STRINGS, TWO-DIMENSIONAL GRAVITY, AND MATRIX MODELS

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ABSTRACT

Two-dimensional models of quantum gravity have been solved using matrix model techniques. Furthermore, these solutions have turned out to be encoded in integrable nonlinear PDEs belonging to the KdV hierarchy. This thesis presents a new KdV recursion relation, distinct from one found previously by Dijkgraaf and Witten, for a certain class of theories known as the two-matrix models. The two recursion relations together are used to relate arbitrary correlation functions containing a puncture operator P (at any genus) to the three basic correlators $\langle PP \rangle$, $\langle PQ \rangle$, and $\langle QQ \rangle$ by unique algebraic expressions. (Q is the dilaton operator.) The derivation requires assuming a certain scaling law, whose justification is discussed.

Other KdV recursion relations, given by Virasoro or W-algebra constraints, are possible for multi-matrix models when an infinite number of couplings are added. These constraints have been presented for A_n-type models by Fukuma *et al.* and Dijkgraaf *et al.* We derive analogous Virasoro constraints for the multi-matrix models associated with the other simply-laced Lie algebras D_{2n+1}, E_6, E_7, and E_8. As a check, it is verified that the proposed constraints imply operator scaling dimensions identical to those found by Kostov. It is then demonstrated that these Virasoro constraints (or, more generally, W-algebra constraints) can be used to derive expressions for correlation functions containing a non-primary operator in terms of correlation functions that only contain primary operators.

The second subject of this thesis concerns the underlying symmetries of string theory as probed by fixed-angle scattering at very high energy. The asymptotic behavior depends sensitively on the choice of the string vacuum. Therefore, we examine the effect of modifying the vacuum on the behavior of high-energy scattering amplitudes. In particular, high-energy fixed-angle elastic scattering of open-string tachyons is studied explicitly. Tadpole corrections to the tree-level formulas are included. The main conclusion of the analysis is that symmetry relations among

amplitudes at high energy seem to be unaffected by modifications of the vacuum, even though the amplitudes themselves do change.

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1. Introduction

Presently, there are four fundamental and outstanding problems in elementary particle physics. Two of the problems are related to phenomenology and two of the problems are related to deeper understanding of quantum field theory. First, there is an urgent need to find the Higgs particle. Second, there is an equally urgent need to find a supersymmetry particle. Third, there is the longstanding problem of understanding strong QCD, and in particular, the demonstration of confinement. Fourth, there is the problem of understanding quantum gravity.

String theories directly address the last two problems (and, less directly, the first two, as well). Superstring theories are the only consistent theories of quantum gravity known. Indeed, the framework of superstring theories is such that the unification of the strong force, the weak force, the electromagnetic force, and gravity can come about naturally. Also, a string theory in four dimensions may be the appropriate description of strong QCD.

The $E_8 \times E_8$ heterotic string theory with six dimensions compactified on a Calabi-Yau manifold has been very successful in giving nearly realistic phenomenology. The excitement over string theory in recent years has been that, for the first time, we have candidates for consistent quantum theories, with finite renormalizations of physical quantities, which can incorporate all known forces. In other words, we have found possible candidates for the "theory of everything."

The attempts to find realistic phenomenology for superstrings have stimulated the growth of investigations into conformal field theories and their classification. The consistency condition for strings to propagate in a particular background is the same as the condition of conformal invariance of the corresponding 2-D quantum field theory on a 2-D string world sheet, so that the classification of conformal field theories has enabled one to find many candidates for the classical string vacuum.

The distant goal is to find a non-perturbative mechanism in a superstring theory which will choose the unique vacuum out of the many candidates. One knows that non-perturbative effects must be important in superstring theories, since supersymmetry can not be broken perturbatively, and yet one knows that, in our world, supersymmetry is broken. Recently, $c < 1$ non-critical strings have been solved exactly via matrix model techniques, and non-perturbative effects examined. The hope is that the understanding of non-perturbative effects in non-critical strings will shed some light on non-perturbative effects in critical strings. Also, $c < 1$ non-critical strings can be thought of as $c < 1$ conformal matter coupled to 2-D gravity, so one might learn from these models something about quantum gravity in four dimensions.

In the matrix model approach to 2-D gravity one finds that the space of theories obtained by taking the continuum limit of matrix models is organized by KP flows. Furthermore, the KdV equations and the string equation for each resulting theory in the continuum limit can be reformulated in terms of equations for propagation of loops, which can be elegantly expressed as Virasoro constraints and W-algebra constraints. The major part of this thesis will be concerned with the KdV equations and the Virasoro constraints.

In chapter two we present a review of the derivation of the KdV equations and the string equation to establish a general framework for issues in chapters three and four. We explain the connection between a discrete version of 2-D gravity and matrix models. We take the continuum limit of the exact results found in matrix models to obtain the KdV equations and the string equation.

In chapter three we examine the KdV equations for the two-matrix models and find an identity from which we can derive KdV recursion relations for correlations, which express all correlations with an insertion of P in each genus in terms of $\langle PP \rangle$, $\langle PQ \rangle$, and $\langle QQ \rangle$ in a unique way. Here P and Q are the puncture operator and the dilaton operator, and also we assume the scaling ansatz. The importance of the KdV recursion relations is that they do not involve explicitly the infinite number of coupling constants as happens in the case of the recursion relations given by the

Virasoro constraints and the W-constraints. Thus, we offer an alternative way to determine the correlations of the theory.

In chapter four we find Virasoro constraints for D_{2n+1}, E_6, E_7, E_8-type models analogous to the Virasoro constraints recently discovered for A_n-type models by Fukuma *et al.*, and Dijkgraaf *et al.* We verify that the proposed Virasoro constraints give operator scaling dimensions identical to those found by Kostov. We check that these Virasoro constraints and, more generally, W-algebra constraints can be used to express correlation functions with non-primary operators in terms of correlation functions of primary operators only. The importance of the Virasoro constraints for D_{2n+1}-type models is that they make the correlations much easier to compute than as given by the KdV equations and the string equation. The importance of the Virasoro constraints for E_6, E_7, E_8-type models is even more pronounced, since presently there does not exist any description of these theories in the framework of the KdV equations and the string equation.

In chapter five we turn to critical bosonic strings in 26 dimensions, and we examine the effect of modifying the vacuum on high energy scattering of strings. The fixed-angle asymptotic behavior depends sensitively upon the choice of string vacuum. However, the underlying high energy symmetries of string theory seem to be independent of vacuum modifications. We explicitly calculate high-energy fixed-angle scattering of four open string tachyons at the tree level modified by a tadpole insertion, which is described by an annulus diagram with one Neumann boundary condition and one Dirichlet boundary condition. The importance of the calculation is that we have modified the vacuum by insertions of string condensates generalizing the point particle condensates usual in conventional field theory.

PART I

2. Matrix Model Approach to 2-D Gravity

In this chapter we present an introductory review of the matrix models approach to 2-D gravity establishing a general framework for the issues of interest to be examined in chapters three and four.

In section one we state the three different approaches to 2-D gravity to give an account of the recent efforts to understand 2-D quantum gravity. In the subsequent sections we review in detail the development of the matrix model approach to 2-D gravity. In section two we explain how a discrete version of 2-D gravity can be described by matrix models. In section three we take the continuum limit of the results obtained in the matrix models. In section four we obtain non-perturbative results of 2-D gravity from the matrix models. In section five we generalize the results for the matrix models considered in section four by considering multi-matrix models and their continuum limit. We conclude in section six by addressing three unresolved issues.

2.1 Introduction

Using string excitations as collective degrees of freedom of a physical system as an alternative to point particles as elementary excitations in conventional field theory is a powerful idea. String theories will inevitably be the most appropriate descriptions for many physical systems in their various phases. Early attempts to apply strings to the hadron phase of QCD stimulated Polyakov [1] to propose a version of non-critical strings in which the residual degree of freedom of the two dimensional world sheet metric is the Liouville degree of freedom. The importance of understanding non-critical strings is that it may be the key to solving various physical problems including the 3-D Ising model, QCD at strong coupling, etc.

Critical bosonic strings in 26 dimensions can be described as conformal matter with central charge 26 coupled to 2-D gravity. Non-critical strings in less than 26 dimensions correspond to conformal matter with central charge less than 26 coupled to 2-D gravity. Non-critical strings in zero dimensions is therefore pure 2-D gravity. Pure 2-D quantum gravity is non-trivial because of the residual Liouville degree of freedom.

There are three approaches to conformal matter coupled to 2-D gravity that have been studied recently. The first approach is to use Polyakov's Liouville action and couple the conformal matter to the residual gravitational degree of freedom. This was done in light cone gauge by KPZ [2] and in conformal gauge by DDK [3]. The main result was that the conformal dimensions of the operators which were given by a quadratic formula derived by BPZ [4] in the absence of gravity are simplified to a linear formula when there is coupling to gravity.

The second approach is to use a random lattice as a discretization of a curved 2-D surface and study the resulting theory as the lattice spacing approaches zero. Random lattices can be simulated either numerically or analytically. Although numerical simulation can address more diverse models, in some cases analytical approaches give exact results, which facilitate analysis. Analytical simulation of random lattices can be implemented via matrix models. The multi-critical points of the matrix models, where the number of lattice cells diverges with fixed total area, may correspond to the continuum limit of physical theories. Since the lattice is generated dynamically, every genus can contribute, giving rise to non-perturbative effects. (Douglas, Shenker, Kazakov, Brézin, Gross, Migdal [5])

The third approach is topological sigma models coupled to topological gravity pioneered by Witten [6]. Correlation functions of operators calculated at genus zero coincide with those calculated using matrix techniques. One positive sign that recent progress in 2-D quantum gravity is on the right track is that whenever the results from the various approaches overlap, they are in agreement.

2.2 Matrix Models and 2-D Gravity

We are interested in understanding how to formulate a theory in which strings are the elementary excitations. There are various ways to proceed. For example, there is the string theory as given by the Nambu string action and there is the string theory as given by Polyakov string action. At the classical level, the two string theories give the same string propagation, since the equations of motion of the two theories for the string's target space coordinates X^{μ} are identical. At the quantum level, one can proceed much further in the Polyakov string theory in the sense that a path integral measure for the string coordinates is concretely defined and the internal metric is treated as fields living on 2-D world sheet. (Notice that this is quantum in the sense of the 2-D field theory, but not in the sense of the string loop expansion.)

In the critical dimension, the Polyakov string theory simplifies dramatically in that the partition function or scattering amplitudes are given by a finite dimensional integral over the moduli space of Riemann surfaces. The enormous simplification is the result of the cancellation of the conformal anomaly between the Liouville degree of freedom and the contributions from the space coordinates. This enables one to gauge away most of the degrees of freedom of the 2-D metric using the reparameterization invariance and conformal invariance of the partition function. In non-critical dimensions of interest, for example, D=3 fermi strings (which may describe Ising models in three dimensions) or D=4 strings (possibly relevant to QCD), much less is known, because we are unable to treat the Liouville degree of freedom satisfactorily when there is a tachyon in the spectrum. The presence of a tachyon suggests that the vacuum has not been identified correctly.

Consider the D = 2 world sheet, $c = 0$ target space string theory given by the Polyakov partition function

$$Z = \int_{METRICS_g} Dh_{ab}\; e^{-\overline{\mu}\int_{\Sigma} d\sigma_1 d\sigma_2 \sqrt{h}}. \tag{2.2.1}$$

The 2-D world sheet Σ is a genus g Riemann surface, the 2×2 real symmetric matrix h_{ab} is the internal metric of the Riemann surface Σ, and $METRICS_g$ is the space of metrics on a Riemann surface of genus g. The $\int_{\Sigma} d^2\sigma\sqrt{h} \equiv \int_{\Sigma} d^2\sigma\sqrt{det(h_{ab})}$

action gives the coordinate reparameterization invariant intrinsic area of surface Σ. $\overline{\mu}$ is the "cosmological constant" for the 2-D world sheet. Note that the familiar $\int_\Sigma d\sigma_1 d\sigma_2 \sqrt{h} h^{ab} \partial_a X^\mu \partial_b X_\mu$ term, where X^μ is the target space-time coordinate, is not included in the Polyakov string action in eq. (2.2.1), since the dimension of the target space is $c = 0$.

We want to find a suitable discrete version of the continuum string theory given by eq. (2.2.1). Consider

$$Z = \sum_G e^{-\mu \; Area(G)}, \tag{2.2.2}$$

We construct distinct 2-D manifold surfaces by gluing small equilateral triangles together in different configuration and by varying the number of triangles used. A graph G is specified by a set of discrete points and a set of pairs of these points, since we know that a pair of points is separated by a fixed distance and that the surface is tiled by adjacent triangles. Knowing the genus of the surface, a graph G is information out of which such a surface can be reconstructed. μ in eq. (2.2.2) is the cosmological constant for the discrete case, which differs from $\overline{\mu}$ in eq. (2.2.1), since the cosmological constant is non-universal and it changes value in renormalization flows. Finite summation on the finite number of triangulated graphs G is a discrete version of the integration over space of metrics $METRICS_g$ in the continuum Polyakov string theory partition function. We can numerically simulate the discrete graphs and the summation on computers as in lattice QCD studies, and we can try to extract the critical exponents. Alternatively, we can calculate the partition function analytically giving a closed exact expression by the methods of matrix models, and thereafter, extracting the critical exponents. We now give a brief description of the relationships between the discrete version of the continuum string theory and the matrix models. In the next section of this chapter the matrix models will be solved giving an exact expression for the partition function.

Consider a curved 2-D manifold of a given topology, for example, a Riemann surface of genus g, described by a metric $h_{ab}(\sigma_1, \sigma_2)$, where σ_1, σ_2 are local coordinates on the surface. One way to visualize this curved 2-D manifold, is to embed it in some ambient Euclidean space of sufficiently high dimension with the metric induced by the embedding. There is a theorem by Nash that guarantees that an n-dimensional Riemann manifold can be isometrically embedded in an Euclidean space of dimension $2(2n+1)(3n+7)$. Furthermore, we can obtain a discrete approximation to this surface by tiling the surface with equilateral triangles of unit side. We will use the term triangulated surface to refer to the non-smooth surface formed by the tiling triangles together with the graph out of which such a surface can be reconstructed. A graph is specified by a set of discrete points and a set of pairs of these points, since we know that a pair of points is separated by unit distance and that the surface is tiled by adjacent triangles. Curvature is concentrated at the vertices. At the places on the smooth surface where the curvature is zero, there are six adjacent equilateral triangles meeting at a vertex of the triangulated surface. At the places where the curvature is less than zero we will have more than six triangles meeting at a vertex. At the places where the curvature is greater than zero we will have fewer than six triangles meeting at a vertex.

The difficult step is to define the proper measure in the computation of the partition function. In our case this amounts to assigning weights to each graph representing some particular triangulated surface in the computation of the partition function. One possibility is to use the Feynman path integral prescription of assigning equal weights to all inequivalent triangulations. That this prescription would give the same results as the Liouville measure of the continuum theory is not guaranteed *a priori.* However, there is evidence that this is the correct prescription. The major evidence is that the critical exponents obtained from the conformal gauge Polyakov action of the continuum theory are identical to the critical exponents obtained by considering the continuum limit of the discrete theory with the inequivalent triangulations weighted equally. The critical exponents obtained from the discrete theory also agree between the computer lattice simulation results and the matrix models results.

The key to the relationships between the discrete version of the continuum string theory and the matrix models is that there is a one-to-one mapping between triangulations of a 2-D world sheet and ϕ^3 cubic graphs of quantum field theory. The mapping is via a duality transformation. The dual graph is obtained from the triangulation by connecting the centers of adjacent triangles with lines. Thus, the vertices, edges and faces of the triangulation correspond to the loops, lines, and vertices of the ϕ^3 cubic graphs.

Thus, the partition function of the discrete version of the theory is

$$Z = \int dM \; e^{-\lambda^{-1} Tr[\frac{1}{2}M^2 + \frac{M^3}{N}]}, \qquad (2.2.3)$$

where M is an $N \times N$ Hermitian matrix and $\lambda^{-1} = e^{\mu}$. μ is the cosmological constant in eq. (2.2.2). The free energy $F = -\log(Z)$ is the sum of all connected vacuum graphs. We use Hermitian matrices since we are interested in continuum closed string theory in which the surfaces are closed Riemann surfaces (this is explained below). First, we note that the standard path integral graphical expansion of the partition function as defined above would result in each vertex TrM^3 being contracted with other vertices to form a connected graph. The important point is that the internal lines can be considered as thickened to become flat bands because M is a Hermitian matrix. The thickened bands can be imagined to be further thickened until the gaps between the bands close up completely, and the result is an oriented closed two-dimensional surface.

Each discrete closed Riemann surface is obtained uniquely from a vacuum graph of the matrix models and is labeled by a topological invariant, the genus. From 't Hooft's work [7] on large N expansions for QCD, it is clear that expanding the partition function for large N and collecting powers of $1/N$ corresponds to a genus expansion in which each power of $1/N$ corresponds to contributions from the closed Riemann surfaces of definite genus. This technique of doing perturbative expanding in the small parameter $1/N$ is called the large N expansion, which was invented by 't Hooft in the context of QCD. Stimulated by such an interesting problem, the matrix models were studied [8, 9] and exactly solved by using the orthogonal polynomial techniques in the classic paper of Bessis, Itzykson, and Zuber [9].

