ABSORPTION OF SOUND BY SMALL SPHERICAL OBSTACLES

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SUMMARY

The following pages represent an investigation of sound absorption by small spherical obstacles to which we attribute, in different sections, the properties of rigid bodies, viscous fluids, and elastic solids. Both viscosity and thermal conductivity are taken into account. The general equations are derived, subject to the assumption that the size of the obstacles is small compared to the wave length, and are applied to the special cases of water drops in air, air bubbles in water, and to suspensions of elastic solid particles. The theoretical results are compared with the experimental measurements in the case of fogs and found to agree very well within the accuracy to which these measurements can be made.

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LIST OF SYMBOLS

The following symbols are used to indicate elementary quantities:

$\underset{\sim}{A}$ = vector potential

c = velocity of sound

c_v = specific heat

K = isothermal bulk modulus

n = number of obstacles/unit vol.

p = hydrostatic pressure

p_{ij} = stress tensor

r = radial coordinate

R = radius of obstacle

$\underset{\sim}{s}$ = displacement

S = surface

T = absolute temperature

t = time

V = volume

$\underset{\sim}{v}$ = velocity

α = extinction per unit length

α_v = coeff. of volume expansion

γ = ratio of spec. heats.

ϕ_1, ϕ_2 = scalar potentials

λ, μ = elastic constants

η = coeff. of viscosity

ω = angular frequency

ρ = density

σ = thermal conductivity

σ_a = absorption cross-section

τ_{ij} = viscous or elastic stress tensor

θ = polar angle

φ = azimuth angle

$(\ )'$ = quantities pertaining to obstacle

$(\ )_0$ = unperturbed quantities

$(\ )^*$ = complex conjugate

The following symbols are used to indicate compound quantities:

$a_1 = Rk_1$ $\qquad \beta = (\lambda + \frac{2}{3}\mu)\alpha_v$ $\qquad \mathcal{H} = \sigma/\rho c_v$

$a_2 = Rk_2$ $\qquad \delta = \rho/\rho'$ $\qquad \nu = \eta/\rho$

$b = RK$ $\qquad \chi = \sigma/\sigma'$ $\qquad \epsilon = \eta/\eta'$

I. INTRODUCTION

The first investigation of the effect of small spherical obstacles on the propagation of sound was performed by Lord Rayleigh (1). He calculated the scattering effect of the particles in a non-viscous atmosphere and showed that the effect depends upon the ratio of the diameter of the particles to the wave length of sound.

Some time later, Sewell (2) took viscosity into account and calculated the scattering and additional absorption due to a number of randomly distributed cylindrical and spherical obstacles for the case that the wave length was sufficiently long, or the obstacles sufficiently small, so that the phase variation over the obstacle could be neglected. He assumed the suspended particles to be perfectly rough at the surface, perfectly rigid, and fixed in space, i.e. they did not partake of the motion of the surrounding medium. The resulting equations were applied to the particular case of the absorption of sound in fogs.

The assumption of stationary obstacles proved to be quite a bothersome point particularly since Sewell's result yielded a finite attenuation at zero frequency, and in an important paper in 1941, Epstein (3) extended Sewell's theory by employing a method which automatically included the oscillations of the particles in the acoustic field due to their finite density. This step was partly necessitated by the experimental work of Hartman and Focke (4) on aqueous suspensions, since in their case the density ratio was close to unity and Sewell's theory was not at all valid. Epstein considered spherical obstacles and assumed them to be: (a) rigid, (b) viscous fluids, (c) elastic solids, and showed that for the case of fogs, and sufficiently high frequencies so that the oscillations of the drops could be neglected, Sewell's equation was a very close first approximation.

Recent measurements on the absorption of sound in fogs by Knudson (5), using the reverberation chamber technique, indicated, however, that Sewell's formula gives an absorption which is too low but of the right order of magnitude.

Sewell considered the effects of viscosity alone but did not take into account thermal conduction. The same must be said about Epstein's paper referred to above. In a later paper Epstein took heat conduction into account by an indirect and not very accurate method.

Kirchhoff was first to point out in 1866 that the influence of heat conduction is of the same order of magnitude as that of viscosity so that for an accurate solution (or shall we say a more accurate solution) both factors must be taken into account. This was done in the plane wave solution of Kirchhoff (6), and as expected gave a more strongly damped wave than that calculated by Stokes using viscosity alone.

With this in mind, Dr. Epstein suggested to me to try to extend his method as used in (3) so as to include any temperature effects. The fact that the available experimental data gave a greater absorption than that calculated by Sewell's equation had been for some time interpreted as due to heat conduction, and as the results of this essay will show, this interpretation was entirely justified.

A great deal of mathematical detail had to be gone through in order to arrive at the final results. For this reason this paper is divided into two parts: The first part is mainly descriptive in character, containing only the final results and a minimum of mathematics, except for the derivation of fundamental relations (such as II). The second part in the form of appendices contains most of the algebra, and references will be made to them as the need arises.

II. THE BASIC EQUATIONS

(a) Viscous Fluid

The equation of motion of a viscous fluid is given by:

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = \frac{\partial p_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \tag{2.01}$$

Where τ_{ij} is the viscosity stress tensor with components:

$$\begin{aligned} \tau_{ij} &= \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) , \quad i \neq j \\ \tau_{ii} &= 2\eta \frac{\partial v_i}{\partial x_i} - \frac{2}{3} \eta \nabla \cdot \underline{v} \end{aligned} \tag{2.02}$$

It can be shown (7) that (2.02) is actually only a first approximation which neglects additional terms depending on the square of the coefficient of viscosity. These extra terms contain, among other quantities, the thermal stresses which may exist due to inequalities of temperature. The kinetic theory of gases, as developed by S. Chapman and D. Enskog, shows, however, that if the number of gas molecules contained in a cube of dimensions intrinsic to the problem, such as the wave length of sound propagation, is large, then (2.02) is a sufficiently close approximation. Since in our case we shall not be dealing with rarified gases, but gases under ordinary conditions of pressure and temperature, and shall furthermore be concerned with frequencies up to only about one megacycle, we shall supress the additional terms in the stress tensor and adopt (2.02) throughout our whole investigation.

In the acoustic case with which we shall be concerned, the velocities will be rather small so that we can keep only linear terms. This means that we can delete the second term on the left of (2.01) and substituting the expressions for the stress tensor from (2.02), we obtain the usual form of the Navier-Stokes Equation:

$$\rho \frac{\partial \underset{\sim}{v}}{\partial t} = -\nabla p + \eta \nabla^2 \underset{\sim}{v} + \frac{1}{3}\eta \nabla(\nabla \cdot \underset{\sim}{v})$$

which can be transformed into:

$$\rho \frac{\partial \underset{\sim}{v}}{\partial t} = -\nabla p - \eta \nabla \times \nabla \times \underset{\sim}{v} + \frac{4}{3}\eta \nabla(\nabla \cdot \underset{\sim}{v}) \tag{2.03}$$

An additional equation is furnished by the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{v}) \cong \frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \underset{\sim}{v} = 0 \tag{2.04}$$

where again we have discarded second order terms.

