APPLICATIONS OF CURRENT ALGEGRA IN CONFORMAL FIELD THEORY

Dissertation by
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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1991

(Submitted May 14, 1991)

ACKNOWLEDGEMENTS

First, I would like to thank my wife, Alla, for her support and encouragement. I would also like to thank my classmates, in particular Gil Rivlis, Alex Ridgway, John Feiler and Glenn Boyd, for helping to make Caltech an enjoyable place to work, and for many interesting physics discussions. My advisor, John Schwarz, deserves appreciation for his support and for his efforts to keep Caltech at the forefront of research in string theory.

Lastly, I thank my parents, sister, and in-laws for their continuous and generous love and support.

ABSTRACT

In this work, two topics concerning the interplay between current algebra and conformal symmetry in two dimensions are discussed. The construction of a conformal algebra from a current algebra, the Virasoro Master Equation, is presented with analytic and perturbative solutions. Second, N=2 superconformal models based on supersymmetric current algebras with $c > 3$ are coupled to two dimensional topological gravity.

CONTENTS

I. INTRODUCTION

The past decade has witnessed a great deal of interest in 2-d critical phenomena and their application as ground states of critical strings. While the complete classification of such ground states remains an unsolved problem (in fact, the complexity of this problem has greatly increased in some ways), considerable progress has been made, and greater insight into the geometric content of string theory has been achieved. In this thesis, we will demonstrate some of this recent progress.

It was known from the earliest days of the bosonic string that consistent string propagation requires symmetry under conformal transformations, thus leading to the Virasoro algebra. Only in more recent years has the generality of that statement and the computational power which it underlies been clarified and utilized. For example, if one considers string propagation in an arbitrary background, the equations of motion are simply the requirement that the 2-d theory that describes the propagation be conformally invariant.

The utility of conformal invariance was greatly advanced by the pioneering work of Belavin, Polyakov and Zamolodchikov [1] in 1983. In two dimensions, the conformal group is infinite dimensional, with the result that the conformal Ward identities lead to finite dimensional differential equations for the correlation functions. They also defined a set of conformally invariant models known as "minimal" models which possess only a finite number of representations of the symmetry algebra. While the minimal models are the simplest and most elegant examples, they form a very small subset of all conformally invariant theories.

Shortly thereafter, the importance of current algebra in conformal field theory became apparent, and it has continued to play an important role. It was well known that the diagonal bilinear sum of current algebra fields (normal ordered, of course), the so-called Sugawara form, forms a representation of the Virasoro algebra. In such models, commonly known as Wess-Zumino-Witten models,[2] the symmetry algebra therefore includes both the current algebra and the Virasoro algebra. Such extensions of the Virasoro algebra have formed one of the main areas of research in conformal field theory. In particular, the supersymmetric extensions of the Virasoro algebra have played an important role, as it is now known that an N=2 supersymmetric conformal field theory guarantees N=1 *spacetime* supersymmetry, which is one of the more promising theoretical solutions of the hierarchy problem and very elegant.

In this thesis, two cases of the interplay between current algebra and conformal symmetry and its applications to string theory will be discussed. First, we will consider a generalization of the Sugawara construction, first developed by Halpern, Kiritsis, Obers, Porrati and Yamron.[3] They found that bilinear sums of current algebra generators other than the diagonal sum could yield the Virasoro algebra. Taking an arbitrary bilinear and requiring the Virasoro algebra to hold yields a set of coupled quadratic equations collectively known as the Virasoro master equation. In principle, any solution of the Virasoro master equation could be used to represent the dynamics of the compactified dimensions of a critical string theory, which thereby would determine the particle content and interactions of a low energy effective field theory which hopefully would correspond to those observed in nature. The analytic and perturbative solutions of the Virasoro master equation will be discussed in Chapter II. The perturbative solutions presented in section II.D follow previously published work of the author.[4] In order to make the connection between these models and the observed low energy interactions, however, much more remains to be uncovered. In particular, characters and fusion rules must be determined, but these appear to be very difficult problems and are beyond the scope of this work.

The remaining topic to be discussed is that of topological field theory. This area of investigation was recently inaugurated in a series of papers by Witten.[5] In the original formulation, operators in the pure gravity theory correspond to closed forms on the moduli space of a Riemann surface. Sigma models with N=2 supersymmetry and an R parity symmetry can be consistently coupled to this system, in which the physical operators (fields) correspond to the BRST invariant fields of the sigma model.

Besides the connections to ongoing work of mathematicians regarding the structure of such moduli spaces, these models are also interesting because at a sufficiently high temperature (of order the Planck temperature), one might anticipate a new phase in which there is no definite metric. Since string theory contains particles with masses which are arbitrary multiples of the Planck mass, such a phase transition may not exist. Indeed it is argued by some that there is a duality $T \to 1/T$ so that there is a maximum temperature, and therefore no phase transition, but this is very much a point of current debate.

The connection between this formulation of topological field theory and conformal field theory is not at first apparent, but is realized through two important results. As shown by Verlinde and Verlinde,[5] topological gravity can be recast as a conformal field theory, whose powerful techniques yield a full solution for the correlation functions, which had not been possible in the original formulation of Witten. Secondly, Eguchi[6] showed that every N=2 superconformally invariant field theory can be coupled to topological gravity and that one can easily identify the physical fields. Li[7] then coupled the N=2 minimal models to topological gravity and again found recursion relations for the correlation functions. These recursion relations were found to be related to extensions of the Virasoro algebra known as W_n. In Chapter IV.B we will extend this discussion to a larger set of N=2 supersymmetric conformal field theories, namely those developed by Kazama and Suzuki.[8] To do so requires a free field representation of the N=2 superconformal algebra of these models, for which previous work of the author[9] will be utilized, whereupon the coupling to topological gravity follows closely the minimal model case.[7][9]

The importance of these results lies partially in the connection and apparent equivalence between topological field theory and the so-called "matrix" models. In the matrix models, one replaces the usual sum over 2-d metrics by a sum over triangulations of a surface. The dual graph to any such triangulation forms a Feynman diagram of a $U(n)$ invariant one-matrix model, first solved by Brézin.[10] Taking the appropriate continuum limit (taking finer triangulations and adjusting couplings like the cosmological constant), one finds that the correlation functions satisfy recursion relations related to the KdV hierarchy.[11] It is a highly nontrivial and unexpected fact that at least in the case of *pure* topological gravity, these recursion relations are exactly the same as those derived from the topological approach.

Of course, the matrix models represent strings propagating in a zero dimensional background, so they certainly do not correspond to critical strings and do not make predictions regarding low energy physics. However, if one could couple these models to a $c = 3$ conformal field theory, i.e., that of an uncompactified dimension, a solution of string theory, in four dimensions and to all genus, could be derived. Such work is currently under active investigation. In this context, coupling various "internal" theories, such as those developed by Kazama and Suzuki, to 2-d gravity would in principle enable a calculation of the conformal field theory (and therefore, of low energy physics for this string background) to all orders in the genus expansion, a task which is far beyond current analytic techniques. Of course, if the string coupling constant is large, such an expansion will not give an accurate picture of low energy phyics.

II. Introduction to Conformal Field Theory

A. The Virasoro Algebra

Although conformal symmetry has found its most important application in string theory, it was initially studied in connection with certain scaling ideas in the theory of phase transitions. Namely, if one applies the renormalization group so as to approach a fixed point, one naturally arrives at a theory that is scale invariant, which implies that its stress energy tensor is traceless

$$T^a_a(z) = 0. \tag{2.1.1}$$

In the special case of two dimensions, Zamolodchikov[12] showed that scale invariance is equivalent to conformal invariance, that is, coordinate transformations under which the metric tensor transforms into a (position dependent) multiple of itself. More importantly, in two dimensions, such a transformation $x^a \longrightarrow x^a + f^a(x)$ must satisfy

$$\Box \partial \cdot f = 0, \tag{2.1.2}$$

where $\Box$ is the Laplacian, which are equivalent to the Cauchy-Riemann equations, and therefore have as solutions all meromorphic functions. It is the resulting infinite number of conserved quantities that makes conformal field theory so powerful.

For the moment, consider the case of flat (2-d) Euclidean space (the results can be generalized easily), and choose complex coordinates

$$z = x^1 + ix^2, \bar{z} = x^1 - ix^2. \tag{2.1.3}$$

In these coordinates the metric is just $ds^2 = dzd\bar{z}$ and conformal transformations are simply analytic and anti-analytic transformations. These transformations are independent and hence the conformal group factorizes into a direct product of a holomorphic sector times an anti-holomorphic sector. Until confronted with the question of the modular invariance of the one-loop partition function, the analysis can be restricted to the holomorphic sector only.

An infinitesimal holomorphic transformation $z \rightarrow z + \epsilon(z)$ can be represented in terms of the Laurent expansion of $\epsilon(z) = \sum_{-\infty}^{\infty} \epsilon_n z^{n+1}$. Then the corresponding Lie algebra is the algebra of differential operators $l_n = z^{-n-1}\partial_z$, whose commutation relations are

$$[l_n, l_m] = (n-m)l_{n+m}. \tag{2.1.4}$$

In quantizing this system, an anomaly term arises.

Suppose a conformal field theory with an action S is given. The stress tensor (also known as the stress-energy tensor or the energy-momentum tensor) T^{ab} is defined as the variation of the action with respect to the metric,

$$T^{ab} = \frac{\delta S}{\delta g_{ab}}.$$

Let

$$X = \prod_{i=1}^{n} \phi_i(z_i), \tag{2.1.5}$$

where the ϕ_i are local fields of the theory. Then the correlation functions of the fields ϕ_i are defined as

$$\langle X \rangle = \int D\phi e^{-S} X. \tag{2.1.6}$$

Under a coordinate transformation, $x^a \rightarrow x^a + \epsilon^a(x)$ the following Ward identities can be derived

$$\sum_{k=1}^{n} \langle \phi_1 \cdots \delta_\epsilon \phi_k \cdots \phi_n \rangle + \int d^2x \partial^a \epsilon^b \langle T_{ab}(x) X \rangle = 0, \tag{2.1.7}$$

where $\delta_\epsilon \phi_i$ is the variation of the local field under the coordinate transformation. A corollary of this equation is the conservation of the stress tensor

$$\partial_a \langle T^{ab} X \rangle = 0 \tag{2.1.8}$$

everywhere except possibly the positions z_i of the local fields. Combining (2.1.1) and (2.1.8) shows that

$$T \equiv T_{11} - T_{22} + 2iT_{12} \tag{2.1.9}$$

is holomorphic. Its Laurent expansion modes are, for a given action, the quantum versions of the classical generators in (2.1.4). Therefore, for any matter field ϕ, we have

$$[\oint dw \delta z(w) T(w), \phi(z)] = \delta\phi(z). \tag{2.1.10}$$

The w contour in this equation can be taken to be a circle about the origin, but we must be careful about operator ordering. Operator products are defined by time ordering. In taking the commutator, it is common to use radial quantization, namely let $z = re^{i\theta}$ and consider r as "time." In (2.1.10), if w is denoted as $\tilde{w}e^{i\tilde{\theta}}$, then $\tilde{w} > r$ in one term of the commutator and $\tilde{w} < r$ in the other term, and the difference between the two contour integrals can be reduced to a contour integral about z.

The most natural fields to consider are tensor fields which transform under $z \longrightarrow z'(z)$ as

$$\phi(z,\bar{z}) = \phi(z',\bar{z}')(\frac{dz'}{dz})^{\Delta}(\frac{d\bar{z}'}{d\bar{z}})^{\bar{\Delta}}. \tag{2.1.11}$$

Then since the contour integral in (2.1.10) is only sensitive to the singular parts of the operator product expansion (OPE) of T and ϕ, (2.1.11) implies that

$$T(w)\phi(z) \sim \frac{\Delta}{(w-z)^2}\phi(z) + \frac{1}{w-z}\partial\phi(z). \tag{2.1.12}$$

It is perhaps appropriate here to note that in conformal field theory, unlike most other applications of field theory, the OPE is believed to be an exact expression, and not just an asymptotic expansion, as in Yang-Mills theory, where the OPE is true order by order in perturbation theory, but can be spoiled by nonperturbative effects. This is related to the correspondence in conformal field theory between states and operators. The argument for this is that any state on some space-like hypersurface, say $r = constant$, can be evolved by a conformal transformation into one with $r = \epsilon$, with ϵ arbitrarily small, thus creating a state with an operator of arbitrarily small support, *i.e.*, a local operator. Equivalently, inserting a local operator at a point with local coordinate $z = 0$, then the path integral on the surface with the disk $|z| < 1$ cut out defines a state. Then considering a contour around two operators, the state on the contour defines by the above correspondence a local operator, that is just the operator which appears on the right hand side of the OPE.

Naively from its definition, T would be a dimension 2 operator. However, if this were the case, its two-point function could be calculated by the OPE which would vanish by scale invariance (which requires that the one-point function of all operators with non-vanishing scaling dimensions vanish). In a unitary quantum field theory, this would imply that T be identically zero. So a fourth-order pole in the OPE of T with itself must be assumed , *i.e.*,

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{2.1.13},$$

($\partial = \frac{\partial}{\partial z}, \bar{\partial} = \frac{\partial}{\partial \bar{z}}$) where c is a constant known as the central charge or conformal anomaly. In terms of the modes L_n of T,

$$L_n = \oint dz T(z) z^{n+1}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}, \tag{2.1.14}$$

(a factor of $2\pi i$ has been absorbed into the defintion of the contour integral) which is known as the Virasoro algebra.[13]

Since L_0 acts on z as $z \to z + \epsilon(z)$ it is the Hamiltonian of the model. The vacuum is defined as the ground state of the Hamiltonian, which can be taken to have energy eigenvalue zero. In addition, when acting on the vacuum at $z = 0$, the stress tensor should not be singular, i.e.,

$$L_n|0\rangle = 0, \ n \geq 0 \tag{2.1.15}$$

($L_{-1}|0\rangle = 0$ follows from the commutation relation (2.1.14)). Since L_{-1}, L_0, and L_1 generate $SL(2,R)$, this state is known as the $SL(2,R)$ vacuum, and correlation functions are expressed as expectation values in this state, *i.e.*,$\langle X\rangle = \langle 0|X|0\rangle$ for any operator X.

An operator ϕ such as in (2.1.11) satisfies

$$L_0\phi = \Delta\phi \tag{2.1.16}$$

$$L_n\phi = 0, n > 0 \tag{2.1.17}$$

and is commonly known as a "primary" field. Such a field determines a highest weight representation of the Virasoro algebra, sometimes known as a Verma module. The lowering operators L_n for $n < 0$ create new fields known as descendants of ϕ whose correlation functions are determined uniquely by conformal invariance in terms of the correlation functions of ϕ.

It is an interesting question to ask whether any such descendant field can also satisfy the primary field conditions (2.1.16) and (2.1.17). If so, it is easy to show that such a field must create a state of vanishing norm and therefore must decouple from the rest of the theory. Such states are known as null states, and their existence is an important tool in solving for correlation functions. Since the null state can be written as a product of L_n ($n < 0$) operators acting on ϕ, which in turn can be written as differential operators acting on ϕ, the decoupling of a null state results in finite-order differential equations for the correlation functions. An example is the state

$$|\Psi\rangle = (L_{-2} + aL_{-1}^2)\phi(0)|0\rangle. \tag{2.1.18}$$

This is a null state if

$$a = -\frac{3}{2(2\Delta+1)}, \tag{2.1.19}$$

and the dimension Δ of the operator ϕ is

$$\Delta = \frac{1}{16}[5 - c \pm \sqrt{(c-1)(c-25)}]. \tag{2.1.20}$$

Since Ψ decouples from the theory, any correlation function containing it must vanish, so we have

$$\langle\phi_1(z_1)\cdots\phi_n(z_n)|\Psi\rangle = \langle 0|\phi_1(z_1)\cdots\phi_n(z_n)(L_{-2+}aL_{-1}^2)\phi(0)|0\rangle = 0. \tag{2.1.21}$$

Using the Ward identities to move the Virasoro operators to the left (eventually annihilating the out vacuum on the left), we get the following differential equation

$$(a\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n}\frac{\Delta_i}{(z-z_i)^2} - \sum_{i=1}^{n}\frac{1}{(z-z_i)}\frac{\partial}{\partial z_i}) < \phi(z)\phi_1(z_1)\cdots\phi_n(z_n) >= 0 \tag{2.1.22}$$

Also, since the null states have vanishing norm, the existence of null states can signal the onset of negative norm states, contrary to unitarity. An analysis of unitarity by Friedan, Qiu, and Shenker[14] shows that for $c < 1$ only the models with

$$c = 1 - \frac{6}{m(m+1)}\ , m = 2, 3, ... \tag{2.1.23}$$

are unitary. In these models, the dimensions of the primary fields are given by

$$\Delta_{p,q} = \frac{((m+1)p - mq)^2 - 1}{(4m(m+1)}, \ 1 \le p \le m-1, \ 1 \le q \le p \tag{2.1.24}$$

and every Verma module contains at least one null state. For $c > 1$ there are no constraints from unitarity, but it is known, from consideration of modular invariance of the one loop partition function, that there must be an infinite number of primary fields.

B. The Wess-Zumino-Witten Model

The simplest example of a current algebra results from the free bosonic field whose equation of motion is

$$\partial\bar{\partial}\phi = 0, \tag{2.2.1}$$

and therefore has an invariance under the transformation

$$\delta\phi = f(z) + g(\bar{z}) \tag{2.2.2}$$

generated by the conserved $U(1)$ currents (one-forms)

$$J = \partial\phi dz \tag{2.2.3}$$

$$\bar{J} = \bar{\partial}\phi d\bar{z}. \tag{2.2.4}$$

It is also easy to show that the stress energy tensor for the free bosonic field is just

$$T(z) = \frac{1}{2} : (\partial\phi)^2 : (z), \tag{2.2.5}$$

where the colons denote normal ordering defined by

$$: AB : (z) = \oint dw \frac{A(z)B(w)}{z-w} \tag{2.2.6}$$

where the integration contour surrounds the point z. (other normal ordering conventions abound in the literature). The field ϕ satisfies the usual free boson OPE

$$\partial\phi(z)\partial\phi(w) \sim \frac{1}{(z-w)^2}, \tag{2.2.7}$$

and has modes

$$\alpha_n = \int dz z^n \partial\phi(z). \tag{2.2.8}$$

Since the symmetry algebra has been extended to include $U(1)$, the fields of the theory should be eigenstates of J_0, the zero mode of the $U(1)$ generator. This turns out to be equivalent to finding solutions of (2.1.16) and (2.1.17). Therefore highest weight states satisfy

$$\alpha_0|k> = k|k>$$

$$\alpha_n|k> = 0, n > 0,$$

and therefore are created by operators $V_k(w)$ of conformal dimension $k^2/2$ which satisfy

$$\partial\phi(z)V_k(w) \sim \frac{ik}{z-w}V_k(w). \tag{2.2.9}$$

This equation has so-called vertex operator solutions

$$V_k(z) = :e^{ik\phi(z)}:. \tag{2.2.10}$$

It is possible to understand the free boson completely as a generic system with conformal as well as $U(1)\times U(1)$ symmetry. In two dimensions one can interchange a free boson for two free fermions, and therefore a free fermion system is actually the same system as this one. It is natural to ask whether this structure can be extended to larger Lie algebras. One finds that for any Lie algebra G, there is an infinite series of conformal models labelled by an integer k, known as the level, which in addition to the conformal symmetry, have a $\hat{G}_{left}\times\hat{G}_{right}$ symmetry, where $\hat{G}$ denotes the affine algebra based on the Lie algebra G (see the Appendix A for a summary of Lie algebra and affine algebra notation). These models are known as Wess-Zumino-Witten models.[15]

To begin, generalize (2.2.7) to

$$J^a(z)J^b(w) \sim \frac{(k/2)\eta^{ab}}{(z-w)^2} + \frac{if^{abc}J^c(w)}{(z-w)}, \tag{2.2.11}$$

where the label a runs over the adjoint of G, η^{ab} is the metric of G, and f^{abc} are its structure constants (these fields are no longer one-forms). There is an identical structure in the anti-holomorphic sector. These equations should be derived from an action, so start with a G-valued function γ. The natural σ model action is

$$S_0 = -\frac{1}{4\lambda^2}\int dx \mathrm{tr}(\gamma^{-1}\partial_\mu\gamma)^2, \tag{2.2.12}$$

where λ is a coupling constant. The equations of motion resulting from S_0 are

$$\partial^\mu(\gamma^{-1}\partial_\mu\gamma) = 0. \tag{2.2.13}$$

Defining

$$J_\mu = \gamma^{-1}\partial_\mu\gamma, \tag{2.2.14}$$

then

$$\partial^\mu J^a_\mu = 0.$$

We would like, however, to find holomorphic and anti-holomorphic currents, in analogy with (2.2.3) and (2.2.4), which would satisfy

$$\partial_- J^a_+ = \partial_+ J^a_- = 0 \qquad (2.2.15).$$

Witten[2] found the currents which satisfy (2.2.15) by noticing that if

$$J_+ = \gamma^{-1}\partial_+\gamma \qquad (2.2.16)$$

satisfies the first equation in (2.2.15) then

$$J_- = (\partial_-\gamma)\gamma^{-1} \qquad (2.2.17)$$

satisfies the second. But these currents are not the generators of the $G_{left} \times G_{right}$ symmetry resulting from the action S_0. So a term of the form

$$S_1 = K\Gamma(B)$$

$$\Gamma(B) = \frac{1}{24\pi}\int_B \epsilon^{\lambda\mu\nu} tr(\gamma^{-1}(\partial_\lambda\gamma)\gamma^{-1}(\partial_\mu\gamma)\gamma^{-1}\partial_\nu\gamma)d^3y. \qquad (2.2.18)$$

must be added to the action.

The integral in (2.2.18), expressed as a three dimensional integral in a two dimensional theory may look unusual, indeed not local. Suppose that the two-dimensional world is the two sphere S^2, the boundary of a spatial ball B in $\mathbb{R}$. γ is extended into B by analytic continuation to calculate (2.2.18). Such an extension is not unique, but the difference between any two such values of S_1 can be written as in (2.2.18) only with B replaced by the three sphere. This quantity, for any algebra, is a sort of winding number and is equal to an integer multiple of 2π. Quantum mechanically, in order to keep $e^{iS/\hbar}$ well defined we must have $K = k\hbar$, where k is an integer. It is precisely this k that was introduced in (2.2.11). Furthermore, in order to find (2.2.16) and (2.2.17), one finds that $\lambda^2 = \frac{4\pi}{K}$.