2.3 Continuum Limit of Matrix Models

Let us define more precisely the physical problem that we are interested in. We are interested in the physics of Euclidean signature 2-D gravity coupled to various kinds of conformal matter. We hope it will give insights into Lorentz signature 4-D gravity and critical strings. Work on Lorentz signature 2-D gravity is still being developed at this time. We will simplify the considerations by examining pure Euclidean signature 2-D gravity without coupling to matter given by a 2-D world sheet and $c = 0$ target space. We will follow Witten's review paper [10]. Witten took the classical action in the Feynman path integral to be the Polyakov's string action for $c = 0$ target space, and defined the path integrals:

$$F(g) = \int\limits_{METRICS_g} Dh \; e^{-\lambda_1 \int_\Sigma d^2\sigma\sqrt{h} - \lambda_2 \int_\Sigma d^2\sigma\sqrt{h}\frac{R}{2\pi}}, \tag{2.3.1}$$

$$F(g,A) = \int\limits_{METRICS_{g,A}} Dh \; e^{-\lambda_1 A - \lambda_2 \int_\Sigma d^2\sigma\sqrt{h}\frac{R}{2\pi}}. \tag{2.3.2}$$

Since the 2-D world sheet Σ is a connected genus g Riemann surface, one is evaluating the contribution of genus g surfaces to the free energy F rather than the partition function $Z = e^{-F}$, which includes contributions from disconnected surfaces. The 2×2 real symmetric matrix h_{ab} is the internal metric of the Riemann surface Σ. The $\int_\Sigma d^2\sigma\sqrt{h} \equiv \int_\Sigma d^2\sigma\sqrt{det(h_{ab})}$ term gives the coordinate reparameterization invariant intrinsic area of the Riemann surface Σ. $METRICS_g$ is the space of metrics on a Riemann surface of genus g, and $METRICS_{g,A}$ is the space of metrics on a Riemann surface of genus g and area A. R, the scalar curvature on the Riemann surface Σ, is given by $R = -h^{ab}\partial_a h^{cd}\partial_b h_{cd}$.

$$\chi(\Sigma) = \frac{1}{2\pi}\int\limits_\Sigma d^2\sigma\sqrt{h}R = 2 - 2g \tag{2.3.3}$$

is a topological invariant, the Euler number of the Riemann surface Σ. Higher derivative terms made out of the products of the R's are irrelevant when we take the limit in which the length of the edges of the triangles in the triangulation of the Riemann surface Σ approaches zero. These higher-dimensional terms have coupling constants which are proportional to a negative power of a mass scale. The only mass scale in the theory is provide by the inverse of the lattice length. Therefore, as the lattice length approaches zero, these "non-renormalizable" terms become irrelevant in the continuum limit.

The key problem in defining the continuum limit of a discrete lattice approximation to a continuum theory is to obtain finite answers. Thus, the problem is to adjust the real parameters λ_1 and λ_2 as a cutoff $\epsilon \to 0$ so that $F(g, A)$ converges to well defined function of g and A. ϵ is the area of small equilateral triangles used to tile Riemann surfaces of fixed area A. We can rewrite eq. (2.3.2) as

$$F(g, A) = Vol(g, A)e^{-\lambda_1 A - \lambda_2 \chi(\Sigma)}. \tag{2.3.4}$$

The procedure of triangulating a Riemann surface Σ was described in the previous section. Every triangulation of Σ determines a metric. Suppose one triangulates a Riemann surface of genus g and area A with n triangles of small area ϵ. For the number triangles n large, it is reasonable to assume that metrics corresponding to the triangulations are points distributed randomly in $METRICS_{g,A}$, the space of metrics on a Riemann surface of genus g and area A. With a large number of points distributed randomly in $METRICS_{g,A}$, one takes the prescription that integration over $METRICS_{g,A}$ in eq. (2.3.2) can be approximated by summation over the inequivalent triangulations. The key assumption is that the random distribution is uniform. This is basically the Monte Carlo method for integration. The prescription above is precisely the same as described in the previous section, in which equal weights are assigned to inequivalent graphs in the definition of a partition function.

Let $V(g, n)$ be the number of isomorphism classes of triangulations of a genus g surface with n triangles. The mathematics of graph counting gives the result that for large n:

$$V(g, n) = e^{cn} n^{\gamma(2-2g)-1} b_g (1 + O\frac{1}{n})). \tag{2.3.5}$$

The constant γ is universal, even if squares or pentagons were used to tile a surface, whereas c is not universal. Like the cosmological constant, the value of c changes in renormalization flows. Let us triangulate a surface of area A using n triangles of small area ϵ and interpret $Vol_\epsilon(g, A) \equiv V(g, n)/\epsilon$ as the ϵ cutoff approximation to $Vol(g, A)$. So, we have that

$$F_\epsilon(g, A) = Vol_\epsilon(g, A)\ e^{-\lambda_1 A - \lambda_2(2-2g)}. \tag{2.3.6}$$

This gives the finite and well-defined quantity in the limit $\epsilon \to 0$ (with $A = n\epsilon$ held fixed):

$$F(g, A) = \frac{1}{A}(\frac{A}{A_0})^{\gamma(2-2g)} b_g, \tag{2.3.7}$$

with $\lambda_1 = c/\epsilon$ and $\lambda_2 = \gamma \log(A_0/\epsilon)$.

2.4 Non-Perturbative 2-D Gravity

In this section we will review the solution of the matrix models, the continuum limit of the matrix models, and the non-perturbative solutions obtained in [6]. It is possible to compute analytically the partition function [7-9]:

$$Z_N = \int dM\ e^{-\lambda'\ ^1 TrV(M)} \text{ for } V(M) = \sum_{p\geq 1}^{m} \frac{g_p}{N^{p-1}} M^{2p}, \tag{2.4.1}$$

where M is an $N \times N$ Hermitian matrix. (g_1 is set to $\frac{1}{2}$.) Each Hermitian matrix M can be written as $M = U\Lambda U^\dagger$ by some unitary transformation of a diagonal matrix Λ, where U is an $N \times N$ unitary matrix. The integration over the Hermitian matrices in eq. (2.4.1) becomes, by variable substitutions, integration over unitary matrices and diagonal matrices. The integrand in eq. (2.4.1) does not involve the matrix elements of the unitary matrices. The unitary matrices can be thought of as "angular " variables, and the diagonal matrices can be thought of as "radial" variables. The situation here is similar to that in gauge fixing, and the important thing is to remember to compute the Jacobian. Denoting the diagonal matrix elements of a $N \times N$ diagonal matrix Λ as x_i, for $i = 1$ to N, ref. [9] gets the result that

$$Z_N = \int \left(\prod_{i=1}^{N} dx_i \right) e^{-\lambda'^{\,1}(\sum_{i=1}^{N} V(x_i))} \Delta(\Lambda)^2. \tag{2.4.2}$$

Where

$$\Delta(\Lambda) = \prod_{i>j}(x_i - x_j) = det(x_i^{j-1}) = det(P_{j-1}(x_i)) \tag{2.4.3}$$

is called the Vandermonde determinant. ($j = 1$ to N.) $P_n(x) = x^n + \ldots$ is an n^{th} order polynomial, for n non-negative integers, defined by the authors of ref. [9] as orthogonal polynomials such that (m takes the non-negative integer values):

$$h_n \delta_{nm} = \int dx \; e^{-\lambda'^{\,1} V(x)} P_n(x) P_m(x). \tag{2.4.4}$$

The determinant $det(P_{j-1}(x_i))$ in eq. (2.4.3) can be written as alternating sums of products of the matrix elements $P_{j-1}(x_i)$. The result can be plugged into eq. (2.4.2) and most of the terms will not contribute to Z_N because of the orthogonal property of the polynomials. It follows that

$$Z_N = (N!) h_0 h_1 \ldots h_{N-1} = (N!) h_0^N R_1^{n-1} \ldots R_{N-2}^2 R_{n-1}, \tag{2.4.5}$$

where the R_n's are defined recursively by $xP_n(x) = P_{n+1}(x) + R_n P_{n-1}(x)$.

By partial integration:

$$n h_n = \int dx e^{-\lambda'^{\,1} V} x P_n' P_n = R_n \int dx e^{-\lambda'^{\,1} V} (\lambda^{-1} V') P_n P_{n-1}. \tag{2.4.6}$$

A recursion relation for the R_n's is given by substituting the definition of potential V in eq. (2.4.1) into eq. (2.4.6):

$$\lambda \frac{n}{N} = \frac{R_n}{N} [1 + \sum_{p \geq 1} 2(p+1) g_{p+1} \sum_{paths} \frac{R_{d_1}}{N} \frac{R_{d_2}}{N} \ldots \frac{R_{d_p}}{N}]. \tag{2.4.7}$$

The paths can be described as stair climbing traversals from height $n-1$ to n in

$2p+1$ steps, and a R_d/N factor occurs in the R.H.S. of eq. (2.4.7) for each descending step from height d, whereas the factor 1 occurs for each ascending step.

It is convenient to define

$$R(\lambda y) = R_n(\lambda)/N \text{ and } y = n/N. \tag{2.4.8}$$

(Note that λ dependence of R_n can be seen in eq. (2.4.4) and more explicitly in eq. (2.4.7)). For y near one and N finite, perform a Taylor expansion:

$$\frac{R_n(\lambda)}{N} = R(\lambda) + \lambda\frac{n-N}{N}R'(\lambda) + \frac{\lambda^2}{2}(\frac{n-N}{N})^2 R''(\lambda) + \ldots. \tag{2.4.9}$$

To each potential $V(M)$ in eq. (2.4.1) one associates a function:

$$W(R) = 2\sum_{p=0}^{\infty}\frac{2p+1}{(p!)^2}g_{p+1}R^{2p+1}. \tag{2.4.10}$$

Set $n = N$ in eq. (2.4.7), use eq. (2.4.9) in eq. (2.4.7), and take continuum limit $N \longrightarrow \infty$. Ignore $O(1/N)$ terms, and the result is

$$\lambda = W(R(\lambda)). \tag{2.4.11}$$

A multi-critical point $\lambda_c = W(R_c)$ of order m is defined to be potential $V(M)$ with g_p such that

$$W'(R_c) = W''(R_c) = \ldots = W^{(m-1)}(R_c). \tag{2.4.12}$$

In the continuum limit $N \longrightarrow \infty$, different multi-critical potentials give rise to continuum theories with different conformal matter fields coupled to 2-D gravity [11].

To consider pure 2-D quantum gravity, consider the simple quadratic potential $V(M) = g_1 M^2 + g_2 M^4/N$ and the associated function $W(R) = 2g_1 R + 12g_2 R^2$. ($g_1$ is set to $1/2$) Using eq. (2.4.12), the condition of being at a multi-critical point is $W(R) - \lambda_c \propto (R - R_c)^2$. It follows that $\lambda_c = -g_1^2/12g_2$, $R_c = -g_1/12g_2$, and the proportionality constant is $12g_2$. For a simple quadratic potential, one adjusts the parameter λ in eq. (2.4.1) to reach a multi-critical point. With a higher order polynomial potential, it is necessary to tune the coupling constants g_i, as well as λ, to reach a higher multi-critical point.

Setting $n = N$ in eq. (2.4.7), one gets

$$\lambda = 2\frac{R_N}{N}[g_1 + 2g_2(\frac{R_{N+1}}{N} + \frac{R_N}{N} + \frac{R_{N-1}}{N})]. \tag{2.4.13}$$

Using eq. (2.4.9) in eq. (2.4.13), and considering large N, one gets

$$0 = -\lambda + W(R) + N^{-2}4g_2\lambda^2 RR''(\lambda) + O(N^{-4}). \tag{2.4.14}$$

For λ near λ_c, eq. (2.4.14) can be rewritten as

$$0 = -(\lambda - \lambda_c) + 12g_2(R - R_c)^2 + N^{-2}4g_2\lambda_c^2 R_c R''(\lambda) + O(N^{-4}). \tag{2.4.15}$$

Note that a continuum limit in which every genus contributes to the partition function requires the double limit of $N \to \infty$ and $\lambda \to \lambda_c$ resulting in eq. (2.4.15). Whereas, eq. (2.4.11), with the continuum limit $N \to \infty$, results in only genus zero contributing to the partition function.

Now assume the scaling ansatz $R - R_c = N^{-\mu}f(N^{\nu}\Delta)$ where $\Delta = (\lambda - \lambda_c)$. Finite genus calculations give the values $\mu = \frac{2}{5}$ and $\nu = \frac{4}{5}$ for pure 2-D quantum gravity [12]. Eq. (2.4.15) becomes

$$0 = -\Delta + 12g_2(N^{-\frac{2}{5}}f(N^{\frac{4}{5}}\Delta))^2 + N^{-2}4g_2\lambda_c^2 R_c N^{\frac{6}{5}}f''(N^{\frac{4}{5}}\Delta) + O(N^{-4}). \tag{2.4.16}$$

The $O(N^{-4})$ terms are negligible in comparison with the first three terms of eq. (2.4.16). Letting $x = N^{\frac{4}{5}}\Delta$, eq. (2.4.16) becomes

$$x = 12g_2 f^2 + 4g_2\lambda_c^2 R_c f''. \tag{2.4.17}$$

Set $g_2 = 9^{-\frac{1}{3}}/48$ and rescale f by $f \to (-36g_2/g_1^3)f$. Thus, the scaling function $f(x)$

satisfies the non-linear differential equation:

$$x = f^2 + f''. \tag{2.4.18}$$

Eq. (2.4.14) has a solution which is a Painlevé transcendental of the first kind. There are two integration constants, of which one is fixed by matching up the solution at large x with results from perturbative genus expansion, but the non-linear differential equation also has some non-perturbative solutions at finite x. It is clear that eq. (2.4.18) has solutions of the form

$$f \sim -\frac{6}{(x-k)^2}, \tag{2.4.19}$$

for x near k. Parameter k is presently undetermined [5]. Assume k is a finite positive constant, the model has a singularity at $x = k$, and there should be some kind of phase transition, such as a condensation of handles, when x goes below k.

2.4 Multi-matrix Models

The multi-matrix model partition function is defined in ref. [13] as

$$Z_N = \int \Big(\prod_{t=1}^{T} dM(t)\Big)\, e^{-\left(\sum_{t=1}^{T} Tr[V_t(M(t))] + \sum_{t=1}^{T-1} c_t Tr[M(t)M(t+1)]\right)}$$

$$= \int \Big(\prod_{i=1,t=1}^{N,T} dx_i(t)\Big)\, \Delta(\Lambda(1)) e^{-\sum_{i=1,t=1}^{N,T} S_t[x_i]} \Delta(\Lambda(T)). \tag{2.5.1}$$

Where $M(t)$ for $t = 1$ to T are $N \times N$ Hermitian matrices, $\Delta(\Lambda) = \prod_{i>j}(x_i - x_j)$ as before, and $S_t[x_i] = V_t(x_i(t)) + c_t x_i(t) x_i(t+1)$. (Define $c_T = 0$.) $\Lambda(t)$ for $t = 1$ to T are $N \times N$ diagonal matrices with diagonal matrix elements $x_i(t)$ for $t = 1$ to T and $i = 1$ to T. In analogy with the one-matrix models, ref. [13] define orthogonal polynomials:

$$\int \left(\prod_{t=1}^{T} dx(t)\right) P_m^{(1)}(x(1)) e^{-\sum_{t=1}^{T} S_t[x(t)]} P_n^{(T)}(x(T)) = \delta_{m,n} = \langle m|n\rangle, \tag{2.5.2}$$

$$\int \left(\prod_{t=1}^{T} dx(t)\right) P_m^{(1)}(x(1)) x(u) e^{-\sum_{t=1}^{T} S_t[x(t)]} P_n^{(T)}(x(T)) = \langle m|Q(u)|n\rangle, \tag{2.5.3}$$

$$\int \left(\prod_{t=1}^{T} dx(t)\right) P_m^{(1)}(x(1)) e^{-\sum_{t=1}^{u-1} S_t[x(t)] - \frac{V_u[x(u)]}{2}}$$

$$\frac{d}{dx(u)} \left[e^{-\sum_{t=u}^{T} S_t[x(t)] + \frac{V_u[x(u)]}{2}} P_n^{(T)}(x(T)) \right] = \langle m|P(u)|n\rangle. \tag{2.5.4}$$

n and m are positive integers and u is an integer from 1 to T. (Note that, for the purpose of defining an operator $P(u)$, the way we inserted $\frac{d}{dx(u)}$ in eq. (2.5.4) is one choice amongst many possible.) Eq. (2.5.4) can be integrated by parts to give discrete time equation of motion by which $P(t)$ and $Q(t)$ are determined by $P(1)$ and $Q(1)$. Thus, the complete theory (given by the partition function and correlation functions, which are obtained by taking derivatives of the partition function) is solved once $P(1)$ and $Q(1)$ are known. Furthermore, operators $P(1)$ and $Q(1)$ are subjected to the condition $[P(1), Q(1)] = 1$.

Expand the operator $Q(1)$ in the complete basis $\{|n\rangle\}$ as

$$Q(1)|m\rangle = \sum_n Q_{m,n}|n\rangle. \tag{2.5.5}$$

Form another complete basis $\{|\overline{n/N}\rangle\}$ by relabelling the original basis $\{|n\rangle\}$. Eq. (2.5.2) becomes $\langle \overline{m/N}|n\rangle = \delta_{m,n}$ Let N be large, so that the labellings n/N on the new basis approximates the continuum real line (the labellings take on positive values only). Define a function $Q_{m-n}(n/N) = Q_{m,n}$ as the continuum version of $Q_{m,n}$ in eq. (2.5.5).

