ON SPECTRAL PROPERTIES OF POSITIVE OPERATORS

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Abstract

This thesis deals with the spectral behavior of positive operators and related ones on Banach lattices. We first study the spectral properties of those positive operators that satisfy the so-called condition (c). A bounded linear operator T on a Banach space is said to satisfy the condition (c) if it is invertible and if the number 0 is in the unbounded connected component of its resolvent set $\rho(T)$. By using techniques in complex analysis and in operator theory, we prove that if T is a positive operator satisfying the condition (c) on a Banach lattice E then there exists a positive number a and a positive integer k such that $T^k \geq a \cdot I$, where I is the identity operator on E. As consequences of this result, we deduce some theorems concerning the behavior of the peripheral spectrum of positive operators satisfying the condition (c). In particular, we prove that if T is a positive operator with its spectrum contained in the unit circle Γ then either $\sigma(T) = \Gamma$ or $\sigma(T)$ is finite and cyclic and consists of k-th roots of unity for some k. We also prove that under certain additional conditions a positive operator with its spectrum contained in the unit circle will become an isometry. Another main result of this thesis is the decomposition theorem for disjointness preserving operators. We prove that under some natural conditions if T is a disjointness preserving operator on an order complete Banach lattice E such that its adjoint T' is also a disjointness preserving operator then there exists a family of T-reducing bands $\{E_n : n \geq 1\} \cup \{E_\infty\}$ of E such that $T|_{E_n}$ has strict period n and that $T|_{E_\infty}$ is aperiodic. We also prove that any disjointness preserving operator with its spectrum contained in a sector of angle less than π can be decomposed into a sum of a central operator and a quasi-nilpotent operator. Among other things we give some conditions under which an operator T lies in the center of the Banach lattice. Also discussed in this thesis are certain conditions under which a positive operator T with $\sigma(T) = \{1\}$ is greater than or equal to the identity operator I.

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Introduction

Positive operators originated at the beginning of the nineteenth century. They were closely related to integral operators (whose study led to the development of modern functional analysis) and to matrices with nonnegative entries. However, positive operators were not investigated in a systematic manner until the theory of Riesz spaces (vector lattices) was well established. The study of the spectral properties of positive operators began much later and followed closely the development of the theory of normed Riesz spaces (Banach function spaces, or Banach lattices). Although there were some earlier extensions to infinite dimensions of the classical Perron-Frobenius theorems (see [SH1]) for positive matrices (matrices with nonnegative entries), a systematic study of the spectral theory of positive operators was not initiated until the early 1960's. Thereafter, the latter theory developed rather rapidly and many important results were obtained. In 1974, H. H. Schaefer published his monograph *Banach Lattice and Positive Operators* [SH1], which is entirely devoted to the theory of positive operators and their spectral analysis. This book contains many important results up to the early 1970's. Since then, the spectral theory of positive operators has received considerable attention and has found many applications in pure and applied mathematics. For instance, the study of ergodic properties of positive operators is one of the important topics in ergodic theory (see [A], [N1] and [N2]), and the theory of semigroups of positive operators, which was established a decade ago, has found many applications in the theory of differential equations and the theory of probability (see [DA], [N1], [N2] and [OS]). Also, as a branch of operator theory in functional analysis, the spectral theory of positive operators is closely related to the theory of operator algebras, namely Banach algebras and C^*-algebras (see [N1] and [N2]).

The central problem in the study of spectral behavior of positive operators is to find out how the positivity interacts with other properties of the operators. It started with the Perron-Frobenius theory on positive matrices (matrices with non-negative entries), which was established around the turn of this century. One of the important results in this theory is the fact that the peripheral spectrum of a positive matrix A is cyclic in the sense that if λ is an eigenvalue such that $|\lambda| = r(A)$ (the spectral radius of A) and if $\lambda = |\lambda|e^{i\theta}$ then $|\lambda|e^{in\theta}$ is also an eigenvalue of A for any integer n. Now consider a positive operator T on a Banach lattice E. The subset $\mathrm{Per}\sigma(T) = \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$, where $r(T)$ denotes the spectral radius of T, is called the peripheral spectrum of T. It can be shown that $r(T)$ is always in $\mathrm{Per}\sigma(T)$. Many efforts have been made to show that the peripheral spectrum of a positve operator on a Banach lattice is cyclic. So far this problem is still open in its full generality. But some important partial results have been obtained. It is a deep result in this aspect that the whole spectrum of a lattice homomorphism is cyclic. Also if a positive operator satisfies the so-called growth condition then its peripheral spectrum is cyclic. We refer to [SH1] for proofs and some other information. In this thesis we show that if a positive operator T with $r(T) = 1$ satisfies the so-called condition (c) then there exists a positive integer k such that $\sigma(T^k) \cap \{z : |z| = 1\} = \{1\}$. From this result and the spectral mapping theorem we see that the peripheral spectrum of T is contained in the set of all k-th roots of unity. In particular, we show that if T is a positive operator such that $\sigma(T) \subseteq \Gamma$ (the unit circle) then either $\sigma(T) = \Gamma$ or $\sigma(T)$ is finite and cyclic and consists of k-th roots of unity for some k. So $\sigma(T)$ is cyclic in this case.

Another way to investigate the spectral properties of positive operators is to decompose the operators into relatively simpler parts. Some results of this aspect were already obtained in [A] and [AH]. The main result of [AH] asserts that any quasi-invertible disjointness preserving operator can be decomposed into a direct sum of its strictly periodic and aperiodic components. In this thesis we obtain this result in a more general setting by different approaches. Using this result we can prove some deep properties of disjointness preserving operators. We also show that a disjointness preserving operator with its spectrum contained in a sector of angle less than π can be decomposed into a sum of a central operator and a quasi-nilpotent operator. As consequences of these results we derive various properties of disjointness preserving operators.

Chapter 1

Banach Lattices and Positive Operators

This chapter contains an account of those basic aspects of the theory of Banach lattices and positive operators that are needed later in the study of the spectral properties of positive operators. All the results in the first two sections are well known and can be found in any standard textbook, for example, [SH1], [LZ2] and [Z]. One of the results in Section 3 on the contractivity of the projection associated with the center is new and the proof can be found in the indicated reference. In Section 4 we introduce the sign operator associated with a fixed element in a Banach lattice. The sign operator plays an important role in the study of the cyclic properties of the peripheral spectrum of positive operators.

§1.1. Basic Theory of Riesz Spaces

In this section we introduce some basic concepts in the theory of Riesz spaces that we need in the sequel. All these can be found in [LZ2] and [SH1]. Recall that an ordered vector space is a real vector space E equipped with an order relation $\leq$ (i.e., with a transitive, reflexive and antisymmetric relation) which is compatible with the algebraic structures of E in the sense that it satisfies the following two properties:

(1) If $x, y \in E$ satisfy that $x \leq y$, then $x + z \leq y + z$ for all $z \in E$;

(2) If $x, y \in E$ satisfy that $x \leq y$, then $\lambda x \leq \lambda y$ for all real numbers $\lambda \geq 0$.

Occasionally, the relation $y \geq x$ will be used to mean that $x \leq y$. The set $E_+ := \{x \in E : x \geq 0\}$ is called the positive cone of E.

From now on we assume E to be an ordered vector space. A nonempty subset A of E is said to have a supremum or a least upper bound if there exists an element u in E such that $a \leq u$ for all $a \in A$ and such that if v is some other element in E with the property that $a \leq v$ for all $a \in A$, then $u \leq v$. A similar definition can be given for the infimum or the greatest lower bound.

A real Riesz space (or a vector lattice) is an ordered vector space E with the additional property that the supremum (hence, the infimum) of every finite subset of E exists. We denote the supremum and the infimum of the two elements x, y by $x \vee y$ and $x \wedge y$ respectively.

For an element $x \in E$, the positive part of x is given by $x^+ = x \vee 0$, the negative part by $x^- = (-x) \vee 0$, and the absolute value by $|x| = x \vee (-x)$. Two elements $x, y \in E$ are said to be orthogonal or disjoint (denoted by $x \perp y$) if $|x| \wedge |y| = 0$. If A is a subset of E, then the orthogonal complement of A is given by $\{A\}^d = \{x \in E : x \perp y, \forall y \in A\}$.

A Riesz subspace of E is a linear subspace of E which is itself a Riesz space under the ordering induced by that of E. A linear subspace J of E is called an ideal of E if $y \in J, x \in E$ and $|x| \leq |y|$ imply that $x \in J$. A linear subspace of E is called a band if $B^{dd} = B$. Any band is an ideal, and any ideal is a Riesz subspace.

Definition 1.1.1. *A Riesz space E is called order complete (or Dedekind complete) if every nonempty order bounded above subset A (i.e., $\exists u \in E$ such that $x \leq u$ for all $x \in A$) has a supremum.*

A linear operator (or mapping) from a Riesz space E to another F is called positive if $Tx \geq 0$ in F whenever $x \geq 0$ in E. A linear operator from E to F is called regular if it is a difference of two positive linear operators. A linear operator from E to F is called order bounded if it maps order bounded subsets (i.e., order bounded from above and from below) in E into order bounded subsets in F. We will use $L^r(E,F)$ to denote the set of all regular operators. When $E = F$ we simply use $L^r(E)$. It is clear that any positive operator is regular, and any regular operator is order bounded. In general, order bounded operators are not regular, but if the range space F is order complete, then the situation will be different as the following result indicates. First notice that a natural order relation can be defined in $L^r(E,F)$ by saying that $T_1 \geq T_2$ if and only if $T_1 x \geq T_2 x$ for all $x \geq 0$ in E.

Proposition 1.1.2. *Let E and F be any two Riesz spaces such that F is order complete. Then any order bounded operator is regular, and $L^r(E,F)$ is an order complete Riesz space under the natural ordering just introduced above, and the following hold: for all $T_1, T_2 \in L^r(E,F)$ and for all $0 \leq u \in E$,*

(i) $(T_1 \vee T_2)(u) = \sup\{T_1(v) + T_2(w) : v, w \geq 0 \textit{ and } v + w = u\}$;

(ii) $(T_1 \wedge T_2)(u) = \inf\{T_1(v) + T_2(w) : v, w \geq 0 \textit{ and } v + w = u\}$.

A net $\{u_\alpha\}$ in E is called increasing (in symbols $u_\alpha \uparrow$) if $u_\beta \geq u_\alpha$ whenever $\beta \geq \alpha$. The symbol $u_\alpha \uparrow u$ means that $\{u_\alpha\}$ is an increasing net such that $\sup\{u_\alpha\} = u$. The symbol $u_\alpha \downarrow u$ has a similar meaning. Recall that a net $\{u_\alpha\}$ in E is said to be order convergent to $u \in E$ if there exists a decreasing net $\{v_\alpha\}$ with the same index and $v_\alpha \downarrow 0$ such that $|u_\alpha - u| \leq v_\alpha$ for all α. A linear operator from E into F is called order continuous if $\{Tu_\alpha\}$ converges to 0 in order in F whenever $\{u_\alpha\}$ converges to 0 in order in E.

A linear operator from E to F is called a lattice homomorphism (or a Riesz homomorphism) if $|Tx| = T|x|$ for all $x \in E$. Lattice homomorphisms can be characterized by some other conditions. Any of the following conditions is equivalent to the condition that T is a lattice homomorphism.

(1) $T(u \vee v) = T(u) \vee T(v)$ for all $u, v \in E$;

(2) $T(u \wedge v) = T(u) \wedge T(v)$ for all $u, v \in E$;

(3) T is positive and $Tu \perp Tv$ whenever $u \perp v$.

§1.2. Banach Lattices and Positive Operators

In this section we introduce some basic concepts concerning Banach lattices and positive operators. Most of the materials presented in this section are standard and can be found in [SH1].

Recall that a real Banach lattice E is a real Banach space and a Riesz space at the same time such that $|x| \leq |y|$ in E implies that $||x|| \leq ||y||$. To study the spectral properties of positive operators, we need to consider complex Banach lattices. They are defined to be the complexifications of the underlying real Banach lattices. Let E be a real Banach lattice and let $E_{\mathbf{C}} = E + iE$. Then $|z| = \sup\{|\cos\theta x + \sin\theta y| : 0 \leq \theta \leq 2\pi\}$ for each $z = x + iy \in E_{\mathbf{C}}$. For details, we refer to [SH1], pp.133-138. Let T be any linear map on $E_{\mathbf{C}}$. Then $T = T_1 + iT_2$, where T_1, T_2 map E into E and are said to be the real part and imaginary part of T respectively. In this case we write $T_1 = Re(T)$ and $T_2 = Im(T)$.

From now on all the Banach lattices will be complex Banach lattices, and the notions and concepts introduced for real cases will extend to the complex cases naturally. For details, we refer to [SH1]. From now on E will always denote a complex Banach lattice.

A Banach lattice E is said to have: (1) a Fatou norm if $0 \leq x_\alpha \uparrow x$ implies $||x_\alpha|| \uparrow ||x||$; (2) an order continuous norm if $x_\alpha \downarrow 0$ implies $||x_\alpha|| \downarrow 0$. It is easy to see that if E has an order continuous norm, then E has a Fatou norm. Moreover, the dual Banach lattice E' of any Banach lattice E has a Fatou norm and any $C(X)$-space has Fatou norm (see [SH1]).

Let B be a band in E. Then B is a closed ideal of E. If E is order complete, then B is also a complementary subspace of E such that $E = B \oplus B^d$, where the direct sum is topological. For details, we refer to [SH1].

Let $L(E)$ denote the Banach algebra of all bounded operators on E. Since any positive operator on E is bounded (see [SH1]), it is easy to see that $L^r(E)$ is a subalgebra of $L(E)$, which is not closed in general in the operator topology. If E is order complete, then it follows from Proposition 1.1.2 that $L^r(E)$ is an order complete Riesz space. If we define a new norm $||\cdot||_r$ on $L^r(E)$ by $||T||_r = ||\,|T|\,||$, then $L^r(E)$ will become a Banach lattice. This new norm is called the regular operator norm.

Let T be a bounded operator on E. We use $\rho(T)$, $\sigma(T)$ and $r(T)$ to denote the resolvent set, the spectrum and the spectral radius of T respectively. If E is order complete, then $L^r(E)$ is a Banach algebra under the regular norm, and the spectrum of an element $T \in L^r(E)$ in this Banach algebra is called the regular spectrum of T, which will be denoted by $\sigma_0(T)$. For more information about the regular spectrum, we refer to [SH2], in which this concept was first introduced. One can easily see that $\sigma_0(T) \supseteq \sigma(T)$. In general, the inclusion is strict. Later we will give some conditions under which they are equal. Finally let us quote a result which will be needed later in the study of the spectral properties of positive operators. It is one of the important features that positive operators possess. For a proof, we refer to [SH1], pp. 323.

Proposition 1.2.1. *Let T be a positive operator on a Banach lattice. Then $r(T) \in \sigma(T)$, i.e., the spectral radius of a positive operator is always in its spectrum.*

To study the cyclicity properties of the spectrum of positive operators we need the following well-known results.

Proposition 1.2.2. *Let T be a bounded linear operator on a Banach space X. If z_0 is a boundary point of the spectrum $\sigma(T)$ of T, then $\lim_{z\to z_0, z\in\rho(T)} ||R(z,T)|| = \infty$.*

Theorem 1.2.3. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with coefficients a_n in a Banach lattice E. If $a_n \geq 0$ in E for all n and if $r_0 > 0$ is the convergence radius of this power series, then $z = r_0$ is a singular point of the analytic function given by the power series.*

For a proof of Proposition 1.2.2, we refer to [BE], pp. 24. Theorem 1.2.3 is the vector valued version of the well-known Pringsheim's theorem. We refer to the Appendix of [SH4] for a proof. Next we collect some basic properties of the ultraproducts of Banach spaces and Banach lattices.

Let G be a (complex) Banach space and let T be a bounded linear operator on G. We use $\hat{G}$ to denote the ultraproduct of G with respect to a fixed filter finer than the Fréchet filter, while $\hat{T}$ denotes the canonical extension of T to $\hat{G}$. The following proposition collects some basic properties of the ultraproduct. For the proofs, we refer to [SH1], pp. 309-311. In the following $A\sigma(T)$ and $P\sigma(T)$ denote the approximate point spectrum and point spectrum of T respectively.

Proposition 1.2.4. *Let G be a nonzero Banach space and let $\hat{G}$ be any ultraproduct of G. For each bounded linear operator T on G and its canonical extension $\hat{T}$ on $\hat{G}$, the following hold:*

(i) $\sigma(T) = \sigma(\hat{T})$;

(ii) $A\sigma(T) = A\sigma(\hat{T}) = P\sigma(\hat{T})$;

(iii) $[(\lambda I - T)^{-1}]^{\wedge} = (\lambda I - \hat{T})^{-1}$ *for all* $\lambda \in \rho(T) = \rho(\hat{T})$;

(iv) *A number λ_0 is an (isolated) singularity (respectively, a pole of order k) of the resolvent $R(\lambda, T)$ if and only if the same is true of the resolvent $R(\lambda, \hat{T})$;*

(v) *If G is a Banach lattice, then $\hat{G}$ is also a Banach lattice, and the canonical map $G \to \hat{G}$ is an isometric lattice homomorphism. Moreover, T is a positive operator on*

G if and only if $\hat{T}$ is a positive operator on $\hat{G}$.