Having found the correct action and conserved currents, it is now easy to find that the stress tensor is[16]

$$T_G = \frac{1}{k+Q}\sum_{a=1}^{\dim(G)} : J^a_+ J^a_+ : \qquad (2.2.19)$$

(here a basis has been chosen in which $\eta^{ab} = \delta^{ab}$ which is possible if G is compact and semi-simple).* Similarly, the central charge for the resulting conformal algebra is

$$c = \frac{k \dim(G)}{k+Q}. \qquad (2.2.20)$$

*The construction of the stress tensor as a bilinear in currents was first proposed by Gell-Mann.[16]

The currents J^a are primary fields of dimension one with respect to the above form of the stress tensor, as expected for gauge symmetry generators. In terms of the modes

$$J_n^a = \int dz z^n J^a(z) \tag{2.2.21}$$

and therefore

$$[L_n, J_m^a] = -mJ_{m+n}^a. \tag{2.2.22}$$

In finding representations of the joint current algebra-conformal algebra, the usual proceedure is followed; find a maximal set of commuting generators, and diagonalize them. From (2.2.22), we see that in addition to L_0, J_0^a can also be diagonalized, for which the label a runs over the Cartan subalgebra of G. This plus the fact that the modes J_0^a for all a generate the usual Lie algebra G, means that any primary field is labelled by a highest weight Λ of the Lie algebra G, *i.e.*,

$$J_0^a \phi_\lambda^\Lambda(z) = (\tau^a)_\lambda^{\tilde{\lambda}} \phi_{\tilde{\lambda}}^\Lambda(z), \tag{2.2.23}$$

where λ and $\tilde{\lambda}$ are weights in the Λ representation, and τ^a is the representation matrix of the generator a in the representation Λ. The conformal dimension of such a field is

$$\Delta_\Lambda = \frac{Q_\Lambda}{k+Q}, \tag{2.2.24}$$

where Q_Λ is the value of the quadratic Casimir of the representation Λ. This can be rewritten as

$$\Delta_\Lambda = \frac{\Lambda^2 + 2\rho_G \cdot \Lambda}{2(k+Q)} \tag{2.2.25}$$

where ρ_G is one half the sum of all the positive roots. The question here is what values of Λ are allowed, given the value of k? Let α be a root of G, and note that $J_1^{-\alpha}$, J_{-1}^{α} and $k - \alpha \cdot H$ (where H represents the vector of currents in the Cartan subalgebra) form an SU(2) algebra. Then, if $|\mu\rangle$ is a primary state, we have

$$J_1^{-\alpha}|\mu\rangle = 0 \tag{2.2.26}$$

and therefore

$$||J_{-1}^{\alpha}|\mu\rangle||^2 = \langle\mu|J_1^{-\alpha}J_{-1}^{\alpha}|\mu\rangle = \langle\mu|(k-\alpha\cdot H)|\mu\rangle = (k-\alpha\cdot\mu)|||\mu\rangle||^2. \tag{2.2.27}$$

Requiring that this be positive for unitarity implies $k \geq \alpha \cdot \mu$. Considering all the states in the representation Λ and all roots α yields the requirement

$$k \geq \psi \cdot \Lambda, \tag{2.2.28}$$

where ψ is the highest root of G.

Before going on to discuss the supersymmetric generalizations of the conformal algebra and of the Kac-Moody algebra, it is appropriate at this point to discuss an important construction of conformal field theories, known as the coset or G/H construction. To begin, consider a Lie algebra G and a subalgebra H. The Sugawara stress tensors T_G and T_H for G and for H are constructed as in (2.2.19).

Since the generators J^a $(a \in H)$ of H are also generators of G, they are primary fields of dimension one with respect to both stress tensors and hence obey (2.2.22). Then clearly they obey

$$[T_G - T_H, J^a] = 0. \tag{2.2.29}$$

Since T_H is bilinear in the J^a, we therefore have

$$[T_G - T_H, T_H] = 0, \tag{2.2.30}$$

so an orthogonal decomposition of the affine G theory into the affine H theory and the coset theory has been achieved. This decomposition extends throughout the structure of the conformal field theory, namely, primary fields of G can be decomposed into a sum of terms, each of which factorizes into a field of H and a field of the coset theory. Similarly, correlation functions and modular invariance of the coset theory can be derived from the corresponding information in the G and H theories.

Such coset theories have turned out to be of great importance in conformal field theory. It is widely conjectured that all (rational) conformal field theories are either of the Sugawara form or of the coset form, though this is far from proven. In support of this conjecture, it was found early on[17] that the conformal minimal models mentioned above are equivalent to the coset $\frac{SU(2)_1 \times SU(2)_k}{SU(2)_{k+1}}$ where the subscripts denote the levels of the algebras. Similar equivalences can be made for other systems, such as the minimal superconformal models and models with Z_n symmetry, known as W_n algebras,[18] which will arise shortly.

C. The Superconformal and Super Kač-Moody Algebras

There are well-known supersymmetric extensions of the conformal (Virasoro) algebra with $N = 1,2,3,4$ supersymmetry. We will have need of only the cases with $N = 1,2$, which are (in addition to (2.1.13)), for $N = 1$

$$T(z)G^0(w) \sim \frac{3/2G^0(w)}{(z-w)^2} + \frac{\partial G^0(w)}{(z-w)} \tag{2.3.1}$$

and

$$G^0(z)G^0(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w} \tag{2.3.2}$$

where $G^0(w)$ is the supercurrent. These are the OPE forms of the Ramond-Neveu-Schwarz algebra[19]. For $N = 2$, there are two supercurrents (both of dimension 3/2 as above) and a $U(1)$ generator, which will be denoted J. A basis of the supercurrents in which the supercurrents have $U(1)$ charges ± 1 can be chosen, namely,

$$J(z)G_\pm(w) \sim \frac{\pm G_\pm(w)}{z-w}. \tag{2.3.3}$$

In this basis,

$$G_+(z)G_-(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{z-w}. \tag{2.3.4}$$

Clearly, the $N = 1$ algebra should be a subalgebra of the $N = 2$ algebra, and it is easy to see that the field $(1\sqrt{2})(G_+ + G_-)$ satisfies both (2.3.1) and (2.3.2).

The $N = 1$ and $N = 2$ algebras have "minimal" models, in which the number of primary fields is finite and null states imply differential equations for the correlation functions, as in the non-supersymmetric case. For $N = 1$, the minimal models have central charge values of

$$c = \frac{3}{2}(1 - \frac{8}{(m+2)(m+4)}) \ , \ m = 2,3,\ldots$$

and the models are related to the coset $\frac{SU(2)_2 \times SU(2)_k}{SU(2)_{k+2}}$, whereas for $N = 2$, the minimal models have central charge

$$c = \frac{3k}{k+2} \ , \ k = 1,2,\cdots$$

and the models are equivalent to the coset $\frac{SU(2)_k}{U(1)} \times U(1)$.

The $N = 1$ superconformal algebra can be derived from a supersymmetric extension of the current algebra in (2.2.11). This extension involves a set of fermions

j^a, where a again labels the adjoint representation of the Lie algebra G. The new OPE's are

$$j^a(z)j^b(w) \sim \frac{k/2\delta^{ab}}{z-w} \tag{2.3.5}$$

and

$$J^a(z)j^b(w) \sim \frac{if_{abc}j^c(w)}{z-w}. \tag{2.3.6}$$

The form of (2.3.5) is that of a set of free fermions, a condition spoiled somewhat by (2.3.6). So, doing a field redefinition

$$J^a = \hat{J}^a + J^a_f \tag{2.3.7}$$

with

$$J^a_f = -\frac{i}{k} f_{abc} : j^b j^c : \tag{2.3.8}$$

It is easy to check that $\hat{J}^a(z)j^b \sim 0$ and that both $\hat{J}^a$ and J^a_f satisfy (2.2.11) with levels $\hat{k} = k - Q$ and $k_f = Q$ respectively. In terms of the $\hat{J}^a$, the $N = 1$ superconformal generators are given by

$$T(z) = \frac{1}{k}(: \hat{J}^a \hat{J}^a : - : j^a \partial j^a :) \tag{2.3.9}$$

$$G^0(z) = \frac{2}{k}(: j^a \hat{J}^a : -\frac{i}{3k} f_{abc} : j^a j^b j^c :). \tag{2.3.10}$$

The central charge for the combined system is

$$c = \frac{1}{2} dimG + \frac{\hat{k} dimG}{\hat{k} + Q}, \tag{2.3.11}$$

where the first term comes from the free fermions and the second term from the $\hat{J}^a$ fields, as in (2.2.20). Now it is possible to go through the same coset space manipulations as before; namely, defining

$$T_{G/H} = T_G - T_H \tag{2.3.12}$$

$$G^0_{G/H} = G^0_G - G^0_H, \tag{2.3.13}$$

which have vanishing OPE with all the currents, both bosonic and fermionic, in the H theory. Henceforth, let us use the following group index notation: generators of H have latin indices $a, b, \cdots$, while G/H generators are denoted by barred indices

$\bar{a}, \bar{b}, \cdots$. Generators of G, which take both kinds of values, are denoted by capital letters $A, B, \cdots$. Then substituting in (2.3.9) to form $T_{G/H}$ and $G_{G/H}$ yields

$$T_{G/H} = \frac{1}{k}(\hat{J}^{\bar{a}}\hat{J}^{\bar{a}} - \frac{\hat{k}}{k}j^{\bar{a}}\partial j^{\bar{a}} + \frac{2i}{k}\hat{J}^{a}f_{a\bar{b}\bar{c}}j^{\bar{b}}j^{\bar{c}} - \frac{2}{k}f_{\bar{a}\bar{p}\bar{q}}f_{\bar{b}\bar{p}\bar{q}}j^{\bar{a}}\partial j^{\bar{b}} - \frac{1}{k^2}f_{\bar{a}\bar{b}\bar{c}}f_{\bar{a}\bar{d}\bar{e}}j^{\bar{b}}j^{\bar{c}}j^{\bar{d}}j^{\bar{e}}) \tag{2.3.14}$$

and

$$G_{G/H} = \frac{2}{k}(j^{\bar{a}}\hat{J}^{\bar{a}} - \frac{i}{3k}f_{\bar{a}\bar{b}\bar{c}}j^{\bar{a}}j^{\bar{b}}j^{\bar{c}}). \tag{2.3.15}$$

An interesting question is whether these constructions can be extended to the $N=2$ algebra. It is not difficult to show[20] that there is no self consistent (in terms of associativity of the algebra, or equivalently, requiring Jacobi identity relations), linear extension of the Kač-Moody algebra with $N=2$ supersymmetry (except for the somewhat trivial and well-known $U(1)$ case, which is just the NSR superstring) using chiral fields alone. The most elegant argument for this result is that such a current algebra would have to result from a Wess-Zumino-Witten action on a group manifold (as discussed above) with $N=2$ supersymmetry. But requiring $N=2$ supersymmetry is equivalent to requiring that the group manifold be Kähler. There are no non-Abelian Kähler group manifolds, and therefore no $N=2$ (non-Abelian) supersymmetric current algebras. Using twisted fields, however, an $N=2$ Kač-Moody algebra for $SU(2)\times U(1)$ has very recently been found, but the generality of this result is as yet unknown.

However, an important advance was made by Kazama and Suzuki,[8] who found that one could take the $N=1$ construction in (2.3.14) and (2.3.15) and extend it to the $N=2$ superconformal algebra in certain special cases of G and H. Since $T(z)$ and $G^0(z) = \frac{1}{\sqrt{2}}(G_+ + G_-)$ have already been found, all that is needed is to find the $U(1)$ current $J(z)$ and the other supercurrent $G^1(z) = \frac{1}{\sqrt{2}}(G_+ - G_-)$. Since both supercurrents are dimension 3/2 operators, and it is desireable to construct them from only the generators in G/H, they must have the form

$$G^i(z) = \frac{2}{k}[h^i_{\bar{a}\bar{b}}j^{\bar{a}}\hat{J}^{\bar{b}} - \frac{i}{3k}S^i_{\bar{a}\bar{b}\bar{c}}j^{\bar{a}}j^{\bar{b}}j^{\bar{c}}] \tag{2.3.16}$$

where h^i and S^i are sets of coefficients. From (2.3.15),

$$h^0_{\bar{a}\bar{b}} = \delta_{\bar{a}\bar{b}} \tag{2.3.17}$$

and

$$S^0_{\bar{a}\bar{b}\bar{c}} = f_{\bar{a}\bar{b}\bar{c}}. \tag{2.3.18}$$

$J(z)$ does not have to be specified, as it will be determined by the OPE of the supercurrents. Requiring that this OPE gives $T(z)$ with the appropriate coefficient yields the requirements

$$h^1_{\bar{a}\bar{b}} + h^1_{\bar{b}\bar{a}} = 0 \tag{2.3.19}$$

and

$$h^1_{\bar{p}\bar{a}} h^1_{\bar{p}\bar{b}} = \delta_{\bar{a}\bar{b}}. \tag{2.3.20}$$

Putting (2.3.19) and (2.3.20) together, we see that h^1 defines a complex structure on the coset space.

Secondly, there are the requirements

$$h^1_{\bar{a}\bar{d}} f_{\bar{d}\bar{b}e} = f_{\bar{a}\bar{d}e} h^1_{\bar{d}\bar{b}}, \tag{2.3.21}$$

$$S^1_{\bar{a}\bar{b}\bar{c}} = h^1_{\bar{a}\bar{p}} f_{\bar{p}\bar{b}\bar{c}} + h^1_{\bar{b}\bar{p}} f_{\bar{p}\bar{c}\bar{a}} + h^1_{\bar{c}\bar{p}} f_{\bar{p}\bar{a}\bar{b}}, \tag{2.3.22}$$

and

$$S^1_{\bar{a}\bar{b}\bar{c}} = h^1_{\bar{a}\bar{p}} h^1_{\bar{b}\bar{q}} h^1_{\bar{c}\bar{r}} f_{\bar{p}\bar{q}\bar{r}}. \tag{2.3.23}$$

For example, a simple solution of these constraints is

$$f_{\bar{a}\bar{b}\bar{c}} = 0 \tag{2.3.24}$$

in which case let $S^1_{\bar{a}\bar{b}\bar{c}} = 0$ (a solution for h^1 can always be found in this case). Manifolds which satisfy (2.3.24) are known as hermitian symmetric spaces, and have been completely classified.[21] Some examples are the complex Grassmannian manifolds $CG(m,n) = \frac{SU(m+n)}{SU(m)\times SU(n)\times U(1)}$ and $\frac{SO(n+2)}{SO(n)\times SO(2)}$. In fact, not all of these models are independent; for example, the $CG(m,n)$ model at level k is equivalent to the $CG(k,m)$ model at level n. This is most easily seen by expressing the entire algebra in the Weyl-Cartan basis, and noticing that the expressions for T and G^i are appropriately symmetric, but the details will not be presented here.

More generally, assume that $rank(G) = rank(H)$. If not, consider $\frac{G}{H\times U(1)^d} \times U(1)^d$, (where $d = rank(G) - rank(H)$), which is $N = 2$ supersymmetric if the first factor is, since the second factor certainly has $N = 2$ supersymmetry. Then the general result is that if G/H is a Kähler manifold, the $N = 1$ superconformal algebra can be enlarged to $N = 2$. The hermitian symmetric spaces and their tensor products are but a small segment of the set of such manifolds. What is special about the hermitian symmetric spaces is that they are the only G/H models in which H has only a single $U(1)$ factor.

As discussed in section B, the representations of the $N=2$ superconformal G/H algebra are determined by decomposing primary fields of $G \times SO(n)$ into fields in the H algebra and the coset space. The $SO(n)$, $n = \dim(G/H)$ is at level 1 and is derived from the free fermions. Therefore, the representations of the $N=2$ algebra are determined by a highest weight Λ of G, a highest weight $\tilde{\Lambda}$ of $SO(n)$, and a highest weight λ of H. At level 1, there are only four allowed highest weights of $SO(n)$, the singlet, vector, spinor and anti-spinor. The allowed values of Λ are determined by the level as in (2.2.28). The allowed values of λ are also restricted by (2.2.28), but depend in addition on Λ and $\tilde{\Lambda}$ and must be determined anew for each G and H.

There is, however, one caveat to this construction,[22] which was not realized at first. It concerns the modular invariance of the partition function. In general, for each representation of the (super-)conformal algebra, one forms the character

$$\chi_{\Lambda,\tilde{\Lambda},\lambda}(\tau) = Tr[e^{2\pi i L_0 \tau}], \tag{2.3.25}$$

The trace is taken over all the states in the representation. Expressing τ as $\tau = \sigma_1 + i\sigma_2$ where σ_1 and σ_2 are real, consider the bilinear sum of characters

$$Z = \sum_{i,j} N_{ij} \chi_i(\tau) \chi_j^{\dagger}(\bar{\tau},), \tag{2.3.26}$$

where N_{ij} are integers that count how many times the primary field $\phi_i(z)\bar{\phi}_j(\bar{z})$ appears in the spectrum of the theory. Using (2.3.25) this can be written as

$$Z = Z(\sigma_1, \sigma_2) = \sum_{i,j} N_{ij} Tr e^{2\pi\sigma_1 H + 2\pi\sigma_2 P} \tag{2.3.27}$$

where $H = L_0 + \bar{L}_0$ is the Hamiltonian and $P = L_0 - \bar{L}_0$ is the momentum operator. This is nothing but the partition function for Euclidean time $2\pi\sigma_1$, or equivalently, the one loop (i.e., torus) contribution of the zero point function, in which a twist of the final string by an angle $2\pi\sigma_2$ has been allowed. As such, it should be invariant under the modular group, *i.e.*, the set of mappings of τ that take a torus into a conformally equivalent torus. This group is generated by the transformations $T : \tau \to \tau + 1$ and $S : \tau \to -\frac{1}{\tau}$.

Generally, in the coset construction the function

$$\sum_i |\chi_i|^2 \qquad (2.3.28),$$

where the sum runs over all allowed representations, is modular invariant, and is the partition function of the model. So the partition function for the $N = 2$ models which we have been discussing is postulated to be

$$Z = \sum_{\Lambda,\bar{\Lambda},\lambda} |\chi_{\Lambda,\bar{\Lambda},\lambda}|^2. \tag{2.3.29}$$

It is, in fact, modular invariant. However, in the process of forming the coset space, it often occurs that multiple copies of the identity representation are present in (2.3.29). Since the partition function basically counts the number of states at each dimension, this means that the theory has a degenerate vacuum, whereas the Virasoro algebra has a unique vacuum, the $SL(2,C)$ vacuum (for closed strings) discussed earlier. We would like to simply take the partition function and divide by this multiplicity, call it p (which is always an integer), but there may be other fields that only appear once in (2.3.29), or more generally, a number of times that is not divisible by p. Since no state can appear a fractional number of times in the physical spectrum, there is an inconsistency. For example, in the $\frac{SU(m+n)}{SU(m)\times SU(n)\times U(1)}$ models, there is such an inconsistency unless the level k of the numerator and the integers m and n are all mutually prime.

Of course, much more has to be done in order to use these models in string compactification. In particular, the GSO projection[23] has to be generalized to eliminate the states that would violate the spin statistics theorem. This projection automatically eliminates any tachyons, which normally appear in the NS sector of the string. The problem with doing such a projection lies in the possible breakdown of modular invariance, but as shown by Gepner[24] for the $N = 2$ minimal models, such a generalization can be constructed. Similar methods also work for heterotic string compactifications.

III. THE VIRASORO MASTER EQUATION

A. Introduction

As has been seen above, affine Lie algebras are extremely important in the classification of conformal field theories (CFT's).[15] We should be careful to note that thus far, we have been concentrating on conformal field theories with a finite number of primary fields. It is in these cases that there are null vectors that yield differential equations for correlation functions. It is also in these cases that the restrictions of modular invariance can be most easily analyzed,[25–27] although modular invariance is a requirement for any conformal model. These models with a finite number of primary fields are frequently known as rational conformal field theories (RCFT's) as it is known[28] that the dimensions of the primary fields and the value of the central charge are rational.

Until recently, one has always used the Sugawara construction to go from an affine Lie algebra to a conformal field theory, i.e., the stress tensor has the form $\beta J^a J^a$ where J^a are the affine (1,0) currents and β is a normalization factor. This construction leads to a rational central charge of $c = 2kd/(2k + Q)$, where k is the central charge of the affine algebra (which is quantized in units of the root length), d is its dimension, and Q the quadratic Casimir.

Halpern et al.[3] and Morozov et al.[29] have tried to extend this method by allowing the stress tensor to be a general quadratic sum of the affine currents. The requirement that the stress tensor have the correct operator product expansion with itself yields a self-consistency condition on the coefficients in this sum, which is known as the Virasoro master equation.

To date, the only solutions to the master equation are 1) the so called spin-orbit constructions [30] on $g \times U(1)^{g/h}$ where g/h is a symmetric space, 2) solutions restricted to the Cartan subalgebra of any simple algebra g, which are equivalent to $U(1)^{rank\ g}$ 3) a single solution for $SU(2)$ at level 4, and a set of solutions of $SU2) \times SU(2)$, and 4) maximally symmetric solutions for any simply-laced algebra.[31]

In the following, we will review the analytic solutions and then proceed to develop a procedure to find perturbative solutions of the master equation, by expanding stress tensor in inverse powers of the affine central charge.[4,32] This has the advantage of finding more solutions for a given algebra (and not restricting the algebra to be simple, or simply laced), but the disadvantage, as we will see, that the series cannot easily be summed, i.e., in practice we really can only find approximate solutions to the master equation (which should be a good approximation for large values of the level).

The holomorphic affine currents $J^a(z)$ (the z dependence is often omitted in what follows) have the operator product expansion (OPE) in (2.2.11), which is repeated here (with a slight change of notation) for convenience,

$$J_a(z)J_b(w) \sim \frac{k\eta_{ab}}{(z-w)^2} + \frac{if_{ab}{}^c J_c(w)}{(z-w)}, \tag{3.1.1}$$

where the index a labels the adjoint representation of the algebra (assumed to be compact and semi-simple), $f_{ab}{}^c$ are its structure constants, k is the level, and η is the metric of the underlying Lie algebra. As noted above, in the usual construction, where this OPE arises from the Wess-Zumino-Witten action [2] k is an integer in units of the root length. Here, the form of the underlying action is still is unknown, so in principle k should be regarded as a potentially continuous parameter, though whether it is or not will usually not be relevant in what follows.

From the J_a the stress tensor is now constructed with the form

$$T(z) = L^{ab} J_a J_b \tag{3.1.2}$$

with symmetric normal ordering of the right-hand side assumed. In a basis where η is the identity matrix, unitarity requires that L^{ab} be symmetric and real. Requiring that $T(z)$ have the usual OPE with itself yields the Virasoro master equation

$$L^{ab} = 2kL^{ac}L^{cb} - L^{cd}L^{ef} f_{ce}{}^a f_{df}{}^b - L^{cd} f_{ce}{}^f f_{df}{}^{(a} L^{b)e}, \tag{3.1.3}$$

where the parentheses denote symmetrization. An alternative form of the Virasoro master equation is

$$2L^{ab} = L^{cd}L^{ef} R^{ab}_{cd,ef} \tag{3.1.4},$$

where

$$R^{ef}_{ab,cd} = (R_1 - R_2 - R_3)^{ef}_{ab,cd}, \tag{3.1.5}$$

$$(R_1)^{ef}_{ab,cd} = \frac{k}{2}[\eta_{ac}(\delta^e_b \delta^f_d + \delta^e_d \delta^f_b) + \eta_{ad}(\delta^e_b \delta^f_c + \delta^e_c \delta^f_b)] + (a \leftrightarrow b), \tag{3.1.6}$$

$$(R_2)^{ef}_{ab,cd} = \frac{1}{2}(f_{ac}{}^e f_{bd}{}^f + f_{ad}{}^e f_{bc}{}^f) + (a \leftrightarrow b), \tag{3.1.7}$$

$$(R_3)^{ef}_{ab,cd} = \frac{1}{4}[\delta^f_d (f_{ac}{}^g f_{bg}{}^e + f_{bc}{}^g f_{ag}{}^e) + \delta^f_c (f_{ad}{}^g f_{bg}{}^e + f_{bd}{}^g f_{ag}{}^e)$$

$$+\delta_a^f(f_{cb}{}^g f_{dg}{}^e + f_{db}{}^g f_{cg}{}^e) + \delta_b^f(f_{ca}{}^g f_{dg}{}^e + f_{da}{}^g f_{ag}{}^e)] + (c \leftrightarrow f). \quad (3.1.8)$$

If a solution to this equation is found, the value of the central charge is

$$c = \sum_a 2kL^{aa}. \quad (3.1.9)$$

Previously, non-trivial solutions to (3.1.3) have been found by going to the root basis and then choosing in a self consistent way which L^{ab} are non-zero.