It will be shown below that in terms of the new basis $|\overline{x}\rangle$ one can write

$$Q(1) = \sum_k Q_k(x) e^{\frac{kD}{N}}, \tag{2.5.6}$$

where $D = d/dx$. It will be first argued that eq. (2.5.6) is consistent with the definition $Q_{m-n}(n/N) = Q_{m,n}$. Consider matrix elements of $Q(1)$, one gets using eq. (2.5.6):

$$\langle\overline{y}|Q(1)|m\rangle = \int dx \ \langle\overline{y}|Q(1)|\overline{x}\rangle\langle\overline{x}|m\rangle$$

$$= \int dx \ \delta(y-x) \sum_k Q_k(x) e^{\frac{kD}{N}} \langle\overline{x}|m\rangle = \int dx \ \delta(y-x) \sum_k Q_k(x) \langle\overline{x + \frac{k}{N}}|m\rangle$$

$$= \sum_k Q_k(y) \langle\overline{y + \frac{k}{N}}|m\rangle. \tag{2.5.7}$$

Using eq. (2.5.7), one has

$$Q_{m,n} = \langle n|m\rangle = \int dy \ \langle n|\overline{y}\rangle\langle\overline{y}|m\rangle = \int dy \ \langle n|\overline{y}\rangle \sum_k Q_k(y) \langle\overline{y + \frac{k}{N}}|m\rangle$$

$$= \sum_k Q_k(\frac{n}{N}) \langle\overline{\frac{n}{N} + \frac{k}{N}}|m\rangle = Q_{m-n}(\frac{n}{N}). \tag{2.5.8}$$

It is clear that one can reverse the argument above by starting out with eq. (2.5.8). Then, eq. (2.5.7) follows, and orthonormal property of the bases $|n\rangle$ and $|\overline{n/N}\rangle$ can be used to obtain eq. (2.5.6) from eq. (2.5.7).

The matrix elements $Q_{m,n}$ are nonzero for $|m-n| \leq B$ for some finite B, if the potential $V_t(M(t))$ in eq. (2.5.1) are finite order polynomials. This is clear, since $Q(1)$ simply inserts a factor of $x(1)$, so that, for $0 \leq n \leq m+1$, we have that $Q_{m,n}$ are non-zero, and therefore $m-n$ is bounded below. For the other direction of bounding above, consider the matrix elements of $Q(T)$ and $P(T)$ which can be shown to be bounded above, and since $Q(1)$ is related to $Q(T)$ and $P(T)$ by a polynomial relation given by the equation of motion, so $m-n$ is bounded above for matrix elements $Q_{m,n}$ to be non-zero.

One can tune the couplings in $V_t(M(t))$ and c_t's to reach a critical point at which $Q_k(x)$ is non-analytic in x. Only the non-analytic contributions, which correspond to surfaces tiled by divergent number of small triangles, will survive the continuum limit. These surfaces will have finite area even when the size of the small triangle approaches zero. The analytic contributions which corresponds to surfaces with a finite number of infinitesimally small triangles become irrelevant at the continuum limit. Letting $N \to \infty$, $Q(1)$ becomes a finite order differential operator, since $Q_{m,n}$ are non-zero for $|m-n| \leq B$ for some finite B. Therefore, one has that

$$Q(1) = \sum_{i=0}^{q} u_i(x) D^i. \tag{2.5.9}$$

The coefficient u_q can be set to 1 by rescaling. u_{q-1} can be set to zero by changing the norm of $\{|n\rangle\}$. u_{q-2} gives the leading contribution to the free energy F, and $u_{q-2} \sim \frac{\partial^2 F}{\partial x^2} = \langle PP \rangle$. P is the puncture operator as defined later in eq. (2.5.16). The coupling constant of the puncture operator is the cosmological constant x.

A very rough analogy of the double scaling limit of multi-matrix models to the double scaling limit of one-matrix models is given as follows. The operator $Q(1)$ is analogous to the operator R_n of section 2.4. When eq. (2.4.9) is substituted in eq. (2.4.13), the result is eq. (2.4.15) with an infinite number of higher derivative terms. Taking the double scaling limit, one only retains the three lowest derivative terms in eq. (2.4.15). The result is eq. (2.4.18). Similarly, in the case of the multi-matrix models, one starts out with eq. (2.5.6), where the operator $Q(1)$ has an infinite number of higher derivative operators. Taking the double scaling limit, by letting $N \to \infty$ and tuning the coupling to reach a critical point, one obtains the result that the operator $Q(1)$ becomes a finite order differential operator.

In the double scaling continuum limit of multi-matrix models, one expects $P(1)$ to become a finite order differential operator in the same way as $Q(1)$. One demands that the condition

$$[P(1), Q(1)] = 1 \tag{2.5.10}$$

holds in the continuum limit, and that it uniquely determine u_0, $u_1, \ldots, u_{q-2}$ in eq. (2.5.9). So $q-1$ independent equations is needed. Notice that $(Q(1)^{\frac{p}{q}})_-$ commutes with $Q(1)$ to give an order $q-2$ operator for $p > q$ relative prime, so one lets $P(1) = (Q(1)^{\frac{p}{q}})_+$. The $+$ subscript denotes taking the differential operator part of the pseudo-differential operator and the $-$ subscript denotes dropping the differential operator part of the pseudo-differential operator. Taking fractional powers of a differential operator, in general, requires one to consider pseudo-differential operators. Pseudo-differential operator algebra is the closure of the differential operator algebra and an operator D^{-1}. The action of D^{-1} is given by eq. (2.C.3) in appendix 2.C.

To find the spectrum of operators in the theory, ref. [13] proposed that they are the maximal sets of commuting flows generated by the fractional powers of $Q(1)$:

$$\frac{\partial Q(1)}{\partial t_i} = [(Q(1)^{\frac{i}{q}})_+, Q(1)]. \tag{2.5.11}$$

Therefore, the coefficients u_i in eq. (2.5.9) are functions of infinite number of variables, $x, t_1, t_2, t_3, \ldots$. Setting $i = 1$ in eq. (2.5.11) gives $x = t_1$.

The continuum limit of the multi-matrix models, as given in ref. [13], is conjectured to be (A_{q-1}, A_{p-1})-type conformal field theories coupled to 2-D gravity. Let $L = Q$; then, the theories are formulated compactly in terms of pseudo-differential operators by rewriting eqs. (2.5.9), (2.5.10), and (2.5.11):

$$L = \sum_{i=0}^{q} u_i(t_1, t_2, \ldots) D^i. \tag{2.5.12}$$

$$[L, (L^{\frac{p}{q}})_+] = 1. \tag{2.5.13}$$

$$\frac{\partial L}{\partial t_i} = [(L^{\frac{i}{q}})_+, L]. \tag{2.5.14}$$

Eqs. (2.5.12), (2.5.13), and (2.5.14) can be rewritten as an action principle [14]:

$$S = \int dx \; Res[L^{\frac{p}{q}+1} + \sum_{i=1,\; i\neq nq}^{\infty} t_i L^{\frac{i}{q}}]. \tag{2.5.15}$$

n is an integer in eq. (2.5.15), and index i runs over positive integers not divisible by q. $Res(L)$ is defined to be the coefficient of D^{-1} of a pseudo-differential operator L. The t_i's are the sources for operator insertions. Define the operators σ_i and the puncture operator P by

$$\langle\sigma_i\rangle = \int dx \; Res(L^{\frac{i}{q}}),$$

$$P = \sigma_1. \tag{2.5.16}$$

The coupling constant of the puncture operator is $x = t_1$, the cosmological constant. It is useful to define

$$R_i = Res(L^{\frac{i}{q}}) = \langle\sigma_i P\rangle, \tag{2.5.17}$$

to be used in later chapters. It is also shown in ref. [13] that the partition function of the continuum theory is given by

$$Z = e^{-F} = \tau^2, \tag{2.5.18}$$

where $\tau(t_1, t_2, t_3, \ldots)$ is the τ function of KP hierarchy (see appendix 2.C for the definition). Therefore, the one point functions of the operators σ_i are given by

$$\langle\sigma_i\rangle = 2\frac{\partial}{\partial t_i} log \; \tau. \tag{2.5.19}$$

To compare the results in this section with the one matrix models, consider the case $q = 2, p = 3$. $L = D^2 + u(t_1, t_2, \ldots)$, $(L^{\frac{3}{2}})_+ = D^3 + (3/4)u, D$, and $[L, (L^{\frac{3}{2}})_+] = 1$

in eq. (2.5.13). Integrate eq. (2.5.13) once in x, to get $(15/32)u^2 + (5/32)u'' = x + c$. x, the cosmological constant, is non-universal (its value changes in renormalization flows). Redefine $x + c$ as x, and rescale $u \to 2^2 3^{-\frac{3}{5}} 5^{\frac{2}{5}} u$ and $x \to 2^{-1} 3^{-\frac{1}{5}} 5^{\frac{1}{5}} x$), the result is

$$x = u^2 + u''. \tag{2.5.20}$$

Eq. (2.5.20) is the same as eq. (2.4.18), and we can identify $u = f = \langle PP \rangle$.

2.6 Conclusion

We conclude by pointing out three issues which are presently still unresolved. So far, the exactly solved models of conformal matter coupled to 2-D quantum gravity all have $c < 1$. The case of $c = 1$ conformal matter coupled to 2-D quantum gravity is of great interest since this is one of the values of the central charge at which one expects a phase transition to occur. In particular, the continuum analysis of conformal matter coupled to 2-D quantum gravity in conformal gauge by DDK [3] gives a formula for the string susceptibility that is sensible for $c < 1$ and $c > 25$ only. The unresolved issue is to obtain well-defined theories with $1 < c < 25$ or to understand why this is not possible.

One of the unsatisfying features of the matrix model approach to studying conformal matter coupled to 2-D quantum gravity is that the resulting continuum theories of $(q-1)$-matrix models, which are labeled by (A_{q-1}, A_{p-1}) minimal models coupled to 2-D gravity, have $q < p$. Ultimately, one would like a formulation in which q and p are treated on an equal footing.

It is still a mystery why, in the treatment of the one matrix models which can be mapped directly to discretized pure 2-D gravity, it is possible to obtain non-unitary conformal models coupled to 2-D gravity by considering higher multi-critical points in taking the continuum limit. One would like to understand in detail how the conformal matter arises at the higher multi-critical points.

Appendix 2.A. Lie Algebras, Dynkin Diagrams, and Coxeter Numbers

Lie groups are continuous groups with an infinite number of elements. The local information is contained in the Lie algebra:

$$[T_i, T_j] = f^k_{i,j} T_k. \tag{2.A.1}$$

T_i are generators, and $e^{x_i T_i}$ parameterizes by coordinates x_i the group elements near the identity element. A maximal set of linear combinations of generators T_i that mutually commute is called a Cartan subalgebra. The dimension of the Cartan subalgebra is the rank r of the Lie algebra. Let H_i be a basis of the Cartan subalgebra and E_α be a basis of the elements of the Lie algebra not in the Cartan subalgebra. α is an r- dimensional vector with components α_i. It is possible to choose the basis such that

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{2.A.2}$$

Then α is called a root vector. Each root vector can be spanned by r linearly independent root vectors:

$$\alpha = \sum_{i=1}^{r} c_i \alpha^{(i)}. \tag{2.A.3}$$

A root vector is called a positive root if the first nonzero c_i is positive. A simple root is a positive root that cannot be written as the sum of two positive roots. For a Lie algebra of rank r, there are r simple roots, which are just the $\alpha^{(i)}$.

A Cartan matrix is an $r \times r$ matrix defined by

$$C_{i,j} = 2\frac{(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(j)}, \alpha^{(j)})}. \tag{2.A.4}$$

Here the $\alpha^{(i)}$ are the r simple roots, and the inner product amongst the simple roots is the ordinary r-dimensional vector inner product. The diagonal elements of the Cartan matrix are all equal to 2, and it can be shown that the only possible values of the off-diagonal matrix elements are $0, -1, -2, -3$.

A Cartan matrix, which contains compressed but complete description of a Lie algebra, can be graphically represented by a Dynkin diagram. To construct a Dynkin diagram, first draw r dots representing the r simple roots, where r is the rank of the Lie algebra. Next, connect the ith and jth dot with n lines, where $n = C_{i,j}C_{j,i}$. (i and j indices are not summed.) It can be shown that n can only take values $n = 0, 1, 2, 3$, which represents the angles θ between the simple roots with values $\theta = \pi/2, 2\pi/3, 3\pi/4, 5\pi/6$, respectively. Lastly, let black dots represent short simple roots and white dots represent long simple roots. (The length square of a simple root α_j is $\langle\alpha_j, \alpha_j\rangle$.)

In figure 2.A.1, the Lie algebras A_n, B_n, C_n, D_n of rank n are the Lie algebras of the Lie groups $SU(n+1), SO(2n+1), Sp(2n), SO(2n)$, respectively. Lie algebras A_n, D_n, E_6, E_7, E_8 are called simply laced, since their Dynkin diagrams consist of only white dots, which are simple roots of equal length.

The n polynomials in the generators T_i which commute with every element of a rank n Lie algebra are called the Casimirs. The degrees of the Casimirs minus one are called the Coxeter exponents. The Coxeter number of a Lie algebra is the largest Coxeter exponent plus one. In table 2.A.1, Coxeter exponents for simply laced Lie algebras are presented.

Figure 2.A.1 Dynkin Diagrams for Various Lie Algebras

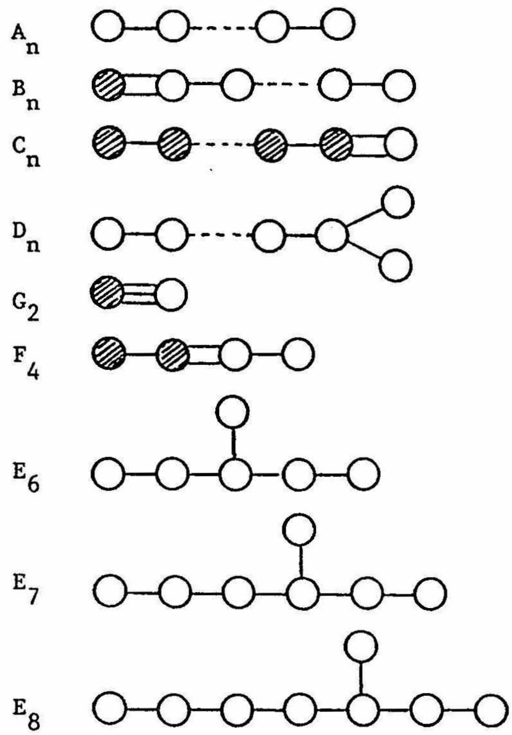

Table 2.A.1 Coxeter Exponents for Simply Laced Algebras

Lie Algebra	Coxeter Exponents	Coxeter Number
A_n	$1, 2, 3, \ldots, n$	$n+1$
D_n	$1, 3, 5, \ldots, 2n-3, n-1$	$2n-2$
E_6	$1, 4, 5, 7, 8, 11$	12
E_7	$1, 5, 7, 9, 11, 13, 17$	18
E_8	$1, 7, 11, 13, 17, 19, 23, 29$	30

Appendix 2.B. Classification of Minimal Conformal Field Theories by Simply Laced Lie Algebras

The work of ref. [15] is reviewed, where systematic analysis of modular invariant partition functions leads to a complete classification of minimal $c \leq 1$ conformal theories with solutions labeled by simply laced Lie algebras. The minimal two-dimensional conformal invariant field theories [4] carry a set of representations of two Virasoro algebras of common central charge

$$c = 1 - \frac{6(p-p')^2}{pp'} \tag{2.B.1}$$

with (p', p) a pair of relative prime positive integers. BPZ [4] have shown that it is consistent to retain only a finite number of primary fields $\phi_{h,\overline{h}}$ of conformal dimensions h and $\overline{h}$ chosen among the Kac values [16]

$$h_{r,s} = \frac{(rp - sp')^2 - (p-p')^2}{4pp'} = h_{p'-r,p-s}, \tag{2.B.2}$$

with

$$1 \leq r \leq p' - 1, \quad 1 \leq s \leq p - 1.$$

An important subset of these minimal theories consists of the unitary $c < 1$ conformal theories, for which p and p' must be consecutive integers:$|p - p'| = 1$ [17].

Cardy [18] has shown that putting such a conformal theory in a finite box with periodic boundary conditions, i.e., on a torus, gives stringent constraints on its operator content. These constraints arise from the requirement of modular invariance of the partition function which has the general form:

$$Z = \sum N_{h,\overline{h}} \chi_h(\tau) \chi^*_{\overline{h}}(\tau). \tag{2.B.3}$$

The conformal characters $\chi_h(\tau) = tr(e^{2\pi i \tau(L_0 - c/24)})$ are explicit functions of τ, the modular ratio of the torus; $N_{h,\overline{h}} = N_{\overline{h},h}$ are non-negative integers, arising from the decomposition of the representation of the Virasoro algebras carried by the space of states into irreducible representations.

Gepner and Witten [19] have studied modular invariants sesquilinear in characters of an affine Lie algebra:

$$\sum N_{l,l^*} \chi_l(\tau) \chi^*_{l^*}(\tau), \tag{2.B.4}$$

with N_{l,l^*} non-negative integers. Take the case of $A_1^{(1)}$ representation of level k. The integrality and positivity conditions on the coefficients N_{l,l^*} reduce drastically the acceptable affine modular invariants. There are only three classes of solutions [15]. The first, which exists for $k \geq 1$, corresponds to the trivial diagonal invariant $\sum |\chi_h|^2$, while the second appears only for even $k \geq 4$. In addition, there are three exceptional cases for $k + 2 = 12$, 18, and 30, which are the Coxeter numbers of the exceptional Lie algebras E_6, E_7, E_8. These two infinite series and three exceptional solutions are in correspondence with the simply laced Lie algebras A_{k+1}, $D_{k/2+2}$, E_6, E_7, and E_8.