The third equation which we need for the mathematical formulation of the problem will pertain to the conduction of heat. Let us consider at first a unit mass of fluid. If the temperature is changed by dT and the volume by dV, then the internal energy will be changed by:

$$d\mathcal{U} = \left(\frac{\partial \mathcal{U}}{\partial T}\right)_V dT + \left(\frac{\partial \mathcal{U}}{\partial V}\right)_T dV$$

Making use of the two well known thermodynamic relations:

$$\left(\frac{\partial \mathcal{U}}{\partial T}\right)_V = c_v \quad , \quad \left(\frac{\partial \mathcal{U}}{\partial V}\right)_T = -p + T\left(\frac{\partial p}{\partial T}\right)_V$$

we obtain for the change in internal energy:

$$d\mathcal{U} = c_v\,dT - p\,dV + T\left(\frac{\partial p}{\partial T}\right)_V dV$$

By the First Law, the quantity of heat absorbed will be:

$$DQ = d\mathcal{U} + p\,dV = c_v\,dT + T\left(\frac{\partial p}{\partial T}\right)_V dV =$$

$$= c_v\,dT - \frac{T}{\rho^2}\left(\frac{\partial p}{\partial T}\right)_V d\rho$$

Hence, the amount of heat absorbed per unit time by a unit volume of fluid will be:

$$\rho\frac{DQ}{dt} = \rho c_v\frac{dT}{dt} - \frac{T}{\rho}\left(\frac{\partial p}{\partial T}\right)_V\frac{d\rho}{dt}$$

This heat must come from two sources:

(a) heat will be conducted into the volume element from outside,

(b) heat will be produced inside the volume element due to viscosity. Denote this rate of production of heat inside per unit volume by F. Then, for energy balance we must have:

$$\rho c_v\frac{dT}{dt} - \frac{T}{\rho}\left(\frac{\partial p}{\partial t}\right)_V\frac{d\rho}{dt} = \sigma\nabla^2 T + F$$

We can write this as:

$$\frac{d}{dt}(\rho c_v T) - \left[T c_v + \frac{T}{\rho}\left(\frac{\partial p}{\partial T}\right)_V\right]\frac{d\rho}{dt} = \sigma\nabla^2 T + F \qquad (2.05)$$

But since:

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\underset{\sim}{v} \quad , \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\nabla$$

we can bring (2.05) into the form:

$$\frac{\partial}{\partial t}(\rho c_v T) + \nabla\cdot(\rho c_v T \underset{\sim}{v}) + T\left(\frac{\partial p}{\partial T}\right)_v \nabla\cdot\underset{\sim}{v} = \sigma\nabla^2 T + F \tag{2.06}$$

We shall show later in VIII that:

$$F = \frac{1}{2\eta}\sum_{i,j}\tau_{ij}\tau_{ij}$$

so that it consists of terms quadratic in the velocities. Hence, if we consistently neglect second order terms, we can write (2.06) as:

$$\rho c_v \frac{\partial T}{\partial t} + c_v T_0 \frac{\partial \rho}{\partial t} + \rho_0 c_v T_0 \nabla\cdot\underset{\sim}{v} + \underset{\sim}{v}\cdot\nabla(\rho_0 c_v T_0) + T_0\left(\frac{\partial p}{\partial T}\right)_v \nabla\cdot\underset{\sim}{v} = \sigma\nabla^2 T$$

But the fourth term vanishes, and from (2.04) the second and third terms cancel, so that:

$$\frac{\partial T}{\partial t} = \mathcal{K}\nabla^2 T - \frac{T_0}{\rho_0 c_v}\left(\frac{\partial p}{\partial T}\right)_v \nabla\cdot\underset{\sim}{v} \tag{2.07}$$

where we have let:

$$\mathcal{K} = \frac{\sigma}{\rho_0 c_v}$$

Equ. (2.07) gives us the relation between the temperature and the acoustic field. The system of equations which we must now solve simultaneously is given by (2.03) (2.04) and (2.07).

If we differentiate (2.03) with respect to the time and make use of the fact that the pressure is a function of both ρ and T so that:

$$\nabla p = \left(\frac{\partial p}{\partial \rho}\right)_T \nabla\rho + \left(\frac{\partial p}{\partial T}\right)_v \nabla T$$

then we obtain:

$$\rho_0 \frac{\partial^2 \underset{\sim}{v}}{\partial t^2} = -\eta \nabla\times\nabla\times\dot{\underset{\sim}{v}} + \frac{4}{3}\eta\nabla(\nabla\cdot\dot{\underset{\sim}{v}}) - \left(\frac{\partial p}{\partial \rho}\right)_T \nabla\frac{\partial \rho}{\partial t} - \left(\frac{\partial p}{\partial T}\right)_v \nabla\frac{\partial T}{\partial t}$$

and eliminating $\frac{\partial \rho}{\partial t}$ by (2.04):

$$\rho_0 \frac{\partial^2 \underset{\sim}{v}}{\partial t^2} = -\eta \nabla\times\nabla\times\dot{\underset{\sim}{v}} + \frac{4}{3}\eta\nabla(\nabla\cdot\dot{\underset{\sim}{v}}) + \left(\frac{\partial p}{\partial \rho}\right)_T \rho_0\nabla(\nabla\cdot\underset{\sim}{v}) - \left(\frac{\partial p}{\partial T}\right)_v \nabla\frac{\partial T}{\partial t} \qquad (2.08)$$

We now express the velocity in terms of a scalar potential ϕ and a vector potential $\underset{\sim}{A}$ by the relations:

$$\underset{\sim}{v} = -\nabla\phi + \nabla\times\underset{\sim}{A} \quad , \qquad \nabla\cdot\underset{\sim}{A} = 0 \qquad (2.09)$$

ϕ representing the longitudinal vibrations and $\underset{\sim}{A}$ the transverse ones. Substituting (2.09) into (2.08) and (2.07) we obtain the three equations:

$$\rho_0 \frac{\partial^2 \underset{\sim}{A}}{\partial t^2} = \eta\nabla^2\dot{\underset{\sim}{A}}$$

$$\rho_0 \frac{\partial^2 \phi}{\partial t^2} = \frac{4}{3}\eta\nabla^2\dot{\phi} + \left(\frac{\partial p}{\partial \rho}\right)_T \rho_0\nabla^2\phi + \left(\frac{\partial p}{\partial T}\right)_v \frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t} = \varkappa\nabla^2 T + \frac{T_0}{c_v\rho_0}\left(\frac{\partial p}{\partial T}\right)_v \nabla^2\phi$$

