TWO TOPICS
IN
2D QUANTUM FIELD THEORY

Dissertation by
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for my parents

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ABSTRACT

Two topics in two-dimensional quantum field theory are presented. The first is a classification of 2- and 3-field rational conformal field theories. Using the fact that the fusion algebra of a RCFT is specified in terms of integers that are related to modular transformation properties, we classify 2- and 3-field chiral RCFT's. We show that the only possibilities for the non-trivial fusion rule in the 2-field case are $\phi \times \phi = 1$ or $\phi \times \phi = 1 + \phi$. Similar results are obtained for the 3-field case. A partial classification of possible conformal dimensions and central charges for these theories is also obtained. The second topic is in two-dimensional quantum gravity. Explicit computation of the non-perturbative correlation functions of the $(1, q)$ models of KdV-gravity is presented. This computation includes contributions from high genus as well as correlation functions of descendant fields. A ghost number conservation law for these models is derived from purely algebraic considerations. A hint of further selection rules is found.

CONTENTS

Part I
INTRODUCTION

1. Introduction to 2D Physics

The world around us is four-dimensional, but lower dimensional physics has proved invaluable in its investigation. First, lower dimensions provide simpler examples and toy models that share the properties, or similar properties, of the four-dimensional theory. Thus, one may gain insight into the more complex problem by dealing with a simpler case. Second, the basic ingredient of string theory, which is still the major candidate for a 'theory of everything,' is a two-dimensional world-sheet. Hence, in a sense, string theory is a two-dimensional field theory. Third, two-dimensional theories have some properties that are *not* shared by theories in higher dimensions. This opens up new areas of investigation both in mathematics and physics that may be very important for further developments.

In this thesis, which is based on published[1,2] and unpublished work, we will investigate two aspects of two-dimensional quantum field theory. The first is a classification of conformal field theories and the second is a solution of certain models of two-dimensional quantum gravity. These will be presented in full detail in Parts II and III.

Conformal field theory is a subject that demonstrates the points above. First, it has been argued that in statistical mechanics, two-dimensional systems at critical points are conformal field theories. This is because a statistical model at a critical point (of a second-order phase transition) is scale invariant in two dimensions (*i.e.*, there are fluctuations of the order parameter on any length scale). It has been proven that scale invariance is equivalent to conformal invariance in two dimensions[3]. For example, the famous Ising model in two dimensions is equivalent, at its critical point, to the $(4,3)$ minimal model of Belavin, Polyakov and Zamolodchikov[4]. The exact solution of two-dimensional statistical mechanical models might shed light on higher dimension systems.

Second, in bosonic string theory one usually writes the action[5]

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{h}h^{\alpha\beta}(\sigma)g_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu + \cdots , \qquad (1.1.1)$$

where σ^0 and σ^1 are the world-sheet coordinates, $h^{\alpha\beta}$ is the inverse of $h_{\alpha\beta}$ which is the metric on the world-sheet and $h = \det h_{\alpha\beta}$. X^μ and $g_{\mu\nu}$ are, respectively, the

coordinates and the metric of space-time. The metric in two dimensions has three independent components and, since the action is reparametrization invariant, one can eliminate two of them. This allows a gauge choice in which the metric takes the so-called conformal form, $h_{\alpha\beta} = f(\sigma)\eta_{\alpha\beta}$, where f is some function. However, there is one more symmetry at our disposal—Weyl symmetry. One can easily check that the classical string action is also invariant under the Weyl rescaling

$$h_{\alpha\beta} \to \Lambda(\sigma)h_{\alpha\beta}. \tag{1.1.2}$$

All these symmetries amount to the fact that the world-sheet theory is conformally invariant, at least at the classical level. This means that a vacuum state of string theory *is* a conformal field theory, at least to leading order in the loop expansion.

Third, the conformal group is infinite dimensional in two dimensions, as will be shown in the Part II. This property is not shared with a conformal group in higher dimensions*. In the last decade conformal field theory has proved to be a useful tool in the investigation of many areas in mathematics, such as infinite dimensional algebras (Kač-Moody algebras[6]), loop groups[7] and more. Indeed, Kač-Moody algebras have proved to be of importance in physics, especially in string theory, and in the construction of new conformal field theories[8,9].

The second topic of this thesis also demonstrates the points above. Quantum gravity is a problem of vast interest, and knowledge of the two-dimensional theory should help us understand the four-dimensional counterpart. The world sheet in string theory is reparametrization invariant and hence may be viewed as a gravitating two-dimensional system. A connection has been found between the KdV hierarchy, topology and certain infinite dimensional algebras using random matrix models—which in turn have been used to derive more connections and to get more insight into the two-dimensional problem†.

In the next two chapters the two topics of this thesis will be discussed in a general way, and some more specific motivation for their investigation will be presented. The main results will also be presented without too many details.

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* In fact, in d dimensions the local conformal group has dimension $\frac{1}{2}(d+1)(d+2)$. The global conformal group might be of smaller dimension. In two dimensions, the global conformal group on the sphere is finite dimensional!

†References will be presented in a later section.

2. Classification of Conformal Field Theories

Two-dimensional quantum field theory with conformal invariance[4,10] is a system with an infinite number of conserved currents, since the conformal group in two dimensions is infinite dimensional. The symmetry algebra, the Virasoro algebra[11], is defined by the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{1.2.1a}$$

$$[L_m, \bar{L}_n] = 0 \tag{1.2.1b}$$

$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0}, \tag{1.2.1c}$$

where m and n are integers. This algebra splits into a direct product of a holomorphic part (spanned by the L_n's) and anti-holomorphic part (spanned by the $\bar{L}_n$'s). We will denote this algebra $\mathcal{V} \otimes \bar{\mathcal{V}}$. Because of this very large symmetry algebra, some conformal field theories can be solved exactly—meaning that one can explicitly calculate correlation functions of the various fields.

The fields in the theory decompose into irreducible highest-weight representations of the Virasoro algebra. Given such a representation, the highest weight field itself is known as a primary field and is specified by a pair of numbers—the so-called conformal dimensions $(h, \bar{h})$. The rest of the fields (*i.e.*, non-primary fields) are known as descendants.

Classification of conformal field theories (CFT's) has been investigated extensively in the last few years[12–16]. A complete classification does not exist yet, but one does know, at least conceptually, how to classify rational conformal field theories. Moreover, rational conformal field theories are the ones that can be explicitly solved, as was done for the special case of minimal models in the fundamental paper by Belavin, Polyakov and Zamolodchikov[4].

A rational conformal field theory (RCFT) is defined as a CFT that has a finite number of primary fields. It has been shown by Kač[17] that in the representation theory of the Virasoro algebra, the only RCFT's are the minimal models of Belavin, Polyakov and Zamolodchikov[4]. The central charge, c, of these minimal models is in a list of discrete values that are bounded from above by 1.

The problem now was to construct RCFT's with c larger than 1. At first this seems impossible—it has been proved[18] that CFT's that are invariant under the Virasoro algebra and have $c > 1$ must have an infinite number of primary fields. The solution was to extend the symmetry algebra with the hope the under the new enlarged symmetry the fields will decompose into fewer representations, maybe even a finite number. Indeed, this was accomplished in different ways, by including supersymmetry and thus getting the superconformal algebras, by extending the algebra with spin one currents and getting the Kač-Moody algebras, by W-algebras[19] and more. Today there is an extensive list of possible extended symmetry algebras.

Let us denote the extended algebra by $\mathcal{A} \otimes \bar{\mathcal{A}}$ and, as is apparent in the notation, we assume that it decomposes into a direct product of a holomorphic part (sometimes known as the chiral part) and an anti-holomorphic part (antichiral)*. This algebra contains the Virasoro algebra $\mathcal{V} \otimes \bar{\mathcal{V}}$ as a subalgebra. The (infinite number of) fields of the theory decompose into irreducible highest-weight representations of this algebra. Each associated highest-weight field is a primary field, and it has specific conformal dimensions, $(h, \bar{h})$, as before. However, each primary field has some additional 'quantum' numbers that characterize it in terms of the extended algebra. RCFT's are defined, as before, by demanding that the theory has a finite number of primary fields.

One might ask the following question: Is there a relation between the chiral and anti-chiral parts of the symmetry algebra? At first glance, looking only at the representation theory, one can find a chiral CFT (*i.e.*, one looks only at the chiral part), and this theory seems to be independent of the antichiral part—the symmetry algebra is a direct product, and hence the representation theory decomposes. Hence, the answer to the question seems to be no. However, this answer is true only when the Riemann surface on which the theory is defined is the sphere. Motivated by string theory loop expansions, one wishes to define CFT's on higher genus surfaces and then other considerations must be applied.

For example, on the torus, we encounter modular invariance. It turns out that the conformally inequivalent tori are specified by a complex parameter, τ, known as the modular parameter. However, there are different τ's, connected by so-called modular transformations, that specify the same torus, *i.e.*, two tori that are connected by a modular transformation are conformally equivalent. The symmetry group generated

*Sometimes the whole algebra is referred to as a 'chiral algebra' and sometimes 'chiral' refers only to the holomorphic part.

by these transformations is the modular group. Demanding that the CFT under investigation be modular invariant leads to severe restrictions on the possible chiral-antichiral combinations in the symmetry algebra and also on the field content of the theory. One may think that one needs to take into account the modular structure of higher genus Riemann surfaces (and in general their moduli space is not even known), but it has been proved that it is sufficient to consider only modular invariance on the torus[20].

To classify RCFT's, one first has to classify all possible chiral algebras that have a finite number of irreducible representations. After doing that, one has to find all possible modular invariant combinations of chiral and antichiral parts. This method has a few drawbacks. First, there is no classification of chiral algebras. Second, even for known chiral algebras, the representation theory is not always known.

The motivation for this work is to see what one can say about a RCFT without fully knowing the symmetry algebra. The work will be restricted to chiral rational conformal field theories, *i.e.*, the anti-holomorphic part, $\bar{\mathcal{A}}$, of $\mathcal{A}\otimes\bar{\mathcal{A}}$, will be ignored. In any case, the left-right combinations of fields is quite trivial for the two- and three-field theories that will be dealt with.

A chiral rational conformal field theory (CRCFT) has a finite number of primary fields. These fields satisfy the fusion rules

$$\phi_i \times \phi_j = \sum_{k=0}^{N-1} {N_{ij}}^k \phi_k \,, \tag{1.2.2}$$

where N is the number of primary fields and ${N_{ij}}^k$ are non-negative integers that specify the number of channels to get ϕ_k when fusing ϕ_i and ϕ_j[21,22]. These integers must satisfy certain relations because of modular invariance.

In Part II the constraints of modular invariance on the fusion algebra, the conformal dimensions of the primary fields, and the central charge of two- and three-field rational conformal field theories will be investigated. The main tool we will use is a formula found by Verlinde[21] that connects the integers ${N_{ij}}^k$ to a certain matrix that describes the modular transformation properties of the theory. It will be shown that the two-field case can be classified, *i.e.*, a list of possible theories with their conformal dimensions and central charge will be presented. The three-field case is harder to deal with but similar results still hold, although the classification is not yet complete.

The analysis demonstrates the crucial role of modular invariance but also shows that this symmetry is not enough to completely classify rational conformal field the-

ories. However, in the cases investigated, the fusion rules themselves, (1.2.2), do get classified. It is tempting to conjecture that this is true in general, *i.e.*, for any RCFT our method will classify the fusion rules.

3. (1,q) Models of KdV Gravity

In the past year and a half there has been a lot of progress in the investigation of two-dimensional quantum gravity. A quantum theory of gravity has proved over the years to be very elusive and many attempts to formulate it have been unsuccessful. Superstring theory provides a theory of gravity in space-time with dimension $D \leq 10$, but we still lack complete understanding of all the special features that are present in string theory, and there is a lot of ambiguity concerning its vacuum and its low energy limit.

In a brilliant set of papers by various authors, a formulation of two-dimensional quantum gravity was presented based on random matrix models[23–28]. It has been demonstrated that the continuum limit of certain random matrix models at suitable critical points can be solved[29–31]. That is, a non-perturbative solution can be found. It has also been conjectured, and a lot of evidence has been presented, that the solution satisfies the KdV-hierarchy[32].

The KdV hierarchy is a hierarchy of non-linear, partial differential equations that is completely integrable[33]. The first non-trivial equation in the hierarchy is the original Korteweg-deVries equation. It is best presented in terms of pseudo-differential operators,

$$\mathcal{O}^{(k)} = \sum_{i=0}^{\infty} o_i(x) D^{k-i}, \tag{1.3.1}$$

where $D = \frac{\partial}{\partial x}$ and k is a certain integer that specifies the highest-order derivative. With a definition of the way the operator D^{-1} operates, the set of such operators form an algebra. The KdV hierarchy is represented by the equations

$$\frac{\partial Q}{\partial t_i} = \left[Q_+^{i/q}, Q \right], \tag{1.3.2}$$

where Q is a *differential* operator of order q and certain fractional powers of Q have been introduced. (The '+' picks out the differential operator part.) The details will be presented in Part III.

In KdV gravity the operators $\{Q^{i/q}\}_{i=1}^{\infty}$ are associated with the fields of the theory and, adding some extra conditions (the 'string equation'), we can compute correlation

functions, at least in principle. The parameters t_i introduced in equation (1.3.2) are called flow parameters. One may think of them, in a field theoretical context, as the coefficient of perturbations that may be added to the action. In this way, taking partial derivatives of a correlation function with respect to these parameters corresponds to inserting additional operators into the correlation function. In the end these parameters are set to zero (or some other appropriate critical values).

The string equation, defined with the help of another differential operator P of order p, is

$$[P, Q] = 1. \tag{1.3.3}$$

The operator P is connected with the potential chosen by the matrix model, and its order p is related to the order of this potential. The operator Q is related to the number of matrices in the model. One may think of P and Q as 'momentum' and 'position' operators in the space of eigenvalues of the matrices used to define the random matrix model. The string equation can be identified with the canonical commutation relation. By choosing different operators Q and P, one chooses different models. If $P = (Q^{p/q})_+$, one gets the so-called (p, q) model.

Other approaches to two-dimensional quantum gravity are based on quantum Liouville theory[34–37] and topological field theory[38]. The latter approach provides a simple understanding of some aspects of two-dimensional gravity. Indeed, a certain class of KdV gravity models are topological field theories. Topological field theory also provides the hints that are needed in order to identify the fields in the theory. The matrix model approach, on the other hand, is somewhat ambiguous in dealing with descendant fields.

In Part III the matrix models and the KdV hierarchy will be defined and an explicit solution will be presented to the $(1, q)$ models of KdV gravity. A 'ghost' number selection rule for the computation of correlation functions will be proven. It will be shown that one can assign a ghost charge to each field,

$$\mathrm{gh}\,(\mathcal{P}_i) = i - q - 1, \tag{1.3.4}$$

and that the correlation function of n fields vanishes unless

$$\sum_{i=1}^{n} \mathrm{gh}\,(\mathcal{P}_i) = 2(q+1)(1-g). \tag{1.3.5}$$

In the above, g is a non-negative integer multiple of one-half, q is a positive integer that specifies the $(1, q)$ model and $\mathcal{P}_i$ are the various fields. It turns out that g

represents the genus of the Riemann surface on which the correlation function does not vanish.

Part II
CLASSIFICATION OF TWO- AND THREE-FIELD RATIONAL CONFORMAL FIELD THEORIES

1. Introduction to Conformal Field Theory

Consider a d-dimensional flat space ($\mathbb{R}^d$) with the Minkowski type metric $g_{\mu\nu} = \eta_{\mu\nu}$ of signature (p, q). Under a change of coordinates, $x \to x'$, the metric transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x). \tag{2.1.1}$$

The conformal group is defined as the subgroup of coordinate transformations having the property that the metric is invariant up to a scale factor,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{2.1.2}$$

Note that the Poincaré group is a subgroup of the conformal group, as it leaves the metric invariant.

To find the generators of the conformal group, consider an infinitesimal coordinate transformation, $x^\mu \to x^\mu + \epsilon^\mu(x)$, under which the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ transforms by

$$ds^2 \to ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu. \tag{2.1.3}$$

To satisfy (2.1.2), $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ must be proportional to $g_{\mu\nu}$,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial \cdot \epsilon) g_{\mu\nu}, \tag{2.1.4}$$

where the proportionality constant is determined by tracing (2.1.4) with $g^{\mu\nu} = \eta^{\mu\nu}$. It also follows from this equation that

$$(g_{\mu\nu} \Box + (d-2) \partial_\mu \partial_\nu) \partial \cdot \epsilon = 0. \tag{2.1.5}$$

This can be derived by first applying ∂^μ to (2.1.4) (summing the index μ) and then applying ∂_μ (there is no μ index left). Repeating with the index ν (*i.e.*, applying ∂^ν and then ∂_ν) and adding the two resulting equations one gets the required result (up to substitution of (2.1.4) again).

The difference between two dimensions and $d > 2$ dimensions is now obvious. When $d > 2$, the dimension of the conformal group is finite since ϵ^μ can be at most

quadratic in x. In two dimensions, however, choosing Euclidean signature ($\eta_{\mu\nu} = \delta_{\mu\nu}$), equation (2.1.4) reduces to the Cauchy-Riemann equations

$$\partial_1\epsilon_1 = \partial_2\epsilon_2 \quad \text{and} \quad \partial_1\epsilon_2 = -\partial_2\epsilon_1. \tag{2.1.6}$$

It is then natural to transform to the complex coordinate $z = x_1 + ix_2$ and its complex conjugate $\bar{z} = x_1 - ix_2$. We similarly define $\epsilon(z) = \epsilon_1 + i\epsilon_2$ and $\bar{\epsilon}(z) = \epsilon_1 - i\epsilon_2$. The two-dimensional coordinate transformations thus coincide with the analytic coordinate transformations

$$z \to f(z) \quad \text{and} \quad \bar{z} \to \bar{f}(\bar{z}). \tag{2.1.7}$$

The local algebra of these transformation is infinite dimensional. For this reason, *i.e.*, the existence of an infinite number of conserved currents, one can get a lot of information about theories that have conformal symmetry in two dimensions.

1.1. The Virasoro Algebra, Primary Fields, and All That

A two-dimensional conformal field theory is defined as a complete set of fields, $\{\phi_i(z,\bar{z})\}$, together with a symmetry algebra, $\mathcal{A} \otimes \bar{\mathcal{A}}$, defined on some Riemann surface. We demand that the symmetry algebra decomposes into a direct product of chiral (*i.e.*, holomorphic) and anti-chiral parts, and that it contains the Virasoro algebra, $\mathcal{V} \otimes \bar{\mathcal{V}}$, as a subalgebra. We also demand a strong version of the operator product expansion (OPE). OPE in the usual definition is only an asymptotic expansion; however, in conformal field theory the expansion is exact—that is, it converges. One can argue that since the theory has no scale, the OPE cannot obtain terms that are proportional to $e^{\frac{-l}{z-w}}$; such terms would generically ruin the exactness of the OPE.

The Virasoro algebra is described in terms of the stress-energy tensor, $T(z)$*, which satisfies the following operator product expansion (OPE),

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{non singular terms.} \tag{2.1.8}$$

c is called the central charge of the algebra. One may understand this OPE as encoding in a convenient way the commutation relations of the generators, L_n, of the Virasoro algebra. These generators are the modes of the stress-energy tensor

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}, \tag{2.1.9}$$

*We will suppress writing the anti-holomorphic operator $\bar{T}(\bar{z})$, as it behaves in a similar way.

and satisfy the commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}. \tag{2.1.10}$$

The fields of a 2D CFT can be arranged in irreducible highest-weight representations of the chiral algebra. The highest weight field is called a primary field and, if the chiral algebra is just the Virasoro algebra, it is defined by the OPE

$$T(z)\phi_h(w,\bar{w}) \sim \frac{h\phi_h(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\phi_h(w,\bar{w})}{z-w}, \tag{2.1.11}$$

where the non-singular terms are not written, as will be the practice from now on. The number h is called the conformal dimension of the primary field. A similar expression with $\bar{h}$ holds for the anti-holomorphic part, $\bar{T}(\bar{z})$. This means that the primary fields transform an $(h,\bar{h})$ tensor under coordinate transformations

$$\phi_{h,\bar{h}}(z,\bar{z})dz^h d\bar{z}^{\bar{h}} = \phi'_{h,\bar{h}}(z',\bar{z}')dz'^h d\bar{z}'^{\bar{h}}. \tag{2.1.12}$$

We stress that non-primary fields do *not* transform in such a simple way. In a general chiral algebra this condition, (2.1.11), is still satisfied, but it is no longer the complete definition. One needs to add the condition that the primary field is the highest weight of a highest-weight representation of the whole chiral algebra and not just the Virasoro subalgebra. The details depend, of course, on the knowledge of the chiral algebra.

The above statements can also be written in mode expansions. The vacuum state, $|0\rangle$, is defined as the state with zero energy that is annihilated by all lowering operators[†]

$$L_0|0\rangle = 0 \tag{2.1.13}$$

$$L_n|0\rangle = 0 \quad n > 0. \tag{2.1.14}$$

Then a primary state, $|h\rangle$, is defined by

$$|h\rangle = \phi_h(0)|0\rangle. \tag{2.1.15}$$

It follows that

$$L_0|h\rangle = h|h\rangle \tag{2.1.16}$$

[†]It follows from (2.1.10) that also $L_{-1}|0\rangle = 0$. This is the source of an $SL(2,R)$ symmetry of the vacuum.

$$L_n|h\rangle = 0 \quad \text{for} \quad n > 0. \tag{2.1.17}$$

That is, $|h\rangle$ is an eigenvector of L_0 with eigenvalue h. The second line means that $|h\rangle$ is a highest-weight state. One gets the rest of the states in the representation by using the raising operators (L_{-n} for positive n in the case that the chiral algebra is just the Virasoro algebra). These fields (or states) are usually called descendant fields, or descendants for short. The exact form depends on the chiral algebra, but supposing for now that the chiral algebra is just the Virasoro algebra, one can write an expression for a generic descendant state

$$|\mathbf{n}, h\rangle = L_{-1}^{n_1} L_{-2}^{n_2} \cdots L_{-k}^{n_k} \cdots |h\rangle, \tag{2.1.18}$$

where $\mathbf{n}$ is the list $\{n_1, n_2, \ldots, n_k, \ldots\}$. It is not hard to show that the descendants are also eigenvectors of L_0 with eigenvalue $h + \sum_k k n_k$

$$L_0|\mathbf{n}, h\rangle = (h + \sum_k k n_k)|\mathbf{n}, h\rangle. \tag{2.1.19}$$

The value $\sum_k k n_k$ is usually called the level (not to be confused with the level of Kač-Moody algebras that will be mentioned later).

