω, ω^2 according to $g_3^6=I$, $\omega^2 I$, ωI, respectively. Let β be the dimension of this eigenspace. Thus we have $(\frac{1}{2}+\frac{1}{4}+\frac{1}{6})+\frac{1}{4^n-1}\{\frac{1}{2}(4^{\nu_1}-1)+\frac{1}{4}(4^{\alpha}-1)+\frac{2}{4}(4^{\nu_2}-1)+\frac{1}{6}(4^{\beta}-1)+\frac{2}{6}(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}}-3)+\frac{2}{6}(4^{\mu_1}+4^{\mu_\omega}+4^{\mu_{\omega^2}}-3)\}=1+\frac{6}{4^n-1}$. This identity can be transformed into $6(4^{\nu_1}+4^{\nu_2})+3\cdot 4^{\alpha}+2\cdot 4^{\beta}+4(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}}+4^{\mu_1}+4^{\mu_\omega}+4^{\mu_{\omega^2}})=4^n+7\cdot 4^2$. Denote the left hand side of this equation as x. As $\nu_1 \geq \lceil \frac{n}{2} \rceil \geq 5$, we have $\nu_2 \leq n-5$, and each $\mu_\lambda \leq n-5$. As $\nu_2 \geq \lceil \frac{n}{4} \rceil \geq 3$, we have $\nu_1 \leq n-3$, and each $\gamma_\lambda \leq n-3$. Suppose $\alpha=n-1$. Then each $\mu_\lambda \leq 1$, which gives $4(4^{\mu_1}+4^{\mu_\omega}+4^{\mu_{\omega^2}}) \leq 3\cdot 4^2$. Also clearly $\beta \leq n-2$. Since $\langle g_3^3, \ g_3^2\rangle=\langle g_3\rangle$, we have $\gamma_\lambda+\beta \leq n+max\{\mu_\lambda\} \leq n+1$, which gives $2\cdot 4^{\beta}+4(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}}) \leq 2\cdot 4^{n-2}+3\cdot 4^4$. Thus $x \leq 6(4^{n-3}+4^{n-5})+3\cdot 4^{n-1}+(2\cdot 4^{n-2}+3\cdot 4^4)+3\cdot 4^2<4^n+7\cdot 4^2$. So $\alpha \leq n-2$. Then $3\cdot 4^{\alpha}+2\cdot 4^{\beta} \leq 3\cdot 4^{n-2}+2\cdot 4^{n-1}$. Since $6(4^{\nu_1}+4^{\nu_2})+4(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}}+4^{\mu_1}+4^{\mu_\omega}+4^{\mu_{\omega^2}}) \leq 6(4^{n-3}+4^{n-5})+4(4^{n-3}+4^3+1)+4(4^{n-5}+4^5+1)$, we have $x \leq 3\cdot 4^{n-1}+4^{n-2}+2\cdot 4^{n-3}+2\cdot 4^{n-4}+2\cdot 4^{n-5}+4^6+4^4+2\cdot 4 <4^n+7\cdot 4^2$. So S is not of type (2,4,6).

(8.25) If $d=1$ and $q=4$, then S is not of type (2,3,8).

Proof. For $i=1$ and 3, we can choose a preimage g_i of $\overline{g}_i$ so that $g_i^k=1$, where $k=|\overline{g}_i|$. Let ν_i be the dimension of $C_V(g_i)$ for $i=1$ and 3. Let α, β be the dimension of $C_V(g_3^4)$ and $C_V(g_3^2)$ respectively. For g_2, let μ_λ be the dimension of the eigenspace corresponding to the eigenvalue $\lambda \in GF(4)^{\#}=\langle\omega\rangle$. So if $g_2^3=\omega^2 I$ or ωI, then all μ_λ are 0; and $\mu_1+\mu_\omega+\mu_{\omega^2}=n$ if $g_2^3=I$. We have $(\frac{1}{2}+\frac{1}{3}+\frac{1}{8})+\frac{1}{4^n-1}\{\frac{1}{2}(4^{\nu_1}-1)+\frac{2}{3}(4^{\mu_1}+4^{\mu_\omega}+4^{\mu_{\omega^2}}-3)+\frac{1}{8}(4^{\alpha}-1)+\frac{2}{8}(4^{\beta}-1)+\frac{4}{8}(4^{\nu_3}-1)\}=1+\frac{6}{4^n-1}$. This identity can be transformed into $12(4^{\nu_1}+4^{\nu_3})+3\cdot 4^{\alpha}+6\cdot 4^{\beta}+$

$4^2(4^{\mu_1}+4^{\mu_\omega}+4^{\mu_{\omega^2}})=4^n+7\cdot 2^5$. Since $\nu_1\geq\lceil\frac{n}{2}\rceil\geq 5$, $\nu_3\geq\lceil\frac{n}{8}\rceil\geq 2$, $\alpha\geq\lceil\frac{n}{2}\rceil\geq 5$, and $\beta\geq\lceil\frac{n}{4}\rceil\geq 3$, we have 2^2 divides $(4^{\mu_1}+4^{\mu_\omega}+4^{\mu_{\omega^2}})-7\cdot 2$. This can be satisfied only when there are exactly two zeroes among μ_1, μ_ω, μ_{ω^2}, which contradicts to that g_2 is not a scalar. So S is not of type (2,3,8).

(8.26) If $d=1$ and $q=4$, then S is not of type (2,3,9).

Proof. Let ν_1 have the same meaning as in (8.23). For g_2, g_3^3, g_3, let α_λ, β_λ, γ_λ, be the dimension of the eigenspace corresponding to the eigenvalue $\lambda\in GF(4)^{\#}=\langle\omega\rangle$ respectively. We have $(\frac{1}{2}+\frac{1}{3}+\frac{1}{9})+\frac{1}{4^n-1}\{\frac{1}{2}(4^{\nu_1}-1)+\frac{2}{3}(4^{\alpha_1}+4^{\alpha_\omega}+4^{\alpha_{\omega^2}}-3)+\frac{2}{9}(4^{\beta_1}+4^{\beta_\omega}+4^{\beta_{\omega^2}}-3)+\frac{6}{9}(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}}-3)\}=1+\frac{6}{4^n-1}$. This identity can be transformed into $9\cdot 4^{\nu_1}+12(4^{\alpha_1}+4^{\alpha_\omega}+4^{\alpha_{\omega^2}})+4(4^{\beta_1}+4^{\beta_\omega}+4^{\beta_{\omega^2}})+12(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}})=4^n+25\cdot 2^3$. Denote the left hand side of this equation as x. Since $\nu_1\geq\lceil\frac{n}{2}\rceil\geq 5$, we have all α_λ, γ_λ are $\leq n-5$, which gives $12(4^{\alpha_1}+4^{\alpha_\omega}+4^{\alpha_{\omega^2}})+12(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}})\leq 12\cdot 6\cdot 4^{n-5}=4^{n-2}+6\cdot 4^{n-5}$. Also clearly we have ν_1 and all β_λ are $\leq n-2$, which gives $9\cdot 4^{\nu_1}+4(4^{\beta_1}+4^{\beta_\omega}+4^{\beta_{\omega^2}})\leq 9\cdot 4^{n-2}+4(4^{n-2}+4^2+1)=13\cdot 4^{n-2}+4^3+4^2$. Hence $x\leq 14\cdot 4^{n-2}+6\cdot 4^{n-5}+4^3+4^2<4^n+25\cdot 2^3$. So S is not of type (2,3,9).

(8.27) If $d=1$ and $q=4$, then S is not of type (2,3,12).

Proof. Let ν_1, β, δ be the dimension of the eigenspace of g_1, g_3^6, g_3^3 corresponding to the eigenvalue in $GF(4)$ respectively. Let α_λ, γ_λ, ξ_λ, μ_λ be the dimensions of eigenspces of g_2, g_3^4, g_3^2, g_3 corresponding to the eigenvalue $\lambda\in GF(4)^{\#}=\langle\omega\rangle$ respectively. We have $(\frac{1}{2}+\frac{1}{3}+\frac{1}{12})+\frac{1}{4^n-1}\{\frac{1}{2}(4^{\nu_1}-1)+\frac{2}{3}(4^{\alpha_1}+4^{\alpha_\omega}+4^{\alpha_{\omega^2}}-3)+\frac{1}{12}(4^{\beta}-1)+\frac{2}{12}(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}}-3)+\frac{2}{12}(4^{\delta}-1)+\frac{2}{12}(4^{\xi_1}+4^{\xi_\omega}+4^{\xi_{\omega^2}}-3)+\frac{4}{12}(4^{\mu_1}+4^{\mu_\omega}+4^{\mu_{\omega^2}}-3)\}=1+\frac{6}{4^n-1}$. This identity can be transformed into $6\cdot 4^{\nu_1}+8(4^{\alpha_1}+4^{\alpha_\omega}+4^{\alpha_{\omega^2}})+4^{\beta}+2(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}})+2\cdot 4^{\delta}+2(4^{\xi_1}+4^{\xi_\omega}+4^{\xi_{\omega^2}})+4(4^{\mu_1}+4^{\mu_\omega}+4^{\mu_{\omega^2}})=4^n+2^7$. Denote the left hand side of this equation as x. We have $\nu_1\geq\lceil\frac{n}{2}\rceil\geq 5$. As $|\overline{g}_3^3|=4$, we have $3\leq\lceil\frac{n}{4}\rceil\leq\delta\leq n-2$. Also clearly $\gamma_{\lambda^2}\geq\xi_\lambda$. Thus $max\{\xi_\lambda\}\leq n-2$. It is clear that $\nu_1\leq n-2$. Suppose $\nu_1=n-2$. Then $max\{\alpha_\lambda\}$ and $max\{\mu_\lambda\}$ are both ≤ 2; which implies each

$\alpha_\lambda=0$, and $4(4^{\mu_1}+4^{\mu_\omega}+4^{\mu_{\omega^2}})\leq 3\cdot 4^3$. If $max\{\gamma_\lambda\}=n-1$, then $6\cdot 4^{\nu_1}+2(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}})>14\cdot 4^{n-2}$; which implies that $max\{\xi_\lambda\}\leq n-3$ and $\beta\leq n-2$. Also since $max\{\gamma_\lambda\}+\delta\leq n+max\{\mu_\lambda\}\leq n+2$, we have $\delta=3$. Then $x\leq 6\cdot 4^{n-2}+8\cdot 3+4^{n-2}+2(4^{n-1}+4+1)+2\cdot 4^3+2(4^{n-3}+4^3+1)+3\cdot 4^3<4^n+2^7$. So $max\{\gamma_\lambda\}\leq n-2$. Then $2(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}})\leq 2(4^{n-2}+4^2+1)$. Also as $\delta+max\{\xi_\lambda\}\leq n+max\{\mu_\lambda\}\leq n+2$, we have $2\cdot 4^\delta+2(4^{\xi_1}+4^{\xi_\omega}+4^{\xi_{\omega^2}})\leq 2(4^{n-2}+3\cdot 4^4)$. Thus $x\leq 6\cdot 4^{n-2}+8\cdot 3+4^{n-1}+2(4^{n-2}+4^2+1)+2(4^{n-2}+3\cdot 4^4)+3\cdot 4^3<4^n+2^7$. *Hence* $\nu_1\leq n-3$. If $max\{\gamma_\lambda\}=n-1$, then as $max\{\gamma_\lambda\}+\delta\leq n+max\{\mu_\lambda\}\leq 2n-5$, we have $\delta\leq n-4$. So $2(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}})+2\cdot 4^\delta\leq 2(4^{n-1}+4+1)+2\cdot 4^{n-4}$. If $max\{\gamma_\lambda\}\leq n-2$, then $2(4^{\gamma_1}+4^{\gamma_\omega}+4^{\gamma_{\omega^2}})+2\cdot 4^\delta\leq 2(4^{n-2}+4^2+1)+2\cdot 4^{n-2}<2(4^{n-1}+4+1)+2\cdot 4^{n-4}$. Also as $\nu_1\geq\lceil\frac{n}{2}\rceil\geq 5$, $max\{\alpha_\lambda\}$ and $max\{\mu_\lambda\}$ *are* both $\leq n-5$; which gives $8(4^{\alpha_1}+4^{\alpha_\omega}+4^{\alpha_{\omega^2}})+4(4^{\mu_1}+4^{\mu_\omega}+4^{\mu_{\omega^2}})\leq 12(4^{n-5}+4^5+1)=3\cdot 4^{n-4}+3\cdot 4^6+12$. So whether $max\{\gamma_\lambda\}$ equal to $n-1$ or not, we have $x\leq 6\cdot 4^{n-3}+(3\cdot 4^{n-4}+3\cdot 4^6+12)+4^{n-1}+2(4^{n-1}+4+1)+2\cdot 4^{n-4}+2(4^{n-2}+4^2+1)<4^n+2^7$. Hence S is not of type (2,3,12).

(8.28) If $d=1$ and $q=5$, then S is not of type (2,4,5).

Proof. Since $(5,q-1)=1$, we can choose a preimage g_3 of $\overline{g}_3$ so that $g_3^5=1$. Let ν_3 be the dimension of $C_V(g_3)$. Clearly g_1 can have only at most two eigenvalues in $GF(5)$; let α, β be the dimensions of their eigenspaces respectively. Similarly, let γ_λ be the dimension of the eigenspace of g_2 corresponding to $\lambda\in GF(5)^{\#}=\langle\omega\rangle$. Let δ, ξ be the dimensions of the eigenspaces of g_2^2 with corresponding eigenvalues in $GF(5)$. Since $|\bar{\Omega}|=\left[\begin{smallmatrix} n \\ 1\end{smallmatrix}\right]_5=\frac{1}{4}(5^n-1)$, we have $(\frac{1}{2}+\frac{1}{4}+\frac{1}{5})+\frac{1}{5^n-1}\{\frac{1}{2}(5^\alpha+5^\beta-2)+\frac{1}{4}(5^\delta+5^\xi-2)+\frac{2}{4}(5^{\gamma_1}+5^{\gamma_\omega}+5^{\gamma_{\omega^2}}+5^{\gamma_{\omega^3}}-4)+\frac{4}{5}(5^{\nu_3}-1)\}=1+\frac{8}{5^n-1}$. This identity can be transformed into $10(5^\alpha+5^\beta)+5(5^\delta+5^\xi)+10(5^{\gamma_1}+5^{\gamma_\omega}+5^{\gamma_{\omega^2}}+5^{\gamma_{\omega^3}})+16\cdot 5^{\nu_3}=5^n+49\cdot 5$. Denote the left hand side of this equation as x. Clearly $\nu_3\leq n-2$. Suppose $\nu_3=n-2$. Then $max\{\alpha,\ \beta\}$, $max\{\delta,\ \xi\}$, and $max\{\gamma_\lambda\}$ are all ≤ 2. This implies that $x\leq(20+10+40)5^2+16\cdot 5^{n-2}=3\cdot 5^{n-1}+5^{n-2}+2\cdot 5^4+4\cdot 5^3<5^n+49\cdot 5$. So $\nu_3\leq n-3$. We have $\nu_3\geq\lceil\frac{n}{5}\rceil\geq 2$. So $16\cdot 5^{\nu_3}>49\cdot 5$, which implies that $max\{\alpha,\ \beta\}$, $max\{\delta,\ \xi\}$, and $max\{\gamma_\lambda\}$ are all $\leq n-2$; and it is easy to see that these three maximum cannot all equal to $n-2$ at the same time. Then $10(5^\alpha+5^\beta)+5(5^\delta+5^\xi)+10(5^{\gamma_1}+5^{\gamma_\omega}+5^{\gamma_{\omega^2}}+5^{\gamma_{\omega^3}})\leq 10(5^{n-2}+5^2)+5(5^{n-3}+5^3)+10(5^{n-2}+5^2+1+1)=4\cdot5^{n-1}+5^{n-2}+5^4+4\cdot5^3+4\cdot5$. Hence $x\leq4\cdot5^{n-1}+5^{n-2}+5^4+4\cdot5^3+4\cdot5+16\cdot5^{n-3}=4\cdot5^{n-1}+4\cdot5^{n-2}+5^{n-3}+4\cdot5^3+4\cdot5<5^n+49\cdot5$. So S is not of type (2,4,5).

(8.29) If $d=1$ and $q=5$, then S is not of type (2,4,6).

Proof. Let α, β, γ_λ, δ, and ξ have the same meaning as in (8.28). It is easy to see that there are at most 2, 1, 2 eigenvalues in $GF(5)$ for g_3^3, g_3^2, g_3 respectively. Denote the dimensions of their corresponding eigenspaces by s, t, u, v, w in that order. Then we have $(\frac{1}{2}+\frac{1}{4}+\frac{1}{6})+\frac{1}{5^n-1}\{\frac{1}{2}(5^\alpha+5^\beta-2)+\frac{1}{4}(5^\delta+5^\xi-2)+\frac{2}{4}(5^{\gamma_1}+5^{\gamma_\omega}+5^{\gamma_{\omega^2}}+5^{\gamma_{\omega^3}}-4)+\frac{1}{6}(5^s+5^t-2)+\frac{2}{6}(5^u-1)+\frac{2}{6}(5^v+5^w-2)\}=1+\frac{8}{5^n-1}$. This identity can be transformed into $6(5^\alpha+5^\beta)+3(5^\delta+5^\xi)+6(5^{\gamma_1}+5^{\gamma_\omega}+5^{\gamma_{\omega^2}}+5^{\gamma_{\omega^3}})+2(5^s+5^t)+4\cdot5^u+4(5^v+5^w)=5^n+153$. Denote the left hand side of this equation as x. Clearly $max\{\alpha,\ \beta\}\leq n-2$ and $max\{\gamma_\lambda\}\leq n-2$. Since g_3^2, g_3 both have a simple submodule of dimension ≥2, we have $u\leq n-2$ and $max\{v,w\}\leq n-2$. Also either all $\gamma_\lambda=0$ or their sum equal to n. For each of the pairs $\{\alpha,\beta\}$, $\{s,t\}$, $\{\delta,\xi\}$, either both numbers in the pair are 0, or both non zero and their sum equal to n. Suppose $max\{\delta,\xi\}=n-1=max\{s,t\}$. Then $\alpha=\beta=0$ and $\gamma_\lambda=0\ \forall\gamma$. Thus $6(5^\alpha+5^\beta)+3(5^\delta+5^\xi)+6(5^{\gamma_1}+5^{\gamma_\omega}+5^{\gamma_{\omega^2}}+5^{\gamma_{\omega^3}})+2(5^s+5^t)=5^n+61$. This implies $5^u+5^v+5^w=23$, which can be seen easily impossible. Suppose $max\{\delta,\xi\}=n-1$. So $max\{s,t\}\leq n-2$. Also $max\{v,w\}\leq1$. If $max\{\alpha,\beta\}=n-2=u$, then $6(5^\alpha+5^\beta)+3(5^\delta+5^\xi)+4\cdot5^u=5^n+165$, which is a contradiction. If $max\{\alpha,\beta\}=n-2$ and $u\leq n-3$, then $\gamma_\lambda=0\ \forall\gamma$ and $max\{s,t\}+u\leq n+max\{v,w\}\leq\ n+1$ implies $2(5^s+5^t)+4\cdot5^u\leq2(5^{n-2}+5^2)+4\cdot5^3$. This leads to $x\leq6(5^{n-2}+5^2)+3(5^{n-1}+5)+24+2(5^{n-2}+5^2)+4\cdot5^3+40<5^n+153$. If $max\{\alpha,\beta\}\leq n-3$ and $u\geq n-3$, then $\gamma_\lambda\leq3\ \forall\gamma$. Then $x\leq6(5^{n-3}+5^3)+3(5^{n-1}+5)+6\cdot4\cdot5^3+2(5^{n-2}+5^2)+4\cdot5^{n-2}+40<5^n+153$. If $max\{\alpha,\beta\}\leq n-3$ and $u\leq n-4$, then $x\leq6(5^{n-3}+5^3)+3(5^{n-1}+5)+6(5^{n-2}+5^2+2)+2(5^{n-2}+5^2)+$

$4\cdot5^{n-4}+40<5^n+153$. Hence $max\{\delta,\xi\}\le n-2$. Suppose $max\{s,t\}=n-1$. Since $max\{\alpha,\beta\}+max\{\gamma_\lambda\}\le n$, we have $6(5^\alpha+5^\beta)+6(5^{\gamma_1}+5^{\gamma_\omega}+5^{\gamma_{\omega^2}}+5^{\gamma_{\omega^3}})\le 6(5^{n-2}+5^3)$. If $max\{\delta,\xi\}=n-2$, then $max\{v,w\}\le 2$; which gives $x\le 6(5^{n-2}+5^3)+3(5^{n-2}+5^2)+2(5^{n-1}+5)+4\cdot5^{n-2}+4\cdot2\cdot5^2<5^n+153$. If $max\{\delta,\xi\}\le n-3$, then $x\le 6(5^{n-2}+5^3)+3(5^{n-3}+5^3)+2(5^{n-1}+5)+4\cdot5^{n-2}+4(5^{n-2}+5^2)<5^n+153$. Hence $max\{s,t\}\le n-2$. Then as $max\{\delta,\xi\}\le n-2$, $u\le n-2$ and $max\{v,w\}\le n-2$, we have $x\le 6(5^{n-2}+5^3)+3(5^{n-2}+5^2)+2(5^{n-2}+5^2)+4\cdot5^{n-2}+4(5^{n-2}+5^2)<5^n+153$. So S is not of type (2,4,6).

(8.30) S has no involution.

Proof. This follows from (8.16) to (8.29).

(8.31) S has no element of order 3.

Proof. Suppose without loss of generality that $|\overline{g}_1|=3$. Consider $d\ge 2$ first. If $\nu(\overline{g}_1)\ge n-2$, then by (3.3) we have $|\overline{g}_2|$ and $|\overline{g}_3|$ are both $\ge\lceil\frac{n}{2}\rceil\ge 5$. Similar as in (8.16), for $|\overline{g}_i|\ge 9$, $i=2$ or 3, we have $\mathfrak{U}(\overline{g}_i)\le\frac{225}{128}\cdot\frac{1}{9}+\frac{1}{4^5}$. Thus for $|\overline{g}_i|\ge 5$, we have $\mathfrak{U}(\overline{g}_i)\le max\{\frac{1}{5}+\frac{2}{5\cdot4^3}, \frac{56}{243}+\frac{1}{2\cdot3^7}, \frac{55}{343}+\frac{6}{7^8}, \frac{17}{108}+\frac{1}{8\cdot3^7}, \frac{225}{128}\cdot\frac{1}{9}+\frac{1}{4^5}\}=\frac{56}{243}+\frac{1}{2\cdot3^7}$. This gives $\sum_{i=1}^{3}\mathfrak{U}(\overline{g}_i)\le\frac{2675}{6561}+2\cdot(\frac{56}{243}+\frac{1}{2\cdot3^7})\le 1$. Hence $\nu(\overline{g}_1)\le n-3$, which gives that $\mathfrak{U}(\overline{g}_1)\le\frac{1}{3}+\frac{2}{3^6}$ by (8.1). Similarly if $|\overline{g}_i|=3$ for $i=2$ or 3, we have $\mathfrak{U}(\overline{g}_i)\le\frac{1}{3}+\frac{2}{3^6}$. Since one of $\overline{g}_2$, $\overline{g}_3$ is of order ≥ 4, say $|\overline{g}_3|\ge 4$, we have $\mathfrak{U}(\overline{g}_3)\le max\{\frac{47}{162}+\frac{1}{4\cdot3^7}, \frac{56}{243}+\frac{1}{2\cdot3^7}\}=\frac{47}{162}+\frac{1}{4\cdot3^7}$. Thus we have $\sum_{i=1}^{3}\mathfrak{U}(\overline{g}_i)\le 2\cdot(\frac{1}{3}+\frac{2}{3^6})+(\frac{47}{162}+\frac{1}{4\cdot3^7})\le 1$. Therefore we must have $d=1$.

Suppose $\nu(\overline{g}_1)=n-1$. Thus $|\overline{g}_2|$ and $|\overline{g}_3|$ are both $\ge n\ge 9$. For $i=2$ and 3, if $9\le|\overline{g}_i|\le 14$, then we have $\mathfrak{U}(\overline{g}_i)\le max\{\frac{17}{81}+\frac{2}{3^{10}}+\frac{2}{3^7}, \frac{27}{125}+\frac{1}{2\cdot5^7}, \frac{21}{121}+\frac{10}{11^9}, \frac{2}{13}+\frac{11}{12\cdot13^8}, \frac{25}{169}+\frac{12}{13^9}, \frac{52}{343}+\frac{1}{2\cdot7^7}\}=\frac{27}{125}+\frac{1}{2\cdot5^7}$. For $|\overline{g}_i|\ge 15$, we have $\mathfrak{U}(\overline{g}_i)\le max\{\frac{460}{243}\cdot\frac{1}{15}+\frac{1}{27}+\frac{1}{3^6}, (\frac{7}{4}+\frac{3}{4^8})\frac{1}{15}+(\frac{1}{4^2}+\frac{1}{4^7}), (\frac{12}{5}+\frac{7}{5^8})\frac{1}{15}+(\frac{1}{5^2}+\frac{1}{5^7}), \frac{37}{13}\cdot\frac{1}{15}+(\frac{1}{7^2}+\frac{1}{7^7})\}=\frac{37}{13}\cdot\frac{1}{15}+(\frac{1}{7^2}+\frac{1}{7^7})<\frac{27}{125}+\frac{1}{2\cdot5^7}$. Then $\sum_{i=1}^{3}\mathfrak{U}(\overline{g}_i)\le\frac{10937}{19683}+2(\frac{27}{125}+\frac{1}{2\cdot5^7})\le 1$. Hence we must have $\nu(\overline{g}_1)\le n-2$; which implies that $\mathfrak{U}(\overline{g}_1)\le\frac{11}{27}+\frac{2}{3^8}$. Assume without loss of generality that in the following $|\bar{g}_2|\le|\bar{g}_3|$. Now if $|\bar{g}|\ge 7$, then $\mathcal{U}(\bar{g})\le max\{\frac{13}{49}+\frac{6}{7^9},\ \frac{1}{4}+\frac{13}{8\cdot 3^8},\ \frac{27}{125}+\frac{1}{2\cdot 5^7}\}=\frac{13}{49}+\frac{6}{7^9}$. Since $\nu(\bar{g}_1)\ge\lceil\frac{n}{3}\rceil\ge 3$, both $\nu(\bar{g}_2)$ and $\nu(\bar{g}_3)$ are $\le n-3$. Thus $\mathcal{U}(\bar{g}_2)\le max\{\frac{41}{128}+\frac{33}{4^9},\ \frac{31}{135}+\frac{4}{5\cdot 3^6},\ \frac{25}{81}+\frac{23}{2\cdot 3^9},\ \frac{13}{49}+\frac{6}{7^9}\}=\frac{41}{128}+\frac{33}{4^9}$ if $|\bar{g}_2|\ge 4$; which gives $\mathcal{U}(\bar{g}_1)+\mathcal{U}(\bar{g}_2)\le(\frac{11}{27}+\frac{2}{3^8})+(\frac{41}{128}+\frac{33}{4^9})$. If $|\bar{g}_2|=3$, then both $\nu(\bar{g}_1)$ and $\nu(\bar{g}_2)$ are $\le n-3$, which implies $\mathcal{U}(\bar{g}_1)+\mathcal{U}(\bar{g}_2)\le 2(\frac{29}{81}+\frac{2}{3^7})<(\frac{11}{27}+\frac{2}{3^8})+(\frac{41}{128}+\frac{33}{4^9})$. So if $|\bar{g}_3|\ge 7$, then we have $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\le(\frac{11}{27}+\frac{2}{3^8})+(\frac{41}{128}+\frac{33}{4^9})+(\frac{13}{49}+\frac{6}{7^9})\le 1$. Hence we must have both $|\bar{g}_2|$ and $|\bar{g}_3|$ are ≤ 6.

Suppose $|\bar{g}_2|=|\bar{g}_3|=6$. If $q\ge 4$, then as $\nu(\bar{g}_1)\le n-2$, we have $\mathcal{U}(\bar{g}_1)\le\frac{3}{8}+\frac{2}{3\cdot 4^7}$. Then $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\le(\frac{3}{8}+\frac{2}{3\cdot 4^7})+2(\frac{25}{81}+\frac{23}{2\cdot 3^9})\le 1$. So $q=3$. Thus $\nu(\bar{g}_1)\ge\lceil\frac{n}{3}\rceil\ge 5$, which implies $\nu(\bar{g}_2)\le n-5$, and thus $\nu(\bar{g}_2^3)+\nu(\bar{g}_2^2)\le n+\nu(\bar{g}_2)\le 2n-5$. Then $\mathcal{N}(\bar{g}_2^3)+2\mathcal{N}(\bar{g}_2^2)\le(\frac{1}{3^4}+\frac{1}{3^9})+2(\frac{1}{3}+\frac{1}{3^{12}})$, which gives $\mathcal{U}(\bar{g}_2)\le\frac{1}{6}\{1+(\frac{1}{3^4}+\frac{1}{3^9})+2(\frac{1}{3}+\frac{1}{3^{12}})+2(\frac{1}{3^5}+\frac{1}{3^8})\}=\frac{5}{18}+\frac{5}{2\cdot 3^6}+\frac{7}{2\cdot 3^{10}}+\frac{1}{3^{13}}$. It is similar for $\mathcal{U}(\bar{g}_3)$. Hence $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\le(\frac{11}{27}+\frac{2}{3^8})+2(\frac{5}{18}+\frac{5}{2\cdot 3^6}+\frac{7}{2\cdot 3^{10}}+\frac{1}{3^{13}})\le 1$. So S is not of type (3,6,6).

Suppose $|\bar{g}_2|=5$. Then $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\le(\frac{11}{27}+\frac{2}{3^8})+(\frac{31}{135}+\frac{4}{5\cdot 3^6})+(\frac{25}{81}+\frac{23}{2\cdot 3^9})\le 1$.

Suppose $|\bar{g}_2|=4$. Then as $\nu(\bar{g}_2)\ge\lceil\frac{n}{4}\rceil\ge 3$, we have $\nu(\bar{g}_1)\le n-3$; which implies that $\mathcal{U}(\bar{g}_1)\le\frac{29}{81}+\frac{2}{3^7}$. Then $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\le(\frac{29}{81}+\frac{2}{3^7})+2(\frac{41}{128}+\frac{33}{4^9})\le 1$.

Suppose $|\bar{g}_2|=3$. Then both $\nu(\bar{g}_1)$ and $\nu(\bar{g}_2)$ are $\le n-3$ if $q\ge 4$; and both $\le n-5$ if $q=3$. Consider $q\ge 5$ first. Then by (8.11), $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\le 2(\frac{127}{375}+\frac{2}{2\cdot 5^6})+(\frac{41}{128}+\frac{33}{4^9})\le 1$. Suppose $q=4$. Since $\nu(\bar{g}_1)+\nu(\bar{g}_2)\le n$ and both of them are $\le n-3$, we have $\mathcal{N}(\bar{g}_1)+\mathcal{N}(\bar{g}_2)\le(\frac{1}{4^3}+\frac{1}{4^6})+\frac{3}{4^6}$; which implies $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\le\frac{2}{3}+\frac{2}{3}\{(\frac{1}{4^3}+\frac{1}{4^6})+\frac{3}{4^6}\}+(\frac{41}{128}+\frac{33}{4^9})\le 1$. Suppose $q=3$. Then both $\mathcal{U}(\bar{g}_1)$ and $\mathcal{U}(\bar{g}_2)$ are $\le\frac{1}{3}\{1+2(\frac{1}{3^5}+\frac{1}{3^8})\}=\frac{1}{3}+\frac{2}{3^6}+\frac{2}{3^9}$. Then $\sum_{i=1}^{3}\mathcal{U}(\bar{g}_i)\le 2(\frac{1}{3}+\frac{2}{3^6}+\frac{2}{3^9})+(\frac{41}{128}+\frac{33}{4^9})\le 1$.

Therefore S has no element of order 3.

(8.32) $d=1$ and S has no element of order 4.

Proof. Suppose $d\ge 2$. From above we saw that all $|\bar{g}_i|\ge 4$. Also we have seen in previous entry that for $|\overline{g}_i|\geq 5$, we have $\mathcal{U}(\overline{g}_i)\leq\frac{56}{243}+\frac{1}{2\cdot 3^7}$. Thus for any $\overline{g}_i\in S$, we have $\mathcal{U}(\overline{g}_i)\leq max\{\frac{47}{162}+\frac{1}{4\cdot 3^7}, \frac{56}{243}+\frac{1}{2\cdot 3^7}\}=\frac{47}{162}+\frac{1}{4\cdot 3^7}$. Then $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq 3(\frac{47}{162}+\frac{1}{4\cdot 3^7})\leq 1$. Hence $d=1$.

Suppose without loss of generality that $|\overline{g}_1|=4$ and $|\overline{g}_2|\leq|\overline{g}_3|$. Since $\nu(\overline{g}_1)\geq\lceil\frac{n}{4}\rceil\geq 3$, both $\nu(\overline{g}_2)$ and $\nu(\overline{g}_3)$ are $\leq n-3$. If $|\overline{g}_3|\geq 7$, then $\mathcal{U}(\overline{g}_3)\leq max\{\frac{11}{63}+\frac{2}{7\cdot 3^5}, \frac{7}{36}+\frac{43}{8\cdot 3^8}, \frac{27}{125}+\frac{1}{2\cdot 5^7}\}=\frac{27}{125}+\frac{1}{2\cdot 5^7}$. Also as $\mathcal{U}(\overline{g}_2)\leq\frac{41}{128}+\frac{33}{4^9}$, we have $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq(\frac{2}{5}+\frac{3}{4\cdot 5^8})+(\frac{41}{128}+\frac{33}{4^9})+(\frac{27}{125}+\frac{1}{2\cdot 5^7})\leq 1$. Hence $|\overline{g}_3|\leq 6$. So $\nu(\overline{g}_1)\leq n-2$. If one of $\overline{g}_2$, $\overline{g}_3$ is of order 5, then $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq(\frac{7}{18}+\frac{7}{4\cdot 3^8})+(\frac{41}{128}+\frac{33}{4^9})+(\frac{31}{135}+\frac{4}{5\cdot 3^6})\leq 1$. If $|\overline{g}_2|=4$, then both $\nu(\overline{g}_1)$ and $\nu(\overline{g}_2)$ are $\leq n-3$; which gives $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq 3(\frac{41}{128}+\frac{33}{4^9})\leq 1$. It leaves only type (4,6,6). Suppose $q\geq 4$. Then $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq(\frac{11}{32}+\frac{9}{4^9})+2(\frac{25}{81}+\frac{23}{2\cdot 3^9})\leq 1$. For $q=3$, we have $\nu(\overline{g}_1)\geq\lceil\frac{n}{4}\rceil\geq 4$, which implies both $\nu(\overline{g}_2)$ and $\nu(\overline{g}_3)$ are $\leq n-4$. Then $\nu(\overline{g}_2^3)+\nu(\overline{g}_2^2)\leq n+\nu(\overline{g}_2)\leq 2n-4$. This gives that $\mathcal{N}(\overline{g}_2^3)+2\mathcal{N}(\overline{g}_2^2)\leq(\frac{1}{3^3}+\frac{1}{3^{10}})+2(\frac{1}{3}+\frac{1}{3^{12}})$. Hence $\mathcal{U}(\overline{g}_2)\leq\frac{1}{6}\{1+(\frac{1}{3^3}+\frac{1}{3^{10}})+2(\frac{1}{3}+\frac{1}{3^{12}})+2(\frac{1}{3^4}+\frac{1}{3^9})\}=\frac{5}{18}+\frac{5}{2\cdot 3^5}+\frac{65}{2\cdot 3^{13}}$. Similar for $\mathcal{U}(\overline{g}_3)$. Then $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq(\frac{7}{18}+\frac{7}{4\cdot 3^8})+2(\frac{5}{18}+\frac{5}{2\cdot 3^5}+\frac{65}{2\cdot 3^{13}})\leq 1$. Therefore S has no element of order 4.

(8.33) S has no element of order 5.

Proof. Suppose without loss of generality that $|\overline{g}_1|=5$ and $|\overline{g}_2|\leq|\overline{g}_3|$. If $|\overline{g}_3|\geq 7$, then as we have seen previously that $\mathcal{U}(\overline{g}_3)\leq\frac{13}{49}+\frac{6}{7^9}$; which gives $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq(\frac{9}{25}+\frac{4}{5^9})+(\frac{10}{27}+\frac{1}{2\cdot 3^7})+(\frac{13}{49}+\frac{6}{7^9})\leq 1$. So $|\overline{g}_2|\leq|\overline{g}_3|\leq 6$. Then $\nu(\overline{g}_1)\leq n-2$; which implies $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq(\frac{1}{4}+\frac{1}{5\cdot 4^6})+2(\frac{10}{27}+\frac{1}{2\cdot 3^7})\leq 1$. Therefore S has no element of order 5.

(8.34) S has no element of order 6.