Before proceeding to discuss the analytic and perturbative solutions of the Virasoro master equation, let us mention some general features of the equation. First, the Virasoro master equation is a set of dim $g(\text{dim } g + 1)/2$ quadratic equations on an equal number of components. After modding out by the dim g inner automorphisms of g and subtracting the trivial Sugawara solution, the number of expected solutions $N(g)$ is approximately

$$N(g) = 2^{e(g)} - 1, \; e(g) = \frac{1}{2}\text{dim } g(\text{dim } g - 1) - 1. \quad (3.1.10)$$

This is a large number even for a relatively small group, for example, for $SU(3)$, $N(g) \sim 2^{27}$. The actual number may be significantly less, due to accidental degeneracies.

In (3.1.10), the final -1 in $e(g)$ is due to K-conjugation invariance, defined as follows. If a solution L^{ab} is found, then it is easy to show that

$$K^{ab} = L_g^{ab} - L^{ab}$$

is also a solution, where L_g^{ab} is the Sugawara stress tensor

$$L_g^{ab} = \frac{\eta^{ab}}{2k + Q}. \quad (3.1.11)$$

This follows from the fact that the matrix $R_{ab,cd}^{ef}$ satisfies

$$U^{cd} L_g^{ef} R_{cd,ef}^{ab} = 2U^{ab} \quad (3.1.12)$$

for any symmetric matrix U. In fact, this is an orthogonal decomposition; the stress tensors associated with L^{ab} and K^{ab} commute, which follows in part from (3.1.12).

This procedure of K-conjugation can be used repeatedly to arrive at nests of solutions. For example, given a collection of algebras such that $g_m \supset g_{m-1} \supset \cdots g_1 \supset g$ suppose that L_g^* is a solution of the Virasoro master equation with central charge

c^* on affine g. Then by repeated embedding with K-conjugation there are additional solutions on g_n,

$$L_{(n)}[L_g^*] = L_{g_n} - L_{(n-1)}[L_g^*] = \sum_{i=1}^{n}(-1)^{n+i}L_{g_i} + (-1)^n L_g^*. \tag{3.1.13}$$

This procedure includes all the known coset models, which are based on the Sugawara construction, but there are many additional examples.

Of course, there is much more to a general conformal field theory than just the form of the Virasoro algebra. One of the most important pieces of information which one needs is the spectrum of primary fields. In fact, caution should be excercised when using the term "primary" field. Normally, what is meant by a primary field is one that is annihilated by all the positive-mode frequencies of the symmetry algebra generators. The latter must include the Virasoro generators, but may include further holomorphic fields, such as the Kač-Moody generators, supersymmetry generators, etc. It seems that in the case of the non-Sugawara solutions of the Virasoro master equations, no such additional fields arise, but there is no general proof of this conjecture.

Assuming this is true, however, one does have a procedure for finding primary fields. Namely, in the Sugawara construction, it is known that the primary fields are associated with unitary representations of the underlying Lie algebra g, see (2.2.23). Considering these same fields ϕ_λ^Λ, it is therefore known that these are annihilated by the raising operators J_n^a, $n > 0$, which is one requirement for a primary field. The other requirement is that the primary field be an eigenstate of L_0. If our primary field are written as

$$\Phi = a_\lambda \phi_\lambda^\Lambda,$$

then Φ is an eigenstate of L_0 with dimension Δ if

$$L^{ab}(\tau^a\tau^b)_\lambda^{\tilde{\lambda}} a_\lambda = \Delta a_{\tilde{\lambda}}. \tag{3.1.14}$$

When L^{ab} is real and g is compact, the matrix that appears in (3.1.14) is hermitian, which implies that the conformal weights Δ are real. Furthermore, since the primary states are unitary transforms of the Sugawara states, which defined unitary representations of the Virasoro algebra, the new primary states also yield unitary representations. It is widely conjectured, but not proven, that the states derived via this procedure comprise *all* the primary fields of the new Virasoro algebra.

B. Geometric Form of the Virasoro Master Equation

It is clear that the Virasoro master equation (3.1.3) is covariant under local group transformations in the adjoint representation of G, i.e.,

$$L^{ab} \rightarrow L^{cd}\Omega_c^a\Omega_d^b$$

$$\Omega_a^c\eta_{cd}\Omega_b^d = \eta_{ab}, \ \Omega(x) \in G. \tag{3.2.1}$$

It is therefore natural to interpret the master equation as living on the tangent space of the group manifold G. This has yielded a geometric form of the master equation.[33]

The Riemann metric g_{ij} on G is defined via

$$g_{ij} = e_i^a L_{ab} e_j^b, \tag{3.2.2}$$

where L_{ab} is the inverse of L^{ab}, and the vierbein e_i^a is given by

$$e_i = -i\gamma^{-1}\partial_i\gamma = e_i^a T_a, \ \gamma(x) \in G, \tag{3.2.3}$$

with T_a an arbitrary hermitian representation of G. Note that the metric as defined here is invariant under left multiplication $\gamma(x) \rightarrow \gamma_0\gamma(x)$. It is this metric that appears in the sigma model lagrangian on G, for example in terms like

$$g_{ij}\partial_\mu x^i \partial^\mu x^j.$$

Multiplying the master equation (3.1.3) by $T_a T_b$, it is not difficult to show that this reduces to

$$0 = e^i(-g_{ij} + 2G_{ij} + 2[e_i, e_j])e^j, \ e^i = g^{ij}e_j, \tag{3.2.4}$$

where $G_{ij} = k\eta_{ij}$. To arrive at an Einstein-like form of this equation, introduce the fundamental structure $\omega_{ia}^b = e_i^c\omega_{ca}^b$, where

$$\omega_{cab} = -\omega_{cba} = \frac{1}{2}(f_{ab}^d L_{cd} - f_{c[a}^d l_{b]d}) \tag{3.2.5}$$

$$[e_i, e_j] = -i\omega_{[ij]}^k e_k \tag{3.2.6}$$

$$\omega_{[ab]}^c = -f_{ab}^c. \tag{3.2.7}$$

ω can be identified as the portion of the spin connection that transforms as a tensor at zero torsion. In the general case of L^{ab} that is under consideration, it is the left-invariant generalization of the usual left- and right-invariant connection obtained from the Killing metric, which is used in the Sugawara form. Using ω, (3.2.4) can be rewritten as

$$0 = -g_{ij} + 2G_{ij} + 2\omega_{kl(i}\omega_{j)}^{kl}. \tag{3.2.8}$$

In terms of ω, the Ricci tensor R_{ij} and curvature scalar (at zero torsion) can be written as

$$R_{ij} = -\omega_{kli}\omega_{j}^{lk} \tag{3.2.9}$$

$$R = -\omega_{klm}\omega^{lkm} = \omega_{klm}\omega^{mkl}. \tag{3.2.10}$$

Now the most general torsion T_{abc} linear in L_{ab} is

$$T_{abc} = \alpha(F_{ab}^{d}L_{cd} + \lambda f_{c[a}^{d}L_{b]d}), \tag{3.2.11}$$

which enters the spin connection via the associated contortion τ_{cab}

$$\tau_{cab} = \frac{1}{2}(T_{c[ab]} - T_{abc}) = \frac{\alpha}{2}[(2\lambda - 1)f_{ab}^{d}L_{cd} + f_{c[a}^{d}L_{b]d}]. \tag{3.2.12}$$

The most natural or connection-like torsion is given by $\alpha = \lambda = -1$, which yields

$$T_{abc} = -2\omega_{cab}, \; \tau_{cab} = \omega_{cab} - \omega_{[ab]c}. \tag{3.2.13}$$

Then (3.2.8) becomes

$$\hat{R}_{ij} + g_{ij} = G_{ij} \tag{3.2.14}$$

$$\hat{R}_{ij} \equiv R_{ij} - \frac{1}{2}\tau_{kl(i}\tau_{j)}^{kl}. \tag{3.2.15}$$

The central charge c is given by

$$c = \dim g - 4R.$$

At this point it is perhaps appropriate to comment that the form of the stress tensor that has been chosen is not the most general one that could be constructed out of the affine currents J^a. The most general form is

$$T = L^{ab}J^aJ^b + D^a\partial J^a, \tag{3.2.16}$$

where D^a are an additional set of constants. With this definition, the master equation is

$$2L^{ab} = L^{cd}L^{ef}R_{cd,ef}^{ab} + 2\Delta^{ab}, \tag{3.2.17}$$

where

$$\Delta^{ab} = i(L^{ac} f^{b}_{ce} + L^{bc} f^{a}_{ce}) D^{e} \tag{3.2.18}$$

and the central charge is given by

$$c = 2G_{ab}(L^{ab} - \Delta^{ab} - 6D^{a}D^{b}). \tag{3.2.19}$$

For this system, the geometric form of the master equation is an Einstein-Maxwell system

$$\hat{R}_{ij} + g_{ij} + i\mathbf{L}_{D} g_{ij} = g^{(g)}_{ij} \tag{3.2.20}$$

$$\omega_{i}^{[kl]} \omega_{(kj)l} D^{i} = i D^{i} \mathbf{L}_{D} g_{ij}, \tag{3.2.21}$$

where $\mathbf{L}_D$ is the Lie derivative in the D direction given by

$$\mathbf{L}_{D} g_{ij} \equiv g_{k(i}\partial_{j)} D^{k} + D^{k}\partial_{k} g_{ij} = -\omega^{k}_{(ij)} D_{k}. \tag{3.2.22}$$

Although this procedure has reduced the master equation to something like an Einstein equation on a group manifold, there is still not an action for this system. It has not been proven that such an action does not exist, but if it does exist, it is certainly not a simple generalization of the Wess-Zumino-Witten action. Furthermore, a number of the properties of the master equation are unclear from this point of view, K-conjugation in particular.

C. Self-Consistent Ansätze

The master equation is a large number of coupled quadratic equations, whose general solution is very difficult to determine. In order to determine at least some non-trivial solutions, it has proven useful to consider self-consistent ansätze.[3] That is, L^{ab} are non-zero only for certain a and b, in such a way that the number of equations and the number of unknowns still agree.

To do so, it has proven convenient to go to the Cartan-Weyl basis, in which the metric η_{ab} is given by

$$\eta_{AB} = \delta_{AB}, \ \eta_{\alpha\beta} = \delta_{\alpha+\beta,0}, \tag{3.3.1}$$

where capital letters denote elements of the Cartan subalgebra, and greek letters denote roots of g. The latter of these two formulas, for the case of simply-laced algebras, implies that the simple roots have been normalized to $(\text{length})^2$ 2. In this basis, the structure constants are

$$f_{A\alpha}{}^{\alpha} = f_{\alpha-\alpha}{}^{A} = -i\alpha^A, \; f_{\alpha\beta}{}^{\gamma} = -iN_\gamma(\alpha,\beta), \tag{3.3.2}$$

where

$$N_\gamma(\alpha,\beta) = N_{-\beta}(-\gamma,\alpha) = -N_{-\gamma}(-\alpha,-\beta)$$

$$\sum_\alpha \alpha^A\alpha^B = Q\delta^{AB}, \quad \sum_{\{\beta,\gamma|\gamma+\beta=\alpha\}} N_\alpha^2(\beta,\gamma) = Q - 2\alpha^2. \tag{3.3.3}$$

In the new basis, if the stress tensor is denoted as

$$T = L^{AB}J_AJ_B + L^{\alpha\beta}J_\alpha J_\beta + L^{A\alpha}J_AJ_\alpha, \tag{3.3.4}$$

then for unitarity,

$$L^{AB} = L^{BA}, \; L^{A\alpha} = L^{\alpha A} = (L^{A-\alpha})^*, \; L^{\alpha\beta} = L^{\beta\alpha} = (L^{-\alpha-\beta})^* \tag{3.3.5}$$

is required.

The simplest self-consistent ansatz is to allow only the L^{AB} to be non-zero. In this case, the master equation reads

$$L^{AB} = 2k\sum_C L^{AC}L^{CB}.$$

It is clear that this is simply a rotation (and/or truncation) of a $U(1)^{\text{rank}(g)}$ current algebra, and yields no new solutions.

A more complicated ansatz is to allow only the L^{AB} and $L^{\rho\pm\rho}$ to be nonzero, for some subset $\{\rho\}$ of the roots of g. The self-consistency condition then requires that no four of the roots in $\{\rho\}$ satisfy

$$\alpha + \beta = \gamma \text{ and } \alpha - \beta = \delta. \tag{3.3.6}$$

For simply-laced g, this is actually not a restriction at all, but for other groups such as $SO(2n+1)$ for example, it is.

After some algebra, the master equation takes the form of three equations, namely

$$L^{AB} = 2k\sum_C L^{AC}L^{CB} + \sum_\rho (|L^{\rho\rho}|^2 - (L^{\rho-\rho})^2)\rho^A\rho^B + \sum_{\rho C} L^{\rho-\rho}\rho^C L^{C(A}\rho^{B)}$$

$$L^{\rho\rho}[1-4kL^{\rho-\rho}-4\chi_\rho-2\sum_{\{\alpha,\beta|\alpha+\beta=\rho\}}L^{\beta-\beta}N_\rho^2(\alpha,\beta)]=\sum_{\{\alpha,\beta|\alpha+\beta=\rho\}}L^{\alpha\alpha}L^{\beta\beta}N_\rho^2(\alpha,\beta)$$

$$L^{\rho-\rho}=2(k-\rho^2)|L^{\rho\rho}|^2+2(k+\rho^2)(L^{\rho-\rho})^2+\sum_{\{\alpha,\beta|\alpha+\beta=\rho\}}(2L^{\rho-\rho}-L^{\beta-\beta})L^{\alpha-\alpha}N_\rho^2(\alpha,\beta), \tag{3.3.7}$$

where

$$\chi_\rho=\sum_{AB}L^{AB}\rho^A\rho^B. \tag{3.3.8}$$

Another self-consistent ansatz allows L^{AB} to be non-zero, and $L^{\alpha\beta}\neq 0$ for roots that satisfy $\alpha\cdot\beta=0$ (this implies that $\alpha\pm\beta$ is not another root of g). In this case

$$L^{AB}=2k\sum_C L^{AC}L^{CB}+\sum_{\alpha\beta}|L^{\alpha\beta}|^2\alpha^A\beta^B+\sum_{\alpha C}L^{\alpha-\alpha}\alpha^C L^{C(A}\alpha^{B)}$$

$$L^{\alpha\beta}[1-\sum_{AB}L^{AB}(\alpha+\beta)^A(\alpha+\beta)^B]=\sum_\rho(2k-(\alpha-\beta)\cdot\rho)L^{\alpha\rho}L^{-\rho\beta}. \tag{3.3.9}$$

This is equivalent to considering a $SU(2)^q\times U(1)^p$ subgroup of g.

An even simpler self consistent ansatz is the intersection of the last two. In this case, (3.3.7) can be solved for $L^{\rho\rho}$ and $L^{\rho-\rho}$ in terms of L^{AB}, which leaves only one equation that determines the latter. Specifically, for $k\neq\rho^2$,

$$L^{\rho-\rho}=\frac{1}{4k}(1-4\chi_\rho),\ L^{\rho\rho}=\frac{1}{16k^2}(1-4\chi_\rho)(1+\frac{k+\rho^2}{k-\rho^2}4\chi_\rho)$$

$$L^{AB}=2k\sum_C L^{AC}L^{CB}+\sum_{\rho>0}\frac{1-4\chi_\rho}{2k}(\frac{2\rho^A\rho^B}{k-\rho^2}\chi_\rho+\sum_C\rho^C L^{C(A}\rho^{B)}). \tag{3.3.10}$$

As a simple example, consider the case of $SU(2)$.[3] In this case, the last ansatz we considered, (3.3.10), is satisfied since $SU(2)$ has only one positive root. Taking $L^{AA}=\lambda$, where A represents J^3, then (3.3.10) reduces to an equation for λ

$$\lambda(k-4)(1-2\lambda(k+2))=0 \tag{3.3.11}$$

as long as $k\neq 2$. This yields $k=4$. Choosing the other factor in (3.3.11) to be zero would yield a non-unitary solution due to (3.3.5). Furthermore, from (3.3.10),

$$L^{\alpha-\alpha}=\frac{1}{8\alpha^2}(1-4\lambda\alpha^4),\ (L^{\alpha\alpha})^2=\frac{1}{64\alpha^4}(1-4\lambda\alpha^4)(1+12\lambda\alpha^4), \tag{3.3.12}$$

where α represents the single positive root, so that unitarity requires

$$-\frac{1}{12} \leq \lambda\alpha^4 \leq \frac{1}{4} \tag{3.3.13}$$

Note that, independent of λ, these solutions have $c = 1$. In addition, if the general procedure for determining the conformal weights of the primary fields, (3.1.14), is applied, the spin one representation of $SU(2)$ for example yields primary fields of the new Virasoro algebra of dimension

$$\Delta = \frac{1}{8}\{2(1-4\lambda\alpha^4), 1 + 4\lambda\alpha^4 \pm \eta((1-4\lambda\alpha^4)(1+12\lambda\alpha^4))^{1/2}\}, \tag{3.3.14}$$

where η is the sign of $L^{\alpha\alpha}$ in (3.3.12).

Using such self-consistent ansatz, one can find solutions to the Virasoro master equation for many systems, particularly ones with large symmetries, such as tensor products $G\times G \cdots \times G$. For example, by looking at the tensor product $G\times SO(\dim g)_1$, new superconformal algebras can be found.[34] In these more general cases, unlike the simple $SU(2)$ case demonstrated here, irrational values of the central charge are generally found. In all cases, the central charge is greater than or equal to one, so the unitarity constraints of Friedan, Qiu, and Shenker[14] which we discussed in Chapter I are not violated.

For $c \geq 1$, modular invariance requires an infinite number of primary fields, assuming that the symmetry algebra (the set of holomorphic fields) is only the Virasoro algebra and nothing more. But the procedure for finding primary fields starts from a finite-dimensional space, so only find a finite number of primary fields can be found. On the other hand, there do not seem to be any additional holomorphic fields with which to enlarge the symmetry algebra. In fact, it is easy to see that even the affine currents are not primary with respect to the Virasoro algebra (for non-Sugawara solutions of the master equation), and that the combined Virasoro-affine algebra only closes in the enveloping algebra of the current algebra. This leads to considerations of infinite extensions of the Virasoro algebra, such as W_∞, but it is difficult to see how such a structure as this can arise from complicated solutions of the master equation.

D. High Level Analysis

In this section, instead of studying the exact solutions discussed above, we follow the fact, noted in [3], that at high level k, the master equation is approximately

$$L^{ab} = 2kL^{ac}L^{cb} \tag{3.4.1}$$

or, more schematically, $L^2 = L$. This fact allows us to derive perturbative solutions to the master equations that were previously unknown.

So let us attempt to use the quantity $x \equiv 1/(2k)$ as an expansion parameter. This is a semiclassical expansion, since in the last chapter it was seen that the level k multiplies the entire WZW action, and so high level corresponds to small $\hbar$. Also, at high level, the current algebra approaches a $U(1)^{\dim g}$ algebra. So let

$$L^{ab} = xL_1^{ab} + x^2 L_2^{ab} + \ldots \tag{3.4.2}$$

Substituting this in (3.1.3), it is easily seen that

$$L_1^{ab} = L_1^{ac} L_1^{cb}, \tag{3.4.3}$$

$$L_2^{ab} = L_1^{ac} L_2^{cb} + L_2^{ac} L_1^{cb} - L_1^{cd} L_1^{ef} f^{cea} f^{dfb} - L_1^{cd} f^{cef} (f^{dfa} L_1^{be} + f^{dfb} L_1^{ae}), \tag{3.4.4}$$

etc. The first of these is trivial to solve; L_1 is a diagonal matrix of 1's and zeroes, up to a (real) rotation, i.e.,

$$L_1^{ab} = \Omega^{ac} \Omega^{cb} \Theta^c \tag{3.4.5}$$

where $\Omega \in SO(\dim g)$ and $\Theta^c = 0$ or 1. Since the master equation is invariant under a change of basis, without loss of generality the basis can be chosen so that L_1 is precisely diagonal,

$$L_1^{ab} = v^a \delta^{ab} \quad , \quad v^a = 1 \text{ or } 0. \tag{3.4.6}$$

The vector $\vec{v}$ is said to be in standard form if and only if it is a series of 1's followed by a series of zeroes, i.e.,

$$v^a = 1, a \leq n, v^a = 0, a > n.$$

Let us now look more closely at equation (3.4.4). If $\vec{v}$ is not in standard form, carry out a permutation so that it is in standard form. There are three cases to consider:

1) $a \leq n$, $b \leq n$. In this case, $L_1^{ac} = \delta^{ac}$ and $L_1^{cb} = \delta^{cb}$ for all c so that

$$L_2^{ab} = \sum_{e,c} (v^c v^e f^{cea} f^{ceb} + 2v^c f^{cbe} f^{cea}). \tag{3.4.7}$$

2) $a > n$, $b > n$. In this case,

$$L_2^{ab} = -\sum_{c,e} v^c v^e f^{cea} f^{ceb}. \tag{3.4.8}$$

3) $a \leq n$, $b > n$. Here is where the problems set in, for now L_2^{ab} drops out of the equation, leaving only

$$0 = -\sum_{c,e}(v^c v^e f^{cea} f^{cdb} + v^c f^{ceb} f^{cae}). \tag{3.4.9}$$

If $\vec{v}$ does not satisfy this equation, then there is no solution of this type. This is a situation reminiscent of degenerate perturbation theory in quantum mechanics. If $\vec{v}$ does satisfy this equation, the problem of finding L_2^{ab} for $a \leq n, b > n$ remains. For such an a and b, look at the equation at the next level of the perturbation series. In exactly the same way as before, L_3^{ab} drops out, leaving

$$0 = L_2^{ac} L_2^{cb} - (L_1^{cd} L_2^{ef} + L_2^{cd} L_1^{ef}) f_{ce}{}^a f_{df}{}^b - L_1^{cd} f_{ce}{}^g (f_{dg}{}^a L_2^{be} + f_{dg}{}^b L_2^{ae}) - L_2^{cd} f_{ca}{}^g f_{dg}{}^b, \tag{3.4.10}$$

where summation over repeated indices is implied. At first glance, this looks like a set of quadratic equations in L_2^{ab}, which is no simpler than the original master equation. However, denote L_2 as

$$L_2 = \begin{pmatrix} M_1 & M_3 \\ M_3^T & M_2 \end{pmatrix},$$

where M_1 is an n by n matrix, M_2 is (dim $g - n$) by (dim $g - n$), etc. Then M_1 and M_2 are already known from equations (3.4.7) and (3.4.8), respectively. Then, for example, the first term in (3.4.10) is just $(M_1 M_3 + M_3 M_2)^{ab}$ which is linear in the elements of M_3, the unknowns we are trying to find. Similarly, (3.4.10) reduces to a system of linear equations in these variables.