§1.3. The Center of a Banach Lattice

In this section we introduce the center of a Banach lattice and give without proofs some important properties of the center. Let E be a Banach lattice. The center of E, denoted by $Z(E)$, is the collection of all those operators $T \in L(E)$ for which there exists a positive constant c such that $|Tx| \leq c|x|$ for all $x \in E$. It is known that $Z(E)$ is Banach lattice under the natural ordering and a closed commutative subalgebra of $L(E)$. In fact, the following hold:

Theorem 1.3.1. *The center $Z(E)$ is a closed commutative subalgebra of $L(E)$, algebraically and order isomorphic to the Banach algebra and Banach lattice $C(X)$ for a suitable compact Hausdorff space X. Moreover, $Z(E)$ is a full subalgebra of $L(E)$ in the sense that whenever $T \in Z(E)$ is invertible in $L(E)$, then $T^{-1} \in Z(E)$. Thus the spectrum of $T \in Z(E)$ coincides with the spectrum of the corresponding function in the Banach algebra $C(X)$.*

The proof for the above theorem can be founded in [L] and [SP]. When the Banach lattice E is order complete, $Z(E)$ is exactly the band generated by the identity operator I on E in $L^r(E)$. In this case, the operator norm and the regular norm in $Z(E)$ coincide, and $||T|| = ||T||_r = \inf\{c > 0 : |Tx| \leq c|x| \text{ for all } x \in E\}$. Let Φ denote the band projection associated with $Z(E)$ when E is order complete. It is obvious that Φ is a contraction with respect to the regular norm. It is a recent result that Φ is also a contraction with respect to the operator norm as the following theorem indicates. We refer to [V] for a proof.

Theorem 1.3.2. *Let E be an order complete Banach lattice. Then $||\Phi(T)|| \leq ||T||$ for all $T \in L^r(E)$.*

Since Φ is a contraction with respect to the operator norm on $L^r(E)$, Φ can be extended to $\overline{L^r(E)}$, the operator norm closure of $L^r(E)$ in $L(E)$. We will still use Φ to denote its extension. From now on Φ is defined on $\overline{L^r(E)}$.

§1.4. The Sign Operator

In this section we introduce the sign operator associated with a fixed element in an order complete Banach lattice. The sign operator will be needed later in the study of the cyclicity of the peripheral spectrum of positive operators. The materials presented in this section are well-known and can be found, for instance, in [KR].

Proposition 1.4.1. *Let E be an order complete Banach lattice and let x_0 be a fixed element in E. Then there exists a surjective isometry D such that*

(i) $D \in Z(E)$ *and* $|Dx| = |x| = |D^{-1}x|$ *for all* $x \in E$;

(ii) $D|x_0| = x_0$.

Proof. We may assume that x_0 is a nonzero element in E. Let E_0 be the ideal generated by x_0 in E. That is $E_0 = \{x \in E : |x| \leq c|x_0| \text{ for some number } c\}$. It is well-known that E_0 is an AM-space with unit $|x_0|$ under the norm $||x||_0 = \inf\{c : |x| \leq c|x_0|\}$. So E_0 can be identified with the Banach lattice $C(X)$ for some compact space X. Moreover, under this identification $|x_0|$ corresponds to the constant one function 1_X on X and x_0 corresponds to a function $h \in C(X)$ such that $|h(t)| = 1$ for all $t \in X$. We now define a linear operator D on $E_0 \equiv C(X)$ by $Df = h \cdot f$ for all $f \in C(X)$. It is easy to see that (i) $D \in Z(E_0)$ and D is invertible; (ii) $|Dx| = |x| = |D^{-1}x|$ for all $x \in E_0$ and $D|x_0| = x_0$. Since D is order continuous on E_0 and since E_0 is order dense in the band $\{x_0\}^{dd}$ generated by x_0 in E, D can extend to $\{x_0\}^{dd}$. Finally we define $Dy = y$ for all $y \in \{x_0\}^d$. It is easy to check that D has the required properties.

The operator given in the above proposition is called the sign operator associated with x_0. In the following two propositions we collect some properties of the sign operator.

Proposition 1.4.2. Let E be an order complete Banach lattice and let T be a positive operator on E. Suppose that D is a linear operator on E such that $|Dx| = |x| = |D^{-1}x|$ for all $x \in E$. Let $T_D = a^{-1}D^{-1}TD = A + iB$, where a is a complex number such that $|a| = 1$ and $A = Re(T_D), B = Im(T_D)$. Then $A \le T$ and

$$|B| = B \vee (-B) \le \frac{T - A}{|\sin\theta|} + \frac{1 - \cos\theta}{|\sin\theta|}T$$

for any $\pi \ne \theta \in (0, 2\pi)$.

Proof. First we observe that $|T_D| \le T$. On the other hand,

$$|T_D| = \sup\{|\cos\theta A + \sin\theta B| : 0 \le \theta \le 2\pi\} \ge A.$$

So $A \le T$ and $T \ge \cos\theta A + \sin\theta B$ for any $\theta \in [0, 2\pi]$. From the latter we obtain

$$\frac{T - \cos\theta A}{\sin\theta} \ge B \text{ for } \theta \in (0, \pi)$$

$$\text{and } \frac{T - \cos\theta A}{\sin\theta} \le B \text{ for } \theta \in (\pi, 2\pi).$$

This implies that

$$\begin{aligned} |B| = B \vee (-B) &\le \frac{T - \cos\theta A}{|\sin\theta|} \\ &\le \frac{T - A}{|\sin\theta|} + \frac{1 - \cos\theta}{|\sin\theta|}T \end{aligned}$$

for all $\pi \ne \theta \in (0, 2\pi)$ since $A \le T$. The proof is finished.

Proposition 1.4.3. *Let E be an order complete Banach lattice and let T be a positive operator on E. Suppose that $Tx_0 = ax_0$ and $T|x_0| = |x_0|$, where $|a| = 1$ and x_0 is a nonzero element in E. If D is the sign operator associated with x_0, then the* *following hold:*

(i) $Tx = a^{-1}D^{-1}TDx$ *for all* $x \in \{x : |x| \leq c|x_0|$ *for some number* $c\}$, *the ideal generated by* x_0 *in* E;

(ii) $TD^l|x_0| = a^l D^l|x_0|$ *for* $l = 0, \pm 1, \pm 2, \cdots$.

Proof. Let $T_D = a^{-1}D^{-1}TD = A + iB$, where $Im(A) = Im(B) = 0$. Then

$$T_D|x_0| = a^{-1}D^{-1}TD|x_0| = a^{-1}D^{-1}Tx_0 = |x_0| = T|x_0|$$

and

$$T_D|x_0| = A|x_0| + iB|x_0|.$$

So $A|x_0| = T|x_0|$. If $|x| \leq c|x_0|$, then $|(T-A)x| \leq c(T-A)|x_0| = 0$ since $T \geq A$ by Proposition 1.4.2. So $Tx = Ax$ for any x in the ideal generated by x_0. Also, by Proposition 1.4.2 we have

$$|Bx| \leq |B||x| \leq \inf\{\frac{1-\cos\theta}{|\sin\theta|}T|x| : \pi \neq \theta \in (0, 2\pi)\} = 0.$$

Therefore, $T_D x = Tx$. Finally, if $n \geq 0$, then

$$TD^n|x_0| = aD(a^{-1}D^{-1}TD)D^{n-1}|x_0| = aDTD^{n-1}|x_0| = \cdots = a^n D^n|x_0|,$$

and

$$TD^{-n}|x_0| = a^{-1}D^{-1}(aDTD^{-1})D^{-n+1} = a^{-1}D^{-1}(T)D^{-n+1}|x_0| = \cdots = a^{-n}D^{-n}|x_0|.$$

The proof is completed.

Chapter 2

Spectral Theory of Positive Operators

In this chapter we mainly consider invertible positive operators whose inverses are not necessarily positive. Other related operators are also discussed. In the first section we prove some preliminary results that are needed in the rest of this thesis. These results are also independently interesting. Other sections contain the main results of this chapter. The notations we use here are standard.

§2.1. Preliminaries

By an operator we mean a bounded linear operator on a Banach space. Let T be a bounded operator on a Banach space. We use $\sigma(T)$, $\rho(T)$ and $r(T)$ to denote its spectrum, resolvent set and its spectral radius respectively, while $\rho_\infty(T)$ denotes the unbounded connected component of $\rho(T)$. We use $R(z,T)$ to denote $(z\,I - T)^{-1}$ if $z \in \rho(T)$. Recall that Φ denotes the band projection associated with the center of an order complete Banach lattice. We begin with the following definition:

Definition 2.1.1. *A bounded linear invertible operator T on a Banach space is said to satisfy the condition* (c) *if the number* 0 *belongs to the unbounded connected component of $\rho(T)$.*

The class of operators satisfying the condition (c) contains many important operators as the following examples show. But first let us recall that a subset A of the complex plane is said to be cyclic if $|\lambda|e^{in\theta} \in A$ for any integer n whenever $\lambda \in A$ and $\lambda = |\lambda|e^{i\theta}$.

Examples 2.1.2. Let T be an invertible operator on a Banach space. Then any of the following conditions implies that T satisfies the condition (c): (i) $\sigma(T)$ is finite (this is always the case if the Banach space is finite dimensional); (ii) $\rho(T)$ is connected; (iii) There exists $0 < \theta < 2\pi$ such that $\{re^{i\theta} : r > 0\}$ is contained in $\rho(T)$; (iv) $\sigma(T)$ is cyclic and does not contain any circle with center at the origin. For example, when T is a lattice homomorphism on a Banach lattice $\sigma(T)$ is cyclic (see [SH1], pp. 325).

Proposition 2.1.3. *Let T be an order bounded operator on an order complete Banach lattice. Let $\rho_\infty(T)$ be the unbounded connected component of $\rho(T)$. Then the following hold:*

(i) *$(zI - T)^{-1}$ belongs to $\overline{L_r(E)}$ for any $z \in \rho_\infty(T)$;*

(ii) *The complex operator-valued function $F(z)$ defined $F(z) = \Phi[(zI - T)^{-1}]$ is analytic in $\rho_\infty(T)$.*

Proof. (i) For $|z| > r(T)$, $R(z,T) = \sum_{n=0}^{\infty} T^n/z^{n+1}$ and so $R(z,T)$ is in $\overline{L_r(E)}$. Let $D = \{z \in \rho(T) : R(z,T) \in \overline{L_r(E)}\}$. Then D is a closed and open subset of $\rho(T)$. The fact that D is closed and open in $\rho(T)$ follows from the fact that $\overline{L^r(E)}$ is a closed subalgebra of $L(E)$ and the following fact: if $z_0 \in \rho(T)$ and if $|z| < ||R(z_0,T)||^{-1}$ then $z + z_0 \in \rho(T)$, and moreover $R(z + z_0, T) = \sum_{k=0}^{\infty}(-z)^k R(z_0,T)^{k+1}$. Therefore $\rho_\infty(T) \subseteq D$.

(ii) We know that $R(z,T)$ is analytic in $\rho(T)$ and hence in $\rho_\infty(T)$. By (i) $F(z)$ is well defined for all $z \in \rho_\infty(T)$. Since $R(z,T)$ satisfies the Cauchy integral formula in $\rho_\infty(T)$ it follows from (i) and Theorem 1.3.2 that $F(z)$ also satisfies the Cauchy integral formula in $\rho_\infty(T)$. Therefore, $F(z)$ is analytic in $\rho_\infty(T)$.

To prove the main theorem we need to prove several lemmas, which are also independently interesting.

Lemma 2.1.4. *Let T be a positive operator on an order complete Banach lattice E. For each nonnegative integer n let $P_n = \Phi(T^n)$. Then the operator sequence $\{P_n : n \geq 0\}$ has the following properties:*

(i) *$P_0 = I$ and $P_n \cdot P_m \leq P_{n+m}$ for all n, $m \geq 0$;*

(ii) $0 \leq P_1 \leq r(T) \cdot I$,

where I is the identity operator on E.

Proof. (i) follows from the fact that $\Phi(S_1)\Phi(S_2) \leq \Phi(S_1S_2)$ for any two positive operators S_1, S_2. To see this let $S_i = P_i + B_i$ with $P_i \in Z(E)$ and $B_i \in Z(E)^d$, where $i = 1,\ 2$. Then

$$S_1S_2 = P_1P_2 + B_1B_2 + P_1B_2 + B_1P_2.$$

Since P_1B_2 and B_1P_2 are both in $Z(E)^d$ (this follows from the fact that the absolute value of any operator in $Z(E)$ is dominated by a multiple of the identity operator), we have

$$\Phi(S_1S_2) = P_1P_2 + \Phi(B_1B_2) \geq P_1P_2.$$

Now the result follows.

(ii) Let $T = P_1 + B$ have the same meaning as above. Then $0 \leq P_1^n \leq T^n$. This implies that $r(P_1) \leq r(T)$. Since P_1 is in $Z(E)$ it follows from Theorem 1.3.1 that $0 \leq P_1 \leq r(P_1) \cdot I$. Now the result follows.

Lemma 2.1.5. *Let T be an invertible order bounded operator on an order complete Banach lattice. Assume that T satisfies the condition* (c). *Let $P_n = \Phi(T^n)$ for all nonnegative integers n and let $F(z)$ be as in Proposition* 2.1.3 *for all $z \in \rho_\infty(T)$. Then the following hold:*

(i) *$F(z)$ is analytic in $\rho_\infty(T)$ and $\lim_{z\to\infty} F(z) = 0$ and $F(z)$ is not identically equal to zero;*

(ii) *If we identify $Z(E)$ with $C(X)$ for some compact space X and identify P_n as functions on X then* $\overline{\lim}_{n\to\infty}[|P_n(x)|]^{1/n} \geq r(T^{-1})^{-1}$ *for any* $x \in X$.

Proof. (i) The first statement follows from Proposition 2.1.3. For $|z| > r(T)$, $F(z) = \Phi[R(z,T)] = \Phi(\sum_{k=0}^{\infty} T^k/z^{k+1}) = \sum_{k=0}^{\infty} P_k/z^{k+1}$. So we see that $\lim_{z\to\infty} F(z) = 0$ and $F(z)$ is not identically equal to zero.

(ii) Suppose that there exists a point $x_0 \in X$ such that $\overline{\lim}_{n\to\infty}[|P_n(x_0)|]^{1/n} \leq r(T^{-1})^{-1} - \varepsilon$ for some $\varepsilon > 0$. Let $f(z) = F(z)(x_0)$, where $F(z)$ is regarded as an element of $C(X)$. By (i) we see that $f(z)$ is analytic in $\rho_\infty(T)$, which contains $\{z : |z| < r(T^{-1})^{-1}\}$ since T satisfies the condition (c). Now for $|z| > r(T)$, $f(z) = \sum_{k=0}^{\infty} P_k(x_0)/z^{k+1}$. By hypothesis this series can be extended to $\{z : |z| > r(T^{-1})^{-1} - \varepsilon\}$. Therefore, $f(z)$ can be extended to an entire function. By (i) $f(z)$ has to be the zero function. This is a contradiction since $P_0(x_0) = 1$.

§2.2. Positive Operators Satisfying the Condition (c)

In this section we study various properties of those positive operators satisfying the condition (c) on a general Banach lattice E. Since we can consider the dual space and the adjoint operator if needed, we may assume that E is order complete. The following theorem is the first main result of this thesis.

Theorem 2.2.1. *Let T be an invertible positive operator on a Banach lattice E. Assume that T satisfies the condition* (c). *Then there exist a positive number a and a positive integer k such that $T^k \geq a \cdot I$. More precisely, for any $0 < \varepsilon < r(T^{-1})^{-1}$ there exists a positive integer n_0 such that $T^n \geq [r(T^{-1})^{-1} - \varepsilon]^n \cdot I$ whenever n is a multiple of n_0.*

Proof. We may assume that E is order complete. Fix $0 < \varepsilon < r(T^{-1})^{-1}$. We will use the previous notations. It follows from (ii) of Lemma 2.1.5 that for any point $x \in X$

there exists a positive integer N_x depending on x such that $P_{N_x}(x) > [r(T^{-1})^{-1} - \varepsilon]^{N_x}$. Hence there is a neighborhood of x , say $U(x)$, such that this inequality holds for all points in this neighborhood. Now $\{U(x) : x \in X\}$ is an open covering of X. By the compactness of X , there are finitely many points $x_1, x_2, \cdots, x_m$ in X such that $\{U(x_i) : 1 \leq i \leq m\}$ is still a covering of X. Let $N_1, N_2, \cdots, N_m$ be the corresponding integers. Let $n_0 = N_1 N_2 \cdots N_m$. Then for any $x \in X$ there is a $U(x_i)$ such that $x \in U(x_i)$ and so $P_{N_i}(x) > [r(T^{-1})^{-1} - \varepsilon]^{N_i}$. It follows from (i) of Lemma 2.1.4 that

$$P_{n_0}(x) \geq [P_{N_i}(x)]^{n_0/N_i} > [r(T^{-1})^{-1} - \varepsilon]^{n_0}$$

Therefore, $P_{n_0} \geq [r(T^{-1})^{-1} - \varepsilon]^{n_0} \cdot I$. If n is a multiple of n_0 then

$$P_n \geq (P_{n_0})^{n/n_0} \geq [r(T^{-1})^{-1} - \varepsilon]^n \cdot I$$

Notice that $P_n = \Phi(T^n)$. So the theorem follows.