Let us now assume a periodic state with time dependence e^{-iwt}. Then the three equations above become ($\underset{\sim}{A}$, ϕ and T now refer to the space dependent part only):

$$\rho_0 \omega \underset{\sim}{A} = i\eta \nabla^2 \underset{\sim}{A} \tag{2.10}$$

$$\rho_0 \omega^2 \phi = \frac{4}{3} i\omega\eta\nabla^2\phi - \left(\frac{\partial p}{\partial \rho}\right)_T \rho_0 \nabla^2\phi + i\omega\left(\frac{\partial p}{\partial T}\right)_V T \tag{2.11}$$

$$-i\omega T = \mathcal{H}\nabla^2 T + \frac{T_0}{C_v \rho_0}\left(\frac{\partial p}{\partial T}\right)_V \nabla^2\phi \tag{2.12}$$

From (2.11) we obtain:

$$T = \frac{-i\rho_0\omega\phi - \left[\frac{4}{3}\eta + \frac{i\rho_0}{\omega}\left(\frac{\partial p}{\partial \rho}\right)_T\right]\nabla^2\phi}{\left(\frac{\partial p}{\partial T}\right)_V} \tag{2.13}$$

and substituting this into (2.12):

$$\mathcal{H}\left[\frac{4}{3}\eta + \frac{i\rho_0}{\omega}\left(\frac{\partial p}{\partial \rho}\right)_T\right]\nabla^4\phi + \left[\frac{4}{3}i\omega\eta - \rho_0\left(\frac{\partial p}{\partial \rho}\right)_T + i\rho_0\mathcal{H}\omega - \frac{T_0}{C_v\rho_0}\left(\frac{\partial p}{\partial T}\right)_V^2\right]\nabla^2\phi -$$

$$-\rho_0\omega^2\phi = 0 \tag{2.14}$$

But:

$$C_p - C_v = T\left(\frac{\partial p}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_p = -T\left(\frac{\partial p}{\partial T}\right)_V^2\left(\frac{\partial V}{\partial p}\right)_T$$

so that:

$$T\left(\frac{\partial p}{\partial T}\right)_V^2 = -\left(C_p - C_v\right)\left(\frac{\partial p}{\partial V}\right)_T$$

and hence:

$$-\rho_0\left(\frac{\partial p}{\partial \rho}\right)_T - \frac{T_0}{C_v\rho_0}\left(\frac{\partial p}{\partial T}\right)_V^2 = -\rho_0\left(\frac{\partial p}{\partial \rho}\right)_T + \frac{\gamma}{\rho_0}\left(\frac{\partial p}{\partial V}\right)_T - \frac{1}{\rho_0}\left(\frac{\partial p}{\partial V}\right)_T$$

But since:

$$\left(\frac{\partial p}{\partial V}\right)_T = -\rho_0\left(\frac{\partial p}{\partial \rho}\right)_T$$

we can write:

$$-\rho_0\left(\frac{\partial p}{\partial \rho}\right)_T - \frac{T_0}{c_v\rho_0}\left(\frac{\partial p}{\partial T}\right)_V^2 = -\gamma\rho_0\left(\frac{\partial p}{\partial \rho}\right)_T$$

Substituting this into (2.14), introducing the kinematic viscosity ν by $\nu = \eta/\rho_0$ and letting:

$$\gamma\left(\frac{\partial p}{\partial \rho}\right)_T = c^2$$

(the longitudinal velocity of sound), we can bring (2.14) into the form:

$$\mathcal{H}\left[\frac{4}{3}\nu + \frac{ic^2}{\omega\gamma}\right]\nabla^4\phi + \left[\frac{4}{3}i\omega\nu + i\omega\mathcal{H} - c^2\right]\nabla^2\phi - \omega^2\phi = 0 \qquad (2.14)$$

We can write this as:

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\phi = 0 \qquad (2.15)$$

so that if $k_1^2 \neq k_2^2$, a solution of (2.15) will be

$$\phi = \phi_1 + \phi_2$$

where:

$$\left.\begin{aligned} \nabla^2\phi_1 + k_1^2\phi_1 &= 0 \\ \nabla^2\phi_2 + k_2^2\phi_2 &= 0 \end{aligned}\right\} \qquad (2.16)$$

also, from (2.10):

$$\nabla^2 \underset{\sim}{A} + K^2 \underset{\sim}{A} = 0 , \qquad K^2 = \frac{i\omega}{\nu} \tag{2.17}$$

We have thus reduced the problem to the solution of two scalar --and one vector wave equation, subject to certain bounday conditions.

In this form the problem was presented by Professor Epstein in his lectures on "Mechanics of the Continuum" in the fall term of 1946.

(b) Elastic Solid.

The equation of motion of an elastic solid is given by:

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} \tag{2.18}$$

where τ_{ij} is the elastic stress tensor with components

$$\tau_{ij} = \mu \left(\frac{\partial s_i}{\partial x_j} + \frac{\partial s_j}{\partial x_i} \right)$$

$$\tau_{ii} = 2\mu \frac{\partial s_i}{\partial x_i} + \lambda \nabla \cdot \underset{\sim}{s} \tag{2.19}$$

These expressions for the stress tensor components neglect any additional thermal stresses which may exist due to inequalities in temperature. In the previous case, these could be neglected, since they were very small compared to the viscous stresses. In the case of a solid, however, this may not be the case and we shall not be able to neglect them. It can be shown (8) that if temperature effects are taken into account, the normal stresses become:

$$p_{ii} = 2\mu \frac{\partial s_i}{\partial x_i} + \lambda \nabla \cdot \underset{\sim}{s} - \beta T = \tau_{ii} - \beta T \qquad (2.20)$$

where

$$\beta = \left(\lambda + \tfrac{2}{3}\mu\right)\alpha_v$$

and α_v is the coefficient of volume expansion. The shearing stresses are not effected by temperature differences. Substituting (2.19) and (2.20) into (2.18) and again neglecting second order terms in the dependent variable, we obtain for the equation of motion:

$$\rho_0 \frac{\partial^2 \underset{\sim}{s}}{\partial t^2} = \mu \nabla^2 \underset{\sim}{s} + (\lambda+\mu)\nabla(\nabla \cdot \underset{\sim}{s}) - \beta \nabla T$$

or:

$$\rho_0 \frac{\partial^2 \underset{\sim}{s}}{\partial t^2} = -\mu \nabla \times \nabla \times \underset{\sim}{s} + (\lambda + 2\mu)\nabla(\nabla \cdot \underset{\sim}{s}) - \beta \nabla T \qquad (2.21)$$

Since the derivation of (2.07) applies to a solid as well as to a fluid, we can write:

$$\frac{\partial T}{\partial t} = \mathcal{H} \nabla^2 T - \frac{T_0}{\rho_0 c_v}\left(\frac{\partial p}{\partial T}\right)_v \nabla \cdot \frac{\partial \underset{\sim}{s}}{\partial t} \qquad (2.22)$$