Of course, this system of equations might have no solutions, an infinite number of solutions, or a unique solution. It does not seem possible to prove inductively, for example, that a unique solution up to level n guarantees a unique solution or even a finite number of solutions at the next level. Nor can can it be easily proven that the resulting series converges. Instead the method has been tested on some simple systems, which display all the possible problems.

I. SU(2): This case is particularly simple, for it is easy to show that if a solution up to level n is diagonal, then L_{n+1}^{ab} is also diagonal. By the symmetry of the structure constants (ϵ^{abc}), there are clearly only three choices for $\vec{v}$, namely (1,0,0), (1,1,0), and (1,1,1), up to permutations. Using the pertubative techniques outlined above, it can be checked easily that the first of these generates the $U(1)$ Sugawara form, the last of these generates the full $SU(2)$ Sugawara, and the second generates the $SU(2)/U(1)$ Sugawara.

II. SU(3): The counting of the possible vectors $\vec{v}$ grows very rapidly with the dimension of the underlying group. Naively, the number of vectors with 1's and 0's is just $2^{dim g} = 2^8 = 256$ for $SU(3)$. We can immediately get rid of one factor of 2 by recalling that solutions of the master equation always come in pairs; if L^{ab} is a solution, so is $L^{ab}_{Sug} - L^{ab}$ where $L^{ab}_{Sug} = \delta^{ab}/(2k+Q)$. In our formulation, for every vector $\vec{v}$ which generates a solution, the vector $(1,1,...) - \vec{v}$ also generates a solution.

Of the remaining choices for $\vec{v}$, many (almost half) can be eliminated as corresponding to the Sugawara forms of the $U(1)$, $U(1)^2$, $SU(2)$, or $SU(2) \times U(1)$ subgroups of $SU(3)$. Many of those vectors that remain after this cut can be identified with one another via permutations of the generators. This procedure leaves only six inequivalent vectors $\vec{v}$, which we take to be

$$\vec{v}_1 = (1,1,0,1,0,0,0,0)$$

$$\vec{v}_2 = (0,1,0,0,1,0,0,0)$$

$$\vec{v}_3 = (0,1,1,1,1,0,0,0)$$

$$\vec{v}_4 = (1,0,0,1,0,1,0,0)$$

$$\vec{v}_5 = (1,1,0,1,0,1,0,0)$$

$$\vec{v}_6 = (0,1,0,0,1,0,1,0). \qquad (3.4.11)$$

Of these six, it turns out that $\vec{v}_1$ and $\vec{v}_5$ yield unique solutions for L_2^{ab}; we will not display them as the results are not particularly illuminating. The corresponding values of c have been computed up to third order using the analogs of equations (3.4.7) and (3.4.8) for L_3, and are found to be, respectively,

$$c_1 = 3 + \frac{.1875}{k} + \frac{.477}{k^2} + o(1/k^3)$$

$$c_5 = 4 - \frac{.625}{k} + \frac{2.44}{k^2} + o(1/k^3)$$

It turns out that the vector $\vec{v}_3$ does not allow any solution for L_2, while the other three vectors all allow multiple (i.e. infinite) solutions. For example, using the vector $\vec{v}_6$ yields the equations

$$M_3^{11} = M_3^{21} = M_3^{23} = M_3^{31} = 0$$

with all other elements of M_3 unknown, i.e. an 11 parameter solution. These parameters could possibly be restricted by finding the central charge to third order and requiring that it be positive, but this would not be a conclusive test in any case. It should be noted that there is one known solution in this 11 parameter space, namely the symmetric space solution $SU(3)_k/SU(2)_{4k}$, in which L^{ab} is proportional to $\vec{v}_6$. Thus, our 11 parameter space of solutions might be regarded as deformations about this symmetric space, though this connection has not been made precise.

The other remaining cases, $\vec{v}_2$ and $\vec{v}_4$, yield 8 and 4 parameter solutions respectively. It appears, however, that if no free parameters are required at this first level, then none will appear at later levels, as the structure constants in the master equation tend to make more and more elements non-zero, and a zero determinant in the system of equations determining M_3 (more precisely, its analogs at higher orders in the perturbation series) becomes very unlikely. This has been checked this to third order in the pertubation series for the two discrete solutions mentioned above, but there is no general proof of this assertion.

Since we have explicitly evaluated only the first few orders of the perturbative expansion, we cannot be sure that the expansion converges. However, looking at the high order form of the expansion (for example, substituting $L_m \sim m^{\alpha(m)}$) it is not difficult to show that the expansion should converge for $x < 1$, i.e., $k > 1/2$ which is the region of interest.

Another algebra for which this high level analysis is particularly interesting is $SO(2n) = D_n$[32]. It is known that this algebra has a basis in which, for fixed a and b, the structure constant $f_{ab}{}^{c}$ is nonzero for only a single value of c. It follows from this fact that an ansatz in which only the diagonal components of L^{ab} are nonzero is self consistent.

The generators of D_n can be denoted $\alpha^{ij} = e_i - e_j$, where the e_i are unit $(n+1)$-dimensional vectors. We can then associate a zeroth order solution of the master equation, as in (3.4.6), with a graph by denoting each value of i with a dot. If $v^a = v^{ij} \neq 0$, connect the dots corresponding to i and j with a line. It is then easy to recognize equivalent perturbative solutions; they correspond to graphs that are themselves equivalent, when we allow an arbitrary relabelling, or permutation, of the dots in the graph. Similarly, disconnected graphs correspond to stress tensors which could be decomposed into two commuting parts. Also, solutions that are K-conjugate to each other correspond to graphs that are conjugate to each other (the conjugate of a graph is obtained by removing all the lines in a graph and placing a line between all pairs that were not connected in the original graph). Numerous quantities, such as the leading terms for the conformal weights and the L^{ab}, can be computed in terms of the corresponding graph given the zeroeth order solution. Similar methods can be used for certain other algebras, such as $SU(4)$, but not in general.

Our perturbative method for solving the Virasoro master equation has serious limitations; the series cannot be summed easily (or even proven to converge), nor is there a unique solution for a given zeroth order solution. This is not very surprising given the large number of solutions of the master equation. Nevertheless, many more approximate solutions can be found than can be derived based on self-consistent ansatze, especially for large groups, where only maximally symmetric ansatze can be analyzed easily. It might have been hoped that by finding more solutions, some kind of organization of solutions would have arisen to help in understanding the deeper structure that underlies the master equation. In our method, no such organization seems to be present; there seems to be no reason why one zeroeth order solution yields a unique solution, while others result in many parameter solutions. One possible approach to improving our methods would be, for example, to rediagonalize L at every level of the perturbation theory. This was not really necessary when discussing only the first few levels, but could be useful in more extended calculations. In any case, there are many properties of the master equation and its solutions that are still mysterious and deserve further attention.

IV. Current Algebra and Topological Field Theory

A. Introduction to Topological Field Theory

1. Cohomological Formulation

Topological gravity and topological field theory were originally developed to study moduli spaces related to manifolds using field theoretic methods and ideas. For example, in Donaldson theory,[35] one begins with a four dimensional manifold and a gauge group G. Let I be the space of Yang-Mills instantons (solutions of the Euclidean classical equations of motion) with some fixed instanton number on the manifold M. If this is a finite set, define the "partition" function Z as simply the number of points in this set, or equivalently $Z = \#(I)$. In a field theoretic language,[36] the partition function is expressed as a path integral which can be evaluated in the small coupling limit (see below). In this limit, keeping only the quadratic terms in an expansion about an instanton solution results in a partition function of the form

$$Z = \sum_i \frac{det(D_F)}{\sqrt{det(\Box)}}$$

(i denotes different instantons) where D_F essentially represents the Dirac operator and $\Box$ the Klein-Gordon operator (see [36] for the details that are being omitted here). These operators are related by supersymmetry, so that Z reproduces Donaldson's result, $Z = \sum_i 1 = \#(I)$.

Generically, I has a non-zero dimension, so this will not suffice. Call its dimension $n = 2k$. Now, let Σ be a submanifold of M. Generically, the Dirac operator (restricted to Σ) D_Σ has no zero modes, but there is a submanifold H_Σ of I of codimension two that consists of instantons for which the Dirac operator does have a zero mode. The intersection number $\#(H_{\Sigma_i})$ can then be interpreted as a correlation function of operators O_i in some quantum field theory. In field theory, $\dim I \neq 0$ means that there are fermion zero modes, which would cause the partition function to automatically vanish. Therefore, some operator O is inserted in the path integral to absorb the zero modes. Requiring that the new path integral is independent of the metric and independent of the location of the operator insertion then implies (see [36]) that $\{Q, O\} = 0$, but $O \neq Q\chi$, where Q, the BRST operator, satisfies $Q^2 = 0$. As shown in [36], this operator O can be written as a product of operators O_i, each associated with a homology cycle (equivalently, a submanifold) of M as in Donaldson theory.

The field theoretic method by which such moduli spaces can be studied is to arrange, via a particular choice of gauge multiplets, that the action be a BRST variation. The BRST variation of an operator O is denoted $\{Q,O\}$. For example, suppose that the partition function can be written as

$$Z=\int(DX)exp(-S/e^2), \tag{4.1.1}$$

where X represents all the fields in the theory, and the coupling e does not appear in S or the measure. Consider how this changes with e,

$$\delta Z=\delta(-\frac{1}{e^2})\int(DX)exp(-S/e^2)\{Q,V\}=0, \tag{4.1.2}$$

where we have used the fact that the expectation value of (or in fact any correlation function with) a BRST commutator vanishes. That fact results from the BRST invariance of S and of the measure. Given the invariance of the measure, it follows that the path integral

$$Z_\epsilon(O)=\int(DX)e^{\epsilon Q}e^{-S/e^2}O$$

(where O is an arbitrary operator) is independent of ϵ. Expanding in ϵ, using $Q^2=0$, and subtracting off $Z_0(O)$ yields

$$0=\int(DX)e^{-S/e^2}\{Q,O\}$$

as desired. Given (4.1.2), take the limit where e is very small, in which the path integral is completely determined by configurations of minimal action. In Donaldson theory,[36] the gauge field terms in the action are

$$S=\int_M\sqrt{g}Tr[(F_{\alpha\beta}+\tilde{F}_{\alpha\beta})(F^{\alpha\beta}+\tilde{F}^{\alpha\beta})] \tag{4.1.3}$$

where $F_{\alpha\beta}$ is the field strength, $\tilde{F}_{\alpha\beta}=\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}$. Therefore the minimum action configurations satisfy

$$F_{\alpha\beta}+\tilde{F}_{\alpha\beta}=0$$

which are just the instanton configurations.

In the case of Donaldson theory,[36] the full action is derived by starting with a multiplet (A_μ,ψ_μ,ϕ), all in the adjoint of the gauge group, with transformation laws

$$\delta A_\mu=\epsilon\psi_\mu$$

$$\delta\psi_\mu = -\epsilon\partial_\mu\phi$$

$$\delta\phi = 0. \tag{4.1.4}$$

Other multiplets must then be added to make an appropriate BRST invariant Lagrangian, whose minimum action configurations are the instantons, but the details, given in reference [36], will not be repeated here. From (4.1.4), one immediately sees that gauge and BRST invariant operators are $O_{k(0)}(x) = (Tr\phi^2)^k$, which are zero forms. Then it is possible to recursively solve the equations

$$dO_{k(0)} = \{Q, O_{k(1)}\}$$

$$dO_{k(1)} = \{Q, O_{k(2)}\}$$

$$dO_{k(2)} = \{Q, O_{k(3)}\}$$

$$dO_{k(3)} = \{Q, O_{k(4)}\}$$

$$dO_{k(4)} = 0, \tag{4.1.5}$$

which are known as the descent equations. The subscript in parentheses denotes the degree of the operator considered as a form on the surface. Given an r dimensional submanifold Σ, a BRST invariant operator

$$O_k = \int_\Sigma O_{k(r)} \tag{4.1.6}$$

can be defined, which can then be considered as a perturbation to the original action. It turns out that correlation functions of the above operators with $k = 1$ are precisely the intersection numbers studied in Donaldson theory.

A similar field theoretic formulation of two dimensional gravity can be made, as will be discussed in the next section. While this choice is certainly not unique, the nature of the physical observables is relatively clear. Instead of the space of instantons, we are interested in the moduli space $M_{g,s}$ of a Riemann surface with genus g and s punctures, located at positions $(x_1, x_2, \ldots, x_s)$, which are studied by investigating its cohomology classes.

Rather than choose a particular field theoretic formulation of topological gravity and its BRST invariant operators σ_n, which is deferred to section (A.4) of this chapter, at this stage the theory can be defined by its correlation functions, which are taken as[5]

$$\langle \prod_i^s \sigma_{d_i} \rangle = \int_{M_{g,s}} \wedge_i \lambda_{(i)}, \tag{4.1.7}$$

($d_i \geq 0$ are integers) where $M_{g,s}$ is the moduli space of Riemann surfaces of genus g with s punctures, and the $\lambda_{(i)}$ are $2d_i$ forms defined as

$$\lambda_{(i)} = \alpha_{(i)}^{d_i} d_i!. \tag{4.1.8}$$

The $\alpha_{(i)}$ are two forms which are defined as follows. Suppose the punctures (operator insertions) are located at positions x_i on the Riemann surface Σ. Then the cotangent space at x_i, which is a two-dimensional real vector space, varies smoothly in the moduli space, and therefore, is the fiber of a real bundle $W_{(i)}$. Such a two dimensional bundle has an Euler class, which can be represented as a two form on the moduli space, and is defined to be the two form $\alpha_{(i)}$.

There is an alternative definition of $\alpha_{(i)}$, which turns out to be convenient in finding recursion relations for the correlation functions. Namely, let $L_{(i)}$ be the line bundle over $M_{g,s}$ whose fibers are the cotangent space (a one complex-dimensional space) at a puncture x_i, and let w be a meromorphic section of $L_{(i)}$. If w_z and w_p denote, respectively, the set of zeroes and poles of w, then $\alpha_{(i)}$ can be represented as the divisor $W_{(i)} = w_z - w_p$, and the wedge product $\alpha_{(i)}^d$ as an intersection of $W_{(i)}^j$, $j = 1, \ldots, d$, where the superscript labels d different sections w chosen independently. The correlation function is then defined as the intersection of all the $W_{(i)}^j$.

Up to this point we have not been very careful about the integration over moduli space. Since $M_{g,s}$ has dimension $6g - 6 + 2s$, and $\lambda_{(i)}$ is a $2d_i$ form, the correlation function automatically vanishes unless

$$\sum_i d_i = 3g - 3 + s. \tag{4.1.9}$$

In addition, some compactification of moduli space is necessary. The natural choice in string theory, which is used here, is "stable" compactification. In this compactification, in addition to the usual smooth manifolds, surfaces that develop a node. The node may separate the surface into two branches (see Fig. 1) or just pinch off a handle (see Fig. 2, respectively). Operator insertions, however, always are required to remain at points distinct from each other and from the nodes.

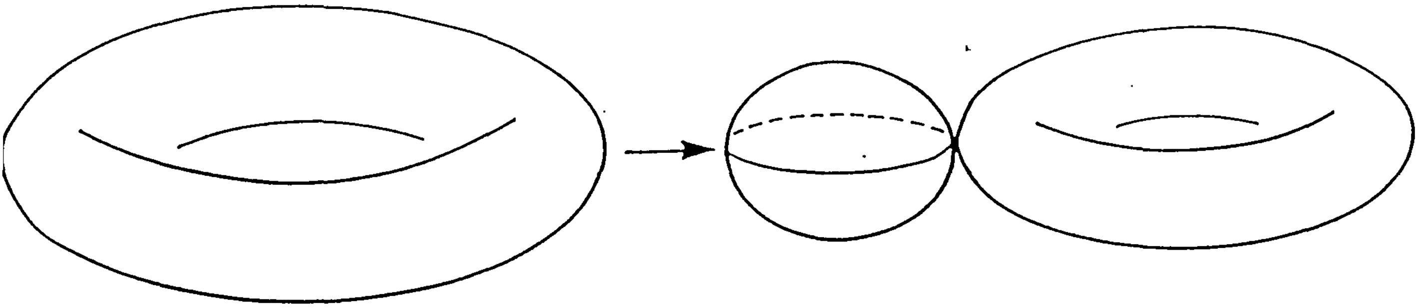

Figure 1: A surface separating into two branches.

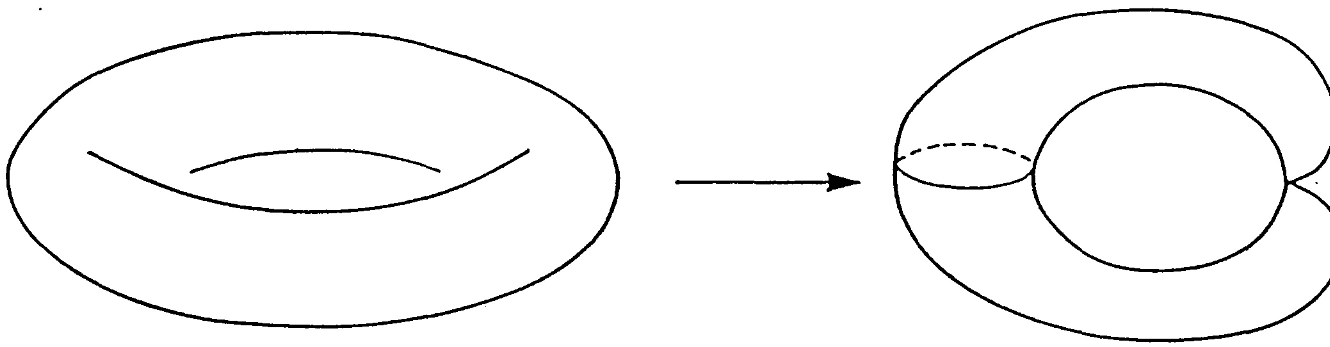

Figure 2: Pinching a handle.

The alternative definition of $\alpha_{(i)}$ is now useful because, by a judicious choice of a section w, a correlation function can be reduced to a sum of correlation functions each involving fewer operators. However, such a reduction is only valid for genus zero (the recursion relations will be extended to all genuses shortly). So consider a genus zero surface that develops a node. A section w is chosen so that $W^1_{(1)}$ is

$$w = dx_1\left(\frac{1}{x_1 - x_{s-1}} - \frac{1}{x_1 - x_s}\right). \tag{4.1.10}$$

This is invariant under $SL(2, C)$ transformations, so it is a well defined section on the sphere. In calculating the intersection index, we have to consider all the possibilities of whether x_1, x_s, and x_{s-1} are on the same branch or different branches of the surface. It turns out that the only contribution is when x_1 is on one branch, call it Σ_1, while x_s and x_{s-1} are on another branch Σ_2. When this is the case, w must vanish identitically on Σ_1, since the only pole it could have would be a simple pole at the node, but by the Riemann-Roch theorem, w is of degree two, and hence it must have at least two poles.

Then, for any decomposition of the set $S = \{x_2, \cdots, x_{s-2}\}$ as a union of subsets

X and Y, let $\Delta_{X,Y}$ be the divisor in $M_{0,s}$ consisting of configurations in which the surface has degenerated into two branches with $x_1 \cup X$ in one branch and $Y \cup \{x_s, x_{s-1}\}$ in the other branch. In this case, the divisor of zeros of w is

$$W^1_{(1)} = \sum_{S=X\cup Y} \Delta_{X,Y}, \tag{4.1.11}$$

as w has no poles to subtract off.

The key fact is that $\Delta_{X,Y}$ naturally decomposes into the Cartesian product of divisors in $M_{0,r+2} \times M_{0,s-r}$, where it is assumed that X contains r points, and where care has been taken to include the node as a distinguished point on each branch. So the correlation function decomposes, for fixed X and Y, into a product of correlation functions, only with σ_{d_1} changed to σ_{d_1-1}, since the intersection has already been restricted to the divisor $W^1_{(1)}$. That is, the full recursion relation is

$$\langle \prod_{i=1}^{n} \sigma_{d_i} \rangle = d_1 \sum_{S=X\cup Y} \langle \sigma_{d_1-1} \prod_{j\in X} \sigma_{d_j} P \rangle \langle P \prod_{k\in Y} \sigma_{d_k} \sigma_{d_{s-1}} \sigma_{d_s} \rangle. \tag{4.1.12}$$

The presence of the puncture operator $P = \sigma_0$ in this equation represents the fact that the node on each branch is a distinguished point but one at which no operator is inserted.

Thus far, we have not made the connection with the field theoretic formulation of topological gravity which was alluded to earlier. In fact, thus far all the possible perturbations to the underlying Lagrangian (which is generally a topological invariant such as the Euler characteristic, and can in fact be taken to be identically zero) as in equation (4.1.6) are absent. It is only at this point in the phase space of couplings that the correlation functions are intersection indices of forms on moduli space.

Suppose that the action is perturbed $S \to S - \sum_i t_i \int \sigma_{i(2)}$ with coupling constants t_i. It is desirable that the recursion relations above should continue to hold. To do so, note that expanding the factor e^{-S} in the field theoretic path integral just brings down factors of the perturbations, or in other words (using (4.1.6))

$$\frac{\partial}{\partial t_i} \langle X \rangle = \langle X \sigma_i \rangle \tag{4.1.13}$$

for any product of operators X.

Let us consider the recursion relation (4.1.12) for the case of three operators, which is

$$\langle \sigma_{d_1} \sigma_{d_2} \sigma_{d_3} \rangle = d_1 \langle \sigma_{d_1-1} P \rangle \langle P \sigma_{d_2} \sigma_{d_3} \rangle. \tag{4.1.14}$$

The derivative of the left-hand side with respect to t_r is just $\langle\sigma_{d_1}\sigma_{d_2}\sigma_{d_3}\sigma_r\rangle$, while the derivative of the right-hand side is

$$d_1\langle\sigma_{d_1-1}\sigma_r P\rangle\langle P\sigma_{d_2}\sigma_{d_3}\rangle + d_1\langle\sigma_{d_1-1}P\rangle\langle P\sigma_{d_2}\sigma_{d_3}\sigma_r\rangle. \qquad (4.1.15)$$

But the equality of these two expressions is just the recursion relation for the four point function! This process can be continued order by order in the expansion of the action, proving that the recursion relations are true for any perturbation of the Lagrangian.