To produce positive conformal modular invariants, a pair of affine invariants of levels $k = p - 1$ and $k' = p' - 2$ is needed. p and p' are relative primes, so they cannot be both even, and this forces one of the two algebras to be an A algebra. Modular invariant partition functions in terms of conformal characters and the corresponding labels by pairs of simply laced Lie algebra are summarized in table 2.B.1 [15].

Table 2.B.1 Classification of Minimal Conformal Field Theories by Simply Laced Lie Algebras

$(A_{p'-1}, A_{p-1})$	$\frac{1}{2}\sum_{s=1}^{p-1}\sum_{r=1}^{p'-1}\lvert\chi_{rs}\rvert^2$	$p', p \geq 2$
$(D_{2\rho+2}, A_{p-1})$	$\frac{1}{2}\sum_{s=1}^{p-1}\{\sum_{r\ \mathrm{odd}=1,\ r\neq 2\rho-1}^{4\rho+1}\lvert\chi_{rs}\rvert^2$ $+2\lvert\chi_{2\rho+1,s}\rvert^2$ $+\sum_{r\ \mathrm{odd}=1}^{2\rho-1}(\chi_{rs}\chi^*_{r,p-s} + c.c.)\}$	$p' = 4\rho + 2$ $p \geq 2$
$(D_{2\rho+1}, A_{p-1})$	$\frac{1}{2}\sum_{s=1}^{p-1}\{\sum_{r\ \mathrm{odd}=1}^{4\rho-1}\lvert\chi_{rs}\rvert^2 + \lvert\chi_{2\rho,s}\rvert^2$ $+\sum_{r\ \mathrm{even}=2}^{2\rho-2}(\chi_{rs}\chi^*_{p'-r,s} + c.c.)\}$	$p' = 4\rho$ $p \geq 2$
(E_6, A_{p-1})	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1s}+\chi_{7s}\rvert^2 + \lvert\chi_{4s}+\chi_{8s}\rvert^2$ $+\lvert\chi_{5s}+\chi_{11s}\rvert^2\}$	$p' = 12$ $p \geq 2$
(E_7, A_{p-1})	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1s}+\chi_{17s}\rvert^2 + \lvert\chi_{5s}+\chi_{13s}\rvert^2$ $+\lvert\chi_{7s}+\chi_{11s}\rvert^2 + \lvert\chi_{9s}\rvert^2$ $+[(\chi_{3s}+\chi_{15s})\chi^*_{9s} + c.c]$	$p' = 18$ $p \geq 2$
(E_8, A_{p-1})	$\frac{1}{2}\sum_{s=1}^{p-1}\{\lvert\chi_{1s}+\chi_{11s}+\chi_{19s}+\chi_{29s}\rvert^2$ $+\lvert\chi_{7s}+\chi_{13s}+\chi_{17s}+\chi_{23s}\rvert^2$	$p' = 30$ $p \geq 2$

Appendix 2.C KdV Equations, Pseudo-Differential Operators, and the τ Function

The KP hierarchy [20] is defined by the equations for commuting flows:

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \text{ for } n \geq 1. \tag{2.C.1}$$

The pseudo-differential operator

$$L = D + u_2(t)D^{-1} + u_3(t)D^{-2} + \dots \tag{2.C.2}$$

has coefficients u_i, which are functions of an infinite number of variables $t_1, t_2, t_3, \dots$. t denotes the variables $t_1, t_2, t_3, \dots$. $D = \frac{\partial}{\partial x}$, so that $n = 1$ in eq. (2.C.1) implies that $x = t_1$. The $+$ subscript denotes taking the differential operator part of the pseudo-differential operator and the $-$ subscript will denote dropping the differential operator part of the pseudo-differential operator. The closure of the differential operator algebra and D^{-1} defines a pseudo-differential operator algebra in which fractional powers of operators are well defined. The action of D^{-1} is

$$D^{-1}f(x) = \sum_{i=0}^{\infty}(-1)^i(D^{(i)}f(x))D^{(-1-i)}. \tag{2.C.3}$$

The τ function is defined by the following equations:

$$L = PDP^{-1},$$

$$P = 1 + w_1(t)D^{-1} + w_2(t)D^{-2} + \dots,$$

$$1 + w_1(t)k^{-1} + w_2(t)k^{-2} + \dots = \frac{\tau(t_1 - \frac{1}{k}, t_2 - \frac{1}{2k^2}, t_3 - \frac{1}{3k^3}, \dots)}{\tau(t_1, t_2, t_3, \dots)}. \tag{2.C.4}$$

(k is a real number.)

The p-reduction of the KP hierachy is defined by imposing an additional constraint on L:

$$(L^q)_- = 0. \qquad (2.C.5)$$

This means that the coefficients u_i in eq. (2.C.2) have no dependence on the variables $t_q, t_{2q}, t_{3q}, \ldots$. Eq. (2.C.5) also implies that the coefficients u_i can be expressed in terms of $u_2, u_3, \ldots, u_q$. The two-reduction and three-reduction of the KP hierarchy are called the KdV and Boussinesq hierarchy, respectively.

Appendix 2.D W-algebra

W-algebra [21], which is associated with higher spin fields, is a generalization of Virasoro algebra, which is associated with spin two fields. See ref. [4, 21] for definition of spin and scaling dimension of a field. For example, in the case of spin three, the W-algebra is denoted $W^{(3)}$. Write the generators of $W^{(3)}$ as $W_n^{(3)}$, where n is an integer. The algebra of $W^{(3)}$ is defined by [15]:

$$[L_n, W_m^{(3)}] = (2n - m)W_{n+m}^{(3)},$$

$$[W_n^{(3)}, W_m^{(3)}] = \frac{3}{2}(n-m)\Big((n^2 + 4nm + m^2) + 9(n + m + 14\Big)L_{n+m}$$

$$-9(n-m)U_{n+m} - \frac{1}{10}n(n^2-1)(n^2-4)\delta_{n,-m}, \qquad (2.D.1)$$

where $U_n = \sum_{k\leq -2} L_k L_{n-k} + \sum_{k\geq -1} L_{n-k}L_k$ and L_n are the generators of Virasoro algebra. Note that $W_n^{(3)}$ does not form a Lie algebra, but rather it is a more general algebra with quadratic determining relations, eq. (2.D.1).

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3. KdV Recursion Relations for the Two-Matrix Models

The continuum limit of the one-matrix models is given by eqs. (2.5.12), (2.5.13), (2.5.14), and (2.5.15), for $q = 2$. Using the KdV equations (2.5.14) for $q = 2$, Dijkgraaf and Witten [1] found a KdV recursion relation between correlation functions of the one-matrix models. (See section 3.3.) Ref. [1] also found a KdV recursion relation between correlation functions of the two-matrix models. In this chapter we find a new KdV recursion relation for the two-matrix models that is not equivalent to the one in Ref. [1].

In section one, we summarize the recent developments in matrix models. In section two, a useful identity is derived, and we use it to find two KdV recursion relations for the two-matrix models. A technical assumption used, called "the scaling ansatz", is explained. As a warm-up exercise, in section three, we consider the applications of the KdV recursion relation for the one-matrix models. It turns out that every correlation function containing an insertion of P can be expressed in terms of $\langle PP \rangle$. P is the puncture operator defined by eq. (2.5.16). Finally, in section four, we consider the KdV recursion relations for the two-matrix models. It can be shown that every correlation function with an insertion of P can be expressed in terms of $\langle PP \rangle$ and two other correlation functions. In section five, KdV recursion relations for the three-matrix models are considered, and we give the conclusion in section six.

3.1 Introduction

Recently there has been much interest in understanding the role of the KdV recursion relations and the string equation within the context of the multi-matrix models approach to conformal matter coupled to 2-D gravity [1-4]. By recasting the KdV equations for the one-matrix models and the string equation, Dijkgraaf *et al.* [5] derived the loop equations which, remarkably, have the algebraic structure of Virasoro constraints. Fukuma *et al.* [6] also derived the Virasoro constraints, but in a different way. (See section 4.2 for a review of the results obtained in ref. [5].) We are interested in treating KdV recursion relations as algebraic recursion relations for the correlation functions. For the one-matrix models, we will find that all correlation functions with an insertion of P at any genus can be determined in terms of $\langle PP \rangle$. We will show the above statement by using the KdV recursion relation for the one-matrix models and assuming a scaling ansatz for correlation functions at each genus.

For the two-matrix models, we will find a KdV recursion relation in addition to that in [1]. Writing the two-matrix models in the compact language of pseudo-differential operators, we will be able to derive a simple and useful identity. From this identity, we can obtain two KdV recursion relations for the two-matrix models. The procedure is simple but laborious, so the algebra needs to be carried out on a computer.

It will be clear that the method we use to find the two KdV recursion relations for the two-matrix models can also be applied to finding the KdV recursion relations for the three-matrix models and, in general, all of the multi-matrix models proposed by Douglas [3]. The multi-matrix models have interaction terms of the action with nearest neighbor matrices only forming a simple linear chain. The diagrams of the linear chains are the same as the Dynkin diagrams of the A_n Lie groups. Other multi-matrix models have interaction terms forming diagrams that are the same as the Dynkin diagrams of the D_n, E_6, E_7, E_8. However, it is not clear whether the method we use for the two-matrix case generalizes to the matrix models related to the D_n, E_6, E_7, E_8.

3.2 Two-Matrix Model KdV Recursion Relations

The KdV equations of the continuum two-matrix models, formulated in terms of pseudo-differential operators, are given by eq. (2.5.14) for $q = 3$ [1,3,4]:

$$\frac{\partial L}{\partial t_n} = [L_+^{\frac{n}{3}}, L], \quad n \in \{3j+1, 3j+2 | j \in Z, j \geq 0\}, \tag{3.2.1}$$

where $L = D^3 + 3\{u_1, D\} + 3u_2$ is given by eq. (2.5.12), and $D = \frac{\partial}{\partial x}$. (We have rescaled and redefined u_1 and u_2 for later notational convenience.) Coefficients u_1

and u_2 are functions of an infinite number of variables $x, t_1, t_2, t_3, \ldots$. Setting $n = 1$ in eq. (3.2.1), we get that $x = t_1$. Also, the 3-reduction condition (eq. (2.C.5)) applies to the two-matrix case, and this fact is reflected in eq. (3.2.1). Therefore, the coefficients u_1 and u_2 are functions of t_n for n positive integer not divisible by three. The string equation of the two-matrix models is given by eq. (2.5.14) for $q = 3$. The string equation will not be needed again in the rest of this chapter, but it is mentioned for the sake of completeness.

The action of the two-matrix models is given by eq. (2.5.15) for $q = 3$:

$$S = \int dx \; Res[L^{\frac{p}{3}+1} + \sum_{i=1,\; i\neq 3i}^{\infty} t_n L^{\frac{n}{3}}]. \tag{3.2.2}$$

i is an integer in eq. (3.2.2), and index n runs over positive integers not divisible by three. p labels the multi-critical points of the two-matrix models. Distinct multi-critical points correspond to distinct conformal matter coupled to 2-D gravity. For example, Ising conformal model coupled to 2-D gravity corresponds to $q = 3$ and $p = 4$. $Res(L)$ is defined to be the coefficient of D^{-1} of a pseudo-differential operator L. The operators $\int dx \; Res(L^{\frac{n}{q}})$ will be denoted by $\langle \sigma_n \rangle$. The puncture operator is defined to be $P = \sigma_1$. The dilaton operator is defined to be σ_2. t_n are the coupling constants of the scaling operators σ_n, t_1 is the coupling constant of the puncture operator P, and t_2 is the coupling constant of the dilaton operator Q. To insert an operator σ_n in a correlation function, differentiate the correlation function with respect to t_n.

The $+$ subscript denotes taking the differential operator part of the pseudo-differential operator and the $-$ subscript will denote dropping the differential operator part of the pseudo-differential operator. Wherever the index n appears in this chapter, we will implicitly assume that n is a positive integer that is not divisible by three. Lastly, we gather from section 2.5 the important fact that $u_1 = \langle PP \rangle$, and it will be shown below that $u_2 = \langle PQ \rangle$.

We also need a technical assumption called the "scaling ansatz": we assume that every correlation function at every genus scales as a power of $t_1 = x$, when all the other couplings t_i are set to zero, and that a higher genus correlation function has x to a higher power than the same correlation function at a lower genus. The scaling ansatz can be shown to be true for simple correlation functions at low genus using matrix model techniques. For example, the partition function for the pure 2-D gravity critical point of the matrix model scales as $x^{5\chi/4}$, $\chi = 2 - 2g$.

We will now proceed to derive the two KdV recursion relations for the two-matrix models. We define $R_{n,i}$, for $i = 1$ and 2, by $L_-^{\frac{n}{3}} = \{R_{n,1}, D^{-1}\} + R_{n,2}D^{-2} + O(D^{-3})$. Then, eq. (3.2.1) translates to:

$$(6\frac{\partial u_1}{\partial t_n})D + 3\frac{\partial u_1'}{\partial t_n} + 3\frac{\partial u_2}{\partial t_n} = [D^3, \{R_{n,1}, D^{-1}\} + R_{n,2}D^{-2} + \ldots]$$

$$= (6R_{n,1}')D + 3R_{n,1}'' + 3R_{n,2}'. \qquad (3.2.3)$$

Eq. (3.2.3) simplifies in compact notation as:

$$\frac{\partial u_i}{\partial t_n} = \frac{\partial R_{n,i}}{\partial x}, \quad i = 1 \text{ and } 2. \qquad (3.2.4)$$

Starting with $L = D^3 + 3\{u_1, D\} + 3u_2$, we can compute $L^{\frac{1}{3}}$ to find that $L_-^{\frac{1}{3}} = \{u_1, D^{-1}\} + u_2D^{-2} + O(D^{-3})$. We can compute $L^{\frac{2}{3}}$ by squaring $L^{\frac{1}{3}}$, and we will find that $L_-^{\frac{2}{3}} = \{u_2, D^{-1}\} + O(D^{-2})$. So $R_{1,1} = u_1$ and $R_{1,2} = u_2$ implies $t_1 = x$. So $R_{2,1} = u_2$ implies $u_2 = \langle PQ \rangle$. In general, we get $R_{n,1} = \langle P\sigma_n \rangle$ and $R_{n,2} = \langle Q\sigma_n \rangle$. Note that $R_{1,2} = u_2 = R_{2,1}$.

We now find an identity [7] from which we can derive the KdV recursion relations. On the one hand we have:

$$[L_+^{\frac{n}{3}+1}, L] = [L, L_-^{\frac{n}{3}+1}] = (6R_{n+3,1}')d + 3R_{n+3,1}'' + 3R_{n+3,2}'. \qquad (3.2.5)$$

On the other hand we have:

$$L_+^{\frac{n}{3}+1} = \frac{1}{2}\{L_+^{\frac{n}{3}}, L\} + \frac{1}{2}\{L_-^{\frac{n}{3}}, L\}_+,$$

$$[L_+^{\frac{n}{3}+1}, L] = \frac{1}{2}[\{L_+^{\frac{n}{3}}, L\}_+, L] + \frac{1}{2}[\{L_-^{\frac{n}{3}}, L\}_+, L]$$

$$= \frac{1}{2}\{L, [L, L_-^{\frac{n}{3}}]\} - \frac{1}{2}[\{L_-^{\frac{n}{3}}, L\}_+, L]. \tag{3.2.6}$$

Thus, we have the identity:

$$0 = [L, L_-^{\frac{n}{3}+1}] - \frac{1}{2}\{L, [L, L_-^{\frac{n}{3}}]\} + \frac{1}{2}[\{L_-^{\frac{n}{3}}, L\}_+, L]. \tag{3.2.7}$$

The identity above will have coefficients of D^1 and D^0 only, and we set them to zero to obtain the two KdV recursion relations. Note that $[L, L_-^{\frac{n}{3}}] = [L, L_-^{\frac{n}{3}}]_+$ can be used to simplify the computations. The D^1 coefficient gives:

$$6R'_{n+3,1} = 18u_2R'_{n,1} + 12u'_2R_{n,1} + 12u_1R'_{n,2} + 6u'_1R_{n,2} + 2R'''_{n,2}. \tag{3.2.8}$$