Proof. Since there are at least two among $\nu(\overline{g}_1)$, $\nu(\overline{g}_2)$, $\nu(\overline{g}_3)$ which are $\leq n-3$; and if $\nu(\overline{g}_i)\leq n-3$, then $\mathcal{U}(\overline{g}_i)\leq max\{\frac{25}{81}+\frac{23}{2\cdot 3^9}, \frac{13}{49}+\frac{6}{7^9}\}=\frac{25}{81}+\frac{23}{2\cdot 3^9}$; we have $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq(\frac{10}{27}+\frac{1}{2\cdot 3^7})+2(\frac{25}{81}+\frac{23}{2\cdot 3^9})\leq 1$. Therefore S has no element of order 6.

(8.35) S has at least one element of order less than or equal to 6.

Proof. We know already that $|S|=3$ and $d=1$. If each $|\overline{g}_i|\geq 7$, then $\sum_{i=1}^{3}\mathcal{U}(\overline{g}_i)\leq 3(\frac{13}{49}+\frac{6}{7^9})\leq 1$.

Section 9. The Cases: $16\leq q\leq 167$.

In the following, if g has an eigenspace of dimension $n-1$, then we say that g is of type B. The result of this section is:

Proposition: *$\overline{G}$ is not a group of genus zero unless one of the following holds: (a) $16\leq q\leq 83$ and $n\leq 3$. (b) $89\leq q\leq 167$ and $n=2$.*

Proof. This is the combination of (8.1) and (8.7).

(9.1) If $89\leq q\leq 167$ and $n\geq 3$, then $\overline{G}$ is not group of genus zero.

Proof. By (6.6), we have for any $g\in G^{\#}$, $\mathcal{N}(g)\leq\frac{1}{q}+\frac{1}{q^{n-1}}\leq\frac{1}{q}+\frac{1}{q^2}\leq\frac{1}{89}+\frac{1}{89^2}\leq\frac{1}{85}$. Thus $\overline{G}$ is not group of genus zero.

In the following, we assume that $16\leq q\leq 83$ and $n\geq 4$.

(9.2) (a) Suppose $25\leq q\leq 83$. Then for any $g\in G^{\#}$, $\mathcal{N}(g)\leq\frac{1}{24}$.

(b) For any $g\in G^{\#}$, $\mathcal{N}(g)\leq\frac{257}{4096}$. For any g not of type B, $\mathcal{N}(g)\leq\frac{1}{128}$.

(c) If $|g|=2$, then $\mathcal{U}(g)\leq\frac{4353}{8192}$. If $|g|\geq 6$, then $\mathcal{U}(g)\leq\frac{5381}{24576}$.

Proof. For (a), as in (6.6) we have $\mathcal{N}(g)\leq\frac{1}{q}+\frac{1}{q^{n-1}}\leq\frac{1}{q}+\frac{1}{q^3}\leq\frac{1}{25}+\frac{1}{25^3}\leq\frac{1}{24}$. Part (b) follows from (6.6) similarly. The first part of (c) follows from (b). For $|g|=6$, $\mathcal{U}(g)\leq\frac{1}{6}(1+5\cdot\frac{257}{4096})=\frac{5381}{24576}$; and for $|g|\geq 7$, $\mathcal{U}(g)\leq\frac{1}{7}+\frac{257}{4096}<\frac{5381}{24576}$. So (c) holds.

(9.3) $|S|=3$.

Proof. If $25\leq q\leq 83$, then the conclusion follows from (9.2)(a) and (2.4)(c).

For $q=16$, 17, 19, or 23, (9.2)(b) implies $|S|\leq 4$ by (2.4)(b). Suppose $|S|=4$. As there are at most 3 involutions in S, we have the contradiction $\sum_{i=1}^{4}\mathcal{U}(g_i)\leq 3\cdot\frac{4353}{8192}+(\frac{1}{3}+\frac{257}{4096})<2.$ Therefore $|S|=3$.

(9.4) S has exactly one involution.

Proof. If the smallest order of elements in S is ≥ 4, then we have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq 3\cdot(\frac{1}{4}+\frac{257}{4096})<1.$

Suppose the smallest order of elements in S is 3. Say $|g_1|=3$. Then neither g_2 nor g_3 can be of type B, because $G=\langle g_1,g_i\rangle$, $i=2$ or 3, and $n\geq 4$ implies that g_1 has an eigenspace of dimension ≥ 2. If $|g|\geq 5$, then $\mathcal{U}(g)\leq\frac{1}{5}+\frac{257}{4096}<\frac{4417}{16384}$. If $|g_i|=4$, $i=2$ or 3, then $\mathcal{U}(g_i)\leq\frac{1}{4}(1+\frac{257}{4096}+2\cdot\frac{1}{128})=\frac{4417}{16384}$. If $|g_i|=3$, $i=2$ or 3, then $\mathcal{U}(g_i)\leq\frac{1}{3}(1+2\cdot\frac{1}{128})=\frac{65}{192}$. Since at most one of g_2, g_3 can be of order 3, we have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{1}{3}(1+2\cdot\frac{257}{4096})+\frac{65}{192}+\frac{4417}{16384}<1$. Hence S has exactly one involution.

(9.5) S is of the type $(2,3,k)$ for some $k\geq 7$.

Proof. By (9.4) we assume without loss of generality that $|g_1|=2$. As in (9.4), neither g_2 nor g_3 can be of type B. For $|g|\geq 6$, then by (9.2)(c), we have $\mathcal{U}(g)\leq\frac{5381}{24576}$. For $|g_i|=5$, $i=2$ or 3, we have $\mathcal{U}(g_i)\leq\frac{1}{5}(1+4\cdot\frac{1}{128})=\frac{33}{160}<\frac{5381}{24576}$. So if both $|g_2|$ and $|g_3|$ are ≥ 5, then we have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{4353}{8192}+2\cdot\frac{5381}{24576}<1$. So one of g_2, g_3, is of order 3, or 4.

Suppose $|g_2|=4$. As g_2 is not of type B, as in (9.4), we have $\mathcal{U}(g_2)\leq\frac{4417}{16384}$. If $|g_3|\geq 8$, then $\mathcal{U}(g_3)\leq\frac{1}{8}+\frac{257}{4096}$. If $|g_3|=7$, then $\mathcal{U}(g_3)\leq\frac{1}{7}(1+6\cdot\frac{1}{128})=\frac{67}{448}<\frac{1}{8}+\frac{257}{4096}$. So if $|g_3|\geq 7$, then we have $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{4353}{8192}+\frac{4417}{16384}+(\frac{1}{8}+\frac{257}{4096})<1$, a contradiction. So $|g_3|=5$ or 6. If g_1 is not of type B, then we have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{1}{2}(1+\frac{1}{128})+\frac{4417}{16384}+x<1$, where $x=\frac{1}{5}(1+4\cdot\frac{1}{128})$, $\frac{1}{6}(1+\frac{257}{4096}+2\cdot\frac{257}{4096}+2\cdot\frac{1}{128})$ for $|g_3|=5$, 6 respectively. So g_1 is of type B. Then each eigenspace of g_i, $i=2$ or 3, is of dimension 1. Thus we have $\mathcal{N}(g_i)\leq\frac{|g_i|}{q^{n-1}}\leq\frac{|g_i|}{q^3}$, and the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{4353}{8192}+\frac{1}{4}(1+\frac{257}{4096}+2\cdot\frac{4}{16^3})+y<1$, where $y=\frac{1}{5}(1+4\cdot\frac{5}{16^3})$, $\frac{1}{6}(1+\frac{257}{4096}+2\cdot\frac{257}{4096}+2\cdot\frac{6}{16^3})$ for $|g_3|=5$, 6 respectively. So S is neither of type (2,4,5) nor of type (2,4,6). Therefore one of g_2, g_3 is of order 3; and thus without loss of generality, S is of the type $(2,3,k)$ for some $k\geq 7$.

(9.6) S is not of type $(2,3,k)$ for any k.

Proof. Suppose g_i, where $i=1$ or 3, is of type B. Thus g_i has an eigenspace W of dimension $n-1$. Since $n\geq 4$, if g_2 is unipotent, then g_2 has at least 2 blocks, which implies $U=C_V(g_2)$ is of dimension ≥ 2; if g_2 is semisimple, then g_2 has at least one eigenspace U of dimension ≥ 2. Then $U\cap W\neq 0$, which contradicts to that $G=\langle g_i,g_2\rangle$ is absolutely irreducible. So g_i is not of type B for $i=1$ and 3. Similarly, g_2 is not of type B. That is there is no element of type B among g_1, g_2, g_3. For $|g_3|\geq 11$, we have $\mathcal{U}(g_3)\leq\frac{1}{11}+\frac{257}{4096}$. For $|g_3|=10$, 9, 8, 7, we have that $\mathcal{U}(g_3)\leq\frac{1}{10}(1+\frac{257}{4096}+4\cdot\frac{257}{4096}+4\cdot\frac{1}{128})$, $\frac{1}{9}(1+2\cdot\frac{257}{4096}+6\cdot\frac{1}{128})$, $\frac{1}{8}(1+\frac{257}{4096}+2\cdot\frac{257}{4096}+4\cdot\frac{1}{128})$, $\frac{1}{7}(1+6\cdot\frac{1}{128})$ respectively. As each of these 4 numbers is less than $\frac{1}{11}+\frac{257}{4096}$, we have for $|g_3|\geq 7$, $\mathcal{U}(g_3)\leq\frac{1}{11}+\frac{257}{4096}$. Then we have the contradiction $\sum_{i=1}^{3}\mathcal{U}(g_i)\leq\frac{1}{2}(1+\frac{1}{128})+\frac{1}{3}(1+2\cdot\frac{1}{128})+\frac{1}{11}+\frac{257}{4096}<1$. So S is not of type $(2,3,k)$.

(9.7) If $16\leq q\leq 83$ and $n\geq 4$, then $\overline{G}$ is not group of genus zero.

Proof. This is the combination of (9.5) and (9.6).

Chapter III
Irreducible Maximal Subgroups Containing a Transvection

Section 10. Conjugacy Classes of Transvections.

Assume that H is an irreducible maximal subgroup of G, where $SL_n(q) \leq G \leq GL_n(q)$. Let T be the set of all transvections of H and $M=\langle T\rangle$. Assume $T \neq \emptyset$. Lemmas (10.1) through (10.6) use some of the ideas in the proof of chapter 6 in [As3].

(10.1) Let s, t be transvections, and $S=\langle s,t\rangle$. Then one of the following holds:

(a) $[s,t]=1$ if and only if $[V,s] \leq C_V(t)$ and $[V,t] \leq C_V(s)$.

(b) $[V,s] \oplus C_V(t) = V = [V,t] \oplus C_V(s)$. S is irreducible on $[V,S]$ with $dim[V,S]=2$. $[s,t]$ is not a transvection. For p odd, either (a) $S \simeq SL_2(GF(p)(\lambda))$ for some $\lambda \in GF(q)$ and S has 2 conjugacy classes of transvections; or (b) $S \simeq SL_2(5)$ and $p=3$. In both (a) and (b), $\langle s^S\rangle = S = \langle t^S\rangle$. For $p=2$, $S \simeq D_{2k}$, where $k=|st| \geq 3$ is odd, and s, t are conjugates in S.

(c) Either $[V,s] \leq C_V(t)$ and $[V,t] \nleq C_V(s)$, or $[V,t] \leq C_V(s)$ and $[V,s] \nleq C_V(t)$. S is extraspecial of order p^3. s and t are not conjugates in S, and $r=[s,t]$ is a transvection. If $[V,s] \leq C_V(t)$ and $[V,t] \nleq C_V(s)$, then $C_V(r)=C_V(t)$ and $[V,r]=[V,s]$. If $[V,t] \leq C_V(s)$ and $[V,s] \nleq C_V(t)$, then $C_V(r)=C_V(s)$ and $[V,r]=[V,t]$. Moreover, if $p=2$, then $S \simeq D_8$; if p odd, then S is of exponent p.

Proof. (a) is well known. So suppose $st \neq ts$. Thus at least one of $[V,s] \nleq C_V(t)$ or $[V,t] \nleq C_V(s)$ holds. First consider the case that both of them hold. Then we have $V = U \oplus [V,s] \oplus [V,t]$, where $U = C_V(s) \cap C_V(t)$ and $[V,S]=[V,s] \oplus [V,t]$. So $dim[V,S]=2$. Also it is easy to check that S is irreducible on $[V,S]$. S has a faithful representation on $[V,S]$. For a suitable basis X of $[V,S]$, we have the matrix representations $M_X(s)=\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$, $M_X(t)=\left[\begin{smallmatrix}1&\lambda\\0&1\end{smallmatrix}\right]$, for some $\lambda \in GF(q)$; and easy to check that $[s,t]$ is not a transvection. If p is odd, by Dickson's theorem, we have either (1) $S \simeq SL_2(GF(p)(\lambda))$, and in this case it is well known that S has 2 conjugacy classes of transvections; or (2) $S \simeq SL_2(5)$ and $p=3$. In both (1) and (2), $\langle s^S \rangle$ and $\langle t^S \rangle$ are normal in S, and neither of them is contained in $Z(S)$. So if $S \not\simeq SL_2(3)$, then $S/Z(S)$ simple implies that $\langle s^S \rangle Z(S) = S = \langle t^S \rangle Z(S)$. Moreover as S is perfect if $S \not\simeq SL_2(3)$, so we have $\langle s^S \rangle = S = \langle t^S \rangle$. In the case that $S \simeq SL_2(3)$, $M_X(s^2) = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$; and as 2 is not a square in $GF(3)$, s^2 and s are not conjugate in $SL_2(3)$. Then $s^2 \in \langle s^S \rangle$ implies that all transvections of S are contained in $\langle s^S \rangle$. Since $SL_2(3)$ is generated by transvections, we have $\langle s^S \rangle = S$. Similarly, $S = \langle t^S \rangle$. If $p=2$, then s and t generate the dihedral group D_{2k}, where $k = |st|$. Since $S \leq SL([V,S])$, if k is even, then $k=2$. But $k=1$ or 2 implies that $st=ts$, a contradiction. So k is odd and $k \geq 3$. Say $k=2a+1$, $a \geq 1$. Denote $r=st$. Then $r^s = r^{-1}$. Thus $s^{r^{a+1}} = sr^{2(a+1)} = sr = t$, i.e., s and t are conjugates in S. It remains to consider the case that exactly one of $[V,s] \not\leq C_V(t)$, $[V,t] \not\leq C_V(s)$ holds. Say $[V,t] \not\leq C_V(s)$ but $[V,s] \leq C_V(t)$, the other case is similar. So $V = [V,t] \oplus C_V(s)$. Let $U = C_V(s) \cap C_V(t)$. Then $dim(U) = n-2$. We can choose basis $X = \{v_1, \ldots, v_n\}$ of V such that $U = \langle v_1, \ldots, v_{n-2} \rangle$, $[V,s] = \langle v_{n-2} \rangle$, $C_V(s) = \langle U, v_{n-1} \rangle$, $[V,t] = \langle v_n \rangle$, and $M_X(s) = \begin{bmatrix} I_{n-3} & 0 \\ 0 & A \end{bmatrix}$, $M_X(t) = \begin{bmatrix} I_{n-3} & 0 \\ 0 & B \end{bmatrix}$ with $A = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{smallmatrix}\right]$ for some $\lambda \in GF(q)$. So $S \simeq \langle A, B \rangle$. Let $r = [s,t]$. Then $M_X(r) = \begin{bmatrix} I_{n-3} & 0 \\ 0 & C \end{bmatrix}$ with $C = A^{-1}B^{-1}AB = \left[\begin{smallmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$. So r is a transvection with $C_V(r) = C_V(t)$, $[V,r] = [V,s]$. Also as $CA = \left[\begin{smallmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right] = AC$ and $CB = \left[\begin{smallmatrix} 1 & 0 & 0 \\ -\lambda & 1 & \lambda \\ 0 & 0 & 1 \end{smallmatrix}\right] = BC$, $[r,s] = 1 = [r,t]$. So $\langle r \rangle \leq Z(S)$. Let $\overline{S} = S/\langle r \rangle$. Then $\overline{s}\,\overline{t} = \overline{t}\,\overline{s}$ and S is not abelian imply $\overline{S} = \langle \overline{s}, \overline{t} \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. So $S^{(1)} = \langle r \rangle = \Phi(S)$. If $\langle r \rangle < Z(S)$, then $|Z(S)| = p^2$; and thus $S = \langle s \rangle Z(S)$ which implies S is abelian, a contradiction. Hence $\langle r \rangle = Z(S)$. So S is extraspecial of order p^3. If $t = s^\alpha$ for some $\alpha \in S$, then $\overline{t} = \overline{s}^{\overline{\alpha}} = \overline{s}$ as $\overline{S}$ is abelian. Then this implies $\overline{S} \simeq \mathbb{Z}_p$, a contradiction. So s, t are not conjugate in S. As S is of class 2 and $\Omega_1(S) = S$, S is of exponent p for p odd by (23.11) of [As1]. If $p=2$, then $|st| = 4$ as $r = (st)^2$ has order 2. Thus $S \simeq D_8$.

(10.2) For each $t \in T$, there exist $m \in M$ such that $\langle t, t^m \rangle$ is not a p-group.

Proof. Since H is irreducible, $O_p(H) = 1$. Then $O_p(M) = 1$, as $M \trianglelefteq H$. Thus the Baer-Suzuki theorem, (39.6) of [As1], completes the proof.

(10.3) If r, t are transvections in H with $C_V(r)=C_V(t)$, then $\langle r^M\rangle=\langle t^M\rangle$.

Proof. By (10.2) and (10.1)(b), there exists $t_1=t^m$ for some $m\in M$ such that $\langle t,t^m\rangle\simeq SL_2(\mathbb{K})$ for some $\mathbb{K}$, or D_{2k} for some odd k. In either case, we have $C_V(t)\cap[V,t_1]=0=C_V(t_1)\cap[V,t]$. If $[r,t_1]=1$, then $[V,t_1]\leq C_V(r)=C_V(t)$, a contradiction. Thus $S=\langle r,t_1\rangle\simeq SL_2(\mathbb{K})$, D_{2k} with k odd, or extraspecial of order p^3. In the first two cases, $r\in\langle t_1^S\rangle\leq\langle t_1^M\rangle=\langle t^M\rangle\trianglelefteq M$, which implies $\langle r^M\rangle\leq\langle t^M\rangle$. If S is extraspecial of order p^3, then $r_1=[r,t_1]\in T$. Either $C_V(r_1)=C_V(r)$ and $[V,r_1]=[V,t_1]$, or $C_V(r_1)=C_V(t_1)$ and $[V,r_1]=[V,r]$. The first case implies that $[V,t_1]\leq C_V(r)=C_V(t)$, a contradiction. So it is the case that $C_V(r_1)=C_V(t_1)$, $[V,r_1]=[V,r]$. Denote $\alpha=[t,r_1]$. If $\alpha=1$, then $[V,t]\leq C_V(r_1)=C_V(t_1)$, a contradiction. Also $[V,r_1]=[V,r]\leq C_V(r)=C_V(t)$ implies that $\langle t,r_1\rangle$ is not a group in (10.1)(b). So $\langle t,r_1\rangle$ is extraspecial of order p^3, and hence $\alpha\in T$. Then either $C_V(\alpha)=C_V(r_1)$, $[V,\alpha]=[V,t]$ or $C_V(\alpha)=C_V(t)$, $[V,\alpha]=[V,r_1]$. The first case gives $[V,t]=[V,\alpha]\leq C_V(\alpha)=C_V(r_1)=C_V(t_1)$, a contradiction. So it is the second case which gives $C_V(\alpha)=C_V(t)=C_V(r)$, $[V,\alpha]=[V,r_1]=[V,r]$. As $\alpha\in T$, there exist some $\beta\in M$ such that $\langle\alpha,\alpha^\beta\rangle$ is not a p-group; this is the case (b) in (10.1) which corresponds to the situation that $C_V(\alpha)\cap[V,\alpha^\beta]=0=C_V(\alpha^\beta)\cap[V,\alpha]$. Thus $C_V(r)\cap[V,\alpha^\beta]=0=C_V(\alpha^\beta)\cap[V,r]$ and it again corresponds to case (2) in (11), i.e., $X=\langle r,\alpha^\beta\rangle$ is not a p-group and $r\in\langle(\alpha^\beta)^X\rangle$. But $\alpha=t^{-1}t^{r_1}\in\langle t^M\rangle\trianglelefteq M$, so $\langle(\alpha^\beta)^X\rangle\leq\langle t^M\rangle$. Hence $r\in\langle t^M\rangle$, which implies that $\langle r^M\rangle\leq\langle t^M\rangle$; i.e., in any case, we have that $\langle r^M\rangle\leq\langle t^M\rangle$. By the symmetry of the assumption, we also have $\langle t^M\rangle\leq\langle r^M\rangle$. Therefore $\langle r^M\rangle=\langle t^M\rangle$.

(10.4) If r,s are transvections in H with $[V,r]=[V,s]$, then $\langle r^M\rangle=\langle s^M\rangle$.

Proof. The proof here is dual to that of (10.3).

Let T_i, $i \in I$ be the distinct M-orbits on T and $M_i = \langle T_i \rangle$. Then

(10.5) For $i \neq j$, either $M_i = M_j$ or $[M_i, M_j] = 1$.

Proof. If $[T_i, T_j] = 1$, then $[M_i, M_j] = 1$. So suppose there exist $s \in T_i$, $t \in T_j$ with $st \neq ts$. Then by (10.1), either $S = \langle s,t \rangle \simeq SL_2(GF(p)(\lambda))$, p odd and $\lambda \in GF(q)$; or S extraspecial of order p^3 and $r = [s,t] \in T$. In the first case, by (10.1)(b), $t \in S = \langle s^S \rangle \leq \langle T_i \rangle = M_i \trianglelefteq M$, so $T_j = t^M \subseteq M_i$, and $M_j \leq M_i$. Similarly, $M_i \leq M_j$. Hence $M_i = M_j$. In the second case, we have that either $C_V(r) = C_V(s)$ and $[V,r] = [V,t]$ or $C_V(r) = C_V(t)$ and $[V,r] = [V,s]$. Then by (10.3) and (10.4), $M_i = \langle s^M \rangle = \langle r^M \rangle = \langle t^M \rangle = M_j$. Therefore either $M_i = M_j$ or $[M_i, M_j] = 1$.

Now let $M_1, M_2, \ldots, M_l$ be all the distinct M_i's, thus $M = M_1 M_2 \cdots M_l$. Also let $V_j = [V, M_j]$ for $1 \leq j \leq l$.

(10.6) (a) $V = V_1 \oplus V_2 \oplus \cdots \oplus V_l$, $dim(V_j) = k$ $\forall j$, $n = kl$.

(b) H is the stabilizer of decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_l$ in G if $l \geq 2$, i.e., $H \simeq G \cap (GL_k(q) wr S_l)$.

Proof. Since $M \trianglelefteq H$, H acts on $[V,M]$ and $C_V(M)$. But H is irreducible on V, so $[V,M] = V$ and $C_V(M) = 0$. Let $U_j = \langle V_i : i \neq j \rangle$. Thus $V = V_j + U_j$. Also let $N_j = M_1 \cdots M_{j-1} M_{j+1} \cdots M_l$. Then N_j centralizes V_j and M_j centralizes U_j. So $V_j \cap U_j \leq C_V(M) = 0$. Hence $V = V_1 \oplus V_2 \oplus \cdots \oplus V_l$. Clearly H stabilizes the decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_l$. As H is irreducible on V, H is transitive on $V_1, V_2, \ldots, V_l$. Hence $dim(V_j) = k$ $\forall j$. The stabilizer of decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_l$ in $GLn(q)$ is $GL_k(q) wr S_l$. Since H is maximal in G, we have $H \simeq G \cap (GL_k(q) wr S_l)$ if $l \geq 2$.

Section 11. Stabilizers of Direct Sum Decompositions.

In this section, we assume that $l \geq 2$. Let $\Lambda_{k,l}$ be the same as in Section 4. Let $f(g, \Lambda_{k,l})$

be the number of $\{V_1, V_2, \ldots, V_l\}$ in $A_{k,l}$ fixed by g.

(11.1) $\mathcal{N}(g) = f(g, A_{k,l}) / |A_{k,l}|$.

Proof. G is transitive on $A_{k,l}$, as $SL(V)$ is and $SL(V) \subseteq G$. Since $H = Stab_G(\{V_1, V_2, \ldots, V_l\})$ for some $\{V_1, V_2, \ldots, V_l\}$ in $A_{k,l}$, the representation of G on $\Omega = G/H$ and on $A_{k,l}$ are equivalent.

Now assume $g \in GL_n(q)$ with $(|g|, p) = 1$. Let $f = min(g) = f_1 f_2 \cdots f_\alpha$, where each f_μ is irreducible in $\mathbb{F}[x]$. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_\alpha$, where each V_μ is the homogeneous component corresponding to the irreducible factor f_μ. Denote $c_\mu = deg(f_\mu)$. So $dim_{\mathbb{F}}(V_\mu) = d_\mu c_\mu$, and $n = \sum_{\mu=1}^{\alpha} d_\mu c_\mu$. Also we label that $f_1 = x - 1$, if $x - 1$ is a factor of f.

For $X = \{W_1, \ldots, W_l\} \in A_{k,l}$, define $S(X) = \{Y \in A_{k,l} : |X \cap Y| \geq 1\}$. Let $r(l) = \frac{1}{q^{(l-1)k}} \cdot \frac{1}{\prod_{i=1}^{l-1}\left(1 - \frac{i^2}{2qk^{2}i}\right)}$.

(11.2) $|S(X)| \geq l\{q^{k(n-k)} - \frac{(l-1)^2}{2}\} |A_{k,l-1}|$.

Proof. Since the number of U such that $W_1 \oplus U = V$ is $q^{k(n-k)}$, the number of $Y \in S(X)$ with $W_1 \in X \cap Y$ is $q^{k(n-k)} |A_{k,l-1}|$. Similarly, the number of $Y \in S(X)$ with $\{W_1, W_2\} \in X \cap Y$ is $q^{2k(n-2k)} |A_{k,l-2}|$. Thus by the inclusion-exclusion principle, we have $|S(X)| \geq \binom{l}{1} q^{k(n-k)} |A_{k,l-1}| - \binom{l}{2} q^{2k(n-2k)} |A_{k,l-2}| = \left\{ l q^{k(n-k)} - \frac{l(l-1)^2}{2} q^{2k(n-2k)-k(n-2k)} \Big/ \begin{bmatrix} n-k \\ k \end{bmatrix}_q \right\} |A_{k,l-1}| \geq l\{q^{k(n-k)} - \frac{(l-1)^2}{2}\} |A_{k,l-1}|$.

(11.3) (a) $\mathcal{N}(g) \leq r(l)$.

(b) $\mathcal{N}(g) \leq \frac{4}{q^{(l-1)k}}$, if $q \geq 3$.

(c) $\mathcal{N}(g) \leq \frac{56}{2^{(l-1)k}}$, if $q = 2$.

Proof. We show (a) by induction. We claim that for any $Y\in A_{k,l}$, the number of X with $X^g=X$ and $Y\in S(X)$ is at most $a=lq^{k(n-k-1)}|A_{k,l-1}|r(l-1)$. Suppose this is true for the moment.

Then $\quad \mathcal{N}(g)=f(g,A_{k,l})/|A_{k,l}|\le\frac{af(g,A_{k,l})}{\sum_{X^g=X}|S(X)|}\le\max_{X^g=X}\left\{\frac{a}{|S(X)|}\right\}\le\frac{lq^{k(n-k-1)}|A_{k,l-1}|r(l-1)}{l\{q^{k(n-k)}-\frac{(l-1)^2}{2}\}|A_{k,l-1}|}=$

$\frac{r(l-1)}{q^k\left(1-\frac{(l-1)^2}{2q^{k^2(l-1)}}\right)}=r(l)$. So it remains to prove the claim. Let $Y=\{U_1,\dots,U_l\}$ be fixed and suppose $X=\{W_1,\dots,W_l\}$ is such that $X^g=X$ and $U_1=W_1\in X\cap Y$. It suffices to show that the number of such X is at most $q^{k(n-k-1)}|A_{k,l-1}|r(l-1)$. If $W_1^g=W_1$, then g acts on $W=W_2\oplus\cdots\oplus W_l$. So $W_1=(W_1\cap V_1)\oplus\cdots\oplus(W_1\cap V_\alpha)$, and $W=(W\cap V_1)\oplus\cdots\oplus(W\cap V_\alpha)$. For any fixed A such that $A\le V_\mu$, $A^g=A$ with $dim_{\mathbb{F}}(A)=sc_\mu$, the number of $B\le V_\mu$ such that $A\cap B=0$, $B^g=B$, $dim_{\mathbb{F}}(B)=rc_\mu$ is equal to $\left[\begin{smallmatrix} d_\mu-s \\ r\end{smallmatrix}\right]_{q^{c_\mu}}(q^{c_\mu})^{rs}$. Hence the number of W such that $W_1\oplus W=V$ and $W^g=W$ is q^h, where $h=\sum_{\mu=1}^{\alpha}c_\mu s_\mu(d_\mu-s_\mu)$, and s_μ is such that $dim_{\mathbb{F}}(W_1\cap V_\mu)=c_\mu s_\mu$, $1\le\mu\le\alpha$. If g is not trivial on one of the W, then g is not trivial on every W. In this case, the number of X is at most $q^h|A_{k,l-1}|r(l-1)$ by induction. Since $\sum_{\mu=1}^{\alpha}c_\mu s_\mu=k$ and $\sum_{\mu=1}^{\alpha}c_\mu d_\mu=n$; if any one s_μ is such that $c_\mu s_\mu=k$, then $h=c_\mu s_\mu(d_\mu-s_\mu)=k(d_\mu-s_\mu)\le kc_\mu(d_\mu-s_\mu)=k(c_\mu d_\mu-k)\le k(n-k-1)$; if every $c_\mu s_\mu\le k-1$, then $h\le(k-1)\sum_{\mu=1}^{\alpha}(d_\mu-s_\mu)\le(k-1)\sum_{\mu=1}^{\alpha}c_\mu(d_\mu-s_\mu)=(k-1)(n-k)=k(n-k)-(n-k)\le k(n-k)-k$. So the number of X in this case is at most $q^{k(n-k-1)}|A_{k,l-1}|r(l-1)$. If g acts as the identity on W, then $s_\mu=d_\mu$, $\forall\ 2\le\mu\le\alpha$. In this case, the number of X is exactly $q^h|A_{k,l-1}|$, which is at most $q^{(k-1)k(l-1)}|A_{k,l-1}|$, because $h=s_1(d_1-s_1)=s_1(n-k)\le(k-1)k(l-1)$. Since $k(n-k-1)-k(l-2)$ $=(k-1)k(l-1)$ implies that $q^{(k-1)k(l-1)}\le q^{k(n-k-1)}\frac{1}{q^{k(l-2)}}\cdot\frac{1}{\prod_{i=1}^{l-2}\left(1-\frac{i^2}{2qk^2i}\right)}=q^{k(n-k-1)}r(l-1)$, we still have that the number of X is at most $q^{k(n-k-1)}|A_{k,l-1}|r(l-1)$.

If $W_1^g\ne W_1$, then g has an s-cycle on X, say $(W_1,W_2,\dots,W_s)$. Denote $dim_{\mathbb{F}}((W_1\oplus\cdots\oplus W_s)\cap V_\mu)=c_\mu s_\mu$, $1\le\mu\le\alpha$. So $\sum_{\mu=1}^{\alpha}c_\mu s_\mu=sk$. The number of W such that

$W\oplus(W_1\oplus\cdots\oplus W_s)=V$ and $W^g=W$ is q^h with $h=\sum_{\mu=1}^{\alpha} c_\mu s_\mu(d_\mu-s_\mu)$. If any one $c_\mu s_\mu=sk$, then all other $s_\mu=0$. That is $W_1\oplus\cdots\oplus W_s\leq V_\mu$ for this particular μ. Since $s\geq 2$, we have $c_\mu\geq 2$, which implies that $s_\mu=\frac{sk}{c_\mu}\leq\frac{sk}{2}\leq sk-1$. Thus $h=c_\mu s_\mu(d_\mu-s_\mu)=s_\mu(c_\mu d_\mu-c_\mu s_\mu)\leq (sk-1)(n-sk)$. If every $c_\mu s_\mu\leq sk-1$, then $h\leq(sk-1)\sum_{\mu=1}^{\alpha}(d_\mu-s_\mu)\leq(sk-1)\sum_{\mu=1}^{\alpha}c_\mu(d_\mu-s_\mu)=(sk-1)(n-sk)$. Thus in the case that $W_1^g\neq W_1$, the number of X is at most $q^{(sk-1)(n-sk)}|\Lambda_{k,l-s}|$ for some $s\geq 2$. Now we will be done if we can show that for any t with $1\leq t\leq l$, $q^{(tk-1)(n-tk)}|\Lambda_{k,l-t}|\leq q^{(k-1)k(l-1)}|\Lambda_{k,l-1}|$, because the latter term does not exceed $q^{k(n-k-1)}|\Lambda_{k,l-1}|r(l-1)$ as we have already seen. Choose a fixed $0\neq v\in V$ and a fixed $E\leq V$ so that $\langle v\rangle\oplus E=V$. So $dim_{\mathbb{F}}(E)=n-1$. Also choose fixed A, $E_2,\ldots,E_t$ so that $A\oplus E_2\oplus\cdots\oplus E_t=E'\leq E$, and $dim_{\mathbb{F}}(A)=k-1$, $dim_{\mathbb{F}}(E_j)=k$ $\forall$ $2\leq j\leq t$. Let $E_1=\langle v\rangle\oplus A$. The number of D such that $A\oplus D=E$ is $q^{(k-1)(n-k)}$. Thus $q^{(k-1)k(l-1)}|\Lambda_{k,l-1}|$ is the number of $\{E_1,D_2,\ldots,D_l\}\in\Lambda_{k,l}$ such that $A\oplus D_2\oplus\cdots\oplus D_l=E$. The number of D' such that $E'\oplus D'=E$ is $q^{(tk-1)(n-tk)}$. Thus $q^{(tk-1)(n-tk)}|\Lambda_{k,l-t}|$ is the number of $\{E_1,E_2,\ldots,E_t,D_{t+1},\ldots,D_l\}\in\Lambda_{k,l}$ such that $E'\oplus D_{t+1}\oplus\cdots\oplus D_l=E$. As $A\oplus E_2\oplus\cdots\oplus E_t=E'$, we have that $q^{(tk-1)(n-tk)}|\Lambda_{k,l-t}|\leq q^{(k-1)k(l-1)}|\Lambda_{k,l-1}|$.

For part (b), if $q\geq 3$, then $\frac{i^2}{2q^{k^2 i}}\leq\frac{i^2}{2\cdot 3^i}\leq\frac{2}{i(i+1)}$ $\forall$ $i\geq 1$. So $\prod_{i=1}^{l-1}\left(1-\frac{i^2}{2q^{k^2 i}}\right)\geq \prod_{i=1}^{\infty}\left(1-\frac{i^2}{2q^i}\right)\geq\left(1-\frac{1}{2q}\right)\prod_{i=2}^{\infty}\left(1-\frac{2}{i(i+1)}\right)=\left(1-\frac{1}{2q}\right)\frac{1}{3}\geq\frac{5}{6}\cdot\frac{1}{3}>\frac{1}{4}$, which implies that $r(l)\leq\frac{4}{q^{(l-1)k}}$, if $q\geq 3$. Now suppose $q=2$. Since for $i\geq 13$, we have $2^{i+2}\geq i^3(i+1)$, which gives $\prod_{i=1}^{l-1}\left(1-\frac{i^2}{2\cdot 2^{k^2 i}}\right)\geq \prod_{i=1}^{\infty}\left(1-\frac{i^2}{2^{i+1}}\right)\geq\prod_{i=1}^{12}\left(1-\frac{i^2}{2^{i+1}}\right)\prod_{i=13}^{\infty}\left(1-\frac{2}{i(i+1)}\right)\geq\frac{21}{1000}\cdot\frac{12}{14}=\frac{9}{500}$. So $r(l)\leq\frac{56}{q^{(l-1)k}}$, if $q=2$. Thus (c) holds.

Now assume g is such that $|g|=p^e$, where $p=char(\mathbb{F})$. Define $S(X)$ for $X\in\Lambda_{k,l}$ and $r(l)$ in the same way as before.

(11.4) Suppose $V=U_1\oplus U$ with $U_1^g=U_1$, $U^g=U$. Let $a_1,\ldots,a_\alpha$; $b_1,\ldots,b_\beta$ be the dimensions of those Jordan blocks of U_1, U respectively. Then the number of W such that $V=U_1\oplus W$ with $W^g=W$ is equal to q^h, where $h=\sum_{\mu=1}^{\alpha}\sum_{\nu=1}^{\beta} min(a_\mu,b_\nu)$.