The above result and the method have several important consequences. We mention some of them in the following, and other consequences will appear in other sections. The following result is an easy consequence of the above theorem. It is one of the important features that positive operators possess.

Corollary 2.2.2. *Let T be a lattice isomorphism on an arbitrary Banach lattice E such that $\sigma(T) = \{1\}$. Then $T = I$, the identity operator on E.*

Proof. Since T satisfies the condition (c), it follows from Theorem 2.2.1 that $T^k \geq a \cdot I$ for some positive integer k and some positive number a. Notice that $T^{-1} \geq 0$. So by the above we have

$$a \cdot T^{-k} \leq T^k \cdot T^{-k} = I$$

This implies that $T^{-k} \in Z(E)$. By Theorem 1.3.1 and by the spectral mapping theorem we obtain $T^{-k} = I$. Consequently, $T^k = I$. Now $(T - I)(T^{k-1} + \cdots + I) = 0$

and the second factor is invertible by the spectral mapping theorem. Hence, $T = I$.

Corollary 2.2.3. *If T is a lattice isomorphism on an arbitrary Banach lattice E and if T satisfies the condition* (c), *then there exists a positive integer k such that $T^k \in Z(E)$.*

Proof. The proof is similar to the first part of the above proof, where we showed that $T^{-k} \in Z(E)$ for some positive integer k. So $T^k \in Z(E)$ since $Z(E)$ is a full subalgebra of $L(E)$ by Theorem 1.3.1.

Remark. Corollary 2.2.2 is due to [SWA] in which a more general result is obtained. We will give a proof for this result in the following corollary. We put Corollary 2.2.2 in a separate place because the proof is straightforward and unique. Corollary 2.2.3 is essentially due to [AH]. In Chapter 3 we will generalize all of these results for disjointness preserving operators under less strict conditions.

Sometimes it is important to know when an operator will belong to the center of the Banach lattice. The following result, due to [SWA], gives an answer to this question for lattice homomorphisms. The proof we give here is different from the original one. We will continue to discuss this question in Chapter 3 for some other types of operators.

Corollary 2.2.4. *Let T be a lattice isomorphism on E such that $\sigma(T) \subset (0, \infty)$. Then $T \in Z(E)$.*

Proof. By the above corollary, $T^k \in Z(E)$ for some positive integer k. Now consider the complex function $f(z) = \exp(1/k \log z)$, which is defined and analytic in $D = \{z : |arg(z)| < \pi\}$. Notice that $\sigma(T) \cup \sigma(T^k) \subset D$. So there exists an operator $S \in L(E)$ such that $S = f(T^k)$. In fact,

$$S = \frac{1}{2\pi i} \int_C f(z) R(z, T^k) dz$$

where C is a simple closed curve in D with the spectrum of T^k contained in its interior. Since $R(z, T^k) \in Z(E)$ for all $z \in C$ (because $T^k \in Z(E)$ and $Z(E)$ is a closed full subalgebra of $L(E)$, see Theorem 1.3.1 of Chapter 1), it follows that $S \in Z(E)$. On the other hand, since $f(z^k) = z$ for all $z \in G = \{z : |arg(z)| < \pi/k\}$ and since $\sigma(T^k) \subset G$, we have $S = f(T^k) = T$. Therefore, $T \in Z(E)$.

Another important consequence of Theorem 2.2.1 is the following result on the peripheral spectrum of positive operators satisfying the condition (c). First let us recall that the subset $\mathrm{Per}\sigma(T) = \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$ is called the peripheral spectrum of T.

Theorem 2.2.5. *Let T be a positive operator on a Banach lattice E. Assume that T is invertible and satisfies the condition* (c). *Then there exists a positive integer k such that $Per\sigma(T^k) = \{[r(T)]^k\}$. So if $r(T) = 1$ then $\sigma(T^k) \cap \{z : |z| = 1\} = \{1\}$.*

Proof. We may assume that $r(T) = 1$. It follows from Theorem 2.2.1 that $T^k \geq a \cdot I$ for some positive number a and some positive integer k. Let $T^k = a \cdot I + B$. Then B is a positive operator on E. Notice that $r(B) = 1 - a$ since the spectral radius of a positive operator is an element of its spectrum (see Proposition 1.2.1) and since $r(T^k) = 1$. By the spectral mapping theorem, it holds that

$$\sigma(T^k) = \{a + z : z \in \sigma(B)\} \subseteq \{a + z : |z| \leq 1 - a\}.$$

Hence, $\sigma(T^k) \cap \{z : |z| = 1\} = \{1\}$.

Theorem 2.2.6. *Let T be a positive operator on a Banach lattice E. Suppose that there exists a sequence $\{a_n\}$ of nonnegative numbers such that*

(i) *$T^n \geq a_n I$ for all n, where I is the identity operator on E, and*

(ii) $\overline{\lim}_{n\to\infty}(a_n)^{1/n} = r(T)$.

Then the peripheral spectrum of T is cyclic.

To prove the theorem we need some preparation. The ideas and the techniques we are going to use come from [KR] and to which the following lemma is due.

Lemma 2.2.6a. *Let E be a Banach lattice and let T be a positive operator on E such that $r(T) = 1$. Assume that $Tx_0 = ax_0$, where $0 \neq x_0 \in E$ and $|a| = 1$. Fix any $z_0 > 1$ and let E_0 be the ideal generated by $R(z_0, T)|x_0|$. Then the following hold:*

(i) *E_0 is an AM-space with unit $R(z_0, T)|x_0|$ under a new lattice norm. E_0 is T-invariant and $R(T|_{E_0}) \leq z_0$ and $|x_0| \in E_0$;*

(ii) *If $E_0 \equiv C(X)$ for some compact space X, then there exists $q_0 \in X$ such that $T^n|x_0|(q_0) = z_0 - 1$ for $n = 0, 1, 2, \cdots$.*

Proof. To simplify notations we let $R(z) = R(z, T)$ and $u_0 = R(z_0)|x_0|$. Then

$$E_0 = \{x \in E : \ |x| \leq cu_0 \text{ for some } c \geq 0\}.$$

E_0 is an AM-space with unit u_0 under the norm

$$||x||_0 = \inf\{c \geq 0 : \ |x| \leq cu_0\}.$$

Hence E_0 can be identify with $C(X)$ for some compact space X. Moreover, u_0 corresponds to the constant one function 1_X on X. For proofs of these facts, we refer to [SH1]. Since $|Tu_0| = |-|x_0| + z_0R(z_0)|x_0|| \leq z_0R(z_0)|x_0| = z_0u_0$, (i) follows immediately.

We now prove (ii). We identify E_0 with $C(X)$. First we notice that

$$|x_0| \leq T|x_0| \leq T^2|x_0| \leq \cdots$$

and that

$$T^k 1_X = \sum_{n=0}^{\infty} z_0^{-n-1} T^{k+n}|x_0|$$

$$\geq \sum_{n=0}^{\infty} z_0^{-n-1} T^n |x_0| = 1_X.$$

If $(T^{k_0} 1_X)(q) > 1$ for all $q \in X$, then by the compactness of X there exists a positive number a such that $T^{k_0} 1_X \geq (1+a) 1_X$. This implies that

$$R(z, T^{k_0}) 1_X = \sum_{n=0}^{\infty} z^{-n-1} T^{n k_0} 1_X$$

$$\geq \sum_{n=0}^{\infty} z^{-n-1} (1+a)^n 1_X = (z - (1+a))^{-1} 1_X$$

for $z > 1 + a$. So $r(T^{k_0}) \geq 1 + a$, which is a contradiction. Hence

$$Q_k = \{q \in X : T^k 1_X(q) = 1\} \neq \emptyset \text{ for any } k.$$

Since $1_X \leq T 1_X \leq T^2 1_X \leq \cdots$, we have

$$Q_1 \supseteq Q_2 \supseteq \cdots.$$

Now each Q_k is compact, so $\bigcap_{k=1}^{\infty} Q_k \neq \emptyset$. Let q_0 be a point in this intersection. Then $T^k 1_X(q_0) = 1$ for all k. Now

$$1 = T^k 1_X(q_0) = (-T^{k-1}|x_0|)(q_0) + (z_0 T^{k-1} R(z_0)|x_0|)(q_0)$$

$$= -T^{k-1}|x_0|(q_0) + z_0$$

Hence, $T^{k-1}|x_0|(q_0) = z_0 - 1$ for all k. The proof is finished.

Lemma 2.2.6b. *Let T be a positive operator on a Banach lattice E. Suppose that T satisfies the conditions in Theorem 2.2.6. Then for any nonzero positive operator $Q \in Z(E)$ and for any positive element $x_0 \in E$ with $Q^2 x_0 \neq 0$, the point $z = r(T)$ is a singularity of the E-valued analytic function $f(z) = QR(z,T)Qx_0$ defined in $\rho(T)$.*

Proof. For $|z| > r(T)$, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{QT^n Q x_0}{z^{n+1}}.$$

Notice that $QT^nQx_0 \geq a_nQ^2x_0 \geq 0$ for any n by assumption. So

$$||Q||^2 \cdot ||T^n|| \cdot ||x_0|| \geq ||QT^nQx_0|| \geq a_n||Q^2x_0||.$$

This implies that $\overline{\lim}_{n\to\infty}||QT^nQx_0||^{1/n} = r(T)$. Now it follows from Theorem 1.2.3 (the Pringsheim's theorem) that $r(T)$ is a singularity of $f(z)$.

Proof of Theorem 2.2.6. Let $r(T) = 1$ and let $a \in \sigma(T)$ with $|a| = 1$. By considering the ultraproduct space of the space E and the double dual space of the ultraproduct space (see Proposition 1.2.4), we may assume that E is order complete and a is an eigenvalue of T. Let $Tx_0 = ax_0$ with $0 \neq x_0 \in E$ and fix $z_0 > 1$. To simplify notations we let $R(z) = R(z, T)$ when the meaning of $R(z)$ is clear.

Let E_0 be the ideal generated by $R(z_0)|x_0|$. By Lemma 2.2.6a we have $T(E_0) \subseteq E_0$ and $r(T|_{E_0}) \leq z_0$. In this paragraph we will work in the AM-space E_0, which is identified with $C(X)$ for some compact space X. For $z_1 > z_0$, let

$$R(z_1) - aD^{-1}R(az_1)D = A + iB,$$

where $Im(A) = Im(B) = 0$, i.e., both A and B are real operators, and D is the sign operator associated with x_0 (see Section 4 of Chapter 1). For any positive integer n, let

$$a^{-n}D^{-1}T^nD = C_n + iD_n$$

with $Im(C_n) = Im(D_n) = 0$. By Proposition 1.4.2, we see that $T^n \geq C_n$. Now let

$$A_n = T^n - C_n = T^n - Re(a^{-n}D^{-1}T^nD).$$

Then

$$0 \leq z_1^{-n-1}A_n \leq \sum_{k=0}^{\infty} z_1^{-k-1}A_k$$

$$= R(z_1) - Re[aD^{-1}R(z_1a)D] = A.$$

By Lemma 2.2.6a there exists a point $q_0 \in X$ such that $T^n|x_0|(q_0) = z_0 - 1$ for all nonnegative integers n. Thus

$$0 \leq A|x_0|(q_0) = \sum_{n=0}^{\infty} z_1^{-n-1}(A_n|x_0|)(q_0)$$

$$= \sum_{n=0}^{\infty} z_1^{-n-1}(T^n|x_0| - |x_0|)(q_0) = 0$$

since $A_n|x_0| = T^n|x_0| - Re(a^{-n}D^{-1}T^n D|x_0|) = T^n|x_0| - |x_0|$. We know that $A|x_0|$ can be identified as a nonnegative function on X, say $f \equiv A|x_0|$. Let

$$U_k = \{q \in X : f(q) < \frac{z_0 - 1}{2k}\} \cap \{q \in X : |x_0|(q) > \frac{z_0 - 1}{2}\}.$$

Then U_k is an open neighborhood of q_0. Since E_0 is an order complete AM-space (since E is order complete), X is extremely disconnected (see [SH1], pp.107-108.). So $\overline{U_k}$ (the closure of U_k) is open and closed in X. Let Q_k be the band projection associated with this open and closed subset. Then we have

$$0 \leq z_1^{-n-1} Q_k A_n Q_k |x_0| \leq Q_k A Q_k |x_0|$$

$$\leq Q_k A|x_0| \leq \frac{1}{k} Q_k |x_0|.$$

Since Q_k is in the center of the AM-space E_0 and since E_0 as a subset of E is an ideal of E, Q_k can extend to be a band projection on E by setting $Q_k x = 0$ for all $x \in \{R(z_0)|x_0|\}^d$. Finally, notice that $Q_k|x_0| \neq 0$ by the choice of Q_k.

Now consider any ultraproduct space $\hat{E}$ of E. Let $\hat{D} = (D), \widehat{D^{-1}} = (D^{-1})$ and $\hat{T} = (T)$ be the canonical extensions of D, D^{-1} and T respectively. Let $Q = (Q_k)$ and let $X_0 \in \hat{E}$ be given by

$$X_0 = \left(\frac{Q_k|x_0|}{||Q_k|x_0|||} \right).$$

Then $QX_0 = X_0 \neq 0$. By the last paragraph we have

$$Q\hat{A}_n Q X_0 = 0 \text{ in } \hat{E} \text{ for all positive integers } n$$

i.e., $Q\hat{T^n}QX_0 = Q\hat{C}_nQX_0$. Consider the ideal generated by X_0 in $\hat{E}$, which is given by

$$L = \{Y \in \hat{E} : |Y| \leq cX_0 \text{ for some } c \geq 0\}.$$

Then for any $Y \in L$, we have

$$|Q\hat{A}_nQY| \leq cQ\hat{A}_nQX_0 = 0.$$

This implies that $Q\hat{T^n}QY = Q\hat{C}_nQY$. Now by Proposition 1.4.2 we obtain

$$|Q|\hat{D}_n|QY| \leq cQ|\hat{D}_n|QX_0 \leq c \bigwedge_{\pi \neq \theta \in (0,2\pi)} \frac{(1-\cos\theta)}{|\sin\theta|} Q\hat{T^n}QX_0 = 0.$$

Therefore, we obtain

$$Q\hat{T^n}QY = Qa^{-n}\hat{D^{-1}}\hat{T^n}\hat{D}QY$$

for all positive integers n and for all $Y \in L$. Since $D, Q_k \in Z(E)$, we have $Q\hat{D} = \hat{D}Q$. So the above identity can be written as

$$Q\hat{T^n}QY = \hat{D^{-1}}a^{-n}(Q\hat{T^n}Q)\hat{D}Y.$$

By using this identity repeatedly we obtain

$$Q\hat{T^n}QX_0 = a^{-kn}\hat{D^{-k}}(Q\hat{T^n}Q)\hat{D^k}X_0$$

for any integer k (since L is invariant under $Q, \hat{D}$ and $\hat{D^{-1}}$).

Finally, we show that the peripheral spectrum of T is cyclic. That is to show $a^k \in \sigma(T)$ for any integer k. From the last paragraph we have, for $|z| > 1$,

$$Q\hat{R}(z)QX_0 = \sum_{n=0}^{\infty} z^{-n-1}Q\hat{T^n}QX_0$$

$$= \sum_{n=0}^{\infty} z^{-n-1}\hat{D^{-k}}(a^{-kn}Q\hat{T^n}Q)\hat{D^k}X_0$$

$$= a^k\hat{D^{-k}}Q\hat{R}(za^k)Q\hat{D^k}X_0.$$

By Lemma 2.2.6b we know that $z = 1$ is a singular point of $Q\hat{R}(z)QX_0$. Now the above identity implies that $z = 1$ is also a singular point of $Q\hat{R}(za^k)Q\hat{D}^kX_0$. In particular, $z = 1$ is a singular point of $\hat{R}(za^k)$. This implies that a^k is a singular point of $\hat{R}(z)$. This shows that $a^k \in \sigma(\hat{T}) = \sigma(T)$. The proof is finished.

Corollary 2.2.7. *Let T be a positive operator on a Banach lattice E such that $r(T) > 0$. Assume that*

(i) $[0, r(T)) \subset \rho(T)$;

(ii) *$z = r(T)$ is an isolated point of $\sigma(T)$.*

Then there exists a sequence $\{a_n\}$ of nonnegative numbers such that $T^n \geq a_n I$ and $\overline{\lim}_{n\to\infty}(a_n)^{1/n} = r(T)$. Thus the peripheral spectrum of T is cyclic.