Again, let us introduce a scalar--and vector potential (for the longitudinal and transverse vibrations respectively) by the relation:

$$\underset{\sim}{S} = -\nabla\phi + \nabla\times\underset{\sim}{A} \quad , \qquad \nabla\cdot\underset{\sim}{A} = 0$$

Substituting this into (2.21) and (2.22) we obtain the three equations:

$$\rho_0 \frac{\partial^2 \underset{\sim}{A}}{\partial t^2} = \mu \nabla^2 \underset{\sim}{A}$$

$$\rho_0 \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu)\nabla^2\phi + \beta T$$

$$\frac{\partial T}{\partial t} = \mathcal{H}\nabla^2 T + \frac{T_0}{\rho_0 c_v}\left(\frac{\partial p}{\partial T}\right)_V \nabla^2 \frac{\partial \phi}{\partial t}$$

Assuming a periodic state with time dependence e^{-iwt}, these equations become ($\underset{\sim}{A}$, ϕ and T now refer to the spacially dependent part only):

$$-\rho_0 \omega^2 \underset{\sim}{A} = \mu \nabla^2 \underset{\sim}{A} \tag{2.23}$$

$$-\rho_0 \omega^2 \phi = (\lambda + 2\mu)\nabla^2\phi + \beta T \tag{2.24}$$

$$-i\omega T = \mathcal{H}\nabla^2 T - \frac{i\omega T_0}{\rho_0 c_v}\left(\frac{\partial p}{\partial T}\right)_V \nabla^2\phi \tag{2.25}$$

Solving (2.24) for T and substituting into (2.25) we obtain:

$$\mathcal{H}(\lambda + 2\mu)\nabla^4\phi + \left[\mathcal{H}\rho_0\omega^2 + i\omega\left(\lambda + 2\mu + M\right)\right]\nabla^2\phi + i\omega^3\rho_0\phi = 0 \tag{2.26}$$

where:

$$M = \beta\frac{T_0}{\rho_0 c_v}\left(\frac{\partial p}{\partial T}\right)_V = \left(\lambda + \tfrac{2}{3}\mu\right)\frac{\alpha_V T_0}{\rho_0 c_v}\left(\frac{\partial p}{\partial T}\right)_V$$

We can again factor this into:

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\phi = 0$$

so that if $k_1^2 \neq k_2^2$ a solution will be as before:

$$\phi = \phi_1 + \phi_2$$

where:

$$\left.\begin{aligned} \nabla^2\phi_1 + k_1^2\phi_1 &= 0 \\ \nabla^2\phi_2 + k_2^2\phi_2 &= 0 \end{aligned}\right\} \tag{2.27}$$

also, from (2.23):

$$\nabla^2 \underset{\sim}{A} + K^2 \underset{\sim}{A} = 0 \quad , \qquad K^2 = \frac{\omega^2 \rho_0}{\mu} \tag{2.28}$$

Thus we see that the equations which we must solve in the case of a viscous fluid or an elastic solid are identical in form, differing only in the constant coefficients.

We note from (2.14) and (2.26) that if we had not taken thermal conduction into account (which corresponds to letting $\mathcal{H} \to 0$) then we would have come out with only a single scalar potential. The existence of the second scalar potential is something new and must be due to the effects of thermal conductivity. The deeper physical significance of the second scalar potential will become clearer later.

III. CALCULATION OF k_1 AND k_2

(a) Viscous Fluid

From (2.15) we get:

$$\nabla^4\phi + (k_1^2 + k_2^2)\nabla^2\phi + k_1^2 k_2^2 \phi = 0$$

and comparing this with (2.14) we obtain the two equations:

$$k_1^2 + k_2^2 = \frac{\frac{4}{3}i\omega\nu + i\omega\mathcal{H} - c^2}{\mathcal{H}\left(\frac{4}{3}\nu + \frac{ic^2}{\omega\gamma}\right)} \quad , \quad k_1^2 k_2^2 = -\frac{\omega^2}{\mathcal{H}\left(\frac{4}{3}\nu + \frac{ic^2}{\omega\gamma}\right)} \tag{3.01}$$

Solving for k_1^2 and k_2^2, we see that they will be the two solutions of the equation:

$$k^4\mathcal{H}\left(\frac{4}{3}\nu + \frac{ic^2}{\omega\gamma}\right) - k^2\left(\frac{4}{3}i\omega\nu + i\omega\mathcal{H} - c^2\right) - \omega^2 = 0 \tag{3.02}$$

It is shown in Appendix I that for frequencies up to about one megacycle, we can write with accuracy sufficient for our purposes:

$$k_1 \cong \frac{\omega}{c}\left[1 + \frac{i\omega}{c^2}\left(\frac{2}{3}\nu + \frac{\mathcal{H}}{2}\left(1 - \frac{1}{\gamma}\right)\right)\right] \tag{3.03}$$

$$k_2 \cong (1+i)\sqrt{\frac{\omega\gamma}{2\mathcal{H}}} \tag{3.04}$$

We see that except for the case of zero frequency, k_1 is far from being equal to k_2 so that no degeneracy exists and our two solutions of (2.15) will be independent.

(b) Elastic Solid

In this case we obtain from (2.26) the two equations:

$$k_1^2 + k_2^2 = \frac{\mathcal{H}\rho_0 w^2 + iw(\lambda + 2\mu + M)}{\mathcal{H}(\lambda + 2\mu)}\ , \qquad k_1^2 k_2^2 = \frac{i\rho_0 w^3}{\mathcal{H}(\lambda+2\mu)} \tag{3.05}$$

so that k_1^2 and k_2^2 will be the two solutions of:

$$k^4 \mathcal{H}(\lambda+2\mu) - k^2\left[\mathcal{H}\rho_0 w^2 + iw(\lambda+2\mu+M)\right] + i\rho_0 w^3 = 0 \tag{3.06}$$

It is shown in Appendix II that for frequencies up to about 100 megacycles, we have with sufficient accuracy:

$$k_1 \cong w\sqrt{\frac{\rho_0}{\lambda+2\mu}}\left(1 + \frac{iw\mathcal{H}\rho_0 M}{4(\lambda+2\mu)^2}\right) \tag{3.07}$$

$$k_2 \cong (1+i)\sqrt{\frac{w}{2\mathcal{H}}} \tag{3.08}$$

Again we see that except for the case of zero frequency, no degeneracy will exist so that our two solutions to (2.26) will be independent.

The values of k_1, k_2 and K completely specify the nature of the potential functions. The physical significance of the results obtained in this paragraph will be discussed in the next section.