Thus far, all the correlation functions in the above equations apply only to genus zero. Similar equations can be written in which correlation functions at genus one are reduced to correlation functions with either fewer operators or correlation functions on the sphere. Continuing in this vein is difficult, however, as the details depend very much on the special topology of surfaces with low genus. Instead, recursion relations that are valid for all genus are desired. The first such recursion relation to be derived[37] is the "puncture" equation, which reads

$$\langle P\prod_{i=1}^{s-1}\sigma_{n_i}\rangle = \sum_{i=1}^{s-1} n_i\langle\prod_{k=1}^{s-1}\sigma_{n_k-\delta_{ik}}\rangle. \qquad (4.1.16)$$

The proof relies on the natural projection map $\pi : M_{g,s} \longrightarrow M_{g,s-1}$ which roughly consists of forgetting the location of the puncture operator in (4.1.16). In doing so, any genus zero component of the surface with three marked points, one of which is the location of the puncture operator, must be collapsed to a point as such a component has no moduli. Once this is realized, the proof relies on very much the same reasoning as in the genus zero recursion relation above, and the details will not be repeated here.

Care must be taken in using (4.1.16) for low values of the genus and for correlation functions with only a few operators. These cases must be treated separately. In particular, $\langle PPP\rangle = 1$ since the sphere with three punctures in completely rigid, *i.e.*, its moduli space consists of a single point.

It is important to note, that unlike the previous genus-zero and genus-one recursion relations, (4.1.16) is only valid at the zero coupling point. However, (4.1.16) can be written as a differential equation for the partition function Z; recalling that derivatives with respect to the couplings correspond to operator insertions, we see that (4.1.16) is equivalent to the differential equation

$$\frac{\partial Z}{\partial t_0} = \frac{t_0^2}{2} + \sum_{i=0}^{\infty}(i+1)t_{i+1}\frac{\partial}{\partial t_i}Z. \qquad (4.1.17)$$

Taking the derivative of (4.1.17) with respect to t_0 and denoting $u = \langle PP \rangle$, $u_i = \langle P\sigma_i \rangle$,

$$u = t_0 + \sum (i+1) t_{i+1} u_i. \tag{4.1.18}$$

This equation is remarkably close to the so-called "string" equation of the recent matrix models,[38] which purport to provide an exact solutions of 2-d gravity. It is this correspondence, which has been extended to an exact equivalence between these two methods, that has driven the great interest in topological gravity and topological field theory.

2. Matrix Models of 2-d Gravity

In analogy with Yang-Mills theory, where a path integral over all connections modulo gauge equivalence is calculated, in quantum gravity an integration over all possible metrics modulo diffeomorphisms is necessary. One method of doing so is to discretize the surface, that is, divide the surface into triangles say, and sum over all such decompositions. For simplicity, assume that a surface is covered with a set of n_2 equilateral triangles, with sides of length a. If n_1 is the number of edges and n_0 the number of vertices in such a triangulation, then the Euler characteristic is given by the well-known formula $\chi = n_2 - n_1 + n_0$. Then the partition function of quantum gravity, including a cosmological constant term, is approximated by

$$Z = \int_M Dg e^{-\int d^2x g^{\frac{1}{2}} (\Lambda - \frac{1}{16\pi G_0} R)} \to \sum_T \frac{1}{C(T)} e^{-\Lambda a^2 n_2(T) + \frac{1}{4G_0}\chi(T)}, \tag{4.1.19}$$

where $C(T)$ is the symmetry factor of a given triangulation T.

The essential trick used to do the sum here is to notice that the dual graph to any triangulation (defined by connecting by a line any triangles that share an edge) is a Feynman diagram in a ϕ^3 field theory. Consider a $U(N)$ invariant matrix model,[10] whose partition function is

$$Z = \int d\phi e^{-NTr(\frac{\phi^2}{2} - g\frac{\phi^3}{3})} \equiv \int d\phi e^{-NTr(V(\phi)}, \tag{4.1.20}$$

where ϕ is a Hermitian $N \times N$ matrix and the measure is $d\phi \equiv \prod_{i,j} d\phi_{ij}$, $i, j = 1, \ldots n$ (each ϕ_{ij} is a complex coordinate). Then the perturbation expansion of the vacuum energy $E_0 = -ln(Z)$ can be written as a sum over oriented graphs, dual to the oriented triangulations. One finds that

$$E_0 = \sum_T \frac{1}{C(T)} g^{n_2(T)} N^{\chi(T)}, \tag{4.1.21}$$

so that this can be identified with the partition function (4.1.19) by putting $N = e^{\frac{1}{4G_0}}$, and $g = e^{-\Lambda a^2}$. The same methods can be used to discuss models based on symmetric or symplectic matrices, which would correspond to non-orientable surfaces.

For the moment, assume the potential V is quartic rather than cubic, so that the potential is an even function of ϕ. This is equivalent to covering the surface with rectangles rather than triangles. Then take

$$V(\phi) = \frac{1}{g}(\frac{\phi^2}{2} - \frac{\phi^4}{4}).$$

The most commonly used method to analyze the matrix models is known as the method of orthogonal polynomials. First, the $U(N)$ invariance is used to diagonalize ϕ via $\phi = U\Lambda U^\dagger$, with $\Lambda_{ij} = \delta_{ij}\lambda_i$ and $UU^\dagger = 1$, and integrate out the "angular" degrees of freedom described by U. Therefore

$$Z = \int (\prod_{i=1}^{N} d\lambda_i)\Delta^2(\lambda)e^{-N\sum_i V(\lambda_i)}, \tag{4.1.22}$$

where Δ is the Van der Monde determinant

$$\Delta(\lambda) = \prod_{i>j}(\lambda_i - \lambda_j). \tag{4.1.23}$$

It can be shown from the continuum or Liouville formulation of 2-d gravity,[39] that as $G_0 \to 0$, the renormalized gravitational constant G_R behaves as

$$G_R^{-1} = G_0^{-1} + 2\beta \ln(\frac{\lambda_R}{\mu^2}), \tag{4.1.24}$$

where μ is the subtraction mass scale that can be associated with $1/a$, and $\lambda_R = \lambda - \lambda_c$ is the renormalized cosmological constant, with λ the bare cosmological constant. λ_c is the critical point in the renormaliztion group flow, which can be associated with $g_c = e^{-\lambda_c a^2}$ where g_c is the value for which the eigenvalues of the potential begin to exceed to height of the potential well so that the eigenfunctions extend to infinity. β is given by

$$\frac{1}{12}[25 - c + \sqrt{(25-c)(1-c)}] \tag{4.1.25}$$

for central charge c of any matter fields. The question is whether this behavior can be found in the matrix models. By the correspondence above, with no matter fields $\beta = 5/2$. Requiring that G_R be independent of the cutoff a, this corresponds to

$N \to \infty$ and $g \to g_c = 1/12$, with $N^2(g - g_c)^5 = \text{constant}$. So, given (4.1.22), introduce a series of polynomials

$$P_n(\lambda) = \lambda^n + \cdots, \; n \geq 0 \tag{4.1.26}$$

which are orthogonal with respect to the measure

$$d\mu(\lambda) = d\lambda e^{-NV(\lambda)}. \tag{4.1.27}$$

Explicitly,

$$\int d\mu P_n(\lambda) P_m(\lambda) = \delta_{mn} S_m, \tag{4.1.28}$$

and the partition function is

$$Z = N! \prod_{i=0}^{N-1} S_i. \tag{4.1.29}$$

Recursion relations obtained by considering multiplication by λ and differentiation with respect to λ are used to compute the S_n. Define coefficients a_{mn} and b_{mn} via

$$\lambda P_n = \sum_{m=0}^{n+1} b_{nm} P_m, \; \frac{\partial}{\partial \lambda} P_n = \sum_{m=0}^{n-1} a_{nm} P_m. \tag{4.1.30}$$

From the orthogonality relations,

$$b_{n,n+1} = 1, \; b_{n,n-1} \equiv b_n = \frac{S_n}{S_{n-1}}, \; b_{mn} = 0 \text{ otherwise}. \tag{4.1.31}$$

Conversely,

$$0 = \int d\lambda \frac{\partial}{\partial \lambda} (e^{-NV(\lambda)} P_n P_m) = a_{nm} S_m + a_{mn} S_n - N[V'(B)]_{nm} S_m, \; B_{mn} = b_{mn}. \tag{4.1.32}$$

Since $a_{n,n-1} = n$, we get the recursion relation

$$\frac{n}{N} = [V'(B)]_{n,n-1} \tag{4.1.33}$$

through which the b_m and therefore the S_n can be computed starting from b_1.

For example, the quartic potential yields

$$g\frac{n}{N} = b_n(1 - b_{n-1} - b_n - b_{n+1}). \tag{4.1.34}$$

The large N limit is accomplished by assuming that b_n becomes a smooth function of the continuous parameter $x = n/N$. Then near g_c (for which $b = b_c = 1/6$) rescale

$$z = N^{4/5}(\frac{g_c - g}{g_c}), \ x = N^{4/5}(1 - \frac{n}{N}). \tag{4.1.35}$$

Now let $N \to \infty$, with z fixed and x fixed. The solution of (4.1.34) scales as

$$b_n = b_c[1 - N^{-2/5}u(x,z) + \cdots], \tag{4.1.36}$$

and (4.1.34) becomes, to leading order,

$$u^2 - \frac{1}{3}\frac{\partial^2 u}{\partial x^2} = x + z. \tag{4.1.37}$$

Now, the vacuum energy $E_0 = -\ln Z$ has a finite part given by

$$F = fp \int_0^\infty dx x u(x,z), \tag{4.1.38}$$

where the fp in front of the integral is an instruction to ignore the part of integral which goes as $N^2 + N$ as $N \to \infty$. Therefore the string susceptibility f is given by

$$f(z) = \frac{\partial^2 F}{\partial z^2} = u(0,z). \tag{4.1.39}$$

This is the same function as the two point correlator of the puncture operator we introduced before. From (4.1.37), the string susceptibility satisfies the equation

$$f^2 - \frac{1}{3}\frac{\partial^2 f}{\partial z^2} = z, \tag{4.1.40}$$

which is the first equation in the KdV hierarchy.

The fact that the KdV hierarchy has come into play is a general feature, and not special to the choice of a quartic potential. For any (even) potential, we can still consider the operations of multiplying and differentiating by λ. Denote these operators Q and P, respectively, and denote their matrix elements P_{mn} and Q_{mn}. If the potential is of order $2m$, it is easy to see that

$$Q_{np} = 0 \text{ if } |n - m| > 1$$

$$P_{np} = 0 \text{ if } |n - m| > 2m - 1.$$

In the continuum limit, P and Q become differential operators, with Q having degree 2 and P having degree $2m - 1$. The canonical commutation relation $[P, Q] = 1$ is then seen to yield (4.1.37) in the special case of $m = 2$.

For a general value of m, the potential must be tuned so as to reach a critical point, which is tantamount to tuning the operator P while Q remains a second order operator which we write as

$$Q = d^2 + u(x). \tag{4.1.41}$$

By explicit calculation for the lowest values of m, it has been found that the relevant operators can be written as

$$P = (Q^{k-1/2})_+, \tag{4.1.42}$$

where $Q^{k-1/2}$ is a pseudo-differential operator, *i.e.*, it contains both positive and negative powers of $d = \frac{\partial}{\partial x}$, where d^{-1} satisfies

$$d^{-1}f = \sum_{j=0}^{\infty}(-1)^j f^{(j)} d^{-j-1}. \tag{4.1.43}$$

The + subscript in (4.1.42) means that only the part of $Q^{k-1/2}$ that contains nonnegative powers of d is kept. Since $[Q^{k-1/2}, Q] = 0$, it is easily seen that

$$[P, Q] = (k + 1/2)R'_k[u], \tag{4.1.44}$$

where the polynomial $R_k[u]$ is the coefficient of d_{-1} in the expansion of $Q^{k-1/2}$. For the present case, where Q is given by (4.1.41), these polynomials are known as Gelfand-Dikii polynomials[40], the first few examples of which are

$$R_0 = 1/2, \ R_1 = -\frac{u}{4}, \ R_2 = \frac{3}{16}(u^2 - \frac{u''}{3}).$$

In addition, the $R_i[u]$ satisfy a recursion relation

$$R'_{m+1}[u] = \frac{1}{4}R'''_m[u] - \frac{1}{2}u'R_m[u]. \tag{4.1.45}$$

Therefore, if we have the general form for P, *i.e.*,

$$P = \sum_{k=0}^{\infty} t_k [Q^{k-1/2}]_+$$

then the string equation $[P, Q] = 1$ becomes

$$x = \sum_{k=0}^{\infty} t_k R_k[u], \tag{4.1.46}$$

which looks remarkably like the recursion relation (4.1.18), which was derived from topological considerations, with the identification

$$R_k[u] = \langle P\sigma_k \rangle.$$

The only difference between these two recursion relations is the presence of derivative terms in $R_k[u]$. These apparently come from genus $g > 1$ contributions which are not accounted for in the topological recursion relations (the reasons for this absence are not known). It can also be shown that the genus 1 topological recursion relations are equivalent to those derived from the matrix models.

In the matrix models, it is not necessary to consider just a single matrix, but rather, an arbitrary number of matrices can be included. One can even associate each matrix with a point on a line and introduce nearest neighbor interactions as a discrete version of a line. It has been found that when tuning to a critical point, the resulting scaling dimensions of the physical operators correspond to those of certain conformal models, though not exactly the minimal and unitary models discussed earlier. If the equivalence between topological methods and the matrix models is to hold, we would certainly like to extend the equivalence to these systems. That is to say, topological gravity must be coupled to something akin to "matter" fields.

3. Topological Sigma Models

To define a topological sigma model, or topological field theory,[41] consider a 2-d flat Euclidean space with a general $N = 2$ supersymmetric (but not necessarily conformal) field theory. Assume further an R parity symmetry generated by a current R^μ, in which the two supercharges $Q_{\alpha\pm}$ have R quantum numbers $\pm\frac{1}{2}$ (α is a 2-d spinor index). The supersymmetry algebra is

$$\{Q_{\alpha+}, Q_{\beta-}\} = \gamma^\mu_{\alpha\beta} P_\mu, \; \{Q_{\alpha+}, Q_{\beta+}\} = \{Q_{\alpha-}, Q_{\beta-}\} = 0, \tag{4.1.47}$$

where P_μ are the translation generators. Since, in two dimensions, the rotation group is $U(1)$ and has a single generator J, the values of α can be taken as the helicities $\pm\frac{1}{2}$ or simply $\pm$, so we label the components of the supercharges as $Q_{\pm,\pm}$.

Now, let $T_{\mu\nu}$ be the stress tensor of the theory. Consider a new theory with stress tensor $T'_{\mu\nu} = T_{\mu\nu} + \epsilon_{\mu\sigma}\partial^\sigma R_\nu + \epsilon_{\nu\sigma}\partial^\sigma R_\mu$, and new rotation generator $J' = J + R$. From the supersymmetry algebra (4.1.47), it now follows that $Q_L = Q_{-+}$ and $Q_R = Q_{+-}$ are nilpotent and anticommute. We will regard $Q = Q_L + Q_R$ as the BRST generator in the new theory.

It follows from the redefinition of the rotation generator that the two supercharges have spin zero. Therefore a spin structure is not needed, so that the theory can be formulated on any Riemann surface. A second hint is that the new stress energy tensor is a BRST commutator

$$T_{\mu\nu} = \{Q_\alpha, S_{\mu\beta}\}\gamma_\mu{}^{\alpha\beta},$$

where $S_{\mu\beta}$ is the supercurrent. Since the stress tensor generates changes in the metric, this indicates that the metric is irrelevant to the calculation of correlation functions.

Let us first discuss these topological sigma models before coupling them to topological gravity. The advantage is that one obtains a basic factorization law (explained below), in which amplitudes can be evaluated by cutting a surface in some channel and summing over intermediate states. This is no longer true after coupling to gravity, because (i) the position of the cut must be integrated, and (ii) cutting in a particular channel conflicts with the action of the mapping class group of the punctured surface.

To be explicit, correlation functions can be built out of essentially two ingredients. Denote the BRST invariant local operators as O_α. The first of these ingredients is the two point function on the sphere

$$\langle O_\alpha O_\beta \rangle = \eta_{\alpha\beta} \tag{4.1.48}$$

and the second is the operator product algebra

$$O_\alpha(x) O_\beta(y) \sim \sum_\gamma c^\gamma_{\alpha\beta} O_\gamma(y). \tag{4.1.49}$$

Note that there are no powers of $(x-y)$ on the right hand side, as they would require some notion of distance (i.e., metric dependence). Then, for example, the three point function $\langle O_\alpha O_\beta O_\gamma \rangle$ equals $c^\sigma_{\alpha\beta}\eta_{\sigma\gamma}$ (or $c^\sigma_{\beta\gamma}\eta_{\alpha\sigma}$ or $c^\sigma_{\alpha\gamma}\eta_{\beta\sigma}$).

To investigate these models in a field theoretic manner, begin with a set of bosonic fields X^i that correspond to local coordinates on a target space K and describe maps $X : \Sigma \to K$. We then add a set of fermionic partners ψ^i with BRST transformation laws

$$\delta X^i = i\epsilon\psi^i$$

$$\delta\psi^i = 0. \tag{4.1.50}$$

Other multiplets would then be added (such multiplets are already present when twisting an $N = 2$ supersymmetric field theory) so that a Lagrangian could be written whose BRST invariant configurations would be the instantons, that is, the holomorphic maps from Σ to K. Equivalently, these multiplets are auxiliary fields that are merely added so that $Q^2 = 0$ is valid without invoking the equations of motion. One finds that the observables can be written as $O_A = A_{i_1 i_2 \ldots i_n}(X)\psi^{i_1} \cdots \psi^{i_n}$ where A is an n-form that is BRST closed but not exact. O_A is a zero form on the Σ, but as in (4.1.5), one has descent equations that relate O_A to one and two forms. By Poincaré duality, the form A can be related to a homology cycle M_A in K on which A has delta function support.

For a given correlation function $\langle \prod_i O_{A_i}(x_i) \rangle$, we let $L_{(i)}$ be the submanifold of the moduli space of instantons Φ of a particular homotopy type M such that $\Phi(x_i) \in M_i$, where M_i is the Poincaré dual of A_i. A nonvanishing contribution to the correlation function can only come from the intersection of the $L_{(i)}$, as $O_{A_i}(x_i)$ is proportional to $A(\Phi(x_i))$. So define the correlation functions as simply the intersection number of the L_i, much like the definition of the correlation functions in pure topological gravity.

Admittedly, the above discussion has been rather brief and sketchy. The more interesting case is when such models are coupled to topological gravity. After this coupling, there are an infinite number of operators $O_{n,\alpha} = O_\alpha \phi^{2n}$ where O_α comes from the sigma model sector and ϕ from the gravity sector. The operators with $n > 0$ are known as gravitational descendants of the "primary" fields with $n = 0$. The two-form version of these operators can, as before, provide possible perturbations of the underlying action

$$S = S_0 + \sum_{n,\alpha} t_{n,\alpha} \int_\Sigma O_{n,\alpha(2)}. \tag{4.1.51}$$

The derivation of the genus 0 recursion relations is almost identical to that leading to (4.1.12). The result is

$$\langle \prod_{i=1}^{n} \sigma_{d_i,\alpha_i} \rangle = d_1 \sum_{A=X\cup Y} \langle \sigma_{d_1-1,\alpha_1} \prod_{j\in X} \sigma_{d_j,\alpha_j} O_{0,\rho} \rangle \eta^{\rho\tau} \langle O_{0,\tau} \prod_{k\in Y} \sigma_{d_k,\alpha_k} \sigma_{d_{s-1},\alpha_{s-1}} \sigma_{d_s,\alpha_s} \rangle, \tag{4.1.52}$$

where again $A = \{2, 3, \cdots, s-2\}$, and sums over ρ and τ are implied.

These recursion relation were originally derived without all the perturbations in (4.1.51), but the form of (4.1.52) guarantees that it is true for all values of the couplings. Similarly, the generalization of the puncture equation (4.1.16) for the coupled system is

$$\langle \prod_{i=i}^{s-1} O_{d_i,\alpha_i} O_{0,\alpha_s} \rangle = \sum_{i=1}^{s-1} d_i \langle \prod_{k=1}^{s-1} O_{d_k - \delta_{ik},\alpha_k} \rangle. \tag{4.1.53}$$

Henceforth, $O_{0,\alpha}$ will be denoted O_α.

The new puncture equation can be rewritten as a differential equation for the partition function Z, namely

$$\frac{\partial Z}{\partial t_{0,0}} = \frac{1}{2} t_{0,\alpha} t_{0,\beta} \eta^{\alpha\beta} + \sum_{i=0}^{\infty} \sum_{\alpha} (i+1) t_{i+1,\alpha} \frac{\partial}{\partial t_{i,\alpha}} Z. \tag{4.1.54}$$

It is interesting and useful to note that when $t_{i,\alpha} = 0$ for $i > 0$ (the so-called "small phase space") that this equation relates the parameters $t_{0,\alpha}$ and the two point functions $u_\alpha = \langle P O_\alpha \rangle$ via

$$u_\alpha = \eta_{\alpha\beta} t_{0,\beta}, \tag{4.1.55}$$

so that the two point functions u_α can be used as coordinates in this region of phase space.

More generally, at least for genus zero, the puncture equation turns out to have a form identical to that of the "string" equation for the general multi-matrix model, thus implying the equivalence between the matrix models and topological gravity coupled to particular topological sigma models. We will discuss these issues in more detail in the following sections.

4. Conformal Field Theory Formulation

Topological gravity in two dimensions can be formulated as the topological version of a gauge theory, a topological field theory, based on the gauge group $SL(2,R)$.[42] $SL(2,R)$ has three generators, which we will call L_0, and $L_\pm$, with commutation relations

$$[L_0, L_\pm] = \pm L_\pm, \; [L_+, L_-] = L_0. \tag{4.1.56}$$

Expanding the gauge field A as (suppressing vector indices for now)

$$A = \omega L_0 + e^+ L_+ + e^- L_-, \tag{4.1.57}$$

where ω is interpreted as the spin connection and $e^\pm$ as the zweibein. Then the field strength $F = dA + AA$ is

$$F = F^0 L_0 + F^+ L_+ + f^- L_- = (d\omega + e^+ \wedge e^-) L_0 + (de^+ + \omega e^+) L_+ + (de^- - \omega e^-) L_-. \tag{4.1.58}$$

Requiring that the field strength vanish is equivalent to the torsion free condition (from the coeffiecients of L_+ and L_- in (4.1.58)) and the constant curvature condition

$$R = \frac{d\omega}{e^+ \wedge e^-} = -1 \tag{4.1.59}$$

(this division is well defined, as in 2-d both $d\omega$ and $e^+ \wedge e^-$ are proportional to the top form) appropriate to the moduli space of metrics of Riemann surfaces with genus larger than one.