Proof. We may assume that E is order complete. As before we identify the center $Z(E)$ with $C(X)$ for some compact space X. Let $r(T) = 1$. Let $P_n = \Phi(T^n)$ and let $F(z) = \Phi(R(z, T))$. By Proposition 2.1.3 we know that $F(z)$ is analytic in $\rho_\infty(T)$. In this case, $[0, 1) \subset \rho_\infty(T)$. Fix any $x \in X$. Then for $|z| > 1$, we have

$$f(z) = F(z)(x) = \sum_{n=0}^{\infty} \frac{P_n(x)}{z^{n+1}}.$$

By assumptions and by Theorem 1.2.3 (the Pringsheim's theorem), $z = 1$ must be a singular point of $f(z)$. So we must have

$$\overline{\lim}_{n\to\infty}(P_n(x))^{1/n} = 1.$$

As in the proof of Theorem 2.2.1 we can prove that for any $\varepsilon > 0$ there exists a positive integer N such that $P_N \geq (1-\varepsilon)^N I$. Now choose $n_1, n_2, \cdots$ such that $P_{n_k} \geq (1 - 1/k)^{n_k} I$. Now define

$$a_n = \begin{cases} (1-1/k)^{n_k} & \text{if } n = n_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $T^n \geq a_n I$ for any n and

$$1 \leq \overline{\lim}_{n\to\infty}(a_n)^{1/n} \leq \lim_{n\to\infty} ||T^n||^{1/n} = 1.$$

By Theorem 2.2.6, the peripheral spectrum of T is cyclic.

Remark. The last statement of Corollary 2.2.7 is due to Krieger (see [KR]). The techniques we used in the proof of Theorem 2.2.6 come from this paper.

It is known that the whole spectrum of a lattice homomorphism is cyclic (see [SH1], pp. 325). E. Scheffold showed that any bounded closed cyclic subset of the complex plane is the spectrum of some lattice homomorphism (see [SF]). We now prove the following theorem, which asserts that if the spectrum of a positive operator is contained in its spectral circle then the spectrum of this operator is either the whole spectral circle or finite and cyclic.

Theorem 2.2.8. *Let T be a positive operator on a Banach lattice such that $r(T) = 1$. Suppose that $\sigma(T)$ is contained in the unit circle Γ. Then either $\sigma(T) = \Gamma$ or there exists a positive integer k such that* (i) *$\sigma(T)$ consists of k-th roots of unity and* (ii) *$\sigma(T)$ is cyclic.*

Proof. If $\sigma(T)$ is not the whole unit circle, then T satisfies the condition (c). By assumption and by Theorem 2.2.5 there exists a positive integer k such that $\sigma(T^k) = \{1\}$. By spectral mapping theorem we have $\sigma(T) \subseteq \{z : z^k = 1\}$. So $z = 1$ is an isolated point of $\sigma(T)$. Now by Corollary 2.2.7 we conclude that the peripheral spectrum of T is cyclic.

Remark. In Theorem 2.2.8 the spectrum of T is not necessarily equal to $\{z : z^k = 1\}$ for some positive integer k. For example, let $E = L^p(\Gamma)$, where $1 \leq p \leq \infty$ and Γ is the unit circle equiped with the Lebesgue measure. Define two operators T_1 and T_2 on E by $T_1 f(z) = f(-z)$ and $T_2 f(z) = f(e^{2\pi/3i}z)$ for any $z \in \Gamma$. Now consider the product space $E \times E$ and the operator $T = T_1 \times T_2$. Then $T^6 = I$ and $\sigma(T) = \sigma(T_1) \cup \sigma(T_2) = \{1, -1, e^{2\pi/3i}, e^{4\pi/3i}\}$. Moreover, 6 is the smallest positive integer k such that $\sigma(T^k) = \{1\}$. It is easy to see that some 6-th root of unity is not in $\sigma(T)$.

We already see that if T is a positive operator such that its spectrum is properly contained in the unit circle then T satisfies the condition (c), and so by Theorem 2.2.1 the projection of some power of T into the center $Z(E)$ dominates a positive multiple of the identity operator and thus a nonsingular element in the center. We now show that the opposite is also true.

Proposition 2.2.9. *Let T be a positive operator on an order complete Banach lattice E such that $\sigma(T) \subseteq \{z : |z| = 1\}$. Then the following hold:*

(i) *$\sigma(T) = \{z : |z| = 1\}$ if and only if $\Phi(T^n)$ is a singular element in $Z(E)$ for each positive integer n;*

(ii) *If $\sigma(T) = \{z : |z| = 1\}$ and if $Z(E) \equiv C(X)$ for some compact space X, then there exists a point $p_0 \in X$ such that $\Phi(T^n)(p_0) = 0$ for all positive integers n.*

Proof. (i) We have already shown that if $\sigma(T)$ is not the whole unit circle then the projection of some power of T into the center is a nonsingular element. It remains to show that if $\sigma(T)$ is the whole circle then each $\Phi(T^n)$ is a singular element in $Z(E)$. If this were not the case then we could find some positive integer n such that $\Phi(T^n)$ is nonsingular in $Z(E)$. So by Theorem 1.3.1 there exists a positive number a such that $\Phi(T^n) \geq a \cdot I$. Thus $T^n \geq a \cdot I$. As in the proof of Theorem 2.2.5 we can show that $\sigma(T^n) = \{1\}$. This is a contradiction, since it follows from the assumption and the spectral mapping theorem that $\sigma(T^n)$ is the whole unit circle.
(ii) For any positive integer n let $X_n = \{p \in X : \Phi(T^{n!})(p) = 0\}$. By what we just proved, X_n is nonempty. Moreover, X_n is compact and $X_{n+1} \subseteq X_n$ for any n

by Lemma 2.1.4. So it follows from the compactness of X that there exists a point $p_0 \in X$ such that $p_0 \in X_n$ for all n. By Lemma 2.1.4 again, we see that p_0 is the required point.

§2.3. Some Consequences

In this section we use the techniques developed in the previous sections to investigate the spectral properties of certain types of positive operators. Also we will give some conditions for which the regular spectrum is equal to the spectrum. We begin with the following definition (see also [AH]).

Definition 2.3.1. *Let T be an order (= regular) operator on an order complete Banach lattice E.* (i) *T has strict period n if $T^n \in Z(E)$ and $|T|^k \perp I$ for all $k = 1, 2, \cdots, n-1$;* (ii) *T is aperiodic if $|T|^k \perp I$ for all positive integers k.*

Theorem 2.3.2. *Let T be a positive operator on an order complete Banach lattice E. Suppose that T has strict period n for some positive integer n. Then for any n-th root of unity α, $\sigma(T) = \alpha\sigma(T)$. So $\lambda \in \sigma(T)$ if and only if $|\lambda| \in \sigma(T)$ and $\lambda = |\lambda|\alpha$, for some n-th root of unity α. In this case, the whole spectrum of T is cyclic.*

Proof. By assumption, $T^n \in Z(E)$ and $|T|^k \wedge I = 0$ for $k = 1, 2, \cdots, n-1$. For $|z| > r(T)$, we have

$$F(z) := \Phi[R(z,T)] = \sum_{k=0}^{\infty} \frac{\Phi(T^k)}{z^{k+1}} = \frac{I}{z} + \frac{T^n}{z^{n+1}} + \frac{T^{2n}}{z^{2n+1}} + \cdots = z^{n-1}(z^n - T^n)^{-1}.$$

We know that $F(z)$ is analytic in $\rho_\infty(T)$ by Proposition 2.1.3. Hence $z^{n-1}(z^n - T^n)^{-1}$ can be extended to be an analytic function in $\rho_\infty(T)$. So, for any $z \in \rho_\infty(T)$, we have that $(z^n - T^n)^{-1}$ exists and thus $z^n \in \rho(T^n)$. Now let $z_0 \in \sigma(T)$. Then $z_0^n \in \sigma(T^n)$ and also $(\alpha z_0)^n \in \sigma(T^n)$. Hence, αz_0 is not in $\rho_\infty(T)$.

Since T is positive and since $T^n \in Z(E)$ (thus $\sigma(T^n)$ consists of nonnegative numbers), we have $\sigma(T) \subset \{z : z^n \geq 0\}$ by the spectral mapping theorem. Hence, $\rho_\infty(T) = \rho(T)$. It follows from the above paragraph that αz_0 is not in $\rho(T)$, i.e., $\alpha z_0 \in \sigma(T)$.

We now prove the last part of the theorem. Let $\lambda \in \sigma(T)$ be a nonzero element. Then $|\lambda| = \alpha\lambda$ for some α with $|\alpha| = 1$. Since $\lambda^n \geq 0$ by above, $\alpha^n = 1$ and the conclusion follows by the first part of the theorem. The converse can be proved similarly. Finally, the fact that $\sigma(T) = \alpha\sigma(T)$ for any n-th root of unity α and the fact that $\sigma(T) = \{z : z^n \geq 0\}$ imply that $\sigma(T)$ is cyclic.

Theorem 2.3.3. *Let T be an aperiodic operator on E. Suppose that $\sigma(T) \subseteq \{z : |z| = r(T)\}$. Then $\sigma(T) = \{z : |z| = r(T)\}$.*

Proof. We use the same notations as before. Since T is aperiodic, we have $F(z) = I/z$ for all $z \in \rho_\infty(T)$. We know that $F(z)$ is analytic in $\rho_\infty(T)$, and so 0 is not in $\rho_\infty(T)$. From this, we conclude that $\sigma(T) = \{z : |z| = r(T)\}$.

Example 2.3.4. Let $E = L_p(\mathbf{R})$ with $1 \leq p \leq \infty$. Let $h(x)$ be any bounded measurable function on $\mathbf{R}$ such that $|h(x)| = 1$ for all $x \in \mathbf{R}$. Define an operator T on E by $Tf(x) = h(x)f(x+1)$ for all $x \in \mathbf{R}$. It is easy to see that T is an order bounded operator and that $|T|^n \perp I$ for all positive integer n. Also it is easy to show that T is double power bounded, from which we conclude that $\sigma(T) \subseteq \{z : |z| = 1\}$. Now it follows from the above result that $\sigma(T) = \{z : |z| = 1\}$.

Theorem 2.3.5. *Let T be a positive operator on an order complete Banach lattice E. If T has strict period n for some positive integer n, then $\sigma_0(T) = \sigma(T)$, i.e., the regular spectrum of T is equal to its spectrum.*

Proof. Since $T^n \in Z(E)$, we have

$$\sigma_0(T^n) = \sigma(T^n) \subset [0, \infty).$$

Let $\lambda_0 \in \sigma_0(T)$. Then by the spectral mapping theorem $\lambda_0^n \in \sigma_0(T^n) = \sigma(T^n)$. Hence there exists $\lambda \in \sigma(T)$ such that $\lambda_0^n = \lambda^n$. Let $\lambda = \alpha|\lambda|$ for some α such that $\alpha^n = 1$. By Theorem 2.3.2, $|\lambda| \in \sigma(T)$. Now $\lambda_0 = \beta|\lambda|$ for some β such that $\beta^n = 1$. By Theorem 2.3.2 again we obtain $\lambda_0 \in \sigma(T)$. Therefore, $\sigma_0(T) \subseteq \sigma(T)$. Since $\sigma(T) \subseteq \sigma_0(T)$ is always true, we have shown that $\sigma_0(T) = \sigma(T)$.

Theorem 2.3.6. *Let E be an order complete Banach lattice and let T be a positive operator on E. If $T^i \perp T^j$ for any nonnegative integers $i \neq j$, then $\{z : |z| = r(T)\} \subseteq \sigma_0(T)$.*

Proof. Without loss of generality we may assume that $r(T) = 1$. Let $R(z,T) = R(z)$. For $|z| > 1$, we have

$$R(z) = \sum_{k=0}^{\infty} \frac{T^k}{z^{k+1}},$$

where the series converges in the regular norm of $L^r(E)$. By assumption, we obtain

$$|R(z)| = |\sum_{k=0}^{\infty} \frac{T^k}{z^{k+1}}| = \sum_{k=0}^{\infty} \frac{T^k}{|z|^{k+1}} = R(|z|).$$

So if $|z_0| = 1$, then

$$\lim_{z \to z_0, |z|>1} ||R(z)||_r = +\infty.$$

This implies that z_0 is a singular point of $R(z)$ since $R(z)$ is analytic in the topology induced by the regular norm. Thus $z_0 \in \sigma_0(T)$.

Corollary 2.3.7. *Let X be an extremely disconnected compact space and let $E = C(X)$. Suppose that T is a positive operator on E such that $T^i \perp T^j$ for any nonnegative integers $i \neq j$. Then the peripheral spectrum of T is the whole spectral circle, i.e., $\{z : |z| = r(T)\} \subseteq \sigma(T)$.*

Proof. Since a linear operator on $C(X)$ is bounded if and only if it is order bounded, we have $L^r(E) = L(E)$ and the regular norm is equal to the usual operator norm (see [SH1], pp. 232). So $\sigma_0(T) = \sigma(T)$ and the result follows from the above theorem.

§2.4. Positive Isometries and Lattice Isomorphisms

In this section we will use the results obtained in the previous sections to discuss some aspects of the properties of positive isometries and lattice isomorphisms and their relations.

We begin with the finite dimensional case. Recall that if E is an n-dimensional complex Banach lattice then E is topologically and order isomorphic to $\mathbf{C}^n$. Hence any operator on E can be represented as an $n \times n$ matrix. Moreover, positive operators on E correspond to matrices with nonnegative entries. First we prove the following interesting result, which is originally due to F. Beukers (by communication with C. B. Huijsmans).

Theorem 2.4.1. *If T is a positive operator on a finite dimensional Banach lattice such that $\sigma(T) = \{1\}$, then $T \geq I$.*

Proof. Let $\dim(E) = n$. Then T can be represented as an $n \times n$ matrix with nonnegative entries. Let $A = T - I$. Then $A \geq -I$. Now let A be represented as an $n \times n$ matrix $(a_{ij})_{n \times n}$. Then $a_{ij} \geq 0$ for $i \neq j$ and $a_{ii} \geq -1$.

Consider A^2. Since $\sigma(A^2) = \{0\}$, the trace of A^2 is zero. But the trace of A^2 is the sum of its diagonal elements, all of which are nonnegative. In fact, the i-th diagonal element of A^2 is $\sum_{k=1}^{n} a_{ik}a_{ki}$ and each $a_{ik}a_{ki}$ is nonnegative. Therefore, all the diagonal elements of A^2 are zero. In particular, we obtain $a_{ii} = 0$. Hence all the entries of A are nonnegative and so $T - I \geq 0$. The proof is completed.

Corollary 2.4.2. *Let T be a positive operator on a finite dimensional Banach lattice such that $\sigma(T) \subseteq \{z : |z| = 1\}$. Then there exists a positive integer k such that $T^k \geq I$.*

Proof. Since T satisfies the condition (c), it follows from Theorem 2.2.1 that $T^k \geq a \cdot I$

for some positive integer k and some positive number a. Let $T^k = a \cdot I + B$. Since B is a positive operator, its spectral radius is an element of its spectrum. So we have $r(B) \leq 1 - a$. Hence $\sigma(T^k) \subset \{a + z : |z| \leq 1 - a\}$. On the other hand $\sigma(T^k) \subset \{z : |z| = 1\}$ by the spectral mapping theorem. So $\sigma(T^k) = \{1\}$. Now the result follows from the above theorem.

Remark. In the above proof we did not use the fact that $\sigma(T)$ is cyclic. From this fact we can conclude that there is some positive integer k such that $\sigma(T^k) = \{1\}$ and then the result follows from Theorem 2.4.1. For cyclic properties of the spectrum of positive operators on finite dimensional Banach lattices, we refer to [SH1]. The following theorem was proved for finite dimensional L^p-spaces in [SH3].

Theorem 2.4.3. *Let T be a positive contraction on a finite dimensional Banach lattice such that $\sigma(T) \subset \{z : |z| = 1\}$. Then T is an isometry.*

Proof. From Corollary 2.4.2 we see that there is a positive integer k such that $T^k \geq I$. Let $T^k = I + A$. Then A is a positive operator. By assumption T^k is a contraction and hence $||(I + A)^n|| \leq 1$ for all nonnegative integers n. Notice that $0 \leq n \cdot A \leq (I + A)^n$, from which we conclude that $A = 0$. Therefore, $T^k = I$ and so $T^{-1} = T^{k-1}$. So T^{-1} is also a contraction. Hence T is an isometry.

Corollary 2.4.4. *If T is a positive contraction on a finite dimensinal Banach lattice such that $\sigma(T) = \{1\}$ then $T = I$.*

In general Theorem 2.4.3 is not true for infinite dimensional Banach lattices even though T is a lattice homomorphism. We will give a counterexample to explain this in the following. But if T is a lattice homomorphism and a contraction at the same time and if its spectrum is not the whole unit circle then it is easy to show that T is an isometry. We put this result into the following proposition.