The D^0 coefficient gives:

$$3R'_{n+3,2} = 9u_2R'_{n,2} + 3u'_2R_{n,2} + 3u'_1R'_{n,2} - 48u_1^2R'_{n,1} - 48u_1u'_1R_{n,1} + 6u'_2R'_{n,1}$$

$$-3u'_2R'_{n,2} - 15u'_1R''_{n,1} + 3u''_1R_{n,2} - 9u''_1R'_{n,1} + 6u''_2R_{n,1} - 3u''_2R_{n,2}$$

$$-10u_1R^{(3)}_{n,1} - 2u'''_1R_{n,1} - \frac{1}{3}R^{(5)}_{n,1}. \tag{3.2.9}$$

We translate eq. (3.2.8) and eq. (3.2.9) by using eq. (3.2.4), and we obtain unique assignments of the genus for the correlation functions by using scaling arguments described below:

$$\langle\sigma_{n+3}PP\rangle_g = \sum_{g_1+g_2=g}\Big(3\langle\sigma_nPP\rangle_{g_1}\langle PQ\rangle_{g_2} + 2\langle PP\rangle_{g_1}\langle\sigma_nPQ\rangle_{g_2}$$

$$+\langle PPP\rangle_{g_1}\langle\sigma_nQ\rangle_{g_2} + 2\langle QPP\rangle_{g_1}\langle\sigma_nP\rangle_{g_2}\Big) + \frac{1}{3}\langle\sigma_nQPPP\rangle_{g-1}. \tag{3.2.10}$$

$$\langle\sigma_{n+3}PQ\rangle_g = \sum_{g_1+g_2=g}\Big(-\langle QPP\rangle_{g_1}\langle\sigma_n QP\rangle_{g_2} - \langle QPPP\rangle_{g_1}\langle\sigma_n Q\rangle_{g_2}\Big)$$

$$+\sum_{g_1+g_2=g-1}\Big(3\langle PQ\rangle_{g_1}\langle\sigma_n PQ\rangle_{g_2} + \langle QPP\rangle_{g_1}\langle\sigma_n Q\rangle_{g_2} + \langle PPP\rangle_{g_1}\langle\sigma_n PQ\rangle_{g_2}$$

$$+2\langle QPP\rangle_{g_1}\langle\sigma_n PP\rangle_{g_2} + \langle PPPP\rangle_{g_1}\langle\sigma_n Q\rangle_{g_2} + 2\langle QPPP\rangle_{g_1}\langle\sigma_n P\rangle_{g_2}\Big)$$

$$+\sum_{g_1+g_2+g_3=g}\Big(-16\langle PP\rangle_{g_1}\langle PP\rangle_{g_2}\langle\sigma_n PP\rangle_{g_3} - 16\langle PP\rangle_{g_1}\langle PPP\rangle_{g_2}\langle\sigma_n P\rangle_{g_3}\Big)$$

$$+\sum_{g_1+g_2=g-1}\Big(-\frac{10}{3}\langle PP\rangle_{g_1}\langle\sigma_n PPPP\rangle_{g_2} - 5\langle PPP\rangle_{g_1}\langle\sigma_n PPP\rangle_{g_2}$$

$$-3\langle PPPP\rangle_{g_1}\langle\sigma_n PP\rangle_{g_2} - \frac{2}{3}\langle PPPPP\rangle_{g_1}\langle\sigma_n P\rangle_{g_2}\Big) - \frac{1}{9}\langle\sigma_n PPPPPP\rangle_{g-2}. \quad (3.2.11)$$

The scaling ansatz is used to construct scaling arguments used to obtain the assignments of the genus subscripts in eqs. (3.2.10) and (3.2.11). We observe two simple rules by applying the scaling ansatz. The first rule is that from DDK scaling dimensions, and earlier works in matrix models, we have the result that a higher genus correlation function scales with a lower power of x, the scaling variable, than the same correlation function at a lower genus. This rule implies that, given two correlation functions which scale with the same power of x and differ only in the number of insertions of P, the correlation function containing more P insertions must be at a lower genus.

Each term in the R.H.S. of eqs. (3.2.10) and (3.2.11) is a correlation function or a product of correlation functions. The second rule is that we may distribute insertions of P's to each factor within a term which is product of correlation functions, since that is just taking derivatives $\frac{\partial}{\partial x}$ of the term. Therefore, terms which differ only in the distribution of insertion of P's must correspond to the same total genus. The total genus of a term is defined to be the sum of the subscript genus assigned to the correlation functions which multiply to make up that term. It follows that similar comment holds for distributing the insertions of σ_n's, and, in particular, we will use the fact that terms which differ only in the distribution of insertions of Q's must correspond to the same total genus.

We now apply the two rules above to eq. (3.2.11). First of all, there are the terms without any explicit insertion of Q, which are the last seven terms of eq. (3.2.11). Using the first rule, we deduce that the terms which are products of three correlation functions must have total genus of g, the terms which are products of two correlation functions must have total genus of $g-1$, and the term which is one correlation must have total genus of $g-2$. The genus of the correlation functions in the other terms can be similarly determined.

3.3 One-Matrix Model Correlation Functions

For the one-matrix models at the $(k-1)^{th}$ multi-critical points, $k \geq 2$, we will determine all correlation functions with an insertion of P in every genus in terms of $\langle PP \rangle$ of that genus and lower genus. We assume that every correlation function scales as a power of x, the coupling constant of the puncture operator. The procedure will demonstrate the method to be followed for the more complicated two-matrix models.

The one-matrix model KdV recursion relation can be derived from KdV equations for the one-matrix models given by eq. (2.5.14) for $q = 2$. (We have seen that for $q = 2$ and $p = 3$, the theory is identified by eq. (2.5.20) to be pure 2-D gravity. For the one-matrix models, we have $p = 2k - 1$.) Using an identity similar to eq. (3.2.7) and following the same arguments as in section 3.2, one can find the KdV recursion relation given by ref. [1]:

$$\langle \sigma_m PP \rangle_g = \sum_{g_1+g_2=g} \Big(\langle \sigma_{m-2} P \rangle_{g_1} \langle PPP \rangle_{g_2} + 2 \langle \sigma_{m-2} PP \rangle_{g_1} \langle PP \rangle_{g_2} \Big)$$

$$+ \langle \sigma_{m-2} PPPP \rangle_{g-1}, \quad m \geq 3. \tag{3.3.1}$$

(m is an odd integer through out this section.)

We will argue inductively genus by genus and build up correlation functions with more and more operator insertions. Let us first examine the genus zero case. The one-matrix model KdV recursion relation for genus zero is

$$\langle\sigma_m PP\rangle_0 = \langle\sigma_{m-2}P\rangle_0\langle PPP\rangle_0 + 2\langle\sigma_{m-2}PP\rangle_0\langle PP\rangle_0. \tag{3.3.2}$$

At the $(k-1)^{th}$ multi-critical point, the matrix model results give $\langle PP\rangle_0 = x^{\frac{1}{k}}$ [2]. The correlation function $\langle PP\rangle_0$ is calculated using the string equation (2.5.13), and it will serve as the initial data of the KdV recursion relation, eq. (3.3.2). The scaling ansatz gives $\langle\sigma_m\rangle_0 = a_m x^{b_m}$. Plugging them into the KdV recursion relation eq. (3.3.2), we get that $b_m = 1 + \frac{m+1}{2k}$ and $a_m = 2/(1+\frac{m+1}{2k})(m+1)$.

Differentiate the genus zero KdV recursion relation eq. (3.3.2):

$$\langle\sigma_l\sigma_m PP\rangle_0 = \langle\sigma_l\sigma_{m-2}P\rangle_0\langle PPP\rangle_0 + \langle\sigma_{m-2}P\rangle_0\langle\sigma_l PPP\rangle_0$$

$$+2\langle\sigma_l\sigma_{m-2}PP\rangle_0\langle PP\rangle_0 + 2\langle\sigma_{m-2}PP\rangle_0\langle\sigma_l PP\rangle_0. \tag{3.3.3}$$

The R.H.S. is determined once $\langle\sigma_l\sigma_{m-2}PP\rangle_0$ is determined. Also, $\langle\sigma_l\sigma_{m-2}P\rangle_0$ is determined by integrating $\langle\sigma_l\sigma_{m-2}PP\rangle_0$ with respect to x. Since $\langle\sigma_l PPP\rangle_0$ is determined by differentiating $\langle\sigma_l\rangle_0$ with respect to x, so $\langle\sigma_l\sigma_m PP\rangle_0$ is determined via induction. Differentiate twice the genus zero KdV recursion relation, eq. (3.3.2), we get that $\langle\sigma_i\sigma_l\sigma_m PP\rangle_0$ is determined once $\langle\sigma_i\sigma_l\sigma_{m-2}PP\rangle_0$ is determined. So $\langle\sigma_i\sigma_l\sigma_m PP\rangle_0$ is determined, and so on. (i and l are positive odd integers.)

For genus one, the scaling ansatz gives $\langle\sigma_m\rangle_1 = c_m x^{d_m}$. Using the genus one KdV recursion relation gives $d_m = -1 + (m-1)/2k$ and c_m's determined by c_0. c_0 is undetermined, and this is the reason that the initial data of the recursion relations are the values of $\langle PP\rangle$ for every genus. For higher genus, there are no new features, and the same arguments follows.

There is, however, one crucial point in our arguments which needs more careful attention. In order to use the KdV recursion relation to build up the correlation functions in terms of lower ones, we had to integrate the L.H.S. of both eqs. (3.3.2) and (3.3.3) once to eliminate one insertion of P from the insertion of PP, and plug the results back to the R.H.S. of eqs. (3.3.2) and (3.3.3), respectively. Usually, the integrated correlation scales with x to a non-zero power, so that the integration constant can be set to zero using the scaling ansatz. In the special case that the integrated correlation scales as x to the zero, let us examine the problem involved.

R_m is a homogeneous polynomial in $u, u', u'', \ldots$ of degree $(m+1)/2$, where $degree(u) = 1, degree(u') = 3/2, degree(u'') = 2, \ldots$ [8]. From section 2.5, we have $R_m = \langle \sigma_m P \rangle$ is defined by eq. (2.5.17) and $u = \langle PP \rangle$. The homogeneity of the polynomial is related to scaling behavior. For example, in genus zero $R_m = u^{(m+1)/2}$, so that $\langle \sigma_m P \rangle_0$ scales as $x^{(m+1)/2k}$. We can use the homogeneity property to determine the integration constants:

$$R'_m = f[u, u', u'', \ldots], \qquad R_m = F[u, u', u'', \ldots] + const. \tag{3.3.4}$$

For each $m \geq 1$, R_m is a homogeneous polynomial in $u, u', u'', \ldots$ of degree $(m+1)/2$, and this determines *const.* Namely, if $F[u, u', u'', \ldots]$ is a homogeneous polynomial in $u, u', u'', \ldots$ of degree m, then *const.* is set to zero, else R_m is not homogeneous since a constant has degree zero. Similarly, we can integrate $((\prod_{l \in S} \frac{\partial}{\partial t_l}) R_m)'$ to get $(\prod_{l \in S} \frac{\partial}{\partial t_l}) R_m$. Requiring that each $(\prod_{l \in S} \frac{\partial}{\partial t_l}) R_m$ be a homogeneous polynomial in $u, u', u'', \ldots$ of degree $(m+1)/2$, determines the integration constants. The result is that we first express every correlation in terms of $u = \langle PP \rangle$ and its derivatives $u', u'', \ldots$. Then, we substitute in $u = \langle PP \rangle$ expressed in terms of x, the scaling variable, to determine every correlation in terms of x.

In passing, we will mention that a non-linear recursion relation exists for R_m [8]:

$$R_m = \sum_{i=-1}^{m-2} R_i R''_{m-3-i} + \frac{5}{2} \sum_{i=-1}^{m-2} R'_i R'_{m-3-i}$$

$$+\frac{u}{2}\sum_{i=-1}^{m-2} R_i R_{m-3-i} + \sum_{i=1}^{m-2} R_i R_{m-1-i}, \quad m \geq 3. \tag{3.3.5}$$

The summation index i runs over odd integers. Also $R_{-1} \equiv 1$ and $R_1 \equiv u$. But the non-linear recursion relation above gives the same results as the recursion relation eq. (3.3.1), which, using the fact that $R_m = \langle P\sigma_m \rangle$, can be expressed as:

$$R'_m = R'''_{m-2} + 2uR'_{m-2} + u'R_{m-2}. \tag{3.3.6}$$

Note that eq. (3.3.6) can be solved to give $R_m = R''_{m-2} + 2uR_{m-2} - (\frac{d}{dx})^{-1}u'R_{m-2}$, where $(\frac{d}{dx})^{-1}$ does not have any integration constant [8]. The key point is that eq.(3.3.6) can be used to determine R_m recursively.

3.4 Two-Matrix Model Correlation functions

We proceed by induction genus by genus and build up correlation functions in terms of lower ones as in the case of the one-matrix models. Consider genus zero, the KdV recursion relations are:

$$\langle \sigma_{n+3}PP \rangle_0 = 3\langle \sigma_n PP \rangle_0 \langle PQ \rangle_0 + 2\langle PP \rangle_0 \langle \sigma_n PQ \rangle_0$$

$$+\langle PPP \rangle_0 \langle \sigma_n Q \rangle_0 + 2\langle QPP \rangle_0 \langle \sigma_n P \rangle_0. \tag{3.4.1}$$

$$\langle \sigma_{n+3}PQ \rangle_0 = -\langle QPP \rangle_0 \langle \sigma_n QP \rangle_0 - \langle QPPP \rangle_0 \langle \sigma_n Q \rangle_0$$

$$-16\langle PP \rangle_0 \langle PP \rangle_0 \langle \sigma_n PP \rangle_0 - 16\langle PP \rangle_0 \langle PPP \rangle_0 \langle \sigma_n P \rangle_0. \tag{3.4.2}$$

We assume that we know $\langle PP \rangle_0$, $\langle PQ \rangle_0$, and $\langle QQ \rangle_0$, and we assume the scaling ansatz. Setting $n = 1$ in eq. (3.4.1), we can determine $\langle \sigma_4 P \rangle_0$ after we integrate out one insertion of P and using the scaling ansatz. Setting $n = 1$ in eq. (3.4.2), we can determine $\langle \sigma_4 Q \rangle_0$ after integrating out one P. Setting $n = 2$ in eq. (3.4.1), we can determine $\langle \sigma_5 P \rangle_0$. Setting $n = 2$ in eq. (3.4.2), we can determine $\langle \sigma_5 Q \rangle_0$. We can proceed with $n = 4$, $n = 5$, $n = 7$, $n = 8$, ... in eq. (3.4.1) and eq. (3.4.2) to find all $\langle \sigma_n P \rangle_0$. Note that it is clear that eq. (3.4.1) by itself would be incomplete as a recursion relation. Other correlation functions and higher genus proceed as in the one-matrix models. The result is that we have shown that the two KdV recursion relations for continuum two-matrix models can be used to express all correlation functions containing an insertion of P at any genus in terms of $\langle PP \rangle$, $\langle PQ \rangle$, and $\langle QQ \rangle$ in a unique way. P and Q are the puncture operator and the dilaton operator, respectively. A technical assumption called "the scaling ansatz" was also required. (See section 3.2 for the definition.)

3.5 Three-Matrix Model KdV Recursion Relations

The procedure by which we derive the two KdV recursion relations for the two-matrix models is systematic so that it can be generalized to higher multi-matrix models. In this section, we will derive the three KdV recursion relations for the three-matrix models. For the three-matrix models, we can find KdV recursion relations by using an identity similar to eq. (3.2.7):

$$0 = [L, L_-^{\frac{m}{4}+1}] - \frac{1}{2}\{L, [L, L_-^{\frac{m}{4}}]\} + \frac{1}{2}[\{L_-^{\frac{m}{4}}, L\}_+, L],$$

$$m \in \{4j+1, 4j+2, 4j+3 | j \in Z, j \geq 1\}. \tag{3.5.1}$$

The identity above has coefficients of D^2, D^1, and D^0 only, and we set them to zero to obtain three KdV recursion relations for the three-matrices models. It is clear that we can proceed systematically to find KdV recursion relations for $(q-1)$-matrix models, and that we will find that there are $(q-1)$ KdV recursion relations given by identities similar to eqs. (3.2.7) and (3.5.1).

Lastly, we comment that the KdV recursion relations found for the two-matrix models, eqs. (3.2.10) and (3.2.11), do not involve explicitly the infinite number of coupling constants in contrast to the recursion relations given by the Virasoro constraints and the W-constraints [5,6]. The same comment also holds in the cases of the three-matrix models and, in general, $(q-1)$-matrix models.

3.6 Conclusion

We conclude by discussing some questions and speculations. As mentioned, we can proceed systematically to find that there are $(q-1)$ KdV recursion relations for $(q-1)$-matrix models. In the one-matrix models, the KdV recursion relation can be used to perturb away from the string equation to obtain the Virasoro constraints [5]. In the two-matrix models, one can speculate that the KdV recursion relation in [1] can be used to perturb away from the string equation to obtain the Virasoro constraints, whereas the KdV recursion relation found in this paper can be used to perturb away from the string equation to obtain the W-constraints. In general, for $(q-1)$-matrix models, one can speculate that one of the KdV recursion relations is related to the Virasoro constraints and $(q-2)$ remaining ones are related to the $(q-2)$ sets of W-constraints. Recently, ref. [9] has derived the W-constraints from the KdV equations and the string equation.

There is a question about whether one can recover the KdV equation from the Virasoro constraints [5]. As we have seen, using the KdV equations and assuming a scaling ansatz, one can determine all correlation functions with an insertion of P in terms of $\langle PP \rangle$. At the $(k-1)^{th}$ multi-critical point the Virasoro constraints determine correlation functions with non-primary fields in terms of primary fields, but it seems that the correlation functions of primary fields are undetermined. Therefore, it seems unlikely that the KdV equations can be recovered from the Virasoro constraints at the $(k-1)^{th}$ multi-critical point.

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4. Virasoro Constraints for D_{2n+1}, E_6, E_7, E_8-Type Minimal Models Coupled to 2-D Gravity

The continuum limit of the multi-matrix models considered in section 2.5 gives rise to eqs. (2.5.12), (2.5.13), (2.5.14), and (2.5.15). These equations can be used to compute scaling dimensions of the operators in the theory. It turns out that the scaling dimensions obtained are identical to those of (A_{q-1}, A_{p-1})-type minimal conformal field theories coupled to 2-D gravity as calculated in the continuum Liouville formulation. (See appendix 2.B for the classification of minimal conformal field theories.)