A procedure has been developed[42] to form a topological field theory for any gauge group G. A "topological" symmetry is introduced, under which the gauge field transforms as

$$\delta A^a_\mu = \psi^a_\mu, \tag{4.1.60}$$

where ψ varies as A does under G but may differ in boundary conditions (that is, statistics). In order that the equations of motion include the vanishing of the stress tensor, anti-ghosts χ^a and auxiliary fields π^a with $\delta\chi^a = \pi^a$ are introduced and the action is taken to be

$$S_0 = \delta \int d^2x (\epsilon^{\alpha\beta} \chi^a F_{\alpha\beta}(A)^a), \tag{4.1.61}$$

$$= \int d^2x \pi^a \epsilon^{\alpha\beta} F^a_{\alpha\beta} - \chi^a (D_\alpha \psi_\beta)^a \epsilon^{\alpha\beta}, \tag{4.1.62}$$

so that integrating out π^a enforces the vanishing of the field strength. Now χ is chosen so that S_0 is a singlet under G, and this implies that there is an additional symmetry, which is called the "ghostly" Yang-Mills symmetry, in which ψ transforms as A does under the regular Yang-Mills symmetry, and in which π transforms as χ, while χ and A are invariant (compare the effects of the fields c^a and Φ in the right hand side of the next equation). In addition, there is the original Yang-Mills symmetry which has not yet been included. Including all three symmetries in the BRST transformation laws, and adding terms when necessary to maintain $Q^2 = 0$, the complete BRST algebra is

$$\delta A^a_\alpha = \psi^a_\alpha + (D_\alpha c)^a$$

$$\delta \psi^a_\alpha = -([c, \psi_\alpha])^a - (D_\alpha \Phi)^a$$

$$\delta \chi^a = -([c, \chi])^a + \pi^a$$

$$\delta \pi^a = -([c, \pi])^a + ([\Phi, \chi])^a$$

$$\delta c^a = \Phi^a - (1/2)([c,c])^a$$
$$\delta\Phi^a = -([c,\Phi])^a \qquad (4.1.63)$$

where c^a is the Yang-Mills ghost field, Φ^a is the ghostly Yang-Mills ghost field. Gauge fixing terms must be added to S_0 for these additional symmetries, but of course there is considerable latitude in doing so.

This $SL(2,R)$ theory can be extended[43] to $SL(n,R)$, which is related to so-called W_n algebras (non-linear extensions of the Virasoro algebra by operators of dimension $3,4,\cdots,n$). In all of these cases, the resulting action is rather complicated and difficult to use for the computation of correlation functions, especially at higher genus.

This complication is largely a result of having chosen constant curvature metrics for the instanton configurations. In fact, the curvature could in principle have been any fixed function, since the theory is topological and independent of the metric. Perhaps the simplest choice for the curvature would be zero; of course, this cannot be done globally since the Riemann-Roch theorem would be violated, but all the curvature can be lumped into a few isolated points, *i.e.*, when correlation functions are calculated operators that create curvature are inserted. The benefit is that such complications can be delayed until after the action has been derived, physical states and operators found, etc. This is the approach taken by Verlinde and Verlinde,[5] and it allows writing topological gravity as a conformal field theory, which in turn, allows the derivation of recursion relations for amplitudes at arbitrary genus.*

Zero curvature can be achieved by simply modifying the $SL(2,R)$ algebra (4.1.56) to

$$[L_0, L_\pm] = \pm L_\pm,\ [L_+, L_-] = \lambda^2 L_0 \qquad (4.1.64)$$

in the limit of $\lambda \longrightarrow 0$. This limit has to be done very carefully. For example, frequently the equations of motion must be determined and used before the $\lambda \longrightarrow 0$ limit is taken.

The starting point for the action now becomes

$$S_0 = \delta \int \chi^0 F^0 + \lambda^2(\chi^+ F^- + \chi^- F^+). \qquad (4.1.65)$$

Upon substituting the BRST transformations (see above), this is

$$S_0 = \int \pi^0 F^0 - \chi^0 (D\psi)^0 + \lambda^2(\pi^+ F^- + \pi^- F^+ - \chi^+(D\psi)^- - \chi^-(D\psi)^+), \quad (4.1.66)$$

*Verlinde and Verlinde[5] did not proceed in this manner. The derivation below is that of the author, done in collaboration with J. Hughes and based partially on discussions with K. Li.

where the covariant derivative is given by

$$D\psi = (d\psi^0 + \lambda^2(e^+\psi^- + \psi^+e^-))L_0 + (d\psi^+ + \psi^0e^+ + \omega\psi^+)L_+ + (d\psi^- - \psi^0e^- - \omega\psi^-)L_-. \tag{4.1.67}$$

Integrating out $\pi^\pm$ and $\chi^\pm$ (before letting λ go to zero), gives δ functions that set $F^\pm = 0$ and $(D\psi)^\pm = 0$. Now, letting $\lambda \to 0$, the first two terms in (4.1.66) become

$$S_0 = \int \pi_0 d\omega - \chi_0 d\psi^0. \tag{4.1.68}$$

Of course, the Yang-Mills and ghostly Yang-Mills symmetries have not yet been fixed. Instead of choosing a gauge such as Coulomb or Lorentz gauge, in this case it is more useful if an algebraic gauge is chosen, namely

$$e^+_+ = e^-_- = e^\phi, \; e^+_- = e^-_+ = 0, \tag{4.1.69}$$

which can be achieved by adding the following term to the action

$$S_1 = \delta_{YM} \int d^2x b^+_- e^-_+ + b^-_+ e^+_- + b_0(e^+_+ - e^-_-), \tag{4.1.70}$$

where δ_{YM} is the BRST variation involving the field c^a. Now integrate out $\delta_{YM} b^+_-$, $\delta_{YM} b^-_+$, $\delta_{YM} b_0$, and b_0. This yields

$$S_1 = \int d^2x [b^+_-(\partial c^- - \omega_+ c^- + e^-_+ c_0) + b^-_+(\bar\partial c^+ + \omega_- c^+ - e^+_- c_0)] \tag{4.1.71}$$

while integrating out b_0 requires

$$\partial c^+ + \omega_+ c^+ - e^+_+ c_0 - \bar\partial c^- + \omega_- c^- e^-_- c_0 = 0. \tag{4.1.72}$$

Note that once the gauge has been fixed as in (4.1.69), the vanishing field strength condition $F^\pm = 0$ (see (4.1.58)) can be solved for the spin connection ω,

$$\omega = -\bar\partial\phi d\bar z + \partial\phi dz, \tag{4.1.73}$$

so that the first term in (4.1.68) is just

$$\int d^2x \pi \partial\bar\partial\phi, \tag{4.1.74}$$

(where π_0 has been renamed π). Furthermore, defining $c_+ = c^-/e^-_-$, $c_- = c^+/e^+_+$, and solving (4.1.72) for c_0, yields

$$c_0 = (1/2)(\partial c_- - \bar\partial c_+ + 2\omega_+ c_- + 2\omega_- c_+). \tag{4.1.75}$$

Substituting into (4.1.71), again defining $b_{++} = e^{+}_{+} b^{-}_{+}$, $b_{--} = e^{-}_{-} b^{+}_{-}$ yields

$$S_1 = \int d^2x (b_{--}\partial c_+ + b_{++}\bar{\partial} c_-) \equiv \int d^2x (b\bar{\partial} c + \bar{b}\partial\bar{c}), \tag{4.1.76}$$

which is just the well known bc ghost system action.

We now do the same for the ghostly Yang-Mills symmetry, by setting

$$\psi^{+}_{+} = \psi^{-}_{-} = \eta e^{\phi},\ \psi^{+}_{-} = \psi^{-}_{+} = 0. \tag{4.1.77}$$

This is accomplished by introducing a set of fields $\beta^{+}_{-}, \beta^{-}_{+}$, and β_0, exactly analogous to the various b fields above. Repeating the steps above, and using the conditions that $(D\psi)^{\pm} = 0$, one finds that

$$\psi^{0}_{-} = -\bar{\partial}\eta,\ \psi^{0}_{+} = \partial\eta, \tag{4.1.78}$$

which implies that the second term of (4.1.68) is

$$\int d^2x \chi_0 \partial\bar{\partial}\eta.$$

There is a contribution to the action from the analog of (4.1.71)

$$S_2 = \int d^2x [\beta_{--}(c_+\partial\eta + \partial\phi_+) + \beta_{++}(c_-\bar{\partial}\eta + \bar{\partial}\phi_-)], \tag{4.1.79}$$

where again we have used the rescaled variables

$$c^{\pm} = e^{\phi} c_{\mp},\ \phi^{\pm} = e^{\phi}\phi_{\mp},\ \beta^{\pm}_{\mp} = e^{-\phi}\beta_{\mp\mp},\ \psi^{\pm}_{\mp} = e^{\phi}\psi_{\mp\mp}. \tag{4.1.80}$$

Now, the action (4.1.79) is almost, but not quite, the action for the usual $\beta\gamma$ system. We can make a field redefinition

$$\phi_{\pm} \to \phi_{\pm} + c_{\pm}\eta \equiv \gamma, \bar{\gamma}, \tag{4.1.81}$$

which will make (4.1.79) just the $\beta\gamma$ ghost system, at the cost of introducing a term $\eta\partial_{\pm} c_{\pm}$ in the action. This, of course, can be absorbed into a redefinition of $b_{\mp\mp}$ in (4.1.76). In terms of these redefined fields, the condition $\delta\psi^{+}_{+} - \delta\psi^{-}_{-} = 0$ (which comes from integrating out β_0) can be solved for ϕ^0, which yields

$$\gamma^0 \equiv \phi^0 = (1/2)(\partial\gamma + c\partial\psi + \gamma\partial\phi - \text{complex conjugate}), \tag{4.1.82}$$

where the fields have been renamed

$$\psi \equiv \eta,\ c \equiv c_-,\ \bar{c} \equiv c_+ \tag{4.1.83}$$

in order to agree with the notation in [5].

It is not difficult to check that the BRST invariant operators are

$$\sigma_n = (\gamma^0)^n. \tag{4.1.84}$$

These operators have ghost number equal to $2n$. Similar formulas for the one-form and two-form versions of these operators can be derived simply, as discussed earlier. These have ghost number $2n-1$ and $2n-2$, respectively.

It is crucial that upon setting $\psi(z,\bar{z}) = \psi(z) + \bar{\psi}(\bar{z})$ (which can be done at least locally, since the equation of motion for ψ is $\partial\bar{\partial}\psi(z,\bar{z}) = 0$), that γ^0 can be written as a double BRST variation

$$\gamma^0 = \frac{1}{2}\{Q, \{Q_s - \bar{Q}_s, \phi\}\}, \tag{4.1.85}$$

where Q_s (which also squares to zero) and its complex conjugate are the pieces of the BRST generator that come from the supersymmetry-like variation $\delta A \sim \psi$ discussed at the beginning of this section in (4.1.63).

More importantly, from the BRST algebra, the form of the BRST generator Q can be derived. It turns out that the most important piece of Q relates to the BRST variation of the field ϕ, which is

$$\delta\phi = \partial c + c\partial\phi,$$

This comes from the component of Q which is

$$Q = c(\partial\pi\partial\phi + \partial^2\pi) + \cdots. \tag{4.1.86}$$

The term in parentheses is part of the stress tensor T, and its importance lies in the second term, which is not evident from the π,ϕ portion of the action displayed in (4.1.74). It is well known that this form of the stress tensor results from an action in which the field ϕ is coupled to the background curvature. Therefore, any correlation function must have curvature insertions of the form $e^{q_i\pi(x_i)}$ to saturate the curvature, with the q_i satisfying

$$g^{\frac{1}{2}}R(x) = \sum_i q_i\delta(x - x_i). \tag{4.1.87}$$

When this is integrated, it yields $\sum q_i = 2g - 2$ for genus g. In (4.1.87), it does not matter where the insertions (the x_i) are located, since $d(e^{q\pi}) = \{Q, d\chi e^{q\pi}\}$.

The correlation functions are defined by

$$\langle \sigma_{n_1} \cdots \sigma_{n_s} \rangle = \int_\mu \int_{\text{fields}} e^{-S} \prod_{i=1}^{3g-3} G_i \bar{G}_i \prod_j e^{q_j \pi(x_j)} \prod_{k=1}^{s} \int_\Sigma \sigma_{n_k}^{(2)}, \tag{4.1.88}$$

where the μ integral is over moduli space, and the G_i are the supercurrents that come from the integrations over the supermoduli, $\hat{m}_i$, *i.e.*, from $\frac{\partial S}{\partial \hat{m}_i} = G_i$. The vanishing ghost number condition implied by this definition of the correlation function is exactly the same as that in Witten's formulation(4.1.9), giving us confidence that this is indeed the correct definition.

Note that in this definition of the correlation functions all the operators are integrated over the surface, as they should be, since the positions of the operators are moduli of the punctured surface and therefore should be integrated. The only cases in which that this is not appropriate is for contact terms, *i.e.*, when two operators approach each other. Recall, for example, in bosonic string theory, either the integrated vertex operator $\int V(z,\bar{z})d^2z$ or the field $c\bar{c}V(z,\bar{z})$ (on the sphere, three insertions of the latter type are required to absorb the ghost anomaly) can be used. Similarly, in the present context, either $\int \sigma_{n(2)}$ or $c\bar{c}\delta(\gamma)\delta(\bar{\gamma})2^{-n}(\gamma^0)^n$ can be used. Verlinde and Verlinde [5] call these the 0 picture and -1 picture form of the operators, in accordance with the corresponding string theory notation.

Consider the situation where two operators approach each other. To do so, recall some facts concerning the factorization of amplitudes in string theory.[44] In string theory, an analog of the two operators approaching is the development of a pinch in the surface. At the pinch the two pieces of the surface (denoted Σ_1 and Σ_2) can be connected by a so-called "plumbing fixture." Local coordinates z_1 and z_2 about the positions of the two operators are introduced. At the pinch the two pieces are connected by associating points that satisfy $z_1 z_2 = q$. The resulting surface is denoted $\Sigma(q)$. Furthermore, in connecting the two pieces, a sum over a complete set of intermediate states, which by the state-operator correspondence is equivalent to a sum over operators, is inserted at each end of the plumbing fixture. The result for the path integral over the surface $\Sigma(q)$ is

$$\int^{\Sigma(q)} = \sum_a \int^{\Sigma_1} \phi_a(z_1) \int^{\Sigma_2} \phi^a(z_2) q^{L_0} \bar{q}^{L_0}, \tag{4.1.89}$$

where L_0 is the scaling dimension of ϕ_a. Of course, q and $\bar{q}$ are now moduli, and therefore the measure for their integration is needed. The insertion for any modulus is given by

$$\int d^2z(\mu_{\bar{z}}^{z} b_{zz} + \mu_{z}^{\bar{z}} b_{\bar{z}\bar{z}}), \tag{4.1.90}$$

where b_{zz} is the usual ghost, and the Beltrami differential $\mu_{\bar{z}}^{z}$ is defined as

$$\mu_{\bar{z}}^{z} = \frac{1}{2} g^{z\bar{z}} \frac{\partial g_{\bar{z}\bar{z}}}{\partial m_i}. \tag{4.1.91}$$

Equivalently, instead of picturing the moduli as representing a change in the metric for a fixed coordinate system, they can be pictured as a change in the coordinate system for a fixed metric. If $z \to z + v^z \delta m^i$, then $\mu_{\bar{z}}^{z} = \bar{\partial} v^z$. For the modulus q above, where $z_1 z_2 = q$, $q \to q + \delta q$ can be achieved by changing $z_1 \to z_1(1 - \frac{\delta q}{q})$. Substituting this into (4.1.90), the net insertion is (ignoring ϕ_a, which is not relevant to our discussion, as we are really only interested in the measure for the integration over moduli)

$$\int \frac{d^2q}{|q|^2} G_0 \bar{G}_0 q^{L_0} \bar{q}^{L_0} b_0 \bar{b}_0. \tag{4.1.92}$$

With this analogy in hand, return to the case of topological gravity and now consider the case where the puncture operator σ_0 approaches another operator, say σ_n. The contribution from this region can be expressed as

$$\int_{D_\epsilon} P|\sigma_n\rangle = \int_{|q|<\epsilon} \frac{d^2q}{|q|^2} G_0 \bar{G}_0 q^{L_0} \bar{q}^{L_0} |\sigma_n\rangle, \tag{4.1.93}$$

(the factor of $b_0 \bar{b}_0$ has been taken care of by the factor of $c\bar{c}$ in the -1 form of the puncture operator) where

$$|\sigma_n\rangle = \sigma_n^{(0)}(0) P(0)|0\rangle. \tag{4.1.94}$$

To evaluate this, use the fact that

$$|\sigma_n\rangle = (1/2)\gamma_0|\sigma_{n-1}\rangle = (1/2)\{Q,\{Q_s - \bar{Q}_s, \phi\}\}|\sigma_{n-1}\rangle = (1/2)Q(Q_s - \bar{Q}_s)\phi(0)|\sigma_{n-1}\rangle, \tag{4.1.95}$$

and try to move the $Q's$ over to the left. As $\epsilon \to 0$, the only terms that contribute are $\{Q, G_0\} = L_0 = \{Q_s, G_0\}$, so that (4.1.93) becomes

$$\int_{|q|<\epsilon} d^2q \partial_q \partial_{\bar{q}} (q^{L_0} \bar{q}^{\bar{L}_0} \phi(0))|\sigma_{n-1}\rangle. \tag{4.1.96}$$

To obtain a non-trivial result from this integral, the term in parentheses must behave as $log(|q|^2)$. To show that this is in fact the case, first note the form of the π-ϕ piece of the stress tensor in (4.1.86) gives the OPE

$$T(z)\phi(w) \sim \frac{1}{(z-w)^2} + \frac{\partial\phi}{(z-w)}. \tag{4.1.97}$$

Therefore

$$[L_0, \phi(0)] = \int \partial\phi + 1, \tag{4.1.98}$$

which implies

$$L_0\phi(0)|\sigma_{n-1}\rangle = [L_0, \phi(0)]|\sigma_{n-1}\rangle = |\sigma_{n-1}\rangle. \tag{4.1.99}$$

To derive this last equation, note that γ^0 and therefore $|\sigma_n\rangle$ contain no factors of π, so that $\int \partial\phi|\sigma_{n-1}\rangle = 0$, while it can easily be checked from the form of L_0 that $L_0|\sigma_{n-1}\rangle = 0$. Therefore

$$q^{L_0}\bar{q}^{L_0}\phi(0)|\sigma_{n-1}\rangle \sim log|q|^2|\sigma_{n-1}\rangle + \text{reg}. \tag{4.1.100}$$

Putting this information together yields

$$\int P|\sigma_n\rangle = |\sigma_{n-1}\rangle, \tag{4.1.101}$$

which implies

$$\langle P\sigma_{n_1}\cdots\sigma_{n_k}\rangle = \sum_j \langle\sigma_{n_j-1}\prod_{i\neq j}\sigma_{n_i}\rangle. \tag{4.1.102}$$

This equation is just the puncture equation, which was derived in section 1 of this chapter from the cohomological point of view. Similarly, it is possible to show that

$$\langle\sigma_1\prod_{i=1}^{s}\sigma_{n_i}\rangle = (2g-2+s)\langle\prod_{i=1}^{s}\sigma_{n_i}\rangle. \tag{4.1.103}$$

which is not surprising, since $\sigma_1 \sim d\omega \sim R$, so σ_1 should just compute the curvature, which is exactly the factor that appears in this equation. This equation is known as the dilaton equation.

In the above, we have not specified the locations of the curvature insertions $e^{q\pi}$. In some previous studies, such insertions are made at special points (such as Weierstrass points) on the surface in question, but here, following reference [5] a different choice will be made, one also made in certain string field theory approaches, which is to make the insertions at the points at which the operators (in this case the σ_n) are located. By using the ghost number condition

$$\sum(n_i - 1) = 3g - 3 \tag{4.1.104}$$

together with $\sum q_i = 2g - 2$, it is clear that an appropriate choice is

$$\sigma_n \rightarrow \hat{\sigma}_n = \sigma_n e^{\frac{2}{3}(n-1)\hat{\pi}}, \tag{4.1.105}$$

where π has been modified to $\hat{\pi}$, which is BRST invariant (its exact form is not important here).

Having made this field redefinition, the contributions to the puncture and dilaton equations come only from contact interactions, *i.e.*, only when two operators approach each other. It is tempting to conjecture that this continues to be the case for the other operators as well. The most general form of the contact terms, or the 'contact algebra' as reference [5] calls it, consistent with ghost number counting, is

$$(\int \hat{\sigma}_m)|\hat{\sigma}_n\rangle = A^n_m |\hat{\sigma}_{n+m-1}\rangle. \tag{4.1.106}$$

Note that $\hat{\sigma}_1$ does not change the ghost number, while all the other operators do change the ghost number, so the contact algebra is necessarily non-commutative. However, in the definition of correlation functions, it clearly does not matter in what order the operators are integrated over the surface. Therefore,

$$(\int \hat{\sigma}_n \int \hat{\sigma}_m)|\hat{\sigma}_k\rangle = (\int \hat{\sigma}_m \int \hat{\sigma}_n)|\hat{\sigma}_k\rangle. \tag{4.1.107}$$

Both the left- and right-hand sides are proportional to $|\hat{\sigma}_{m+n+k-2}\rangle$, so equating coefficients yields the requirement

$$A^{k+m-1}_n A^k_m + A^m_n A^k_{m+n-1} = A^{k+n-1}_m A^k_n + A^m_n A^k_{m+n-1}. \tag{4.1.108}$$

It is not difficult to show that the only polynomial solution to this equation is

$$A^m_n = A_n = \frac{1}{3}(2n+1). \tag{4.1.109}$$

What now needs to be done is to extend this local statement into a global relation for the correlation functions. So consider a general correlation function $I = \langle \sigma_n \sigma_m \prod_i \sigma_{n_i} \rangle$ and require that it be independent of the order of, say, the first two operators. When integrating over the position x_n of σ_n, there are three regions of interest, namely, where x_n approaches the position of another operator, where x_n approaches a node in the surface, and the rest of the surface. These contributions are called the contact, factorization, and surface contributions, respectively. In particular, the contact contribution has already been calculted above.

For the factorization contribution, there are two possibilities. Either the node that is being approached divides the surface into two disconnected components, or the node is on a handle so that removing the node creates a single surface with the genus reduced by one. In the first case, the correlation function simply factorizes into a product of correlation functions on the two surfaces, with operator insertions on either side of the node. The other operators in I may be on either surface, and the various possibilities must be summed. So in a correlation function $\langle \prod_{i \in S} \sigma_{n_i} \rangle$, the contribution from σ_{n_1} approaching a node is

$$\sum_{g=g_1+g_2, S=X\cup Y} \hat{B}^k_{n_1} \langle \sigma_{k-2} \prod_{i\in X} \sigma_{n_i} \rangle_{g_1} \langle \sigma_{n_1-k} \prod_{j\in Y} \prod \sigma_{n_j} \rangle_{g_2}, \tag{4.1.110}$$

where ghost number conservation has been taken into account, and where $\hat{B}^i_j$ are numerical coefficients to be determined. A similar formula for the second kind of node, with coeffiecients B^i_j and a single correlation function on a surface of genus $g-1$, is easily obtained.