IV PLANE WAVE SOLUTION

(a) Viscous Fluid

For the case of a plane wave moving in the positive x direction the longitudinal waves are expressed by $\phi = Ae^{ik_1x} + Be^{ik_2x}$ where A, B are constants, since we have the possibility of two different longitudinal waves and the general solution will be a linear combination of them. Since k_1 and k_2 are complex, the amplitude of the two waves will decline exponentially. In order to get a quantitative measure of the rapidity of absorption, let us define the quantity l to be the distance in which the amplitude declines to 1/e of its initial value. Then, from (3.03) and (3.04), we obtain:

$$l_{L_1} = \frac{c^3}{\omega^2\left[\frac{2}{3}\nu + \frac{\mathcal{H}}{2}\left(1 - \frac{1}{\gamma}\right)\right]} \tag{4.01}$$

$$l_{L_2} = \sqrt{\frac{2\mathcal{H}}{\omega\gamma}} \tag{4.02}$$

where the subscripts L_1 and L_2 refer to the two different longitudinal waves. We see that the absorption increases with increasing frequency, but not quite as rapidly for ϕ_2 as for ϕ_1. The quantities l_{L_1} and l_{L_2} are tabulated in Table II as a function of frequency for air and water, and it is evident that they are of entirely different orders of magnitude, ϕ_1 being damped only slightly but for ϕ_2 the damping is appreciable.

In the absence of conduction effects, we let $\mathcal{H} \to 0$ and obtain:

$$l_{L_1} \xrightarrow[\mathcal{H} \to 0]{} \frac{3c^3}{2\omega^2\nu} \,, \qquad l_{L_2} \xrightarrow[\mathcal{H} \to 0]{} 0$$

so that in this case we get only one longitudinal wave instead of two, since ϕ_2 suffers infinite absorption and hence does not exist. l_{L_1} agrees with equ. (9) of (3) in the limit as $\mathcal{H} \to 0$, as it should, because conduction effects were there neglected. Equ. (4.01) is identical with the result of Kirchhoff (6) who used a slightly different method of attack.

For the case of air, V and $\mathcal{H}$ are about equal, so that, from (4.01), the additional absorption of ϕ_1 due to thermal conduction is of the same order of magnitude as that due to viscosity alone, a result which was already anticipated by Kirchhoff.

But in either case, whether conduction is taken into account or not, the absorption for ϕ_1 is small, so small that the damping taking place in distances comparable to the sizes of the obstructing spheres we shall consider in the later sections is negligible. The absorption of ϕ_2 and, in fact, the existence of ϕ_2 is due entirely to the effects of thermal conduction, and it would therefore seem appropriate to call ϕ_2 the "conduction wave". We shall denote ϕ_1 by "longitudinal wave". We see from Table II that the conduction wave is very strongly absorbed in distances comparable to the sizes of the spheres we shall consider, and we shall not be able to neglect this.

We note from (2.17) that the inclusion of thermal conduction does not affect the transverse component. We obtain:

$$l_T = \sqrt{\frac{2\nu}{\omega}} \qquad (4.03)$$

Thus the absorption of the transverse-- and the conduction wave are of the same order of magnitude.

It therefore appears that the main energy loss occurring in the diffraction of acoustic waves by small obstacles is due not only to the partial conversion of the incident longitudinal wave into a transverse wave which is quickly absorbed, but also to the partial conversion into a longitudinal conduction wave which also suffers a strong absorption.

(b) Elastic Solid

As before we have $\phi = Ae^{ik_1x} + Be^{ik_2x}$, and since k_1 and k_2 are complex, the two waves will again be exponentially damped. Defining l in the same manner as before, we obtain from (3.07) and (3.08):

$$l_{L_1} = \frac{4\rho}{\omega^2 \mathcal{H} M}\left(\frac{\lambda+2\mu}{\rho}\right)^{5/2} \tag{4.04}$$

$$l_{L_2} = \sqrt{\frac{2\mathcal{H}}{\omega}} \tag{4.05}$$

Again the absorption increases with increasing frequency but not quite as rapidly for ϕ_2 as for ϕ_1 . From Appendix II we see that l_{L_1} is very large, indicating a very slight absorption of ϕ_1 , while the absorption of ϕ_2 will not be negligible. The absorption of ϕ_1 is due entirely to the effects of thermal conduction. For zero thermal conductivity the longitudinal wave is not absorbed at all while the conduction wave undergoes infinite absorption and will hence not exist.

The essential difference between the case of a viscous fluid and an elastic solid appears in the transverse component. Whereas in the former case the transverse wave was strongly absorbed, in the latter case it will not be absorbed at all, since from (2.28) K is real. Furthermore we see from (3.07) and (2.28) that in an elastic solid the wave lengths of the longitudinal-- and transverse waves are of the same order while for a viscous fluid the wave length of the transverse component is very much smaller than that of the longitudinal one.

Summarizing we may say that for either an elastic solid or a viscous fluid, the absorption of the longitudinal wave is very small and in all our later applications we shall be able to neglect it. In either case the absorption of the conduction wave is appreciable and must be taken into account. For the transverse component we have strong absorption in a viscous fluid and no absorption at all in an elastic solid. To our degree of approximation, the velocity of propagation of an acoustic disturbance is not effected by the inclusion of viscosity or thermal conduction.

V. RIGID, ROUGH SPHERE WITH ZERO THERMAL CONDUCTIVITY

Before tackling the more complicated situations, it will be instructive to consider at first the simplest of all possible cases. Consider a plane wave, propagating in a viscous fluid and impinging on a spherical obstacle. We shall assume the obstacle to be heavy and rigid, and rough at the surface and composed of heat insulating material. The surrounding fluid, however, has a finite thermal conductivity, and our problem will be to determine the acoustic field. Since the obstacle is assumed to be rigid, we can say immediately that inside the obstacle, all the potentials will vanish. Outside the obstructing sphere we have the incident wave, represented by a potential ϕ_i and the scattered wave represented by the scalor potentials ϕ_1 and ϕ_2 and by the vector potential $\underset{\sim}{A}$. Adopt a spherical coordinate system (r, θ, φ) and let the incident plane wave travel in the direction of the positive polar axis. We see from symmetry that we shall have an A_φ component only and that there will be no dependence on φ. Since ϕ_1 and ϕ_2 will be solutions of (2.16) and A_φ will be a solution of (2.17), we can write down the following expansions for the potentials (9):

$$\left.\begin{aligned}
\phi_i &= e^{ik_1 r\cos\theta} = \sum_0^\infty i^n(2n+1) j_n(k_1 r) P_n(\cos\theta) \\
\phi_1 &= \sum_0^\infty i^n(2n+1) B_n h_n^{(1)}(k_1 r) P_n(\cos\theta) \\
\phi_2 &= \sum_0^\infty i^n(2n+1) C_n h_n^{(1)}(k_2 r) P_n(\cos\theta) \\
A_\varphi &= \sum_1^\infty i^n(2n+1) D_n h_n^{(1)}(Kr) P_n^1(\cos\theta)
\end{aligned}\right\} \qquad (5.01)$$

where, from (3.04) (3.03) (2.17):

$$k_1 = \frac{\omega}{c}\ , \qquad k_2 = (1+i)\sqrt{\frac{\omega\gamma}{2\chi}}\ , \qquad K = (1+i)\sqrt{\frac{\omega}{2\nu}}$$