A question that one may ask is whether all the descendants are independent. Another question, that turns out to be equivalent to the first one, is whether one can get a descendant that behaves like a primary field—that is, it satisfies (2.1.16) and (2.1.17). The answer is usually no. However, there are models with such states. This was exploited in the paper of Belavin, Polyakov and Zamolodchikov[4] to find the so-called minimal models. A bit more on these models will be presented later.

The fields of a CFT satisfy OPEs

$$\phi_i(z, \bar{z})\phi_j(0,0) = \sum_k C_{ijk} z^{-\Delta_{ijk}} \bar{z}^{-\bar{\Delta}_{ijk}} \phi_k(0,0)\,, \tag{2.1.20}$$

where Δ_{ijk} and $\bar{\Delta}_{ijk}$ are simple combinations of the conformal dimensions of the fields,

$$\Delta_{ijk} = h_i + h_j - h_k \quad \text{and} \quad \bar{\Delta}_{ijk} = \bar{h}_i + \bar{h}_j - \bar{h}_k. \tag{2.1.21}$$

The fields in the above equation may be primary or not; their dimensions are their eigenvalues under the generators L_0 and $\bar{L}_0$ (see (2.1.19)) in any case.

Let us note that knowledge of the coefficients C_{ijk} actually provides a complete solution of the theory. To show that, one uses the fact that the one-point correlation

function (on the sphere) vanishes unless the field happens to be the identity (or a descendant of the identity). Then the two-point function is

$$\langle \phi_i(z_1)\phi_j(z_2)\rangle = \langle \sum_k C_{ijk} z_{12}^{-\Delta_{ijk}} \phi_k(z_2)\rangle = \frac{C_{ij0}}{z_{12}^{h_1+h_2}}, \tag{2.1.22}$$

where the $\bar{z}$ dependence has been omitted. Moreover, it can be shown[10] that this correlation vanishes unless $h_1 = h_2$. Usually one renormalizes the fields such that $C_{ij0} = \delta_{ij}$[‡]. Computation of the three-point functions follows in a similar way, and the details will not be presented here. However, the three-point functions determine the rest of the coefficients C_{ijk}. No further information is needed to compute higher-point correlation functions. This is because the OPE is associative and, hence, when one computes correlation functions, one can use (2.1.20) to reduce the correlation to a function of the C_{ijk}'s. Indeed, some methods of classification involve the classification of different possible values for these coefficients allowed by crossing symmetry of the four-point functions[39–42,15,16]. That is, performing the OPE (2.1.20) on different fields in the four-point function provides alternative ways to get the result. Demanding that these results be identical provides constraints on the values of C_{ijk}. This is demonstrated in Fig. 1.

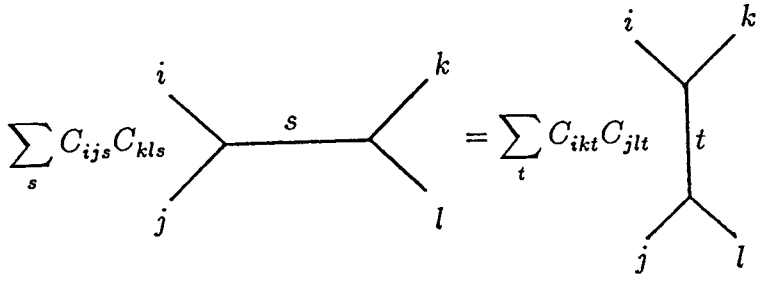

Figure 1: The crossing symmetry of the four-point function $\langle \phi_i\phi_j\phi_k\phi_l\rangle$.

Essential information that is encoded in the OPE is the answer to the following question. Given two primary fields on the left hand-side of equation (2.1.20), which representations appear on the right-hand side? The answer is encoded in the so-called fusion algebra

$$\phi_i \times \phi_j = \sum_k N_{ij}{}^k \phi_k. \tag{2.1.23}$$

[‡]However, there is a slight subtlety that we will deal with in Chapter 3.

Here, one understands the fields as primaries that represent the whole representation, and the sum runs over all the primary fields in the given theory. These are just the rules for decomposing Kronecker tensor products into irreducible representations, analogous to $\mathbf{8} \times \mathbf{8} = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{10} + \bar{\mathbf{10}} + \mathbf{27}$ in the case of $SU(3)$. It follows that the coefficients ${N_{ij}}^k$ are non-negative integers.

Generically, there is an infinite number of primary fields, and a detailed analysis is very difficult. However, there are some models in which the number of primary fields is finite. These models are known as rational conformal field theories. Of course, the finiteness of the number of primaries depends on the chiral algebra. That is, a RCFT with respect to some chiral algebra is usually not a RCFT with respect to the Virasoro subalgebra. Indeed, it has been shown that if the chiral algebra is just the Virasoro algebra, then the number of primary fields is finite only if the central charge, c, is in a discrete list of values

$$c = 1 - \frac{6(r-s)^2}{rs}. \tag{2.1.24}$$

These are the minimal models mentioned above. In equation (2.1.24) r and s are relatively prime (≥ 2), the number of primary fields is $(s-1)(r-1)/2$, and the conformal dimensions of the primary fields are

$$h_{p,q} = \frac{(ps - qr)^2 - (s-r)^2}{4rs}, \tag{2.1.25}$$

where $1 \leq p \leq r-1$ and $1 \leq q \leq s-1$[§].

One immediately sees from equation (2.1.24) that for all the minimal models $c \leq 1$. This suggests the problem of finding solvable conformal field theories with $c > 1$. This is especially desired in the context of string theory. In the bosonic string theory, for example, the reparametrization ghosts contribute central charge -26, and, to avoid conformal anomalies at the quantum level, one needs a vanishing total central charge. The solution is to add a conformal field theory of central charge 26. Usually this CFT is composed of four free bosons (that represent space-time coordinates) and another $c = 22$ CFT that represents the 'internal' degrees of freedom. Preferably, we would like a finite number of internal degrees of freedom, and that is the reason to look for RCFTs with $c > 1$. One solution to this problem is to extend the chiral algebra, for example by supersymmetry or by Kač-Moody algebras. Nowadays there is a long list of extended chiral algebras.

[§]Actually, these values count each primary twice.

1.2. Extended Chiral Algebras

In this section extended chiral algebras are briefly discussed. The idea is to extend the Virasoro algebra by chiral fields so that under the extended symmetry the fields in the theory decompose into fewer representations. Historically, the first such extension was the superconformal algebra[43,44]. This algebra contains a supercurrent, $G(z)$, in addition to the energy-momentum tensor. The resulting algebra has the following OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{2.1.26a}$$

$$T(z)G(w) \sim \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} \tag{2.1.26b}$$

$$G(z)G(w) \sim \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2T(w)}{z-w}. \tag{2.1.26c}$$

This superconformal algebra has (super)-minimal models with the following values for the central charge[45,46]

$$c = \frac{3}{2}\left(1 - \frac{2(s-r)^2}{rs}\right). \tag{2.1.27}$$

Here, the limiting value for c is $\frac{3}{2}$. A similar $N = 2$ superconformal algebra[47–49] (with two supercurrents and an additional $U(1)$ current) yields a limiting c of 3[50–52]. The supersymmetric extensions to the Virasoro algebra have been classified by Ramond and Schwarz[53].

There is one subtlety in dealing with superconformal algebras that will be demonstrated for the ($N = 1$) superconformal algebra, (2.1.26). The OPEs only describe the local behavior of the algebra, and the global behavior is linked to boundary conditions on the operators. $T(z)$ always has periodic boundary condition since otherwise conformal symmetry would be broken. The other generators, however, can have antiperiodic boundary conditions. In the superconformal case there are only two possibilities, the Ramond sector, in which $G(z)$ has integer modes, and the Neveu-Schwarz sector, in which $G(z)$ has half-integral modes. These two algebras are not equivalent, *i.e.*, there is no automorphism between them. Similarly, in the $N = 2$ case, there are two sectors, the twisted sector (T-sector), which is usually not encountered in CFT, and the NSR sector. The fact that the modes are not necessarily integral can yield, in equation (2.1.19), half-integral levels.

We will discuss Kač-Moody algebras[54–56,6] in a more detailed way as these have proved to be of extreme importance in CFT**. One may think of them as central extensions of the algebra of the loop group of a Lie group. A Kač-Moody algebra, $\hat{g}_k$, based on the compact finite dimensional Lie algebra g, is defined by $\dim(g)$ currents, $J^a(z)$, satisfying the OPE

$$J^a(z)J^b(w) \sim \frac{\frac{k}{2}\delta^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^c(w)}{z-w}. \tag{2.1.28}$$

k is known as the level of the algebra (not to be confused with the level of a descendant) and $f^{ab}{}_c$ are the structure constants of the Lie algebra g. The Lie algebra g could be simple, semi-simple, or it could have $U(1)$ factors. We will assume for simplicity that the algebra g is simple.

Using the Sugawara[57,58] construction, we define a stress-energy tensor

$$T(z) = \frac{1}{k+Q_\psi} \sum_{a=1}^{\dim g} :J^a(z)J^a(z):, \tag{2.1.29}$$

where Q_ψ is the quadratic casimir of the adjoint representation defined by $f_{cd}{}^a f^{cdb} = Q_\psi \delta^{ab}$, and ': :' is a normal ordering prescription that removes the singularity that results from multiplying two operators at the same point††. This $T(z)$ satisfies the Virasoro algebra OPE, (2.1.8), with central charge

$$c_g = \frac{k \dim g}{k+Q_\psi}. \tag{2.1.30}$$

Note that generically $c > 1$.

Conformal field theories that are based on this construction have a finite number of primary fields, labeled by the highest weights of the representations of the underlying Lie algebra. The possible representations are restricted by the condition

$$0 \leq \frac{2\Lambda \cdot \psi}{\psi^2} \leq \frac{k}{\psi^2} \in \mathbf{Z}, \tag{2.1.31}$$

**For a review, see Goddard and Olive[8].

†† For example,

$$:J^a(z)J^b(z): := \lim_{w\to z}\left(J^a(z)J^b(w) - \frac{\frac{k}{2}\delta^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^c(w)}{z-w}\right).$$

where Λ is the highest weight of a representation and ψ is the highest root of the Lie algebra g. The origin of this formula is as follows. In the mode expansion ($J(z) = \sum_{\mathbf{Z}} J_n z^{-n-1}$) one can choose the analogue of the Cartan-Weyl basis. In that basis, the generators $E_1^{-\alpha}$, E_{-1}^{α} and $\frac{2}{\alpha^2}\left(\frac{k}{2} - \alpha \cdot H_0\right)$ form a subalgebra that is isomorphic to $SU(2)$. Here α is a root, E_n^{α} are the raising and lowering operators and the H_0^i's belong to the Cartan subalgebra. Since those operators form an $SU(2)$ algebra, the 'spin' must be quantized to be half integral. Since $\frac{2\alpha \cdot H_0}{\alpha^2}$ is quantized (this is a basic fact from the theory of Lie algebras), $\frac{k}{\alpha^2}$ must be quantized as well. We can extremize this quantization rule by selecting the highest root of g, ψ.

The other part of the formula follows from demanding that the vector $E_{-1}^{\alpha}|\mu\rangle$ be of positive norm. Here μ is a weight in the representation. The requirement that this vector be of positive norm is a consequence of unitarity—that is, we build the Kač-Moody algebra representation on unitary representations of the underlying Lie algebra. Since this algebra is compact (as chosen), those representations are finite dimensional[‡‡].

The primary fields have conformal dimensions

$$h_{\Lambda} = \frac{Q_{\Lambda}}{k + Q_{\psi}}, \tag{2.1.32}$$

where Q_{Λ} is the quadratic Casimir of the representation Λ. For example, the conformal field theory based on $SU(2)$ at level k has primaries with dimensions (of the $SU(2)$ representation) $1 \leq 2j + 1 \leq k$, where j is the highest weight (spin). The conformal dimensions are

$$h_j = \frac{j(j+1)}{k+2}. \tag{2.1.33}$$

We will return to $SU(2)^{(k)}$ in the next chapter. All these theories can be constructed as Wess-Zumino-Witten (WZW) models[59–61] based on non-linear sigma models.

An important construction, known as the coset or GKO construction[62,9], is used to generate many more CFT's by taking cosets of g and one of its subalgebras, say h. If $T_G(z)$ and $T_H(z)$ are the Virasoro currents for g and h, respectively, then $T_{G/H}(z) = T_G(z) - T_H(z)$ also satisfies the Virasoro algebra with central charge

$$c_{G/H} = c_G - c_H. \tag{2.1.34}$$

[‡‡]There are extensions of this construction to non-compact or non-unitary groups, but we will not deal with them here.

By substituting (2.1.30) and choosing the algebras g and h carefully, one can get c's that are smaller than 1. For example, by choosing $g = SU(2)^{(k)} \otimes SU(2)^{(1)}$ and $h = SU(2)^{(k+1)}$, one gets the central charges of the minimal models (2.1.24). Moreover, it has been proved that this construction actually gives the minimal models. Similar results hold for the superconformal algebras. It has been conjectured that all rational conformal field theories can be represented as coset models. Certainly it is true for the known models.

The last point we will discuss is the problem of modes. Strictly speaking, a chiral algebra is defined as a set of holomorphic and anti-holomorphic operators that have *integer* dimensions[63]. The superconformal algebra seems to contradict this statement, and there are other examples such as parafermion theories[64] and orbifold theories[65]. These cases are examples in which a complicated chiral algebra is extended with fractional dimension chiral fields and the resulting structure has a chiral algebra-like appearance. We usually project these modes out. Further examples are discussed by Goddard and Schwimmer[66]. We will assume that the chiral algebra has only integer spin currents[§§].

We will conclude this section with a rigorous definition of a rational conformal field theory, as given by Moore and Seiberg[63]. We start with the definition of a conformal field theory.

Definition 1: **Conformal Field Theory**

A conformal field theory is an inner product space $\mathcal{H}$ which can be decomposed into a direct sum

$$\mathcal{H} = \bigoplus_{h,\bar{h}} V(h,c) \otimes \bar{V}(\bar{h},\bar{c}) \tag{2.1.35}$$

of irreducible highest weight modules of $\mathcal{V} \otimes \bar{\mathcal{V}}$ such that

a. (Vacuum) There is a unique $SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})$ invariant state, $|0\rangle$, with $(h,\bar{h}) = (0,0)$.

b. (Operators and Duals) For each vector $\alpha \in \mathcal{H}$ there is an operator $\phi_\alpha(z)$ on $\mathcal{H}$, parametrized by $z \in \mathbb{C}$. Also, for every operator ϕ_α there exists a conjugate operator $\phi_{\alpha\dagger}$ (partially) characterized by the requirement that the OPE $\phi_\alpha \phi_{\alpha\dagger}$ contains a descendant of the unit operator.

c. (Primary Fields) For $\alpha = i$, a highest-weight state, we have $[L_n, \phi_i(z,\bar{z})] = \left(z^{n+1}\frac{d}{dz} + h_i(n+1)z^n\right)\phi_i(z,\bar{z})$.

[§§] Actually, for our purpose, it is enough to demand integer modes.

d. (Duality) The inner products $\langle 0|\phi_{i_1}(z_{i_1},\bar{z}_{i_1})\cdots\phi_{i_n}(z_{i_n},\bar{z}_{i_n})|0\rangle$ exist for $|z_{i_1}| > \ldots > |z_{i_n}| > 0$ and admit an unambiguous real-analytic continuation, independent of ordering, to $\mathbb{C}^n$ minus diagonals.

e. (Modular Invariance) The one-loop partition function and correlation functions, computed as traces, exist and are modular invariant.

Definition 2: **Chiral Algebra**

In a conformal field theory, a closed set (under OPE) of holomorphic (and anti-holomorphic) fields is called a chiral algebra. A maximal set of such operators is a maximal chiral algebra.

Definition 3: **Rational Conformal Field Theory**

Rational conformal field theories are conformal field theories such that the (physical) Hilbert space, $\mathcal{H}$, is given by the *finite* sum

$$\mathcal{H} = \bigoplus_{r,\bar{r}=0}^{N} N_{r,\bar{r}}\mathcal{H}_r \otimes \mathcal{H}_{\bar{r}}, \tag{2.1.36}$$

where $\mathcal{H}_r$ ($\mathcal{H}_{\bar{r}}$) is an irreducible representation of $\mathcal{A}$ ($\bar{\mathcal{A}}$), the (anti-)holomorphic part of the chiral algebra. $N_{r,\bar{r}}$ is an integer counting the multiplicity of $\mathcal{H}_r \otimes \mathcal{H}_{\bar{r}}$ in $\mathcal{H}$.

2. Modular Invariance in CFT

Conformal field theory was initially defined on the sphere (or plane) but it is seen in its full beauty when considered on an arbitrary Riemann surface. Indeed, in string theory, the sphere is just the first term in a genus perturbation expansion and so we are interested also in CFT's on higher genus surfaces. The first of these is the torus, so one goal is to formulate a consistent conformal field theory on the torus. Cardy was the first to discuss the importance of modular invariance in restricting the operator content of a CFT[18].

2.1. CFT on the Torus

A torus is defined by two complex periods, ω_1 and ω_2, with the ratio $\tau = \frac{\omega_2}{\omega_1}$ chosen to be in the upper half of the complex plane. If one chooses coordinates in which $\omega_1 = 1$, one gets a torus with periods 1 and τ (see Fig. 2). τ is known as the modular parameter and each different τ represents a conformally distinct torus, up to a subtlety that will be discussed later.

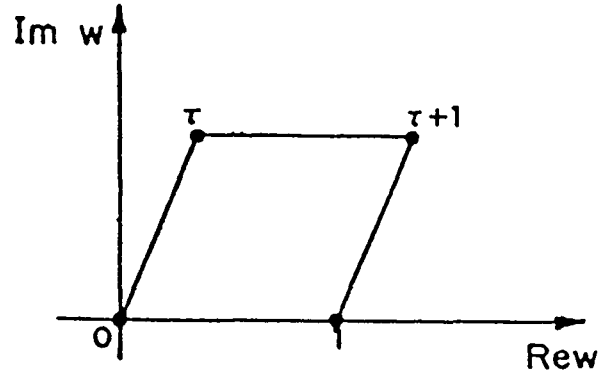

Figure 2: A torus with modular parameter τ.

To define a CFT on the torus one has to define operators, especially the stress-energy tensor, on the torus. We want to make use of the formalism developed for the plane (or sphere). To do that one first cuts the torus along one of its geodesics, say

ω_1. In this way we get a cylinder that can be mapped to an annulus in the plane by the exponential map (see Fig. 3)

$$w \longrightarrow z = e^{\frac{2\pi i}{\omega_1} w}. \tag{2.2.1}$$

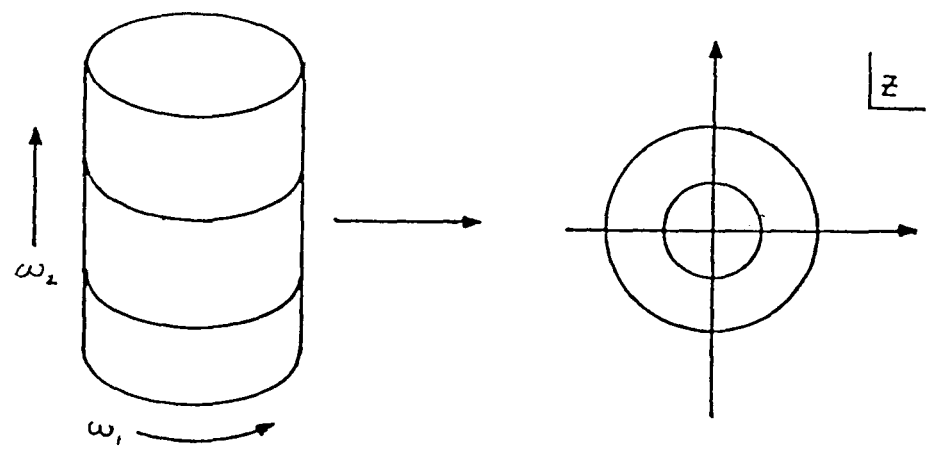

Figure 3: Mapping the cylinder to the plane with (2.2.1).