Proposition 2.4.5. *Let T be a lattice homomorphism on an arbitrary Banach lattice. If T is a contraction and if $\sigma(T)$ is properly contained in the unit circle then T is an isometry.*

Proof. By assumption we see that T satisfies the condition (c), so it follows from Corollary 2.2.3 that $T^k \in Z(E)$ for some positive integer k. So the spectrum of T^k consists of positive numbers. By the spectral mapping theorem $\sigma(T^k)$ is still contained in the unit circle, and so $\sigma(T^k) = \{1\}$. Now it follows from Theorem 1.3.1 that $T^k = I$ and so $T^{-1} = T^{k-1}$, from which we conclude that T^{-1} is also a contraction. Therefore, T is an isometry.

Example 2.4.6. Choose a continuous function $g : [0,1] \to [0,1]$ as follows

$$g(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1/5 \text{ or } 4/5 \le x \le 1, \\ 1/2 & \text{if } 2/5 \le x \le 3/5, \\ \text{linear} & \text{otherwise.} \end{cases}$$

Define a function $h : \mathbf{R} \to [0,1]$ by

$$h(x) = \begin{cases} 1 & \text{if } -\infty < x \le 0, \\ g(x) & \text{if } 0 \le x \le 1, \\ g(x - k^2) & \text{if } k^2 \le x \le k^2 + 1 (k = 1, 2, \cdots), \\ 1 & \text{otherwise.} \end{cases}$$

We now define $T : L_p(\mathbf{R}) \to L_p(\mathbf{R})$, where $1 \le p \le \infty$, by

$$Tf(x) = h(x)f(x+1) \text{ for all } x \in \mathbf{R}.$$

We prove that (1) T is a contraction but not an isometry; (2) T is a lattice isomorphism such that $\sigma(T) \subseteq \{z : |z| = 1\}$. From the way we define h, (1) and the first part of (2) are obvious. To prove that the spectrum of T is contained in the unit circle we need to prove that $r(T^{-1}) \le 1$. First observe that

$$T^{-n}f(x) = h^{-1}(x-1)h^{-1}(x-2)\cdots h^{-1}(x-n)f(x-n)$$

So $||T^{-n}|| \le \sup_{x \in \mathbf{R}} |h^{-1}(x)h^{-1}(x+1)\cdots h^{-1}(x+n-1)|$. For any $x \in \mathbf{R}$, the points $x, x+1, \cdots, x+n-1$ are contained in the interval $[x, x+n-1]$ of length $n-1$.

In $[x, x+n-1]$ there are at most $[\sqrt{n}]+2$ subintervals of the form $[i, i+1]$(i is a nonnegative integer) on which the function h is not constant. Therefore, $||T^{-n}|| \leq 2^{[\sqrt{n}]+2}$. This implies that $r(T^{-1}) \leq 1$. We have finished the proof. Hence, we give a negative answer to a question proposed by H. H. Schaefer in [H3](pp.75). From Proposition 2.4.5 one can see that the spectrum of these operators must be the whole unit circle.

Now we ask such a question: Let T be a positive contraction on a Banach lattice E such that its spectrum is properly contained in the unit circle $\{z : |z| = 1\}$. Is it true that T is an isometry on E? Theorem 2.4.3 shows that answer is positive if the Banach lattice is finite dimensional. In general, we obtain from Theorem 2.2.5 that $\sigma(T^k) = \{1\}$ for some positive integer k. If we can show that $\sigma(T^k) = \{1\}$ implies $T^k \geq I$, then it is easy to see that $T^k = I$, from which we can conclude that $||T^{-1}|| \leq 1$, and so T is an isometry. We will discuss this question in next section, where we prove the following theorem.

Theorem. *Let T be a positive contraction on a Banach lattice E such that $\sigma(T) \subset \{z : |z| = 1\}$ properly. If there exist constants $0 < \alpha < 1/2$ and c such that $||T^{-n}|| = O(\exp(cn^{\alpha}))$ as $n \to +\infty$, then T is an isometry on E.*

§2.5. On conditions under which $T \geq I$

Corollary 2.2.2 shows that if T is a lattice homomorphism on a Banach lattice such that $\sigma(T) = \{1\}$ then T is the identity operator. C. B. Huijsmans and Ben de Pagter asked the following question: Let T be a positive operator on a Banach lattice E such that $\sigma(T) = \{1\}$. Is it true that $T \geq I$, where I is the identity operator on E? So far this question is still open in its full generality. In this section we give some conditions for which this open question has a positive answer. Unless otherwise stated, the Banach lattices in this section are arbitrary (complex) Banach lattices which may be assumed to be order complete. We first prove some lemmas which we need to prove our main results in this section.

Lemma 2.5.1. *Let S be a positive operator on E. Then $0 \leq \Phi(S) \leq r(S) \cdot I$. In particular, if S is a quasi-nilpotent positive operator then $\Phi(S) = 0$.*

Proof. Let $S = P + B$ with $P \in Z(E)$ and $B \in Z(E)^d$. Then $0 \leq P \leq S$ and so $0 \leq P^n \leq S^n$. Hence, $r(P) \leq r(S)$. Since $P \in Z(E)$, we have $0 \leq P \leq r(P) \cdot I$. Now the result follows.

Lemma 2.5.2. *Let A be a regular operator such that $A \geq -I$ and $\sigma(A) = \{0\}$. Let $T(t) = exp(tA)$ be the semigroup generated by A. Then $T(t) \geq 0$ and $\sigma(T(t)) = \{1\}$ for all $t \geq 0$.*

Proof. $T(t) = exp(tA) = exp(t(I + A)) \cdot exp(-t) \geq exp(-t) \cdot I$. It follows from the spectral mapping theorem that $\sigma(T(t)) = \{1\}$ for all $t \geq 0$.

Lemma 2.5.3. *Let A be a regular operator such that $A \geq -I$ and $\sigma(A) = \{0\}$. If there exists a positive integer N such that $\Phi(A^n) = 0$ for all $n \geq N$ then $A \geq 0$.*

Proof. Consider the semigroup $T(t) = exp(tA)$ generated by A. By Lemma 2.5.1 and Lemma 2.5.2 we see that $0 \leq \Phi(T(t)) \leq I$ for all $t \geq 0$ and so

$$0 \leq \sum_{k=0}^{\infty} \frac{t^k \Phi(A^k)}{k!} \leq I.$$

By assumption we get

$$0 \leq \sum_{k=0}^{N-1} \frac{t^k \Phi(A^k)}{k!} \leq I$$

for all $t \geq 0$. Hence $\Phi(A^{N-1}) = \cdots = \Phi(A) = 0$. Let $A = P + B$ with $P \in Z(E)$ and $B \in Z(E)^d$. By the assumption that $A \geq -I$ it follows that $B \geq 0$. $\Phi(A) = 0$ implies that $P = 0$. Hence $A \geq 0$.

Remark. The above semigroup method is originally due to A. R. Schep (by communication with C. B. Huijsmans)

Lemma 2.5.4. *Let $\mathcal{A}$ be the subalgebra of $L(E)$ generated by T and I, where T is a positive operator with $\sigma(T) = \{1\}$. Then the following statements hold:*

(i) *$\mathcal{A}$ is the norm closure of $\{\sum_{i=0}^{n} a_i T^n : a_i \in \mathbf{C}, n \in \mathbf{N}\}$ and $T^{-n} \in \mathcal{A}$ for all $n \geq 1$;*

(ii) *$\mathcal{A}$ is contained in $\overline{L_r(E)}$.*

Proof. (i) The first statement is obvious and the second one follows from the following fact: $T^{-1} = \sum_{k=0}^{\infty}(-1)^k(T-I)^k$.
(ii) follows from (i).

Theorem 2.5.5. *Let T be a positive operator on a Banach lattice such that $\sigma(T) = \{1\}$. Let $\mathcal{A}$ be the subalgebra of $L(E)$ generated by T and I. If there exists an invertible operator $S \in \mathcal{A}$ such that $S(T-I)^k \geq 0$ for some positive integer k. Then $T \geq I$.*

Proof. Let $D = S(T-I)^k$ and let $A = T - I$. We want to show that $A \geq 0$. Notice that $A^k = S^{-1}D$, where $S^{-1} \in \mathcal{A}$. For any power T^i of T, since $\sigma(T^i D) = \{0\}$, it follows from Lemma 2.5.1 that $\Phi(T^i D) = 0$. Therefore, $\Phi(Q \cdot D) = 0$ for any $Q \in \mathcal{A}$ by (i) of Lemma 2.5.4. In particular, we have

$$\Phi(A^{k+l}) = \Phi(A^l \cdot S^{-1}D) = 0$$

for all nonnegative integers l. Now it follows from Lemma 2.5.3 that $A \geq 0$. So $T \geq I$.

We are now in position to prove the following theorem, which is one of the main results in this section:

Theorem 2.5.6. *Let T be a positive operator on a Banach lattice such that $\sigma(T) = \{1\}$. If there exists a nonzero complex function $f(z)$ that is analytic in a neighborhood of the number 1 such that* (i) $f(1) = 0$ *and* (ii) $f(T) \geq 0$. *Then $T \geq I$.*

Proof. Let $f(z) = (z-1)^k g(z)$ with $g(z)$ being analytic in a neighborhood of 1 such that $g(1) \neq 0$. Now $(T-I)^k g(T) \geq 0$ and $g(T)$ is invertible. If we can show that $g(T)$ and its inverse are in the subalgebra generated by T and I then the result follows from Theorem 2.5.5. To show that $g(T)$ and its inverse are in $\mathcal{A}$ we only need to observe the following:

$$g(T) = \frac{1}{2\pi i} \int_C g(z)(z-T)^{-1}\, dz$$

and a similar formula for $g(T)^{-1}$ with $g(z)$ being replaced by $1/g(z)$, where C is a smooth closed simple curve in a neighborhood of 1 with the number 1 in its interior.

From Theorem 2.2.1 we see that $T^n \geq (1-\varepsilon)^n \cdot I$ for some n. If we assume that $T^n \geq I$ for some n then we can show that $T \geq I$. If the Banach lattice is finite dimensional then by the Cayley-Hamilton Theorem the number 1 is a pole of the resolvent of T. Now if we assume that 1 is a pole of the resolvent of T for the general case then we can show that $T \geq I$. We collect all of these results and related ones in the next corollary.

Corollary 2.5.7. *Let T be as in Theorem* 2.5.6. *Then the following hold:*

(i) *If $T^2 \geq T$ or $T \geq T^2$ then $T \geq I$;*

(ii) *If $T^k \geq I$ for some positive integer k then $T \geq I$;*

(iii) *If 1 is a pole of the resolvent of T then $T \geq I$;*

(iv) *If $(T-I)^k \geq 0$ for some positive integer k then $T \geq I$.*

Proof. The results follow from Theorem 2.5.6. All we have to do is to choose the right functions $f(z)$ for all the above cases.

Next we show that if the negative powers of T do not grow too fast then the answer to the open question is positive.

Theorem 2.5.8. *Let T be a positive operator on an arbitrary Banach lattice E such that $\sigma(T) = \{1\}$. If there exist $0 < \alpha < 1/2$ and a constant c such that*

$$||T^{-n}|| = O(\exp(cn^{\alpha})) \text{ as } n \to +\infty$$

then $T \geq I$.

Proof. We may assume that E is order complete. By assumption there exists a bounded linear operator $A \in \overline{L^r(E)}$ such that $T^n = \exp(nA)$ for any integer n. In fact,

$$A = \log(T) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(T-I)^n}{n}.$$

Now consider the operator valued function $f(z)$ which is given by

$$f(z) = \Phi(\exp(zA)) \text{ for all } z \in \mathbf{C}.$$

By Theorem 1.3.2 and the remark that follows this theorem, we can conclude that $f(z)$ is an entire function. Since A is a quasi-nilpotent operator, it is easy to see that $f(z)$ is an entire function of zero exponential type (see [BO] for definitions).

If n is any positive integer, then $||f(n)|| = ||\Phi(T^n)|| \leq 1$ by Lemma 2.1.4. So by Cartwright's theorem (see [BO], pp. 180) there exists a constant M_1 such that $||f(x)|| \leq M_1$ for any real numbers $x \geq 0$. If $x < 0$ and if $x = -n + t$, where n is a positive integer and $0 \leq t \leq 1$, then it follows from Theorem 1.3.2 that

$$||f(x)|| = ||\Phi(T^{-n}\exp(tA))|| \leq ||T^{-n}|| \cdot ||\exp(tA)||$$

$$\leq M_2M_3\exp(cn^{\alpha}) \leq M_2M_3\exp(c)\exp(c|x|^{\alpha}),$$

where $M_2 = \sup_{0\leq t\leq 1} ||\exp(tA)||$ and M_3 is given by the condition in the theorem. Therefore, there exists a constant M such that

$$||f(x)|| \leq M\exp(c|x|^{\alpha}) \text{ for all real numbers } x.$$

Now by theorem 6.6.9 in [BO], pp. 97 there exist constants K and b such that

$$||f(z)|| \leq K\exp(b|z|^{\alpha}) \text{ for all complex numbers } z.$$

Next we consider the entire function $g(z) = f(z^2)$. It is easy to see that for any $\varepsilon > 0$ there exists a constant L depending on ε such that

$$||g(z)|| \leq K\exp(b|z|^{2\alpha}) \leq L\exp(\varepsilon|z|)$$

since $2\alpha < 1$ by assumption. Moreover $||g(\pm n)|| = ||f(n^2)|| = ||\Phi(T^{n^2})|| \leq 1$. So it follows from a well-known theorem in the theory of entire functions (see [BO], pp. 183) that $g(z)$ is a constant function, and so $f(z)$ is a constant function. In particular, we have $f(0) = f(1)$, i.e., $\Phi(T) = I$. Therefore, $T \geq I$. The proof is completed.

Corollary 2.5.9. *Let T be a positive operator on a Banach lattice such that $\sigma(T) = \{1\}$. If there exists a positive number k such that*

$$||\Phi(T^{-n})|| = O(n^k) \text{ as } n \to +\infty,$$

then $T \geq I$. Conversely, if $T \geq I$, then $\Phi(T^n) = I$ for any integer n.

Proof. The proof is similar as above. Using the same notations, we consider the entire function $f(z)$ given by above. Then by assumption we have

$$||f(\pm n)|| = ||\Phi(T^{\pm n})|| = O(n^k).$$

Now it follows from theorem 10.2.11 in [BO], pp. 183 that $f(z)$ is a polynomial of degree not exceeding $[k] + 1$. Observe that $||f(n)|| = ||\Phi(T^n)|| \leq 1$ for any positive integer n. So $f(z)$ must be a constant function. In particular we obtain $\Phi(T) = I$. So $T \geq I$.

Conversely, if $T \geq I$, then $T = I + A$ for some positive quasi-nilpotent operator A. Since any positve quasi-nilpotent operator belongs to $Z(E)^d$ by Lemma 2.5.1, we see that $\Phi(T^n) = I$ for any positive integer n. Now $T^{-1} = I + \sum_{k=1}^{\infty}(-1)^k A^k$ and $\sum_{k=1}^{\infty} A^k$ is a positive quasi-nilpotent operator, so $\Phi(T^{-n}) = I$ for any positive integer n. The proof is finished.

Remark 1. It follows from Theorem 1.3.2 that the condition in Corollary 2.5.9 will be satisfied if $||T^{-n}|| = O(n^k)$. This is the case when T is a lattice homomorphism such that $\sigma(T) = \{1\}$ or when T^{-1} is a power bounded operator. Moreover, if 1 is a pole of the resolvent of T, then it is well known that $||T^{\pm n}|| = O(n^k)$ for some positive integer k.

Remark 2. The invariant subspace problem for those operators on Banach spaces that satisfy the growth condition in Theorem 2.5.8 has been studied in [AT]. Some of the ideas used in the proofs of the above results are borrowed from this paper.

Theorem 2.5.10. *Let T be a positive contraction on a Banach lattice E such that $\sigma(T) \subset \{z : |z| = 1\}$ properly. If T satisfies the condition in Theorem* 2.5.8 *or in Corollary* 2.5.9 *then T is an isometry on E. In particular, if $\sigma(T) = \{1\}$, then T is the identity operator on E.*

Proof. It follows from Theorem 2.2.5 that $\sigma(T^N) = \{1\}$ for some positive integer N. Now notice that T^N also satisfies the growth condition in Theorem 2.5.8 or in Corollary 2.5.9. So by Theorem 2.5.8 or Corollary 2.5.9 we have $T^N \geq I$. Now let $T^N = I + A$. Then A is a positive operator on E. Since T is a contraction, we have $||nA|| \leq ||(I + A)^n|| \leq 1$ for any positive integer n. So $A = 0$ and $T^N = I$. Thus $T^{-1} = T^{N-1}$, from which we see that $||T^{-1}|| \leq 1$. So T is an isometry.

Chapter 3

Decomposition Theorems

In this chapter we prove two different types of decomposition theorems for disjointness preserving operators on Banach lattices. The first one is concerned with the decomposition of a disjointness preserving operator whose spectrum is contained in a sector of angle less than π, and the second one deals with the decomposition of a disjointness preserving operator under the assumption that its adjoint is also a disjointness preserving operator. These results generalize some earlier ones.