The Virasoro algebra is intimately related to the KdV formalism. Dijkgraaf *et al.* [4] reformulated KdV related equations (2.5.12), (2.5.13), and (2.5.14) as constraint equations that obey Virasoro algebra. Fukuma *et al.* [3] also fuck the same constraint equations. An advantage of reformulating the KdV related equations as Virasoro constraints is that correlation functions become easier to compute. Virasoro constraints obtained in refs. [3,4] are associated with (A_{q-1}, A_{p-1})-type minimal conformal field theories coupled to 2-D gravity.

In this chapter we find Virasoro constraints associated with (D_{2n+1}, A_{p-1})-type minimal conformal field theory coupled to 2-D gravity. We also find Virasoro constraints associated with (E_6, A_{p-1}), (E_7, A_{p-1}), and (E_8, A_{p-1})-type minimal conformal field theory coupled to 2-D gravity. We call these theories D_{2n+1}, E_6, E_7, E_8-type models. We verify that the proposed Virasoro constraints for D_{2n+1}, E_6, E_7, E_8-type models give operator scaling dimensions identical to those found by Kostov [11]. We check that these Virasoro constraints and, more generally, W-algebra constraints can be used to express correlation functions containing non-primary operators in terms of correlation functions of primary operators only (see section 4.4).

4.1 Introduction

Matrix models simulating spin interactions on random surfaces have been used recently to study minimal conformal field theories coupled to 2-D quantum gravity [1, 2]. Analytical formulas for these matrix models, when taken to continuum limit, allow non-perturbative computations of all correlation functions extending the domain of exactly solvable models. Surprisingly, the space of resulting theories is organized by the space of KP flows.

In the works of Fukuma *et al.* [3] one-matrix models were studied using the Schwinger-Dyson equation, and the resulting loop equation in the continuum limit was expressed as Virasoro constraints. At the same time, Dijkgraaf *et al.* [4] found the loop equation by combining the string equation and the KdV equation for continuum limit matrix models and obtained identical Virasoro constraints. Both works conjectured that Douglas' [2] chain of matrices associated with the Dynkin diagram of A_n should give rise to W-algebra constraints. In the D_n, E_6, E_7, E_8 [5] case, where there is no analog of the Mehta formula [6] to solve the matrix models, we must rely on the KdV formalism of Drinfel'd and Sokolov [7, 8]. We therefore do not expect a Schwinger-Dyson loop equation derivation of the Virasoro constraints, but what we can do is to find the Virasoro constraints by using a vertex operator representation for D_n, E_6, E_7, E_8. The Virasoro constraints can then be compared with the results of the KdV approach [8].

4.2 Virasoro Constraints for A_n-Type Models

In this section we will review the results obtained in refs. [3, 4]. Recursion relations between correlation functions of A_n-type minimal conformal models coupled to 2-D gravity were found to satisfy elegant Virasoro constraints and W-algebra constraints. In ref. [4], the Virasoro constraints were obtained by reformulating the KdV equation and the string equation into equations for string loops, which have more physical interpretations. It turns out that operators appearing in the loop equations obey commutation relations corresponding to part of a Virasoro algebra.

Recall some results from section 2.5 and 3.3. The KdV recursion relation of the one-matrix model is

$$R'_m = R'''_{m-2} + 2uR'_{m-2} + u'R_{m-2},\ m \geq 3 \tag{4.2.1}$$

using eqs. (2.5.16) and (2.5.17) in eq. (3.3.1). (m is an odd integer throughout this section.) The string equation for the one-matrix model is:

$$0 = \sum_{m=1}^{\infty} mt_m R_m[u], \tag{4.2.2}$$

using eq. (2.5.13) and eq. (2.5.14). The physical meaning of R_m is that $R_m[u] = \langle \sigma_m P \rangle$ as given in eq. (2.5.17). By eqs. (2.5.15) and (2.5.19), we have that the insertion of σ_m in the correlation function corresponds to differentiation with respect to the coupling constant t_m. t_m is identified with the coordinate parameterizing the m^{th} KdV flow of u. Also, we have $\partial u/\partial t_m = \partial R_m/\partial x$, $t_1 = x$, and $u = \langle PP \rangle$ from section 2.5.

To facilitate algebra, define $w(z)$, $j(z)$, and $L(z)$ as:

$$w(z) = \sum_{m=1}^{\infty} \sigma_n z^{-\frac{m}{2}-1}. \tag{4.2.3}$$

$$j(z) = \sum_{m=1}^{\infty} t_n z^{\frac{m}{2}}. \tag{4.2.4}$$

$$L(z) = [j'(z)\langle w(z)\rangle]_< + 2\lambda^2\langle w^2(z)\rangle + \frac{1}{4}\langle w(z)\rangle^2 + \frac{\lambda^2}{2z^2} + \frac{t_1^2}{4z}. \tag{4.2.5}$$

λ is the string coupling constant. The $<$ subscript in eq. (4.2.5) denotes taking terms with negative powers of z. Writing $L(z)$ in terms of a Laurent expansion in z,

L_n is defined by:

$$L(z) = 8\lambda^2 \sum_{n\geq -1} (\frac{L_n\tau}{\tau}) z^{-n-2}, \tag{4.2.6}$$

where τ is the τ function of the KP hierarchy (see appendix 2.C for the definition). Substituting eqs. (4.2.3), (4.2.4), and (4.2.6) in eq. (4.2.5) implies $L(z) = 0$, if and only if

$$L_n\tau = 0 \quad \text{for} \quad n \geq 1, \quad \text{where}$$

$$L_{-1} = \sum_{m=3}^{\infty} \frac{m}{2} t_m \frac{\partial}{\partial t_{m-2}} + \frac{1}{16}\lambda^{-2} t_1^2,$$

$$L_0 = \sum_{m=1}^{\infty} \frac{m}{2} t_m \frac{\partial}{\partial t_m} + \frac{1}{16}, \quad \text{and}$$

$$L_n = \sum_{m=1}^{\infty} \frac{m}{2} t_m \frac{\partial}{\partial t_{m+2n}} + \lambda^2 \sum_{m=3}^{n} \frac{\partial^2}{\partial t_{m-2} \partial t_{2n-m}}. \tag{4.2.7}$$

From eq. (4.2.1), one has that

$$[\lambda^2 D^4 + (2u - z)D^2 + (Du)D]\langle w(z)\rangle + \frac{Du}{\sqrt{z}} = 0. \tag{4.2.8}$$

Using the equation above, eq. (4.2.8), ref. [4] shows that

$$[\lambda^2 D^4 + (2u - z)D^2 + (D(u - \frac{z}{2}))D]L(z) = 0. \tag{4.2.9}$$

Plugging in eq. (4.2.6) , one has that

$$D^2(\frac{L_{n+1}\tau}{\tau}) = [\lambda^2 D^4 + 2uD^2 + (Du)D](\frac{L_n\tau}{\tau}). \tag{4.2.10}$$

Since the string equation can be translated to $L_{-1}\tau = 0$, one can use induction formula above, eq. (4.2.10), to show that $L_n\tau = 0$ for $n \geq -1$. Therefore, one also has that $L(z) = 0$. Thus, the KdV recursion relation and the string equation for the one-matrix models has been reformulated in refs. [3,4] as constraint equations (4.2.7) which obey Virasoro algebra.

4.3 Virasoro Constraints for D_{2n+1}-Type Models

We consider (D_{2q+1}, A_{p-1}) type minimal conformal field theory coupled to 2-D quantum gravity in the context of finding out the Virasoro constraints. For example, the unitary model (8, 9) = $(4q, p)$ corresponds to (D_5, A_8). (The D_{2q} case will not be considered, because one of the exponents occurs twice, whereas in KP flows each operator has a different scaling dimension.) We use $2q+1$ twisted bosons to construct the vertex operator representation of the level one D_{2q+1} Kac-Moody algebra. $2q$ of the bosons have Z_{4q} twisting and one has Z_2 twisting. Associated with the D_{2q+1} Dynkin diagram are Coxeter number = $4q$ and exponents $I = \{2q, 1, 3, 5, \ldots, 4q-1\}$. (See appendix 2.A for a review on Lie algebras.) We also define indices $J = \{j | j = i + 4qn, i \in I, n \in Z, n \geq 0\}$. Specifically, we write the twisted bosons mode expansions as

$$\frac{\partial}{\partial z}\phi^i = \sum_{n \in Z} a_{i+4qn} z^{-(\frac{i}{4q}+n)-1}, \quad i \in I. \tag{4.3.1}$$

$$a_j = \frac{\partial}{\partial t_j}, \; a_{-j} = jt_j, \; j = [\frac{\partial}{\partial t_j}, jt_j] = [a_j, a_{-j}], \quad j \in I. \tag{4.3.2}$$

(The Z_{4q} cyclic nature is reminiscent of the infinite number of identical blocks of $4q \times 4q$ matrices in the reduction of the τ function [12].)

For a level-one simply laced algebra the Sugawara construction coincides with the Virasoro construction using the quantum equivalence theorem [5]. Therefore we can write the stress-energy tensor in terms of the generators of the Cartan subalgebra, H^i, as

$$T = \frac{1}{2}\sum_{i=1}^{2q+1} :(H^i)^2: + z^{-2}(\text{constant}). \tag{4.3.3}$$

Since $\frac{\partial}{\partial z}\phi^i$'s are complex and H^i's are real, we use the linear combinations:

$$\frac{\partial}{\partial z}\phi^{2j-1} = \frac{1}{\sqrt{2}}[H^j + iH^{2q+2-j}],$$

$$\frac{\partial}{\partial z}\phi^{4q-(2j-1)} = (\frac{\partial}{\partial z}\phi^{2j-1})^* = \frac{1}{\sqrt{2}}[H^j - iH^{2q+2-j}],$$

$$\frac{\partial}{\partial z}\phi^{2q} = H^{q+1},\ 1 \leq j \leq p. \tag{4.3.4}$$

Each boson $\frac{\partial}{\partial z}\phi^i$ contributes $\frac{i(4q-i)}{64q^3}$ to the normal ordering constant, so we have:

$$T = \frac{1}{2}\sum_{i\in I} : (\frac{\partial}{\partial z}\phi^i)(\frac{\partial}{\partial z}\phi^{4q-i}) : + z^{-2}(\frac{1}{16} + \frac{1}{96q} + \frac{q}{12}). \tag{4.3.5}$$

(These terms are just the diagonal invariant terms in the CIZ-classification [10].)

The mode expansion of $T = \sum_{n\in Z} L_n z^{-n-2}$ has Virasoro components:

$$L_{-1} = \frac{1}{4q}[\frac{1}{2}\sum_{i,j\in J, i+j=4q} ijt_it_j + \sum_{i\in J} it_i\frac{\partial}{\partial t_{i-4q}}]. \tag{4.3.6}$$

$$L_0 = \frac{1}{4q}[\sum_{i\in J} it_i\frac{\partial}{\partial t_i} + (\frac{q}{4} + \frac{1}{24} + \frac{q^2}{3})]. \tag{4.3.7}$$

$$L_n = \frac{1}{4q}[\sum_{i\in J} it_i\frac{\partial}{\partial t_i} + \frac{1}{2}\sum_{i,j\in J,\ i+j=4qn} \frac{\partial}{\partial t_i}\frac{\partial}{\partial t_j}],\ n \geq 1. \tag{4.3.8}$$

For example, the D_5 case gives:

$$L_{-1} = \frac{1}{8}[7t_1t_7 + 15t_3t_5 + 8t_4t_4 + 9t_9\frac{\partial}{\partial t_1} + 11t_{11}\frac{\partial}{\partial t_3} + 12t_{12}\frac{\partial}{\partial t_4} + \ldots]. \tag{4.3.9}$$

The W-algebra (see appendix 2.D for the definition) components will be given by mode expansions of higher order Casimirs, $W^{(s)}$ of conformal spin s, constructed out of the $\frac{\partial}{\partial z}\phi^i$ fields. We will not construct them explicitly (number of Casimirs =

rank of the group):

$$W^{(s+1)} = \sum_{n\in Z} W_n^{(s+1)} z^{-n-s-1}, \; s \in I. \tag{4.3.10}$$

Generalizing the works of Fukuma *et al.* and Dijkgraaf *et al.* to the present situation gives the Virasoro and W-algebra constraints as

$$L_n \tau = 0, \; n \geq -1. \tag{4.3.11}$$

$$W_n^{(s+1)} \tau = 0, \; s \in I, \; n \geq -s, \tag{4.3.12}$$

where the $\tau(t_1, t_2, t_3, \ldots)$ is related to the partition function of the theory by $Z = e^{-F} = \tau^2$. The operators of the theory are given by $< \sigma_i >= 2\frac{\partial}{\partial t_i} \log \tau, \; i \in J$. Furthermore, τ satisfies additional reduction constraints related to the group D_{2q+1} (see section 4.6). Note that $W_n^{(2)} = L_n$, and that $L_{-1}\tau = 0$ is the non-perturbative string equation of the theory. (For A_n-type models, $L_{-1}\tau = 0$ given in eq. (4.2.7) is obtained by integrating once the string equation, eq. (4.2.2).)

4.4 Scaling Dimensions

We will now deduce the scaling dimensions of the operators. The gravitationally dressed conformal primary fields are identified as (there are $(2q+1)(p-1)$ of them for the unitary models):

$$\phi_{r,s} = \sigma_{-4qs+pr} \text{ where } r \in I, \; 1 \leq s \leq [\frac{pr}{4q}]. \tag{4.4.1}$$

To consider the (D_{2q+1}, A_{p-1}) model, we set all $t_i = 0$ for $i \in J$, except $t_1 = x$, $t_{4q+p} = 1$ to reach the p^{th} multi-critical point, so we get from eqs. (4.3.6-9) and eq. (4.3.11):

$$0 = L_n \tau = \frac{1}{4q}[x < \sigma_{4qn+1} > + < \sigma_{4qn+4q+p} > + \sum_{i\in J, \; i\leq 4qn} < \sigma_i >< \sigma_{4qn-i} >]. \tag{4.4.2}$$

Note that we include the sphere contribution only, so that correlation functions with two operators do not appear, as they are suppressed by an infinite factor. The infinite factor is the result of $1/N^2$ suppression when $N \to \infty$ in taking the continuum limit of the matrix models and $N \times N$ is the size of the Hermitian matrices. We let the scaling be $< \sigma_i > \sim x^{1-\gamma+\Delta_i}$, then $\Delta_{4qn+1} + 1 = \Delta_{4qn+4q+p} = 1 - \gamma + \Delta_i + \Delta_{4qn-i}$. ($\sim$ means proportional. We do not need the proportional constant, since we are only determining the scaling dimensions.) The equalities of the scaling dimensions imply that Δ_i is linear in i and, using $\Delta_1 = 0$ for the identity operator σ_1, we get:

$$< \sigma_i > \sim x^{(1+\frac{2}{4q+p-1})+\frac{i-1}{4q+p-1}}, \; \Delta_i = \frac{i-1}{4q+p-1}. \qquad (4.4.3)$$

For $i \in I$, this is precisely Kostov's formula [11], i.e., scaling dimension = (exponent -1)/(Coxeter number -1). More generally, fields $\phi_{r,s}$ have scaling dimensions:

$$\Delta_{r,s} = \frac{-4qs + pr - 1}{4q + p - 1}, \qquad (4.4.4)$$

which is the KPZ-formula [12].

Kostov [11] considered the A, D, E 2-D interaction-around-face statistical models formulated on a fluctuating planar lattice. The continuum limit of such systems is described by minimal conformal theories coupled to 2-D quantum gravity. These models are formulated as a gas of self-avoiding non-intersecting loops on a random planar graph. The scaling dimensions of the order parameters can be determined at genus zero. The scaling dimensions given by:

$$\Delta_m = \frac{m-1}{h-1}, \qquad (4.4.5)$$

depend linearly on m whose values are the exponents of the groups A, D, E, and h is the Coxeter number of the groups A, D, E.

4.5 Primary Fields and Non-primary Fields

We will now deduce the criteria distinguishing the primary fields from the non-primary fields. It is clear from demanding a Laurent expansion of $W^{(s)}$ in integer powers of z that

$$z^{-n-1}W_{n-s}^{(s+1)} = z^{-n-1}\sum t_{i_1}t_{i_2}\ldots t_{i_s}\frac{\partial}{\partial t_{i_{s+1}}}$$

$$+z^{-n-1}(\text{terms with more than one } \partial' s),\ s\in I,\ (n-s)\geq -s, \qquad (4.5.1)$$

where $i_1, i_2, \ldots, i_{s+1} \in J$, $i_1 + i_2 + \ldots + i_s - i_{s+1} = -4qn$. At the critical point, $t_{i_1} = t_{i_2} = \ldots = t_{i_{s,\ 1}} = t_{4q+p} = 1$ gives the largest integer value for i_{s+1} amongst all of the terms. Therefore, we have

$$0 = \frac{W_{n-s}^{(s+1)}\tau}{\tau} =< \sigma_{4qn+ps} > +$$

$$(\text{correlations made with } \sigma_i,\ i < 4qn+ps),\quad s\in I,\quad n\geq 0. \qquad (4.5.2)$$

Thus, these Virasoro constraints can be used to express correlation functions with non-primary operator in terms of correlation functions of primary operators only. The point is that the first term in the R.H.S. of eq. (4.5.2) is never a primary operator.