Proof. Since the Jordan canonical form of g is unique, for some bases B_1, B_2, B_2' of U_1, U, W respectively, g has the matrix representation $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$ with respect to both $B_1\cup B_2$ and $B_1\cup B_2'$. As $C=C_{GL(V)}(U_1)\cap C_{GL(V)}(V/U_1)$ is regular on complements to U_1 in V, there exists a unique $c\in C$ with $Uc=W$. Hence there exists a matrix $\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}$ corresponding to c such that $\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}=\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}$. This is equivalent to that $CX=YC$. So the number of W such that $V=U_1\oplus W$ with $W^g=W$ is equal to the number of C's satisfying $CX=YC$. Write $X=(X_\mu)$, $Y=(Y_\nu)$, $C=(C_{\nu\mu})$, $1\leq\mu\leq\alpha$, $1\leq\nu\leq\beta$, where X_μ, Y_ν are the Jordan blocks of X and Y. Then $CX=YC$ is equivalent to $C_{\nu\mu}X_\mu=Y_\nu C_{\nu\mu}$, $\forall$ μ, and ν. As X and Y are in the form $\begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}$, we have

$$\begin{bmatrix} 0 & c_{11} & \cdots & c_{1,a-1} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & c_{b-1,1} & \cdots & c_{b-1,a-1} \\ 0 & c_{b1} & \cdots & c_{b,a-1} \end{bmatrix}=$$

$$C_{\nu\mu}(X_\mu-I_a)=(Y_\nu-I_b)C_{\nu\mu}=\begin{bmatrix} c_{21} & c_{22} & \cdots & c_{2a} \\ \vdots & \vdots & \cdots & \vdots \\ c_{b1} & c_{b2} & \cdots & c_{ba} \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

where for simplicity we have written $a=a_\mu$ and $b=b_\nu$. This implies that $c_{ij}=0$ if $i-j>min(0,b-a)$, and $c_{ij}=c_{i+1,j+1}$ if $i-j\leq min(0,b-a)$. So the number of choices for $C_{\nu\mu}$ is $q^{min(a,b)}$, which implies that the number of choices for C is q^h with $h=\sum_{\mu=1}^{\alpha}\sum_{\nu=1}^{\beta} min(a_\mu,b_\nu)$.

(11.5) (a) $\mathcal{N}(g)\leq r(l)$.

(b) $\mathcal{N}(g)\leq\frac{4}{q^{(l-1)k}}$, if $q\geq3$.

(c) $\mathcal{N}(g)\leq\frac{56}{2^{(l-1)k}}$, if $q=2$.

Proof. Exactly as in the proof for (11.3), we only need to show that for any fixed $Y\in\Lambda_{k,l}$, the number of X with $X^g=X$ and $Y\in S(X)$ is at most $a=lq^{k(n-k-1)}|\Lambda_{k,l-1}|r(l-1)$. Pick

$Y=\{U_1,\ldots,U_l\}$ and suppose $X=\{W_1,\ldots,W_l\}$ is such that $X^g=X$ and $U_1=W_1\in X\cap Y$. If $W_1^g=W_1$, then g acts on $W_2\oplus\cdots\oplus W_l$. Let $a_1,\ldots,a_\alpha$; $b_1,\ldots,b_\beta$ be the dimensions of those Jordan blocks of W_1, $W_2\oplus\cdots\oplus W_l$ respectively. Then the number of W such that $V=W_1\oplus W$ with $W^g=W$ is equal to q^h, where $h=\sum_{\mu=1}^{\alpha}\sum_{\nu=1}^{\beta}min(a_\mu,b_\nu)$. We have that $h\leq\sum_{\mu=1}^{\alpha}\sum_{\nu=1}^{\beta}b_\nu=\sum_{\mu=1}^{\alpha}(n-k)=\alpha(n-k)$, and $h\leq\sum_{\nu=1}^{\beta}\sum_{\mu=1}^{\alpha}a_\mu=\sum_{\nu=1}^{\beta}k=\beta k$. Either $\alpha\leq k-1$ or $\beta\leq n-k-1$, so respectively either $h\leq(k-1)(n-k)=k(n-k)-(n-k)\leq k(n-k)-k$ or $h\leq(n-k-1)k$; i.e., we always have $h\leq k(n-k)-k$. Since the Jordan decomposition of g is unique, either g acts as the identity on every W with $V=W_1\oplus W$ and $W^g=W$, or g doesn't act as the identity on everyone of them. For the latter case, by induction, the number of $\{W_2,\ldots,W_l\}$ on which g acts with $W_1\oplus W_2\oplus\cdots\oplus W_l=V$, and thus the number of X, is at most $q^{k(n-k-1)}|A_{k,l-1}|r(l-1)$. If g acts as the identity on W, then every $b_\nu=1$. Thus $\alpha\leq k-1$, which implies that $h=\alpha(n-k)\leq k(n-k)-(n-k)=k(n-k)-k-k(l-2)$ as we have seen. In this case, the number of X is exactly $q^h|A_{k,l-1}|$, which is less than or equal to $q^{k(n-k-1)}|A_{k,l-1}|\cdot\frac{1}{q^{k(l-2)}}\leq q^{k(n-k-1)}|A_{k,l-1}|r(l-1)$. If $W_1^g\neq W_1$, then g has a p^s-cycle on X, say $(W_1,W_2,\ldots,W_{p^s})$. The number of Jordan blocks of g in $W_1\oplus W_2\oplus\cdots\oplus W_{p^s}$ is at most kp^s-1. Let $a_1,\ldots,a_\alpha$; $b_1,\ldots,b_\beta$ be the dimensions of those Jordan blocks of $W_1\oplus W_2\oplus\cdots\oplus W_{p^s}$ and $W_{p^s+1}\oplus\cdots\oplus W_l$ respectively. So $\alpha\leq kp^s-1$. The number of W such that $W\oplus(W_1\oplus\cdots\oplus W_{p^s})=V$ and $W^g=W$ is q^h with $h=\sum_{\mu=1}^{\alpha}\sum_{\nu=1}^{\beta}min(a_\mu,b_\nu)\leq\sum_{\mu=1}^{\alpha}\sum_{\nu=1}^{\beta}b_\nu=\sum_{\mu=1}^{\alpha}(n-kp^s)=\alpha(n-kp^s)\leq(kp^s-1)(n-kp^s)$. So the number of X is less than or equal to $q^h|A_{k,l-p^s}|\leq q^{(kp^s-1)(n-kp^s)}|A_{k,l-p^s}|$, which is less than or equal to $q^{(k-1)k(l-1)}|A_{k,l-1}|\leq q^{k(n-k-1)}|A_{k,l-1}|r(l-1)$ as we saw in the end of proof of (11.3). So in any case, the number of $X=\{W_1,\ldots,W_l\}$ such that $X^g=X$ and $W_1=U_1\in X\cap Y$ is at most $q^{k(n-k-1)}|A_{k,l-1}|r(l-1)$.

(11.6) If $\overline{G}$ is a group of genus zero, then one of the following holds:

(a) $q=2$ and $n\leq 24$.

(b) $q=3$ and $n\leq 10$.

(c) $q=4$ and $n\leq 8$.

(d) $q=5$ and $n\leq 6$.

(e) $7\leq q\leq 17$ and $n\leq 4$.

(f) $19\leq q\leq 83$ and $n=2$.

Proof. For example, when $q=2$ and $n\geq 25$, we have $\mathcal{N}(g)\leq\frac{56}{2^{(l-1)k}}\leq\frac{56}{2^{\lceil\frac{n}{2}\rceil}}\leq\frac{56}{2^{13}}\leq\frac{1}{85}$. When $q\geq 19$ and $n\geq 3$, we have $\mathcal{N}(g)\leq\frac{4}{q^{\lceil\frac{n}{2}\rceil}}\leq\frac{4}{19^2}\leq\frac{1}{85}$. For $q\geq 89$ and $n=2$, we have $l=2$, $k=1$ and $\mathcal{N}(g)\leq r(2)=\frac{1}{q-\frac{1}{2}}\leq\frac{1}{85}$.

Section 12. Primitive Cases.

Now assume that $l=1$, i.e., H is primitive on V. So M is generated by an M-conjugacy class of transvections. As $M\trianglelefteq H$ and H maximal in G, $H=N_G(M)=G\cap N_{GL(V)}(M)$. Since H is irreducible, $O_p(M)=1$ and $C_V(M)=0$.

(12.1) (W. Kantor) Suppose M is a subgroup of $SL(V)$ generated by a conjugacy class of transvections, such that $O_p(M)\leq M'\cap Z(M)$. Then $V=W\oplus U$ with M trivial on U and indecomposable on W, such that M acts on W as one of the following tensored with $GF(q)$.

(a) $M=SL_m(s)$ or $Sp_m(s)$ in $SL_m(s)$, or $M=SU_m(s^{\frac{1}{2}})$ in $SL_m(s)$.

(b) $M=O_m^{\pm}(s)<SL_m(s)$, s even.

(c) $M=S_m<SL_{m-d}(2)$, $d=(2,m)$.

(d) $M=S_{2m}$ in $SL_{2m-1}(2)$ fixing a 1-space, or in $SL_{2m}(2)$ fixing a 1-space and a $(2m-1)$-space.

(e) $M=3\cdot A_6<SL_3(4)$.

(f) $M=SL_2(5)<SL_2(9)$.

(g) $M=3\cdot P\Omega_6^{-,\pi}(3)<SL_6(4)$.

(h) $M=SU_4(2)<SL_5(4)$ fixing a 1-space or a 4-space.

(i) M=the semidirect product of A with S_m belongs to a Borel subgroup of $SL_m(2^i)$, where A is normal in M and isomorphic to the direct product of $m-1$ copies of cyclic group of order $a|q-1$.

Proof. See [Ka].

(12.2) Let V be an n-dimensional vector space over $\mathbb{F}=GF(q)$, $p=char(\mathbb{F})$, G a group such that $SL(V)\leq G\leq GL(V)$, H an irreducible maximal subgroup of G with $Z(GL(V))=Z\leq H$ and $SL(V)\nleq H$. Let T be the set of all transvections of H and $M=\langle T\rangle$. Suppose M is generated by an M-conjugacy class of transvections. Then M and H are one of the following:

(a) $M=SL_n(q^{\frac{1}{r}})$, $q=p^m$, r a prime with $r|m$, $H=G\cap ZGL_n(q^{\frac{1}{r}})$.

(b) $M=Sp_n(q)$, $H=G\cap\Delta(V,f)$, where f is the symplectic form on V and $\Delta(V,f)=Sp(V,f)\langle\rho\rangle$, ρ is defined by $x_{2i-1}\rho=x_{2i-1}$, $x_{2i}\rho=ax_{2i}$ for some hyperbolic basis $X=\{x_i : 1\leq i\leq n\}$ of V, and with $\langle a\rangle=\mathbb{F}^{\#}$. If q is even, then $\Delta(V,f)=Sp(V,f)\times Z$.

(c) $M=SU_n(s)$, $q=s^2$, $H=G\cap\Delta(V,f)$, where f is the unitary form on V and $\Delta(V,f)=GU(V,f)*Z$ with $GU(V)\cap Z=\langle bI\rangle$, $b\in\mathbb{F}^{\#}$ is of order $s+1$.

(d) $M=3\cdot A_6$, $G=SL_3(4)$, and $H=N_G(M)=M$.

(e) $M=SL_2(5)$, $SL_2(9)\leq G\leq ZSL_2(9)$, and $H=N_G(M)=G\cap MZ$ with $Z=Z(GL_2(9))$.

Proof. We use the notation in Kantor's theorem. Then U has to be equal to 0, and thus $V=\mathbb{F}\otimes_K W$, where $\mathbb{F}=GF(q)$, $K=GF(s)$, $q=s^e$. We consider the cases (a) and (b) in Kantor's theorem first. So let $M=SL(W)$, $Sp(W)$, $SU(W)$, *or* $O^{\pm}(W)$.

We first observe that W is an absolutely irreducible KM-module. To show this, consider the center $[W,t]$ of a transvection $t\in M$. For $z\in End_{KM}(W)$, $[W,t]z=[W,t^z]=[W,t]=\langle w\rangle$, so $wz=\lambda w$ for some $\lambda\in K$. For $h\in M$, $whz=wzh=\lambda(wh)$, which implies that z acts as scalar λ on $\langle wh : h\in M\rangle$. But $SL(W)$, $Sp(W)$, $SU(W)$, $O^{\pm}(W,Q)$ are all irreducible on W. So $W=\langle wh : h\in M\rangle$. Hence $End_{KM}(W)=K$, which implies that W is an absolutely irreducible KM-module for $M=SL(W)$, $Sp(W)$, $SU(W)$, or $O^{\pm}(W,Q)$; hence V is also an absolutely irreducible FM-module.

Then by (3.15) in [As2], $N_{GL(V,F)}(M) \leq GL(W,K)\mathbb{F}^{\#}$. Suppose $K < \mathbb{F}$. Let r be a prime such that $e = re_0$, $L = GF(s^{e_0})$, $U = L \otimes_K W$, and $M_0 = SL(U,L)$. By (3.15) in [As2] again, $N_{GL(V,F)}(M_0) \leq GL(U,L)\mathbb{F}^{\#}$. Thus $H = G \cap N_{GL(V,F)}(M) \leq G \cap GL(W,K)\mathbb{F}^{\#} \leq G \cap GL(U,L)\mathbb{F}^{\#}$ $= H_0$ and as H is maximal, we have $K = F$ and $W = V$ for $M = Sp(W)$, $SU(W)$, or $O^{\pm}(W,Q)$. Further $|\mathbb{F}:K| = r$ with r a prime for $M = SL(W)$, and in this case, $H = G \cap ZGL(W,K)$.

We show next that for $M = Sp(V)$ or $SU(V)$, $H = G \cap \Delta(V,f)$, where f is a symplectic or unitary form on V respectively. Let $n \in N_{GL(V)}(M)$. Define $f_n \in L(V, V^{\theta}; \mathbb{F})$ by $f_n(v,u) = f(vn,un)$, where $\theta \in Aut(\mathbb{F})$ is of order 1 or 2 respectively. Then $\forall \alpha \in M$, $f_n(v\alpha,u\alpha) = f(v\alpha n,u\alpha n) = f(vn\alpha^n,un\alpha^n) = f(vn,un) = f_n(v,u)$, which implies that $f_n \in L_M(V,V^{\theta})$. Since $0 \neq f \in L_M(V,V^{\theta})$, by Ex.9.1 in [As1], $f_n = \lambda_n f$ for some $\lambda_n \in \mathbb{F}$. Thus $f(vn,un) = \lambda_n f(v,u)$ which implies that $n \in \Delta(V,f)$. So $N_{GL(V,F)}(M) \leq \Delta(V,f)$. $\Delta(V,f)$ normalizes M is clear. Thus $H = G \cap \Delta(V,f)$.

For $M = O^{\pm}(V,Q)$, as $char(F) = 2$, $O^{\pm}(V,Q) \leq Sp(V,f)$, where f is the symplectic form associated with the quadratic form Q, and as $O^{\pm}(V,Q)$ is absolutely irreducible, by previous argument, $N_{GL(V)}(M) \leq \Delta(V,f)$. Then the maximality of H supplies a contradiction.

For f symplectic, $\Delta(V,f) = Sp(V,f)\langle\rho\rangle$, where ρ is defined by $x_{2i-1}\rho = x_{2i-1}$, $x_{2i}\rho = ax_{2i}$ for some hyperbolic basis $X = \{x_i : 1 \leq i \leq n\}$ of V, and with $\langle a \rangle = \mathbb{F}^{\#}$. If q is even, then $\Delta(V,f) = Sp(V,f) \times Z$. For f unitary, $\Delta(V,f) = GU(V,f) * Z$ with $GU(V) \cap Z = \langle bI \rangle$, $b \in \mathbb{F}^{\#}$ is of order $q^{\frac{1}{2}} + 1$. See (6.3), (6.4), and (6.2) in [As2].

For the cases (c) and (d) in Kantor's theorem, let V_m be the m-dimensional vector space over $GF(2)$, and let $X = \{x_1, \ldots, x_m\}$ be a basis of V_m. So S_m permutes the basis vectors in X. Let $V_1 = \langle \sum x_i \rangle$ and $V_{m-1} = \{\sum a_i x_i : \sum a_i = 0\}$. Then $V_1 \leq V_{m-1}$ if and only if m is even. Also S_m acts on V_{m-1} and centralizes V_1. Thus case (c) corresponds to the action of S_m on $W = V_{m-1}/(V_1 \cap V_{m-1})$. Define a bilinear form on V_m by $f(x_i,x_j) = \delta_{ij}$, the Kronecker delta.

This form f induces a symplectic form $\overline{f}$ on W by $\overline{f}(\overline{u},\overline{v})=f(u,v)$, and S_m preserves $\overline{f}$, see Ex7.7 in [As1], i.e., $S_m \leq Sp(W)=Sp_{m-d}(2)$, where $d=(m,2)$ *and* $n=m-d$. The transposition (ij) acts on W as a transvection with center $\langle\overline{x_i+x_j}\rangle$. So for $z\in End_{KM}(W)$, z acts as scalar λ_{ij} on $\langle\overline{x_i+x_j}\rangle$. Since $\overline{x_{i-1}+x_i}$, $\overline{x_i+x_{i+1}}$, $\overline{x_{i-1}+x_{i+1}}$ generates a 2-dimensional space, the scalar λ_{ij} does not depend on i,j; which implies that z acts as a scalar on W, which implies that S_m is absolutely irreducible. Thus by previous argument, first $W=V$ and then $N_{GL(V)}(S_m)\leq \Delta(V,\overline{f})$. So the case (c) is out. Case (d) corresponds to the action of S_{2m} on V_{2m-1} or the original action on V_{2m}. Since S_{2m} centralizes $V_1 < V_{2m-1}$, and H is irreducible, case (d) is out.

For case (e) and (f), the conclusion can be obtained from the atlas of $PSL_3(4)$ and $PSL_2(9)$ respectively. Case (g) is out as $3\cdot P\Omega_6^{-,\pi}(3)\leq SU_6(2)$.

Case (i) corresponds to a direct decomposition of $V=V_1\oplus\cdots\oplus V_n$ into n 1-dimensional subspaces, and A_0, the direct product of n copies of cyclic group of order a, fixing this decomposition, S_n permutes the V_i's, and $A=[A_0,S_n]$. Since we assume H is primitive on V, this case is out too.

(12.3) Let $\overline{G}$ be a connected algebraic group over an algebraically closed field $\overline{F}$ of characteristic p and σ an endomorphism of $\overline{G}$ with $G=C_{\overline{G}}(\sigma)$ finite. Let g be a unipotent or semisimple element in G, $\overline{C}=C_{\overline{G}}(g)$, and $\overline{C}^{\circ}$ the connected component of $\overline{C}$. Define an equivalence relation $\sim$ on $\overline{C}/\overline{C}^{\circ}$ by $x\sim y$ if there exists $z\in\overline{C}/\overline{C}^{\circ}$ such that $x=z^{\sigma}yz^{-1}$.

(a) (S. Lang) The map $x\mapsto x\sigma(x^{-1})$ is surjection of $\overline{G}$ onto $\overline{G}$.

(b) The G-classes in $G\cap g\overline{G}$ are in one to one crrespondence with the equivalence classes of $\overline{C}/\overline{C}^{\circ}$. In particular, if $\overline{C}$ is connected, then $G\cap g\overline{G}=gG$.

(c) $\overline{G}$ is transitive on $\overline{G}\sigma$ and $H=C_{\overline{G}}(\sigma^m)$ is transitive on $\{\tau\in H\sigma:\tau^m=\sigma^m\}$.

Proof. See [El].

(12.4) Denote by $f(g,G/H)$ the number of fixed points of g on G/H. Let V be an n-dimensional vector space over $GF(q)$, and $Z=Z(GL(V))$. Suppose $H=ZK$. Let $\mu=\frac{|H|}{|K|}$. Then $f(g,G/H)=\frac{1}{\mu}\{f(g,G/K)+f(gz,G/K)+\cdots+f(gz^{\mu-1},G/K)\}$, where z is such that $Z=\langle z^{-1}\rangle$. In particular, if $g\in H$ is such that $|g|=p^e$, where $p=char(F)$, then $f(g,G/H)=\frac{1}{\mu}f(g,G/K)$.

Proof. Since $\frac{H}{K}\simeq\frac{Z}{Z\cap K}$, $Z\cap K=\langle z^{-\mu}\rangle$, and $H=K\cup Kz^{-1}\cup\cdots\cup Kz^{-(\mu-1)}$. Suppose $Hxg=Hx$. Then $Kxg\subseteq Hx$ and thus $Kxg=Kxz^{-\alpha}$ for some $0\leq\alpha\leq\mu-1$. Then Kx, $Kxz^{-1},\ldots,Kxz^{-(\mu-1)}$ are all fixed by gz^{α}. Conversely, if Kx is fixed by gz^{α}, then each of Kx, $Kxz^{-1},\ldots,Kxz^{-(\mu-1)}$ is fixed by gz^{α}, which implies that $Hxg=Hx$. Also if Kx is fixed by gz^{α}, then it is not fixed by any gz^{β} for any $\beta\neq\alpha$ with $0\leq\beta\leq\mu-1$. Hence $f(g,G/H)=\frac{1}{\mu}\{f(g,G/K)+f(gz,G/K)+\cdots+f(gz^{\mu-1},G/K)\}$. The second statement is due to the fact that no element in $H-K$ has the order equal to a power of p.

(12.5) Let V be an n-dimensional vector space over $\mathbb{F}$, $g\in GL(V)$ with $(|g|, p)=1$. Suppose $f=min(g,\mathbb{F},V)=f_1f_2\cdots f_\alpha$, where each f_μ is irreducible in $\mathbb{F}[x]$. Let $V=V_1\oplus V_2\oplus\cdots\oplus V_\alpha$, where each V_μ is the homogeneous component corresponding to f_μ. Denote $c_\mu=deg(f_\mu)$, $n_\mu=dim_{\mathbb{F}}(V_\mu)$. Let $C=C_{GL(V)}(g)$, $C_\mu=\{e\in C$: e acts as the identity on each V_ν with $\nu\neq\mu\}$. Then $C=C_1\times C_2\times\cdots\times C_\alpha$ and $C_\mu\simeq GL_{\frac{n_\mu}{c_\mu}}(q^{c_\mu})$ for each μ.

Proof. For each $e\in C$, e acts on each homogeneous component V_μ. Thus $C=C_1\times C_2\times\cdots\times C_\alpha$, with C_μ defined as above. Then without loss of generality, we can assume that V is a homogeneous $\mathbb{F}\langle g\rangle$-module, i.e., f is irreducible in $\mathbb{F}[x]$. Thus $\sigma:\mathbb{F}\to\mathbb{F}$, $x\mapsto x^q$ is transitive on the roots of f, which are same as the set of all distinct eigenvalues of g. Let $c=deg(f)$ and $L=GF(q^c)$. So f splits in L, and g is diagonalizable in $W=L\otimes_{\mathbb{F}}V$. Let X be a basis of V. Define a field automorphism σ on $GL(W)$ by $M_X(y^\sigma)=(M_X(y))^\sigma$. Let Y be an eigenvector basis of W and λ a root of f. So $D=M_Y(g)=diag(\lambda I_{d_1}, \lambda^q I_{d_2}, \ldots, \lambda^{q^{c-1}}I_{d_c})$, where d_i is the dimension of eigenspace corresponding to the eigenvalue $\lambda^{q^{i-1}}$. We have $M_Y(y)=BM_X(y)B^{-1}$ for some invertible $B\in L^{n\times n}$. So with respect to Y, σ is defined by $M_Y(y^\sigma)=A(M_Y(y))^\sigma A^{-1}$, where $A=BB^{-\sigma}$. Since $g\in GL(V)$, $DA=AD^\sigma$. Write $A=(A_{ij})$ into the block form corresponding to the dimension of the eigenspaces. So $\lambda^{q^{i-1}}A_{ij}=\lambda^{q^j}A_{ij}$, and thus $A_{ij}\neq 0$ iff $\lambda^{q^{i-1}}=\lambda^{q^j}$. Thus we have a permutation ξ on $\{1,2,\ldots,c\}$ defined by $A_{\xi(j)j}\neq 0$. Denote $A_j=A_{\xi(j)j}$. We have $e\in C_{GL(V)}(g)$ iff $e\in C_{GL(W)}(g)$ and $e^{\sigma}=e$. Also $e\in C_{GL(W)}(g)$ iff $E=M_Y(e)=diag(E_1,E_2,\ldots,E_c)$; and $e^{\sigma}=e$ iff $E_iA_{ij}=A_{ij}E_j^{\sigma}$ $\forall i,j$ iff $E_{\xi(j)}A_j=A_jE_j^{\sigma}$ $\forall j$ iff $E_{\xi^t(1)}=(A_{\xi^{t-1}(1)}A^{\sigma}_{\xi^{t-2}(1)}\cdots A_1^{\sigma^{t-1}})E_1^{\sigma^t}(A_{\xi^{t-1}(1)}A^{\sigma}_{\xi^{t-2}(1)}\cdots A_1^{\sigma^{t-1}})^{-1}$ $\forall 0\leq t\leq c$. So $d_1=d_2=\cdots=d_c=\frac{n}{c}$. Since $A=BB^{-\sigma}$, we have $AA^{\sigma}\cdots A^{\sigma^{t-1}}=BB^{-\sigma^t}$, $(BB^{-\sigma^t})_{ij}=0$ if $i\neq\xi^t(j)$ and $(BB^{-\sigma^t})_{ij}=A_{\xi^{t-1}(j)}A^{\sigma}_{\xi^{t-2}(j)}\cdots A_j^{\sigma^{t-1}}$ if $i=\xi^t(j)$. In particular, as $B\in L^{n\times n}$, $BB^{-\sigma^c}=I_n$, thus $A_{\xi^{c-1}(1)}A^{\sigma}_{\xi^{c-2}(1)}\cdots A_1^{\sigma^{c-1}}=I_{\frac{n}{c}}$. As $E_1=E_{\xi^c(1)}$, the condition for $t=c$ is just $E_1=E_1^{\sigma^c}$. Thus $C_{GL(V)}(g)\simeq GL_{\frac{n}{c}}(q^c)$.

(12.6) Let $K=GF(q_1)$, $q=q_1^r$, r a prime, $\mathbb{F}=GF(q)$, W an n-dimensional vector space over K, $V=\mathbb{F}\otimes_K W$, $g\in GL(W)$ with $(|g|,p)=1$.

(a) Let $C_0=C_{GL(W)}(g)$, $C=C_{GL(V)}(g)$. Then $\frac{|C|}{|C_0|}\leq\left(\frac{q_1}{q_1-1}\right)^n\left(\frac{q}{q_1}\right)^{n^2-2n+2}$. If $n=2$, then we have $\frac{|C|}{|C_0|}\leq\left(\frac{q-1}{q_1-1}\right)^2$. If $n=3$, then $\frac{|C|}{|C_0|}\leq\frac{q(q-1)^2(q^2-1)}{q_1(q_1-1)^2(q_1^2-1)}$.

(b) Let $H=GL(W)$, $G=GL(V)$. Then $\mathcal{N}(g,G/H)\leq\left(\frac{q_1}{q_1-1}\right)^n\left(\frac{q_1}{q}\right)^{2(n-1)}$. If $n=2$, then we have $\mathcal{N}(g,G/H)\leq\frac{q_1(q_1+1)}{q(q+1)}$. If $n=3$, then $\mathcal{N}(g,G/H)\leq\frac{q_1^2(q_1^2+q_1+1)}{q^2(q^2+q+1)}$.

(c) Suppose $ZSL(V)\leq G\leq GL(V)$, $H=G\cap ZGL(W)$, where $Z=Z(GL(V))$. Then $\mathcal{N}(g,G/H)\leq(n,q-1)\cdot\frac{q-1}{q}\left(\frac{q_1}{q_1-1}\right)^{n+1}\left(\frac{q_1}{q}\right)^{2n-3}$. If $n=2$, then $\mathcal{N}(g,G/H)\leq\frac{4q_1(q_1+1)}{q(q+1)}$. If $n=3$, then $\mathcal{N}(g,G/H)\leq\frac{9q_1^2(q_1^2+q_1+1)}{q^2(q^2+q+1)}$.

Proof. Let $m=min(g,K,W)=min(g,\mathbb{F},V)=f_1f_2\cdots f_\alpha$, a product of irreducibles in $K[x]$, $c_\mu=deg(f_\mu)$, $W=W_1\oplus W_2\oplus\cdots\oplus W_\alpha$, where W_μ is the $K\langle g\rangle$-homogeneous component corresponding to f_μ, $n_\mu=dim_K(W_\mu)$, $d_\mu=\frac{n_\mu}{c_\mu}$. Then $C_0=C_1\times\cdots\times C_\alpha$, where each $C_\mu\simeq GL_{d_\mu}(q^{\frac{c_\mu}{r}})$. Each f_μ is either (a) irreducible in $\mathbb{F}[x]$, when $(c_\mu,r)=1$; or (b) splits into r irreducibles: $f_\mu=f_\mu^{(1)}\cdots f_\mu^{(r)}$ in $\mathbb{F}[x]$, when $r|c_\mu$; in this case, $deg(f_\mu^{(i)})=\frac{c_\mu}{r}$ $\forall i$. We have that

$C=A_1\times\cdots\times A_\alpha$, where $A_\mu\simeq GL_{d_\mu}(q^{c_\mu})$ in case (a) and $A_\mu\simeq GL_{d_\mu}(q^{\frac{c_\mu}{r}})\times\cdots\times GL_{d_\mu}(q^{\frac{c_\mu}{r}})$, r copies, in case (b). We have that $(s-1)^k s^{k(k-1)}\leq|GL_k(s)|\leq s^{k^2}$. So in case (a), $\frac{|A_\mu|}{|C_\mu|}=\frac{|GL_{d_\mu}(q^{c_\mu})|}{|GL_{d_\mu}(q_1^{c_\mu})|}\leq\frac{q^{c_\mu d_\mu^2}}{(q_1^{c_\mu}-1)^{d_\mu}q_1^{c_\mu d_\mu(d_\mu-1)}}\leq\left(\frac{q_1}{q_1-1}\right)^{n_\mu}\left(\frac{q}{q_1}\right)^{n_\mu d_\mu}$, as $\frac{q_1^{c_\mu}}{q_1^{c_\mu}-1}\leq\left(\frac{q_1}{q_1-1}\right)^{c_\mu}$ and $c_\mu d_\mu=n_\mu$. In case (b), $\frac{|A_\mu|}{|C_\mu|}=|GL_{d_\mu}(q_1^{c_\mu})|^{r-1}\leq q_1^{c_\mu d_\mu^2(r-1)}=\left(\frac{q}{q_1}\right)^{n_\mu d_\mu}$. So in either case, $\frac{|A_\mu|}{|C_\mu|}\leq\left(\frac{q_1}{q_1-1}\right)^{n_\mu}\left(\frac{q}{q_1}\right)^{n_\mu d_\mu}$ and thus $\frac{|C|}{|C_0|}=\prod_{\mu=1}^{\alpha}\frac{|A_\mu|}{|C_\mu|}\leq\left(\frac{q_1}{q_1-1}\right)^{\sum_{\mu=1}^{\alpha}n_\mu}\left(\frac{q}{q_1}\right)^{\sum_{\mu=1}^{\alpha}n_\mu d_\mu}$. We have that $\sum_{\mu=1}^{\alpha}n_\mu=n$; and if every $d_\mu\leq n-2$, then $\sum_{\mu=1}^{\alpha}n_\mu d_\mu\leq(n-2)\sum_{\mu=1}^{\alpha}n_\mu=(n-2)n<n^2-2n+2$; and if one $d_\mu=n-1$, say $d_1=n-1$, then either $c_1=1$, $n_1=n-1$, $\alpha=2$, $c_2=d_2=n_2=1$, i.e., g is a pseudo-reflection; or $c_1=2$, $\alpha=1$, $n_1=n=2$; so in both cases we have $\sum_{\mu=1}^{\alpha}n_\mu d_\mu=(n-1)^2+1=n^2-2n+2$. That is we always have $\sum_{\mu=1}^{\alpha}n_\mu d_\mu\leq n^2-2n+2$. Hence $\frac{|C|}{|C_0|}\leq\left(\frac{q_1}{q_1-1}\right)^n\left(\frac{q}{q_1}\right)^{n^2-2n+2}$.

Suppose $n=2$. If $\alpha=2$, then $C\simeq GF(q)^\#\times GF(q)^\#$ and $C_0\simeq GF(q_1)^\#\times GF(q_1)^\#$; which gives $\frac{|C|}{|C_0|}=\left(\frac{q-1}{q_1-1}\right)^2$. If $\alpha=1$, then $C_0\simeq GL_1(q_1^2)=GF(q_1^2)^\#$. We have that $C\simeq GL_1(q^2)=GF(q^2)^\#$ when $r\geq3$; and $C\simeq GF(q)^\#\times GF(q)^\#$ when $r=2$. Respectively, we have $\frac{|C|}{|C_0|}=\frac{q-1}{q_1-1}\cdot\frac{q+1}{q_1+1}$ and $\frac{|C|}{|C_0|}=\frac{q-1}{q_1-1}\cdot\frac{q-1}{q_1+1}$. Since $\frac{q+1}{q_1+1}<\frac{q-1}{q_1-1}$, in any of these three cases, we have $\frac{|C|}{|C_0|}\leq\left(\frac{q-1}{q_1-1}\right)^2$.

Suppose $n=3$. If $\alpha=3$, then $C\simeq GF(q)^\#\times GF(q)^\#\times GF(q)^\#$ and $C_0\simeq GF(q_1)^\#\times GF(q_1)^\#\times GF(q_1)^\#$. If $\alpha=2$ and $c_1=c_2=1$, then $C\simeq GF(q)^\#\times GL_2(q)$ and $C_0\simeq GF(q_1)^\#\times GL_2(q_1)$. If $\alpha=2$ and $c_1=1$, $c_2=2$, then $C_0\simeq GF(q_1)^\#\times GF(q_1^2)^\#$ and $C\simeq GF(q)^\#\times GF(q^2)^\#$, $C\simeq GF(q)^\#\times GF(q)^\#\times GF(q)^\#$ for $r\neq2$, $r=2$ respectively. If $\alpha=1$, then $C_0\simeq GF(q_1^3)^\#$ and $C\simeq GF(q^3)^\#$, $C\simeq GF(q)^\#\times GF(q)^\#\times GF(q)^\#$ for $r\neq3$, $r=3$ respectively. Thus it is easy to check that in any case we have $\frac{|C|}{|C_0|}\leq\frac{q(q-1)^2(q^2-1)}{q_1(q_1-1)^2(q_1^2-1)}$.

For part (b), let $g^x \in g^G \cap H$. Since g, g^x have the same set of eigenvalues, including the same multiplicity for each eigenvalue, g, g^x have the same Jordan cannonical form in $K^{n\times n}$ with respect to possibly different bases of W. Hence g, g^x are conjugates by some element in $GL(W)$. So $g^G \cap H = g^H$. Thus $\mathcal{N}(g,G/H)=\frac{|g^G \cap H|}{|g^G|}=\frac{|g^H|}{|g^G|}=\frac{|C||H|}{|G||C_0|}$. Since $q_1 \le q$ implies $\frac{q_1^i-1}{q^i-1}\le\left(\frac{q_1}{q}\right)^i$, we have $\frac{|H|}{|G|}=\left(\frac{q_1}{q}\right)^{\frac{1}{2}n(n-1)}\prod_{i=1}^{n}\frac{q_1^i-1}{q^i-1}\le\left(\frac{q_1}{q}\right)^{n^2}$. By (a), $\frac{|C|}{|C_0|}\le\left(\frac{q_1}{q_1-1}\right)^n\left(\frac{q}{q_1}\right)^{n^2-2n+2}$, thus we have $\mathcal{N}(g,G/H)\le\left(\frac{q_1}{q_1-1}\right)^n\left(\frac{q_1}{q}\right)^{2(n-1)}$. If $n=2$, then as $\frac{|H|}{|G|}=\frac{q_1(q_1-1)(q_1^2-1)}{q(q-1)(q^2-1)}$, we have $\mathcal{N}(g,G/H)\le\frac{q_1(q_1+1)}{q(q+1)}$. If $n=3$, then as $\frac{|H|}{|G|}=\frac{q_1^3(q_1-1)(q_1^2-1)(q_1^3-1)}{q^3(q-1)(q^2-1)(q^3-1)}$, we have $\mathcal{N}(g,G/H)\le\frac{q_1^2(q_1^2+q_1+1)}{q^2(q^2+q+1)}$.

For part (c), let $G_1=GL(V)$, $H_1=GL(W)$. Now use (12.4), the μ there is such that $\mu=\frac{|ZH_1|}{|H_1|}=\frac{q-1}{q_1-1}$; and we have $f(g,G_1/ZH_1)=\frac{1}{\mu}\{f(g,G_1/H_1)+f(gz,G_1/H_1)+\cdots+f(gz^{\mu-1},G_1/H_1)\}$, where $Z=\langle z^{-1}\rangle$. Also $|G_1/ZH_1|=\frac{1}{\mu}|G_1/H_1|$, and as each gz^i, $0\le i\le\mu-1$, is semisimple, by part (b), $\mathcal{N}(g,G_1/ZH_1)=\sum_{i=0}^{\mu-1}\mathcal{N}(gz^i,G_1/H_1)\le\mu\cdot\left(\frac{q_1}{q_1-1}\right)^n\left(\frac{q_1}{q}\right)^{2(n-1)}=\frac{q-1}{q}\left(\frac{q_1}{q_1-1}\right)^{n+1}\left(\frac{q_1}{q}\right)^{2n-3}$. Then (3.5) gives the bound in part (c).