By this mapping the translation operator along ω_2, the 'Hamiltonian,' gets transformed into a combination of dilation and rotation operators on the plane. To find this transformation explicitly recall that under conformal transformations, $T(z)$ transforms according to equation (2.1.8). That is, $T(z)$ is *not* a primary field (see equation (2.1.11)) and, hence, writing a similar expression to (2.1.12) one has

$$T(w) = T(z)\left(\frac{\partial z}{\partial w}\right)^2 + \frac{c}{12}\frac{\frac{\partial z}{\partial w}\frac{\partial^3 z}{\partial w^3} - \frac{3}{2}\left(\frac{\partial^2 z}{\partial w^2}\right)^2}{\left(\frac{\partial z}{\partial w}\right)^2}. \tag{2.2.2}$$

The second term is the conformal anomaly term (Schwarzian derivative) which is proportional to the central charge c. Plugging (2.2.1) into (2.2.2) the transformation law for $T(z)$ is obtained

$$T^{cylinder}(w) = -\frac{4\pi^2}{\omega_1^2}\left(T^{plane}(z)z^2 - \frac{c}{24}\right). \tag{2.2.3}$$

Now the Hamiltonian can be found. The generator that corresponds to translations in the mode expansion of $T(z)$

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \tag{2.2.4}$$

is L_{-1}. So computing $L_{-1}^{cylinder}$ one finds

$$\begin{aligned} L_{-1}^{cyl} &= \frac{1}{2\pi i}\oint_{geodesic\ \omega_1} T^{cyl}(w)dw \\ &= \frac{1}{\omega_1}\oint \left(T^{pl}(z)z^2 - \frac{c}{24}\right)\frac{dz}{z} \\ &= \frac{2\pi i}{\omega_1}\left(L_0^{pl} - \frac{c}{24}\right). \end{aligned} \tag{2.2.5}$$

Therefore, the translation along ω_2 is*

$$\begin{aligned} &\omega_2 H + \bar{\omega}_2 \bar{H} = \omega_2 L_{-1}^{cyl} + \bar{\omega}_2 \bar{L}_{-1}^{cyl} \\ &= 2\pi i\left(\tau\left(L_0^{pl} - \frac{c}{24}\right) + \bar{\tau}\left(\bar{L}_0^{pl} - \frac{\bar{c}}{24}\right)\right). \end{aligned} \tag{2.2.6}$$

The next stage is to define the partition function on the torus, or the zero-point function, to be

$$\begin{aligned} Z(\tau) &= \mathrm{Tr}\, e^{\omega_2 H + \bar{\omega}_2 \bar{H}} \\ &= q^{-\frac{c}{24}}\bar{q}^{-\frac{\bar{c}}{24}}\,\mathrm{Tr}\, q^{L_0}\bar{q}^{\bar{L}_0}, \end{aligned} \tag{2.2.7}$$

where the trace is, as usual, over all states, and we have defined

$$q = e^{2\pi i\tau}. \tag{2.2.8}$$

Now, as the Hilbert space decomposes into irreducible representation of the Virasoro algebra, $\mathcal{V}\otimes\bar{\mathcal{V}}$ (or, in general, the chiral algebra $\mathcal{A}\otimes\bar{\mathcal{A}}$), one can decompose the trace in equation (2.2.7) to a sum

$$Z(\tau) = \sum_{h,\bar{h}} N_{h,\bar{h}}\chi_h(\tau)\chi_{\bar{h}}^*(\tau), \tag{2.2.9}$$

where the character $\chi_h(\tau)$ is defined by

$$\chi_h(\tau) = \mathrm{Tr}\, e^{2\pi i\tau\left(L_0 - \frac{c}{24}\right)}. \tag{2.2.10}$$

The trace here is carried over the highest-weight representation with highest weight h. The coefficients $N_{h,\bar{h}}$ must be non-negative integers as they count how many times a given representation appears in the theory. Also, $N_{0,0}$ is always 1, to express the fact the the identity operator (or vacuum) is unique.

*Note that we include the antiholomorphic part here!

We now return to the subtlety alluded to above. In calculating the partition function, the two periods, ω_1 and ω_2, were not treated on the equal footing. The periodicity in ω_2 was taken care of by the trace, but what about ω_1? More generally, one should associate $Z(\tau)$ to a lattice spanned by those periods and not to the periods themselves. (Or, equivalently, the original torus can be defined with other periods that are associated to the original periods.) The result is that this kind of lattice is invariant under unimodular transformations of the periods. Or, equivalently, the partition function should be invariant under modular transformations of the modular parameter τ:

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \tag{2.2.11}$$

where a, b, c and d are integers that satisfy $ad - bc = 1$.

These transformations form a discrete infinite group called the modular group (usually denoted Γ and is isomorphic to $PSL(2,\mathbf{Z}) \cong SL(2,\mathbf{Z})/\mathbf{Z}_2$). This group is generated by the two transformations $T : \tau \to \tau + 1$ and $S : \tau \to -\frac{1}{\tau}$. In Fig. 4 the transformations T and $U = T^{-1}ST^{-1}$ $(\tau \to \tau/(\tau + 1))$ are shown.

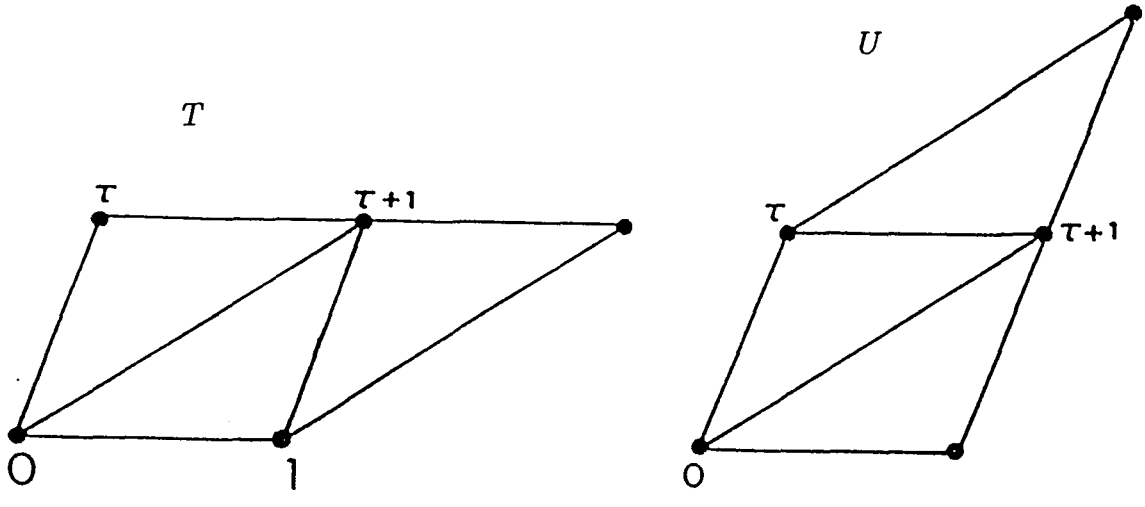

Figure 4: The modular transformations T and U.

It turns out that the requirement of modular invariance of $Z(\tau)$ is very restrictive and, in general, quite hard to satisfy. Specifically, it restricts the $N_{h,\bar{h}}$ that are allowed in (2.2.9).

2.2. Some Simple Consequences of Modular Invariance

To examine the consequences of modular invariance let us see how the characters transform under a modular transformation. It is enough to check the transformation properties under the two basic transformations mentioned at the end of the previous section. Usually those transformations are denoted T and S,

$$T : \tau \to \tau + 1 \tag{2.2.12}$$

$$S : \tau \to -\frac{1}{\tau}. \tag{2.2.13}$$

T will be examined first. When $\tau \to \tau + 1$,

$$\begin{aligned}
\chi_h(\tau) &\to \chi_h(\tau+1) \\
&= \mathrm{Tr}\, e^{2\pi i(\tau+1)\left(L_0 - \frac{c}{24}\right)} \\
&= e^{-\frac{\pi i c}{12}} \sum_i \langle \phi_i | e^{2\pi i \tau \left(L_0 - \frac{c}{24}\right)} e^{2\pi i L_0} | \phi_i \rangle \\
&= e^{-\frac{\pi i c}{12}} \sum_i \langle \phi_i | e^{2\pi i \tau \left(L_0 - \frac{c}{24}\right)} e^{2\pi i (h+n_i)} | \phi_i \rangle \\
&= e^{-\frac{\pi i c}{12} + 2\pi i h} \sum_i \langle \phi_i | e^{2\pi i \tau \left(L_0 - \frac{c}{24}\right)} | \phi_i \rangle \\
&= e^{-\frac{\pi i c}{12} + 2\pi i h} \chi_h(\tau),
\end{aligned} \tag{2.2.14}$$

where the sum is over all the states in the module based on the primary field $|h, \bar{h}\rangle$[†], and n_i is the level of the descendant. As discussed in Section 1.2 it has been assumed that the modes of the symmetry operators that are used to generate the representation are integers.

Substituting (2.2.14) into the partition function (2.2.9),

$$Z(\tau+1) = \sum_{h\bar{h}} N_{h,\bar{h}} e^{-\frac{2\pi i}{24}(c-\bar{c})} e^{2\pi i (h-\bar{h})} \chi_h(\tau) \chi_{\bar{h}}^*(\tau). \tag{2.2.15}$$

Now, since a CFT always has $N_{0,0} = 1$, one gets

$$c - \bar{c} = 0 (\bmod 24) \tag{2.2.16}$$

and for each nonzero $N_{h,\bar{h}}$

$$h - \bar{h} = 0 (\bmod 1). \tag{2.2.17}$$

[†] In general, the chiral algebra is not just the Virasoro algebra and the primaries are primary fields with respect to the full chiral algebra.

Hence, one concludes that if one wants a consistent conformal field theory on the torus, then the only possible primary fields are those with integral difference between their holomorphic and antiholomorphic conformal dimensions. Also, we see that $c-\bar{c}$ must be a multiple of 24. Frequently, one chooses the chiral and anti-chiral parts of the symmetry algebra to be isomorphic, and then their central charge is equal. However, there are examples, based on Kač-Moody algebras, with non-isomorphic algebras. This will be discussed some more later. The heterotic string is another example in which one needs different central charges for the left and right movers.

Next, we look into the consequences of the other generator of modular transformations, S. One cannot immediately say what the transformation rule is, but the characters must transform among themselves. In other words, the characters carry a linear representation of the modular group. That is, when $\tau \to -\frac{1}{\tau}$,

$$\chi_h(\tau) \to \chi_h(-\frac{1}{\tau}) = \sum_{h'} S_h{}^{h'} \chi_{h'}(\tau), \tag{2.2.18}$$

where the sum is over all the characters. Using (2.2.18) one gets, in matrix form, the following condition on the values of $N_{h,\bar{h}}$ in the partition function

$$S^\dagger N S = N. \tag{2.2.19}$$

The next step is to find all the matrices N that satisfy this constraint.

Easier said then done. Usually the form of S is not known and even if known it is very complicated. Indeed, we can find some simple solutions to (2.2.19), but the problem of finding *all* the solutions has been solved in only a few cases. For example, if the underlying symmetry is a Kač-Moody algebra, then the matrix S is known. It is given by the famous Weyl-Kač formula[6]. In that case, S is unitary (indeed, we will explain later that S is always unitary) and, hence, one possibility is always the so-called diagonal solution specified by $N_{h,\bar{h}} = \delta_{h,\bar{h}}$, where δ is the Kronecker delta. 'Sporadic' solutions, which are not diagonal, have been found for various algebras[12,67–72], but a complete classification exists only for $SU(2)$.

2.3. The ADE Classification and Other Theories

The ADE classification[72,73] demonstrates the complexity of finding modular invariant partition functions. The fact that to date only $SU(2)$—the simplest Kač-Moody algebra—has been classified shows how complicated the problem is.

A_{k+1}	$\sum_{\lambda=1}^{k+1}\|\chi_\lambda\|^2$	$k \geq 1$
$D_{2\rho+2}$	$\sum_{\lambda\text{odd}=1}^{2\rho-1}\|\chi_\lambda + \chi_{4\rho+2-\lambda}\|^2 + 2\|\chi_{2\rho+1}\|^2$	$k = 4\rho, \rho \geq 1$
$D_{2\rho+1}$	$\sum_{\lambda\text{odd}=1}^{4\rho-1}\|\chi_\lambda\|^2 + \|\chi_{2\rho}\|^2$ $+ \sum_{\lambda\text{even}=2}^{2\rho-2}(\chi_\lambda\chi^*_{4\rho-\lambda} + c.c.)$	$k = 4\rho - 2,\ \rho \geq 2$
E_6	$\|\chi_1+\chi_7\|^2 + \|\chi_4 + \chi_8\|^2 + \|\chi_5 + \chi_{11}\|^2$	$k = 10$
E_7	$\|\chi_1+\chi_{17}\|^2 + \|\chi_5 + \chi_{13}\|^2 + \|\chi_7 + \chi_{11}\|^2$ $+\|\chi_9\|^2 + [(\chi_3 + \chi_{15})\chi_9^* + c.c]$	$k = 16$
E_8	$\|\chi_1+\chi_{11} + \chi_{19} + \chi_{29}\|^2$ $+\|\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}\|^2$	$k = 28$

Table 1: Modular invariant partition functions for $SU(2)^{(k)}$.

The CFT based on $SU(2)$ at level k has $k+1$ primary fields of dimensions $h = 0, \frac{1}{2}, \ldots, \frac{k}{2}$. Letting $\lambda = 2h+1$ and $N = 2(k+2)$, the characters are

$$\chi_\lambda(\tau) = \frac{1}{\eta^3(\tau)} \sum_{n=-\infty}^{\infty} (nN + \lambda) e^{\pi i \tau \frac{(nN+\lambda)^2}{N}}, \tag{2.2.20}$$

where $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n \tau}\right). \tag{2.2.21}$$

Then the matrix S has the components

$$S_\lambda{}^{\lambda'} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi\lambda\lambda'}{k+2}\right). \tag{2.2.22}$$

It has been shown that the possible modular invariant partition functions for $SU(2)$ are classified in the so-called ADE classification. They are presented in Table 1.

Some more conformal field theories (that are modular invariant) will be presented next. They will be especially useful later when we will try to identify the models derived by the classification method with known conformal field theories.

A useful procedure to derive more CFT's is by taking products of already-known conformal field theories. One should treat such a product as a direct tensor product. The number of primary fields in the product theory is the product of the number of fields in the factor theories, their conformal dimensions is the sum of their conformal dimensions, and the central charge of the product theory is the sum of the central charges of the factor theories.

The first example is E_8-based CFT. At level 1 there is only one primary field—the identity field (with conformal dimension 0). The central charge is 8. It has been proved, though, that χ_0^3 (the character cubed) is a modular invariant by itself![‡] That is, the conformal field theory based on $(E_8)^3$ at level 1 as the chiral part of the chiral algebra and the identity[§] in the antichiral part is a consistent CFT. Note that $c - \bar{c} = 24 - 0 = 0 (\mathrm{mod}\, 24)$, as it must be. An important observation here is that given a conformal field theory, one can generate more conformal field theories with the *same* number of fields with the same conformal dimensions just by multiplying by $(E_8^{(1)})^3_{chiral} \times 1_{antichiral}$. The only difference is that the chiral central charge, c, is increased by 24. Similarly, one can multiply both chiral and antichiral parts by the same power of E_8 level 1, and the number and conformal dimensions of the primary fields will not change (but c and $\bar{c}$ will).

[‡] The character is a modular form, as it must transform into itself under modular transformations (there are no other characters to transform to). Thus, if one cubes the character, it can be shown that the answer is unique—it is the modular invariant function j[74].

[§]The identity theory (the trivial theory) has just the Virasoro algebra as a chiral algebra and has a single primary field—the identity. The difference between E_8 and 1 is the chiral algebra—that is, there are other generators that one can use to get descendants. That means that the identity modules (which are in fact the whole theory) are *not* the same.

3. The Classification Scheme

The classification scheme is based on a remarkable formula found by Verlinde[21].

3.1. Correlation Functions and Verlinde's Formula

When one computes correlation functions on the sphere, one gets, from conformal symmetry, unique expressions. For the two-point function of two primary fields one can show that

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{1,2}\delta_{h_1,h_2}}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}}, \tag{2.3.1}$$

where $C_{1,2}$ is a constant, $z_{12} = z_1 - z_2$, and $h = h_1 = h_2$ (and similarly for $\bar{z}_{12}$ and $\bar{h}$) because of the Kronecker delta. Usually, one normalizes $C_{1,1} = 1$, *i.e.*, a field is conjugate to itself. However, sometimes one gets a conformal field theory with a non-trivial 'charge conjugation' matrix. Consider an example. The CFT based on the Kač-Moody algebra $SU(3)$ at level 1 has 3 primary fields. They are, in the conventional notation for $SU(3)$, **1**, **3** and $\bar{\mathbf{3}}$. As is well known, the product of **3** and $\bar{\mathbf{3}}$ gives the singlet and the octet representations of $SU(3)$. In the conformal field theory case, since there is no octet at level 1, it is just 'ignored.' Similarly, $\mathbf{3}\times\mathbf{3}$ that ordinarily gives a $\bar{\mathbf{3}}$ and a **6** has to be truncated. That is, the fusion rules for $SU(3)^{(1)}$ are

$$\mathbf{3}\times\mathbf{3} = \bar{\mathbf{3}} \tag{2.3.2a}$$

$$\mathbf{3}\times\bar{\mathbf{3}} = \mathbf{1} \tag{2.3.2b}$$

$$\bar{\mathbf{3}}\times\bar{\mathbf{3}} = \mathbf{3}. \tag{2.3.2c}$$

Thus, the **3** and $\bar{\mathbf{3}}$ are conjugate to each other. Generically, one constructs a charge conjugation matrix, C, that shows which fields are conjugate to which. For example, in the $SU(3)$ case,

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.3.3}$$

where the rows and columns correspond to **1**, **3** and $\bar{\mathbf{3}}$. The consequences of charge conjugation symmetry will be obvious in a moment. Let us write, again, the fusion algebra

$$\phi_i \times \phi_j = N_{ij}{}^k \phi_k. \tag{2.3.4}$$

Now, suppose that ϕ_i is conjugate to itself. Then, from the definition of a CFT, one gets

$$\phi_i \times \phi_i = \phi_0 + \text{some other fields.} \tag{2.3.5}$$

ϕ_0 is the identity field and it will be denoted that way from now on. This means that $N_{ii}{}^0 = 1$ if ϕ_i is self-conjugate. If, on the other hand, ϕ_i is not self-conjugate but is conjugate to, say, $\phi_{\bar{\imath}}$, then we get $N_{i\bar{\imath}}{}^0 = 1$. We can use the charge conjugation matrix, C, (and its inverse, which is the same thing) to raise and lower indices of $N_{ij}{}^k$ (and also of $S_i{}^j$).

We are now in the position to quote Verlinde's formula[21]. It gives a relation between S_{ij} and N_{ijk} that reads as follows

$$N_{ijk} = \sum_{n=0}^{N-1} \frac{S_{in} S_{jn} S_{kn}}{S_{0n}}, \tag{2.3.6}$$

where N is the number of primary fields. This has been proved by Dijkgraaf and Verlinde[22] using the so-called pentagonal identity[12]. They also used this formula to prove that the matrix S is symmetric and unitary.

If the RCFT is unitary, there is another fact one can prove. A unitary theory is a theory in which the norms of the states are positive. If one looks at the norm $\|L_{-1}|h\rangle\|^2$, for any primary state $|h\rangle$, then from the commutation relations (2.1.10) we easily get $h > 0$ (if we assumed that $|h\rangle$ has positive norm). It follows that all the conformal dimensions in a unitary theory must be positive (or zero for the identity). Similarly, one can show that c is positive. The fact that was proven by Dijkgraaf and Verlinde[22] is that for a unitary theory the first row (and column) of the S matrix must have the same sign. That is,

$$\frac{S_{i0}}{S_{00}} = \lambda_i^{(0)} > 0 \quad \forall i. \tag{2.3.7}$$

Conversely, if the signs are not the same, the theory is non-unitary. In general, for any S matrix, the first row (and column) cannot vanish (or else, (2.3.6) would not make sense).

3.2. The Scheme and One-Field Theories

We can now present our method of classification. For a given number of primary fields, one writes the most general fusion rules and S matrix possible. Then one uses Verlinde's formula to investigate the constraints on both S_{ij} and N_{ijk}.

A few remarks are in place here. First, if one applies the S transformation twice, then one should get the identity transformation. This is almost true. If the charge conjugation matrix is trivial then, indeed, $S^2 = 1$. However, if $C \neq 1$ then one gets $S^2 = C$. This can be explained by the fact that the characters are identical for a field and its conjugate. (*e.g.*, for $SU(3)^{(1)}$, $\chi_3 = \chi_{\bar{3}}$.) The relation $S^2 = 1$, or its generalization $S^2 = C$, is one of the defining relations of the modular group.

Second, one can make use of the other defining relation of the modular group, which is $(ST)^3 = 1$ (or the generalization $(ST)^3 = C$). This will turn out to be quite important in determining T. One should remember that the knowledge of T is the knowledge of the central charge and the conformal dimensions of the primary fields (modulo integers), as is obvious from the T transformation properties of $\chi_i(\tau)$ (see equation (2.2.14)).

Another point to notice is the following fact due to Vafa (as reported by Verlinde[21]). The point i in the complex τ plane is a fixed point of the S transformation. That is,

$$S : \tau = i \rightarrow -\frac{1}{\tau} = i. \tag{2.3.8}$$

This means that

$$S : \chi_j(i) \rightarrow \chi_j(i) = S_j{}^k \chi_k(i), \tag{2.3.9}$$

or, that the N dimensional vector (of numbers) $(\chi_j(i))$ is an eigenvector of the S matrix with eigenvalue 1. Moreover, the components of the eigenvector must be all positive or all negative! This can be seen by plugging i into the definition of the character, (2.2.10), and noting that $e^{-2\pi(L_0 - c/24)}$ is always positive. This is quite important, as one can just demand that S has such eigenvector. This further restricts the possible S matrices.

Two last simple remarks are the following. In the case that the charge conjugation is trivial, we have $S^2 = 1$ and, because S is a unitary matrix, we have $S^\dagger S = 1$. This means that S is hermitian. So, since S is also symmetric, it must be real! The other fact is that if S and T are a given solution, then $-S$ and $-T$ are also a solution. One of those solutions can be eliminated using Vafa's argument from the previous

paragraph. Usually, it is not easy to know in advance which of the solutions is the correct one. Unitarity is usually also fixed by the classification method, *i.e.*, some solutions are constrained enough to decide whether the theory is unitary or not.