Returning to the correlation function I, consider the difference between integrating out σ_n first and σ_m second and vice versa. It is clear that there is no difference in the surface term, which is independent of the order of integration. This shows that the surface term vanishes in any correlation function. Therefore, a correlation function can always be reduced to one at a lower genus, or at the same genus with fewer operators.

Now consider the case where σ_n approaches a node, and then σ_m approaches one of the new operators in (4.1.110). This is clearly not independent of the order of integration, and therefore we get constraints on the coefficients B^i_j and $\hat{B}^i_j$. In particular,

$$\frac{2}{3}(m-n)B^k_{n+m-1} = \frac{1}{3}(2(n-k)+1)B^k_n - \frac{1}{3}(2(m-k)+1)B^k_m \tag{4.1.111}$$

for $n \le k \le m$, and a similar equation for $\hat{B}$. These constraints only admit the solution B^k_n = constant, the constant being determined in terms of the irreducible correlation functions $\langle PPP \rangle_{g=0}$ and $\langle \sigma_1 \rangle_{g=1}$. Finally, putting all the pieces together, and denoting the summed correlation function as $\langle X \rangle = \sum_g \lambda^{2g-2} \langle X \rangle_g$, we find that

$$\begin{aligned}\langle \sigma_{n+1} \prod_{i\in S} \sigma_{n_i} \rangle = \sum_j \frac{1}{3}(2n_j+1) \langle \sigma_{n+n_j} \prod_{i\neq j} \sigma_{n_i} \rangle + \sum_{k=1}^{n} \frac{b}{a} \langle \sigma_{k-1}\sigma_{n-k} \prod_i \sigma_{n_i} \rangle \\ + \frac{1}{a} \sum_{S=X\cup Y} \langle \sigma_{k-1} \prod_{i\in X} \sigma_{n_i} \rangle \langle \sigma_{n-k} \prod_{j\in Y} \sigma_{n_j} \rangle),\end{aligned} \tag{4.1.112}$$

where $a = 18\langle PPP \rangle$ and $b = 24\langle \sigma_1 \rangle$. It is a remarkable fact that these are exactly the same recursion relations that can be derived from the KdV relations, which arise in the one-matrix model.

B. Coupling $N = 2$ Conformal Field Theories to Topological Gravity

1. Introduction

The rapid developments in solutions of topological gravity[5] just described can be extended to topological gravity coupled to certain topological field theories.[45][7][46] Furthermore, these theories have been proven equivalent to the multi-matrix models by showing that the correlation functions of both theories satisfy the same infinite set of recursion relations.

The general topological approach begins by twisting an $N = 2$ superconformal field theory.[5] The BRST charge is just one of the the supercurrents (partners of the stress tensor) and therefore the BRST invariant operators are simply the chiral primary fields of the original $N = 2$ model. The simplest examples of $N = 2$ theories are the so-called minimal models with $c = 3k/(k+2)$, which have $k+1$ chiral primary fields. Their topological form and coupling to topological gravity have been discussed recently by Li.[7] These results below are reviewed below.

It is well-known that the minimal models exhaust the set of unitary $N = 2$ superconformal theories with $c < 3$, so let us ask about the situation for $c > 3$, for which Kazama and Suzuki have found numerous examples.[8] Normally, $c > 3$ models are considerably more difficult to work with because they must have an infinite number of primary fields,[47] but in the topological case we are only interested in the *chiral* primary fields. For a generic G/H model at level k, it has been shown[22] that the number of such fields is finite and is given by

$$\frac{|W(G)|N_G^k}{|W(H)||Z_G|}, \tag{4.2.1}$$

where $W(G)$ is the Weyl group of G (see Appendix A), Z_G is the center of G, N_G^k is the number of primary fields of G at level k and $|G|$ means the dimension of G. Furthermore, the chiral primary fields of these models have been identified by Gepner.[24] As in the case of the minimal models, it is easiest to couple these models to topological gravity by using a free field formulation of the coset model, as has recently been done for the bosonic case by Gerasimov et al.[48] In Section 2 this construction is reviewed, the chiral primary fields are identified, and selection rule for the existence of a non-vanishing correlation function are derived. In Section 3 the, coupling of these models to topological gravity[9] is considered. As in the minimal model case,[7] this requires only a simple modification of the stress and super-stress tensors. In Section 4, the contact algebra and the recursion relations derived from it is considered. Remaining questions and possible extensions of this work are discussed in the final section.†

2. Free Field Construction and Chiral Primary Fields

As discussed in section (II.C), in an $N = 2$ G/H model, the stress tensor can be expressed as that of the $N = 0$ G/H model tensored with an $SO(n)$ model at level 1 (this is from the free fermions), where $n = dim(G/H)$.

Let us start with the case of $H = U(1)^r$, where $r = \text{rank}G$. It is not difficult to show that the solution for G^+ (see (2.3.16)) in the root basis is

$$G^+ = \frac{1}{\sqrt{k+g}}[\sum_{\alpha>0} j^{-\alpha} J^{\alpha} + i \sum_{-\alpha,\beta,\gamma>0} f^{\alpha\beta\gamma} j^{\alpha} j^{\beta} j^{\gamma}] \tag{4.2.2}$$

and that G^- has the same form with $i \to -i$ and $> 0 \to < 0$. The $U(1)$ current of the $N = 2$ algebra is then

$$J = -\sum_{\alpha>0}(j^{\alpha} j^{-\alpha} + \frac{1}{k+g} h_{\alpha}), \tag{4.2.3}$$

where h_{α} is the element of the Cartan subalgebra dual to the root α (see Appendix A). Note that, by definition, these fields are singlets under the H subalgebra of G.

Now, a chiral primary $C(w)$ field must satisfy

$$G^-(z)C(w) \sim O(\frac{1}{z-w})$$

$$G^+(z)C(w) \sim O(1). \tag{4.2.4}$$

Any such field can be found in the decomposition of a field $\hat{C}$ in the $G \times SO(n)$ theory, *i.e.*, $\hat{C} = C \times h$ for some field h in the H current algebra. But since the supercurrents are singlets under H, the relations in (4.2.4) can be translated into relations on $\hat{C}$, with exactly the same form. It proves easier to find fields satisfying these constraints in the G theory, rather than in the G/H theory. We make the following ansatz for the fields $\hat{C}$;

$$\hat{C} = \prod_{i=1}^{n} j^{-a_i} g, \tag{4.2.5}$$

where the a_i are different positive roots of G and g is some field (not necessarily primary) in the G current algebra.

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†This section is based on previous work of the author.[9]

The constraints in (4.2.4) must be obeyed for each term in $G^{\pm}$ in (4.2.2). Let us start with the second term in G^{+} and denote by $G_{\alpha\beta\gamma}$ the quantity $f^{\alpha\beta\gamma}j^{\alpha}j^{\beta}j^{\gamma}$ for positive roots $\alpha\beta\gamma$. Since we assume $f^{\alpha\beta\gamma}\neq 0$, we have $\alpha=\beta+\gamma$. Let $N_{\alpha}=0\text{or}1$ be the number of times that j^{α} appears in $\hat{C}$. It follows from the OPE of the j^{a} that

$$G_{\alpha\beta\gamma}(z)\hat{C}(w)\sim O[(z-w)^{N_{\beta}+N_{\gamma}-N_{\alpha}}], \tag{4.2.6}$$

and similarly that

$$G^{*}_{\alpha\beta\gamma}(z)\hat{C}(w)\sim O[(z-w)^{-N_{\beta}-N_{\gamma}+N_{\alpha}}]. \tag{4.2.7}$$

Therefore, for $\hat{C}$ to be chiral primary, we must have

$$1\geq N_{\beta}+N_{\gamma}-N_{\beta+\gamma}\geq 0 \tag{4.2.8}$$

for all pairs of positive roots of G. Furthermore, defining $N_{-\beta}=1-N_{\beta}$, (4.2.8) holds for all pairs of roots of G.

Let us denote by $\hat{\Delta}$ the set of roots γ with $N_{\gamma}=0$. Then either $\gamma\in\hat{\Delta}$ or $-\gamma\in\hat{\Delta}$, and if $\beta\in\hat{\Delta}$ and $\gamma\in\hat{\Delta}$, then $\beta+\gamma\in\hat{\Delta}$. Further, let us denote by Δ_{s} the set of indecomposable elements of $\hat{\Delta}$. By indecomposable, we mean, as usual, an element which cannot be written as a sum of elements of $\hat{\Delta}$. From these definitions, it is clear that Δ_{s} is a basis of the entire root system.

It is a standard theorem in the theory of Lie algebras (see Appendix A), that any basis can be obtained as the Weyl transform of some standard basis, by some element $\omega\in W(G)$. These elements are therefore in 1 to 1 correspondence with the solutions of (4.2.8). Thus for any root α, $N_{\alpha}=0$ iff $\omega(\alpha)>0$ and the field $\hat{C}$ has the form

$$\hat{C}=j^{\omega}g,$$

where

$$j^{\omega}=\prod_{\alpha>0,w(\alpha)<0}j^{-\alpha}. \tag{4.2.9}$$

Now the first term in G^{+}, is a sum of terms of the form $j^{-\alpha}J^{\alpha}$ for positive roots α. Suppose that $\omega(\alpha)>0$. Then (4.2.6) requires

$$J^{\alpha}(z)g(w)\sim O(1),\ J^{-\alpha}(z)g(w)\sim O(\frac{1}{z-w}), \tag{4.2.10}$$

while for $\omega(\alpha)<0$

$$J^{-\alpha}(z)g(w)\sim O(1),\ J^{\alpha}(z)g(w)\sim O(\frac{1}{z-w}). \tag{4.2.11}$$

Since all the elements H^i of the Cartan subalgebra can be obtained from commutators $[J^\alpha, J^{-\alpha}]$, these equations yield

$$H^i(z)g(w) \sim O(\frac{1}{z-w}),$$

which is precisely the requirement that g is a primary field of the G current algebra. Primary fields are denoted ϕ_λ^Λ (see (2.2.23)), where λ is a weight in the representation with highest weight Λ. Then the first equation in (4.2.10) is just $J^\alpha \phi_\lambda^\Lambda \sim 0$ iff $\omega(\alpha) > 0$. This merely tells us that $\alpha + w(\lambda)$ is not a weight in the Λ representation, so that $\omega(\lambda)$ must be the highest weight of the representation, *i.e.*, $\lambda = \omega^{-1}(\Lambda)$.

To summarize, what Gepner has found[24] is that the chiral primaries of the $N = 2$ theory can be obtained by decomposing the $G \times SO(n)$ fields

$$C_\omega^\Lambda = j^{-\omega} G_{\omega^{-1}(\Lambda)}^\Lambda, \tag{4.2.12}$$

where G_λ^Λ is the primary field in the affine G theory with highest weight Λ and weight λ, Λ is an integrable highest weight of G at level k, ω is an element of the Weyl group of G, and

$$j^{-\omega} = \prod_{\alpha>0, \omega(\alpha)<0} j^{-\alpha}, \tag{4.2.13}$$

where j^α are the fermions which are used to parametrize the $SO(n)_1$ theory. It is not difficult to show that for H non-Abelian, the same formula applies.

Some of the fields in (4.2.12) are equivalent, due to the fact that we are modding out by the H currents. In fact,

$$C_\omega^\Lambda = C_{\omega\omega_h}^\Lambda = C_{\omega_\sigma \omega}^{\sigma(\Lambda)}, \tag{4.2.14}$$

where ω_h is an element of the Weyl group of H, σ is a proper automorphism of the affine group G and ω_σ is essentially the restriction of σ to the finite algebra G. See Appendix A for a more precise definition of these quantities. For example, for $G = SU(n)$, σ is just a cyclic rotation of the extended Dynkin diagram (see Appendix A), which is just a circle, and $\omega_\sigma = \omega_{\alpha_1} \ldots \omega_{\alpha_{n-1}}$ for the simplest rotation, where each factor represents the Weyl reflection by the appropriate simple root.

Decomposing the field in (4.2.12) as

$$C_\omega^\Lambda = K_\omega^\Lambda H^\lambda, \tag{4.2.15}$$

where H^λ is a field in the H theory at the appropriate level, the chiral primaries fields are just the K_ω^Λ, and the weight λ is[24]

$$\lambda = \omega^{-1}(\Lambda) + \sum_{\alpha \in G/H, \omega(\alpha)<0} \alpha. \tag{4.2.16}$$

The $U(1)$ charge for this field is given by

$$Q_{\Lambda,\omega} = n(\omega) + \frac{2\lambda \cdot (\rho_G - \rho_H)}{k+g}, \tag{4.2.17}$$

where g is the dual Coxeter number of G, ρ_G is one half the sum of the positive roots of G, and $n(\omega)$ is the number of Weyl reflections which comprise ω, or equivalently, the number of positive roots with $\omega(\alpha) < 0$. Thus, we need to concern ourselves with only the $N = 0$ portion of the original $N = 2$ G/H model.

For this, we turn to the recent work of Gerasimov et al.[48] While an in-depth review of their extensive work is not necessary for the present purposes, the essential elements are as follows: one introduces fermionic fields χ_α and $\tilde{W}_\alpha$ for each positive root and scalar fields Φ that take their values in the Cartan torus (see Appendix A), that is to say, to each weight vector $\vec{\mu}_i$ in the Cartan plane, we associate a field Φ_i with postulated operator product expansion (OPE)

$$\Phi_i(z)\Phi_j(0) \sim \vec{\mu}_i \cdot \vec{\mu}_j \frac{q^2}{k^2} log(z), \tag{4.2.18}$$

where $q^2 = k + g$. Defining a field $\vec{\phi}$ by

$$\vec{\mu}_i \cdot \vec{\phi} = i\frac{k}{q}\Phi_i, \tag{4.2.19}$$

then the field $\vec{\phi}_i$ has the more natural OPE

$$(\vec{\alpha} \cdot \vec{\phi})(z)(\vec{\beta} \cdot \vec{\phi})(0) \sim -(\vec{\alpha} \cdot \vec{\beta}) log(z). \tag{4.2.20}$$

The fermionic fields correspond to the raising and lowering operators of G. We will then utilize the so-called Gauss decomposition of G. The Gauss decomposition results from the decomposition of the root space of the corresponding Lie algebra into the Cartan subalgebra (CSA) and the sets of positive and negative roots, which we will call Δ_+ and Δ_-. By the exponential map, we have that an element of G can be written as

$$g = g_L g_D g_U$$

where

$$g_L = exp(a_L),\ a_L \in \Delta_-,\ g_U = exp(a_+),\ a_+ \in \Delta_+,\ g_D = exp(a_D),\ a_D \in CSA. \tag{4.2.21}$$

Actually, this decomposition is true "almost" always, *i.e.*, except on a set of measure zero. But given the association of χ_α and $\tilde{W}_\alpha$ as corresponding to raising and lowering operators, we take $g_L = g_L(\chi)$, $g_U = g_U(\tilde{W})$ and $g_D = g_D(\Phi)$.

Then, the G valued current $g^{-1}\partial g$ which occurs in the WZW action can be written as

$$g^{-1}\partial g \equiv \tilde{J} = g_L^{-1}(\chi)\tilde{J}_{(0)}(\tilde{W}_\alpha, \Phi)g_L(\chi) + g_L^{-1}(\chi)\partial g_L(\chi), \tag{4.2.22}$$

where

$$\tilde{J}_{(0)} = (g_U g_D)^{-1}\partial(g_U g_D) = g_D^{-1}\partial g_D + g_D^{-1}(g_U^{-1}\partial g_U)g_D \tag{4.2.23}$$

The fields used so far might be called "classical" fields, in that there is nothing which specifies the level k of the affine algebra g. In particular, since we trying to find a free field representation, we expect the fermion action to look like $\psi_i\partial\bar{\psi}_i$ for some fermions ψ_i. Both of these problems can be solved at once by a field redefinition. The fields $\tilde{W}_\alpha$ are replaced by fields W_α defined by

$$kTr(g_L^{-1}\partial g_L\tilde{J}) = kTr(g_L^{-1}\partial g_L\tilde{J}_{(0)}) = \sum_\alpha W_\alpha\partial\chi_\alpha. \tag{4.2.24}$$

This redefinition ensures that W_α and χ_α satisfy

$$W_\alpha(z)\chi_\beta(0) \sim \frac{\delta_{\alpha\beta}}{z}, \tag{4.2.25}$$

which is what expected for free fermions. This redefinition has the form

$$\tilde{W}_\alpha \rightarrow W_\alpha(\tilde{W}_\beta) = k\sum_{\beta>0} q_{\alpha\beta}(\chi)\tilde{W}_\beta, \tag{4.2.26}$$

where the $q_{\alpha\beta}$ are polynomial functions of the χ_α. As an example, for $sl(3)$, these redefinitions are just

$$W_{\alpha_1} = k\tilde{W}_{\alpha_1},\ W_{\alpha_2} = k\tilde{W}_{\alpha_2},\ W_{\alpha_3} = k(\tilde{W}_{\alpha_3} + \chi_1\tilde{W}_{\alpha_2}), \tag{4.2.27}$$

where the α_i are the three positive roots of $sl(3)$ (see Appendix A).

Similarly, when looking at the form of the raising and lowering operators $J^{\pm\alpha_i}$ that are found in (4.2.22), the substitution in (4.2.26) must be made. Then the normal ordering implicit in (4.2.22) must be carefully considered, so that the field $\tilde{J}$ is redefined to the Kač-Moody current J. For example, again for $sl(3)$, one finds that

$$J_{-\alpha_1} = -\chi_{\alpha_2} W_{\alpha_3} + \cdots. \tag{4.2.28}$$

Upon making the substitution in (4.2.27) for W_{α_3}, one has

$$\begin{aligned}\chi_{\alpha_2} W_{\alpha_3} &= k \lim_{z\to 0}[\chi_{\alpha_2}(0)(\tilde{W}_{\alpha_3} + \chi_{\alpha_1}\tilde{W}_{\alpha_2})(z) - \text{sing.}] \\ &= k\chi_{\alpha_2}\tilde{W}_{\alpha_3} + k\chi_{\alpha_1}\chi_{\alpha_2}\tilde{W}_{\alpha_2} + \partial\chi_{\alpha_1},\end{aligned} \tag{4.2.29}$$

since

$$k\chi_{\alpha_2}(0)\tilde{W}_{\alpha_2}(z) = \chi_{\alpha_2} W_{\alpha_2} \sim \frac{1}{z}. \tag{4.2.30}$$

In terms of these redefined fields the stress tensor is just

$$T = \frac{1}{2(k+g)} Tr : JJ := \sum_{\alpha\in\Delta^+} W_\alpha \partial\chi_\alpha - (1/2)(\partial\vec{\phi})^2 - \frac{i}{q}\rho_G \cdot \partial^2\vec{\phi}. \tag{4.2.31}$$

In addition, the Cartan currents have a simple form for any algebra

$$H_i = iq\partial\phi^i - \sum_\alpha \alpha^i W_\alpha \chi_\alpha. \tag{4.2.32}$$

The forms of the lowering and raising operators are generally complicated expressions that depend greatly on the algebra in question. It is also a simple matter to bosonize the W_α, χ_α system and get a truly free field realization. It is also possible to write out the forms of the primary fields in this representation explicitly, but their exact form will not be required, so we refer to the references[48][49] for more details.

Specializing again to the case of G simple, $H = U(1)^r$, where $r = rank(G)$,[49] the W_α,χ_α system is bosonized via

$$W_\alpha = -i\partial u_\alpha e^{-u_\alpha + iV_\alpha}$$

$$\chi_\alpha = e^{u_\alpha - iV_\alpha}, \tag{4.2.33}$$

where we have omitted the cocycle factors that are needed to ensure that the fermions χ_α anticommute with each other. Then, using the currents in (4.2.32) for the $U(1)^r$ fields, we merely find other combinations U_α of the u_α and ϕ^i such that

$$U_\alpha(z)U_\beta(0) \sim -\delta_{\alpha\beta} log(z) \tag{4.2.34}$$

$N = 2$ algebra directly using $N = 2$ superfields[51] that give results similar in structure to those above.

Now the twisting $T \to T + \partial J/2$ yields

$$T = -\sum_{\alpha} \partial\phi_\alpha \partial\bar{\phi}_\alpha + i\bar{\beta}_\alpha \partial^2 \phi_\alpha - \psi_\alpha \partial\bar{\psi}_\alpha. \tag{4.2.41}$$

As in the minimal model case,[7] the background charges for the fermion current $\psi_\alpha\bar{\psi}_\alpha$ and bosonic currents $\bar{\beta}_\alpha \partial\phi_\alpha$ and $\beta_\alpha \partial\bar{\phi}_\alpha$ must be cancelled independently. This is done via insertion of contour integrals of screening operators, so that the only remaining selection rule is conservation of ghost charge, with a background charge $c/3$. So, for example, if we consider a correlator of fields K_I^Λ, where for some of the fields (say m of them) their one-form version $G_{-1/2}K_I^\Lambda$ is taken, we find the selection rule

$$\frac{(2\sum_{i=1}^n \Lambda_i + \sum_{j=i}^m \tilde{\Lambda}_j)\cdot \rho_G}{k+g} - m = (1-\tilde{g})(c/3), \tag{4.2.42}$$

on genus $\tilde{g}$, where

$$c = \frac{(k-g)dim(G)}{k} - rank(G) + (1/2)dim(G/H). \tag{4.2.43}$$

Note, that if $\tilde{g} > 1$, we must have $m > 0$ or the correlation function will automatically vanish.

3. Coupling to Topological Gravity

To couple the system (4.2.39) to topological gravity,[9] we use the free field realization of the latter,[5] derived earlier from that of topological ISO(2) gauge theory. After gauge fixing the various symmetries, the action for the pure gravity system is

$$S = \int \pi\partial\bar{\partial}\phi + \chi\partial\bar{\partial}\psi + b\bar{\partial}c + \bar{b}\partial\bar{c} + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}, \tag{4.2.44}$$

while the BRST charge can be written as

$$Q = Q_s + Q_g$$

$$Q_s = \int \partial\pi\psi + b\gamma$$

$$Q_g = \int c(T_L + (1/2)T_{gh}) + \gamma(G_L + (1/2)G_{gh}), \tag{4.2.45}$$

and γ_0 is the basic BRST invariant operator form the pure gravity sector

$$\gamma_0 = \frac{1}{2}(\partial\gamma + \gamma\partial\phi - c\partial\psi + c.c.). \tag{4.2.52}$$

The only selection rule of the coupled system is that of ghost number conservation. The background charge for the combined gravity and matter system, including both left- and right-movers, is $2(g-1)(\frac{c}{3}-3)$, while the ghost charge of the field $\sigma_{n,\Lambda,\omega}$ is $2n - 2 + Q_{\Lambda,\omega}$, so the selection rule is

$$\sum_i 2n_i - 2 + Q_{\Lambda_i,\omega_i} = (1-g)(\frac{2c}{3} - 6). \tag{4.2.53}$$

For example, consider the simple example of $G = SU(3)$ at level 1. Then, using (4.2.14), we can assume that $\Lambda = 0$ without loss of generality. It is easily seen that the chiral primaries have charges $(0, 3/2, 3/2, 7/2, 7/2, 4)$. Then on the sphere, the condition (4.2.53) reads

$$\sum_i (2n_i - 2) + \sum_j m_j Q_j = -4,$$

where m_j=number of fields with charge Q_j. So the only nonvanishing two point function is $n_1 = n_2 = 0, m_0 = 2, m_j = 0$ for $j \neq 0$, i.e., the puncture operator two-point function. One needs a four-point function $< PPP\sigma_{0,4} >$ (where we have labelled the field by its charge) to find a non-zero correlation function for a non-trivial matter content.