$j_n(x)$ and $h_n^{(1)}(x)$ are the spherical Bessel and Hankel functions defined by:

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x)\ , \qquad h_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}}\, H^{(1)}_{n+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\left(J_{n+\frac{1}{2}}(x) + i\, Y_{n+\frac{1}{2}}(x)\right)$$

Where $J_{n+\frac{1}{2}}(x)$ and $Y_{n+\frac{1}{2}}(x)$ are the two ordinary Bessel functions. Some of the important properties of j_n and $h_n^{(1)}$ are listed in Appendix III. We use $h_n^{(1)}$ and not $h_n^{(2)}$ since our time dependence was $e^{-i\omega t}$ and we desire an expanding wave. In all that is to follow, the superscript on $h_n^{(1)}$ will be omitted. B_n, C_n and D_n are constants which have to be determined from the boundary conditions.

Since the sphere is heavy, rigid, and rough at the surface, and has zero thermal conductivity, the velocity must vanish at the boundary of the sphere and so must the normal component of the temperature gradient.

Let the radius of the obstacle be R. Then the boundary conditions become:

$$(v_r)_{r=R} = 0\ , \qquad (v_\theta)_{r=R} = 0\ , \qquad (v_\varphi)_{r=R} = 0\ , \qquad \left(\frac{\partial T}{\partial r}\right)_{r=R} = 0 \qquad (5.02)$$

From (5.01), v_φ is identically zero everywhere. Thus, we have three equations from which the three unknown coefficients B_n, C_n, and D_n can be determined.

We must now express T in terms of the potentials. From

(2.11) we have:

$$T\left(\frac{\partial p}{\partial T}\right)_v = -i\rho_0 w\left(\phi_i+\phi_1+\phi_2\right) - \left(\frac{4}{3}\eta + \frac{i\rho_0 c^2}{w\gamma}\right)\nabla^2\left(\phi_i+\phi_1+\phi_2\right)$$

But since the scalar potentials are solutions of (2.16), this becomes:

$$T\left(\frac{\partial p}{\partial T}\right)_v = -i\rho_0 w\left(\phi_i+\phi_1+\phi_2\right) + \left(\frac{4}{3}\eta + \frac{i\rho_0 c^2}{w\gamma}\right)\left[\left(\phi_i+\phi_1\right)k_1^2 + \phi_2 k_2^2\right]$$

so that

$$T = \alpha_1\left(\phi_i+\phi_1\right) + \alpha_2\phi_2 \tag{5.03}$$

where:

$$\alpha_{1,2} = \frac{-i\rho_0 w + \left(\frac{4}{3}\eta + \frac{i\rho_0 c^2}{w\gamma}\right)k_{1,2}^2}{\left(\frac{\partial p}{\partial T}\right)_v}$$

Now

$$\left(\frac{\partial p}{\partial T}\right)_v = -\left(\frac{\partial p}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p = \rho_0^2\left(\frac{\partial p}{\partial \rho}\right)_T\left(\frac{\partial V}{\partial T}\right)_p = \frac{\rho_0 c^2}{\gamma}\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \frac{\rho_0 c^2\alpha_v}{\gamma} \tag{5.04}$$

so that:

$$\alpha_{1,2} = \frac{\gamma w}{c^2\alpha_v}\left[-i + \left(\frac{4}{3}\frac{\nu}{w} + \frac{ic^2}{w^2\gamma}\right)k_{1,2}^2\right]$$

Hence:

$$\alpha_1 = \frac{\gamma w}{c^2\alpha_v}\left(-i + \frac{4}{3}\frac{w\nu}{c^2} + \frac{i}{\gamma}\right)$$

Since we are neglecting the absorption of the longitudinal wave, we are neglecting terms of order $w\nu/c^2$ and hence we must also neglect them here. So that we can write with sufficient accuracy:

$$\alpha_1 \cong -\frac{i\omega}{c^2\alpha_\gamma}(\gamma-1) \tag{5.05}$$

Also:

$$\alpha_2 = \frac{\gamma\omega}{c^2\alpha_\gamma}\left(-i+\frac{4}{3}i\frac{\nu\gamma}{\mathcal{K}}-\frac{c^2}{\omega\mathcal{K}}\right) \cong -\frac{\gamma}{\alpha_\gamma\mathcal{K}} \tag{5.06}$$

We see that:

$$\left|\frac{\alpha_1}{\alpha_2}\right| = \frac{\omega\mathcal{K}}{c^2}\left(1-\frac{1}{\gamma}\right) \ll 1 \tag{5.07}$$

for the frequency range we are interested in, and we shall make use of this fact later on.

Using (2.09) and (5.03), the boundary conditions (5.02) now become:

$$\left.\begin{aligned}
&-\frac{\partial}{\partial r}(\phi_i+\phi_1+\phi_2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot A_\varphi) = 0\\
&-\frac{1}{r}\frac{\partial}{\partial\theta}(\phi_i+\phi_1+\phi_2)-\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) = 0\\
&\alpha_1\frac{\partial}{\partial r}(\phi_i+\phi_1)+\alpha_2\frac{\partial\phi_2}{\partial r} = 0
\end{aligned}\right\} \text{at } r=R$$

Making use of the fact that:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot P_n^1\right) = n(n+1)P_n\,, \qquad \frac{\partial P_n}{\partial\theta} = -P_n^1\,,$$

indicating differentiation with respect to the argument by a prime, omitting the superscript on $h_n^{(1)}$ and letting:

$$k_1 R = a_1 \quad , \quad k_2 R = a_2 \quad , \quad KR = b \tag{5.08}$$

we obtain, using (5.01):

$$\left.\begin{aligned} a_1 h_n'(a_1) B_n + a_2 h_n'(a_2) C_n - n(n+1) h_n(b) D_n &= -a_1 j_n'(a_1) \\ h_n(a_1) B_n + h_n(a_2) C_n - \left[b h_n'(b) + h_n(b) \right] D_n &= -j_n(a_1) \\ \alpha_1 a_1 h_n'(a_1) B_n + \alpha_2 a_2 h_n'(a_2) C_n + 0 \cdot D_n &= -\alpha_1 a_1 j_n'(a_1) \end{aligned}\right\} \tag{5.09}$$

This holds for $n > 0$. If $n = 0$, only the first and third equations are to be used. The second equation does not exist.