Let us demonstrate the classification method for the somewhat trivial case of one primary field. In this case, one does not gain information from the fusion algebra as it consists only of the trivial rule

$$\phi_0 \times \phi_0 = \phi_0. \tag{2.3.10}$$

That is, $N_{000} = 1$ is the only element. Also, as is immediately clear, $C = 1$ and S is a 1×1 matrix. Then, from the condition $S^2 = C = 1$, we get $S = \pm 1$. Vafa's argument can now be used to restrict S to be 1. (-1 does not have a positive eigenvalue!) Next, one computes T by $(ST)^3 = (1T)^3 = T^3 = C = 1$ and gets $3(h - \frac{c}{24}) = 0 (\mathrm{mod}\, 1)$, and, since $h = 0$ (identity field!), one gets $c = 0 (\mathrm{mod}\, 8)$. These theories can be identified with the trivial theory and powers of the E_8 level 1 theory, as mentioned in previous sections. One notes that there are no other possible values for the central charge in a one-field rational conformal field theory. It is also seen that those theories are unitary.

4. The Two-Field Case

4.1. The S and T Matrices and the Fusion Algebra

When there are two primary fields, one of them must be the identity field with conformal dimension 0. The other should be different with $h \neq 0$. Hence, the charge conjugation matrix must be trivial—the identity matrix. The fusion algebra for a two-field RCFT can be written, in the most general form, as

$$\begin{aligned} \phi_0 \times \phi_0 &= \phi_0 \\ \phi_0 \times \phi_1 &= \phi_1 \\ \phi_1 \times \phi_1 &= \phi_0 + n\ \phi_1, \end{aligned} \tag{2.4.1}$$

where n is a non-negative integer. Most of the N_{ijk} are determined by associativity and commutativity of the fusion algebra and the fact that ϕ_0 is the identity field.

Making use of the symmetry of S_{ij} and the fact that it's real, one has

$$S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{2.4.2}$$

where a, b and c are real numbers. Then, substituting into equation (2.3.6), one gets the following constraints

$$N_{000} = 1 = a^2 + b^2 \tag{2.4.3a}$$

$$N_{001} = 0 = ab + bc \tag{2.4.3b}$$

$$N_{011} = 1 = b^2 + c^2 \tag{2.4.3c}$$

$$N_{111} = n = \frac{b^3}{a} + \frac{c^3}{b}. \tag{2.4.3d}$$

The numbers a and b cannot vanish, as mentioned after (2.3.7), and, hence, from (2.4.3b) we get $c = -a$. Equations (2.4.3a) and (2.4.3c) are then identical. Equation (2.4.3d) is then solved and gives

$$\frac{1}{a^2} = \frac{4+n^2}{2} \pm \frac{n}{2}\sqrt{4+n^2}. \tag{2.4.4}$$

This representation of the solution will be convenient later.

Letting

$$T = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \tag{2.4.5}$$

and substituting into the constraint $(ST)^3 = 1$, one gets

$$(ST)^3 = \begin{pmatrix} aX(2XY - Y^2 + a^2(X-Y)^2) & bY(XY + a^2(X-Y)^2) \\ bX(XY + a^2(X-Y)^2) & -aY(2XY - X^2 + a^2(X-Y)^2 \end{pmatrix} = 1. \tag{2.4.6}$$

From the off-diagonal terms—remembering that a, b, X and Y cannot vanish—one gets $XY = -a^2(X-Y)^2$. Substituting into the diagonal terms yields the single equation $aXY(X-Y) = 1$. Solving for X and Y, we get

$$X = (-1)^{\frac{1}{3}} \left(\frac{1}{2a} \pm i\sqrt{1 - \frac{1}{4a^2}} \right) \tag{2.4.7}$$

and

$$Y = (-1)^{\frac{1}{3}} \left(-\frac{1}{2a} \pm i\sqrt{1 - \frac{1}{4a^2}} \right). \tag{2.4.8}$$

To further restrict the solution, one looks at the eigenvector of S that belongs to the $+1$ eigenvalue. This eigenvector is

$$\vec{v}_{(+1)} = \xi \begin{pmatrix} b \\ 1-a \end{pmatrix}, \tag{2.4.9}$$

where ξ is some number. There must be a same-sign component eigenvector for a consistent theory. This will determine the signs of a and b. For a unitary theory a and b are of the same sign. Then, by looking at (2.4.9), one sees that only if a (and b) are positive one gets $b/(1-a) > 0$*. Conversely, if a and b have opposite signs (*i.e.*, a non-unitary theory) then inspecting (2.4.9) shows that $a < 0$ and $b > 0$ is the consistent choice. We conclude that b must be positive in any case, unitary or not.

4.2. The Solution

For $n = 0$, there's only one solution for b which is $b = \frac{1}{\sqrt{2}}$. Then the S matrix has two solutions,

$$S_{\pm} = \begin{pmatrix} \pm\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \mp\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{2.4.10}$$

*Remember that $|a|$ and $|b|$ are less then 1, as can be seen from $a^2 + b^2 = 1$.

S_- is the case for a non-unitary theory and S_+ applies to unitary theories. Let's start with the unitary case.

Plugging into (2.4.7) we get (remembering that $X = e^{-\frac{\pi i c}{12}}$ from (2.2.14)) $c = 1(\bmod 8)$ for the plus sign and $c = 7(\bmod 8)$ for the minus. Plugging into (2.4.8) gives the corresponding values for the conformal dimension, $h = \frac{1}{4}(\bmod 1)$ for the plus and $\frac{3}{4}(\bmod 1)$ for the minus. A quick search finds known models with these values. The model based on $SU(2)^{(1)}$ has $c = 1$ and $h = \frac{1}{4}$, and the model based on $E_7^{(1)}$ has $c = 7$ and $h = \frac{3}{4}$. The fact that our result is modulo 8 for the central charge is not surprising in view of our remark on powers of the E_8 level 1 theory.

The non-unitary case is handled similarly. We get two possible solutions, either $c = -3$ and $h = -\frac{3}{4}$ or $c = -5$ and $h = -\frac{1}{4}$. These numbers are $\bmod 8$ and $\bmod 1$, respectively, but we suggestively wrote them as negative, reminding ourselves that those theories are non-unitary.

For $n = 1$ we get two possibilities for b,

$$b = \sqrt{\frac{5 \pm \sqrt{5}}{10}}, \tag{2.4.11}$$

and for each b we have a unitary and a non-unitary solution. Starting with the larger b (with the 'plus' sign), we get for the unitary case two solutions:

$$(c, h) = (\frac{14}{5}, \frac{2}{5}) \text{ or } (\frac{26}{5}, \frac{3}{5}). \tag{2.4.12}$$

These models can be identified with $G_2^{(1)} \otimes \left(E_8^{(1)}\right)^l$ and $F_4^{(1)} \otimes \left(E_8^{(1)}\right)^l$, where l is some non-negative integer. For the non-unitary models we get

$$(c, h) = (-\frac{6}{5}, -\frac{3}{5}) \text{ or } (-\frac{34}{5}, -\frac{2}{5}). \tag{2.4.13}$$

For the other value of b, for the unitary models, we get

$$(c, h) = (\frac{2}{5}, \frac{1}{5}) \text{ or } (\frac{38}{5}, \frac{4}{5}), \tag{2.4.14}$$

and for the non-unitary theories we get

$$(c, h) = (-\frac{18}{5}, -\frac{4}{5}) \text{ or } (-\frac{22}{5}, -\frac{1}{5}).$$

The second model can be identified with the Lee-Yang singularity[75]—the (2,5) model of the minimal model series (see (2.1.24) and (2.1.25)).

Let us note that there is a natural pairing of the solutions, as was apparent above. The solutions are paired into couples whose central charges sum to 8 and whose conformal dimensions sum to 1. This can be traced back to the formula for X. One may define an angle α by

$$\cos\alpha = \frac{1}{2a}. \tag{2.4.15}$$

Then X takes the simple form

$$X = (-1)^{\frac{1}{3}} e^{\pm i\alpha}. \tag{2.4.16}$$

The pairing is due to the $\pm$ in the exponent. (By the way, the other factor, $(-1)^{1/3}$, is responsible for the modulo 8 freedom for c.) Based on the observation that most paired models above do indeed exist, one is tempted to conjecture that the partner to the Lee-Yang singularity exists also. We were not able to find this model and the problem seems hard.

Proceeding to $n \geq 2$, one notes immediately that one of the solutions for a presents a problem. The solution with the plus sign now satisfies

$$\frac{1}{2a} = \pm\frac{1}{2}\sqrt{\frac{4+n^2}{2} + \frac{n}{2}\sqrt{4+n^2}} > 1 \quad \text{for} \quad n \geq 2. \tag{2.4.17}$$

This means that X no longer has absolute value 1. The negative solution is fine, but if one tries to compute its central charge and conformal dimension, one apparently gets irrational values. This was proved by Caselle and Ponzano[76].

According to our scheme, there is no reason to reject irrational solutions. However, one can use some more knowledge of conformal field theory to restrict these values further. This is accomplished by investigating the analyticity properties of the four-point function. It can be proved that in rational conformal field theories the conformal dimensions (and, hence, the central charge) must be rational. This was first proved by Vafa[77]. This result can even be extended to give a restriction on the form of h. It was shown by Christe and Ravanini[15] that in a two-field RCFT, the non-trivial conformal dimension must satisfy

$$(n+4)h = 0 (\bmod 1), \tag{2.4.18}$$

where n is the integer that appears in the fusion rule (2.4.1). Thus, in our classification, we will ignore the irrational models, since they do not make sense as rational conformal field theories. We have to remember, however, that we used the analyticity of the four-point function.

One last point is that for $n = \infty$ we again get a rational solution! In this case $a = 1$ so $b = 0$, in contradiction to our demand that a and b will not vanish. However, one may be tempted to identify this as some degeneration of two-field theories. One finds that one of these models has central charge 8 and conformal dimension $\frac{5}{6}$. There is a model with one field and central charge 8, the E_8 level 1 model[13]. However, the conformal dimension of the field is (obviously) 0. In any case, this theory does not make sense as a two-field theory.

Table 2 lists all the rational solutions for the two-field theories. This includes the unphysical solution of $n = \infty$.

n	$h(\mathrm{mod}\,1)$	$c(\mathrm{mod}\,8)$	Identified model
0	$\frac{1}{4}$	1	A_1 level 1
0	$\frac{3}{4}$	7	E_7 level 1
0	$-\frac{1}{4}$	-5	unknown
0	$-\frac{3}{4}$	-3	unknown
1	$\frac{2}{5}$	$\frac{14}{5}$	G_2 level 1
1	$\frac{3}{5}$	$\frac{26}{5}$	F_4 level 1
1	$-\frac{2}{5}$	$-\frac{34}{5}$	unknown
1	$-\frac{3}{5}$	$-\frac{6}{5}$	unknown
1	$\frac{1}{5}$	$\frac{1}{5}$	unknown
1	$\frac{4}{5}$	$\frac{38}{5}$	unknown
1	$-\frac{1}{5}$	$-\frac{22}{5}$	Lee–Yang singularity
1	$-\frac{4}{5}$	$-\frac{18}{5}$	unknown
∞	$\frac{1}{6}$	0	unphysical
∞	$\frac{5}{6}$	8	unphysical

Table 2: Possible two-field rational solutions

The final comment is that the reasoning may be inverted to give a classification of the fusion algebras. It is obvious from the analysis above that $n \geq 2$ gives an inconsistent rational conformal field theory. The bottom line is that there are only two possible fusion algebras for a two-field RCFT. One has the following non-trivial rule

$$\phi_1 \times \phi_1 = \phi_0, \tag{2.4.19}$$

and the other has

$$\phi_1 \times \phi_1 = \phi_0 + \phi_1. \tag{2.4.20}$$

5. The Three-Field Case

In the case of two non-identity fields, we can have a non-trivial charge conjugation matrix of the form

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.5.1}$$

We will deal with the two cases separately, starting with the non-trivial charge conjugation. Surprisingly, this case is actually simpler then the self-conjugate case, even without taking into account the fact that the conformal dimensions of the two non-identity fields are the same.

5.1. Non-trivial Charge Conjugation Matrix ($C \neq 1$)

Writing a symmetric matrix for S we can solve the matrix equation $S^2 = C$. One gets two solutions,

$$S_{\pm} = \begin{pmatrix} a & b & b \\ b & -\frac{a}{2} \pm \frac{i}{2} & -\frac{a}{2} \mp \frac{i}{2} \\ b & -\frac{a}{2} \mp \frac{i}{2} & -\frac{a}{2} \pm \frac{i}{2} \end{pmatrix}, \tag{2.5.2}$$

where a and b are complex numbers subject to

$$a^2 + 2b^2 = 1. \tag{2.5.3}$$

We note that S is *not* hermitian or real. Taking unitarity into account, one gets the extra condition

$$|a|^2 + 2|b|^2 = 1. \tag{2.5.4}$$

These two conditions, (2.5.3) and (2.5.4), are compatible only if a and b are real.

The eigenvector with eigenvalue $+1$ is proportional to

$$\vec{v}_{(+1)} = \begin{pmatrix} 2b \\ 1-a \\ 1-a \end{pmatrix}, \tag{2.5.5}$$

and since $1 - a$ is always positive, b must be positive in order to get a same-sign component eigenvector. (Again, $|a| < 1$ or else b would vanish.) Hence, for a unitary theory a must be positive, and for a non-unitary one a could be negative.

Using (2.3.6) one finds the values of N_{ijk} and, raising the last index, one finds the following form for the fusion algebra

$$\begin{aligned} \phi_1 \times \phi_1 &= m\phi_1 + n\phi_2 \\ \phi_1 \times \phi_2 &= \phi_0 + m\phi_1 + m\phi_2 \\ \phi_2 \times \phi_2 &= n\phi_1 + m\phi_2, \end{aligned} \tag{2.5.6}$$

where

$$m = \frac{1 - 3a^2}{\sqrt{8}a\sqrt{1 - a^2}} \tag{2.5.7}$$

and

$$n = \frac{1 + a^2}{\sqrt{8}a\sqrt{1 - a^2}}. \tag{2.5.8}$$

We note that, as required, these fusion rules are symmetric under charge conjugation, *i.e.*, under $\phi_1 \leftrightarrow \phi_2$.

The coefficients m and n must be non-negative integers, and, thus, we can restrict the solution further. Solving for a in terms of the ratio $\frac{m}{n}$ and substituting the result back into n, one easily sees that there is only one solution: $m = 0$ and $n = 1$. The value of b is $\frac{1}{\sqrt{3}}$*.

We now write a general form for T, remembering that $h_1 = {h_2}^\dagger$,

$$T = \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Y \end{pmatrix}, \tag{2.5.9}$$

and then solve the condition $(ST)^3 = C$. For the unitary case (*i.e.*, positive a) we get only one solution for each S matrix. For S_+ we get $c = 2$ and $h = \frac{1}{3}$ and for S_- we get $c = 6$ and $h = \frac{2}{3}$. These can be identified as the $SU(3)^{(1)}$ and $E_6^{(1)}$ Kač-Moody models, respectively.

The non-unitary case (negative a) is also easy to solve, though we were not able to identify the models. The results are summarized in Table 3.

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*From crossing symmetry of the four-point function one derives the condition $m^2 + 1 = n^2$, which has only one solution, the one given above. So, in fact, modular invariance is not really needed to classify this fusion algebra.

† Actually, one can show that for a general choice, with $h_1 \neq h_2$, we get an inconsistency—*i.e.*, $(ST)^3 \neq C$.

$h_1 \equiv h_2 \pmod 1$	$c \pmod 8$	Identified model
$\frac{1}{3}$	2	A_2 level 1
$\frac{2}{3}$	6	E_6 level 1
$-\frac{2}{3}$	-2	unknown
$-\frac{1}{3}$	-6	unknown

Table 3: The solution for three-field theory—non-trivial C.

We stress that in this case (of non-trivial C) we completely classified the central charge and conformal dimensions and also the fusion algebra. The only fusion algebra possible is

$$\begin{aligned} \phi_1 \times \phi_1 &= \phi_2 \\ \phi_1 \times \phi_2 &= \phi_0 \\ \phi_2 \times \phi_2 &= \phi_1. \end{aligned} \tag{2.5.10}$$

5.2. Trivial Charge Conjugation Matrix ($C = 1$)

When $C = 1$, one has $S^\dagger S = S^2 = 1$, and, hence, S is hermitian. Since S is also symmetric, it must be real. Let us write a general form for a symmetric S, as follows

$$S = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \tag{2.5.11}$$

where a, b, ..., and f are real. This matrix is then subject to the equations $S^2 = 1$. A convenient change of variables is the following[‡]

$$b = 2yz \ , \ c = 2xz \ , \ e = 2xy. \tag{2.5.12}$$

Substituting this change of variables into the equation $S^2 = 1$ immediately leads to the solution

$$\begin{aligned} a &= -x^2 - y^2 + z^2 \\ d &= -x^2 + y^2 - z^2 \\ f &= +x^2 - y^2 - z^2, \end{aligned} \tag{2.5.13}$$

[‡]This change is legitimate. It is easy to check that these equations are independent.

subject to the extra condition

$$\left(x^2+y^2+z^2\right)^2=1. \tag{2.5.14}$$

This means that x, y and z are either all real or all purely imaginary since, by definition, xy, yz and xz must be real. If all are imaginary, it is just equivalent to an overall negative sign in the S matrix—a property we knew of before. Hence, we can rewrite the S matrix as follows

$$S_{\pm}=\pm\begin{pmatrix}2z^2-1 & 2yz & 2xz\\ 2yz & 2y^2-1 & 2xy\\ 2xz & 2xy & 2x^2-1\end{pmatrix}, \tag{2.5.15}$$

where x, y and z are real and are subject to the constraint

$$x^2+y^2+z^2=1. \tag{2.5.16}$$

We note that, without loss of generality, one can choose the sign of one of the variables, say x, to be positive. This is because S does not change if we flip the signs of x, y and z together.

So far it seems that there is no difference between the plus and minus signs in front of S. However, a closer look reveals a difference. When one computes the eigenvalues of S, one finds that they are $+1$ and -1. But, with S_+ the $+1$ eigenspace is one dimensional and with S_- this eigenspace is two dimensional. This should be used to determine the relative signs of x, y and z, as we remember that we must have a same-sign component eigenvector of a $+1$ eigenvalue in order to get a consistent solution.

To be more specific, let us look first at the plus case. After some algebra, we find that the eigenspace is generated by the vector

$$\vec{v}_{(+1)}=\begin{pmatrix}z\\ y\\ x\end{pmatrix}. \tag{2.5.17}$$

It is obvious, then, that if we want x y and z to be of the same sign, and since we chose x to be positive, then y and z are positive as well. Note that this argument does not depend on unitarity.

In the minus case, on the other hand, the $+1$ eigenspace is generated by

$$\vec{v}_{(+1)}=\begin{pmatrix}xv_1\\ xv_2\\ -yv_2-zv_1\end{pmatrix}, \tag{2.5.18}$$

where v_1 and v_2 are real (and span the eigenspace). Since x is positive, v_1 and v_2 must be non-negative. Hence, at least one of y and z must be negative. If one adds the unitarity condition then, from the form of S, y is positive and, hence, z is negative.

We are now in a position to calculate the fusion algebra. Substituting the S matrix into (2.3.6), we get the most general fusion algebra

$$\begin{aligned}
\phi_1 \times \phi_1 &= \phi_0 + k\ \phi_1 + l\ \phi_2 \\
\phi_1 \times \phi_2 &= l\ \phi_1 + m\ \phi_2 \\
\phi_2 \times \phi_2 &= \phi_0 + m\ \phi_1 + n\ \phi_2,
\end{aligned} \tag{2.5.19}$$

where k, l, m and n are given by

$$k = \frac{(1-6y^2)(1-2z^2)+4y^4}{2yz(-1+2z^2)} \tag{2.5.20}$$

$$l = -\frac{x(1-2x^2)}{z(1-2z^2)} \tag{2.5.21}$$

$$m = -\frac{y(1-2y^2)}{z(1-2z^2)} \tag{2.5.22}$$

$$n = \frac{(1-6x^2)(1-2z^2)+4x^4}{2xz(-1+2z^2)}. \tag{2.5.23}$$

These coefficients should be non-negative integers, as usual. Note that since in (2.3.6) one has a sum of quartics in the S matrix elements, these formulae for k, l, m and n apply to both S_+ and S_-.

A remark is in place here. When one computes four-point correlation functions of primary fields, one gets a consistency condition on the fusion algebra. This is the crossing symmetry mentioned in Section 2.1. This is translated into a condition on the coefficients N_{ijk} in the fusion algebra,

$$\sum_m N_{ij}{}^m N_{mk}{}^l = \sum_m N_{ik}{}^m N_{mj}{}^l. \tag{2.5.24}$$

However, Verlinde's formula already ensures that this relation is satisfied. Hence, we get a non-trivial check on the self-consistency of our formulae.

In the three-field case there is one non-trivial crossing relation. That is

$$l(l-n) + m(m-k) = 1. \tag{2.5.25}$$

It is straightforward to check that the formulae for k, l, m and n satisfy this relation. This will be useful in computing the solutions.

The next stage would be to solve the equations for k, l, m and n with the constraint that they are non-negative integers. The general solution is not easy to find in a 'nice' form. In general we should be able to solve for, say, m and n if we know k and l and then we should implement the fact that they are integers. In the next subsection we will present a few simple solutions. We were not able to find the general solution, but it has been proved by Caselle and Ponzano[76] that the solutions we will write in the next section actually exhaust the space of solutions. Their reasoning takes into account the fact that the conformal dimensions must be rational in a rational conformal field theory.