4. Contact Terms

One of the key results in the solution of pure topological gravity[5] (described in section (IV.A.4)) was the derivation of the "contact" algebra and its use in deriving an infinite set of recursion relations for the correlation functions. By similar methods methods, partial results can be obtained for topological matter systems coupled to topological gravity.

Due to the form of the stress energy tensor (4.2.46), insertions of $\exp(q_l\tilde{\pi})$ must be made, where $\sum q_l = 2g - 2$ and $\tilde{\pi} = \pi - c\partial\chi - \bar{c}\bar{\partial}\chi$ is BRST invariant. In [5], these insertions were placed at precisely the positions of the insertions of the physical operators, which we can repeat here. Namely, rewrite (4.2.53) as

$$2g - 2 = \sum_i (3 - c/3)^{-1}(2n_i - 2 + Q_{\Lambda_i,\omega_i}), \tag{4.2.54}$$

so therefore set

$$q_{n_i,\Lambda_i,\omega_i} = (3 - c/3)^{-1}(2n_i - 2 + Q_{\Lambda_i,\omega_i}). \qquad (4.2.55)$$

Consider now the case when the field $\sigma_{m,0,0}$ (i.e., a field purely from the gravity sector) approaches some other field, $\sigma_{n,\Lambda,\omega}$. The contact term $\int_{D_\epsilon} \sigma_{m,0,0}|\sigma_{n,\Lambda,\omega} >$ can be evaluated as in pure gravity since there is no contribution from the matter sector. By ghost number counting and knowing that Λ and ω are unchanged, the result must be proportional to $\sigma_{n+m-1,\Lambda,\omega}$ and the net result is

$$\int \sigma_{m,0,0}|\sigma_{n,\Lambda,\omega} >= (q_{n,\Lambda,\omega} + 1)|\sigma_{m+n-1,\Lambda,\omega} > . \qquad (4.2.56)$$

Similarly, one must also include factorization terms corresponding to cases where an operator approaches a node on the surface. If the node is one that pinches a handle, the contribution is

$$< \sigma_{m,0,0} \prod \sigma_{n_i,\Lambda_i\omega_i} >_{\Delta,g} = \sum_{(1),(2)} B^m_{(1),(2)} < \sigma_{(1)}\sigma_{(2)} \prod \sigma_{n_i,\Lambda_i,\omega_i} >_{g-1}, \qquad (4.2.57)$$

(where (1) denotes the set $(n_1, \Lambda_1, \omega_1)$ etc.) while if the node separates the surface into two parts, the right-hand side becomes

$$\sum_{g=g_1+g_2, S=X\bigcup Y} \sum_{(1),(2)} \tilde{B}^m_{(1),(2)} < \sigma_{(1)} \prod_{i\in X} \sigma_{n_i,\Lambda_i,\omega_i} >_{g_1} < \sigma_{(2)} \prod_{j\in Y} \sigma_{n_j,\Lambda_j,\omega_j} >_{g_2}, \qquad (4.2.58)$$

where B and $\tilde{B}$ are sets of constants to be determined. They are restricted by ghost number counting, i.e., the fields $\sigma_{(1)}$ and $\sigma_{(2)}$ must have charges Q_1 and Q_2 which add up to $2m - 8 + 2c/3$. In previous cases[5] [7] they were also restricted by requiring that any correlation function be independent of the order of integration of any two operators, which required B and $\tilde{B}$ to be independent of their labels. We conjecture that the same is true in the present case, although the non-uniqueness of the ghost numbers and the more general form of the charges makes a proof difficult.

5. Discussion

Although we have used the case of $G/U(1)^r$ as an example, the results are *much* more general. For example, Gerasimov *et al.*[48,49] give a free field realization for the $N = 0$ coset $\frac{G_{k_1}\times G_{k_2}}{G_{k_1+k_2}}$ for G simple. In this case, begin by expressing the stress tensor for G_{k_i}, $i = 1, 2$ with fields $u^{(i)}_\alpha$, $V^{(i)}_\alpha$ and $\vec{\phi}^{(i)}$, as in (4.2.31) and (4.2.33). Then the

Cartan subalgebra of G_k ($k = k_1 + k_2$) consists of the currents $H_i = H_i^{(1)} + H_i^{(2)}$ ($i = 1, \cdots r$). Using (4.2.32) and (4.2.33), these can be expressed as

$$H_i = \partial[\sum_{\alpha\in\Delta_+} \alpha^i u_\alpha^{(2)} + iq\phi^i], \tag{4.2.59}$$

where $q^2 = k + g$, and

$$\phi = \frac{1}{q}(q_1\phi^{(1)} + q_2\phi^{(2)} - i\sum_{\alpha\in\Delta_+} \alpha u_\alpha^{(1)}) \tag{4.2.60}$$

($q_i^2 = k_i + Q$). The raising and lowering operators of G_k consist of the currents J_α ($\alpha > 0$) of G_{k_2}, $\alpha \in \Delta_+$, and similar lowering operators, whose form is not needed. Then find other linear combinations of the fields that have vanishing OPEs with the G_k fields as before. The result is that the coset stress tensor can be written as[49]

$$T_{coset} = T_{PF}(U_\alpha, v_\alpha^{(1)}) + T(\tilde{\phi}), \tag{4.2.61}$$

where T_{PF} is the parafermion stress tensor given in (4.2.37) and

$$T(\tilde{\phi}) = -\frac{1}{2}(\partial\tilde{\phi})^2 + \frac{i\sqrt{k_1}}{qq_2}\rho\cdot\partial^2\tilde{\phi}, \tag{4.2.62}$$

and

$$\tilde{\phi} = \frac{1}{q\sqrt{k_1}}(q_1q_2\phi^{(1)} - k_1\phi^{(2)} - iq_2\sum_{\alpha\in\Delta_+} \alpha u_\alpha^{(1)}. \tag{4.2.63}$$

U_α is given in (4.2.36) with $u_\alpha \to u_\alpha^{(2)}$. Clearly, we can repeat the manipulations demonstrated in the $G/U(1)^r$ case to form the topological model.

Similar bosonic constructions exist for any G/H where G and H are both simple. The latter is straightforward to generalize to the case of H semi-simple, which includes the broad class of models developed by Kazama and Suzuki.[8] For any of these models, the extension to $N = 2$ is straightforward as is the coupling to topological gravity. The former, for say $G = SU(n)$, $k_1 = 1$ leads to W-algebras.[18] It has been suggested[43] that these should couple nicely to W-gravity, the topological theory based on $SL(n, R)$.[43] It should be possible to extend the methods presented here to this case, using the extended Casimir approach.[52] It is also interesting that many of these models do not admit a Landau-Ginzburg description[22] which has proven useful in showing the equivalence[45] between the topological minimal models and the multi-matrix models.[53]

In particular, it would interesting to complete the solution of the contact and factorization terms and to determine whether or not the resulting recursion relations are equivalent to a set of Virasoro constraints[54] (or a suitable extension thereof). This may be difficult for the following reason. Consider a correlation function of primary fields on some genus g. If there is a non-zero n point function on the sphere, then we must allow for a contact term where $n-1$ of the fields approach each other. But for the $c>3$ theories, there are generally fields with charge greater than 1. Looking at (4.2.54), with all the $n_i=0$ and on the sphere, if there is a field with $Q=2$ then that field could appear arbitrarily many times in some product of fields, and (4.2.54) would still be satisfied (assuming that it was satisfied without the presence of that field). In that case, there would be nonzero n point correlators for arbitrarily high n, so in trying to organize the resulting recursion relations into some extension of the Virasoro algebra, that algebra would necessarily have to include generators of arbitrarily high spin or dimension. However, normally there is only one generator for each primary field, of which there are only a *finite* number. However, even in the case of the N-matrix models, which correspond to minimal models coupled to topological gravity, there are very recent results[55] which indicate that the W_N symmetry previously known is actually just a truncation of an underlying $W_{1+\infty}$ algebra. That is to say, the underlying symmetry is actually $W_{1+\infty}$, and W_N only results upon the elimination of redundant variables. It may be that in the case of $c>3$, there are no such redundant variables. Whether or not this is true, it is tempting to speculate that the underlying symmetry of the $c>3$ models coupled to topological gravity is some form of $W_{1+\infty}$ (or its cousins), which would lead to a more complete set of contact relations and therefore recursion relations.

Of course, it may very well be that a complete solution via the contact algebra approach may not be appropriate for these cases. If this is the case, it would certainly be interesting to understand in more depth the reason for this key threshold.

Appendix A: Lie Algebras and Affine Lie Algebras

In this appendix, conventions, notations and some theorems concerning Lie algebras and affine Lie algebras (Kač-Moody algebras) are reviewed. For the most part, we follow Goddard and Olive.[56] Unless otherwise stated, any Lie algebra considered here is assumed to be semi-simple and its corresponding Lie group is assumed to be compact.

1. Lie Algebras.

A Lie algebra g corresponding to a Lie group G is most conveniently described in terms of the commutation relations of its (hermitian) generators T^a

$$[T^a, T^b] = i f^{abc} T^c, \tag{A.1.1}$$

where the structure constants f^{abc} are completely antisymmetric. The index a takes values $1, \ldots n$ where n is known as the dimension of the algebra $n = \dim g$. By the exponential map, the T^a generate the elements $e^{\sum_a iT^a x_a}$ in the connected component of G.

It is convenient to choose a basis for the algebra that contains a maximal set of commuting generators H^i, i.e.,

$$[H^i, H^j] = 0, \;\; 1 \le i, j \le r. \tag{A.1.2}$$

where r is known as the *rank* of g. This set of generators is known as the *Cartan subalgebra* (the corresponding group is the Cartan torus) and is isomorphic to $U(1)^r$. Since the H^i commute, the other generators, known as step operators and denoted E^α can be chosen as simultaneous eigenvectors of the H^i, where the eigenvalues α are known as *roots*,

$$[H^i, E^\alpha] = \alpha^i E^\alpha. \tag{A.1.3}$$

It follows from the Jacobi relations that

$$[E^\alpha, E^\beta] = \epsilon(\alpha, \beta) E^{\alpha+\beta}$$

if $\alpha + \beta$ is a root. $\epsilon(\alpha, \beta)$ are constants, antisymmetric in α and β, which can be chosen as ± 1 if the algebra is simply-laced, that is, if all the roots have equal length squared. If $\alpha + \beta = 0$ then,

$$[E^\alpha, E^\beta] = \frac{2\alpha \cdot H}{\alpha^2}$$

and otherwise $[E^\alpha, E^\beta] = 0$.

For every root α, it can be shown easily that $-\alpha$ is also a root, so that the set of roots, Δ, can be divided into the set of positive roots, Δ_+, and their negatives, Δ_-. The corresponding step operators are related by Hermitian conjugation. Now the number of positive roots in general exceeds the rank of the algebra. A basis for Δ_+ can be chosen in which any root can be express as a linear combination of the basis roots, known as the *simple roots* and denoted $\alpha_{(i)}$, with *integer* coefficients, i.e.,

$$\alpha = \sum_{i=1}^{r} n_i \alpha_{(i)}, \; n_i \in Z, \; n \geq 0.$$

The scalar products

$$K_{ij} = \frac{2\alpha_{(i)} \cdot \alpha_{(j)}}{\alpha_{(j)}^2}, \tag{A.1.4}$$

which must be integers (see below), form an $r \times r$ matrix, known as the *Cartan matrix*, whose diagonal elements are equal to 2 and whose off-diagonal elements are either zero or negative integers. In fact, the algebra is uniquely determined by the form of the Cartan matrix. The information in the Cartan matrix is conveniently encoded in the *Dynkin diagram*, which is formed as follows. For each simple root, draw a circle (or a dot) and connect each pair (i, j) of circles with $K_{ij} K_{ji}$ lines, with an arrow pointing from j to i if $\alpha_{(j)}^2 > \alpha_{(i)}^2$.

A d dimensional representation of a Lie algebra is a map that associates to each generator a linear operator that acts on a d dimensional Hilbert space. The states in the Hilbert space are classified by the eigenvalues of the Cartan subalgebra generators,

$$H^i |\mu\rangle = \mu^i |\mu\rangle. \tag{A.1.5}$$

The vector μ which appears in this equation is known as a *weight*, and the set of all weights defines a lattice Λ_W called the weight lattice. Note that the generators themselves define a dim$g$ dimensional representation, known as the adjoint representation. The weights of the adjoint representation are called roots, and they form a sublattice of the weight lattice known as the root lattice Λ_R.

From the commutation relations above, it is clear that $I_+ = E^\alpha$, $I_- = E^{-\alpha}$, and $I_3 = \alpha \cdot H / \alpha^2$ form an su(2) algebra. Since $2I_3$ has integer eigenvalues only,

$$\frac{2\alpha \cdot \mu}{\alpha^2} \in Z \tag{A.1.6}$$

for any weight μ. Furthermore, in the adjoint representation, the step operators of the form $E^{\beta+m\alpha}$, $m \in Z$ must form an su(2) multiplet, so that

$$2\alpha \cdot \beta/\alpha^2 + 2m = -2\alpha \cdot \beta/\alpha^2$$

for some m for which $\beta + m\alpha$ is a root. Therefore

$$\sigma_\alpha(\beta) \equiv \beta - 2(\alpha \cdot \beta)\alpha/\alpha^2 = \beta + m\alpha. \tag{A.1.7}$$

(A.1.7) shows that the linear operator σ_α generates a reflection in the plane normal to α which permutes the roots. These reflections generate a finite group $W(g)$ called the *Weyl group.* It can be proven that any basis for Δ_+ can be written as the Weyl transform of a fixed basis.

Since they form a basis for the root space, clearly it is sufficient to require (A.1.6) for the simple roots. Therefore, a basis $\{\lambda_{(i)}\}$, $i = 1, \dots r$ for the weight space can be chosen for which

$$2\lambda_{(i)} \cdot \alpha_{(j)}/\alpha_{(j)}^2 = \delta_{ij}. \tag{A.1.8}$$

These weights are known as *fundamental weights.*

In a given finite-dimensional representation R, there is always one weight μ_R whose corresponding state is annihilated by all the raising step operators,

$$E^\alpha|\mu_R\rangle = 0, \ \forall\alpha \in \Delta_+ \tag{A.1.9}$$

μ_R is known as the *highest weight* of the representation R, and the highest weight of the adjoint representation is called the highest root. The representation R then contains all weights μ for which $\mu_R - \sigma(\mu) \in \Delta_+$, $\forall\sigma \in W(g)$.

For any algebra g, there are polynomial combinations of the generators which commute with all the generators. By Schur's lemma, such an operator acts as a constant within any representation. These operators are called *Casimirs.* The quadratic Casimir is the simplest such operator and can be expressed as $\sum_a T^a T^a$. By going to the root basis, it is easily seen that the Casimir takes the value

$$Q_R = \mu_R \cdot (\mu_R + 2\rho), \tag{A.1.10}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha$.

An automorphism of g is a one-to-one transformation of the root space that preserves the inner product encoded in the Cartan matrix. It is a theorem that any automorphism is equal to the product of some element of the Weyl group and a diagram automorphism of the Dynkin diagram, i.e., a permutation of the resulting simple roots.

2. Affine Lie Algebras

Just as a Lie algebra represents the infinitesimal elements of the corresponding Lie group, an affine Lie algebra represents the infinitesimal generators of the group of maps from the circle to a Lie group $S^1 \to G$. By the exponential map, an element γ of G can be written as

$$\gamma(z) = exp[-iT^a\theta_a(z)], \tag{A.2.1}$$

where θ_a, $1 \leq a \leq \dim g$ are parameters for G. Making a Laurent expansion of θ_a,

$$\theta_a(z) = \sum_{n=-\infty}^{n=\infty} \theta_a^{-n} z^n, \tag{A.2.2}$$

we see that the generators $T_n^a \equiv T^a\theta_a^n$ are an infinite set of parameters for G and satisfy

$$[T_m^a, T_n^b] = if^{abc}T_{m+n}^c. \tag{A.2.3}$$

Since integer moding only has been considered, technically (A.2.3) are the commutation relations of an *untwisted* affine Lie algebra.

Quantum mechanically, a Schwinger term, also known as a *central extension*, is expected to modify these commutation relations, and indeed, it can be shown that this algebra has no non-trivial highest weight representations without a central extension. Including the central extension, in the root basis the algebra becomes

$$[H_m^i, H_n^j] = km\delta^{ij}\delta_{m,-n} \tag{A.2.4}$$

$$[H_m^i, E_n^\alpha] = \alpha^i E_{m+n}^\alpha \tag{A.2.5}$$

$$[E_m^\alpha, E_n^\beta] = \epsilon(\alpha,\beta)E_{m+n}^{\alpha+\beta}, \text{ if } \alpha+\beta \text{ is a root} \tag{A.2.6}$$

$$[E_m^\alpha, E_n^\beta] = \frac{2}{\alpha^2}[\alpha\cdot H + km\delta_{m,-n}] \text{ if } \alpha = -\beta \tag{A.2.7}$$

$$[k, E_n^\alpha] = [k, H_n^i] = 0. \tag{A.2.8}$$

k is a number known as the *level* which is real for hermiticity.

To continue as above, find a maximal set of commuting elements. Clearly, the H_0^i (which comprise the Cartan subalgebra of the corresponding Lie algebra) and the element k should be included in this set. This set is not maximal since $[H_0^i, H_n^j] = 0$ and further, the roots $(\alpha, 0)$ are infinitely degenerate since $[H_0^i, E_n^\alpha] = \alpha^i E_n^\alpha$. Both of these problems can be solved by introducing a new element d to the algebra, with commutation relations

$$[d, T_n^a] = nT_n^a, \; [d, k] = 0. \tag{A.2.9}$$

d can be associated with L_0 of the Suguwara construction, but such an identification is not necessary. Including d in the Cartan subalgebra, the positive roots can be expressed as $(\alpha, 0, n)$, where n is an integer and α is a root of g corresponding to the step operator E_n^α, and $(0, 0, n)$ corresponding to the operators H_n^i. The simple roots can be taken as

$$(\alpha_{(i)}, 0, 0) \text{ and } \alpha_{(0)} \equiv (-\psi, 0, 1) \tag{A.2.10}$$

where ψ is the highest root of g.

To construct the analog of the Cartan matrix and Dynkin diagram, an inner product in the root space is necessary. The appropriate definition is

$$(\alpha_1, n_1, m_1) \cdot (\alpha_2, n_2, m_2) = \alpha_1 \cdot \alpha_2 + n_1 m_2 + n_2 m_1. \tag{A.2.11}$$

Since the roots have $n_1 = n_2 = 0$, the inner product of two roots is

$$(\alpha_1, 0, m_1) \cdot (\alpha_2, 0, m_2) = \alpha_1 \cdot \alpha_2. \tag{A.2.12}$$

The Cartan matrix (now an $(r+1) \times (r+1)$ matrix) is defined exactly as in the Lie algebra case, as is the Dynkin diagram. For example, the Dynkin diagram of affine $su(n)$ consists of $n+1$ points each connected to two others by a single line (i.e., a circle with $n+1$ distinguished points).

The affine Weyl group $\hat{W}$ is defined as the finite group generated by the Weyl reflection by the simple roots, as in (A.1.7), except that Weyl reflections by roots of the form $z = (0, 0, n)$ are not included, since $z^2 = 0$. To analyze its structure, let $a = (\alpha, 0, n)$ and $x = (\beta, k, p)$. Then

$$\sigma_a(x) = (\beta - 2[\alpha \cdot \beta + nk]\frac{\alpha}{\alpha^2}, k, p - 2[\alpha \cdot \beta + nk]\frac{n}{\alpha^2}). \tag{A.2.13}$$

Let $\tilde{\beta} = \beta + 2nk\alpha/\alpha^2$. Then (A.2.13) shows that

$$\sigma_a = \sigma_\alpha(t_\alpha)^n \tag{A.2.14}$$

where σ_α is defined in (A.1.7) and t_α is defined by

$$t_\alpha(x) = (\beta + 2k\frac{\alpha}{\alpha^2}, k, p + [\beta^2 - (\beta + 2k\frac{\alpha}{\alpha^2})^2]/(2k)). \tag{A.2.15}$$

t_α can be thought of as a translation by the *coroot* $\alpha^\nu \equiv \frac{\alpha}{\alpha^2}$, as any two such translations commute by (2.1.9). The coroots form a lattice Λ_R^ν called the coroot lattice, which is isomorphic to the root lattice if g is simply laced. Furthermore, from these definitions it follows that

$$\sigma_\beta t_\alpha \sigma_\beta = t_{\sigma_\beta(\alpha)}$$

so that the translations form a normal subgroup of the $\hat{W}$. Therefore, $\hat{W}$ is the semi-direct product of $W(g)$ and Λ_R^ν,

$$\hat{W}(g) = W(g) \ltimes \Lambda_R^\nu. \tag{A.2.16}$$

As discussed in section (2.B), the representations of the affine Lie algebra correspond to representations of the underlying Lie algebra whose highest weights Λ satisfy

$$k \geq \psi \cdot \Lambda.$$

As in the non-affine case, these weights can be expressed as integer linear combinations of fundamental weights, which can be written as

$$l_{(i)} = (\lambda_{(i)}, \frac{1}{2}m_i\psi^2, 0),$$

$$l_{(0)} = (0, \frac{1}{2}m_0\psi^2, 0),$$

where the $m_0 = 1$ and the m_i are the integers that appear in the expansion

$$\frac{\psi}{\psi^2} = \sum_{i=1}^{r} m_i \frac{\alpha_{(i)}}{\alpha_{(i)}^2}. \tag{A.2.17}$$

Lastly, an automorphism σ of an affine Lie algebra is a transformation of the root space that preserves the inner product. Similarly, a diagram automorphism is a permutation of the simple roots, as in the Lie algebra case. The set of diagram automorphisms is isomorphic to the center of G. In particular, under a diagram automorphism, $\alpha_{(i)} \rightarrow \alpha_{(p(i))}$ where $p(i)$ is a permutation. By looking at the first component of new simple roots, such a permutation gives rise to a change of basis $\hat{\sigma}$ of the underlying finite Lie algebra. It can be a non-trivial change because of the form of $\alpha_{(0)}$. By the remarks in section A concerning automorphisms of Lie algebras, this change of basis can be expressed as a Weyl transformation times a diagram automorphism of g. An automorphism σ for which $\hat{\sigma} \in W(g)$, i.e., without a diagram automorphism piece, is called a proper automorphism. For $su(n)$, for example, the proper automorphisms are rotations of the affine Dynkin diagram (which is a circle, as discussed above).

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