4.6 Virasoro Constraints and KdV Approach

Di Francesco and Kutasov [8] developed a KdV approach to D_n-type minimal models coupled to 2-D gravity based on the works of Drinfel'd and Sokolov [7]. The result for a D_{2q+1}-type model at the p^{th} multi-critical point is summarized in the language of pseudo-differential operators:

$$L = D + u^{-1}D^{-1} + u_{-2}D^{-2} + \dots,$$

$$[L^{4q}, (L^p)_+] = \text{constant},$$

$$(L^{4q}D)_- = -u_0 D^{-1} u_0, \tag{4.6.1}$$

where $(L^{4q}D)$ is anti-self-adjoint [7] and has the constant one as the leading coefficient.

The question is how are the W-algebra vertex operator approach to D_{2q+1}-type conformal minimal models coupled to 2-D gravity related to the KdV approach? Let us assume that the conjecture given in refs. [3,4] is indeed correct and applicable to the D_{2q+1}-type models. We answer the question posed above by establishing that the conjecture in refs. [3,4] implies equations defining the D_{2q+1}-type models via the KdV approach.

We take the constraint $L_{-1}\tau = 0$ and differentiate with respect to t_i, $i \in \{1, 2, \dots, 4q-1\}$, then we set all $t_i = 0$ for $i \in J$ except $t_1 = x, t_{4q+p} = 1$. Applying the general relations between the KP τ function and the KP pseudo-differential operator L (note that $\frac{\partial}{\partial t_1} = D$):

$$\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_k} \log \tau = (L^k)_{-1}, \; k \geq 1. \tag{4.6.2}$$

$$\frac{\partial}{\partial t_2}\frac{\partial}{\partial t_k} \log \tau = 2(L^k)_{-2} + \frac{\partial}{\partial t_1}(L^k)_{-1}, \; k \geq 1, \; etc. \tag{4.6.3}$$

Thus $(L^p)_{-1} = \dots = (L^p)_{-(4q-2)} = 0$ and $(L^p)_{-(4q-1)} = (\text{constant})t_1$ which reproduces:

$$[L^{4q}, (L^p)_+] = \text{constant}. \tag{4.6.4}$$

The other remaining conditions of the KdV approach cannot be deduced. We take them as the reduction constraints for D_{2q+1}-type models, imposing relations between

$\frac{\partial}{\partial t_k}\frac{\partial}{\partial t_{4q}} \log \tau, k \geq 1$. We presume that there are no inconsistencies between the reduction constraints and the W-algebra constraints, since the KdV approach is just Drinfel'd et al.'s [7] theory in the specific context of the matrix models. So all we have to do is to show that we have enough degrees of freedom to rig those remaining conditions.

4.6 Virasoro Constraints for E_6, E_7, E_8-Type Models

Associated with the Dynkin diagram E_6 is Coxeter number = 12 and exponents $I = \{1, 4, 5, 7, 8, 11\}$. We also define indices $J = \{j | j = i + 12n,\ i \in I, n \in Z, n \geq 0\}$. Specifically, we write the twisted bosons mode expansions as

$$\frac{\partial}{\partial z}\phi^i = \sum_{n \in Z} a_{i+12n} z^{\frac{i}{12}+n},\ i \in I. \tag{4.7.1}$$

We can write the stress-energy tensor as

$$T = \sum_{i \in I} \frac{1}{2} : (\frac{\partial}{\partial z}\phi^i)(\frac{\partial}{\partial z}\phi^{12-i}) : + z^{-2}(\frac{13}{48}). \tag{4.7.2}$$

All the discussion for D_{2q+1} carries through for E_6. Associated with the Dynkin diagram E_7 and E_8 are Coxeter numbers 18, 30 and exponents $\{1, 5, 7, 9, 11, 13, 17\}$, $\{1, 7, 11, 13, 17, 19, 23, 29\}$. We can define the stress energy tensor in a similar fashion as above and carry out the similar analysis as for D_{2q+1}. (The normal ordering constants for E_7 and E_8 are $\frac{133}{432}$ and $\frac{31}{90}$, respectively.)

4.7 Conclusion

We will conclude this part by discussing some questions and speculations. For A_{q-1}-type models, refs. [3,4] have shown methods to derive the Virasoro constraints in the form of the loop equations. However, to completely prove the conjecture [3,4] that A_{q-1}-type models can be reformulated in terms of W-algebra constraints, one needs to derive the W-algebra constraints explicitly.

Dijkgraaf *et al.* [4] proved that the Virasoro constraints for the one-matrix model are identical to the topological recursion relations. K. Li [13] derived W-algebra constraints for the multi-matrix models by topological considerations.

The reduced Hirota's bilinear identities for the KP τ function [9] in the specific context of the A_{q-1}-type multi-matrix models are expected to simplify to linear W-algebra constraints on the τ function. Establishing this fact directly would provide additional evidence for the correctness of the W-algebra approach.

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PART
II

5. High Energy Scattering and the Unbroken Symmetry of String Theory

The effect of modifying the vacuum on high energy scattering of strings is examined. The fixed-angle asymptotic behavior depends sensitively upon the choice of string vacuum. However, the underlying high energy symmetries of string theory seem to be independent of vacuum modifications.

5.1 Introduction

String theory [1] is a unique candidate for the theory of quantum gravity. To understand the stringy nature of quantum gravity, there has been recent interest [2–4] in string dynamics at energies far beyond the Planck scale. Diffractive scattering ($s \to \infty$, t = fixed) was studied in refs. [2] and [3] and it has been found that the dynamics exhibits an effective strong coupling physics. Thus, it is necessary to sum over contributions at every order in the string perturbation expansion. Fixed-angle scattering has been studied in ref. [4] ($s \to \infty$, $t \to \infty$, s/t = fixed) and exact results of high energy scattering amplitudes were obtained to all orders in the string perturbation expansion. This result enabled Gross [5] to observe that in the $\alpha' \to \infty$ limit there exist underlying symmetries of string theory that are not yet fully understood. A set of linear recursion relations was derived among scattering amplitudes involving different mass string states, which are exact to all orders of the string loop expansion in the limit $\alpha' \to \infty$.

String theories are usually analyzed in some specific vacuum such as the flat 26- or 10-dimensional space-time with no other non-trivial field condensates. This is certainly not the unique vacuum of string theories. Recent efforts to compactify string theories either on a Calabi–Yau space or some other consistent conformal field theory provide more realistic vacua. Whatever the modified vacua may be, they

must be interpreted as different minima of the string field theory effective potential.*
In analogy to point particle field theories, we may expect that symmetry breaking through modification of the vacuum depends only on long-distance (or, more properly, infrared) physics. It is not yet clear to what extent we can interpret compactifications through a Kaluza–Klein type mechanism or conformal field theories as infrared physics of the underlying dynamics. If one temporarily accepts this to be the case, one may expect that ultraviolet (or short-distance) physics such as high energy scattering will not be directly affected by the details of a particular vacuum.

M. B. Green [6] studied the above issue (in a somewhat different context) and his results suggest that, contrary to our expectations above, vacuum structure (infrared physics) and high-energy scattering (ultraviolet physics) are not disconnected at all. The high energy behavior around two different vacua are drastically different in the lowest non-trivial order of string loop perturbation expansion. Even though the results of ref. [2–4] imply that one must sum to all orders of the string loop expansion in order to get the correct high energy behavior of scattering amplitudes, Green's results suggest that one should re-examine the relation between vacuum structure and ultraviolet string dynamics such as in the high energy scattering discussed above. We will return to this issue later.

In the next section we discuss general features of vacuum stability and string symmetry breaking. Next we study high-energy fixed-angle scattering of bosonic string tachyons. From this we will infer the relation between string vacua and high energy scattering behavior and draw our conclusions. For concreteness, we consider the model of open and closed bosonic strings only. For technical simplicity, we will concentrate only on the lowest non-trivial order corrections to high-energy fixed-angle scattering amplitudes coming from the modification of vacuum structure. It would be interesting to also study the equally important higher order terms and extensions to superstring theories. In particular, we are interested in understanding the modification of the supersymmetric vacuum into a non-supersymmetric one.

⋆ Although this is not as straightforward in a theory that includes dynamical gravity as in a theory where the space-time geometry is rigid.

5.2 Vacuum Stability and String Symmetry Breaking

In point particle field theory, the criterion of whether a candidate vacuum $|\Omega\rangle$ is a true vacuum or not is the stability of the effective potential around that vacuum $|\Omega\rangle$. We denote by $\phi(x)$ a generic quantum field and $\Gamma[\overline{\phi}_\Omega(\cdot)]$ an effective action of the given theory. Here,

$$\overline{\phi}_\Omega(x) = \frac{\langle\Omega|\phi(x)|\Omega\rangle}{\langle\Omega|\Omega\rangle} \tag{5.2.1}$$

is an averaged classical field configuration in the vacuum $|\Omega\rangle$. In order for $|\Omega\rangle$ to be a true vacuum, the effective action must satisfy

$$[\delta\Gamma/\delta\overline{\phi}]_{\overline{\phi}=\overline{\phi}_\Omega} = 0. \tag{5.2.2}$$

If the L.H.S. is nonzero, one constructs a new corrected vacuum $|\Omega'\rangle$ by adding tadpole insertions to cancel out the non-vanishing L.H.S. of eq. (5.2.2). In string theory, conventional vertex operators $V_a(x)$ correspond to small fluctuations around a given (flat, for instance) space-time background condensate. In principle, one may calculate a string effective action $\Gamma_{st.}[\overline{\phi}_\Omega(\cdot)]$ from some string field theory. The stability condition, eq. (5.2.2), amounts to saying that

$$\begin{aligned}\langle\Omega|\delta\phi(x)|\Omega\rangle &\equiv \langle\Omega|\phi(x) - \overline{\phi}|\Omega\rangle \\ &\equiv \sum_a g_a\,\langle\Omega|V_a(x)|\Omega\rangle \\ &= 0.\end{aligned} \tag{5.2.3}$$

In the second line, we have identified string field fluctuations with the vacuum average of linear combinations of vertex operators with couplings g_a.* Therefore, vacuum stability is satisfied if

$$\langle\Omega|V_a(x)|\Omega\rangle = 0 \qquad \text{for all } a. \tag{5.2.4}$$

This is the first-quantized string notion of the condition eq. (5.2.2), and as such we may evaluate this one-point amplitude perturbatively in the string loop expansion.

⋆ A subtlety is that in general we must include contributions of auxiliary field vertex operators.

Then eq. (5.2.4) becomes

$$\sum_{g=0}^{\infty} \langle V_a \rangle_g \cdot \kappa^{\chi_E(g)} = 0$$

where

$$\left\langle \widehat{\mathcal{O}} \right\rangle_g = \int (Dg \cdot Dx/\mathcal{M}) \cdot \widehat{\mathcal{O}} \cdot e^{-S_{flat}},$$

$$S_{flat} = \int_{\mathcal{R}_g} d^2\sigma \sqrt{g} g^{ab} \partial_a X^\mu \cdot \partial_b X^\nu \cdot \eta_{\mu\nu}. \tag{5.2.5}$$

$\mathcal{M}$ denotes a symmetry group volume of the path integral, and appropriate powers of the string coupling constant κ are inserted in accordance with the requirements of factorizability and unitarity. $\chi_E = 2 - 2g$ is the Euler characteristic of the genus g world-sheet.

In open and closed bosonic string theory, eq. (5.2.5) is satisfied in the lowest order for closed string states on a sphere, S^2, and for open string states on the boundary of a disc, D^2, due to invariances of S^2 and D^2 under the projective symmetries $SL(2,\mathcal{C})$ and $SL(2,\mathcal{R})$, respectively. However, in the next orders of the string loop expansion, one-point amplitudes are not zero in general. Aside from the the subtlety required to define massive state asymptotic configurations, this is also the case for supersymmetric strings in general. The Fischler–Susskind mechanism [7] modifies the space-time structure of the vacuum, but, as we will discuss later, it is not the most general way to modify the vacuum. Actually, we will find that the Fischler–Susskind mechanism does not change the high-energy scattering behavior at all.

A generic "stringy" symmetry breaking is induced by insertions of tadpoles of zero momentum. In the first quantization method, we cut a disc D^2 out of the original world-sheet configuration and assign a string wave function around its boundary $\partial D^2 = S^1$. On $\partial D^2 = S^1$, $X^\mu(\sigma^a)$ satisfies the Dirichlet boundary condition

$$X^\mu(\sigma^a)\Big|_{\partial D^2 = S^1} = \overline{X}^\mu \text{ (constant)}. \tag{5.2.6}$$

Define $T(\overline{X})$ to be the wave function of a zero momentum point-like string state which is a superposition of point-like string states satisfying $X^\mu(\sigma,\tau)\left|\overline{X}\right\rangle = \overline{X}^\mu \left|\overline{X}\right\rangle$. $T(\overline{X})$ can be thought of as the coupling of the point-like string state to the vacuum $|\Omega\rangle$, so it is a tadpole amplitude. If the vacuum $|\Omega\rangle$ is shifted to $|\Omega'\rangle$ via a tadpole amplitude $T(\overline{X})$,

$$0 = \sum_a \sum_g \kappa^{\chi_E(g)} \cdot g_a \cdot \langle \Omega' | V_a | \Omega' \rangle_g$$

$$= \sum_a \sum_g \sum_t \kappa^{\chi_E(g)} \cdot [T(\overline{X})]^t \cdot g_a \cdot \langle \Omega | V_a | \Omega \rangle_{g,t} . \qquad (5.2.7)$$

Each term of the R.H.S. of eq. (5.2.7) is an amplitude corresponding to a genus g Riemann surface with an insertion of the vertex operator V_a and t insertions of the tadpole amplitude $T(\overline{X})$. Accordingly, scattering amplitudes in the new vacuum are defined by

$$iA(p_1,\ldots,p_N) = (2\pi)^D \delta^{(D)}(p_1+\cdots+p_N)[\prod_{i=1}^N (2\pi)^{D-1}(2p_i^0)]^{-1/2}$$

$$\cdot \sum_g \sum_t \kappa^{\chi_E(g)} [T(\overline{X})]^t \langle V_{I_1}(p_1)\ldots V_{I_N}(p_N)\rangle_{g,t} . \qquad (5.2.8)$$

In order for eq. (5.2.7) to be consistent with factorizability and unitarity, $T(\overline{X})$ must be proportional to κ.* (There are N tachyons and t tadpole insertions so a general loop diagram is proportional to $\kappa^{N+t-2}\kappa^{2g}$, and κ^{N+t} is absorbed into the vertex operators and tadpole amplitudes.)

⋆ Each puncture increases the Euler number of the surface by one unit.

5.3 High-Energy Fixed-Angle Scattering of Open Strings

We will now explicitly calculate high-energy fixed-angle scattering in various vacuum configurations. In the end we will obtain an explicit answer for scattering N closed string tachyons in the modified vacuum in the high energy limit. Let us first consider the four open-string tachyon amplitude on the disc, which is the lowest order diagram. The well-known Veneziano amplitude with $U(1)$ gauge group is

$$A_{D^2}(p_1,p_2,p_3,p_4) = \left(B\left(-\frac{1}{2}s-1,-\frac{1}{2}t-1\right)+\text{perm.}\right)$$

$$= \frac{\Gamma\left(-\frac{1}{2}s-1\right)\Gamma\left(-\frac{1}{2}t-1\right)\Gamma\left(-\frac{1}{2}u-1\right)}{\Gamma\left(\frac{s}{2}+2\right)\Gamma\left(\frac{t}{2}+2\right)\Gamma\left(\frac{u}{2}+2\right)}$$

$$\propto (stu)^{-3}\exp[-(s\log s+t\log t+u\log u)] \qquad as\ |s|,|t|,|u|\to\infty. \tag{5.3.1}$$

Modification of the tree level amplitude due to a tadpole insertion is described by an annulus diagram A^2 with one Neumann boundary condition and one Dirichlet boundary condition. The Dirichlet boundary of the annulus takes a fixed value in the target $D=26$ space-time, corresponding to the emission of a point-string state. The Neumann boundary corresponds to the freely moving ends of an open string.

$$\langle V_T(p_1)\dots V_T(p_N)\rangle_{A^2}^{\text{open}}$$

$$= T(\overline{X})\int_0^1 \frac{dq}{q}\left[\frac{f_1^2(q)}{f_2^D(q)}\right]\int\prod_{i=2}^{N} d\phi_i\ \theta(\phi_i-\phi_{i-1})$$

$$\cdot\left[\frac{f_1(q^2)}{f_4(q^2)}\right]^{2N}\prod_{i<j}\left[\frac{\theta_1(\phi_i-\phi_j|2\gamma)}{\theta_4(\phi_i-\phi_j|2\gamma)}\right]^{2\alpha' p_i\cdot p_j}. \tag{5.3.2}$$

Our notations are as follows: the Neumann boundary of the annulus is fixed at unit radius, the Dirichlet boundary is at radius q, which ranges from 0 to 1, $\gamma=\log(q)/i\pi$,

N is the number of tachyon vertices on the Neumann boundary, the i^{th} tachyon vertex with momentum p_i is situated on the unit circle and labeled by angle $2\pi\phi_i$, which may range from 0 to 2π, $D = 26$ is the space-time dimension, θ_1, θ_4 denote classical Jacobi theta functions, while f_1, f_2, and f_4 are

$$
\begin{aligned}
f_1(q) &= q^{1/12} \prod_{n=1}^{\infty}(1 - q^{2n}), \\
f_2(q) &= q^{1/12} \prod_{n=1}^{\infty}(1 + q^{2n}), \\
f_4(q) &= q^{-1/24} \prod_{n=1}^{\infty}(1 - q^{2n-1}).
\end{aligned}
\tag{5.3.3}
$$

The high energy scattering limit $\alpha' \to \infty$ is controlled by $q \sim 1$, as we will see, and, to this end, we make a Jacobi transformation of eq. (5.3.2). The result is

$$
T(\overline{X}) \int_0^1 \frac{dw}{w} w^{5/8} \left[\frac{f_1^2(w)}{f_4^D(w)}\right] \left[\frac{f_1(w^{1/2})}{f_2(w^{1/2})}\right]^{2N}
$$

$$
\cdot \int \prod_{i=2}^{N} d\phi'_i \; \theta(\phi'_i - \phi'_{i-1}) \prod_{i<j} \left[-i\frac{\theta_1((\phi'_i - \phi'_j)/2|\gamma'/2)}{\theta_2((\phi'_i - \phi'_j)/2|\gamma'/2)}\right]^{2\alpha' p_i \cdot p_j} \tag{5.3.4}
$$

where $w = exp(2\pi^2)/\log(q)), \gamma' = -1/\gamma, \phi'_i = -\phi_i/\gamma$. It is convenient to introduce

$$
\begin{aligned}
G_{ij} &= -i\frac{\theta_1((\phi'_i - \phi'_j)/2|\gamma'/2)}{\theta_2((\phi'_i - \phi'_j)/2|\gamma'/2)} \\
&= \frac{\sqrt{z_i} - \sqrt{z_j}}{\sqrt{z_i} + \sqrt{z_j}} \prod_{n=1}^{\infty} \frac{\left(1 - w^{n/2}\left[\sqrt{z_i/z_j} + \sqrt{z_j/z_i}\right] + w^n\right)}{\left(1 + w^{n/2}\left[\sqrt{z_i/z_j} + \sqrt{z_j/z_i}\right] + w^n\right)},
\end{aligned}
\tag{5.3.5}
$$

where $z_i = \exp[2\pi i\phi'_i]$. Note that the annulus has a rotational symmetry (only differences $\phi_i - \phi_j$ appear), so we use this freedom to set $\phi_1 = 0$. Thus, the integral in eq. (5.3.4) is now over the range $\omega < z_N < z_{N-1} < ... < z_1 = 1$. We define variables

$$\sqrt{z_i/z_{i-1}} = u_i \qquad \text{for } 2 \leq i \leq N$$

and

$$\sqrt{w/z_n} = u_1. \tag{5.3.6}$$

Letting these new variables range from 0 to 1 automatically implements the angular orderings of vertex operators and makes the duality explicit.