Let us now continue with the presentation of the formal solution. Let the T matrix be

$$T = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}. \tag{2.5.26}$$

Then, we need to solve the equations $(ST)^3 = 1$ that appear to be, at first sight, a mess. However, after some algebraic manipulation, we can reduce them to the following three equations. First, we remember that x, y and z and A, B and C are non-zero. Then we look at the off diagonal terms and cancel the non-zero factors. Taking differences of those components we always get the same equation, which is

$$AZ + BY + CX = 0, \tag{2.5.27}$$

where $X = 1 - 2x^2$, $Y = 1 - 2y^2$ and $Z = 1 - 2z^2$. We next solve this equation for A and substitute back into $(ST)^3$. The off diagonal terms are now all equal and demanding that they vanish we get the second equation,

$$BC(2X + 2Y - 1) + XY(B - C)^2 = 0. \tag{2.5.28}$$

Last, we look at the trace of $(ST)^3$, which should be 3, and get the following equation

$$(1 \mp BC^2)X + (1 \mp B^2C)Y = 1, \tag{2.5.29}$$

where the minus (plus) sign corresponds to S_+ (S_-). The differences between diagonal terms are zero by the second equation, (2.5.28).

In principle, all we have to do now is to solve for the S matrix as mentioned above, and then solve for T using equations (2.5.27), (2.5.28) and (2.5.29). Of course,

it helps to remember that $|A| = |B| = |C| = 1$. This should yield all possible three-field theories consistent with our assumptions.

5.3. The Solution

We first investigate the constraints from the diophantine equation (2.5.25). This system has infinitely many solutions with non-negative integers. One notes that l and m must be relative primes[§]. The solutions for $l = 0$, 1 and 2 are presented in Table 4.

k	l	m	n	Extra conditions
0	0	1	n	$n = 0, 1, 2, \ldots$
k	1	m	$m(m-k)$	$m = 1, 2, \ldots\ ; 0 \leq k \leq m$
$2j$	2	$2i+1$	$(2i+1)(i-j)+(2-i)$	$i = 1, 2, \ldots\ ; 0 \leq j \leq i$

Table 4: A few solutions of equation (2.5.25).

We note that there is a symmetry among the solutions, which is obvious by looking at the fusion algebra. We get the same solution by letting

$$\begin{aligned} \phi_1 &\leftrightarrow \phi_2 \\ k &\leftrightarrow n \\ l &\leftrightarrow m. \end{aligned} \tag{2.5.30}$$

Thus, we can restrict the analysis to half of the solution space.

(i) $l = 0$ solution

By letting $l = 0$ we immediately see from (2.5.21) that $x = \frac{1}{\sqrt{2}}$**. Then from (2.5.20) we get $y^2 = \frac{1}{4}$ and from (2.5.22) we get $z^2 = \frac{1}{4}$. We find that n must vanish. For a unitary theory we have $y = \frac{1}{2}$ and then from (2.5.22) we must have $z = -\frac{1}{2}$. This means that we must choose S_- to be the S matrix. We get for the S matrix

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{2.5.31}$$

[§]And so must l and $m-k$ or $l-n$ and m, etc.

**Remember that we assumed that x is positive.

and the fusion algebra

$$\begin{aligned}\phi_1 \times \phi_1 &= \phi_0 \\ \phi_1 \times \phi_2 &= \phi_2 \\ \phi_2 \times \phi_2 &= \phi_0 + \phi_1.\end{aligned} \tag{2.5.32}$$

We then solve for T using the equations in the previous section. We find that A remains arbitrary, $B = -A$ and $C = A^{-2}$. Letting

$$A = e^{-\frac{i\pi c}{12}}, \tag{2.5.33}$$

we get $h_1 = \frac{1}{2} (\mathrm{mod}\, 1)$ and $h_2 = \frac{c}{8} (\mathrm{mod}\, 1)$, while c remains a free parameter. We readily identify many models with such values. The first among them is the Ising model (the $(3,4)$ minimal model, see (2.1.24) and (2.1.25)).

For the non-unitary theory all we can do is change the signs of y and z. In this case, S must once again be S_-, and solving for T we get the same result as for the unitary case. The results are summarized in Table 5.

One thing to emphasize is that in this solution (with $l = 0$) the central charge c remains undetermined. This is different from the two field case, and if we want to further restrict the models we must use other considerations, such as analyticity of the four-point function.

(ii) $l = 1$ *solution*

Plugging the $l = 1$ solution from Table 4 into equations (2.5.20)–(2.5.23), one can show, after some tedious algebra, that there are only two possibilities for the fusion algebra coefficients

$$(k, l, m, n) = (k, 1, 2k, 2k^2) \quad \text{for} \quad k \geq 1 \tag{2.5.34}$$

and

$$(k, l, m, n) = (1, 1, 1, 0). \tag{2.5.35}$$

We will start with (2.5.23). One can solve equation (2.5.23) for z^2 and then from the constraint (2.5.16) we get y^2. From the ratio $k/m = 1$ we get the equation that x must satisfy

$$x^6 - x^4 + \frac{x^2}{4} - \frac{1}{56} = 0. \tag{2.5.36}$$

This equation has 3 positive solutions

$$\begin{aligned} x_1 =& \sqrt{\frac{\cos(\frac{\theta}{3})+1}{3}} \approx 0.814858 \\ x_2 =& \sqrt{\frac{\cos(\frac{\theta}{3}+\frac{2\pi}{3})+1}{3}} \approx 0.362646 \\ x_3 =& \sqrt{\frac{\cos(\frac{\theta}{3}+\frac{4\pi}{3})+1}{3}} \approx 0.452212, \end{aligned} \tag{2.5.37}$$

where $\cos\theta = \frac{13}{14}$.

For each x there is only one consistent solution for y and z

$$\begin{aligned} (y_1, z_1) =& (-x_2, +x_3) \\ (y_2, z_2) =& (+x_3, +x_1) \\ (y_3, z_3) =& (-x_1, -x_2). \end{aligned} \tag{2.5.38}$$

We note that for the second solution we must choose S_+, whereas for the other two we choose S_-. Also, we see that we do not have a choice whether a solution is unitary or not. These solutions are constrained enough to tell us if a unitary solution is possible. Since for a unitary solution we must have positive y, it follows that only the second solution can be unitary. The other two are not, and, indeed, the first one includes a known non-unitary model—the $(2,7)$ model of the minimal Virasoro series (see (2.1.25)). For the values of the conformal dimensions and central charge, see Table 5.

With the other fusion algebra, (2.5.34), one gets two solutions

$$x_\pm = \frac{k_\pm}{2}\ ,\ y_\pm = \pm\frac{k_\mp}{\sqrt{2}} \text{ and } z = -\frac{k_\pm}{2}, \tag{2.5.39}$$

where

$$k_\pm = \sqrt{1 \pm \frac{k}{\sqrt{k^2+2}}}. \tag{2.5.40}$$

The S matrix is (taking S_-[††])

$$S_\pm^{(k)} = \begin{pmatrix} k_\mp^2/2 & (k_\pm^2-1)/k & k_\pm^2/2 \\ (k_\pm^2-1)/k & k_\pm^2-1 & (k_\mp^2-1)/k \\ k_\pm^2/2 & (k_\mp^2-1)/k & k_\mp^2/2 \end{pmatrix}. \tag{2.5.41}$$

[††]The new $\pm$ sign is not related to the old one.

(k,l,m,n)	$c(\mathrm{mod}\,8)$	$h_1(\mathrm{mod}\,1)$	$h_2(\mathrm{mod}\,1)$	Identified Models
$(0,0,1,0)$	c	$\frac{1}{2}$	$\frac{c}{8}$	Ising Model $(c=\frac{1}{2})$
				$SU(2)^{(2)}$ $(c=\frac{3}{2})$
				$SO(2n+1)^{(1)}(c=n+\frac{1}{2})$
				$E_8^{(2)}$ $(c=\frac{31}{2})$
	c	$-\frac{1}{2}$	$\frac{c}{8}$	?
$(1,1,1,0)$	$-\frac{12}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$(2,7)$ Model
	$-\frac{44}{7}$	$-\frac{4}{7}$	$-\frac{5}{7}$	?
	$\frac{48}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	?
	$\frac{8}{7}$	$\frac{2}{7}$	$\frac{6}{7}$	?
	$-\frac{52}{7}$	$-\frac{6}{7}$	$-\frac{4}{7}$	?
	$-\frac{4}{7}$	$-\frac{1}{7}$	$-\frac{3}{7}$	?

Table 5: Some solutions for trivial C—three-field case.

We can now solve equations (2.5.27)–(2.5.29). We get

$$A = Be^{\pm i\alpha} \tag{2.5.42}$$

$$B^3 = -1 \tag{2.5.43}$$

$$C = Be^{\mp i\alpha}, \tag{2.5.44}$$

where

$$\cos\alpha = \frac{k}{k \mp \sqrt{2+k^2}}. \tag{2.5.45}$$

One immediately sees that $S_+^{(k)}$ is not a consistent solution because $|\cos\alpha|$ is larger then 1 (unless $k=0$, which we assumed is not the case). We can also deduce that $h_1 = \mp 2\pi\alpha(\mathrm{mod}\,1)$, $h_2 = 2h_1(\mathrm{mod}\,1)$ and $c = 24h_1 + 4(\mathrm{mod}\,8)$.

This solution is consistent with all our assumptions so far. However, if we take analyticity into account, a numerical computation suggests that (at least for many values of k) the value of h_1 is not rational. It is natural to conjecture that this is the case for all k, and, hence, only the Ising model algebra and the $(2,7)$ algebra are consistent. Indeed, it has been proved by Caselle and Ponzano in work done at the same time as ours that this is the case[76].

For $l > 1$, a numerical computation hints that already the system (2.5.20)–(2.5.23) is not solvable. This has been proved, and it is rather non-trivial, by Caselle

and Ponzano[76].

To summarize, in the three-field case, there are three possible fusion algebras. The $SU(3)$ type

$$\begin{aligned} \phi_1 \times \phi_1 &= \phi_2 \\ \phi_1 \times \phi_2 &= \phi_0 \\ \phi_2 \times \phi_2 &= \phi_1, \end{aligned} \tag{2.5.46}$$

the Ising model type

$$\begin{aligned} \phi_1 \times \phi_1 &= \phi_0 \\ \phi_1 \times \phi_2 &= \phi_2 \\ \phi_2 \times \phi_2 &= \phi_0 + \phi_1, \end{aligned} \tag{2.5.47}$$

and the $(2,7)$ model type

$$\begin{aligned} \phi_1 \times \phi_1 &= \phi_0 + \phi_1 + \phi_2 \\ \phi_1 \times \phi_2 &= \phi_1 + \phi_2 \\ \phi_2 \times \phi_2 &= \phi_0 + \phi_1. \end{aligned} \tag{2.5.48}$$

6. Conclusions

We used modular invariance to classify possible chiral RCFT's without assuming any specific knowledge of the chiral algebra. We found that in the two-field case there are only two possible fusion algebras, which account for all previously known models. However, we also found a few possibilities that we were not able to identify. It is possible, of course, that models do not exist for these cases for some other reasons, but at least one of them—the partner to the $(2,5)$ model (the Lee-Yang singularity)—should exist. It should be an interesting exercise to identify the chiral algebra and to construct it explicitly.

In the three-field case we obtained only three possible fusion algebras—the Ising type algebra, the $(2,7)$ type and the $SU(3)^{(1)}$ type. Again, all known models are accounted for, but the classification still has a free parameter in the case of the Ising-type theories. We know from analyticity arguments that for a RCFT the conformal dimensions and the central charge must be rational and our method does not enforce that requirement. Indeed, one can add this demand to the classification scheme, and it seems that we must do so in order to classify RCFT's.

We conclude that modular invariance alone is not sufficient to classify rational conformal field theories, and we need analyticity of the four-point function also. Indeed, it has been proven by Sonoda[20] that knowledge of the four-point function on the sphere and the one-point function on the torus is enough to reconstruct conformal field theory on an arbitrary Riemann surface. This agrees with our analysis, which demonstrated that modular invariance on the torus is not enough and we need analyticity—four-point function—on the sphere.

However, modular invariance does seem to be sufficient to classify the fusion algebras themselves. This was demonstrated in the two- and three-field cases above. It may be tempting to conjecture that only algebras with fusion coefficients 1 or 0 are possible, but there are counter examples for high-level Kač-Moody algebras. For example, for the $SU(3)$ Lie algebra,

$$\mathbf{8} \times \mathbf{8} = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{10} + \bar{\mathbf{10}} + \mathbf{27}. \tag{2.6.1}$$

So, if we take the Kač-Moody algebra $SU(3)^{(k)}$ at high enough k, the representation $\mathbf{8}$ appears twice. That is, $N_{\mathbf{88}}{}^{\mathbf{8}} = 2$.

Other methods of classification have been used to classify rational conformal field theories. One of them is based on classifying the OPE coefficients C_{ijk} in (2.1.20). This method, since it does not consider modular invariance, is not strong enough. Similarly, Christe and Ravanini[15] used the crossing relations to classify possible fusion algebras. They restricted their classification to $N_{ij}^{k} \leq 1$ (and we showed that these theories do not exist in any case) and found four possible fusion algebras. In Chapter 5 we showed that there are only three.

Other methods that do include modular invariance have been used. For example, Mathur, Mukhi and Sen[13] and also Kiritsis[14] used modular invariance to write differential equations to compute the characters. This is a powerful method, but it is too complicated to apply to the three-field case. Their solution included only the two-field situation. Another difference is that their solution contains two *characters* and not two fields. The difference being that there are three-field theories—the one with non-trivial charge conjugation matrix—that have the same character for two of the fields. Actually, there is even an example of a four-field theory—$SO(8)$ level 1—in which three of the fields have the same character*. Our method works relatively well for the three-field case, although it will probably be hard to solve the four-field theories. The reason for the relative ease of our method is the fact that it is purely algebraic and we do not need to solve for the characters.

*This is a result of the triality symmetry of D_4.

Part III
SOLVING (1,q) KdV GRAVITY

1. Introduction to 2D Gravity and Quantum Gravity

Einstein's field equations for gravity coupled to matter in any dimension are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.1.1}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is Einstein's curvature tensor, Λ is a cosmological constant, $g_{\mu\nu}$ is the metric of space-time, κ is a coupling constant, and $T_{\mu\nu}$ is the stress-energy tensor[78]. In three dimensions the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ has six independent components, exactly the number of components in the Ricci tensor $R_{\mu\nu} = g^{\rho\sigma} R_{\mu\rho\nu\sigma}$. This leads to the well-known result that in three dimensions Einsteinian gravity does not have a propagating degree of freedom—or, in other words, the vacuum solution ($\Lambda = T_{\mu\nu} = 0$) is always locally flat.

1.1. 2D Gravity

In two dimensions the Riemann tensor has only one independent component (and so has the Ricci tensor) that can be expressed in terms of the curvature scalar, $R = g^{\mu\nu} R_{\mu\nu}$, as

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} R \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right), \tag{3.1.2}$$

so R determines completely the local geometry. This is a hint that metric gravity is trivial in two dimensions. Moreover, the Einstein tensor G vanishes identically in two dimensions, rendering the field equations meaningless.

Looking at the vacuum Einstein-Hilbert action (with cosmological constant) in two dimensions

$$S_0 = \frac{1}{2\kappa} \int d^2x \sqrt{|g|} \left(R - 2\Lambda \right), \tag{3.1.3}$$

we can find the reason for this behavior. The first term is proportional to the Euler characteristic class, which is purely topological (*i.e.*, independent of the metric*). Hence, extremizing (3.1.3) with respect to the metric gives, for $\Lambda \neq 0$, the condition

$$\sqrt{|g|} g_{\mu\nu} = 0, \tag{3.1.4}$$

*In the classical theory, S_0 is a constant equal to $\frac{1}{2\kappa}(2(1-g) - \Lambda A)$, where g is the genus and A is the area af the manifold.

which is unacceptable. For $\Lambda = 0$ there are no restrictions on the metric at all.

A modification of the equations of motion is needed. A choice that is possible only in two dimensions is

$$R = \Lambda. \tag{3.1.5}$$

This equation has solutions of space-time with constant curvature and no propagating degrees of freedom. Since all two-dimensional manifolds are conformally flat, one can choose local coordinates in which the metric has the form

$$g_{\mu\nu} = e^{\phi}\eta_{\mu\nu}, \tag{3.1.6}$$

where ϕ is called the Liouville mode. The equation of motion (3.1.5) then becomes

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + \Lambda e^{\phi} = 0. \tag{3.1.7}$$

This is Liouville's equation, which is completely integrable. This equation can be obtained by varying the action

$$S_{Liouville} = \int dx^2(\frac{1}{2}(\partial\phi)^2 - e^{\phi}\Lambda). \tag{3.1.8}$$

1.2. Quantum 2D Gravity and Scaling

When one tries to include quantum fluctuations of the metric, the problem appears to be non-trivial. Quantizing the Liouville action, (3.1.8), is still a problem under investigation. Following Polyakov[79], we consider the partition function

$$Z = \int \mathcal{D}_g[g] \int \mathcal{D}_g[X] e^{-S_M[X,g]+S_0}, \tag{3.1.9}$$

where S_0 is given by (3.1.3) and $S_M(X,g)$ is the matter part of the theory. The integration is carried over all metrics (and matter fields X) for a given topology and given moduli of the manifold. To keep conformal invariance the matter fields X must be chosen to form a conformal field theory. Polyakov suggested to study this action in the conformal gauge, that is, to choose local coordinates in which the metric takes the form

$$g_{\mu\nu}(\sigma) = e^{\phi(\sigma)}g^{(0)}_{\mu\nu}(\sigma), \tag{3.1.10}$$

where $g^{(0)}$ is a background metric that depends, generically, on a finite number of moduli that represent the conformally inequivalent geometries of a given topology.

This is a generalization of (3.1.6). This partition function, for $\Lambda = 0$, is invariant classically under the conformal transformation (of the conformal mode ϕ)

$$\phi(\sigma) \to \phi(\sigma) + \delta\phi(\sigma), \tag{3.1.11}$$

but, because of the conformal anomaly, this invariance is violated at the quantum level. To find the effective action for the Liouville mode, Polyakov integrated out the matter fields and the ghost fields that are introduced by the gauge fixing (3.1.10). This gives

$$\begin{aligned}\Gamma[g] = -\frac{c}{96\pi}\int d^2\sigma\sqrt{|g(\sigma)|}\int d^2\sigma'\sqrt{|g(\sigma')|}R(\sigma)\left(\frac{1}{-\Delta}\right)'_{\sigma\sigma'}R(\sigma') \\ + \lambda_R\int d^2\sigma\sqrt{|g(\sigma)|} + \ldots,\end{aligned} \tag{3.1.12}$$

where c is the central charge ($= c_{matter} - 26$), $\left(\frac{1}{\Delta}\right)'$ is the propagator (without zero modes of the Laplacian, Δ), λ_R is a shifted cosmological constant (renormalized by some UV regulator), and the '...' represents the moduli dependent part. (If we substitute $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$, we will recover (3.1.8).)

The semi-classical regime corresponds to $c \to -\infty$ since in this limit the kinetic term of the Liouville action is positive. However, for a positive effective cosmological constant ($\lambda_R > 0$) the vacuum is anti de Sitter ($R < 0$). It is in the regime $c \to \infty$, that the vacuum is a de Sitter space for positive effective cosmological constant. In this regime the kinetic term is negative. This instability of the vacuum is very similar to the instability in the Einstein theory in four dimensions (or more precisely, $3+1$ dimensions).

The full quantum treatment of the Liouville action includes the quantization of the Liouville field itself, together with the matter fields and the ghost sector. This is a hard problem for $\lambda_R > 0$, since the Liouville theory is interacting in this region (even though it's integrable). This aspect has been pursued mainly by Gervais, Neveu and their collaborators[80–82]. We shall not describe this work here.

Knizhnik, Polyakov and Zamolodchikov[34], Distler and Kawai[35] and David[36] showed the following. Consider the fixed area partition function

$$Z_{A,g} = \int \mathcal{D}[g,A]\int \mathcal{D}[X]e^{-S[g,X]}, \tag{3.1.13}$$

where $\mathcal{D}[g,A]$ means integration over all topologically different metrics on a surface of genus g with fixed area A, and X are matter fields. The total partition function is

(aside from possible nonperturbative effects)

$$Z = \sum_{g=0}^{\infty} \lambda^{2g} \int dA\, e^{-\Lambda A} Z_{A,g}, \tag{3.1.14}$$

where we sum over each different genus weighted by the string coupling constant λ, and Λ is the cosmological constant. It was shown that in the large area (or small coupling constant) limit,

$$Z_{A,g} \sim e^{\Lambda_c A} A^{\gamma(g)-3} c_g, \tag{3.1.15}$$

where c_g is some constant, Λ_c is a critical cosmological constant, and the critical index $\gamma(g)$ depends linearly on the genus

$$\gamma(g) = 2 + \frac{1-g}{\gamma_0 - 2}, \tag{3.1.16}$$

where γ_0 is a constant depending on the matter content of the theory. Inserting this into (3.1.14) gives, in the same limit,

$$Z \sim (\Lambda - \Lambda_c)^{2-\gamma_0} \sum_{g=0}^{\infty} \tilde{c}_g \left(\frac{\lambda^2}{(\Lambda_c - \Lambda)^{2-\gamma_0}} \right)^g. \tag{3.1.17}$$

We see that up to an overall factor, the partition function Z depends only on the combination $\lambda^2/(\Lambda_c - \Lambda)^{2-\gamma_0}$. That is, there is a scaling relation in the partition function, *i.e.*, $Z(\Lambda, \lambda)$ can be written as a function of one variable.