Suppose $n=2$. Without loss of generality, we can assume $g\in ZH_1$. So $g=z^ih$ for some $h\in H_1$. Suppose some $g^x=z^jh'$ with $h'\in H_1$ and $j\ne i$. Since $1=det(g^xg^{-1})=z^{2(j-i)}det(h'h^{-1})$ and $det(h'h^{-1})\in GF(q_1)^{\#}=\langle z^{-\mu}\rangle$, we have $\mu|2(i-j)$. But $1\le|i-j|\le\mu-1$, so $2(i-j)=\mu=q_1^{r-1}+\cdots+q_1+1$. Thus $g^{G_1}\cap ZH_1$ is contained in only one coset z^iH_1 if q_1 even or if q_1 odd and r odd; and $g^{G_1}\cap ZH_1$ is contained in at most two cosets z^iH_1 and z^jH_1 if q_1 odd and r even. Hence by part (b), we have $\mathcal{N}(g,G_1/ZH_1)\le\frac{2q_1(q_1+1)}{q(q+1)}$. Finally, note that when $n=2$, $(n,q-1)\le 2$, so we have the bound in part (c) for $n=2$. For $n=3$, similarly, $g^{G_1}\cap ZH_1$ is contained in at most three cosets of H_1, and $(n,q-1)\le 3$, thus we have the extra factor 9 multiplied to the bound in (b) when $n=3$.

(12.7) Let $g \in G = GL_n(q)$ with $|g| = p^e$, $e \geq 0$, and $C = C_G(g)$. Suppose the Jordan canonical form of g has d_μ blocks of dimension c_μ, where $1 \leq \mu \leq \alpha$, $c_\mu \neq c_\nu$ for $\mu \neq \nu$, and $\sum_{\mu=1}^{\alpha} c_\mu d_\mu = n$. Denote $n_\mu = c_\mu d_\mu$. Then $|C| = q^x \cdot \prod_{\mu=1}^{\alpha} \{(q-1)(q^2-1)\cdots(q^{d_\mu}-1)\}$, where $x = \sum_{\mu=1}^{\alpha} \{n_\mu d_\mu - \frac{1}{2} d_\mu(d_\mu+1)\} + \sum_{\mu \neq \nu} d_\mu d_\nu \min\{c_\mu, c_\nu\}$. In particular, if we arrange c_μ's so that $c_1 > c_2 > \cdots > c_\alpha$, then $x = \sum_{\mu=1}^{\alpha} \{n_\mu d_\mu - \frac{1}{2} d_\mu(d_\mu+1)\} + 2 \sum_{1 \leq \mu < \nu \leq \alpha} d_\mu n_\nu$.

Proof. We can arrange the Jordan canonical form J of g with respect to some basis X so that $J = M_X(g) = diag\{J_1, \ldots, J_\alpha\}$, where α is the number of distint elementary divisors of g, and each J_μ is the direct sum of d_μ Jordan blocks each of which is of the same dimension c_μ, and without loss of generality $c_1 > c_2 > \cdots > c_\alpha$. We have $(J - I_n)A = A(J - I_n)$ iff $c \in C$, where $A = M_X(c)$. Denote $n_\mu = c_\mu d_\mu$. Now $J_\mu - I_{n_\mu} = diag\{S_\mu^{(1)}, \ldots, S_\mu^{(d_\mu)}\}$, where each $S_\mu^{(i)}$ is the $c_\mu \times c_\mu$ matrix $\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$. Writing A in the block form, i.e., $A = (A_{\mu\nu})$, $1 \leq \mu$, $\nu \leq \alpha$; and each $A_{\mu\nu} = (A_{\mu\nu}^{(ij)})$, $1 \leq i \leq d_\mu$, $1 \leq j \leq d_\nu$, where $A_{\mu\nu}$ is an $n_\mu \times n_\nu$ matrix and $A_{\mu\nu}^{(ij)}$ is a $c_\mu \times c_\nu$ matrix. So $cg = gc$ iff $(J - I_n)A = A(J - I_n)$ iff $det(A) \neq 0$ and $S_\mu^{(i)} A_{\mu\nu}^{(ij)} = A_{\mu\nu}^{(ij)} S_\nu^{(j)}$ $\forall$ $1 \leq \mu$, $\nu \leq \alpha$ and $1 \leq i \leq d_\mu$, $1 \leq j \leq d_\nu$. For simplicity, write $a = c_\mu$, $b = c_\nu$. Let $m = \min\{a, b\}$ and T be the $m \times m$ upper triangular matrix of the following form:

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{m-2} & x_{m-1} & x_m \\ 0 & x_1 & x_2 & \cdots & x_{m-3} & x_{m-2} & x_{m-1} \\ 0 & 0 & x_1 & \cdots & x_{m-4} & x_{m-3} & x_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_1 & x_2 & x_3 \\ 0 & 0 & 0 & \cdots & 0 & x_1 & x_2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & x_1 \end{bmatrix}$$

Then it is easy to check that $S_\mu^{(i)} A_{\mu\nu}^{(ij)} = A_{\mu\nu}^{(ij)} S_\nu^{(j)}$ iff $A_{\mu\nu}^{(ij)} = T$ when $a = b$, equivalently $\mu = \nu$; or $A_{\mu\nu}^{(ij)} = \begin{bmatrix} O & T \end{bmatrix}$, when $a < b$, where O is an $a \times (b-a)$ matrix whose entries are all zero; or $A_{\mu\nu}^{(ij)} = \begin{bmatrix} T \\ O \end{bmatrix}$, when $a > b$, where O is an $(a-b) \times b$ matrix whose entries are all zero. For a fixed μ, let $A_\mu = (a_{ij})$, where a_{ij} is the entry on the southeast corner of $A_{\mu\mu}^{(ij)}$. Thus $det(A_{\mu\mu}) \neq 0$ iff $det(A_\mu) \neq 0$. Suppose $det(A) \neq 0$. Then it is easy to see that $det(A_\mu) \neq 0$ $\forall$ μ, which gives

$det(A_{\mu\mu})\neq 0\ \forall\ \mu$. Now suppose $det(A_\mu)\neq 0\ \forall\ \mu$. By induction on α, it can be shown that $det(A)\neq 0$. That is with A in the form described above, $c\in GL_n(q)$ iff each $det(A_\mu)\neq 0$. For each fixed μ, the number of choices for A_μ is equal to $|GL_{d_\mu}(q)|$. Thus the number of choices for $A_{\mu\mu}$ is $|GL_{d_\mu}(q)|\cdot q^{(c_\mu-1)d_\mu^2}=q^{n_\mu d_\mu-\frac{1}{2}d_\mu(d_\mu+1)}(q-1)(q^2-1)\cdots(q^{d_\mu}-1)$. The number of choices for each $A_{\mu\nu}^{(ij)}$ is $q^{min\{c_\mu,c_\nu\}}$, which gives that for each fixed pair $(\mu,\ \nu)$ with $\mu\neq\nu$, the number of choices for $A_{\mu\nu}$ is $q^{d_\mu d_\nu min\{c_\mu,c_\nu\}}$. Hence we have the conclusion.

(12.8) Let $K=GF(q_1)$, $q=q_1^r$, r a prime, $\mathbb{F}=GF(q)$, W an n-dimensional vector space over K, $V=\mathbb{F}\otimes_K W$, $g\in GL(W)$ with $|g|=p^e$, $p=char(K)$.

(a) Let $H=GL(W)$, $G=GL(V)$. Then $\mathcal{N}(g,G/H)\leq\left(\frac{q_1}{q_1-1}\right)^{n-1}\left(\frac{q_1}{q}\right)^n$. If $n=2$, then $\mathcal{N}(g,G/H)\leq\frac{q_1^2-1}{q^2-1}$. If $n=3$, then $\mathcal{N}(g,G/H)\leq\frac{(q_1^2-1)(q_1^2+q_1+1)}{(q^2-1)(q^2+q+1)}$. If $r=2$, then $\mathcal{N}(g,G/H)\leq\frac{5}{2\cdot q_1^n}$.

(b) Suppose $ZSL(V)\leq G\leq GL(V)$, $H=G\cap ZGL(W)$, where $Z=Z(GL(V))$. Then $\mathcal{N}(g,G/H)\leq(n,q-1)\cdot\left(\frac{q_1}{q_1-1}\right)^{n-1}\left(\frac{q_1}{q}\right)^n$. If $n=2$, then $\mathcal{N}(g,G/H)\leq\frac{2(q_1^2-1)}{q^2-1}$. If $n=3$, then $\mathcal{N}(g,G/H)\leq\frac{3(q_1^2-1)(q_1^2+q_1+1)}{(q^2-1)(q^2+q+1)}$. If $r=2$, then $\mathcal{N}(g,G/H)\leq(n,q-1)\cdot\frac{5}{2\cdot q_1^n}$.

Proof. We can suppose without loss of generality that $g\in H$. Since for $g^x\in g^G\cap H$, g^x and g have the same Jordan canonical form with respect to possibly different bases, we have $g^G\cap H=g^H$. So $\mathcal{N}(g,G/H)=\frac{|H||C|}{|G||C_0|}$, where $C_0=C_H(g)$, $C=C_G(g)$. We use the notations in (12.7). By (12.7) and as $(q_1-1)(q_1^2-1)\cdots(q_1^{d_\mu}-1)\geq\left(\frac{q_1-1}{q_1}\right)^{d_\mu}q_1^{\frac{1}{2}d_\mu(d_\mu+1)}$, we have $|C_0|\geq\left(\frac{q_1-1}{q_1}\right)^{\sum_{\mu=1}^{\alpha}d_\mu}q_1^y$ and $|C|\leq q^y$, where $y=\sum_{\mu=1}^{\alpha}n_\mu d_\mu+\sum_{\mu\neq\nu}d_\mu d_\nu min\{c_\mu,c_\nu\}$. As $\sum_{\mu=1}^{\alpha}d_\mu\leq n-1$ and $\sum_{\mu=1}^{\alpha}n_\mu=n$, we have $y\leq\left(\sum_{\mu=1}^{\alpha}d_\mu\right)\left(\sum_{\mu=1}^{\alpha}n_\mu\right)\leq(n-1)n$. Thus $\frac{|C|}{|C_0|}\leq\left(\frac{q_1}{q_1-1}\right)^{n-1}\left(\frac{q}{q_1}\right)^{n(n-1)}$. Then as before $\frac{|H|}{|G|}\leq\left(\frac{q_1}{q}\right)^{n^2}$, so we have $\mathcal{N}(g,G/H)\leq\left(\frac{q_1}{q_1-1}\right)^{n-1}\left(\frac{q_1}{q}\right)^n$. Suppose $n=2$. It is easy to see that we have $C_H(g)\simeq GF(q_1)^{\#}\times E_{q_1}$, $C_G(g)\simeq GF(q)^{\#}\times E_q$. Hence $\mathcal{N}(g,G/H)\leq\frac{q_1^2-1}{q^2-1}$. Suppose $n=3$. Let

$k=dim\{C_W(g)\}$. If $k=1$, then by (12.7), we have $|C_H(g)|=q_1^2(q_1-1)$ and $|C_G(g)|=q^2(q-1)$. If $k=2$, then $|C_H(g)|=q_1^3(q_1-1)^2$ and $|C_G(g)|=q^3(q-1)^2$. So in any case, we have $\frac{|C_G(g)|}{|C_H(g)|}\leq\frac{q^3(q-1)^2}{q_1^3(q_1-1)^2}$. Then as when $n=3$, we have $\frac{|H|}{|G|}=\frac{q_1^3(q_1-1)(q_1^2-1)(q_1^3-1)}{q^3(q-1)(q^2-1)(q^3-1)}$, we have $\mathcal{N}(g,G/H)\leq\frac{(q_1^2-1)(q_1^2+q_1+1)}{(q^2-1)(q^2+q+1)}$. Suppose $r=2$. By (12.7), we have $\frac{|C|}{|C_0|}=q_1^y\cdot\prod_{\mu=1}^{\alpha}\{(1+\frac{1}{q_1})(1+\frac{1}{q_1^2})\cdots(1+\frac{1}{q_1^{d_\mu}})\}$, where $y=\sum_{\mu=1}^{\alpha}n_\mu d_\mu+\sum_{\mu\neq\nu}d_\mu d_\nu min\{c_\mu,c_\nu\}$. Let $s_i=s_i(q_1)=(1+\frac{1}{q_1})(1+\frac{1}{q_1^2})\cdots(1+\frac{1}{q_1^i})$. Then we have $log(s_i)=log(1+\frac{1}{q_1})+\sum_{j=2}^{i}log(1+\frac{1}{q_1^j})\leq log(1+\frac{1}{q_1})+\sum_{j=2}^{i}\frac{1}{q_1^j}\leq log(1+\frac{1}{q_1})+\frac{1}{q_1(q_1-1)}$, which gives $s_i\leq(1+\frac{1}{q_1})e^{\frac{1}{q_1(q_1-1)}}\leq\frac{3}{2}\sqrt{e}\leq\frac{5}{2}$. Hence $\frac{|C|}{|C_0|}\leq q_1^y\left(\frac{5}{2}\right)^\alpha$. Now as before $y\leq\left(\sum_{\mu=1}^{\alpha}d_\mu\right)\left(\sum_{\mu=1}^{\alpha}n_\mu\right)=\left(\sum_{\mu=1}^{\alpha}d_\mu\right)n$, and $n-\left(\sum_{\mu=1}^{\alpha}d_\mu\right)=\sum_{\mu=1}^{\alpha}d_\mu(c_\mu-1)\geq\sum_{\mu=1}^{\alpha}(c_\mu-1)\geq\frac{1}{2}\alpha(\alpha-1)\geq\alpha-1$, as we assume without loss of generality that $c_1>c_2>\cdots>c_\alpha$. If $\alpha=1$, then $c_1\geq2$ and $n-d_1=(c_1-1)d_1\geq1=\alpha$. So if $n-\left(\sum_{\mu=1}^{\alpha}d_\mu\right)=\alpha-1$, then $\alpha-1=\sum_{\mu=1}^{\alpha}d_\mu(c_\mu-1)=\frac{1}{2}\alpha(\alpha-1)$ gives $\alpha=2$ and $c_2=1$, $c_1=2$, $d_1=1$, $d_2=n-2$. Hence $n-\left(\sum_{\mu=1}^{\alpha}d_\mu\right)=\alpha-1$ iff g is a transvection. So if g is not a transvection, then $\sum_{\mu=1}^{\alpha}d_\mu\leq n-\alpha$, and thus $y\leq n(n-\alpha)$. Hence if g is not a transvection, then $\mathcal{N}(g,G/H)\leq\frac{1}{q_1^{n\alpha}}\left(\frac{5}{2}\right)^\alpha\leq\frac{5}{2\cdot q_1^n}$. If g is a transvection, then as $\frac{|C|}{|C_0|}=q_1^x(q_1+1)^2(q_1^2+1)\cdots(q_1^{n-2}+1)$, and $\frac{|H|}{|G|}=\frac{1}{q_1^x(q_1+1)(q_1^2+1)\cdots(q_1^n+1)}$, where $x=\frac{1}{2}n(n-1)$, we have $\mathcal{N}(g,G/H)=\frac{(q_1+1)}{(q_1^{n-1}+1)(q_1^n+1)}\leq\frac{5}{2\cdot q_1^n}$. For part (b), let $G_1=GL(V)$, $H_1=GL(W)$. Now use (12.4), the μ there is such that $\mu=\frac{|ZH_1|}{|H_1|}=\frac{q-1}{q_1-1}$; and we have $f(g,G_1/ZH_1)=\frac{1}{\mu}f(g,G_1/H_1)$, where $Z=\langle z^{-1}\rangle$. Also $|G_1/ZH_1|=\frac{1}{\mu}|G_1/H_1|$, thus by part (a), $\mathcal{N}(g,G_1/ZH_1)=\mathcal{N}(g,G_1/H_1)\leq\left(\frac{q_1}{q_1-1}\right)^{n-1}\left(\frac{q_1}{q}\right)^n$. Then (3.5) gives the bound in part (b). Finally, note that when $n=2$ or 3, $(n,q-1)\leq2$ or 3 respectively, so we have the bounds in part (b) for $n=2$ and 3.

(12.9) Suppose $L_n(q)\le \overline{G}\le PGL_n(q)$ and $\overline{H}=\overline{G}\cap PGL_n(q_1)$, where $q=q_1^r$, r a prime, and $\overline{G}$ is a group of genus zero. Then one of the following holds:

(a) $n=2$ and $q\le 343$.

(b) $n=3$ and $q\le 27$.

(c) $n=4$ and $q\le 25$.

(d) $5\le n\le 8$ and $q\le 9$.

(e) $9\le n\le 11$ and $q=4$.

Proof. Suppose $n=2$ first. By (12.6) and (12.8), as $\frac{2(q_1^2-1)}{q^2-1}\le\frac{2q_1(q_1+1)}{q(q+1)}$ and $q_1\le\sqrt{q}$, we have that for any $\overline{g}\ne\overline{1}$, $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\le\frac{4q_1(q_1+1)}{q(q+1)}\le\frac{4}{q-\sqrt{q}+1}\le\frac{1}{85}$ if $q\ge 358$. Since $q=q_1^r$ with $r\ge 2$, q is not a prime. So if $n=2$ and $\overline{G}$ is a group of genus zero, then $q\le 7^3=343$. Suppose $n=3$. By (12.6) and (12.8), we have that for any $\overline{g}\ne\overline{1}$, $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\le\frac{9q_1^2(q_1^2+q_1+1)}{q^2(q^2+q+1)}\le\frac{9q(q+\sqrt{q}+1)}{q^2(q^2+q+1)}=\frac{9}{q(q-\sqrt{q}+1)}\le\frac{1}{85}$ if $q\ge 31$. So if $n=3$ and $\overline{G}$ is a group of genus zero, then $q\le 27$. Now consider $n\ge 4$. For semisimple g, by (12.6), $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\le(n,q-1)\cdot\frac{q-1}{q}\left(\frac{q_1}{q_1-1}\right)^{n+1}\left(\frac{q_1}{q}\right)^{2n-3}=x$; and for unipotent g, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\le(n,q-1)\cdot\left(\frac{q_1}{q_1-1}\right)^{n-1}\left(\frac{q_1}{q}\right)^{n}=y$. Since $\frac{y}{x}=\left(\frac{q}{q_1}\right)^{n-3}\cdot\frac{(q_1-1)^2}{q_1^2}\cdot\frac{q}{q-1}\ge\frac{q^2(q_1-1)^2}{q_1^3(q-1)}\ge 1$, for any $\overline{g}\ne\overline{1}$, $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\le(n,q-1)\cdot\left(\frac{q_1}{q_1-1}\right)^{n-1}\left(\frac{q_1}{q}\right)^{n}$ $\le n\cdot\left(\frac{q_1}{q_1-1}\right)^{n-1}\left(\frac{q_1}{q}\right)^{n}=a_n$. Now $\frac{a_{n+1}}{a_n}=(1+\frac{1}{n})\cdot\frac{q_1^2}{(q_1-1)q}\le 1$ except when $q_1=2$ and $q=4$. Consider the case that $(q_1,q)\ne(2,4)$ first. So $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\le 4\left(\frac{q_1}{q_1-1}\right)^{3}\left(\frac{q_1}{q}\right)^{4}=a_4$. Suppose $q\ge 27$. If $r=2$, then $q_1\ge 6$ and thus $a_4=\frac{4}{(q_1-1)^3q_1}\le\frac{4}{5^3\cdot 6}\le\frac{1}{85}$. If $r\ge 3$, then $a_4=\frac{4}{(q_1-1)^3q_1^5}\le\frac{4}{2^3\cdot 3^5}\le\frac{1}{85}$ unless $q_1=2$. But when $q_1=2$, if $q\ge 16$, then we have $a_4=\frac{4\cdot 2^3}{8^4}\le\frac{1}{85}$. Hence it leaves only to consider $(q_1,q)=(2,8)$, $(3,9)$, or $(5,25)$. For $q=25$, if $n\ge 5$, then we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\le a_5=\frac{1}{4^4}\le\frac{1}{85}$. For $q=9$, if $n\ge 9$, then we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\le a_9=\frac{9}{2^8\cdot 3}\le\frac{1}{85}$. For $q=8$, if $n\ge 9$, then we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\le(n,q-1)\cdot\frac{2^{n-1}}{4^n}\le\frac{7}{2^{n+1}}\le\frac{7}{2^{10}}\le\frac{1}{85}$. So it remains to consider the last case that $q=4$. In this case, for semisimple g, we have by (12.6) that $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq\frac{9}{2^{n-2}}$. For g unipotent, we have by (12.8) that $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq\frac{15}{2^{n+1}}\leq\frac{9}{2^{n-2}}$. So if $n\geq 12$, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq\frac{1}{85}$.

(12.10) Let $q=q_1^2$, $\mathbb{F}=GF(q)$, V an n-dimensional vector space over $\mathbb{F}$ equipped with a unitary form f, $g\in GU(V,f)$ with $(|g|,p)=1$, $m=min(g,\mathbb{F},V)=f_1f_2\cdots f_\alpha$, where each f_μ is irreducible in $\mathbb{F}[x]$. Let $V=V_1\oplus V_2\oplus\cdots\oplus V_\alpha$, where each V_μ is the homogeneous component corresponding to f_μ. Denote $c_\mu=deg(f_\mu)$, $n_\mu=dim_{\mathbb{F}}(V_\mu)$. For each f_μ, define $\overline{f}_\mu$ to be $(x-\lambda_1^{-q_1})\cdots(x-\lambda_{c_\mu}^{-q_1})$, where $\{\lambda_1,\ldots,\lambda_{c_\mu}\}$ is the set of roots of f_μ. $\overline{f}_\mu$ is called the inverse-conjugate of f_μ. Then each $\overline{f}_\mu$ is an irreducible factor of m, i.e., $\overline{f}_\mu=f_\nu$ for some ν. If $\overline{f}_\mu=f_\mu$, then f_μ is called self-inverse-conjugate. By pairing inverse-conjugates together, we can write $V=U_1\oplus\cdots\oplus U_\beta$, where U_ν is either some V_μ when f_μ is self-inverse-conjugate, or $U_\nu=V_\mu\oplus\overline{V}_\mu$, when f_μ is not self-inverse-conjugate and here $V_\mu,\overline{V}_\mu$ are homogeneous components for f_μ, $\overline{f}_\mu$ respectively. Then each U_ν is a nondegenerate subspace. Let $C=C_{GU(V)}(g)$. Then for each $e\in C$, e stabilizes the decomposition $V=U_1\oplus\cdots\oplus U_\beta$, and $C=C_1\times\cdots\times C_\beta$, where each C_ν act as the identity on all U_μ with $\mu\neq\nu$, and $C_\nu\simeq GU_{\frac{n_\mu}{c_\mu}}(q_1^{c_\mu})$ if $U_\nu=V_\mu$ for some μ when f_μ is self-inverse-conjugate; and $C_\nu\simeq GL_{\frac{n_\mu}{c_\mu}}(q^{c_\mu})$ if $U_\nu=V_\mu\oplus\overline{V}_\mu$ for some μ when f_μ is such that $\overline{f}_\mu\neq f_\mu$.

Proof. Let $X=\{x_i:1\leq i\leq n\}$ be an orthonormal basis of V with respect the unitary form f. Let L be the splitting field of m, $W=L\otimes_{\mathbb{F}}V$, and extend f naturally to W. Define $\theta:L\to L$ by $x\mapsto x^{q_1}$ and $\sigma:GL(W)\to GL(W)$ by $M_X(y^\sigma)=(M_X(y))^{-T\theta}$, where T denotes the transpose of a matrix. Let $Y=\{y_i:1\leq i\leq n\}$ be an eigenvector basis of g in W, so $D=M_Y(g)=diag(\lambda_1I_{d_1},\ \lambda_2I_{d_2},\ \ldots,\ \lambda_\gamma I_{d_\gamma})$, where $\{\lambda_1,\ldots,\lambda_\gamma\}$ is the set of all distinct eigenvalues of g. We have $M_Y(y^\sigma)=A(M_Y(y))^\sigma A^{-1}$, where $A=J(Y,f)=BB^{-\sigma}$ for some $B=(b_{ij})$ defined by $y_i=\sum_j b_{ij}x_j$. Write $A=(A_{ij})$ into the block form corresponding to the dimension of eigenspaces. Since $g^\sigma=g$ iff $DA=AD^\sigma$ iff $\lambda_iA_{ij}=\lambda_j^{-q_1}A_{ij}$ $\forall i,j$ iff $A_{ij}=0$ $\forall$ $\lambda_i\neq\lambda_j^{-q_1}$ and A_{ij} nonsingular $\forall$ $\lambda_i=\lambda_j^{-q_1}$. So for each eigenvalue λ_j, $\lambda_j^{-q_1}$ is also an eigenvalue of g. Thus $\overline{f}_\mu$ is an irreducible factor of m.

Also we have a permutation ξ on $\{1,2,\ldots,\gamma\}$ defined by $A_{\xi(j)j}\neq 0$ or equivalently $\lambda_{\xi(j)}=\lambda_j^{-q_1}$. Since the exponent by $-q_1$ acts either on the roots of f_μ if f_μ is self-inverse-conjugate, or acts on the union of the roots of f_μ and $\overline{f}_\mu$ if f_μ is not self-inverse-conjugate, we see that A has non zero blocks only on the main diagonal corresponding to each U_ν^L, i.e., $A=diag(A^{(1)},\ldots,A^{(\beta)})$. Hence each U_ν defined above is a nondegenerate subspace. Therefore, we can choose X such that X is the union of the orthonormal basis of U_ν's. Then B is in the diagonal block form $B=diag(B_1,\ldots,B_\beta)$, where each B_ν corresponding to U_ν and $B_\nu\in(GF(q^{c_\mu}))^{n_\mu\times n_\mu}$ if U_ν corresponds to f_μ and f_μ self-inverse-conjugate; and $B_\nu\in(GF(q^{c_\mu}))^{2n_\mu\times 2n_\mu}$ if U_ν corresponds to f_μ and f_μ not self-inverse-conjugate. Denote $A_j=A_{\xi(j)j}$. We have $e\in C_{GL(V)}(g)$ iff $e\in C_{GL(W)}(g)$ and $e^\sigma=e$. Also $e\in C_{GL(W)}(g)$ iff $E=M_Y(e)=diag(E_1,E_2,\ldots,E_\gamma)$, and $e^\sigma=e$ iff $E_iA_{ij}=A_{ij}E_j^\sigma$ $\forall i,j$ iff $E_{\xi(j)}A_j=A_jE_j^\sigma$ $\forall j$ iff for each orbit of ξ and a fixed j contained in that orbit, we have that $E_{\xi^t(j)}=(A_{\xi^{t-1}(j)}A^\sigma_{\xi^{t-2}(j)}\cdots A_j^{\sigma^{t-1}})E_j^{\sigma^t}(A_{\xi^{t-1}(j)}A^\sigma_{\xi^{t-2}(j)}\cdots A_j^{\sigma^{t-1}})^{-1}$ $\forall$ $0\leq t\leq l$, where l is the length of the orbit. Hence clearly we have that for each $e\in C$, e stabilizes the decomposition $V=U_1\oplus\cdots\oplus U_\beta$, and $C=C_1\times\cdots\times C_\beta$, where each C_ν act as identity on all U_μ with $\mu\neq\nu$. It remains to determine C_ν. Let C_ν, U_ν be corresponding to f_μ. Suppose f_μ is self-inverse-conjugate first with λ_j a root of f_μ, and thus c_μ is odd. The orbit of ξ on j has length c_μ. E_j needs to satisfy the condition that $E_j=E_{\xi^{c_\mu}(j)}=(A_{\xi^{c_\mu-1}(j)}A^\sigma_{\xi^{c_\mu-2}(j)}\cdots A_j^{\sigma^{c_\mu-1}})E_j^{\sigma^{c_\mu}}(A_{\xi^{c_\mu-1}(j)}A^\sigma_{\xi^{c_\mu-2}(j)}\cdots A_j^{\sigma^{c_\mu-1}})^{-1}$ in which $N_j=A_{\xi^{c_\mu-1}(j)}A^\sigma_{\xi^{c_\mu-2}(j)}\cdots A_j^{\sigma^{c_\mu-1}}=(BB^{-\sigma^{c_\mu}})_{jj}$; and N_j is a block in $A^{(\nu)}A^{(\nu)\sigma}\cdots A^{(\nu)\sigma^{c_\mu-1}}=B_\nu B_\nu^{-\sigma^{c_\mu}}=B_\nu B_\nu^{T_{q_1}^{c_\mu}}$ which is a diagonal block matrix. But $B_\nu\in(GF(q^{c_\mu}))^{n_\mu\times n_\mu}$, so $(B_\nu B_\nu^{T_{q_1}^{c_\mu}})^{T_{q_1}^{c_\mu}}=B_\nu^{T^2 q_1^{2c_\mu}}B_\nu^{T_{q_1}^{c_\mu}}=B_\nu^{q^{c_\mu}}B_\nu^{T_{q_1}^{c_\mu}}=B_\nu B_\nu^{T_{q_1}^{c_\mu}}$, which implies that $N_j^{T_{q_1}^{c_\mu}}=N_j$. Thus $E_j=N_jE_j^{\sigma^{c_\mu}}N_j^{-1}$ iff $E_j\in GU_{\frac{n_\mu}{c_\mu}}(q_1^{c_\mu})$. So $C_\nu\simeq GU_{\frac{n_\mu}{c_\mu}}(q_1^{c_\mu})$ in this case. Now suppose that f_μ is not self-inverse-conjugate. Then the length of the orbit of ξ on j is $2c_\mu$. E_j needs to satisfy the condition that $E_j=E_{\xi^{2c_\mu}(j)}=N_jE_j^{\sigma^{2c_\mu}}N_j^{-1}$ in which $N_j=A_{\xi^{2c_\mu-1}(j)}A^\sigma_{\xi^{2c_\mu-2}(j)}\cdots A_j^{\sigma^{2c_\mu-1}}=$

$(BB^{-\sigma^{2c_\mu}})_{jj}$. But $B_\nu \in (GF(q^{c_\mu}))^{2n_\mu \times 2n_\mu}$ implies that $B_\nu \bar{B}_\nu^{-\sigma^{2c_\mu}} = B_\nu \bar{B}_\nu^{-1} = I$. Hence $N_j = I$, which implies the condition $E_j = E_j^{\sigma^{2c_\mu}}$. So in this case $C_\nu \simeq GL_{\frac{n_\mu}{c_\mu}}(q^{c_\mu})$.

(12.11) Let $\mathbb{F} = GF(q)$, $q = q_1^2$, V an n-dimensional vector space over $\mathbb{F}$ with $n \geq 3$, f a unitary form on V, $g \in GU(V,f)$ with $(|g|, p) = 1$.

(a) Let $H = GU(V)$, $G = GL(V)$. Then $\mathcal{N}(g, G/H) \leq \frac{1}{(q_1-1)^{2(n-1)}}$. If $q=4$, then $\mathcal{N}(g, G/H) \leq \frac{5}{2^{n-1}}$. If $n=3$, then $\mathcal{N}(g, G/H) \leq \frac{1}{q_1^2(q_1^2+q_1+1)}$.

(b) Suppose $ZSL(V) \leq G \leq GL(V)$, $H = G \cap ZGU(V)$, where $Z = Z(GL(V))$. Then $\mathcal{N}(g, G/H) \leq \frac{(n, q_1^2-1)}{(q_1-1)^{2n-3}}$. If $q=4$, then $\mathcal{N}(g, G/H) \leq \frac{15}{2^{n-1}}$. If $n=3$, then $\mathcal{N}(g, G/H) \leq \frac{3\cdot(3, q_1^2-1)}{q_1^2(q_1^2+q_1+1)}$.