We will now study the fixed-angle high-energy limit of eq. (5.3.4). Define $S_{ij} = -p_i \cdot p_j$ and $S_i = -p_i \cdot p_{i+1}$ where $N+1$ is identified with 1. Treat particles 1,2 as incoming and the rest as outgoing, so the center of mass energy squared is $s = S_1$. The high energy scattering limit we are interested in is $s \to \infty$ and $S_{ij} = \lambda_{ij} s \to -\infty$ for $i \neq j$ and $S_{ij} \neq s$, where λ_{ij} are essentially the fixed-angles. We will compute the $s \to -\infty$ limit and analytically continue to $s \to \infty$ to obtain physical amplitudes. Substituting eq. (5.3.6) and eq. (5.3.5) in eq. (5.3.4), one finds that the amplitude is dominated by the end-point where all $u_i \to 0$. Saddle point evaluation of eq. (5.3.4) at this endpoint shows that eq. (5.3.4) is peaked only for $i+1 = j$,

$$\prod_{i<j} [G_{ij}]^{2\alpha' p_i \cdot p_j} \sim \prod_{i=1}^{N} [G_{i,i+1}]^{-2\alpha' S_i}. \tag{5.3.7}$$

Note that $N+1$ is identified with 1 in eq. (5.3.7). Finally, eq. (5.3.4) becomes

$$\begin{aligned} &\langle V_T(p_1) \dots V_T(p_N) \rangle^{\text{open}}_{A^2} \\ &\sim T(\overline{X}) \int_0^\infty \prod_{i=1}^{N} \left(\frac{du_i}{u_i} \right) (u_i)^{5/4} \exp[-2\alpha' S_i \cdot u_i] \\ &= T(\overline{X}) [\Gamma(5/4)]^N \prod_{i=1}^{N} (2\alpha' S_i)^{-5/4}. \end{aligned} \tag{5.3.8}$$

This is the N-point amplitude generalization of high energy ($\alpha' \to \infty$ limit) fixed angle scattering of the four-point amplitude. We note that eq. (5.3.8) has a power-law decrease different from the exponential decrease of conventional scattering amplitudes so that, when $\alpha' \to \infty$, amplitudes with tadpole insertions dominate over those without the insertions.

5.4 High-Energy Fixed-Angle Scattering of Closed Strings

Similarly, one may consider the effect of tadpoles on closed string tachyon N-point amplitudes. Again it is necessary to examine carefully an annulus diagram with mixed Neumann and Dirichlet boundary conditions.

The Green function is found to be

$$G(z_i, z_j) = \frac{f_1(q^4)}{f_4(q^4)} G_{ij} \cdot \tilde{G}_{ij} \tag{5.4.1}$$

where

$$G_{ij} = \frac{\theta_1(z_i - z_j|2\tau)}{\theta_4(z_i - z_j|2\tau)}, \qquad \tilde{G}_{ij} = \frac{\theta_1(z_i - \bar{z}_j|2\tau)}{\theta_4(z_i - \bar{z}_j|2\tau)}.$$

The N-point closed string tachyon amplitude is, with $z_1 =$ real,

$$\langle V_T(p_1) \ldots V_T(p_N) \rangle_{A^2}^{closed} = T(\overline{X}) \int_0^1 \frac{dq}{q} \left[\frac{f_1^2(q)}{f_2^D(q)} \right] \left[\frac{f_1(q^2)}{f_4(q^2)} \right]^{8N}$$

$$\cdot \int \prod_{i=1}^{N} \frac{d^2 z_i}{|z_i|^2} \left[\prod_{i<j} \left| G_{ij} \cdot \tilde{G}_{ij} \right|^{\alpha' p_i \cdot p_j} \right]. \tag{5.4.2}$$

Using the Jacobi transformation, we parametrize vertex operator positions by

$$z_i = e^{2\pi i (\theta'_i + \gamma' w'_i)} \tag{5.4.3}$$

so that

$$G_{ij} = \frac{i\theta_1((\nu'_i - \nu'_j)/2|(\gamma'/2))}{\theta_2((\nu'_i - \nu'_j)/2|(\gamma'/2))}$$

$$= \frac{\sqrt{z_i}-\sqrt{z_j}}{\sqrt{z_i}+\sqrt{z_j}} \prod_{n=1}^{\infty} \frac{\left(1-w^{n/2}\left(\sqrt{z_i/z_j}+\sqrt{z_j/z_i}\right)+w^n\right)}{\left(1+w^{n/2}\left(\sqrt{z_i/z_j}+\sqrt{z_j/z_i}\right)+w^n\right)} \tag{5.4.4}$$

and

$$\begin{aligned} \tilde{G}_{ij} &= \frac{i\theta_1((\nu_i-\bar{\nu}_j)'/2|\gamma'/2)}{\theta_2((\nu_i-\bar{\nu}_j)'/2|\gamma'/2)} \\ &= \frac{\sqrt{z_i}-\sqrt{\bar{z}_j}}{\sqrt{z_i}+\sqrt{\bar{z}_j}} \prod_{n=1}^{\infty} \frac{\left(1-w^{n/2}\left(\sqrt{z_i/\bar{z}_j}+\sqrt{\bar{z}_j/z_i}\right)+w^n\right)}{\left(1+w^{n/2}\left(\sqrt{z_i/\bar{z}_j}+\sqrt{\bar{z}_j/z_i}\right)+w^n\right)}. \end{aligned} \tag{5.4.5}$$

(Note that $(\bar{\nu}') \neq \overline{(\nu')}$ since γ' is an imaginary variable.)

We now look for a saddle point of eq. (5.4.2). We find it convenient to make a change of variables

$$\begin{aligned} \mu_i &= (z_i/z_{i-1})^{1/2} \qquad \text{for } 2 \le i \le N, \qquad \mu_1 = \sqrt{w/z_N} \\ \tilde{\mu}_i &= (\bar{z}_i/z_{i-1})^{1/2} \qquad \text{for } 2 \le i \le N, \qquad \tilde{\mu}_1 = \sqrt{w/z_N} = \mu_1, \end{aligned} \tag{5.4.6}$$

so that

$$\tilde{\mu}_i = \mu_i \left(\frac{\mu_{i+1} \cdot \mu_{i+2} \cdots \mu_N \mu_1}{\bar{\mu}_{i+1} \cdot \bar{\mu}_{i+2} \cdots \bar{\mu}_N \bar{\mu}_1} \right). \tag{5.4.7}$$

A steepest descent saddle point is at $\mu_1, \ldots, \mu_n \to 0$, just as in the open string case. To see this, we keep terms only up to linear order in μ_i and $\tilde{\mu}_i$ in the scattering amplitude,

$$\begin{aligned} \langle V_T(p_1) \ldots V_T(p_N) \rangle_{A^2}^{closed} &\sim T(\overline{X}) \int_0^1 \frac{dq}{q^3} \left[\frac{f_1^2(q)}{f_2^D(q)} \right] \left[\frac{f_1(q^2)}{f_4(q^2)} \right]^{8N} \\ &\cdot \int \prod_{i=1}^{N} \frac{d^2\mu_i}{|\mu_i|^2} |\mu_i|^{5/4} \prod_{j=1}^{N} |G_{j,j+1} \cdot \tilde{G}_{j,j+1}|^{-\alpha' S_j}. \end{aligned} \tag{5.4.8}$$

The last integral is approximated by

$$\int \prod_{i=1}^{N} d^2\mu_i |\mu_i|^{-3/4} \prod_{j=1}^{N} |(1+2\mu_j)(1+2\tilde{\mu}_j)|^{-\alpha' S_i}. \qquad (5.4.8)$$

In the limit that $\alpha' S_i \to \infty$, we find that $|\mu_i|, \ldots, |\mu_N| \to 0$ is the saddle point while no saddle point exists in the angular directions. So we may integrate the angular variables directly over the range. The amplitude eq. (5.4.8) then becomes

$$T(\overline{X}) \prod_i \int d^2\mu_i |\mu_i|^{-3/4} \prod_i \exp[-2Re\{\mu_i + \tilde{\mu}_i\}\alpha' S_i] = T(\overline{X})[\Gamma(5/4)]^N A \prod_i [\alpha' S_i]^{-5/4} \qquad (5.4.9)$$

where

$$A = \int_{-\pi/2}^{\pi/2} \prod_{i=1}^{N} d\theta_i [2Re(e^{i\theta_i} + e^{i\theta_i + 2i(\theta_{i+1} + \cdots + \theta_N + \theta_1)})]^{-5/4}.$$

This is the desired high-energy scattering amplitude formula.

A striking feature of the result above is that the saddle point is such that all the vertex operators approach the Neumann boundary. This means that if we calculate mixed amplitudes with both open string tachyons and closed string tachyons, the high energy behavior is essentially the same as eq. (5.4.9), since the saddle point evaluation of the amplitude is again dominated by contributions coming from the $|\mu_i| \to 0$ limit in which the closed string vertex operators approach each other. More generally, had we considered scattering amplitudes with arbitrary combinations of open and closed string states (for example, a scattering process involving gauge bosons and dilatons), we would have gotten the same answer except for the details of the respective vertex operators. Therefore, at least to this order, the scattering amplitudes are recursively related in a similar manner to the results of closed bosonic string theory that Gross derived [4,5]. It is important to notice that our lowest order recursion relations are independent of the string tadpole amplitude $T(\overline{X})$, and therefore the choice of vacuum. Of course we don't know whether this is true in the next order or to all orders in general. But the lowest order result suggests that in the high energy limit open and closed string amplitudes have similar polynomial behavior. It would be worthwhile to explore whether this is the case even after we include higher order terms of the string loop expansion.

5.5 Conclusion

We will now address the issue of the relationship between high energy scattering and symmetry breaking. The sensitive dependence of high energy scattering amplitudes on the choice of vacuum that we have found means that the ultraviolet and infrared behaviors of string dynamics are not independent. The stringy tadpole is a singular mapping from $\partial D^2 = S^1$ of the world-sheet to a single point $\overline{X}^\mu$ in space-time, and the non-locality of this mapping is perhaps the reason we find the non-exponential high-energy asymptotics of the scattering amplitudes in eq. (5.4.9). The stringy tadpoles considered in this paper are effectively static (*no* momentum transfer) *string* condensates, so particles are expected to scatter with large-angle deflections. Following the procedures in the papers of D. Gross [4,5], we can find exact recursion relations among the scattering amplitudes in the case of symmetry breaking phase. It seems that we get exact recursion relations similar to those found by D. Gross even after the symmetry breaking. Thus, we learn truly high-energy dynamical information that is independent of the vacuum selected. The point is that, unlike point particle field theories, scattering amplitudes *do* depend upon the choice of vacuum. Different vacua are described by different tadpole amplitudes, and the prescription amounts to a modification of the scalar field content of the string field expanded in mass levels.

This conclusion may seem contradictory to the usual concepts of low energy effective Lagrangians of string theories. If the characteristic energy scale one focuses on is much less than that of the string tension, only mild fluctuations in the string field or, equivalently, massless mode dominance is expected, and we get low energy effective Lagrangians involving only massless modes (graviton, antisymmetric tensor field, dilaton and gauge bosons). If space-time is curved slightly (for instance through the Fischler–Susskind mechanism [7]), do the above arguments mean that the tiny bit of difference in curvature makes high-energy scattering behavior drastically different? The answer to this is no. The important point is that the Fischler-Susskind mechanism is not the most general "stringy" symmetry breaking. It is possible to explicitly calculate N-point scattering amplitudes in a slightly curved background generated by the Fischler–Susskind mechanism, and the tadpole included there only shifts the dilaton field background. When one uses perturbation expansions in a curved space-time background, the high-energy scattering behavior in that background agrees with the scattering behavior in conventional flat space-time. This is what we would expect from low energy effective action concepts. The Fischler–Susskind mechanism is still not fully stringy, because only the finite number of fields associated with the massless particles acquire vacuum expectation values.

Our tadpoles are non-local on the world-sheet, hence really stringy, whereas the Fischler–Susskind tadpole is local on the world-sheet. As mentioned above, this non-locality on the world-sheet is crucial to the drastically different high-energy scattering behavior. Whereas a point-like modification of the vacuum will not affect ultraviolet dynamics, a stringy modification (as in our case) results in a change in ultraviolet dynamics.

Our explicit calculations suggest that the relations among high-energy asymptotic scattering amplitudes are independent of the choice of vacuum. (Gross notes in refs. [4, 5] that this is an unbroken phase of string theory, where all the particles are gauge fields.) This raises many related questions: Is there a low energy effective action description of string field theory when we modify the vacuum? Can we find a truly stringy four-dimensional vacuum of string theory? Does conformal field theory compactification allow a truly stringy solution such as monopoles or vortex configurations? Further research into these issues could help to acquire a deeper understanding of the vacuum structure of string theory.

Appendix 5.A Jacobi Theta Functions

The four Jacobi theta functions, θ_k for $k = 1$ to 4, satisfy the heat equation

$$\frac{i}{\pi}\frac{\partial^2\theta_k(\nu|\tau)}{\partial\nu^2} + 4\frac{\partial\theta_k(\nu|\tau)}{\partial\tau} = 0. \qquad (5.A.1)$$

They are defined by

$$\begin{aligned}\theta_1(\nu|\tau) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-\frac{1}{2})^2}e^{i\pi(2n-1)\nu}\\ &= 2f(q^2)q^{\frac{1}{4}}\sin\pi\nu\prod_{n=1}^{\infty}(1-2q^{2n}\cos 2\pi\nu+q^{4n}),\end{aligned} \qquad (5.A.2)$$

$$\begin{aligned}\theta_2(\nu|\tau) &= \theta_1(\nu+\frac{1}{2}|\tau) = \sum_{n=-\infty}^{\infty} q^{(n-\frac{1}{2})^2}e^{i\pi(2n-1)\nu}\\ &= 2f(q^2)q^{\frac{1}{4}}\cos\pi\nu\prod_{n=1}^{\infty}(1+2q^{2n}\cos 2\pi\nu+q^{4n}),\end{aligned} \qquad (5.A.3)$$

$$\begin{aligned}\theta_3(\nu|\tau) &= e^{i\pi(\nu+\tau/4)}\theta_1(\nu+\frac{1}{2}+\frac{1}{2}\tau|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}e^{i\pi 2n\nu}\\ &= f(q^2)\prod_{n=1}^{\infty}(1+2q^{2n-1}\cos 2\pi\nu+q^{4n-2}),\end{aligned} \qquad (5.A.4)$$

$$\begin{aligned}\theta_4(\nu|\tau) &= -ie^{i\pi(\nu+\tau/4)}\theta_1(\nu+\frac{1}{2}\tau|\tau) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{i\pi 2n\nu}\\ &= f(q^2)q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2\pi\nu+q^{4n-2}),\end{aligned} \qquad (5.A.5)$$

where

$$q = e^{i\pi\tau} \tag{5.A.6}$$

and

$$f(q^2) = \prod_{n=1}^{\infty}(1 - q^{2n}). \tag{5.A.7}$$

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