This scaling behavior also shows up in the discrete version of quantum gravity that will be discussed in the next chapter. This scaling behavior allows one to take the 'double scaling' limit, which is defined by

$$\Lambda_c - \Lambda \to 0 \quad \text{and} \quad \lambda^2 \to 0, \tag{3.1.18}$$

while keeping $\lambda^2/(\Lambda_c - \Lambda)^{2-\gamma_0}$ fixed. Since λ is the bare string coupling constant, the limit $\lambda^2 \to 0$ is a 'weak' coupling limit allowing an exact evaluation of the partition function. On the other hand, the effective coupling, $\lambda^2/(\Lambda_c - \Lambda)^{2-\gamma_0}$, is kept fixed, and, hence, no information about the partition function is lost in this limit. This is the trick that allows the matrix model to be analyzed exactly and, thus, one derives non-perturbative results. (At least those that follow from assuming that (3.1.14) is exact.)

2. The Matrix Models

2.1. Discrete 2D Gravity and Matrix Models

In the discretized version of Euclidean 2D gravity we consider a discretized version of the partition function (3.1.14). Assume that we triangulate a closed Riemann surface with n_F triangles. One can compute the Euler characteristic χ or the genus g by Euler's theorem

$$n_F - n_V + n_E = \chi = 2(1-g), \tag{3.2.1}$$

where n_F, n_E and n_V are the numbers of faces, edges and vertices, respectively. Actually, in general one can discretize the surface with arbitrary n-gons rather than just triangles and (3.2.1) will still hold.

Given a surface discretized with n-gons, one associates an area $\frac{a^2}{2}$ to each triangle and $\frac{a^2}{2}(n-2)$ to each n-gon (thinking of an n-gon as composed of $n-2$ triangles and assuming that all edges have the same length). The total area is then

$$A = a^2 \sum_i n_{F_i} \frac{i-2}{2}, \tag{3.2.2}$$

where n_{F_i} is the number of i-gons*. The discrete version of (3.1.14) is then

$$Z \sim \lambda^2 \sum_{discretizations} \lambda^{-n_F+n_E-n_V} e^{(\Lambda_c-\Lambda)A} e^{-S}, \tag{3.2.3}$$

where S is the part of the action that depends on the matter content and possible weight factors for different n-gons.

The basic trick to solve the random discrete model is to realize that the dual graph to a triangulated surface is a Feynman diagram of a ϕ^3 scalar field theory. In a general discretization into different polygons there are higher order vertices with a ϕ^n interaction term corresponding to each n-gon. One thing, however, which is still needed is another parameter (besides Planck's constant) that will control the

*Clearly, $n_F = \sum_i n_{F_i}$.

topology—the genus. The way to do it is to look at an $N \times N$ hermitian matrix as the field instead of a single scalar field. N will be that extra parameter.

The partition function for the one-matrix model reads

$$Z_{matrix} = \int d[\phi] e^{-\beta \operatorname{Tr} U(\phi)}, \tag{3.2.4}$$

where

$$U(\phi) = \frac{1}{2}\phi^2 + \sum_{n=3}^{M} u_n \phi^n \tag{3.2.5}$$

and $\beta = \frac{1}{\hbar}$. ϕ is now an $N \times N$ hermitian matrix and we choose the flat measure

$$d[\phi] = \prod_i d\phi_{ii} \prod_{i<j} d[\Re(\phi_{ij})] d[\Im(\phi_{ij})]. \tag{3.2.6}$$

When one rescales the matrix field $\phi \to \tilde{\phi} = \left(\frac{\beta}{N}\right)^{\frac{1}{2}} \phi$, one gets

$$\begin{aligned} \beta U(\phi) &= \frac{N}{2}\tilde{\phi}^2 + \sum_{n=3}^{M} u_n N \left(\frac{N}{\beta}\right)^{\frac{n-2}{2}} \tilde{\phi}^n \\ &= N\left(\frac{1}{2}\tilde{\phi}^2 + \sum_{n=3}^{M} \tilde{u}_n \tilde{\phi}^n\right) \\ &= N\tilde{U}(\tilde{\phi}). \end{aligned} \tag{3.2.7}$$

One can compute the propagator on the sphere

$$\langle \tilde{\phi}_{ab} \tilde{\phi}_{cd} \rangle_0 = \frac{1}{N} \delta_{ad} \delta_{bc}, \tag{3.2.8}$$

and then read off the coefficients that go into the perturbation expansion. Each propagator contributes $\frac{1}{N}$, a closed index loop yields a factor of N (tracing (3.2.8)) and each $\tilde{\phi}^n$ vertex gives $N\left(\frac{N}{\beta}\right)^{\frac{n-2}{2}} u_n = N\tilde{u}_n$. One can now compute Z_{matrix} using dual description of the discretized Riemann surface. For the matrix model, for a given Feynman diagram, we have n_V loops, n_E propagators, and n_{F_i} vertices of type $\tilde{\phi}^i$. Thus we get

$$Z_{matrix} \sim \sum_{graphs} N^{n_V - n_E + n_F} \left(\frac{N}{\beta}\right)^{\sum_i \frac{i-2}{2} n_{F_i}} W(u), \tag{3.2.9}$$

where

$$W(u) = \prod_{i=3}^{M} u_i^{n_{F_i}}. \tag{3.2.10}$$

Comparing (3.2.9) with (3.2.3), one identifies the bare string coupling constant λ with $\frac{1}{N}$ and $\frac{N}{\beta}$ with $e^{(\Lambda_c-\Lambda)a^2}$. We now wish to take the continuum limit, but before we can do that we need to explain the method of orthogonal polynomials[24,83].

2.2. The Method of Orthogonal Polynomials and the String Equation

Let us first note that $\mathrm{Tr}\,\tilde{U}(\tilde{\phi})$ in the partition function (3.2.4) (using (3.2.7)) is invariant under unitary transformations—that is, under matrices in $U(N)$. We can use this invariance to change variables and integrate out the irrelevant 'phases.' To be more specific, let $\tilde{\phi} = O\Lambda O^\dagger$ where $OO^\dagger = 1$ and

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}. \tag{3.2.11}$$

Integrating out the O dependence in the partition function gives the result

$$Z_{matrix} = \int \prod_{i=1}^{N} d\lambda_i \, \Delta(\lambda)^2 e^{-N\sum_{i=1}^{N}\tilde{U}(\lambda_i)}, \tag{3.2.12}$$

where $\Delta(\lambda)$ is the Vandermonde determinant

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j). \tag{3.2.13}$$

We introduce now the 'monic' polynomials

$$\pi_n(\lambda) = \lambda^n + \ldots, \tag{3.2.14}$$

which are orthogonal with respect to the measure $d\mu(\lambda) = e^{-N\tilde{U}(\lambda)}d\lambda$. That is,

$$\int \pi_m(\lambda)\pi_n(\lambda)d\mu(\lambda) = \delta_{m,n}S_n, \tag{3.2.15}$$

where S_n are (clearly) positive numbers to be determined. We can now compute the partition function (3.2.12) by noting that

$$\Delta(\lambda) = \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \cdots & \lambda_N^{N-1} \end{vmatrix} = \begin{vmatrix} \pi_0(\lambda_1) & \pi_1(\lambda_1) & \cdots & \pi_{N-1}(\lambda_1) \\ \pi_0(\lambda_2) & \pi_1(\lambda_2) & \cdots & \pi_{N-1}(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_0(\lambda_N) & \pi_1(\lambda_N) & \cdots & \pi_{N-1}(\lambda_N) \end{vmatrix}. \tag{3.2.16}$$

This can be shown by standard manipulations of the determinant $\det(\pi_{j-1}(\lambda_i))$. Plugging this into (3.2.12) one gets

$$Z_{matrix} = N! \prod_{i=0}^{N-1} S_i. \tag{3.2.17}$$

To compute the S_n's, one uses recursion relations, which are obtained by considering how multiplication and differentiation by λ act on the polynomials π_n. Let

$$\lambda\pi_n = \sum_{m=0}^{n+1} b_{nm}\pi_m \tag{3.2.18a}$$

$$\frac{\partial}{\partial\lambda}\pi_n = \sum_{m=0}^{n-1} a_{nm}\pi_m. \tag{3.2.18b}$$

By inspection, one sees that $b_{nm} = 0$ for $m \geq n+2$ and that $a_{nm} = 0$ for $m \geq n$. Also $\lambda\pi_n = \lambda^{n+1} + \ldots$ means that $b_{n,n+1} = 1$ and $\frac{\partial}{\partial\lambda}\pi_n = n\lambda^{n-1} + \ldots$ shows that $a_{n,n-1} = n$.

Next use the orthogonality relation (3.2.15) on (3.2.18a),

$$\int \lambda\pi_m(\lambda)\pi_n(\lambda)d\mu(\lambda) = b_{nm}S_m = b_{mn}S_n, \tag{3.2.19}$$

and get, from the symmetry, that $b_{nm} = 0$ for $|m-n| > 1$ and

$$b_{n,n-1} \equiv b_n = \frac{S_n}{S_{n-1}}. \tag{3.2.20}$$

We also denote $b_{n,n} = \tilde{b}_n$.

We can now write Z_{matrix} in terms of the b_n (and S_0). For convenience, we write the free energy

$$F = -\ln Z = -ln(N!\, S_0^N) - \sum_{n=1}^{N-1}(N-n)\ln b_n. \tag{3.2.21}$$

From (3.2.18b) and (3.2.19) we derive

$$0 = \int \frac{\partial}{\partial\lambda}\left(\pi_m(\lambda)\pi_n(\lambda)e^{-N\tilde{U}(\lambda)}\right)d\lambda = a_{nm}S_m + a_{mn}S_n - N\left(\tilde{U}'(B)\right)_{nm} S_m, \tag{3.2.22}$$

and hence

$$\tilde{U}'(B)_{nm} = \frac{1}{NS_m}\left(a_{nm}S_m + a_{mn}S_n\right), \tag{3.2.23}$$

where B is the matrix whose elements are b_{mn}. Substituting $m = n$ in (3.2.23) we get

$$\left(\tilde{U}'(B)\right)_{n,n} = 0, \tag{3.2.24}$$

and substituting $m = n - 1$ we get

$$\left(\tilde{U}'(B)\right)_{n,n-1} = \frac{n}{N}. \tag{3.2.25}$$

Equations (3.2.24) and (3.2.25) are, in fact, recursion relations that determine S_n, b_n and $\tilde{b}_n$ in terms of the initial conditions.

As a simple example, consider the potential $\tilde{U}(\lambda) = \frac{\lambda^2}{2} + \alpha\frac{\lambda^3}{3}$. Then one gets the recursion relations

$$\tilde{b}_n + \alpha(b_{n+1} + \tilde{b}_n^2 + b_n) = 0, \tag{3.2.26}$$

and

$$\frac{n}{N} = b_n\left(1 + \alpha(\tilde{b}_b + \tilde{b}_{n-1})\right). \tag{3.2.27}$$

Another simple example is the potential $\tilde{U}(\lambda) = \frac{1}{g}\left(\frac{\lambda^2}{2} - \frac{\lambda^4}{4}\right)$. In this case one can argue that because $\tilde{U}(\lambda)$ is even, the polynomials $\pi(\lambda)$ must be even or odd. Thus, $\tilde{b}_n = 0$, and we need only the one recursion relation

$$g\frac{n}{N} = b_n(1 - (b_{n-1} + b_n + b_{n+1})). \tag{3.2.28}$$

The large N limit is obtained as follows. We assume that b_n (and $\tilde{b}_n$ in general) becomes a smooth function of the continuous parameter $x = \frac{n}{N}$. In the limit one can easily get from (3.2.28)

$$gx = V(b) = b(x) - 3b(x)^2, \tag{3.2.29}$$

and from (3.2.21) the free energy

$$F = -N^2\int_0^1 (1-x)\ln(b(x))\,dx, \tag{3.2.30}$$

where we have dropped terms that are subleading in powers of $\frac{1}{N}$.

This free energy has a critical point. That is, it is ill defined when $b(x)$ becomes negative (or zero). This point can be found by solving $V'(b) = 0$ at $x = 1$. One gets

$$g_c = V(b_c) = \frac{1}{12} \text{ and } b_c = \frac{1}{6}. \tag{3.2.31}$$

The double scaling limit is then obtained as follows. Consider g close to g_c and n close to N in such a way that

$$t = N^{\frac{4}{5}} \left(\frac{g_c - g}{g_c} \right) \text{ and } x = N^{\frac{4}{5}} \left(1 - \frac{n}{N} \right) \tag{3.2.32}$$

are fixed when we let $N \to \infty$. b_n scales as

$$b_n = b_c \left(1 - N^{-\frac{2}{5}} u(x,t) + O\left(N^{-\frac{3}{5}} \right) \right), \tag{3.2.33}$$

and (3.2.28) becomes, after substituting (3.2.33) and (3.2.32) and using the criticality values,

$$u^2 - \frac{1}{3} \frac{\partial^2 u}{\partial x^2} = x + t. \tag{3.2.34}$$

This is the celebrated 'string equation.'

The free energy (3.2.21) has divergent parts proportional to $N^2 + N$ and $N \to \infty$. However, its finite part is

$$F(t) = \text{fp} \int_0^\infty x u(x,t)\, dx, \tag{3.2.35}$$

where 'fp' stands for 'finite part.' One can show that $u(0,t)$ is the 'specific heat'—that is,

$$u(0,t) = -\frac{\partial^2 F}{\partial t^2}. \tag{3.2.36}$$

An interesting thing to note is that the string equation is the first equation (integrated) of the KdV hierarchy. We will show this in greater detail in the next section, since this rich structure is the main point of our computation.

A few remarks are in place here. The string equation (3.2.34) is relatively simple. Other interaction potentials give higher-order differential equations and, as we will see later, they are also equations in the KdV hierarchy.

The second point concerns the so-called multi-matrix models. We could generalize the matrix model partition function (3.2.4) by considering $q-1$ hermitian matrices and the partition function

$$Z_{multi} = \int \prod_{i=1}^{q-1} dM_i e^{-N\left(\sum_{i=1}^{q-2} C_i M_i M_{i+1} + \sum_{i=1}^{q-1} V_i(M_i)\right)}, \tag{3.2.37}$$

where C_i are some constants, M_i are the matrices and $V_i(x)$ are some potentials. These models have been conjectured to give as string equations, equations from the q-th generalized KdV hierarchy[32]. Evidence has been brought to justify that claim and, indeed, from the topological approach to quantum gravity it seems to be true[84]. More on this later.

3. KdV Hierarchy and KdV Gravity

In this section we will present the KdV approach to two-dimensional quantum gravity[32,85,86]. But first, we introduce the KdV hierarchy[33] itself.

3.1. The Generalized KdV Hierarchy

We will describe the generalized KdV hierarchy in the so-called Lax form, as it is the most useful for our purposes. This form is based on the algebra of pseudodifferential operators. A pseudodifferential operator, $\mathcal{O}$, is the formal sum

$$\mathcal{O} = \sum_{i=0}^{\infty} a_i(x) D^{n-i}, \tag{3.3.1}$$

where $a_i(x)$ are formal power series (or analytic functions), n is some integer, and $D = \frac{\partial}{\partial x}$. In order to multiply these operators we need to know how D^{-1} operates. We get this by demanding that $DD^{-1}f(x) = f(x)$. One can show that the generalized Liebniz rule is

$$D^{-n}f(x) = \sum_{i=0}^{\infty}(-1)^i \frac{(m+i-1)!}{i!(m-1)!} f^{(i)} D^{-n-i} \; ; \quad n > 0, \tag{3.3.2}$$

and

$$D^n f(x) = \sum_{i=0}^{n} \binom{n}{i} f^{(i)} D^{n-i} \; ; \quad n \geq 0. \tag{3.3.3}$$

In the above, $D^{\pm n}$ is an operator that continues to operate through other operators. That is, $Df(x) = f'(x) + f(x)D$, and, thus, $D^{\pm n} f \cdot g = D^{\pm n}(fg)$.

We define the order of the pseudodifferential operator, $\mathcal{O}$, to be

$$\mathrm{ord}(\mathcal{O}) = n, \tag{3.3.4}$$

the order of the highest derivative.

Given a differential operator of the form*

$$Q = D^q + u_{q-2}(x)D^{q-2} + u_{q-3}(x)D^{q-3} + \cdots + u_1(x)D + u_0(x), \tag{3.3.5}$$

*The generalization to pseudodifferential operators is straightforward.

we can define the q-th root in the following way. We formally write

$$Q^{\frac{1}{q}} = D + a_0(x) + a_{-1}(x)D^{-1} + a_{-2}(x)D^{-2} + \cdots . \tag{3.3.6}$$

We can recursively get all the unknown functions $a_i(x)$ as polynomials in the $u_i(x)$ and their derivatives by demanding that $(Q^{\frac{1}{q}})^q = Q$. The fact that we do not have a term in Q that is proportional to D^{q-1} is the reason that this root is unique.

The q-th generalized KdV hierarchy is defined in the following way. First, assume that in addition to the variable x, the functions $u_i(x)$ in Q (of equation (3.3.5)) are also functions of an infinite number of 'time' variables t_i, $i = 1, 2, \ldots$. We then define the flow of the operator Q (in the space of operators) by the following set of partial differential equations

$$\frac{\partial Q}{\partial t_i} = \left[\left(Q^{\frac{1}{q}}\right)_+^i, Q\right], \tag{3.3.7}$$

where $\mathcal{O}_+$ is the differential part of $\mathcal{O}$ (*i.e.*, dropping all negative derivatives). This truncation operation is performed after raising to powers unless specifically written otherwise (*i.e.*, $\mathcal{O}_+^i = (\mathcal{O}^i)_+$ unless written as $(\mathcal{O}_+)^i$).

As an example, consider the simplest non-trivial Q possible, $Q = D^2 + u(x,\mathrm{t})$. Then

$$Q^{\frac{1}{2}} = D + \frac{u(x,\mathrm{t})}{2}D^{-1} - \frac{u'(x,\mathrm{t})}{4}D^{-2} + \cdots \tag{3.3.8}$$

is its square root. The $i = 1$ equation in (3.3.7) is then

$$\frac{\partial u(x,\mathrm{t})}{\partial t_1} = \left[D, D^2 + u(x,\mathrm{t})\right] = u'(x,\mathrm{t}), \tag{3.3.9}$$

where the prime will denote (from now on) differentiation with respect to x. We see that $u(x, t_1, t_2, \ldots)$ depends only on $x + t_1$ and the other t's and, hence, we can, if we want, shift t_1 (or x) to remove one variable from the equations. It is easy to see that this is always the case for the first equation of the generalized KdV hierarchy. This is obvious because $Q_+^{\frac{1}{q}} \equiv D$ for any q.

The second equation, and actually any equation for which i is even (when $q = 2$), is trivial. This can be easily seen by

$$\left(Q^{\frac{1}{2}}\right)_+^{2j} = Q_+^j = Q^j, \tag{3.3.10}$$

and, hence, since $[Q^j, Q] = 0$, we get

$$\frac{\partial u}{\partial t_{2j}} = 0. \tag{3.3.11}$$

This means that the even flows are trivial. This, again, can be generalized to any q. In this case, all the equations of order j, where j is a multiple of q, are trivial.

The third equation in the $q = 2$ hierarchy is

$$\frac{\partial u}{\partial t_3} = \left[D^3 + \frac{3}{2}uD - \frac{3u'}{4}, D^2 + u \right] = \frac{3}{2}uu' + \frac{1}{4}u'''. \tag{3.3.12}$$

This is the original Korteweg-de Vries equation that gives its name to the entire hierarchy. The higher have higher orders and higher non-linearities. In the generalized KdV hierarchy the situation is similar. The only difference being the fact that the equations involve $q - 1$ unknown functions.

Next, we quote a few facts about the KdV hierarchy. The first and most important is that this system of differential equations is integrable[87]. A system of differential equations is integrable when there is the same number of degrees of freedom as integrals (*i.e.*, constants) of motion. If the system is finite the problem (of proving that a system is integrable) is not very hard. However, for an infinite system (like the KdV hierarchy) this is not easy to prove. In physics terms, the idea is to find a canonical transformation that transforms the theory to a free one (as has been done with the sine-Gordon equation[33]). For the KdV hierarchy this has been proved, but the proof will not be presented here. However, one of the consequences is that the flows commute. That is the basic reason why we wrote the KdV hierarchy in the form (3.3.7).

Another fact is the following. If a differential operator $\mathcal{O}$ commutes with Q, then the flow of this operator is given by the equations

$$\frac{\partial \mathcal{O}}{\partial t_i} = \left[\left(Q^{\frac{1}{q}} \right)_+^i, \mathcal{O} \right]. \tag{3.3.13}$$

To prove this one differentiates $[\mathcal{O}, Q]$ with respect to t_i and then makes use of the KdV hierarchy to substitute $\partial Q/\partial t_i$ with a commutator. Next, the Jacobi identity is used to move $\mathcal{O}$ around. Then, using $[Q, \mathcal{O}] = 0$, one gets

$$\left[\frac{\partial \mathcal{O}}{\partial t_i} - \left[\left(Q^{\frac{1}{q}} \right)_+^i, \mathcal{O} \right], Q \right] = 0. \tag{3.3.14}$$

One can then show that this commutator vanishes only if equation (3.3.13) is satisfied[88].

We will end this subsection with some notations and definitions. Given a pseudodifferential operator $\mathcal{O}$, as in (3.3.1), we define $\operatorname{Res} \mathcal{O}$ as the coefficient of D^{-1},

i.e., let $\mathcal{O} = \sum_{-n}^{\infty} a_i(x) D^{-i}$, then

$$\operatorname{Res} \mathcal{O} = a_1. \tag{3.3.15}$$

The order of the operator $\operatorname{ord} \mathcal{O}$ has been defined in (3.3.4) to be n.

If the coefficient of the highest derivative is 1[†], then we can define scaling dimensions for the operator $\mathcal{O}$ and the coefficient functions $a_i(x)$[‡],

$$\dim \mathcal{O} = \operatorname{ord} \mathcal{O} = n \tag{3.3.16a}$$

$$\dim D = 1 \tag{3.3.16b}$$

$$\dim a_i(x) = i, \tag{3.3.16c}$$

where n and $a_i(x)$ refer to the symbols used in (3.3.1). The idea is that under the rescaling $D \to \lambda D$ and $a_i(x) \to \lambda^i a_i(x)$ then $\mathcal{O} \to \lambda^n \mathcal{O}$i. That means, especially, that each term in $\mathcal{O}$ transforms in the same way. That will be very useful later.