§3.1. Basic Properties of Disjointness Preserving Operators

In this section we introduce disjointness preserving operators and give, without proofs, some basic properties of these types of operators. Proofs for these properties can be found in the indicated references. As in [A] and [AH], we introduce the following definition.

Definition 3.1.1. *An order bounded operator T on a Banach lattice E is called a disjointness preserving or a Lamperti operator if $Tx \perp Ty$ whenever $x, y \in E$ such that $x \perp y$.*

The class of disjointness preserving operators plays an important role in the study of the spectral properties of operators, and it contains many important operators as the following examples show.

Example 3.1.2. Let T be an $n \times n$ matrix over $\mathbf{C}$, the set of all complex numbers. If each row as well as each column of T contains at most one nonzero number, then

T is a disjointness preserving operator.

Example 3.1.3. Let X, Y be two compact spaces and let $T : C(X) \to C(Y)$ be a linear mapping. Then T is a disjointness preserving operator if and only if there exists a map $\phi : Y \to X$ and a function $h \in C(Y)$ such that

$$Tf(t) = h(t)f(\phi(t)) \text{ for all } t \in Y \text{ and all } f \in C(X),$$

where $h = T1_X$ (1_X is the constant one function on X) and ϕ is uniquely determined and continuous on $Y_0 = \{t \in Y : h(t) \neq 0\}$. This result was proved for lattice homomorphisms in [WO] and generalized to this setting in [A].

Example 3.1.4. (i) Weighted shift operators on $E = l_p(\mathbf{N})$ or $l_p(\mathbf{Z})$ with $1 \leq p \leq \infty$ are disjointness preserving operators; (ii) All isometries on $L_p(X)$ ($1 \leq p \leq \infty, p \neq 2$, and X is a σ-finite measure space) are disjointness preserving operators. The proof for (i) is straightforward by the definition. (ii) was shown by Banach ([BA], pp.175) for $X = [0, 1]$ and by Lamperti ([LA]) for the σ-finite case.

Disjointness preserving operators have close relations with lattice homomorphisms. In fact, lattice homomorphisms are exactly those positive disjointness preserving operators. Also notice that if T is a disjointness preserving operator then T^n is also a disjointness preserving operator for any positive integer n. The following proposition collects some basic properties of disjointness preserving operators.

Proposition 3.1.5. 1. *Let T be an order bounded operator on a Banach lattice E. Then the following assertions are equivalent:*

(i) *T is a disjointness preserving operator;*

(ii) *$|T|$ exists and satisfies $|Tz| = ||T|z| = |T||z|$ for all $z \in E$. In particular, $|T|$ is a lattice homomorphism.*

2. *Let T be a disjointness preserving operator on E. Then the following statements hold:*

(i) *T' has a modulus and $|T'| = |T|'$;*

(ii) *T is invertible if and only if $|T|$ is invertible. Moreover, if T is invertible, then T^{-1} and T' are disjointness preserving operators and $|T^{-1}| = |T|^{-1}$.*

3. *If T is a disjointness preserving operator, then $|T^n| = |T|^n$ for any positive integer n.*

For the proofs of the statement 1 and 2, we refer to [A]. Statement 3 follows from (ii) of statement 1 and mathematical induction.

Using the techniques that we developed in Section 1 of Chapter 2 and the above proposition, we can prove the following result, which is a generalization of Corollary 2.2.3, and which is essentially due to [AH] (Proposition 5.4). The proof we give here is completely different.

Proposition 3.1.6. *If T is a disjointness preserving operator on a Banach lattice E such that T satisfies the condition* (c), *then there exists a positive integer k such that $T^k \in Z(E)$. In particular, if T is a disjointness preserving operator with $\sigma(T) = \{1\}$, then $T = I$.*

Proof. Without loss of generality we may assume that E is order complete. It follows from Proposition 2.1.3 of Chapter 2 that $F(z) := \Phi[(zI - T)^{-1}]$ is analytic in $\rho_\infty(T)$. Now, for $|z| > r(T)$, $F(z) = \sum_{n=0}^{\infty} P_n/z^{n+1}$, where $P_n = \Phi(T^n)$. If we identify the center $Z(E)$ with $C(X)$ for some compact space X and identify P_n as functions on X, then as in the proof of Lemma 2.1.5 of Chapter 2 we can show that

$$\overline{\lim}_{n\to\infty}[|P_n|(x)]^{1/n} \geq r(T^{-1})^{-1} \text{ for all } x \in X.$$

Since Φ is a lattice homomorphism it follows from Proposition 3.1.5 that

$$|P_n| = |\Phi(T^n)| = \Phi(|T^n|) = \Phi(|T|^n).$$

Hence, we have

$$|P_n| \cdot |P_m| = \Phi(|T|^n) \cdot \Phi(|T|^m) \leq \Phi(|T|^{n+m}) = |P_{n+m}|$$

for all nonnegative integers n and m. As in the proof of Theorem 2.2.1 of Chapter 2 we can show that there exists a positive number a and a positive integer k such that $|P_k| \geq a \cdot I$. Therefore, $|T|^k \geq a \cdot I$. Now by Proposition 3.1.5 we obtain the following

$$|T^{-k}| \leq a^{-1} \cdot |T|^k \cdot |T^{-k}| = a^{-1} \cdot I$$

Thus $T^{-k} \in Z(E)$ and so $T^k \in Z(E)$ by Theorem 1.3.1 of Chapter 1.

If $\sigma(T) = \{1\}$ then by what we just proved and by the spectral mapping theorem we have $T^k = I$. Now it is routine to conclude that $T = I$. The proof is completed.

The following proposition is an immediate consequence of Proposition 3.1.5(ii) and the definition of compactness. We will need this result in Section 4 for the study of reducibility of disjointness preserving operators.

Proposition 3.1.7. *Let T be a disjointness preserving operator on an arbitrary Banach lattice E. Then T is compact if and only if $|T|$ is compact.*

Proof. First assume that T is compact. Let $\{x_n\}$ be a norm bounded sequence in E. Then it follows from the compactness of T that there exists a subsequence of $\{x_n\}$, say $\{x_n\}$, such that $||Tx_n - Tx_m|| \to 0$ as $n, m \to \infty$. By Proposition 3.1.5(ii), we have

$$|\,|T|(x_n - x_m)\,| = |T(x_n - x_m)|$$

This implies that $||\,|T|(x_n - x_m)\,|| \to 0$ as $n, m \to \infty$. So $|T|$ is compact. Similarly, we can prove the other direction.

§3.2. Decomposition Theorems of Type I

In this section we will show that if T is an order continuous disjointness preserving operator on a Banach lattice having the Fatou norm and if its spectrum is contained in a sector of angle less than π then T can be decomposed into the sum of its central part and its quasi-nilpotent part. As consequences of this result we deduce various properties of disjointness preserving operators. In this section E will always be an order complete Banach lattice unless otherwise is stated.

Proposition 3.2.1. *Let T be an order continuous lattice homomorphism on E. Then the following statements hold:*

(i) $T(T_1 \wedge T_2) = (TT_1) \wedge (TT_2)$ *for any* $T_1, T_2 \in L_r(E)$. *In particular,* $(TT_1) \wedge (TT_2) = 0$ *when* $T_1 \wedge T_2 = 0$;

(ii) *If* $T = P + B$ *with* $P \in Z(E)$ *and* $B \in Z(E)^d$, *then* $PB = 0$. *In general,* $BP \neq 0$. *But if, in addition, the adjoint* T' *is also a lattice homomorphism, then* $BP = 0$ *and* $P' \perp B'$ *in* $L^r(E')$.

Proof. The proof for (i) is standard and can be found, for example, in [AB], pp.91, Theorem 7.5. The proof for (ii) is as follows. By (i), we have

$$0 = T(I \wedge B) = T \wedge TB = (P + B) \wedge (PB + B^2) \geq B \wedge PB \geq (a \cdot P)B \geq 0,$$

where a is a small positive number such that $0 \leq aP \leq I$. Therefore, $PB = 0$.

We will give an example to show that $BP \neq 0$ in general (see below). Now let us prove the last statement. Assume that T' is also a lattice homomorphism. Then $T' = P' + B'$ with $P' \in Z(E')$ since the center is the band generated by the identity operator. Let $B' = C + D$ with $C \in Z(E')$ and $D \in Z(E')^d$. Then $T' = (P' + C) + D$. Since an adjoint operator is always order continuous it follows from the first statement of (ii) that $(P' + C)D = 0$. Notice that all these operators are positive, and so we have

$P'D = 0$ and $CD = 0$. Therefore, $(BP)' = P'B' = P'C + P'D = P'C$. Notice that $PB = 0$ implies $B'P' = 0$ and so $CP' = 0$. Since $Z(E')$ is commutative, $P'C = CP'$. Hence, $(BP)' = 0$ and so $BP = 0$. Finally, since $P'C = 0$ and since P', C are in the center, we have $P' \perp C$. Now $D \in Z(E')^d$ implies that $P' \perp B'$.

Corollary 3.2.2. *Let T be an order continuous disjointness preserving operator on E. Suppose that $T = P + B$ with $P \in Z(E)$ and $B \in Z(E)^d$. Then $|P| \cdot |B| = 0$. In particular, $P \cdot B = 0$. If, in addition, T' is also a disjointness preserving operator, then $|B| \cdot |P| = 0$ and $P' \perp B'$.*

Proof. By Statement 1 of Proposition 3.1.5 we know that $|T|$ is an order continuous lattice homomorphism. Since $|T| = |P| + |B|$ with $|P| \in Z(E)$ and $|B| \in Z(E)^d$, it follows from Proposition 3.2.1 that $|P| \cdot |B| = 0$. Finally, by Proposition 3.1.5 again, $|T|' = |T'|$ is also a lattice homomorphism. So by Proposition 3.2.1 one has $|B| \cdot |P| = 0$ and $P' \perp B'$.

Next we will give some examples of lattice homomorphisms and show that in (ii) of Proposition 3.2.1 $BP \neq 0$ in general and that the order continuity of T cannot be omitted in (i) of the same proposition.

Example 3.2.3a. Let $E = \mathbf{C}^2$ and choose

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then T is an order continuous lattice homomorphism on E. In this case,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that $PB = 0$, but $BP \neq 0$.

Example 3.2.3b. Let X be a compact Hausdorff space that is extremely disconnected and whose topology is not discrete (this is possible, see [SH1], pp. 107-108).

Let $E = C(X)$. Then E is an order complete Banach lattice. Let $x_0 \in X$ be a nondiscrete point, i.e., any neighborhood of x_0 contains more than one point. Let $\phi \in E'$ be the Dirac measure at x_0, that is $\phi(f) = f(x_0)$ for all $f \in E$. Now define a linear operator on E by $Tf(x) = \phi(f)1_X$ for any $f \in E$ and any $x \in X$, where 1_X is the constant one function on X. Then one has the following: (i) T is a lattice homomorphism; (ii) $T^2 = T$; and (iii) $T \perp I$.

The proof for (i) and (ii) are straightforward. We now prove (iii). Since X is extremely disconnected, there exists a decreasing net $\{U_\alpha\}$ of neighborhoods of x_0 such that each U_α is open and closed in X and that $\cap_\alpha U_\alpha = \{x_0\}$. Let h_α be the characteristic function of U_α. Then $h_\alpha \in C(X)$. Now by Proposition 1.1.2 of Chapter 1 one has

$$T \wedge I(1_X) = \inf\{T(f) + g : \ f, g \geq 0 \text{ and } f + g = 1_X\}$$

$$\leq \inf_\alpha\{T(1_X - h_\alpha) + h_\alpha\} = \inf_\alpha\{h_\alpha\} = 0,$$

since x_0 is a nondiscrete point of X. Therefore, $T \wedge I = 0$. Finally, we notice that $T^2 \wedge T = T$, so the conclusion of (i) of Proposition 3.2.1 fails.

Before we state the main result of this section we need to fix some notations. For $r_0 > 0, \theta_0, \theta_1 \in [0, 2\pi)$, we call the subset

$$\Delta(r_0, \theta_0, \theta_1) = \{re^{i\theta} : \ 0 \leq r \leq r_0, \theta_0 \leq \theta \leq \theta_0 + \theta_1\}$$

of the complex plane **C** a sector. Here θ_1 is called the angle of the sector.

Proposition 3.1.6 shows that if a disjointness preserving operator T satisfies the condition (c) then some power of T will be in the center $Z(E)$. Now we would like to know under what conditions a disjointness preserving operator itself will be in the center. It is shown in [A] that if T is an invertible disjointness preserving operator such that its spectrum is included in a sector of angle less then $2\pi/3$ then $T \in Z(E)$. Some generalization of this result has been obtained in [AH] for quasi-invertible disjointness preserving operators. Here we obtain the following more general result. Remember that any order continuous Banach lattice as well as any dual Banach lattice has the Fatou norm. Also, any $C(X)$-space has the Fatou norm.

Theorem 3.2.4. (Decomposition Theorem I) *Let T be an order continuous disjointness preserving operator on an order complete Banach lattice E with Fatou norm. If $\sigma(T)$ is included in a sector Δ of angle less than π then T can be decomposed into $T = P + B$ such that*

(i) *$P \in Z(E)$ and $|B|$ (hence B) is a quasi-nilpotent operator such that $|P| \cdot |B| = 0$;*

(ii) *$\sigma(T) \subseteq \sigma(P) \subseteq \sigma_\infty(T)$, where $\sigma_\infty(T) = \mathbf{C} \setminus \rho_\infty(T)$. So $P \neq 0$ if and only if T is not quasi-nilpotent.*

To prove the above theorem we need to do some preparation. As already mentioned, the center $Z(E)$ of E can be identified with $C(X)$ for some compact space X. So each $P \in Z(E)$ can be regarded as a continuous function on X, and thus we can write $P(x)$ for any $x \in X$ to denote the value of the function P at the point $x \in X$. From now on we will keep this in mind.

In the following lemmas we will assume that T is an order continuous disjointness preserving operator on an order complete Banach lattice E.

Lemma 3.2.4a. *Suppose that E has the Fatou norm and suppose that $\sigma(T)$ is included in a sector Δ of angle less than 2π. If T is not quasi-nilpotent, then there exists a positive integer k such that $\Phi(T^k) \neq 0$, i.e. $|T^k| \wedge I \neq 0$.*

Proof. Assume on the contrary that $|T^k| \wedge I = 0$ for all positive integers k. By Proposition 3.1.5 this implies that $|T|^k \wedge I = 0$ for all k. Then by Proposition 3.2.1, we have $|T|^i \wedge |T|^j = 0$ for $i \neq j$. Now it follows from a result in [HP] that the peripheral spectrum of T is equal to $\{z : |z| = r(T)\}$. Since $r(T) > 0$, this contradicts to the assumption.

Lemma 3.2.4b. *Let $T^k = P_k + B_k$ with $P_k \in Z(E)$ and $B_k \in Z(E)^d$. Identify $Z(E)$ with $C(X)$. If $P_1(x_0) \neq 0$ for some $x_0 \in X$, then $P_k(x_0) = [P_1(x_0)]^k$ for all positive integers k.*

Proof. We prove the Lemma by induction. First we observe that both $|C| \cdot |D|$ and $|D| \cdot |C|$ belong to $Z(E)^d$ for any $C \in Z(E)$ and any $D \in Z(E)^d$. Let $k = 2$. Then $P_2 = P_1^2 + \Phi(B_1^2)$. It follows from Proposition 3.2.1 (i) that

$$0 = |T||P_1| \wedge |T||B_1| = (|P_1|^2 + |B_1||P_1|) \wedge (|P_1||B_1| + |B_1|^2)$$

$$\geq |P_1|^2 \wedge \Phi(|B_1|^2) \geq 0.$$

So $P_1^2 \perp \Phi(B_1^2)$ in $Z(E)$. Since $P_1^2(x_0) \neq 0$ by assumption, we must have $\Phi(B_1^2)(x_0) = 0$. Now $P_2(x_0) = P_1^2(x_0) + \Phi(B_1^2)(x_0) = P_1^2(x_0)$.

Assume that the statement is true for k, i.e., $P_k(x_0) = P_1^k(x_0)$. First observe that $P_{k+1} = P_1 P_k + \Phi(B_1 B_k)$. By Proposition 3.2.1 (i) again, we have

$$0 = |T||P_k| \wedge |T||B_k| = (|P_1||P_k| + |B_1||P_k|) \wedge (|P_1||B_k| + |B_1||B_k|)$$

$$\geq |P_1||P_k| \wedge \Phi(|B_1||B_k|) \geq 0.$$

So $P_1 P_k \perp \Phi(B_1 B_k)$ in $Z(E)$. Now $P_1 P_k(x_0) \neq 0$ implies that $\Phi(B_1 B_k)(x_0) = 0$. Hence, $P_{k+1}(x_0) = P_1 P_k(x_0) = P_1^{k+1}(x_0)$. The proof is finished.