Proof. The reason that we assume $n \geq 3$ is that $GU_2(q_1) = SL_2(q_1)$, which has been considered already. Let f_μ, c_μ, n_μ, α, β be defined in the same way as in (12.10), and $d_\mu = \frac{n_\mu}{c_\mu}$. From (12.5) and (12.10), we have that $C_0 = C_H(g) = D_1 \times \cdots \times D_\beta$, $C = C_G(g) = C_1 \times \cdots \times C_\beta$, where for each $1 \leq \mu \leq \beta$, either (1) $D_\mu \simeq GU_{d_\mu}(q_1^{c_\mu})$ and $C_\mu \simeq GL_{d_\mu}(q^{c_\mu})$; or (2) $D_\mu \simeq GL_{d_\mu}(q^{c_\mu})$ and $C_\mu \simeq GL_{d_\mu}(q^{c_\mu}) \times GL_{d_\mu}(q^{c_\mu})$, which is the case corresponding to f_μ whose conjugate $\bar{f}_\mu \neq f_\mu$. We have that $(q_1-1)^k q_1^{k(k-1)} \leq |GU_k(q_1)| \leq (q_1+1)^{\frac{k+1}{2}} q_1^{k^2}$. So in case (1), $\frac{|C_\mu|}{|D_\mu|} = \frac{|GL_{d_\mu}(q^{c_\mu})|}{|GU_{d_\mu}(q_1^{c_\mu})|} \leq \frac{q^{c_\mu d_\mu^2}}{(q_1^{c_\mu}-1)^{d_\mu} q_1^{c_\mu d_\mu (d_\mu - 1)}} \leq \left(\frac{q_1}{q_1-1}\right)^{c_\mu d_\mu} q_1^{n_\mu d_\mu}$; and in case (2), $\frac{|C_\mu|}{|D_\mu|} = |GL_{d_\mu}(q^{c_\mu})| \leq q^{c_\mu d_\mu^2} = q_1^{2n_\mu d_\mu}$. Hence $\frac{|C|}{|C_0|} = \prod_{\mu=1}^{\beta} \frac{|C_\mu|}{|D_\mu|} \leq \left(\frac{q_1}{q_1-1}\right)^{\sum_{\mu=1}^{\alpha} c_\mu d_\mu} q_1^{\sum_{\mu=1}^{\alpha} n_\mu d_\mu}$. We still have that $\sum_{\mu=1}^{\alpha} c_\mu d_\mu = n$; and exactly as in (12.6), since $n \geq 3$, we have $\sum_{\mu=1}^{\alpha} n_\mu d_\mu \leq (n-2)n$ unless g is a pseudo-reflection. Hence $\frac{|C|}{|C_0|} \leq \left(\frac{q_1}{q_1-1}\right)^n q_1^{n^2-2n}$ if g is not a pseudo-reflection. If g is a pseudo-reflection, then as $n \geq 3$, both f_1 and f_2 are self-inverse-conjugate. So $C_0 \simeq GU_{n-1}(q_1) \times GU_1(q_1)$, and thus

$\frac{|C|}{|C_0|}=q_1^{\frac{1}{2}(n-1)(n-2)}(q_1-1)^2(q_1^2+1)\cdots(q_1^{n-1}+(-1)^{n-1})$. Suppose $q=4$. Then in case (1), $\frac{|C_\mu|}{|D_\mu|}=\frac{|GL_{d_\mu}(q^{c_\mu})|}{|GU_{d_\mu}(q_1^{c_\mu})|}=2^{\frac{1}{2}c_\mu d_\mu(d_\mu-1)}(2^{c_\mu}-1)(2^{2c_\mu}+1)\cdots(2^{d_\mu c_\mu}+(-1)^{d_\mu})\leq 2^{d_\mu n_\mu}s_{d_\mu}\leq\left(\frac{5}{2}\right)2^{d_\mu n_\mu}$,

where $s_{d_\mu}=(1+\frac{1}{2^{c_\mu}})(1+\frac{1}{2^{2c_\mu}})\cdots(1+\frac{1}{2^{d_\mu c_\mu}})\leq\frac{5}{2}$ as before; and in case (2), we still have $\frac{|C_\mu|}{|D_\mu|}\leq 2^{2n_\mu d_\mu}$. So $\frac{|C|}{|C_0|}\leq\left(\frac{5}{2}\right)^\alpha 2^{\sum_{\mu=1}^{\alpha}n_\mu d_\mu}$. Suppose $n=3$. If $\alpha=1$, then $C\simeq GF(q^3)^{\#}$, $C_0\simeq GU_1(q_1^3)$; and thus $\frac{|C|}{|C_0|}=q_1^3-1$. If $\alpha=2$ and $c_1=2$, $c_2=1$, then $C\simeq GF(q^2)^{\#}\times GF(q)^{\#}$, $C_0\simeq GU_1(q_1^2)\times GU_1(q_1)$; and thus $\frac{|C|}{|C_0|}=(q_1^2-1)(q_1-1)$. If $\alpha=2$ and $c_1=c_2=1$, then $C\simeq GL_2(q)\times GF(q)^{\#}$, $C_0\simeq GU_2(q_1)\times GU_1(q_1)$; and thus $\frac{|C|}{|C_0|}=q_1(q_1-1)^2(q_1^2+1)$. If $\alpha=3$, then $C\simeq GF(q)^{\#}\times GF(q)^{\#}\times GF(q)^{\#}$, $C_0\simeq GU_1(q_1)\times GU_1(q_1)\times GU_1(q_1)$; and thus $\frac{|C|}{|C_0|}=(q_1-1)^3$. So for $n=3$, in any case, we have $\frac{|C|}{|C_0|}\leq q_1(q_1-1)^2(q_1^2+1)$. Now let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$ and $\overline{V}=\overline{\mathbb{F}}\otimes_{\mathbb{F}}V$, $\overline{G}=GL(\overline{V})$. For an orthonormal basis X of V with respect to f, define $\sigma:\overline{G}\to\overline{G}$ by $M_X(y^\sigma)=(M_X(y))^{-T_{q_1}}$. So $H=C_{\overline{G}}(\sigma)$. Since g is semisimple, $\overline{C}=C_{\overline{G}}(g)$ is connected. Thus by (12.3)(b), $g^H=g^{\overline{G}}\cap H\supseteq g^G\cap H\supseteq g^H$, so $g^G\cap H=g^H$. Thus $\mathcal{N}(g,G/H)=\frac{|g^H|}{|g^G|}=\frac{|C||H|}{|G||C_0|}$. Since $\frac{|H|}{|G|}=1\Big/\left\{q_1^{\frac{1}{2}n(n-1)}(q_1-1)(q_1^2+1)\cdots(q_1^n+(-1)^n)\right\}\leq\frac{1}{(q_1-1)^n q_1^{n(n-1)}}$, we have that $\mathcal{N}(g,G/H)\leq\frac{1}{(q_1-1)^{2n}}$ if g is not a pseudo-reflection, and $\mathcal{N}(g,G/H)=\frac{q_1-1}{q_1^{n-1}(q_1^n+(-1)^n)}\leq\frac{1}{q_1^{2(n-1)}}$ if g is a pseudo-reflection. In both cases, we have $\mathcal{N}(g,G/H)\leq\frac{1}{(q_1-1)^{2(n-1)}}$. For $q=4$, denote $\xi=\sum_{\mu=1}^{\alpha}d_\mu n_\mu$. If $\alpha=1$, then $c_1\geq 2$, $n_1=n\geq 4$, $d_1\leq\frac{n}{2}$, and thus $d_1n_1\leq\frac{n^2}{2}$; which gives $\mathcal{N}(g,G/H)\leq\left(\frac{5}{2}\right)\frac{2^{\frac{n^2}{2}}}{2^{n(n-1)}}=\left(\frac{5}{2}\right)\frac{1}{2^{n(\frac{n}{2}-1)}}\leq\frac{5}{2^{n+1}}\leq\frac{5}{2^{n-1}}$. So suppose $\alpha\geq 2$. We have $d_\mu\leq n-\alpha+1$ $\forall$ μ. If $d_\mu\leq n-\alpha$ $\forall$ μ, then $\xi\leq(n-\alpha)\sum_{\mu=1}^{\alpha}n_\mu=(n-\alpha)n$, which gives $\mathcal{N}(g,G/H)\leq\left(\frac{5}{2}\right)^\alpha\frac{1}{2^{n(\alpha-1)}}\leq\left(\frac{5}{2}\right)\left(\frac{5}{2^{n+1}}\right)^{\alpha-1}\leq\frac{25}{2^{n+2}}\leq\frac{5}{2^{n-1}}$. If some μ, say $\mu=1$, is such that $d_1=n-\alpha+1$, then $c_\mu=1$ $\forall$ μ and $d_\mu=1$ $\forall$ $\mu\geq 2$; and also in this case g is a pseudo-reflection iff $\alpha=2$. Then $\xi=(n-\alpha+1)^2+(\alpha-1)$ and $n(n-1)-\xi=(n-2)\alpha+(n-\alpha)(\alpha-3)\geq$

$(n-2)\alpha$ if $\alpha\geq 3$. So in this case, if g is not a pseudo-reflection, then $\mathcal{N}(g,G/H)\leq\left(\frac{5}{2}\right)^{\alpha}\frac{1}{2^{(n-2)\alpha}}\leq\left(\frac{5}{2^{n-1}}\right)^{\alpha}\leq\frac{5}{2^{n-1}}$ if $n\geq 4$; and if $n=3$, then the bound $\mathcal{N}(g,G/H)\leq\frac{5}{2^{n-1}}$ clearly holds but is useless. If g is a pseudo-reflection, then as above $\mathcal{N}(g,G/H)\leq\frac{1}{2^{2(n-1)}}\leq\frac{5}{2^{n-1}}$. So in any case, we have $\mathcal{N}(g,G/H)\leq\frac{5}{2^{n-1}}$ for $q=4$. For $n=3$, as $\frac{|H|}{|G|}=\frac{1}{q_1^3(q_1-1)(q_1^2+1)(q_1^3-1)}$, we have $\mathcal{N}(g,G/H)\leq\frac{1}{q_1^2(q_1^2+q_1+1)}$. For part (b), let $G_1=GL(V)$, $H_1=GU(V)$. Now use (12.4), the μ there is such that $\mu=\frac{|ZH_1|}{|H_1|}=q_1-1$; and we have $f(g,G_1/ZH_1)=\frac{1}{\mu}\{f(g,G_1/H_1)+f(gz,G_1/H_1)+\cdots+f(gz^{\mu-1},G_1/H_1)\}$. Also $|G_1/ZH_1|=\frac{1}{\mu}|G_1/H_1|$, and as each gz^i, $0\leq i\leq\mu-1$, is semisimple, thus by part (a), $\mathcal{N}(g,G_1/ZH_1)=\sum_{i=0}^{\mu-1}\mathcal{N}(gz^i,G_1/H_1)\leq\frac{1}{(q_1-1)^{2n-3}}$. Then (3.5) gives the first bound in part (b). For $q=4$, as $\mu=1$ and $(n,q_1^2-1)\leq 3$, we have $\mathcal{N}(g,G/H)\leq\frac{15}{2^{n-1}}$. For $n=3$, similarly as in (12.6), $g^{G_1}\cap ZH_1$ is contained in at most three cosets of H_1, thus we have the extra factor $3(3,q_1^2-1)$ multiplied to the bound in (a) when $n=3$.

(12.12) Let $\mathbb{F}=GF(q)$, $q=q_1^2$, V an n-dimensional vector space over $\mathbb{F}$ with $n\geq 3$, f a unitary form on V, $g\in GU(V,f)$ with $|g|=p^e$, where $p=char(\mathbb{F})$.

(a) Let $H=GU(V)$, $G=GL(V)$. Then $\mathcal{N}(g,G/H)\leq\left(\frac{5}{2}\right)\cdot\frac{1}{q_1^{2(n-2)}}$. If $n\neq 4$, then $\mathcal{N}(g,G/H)\leq\left(\frac{5}{2}\right)\cdot\frac{1}{q_1^{2n-3}}$. If $n=3$, then $\mathcal{N}(g,G/H)\leq\frac{(q_1+1)^2}{(q_1^2+1)(q_1^3-1)}$.

(b) Suppose $ZSL(V)\leq G\leq GL(V)$, $H=G\cap ZGU(V)$, where $Z=Z(GL(V))$. Then $\mathcal{N}(g,G/H)\leq\left(\frac{5}{2}\right)\cdot\frac{(n,q_1^2-1)}{q_1^{2(n-2)}}$. If $n\neq 4$, then $\mathcal{N}(g,G/H)\leq\left(\frac{5}{2}\right)\cdot\frac{(n,q_1^2-1)}{q_1^{2n-3}}$. If $n=3$, then $\mathcal{N}(g,G/H)\leq\frac{(3,q_1^2-1)(q_1+1)^2}{(q_1^2+1)(q_1^3-1)}$.

Proof. For part (a), let $W=C_V(g)\cap[V,g]$, $U=C_V(g)+[V,g]$. Since $g\in GU(V)$, $C_V(g)=[V,g]^{\perp}$. Thus W is totally singular, and $U\leq W^{\perp}$. But $dim(U)=dim(C_V(g))+dim([V,g])-dim(W)=$

$n-dim(W)=dim(W^{\perp})$, thus $U=W^{\perp}$. Let W' be a complement to W in $W^{\perp}$. So W' is a non-degenerate subspace. Let $R=\{r_i:1\leq i\leq m\}$ be a basis of W. Then there exists $S=\{s_i:1\leq i\leq m\}\subseteq V$ such that for each i, $\{r_i,s_i\}$ is a hyperbolic pair of the hyperbolic plane $W_i=\langle r_i,s_i\rangle$, and $(W')^{\perp}$ is the orthogonal direct sum of W_i's. Let T be a basis for W', and thus $X=R\cup T\cup S$ is a basis of V. With respect to X, we have that $J=J(X,f)=\begin{bmatrix} 0 & 0 & I_m \\ 0 & \alpha & 0 \\ I_m & 0 & 0 \end{bmatrix}$, and thus $\alpha^{T\theta}=\alpha$, where θ: $GF(q)\to GF(q)$ by $x\mapsto x^{q_1}$. Also $m\geq 1$, as $|g|=p^e$, and $g\neq 1$. Let $C=C_G(g)$. Then C acts on W and $W^{\perp}$, as C acts both on $C_V(g)$ and $[V,g]$. Hence C is contained in the parabolic subgroup P of G stabilizing the flag $0<W\leq W^{\perp}<V$. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$, $\overline{V}=\overline{\mathbb{F}}\otimes_{\mathbb{F}}V$, $\overline{G}=GL(\overline{V})$, and $\overline{P}$ the parabolic in $\overline{G}$ stabilizing the flag $0<\overline{W}\leq\overline{W^{\perp}}<\overline{V}$. $\overline{P}$ is a connected algebraic group. We have that $y\in\overline{P}$ iff $M_X(y)=\begin{bmatrix} A & 0 & 0 \\ B & C & 0 \\ D & E & F \end{bmatrix}$.

Define $\sigma:\overline{G}\to\overline{G}$ by $M_X(y^{\sigma})=J(M_X(y))^{-T\theta}J^{-1}$. So $H=C_{\overline{G}}(\sigma)$, and σ acts on G. Let $y\in\overline{P}$. Then

$$M_X(y^{-\sigma})=\begin{bmatrix} 0 & 0 & I_m \\ 0 & \alpha & 0 \\ I_m & 0 & 0 \end{bmatrix}\begin{bmatrix} A & 0 & 0 \\ B & C & 0 \\ D & E & F \end{bmatrix}^{T\theta}\begin{bmatrix} 0 & 0 & I_m \\ 0 & \alpha^{-1} & 0 \\ I_m & 0 & 0 \end{bmatrix}=$$

$$\begin{bmatrix} F^{T\theta} & 0 & 0 \\ \alpha E^{T\theta} & \alpha C^{T\theta}\alpha^{-1} & 0 \\ D^{T\theta} & B^{T\theta}\alpha^{-1} & A^{T\theta} \end{bmatrix}$$

, which implies that $y^{-\sigma}\in\overline{P}$. So $y^{\sigma}\in\overline{P}$, and $\overline{P}^{\sigma}=\overline{P}$. Also we can see that σ acts on the Levi factor L of P corresponding to B, D, E being zero matrices; and on the unipotent radical U of P corresponding to A, C, F being identity matices; which implies that $P_1=C_P(\sigma)=C_L(\sigma)C_U(\sigma)$. The representation of G on G/H is equivalent to the representation of G on σ^G. We have $\sigma^{xg}=\sigma^x$ iff $\sigma^x\in\sigma^G\cap\sigma C_G(g)\subseteq\sigma^G\cap\sigma P$. Since σ^2 acts as the identity on G, we have $(\sigma^x)^2=(\sigma^2)^x=\sigma^2$; and as $\overline{P}^{\sigma}=\overline{P}$, by (12.3)(c), $P=C_{\overline{P}}(\sigma^2)$ is transitive on $\sigma^G\cap\sigma P$. Thus we conclude that $\sigma^G\cap\sigma P=\sigma^P$, and $\mathcal{N}(g,G/H)=\frac{|\sigma^G\cap\sigma C|}{|\sigma^G|}\leq\frac{|\sigma^G\cap\sigma P|}{|\sigma^G|}=\frac{|\sigma^P|}{|\sigma^G|}=\frac{|H||P|}{|G||P_1|}$. We have that $y\in C_P(\sigma)$ iff $MJM^{T\theta}=J$, where $M=M_X(y)=\begin{bmatrix} A & 0 & 0 \\ B & C & 0 \\ D & E & F \end{bmatrix}$. But $MJM^{T\theta}=$

$$\begin{bmatrix} 0 & 0 & AF^{T\theta} \\ 0 & C\alpha C^{T\theta} & C\alpha E^{T\theta}+BF^{T\theta} \\ FA^{T\theta} & FB^{T\theta}+E\alpha C^{T\theta} & FD^{T\theta}+E\alpha E^{T\theta}+DF^{T\theta} \end{bmatrix}$$, hence $y\in C_L(\sigma)$ iff B, D, E all zero matrices, $F=A^{-T\theta}$ and $C\alpha C^{T\theta}=\alpha$; thus $C_L(\sigma)\simeq GL_m(q)\times GU_{n-2m}(q_1)$. Also $y\in C_U(\sigma)$ iff A, C, F all identity matrices, $\alpha E^{T\theta}+B=0$, and $D^{T\theta}+E\alpha E^{T\theta}+D=0$. For any given $E\in(GF(q))^{m\times(n-2m)}$, B is determined by $B=-\alpha E^{T\theta}$; and $E_\alpha=-E\alpha E^{T\theta}=(e_{ij})$ is then fixed too. We have that $E_\alpha^{T\theta}=E_\alpha$, i.e., $e_{ij}^\theta=e_{ji}$ $\forall$ i,j. Let $\rho:GF(q)\to GF(q_1)$ by $x\mapsto x+x^\theta$. Thus ρ is a surjective $GF(q_1)$-linear transformaton. So $|ker(\rho)|=q_1$. Denote $D=(d_{ij})$. The condition $D+D^{T\theta}=E_\alpha$ is equivalent to $d_{ij}+d_{ji}^\theta=e_{ij}$ $\forall$ i, j. Thus as each $e_{ii}\in GF(q_1)$, there exist q_1 choices for d_{ii}. For $i<j$, we select $d_{ij}\in GF(q)$ arbitrarily and let $d_{ji}=(e_{ij}-d_{ij})^\theta$, then $d_{ij}+d_{ji}^\theta=e_{ij}$ is automatically satisfied $\forall$ $i\neq j$. Therefore, for each fixed choice of $E\in(GF(q))^{m\times(n-2m)}$, there are at least $q_1^m q^{\frac{1}{2}m(m-1)}=q_1^{m^2}$ chioces of D such that $D^{T\theta}+E\alpha E^{T\theta}+D=0$ is satisfied. So $|C_U(\sigma)|\geq q^{m(n-2m)}q_1^{m^2}=q_1^{(2n-3m)m}$. Also $P=LU$, where $L\simeq GL_m(q)\times GL_{n-2m}(q)\times GL_m(q)$ and U is a semidirect of N by A, where $N\simeq E_{q^{(n-m)m}}$ is normal in U and $A\simeq E_{q^{(n-2m)m}}$. Thus $\frac{|P|}{|P_1|}\leq|GL_m(q)|\cdot\frac{|GL_{n-2m}(q)|}{|GU_{n-2m}(q_1)|}\cdot q_1^{(2n-3m)m}=$ $q_1^{\frac{1}{2}n(n-1)}\{(q-1)(q^2-1)\cdots(q^m-1)\}\{(q_1-1)(q_1^2+1)\cdots(q_1^{n-2m}+(-1)^{n-2m})\}$. Since $\frac{|H|}{|G|}=1\Big/\Big\{q_1^{\frac{1}{2}n(n-1)}(q_1-1)(q_1^2+1)\cdots(q_1^n+(-1)^n)\Big\}$, we have that

$$\frac{|H||P|}{|G||P_1|}\leq\frac{\{(q_1-1)(q_1^2-1)\cdots(q_1^m-1)\}\{(q_1+1)(q_1^2+1)\cdots(q_1^m+1)\}}{(q_1^{n-2m+1}+(-1)^{n-2m+1})\cdots(q_1^n+(-1)^n)}=a.$$

Let $b=(q_1+1)(q_1^2+1)\cdots(q_1^m+1)$, $c=\frac{(q_1-1)(q_1^2-1)\cdots(q_1^m-1)}{(q_1^{2(k-m)+1}-1)(q_1^{2(k-m)+3}-1)\cdots(q_1^{2k-1}-1)}$, $d=(q_1^{2(k-m)}+1)(q_1^{2(k-m)+2}+1)\cdots(q_1^{2k-2}+1)$, $e=(q_1^{2(k-m)+2}+1)(q_1^{2(k-m)+4}+1)\cdots(q_1^{2k}+1)$. Then $a=\frac{cb}{d}$ if $n=2k-1$, and $a=\frac{cb}{e}$ if $n=2k$. Since $\frac{q_1^i-1}{q_1^j-1}\leq\frac{1}{q_1^{j-i}}$ for $j\geq i$, we have $c\leq\frac{1}{q_1^{2m(k-m)+\frac{1}{2}m(m-1)}}$. Similar as before, we have $b\leq\left(\frac{5}{2}\right)q_1^{\frac{1}{2}m(m+1)}$. Also clearly, $d\geq q_1^{2m(k-m)+m(m-1)}$ and

$e \geq q_1^{2m(k-m)+m(m+1)}$. So $\frac{cb}{d} \leq \left(\frac{5}{2}\right)\frac{1}{q_1^{m(4k-3m-2)}} = \left(\frac{5}{2}\right)\frac{1}{q_1^{m(2n-3m)}}$ as $n=2k-1$, and $\frac{cb}{e} \leq \left(\frac{5}{2}\right)\frac{1}{q_1^{m(4k-3m)}} = \left(\frac{5}{2}\right)\frac{1}{q_1^{m(2n-3m)}}$ as $n=2k$. Since $1 \leq m \leq \left[\frac{n}{2}\right]$, it is easy to check that $m(2n-3m) \geq 2n-3$ except when $n=4$ and $m=2$, in which case $m(2n-3m)=2n-4$. Hence we have $\mathcal{N}(g,G/H) \leq \left(\frac{5}{2}\right)\frac{1}{q_1^{2(n-2)}}$; and if $n \neq 4$, then $\mathcal{N}(g,G/H) \leq \left(\frac{5}{2}\right)\frac{1}{q_1^{2n-3}}$. If $n=3$, then $m=1$, and we have $|P_1| = q_1^3(q_1-1)^2(q_1+1)$, $|P| = q^3(q-1)^3 = q_1^6(q_1^2-1)^3$. Since $\frac{|H|}{|G|} = \frac{1}{q_1^3(q_1-1)(q_1^2+1)(q_1^3-1)}$, we have $\mathcal{N}(g,G/H) \leq \frac{(q_1+1)^2}{(q_1^2+1)(q_1^3-1)}$. For part (b), let $G_1 = GL(V)$, $H_1 = GU(V)$. Now use (12.4), the μ there is such that $\mu = \frac{|ZH_1|}{|H_1|} = \frac{q_1^2-1}{q_1+1} = q_1 - 1$; and we have $f(g,G_1/ZH_1) = \frac{1}{\mu} f(g,G_1/H_1)$, where $Z = \langle z^{-1} \rangle$. Also $|G_1/ZH_1| = \frac{1}{\mu}|G_1/H_1|$, thus by part (a), $\mathcal{N}(g,G_1/ZH_1) = \mathcal{N}(g,G_1/H_1) \leq \left(\frac{5}{2}\right)\frac{1}{q_1^{2(n-2)}}$. Then (3.5) gives the bound in part (b).

(12.13) Suppose $L_n(q) \leq \overline{G} \leq PGL_n(q)$ with $n \geq 3$ and $\overline{H} = \overline{G} \cap PGU_n(q_1)$, where $q = q_1^2$, and $\overline{G}$ is a group of genus zero. Then one of the following holds:

(a) $n=3$ and $q \leq 25$.

(b) $n=4$ and $q \leq 25$.

(c) $n=5$ and $q \leq 9$.

(d) $6 \leq n \leq 11$ and $q=4$.

Proof. Suppose $n=3$ first. For g semisimple, by (12.11), $\mathcal{N}(\overline{g},\overline{G}/\overline{H}) \leq \frac{9}{q_1^2(q_1^2+q_1+1)} \leq \frac{3(q_1+1)^2}{(q_1^2+1)(q_1^3-1)}$. For g unipotent, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H}) \leq \frac{3(q_1+1)^2}{(q_1^2+1)(q_1^3-1)} \leq \frac{1}{85}$ if $q_1 \geq 7$. If $q_1 = 6$, then as $(3, q_1^2-1) = 1$, we have for g either semisimple or unipotent that $\mathcal{N}(\overline{g},\overline{G}/\overline{H}) \leq \frac{(q_1+1)^2}{(q_1^2+1)(q_1^3-1)} = \frac{49}{7955} \leq \frac{1}{85}$. Hence if $n=3$, then $q \leq 25$. Suppose $q=4$ and $n \geq 12$. For semisimple g, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H}) \leq \frac{15}{2^{n-1}}$; and for unipotent g, $\mathcal{N}(\overline{g},\overline{G}/\overline{H}) \leq \left(\frac{5}{2}\right)\cdot\frac{3}{2^{2n-3}} \leq \frac{15}{2^{n-1}} \leq \frac{1}{85}$. Hence if $q=4$, then $n \leq 11$. Now consider $n \geq 4$ and $q \geq 9$. For semisimple g, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H}) \leq \frac{n}{(q_1-1)^{2n-3}} = a_n$. Since $\frac{a_{n+1}}{a_n} = (1+\frac{1}{n})\cdot\frac{1}{(q_1-1)^2} \leq 1$, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq a_4=\frac{4}{(q_1-1)^5}\leq\frac{1}{85}$ if $q_1\geq 5$. For unipotent g, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq\binom{5}{2}\cdot\frac{n}{q_1^{2(n-2)}}=b_n$. Since $\frac{b_{n+1}}{b_n}=(1+\frac{1}{n})\cdot\frac{1}{q_1^2}\leq 1$, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq b_4=\binom{5}{2}\cdot\frac{4}{q_1^4}\leq\frac{1}{85}$ if $q_1\geq 6$. So if $n\geq 4$, then $q\leq 25$. Thus it remains to consider the cases in which $q=9$, 16, 25. For $q=9$ and $n\geq 6$, we have $a_6=\frac{6}{2^9}\leq\frac{1}{85}$ and $b_6=\binom{5}{2}\cdot\frac{6}{3^8}\leq\frac{1}{85}$. Hence if $q=9$, then $n\leq 5$. For $q=16$ or 25, if $n\geq 5$, then we have $a_5\leq\frac{5}{3^7}\leq\frac{1}{85}$ and $b_5\leq\binom{5}{2}\cdot\frac{5}{4^6}\leq\frac{1}{85}$. Hence if $q=16$ or 25, then $n\leq 4$. In summary, we have the entire conclusion.

(12.14) Let V an n-dimensional vector space over $\mathbb{F}=GF(q)$, $p=char(\mathbb{F})$, $n=2l$ with $l\geq 2$, and f_0 a symplectic form on V, $g\in\Delta(V,f_0)$.

(a) Let $H=\Delta(V,f_0)$, $G=GL(V)$. If $(|g|,p)=1$, then $\mathcal{N}(g,G/H)\leq\frac{1}{(q-1)^{l-1}q^{l-1}}$ for $l\geq 3$; and $\mathcal{N}(g,G/H)\leq\frac{(q+1)(q^2+2)}{q^2(q^3-1)}$ for $l=2$. If $|g|=p^e$, then $\mathcal{N}(g,G/H)\leq\frac{1}{(q-1)^l q^{l-2}}$.

(b) Suppose $ZSL(V)\leq G\leq GL(V)$, and $H=G\cap\Delta(V,f_0)$. If $(|g|,p)=1$, then $\mathcal{N}(g,G/H)\leq\frac{(2l,q-1)}{(q-1)^{l-1}q^{l-1}}$ for $l\geq 3$; and $\mathcal{N}(g,G/H)\leq\frac{(4,q-1)(q+1)(q^2+2)}{q^2(q^3-1)}$ for $l=2$. If $|g|=p^e$, then $\mathcal{N}(g,G/H)\leq\frac{(2l,q-1)}{(q-1)^l q^{l-2}}$.

Proof. The reason that we assume $l\geq 2$ is that $Sp_2(q)=SL_2(q)$. Let $A=A_r=\{f\in L(V,V;\mathbb{F})$: $f(u,v)=-f(v,u)$ and $f(v,v)=0\ \forall u,v\in V\}$. Thus $f_0\in A$ and A is an $\binom{n}{2}$-dimensional subspace of $L(V,V;\mathbb{F})$. Define $\pi:GL(V)\rightarrow GL(A)$ by $((g\pi)f)(u,v)=f(ug,vg)$ $\forall$ u, $v\in V$, and here A is considered as a left $\mathbb{F}G$-module while V is considered as a right $\mathbb{F}G$-module. That is $(((g_1\pi)(g_2\pi))f)(u,v)=((g_1\pi)((g_2\pi)f))(u,v)=((g_2\pi)f)(ug_1,vg_1)=f(ug_1g_2,vg_1g_2)=(((g_1g_2)\pi)f)(u,v)$, so π is a group homomorphism and G is represented on A. We have $H=\Delta(V,f_0)=Stab_{GL(V)}(\langle f_0\rangle)$ and thus the representation of G on G/H is equivalent to the representation of G on ${}^G\langle f_0\rangle$, where ${}^G\langle f_0\rangle$ is the orbit of the 1-dimensional subspace $\langle f_0\rangle$ under G. Suppose $\langle f\rangle\in{}^G\langle f_0\rangle$ with $g\pi\langle f\rangle=\langle f\rangle$. Then $g\pi f=\omega f$ for some $\omega\in\mathbb{F}^{\#}$, which implies that

$\langle f\rangle \leq E(g\pi,\omega,A)$, the eigenspace of $g\pi$ in A with respect to the eigenvalue ω. So $f(g,{}^{G}\langle f_0\rangle)$, the number of fixed points by $g\pi$ in ${}^{G}\langle f_0\rangle$, is such that $f(g,{}^{G}\langle f_0\rangle)\leq \sum_{\omega\in\mathbb{F}^{\#}} n(\omega)$, where $n(\omega)$ is the number of 1-dimensional subspaces in $E(g\pi,\omega,A)$. Denote $d(\omega)=dim_{\mathbb{F}}(E(g\pi,\omega,A))$, then $n(\omega)=\frac{q^{d(\omega)}-1}{q-1}$. Now consider the case that $(|g|,p)=1$ first. Let K be the splitting field of $m=min(g,\mathbb{F},V)$, $\Lambda(g)=\{\lambda_1,\ldots,\lambda_\alpha\}$ be the set of roots of m in K, and m_μ be the multiplicity of λ_μ. Let $A^K=K\otimes_{\mathbb{F}}A$. Then A^K is isomorphic naturally to $A_r(V^K)$ as K-spaces. Since $\omega\in\mathbb{F}^{\#}$, $dim_{\mathbb{F}}(E(g\pi,\omega,A))=dim_K(E(1\otimes g\pi,\omega,A^K))$. Let $X=\{x_i:1\leq i\leq n\}$ be an eigenvector basis of g in V^K. Denote $\lambda(x_i)$ the eigenvalue of g corresponding to x_i. For $f\in A^K$, $(g\pi)f=\omega f$ iff $f(x_ig,x_jg)=\omega f(x_i,x_j)$ $\forall i<j$, iff $(\lambda(x_i)\lambda(x_j)-\omega)f(x_i,x_j)=0$ $\forall i<j$, iff $f(x_i,x_j)=0$ $\forall i<j$ with $\lambda(x_i)\lambda(x_j)\neq\omega$. Thus $d(\omega)$ is equal to the number of pairs (i,j) with $i<j$ and $\lambda(x_i)\lambda(x_j)=\omega$. Given $\lambda_\mu\in\Lambda(g)$, denote $\tilde{m}_\mu$ the multiplicity of $\tilde{\lambda}_\mu\in\Lambda(g)$ where $\tilde{\lambda}_\mu$ is such that $\lambda_\mu\tilde{\lambda}_\mu=\omega$; if no such $\tilde{\lambda}_\mu\in\Lambda(g)$ exists, let $\tilde{m}_\mu=0$. Thus $d(\omega)=\frac{1}{2}\left\{\sum_{\lambda_\mu\in\Lambda_1} m_\mu(m_\mu-1)+\sum_{\lambda\in\Lambda_2} m_\mu\tilde{m}_\mu\right\}$, where $\Lambda_1=\{\lambda\in\Lambda(g):\lambda^2=\omega\}$ and $\Lambda_2=\{\lambda\in\Lambda(g)\colon \exists\ \lambda'\in\Lambda(g)$ such that $\lambda\lambda'=\omega$ and $\lambda\neq\lambda'\}$. There is a group homomorphism $\tau:\Delta(V,f_0)\rightarrow\mathbb{F}^{\#}$ defined by $x\mapsto\tau(x)$, where $\tau(x)$ is determined by $f_0(ux,vx)=\tau(x)f_0(u,v)$ for $x\in\Delta(V,f_0)$; and $ker(\tau)=Sp(V)$. This gives $|H|\leq|Sp(V)|(q-1)$. So $\mathcal{N}(g,G/H)\leq\frac{|H|}{|G|}\sum_{\omega\in\mathbb{F}^{\#}}\frac{q^{d(\omega)}-1}{q-1}\leq\frac{|Sp(V)|}{|GL(V)|}\sum_{\omega\in\mathbb{F}^{\#}}(q^{d(\omega)}-1)\leq\frac{|Sp(V)|}{|GL(V)|}\sum_{\omega\in\mathbb{F}^{\#}}q^{d(\omega)}$. From the order formular for $Sp_{2l}(q)$, we have $\frac{|Sp_{2l}(q)|}{|GL_{2l}(q)|}=1\Big/\left\{q^{l(l-1)}(q-1)(q^3-1)\cdots(q^{2l-1}-1)\right\}\leq \frac{1}{(q-1)^l q^{2l(l-1)}}$. Now $d(\omega)\leq\frac{1}{2}\sum_{\lambda\in\Lambda}m_\mu\tilde{m}_\mu\leq\frac{1}{2}\sum_{\lambda\in\Lambda}m_\mu(2l-3)=2l^2-3l$ if all $m_\mu\leq 2l-3$. If some $m_\mu=2l-1$, then $d(\omega)\leq max\{\frac{1}{2}(2l-1)(2l-2),\ 2l-1\}=2l^2-3l+1$. Suppose some m_μ, say $m_1=2l-2$. Then either $m_2=1=m_3$, or $m_2=2$. We have similarly that $d(\omega)\leq\frac{1}{2}(2l-2)(2l-3)+1\leq 2l^2-3l+1$ unless $m_2=2$ and $l=2$. So for $l\geq 3$, we always have $d(\omega)\leq 2l^2-3l+1$, which gives $\mathcal{N}(g,G/H)\leq\frac{1}{(q-1)^{l-1}q^{l-1}}$. Now suppose $l=2$. Denote

$\chi=\sum_{\omega\in\mathbf{F}^{\#}}(q^{d(\omega)}-1)$. By writing g in diagonal form in V^K, it is easy to see that $d(\omega)\leq 4$ for any $\omega\in\mathbf{F}^{\#}$. Also it is clear that if some $d(\omega)=4$, then $\chi\leq(q^2-1)+(q^4-1)$; and if some $d(\omega)=3$, then $\chi\leq 2(q^3-1)$. If every $d(\omega)\leq 2$, then $\chi\leq t(q^2-1)+(6-2t)(q-1)=t(q^2-1)+(3-t)\{2(q-1)\}\leq t(q^2-1)+(3-t)(q^2-1)=3(q^2-1)$, where t is the number of ω's with $d(\omega)=2$. So in any case, we have $\chi\leq(q^2-1)+(q^4-1)$. Since $\frac{|Sp_4(q)|}{|GL_4(q)|}=\frac{1}{q^2(q-1)(q^3-1)}$, we have $\mathcal{N}(g,G/H)\leq\frac{(q^2-1)+(q^4-1)}{q^2(q-1)(q^3-1)}=\frac{(q+1)(q^2+2)}{q^2(q^3-1)}$. Now consider the case that $|g|=p^e$. Let $X=\{x_1,\ldots,x_n\}$ be the Jordan cannonical basis of g in V, and denote d_μ the dimension of μ-th Jordan block of g, $1\leq\mu\leq\alpha$, where α is the number of Jordan blocks of g. So either $x_ig=x_i+x_{i+1}$ or $x_ig=x_i$. Denote $f(x_i,x_j)=\alpha_{ij}$. The condition $f(x_ig,x_jg)=\omega f(x_i,x_j)$ is equivalent to one of the following: (1) $(\omega-1)\alpha_{ij}=\alpha_{i,j+1}+\alpha_{i+1,j}+\alpha_{i+1,j+1}$; (2) $(\omega-1)\alpha_{ij}=\alpha_{i,j+1}$; (3) $(\omega-1)\alpha_{ij}=\alpha_{i+1,j}$; (4) $(\omega-1)\alpha_{ij}=0$. If $\omega\neq 1$,then these four conditions force $\alpha_{ij}=0$ $\forall i,j$. Thus $d(\omega)=0$ $\forall\omega\neq 1$. For $\omega=1$, these four conditions imply that $d(1)=\frac{1}{2}\sum_{\mu\neq\nu}min(d_\mu,d_\nu)+\sum_{\mu=1}^{\alpha}\left[\frac{d_\mu}{2}\right]$. We have that $d(1)=\frac{1}{2}\sum_{\mu\neq\nu}min(d_\mu,d_\nu)+\sum_{\mu=1}^{\alpha}\left[\frac{d_\mu}{2}\right]\leq\frac{1}{2}\sum_{\mu=1}^{\alpha}\sum_{\nu=1}^{\alpha}min(d_\mu,d_\nu)\leq\frac{1}{2}\sum_{\mu=1}^{\alpha}\sum_{\nu=1}^{\alpha}d_\nu=\frac{1}{2}\sum_{\mu=1}^{\alpha}2l=\alpha l\leq(2l-3)l$ if $\alpha\leq 2l-3$. If $\alpha=2l-2$, then either $d_1=\cdots=d_{\alpha-2}=1$, $d_{\alpha-1}=2=d_\alpha$ or $d_1=\cdots=d_{\alpha-1}=1$, $d_\alpha=3$. In the first case $d(1)=\frac{1}{2}\{(\alpha-2)^2-(\alpha-2)+4(\alpha-2)+4\}+2=2l^2-5l+6=(2l^2-3l+2)-2(l-2)\leq 2l^2-3l+2$ as $l\geq 2$. In the second case $d(1)=\frac{1}{2}\{(\alpha-1)^2-(\alpha-1)+2(\alpha-1)\}+1=2l^2-5l+4=(2l^2-3l)-2(l-2)\leq 2l^2-3l$. If $\alpha=2l-1$, then $d_1=\cdots=d_{\alpha-1}=1$, $d_\alpha=2$ and $d(1)=\frac{1}{2}\{(\alpha-1)^2-(\alpha-1)+2(\alpha-1)\}+1=2l^2-3l+2$. So in any case, we have that $d(1)\leq 2l^2-3l+2$. Thus $\mathcal{N}(g,G/H)\leq\frac{|H|}{|G|}\sum_{\omega\in\mathbf{F}^{\#}}\frac{q^{d(\omega)}-1}{q-1}\leq\frac{|Sp(V)|}{|GL(V)|}q^{d(1)}\leq\frac{q^{2l^2-3l+2}}{(q-1)^lq^{2l(l-1)}}=\frac{1}{(q-1)^lq^{l-2}}$, i.e., the second part of (a) holds. Part (b) follows easily from (a).

(12.15) Suppose $L_n(q)\leq\overline{G}\leq PGL_n(q)$ with $n=2l\geq 4$ and $\overline{H}=\overline{G}\cap\overline{\Delta}$, where $\overline{\Delta}=\Delta(\mathrm{V},\mathrm{f}_0)/\mathrm{Z}$,

$Z=Z(GL_n(q))$, f_0 a symplectic form on a $2l$-dimensional vector space V over $GF(q)$, and $\overline{G}$ is a group of genus zero. Then one of the following holds:

(a) $n=4$ and $q\leq 17$.

(b) $n=6$ and $q\leq 5$.

(c) $n=8$ and $q\leq 3$.

(d) $10\leq n\leq 26$ and $q=2$.