3.2. From Matrix Models to KdV

Returning to quantum gravity, we will show in this subsection how the KdV hierarchy arises. We start with the partition function for the matrix models (3.2.12), and assume, for convenience, that the potential $U(\lambda)$ is an even of order function $2m$. We then define orthonormal functions[89]

$$f_n(\lambda) = \frac{1}{\sqrt{S_n}} e^{-\frac{N}{2}U(\lambda)} \pi_n(\lambda), \tag{3.3.17}$$

which generate an abstract Hilbert space with an inner product

$$\langle f_m | f_n \rangle = \int f_m(\lambda) f_n(\lambda)\, d\lambda = \delta_{m,n}. \tag{3.3.18}$$

We define two operators, Q and P, acting in this Hilbert space by[§]

$$Q : f_m \to \lambda f_m = \sum_n Q_{mn} f_n, \tag{3.3.19}$$

and

$$P : f_m \to \frac{\partial}{\partial \lambda} f_m = \sum_n P_{mn} f_n. \tag{3.3.20}$$

[†] $a_0(x) = 1$ in equation (3.3.1).

[‡] Note that $\partial/\partial x$ does not necessarily transform as x^{-1}. One should treat D and x as independent when dealing with scaling dimensions.

[§] These are like 'position' and 'momentum' operators.

These matrices satisfy

$$Q = Q^t \quad \text{and} \quad P = -P^t, \tag{3.3.21}$$

as well as the commutation relation

$$[P, Q] = 1. \tag{3.3.22}$$

It can also be shown that

$$Q_{pq} = 0 \quad \text{if} \quad |p - q| > 1 \tag{3.3.23}$$

and

$$P_{pq} = 0 \quad \text{if} \quad |p - q| > 2m - 1. \tag{3.3.24}$$

For the $m = 2$ model in the scaling limit, n/N becomes a continuous variable x and (3.3.23) and (3.3.24) imply that P and Q become differential operators of orders 3 and 2, respectively. In fact, their explicit form is

$$Q = D^2 - u(x), \tag{3.3.25}$$

and

$$P = 4D^3 - 4u(x)D - 3u'(x), \tag{3.3.26}$$

where $D = \partial/\partial x$ as defined before, and $u(x)$ is the 'specific heat,' (3.2.36). The commutator, (3.3.22), implies an extra constraint on $u(x)$,

$$[P, Q] = 2uu' - \frac{u'''}{3} = 1. \tag{3.3.27}$$

This is the string equation (3.2.34) differentiated once. On the other hand, one recognizes $[P, Q]$ as the right-hand side of the third flow in the $q = 2$ generalized KdV hierarchy (see, (3.3.7) and (3.3.12)), up to a change of sign of u and multiplication by a constant.

This remarkable connection holds also for larger m. In general, (3.3.23) and (3.3.24) imply that Q and P are differential operators of orders 2 and $2m - 1$ (at most). Hence, P can be identified as $\left(Q^{\frac{1}{2}}\right)_+^{2m-1}$, and the string equation is the constraint of the canonical commutation. More precisely, we should identify P with a linear combination of operators

$$P = \sum_{i=1}^{m} t_i \left(Q^{\frac{1}{2}}\right)_+^{2i-1}. \tag{3.3.28}$$

Setting the coefficients t_i to specific values picks out a critical point of the theory.

In the multi-matrix models, (3.2.37), the operators P and Q are differential operators of orders q and p, where $q-1$ is the number of matrices in the model and p is the order of the potential used. It has been conjectured (and shown for $q=3$ and $p=4$[86]) that Q and P are operators in the generalized KdV hierarchy. Q is the basic operator and P is a combination as in equation (3.3.28), except that $\frac{1}{2}$ should be replaced by $\frac{1}{q}$.

In fact, the connection to the KdV hierarchy is even stronger. From the topological approach to quantum gravity, one identifies $u(x)$ (or u_{q-2} for multi-matrix models) as the two-point correlation function of the so-called puncture operator, $\mathcal{P}$. This is because the puncture operator arises from differentiation of the partition function with respect to x, which is identified with the cosmological constant in the continuum limit. This identification will be extended in the next section to a map between all the fields in the matrix models (or in the topological approach) and operators in the KdV hierarchy.

3.3. KdV Gravity

In this subsection we will define quantum gravity models in terms of the generalized KdV hierarchy and present a method to compute correlation functions, which, in some sense, solves the theory.

We start with a qth order differential operator Q,

$$Q = D^q + u_{q-2}D^{q-2} + u_{q-3}D^{q-3} + \cdots + u_0, \tag{3.3.29}$$

with $q-1$ (as yet) unknown functions. We also construct the various powers of the q-th root of Q,

$$Q^{i/q} \equiv \left(Q^{\frac{1}{q}}\right)^i = D^i + \sum_{j=2}^{\infty} v_j^{(i)} D^{i-j}, \tag{3.3.30}$$

where the $v_j^{(i)}$ are functionals of the u_i in (3.3.29) and their derivatives. We then define P as the sum

$$P = \sum_{i=1}^{p} t_i \left(Q^{i/q}\right)_+ . \tag{3.3.31}$$

This is a p-th order differential operator.

Next, we write the string equation

$$[P, Q] = 1. \tag{3.3.32}$$

This is a system of differential equations that determines the functions $u_i(x)$ (up to constants of integration). In a sense, this equation (or, actually the choice of P) determines which critical point we must choose—or which model we are defining. We define the (p,q)-model as the model for which P is the single operator

$$P = \left(Q^{p/q}\right)_+ . \tag{3.3.33}$$

(Later we will explicitly solve the $(1,q)$-models and prove some interesting selection rules for them.)

To continue, we now give the map between fields in the theory and operators in the KdV-hierarchy. For each 'primary field' or matrix $\mathcal{P}_\alpha$ in the original matrix model there is a corresponding pseudodifferential operator $Q^{\alpha/q}$ in the KdV hierarchy[84,85]

$$\mathcal{P}_\alpha \leftrightarrow Q^{\alpha/q} \quad \alpha = 1, \ldots, q-1. \tag{3.3.34}$$

The 'descendant' fields $\mathcal{P}_i$ correspond to pseudodifferential operators as follows

$$\mathcal{P}_i = \mathcal{P}_{\alpha+mq} \equiv \sigma_m(\mathcal{P}_\alpha) \leftrightarrow Q^{m+\alpha/q} \quad i > q. \tag{3.3.35}$$

It should be noted that the fields come in natural groupings of $q-1$ fields. This is because $\mathcal{P}_{qm}$ is mapped to a *differential* operator, Q^m, and, as we will see shortly, it does not contribute to correlation functions because of the trivial way in which it flows under the KdV hierarchy. We also note that, by the mapping (3.3.35), we can associate

$$\mathrm{ord}\left(\sigma_m\left(\mathcal{P}_\alpha\right)\right) = \dim\left(\sigma_m\left(\mathcal{P}_\alpha\right)\right) = \alpha + qm, \tag{3.3.36}$$

where ' ord ' is defined in (3.3.4).

The correlation functions in these (KdV gravity) models are determined as follows. We identify the function of lowest scaling dimension (defined in equation (3.3.16c)), u_{q-2}, with the two-point function of lowest dimension primary fields. The lowest dimension primary, $\mathcal{P}_1$, called the puncture operator, is mapped to $Q^{\frac{1}{q}}$. The identification of u_{q-2} with the specific heat is correct up to a proportionality constant. Choosing this constant to be q** we can write

$$\langle \mathcal{P}_1 \mathcal{P}_1 \rangle = \mathrm{Res}\, Q^{1/q}. \tag{3.3.37}$$

The higher point correlation functions are determined by the flows of the KdV hierarchy. Each operator has a flow parameter, t_i, and differentiating a correlation function

**One can rescale the fields to get rid of this proportionality constant.

with respect to this parameter is equivalent to the insertion of the corresponding operator in the correlation function. Using the flows of the KdV hierarchy, (3.3.7), we have a prescription to compute the higher point functions. To compute correlation functions with less than two punctures one has to integrate with respect to the cosmological constant. There is some ambiguity as to the nature of the constants of integration. We will return to this ambiguity when we discuss the computation in greater detail in Chapter 5. For the time being, we will only consider correlation functions with at least two punctures.

One last remark about the correlation functions. Suppose that we had an action for our models. Then we might imagine adding, as a perturbation to the action, all the fields in the theory with coefficients t_i. Differentiating the partition function with respect to one of these parameters is equivalent to inserting the corresponding operator (field) into the partition function—making it a one point correlation function. Taking more derivatives brings down more fields. In the end, we should set all the parameters to zero to obtain the correlation functions. The idea in KdV gravity is that the extra information carried by these derivatives corresponds to flows of the KdV hierarchy whose solution, therefore, constitutes a solution to the original problem.

4. The Structure of (1,q) Models

In this section we will explain the structure of $(1,q)$ models—their advantage being that one can explicitly write the operators and compute the exact non-perturbative correlation functions. In general (p,q) models, the solution depends on the exact solution of the string equation, and this equation usually cannot be solved analytically. For example, Di Francesco and Kutasov[85] have computed some correlation functions of the $(q+1,q)$ models, but their calculation is only complete to the first order in the genus expansion (*i.e.*, for the sphere).

4.1. Q, P, and Other Operators

The $(1,q)$ models are defined by the identification

$$P = Q_+^{\frac{1}{q}} = D. \tag{3.4.1}$$

The string equation is then given by

$$[P,Q] \equiv [D,Q] = 1, \tag{3.4.2}$$

which implies (see (3.3.5))

$$u_l' = \delta_{l0} \quad \Rightarrow \quad u_l = x\delta_{l0}. \tag{3.4.3}$$

The integration constants are set to zero in order to have well-defined scaling dimensions. This is required for the following reason. The functions u_i have scaling dimensions (compare (3.3.1), (3.3.5), and (3.3.16c))

$$\dim u_i(x) = q - i\,, \quad i = 0,1,\ldots,q-2. \tag{3.4.4}$$

If $u_i(x) = c_i$ for $i \neq 0$ where c_i are constants, then, clearly, (3.4.4) requires setting these constants to zero. Similarly, in the above, $x = u_0$ has scaling dimension q. Thus, the derivative of x, a constant, must have a scaling dimension of $q+1$*, but this is inconsistent with the scaling behavior the constants of integration would have

*Note that x and $\partial/\partial x$ are scaled differently as mentioned before.

in (3.4.4). Thus, the constants of integration must all be set to zero. This is also consistent with the topological field theory interpretation of these models.

Equation (3.4.4) implies that the $(1,q)$ models are defined by the Q-operator,

$$Q = D^q + x. \tag{3.4.5}$$

These models exhibit some interesting structure due to the simple form of Q. Since $u_0 = x$, with $\dim u_0 = q$, is the only building block (apart from derivatives), the operators in the theory have a form that we will call p-type[†].

Definition: **P-Type Operator**

A pseudodifferential operator, $\mathcal{O}$,

$$\mathcal{O} = \sum_{i=0}^{\infty} o_i(x) D^{-i+n} \tag{3.4.6}$$

is a *p-type operator of index* q if, for some positive integer q, $o_i(x)$ are polynomials in x of the form

$$o_i(x) = \begin{cases} o_0^i x^k + o_1^i x^{k-(q+1)} + \cdots + o_l^i x^{k-l(q+1)} + \cdots + o_{[k/q+1]}^i x^{k \bmod (q+1)}, & \text{if } k \geq 0; \\ 0, & \text{if } k < 0. \end{cases} \tag{3.4.7}$$

where the integer k is given by

$$k = [\frac{i}{q}] - (i \bmod q), \tag{3.4.8}$$

and $[x]$ is the greatest integer less than or equal to x.

For example, let $Q^{1/q} = D + \sum_{i=1}^{\infty} a_i(x) D^{-i}$. Then the scaling dimension of $a_i(x)$ is $i+1$. So, if (say) $q = 4$, $a_{28}(x) = \alpha x^6 + \beta x$ is the only form $a_{28}(x)$ can have.

Let us explain the basic structure that distinguishes p-type operators from other operators. If the operator is of order n then the coefficient of D^n (the highest term) is a constant. The next term possible is of order $n-q$ and its coefficient is proportional to $u_0 = x$. That is, we have a gap of q terms. The next one is of order $n-q-1$ and its coefficient is a constant (equal to a derivative of $u_0 = x$). Then we have another gap, this time of $q-1$ terms, and the next order is D^{n-2q}, with coefficient proportional to $u_0^2 = x^2$, and so forth. The other thing to notice is that when we get to order $q(q+1)$ we start getting polynomials in x as the coefficients. This is because $(u_0')^q$ is a constant of the same dimension as $u_0^{q+1} = x^{q+1}$.

[†]This name was chosen because in the first formulation of this solution we denoted the q-th root of Q by P and we found that all the powers of P had the same general form.

It is easy now to convince oneself that $Q^{\frac{1}{q}}$ is a p-type operator. For this reason one can compute many terms of this q-th root, the first few being

$$Q^{\frac{1}{q}} = D + \frac{x}{q}D^{1-q} - \frac{q-1}{2q}D^{-q} - \frac{q-1}{2q^2}x^2D^{1-2q} + \cdots . \tag{3.4.9}$$

In the Appendix we present all the terms we have computed for the more general operator $Q^{i/q}$.

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are p-type, then so is the product $\mathcal{O}_1\mathcal{O}_2$. Hence, $Q^{\alpha/q}$ are all p-type. However, the sum $\mathcal{O}_1 + \mathcal{O}_2$ is not always of this type. Only when the orders of the operator agree can we sum them (up to some shift modulo $q+1$, as will be proved in Lemma 1 in the next section). As we will see later, we need a way to sum such operators, but we have to do it only in commutators. For this we need a theorem that we will present in the next section.

4.2. The Basic Theorem

In this section we prove a theorem, which we will use in Chapter 5 to derive selection rules and correlation functions.

Theorem

Let $\mathcal{O}^{(l)}$ and $\mathcal{O}^{(m)}$ be p-type operators of orders l and m and same index q, respectively. Then the commutator $\left[\mathcal{O}^{(l)}, \mathcal{O}^{(m)}\right]$ is a p-type operator of index q and order less than or equal to $m + l - (q+1)$.

Before we give the proof, let us present two useful lemmas.

Lemma 1

Let $\mathcal{O}^{(m)}$ and $\mathcal{O}^{(n)}$ be p-type operators of index q and orders m and n. Then the operator $\mathcal{O}^{(m)} + \mathcal{O}^{(n)}$ is a p-type operator of index q if $m = n (\mathrm{mod}\, q+1)$. The order of this operator is $\max(m, n)$.

Proof of Lemma 1

Without loss of generality, let $n = m + l(q+1)$, where l is some positive integer. Letting (change summation index in (3.4.6))

$$\mathcal{O}^{\alpha} = \sum_{i=-\alpha}^{\infty} {}^{(\alpha)}o_i(x)D^{-i}, \tag{3.4.10}$$

where $\alpha = m, n$, the coefficient of D^{-i} in $\mathcal{O}^{(m)} + \mathcal{O}^{(n)}$ is ${}^{(m)}o_i(x) + {}^{(n)}o_i(x)$. We can compute the dimensions

$$\dim {}^{(\alpha)}o_i(x) = \alpha + i, \tag{3.4.11}$$

and, hence,

$$\dim^{(n)} o_i(x) - \dim^{(m)} o_i(x) = n - m = l(q+1) = 0 \bmod (q+1). \tag{3.4.12}$$

Thus, the polynomials agree since the powers of x that appear in both are the same modulo $q+1$. (Formally, $x^{\dim^{(n)} o_i(x) - \dim^{(m)} o_i(x)} = x^{n-m} = x^{l(q+1)}$.) Obviously, the order of the resulting operator $\mathcal{O}^{\alpha}$ is n. QED.

Lemma 2

Let $\mathcal{O}^{(n)}$ be an order n p-type operator of index q,

$$\mathcal{O}^{(n)} = \sum_{i=0}^{\infty} o_i(x) D^{n-i}. \tag{3.4.13}$$

Then the operator $\mathcal{O}^{(n)\prime}$ defined by

$$\mathcal{O}^{(n)\prime} = \sum_{i=0}^{\infty} o_i'(x) D^{n-i} \tag{3.4.14}$$

is a p-type operator of index q and order $n-q$.

Proof of Lemma 2

Using (3.4.7) we get $o_0(x) \propto 1$ and $o_i(x) = 0$ for $1 \leq i < q$. Thus,

$$\begin{aligned} \mathcal{O}^{(n)\prime} &= \sum_{i=0}^{\infty} o_i'(x) D^{n-i} \\ &= \sum_{i=q}^{\infty} o_i'(x) D^{n-i} \\ &= \sum_{j=0}^{\infty} o_{j+q}'(x) D^{n-j-q} \\ &\equiv \sum_{j=0}^{\infty} \tilde{o}_j(x) D^{n-j-q}, \end{aligned} \tag{3.4.15}$$

where we changed the summation index and defined $\tilde{o}_i(x)$. We see that $\operatorname{ord} \mathcal{O}^{(n)\prime} = n - q$. Next, since $\tilde{o}_j(x) = o_{j+q}'(x)$, we have from (3.4.7)

$$\begin{aligned} \tilde{o}_j(x) &= (o_0^i x^k + o_1^i x^{k-q-1} + \cdots)' \\ &= k o_0^i x^{k-1} + (k-q-1) o_1^i x^{k-q-2} + \cdots, \end{aligned} \tag{3.4.16}$$

where k is given by (3.4.8)

$$k = [\frac{j+q}{q}] - ((j+q) \bmod q). \tag{3.4.17}$$

Hence,

$$k-1=[\frac{j}{q}]-(j \bmod q). \tag{3.4.18}$$

This completes the proof. In Table 6 we illustrate this Lemma.

Order n	Coeff. in $\mathcal{O}^n$	Coeff. in $\mathcal{O}^{n'}$	Order $m=n-q$
n	1	0	–
$n-1$	0	0	–
$n-2$	0	0	–
$\vdots$	$\vdots$	$\vdots$	–
$n-q+1$	0	0	–
$n-q$	x	1	m
$n-q-1$	1	0	$m-1$
$n-q-2$	0	0	$m-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2q+1$	0	0	$m-q+1$
$n-2q$	x^2	x	$m-q$
$n-2q-1$	x	1	$m-q-1$
$n-2q-2$	1	0	$m-q-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-q(q+1)$	$x^{q+1},1$	$x^q,0$	$m-q^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 6: Differentiation maps a p-type operator of order n into one of order $m=n-q$.

Proof of Theorem

The commutator $\left[\mathcal{O}^{(l)},\mathcal{O}^{(m)}\right]$ is a sum of commutators of the form

$$\left[f(x)D^a,g(x)D^b\right], \tag{3.4.19}$$

where $f(x)$ and $g(x)$ are some polynomials as in the definition of p-type operators. We can treat $f(x)D^a$ as if it's an order l p-type operator (of index q) and, similarly,

$g(x)D^b$ is order m p-type. To prove the theorem we have to show that the commutator (3.4.19) is a p-type operator of order $m+l-q-1$ and then, by Lemma 1, we can sum these commutators and get the required result.

So, we decompose (3.4.19) as follows

$$\left[f(x)D^a, g(x)D^b\right] = f(x)D^{a-1}\left[D, g(x)D^b\right] + \left[f(x)D^{a-1}, g(x)D^b\right]D. \tag{3.4.20}$$

On the right-hand side, $f(x)D^{a-1}$ is order $l-1$ (p-type), and $\left[D, g(x)D^b\right]$ is order $m-q$ by Lemma 2. Thus, the first term on the right-hand side is a p-type operator of order $l-1+m-q = l+m-q-1$. The second term still appears to be of order $l+m$. We now continue, recursively, in the same decomposition as in (3.4.20), with $\left[f(x)D^{a-1}, g(x)D^b\right]D$, and at each stage we get an order $l+m-q-1$ operator with an $m+l$ one. The last stage yields

$$\left[f(x)D^a, g(x)D^b\right] = \mathcal{O}^{l+m-q-1} + \left[f(x), g(x)D^b\right]D^a, \tag{3.4.21}$$

where we used Lemma 1 to sum the operators and $\mathcal{O}^{l+m-q-1}$ is some p-type operator of order $m+l-q-1$.

Proceeding, in the same way, with the difference being that we use $g(x)D^b$ and operates on $f(x)$, the end result is

$$\left[f(x)D^a, g(x)D^b\right] = \tilde{\mathcal{O}}^{l+m-q-1} + \left[f(x), g(x)\right]D^{a+b}, \tag{3.4.22}$$

where $\tilde{\mathcal{O}}^{l+m-q-1}$ is an order $m+l-q-1$ p-type. But, obviously,

$$\left[f(x), g(x)\right] = 0, \tag{3.4.23}$$

and, thus, we proved the theorem.

Corollary

Let $\mathcal{O}^{(l)}$ and $\mathcal{O}^{(m)}$ be p-type operators of orders l and m and same index q. Then the commutator $\left[\mathcal{O}_+^{(l)}, \mathcal{O}^{(m)}\right]$ is a p-type operator of index q and order less than or equal to $m+l-(q+1)$.

Proof : $\mathcal{O}_+^{(l)}$ is also p-type.