Lemma 3.2.4c. *Let $T^k = P_k + B_k$ be the same as above. If E has the Fatou norm and if $\sigma(T)$ is included in a sector Δ of angle less than π, then $P_k = P_1^k$ for all positive integers k.*

Proof. Fix any $x \in X$. If $P_k(x) = 0$, then $P_1(x) = 0$, since $|P_k| = \Phi(|T^k|) = \Phi(|T|^k) \geq [\Phi(|T|)]^k = |P_1|^k$ by Statement 3 of Proposition 3.1.5. Now assume that $P_k(x) \neq 0$ for some k. If we can show that $P_1(x) \neq 0$ then it follows from

Lemma 3.2.4b that $P_k(x) = P_1^k(x)$, and the proof will be finished. If $P_1(x) = 0$ then there is a positive integer $N \geq 2$ such that $P_1(x) = \cdots = P_{N-1}(x) = 0$ and $P_N(x) \neq 0$. Then we claim that if n is not a multiple of N then $P_n(x) = 0$. The proof is as follows. Let $lN + 1 \leq n \leq (l+1)N - 1$ for some interger l. Then $P_{n-lN}(x) = 0$ by assumption. It follows from Proposition 3.2.1(i) and Proposition 3.1.5 that

$$0 = |T^{lN}|(|B_{n-lN}| \wedge I) = |T^{lN}||B_{n-lN}| \wedge |T^{lN}| \geq \Phi(|B_{lN}||B_{n-lN}|) \wedge |P_{lN}| \geq 0.$$

Since $|P_{lN}|(x) \geq |P_N|^l(x) > 0$, we have $\Phi(B_{lN}B_{n-lN})(x) = 0$. Notice that

$$P_n = P_{lN}P_{n-lN} + \Phi(B_{lN}B_{n-lN}).$$

Thus $P_n(x) = 0$ from above. So we have proved our claim.

Next we consider the function $f(z) := \Phi[(zI - T)^{-1}](x)$ defined in $\rho_\infty(T)$ (which contains $\mathbf{C} \setminus \Delta$). By Proposition 2.1.3 of Chapter 2, we see that $f(z)$ is analytic in $\rho_\infty(T)$, and for $|z| > r(T)$,

$$f(z) = \sum_{n=0}^{\infty} \frac{P_n(x)}{z^{n+1}}.$$

Since $P_n(x) = 0$ for any n which is not a multiple of N, we have, for $|z| > r(T)$,

$$f(z) = \frac{z^{N-1}}{z^N - P_N(x)}.$$

Notice that $\mathbf{C} \setminus \Delta \subseteq \rho_\infty(T)$ in which $f(z)$ is analytic. So the function $z^{N-1}(z^N - P_N(x))^{-1}$, which is equal to $f(z)$ for $|z| > r(T)$, can be extended to $\rho_\infty(T)$. This is a contradiction, since $z^N - P_N(x) = 0$ has N solutions, some of which will be in $\rho_\infty(T)$ when $N \geq 2$. Therefore, we proved that $P_1(x) \neq 0$.

Lemma 3.2.4d. *Let B be an order bounded operator on E. Suppose that $|B|$ is a quasi-nilpotent operator. Then PB and BP are quasi-nilpotent operators for any $P \in Z(E)$.*

Proof. Fix $P \in Z(E)$. Then there exists a positive number a such that $|P| \leq a \cdot I$. So

$|(PB)^n| \leq a^n|B|^n$. Since $|B|$ is quasi-nilpotent, PB is also quasi-nilpotent. Similarly, we can show that BP is a quasi-nilpotent operator.

Let P be a positive operator belonging to the center $Z(E)$. Since $Z(E)$ is order complete, the operator $P_0 := \sup\{nP \wedge I : n = 1, 2, \cdots\}$ exists. It is easy to see that $P_0^2 = P_0$, so P_0 is a band projection. Actually, $Z(E)$ can be identified with $C(X)$ for some compact extremelly disconnected space (i.e., every open subset has an open closure, see [SH1] pp. 107-108), and so P is a continuous function on X. The closure of the open subset $\{x \in X : P(x) \neq 0\}$ is closed and open in X, so its characteristic function is a continuous function on X, whose corresponding operator in $Z(E)$ is exactly P_0. The operator P_0 has the following property:

Proposition 3.2.5. *Let $P \in Z(E)$ be a positive operator and let S be any positive operator on E. Then $PS = 0$ if and only if $P_0S = 0$. Similarly, if S is order continuous, then $SP = 0$ if and only if $SP_0 = 0$.*

Proof. Assume that $PS = 0$. Then for any $f \in E_+$, we have $0 \leq (nP \wedge I)Sf \leq nP(Sf) = 0$. Since $P_0(Sf) = \lim_{n\to\infty} nP \wedge I(Sf) = 0$, we see that $P_0S = 0$. On the other hand, if we assume that $P_0S = 0$, then $(nP \wedge I)S = 0$ for all n. In particular, $(P \wedge I)S = 0$. Choose a small positive integer a such that $0 \leq aP \leq I$. Then $P \wedge I \geq aP$ and so $aPS = 0$. The proof for the second part of the Proposition is similar, and hence will be omitted.

Finally, we prove a lemma that we will use later. Basically, it means that under certain conditions the spectrum of the restriction of an operator to its non-trivial invariant subspaces will be closely related to the original spectrum of the operator. The following lemma is not stated in its most general form.

Lemma 3.2.6. *Let Y be a Banach space and let T be a bounded operator on Y.*

Let Y_0 be a non-trivial invariant subspace of T. If $\sigma(T)$ is contained in a sector Δ of angle less than 2π, then $\sigma(T|_{Y_0}) \subseteq \Delta$, where $T|_{Y_0}$ is the restriction of T to Y_0.

Proof. If the statement were not true then $\sigma(T|_{Y_0})$ would have a boundary point, say λ, which does not belong to Δ. Since boundary points of the spectrum of a bounded operator are approximate eigenvalues (see [BE] pp.24), λ is an approximate eigenvalue of $T|_{Y_0}$, and thus an approximate eigenvalue of T. This is a contradiction to the assumption that $\sigma(T) \subseteq \Delta$.

Proof of Theorem 3.2.4. (i) If T is quasi-nilpotent, then $|T|$ is also quasi-nilpotent by Statement 3 of Proposition 3.1.5, and so $P = 0$ and $B = T$. In the following we assumue that T is not quasi-nilpotent. Let $T = P + B$ with $P \in Z(E)$ and $B \in Z(E)^d$. By Corollary 3.2.2, we have $|P||B| = 0$.

Let $P_0 := \sup\{n|P| \wedge I : n = 1, 2, \cdots\}$. Then P_0 is a band projection by the remark before Proposition 3.2.5, and it follows from Proposition 3.2.5 that $P_0|B| = 0$. Let E_0 be the band of E associated with P_0. Then we have $P_0|B|(E) = \{0\}$, which implies that $|B|(E) \subseteq E_0^d$.

We may assume that $E_0^d \neq \{0\}$, otherwise $B = 0$. Now consider the operator $B|_{E_0^d}$. Since $0 \leq |B| \leq |T|$, we see that B is also an order continuous disjointness preserving operator on E, and therefore, $B|_{E_0^d}$ is an order continuous disjointness preserving operator on E_0^d. Moreover, since $P(E_0^d) = \{0\}$, we see that E_0^d is an invariant band of T and $T = B$ on E_0^d. Therefore, $\sigma(B|_{E_0^d})$ is also included in the sector Δ by Lemma 3.2.6. Since $|B| \wedge I = 0$ (i.e. $\Phi(B) = 0$), by applying Lemma 3.2.4c to B we see that $B^k \perp I$ for all positive integers k, and hence $r(B|_{E_0^d}) = 0$ by Lemma 3.2.4a. So we showed that the restriction of B to E_0^d is a quasi-nilpotent operator. In the above we proved that $B(E) \subseteq E_0^d$, so for any $f \in E$, we have

$$||B^{n+1}f|| = ||B^n(Bf)|| \leq ||B^n||_0||Bf||,$$

where $||B^n||_0$ means the norm of B^n when restricted to E_0^d. Hence, $||B^{n+1}|| \le ||B^n||_0 \cdot ||B||$. From this we see that B is a quasi-nilpotent operator on E. By Statement 3 of Proposition 3.1.5, we see that $|B|$ is also a quasi-nilpotent operator. So we completed the proof for (i).

(ii) We now prove that $\sigma(T) \subseteq \sigma(P) \subseteq \sigma_\infty(T)$. First we show that $\sigma(P) \subseteq \sigma_\infty(T)$. As in the proof of Lemma 3.2.4c, we consider the analytic function $f(z) := \Phi[(zI - T)^{-1}](x)$ defined in $\rho_\infty(T)$ for any fixed $x \in X$. By Lemma 3.2.4c, we have, for $|z| > r(T)$,

$$f(z) = (z - P(x))^{-1}.$$

Hence, $(z - P(x))^{-1}$ is analytic in $\rho_\infty(T)$, from which we conclude that $P(x)$ is not in $\rho_\infty(T)$, i.e., $P(x) \in \sigma_\infty(T)$. Since $\sigma(P) = \{P(x) : x \in X\}$, we have $\sigma(P) \subseteq \sigma_\infty(T)$.

Now we prove the other inclusion. To prove that $\rho(P) \subseteq \rho(T)$, we let $z \in \rho(P)$. Then we have

$$zI - T = (zI - P)[I - (zI - P)^{-1}B].$$

By Lemma 3.2.4d and by what we just proved, $(zI - P)^{-1}B$ is a quasi-nilpotent operator. So $zI - T$ is invertible. Hence, we proved that $\rho(P) \subseteq \rho(T)$. Therefore, $\sigma(T) \subseteq \sigma(P)$. The proof is finished.

Remark. In the above decomposition theorem, $BP \neq 0$ in general. We already gave an example (see Example 3.2.3a) to show this. Moreover, the angle π cannot be replaced by any bigger number. For instance, Let $E = L^p(\mathbf{R})$ with $1 \le p \le \infty$ and define a linear map T on E by $Tf(x) = f(-x)$ for any $f \in E$ and any $x \in \mathbf{R}$. Then T is a lattice isomorphism such that (i) $T^2 = T$ and (ii) $T \perp I$, and thus $\sigma(T) = \{-1, 1\}$. Here $\sigma(T)$ is contained in a sector of angle π. If we write $T = P + B$ with $P \in Z(E)$ and $B \in Z(E)^d$, then $P = 0$, $B = T$ and B is not quasi-nilpotent.

If the adjoint T' of T is also a disjointness preserving operator, then Proposition 3.2.1(ii) shows that $BP = 0$, and in this case we have the following decomposition theorem. First recall that a band H of E is said to T-reducing if both H and H^d are T-invariant.

Theorem 3.2.7. *Let T be an order continuous disjointness preserving operator on E with the Fatou norm. Suppose that $\sigma(T)$ is contained in a sector Δ of angle less than π and suppose that the adjoint T' is also a disjointness preserving operator. Then there exist T-reducing bands H and K such that*

(i) *$E = H \oplus K$;*

(ii) $T|_H \in Z(H)$ *and* $T|_K$ *is quasi-nilpotent.*

Proof. By Theorem 3.2.4, $T = P + B$ with $P \in Z(E)$ and $B \in Z(E)^d$ such that $|P||B| = 0$ and that $|B|$ is a quasi-nilpotent operator. Consider the adjoint T' of T. Then $T' = P' + B'$ with $P' \in Z(E')$ and $B' \in Z(E')^d$ since $|B'| = |B|'$ (by Proposition 3.1.5) is also quasi-nilpotent. On the other hand, since E' has the Fatou norm and since T' is an order continuous disjointness preserving operator such that $\sigma(T') = \sigma(T)$, by applying Theorem 3.2.4 to T' we see that $T' = C + D$ with $C \in Z(E')$ and $D \in Z(E')^d$ such that $|C||D| = 0$. Therefore, $P' = C$ and $B' = D$, i.e., $|P'||B'| = 0$. Now notice that $(|B||P|)' = |P|'|B|' = |P'||B'| = 0$ by Proposition 3.1.5. Hence, $|B||P| = 0$.

Let $P_0 = \sup\{n|P| \wedge I : n = 1, 2, \cdots\}$. Then we know that P_0 is a band projection on E. By Proposition 3.2.5, we have $P_0 B = BP_0 = 0$. Let H be the associated band with P_0. Then the above shows that H and $K := H^d$ are T-reducing. Now it is easy to see that $T|_H = P|_H$ and $T|_K = B|_K$. So the proof is finished.

Corollary 3.2.8. *Let T be a disjointness preserving operator on an arbitrary Banach lattice F such that* (i) *T is invertible and* (ii) *$\sigma(T)$ is contained in a sector Δ of angle less than π. Then $T \in Z(F)$.*

Proof. (i) It follows from Proposition 3.1.5 that T' is an invertible disjointness preserving operator on F', which has the Fatou norm. Also, $\sigma(T')$ is contained in the same sector. Now by Theorem 3.2.4(i) we have $T' = C + D$, where $C \in Z(F')$ and $D \in Z(F')^d$ such that $CD = 0$. Since in this case $0 \in \rho_\infty(T')$, it follows from (ii) of the same theorem that $0 \in \rho(C)$, which implies that C is invertible. Now $CD = 0$ implies that $D = 0$. Hence, $T' = C \in Z(F')$ and $T'' \in Z(F'')$. Since F is a closed sublattice of F'', we have $T \in Z(F)$.

Remark 1. In Theorem 3.2.7 the order completeness is required. Consider the Banach lattice c, the space of all convergent sequences. It is easy to see that c is not order complete. Define a lattice homomorphism T on c by

$$(Tx)_n = \begin{cases} (x_n/n) & \text{if } n \text{ is even,} \\ (x_{n+2}/n^n) & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to verify that T' is a lattice homomorphism. But, as shown in [WI], T can not be decomposed in the way as in Theorem 3.2.7.

Remark 2. Theorem 3.2.7 was proved in [WI] for compact lattice homomorphisms on order continuous Banach lattices under the assumption that the spectrum of the lattice homomorphisms consists of nonnegative real numbers, and Corollary 3.2.8 was proved in [A] under the assumption that $\sigma(T)$ is included in a sector of angle less than $2\pi/3$.

Next we give some conditions on T such that the regular spectrum $\sigma_0(T)$ (see Section 3 of Chapter 1 for definition) is equal to $\sigma(T)$.

Proposition 3.2.9. *Let T be an order continuous disjointness preserving operator on E with the Fatou norm. If $\sigma(T)$ is contained in a sector Δ of angle less than π, then any of the following conditions implies that $\sigma_0(T) = \sigma(T)$.*

(i) *$\rho(T)$ is connected;*

(ii) *T' is also a disjointness preserving operator.*

Proof. Let $T = P + B$ be as usual, i.e., $P \in Z(E)$ and $B \in Z(E)^d$. Then by Theorem 3.2.4 we know that $|B|$ is a quasi-nilpotent operator.

(i) By assumption, we have $\rho_\infty(T) = \rho(T)$. It follows from Theorem 3.2.4 that $\sigma(P) = \sigma(T)$. Since $\sigma(T) \subseteq \sigma_0(T)$ is always true, it is enough to show that $\sigma(P) \supseteq \sigma_0(T)$. To do this let $z \in \rho(P)$ and observe that $zI - T = (zI - P)(I - (zI - P)^{-1}B)$. Since $Z(E)$ is a full subalgebra of $L(E)$ and since $|(zI - P)^{-1}B|$ is quasi-nipoltent by Lemma 3.2.4d, we see that $zI - T$ has a bounded inverse and the inverse is an order bounded operator on E. Therefore, z is not in $\sigma_0(T)$. This shows that $\sigma(P) \supseteq \sigma_0(T)$.

(ii) Assume that T' is also a disjointness preserving operator. Then by Theorem 3.2.7 we have

$$\sigma_0(T) = \sigma_0(P) \cup \sigma_0(B) = \sigma_0(P) \cup \{0\} = \sigma(P) \cup \{0\} = \sigma(T).$$

The proof is finished.

§3.3. Decomposition Theorems of Type II

In this section we prove another type of decomposition theorems for disjointness preserving operators without any restrictions on the spectrum of the operators. The main result asserts that under certain natural conditions a disjointness preserving operator can be decomposed into a direct sum of its strictly periodic and aperiodic components. This result generalizes an earlier one in [AH]. Our approach is different from that of [AH] in that we directly work on the given operators and related ones rather than the operators induced by the given operators on $Z(E)$ as was done in [AH].

Through this section the Banach lattice E is assumed to be order complete unless otherwise stated. Let T be an order bounded operator on E. Recall that (i) T is said to have strict period n if $T^n \in Z(E)$ and $|T|^k \wedge I = 0$ for $k = 1, 2, \cdots, n-1$; (ii) T is aperiodic if $|T|^k \wedge I = 0$ for all $k = 1, 2, \cdots$.

A positive operator on E is called (i) interval preserving if $T[0,x] = [0,Tx]$ for all $0 \le x \in E$; (ii) almost interval preserving if $T[0,x]$ is norm dense in $[0,Tx]$ for all $0 \le x \in E$. Interval preserving operators are also said to have the Maharam property (see [L]). The following results are well-known and we refer to [SW], pp. 74 for proofs.