Let us now examine the arguments of the Bessel Functions a little more closely. We have:

$$a_1 = \frac{\omega R}{c} \quad , \quad a_2 = R(1+i)\sqrt{\frac{\omega \gamma}{2\mathcal{H}}} \quad , \quad b = R(1+i)\sqrt{\frac{\omega}{2\nu}}$$

Since $\frac{c}{\omega} = \frac{\lambda_L}{2\pi}$, where λ_L is the wave length of the longitudinal wave, we see that a_1 represents the ratio of the circumference of the obstacle to the wave length of the incident wave. In all that is to follow we shall consider this ratio to be small. (In the case of fog particles, R is about 10^{-3} cm, so that even for w as high as 10^6 sec^{-1}, a_1 for air would be of order 10^{-2}). The assumption that $\frac{\omega R}{c} \ll 1$ imposes an upper limit on the frequency beyond which our approximation will not hold. The quantities a_2 and b may not be small so-we cannot make any approximations there.

It is shown in Appendix IV that subject to the assumption that $a_1 \ll 1$, and alsq making use of the fact that $\left|\frac{\alpha_1}{\alpha_2}\right| \ll 1$, the coefficients turn out to be:

$$B_0 = -\frac{1}{3} i a_1^3 \quad , \quad B_n = -i \frac{2^{2n} n! n!}{(2n+2)!(2n-1)!} \frac{h_{n+1}(b)}{h_{n-1}(b)} a_1^{2n+1} \tag{5.10}$$

$$C_0 = 0 \quad , \quad C_n = \frac{\alpha_1}{\alpha_2} \frac{2^n n! n}{(2n)!} \frac{h_n(b) a_1^n}{\left[(n+1) h_n(a_2) - a_2 h_{n-1}(a_2)\right] b h_{n-1}(b)} \tag{5.11}$$

$$D_n = \frac{2^n n!}{(2n)!} \frac{a_1^n}{b h_{n-1}(b)} \tag{5.12}$$

Equs. (5.10) and (5.12) are identical with the solutions obtained in (3). This is of course as it should be since to our approximation B_n and D_n are independent of the thermal conductivity, so that as far as these two coefficients are concerned, it should not make any difference whether thermal conductivity is included or not. What is essentially new is C_n pertaining to the conduction wave. Since ϕ_2 is strongly damped we see that at points far away from the obstacle, the acoustic field is essentially the same as the one we would have obtained if thermal effects had been neglected. As $\mathcal{H} \rightarrow 0$, C_n does not go to zero but ϕ_2 will and we are then left with the solution given in (3). Since a_1 is assumed to be small, the series for the potentials will converge rapidly and it will be sufficient within the accuracy at which we aspire to maintain only the terms with n = 0 and n = 1.

Equs. (5.10)-(5.12) coupled with (5.01) completely describe the acoustic field and represent the solution of the problem.

VI. VISCOUS FLUID SPHERE

We shall now assume the obstacle to be composed of a viscous fluid of finite thermal conductivity. The outer medium is the same as before. Here we shall have to deal with seven different waves: four waves outside the obstacle and three waves inside. If primed letters pertain to the medium inside the sphere and unprimed ones to the medium outside, then the expansions for the potentials take the following form:

$$\begin{aligned}
\phi_i &= \sum_0^\infty i^n(2n+1) j_n(k_1 r) P_n(\cos\theta) \\
\phi_1 &= \sum_0^\infty i^n(2n+1) B_n h_n(k_1 r) P_n(\cos\theta) \\
\phi_2 &= \sum_0^\infty i^n(2n+1) C_n h_n(k_2 r) P_n(\cos\theta) \\
A_\varphi &= \sum_1^\infty i^n(2n+1) D_n h_n(Kr) P_n(\cos\theta) \\
\phi_1' &= \sum_0^\infty i^n(2n+1) B_n' j_n(k_1' r) P_n(\cos\theta) \\
\phi_2' &= \sum_0^\infty i^n(2n+1) C_n' j_n(k_2' r) P_n(\cos\theta) \\
A_\varphi' &= \sum_1^\infty i^n(2n+1) D_n' j_n(K' r) P_n^1(\cos\theta)
\end{aligned} \tag{6.01}$$

Since the Hankel function diverges at the origin, we must use j_n inside the sphere (standing waves).

We now must determine six unknown coefficients instead of only three, as in the previous paragraph, and this will make matters a little more complicated.

The boundary conditions we must satisfy are that across the boundary of the obstacle the velocity, the temperature, the heat flow, and the stresses p_{rr}, $p_{r\theta}$ and $p_{r\varphi}$ must be continuous.

It can be shown that in our case, where there is no dependence on φ and only an A_φ component, these stresses are given by:

$$p_{rr} = -p + 2\eta\left\{-\frac{\partial^2\phi}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{1}{r}\frac{\partial A_\varphi}{\partial r} - \frac{1}{r^2}A_\varphi\right)\right]\right\} \tag{6.02}$$

$$p_{r\theta} = \eta\left\{-2\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{1}{r^2}\phi\right) - \left(\frac{\partial^2 A_\varphi}{\partial r^2} - \frac{2}{r^2}A_\varphi\right) + \right.$$

$$\left. + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot A_\varphi\right)\right]\right\} \tag{6.03}$$

$$p_{r\varphi} = 0$$

The stress condition thus gives us two equations, so that we shall get a total of six equations to solve for the six unknown coefficients appearing in (6.01).

The term p in (6.02) is the hydrostatic pressure and we must now express it in terms of the potentials. In view of the periodicity, we have:

$$p = \frac{i}{\omega}\frac{\partial p}{\partial t} \tag{6.04}$$

But:

$$\frac{\partial p}{\partial t} = \left(\frac{\partial p}{\partial \rho}\right)_T \frac{\partial \rho}{\partial t} + \left(\frac{\partial p}{\partial T}\right)_V \frac{\partial T}{\partial t} = -\left(\frac{\partial p}{\partial\rho}\right)_T \rho_0 \nabla\cdot\underset{\sim}{v} + \left(\frac{\partial p}{\partial T}\right)_V \frac{\partial T}{\partial t} =$$

$$= \left(\frac{\partial p}{\partial\rho}\right)_T \rho_0\nabla^2\phi - i\omega T\left(\frac{\partial p}{\partial T}\right)_V$$

and using (2.11), this becomes:

$$\frac{\partial p}{\partial t} = -\rho_0 \omega^2 \phi + \frac{4}{3} i\omega \eta \nabla^2 \phi$$

so that from (6.04):

$$p = -i\rho_0 \omega \phi - \frac{4}{3}\eta \nabla^2 \phi$$

Since: $\phi = \phi_1 + \phi_2$ then from (2.16):

$$p = \left(\frac{4}{3}\eta k_1^2 - i\rho_0\omega\right)\phi_1 + \left(\frac{4}{3}\eta k_2^2 - i\rho_0\omega\right)\phi_2$$