5. Correlation Functions and Selection Rules of (1,q) Models

5.1. Selection Rules

As was mentioned in Chapter 3, for *any* KdV gravity model, all correlation functions with two punctures are given by the KdV flows of the two-puncture correlation function,

$$\langle \mathcal{P}_1 \mathcal{P}_1 \rangle = \frac{1}{q} u_{q-2} = \operatorname{Res} Q^{\frac{1}{q}} = \operatorname{Res} R, \tag{3.5.1}$$

where we have denoted, for simplicity, $Q^{\frac{1}{q}} \equiv R$ (for Root). Then, one can compute higher point functions as follows,

$$\langle \mathcal{P}_1 \mathcal{P}_1 \mathcal{P}_i \rangle = \frac{\partial}{\partial t_i} \langle \mathcal{P}_1 \mathcal{P}_1 \rangle = \frac{\partial}{\partial t_i} \operatorname{Res} R, \tag{3.5.2}$$

where the last step follows from (3.5.1). We now use the KdV flows (3.3.7) and get

$$\frac{\partial}{\partial t_i} \operatorname{Res} R = \operatorname{Res} \left[R_+^i, R \right] = \operatorname{Res} [R, R_-^i]. \tag{3.5.3}$$

We have used the fact that

$$\left[R_+^i, R^j \right] = \left[R^i - R_-^i, R^j \right] = \left[R^j, R_-^i \right], \tag{3.5.4}$$

where $\mathcal{O}_- \equiv \mathcal{O} - \mathcal{O}_+$. Then we note that $\left[\mathcal{O}_-^{(1)}, \mathcal{O}_-^{(2)} \right]$ does not have a residue (first non-vanishing order is at least -2), so we can change R^j in the right-hand expression of (3.5.4) into R_+^j. Also, $\left[\mathcal{O}_+^{(1)}, \mathcal{O}_+^{(2)} \right]$ does not have a residue (lowest order is at most 2) so we change R_-^i into R^i and then into R_+^i. Thus, using $R_+ \equiv D$, we get,

$$\langle \mathcal{P}_1 \mathcal{P}_1 \mathcal{P}_i \rangle = \operatorname{Res} \left[D, R^i \right] = (\operatorname{Res} R^i)'. \tag{3.5.5}$$

To prove the last step, let $R^i = \sum_{j=0}^{\infty} r_j(x) D^{i-j}$. Thus,

$$\left[D, R^i \right] = \sum_{j=0}^{\infty} \left[D, r_j(x) D^{i-j} \right] = \sum_{j=0}^{\infty} r_j'(x) D^{i-j}, \tag{3.5.6}$$

and, hence, $\operatorname{Res} \sum_{j=0}^{\infty} r_j'(x) D^{i-j} = r_{i-1}'(x) = (\operatorname{Res} R^i)'$.

To compute the two-point function $\langle \mathcal{P}_1 \mathcal{P}_i \rangle$, we integrate (3.5.5) with respect to $t_1 = x$ and, as explained earlier, choose the constant of integration to be zero, in agreement with references [84,86]. Hence, we get

$$\langle \mathcal{P}_1 \mathcal{P}_i \rangle = \operatorname{Res} R^i, \tag{3.5.7}$$

which is consistent with (3.5.1). Similarly, we get formulae for higher correlation functions*,

$$\langle \mathcal{P}_1 \mathcal{P}_i \mathcal{P}_j \rangle = \frac{\partial}{\partial t_i} \operatorname{Res} R^j = \operatorname{Res} \left[R^i_+, R^j \right], \tag{3.5.8}$$

$$\langle \mathcal{P}_1 \mathcal{P}_i \mathcal{P}_j \mathcal{P}_k \rangle = \operatorname{Res} \left(\left[\left[R^k_+, R^i \right]_+, R^j \right] + \left[R^i_+, \left[R^k_+, R^j \right] \right] \right), \tag{3.5.9}$$

and so on. The general structure is as follows. For every new operator insertion in the correlation function, we replace, in the residue argument, each possible operator by a commutator of the form $[R^i_+, R^j]$ or $[R^i_+, R^j]_+$, *e.g.*, we replace $\left[R^i_+, R^j \right]$ in equation (3.5.8) by the sum in (3.5.9).

Now comes the important step. According to the basic theorem of Chapter 4, when we compute the operator inside the residue, in each stage of the computation we have, recursively, a commutator of a truncated p-type operator with another p-type operator. Thus, again according to the theorem, each new flow changes the order of the operator inside the residue by $(\operatorname{ord} \mathcal{O}) - (q+1)$. Hence, for a general correlation function, we have

$$\langle \mathcal{P}_1 \mathcal{P}_{i_1} \mathcal{P}_{i_2} \cdots \mathcal{P}_{i_n} \rangle = \operatorname{Res} \mathcal{O}, \tag{3.5.10}$$

where $\mathcal{O}$ is a p-type operator, and we can easily compute the order of this operator. It is (at most†)

$$\operatorname{ord} \mathcal{O} = i_n - \sum_{j=1}^{n-1} (q + 1 - i_j), \tag{3.5.11}$$

where we choose R^{i_n} as the 'first' operator on which we begin to compute, and at each subsequent insertion we reduce the order of the resulting operator by the amount $q + 1 - i_j$.

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*No need for integration here, since we can use the KdV flows on equation (3.5.7).

†It may happen, by 'accident,' that the highest order (or few highest) will disappear. The operator is still *p-type* of the order given, but as a regular operator it might be of lower order.

It is natural to associate a 'ghost' number charge with each field, $\mathcal{P}_i$

$$\mathrm{gh}\,\mathcal{P}_i = q + 1 - i. \tag{3.5.12}$$

In order that $\mathcal{O}$ have a nonzero residue, we must have

$$q + 1 - \sum_{j=1}^{n} \mathrm{gh}\,\mathcal{P}_{i_j} \geq -1, \tag{3.5.13}$$

or, adding $\mathrm{gh}\,\mathcal{P}_{i_0} \equiv \mathrm{gh}\,\mathcal{P}_1 = q$,

$$\sum_{j=0}^{n} \mathrm{gh}\,\mathcal{P}_{i_j} \leq 2(q+1). \tag{3.5.14}$$

This is the first selection rule.

The second selection rule is easier to derive, but it is true only in the 'topological limit,' *i.e.*, when $x \to 0$[‡]. Since we associate a scaling dimension, $\dim \mathcal{P}_i = i$, with each field (primary or descendant), and since x has scaling dimension q, we must have

$$\sum_{j=0}^{n} i_j = 0 \bmod (q+1). \tag{3.5.15}$$

Otherwise, the correlation function in equation (3.5.10) cannot be proportional to a constant and, thus, must vanish in the topological limit.

By using equations (3.5.14), (3.5.15) and (3.5.12), we can write the following conservation law,

$$\sum_{j=0}^{n} \mathrm{gh}\,\mathcal{P}_{i_j} = 2(1-g)(q+1), \tag{3.5.16}$$

where g can be any non-negative half-integer. It is now tempting to identify (3.5.16) with the contributions to the correlation functions coming from different genera. We see that each correlation function receives a contribution only from one genus, which is given by this equation. The selection rule does not exclude half-integer g, which would correspond to bordered Riemann surfaces[§]. In the $(1, q)$ models, direct computation

[‡]This limit is called 'topological' for the following reason. In the topological field theory approach to quantum gravity, the Lagrangian is topological. That is, it is the anti-commutator of the BRST operator with some other operator, $\mathcal{L} = \{Q_{BRST}, \Lambda\}$. We can add perturbations to the Lagrangian such as $x \int \Sigma(\mathcal{P}_1)$, where $\Sigma(\mathcal{P}_1)$ is the 2-form version of $\mathcal{P}_1$. But then $\mathcal{L}$ is not topological any more[59,22].

[§]We assume that KdV gravity corresponds to the hermitian matrix model and, thus, to orientable surfaces. Otherwise we can add crosscaps.

of the correlation functions (that will be presented in the next section and in the appendix) indicates that only integer valued genera contribute. This hints at the existence of more symmetry.

Let us note that this conservation law, or even the slightly weaker selection rule (3.5.14), is very restrictive. If we consider correlation functions of primary fields only, the ghost numbers are all positive, as seen by the definition (3.5.12) and the fact that the scaling dimension is less than q for primary fields. The limit in (3.5.14) is therefore saturated quickly. For example, if $q = 4$ (say), then the only primary field correlation functions that can be non-zero in the limit $x \to 0$ are $\langle \mathcal{P}_1^2 \mathcal{P}_2 \rangle$, $\langle \mathcal{P}_1 \mathcal{P}_2^2 \rangle$, $\langle \mathcal{P}_1 \mathcal{P}_3^3 \rangle$, $\langle \mathcal{P}_2^2 \mathcal{P}_3^2 \rangle$ and $\langle \mathcal{P}_3^5 \rangle$. The last two correlation functions do not contain the puncture operator, which means that we would need to integrate a higher correlation function (once) to compute them.

5.2. Correlation Functions

To explicitly compute some correlation functions we need an expression for R^i. A direct computation, using $R \equiv Q^{1/q}$ and $Q = D^q + x$, leads to the following form for R^i**,

$$\begin{aligned} R^i =& D^i + \frac{i}{q} x D^{i-q} - \frac{i(q-i)}{2q} D^{i-q-1} - \frac{i(q-i)}{2q^2} x^2 D^{i-2q} + \\ & \frac{i(q-i)(2q-i)}{2q^2} x D^{i-2q-1} - \frac{i(q-i)(2q-i)(5q-3i+4)}{24q^2} D^{i-2q-2} + \\ & \frac{i(q-i)(2q-i)}{6q^3} x^3 D^{i-3q} - \frac{i(q-i)(2q-i)(3q-i)}{4q^3} x^2 D^{i-3q-1} + \\ & \frac{i(q-i)(2q-i)(3q-i)(8q-3i+4)}{24q^3} x D^{i-3q-2} + \dots . \end{aligned} \tag{3.5.17}$$

One can verify this formula by noting that $R^i R^j = R^{i+j}$ and that $R^q = D^q + x \equiv Q$. Hence, we can now compute the correlation functions explicitly. We restrict ourselves to correlation functions of primary fields only. In the Appendix we provide some expressions for correlation functions of descendants. When computing correlation functions with two punctures, which are well defined without integration, one sees that they vanish for four-point and higher as a consequence of ghost number conservation. For the one puncture correlation functions, one gets

$$\langle \mathcal{P}_1 \mathcal{P}_i \rangle = \frac{q-1}{q} x \delta_{i,q-1} \tag{3.5.18}$$

** In the Appendix we present the full expression, to the order we have computed, together with some descendant correlation functions.

$$\langle \mathcal{P}_1 \mathcal{P}_i \mathcal{P}_j \rangle = \frac{ij}{q} \delta_{i+j,q} \tag{3.5.19}$$

$$\langle \mathcal{P}_1 \mathcal{P}_i \mathcal{P}_j \mathcal{P}_k \rangle = 0 \tag{3.5.20}$$

$$\langle \mathcal{P}_1 \mathcal{P}_i \mathcal{P}_j \mathcal{P}_k \mathcal{P}_l \rangle = 0. \tag{3.5.21}$$

$$\langle \mathcal{P}_1 \mathcal{P}_i \mathcal{P}_j \mathcal{P}_k \mathcal{P}_l \mathcal{P}_m \rangle = 0. \tag{3.5.22}$$

One immediately notices the interesting result that the four-, five- and six-point functions with one puncture vanish identically[††] (even for $x \neq 0$) when they obey the selection rule (3.5.16). This may be traced back to the choice of integration constants made in computing the two-point function $\langle \mathcal{P}_1 \mathcal{P}_i \rangle$. This implies that in the integration one introduced a new selection rule beyond ghost number conservation. A different choice, some analytic function of the flow parameters t_i, $i \neq 0$, would lead to other values. For example, if we choose the two-point function to be

$$\langle \mathcal{P}_1 \mathcal{P}_i \rangle = \operatorname{Res} R^i + a_i + \frac{1}{2} b_i{}^j t_j + \frac{1}{6} c_i{}^{jk} t_j t_k + \dots , \tag{3.5.23}$$

we get corrections to the higher point functions. In particular, the four-point would be

$$\langle \mathcal{P}_1 \mathcal{P}_i \mathcal{P}_j \mathcal{P}_k \rangle = c_{ijk}. \tag{3.5.24}$$

We do not have a good understanding as to which is the correct choice, and it might be that any choice is suitable, or that the only 'physically' meaningful quantities are those derived from the specific heat. This would imply, for example, that the only 'physical' correlation functions are those with two puncture operators. One could determine which of the constants of integration do not vanish, for example, by requiring that correlation functions that conserve ghost number do not vanish, *i.e.*, that no new selection rules are introduced. We will have more to say on this issue in the next chapter.

[††] Of course, one tends to conjecture that all higher point functions vanish also. Some nonvanishing correlation functions of descendants are given in the Appendix. These receive contributions at higher genus.

6. Summary and Discussion

We have explicitly computed correlation functions of fields in the (1,q) model of KdV gravity. As has been explained, these correspond to the critical topological point of the KdV hierarchy. Though these models have a very simple string equation, their solutions possess very interesting properties. We have shown how ghost number conservation, a property of a topological field theory, arises algebraically in the (1,q) series. It was also easy to see how the genus expansion follows from this ghost number conservation. Some of these properties are special to the (1,q) models. Ghost number is not conserved at the higher critical points, since these points are reached by perturbing with operators carrying ghost number (in the topological sense). It is natural to suppose that solutions of the first critical theory are sufficient to derive, by perturbation, the solutions to the higher critical point theories.

We have been somewhat conservative in our discussion of the (1,q) models. It has been claimed, at least for the one matrix model, that the partition function may be a special τ function of the KdV hierarchy[90,91]. This τ function would the Virasoro constraints and, thus, be determined uniquely. For the multi-matrix models one has to impose further constraints to fix τ uniquely, *i.e.*, W-algebra constraints. For the one matrix model, the constraint $L_{-1}\tau = 0$ can be derived by integrating the string equation. The other constraints are determined using the KdV flows. This procedure would then provide an unambiguous way to determine any correlation function with or without punctures, given the τ function. From KdV gravity we only have information about the specific heat, $u_{q-2} = D^2 \log \tau$. Even the string equation is only a constraint on the specific heat. Integrating this equation in order to derive the first Virasoro constraint requires dropping a constant of integration, which is, in general, a function of the $t_i, i \neq 0$. Indeed, since the KdV and string equations only involve the specific heat, the partition function is only determined up to which differ by multiplication by a function independent of t_1. If one wishes to maintain ghost number conservation, then it is further constrained. We have thus followed a more conservative attitude and basically only discussed correlators derived from the specific heat and not the partition function.

On the other hand, it might well be that the definition of KdV gravity is not

restricted enough. The natural extension is to impose the Virasoro constraints in the one-matrix model or the W_n constraints in the $n-1$ matrix models. In the one-matrix case this indeed amounts to integrating the string equation and fixing the integration constants to be zero.

Appendix A. Some Explicit Formulae

In this appendix we present explicit expressions for R^i (given $R^q = D^q + x$) to all the orders that we have computed and for some correlation functions involving descendants.

R^i can be written in the following compact form

$$R^i = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \alpha^i{}_{j,k} x^{j-k} D^{i-jq-k}. \tag{3.A.1}$$

The coefficients are as follows:

$$\alpha^i{}_{j,0} = \frac{1}{j!} \prod_{n=0}^{j-1} (i - nq) \quad j \geq 0 \tag{3.A.2}$$

$$\alpha^i{}_{j,1} = \frac{1}{2(j-1)!} \prod_{n=0}^{j} (i - nq) \quad j \geq 1 \tag{3.A.3}$$

$$\alpha^i{}_{j,2} = \frac{3i - 4 - (3j-1)q}{24(j-2)!} \prod_{n=0}^{j} (i - nq) \quad j \geq 2 \tag{3.A.4}$$

$$\alpha^i{}_{j,3} = \frac{(i-2-jq)(i-2-(j-1)q)}{48(j-3)!} \prod_{n=0}^{j} (i - nq) \quad j \geq 3 \tag{3.A.5}$$

$$\begin{aligned} \alpha^i{}_{4,4} = & \frac{1}{5760}(-288 + 320i - 120i^2 + 15i^3 - 1072q + 800iq - 150i^2 q \\ & - 1288q^2 + 485iq^2 - 502q^3) \prod_{n=0}^{4} (i - nq) \end{aligned} \tag{3.A.6}$$

$$\begin{aligned} \alpha^i{}_{5,4} = & \frac{1}{5760}(-288 + 320i - 120i^2 + 15i^3 - 1392q + 1040iq - 195i^2 q \\ & - 2208q^2 + 830iq^2 - 1152q^3) \prod_{n=0}^{5} (i - nq) \end{aligned} \tag{3.A.7}$$

$$\begin{aligned} \alpha^i{}_{6,4} = & \frac{1}{11520}(-288 + 320i - 120i^2 + 15i^3 - 1712q + 1280iq - 240i^2 q \\ & - 3368q^2 + 1265iq^2 - 2192q^3) \prod_{n=0}^{6} (i - nq) \end{aligned} \tag{3.A.8}$$

$$\alpha^i{}_{5,5} = \frac{(i-4-5q)(i-4-4q)(24-16i+3i^2+64q-23iq+38q^2)}{11520}\prod_{n=0}^{5}(i-nq) \tag{3.A.9}$$

$$\alpha^i{}_{6,5} = \frac{(i-4-6q)(i-4-5q)(24-16i+3i^2+80q-29iq+64q^2)}{11520}\prod_{n=0}^{6}(i-nq) \tag{3.A.10}$$

$$\begin{aligned}\alpha^i{}_{6,6} =& \frac{1}{2903040}(-69120+82656i-40544i^2+10080i^3-1260i^4+63i^5-420768q+\\ & 408576iq-151536i^2q+25200i^3q-1575i^4q-1002976q^2+741384iq^2\\ & -184884i^2q^2+15435i^3q^2-1172904q^3+586656iq^3-73801i^2q^3\\ & -674256q^4+171150iq^4-152696q^5)\prod_{n=0}^{6}(i-nq).\end{aligned} \tag{3.A.11}$$

The above expression has been proved to order $i-7q-3$. The general expressions for $\alpha^i{}_{jk}$ for $k = 0$, 1, 2 and 3 are only a conjecture that works for the orders we computed.

One can now use this expression for R^i to compute some correlation functions. For convenience, we computed them in the topological limit, $x \to 0$, although one can compute them almost as easily for a general point. The only difference is that their form is not as elegant. In the following, $1 \le i \le q-1$ (primary field range) unless something is a multiple of q (that is, $q-i+2$, say, could be a multiple of q and then the correlation function must vanish) . Also, k ranges over all non-negative integers such that we have at least one puncture operator, $\mathcal{P}_1$.

$$\langle \mathcal{P}_1^{k+2}\sigma_k(\mathcal{P}_{q-1})\rangle = \frac{1}{q^{k+1}}\prod_{n=1}^{k+1}(nq-1) \tag{3.A.12}$$

$$\langle \mathcal{P}_1^{k-1}\sigma_k(\mathcal{P}_1)\rangle = \frac{1}{24q^k}(q-1)\prod_{n=1}^{k}(nq+1) \tag{3.A.13}$$

$$\langle \mathcal{P}_1^{k+1}\mathcal{P}_i\,\sigma_k(\mathcal{P}_{q-i})\rangle = \frac{(-1)^{k+1}}{q^{k+1}}\prod_{n=0}^{k+1}(i-nq) \tag{3.A.14}$$

$$\langle \mathcal{P}_1^{k-1}\mathcal{P}_i\,\sigma_k(\mathcal{P}_{q-i+2})\rangle = \frac{(-1)^{k+1}}{24q^{k+1}}i(i-q)\prod_{n=0}^{k+1}(i-nq-2) \tag{3.A.15}$$

$$\langle \mathcal{P}_1^{k+2}\sigma_1(\mathcal{P}_i)\sigma_k(\mathcal{P}_{q-i})\rangle = \frac{(k+1)(-1)^{k+1}}{q^{k+2}}\prod_{n=0}^{k+2}(i-nq) \tag{3.A.16}$$

$$\langle \mathcal{P}_1^{k}\sigma_1(\mathcal{P}_i)\sigma_k(\mathcal{P}_{q-i+2})\rangle = \frac{(k+1)(-1)^{k+1}}{24q^{k+2}}i(i-q)(i-2q)\prod_{n=1}^{k+2}(i-nq-2). \tag{3.A.17}$$

It should be noted that correlation functions with descendants receive contributions at higher genus. The genus at which the above correlation functions are nonvanishing can be determined by computing the ghost number and using (3.5.16). Thus, for the correlation functions above, (3.A.12), (3.A.14) and (3.A.16) are at genus 0 and (3.A.13), (3.A.15) and (3.A.17) are at genus 1. For correlations that receive contribution at higher genus we have, for example,

$$\langle \mathcal{P}_1\sigma_4(\mathcal{P}_3)\rangle = -\frac{1}{640q^4}(q^2-9)(q^2-1)(2q+1)(2q+3)(4q+3) \tag{3.A.18}$$

at genus 2, and

$$\begin{aligned}\langle \mathcal{P}_1\sigma_6(\mathcal{P}_5)\rangle =& \frac{5}{580608q^6}(q^2-25)(q^2-1)(2q+1)(2q+5)(3q+5)\\ & (4q+5)(6q+5)(8q^2-13q-13)\end{aligned} \tag{3.A.19}$$

at genus 3.

We can prove that for a correlation function of one puncture, $\mathcal{P}_1$, and n descendants, the contribution is always from genus higher than zero. The proof is as follows. The sum of the ghost numbers of the fields is

$$2(1-g)(q+1) = q + \sum_{j=1}^{n}(q+1-i_j), \tag{3.A.20}$$

where i_j is the dimension of the j-th descendant. q is the ghost number of $\mathcal{P}_1$. Since $i_j \geq q+1$ (the fields are all descendants), we have

$$2(1-g)(q+1) = q + n - \sum_{j=1}^{n}(i_j - q) \leq q + n - n = q. \tag{3.A.21}$$

Thus, g is larger than zero. Similar arguments apply to level k descendants where the genus is larger than k.

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