Proof. Suppose $l=2$ first. For semisimple g, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq\frac{4(q+1)(q^2+2)}{q^2(q^3-1)}=4\left\{\frac{1}{q^3-1}\left(1+\frac{1}{q}+\frac{1}{q^2}\right)+\frac{1}{q^2(q-1)}+\frac{1}{q^2+q+1}\right\}$, which is decreasing when q is increasing, and thus it implies $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq\frac{1}{85}$ if $q\geq 19$. For unipotent g, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq\frac{4}{(q-1)^2}\leq\frac{1}{85}$ if $q\geq 19$. Hence if $n=4$, then $q\leq 17$. Now suppose $l\geq 3$. For semisimple g, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq\frac{2l}{(q-1)^{l-1}q^{l-1}}=a_l$. For unipotent g, we have $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq\frac{2l}{(q-1)^l q^{l-2}}=b_l=b_l(q)$. Since $\frac{a_l}{b_l}=\frac{q-1}{q}\leq 1$, and $\frac{b_{l+1}}{b_l}=(1+\frac{1}{l})\cdot\frac{1}{(q-1)q}\leq 1$, we have for both cases that $\mathcal{N}(\overline{g},\overline{G}/\overline{H})\leq b_3=\frac{6}{(q-1)^3q}\leq\frac{1}{85}$ if $q\geq 7$. So for $l\geq 3$, we must have $q=2$, 3, 4, or 5. Since $b_{14}(2)=\frac{7}{1024}$, $b_5(3)=\frac{5}{432}$, $b_4(4)=\frac{1}{162}$, $b_4(5)=\frac{1}{800}$, and each one of them is $\leq\frac{1}{85}$. Hence for $q=2$, 3, 4, 5, we have respectively that $l\leq 13$, 4, 3, 3. Therefore we have the conclusion.

Chapter IV
Irreducible Maximal Subgroups Containing No Transvection

Section 13. Initial Reduction.

Now assume that H is a maximal subgroup of G such that H contains no transvection and H is irreducible and primitive on V, where H and G are the preimages of $\overline{H}$ and $\overline{G}$ respectively. Note that since $SL_n(q) \leq G$, the unipotent radical denoted as A in (13.1) through (13.4) is contained in G. The result of this section is:

Proposition. *If $\overline{G}$ is a group of genus zero, then one of the following holds:*

(a) $n=2,3$ or 4, and $q \leq 83$.

(b) $n \geq 5$ and $q \leq 9$.

Proof. This follows from (2.5), (13.6) and (13.7) directly.

(13.1) (J. Thompson) (a) Suppose g fixes some hyperplane U. Let A be the group of all transvections with axis U. Then $\mathcal{N}(g) \leq \frac{|C_A(g)|}{|A|}$.

(b) Suppose $|g|=p^e$, where $p=char(\mathbb{F})$. Then $\mathcal{N}(g) \leq \frac{1}{q^{n-\nu}}$, if the smallest dimension of Jordan blocks of g is 1; and $\mathcal{N}(g) \leq \frac{1}{q^{n-\nu-1}}$, if all blocks have dimension at least 2; where ν is the number of Jordan blocks of g.

(c) If g is a transvection, then $\mathcal{N}(g)=0$.

Proof. This can be obtained as a special case from [Th]. For easy reference, we include a proof here. Let $O_1, O_2, \ldots, O_k$ be the A-orbits on Ω, and f_i be the number of fixed points of g in O_i. As $H \cap A^y = 1$ $\forall y \in G$, A is regular on each O_i. Also $U^g = U$ implies $A^g = A$. Suppose $f_i \geq 1$, i.e., there exists $x \in O_i$ with $xg=x$. Now $xag=xa$ if and only if $xg^{-1}aga^{-1}=x$ if and only if $g^{-1}aga^{-1}=1$. Thus either $f_i=0$ or $f_i=|C_A(g)|$. Then $\mathcal{N}(g)=\frac{f(g)}{|\Omega|}=\frac{f_1+f_2+\cdots+f_k}{|O_1|+|O_2|+\cdots+|O_k|} \leq$

$max\{\frac{f_i}{|O_i|}\}\leq\frac{|C_A(g)|}{|A|}$. So (a) holds. Let $\{v_1,\ldots,v_n\}$ be the basis of the Jordan canonical form of g. We can arrange $\{v_1,\ldots,v_n\}$ so that $\{v_1,\ldots,v_\mu\}$ is the basis for a Jordan block of g of the smallest dimension. So $v_1g=v_1+v_2$ if $\mu\geq 2$ and $v_1g=v_1$ if $\mu=1$. Let $U=\langle v_2,\ldots,v_n\rangle$ and A be the group of all transvections with axis U. Let $t\in C_A(g)$ with $v_1t=v_1+u$, where $u\in U$. Since $v_1g^{-1}=v_1+w$ for some $w\in U$, $v_1g^{-1}tg=(v_1+w)tg=(v_1+u+w)g= v_1+ug$. So $u\in C_U(g)$ and thus $|C_A(g)|\leq|C_U(g)|$. But $C_U(g)=C_V(g)$ if $\mu\geq 2$ and $C_U(g)\oplus\langle v_1\rangle= C_V(g)$ if $\mu=1$; and $dim_{\mathbb{F}}(C_V(g))$ is equal to ν, the number of Jordan blocks of g. So $|C_A(g)|\leq q^\nu$ if $\mu\geq 2$ and $|C_A(g)|\leq q^{\nu-1}$ if $\mu=1$. Then as $|A|=q^{n-1}$, (b) holds. (c) is evident.

(13.2) Suppose $V=U\oplus W$ with $U^g=U$, and $W^g=W$. Let $\mu=dim_{\mathbb{F}}(U)$, and $A=C_{GL(V)}(W)\cap C_{GL(V)}(V/W)$. Let O_1, $O_2,\ldots$, O_k be the A-orbits on Ω, and f_i be the number of points in O_i fixed by g. Then

(a) Either $f_i=0$ or $f_i=|C_{\tilde{A}_i}(g)|$, where $\tilde{A}_i=\frac{A}{A_{x_i}}$, and $x_i\in O_i$ with $x_ig=x_i$. Also $A_{x_i}=A\cap G_{x_i}= A\cap H^{y_i}$ for some $y_i\in G$.

(b) $\mathcal{N}(g)\leq \max_i\left\{\frac{|C_{\tilde{A}_i}(g)|}{|\tilde{A}_i|}\right\}$, where $\tilde{A}_i$ is as in (a) for each i.

(c) Let A_{x_i}, $\tilde{A}_i$ be as in (a). Suppose $E\leq A$ with $E\cap A_{x_i}=1$, and $C_{\tilde{E}}(g)=1$. Then $\frac{|C_{\tilde{A}_i}(g)|}{|\tilde{A}_i|}\leq\frac{1}{|E|}$.

Proof. Suppose $f_i\neq 0$. Let $x_i\in O_i$ with $x_ig=x_i$. Then $(A_{x_i})^g=A_{x_i}$, so g induces an automorphism on $\tilde{A}_i=A/A_{x_i}$. Since $x_iag=x_ia$ if and only if $x_ig^{-1}aga^{-1}=x_i$ if and only if $g^{-1}aga^{-1}\in A_{x_i}$ if and only if $A_{x_i}a\in C_{\tilde{A}_i}(g)$, we have $f_i=|C_{\tilde{A}_i}(g)|$. So (a) holds. As $\mathcal{N}(g)=\frac{f(g)}{|\Omega|}=\frac{f_1+f_2+\cdots+f_k}{|O_1|+|O_2|+\cdots+|O_k|}\leq \max_i\{\frac{f_i}{|O_i|}\}$, but $|O_i|=|\tilde{A}_i|$; so by (a), we have (b). As $E\cap A_{x_i}=1$, $|\tilde{E}|=|E|$. Since $C_{\tilde{E}}(g)=1$, $\tilde{E}\cap C_{\tilde{A}_i}(g)=1$. So $|\tilde{E}||C_{\tilde{A}_i}(g)|=|\tilde{E}C_{\tilde{A}_i}(g)|\leq|\tilde{A}_i|$. Hence $\frac{|C_{\tilde{A}_i}(g)|}{|\tilde{A}_i|}\leq\frac{1}{|\tilde{E}|}=\frac{1}{|E|}$.

(13.3) Suppose $(|g|,char(\mathbb{F}))=1$, and $min(g)=f_1f_2\cdots f_\alpha$ with $\alpha\geq 2$, where each f_ν, $1\leq\nu\leq\alpha$, is irreducible in $\mathbb{F}[x]$. Let $V=V_1\oplus V_2\oplus\cdots\oplus V_\alpha$, where each V_ν is the homogeneous component corresponding to f_ν. Let $A=C_{GL(V)}(W)\cap C_{GL(V)}(V/W)$, where $W=V_2\oplus\cdots\oplus V_\alpha$. Then $C_A(g)=1$.

Proof. $C_A(g)$ acts on V_1 and centralizes V/W and W; which implies that $C_A(g)\leq C_A(V)=1$.

(13.4) Suppose g satisfies the conditions in (13.3). Using the notations in (13.3), let μ be the smallest dimension of V_ν. Then $\mathcal{N}(g)\leq\frac{1}{q^{n-\mu}}$. In particular, if $min(g)$ spits in $\mathbb{F}[x]$. Then μ is the smallest dimension of all the eigenspaces of g.

Proof. Without loss of generality, assume $\mu=dim_{\mathbb{F}}(V_1)$. Let A and W be the same as in (13.3). $\phi:A\rightarrow A$ is an isomorphism by (13.3), where $\phi:x\mapsto[x,g]$. Let H be a hyperplane of V_1, $B=C_A(H)$, and $E=\phi^{-1}(B)$. So $|E|=|B|=q^{n-\mu}$, as B is a group of transvections with axis $W+H$ and centers in W. For $x\in E\cap Ax_i$ or $Ax_ix\in C_{\tilde{E}}(g)$, we have $[x,g]\in B\cap Ax_i=1$, as Ax_i contains no transvection. So (13.3) implies that $E\cap Ax_i=1$, and $C_{\tilde{E}}(g)=1$. Thus by (13.2)(b) and (c), $\mathcal{N}(g)\leq\frac{1}{q^{n-\mu}}$.

Remark. It remains to consider the case in which $(|g|,char(\mathbb{F}))=1$, and $min(g)$ is irreducible in $\mathbb{F}[x]$. Since we are interested in finding the bound for $\frac{f(\overline{g})}{|\Omega|}$, and as $f(\overline{g})\leq f((\overline{g})^i)$ for any integer i, it suffices to bound for $f(\overline{g})$ for $\overline{g}$ with $|\overline{g}|$ a prime. Suppose $\overline{g}=Zg$. Then $\langle\overline{g}\rangle=\frac{Z\langle g\rangle}{Z}\simeq\frac{\langle g\rangle}{\langle g\rangle\cap Z}$. So assume that $|\overline{g}|=s$ is a prime such that $(s,p)=1$ and the representative g is such that $min(g)$ irreducible in $\mathbb{F}[x]$. Then $g^s=\omega^i\in Z$, where $\langle\omega\rangle=GF(q)^{\#}$. We have two cases:

Case 1: $s|i$ or $s\nmid q-1$. If $i=sj$, then replacing g by $\omega^{-j}g$, we can assume that $|g|=s$. If $s\nmid q-1$, then $\langle\omega^s\rangle=\langle\omega\rangle$, which implies that $\omega^i=\omega^{sj}$ for some j. Hence replacing g by $\omega^{-j}g$, we can still assume that $|g|=s$. So in case 1 we always assume that $|g|=s$. The degree c of $min(g)$ equals the order of q in the group of units in $\mathbb{Z}_s$. If λ is an eigenvalue of g, then $min(g)=\prod_{j=0}^{c-1}(x-\lambda^{q^j})$. V is a homogeneous $\mathbb{F}\langle g\rangle$-module and $C_G(g)\simeq GL_{\frac{n}{c}}(q^c)$.

Case 2: $s|q-1$ and $s\nmid i$. In this case, the polynomial $x^s-\omega^i$ has no linear factor in $\mathbb{F}[x]$. Let $q-1=s\gamma$ and let λ be a root of $x^s-\omega^i$. Thus λ, $\lambda\omega^\gamma$, ... , $\lambda\omega^{(s-1)\gamma}$ are all the distinct roots of $x^s-\omega^i$. Let $\alpha\in Aut(\mathbb{F}(\lambda))$ with α: $a\mapsto a^q$. Now α has an orbit of length $l>1$ on $\{\lambda,\lambda^q,\ldots\}$. Thus each irreducible factor of $x^s-\omega^i$ in $\mathbb{F}[x]$ has degree l, which implies $l|s$. As $l>1$ and s is a prime, $l=s$. That is $x^s-\omega^i$ is irreducible in $\mathbb{F}[x]$ and thus $min(g,\ \mathbb{F},\ V)=x^s-\omega^i$; $\mathbb{F}(\lambda)=GF(q^s)$; λ, λ^q, $\ldots$, $\lambda^{q^{s-1}}$ are all the distinct eigenvalues of g, $min(g)=\prod_{j=0}^{s-1}(x-\lambda^{q^j})$. V is a homogeneous $\mathbb{F}\langle g\rangle$-module and $C_G(g)\simeq GL_{\frac{n}{s}}(q^s)$. Let $\langle\xi\rangle=GF(q^s)^{\#}$. So $|\xi|=q^s-1=(q-1)\epsilon$, where $\epsilon=q^{s-1}+\cdots+q+1\equiv 0(mod\ s)$ as $q\equiv 1(mod\ s)$. Thus $\epsilon=s\delta$. We have $\omega=\xi^\epsilon$ and thus $\lambda=\xi^{\delta i}$ is a root of $x^s-\omega^i$. Hence $|g|=|\xi^{\delta i}|=\dfrac{(q-1)s}{(i,(q-1)s)}$ in this case.

In both cases, if $g'\in GL_n(q)$ with $|g'|=|g|$ and $min(g')=min(g)$, then $g'=g^x$ for some $x\in G$. In the following, we assume that g is either in case 1 or in case 2. Also we always denote the degree of $min(g)$ by c, so $c=s$ in case 2.

(13.5) One of the following holds:

(a) $H\simeq GL_{\frac{n}{r}}(q^r)\cap G$, where r is a prime dividing n. H acts on V', where V' is V considered as $\frac{n}{r}$ dimensional vector space over the field $GF(q^r)$. $\mathcal{N}(g)\leq\dfrac{(n,q-1)}{q^{(n-\frac{n}{c}-1)(n-\frac{n}{r})}(q-1)^{n-\frac{n}{r}}}$ if $r\nmid c$; $\mathcal{N}(g)\leq\dfrac{(n,q-1)\binom{m+r-1}{r-1}}{q^{(n-\frac{n}{c}-1)(n-\frac{n}{r})-n}(q-1)^{2n-\frac{n}{r}}}$ if $r|c$, where $m=\frac{n}{c}$. If $n=2$, then $r=c=2$, and $\mathcal{N}(g)\leq\dfrac{4}{q(q-1)}$.

(b) $H\simeq GL(V_1)*GL(V_2)\cap G$, where V_1, V_2 are l, m dimensional vector spaces over $\mathbb{F}$ respectively; and $n=lm$, $l\neq m$, $l\neq 1$, $m\neq 1$, $V=V_1\otimes V_2$. Moreover there is a homomorphism $\pi: GL(V_1)\times GL(V_2)\rightarrow GL(V)$ by $(v_1\otimes v_2)(g_1,g_2)\pi=v_1g_1\otimes v_2g_2$, and $g=(g_1,g_2)\pi$. $\mathcal{N}(g)\leq (n,q-1)\Big/q^{(l^2-2)(m^2-1)(1-\frac{1}{c})-1-(n+l)}(q-1)^{n+l}$ if $c|l$ but $c\nmid m$; $\mathcal{N}(g)\leq (n,q-1)\Big/q^{(m^2-2)(l^2-1)(1-\frac{1}{c})-1-(n+m)}(q-1)^{n+m}$ if $c|m$ but $c\nmid l$; $\mathcal{N}(g)\leq (n,q-1)\Big/q^{(l^2-2)(m^2-1)(1-\frac{1}{c})-1-(n+l)}(q-1)^{n+l}+(n,q-1)\Big/q^{(m^2-2)(l^2-1)(1-\frac{1}{c})-1-(n+m)}(q-1)^{n+m}$ if $c|l$ and $c|m$.

(c) $H=N_G(R)$, where $n=r^m$ is a power of prime $r\neq p$ and R is an r-group of symplectic type such that $|R{:}Z(R)|=r^{2m}$. Also R is of exponent r if r is odd and of exponent 4 if $r=2$. Moreover $|Z(R)|>2$, $q=p^e$, where $e=|p|$ in the group of units U of $\mathbb{Z}_{r^k}$ with $r^k=|Z(R)|$, and e is required to be odd. $\frac{H}{Z}\simeq C_{Aut(R)}(Z(R))\simeq E_{r^{2m}}\cdot Sp_{2m}(r)$. If $r=2$, then $q=p$. If r is odd, then g is in case 2. In any case, $\mathcal{N}(g)\leq(n,q-1)n^{2m+3}\Big/q^{n^2(1-\frac{1}{c})-n}(q-1)^{n-1}$.

(d) $H=DS_m\cap G$, where $D\simeq GL(V_1)*GL(V_2)*\cdots*GL(V_m)$, each V_i is a k-dimensional vector space over $\mathbb{F}$, $V=V_1\otimes V_2\otimes\cdots\otimes V_m$, $n=k^m$, $m>1$, DS_m is the semidirect product of D by S_m, and there exists a homomorphism $\pi{:}GL(V_1)wrS_m\to GL(V)$ by $(v_1\otimes v_2\otimes\cdots\otimes v_m)(x,(g_1,\ldots,g_m))\pi=v_{1x^{-1}}g_1\otimes v_{2x^{-1}}g_2\otimes\cdots\otimes v_{mx^{-1}}g_m$, $x\in S_m$, and $g=(x,(g_1,\ldots,g_m))\pi$. $\mathcal{N}(g)\leq(n,q-1)\Big/(m-1)!q^{(k^{2m}-k^2)(1-\frac{1}{c})-(k^m+k)}(q-1)^{k^m+k}$ for g in case 1; and $\mathcal{N}(g)\leq(n,q-1)m!\Big/q^{k^{2m}(1-\frac{1}{c})-k^2(m-\frac{1}{c})-(k^m+k)}(q-1)^{k^m+k}$ for g in case 2.

(e) $H=O^+_{2m}(q)\mathbb{F}^{\#}\cap G$, p odd, $n=2m$ and c even, g in case 1, and $\mathcal{N}(g)\leq(n,q-1)(q^{\frac{n}{2}}-1)\Big/q^{\frac{1}{2}n^2(1-\frac{2}{c})}(q-1)^{n-1}$.

(f) $\mathcal{N}(g)\leq\dfrac{(n,q-1)}{q^{(n-1)(n-\frac{n}{c}-3)-3}(q-1)^{n-\frac{n}{c}}}$.

Proof. Suppose $H=H_1\cap G$, where H_1 is on the list of C_1 through C_8 in [As2]. We consider only the case that $G=GL_n(q)$, as for general G with $ZSL_n(q)\leq G\leq GL_n(q)$, the bound for $\mathcal{N}(g)$ can be obtained by multiplication of the factor $(n,q-1)$ according to (3.5). Also suppose $\mathcal{N}(g)\neq 0$. Thus $g^G\cap H\neq\emptyset$. Without loss of generality, assume that $g\in H$. As H is irreducible, H is not in C_1.

Suppose $H\in C_2$. Then H is the stabilizer of a direct sum decomposition of V, contradicting to the hypothesis that H is primitive.

Suppose $H\in C_3$. Then $H\simeq GL_{\frac{n}{r}}(q^r)$ where r is a prime dividing n, and H acts on V', where V' is V considered as $\frac{n}{r}$ dimensional vector space over the field $GF(q^r)$. For g, we consider case 1 and case 2 together, thus here let c denotes the degree of $f=min(g,V,\mathbb{F})=\prod_{j=0}^{c-1}(x-\lambda^{q^j})$, i.e., for case 2, $c=s$. Suppose that $r\nmid c$ first. Then the map $\rho^r{:}x\mapsto x^{q^r}$ is transitive on the eigenvalues of g. Thus $min(g,V',GF(q^r))=f$ and is irreducible in

$GF(q^r)[x]$. So V' is a homogeneous $GF(q^r)\langle g\rangle$-module, it follows that $C_H(g)\simeq GL_{\frac{n}{rc}}(q^{rc})$. Now if $g^x\in g^G\cap H$, then $g^x=g^h$ for some $h\in H$ because $min(g^x,V',GF(q^r))=min(g^x,V,\mathbb{F})=min(g,V,\mathbb{F})=min(g,V',GF(q^r))$ and g, g^x both homogeneous on V'. So $g^G\cap H=g^H$. Since $\frac{|GL_{\frac{n}{r}}(q^r)|}{|GL_n(q)|}=1\Big/\Big\{q^{\frac{1}{2}n(n-\frac{n}{r})}(q-1)(q^2-1)\cdots(q^{r-1}-1)(q^{r+1}-1)\cdots(q^{2r-1}-1)(q^{2r+1}-1)\cdots$

$(q^{n-1}-1)\Big\}\le\frac{1}{q^{(n-1)(n-\frac{n}{r})}(q-1)^{n-\frac{n}{r}}}$ and $\frac{|GL_{\frac{n}{c}}(q^c)|}{|GL_{\frac{n}{rc}}(q^{rc})|}=q^{\frac{1}{2}\cdot\frac{n}{c}(n-\frac{n}{r})}(q^c-1)(q^{2c}-1)\cdots(q^{c(r-1)}-1)$

$(q^{c(r+1)}-1)\cdots(q^{c(2r-1)}-1)(q^{c(2r+1)}-1)\cdots(q^{n-c}-1)\le q^{\frac{n}{c}(n-\frac{n}{r})}$, we have $\mathcal{N}(g)=\frac{|H||C_G(g)|}{|C_H(g)||G|}\le$

$\frac{1}{q^{(n-\frac{n}{c}-1)(n-\frac{n}{r})}(q-1)^{n-\frac{n}{r}}}$. Now suppose that $r|c$. So $c=\beta r$. We have $\tilde{f}=min(g,V',GF(q^r))|f=f_1f_2\cdots f_r$, where $f_i=\prod_{k=0}^{\beta-1}(x-\lambda^{q^{i-1+kr}})\in GF(q^r)[x]$. Thus $\tilde{f}=f_1^{\delta_1}f_2^{\delta_2}\cdots f_r^{\delta_r}$, where $\delta_i=0$ or 1, and $\delta_1+\delta_2+\cdots+\delta_r\ge 1$. Let b_i be the number of Jordan blocks of g on V' corresponding to f_i, so if $\delta_i=0$, then $b_i=0$ and if $\delta_i=1$, then $b_i\ge 1$. Thus $b_1+b_2+\cdots+b_r=\frac{n}{c}$ and $C_H(g)\simeq GL_{b_1}(q^c)\times GL_{b_2}(q^c)\times\cdots\times GL_{b_r}(q^c)$. Conversely, if g acts on V' with $min(g,V',GF(q^r))=f_1^{\delta_1}f_2^{\delta_2}\cdots f_r^{\delta_r}$ for some choice of δ_i's, then $min(g,V,\mathbb{F})=f=f_1f_2\cdots f_r$. So $g^G\cap H=\dot{\bigcup}_A x^H$, where A is the set of elements of H such that for $x\in A$, $min(x,V',GF(q^r))=f_1^{\delta_1}f_2^{\delta_2}\cdots f_r^{\delta_r}$ with $\delta_1+\delta_2+\cdots+\delta_r\ge 1$, and for $x_1,x_2\in A$ with $x_1\ne x_2$, $(b_1^{(1)},b_2^{(1)},\dots,b_r^{(1)})\ne(b_1^{(2)},b_2^{(2)},\dots,b_r^{(2)})$. The number of H-conjugacy classes is equal to the number of solutions of $b_1+b_2+\cdots+b_r=\frac{n}{c}=m$; which is equal to the number of strings of the form 1110010⋯011 in which there are m 1's and $r-1$ 0's; for example, the string 1110010⋯011 corresponds to $b_1=3$, $b_2=0$, $b_3=1,\dots,b_r=2$. Hence the number of solutions is $\binom{m+r-1}{r-1}$. Since $|GL_{b_1}(q^c)\times GL_{b_2}(q^c)\times\cdots\times GL_{b_r}(q^c)|\ge q^{c\sum_{i=1}^{r}b_i(b_i-1)}(q^c-1)^{\sum_{i=1}^{r}b_i}\ge q^{n(\frac{n}{rc}-1)}(q^c-1)^{\frac{n}{c}}\ge q^{n(\frac{n}{rc}-1)}(q-1)^n$ gives us $\frac{|C_G(g)|}{|C_H(g)|}\le\frac{q^{\frac{n}{c}(n-\frac{n}{r})+n}}{(q-1)^n}$; and thus $\mathcal{N}(g)\le\frac{\binom{m+r-1}{r-1}}{q^{(n-\frac{n}{c}-1)(n-\frac{n}{r})-n}(q-1)^{2n-\frac{n}{r}}}$. If $n=2$, then

$r=c=2$, and there are two H-conjugacy classes. Thus $\mathcal{N}(g)\leq\frac{2|GL_1(q^2)|}{|GL_2(q)|}=\frac{2}{q(q-1)}$.

Suppose $H\in C_4$. So $H\simeq GL(V_1)*GL(V_2)$, where V_1, V_2 are l, m-dimensional vector spaces over $\mathbb{F}$ respectively; and $n=lm$, $l\neq m$, $l\neq 1$, $m\neq 1$, $V=V_1\otimes V_2$. Moreover there is a homomorphism $\pi:GL(V_1)\times GL(V_2)\rightarrow GL(V)$ by $(v_1\otimes v_2)(g_1,g_2)\pi=v_1g_1\otimes v_2g_2$. Thus if λ_1, λ_2 are eigenvalues of g_1, g_2 respectively, then $\lambda_1\lambda_2$ is an eigenvalue of $g=(g_1,g_2)\pi$. Thus neither g_1 nor g_2 can have more than c distinct eigenvalues, as g has exactly c eigenvalues. Also clearly either $g_1\notin Z(GL(V_1))$ or $g_2\notin Z(GL(V_2))$. Say $g_1\notin Z(GL(V_1))$. First consider g in case 1, so $|g|=s$, and thus $g_i^s\in Z(GL(V_i))$. Then as $(s,q-1)=1$, we can assume without loss of generality that $|g_i|\,|s$ for $i=1$ and 2. So g_1 has an eigenvalue $\lambda_1\notin\mathbb{F}$, and $\lambda_1,\lambda_1^q,\ldots,\lambda_1^{q^{c-1}}$ are all the distinct eigenvalues of g_1. Let μ be an eigenvalue of g_2. Then $\mu\lambda_1$, $\mu\lambda_1^q,\ldots$, $\mu\lambda_1^{q^{c-1}}$ are all the c distinct eigenvalues of g. Hence their product $\mu^c(\lambda_1\lambda_1^q\cdots\lambda_1^{q^{c-1}})\in\mathbb{F}$. But $0\neq\lambda_1\lambda_1^q\cdots\lambda_1^{q^{c-1}}\in\mathbb{F}$, so $\mu^c\in\mathbb{F}$. Since $c|(s-1)$, $s=c\nu+1$ for some ν. So $\mu(\mu^c)^\nu=\mu^s=1$, as $|g_2|\,|s$. Then $\mu=(\mu^c)^{-\nu}\in\mathbb{F}$, which implies $g_2\in Z(GL(V_2))$. We thus have that either $g_1\in Z(GL(V_1))$, g_2 has exactly c eigenvalues or $g_2\in Z(GL(V_2))$, g_1 has exactly c eigenvalues. Correspondingly, either $c|l$, $C_H(g)=A*B$, where the product is central with $A\simeq GL_{\frac{l}{c}}(q^c)$, $B\simeq GL_m(q)$, and $Z(B)\leq Z(A)$; or $c|m$, $C_H(g)=C*D$ where the product is central with $C\simeq GL_l(q)$, $D\simeq GL_{\frac{m}{c}}(q^c)$, and $Z(C)\leq Z(D)$. Under these notations, we have three possibilities: (1) $c|l$ but $c\nmid m$, $g^G\cap H=g^H$ with $C_H(g)=A*B$. (2) $c|m$ but $c\nmid l$, $g^G\cap H=g^H$ with $C_H(g)=C*D$. (3) $c|l$ and $c|m$, $g^G\cap H=x_1^H\,\dot{\cup}\,x_2^H$ with $C_H(x_1)=A*B$, $C_H(x_2)=C*D$. Now consider that g in case 2. Let $\lambda_1,\lambda_1^q,\ldots,\lambda_1^{q^{\alpha-1}}$, $\lambda_2,\lambda_2^q,\ldots,\lambda_2^{q^{\beta-1}}$ be orbits of λ_1, λ_2 under the map $\rho:x\mapsto x^q$, where λ_1, λ_2 are eigenvalues of g_1, g_2 respectively. Since $\lambda_1\lambda_2$ is an eigenvalue of g and has orbit length c under ρ, $c|l.c.m.\{\alpha,\beta\}$. As c=s is a prime, we have that either $c|\alpha$ or $c|\beta$, say $c|\alpha$. Also clearly $\alpha\leq c$. Hence $\alpha=$c and $min(g_1)$ is of degree c and is irreducible in $\mathbb{F}[x]$. So V_1 is a homogeneous $\mathbb{F}\langle g_1\rangle$-module and thus $c|l$. We still have three possibilities: (a) $c|l$ but $c\nmid m$, $|g^G\cap H|\leq|g_1^{GL_l(q)}||GL_m(q)|$. (b) $c|m$ but $c\nmid l$, $|g^G\cap H|\leq|GL_l(q)||g_2^{GL_m(q)}|$. (c) $c|l$ and $c|m$,

$|g^G\cap H|\leq|x_1^{GL_l(q)}||GL_m(q)|+|GL_l(q)||x_2^{GL_m(q)}|$. For cases (1) and (a) above, $\mathcal{N}(g)\leq\frac{|GL_l(q)||GL_m(q)||GL_{\frac{n}{c}}(q^c)|}{|GL_n(q)||GL_{\frac{l}{c}}(q^c)|}\leq q^{l^2+m^2+c(\frac{n}{c})^2}\Big/q^{n(n-1)}(q-1)^n q^{c\frac{l}{c}(\frac{l}{c}-1)}(q^c-1)^{\frac{l}{c}}\leq$ $1\Big/q^{(l^2-2)(m^2-1)(1-\frac{1}{c})-1-(n+l)}(q-1)^{n+l}$. Similarly, for case (2) and (b), $\mathcal{N}(g)\leq 1\Big/q^{(m^2-2)(l^2-1)(1-\frac{1}{c})-1-(n+m)}(q-1)^{n+m}$. For case (3) and (c), $\mathcal{N}(g)\leq$ $\left\{1\Big/q^{(l^2-2)(m^2-1)(1-\frac{1}{c})-1-(n+l)}(q-1)^{n+l}\right\}+\left\{1\Big/q^{(m^2-2)(l^2-1)(1-\frac{1}{c})-1-(n+m)}(q-1)^{n+m}\right\}$.

Suppose $H\in C_5$. Then $H\simeq GL(U)\mathbb{F}^{\#}$, where U is an n-dimensional $\mathbb{K}$-subspace of V and $\mathbb{K}$ is the subfield of $\mathbb{F}$ of prime index r; and $\mathbb{F}^{\#}$ is the center of G. But then H contains a transvection, contradicting to the hypothesis.

Suppose $H\in C_6$. Then $H=N_G(R)$, where $n=r^m$ is a power of prime $r\neq p$ and R is an r-group of symplectic type such that $|R:Z(R)|=r^{2m}$. Also R is of exponent r if r is odd and of exponent 4 if $r=2$. Moreover by ($C_6$1) in [As2], $|Z(R)|>2$, $q=p^e$, where $e=|p|$ in the group of units U of $\mathbb{Z}_{r^k}$ with $r^k=|Z(R)|$; also e is required to be odd. By (4) of Theorem A in [As2], $\frac{H}{\mathbb{F}_q^*}\simeq C_{Aut(R)}(Z(R))$. If g is in case 1 and $s=r$, then $1\equiv q^c=q^{r^l}=q^{s^l}\equiv q(mod\ s)$, where $c=r^l$ for some $l\leq m$; but this is a contradiction as $c=|q|\geq 2$ in $\mathbb{Z}_s$. So $s\neq r$. Suppose r is odd first. Then R is an extraspecial r-group. Since $[\langle g\rangle,Z(R)]=1$, V is a faithful $GF(p)\langle g\rangle R$-module, $C_V(g)=0$, and p, r, s are distinct primes, by (36.1)(1) in [As1], $s=2^\alpha+1$ is a Fermat prime and $r=2$, a contradiction. So $r=2$, then $|Z(R)|=4$, and $(p,2)=1$ implies $p^2\equiv 1(mod\ 4)$. Thus $e|2$. So e being odd implies that $e=1$. So $q=p$. By exercise 8.5 in [As1], a 2-group R of symplectic type of exponent 4 is isomorphic to D^t, $D^{t-1}Q$, or $\mathbb{Z}_4*D^t$, where $D=D_8$, $Q=Q_8$, and D^t is the central product of t copies of D with identified centers. But for the first two cases, $|Z(R)|=2$. Hence we have $R\simeq\mathbb{Z}_4*D^t$, and thus $t=m$ here. By same exercise, $\frac{H}{\mathbb{F}^{\#}}\simeq C_{Aut(R)}(Z(R))\simeq$ $E_{2^{2m}}\cdot Sp_{2m}(2)$. Hence $\mathcal{N}(g)\leq\frac{|H||C_G(g)|}{|G|}\leq(p-1)2^{2m}2^{2m^2+m}p^{c(\frac{n}{c})^2}\Big/p^{n(n-1)}(p-1)^n\leq$ $n^{2m+3}\Big/p^{n^2(1-\frac{1}{c})-n}(p-1)^{n-1}$. Now consider g is in case 2. If $r=2$, we have the same result. If r is odd, then $\frac{H}{\mathbb{F}^{\#}}\simeq C_{Aut(R)}(Z(R))=Inn(R)\cdot Sp(\frac{R}{Z(R)})\simeq E_{r^{2m}}\cdot Sp_{2m}(r)$. Thus

$$\mathcal{N}(g)\leq\frac{|H||C_G(g)|}{|G|}\leq(q-1)r^{2m}r^{2m^2+m}q^{c(\frac{n}{c})^2}\Big/q^{n(n-1)}(q-1)^n\leq n^{2m+3}\Big/q^{n^2(1-\frac{1}{c})-n}(q-1)^{n-1}.$$