(1) T is a lattice homomorphism if and only if T' is interval preserving;

(2) T is almost interval preserving if and only if T' is a lattice homomorphism.

If F is a Riesz space we will use $F_n^\sim$ to denote the order continuous dual of F, i.e., the set of all order bounded order continuous functionals on F. It is known that if F is a Banach lattice then $F_n^\sim$ is a band of F'. We say that $F_n^\sim$ separates the points of F if $(F_n^\sim)^0 = \{x \in F : f(x) = 0 \text{ for all } f \in F_n^\sim\} = \{0\}$. It is easy to show that any Banach lattice with order continuous norm as well as any dual Banach lattice has this property. For any dual Banach lattice E, if $E = F'$, then each element in F induces an order continuous linear functional on E. In particular, all L^p spaces have separating order continuous duals. Let T be an order continuous operator on F. Then it is easy to see that $F_n^\sim$ is invariant under T'. Moreover, if F is order complete, then the map $T \to T'$ from $L^r(F)$ to $L^r(F_n^\sim)$ is a lattice homomorphism. We refer to [AB], pp. 92 for a proof. Using this result we can easily obtain the following.

Lemma 3.3.1a. *Let T be an order continuous lattice homomorphism on E such that T' is also a lattice homomorphism. Let S be an order continuous positive operator on E with the property that $ST = TS$. If $E_n^\sim$ separates the points of E or if T is interval preserving, then $(S \wedge I)T = (ST) \wedge T = T(S \wedge I)$.*

Proof. If T is interval preserving, then the result follows from the following identity and Proposition 3.2.1

$$(S_1 \wedge S_2)T = (S_1T) \wedge (S_2T) \text{ for all } S_1, S_2 \in L^r(E).$$

A proof of this identity can be found in [AB], pp. 90. So in the rest of the proof we assume that E has separating order continuous dual. We restrict the adjoint of T to $E_n^\sim$. By the result we just mentioned (see [AB], pp. 92) and by Proposition 3.2.1 (i), we have

$$[(ST) \wedge T]' = (T'S') \wedge T' = T'(S' \wedge I)$$

$$= T'(S \wedge I)' = [(S \wedge I)T]',$$

where the identity holds on $E_n^\sim$. So for any $x \in E$ and any $f \in E_n^\sim$, we have $f[(ST) \wedge T(x)] = f[(S \wedge I)T(x)]$. Since $E_n^\sim$ separates points of E, we obtain $(ST) \wedge T = (S \wedge I)T$. The proof is finished.

Let $0 \leq P \in Z(E)$. Then $Q = \sup\{(nP) \wedge I : n = 1, 2, \cdots\}$ exists and $Q^2 = Q$ is a band projection (see Section 2 of this chapter). We will call the range of Q the band associated with P or the support of P.

Lemma 3.3.1b. *Let T be an order continuous lattice homomorphism on E such that T' is also a lattice homomorphism. Suppose that T is interval preserving or E has separating order continuous dual. Let $T^k = P_k + B_k$ with $P_k \in Z(E)$ and $B_k \in Z(E)^d$. Then the band E_k associated with P_k is T-reducing. Moreover, $B_k|_{E_k} = 0$.*

Proof. Let $Q_k = \sup\{(nP_k) \wedge I : n = 1, 2, \cdots\}$. Then Q_k is the band projection with range E_k. It is enough to show that $TQ_k = Q_kT$. First we observe that

$$Q_k = \lim_{n \to \infty} (nP_k) \wedge I = \lim_{n \to \infty} (nT^k) \wedge I.$$

So it follows from Lemma 3.3.1a that

$$TQ_k = \lim_{n \to \infty} T[(nT^k) \wedge I] = \lim_{n \to \infty} (nT^kT) \wedge T$$

$$= \lim_{n\to\infty}[(nT^k) \wedge I]T = Q_kT.$$

The above limits are taken with respect to the order convergence. Finally, $B_k|_{E_k} = 0$ follows from Proposition 3.2.1 (ii) and Proposition 3.2.5.

We are now in position to prove the following theorem, which is the main result of this section.

Theorem 3.3.1. (Decomposition Theorem II) *Let T be an order continuous disjointness preserving operator on E such that T' is also a disjointness preserving operator. Suppose that $E_n^\sim$ separates the points of E or $|T|$ is interval preserving. Then there exist a family $\{E_n : n = 1, 2, \cdots\} \cup \{E_\infty\}$ of T-reducing bands of E such that*

(i) $E = \oplus_{n=1}^{\infty} E_n \oplus E_\infty$;

(ii)$T|_{E_n}$ *has strict period n and $T|_{E_\infty}$ is aperiodic,*

where the direct sum in (i) *is understood in the sense that each $x \in E$ can be uniquely represented as $x = x_\infty + o - \lim_{n\to\infty} \sum_{k=1}^{n} x_k$ with $x_k \in E_k$ and $x_\infty \in E_\infty$ and with the convergence being in order.*

Proof. By Proposition 3.1.5 we may assume that T is an order continuous lattice homomorphism such that T' is also a lattice homomorphism, since a decomposition for $|T|$ is also a decomposition for T. First we notice that Lemma 3.3.1b holds under the assumptions in Theorem 3.3.1.

Let $T = P_1 + B_1$ be as usual, i.e., $P_1 \in Z(E)$ and $B_1 \in Z(E)^d$. Now if E_1 is the band associated with P_1 then by Lemma 3.3.1b E_1 is T-reducing. Moreover, $T|_{E_1} \in Z(E_1)$ and $T|_{E_1^d} = B_1|_{E_1^d} \in Z(E_1^d)^d$. So $T|_{E_1}$ has strict period 1. We now work on E_1^d and consider the restriction of T to E_1^d. Let $T^2 = P_2 + B_2$ with $P_2 \in Z(E_1^d)$ and $B_2 \in Z(E_1^d)^d$. Let $E_2 \subseteq E_1^d$ be the band associated with P_2. Since E_1^d and the restriction of T to this band satisfy the conditions in Theorem 3.3.1, it follows from Lemma 3.3.1b that E_2 is T-reducing and $T^2|_{E_2^d} = B_2|_{E_2^d} \in Z(E_2^d)$. It is easy to see that $T|_{E_2}$ has strict period 2.

Now assume that we have constructed T-reducing and disjoint bands $\{E_1, E_2, \cdots, E_n\}$ of E such that

(i) $E_{i-1}^d = E_i + E_i^d$ for $i = 2, 3, \cdots, n$;

(ii) $T|_{E_i}$ has strict period i for $i = 1, 2, \cdots, n$;

(iii) $(T|_{E_n^d})^k \in Z(E_n^d)^d$ for $k = 1, 2, \cdots, n$.

We now consider the Banach lattice E_n^d and the restriction of T to this space. It is easy to verify that all the conditions in Theorem 3.3.1 are satisfied. Let $T^{n+1} = P_{n+1} + B_{n+1}$ with $P_{n+1} \in Z(E_n^d)$ and $B_{n+1} \in Z(E_n^d)^d$. Let E_{n+1} be the band associated with P_{n+1}. Then by Lemma 3.3.1b E_{n+1} is T-reducing. Moreover, $T|_{E_{n+1}}$ has strict period $n+1$ and $(T|_{E_{n+1}^d})^{n+1} = B_{n+1}|_{E_{n+1}^d} \in Z(E_{n+1}^d)^d$. By mathematical induction we can construct a family $\{E_n : n = 1, 2, \cdots\}$ of T-reducing bands having the properties stated in the theorem. Now let

$$E_\infty = \left(\bigcup_{n=1}^{\infty} E_n\right)^d = \bigcap_{n=1}^{\infty} E_n^d.$$

Then we have

$$E = \oplus_{n=1}^{\infty} E_n \oplus E_\infty.$$

Finally, we show that $T|_{E_\infty}$ is aperiodic. Observe that for any positive integer k we have $E_\infty \subseteq E_k^d$. Since $(T|_{E_k^d})^k \in Z(E_k^d)^d$ by the above (iii), we see that $(T|_{E_\infty})^k \in Z(E_\infty)^d$. So $T|_{E_\infty}$ is aperiodic on E_∞. The proof is finished.

Corollary 3.3.2. *Let T be an order continuous disjointness preserving operator on E such that T' is also a disjointness preserving operator. Then any of the following conditions implies that T can be decomposed as in Theorem* 3.3.1.

(i) *E has order continuous norm;*

(ii) *E is a dual Banach lattice;*

In particular, T' can be decomposed on E' as in Theorem 3.3.1.

Proof. For (i) and (ii), it is enough to notice that in either case E has separating order continuous dual. Also, T' and T'' are disjointness preserving operators by Proposition 3.1.5.

Next we show that the decomposition theorem given in [AH] is a special case of Theorem 3.3.1. To do this let us recall that a disjointness preserving operator T on E is said to be quasi-invertible if the following conditions are satisfied (see [AH]):

(i) T is order continuous and T is one-one (injective);

(ii) $\{T(E)\}^{dd} = E$;

(iii) T' is a disjointness preserving operator.

It has been shown in [AH] (see proposition 2.3 of this paper) that a disjointness preserving operator T on E is quasi-invertible if and only if T is one-one and the range of T is an order dense ideal of E. By the use of the latter conditions we can prove the following.

Proposition 3.3.3. *Let T be a quasi-invertible disjointness preserving operator on E. Then $|T|$ is interval preserving.*

Proof. By Proposition 3.1.5 $|T|$ exists and is a lattice homomorphism on E. Moreover, $|T||x| = |Tx|$ for any $x \in E$. So it follows from the above definition that $|T|$ is also one-one (injective) and that the range of $|T|$ is a dense ideal of E. Let $F = |T|(E)$. Then F is also an order complete Riesz space. Since $|T| : E \to F$ is one-one and onto, there exists a linear map $S : F \to E$ such that $S|T|$ and $|T|S$ are the identities on E and on F respectively. It is easy to see that S is positive. Fix any $0 \le x \in E$ and let $z \in [0, |T|x] \subseteq F$. Then we have $0 \le Sz \le S(|T|x) = x$. So $Sz \in [0, x]$ and $|T|(Sz) = z$. Thus $|T|[0, x] = [0, |T|x]$. Therefore $|T|$ is interval preserving.

Corollary 3.3.4.([AH]) *Let T be a quasi-invertible disjointness preserving operator on E. Then T can be decomposed as in Theorem* 3.3.1.

Proof. It is an immediate consequence of Theorem 3.3.1 and Proposition 3.3.3.

§3.4. Some Consequences

In this section we use the results obtained in the previous sections to deduce certain properties of disjointness preserving operators. Through this section we assume that E is order complete and that T is an order continuous disjointness preserving operator on E.

Let $P \in Z(E)$. We already see that $Q = \sup\{(n|P|) \wedge I : n = 1, 2, \cdots\}$ is a band projection. Let E_P be the band associated with Q. Then the following holds.

Lemma 3.4.1. *Let $x \in E$. Then $Px = 0$ if and only if $Qx = 0$.*

Proof. Assume that $Qx = 0$. Then $Q|x| = 0$ since Q is a lattice homomorphism. In particular, we obtain $|P| \wedge I(|x|) = 0$. Choose a small positive number $0 < c < 1$ such that $c|P| \le I$. Then $0 \le c|P|(|x|) \le |P| \wedge I(|x|) = 0$. This implies that $Px = 0$.

Now suppose that $Px = 0$. Then $|P|(|x|) = 0$ since $P \in Z(E)$. So $(n|P|) \wedge I(|x|) = 0$ for all n. Since $Q|x| = \lim_{n\to\infty}(n|P|) \wedge I(|x|)$, we have $Qx = 0$.

Proposition 3.4.2. *Let $P \in Z(E)$ and let E_P be the band associated with P. Then P is one-one on E_P and $\{P(E_P)\}^{dd} = E_P$.*

Proof. Let $x \in E_P$ and let $Px = 0$. Then it follows from Lemma 3.4.1 that $Qx = 0$.

But $Qx = x$ since Q is the band projection on E_P, and we have $x = 0$. This shows that P is one-one on E_P. Now we prove the last statement. Let $x \in E_P$ such that $x \perp P(E_P)$. In particular, we have $|x| \perp |P|(|x|)$. So $|x| \perp (n|P|) \wedge I(|x|)$ for all n. This implies that $|x| \perp Q|x|$, i.e., $|x| \perp |x|$. Hence $x = 0$. Therefore, $\{P(E_P)\}^{dd} = E_P$. The proof is finished.

Theorem 3.4.3. *Let T be a disjointness preserving operator on E such that T' is also a disjointness preserving operator. Assume that either $E_n^{\sim}$ separates points of E or $|T|$ is interval preserving. If T has strict period n, then for any n-th root of unity α it holds that $\sigma(T) = \alpha\sigma(T)$.*

Proof. First observe that under the assumptions given above, Theorem 3.3.1 holds. Let $P_n = T^n$. Then $P_n \in Z(E)$ by assumption. Let E_n be the band associated with P_n. Then by Lemma 3.3.1b E_n is T-reducing. Now consider the decomposition $E = E_n \oplus E_n^d$. It is easy to see that $T|_{E_n}$ has strict period n and $T|_{E_n^d}$ is quasi-nilpotent such that $(T|_{E_n^d})^n = 0$. By Proposition 3.4.2 $T^n = P_n$ is one-one on E_n and $\{P_n(E_n)\}^{dd} = E_n$. In particular, $T|_{E_n}$ is one-one on E_n and $\{T(E_n)\}^{dd} = E_n$. So $T|_{E_n}$ is quasi-invertible on E_n (see [AH] or Section 3 of this Chapter). By Theorem 4.3, pp. 159 in [AH], we have

$$\sigma(T|_{E_n}) = \alpha\sigma(T|_{E_n}).$$

Since $T|_{E_n^d}$ is quasi-nilpotent, the following holds

$$\sigma(T) = \sigma(T|_{E_n}) \cup \sigma(T|_{E_n^d})$$

$$= \alpha\sigma(T|_{E_n}) \cup \alpha\sigma(T|_{E_n^d}) = \alpha\sigma(T).$$

The proof is finished.

Theorem 3.4.4. *Let T and E be as in Theorem 3.4.3. Then $\sigma_0(T) = \sigma(T)$.*

Proof. By assumption $T^n \in Z(E)$. Hence $\sigma_0(T^n) = \sigma(T^n)$ by the fact that

$Z(E)$ is a full subalgebra. Let $\lambda \in \sigma_0(T)$. By the spectral mapping theorem $\lambda^n \in \sigma_0(T^n) = \sigma(T^n)$. Thus there exists $\mu \in \sigma(T)$ such that $\lambda^n = \mu^n$. So there exists an n-th root of unity α such that $\lambda = \alpha\mu$. Now by Theorem 3.4.3 we have $\lambda \in \sigma(T)$. We have shown that $\sigma_0(T) \subseteq \sigma(T)$. Since the other inclusion is always valid, $\sigma_0(T) = \sigma(T)$.

Remark. Theorem 3.4.3 and Theorem 3.4.4 were proved in [AH] for quasi-invertible disjointness preserving operators. These two theorems are also valid for positive operators having strict period n (see Section 3 of Chapter 2).

Next we investigate some aspects of the irreducibility and reducibility of disjointness preserving operators. In the following we assume that $\dim(E) > 1$. Recall that an operator on E is called (i) irreducible if E and $\{0\}$ are the only T-invariant closed ideals of E; (ii) band irreducible if E and $\{0\}$ are the only T-invariant bands.

Proposition 3.4.5. *Let T be an order continuous disjointness preserving operator on E. Suppose that E has Fatou norm. If $\sigma(T)$ is contained in a sector Δ of angle less than π, then either $\sigma(T) = \{0\}$ or E has a proper T-invariant band. In particular, if $r(T) > 0$, then T is band reducible.*

Proof. By Theorem 3.2.4, T can be written as $T = P + B$ with $P \in Z(E)$ and $B \in Z(E)^d$. Moreover, $|B|$ is quasi-nilpotent. Let E_0 be the band associated with P. We already showed that $B(E) \subseteq E_0^d$ (see the second paragraph of the proof of Theorem 3.2.4). We consider several cases. (a) if $E_0^d = \{0\}$, then $B = 0$ and so $T = P$ is reducible (since $\dim(E) > 1$); (b) if $E_0^d = E$, then $P = 0$ and $T = B$ is quasi-nilpotent; (c) if neither (a) nor (b) is true, then E_0^d is a proper T-invariant band.

The following result is related to Proposition 3.4.5, although it is an immediate consequence of some known results.

Proposition 3.4.6. *Let T be a compact disjointness preserving operator on any infinite dimensional Banach lattice F. Then T is reducible, i.e., F has a proper T-invariant closed ideal.*

Proof. By Proposition 3.1.7, we know that $|T|$ is a compact lattice homomorphism. If $|T|$ is irreducible, then $r(|T|) > 0$ by the main result of [P]. By Proposition 3.1.5, $r(T) = r(|T|) > 0$. Since the spectrum of a compact operator is discrete, it follows from Theorem 4.1 of [A] that T has a proper invariant closed ideal. This ideal is also $|T|$-invariant. This is a contradiction. Hence $|T|$ is reducible and so is T.

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