Suppose $H\in C_7$. Then $H=DS_m$, where $D\simeq GL(V_1)*GL(V_2)*\cdots*GL(V_m)$, each V_i is a k-dimensional vector space over $\mathbb{F}$, $V=V_1\otimes V_2\otimes\cdots\otimes V_m$, $n=k^m$, $m>1$, H is the semidirect product of D by S_m, and there exists a homomorphism $\pi:GL(V_1)wrS_m\to GL(V)$ by $(v_1\otimes v_2\otimes\cdots\otimes v_m)(x,(g_1,\ldots,g_m))\pi=v_{1x^{-1}}g_1\otimes v_{2x^{-1}}g_2\otimes\cdots\otimes v_{mx^{-1}}g_m$, $x\in S_m$. We consider that g in case 1 first. Suppose $g\in D$. So $g=(1,(g_1,\ldots,g_m))\pi$. If $\lambda_1,\ldots,\lambda_m$ are eigenvalues of $g_1,\ldots,g_m$ respectively, then $\lambda_1\cdots\lambda_m$ is an eigenvalue of g. Since g has exactly c eigenvalues, a similar argument to the case for C_4 shows that all $g_i\in Z(GL(V_i))$ except one g_j for some j, and this g_j has exactly c eigenvalues. Thus $c|k$ and $C_D(g)=A*B$, where the product $A*B$ is central and $A\simeq GL(\frac{k}{c},q^c)$, B is the central product of $m-1$ copies of $GL_k(q)$ with identified centers, and $Z(B)\leq Z(A)$. Let $h\in H$, $h=(x,(h_1,\ldots,h_m))\pi$. Then $g^h=\{(x^{-1},(h_{1x}^{-1},\ldots,h_{mx}^{-1}))(1,(g_1,\ldots,g_m))(x,(h_1,\ldots,h_m))\}\pi=(1,(g_{1x^{-1}}^{h_1},\ldots,g_{mx^{-1}}^{h_m}))\pi$. If $h\in C_H(g)$, then $(1,(g_1,\ldots,g_m))\pi=(1,(g_{1x^{-1}}^{h_1},\ldots,g_{mx^{-1}}^{h_m}))\pi$. As $g=(1,(z_1,\ldots,z_{j-1},g_j,z_{j+1},\ldots,z_m))\pi$ with $z_i\in Z(GL(V_i))$ for $i\neq j$, we have $jx=j$. Thus $C_H(g)\simeq C_D(g)S_{m-1}$, where the product is semidirect. Also if g_1, $g_2\in g^G\cap D$, then g_1,g_2 are conjugates in H. Now suppose $g^G\cap(H\setminus D)\neq\emptyset$. Without loss of generality, assume that $g\in H\setminus D$. So $g=(x,(g_1,\ldots,g_m))\pi$ with $x=(a)(b)\cdots(\alpha_1\cdots\alpha_s)(\beta_1\cdots\beta_s)\cdots$, x has at least one s-cycle; and $|g|=s$ implies g_a^s, g_b^s, $\ldots$, $g_{\alpha_1}g_{\alpha_2}\cdots g_{\alpha_s}$, $g_{\beta_1}g_{\beta_2}\cdots g_{\beta_s},\ldots$, are all in $\mathbb{F}^{\#}$. As $(v_{\alpha_s}g_{\alpha_1}\otimes v_{\alpha_s}g_{\alpha_1}g_{\alpha_2}\otimes\cdots\otimes v_{\alpha_s}g_{\alpha_1}g_{\alpha_2}\cdots g_{\alpha_{s-1}}\otimes v_{\alpha_s})((\alpha_1\cdots\alpha_s),(g_{\alpha_1},\ldots,g_{\alpha_s}))\pi=v_{\alpha_s}g_{\alpha_1}\otimes v_{\alpha_s}g_{\alpha_1}g_{\alpha_2}\otimes\cdots\otimes v_{\alpha_s}g_{\alpha_1}g_{\alpha_2}\cdots g_{\alpha_{s-1}}\otimes v_{\alpha_s}g_{\alpha_1}g_{\alpha_2}\cdots g_{\alpha_s}=\mu(v_{\alpha_s}g_{\alpha_1}\otimes v_{\alpha_s}g_{\alpha_1}g_{\alpha_2}\otimes\cdots\otimes v_{\alpha_s}g_{\alpha_1}g_{\alpha_2}\cdots g_{\alpha_{s-1}}\otimes v_{\alpha_s})$, where $\mu\in\mathbb{F}^{\#}$. So if x has no 1-cycle, or if for any 1-cycle (a) of x, $g_a\in\mathbb{F}^{\#}$, then g would have an eigenvalue in $\mathbb{F}$, which is a contradiction. Since g has exactly c eigenvalues, x has a 1-cycle (a) such that $g_a\notin\mathbb{F}^{\#}$ and g_a has exactly c eigenvalues; and for any other 1-cycle (b) or any s-cycle $(\alpha_1\cdots\alpha_s)$, $((b),g_b)$, $((\alpha_1\cdots\alpha_s),(g_{\alpha_1},\ldots,g_{\alpha_s}))$ acts as scalars on V_b, $V_{\alpha_1}\otimes V_{\alpha_2}\otimes\cdots\otimes V_{\alpha_s}$ respectively. So without loss of generality, can assume g_b, $g_{\alpha_1},\ldots,g_{\alpha_s}$ are all equal to 1. But then $\mu(v_{\alpha_1}\otimes v_{\alpha_2}\otimes\cdots\otimes v_{\alpha_s})=(v_{\alpha_1}\otimes v_{\alpha_2}\otimes\cdots\otimes v_{\alpha_s})((\alpha_1\cdots\alpha_s),(1,\ldots,1))\pi=v_{\alpha_s}\otimes v_{\alpha_1}\otimes\cdots\otimes v_{\alpha_{s-1}}$ implies that $v_{\alpha_s}=\mu' v_{\alpha_1}$ for some $\mu'\in\mathbb{F}^{\#}$, contradicting to that $dim_{\mathbb{F}}(V_i)=k\geq2$ $\forall i$. So

$g^G \cap (H \setminus D) = \emptyset$. Hence $g^G \cap H = g^G \cap D = g^H$, and $C_H(g) \simeq C_D(g) S_{m-1}$ with $C_D(g) = A*B$, where $A*B$ is as above. So $\mathcal{N}(g) = \dfrac{|GL_k(q)||GL_{\frac{k^m}{c}}(q^c)|}{(m-1)!|GL_{k^m}(q)||GL_{\frac{k}{c}}(q^c)|} \leq$

$q^{k^2+c(\frac{k^m}{c})^2} \Big/ (m-1)! q^{k^m(k^m-1)} (q-1)^{k^m} q^{c\frac{k}{c}(\frac{k}{c}-1)} (q^c-1)^{\frac{k}{c}} \leq 1 \Big/ (m-1)! q^{(k^{2m}-k^2)(1-\frac{1}{c})-(k^m+k)} (q-1)^{k^m+k}$

. Now consider that g in case 2. Let $g = (x, (g_1, \ldots, g_m))\pi$ with $x \in S_m$. Since $min(g) = x^s - \omega^i$ with $\omega^i \in \mathbb{F}^{\#}$ and $k \geq 2$, each cycle of x is either an s-cycle or a 1-cycle. Similar to above, if x has no 1-cycle, or if for any 1-cycle (a) of x, $g_a \in \mathbb{F}^{\#}$, then g would have an eigenvalue in $\mathbb{F}$, which is a contradiction. Since g has exactly s eigenvalues, as in the case for C_4, x has at least one 1-cycle (a) such that the map $\rho: x \mapsto x^q$ has an orbit of length s on the eigenvalues of g_a and g_a has exactly s eigenvalues. Hence $|g^G \cap H| \leq m \dfrac{|GL_k(q)|}{|GL_{\frac{k}{s}}(q^s)|} |GL_k(q)|^{m-1}(m-1)!$, which gives

$$\mathcal{N}(g) \leq \frac{m!|GL_k(q)|^m |GL_{\frac{k^m}{s}}(q^s)|}{|GL_{k^m}(q)||GL_{\frac{k}{s}}(q^s)|} \leq m! q^{mk^2+s(\frac{k^m}{s})^2} \Big/ q^{k^m(k^m-1)} (q-1)^{k^m} q^{s\frac{k}{s}(\frac{k}{s}-1)} (q^s-1)^{\frac{k}{s}} \leq$$

$$m! \Big/ q^{k^{2m}(1-\frac{1}{s})-k^2(m-\frac{1}{s})-(k^m+k)} (q-1)^{k^m+k}.$$

Suppose $H \in C_8$. Then $H = Sp(V)\mathbb{F}^{\#}$; or $GU(V)\mathbb{F}^{\#}$; or p odd and $H = O_n(q)\mathbb{F}^{\#}$ if n odd, $H = O_n^{\pm}(q)\mathbb{F}^{\#}$, if n even. But $Sp(V)$ and $GU(V)$ contain a transvection, so the first two cases are out as we assume that H contains no transvection. So H contains the orthogonal group. We consider both cases together. We will show that c, thus n, has to be even. Let $(\ ,\ )$ be a nondegenerate symmetric bilinear form on V. Let $\mathbb{K} = GF(q^c)$, and $\Lambda = \{\lambda, \lambda^q, \ldots, \lambda^{q^{c-1}}\}$, the set of eigenvalues of g. So $V^{\mathbb{K}} = \bigoplus_{\alpha \in \Lambda} V_\alpha$, where V_α is the eigenspace for α. Now for $u \in V_\alpha$, $v \in V_\beta$, $(u,v) = (ug, vg) = \alpha\beta(u,v)$. Thus either $(u,v) = 0$ or $\beta = \alpha^{-1}$. But $|\alpha| \neq 2$, so $\alpha \neq \alpha^{-1}$. Hence $(\ ,\ )$ is trivial on each V_α. As $(\ ,\ )$ is nondegenerate, for each $\alpha \in \Lambda$, $\alpha^{-1} \in \Lambda$. So $|\Lambda| = c$ is even, say $c = 2a$ and $n = 2m$. Then $\lambda^{-1} = \lambda^{q^a}$ and $\Lambda = \{\lambda, \lambda^q, \ldots, \lambda^{q^{a-1}}, \lambda^{-1}, \lambda^{-q}, \ldots, \lambda^{-q^{a-1}}\}$. If g in case 2, then $s = c = 2$ and $\{\lambda, \lambda^{-1}\}$ are the only eigenvalues of g, which implies that $x^s - \omega^i = min(g) = x^2 - 1$, contradicting to $s \nmid i$. So g is in case 1, and $(\ ,\ )$ is trivial on $\bigoplus_{\alpha \in \Delta} V_\alpha$, where $\Delta=\{\lambda,\lambda^q,\ldots,\lambda^{q^{a-1}}\}$. Thus the Witt index of V is m; which implies that the only remaining possibility is $H=O_{2m}^+(q)\mathbb{F}^\#$. Since $|O_{2m}^+(q)|\leq q^{2m(m-1)}(q^m-1)$, we have $\mathcal{N}(g)\leq \frac{|H||C_G(g)|}{|G|}\leq(q-1)q^{2m(m-1)}(q^m-1)q^{c(\frac{n}{c})^2}\Big/q^{n(n-1)}(q-1)^n\leq(q^{\frac{n}{2}}-1)\Big/q^{\frac{1}{2}n^2(1-\frac{2}{c})}(q-1)^{n-1}$.

Finally, if H is not one of the subgroups in the list C_1 through C_8 of [As2], by the bound for $|H|$ in Liebeck's paper [Li], we have $\mathcal{N}(g)\leq\frac{|H|}{|gG|}\leq\frac{1}{q^{(n-1)(n-\frac{n}{c}-3)-3}(q-1)^{n-\frac{n}{c}}}$.

(13.6) Suppose $|g|=p^e$ with $e\geq 1$. Let ν be the number of Jordan blocks of g. Then one of the following holds:

(a) $\nu=n-1$ iff g is a transvection, in which case $\mathcal{M}(g)=0$.

(b) If g has only one block of dimension $p^{e-1}+1$ and all other blocks are of dimensions at most p^{e-1}, then $\mathcal{M}(g)=0$.

(c) $\nu\leq n-2$ and g has at least one Jordan block of dimension 1, and $\mathcal{N}(g)\leq\frac{1}{q^2}$, which is at most $\frac{1}{85}$ except when $2\leq q\leq 9$.

(d) $\nu\leq n-2$ and all blocks of g have dimensions at least 2, then $\mathcal{N}(g)\leq\frac{1}{q^{\{\frac{n}{2}\}-1}}$, which is at most $\frac{1}{85}$, except in following cases: (1) $q=2$ and $n\leq 14$. (2) $q=3$ and $n\leq 10$. (3) $q=4$ and $n\leq 8$. (4) $5\leq q\leq 9$ and $n\leq 6$. (5) $11\leq q\leq 83$ and $n=3$ or 4.

Proof. These follow from (13.1) by direct computation. For example, in case (b), $\mathcal{M}(g)=\mathcal{N}(g^{p^{e-1}})=0$ as $g^{p^{e-1}}$ is a transvection.

(13.7) Suppose g is semisimple, and assume that it is not one of the following cases: (1) $q=2$ and $n\leq 12$. (2) $q=3$ and $n\leq 8$. (3) $q=4$, 5 and $n\leq 6$. (4) $7\leq q\leq 9$ and $n\leq 4$. (5) $11\leq q\leq 19$ and $n\leq 3$. (6) $23\leq q\leq 83$ and $n=2$. Then we have that either

(a) $\mathcal{N}(g)\leq\frac{1}{q^{\frac{n}{2}}}$ and $\mathcal{N}(g)\leq\frac{1}{85}$; or

(b) $\mathcal{N}(g)\leq\frac{1}{2^8}$.

Proof. If $min(g)$ has at least two irreducible factors in $\mathbb{F}[x]$, then by (13.4) $\mathcal{N}(g)\leq\frac{1}{q^{n-\mu}}\leq\frac{1}{q^{\frac{n}{2}}}$, where μ is the smallest dimension of homogeneous subspaces of g and thus $\mu\le\frac{n}{2}$. Also if it is not one of those cases listed above, it is easy to check that $\frac{1}{q^{\frac{n}{2}}}\le\frac{1}{85}$. So (a) holds in this case. Now suppose $min(g)$ is irreducible in $\mathbb{F}[x]$. Then the conclusion follows from (13.5) by direct calculation. We record several examples here; for the rest the calculations are very similar. If $H\simeq GL_{\frac{n}{r}}(q^r)\cap G$ with r a prime dividing n, then $\mathcal{N}(g)\le\frac{(n,q-1)}{q^{(n-\frac{n}{c}-1)(n-\frac{n}{r})}(q-1)^{n-\frac{n}{r}}}$ if $r\nmid c$; $\mathcal{N}(g)\le\frac{(n,q-1)\binom{m+r-1}{r-1}}{q^{(n-\frac{n}{c}-1)(n-\frac{n}{r})-n}(q-1)^{2n-\frac{n}{r}}}$ if $r|c$, where $m=\frac{n}{c}$. Since $n-\frac{n}{c}-1\ge\frac{n}{2}-1\ge1$ if $n\ge4$, and $n-\frac{n}{r}\ge\frac{n}{2}\ge1$, we have that in the first case $(n-\frac{n}{c}-1)(n-\frac{n}{r})\ge\frac{n}{2}$ if $n\ge4$. Also clearly $n\ne2$, 3, because if it were, then $r=n=c$ contradicting to $r\nmid c$. So in the first case, (a) holds. In the second case, we consider $q\ge3$ first. Thus $\binom{m+r-1}{r-1}\le2^{m+r-1}\le(q-1)^{m+r-1}$. Since $2n-\frac{n}{r}-m-r+1\ge n-r+1\ge1$, we have that for $q\ge3$, $\mathcal{N}(g)\le\frac{1}{q^{(n-\frac{n}{c}-1)(n-\frac{n}{r})-n}}$. If $n\ge8$, then $(n-\frac{n}{c}-1)(n-\frac{n}{r})-n\ge(\frac{n}{2}-3)\frac{n}{2}\ge\frac{n}{2}$, i.e., (a) holds. For $n=7$, $c=r=7$, and thus $(n-\frac{n}{c}-1)(n-\frac{n}{r})-n=23\ge\frac{n}{2}$, i.e., (a) holds. If $n=6$, then $(r,c)=(2,2)$, $(2,6)$, $(3,3)$ or $(3,6)$; and we have that $\binom{m+r-1}{r-1}=4$, 2, 6, or 3; and $2n-\frac{n}{r}=9$, 9, 10 or 10 respectively. Since $(n-\frac{n}{c}-1)(n-\frac{n}{r})-n\ge0$ and $(n,q-1)\le6$, for $q\ge7$ we have that $\mathcal{N}(g)\le\frac{36}{6^9}\le\frac{1}{2^8}$, i.e., (b) holds. If $n=5$, then $c=r=5$, and thus $(n-\frac{n}{c}-1)(n-\frac{n}{r})-n=7\ge\frac{n}{2}$, i.e., (a) holds. If $n=4$, then $(r,c)=(2,2)$ or $(2,4)$; and we have that $\binom{m+r-1}{r-1}=3$ or 2 respectively; and $2n-\frac{n}{r}=6$. Since $(n-\frac{n}{c}-1)(n-\frac{n}{r})-n=-2$ or 0 respectively and $(n,q-1)\le4$, for $q\ge11$ we have that $\mathcal{N}(g)\le\frac{12q^2}{(q-1)^6}\le\frac{2}{(q-1)^3}\le\frac{1}{2^8}$, i.e., (b) holds. If $n=3$, then $c=r=3$, and thus $(n-\frac{n}{c}-1)(n-\frac{n}{r})-n=-1$, $\binom{m+r-1}{r-1}=3$, $2n-\frac{n}{r}=5$. So $\mathcal{N}(g)\le\frac{9q}{(q-1)^5}\le\frac{1}{(q-1)^3}\le\frac{1}{2^8}$ if $q\ge23$, i.e., (b) holds. If $n=2$, then $c=r=2$, and $\mathcal{N}(g)\le\frac{4}{q(q-1)}\le\frac{1}{q}\le\frac{1}{85}$ if $q\ge89$, i.e., (a) holds. Now consider the case $q=2$. So $(n,q-1)=1$. Since $(n-\frac{n}{c}-1)(n-\frac{n}{r})-n-m-r+1\ge(\frac{n}{2}-6)\frac{n}{2}\ge\frac{n}{2}$ if $n\ge14$, and if $n=13$, then $r=c=13$, and $(n-\frac{n}{c}-1)(n-\frac{n}{r})-n-m-r+1=106\ge\frac{n}{2}$, we have that in this case (a) holds. For H in (13.5)(b), i.e., $H\simeq GL(V_1)*GL(V_2)\cap G$, where V_1, V_2 are l, m dimensional vector spaces over $\mathbb{F}$ respectively and $n=lm$, $l\ne m$, $l\ne1$, $m\ne1$, $V=V_1\otimes V_2$, we have that $\mathcal{N}(g)\leq\left\{(n,q-1)\Big/q^{(l^2-2)(m^2-1)(1-\frac{1}{c})-1-(n+l)}(q-1)^{n+l}\right\}+\left\{(n,q-1)\Big/q^{(m^2-2)(l^2-1)(1-\frac{1}{c})-1-(n+m)}(q-1)^{n+m}\right\}$. The condition $n=lm$, $l\neq m$, $l\neq 1$, $m\neq 1$ implies that $n\geq 6$ and $lm\leq\frac{n^2}{6}$, $l^2+m^2\leq(\frac{1}{4}+\frac{1}{9})n^2$, $m^2\leq\frac{n^2}{4}$. Hence $(l^2-2)(m^2-1)(1-\frac{1}{c})-1-(n+l)\geq\frac{1}{2}(n^2-2m^2-l^2)-n-l\geq\frac{7}{36}n^2-\frac{3}{2}n\geq\frac{n}{2}+1$ if $n\geq 11$, in which case $\mathcal{N}(g)\leq\frac{2}{q^{\frac{n}{2}+1}}\leq\frac{1}{q^{\frac{n}{2}}}$, i.e., (a) holds. Clearly $n\neq 7$ and $n\neq 9$. If $n=6$, then $l=2$ and $m=3$ and the formular reduces to $\mathcal{N}(g)\leq(6,q-1)q/(q-1)^8+(6,q-1)/q^{\frac{1}{2}}(q-1)^9\leq 6/(q-1)^7+6/(q-1)^8+6/(q-1)^9$, which is less than or equal to $\frac{1}{2^8}$ if $q\geq 7$. If $n=8$, then $l=2$ and $m=4$ and the formular reduces to $\mathcal{N}(g)\leq(8,q-1)/q^4(q-1)^{10}+(8,q-1)/q^8(q-1)^{12}\leq 1/q^4$ if $q\geq 4$, i.e., (a) holds. If $n=10$, then $l=2$ and $m=5$, and the formular reduces to $\mathcal{N}(g)\leq(10,q-1)/q^{12}(q-1)^{12}+(10,q-1)/q^{\frac{37}{2}}(q-1)^{15}\leq 1/q^5$ if $q\geq 3$, i.e., (a) holds. For H in (13.5)(d), we have that $n=k^m$, $m>1$, and $\mathcal{N}(g)\leq(n,q-1)\Big/(m-1)!\,q^{(k^{2m}-k^2)(1-\frac{1}{c})-(k^m+k)}(q-1)^{k^m+k}$ if g is in case 1. Since $(k^{2m}-k^2)(1-\frac{1}{c})-(k^m+k)\geq\frac{1}{2}(k^m+k)(k^m-k-2)\geq\frac{1}{2}(k^m+k)$ unless $k=2=m$. So $\mathcal{N}(g)\leq\frac{1}{q^{\frac{n}{2}}}$, i.e., (a) holds. If $k=2=m$, then $n=4$ and $\mathcal{N}(g)\leq 4\Big/(q-1)^6$, which is less than or equal to $\frac{1}{2^8}$ if $q\geq 11$.

Section 14. The cases: $2\leq q\leq 9$.

In this section, we keep the hypotheses and the notations in Section 13. The result of this section is:

Proposition. *If $\overline{G}$ is a group of genus zero, then one of the following holds:*

(a) $q=2$ and $n\leq 20$.

(b) $q=3$ and $n\leq 9$.

(c) $4\leq q\leq 9$ and $n\leq 5$.

(d) $11\leq q\leq 83$ and $n\leq 4$.

Proof. This follows from the proposition in Section 13 and (14.4), (14.9), (14.13), (14.17), (14.21), (14.25), and (14.29).

In the following, in view of the proposition stated in the beginning of Section 13, we distinguish the cases $q=2$, 3,..., up to $q=9$.

Case 1: $q=2$.

Unless explicitly specified, we assume that $n \geq 16$ in the following.

(14.1) $\mathcal{N}(g) \leq \frac{1}{85}$, unless g is listed in the following table. Column 3 and 4 list upper bounds for $\mathcal{N}(g)$, $\mathcal{U}(g)$ respectively.

type	$\|g\|$	$\mathcal{N}(g)$	$\mathcal{U}(g)$
$2^1 1^{n-2}$	2	0	1/2
$2^2 1^{n-4}$	2	1/4	5/8
$3^1 1^{n-3}$	4	0	1/4
$2^3 1^{n-6}$	2	1/8	9/16
$3^1 2^1 1^{n-5}$	4	0	1/4
$4^1 1^{n-4}$	4	1/8	3/8
$2^4 1^{n-8}$	2	1/16	17/32
$3^1 2^2 1^{n-7}$	4	0	1/4
$3^2 1^{n-6}$	4	1/16	11/32
$4^1 2^1 1^{n-6}$	4	1/16	11/32
$5^1 1^{n-5}$	8	0	1/8
$2^5 1^{n-10}$	2	1/32	33/64
$3^1 2^3 1^{n-9}$	4	0	1/4
$3^2 2^1 1^{n-8}$	4	1/32	21/64
$4^1 2^2 1^{n-8}$	4	1/32	21/64
$4^1 3^1 1^{n-7}$	4	1/32	19/64
$5^1 2^1 1^{n-7}$	8	0	1/8

$6^1 1^{n-6}$	8	1/32	3/16
$2^6 1^{n-12}$	2	1/64	65/128
$3^1 2^4 1^{n-11}$	4	1/64	1/4
$3^2 2^2 1^{n-10}$	4	1/64	41/128
$3^3 1^{n-9}$	4	1/64	37/128
$4^1 3^1 2^1 1^{n-9}$	4	1/64	37/128
$4^2 1^{n-8}$	4	1/64	35/128
$5^1 2^2 1^{n-9}$	8	0	1/8
$5^1 3^1 1^{n-8}$	8	0	1/8
$6^1 2^1 1^{n-8}$	8	1/64	23/128
$7^1 1^{n-7}$	8	1/64	5/32

Proof. The conclusion follows from (13.6) and (13.7) plus some direct calculation. For example, if g is of type $3^1 2^1 1^{n-5}$, then g^2 is a transvection. So $\mathcal{N}(g)=\mathcal{N}(g^2)=0$, and thus $\mathcal{U}(g)=1/4$. If g is of type $3^2 1^{n-6}$, then g^2 is of type $2^2 1^{n-4}$. So $\mathcal{U}(g)=\frac{1}{4}\{1+\phi(2)\mathcal{N}(g^2)+\phi(4)\mathcal{N}(g)\}\leq \frac{1}{4}(1+\frac{1}{4}+2\cdot\frac{1}{16})=\frac{11}{32}$.

(14.2) We have the following bounds for $\mathcal{U}(g)$:

$\|g\|$	$\mathcal{U}(g)$	$\|g\|$	$\mathcal{U}(g)$
3	43/128	5	13/64
6	81/384	7	131/896
9	11/96	10	41/320
11	133/1408	12	49/384
13	67/832	14	83/896
15	9/128	17	1/16
18	7/96	19	137/2432
20	5/64	21	23/448

Proof. These bounds follow from direct calculations. For example, suppose that $|g|=12$. Then $\mathcal{U}(g)=\frac{1}{12}\{1+\mathcal{N}(g^6)+2\mathcal{N}(g^3)+2\mathcal{N}(g^4)+2\mathcal{N}(g^2)+4\mathcal{N}(g)\}$. Since $|g^6|=2$, $\mathcal{N}(g^6)\leq\frac{1}{4}$. If g^3 is not of type $3^1 1^{n-3}$, then $\mathcal{N}(g^3)\leq\frac{1}{8}$. Thus $\mathcal{N}(g^6)+2\mathcal{N}(g^3)\leq\frac{1}{2}$. If g^3 is of type $3^1 1^{n-3}$, then g^6 is of type $2^1 1^{n-2}$ and $\mathcal{N}(g^3)\leq\frac{1}{4}$, $\mathcal{N}(g^6)=0$. Thus we still have $\mathcal{N}(g^6)+2\mathcal{N}(g^3)\leq\frac{1}{2}$. Also $\mathcal{N}(g)\leq\mathcal{N}(g^2)\leq\mathcal{N}(g^4)\leq\frac{1}{2^8}$. Hence $\mathcal{U}(g)\leq\frac{1}{12}\{1+\frac{1}{2}+(2+2+4)\frac{1}{2^8}\}=\frac{49}{384}$.

(14.3) $|S|=3$.

Proof. Let $g\in S$. If $|g|=2$, then $\mathcal{U}(g)\leq\frac{1}{2}(1+\frac{1}{4})=\frac{5}{8}$. If $|g|\geq3$, then $\mathcal{U}(g)\leq\frac{1}{|g|}+\mathcal{M}(g)\leq\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$. As $max\{\frac{5}{8},\frac{7}{12}\}<\frac{2}{3}$, by (2.4)(a), $|S|\leq5$.

Suppose $|S|=5$. Let α be the number of g's in S such that g is of type $2^2 1^{n-4}$. For $g\in S$ is not of type $2^2 1^{n-4}$, then $\mathcal{U}(g)\leq\frac{1}{5}+\frac{1}{4}=\frac{9}{20}$ if $|g|\geq5$; $\mathcal{U}(g)\leq\frac{3}{8}$ if $|g|=4$; $\mathcal{U}(g)\leq\frac{43}{129}$ if $|g|=3$; $\mathcal{U}(g)\leq\frac{9}{16}$ if $|g|=2$. Since $max\{\frac{9}{20},\frac{3}{8},\frac{43}{129},\frac{9}{16}\}=\frac{9}{16}$, $|S|-2<\sum_{g\in S}\mathcal{U}(g)$ implies that $3<\frac{5}{8}\alpha+\frac{9}{16}(5-\alpha)$. So $\alpha>3$, i.e., there are at least 4 elements in S are of type $2^2 1^{n-4}$. As $G=\langle g_1, g_2, g_3, g_4\rangle$, and for g of type $2^2 1^{n-4}$, $dim[V,g]=2$, (3.2)(a) supplies a contradiction.

Now assume that $|S|=4$. Let α be the number of involutions in S. Then $\alpha\leq3$. If $|g|\geq8$, then $\mathcal{U}(g)\leq\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$. Checking the tables in (14.1) and (14.2), as $max\{\frac{43}{129},\frac{3}{8},\frac{13}{64},\frac{81}{384},\frac{131}{896}\}=\frac{3}{8}$, we have that if $g\in S$ is not an involution, then $\mathcal{U}(g)\leq\frac{3}{8}$. Thus $|S|-2<\sum_{g\in S}\mathcal{U}(g)$ implies that $2<\frac{5}{8}\alpha+\frac{3}{8}(4-\alpha)$; which gives $\alpha>2$. Then $\alpha=3$ and thus (3.2)(a) supplies a contradiction. Therefore $|S|=3$.

In the following, we assume that $|g_1|=k$, $|g_2|=l$, $|g_3|=m$ with $k\leq l\leq m$.

(14.4) One of the following holds:

(a) S is of type (2,4,5) and $n\leq16$.

(b) S is of type (2,4,6) and $n\leq20$.

(c) S is of type (2,4,8) and $n\leq 16$.

(d) S is of type (2,3,8) and $n\leq 18$.

Proof. Since $\mathcal{M}(g)\leq\frac{1}{4}$ $\forall g\in G$, (2.6)(b) gives that $k\leq 8$. By the table in (14.2), we have that for $5\leq|g|\leq 11$, $\mathcal{U}(g)\leq max\{\frac{13}{64}, \frac{81}{384}, \frac{131}{896}, \frac{7}{32}, \frac{11}{96}, \frac{41}{320}, \frac{133}{1408}\}=\frac{7}{32}$. For $|g|\geq 12$, $\mathcal{U}(g)\leq\frac{1}{12}+\frac{1}{4}=\frac{1}{3}$. Hence if $k\geq 5$, we have $\sum_{g\in S}\mathcal{U}(g)\leq 1$, which is a contradiction. So $k=2$, 3, or 4.

Suppose $k=4$. For $|g|=4$, g^2 is of type $2^2 1^{n-4}$ iff g is of type $4^1 2^\alpha 1^\beta$ or $3^2 2^\alpha 1^\beta$, in either cases, the number of blocks $\nu\geq\frac{n}{2}-1$. Hence there are at most 2 g's in S such that their squares are both of type $2^2 1^{n-4}$. Suppose g_1, g_2 are such that g_1^2, g_2^2 are both of type $2^2 1^{n-4}$. Let ν_i be the number of blocks of g_i. Then $\nu_1\geq\frac{n}{2}-1$, $\nu_2\geq\frac{n}{2}-1$. So $n-\nu_1-1\geq\nu_2-1\geq\frac{n}{2}-2\geq 6$, and thus $\mathcal{U}(g_1)\leq\frac{1}{4}(1+\frac{1}{4}+2\frac{1}{2^6})=\frac{41}{128}$. Similarly $\mathcal{U}(g_2)\leq\frac{41}{128}$. If $|g|=4$ and g^2 is not of type $2^2 1^{n-4}$, then g is of type $4^a 3^b 2^c 1^d$, where $(a,b)\neq(0,2)$ and $(1,0)$. If $(a,b)=(0,1)$, then $\mathcal{U}(g)\leq\frac{1}{4}$ by the table in (14.1). If a=0 and $b\geq 3$, then $\mathcal{U}(g)\leq\frac{1}{4}(1+\frac{1}{8}+2\frac{1}{2^6})=\frac{37}{128}$. If $a=1$ and $b\geq 1$, then $\mathcal{U}(g)\leq\frac{1}{4}(1+\frac{1}{8}+2\frac{1}{2^5})=\frac{19}{64}$. If $a\geq 2$, then $\mathcal{U}(g)\leq\frac{1}{4}(1+\frac{1}{16}+2\frac{1}{2^5})=\frac{35}{128}$. So in any case we have that $\mathcal{U}(g_3)\leq\frac{19}{64}<\frac{5}{16}$. If $5\leq|g|\leq 15$, then by the table in (14.2), we have that $\mathcal{U}(g)\leq max\{\frac{13}{64}, \frac{81}{384}, \frac{131}{896}, \frac{7}{32}, \frac{11}{96}, \frac{41}{320}, \frac{133}{1408}, \frac{49}{384}, \frac{67}{832}, \frac{83}{896}, \frac{9}{128}\}=\frac{7}{32}<\frac{5}{16}$. If $|g|\geq 16$, then $\mathcal{U}(g)\leq\frac{1}{16}+\frac{1}{4}=\frac{5}{16}$. So we always have $\mathcal{U}(g_3)\leq\frac{5}{16}$, which implies $\sum\mathcal{U}(g_i)\leq 2\cdot\frac{41}{128}+\frac{5}{16}=\frac{61}{64}$, a contradiction. So we assume that only g_1 is such that g_1^2 of type $2^2 1^{n-4}$. Then $\mathcal{U}(g_1)\leq\frac{3}{8}$, which implies that $\sum\mathcal{U}(g_i)\leq\frac{3}{8}+2\cdot\frac{5}{16}=1$, a contradiction again. So no g_i is such that g_i^2 is of type $2^2 1^{n-4}$. Then $\sum\mathcal{U}(g_i)\leq 3\cdot\frac{5}{16}<1$, a contradiction. Therefore $k=2$ or 3.

Suppose $k=3$. If there is $g\in S$ is such that $|g|=4$, then as at least on eigenspace of g_1 in the splitting field of $min(g_1)$ has dimension greater than or equal to $\frac{n}{3}$, by (3.2), the number ν of blocks of g is at most $\frac{2n}{3}$. Thus $n-\nu-1\geq 5$ and $\mathcal{U}(g)\leq\frac{1}{4}(1+\frac{1}{4}+2\frac{1}{2^5})=\frac{21}{64}$. For g with $|g|\geq 5$, as before we have $\mathcal{U}(g)\leq\frac{5}{16}$. For g with $|g|=3$, $\mathcal{U}(g)\leq\frac{43}{128}$. As we can have at most 2 elements in S of order 3, $\sum\mathcal{U}(g_i)\leq 2\cdot\frac{43}{128}+\frac{21}{64}=1$, a contradiction. So $k=2$.

If $|g|$ odd, then $\mathcal{U}(g)\leq\frac{1}{|g|}+\frac{1}{2^8}$. If $|g|=2s$ with $(2,s)=1$, then $\mathcal{U}(g)=\frac{1}{|g|}\{1+\mathcal{N}(g^s)+\sum_{d|s}\phi(d)\mathcal{N}(g^{\frac{2s}{d}})\}\leq\frac{1}{|g|}(1+\frac{1}{4})+\frac{|g|-2}{|g|}\cdot\frac{1}{2^8}\leq\frac{5}{4}\cdot\frac{1}{|g|}+\frac{1}{2^8}$. If $|g|=4s$ with $(2,s)=1$, then $\mathcal{U}(g)=\frac{1}{|g|}\{1+\mathcal{N}(g^{2s})+2\mathcal{N}(g^s)+\sum_{d|s}\phi(d)\mathcal{N}(g^{\frac{4s}{d}})\}\leq\frac{1}{|g|}(1+\frac{1}{4}+2\cdot\frac{1}{8})+\frac{|g|-4}{|g|}\cdot\frac{1}{2^8}\leq\frac{3}{2}\cdot\frac{1}{|g|}+\frac{1}{2^8}$. If $8||g|$, then $\mathcal{U}(g)=\frac{1}{|g|}\{1+\mathcal{N}(g^{4s})+2\mathcal{N}(g^{2s})+4\mathcal{N}(g^s)+\sum_{d||g|,d\neq 1,2,4,8}\phi(d)\mathcal{N}(g^{\frac{|g|}{d}})\}\leq\frac{1}{|g|}(1+\frac{1}{4}+2\cdot\frac{1}{8}+4\cdot\frac{1}{32})+\frac{|g|-8}{|g|}\cdot\frac{1}{2^8}\leq\frac{13}{8}\cdot\frac{1}{|g|}+\frac{1}{2^8}$, as if d is divisible by some odd number, then $\mathcal{N}(g^{\frac{|g|}{d}})\leq\frac{1}{2^8}$; other $d=2^e$ for some $e\geq 4$, then $g^{\frac{|g|}{d}}$ has at least one block of size greater than or equal to 9; if it is size 9, then $\mathcal{N}(g^{\frac{|g|}{d}})=0$ by (13.6)(b); if it is of size at least 10, then $\mathcal{N}(g^{\frac{|g|}{d}})\leq\frac{1}{2^9}<\frac{1}{2^8}$. So in any case, we have $\mathcal{U}(g)\leq\frac{13}{8}\cdot\frac{1}{|g|}+\frac{1}{2^8}$, in particular, if $|g|\geq 9$, then $\mathcal{U}(g)\leq\frac{13}{8}\cdot\frac{1}{9}+\frac{1}{2^8}=\frac{425}{2304}$. This implies $1<\sum\mathcal{U}(g_i)\leq\frac{5}{8}+\frac{13}{8}(\frac{1}{|g_2|}+\frac{1}{|g_3|})+\frac{2}{2^8}$. Thus $\frac{1}{|g_2|}+\frac{1}{|g_3|}>\frac{47}{208}$. So $|g_2|\leq 8$. We consider the case $5\leq|g_2|\leq 8$ first. Suppose g_1 is of type $2^2 1^{n-4}$. If $|g_2|=8$, then $\frac{n}{8}\leq dim\{C_V(g_2)\}\leq n-dim\{C_V(g_1)\}=2$ implies that $n=16$ and g_2 is of type 8^2. Then $\mathcal{U}(g_2)\leq\frac{1}{8}(1+\frac{1}{2^7}+2\cdot\frac{1}{2^{11}}+4\cdot\frac{1}{2^{13}})=\frac{2067}{2^{14}}<\frac{425}{2304}$. Thus $\sum\mathcal{U}(g_i)\leq\frac{5}{8}+\frac{2067}{2^{14}}+\frac{425}{2304}\leq 1$, a contradiction. It is impossible that $|g_2|=5$ or 7, as $dim\{C_V(g_1)\}=n-2$ and g_2 has an eigenspace of dimension at least 3. Suppose $|g_2|=6$. If g_2^3 is of type $2^2 1^{n-4}$ or $2^3 1^{n-6}$, then $dim\{C_V(g_1)\cap C_V(g_2^3)\}=n-4$, $n-5$ respectively, which contradicts to g_2^{-2} has an eigenspace of dimension at least 6. So g_2^3 is not of type $2^2 1^{n-4}$ or $2^3 1^{n-6}$, and thus $\mathcal{N}(g_2^3)\leq\frac{1}{16}$, which implies that $\mathcal{U}(g_2)\leq\frac{1}{6}(1+\frac{1}{16}+2\cdot\frac{1}{2^8}+2\cdot\frac{1}{2^8})=\frac{23}{128}$. Then $\sum\mathcal{U}(g_i)\leq\frac{5}{8}+2\cdot\frac{23}{128}<1$, a contradiction. So g_1 is not type $2^2 1^{n-4}$, thus $\mathcal{U}(g_1)\leq\frac{9}{16}$, and also $\mathcal{U}(g_2)$ and $\mathcal{U}(g_3)\leq max\{\frac{13}{64},\frac{81}{384},\frac{131}{896},\frac{13}{64},\frac{425}{2304}\}=\frac{81}{384}$, which implies that $\sum\mathcal{U}(g_i)\leq\frac{9}{16}+2\cdot\frac{81}{384}<1$, a contradiction. So $l=3$ or 4.

Suppose $|g_2|=4$. Let ν_i be the number of blocks of g_i, $1\leq i\leq 2$. Since $n-\nu_1\geq\nu_2\geq\frac{n}{4}\geq 4$, $\mathcal{U}(g_1)\leq\frac{1}{2}(1+\frac{1}{2^4})=\frac{17}{32}$. Also as $n-\nu_2\geq\nu_1\geq\frac{n}{2}\geq 8$, $\mathcal{N}(g_2)\leq\frac{1}{2^7}$; which gives $\mathcal{U}(g_2)\leq\frac{1}{4}(1+\frac{1}{4}+\frac{2}{2^7})=\frac{81}{256}$. For $|g_3|\geq 11$, $\mathcal{U}(g_3)\leq\frac{13}{8}\cdot\frac{1}{11}+\frac{1}{2^8}=\frac{427}{2816}$. So for $|g_3|=7$, 9, 10 or $|g_3|\geq 11$, $\mathcal{U}(g_3)\leq max\{\frac{131}{896},\frac{11}{96},\frac{41}{320},\frac{427}{2816}\}=\frac{427}{2816}$, which implies that

$\sum \mathcal{U}(g_i) \leq \frac{17}{32}+\frac{81}{256}+\frac{427}{2816}=\frac{1407}{1408}<1$, a contradiction. So $|g_3|=5$, 6, or 8.

Suppose S is of type (2,4,5) and $n \geq 17$. As before, $n-\nu_1 \geq \nu_2 \geq 5$ implies $\mathcal{U}(g_1) \leq \frac{1}{2}(1+\frac{1}{2^5})=\frac{33}{64}$. Since g_3 has an eigenspace of dimension $d \geq 4$, by (3.2), we have that α, the number of blocks of g_2^2 is such that $n-\alpha \geq d \geq 4$, which implies that $\mathcal{N}(g_2^2) \leq \frac{1}{2^4}$. As $n-\nu_2 \geq \nu_1 \geq 9$, $\mathcal{N}(g_2) \leq \frac{1}{2^8}$. So $\mathcal{U}(g_2) \leq \frac{1}{4}(1+\frac{1}{2^4}+\frac{2}{2^8})=\frac{137}{512}$. Thus $\sum \mathcal{U}(g_i) \leq \frac{33}{64}+\frac{137}{512}+\frac{13}{64}=\frac{505}{512}<1$, a contradiction.

Suppose S is of type (2,4,6) and $n \geq 21$. As before, $n-\nu_1 \geq \nu_2 \geq 6$ implies $\mathcal{U}(g_1) \leq \frac{1}{2}(1+\frac{1}{2^6})=\frac{65}{128}$. Let α, β be the number of blocks of g_2^2, g_3^3 respectively. Since g_3^{-2} has an eigenspace of dimension $d \geq 7$, by (3.2), $n \geq dim\{C_V(g_2^2) \cap C_V(g_3^3)\}+d \geq \alpha+\beta-n+d$. So $(n-\alpha)+(n-\beta) \geq d$. For any t such that $t=(n-\alpha)+(n-\beta) \geq d$, it is easy to check that $\frac{1}{4}\mathcal{N}(g_2^2)+ \frac{1}{6}\mathcal{N}(g_3^3) \leq \frac{1}{4}\cdot\frac{1}{4}+\frac{1}{6}\cdot\frac{1}{2^{t-2}} \leq \frac{1}{16}+\frac{1}{6}\cdot\frac{1}{2^{d-2}} \leq \frac{1}{16}+\frac{1}{6}\cdot\frac{1}{2^5}=\frac{13}{192}$. As $n-\nu_2 \geq \nu_1 \geq 11$, $\mathcal{N}(g_2) \leq \frac{1}{2^{10}}$. So $\mathcal{U}(g_2)+ \mathcal{U}(g_3)=(\frac{1}{4}+\frac{1}{6})+\frac{2}{4}\mathcal{N}(g_2)+\{\frac{1}{4}\mathcal{N}(g_2^2)+\frac{1}{6}\mathcal{N}(g_3^3)\}+\frac{1}{6}\{2\mathcal{N}(g_3^2)+2\mathcal{N}(g_3)\} \leq \frac{5}{12}+\frac{1}{2^{11}}+\frac{13}{192}+\frac{1}{6}\cdot\frac{4}{2^8}=\frac{2995}{6144}$. Hence $\sum \mathcal{U}(g_i) \leq \frac{65}{128}+\frac{2995}{6144}=\frac{6115}{6144}<1$, a contradiction.

Suppose S is of type (2,4,8) and $n \geq 17$. As before, $n-\nu_1 \geq \nu_2 \geq 5$ implies $\mathcal{U}(g_1) \leq \frac{1}{2}(1+\frac{1}{2^5})=\frac{33}{64}$. Let α, β be the number of blocks of g_2^2, g_3^2 respectively. By (3.2), we have that $n-\alpha \geq dim\{C_V(g_3)\} \geq 3$ and $n-\beta \geq dim\{C_V(g_2)\} \geq 5$. Thus $\mathcal{N}(g_2^2) \leq \frac{1}{2^3}$ and $\mathcal{N}(g_3^2) \leq \frac{1}{2^5}$. Also as $n-\nu_2 \geq \nu_1 \geq 9$, and $n-\nu_3 \geq \nu_1 \geq 9$, we have $\mathcal{N}(g_2) \leq \frac{1}{2^8}$ and $\mathcal{N}(g_3) \leq \frac{1}{2^9}$. Thus $\mathcal{U}(g_2) \leq \frac{1}{4}(1+\frac{1}{2^3}+\frac{2}{2^8})=\frac{145}{512}$ and $\mathcal{U}(g_3) \leq \frac{1}{8}(1+\frac{1}{4}+\frac{2}{2^5}+\frac{4}{2^9})=\frac{169}{1024}$. Then $\sum \mathcal{U}(g_i) \leq \frac{33}{64}+\frac{145}{512}+\frac{169}{1024}= \frac{987}{1024}<1$, a contradiction.

Suppose $|g_2|=3$. As g_2 has an eigenspace of dimension $d \geq 6$, $\mathcal{U}(g_1) \leq \frac{1}{2}(1+\frac{1}{2^6})=\frac{65}{128}$. For $|g_3|=7$, 9, 10 or $|g_3| \geq 11$, as before $\mathcal{U}(g_3) \leq \frac{427}{2816}$, which implies that $\sum \mathcal{U}(g_i) \leq \frac{65}{128}+\frac{43}{128}+\frac{427}{2816}=\frac{2803}{2816}<1$, a contradiction. So $|g_3|=8$.

Suppose S is of type (2,3,8) and $n \geq 19$. Since g_2 has an eigenspace of dimension $d \geq 7$, $\mathcal{U}(g_1) \leq \frac{1}{2}(1+\frac{1}{2^7})=\frac{129}{256}$. Let α be the number of blocks of g_3^2. By (3.2), we have that $n-\alpha \geq d \geq 7$. Thus $\mathcal{N}(g_3^2) \leq \frac{1}{2^7}$. Also as $n-\nu_3 \geq \nu_1 \geq 10$, we have $\mathcal{N}(g_3) \leq \frac{1}{2^{10}}$. So

$\mathcal{U}(g_3)\leq\frac{1}{8}(1+\frac{1}{4}+\frac{2}{2^7}+\frac{4}{2^{10}})=\frac{325}{2048}$. Then $\sum\mathcal{U}(g_i)\leq\frac{129}{256}+\frac{43}{128}+\frac{325}{2048}=\frac{2045}{2048}<1$, a contradiction.

(14.5) If $q=2$ and $n\geq 21$, then $\overline{G}$ is not a group of genus zero.

Proof. This follows from (14.4) clearly.

Case 2: $q=3$.

Unless explicitly specified, we assume that $n\geq 10$ in the following.

(14.6) $\mathcal{N}(g)\leq\frac{1}{85}$, unless g is listed in the following table. Column 3 and 4 list upper bounds for $\mathcal{N}(g)$, $\mathcal{U}(g)$ respectively.

type	$\lvert g\rvert$	$\mathcal{N}(g)$	$\mathcal{U}(g)$
$2^1 1^{n-2}$	3	0	1/3
$2^2 1^{n-4}$	3	1/9	11/27
$3^1 1^{n-3}$	3	1/9	11/27
$2^3 1^{n-6}$	3	1/27	29/81
$3^1 2^1 1^{n-5}$	3	1/27	29/81
$4^1 1^{n-4}$	9	0	1/9
$2^4 1^{n-8}$	3	1/81	83/243
$3^1 2^2 1^{n-7}$	3	1/81	83/243
$3^2 1^{n-6}$	3	1/81	83/243
$4^1 2^1 1^{n-6}$	9	0	1/9
$5^1 1^{n-5}$	9	1/81	35/243
2^5	3	1/81	83/243

Proof. These follow from (13.6) and (13.7) directly.

(14.7) We have the following bounds for $\mathcal{U}(g)$.

$\lvert g\rvert$	$\mathcal{U}(g)$	$\lvert g\rvert$	$\mathcal{U}(g)$

2	122/243	4	41/162
5	247/1215	6	50/243
7	83/567	8	125/972
9	35/243	10	14/135
11	23/243	12	17/162
13	85/1053	14	128/1701
15	103/1215	16	43/648
17	259/4131	18	26/243
19	29/513	20	131/2430
21	5/81	22	4/81

Proof. For example, suppose $|g|=9$. Then $\mathcal{U}(g)=\frac{1}{9}\{1+2\mathcal{N}(g^3)+6\mathcal{N}(g)\}$. First g^3 cannot be of type $3^1 1^{n-3}$. Also g^3 is of type $2^2 1^{n-4}$ iff g has one block of dimension 5 and the rest of the blocks are all of dimension less than or equal to 3, in which case, $2\mathcal{N}(g^3)+6\mathcal{N}(g)\leq\frac{2}{9}+\frac{6}{3^4}=\frac{8}{27}$. If g^3 is not of type $2^2 1^{n-4}$, then $2\mathcal{N}(g^3)+6\mathcal{N}(g)\leq\frac{2}{27}+\frac{6}{3^3}=\frac{8}{27}$. Hence we always have $\mathcal{U}(g)\leq\frac{1}{9}(1+\frac{8}{27})=\frac{35}{243}$.

(14.8) $|S|=3$.

Proof. Since for $|g|\geq 3$, we have $\mathcal{U}(g)\leq\frac{1}{3}+\frac{1}{9}=\frac{4}{9}$; for $|g|=2$, $\mathcal{U}(g)\leq\frac{122}{243}$; and $max\{\frac{4}{9}, \frac{122}{243}\}<\frac{3}{5}$, by (2.4)(b), $|S|\leq 4$. If $|S|=4$, then at least one $g\in S$ is not an involution. So $\sum_{g\in S}\mathcal{U}(g)\leq 3\cdot\frac{122}{243}+\frac{4}{9}<2=|S|-2$, a contradiction. Hence $|S|=3$.

In the following, we assume that $|g_1|=k$, $|g_2|=l$, $|g_3|=m$ with $k\leq l\leq m$.

(14.9) If $q=3$ and $n\geq 10$, then $\overline{G}$ is not a group of genus zero.

Proof. Suppose $k\geq 4$. As $\mathcal{N}(g)\leq\frac{1}{9}$, we have $\mathcal{U}(g)\leq\frac{1}{5}+\frac{1}{9}=\frac{14}{45}$ $\forall g\in S$ with $|g|\geq 5$. Also for $|g|=4$, $\mathcal{U}(g)\leq\frac{41}{162}<\frac{14}{45}$. Thus $\sum_{g\in S}\mathcal{U}(g)\leq 3\cdot\frac{14}{45}<1=|S|-2$, a contradiction. So $k=2$, or 3.

Suppose $k=3$. Since for $|g|\geq 6, \mathcal{U}(g)\leq\frac{1}{6}+\frac{1}{9}=\frac{5}{18}$, if $|g|\geq 4$, then $\mathcal{U}(g)\leq max\{\frac{41}{162}, \frac{247}{1215}, \frac{5}{18}\}=\frac{5}{18}$. Thus if $l\geq 4$, then $\sum\mathcal{U}(g_i)\leq\frac{11}{27}+2\cdot\frac{5}{18}=\frac{26}{27}<1$, a contradiction. So $l=3$ and $m\geq 4$.

Suppose $|g_1|=|g_2|=3$. Let ν_i be the number of blocks of g_i. As $n-\nu_i\geq\nu_j\geq 4$, where $\{i,j\}=\{1,2\}$, $\mathcal{N}(g_i)\leq\frac{1}{3^4}$ for $i=1,2$. So $\mathcal{U}(g_i)\leq\frac{1}{3}(1+\frac{2}{3^4})=\frac{83}{243}$ for $i=1$ and 2. Then $\sum\mathcal{U}(g_i)\leq 2\cdot\frac{83}{243}+\frac{5}{18}=\frac{467}{486}<1$, a contradiction. So $k=2$.

Suppose $|g_1|=2$ and $|g_2|\geq 4$. For $|g|\geq 8$, $\mathcal{U}(g)\leq\frac{1}{8}+\frac{1}{9}=\frac{17}{72}$. Hence for $|g|\geq 5$, $\mathcal{U}(g)\leq max\{\frac{247}{1215}, \frac{50}{243}, \frac{83}{567}, \frac{17}{72}\}=\frac{17}{72}$. Since among g_2, g_3, there are at most one of them is order 4, we have $\sum\mathcal{U}(g_i)\leq\frac{122}{243}+\frac{41}{162}+\frac{17}{72}<1$, a contradiction. So $|g_2|=3$.

Suppose $|g_1|=2$ and $|g_2|=3$. Since g_1 has an eigenspace of dimension $d\geq\frac{n}{2}\geq 5$, $n-\nu_2\geq 5$, where ν_2 is the number of blocks of g_2. Thus $\mathcal{N}(g_2)\leq\frac{1}{3^4}$ and $\mathcal{U}(g_2)\leq\frac{1}{3}(1+\frac{2}{3^4})=\frac{83}{243}$. If $|g_3|\geq 23$, then $\mathcal{U}(g_3)\leq\frac{1}{23}+\frac{1}{9}=\frac{32}{207}$. For $7\leq|g_2|\leq 22$, from the table in (14.7), we have $\mathcal{U}(g)\leq max\{\frac{83}{567}, \frac{125}{972}, \frac{35}{243}, \frac{14}{135}, \frac{23}{243}, \frac{17}{162}, \frac{85}{1053}, \frac{128}{1701}, \frac{103}{1215}, \frac{43}{648}, \frac{259}{4131}, \frac{26}{243}, \frac{29}{513}, \frac{131}{2430}, \frac{5}{81}, \frac{4}{81}\}=\frac{83}{567}<\frac{32}{207}$. Thus $\sum\mathcal{U}(g_i)\leq\frac{122}{243}+\frac{83}{243}+\frac{32}{207}<1$, a contradiction.

Case 3: $q=4$.

Unless explicitly specified, we assume that $n\geq 8$ in the following.

(14.10) $\mathcal{N}(g)\leq\frac{1}{85}$, unless g is listed in the following table. Column 3 and 4 list upper bounds for $\mathcal{N}(g)$, $\mathcal{U}(g)$ respectively.

type	$\lvert g\rvert$	$\mathcal{N}(g)$	$\mathcal{U}(g)$
$2^1 1^{n-2}$	2	0	1/2
$2^2 1^{n-4}$	2	1/16	17/32
$3^1 1^{n-3}$	4	0	1/4
$2^3 1^{n-6}$	2	1/64	65/128
$3^1 2^1 1^{n-5}$	4	0	1/4
$4^1 1^{n-4}$	4	1/64	35/128

2^4	2	1/64	65/128

(14.11) We have the following bounds for $\mathcal{U}(g)$.

$\|g\|$	$\mathcal{U}(g)$	$\|g\|$	$\mathcal{U}(g)$
3	43/128	5	13/64
6	23/128	7	131/896
8	275/2048	9	11/96
10	7/64	11	133/1408
12	3/32	13	67/832
14	71/896	15	9/128

Proof. For example, suppose $|g|=8$. Then $\mathcal{U}(g)=\frac{1}{8}\{1+\mathcal{N}(g^4)+2\mathcal{N}(g^2)+4\mathcal{N}(g)\}$. If g is of type $5^1 4^a 3^b 2^c 1^d$, then as g^4 is a transvection, $\mathcal{N}(g)=\mathcal{N}(g^2)=\mathcal{N}(g^4)=0$. So $\mathcal{U}(g)=\frac{1}{8}$. If g is of type $5^x 4^a 3^b 2^c 1^d$ with $x\geq 2$, then $\mathcal{N}(g^4)\leq\frac{1}{16}$, $\mathcal{N}(g^2)\leq\frac{1}{4^5}$, $\mathcal{N}(g)\leq\frac{1}{4^7}$. So $\mathcal{U}(g)\leq\frac{4361}{32768}$. If g is of type $6^x 5^a 4^b\cdots$ with $x\geq 1$, then $\mathcal{N}(g^4)\leq\frac{1}{16}$, $\mathcal{N}(g^2)\leq\frac{1}{4^4}$, $\mathcal{N}(g)\leq\frac{1}{4^5}$. So $\mathcal{U}(g)\leq\frac{275}{2048}$. If g is of type $7^x 6^a 5^b\cdots$ with $x\geq 1$, then $\mathcal{N}(g^4)\leq\frac{1}{64}$, $\mathcal{N}(g^2)\leq\frac{1}{4^5}$, $\mathcal{N}(g)\leq\frac{1}{4^6}$. So $\mathcal{U}(g)\leq\frac{1043}{8192}$. If g is of type $8^x 7^a 6^b\cdots$ with $x\geq 1$, then $\mathcal{N}(g^4)\leq\frac{1}{4^4}$, $\mathcal{N}(g^2)\leq\frac{1}{4^6}$, $\mathcal{N}(g)\leq\frac{1}{4^7}$. So $\mathcal{U}(g)\leq\frac{4115}{32768}$. As $max\{\frac{1}{8}, \frac{4361}{32768}, \frac{275}{2048}, \frac{1043}{8192}, \frac{4115}{32768}\}=\frac{275}{2048}$, we always have $\mathcal{U}(g)\leq\frac{275}{2048}$.

(14.12) $|S|=3$.

Proof. By (2.4)(b), as $\mathcal{N}(g)\leq\frac{1}{16}\,\forall g\in G$, $|S|\leq 4$. Suppose $|S|=4$. As S has at most 3 involutions, and for $|g|\geq 3$, $\mathcal{U}(g)\leq\frac{1}{3}+\frac{1}{16}=\frac{19}{48}$, so $\sum\mathcal{U}(g_i)\leq 3\cdot\frac{17}{32}+\frac{19}{48}=\frac{191}{96}<2=|S|-2$, a contradiction. So $|S|=3$.

In the following, we assume that $|g_1|=k$, $|g_2|=l$, $|g_3|=m$ with $k\leq l\leq m$.

(14.13) If $q=4$ and $n\geq 8$, then $\overline{G}$ is not a group of genus zero.

Proof. Suppose $k\geq 3$. For $|g|=4$, $\mathcal{U}(g)\leq\frac{35}{128}$. As $\mathcal{N}(g)\leq\frac{1}{16}$, we have $\mathcal{U}(g)\leq\frac{1}{5}+\frac{1}{16}=\frac{21}{80}<\frac{35}{128}$ $\forall g\in S$ with $|g|\geq 5$. Since there are at most 2 elements in S which are of order 3, $\sum_{g\in S}\mathcal{U}(g)\leq 2\cdot\frac{43}{128}+\frac{35}{128}=\frac{121}{128}<1=|S|-2$, a contradiction. So $k=2$.

Suppose $k=2$. Since for $|g|\geq 6, \mathcal{U}(g)\leq\frac{1}{6}+\frac{1}{16}=\frac{11}{48}$, if $|g|\geq 5$, then $\mathcal{U}(g)\leq max\{\frac{13}{64}, \frac{11}{48}\}=\frac{11}{48}$. Thus if $l\geq 5$, then $\sum\mathcal{U}(g_i)\leq\frac{17}{32}+2\cdot\frac{11}{48}=\frac{95}{96}<1$, a contradiction. So $l=3$ or 4.

Suppose $|g_2|=4$. Since for $|g|\geq 8, \mathcal{U}(g)\leq\frac{1}{8}+\frac{1}{16}=\frac{3}{16}$, if $|g|\geq 5$, then $\mathcal{U}(g)\leq max\{\frac{13}{64}, \frac{23}{128}, \frac{131}{896}, \frac{3}{16}\}=\frac{13}{64}$. Let ν_i be the number of blocks of g_i. If g_1 is of type $2^2 1^{n-4}$, then $\nu_1+\nu_2\leq n$, $|g_2|=4$ and $n\geq 8$ implies $n=8$ and g_2 is of type 4^2. So in this case $\mathcal{U}(g_1)+\mathcal{U}(g_2)\leq\frac{1}{2}(1+\frac{1}{16})+\frac{1}{4}(1+\frac{1}{4^3}+\frac{2}{4^5})=\frac{1609}{2048}$, which implies $\sum\mathcal{U}(g_i)\leq\frac{1609}{2048}+\frac{13}{64}=\frac{2025}{2048}<1$, a contradiction. So g_1 is not of type $2^2 1^{n-4}$, then $\mathcal{U}(g_1)\leq\frac{1}{2}(1+\frac{1}{4^3})=\frac{65}{128}$, which implies $\sum\mathcal{U}(g_i)\leq\frac{65}{128}+\frac{35}{128}+\frac{13}{64}=\frac{63}{64}<1$, a contradiction again. So $|g_2|=3$.

Suppose $|g_1|=2$ and $|g_2|=3$. For $|g_3|\geq 11$, $\mathcal{U}(g_3)\leq\frac{1}{11}+\frac{1}{16}=\frac{27}{176}$. Hence for $|g_3|\geq 7$, $\mathcal{U}(g_3)\leq max\{\frac{131}{896}, \frac{275}{2048}, \frac{11}{96}, \frac{7}{64}, \frac{27}{176}\}=\frac{27}{176}$. Since g_2 has an eigenspace of dimension $d\geq 3$, $n-\nu_1\geq 3$, where ν_1 is the number of blocks of g_1. Thus $\mathcal{N}(g_1)\leq\frac{1}{4^3}$ and $\mathcal{U}(g_1)\leq\frac{1}{2}(1+\frac{1}{4^3})=\frac{65}{128}$. Then $\sum\mathcal{U}(g_i)\leq\frac{65}{128}+\frac{43}{128}+\frac{27}{176}=\frac{351}{352}<1$, a contradiction.

Case 4: $q=5$.

Unless explicitly specified, we assume that $n\geq 6$ in the following.

(14.14) $\mathcal{N}(g)\leq\frac{1}{85}$, unless g is listed in the following table. Column 3 and 4 list upper bounds for $\mathcal{N}(g)$, $\mathcal{U}(g)$ respectively.

type	$\|g\|$	$\mathcal{N}(g)$	$\mathcal{U}(g)$
$2^2 1^{n-4}$	5	1/25	29/125
$3^1 1^{n-3}$	5	1/25	29/125
2^3	5	1/25	29/125

(14.15) We have the following bounds for $\mathcal{U}(g)$.

$\lvert g\rvert$	$\mathcal{U}(g)$	$\lvert g\rvert$	$\mathcal{U}(g)$
2	63/125	3	127/375
4	32/125	6	13/75
7	131/875	8	33/250
9	133/1125	10	3/25

Proof. For example, suppose $|g|=10$. Then $\mathcal{U}(g)=\frac{1}{10}\{1+\mathcal{N}(g^5)+4\mathcal{N}(g^2)+4\mathcal{N}(g)\}$. Since $\mathcal{N}(g^5)\leq\frac{1}{5^3}$, $\mathcal{N}(g^2)\leq\frac{1}{5^2}$, $\mathcal{N}(g)\leq\frac{1}{5^3}$, we have $\mathcal{U}(g)\leq\frac{3}{25}$.

(14.16) $|S|=3$.

Proof. By (2.4)(b), as $\mathcal{N}(g)\leq\frac{1}{25}\,\forall g\in G$, $|S|\leq 4$. Suppose $|S|=4$. As S has at most 3 involutions, and for $|g|\geq 3$, $\mathcal{U}(g)\leq\frac{1}{3}+\frac{1}{25}=\frac{28}{75}$, so $\sum\mathcal{U}(g_i)\leq 3\cdot\frac{63}{125}+\frac{28}{75}=\frac{707}{375}<2=|S|-2$, a contradiction. So $|S|=3$.

(14.17) If $q=5$ and $n\geq 6$, then $\overline{G}$ is not a group of genus zero.

Proof. Suppose $k\geq 3$. As $\mathcal{N}(g)\leq\frac{1}{25}$, we have $\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{25}=\frac{29}{100}$ $\forall g\in S$ with $|g|\geq 4$. Also there are at most 2 elements in S which are of order 3. Hence $\sum_{g\in S}\mathcal{U}(g)\leq 2\cdot\frac{127}{375}+\frac{29}{100}=\frac{1451}{1500}<1=|S|-2$, a contradiction. So $k=2$.

Suppose $k=2$. Since for $|g|\geq 5, \mathcal{U}(g)\leq\frac{1}{5}+\frac{1}{25}=\frac{6}{25}$, if $l\geq 5$, then $\sum\mathcal{U}(g_i)\leq\frac{63}{125}+2\cdot\frac{6}{25}=\frac{123}{125}<1$, a contradiction. So $l=3$ or 4.

Suppose $|g_2|=4$. Then $|g_3|\geq 5$. Hence $\sum\mathcal{U}(g_i)\leq\frac{63}{125}+\frac{32}{125}+\frac{6}{25}=1$, a contradiction.

Suppose $|g_2|=3$. For $|g_3|\geq 9$, $\mathcal{U}(g)\leq\frac{1}{9}+\frac{1}{25}=\frac{34}{225}$. Hence for $|g_3|\geq 7$, $\mathcal{U}(g)\leq max\{\frac{131}{875}, \frac{33}{250}, \frac{34}{225}\}=\frac{34}{225}$. Thus we have $\sum\mathcal{U}(g_i)\leq\frac{63}{125}+\frac{127}{375}+\frac{34}{225}=\frac{1118}{1125}<1$, a contradiction.

Case 5: $q=7$.

Unless explicitly specified, we assume that $n \geq 6$ in the following.

(14.18) $\mathcal{N}(g) \leq \frac{1}{85}$, unless g is listed in the following table. Column 3 and 4 list upper bounds for $\mathcal{N}(g)$, $\mathcal{U}(g)$ respectively.

type	$\lvert g\rvert$	$\mathcal{N}(g)$	$\mathcal{U}(g)$
$2^2 1^{n-4}$	7	1/49	55/343
$3^1 1^{n-3}$	7	1/49	55/343
2^3	7	1/49	55/343

(14.19) We have the following bounds for $\mathcal{U}(g)$.

$\lvert g\rvert$	$\mathcal{U}(g)$	$\lvert g\rvert$	$\mathcal{U}(g)$
2	257/512	3	43/128
4	259/1024	5	13/64
6	87/512	8	263/2048

Proof. For example, suppose $|g|=6$. Then $\mathcal{U}(g)=\frac{1}{6}\{1+\mathcal{N}(g^3)+2\mathcal{N}(g^2)+3\mathcal{N}(g)\}$. Since $\mathcal{N}(g^3) \leq \frac{1}{2^8}$, $\mathcal{N}(g^2) \leq \frac{1}{2^8}$, $\mathcal{N}(g) \leq \frac{1}{2^8}$, we have $\mathcal{U}(g) \leq \frac{87}{512}$.

(14.20) $|S|=3$.

Proof. By (2.4)(b), as $\mathcal{N}(g) \leq \frac{1}{49} \forall g \in G$, $|S| \leq 4$. Suppose $|S|=4$. As S has at most 3 involutions, and for $|g| \geq 3$, $\mathcal{U}(g) \leq \frac{1}{3}+\frac{1}{49}=\frac{52}{147}$, so $\sum \mathcal{U}(g_i) \leq 3 \cdot \frac{257}{512}+\frac{52}{147}<2=|S|-2$, a contradiction. So $|S|=3$.

(14.21) If $q=7$ and $n \geq 6$, then $\overline{G}$ is not a group of genus zero.

Proof. Suppose $k \geq 3$. As $\mathcal{N}(g) \leq \frac{1}{49}$, we have $\mathcal{U}(g) \leq \frac{1}{4}+\frac{1}{49}=\frac{53}{196}$ $\forall g \in S$ with $|g| \geq 4$. Also there are at most 2 elements in S which are of order 3. Hence $\sum_{g \in S} \mathcal{U}(g) \leq 2 \cdot \frac{43}{128}+\frac{53}{196}<1=|S|-2$, a contradiction. So $k=2$.

Suppose $k=2$. Since for $|g| \geq 5, \mathcal{U}(g) \leq \frac{1}{5}+\frac{1}{49}=\frac{54}{245}$, if $l \geq 5$, then

$\sum\mathcal{U}(g_i)\leq\frac{257}{512}+2\cdot\frac{54}{245}<1$, a contradiction. So $l=3$ or 4.

Suppose $|g_2|=4$. Then $|g_3|\geq 5$. Hence $\sum\mathcal{U}(g_i)\leq\frac{257}{512}+\frac{259}{1024}+\frac{54}{245}<1$, a contradiction.

Suppose $|g_2|=3$. For $|g_3|\geq 8$, $\mathcal{U}(g)\leq\frac{1}{8}+\frac{1}{49}=\frac{57}{392}$. Hence for $|g_3|\geq 7$, $\mathcal{U}(g)\leq max\{\frac{55}{343}, \frac{57}{392}\}=\frac{55}{343}$. Thus we have $\sum\mathcal{U}(g_i)\leq\frac{257}{512}+\frac{43}{128}+\frac{55}{343}<1$, a contradiction.

Case 6: $q=8$.

Unless explicitly specified, we assume that $n\geq 6$ in the following.

(14.22) $\mathcal{N}(g)\leq\frac{1}{85}$, unless g is listed in the following table. Column 3 and 4 list upper bounds for $\mathcal{N}(g)$, $\mathcal{U}(g)$ respectively.

type	$\lvert g\rvert$	$\mathcal{N}(g)$	$\mathcal{U}(g)$
$2^2 1^{n-4}$	2	1/64	65/128
$3^1 1^{n-3}$	4	0	1/4
2^3	2	1/64	65/128

(14.23) We have the following bounds for $\mathcal{U}(g)$.

$\lvert g\rvert$	$\mathcal{U}(g)$	$\lvert g\rvert$	$\mathcal{U}(g)$
3	43/128	4	261/1024
5	13/64	6	11/64
7	131/896	8	1045/8192

Proof. For example, suppose $|g|=8$. Then $\mathcal{U}(g)=\frac{1}{8}\{1+\mathcal{N}(g^4)+2\mathcal{N}(g^2)+4\mathcal{N}(g)\}$. Since $\mathcal{N}(g^4)\leq\frac{1}{64}$, $\mathcal{N}(g^2)\leq\frac{1}{8^3}$, $\mathcal{N}(g)\leq\frac{1}{8^4}$, we have $\mathcal{U}(g)\leq\frac{1045}{8192}$.

(14.24) $|S|=3$.

Proof. By (2.4)(b), as $\mathcal{N}(g)\leq\frac{1}{64}\,\forall g\in G$, $|S|\leq 4$. Suppose $|S|=4$. As S has at most 3 involutions, and for $|g|\geq 3$, $\mathcal{U}(g)\leq\frac{1}{3}+\frac{1}{64}=\frac{67}{192}$, so $\sum\mathcal{U}(g_i)\leq 3\cdot\frac{65}{128}+\frac{67}{192}<2=|S|-2$, a contradiction. So

$|S|=3$.

(14.25) If $q=8$ and $n\geq 6$, then $\overline{G}$ is not a group of genus zero.

Proof. Suppose $k\geq 3$. As $\mathcal{N}(g)\leq\frac{1}{64}$, we have $\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{64}=\frac{17}{64}$ $\forall g\in S$ with $|g|\geq 4$. Also there are at most 2 elements in S which are of order 3. Hence $\sum_{g\in S}\mathcal{U}(g)\leq 2\cdot\frac{43}{128}+\frac{17}{64}<1=|S|-2$, a contradiction. So $k=2$.

Suppose $k=2$. Since for $|g|\geq 5, \mathcal{U}(g)\leq\frac{1}{5}+\frac{1}{64}=\frac{69}{320}$, if $l\geq 5$, then $\sum\mathcal{U}(g_i)\leq\frac{65}{128}+2\cdot\frac{69}{320}<1$, a contradiction. So $l=3$ or 4.

Suppose $|g_2|=4$. Then $|g_3|\geq 5$. Hence $\sum\mathcal{U}(g_i)\leq\frac{65}{128}+\frac{261}{1024}+\frac{69}{320}<1$, a contradiction.

Suppose $|g_2|=3$. For $|g_3|\geq 8$, $\mathcal{U}(g)\leq\frac{1}{8}+\frac{1}{64}=\frac{9}{64}$. Hence for $|g_3|\geq 7$, $\mathcal{U}(g)\leq max\{\frac{131}{896}, \frac{9}{64}\}=\frac{131}{896}$. Thus we have $\sum\mathcal{U}(g_i)\leq\frac{65}{128}+\frac{43}{128}+\frac{131}{896}<1$, a contradiction.

Case 7: $q=9$.

Unless explicitly specified, we assume that $n\geq 6$ in the following.

(14.26) $\mathcal{N}(g)\leq\frac{1}{85}$, unless g is listed in the following table. Column 3 and 4 list upper bounds for $\mathcal{N}(g)$, $\mathcal{U}(g)$ respectively.

type	$\|g\|$	$\mathcal{N}(g)$	$\mathcal{U}(g)$
$2^2 1^{n-4}$	3	1/81	83/243
$3^1 1^{n-3}$	3	1/81	83/243
2^3	3	1/81	83/243

(14.27) We have the following bounds for $\mathcal{U}(g)$.

$\|g\|$	$\mathcal{U}(g)$	$\|g\|$	$\mathcal{U}(g)$
2	257/512	4	259/1024
5	13/64	7	131/896

Proof. For example, suppose $|g|=4$. Then $\mathcal{U}(g)=\frac{1}{8}\{1+\mathcal{N}(g^2)+2\mathcal{N}(g)\}$. Since $\mathcal{N}(g^2)\leq\frac{1}{2^8}$, $\mathcal{N}(g)\leq\frac{1}{2^8}$, we have $\mathcal{U}(g)\leq\frac{259}{1024}$.

(14.28) $|S|=3$.

Proof. By (2.4)(b), as $\mathcal{N}(g)\leq\frac{1}{81}\,\forall g\in G$, $|S|\leq 4$. Suppose $|S|=4$. As S has at most 3 involutions, and for $|g|\geq 3$, $\mathcal{U}(g)\leq\frac{1}{3}+\frac{1}{81}=\frac{28}{81}$, so $\sum\mathcal{U}(g_i)\leq 3\cdot\frac{257}{512}+\frac{28}{81}<2=|S|-2$, a contradiction. So $|S|=3$.

(14.29) If $q=9$ and $n\geq 6$, then $\overline{G}$ is not a group of genus zero.

Proof. Suppose $k\geq 3$. As $\mathcal{N}(g)\leq\frac{1}{81}$, we have $\mathcal{U}(g)\leq\frac{1}{4}+\frac{1}{81}=\frac{85}{324}$ $\forall g\in S$ with $|g|\geq 4$. Also there are at most 2 elements in S which are of order 3. Hence $\sum_{g\in S}\mathcal{U}(g)\leq 2\cdot\frac{83}{243}+\frac{85}{324}<1=|S|-2$, a contradiction. So $k=2$.

Suppose $k=2$. Since for $|g|\geq 5, \mathcal{U}(g)\leq\frac{1}{5}+\frac{1}{81}=\frac{86}{405}$, if $l\geq 5$, then $\sum\mathcal{U}(g_i)\leq\frac{257}{512}+2\cdot\frac{86}{405}<1$, a contradiction. So $l=3$ or 4.

Suppose $|g_2|=4$. Then $|g_3|\geq 5$. Hence $\sum\mathcal{U}(g_i)\leq\frac{257}{512}+\frac{259}{1024}+\frac{86}{405}<1$, a contradiction.

Suppose $|g_2|=3$. For $|g_3|\geq 7$, $\mathcal{U}(g_3)\leq\frac{1}{7}+\frac{1}{81}=\frac{88}{567}$. Thus we have $\sum\mathcal{U}(g_i)\leq\frac{257}{512}+\frac{83}{243}+\frac{88}{567}<1$, a contradiction.

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