Now:

$$\frac{4}{3}\eta k_1^2 - i\rho_0\omega = -i\rho_0\omega\left(1 + \frac{4}{3}\frac{i\nu}{\omega}k_1^2\right) = -K^2\eta\left(1 + \frac{4}{3}\frac{i\nu\omega}{c^2}\right)$$

$$\frac{4}{3}\eta k_2^2 - i\rho_0\omega = -i\rho_0\omega\left(1 + \frac{4}{3}\frac{i\nu}{\omega}k_2^2\right) = -K^2\eta\left(1 - \frac{4}{3}\frac{\nu\gamma}{\mathcal{H}}\right)$$

so that we can write:

$$p = -K^2\eta\left(\beta_1\phi_1 + \beta_2\phi_2\right) \tag{6.05}$$

where:

$$\beta_1 = 1 + \frac{4}{3}\frac{i\omega\nu}{c^2} \cong 1 \tag{6.06}$$

$$\beta_2 = 1 - \frac{4}{3}\frac{\nu\gamma}{\mathcal{H}} \tag{6.07}$$

Using (6.02) (6.03) (6.07) and (5.03), the boundary conditions yield the following six equations (outside the obstacle we must again identify ϕ_1 with $\phi_i + \phi_1$.):

$$B_n a_1 h_n'(a_1) - B_n' a_1' j_n'(a_1') + C_n a_2 h_n'(a_2) - C_n' a_2' j_n(a_2') - D_n \cdot n(n+1) h_n(b) +$$
$$+ D_n' n(n+1) j_n(b') = - a_1 j_n'(a_1) \qquad (6.08)$$

$$B_n h_n(a_1) - B_n' j_n(a_1) + C_n h_n(a_2) - C_n' j_n(a_2') - D_n \left[b h_n'(b) + h_n(b) \right] +$$
$$+ D_n' \left[b' j_n'(b') + j_n(b') \right] = - j_n(a_1) \qquad (6.09)$$

$$B_n \alpha_1 h_n(a_1) - B_n' \alpha_1' j_n(a_1') + C_n \alpha_2 h_n(a_2) - C_n' \alpha_2' j_n(a_2') = - \alpha_1 j_n(a_1) \qquad (6.10)$$

$$B_n \chi \alpha_1 a_1 h_n'(a_1) - B_n' \alpha_1' a_1' j_n'(a_1') + C_n \chi \alpha_2 a_2 h_n'(a_2) - C_n' \alpha_2' a_2' j_n'(a_2') =$$
$$= - \chi \alpha_1 a_1 j_n'(a_1) \qquad (6.11)$$

$$B_n \epsilon \left[a_1 h_n'(a_1) - h_n(a_1) \right] - B_n' \left[a_1' j_n'(a_1') - j_n(a_1') \right] + C_n \epsilon \left[a_2 h_n'(a_2) - h_n(a_2) \right] -$$
$$- C_n' \left[a_2' j_n'(a_2') - j_n(a_2') \right] - \tfrac{1}{2} D_n \epsilon \left[b^2 h_n''(b) + (n^2+n-2) h_n(b) \right] +$$
$$+ \tfrac{1}{2} D_n' \left[b'^2 j_n''(b') + (n^2+n-2) j_n(b') \right] = - \epsilon \left[a_1 j_n'(a_1) - j_n(a_1) \right] \qquad (6.12)$$

$$B_n \epsilon \left[b^2 \beta_1 h_n(a_1) - 2a_1^2 h_n''(a_1) \right] - B_n' \left[b'^2 \beta_1' j_n(a_1') - 2a_1'^2 j_n''(a_1') \right] + C_n \epsilon \left[b^2 \beta_2 h_n(a_2) - 2a_2^2 h_n''(a_2) \right] -$$
$$- C_n' \left[b'^2 \beta_2' j_n(a_2') - 2a_2'^2 j_n''(a_2') \right] + 2D_n \epsilon \cdot n(n+1) \left[b h_n'(b) - h_n(b) \right] -$$
$$- D_n' 2n(n+1) \left[b' j_n'(b') - j_n(b') \right] = - \epsilon \left[b^2 \beta_1 j_n(a_1) - 2a_1^2 j_n''(a_1) \right] \qquad (6.13)$$

where:

$$\epsilon = \eta/\eta', \qquad \chi = \sigma/\sigma'$$

We shall solve these equations subject to the same approximations we made before, namely that:

$$a_1 \ll 1 \ , \quad a_1' \ll 1 \ , \quad \left|\frac{\alpha_1}{\alpha_2}\right| \ll 1 \ , \quad \left|\frac{\alpha_1'}{\alpha_2'}\right| \ll 1 \, .$$

Since we have seen in the previous section that the expansions for the potentials converge rather rapidly, it will be sufficient to evaluate only the coefficients for n = 0 and n = 1. If n = 0 the second and fifth equation do not exist and in the remaining equations the terms involving D_n and D_n' do not appear.

The solution is indicated in Appendix V and the results are as follows:

$$B_0 = -\frac{1}{3} i a_1^3 \left(1 - \delta \frac{a_1'^2}{a_1^2}\right) \tag{6.14}$$

$$B_1 = \frac{1}{3} i a_1^3 (1-\delta) G_{B_1} = \frac{1}{3} i a_1^3 (1-\delta) \frac{\sum\limits_{n,m=0,1} G_{nm}^{B_1} \cdot (b)^{1-m} (b')^{1-n} j_n(b') h_m(b)}{\sum\limits_{n,m=0,1} G_{nm} (b)^{1-m} (b')^{1-n} j_n(b') h_m(b)} \tag{6.15}$$

$$B_0' = \delta \tag{6.16}$$

$$B_1' = \frac{a_1}{a_1'} G_{B_1'} = 3\delta \frac{a_1}{a_1'} G_{B_1} \tag{6.17}$$

$$C_0 = \frac{\alpha_1}{\alpha_2} \frac{\left(\delta \frac{\alpha_1'}{\alpha_1} - 1\right) a_2' j_1(a_2')}{a_2' j_1(a_2') h_0(a_2) - \chi a_2 j_0(a_2') h_1(a_2)} \tag{6.18}$$

$$C_1 = \frac{1}{3} a_1 \frac{\alpha_1}{\alpha_2} \frac{\left[2 + \chi - 3\frac{\alpha_1'}{\alpha_1} G_{B_1'} + 2(1-\chi)(1-\delta) G_{B_1}\right] j_1(a_2') + \left[\frac{\alpha_1'}{\alpha_1} G_{B_1'} - 1 - (1-\delta) G_{B_1}\right] a_2' j_0(a_2')}{2(\chi-1) j_1(a_2') h_1(a_2) + a_2' j_0(a_2') h_1(a_2) - \chi a_2 h_0(a_2) j_1(a_2')} \tag{6.19}$$

$$C_0' = \frac{\alpha_1}{\alpha_2'} \frac{\chi\left(\delta \frac{\alpha_1'}{\alpha_1} - 1\right) a_2 h_1(a_2)}{a_2' j_1(a_2') h_0(a_2) - \chi a_2 j_0(a_2') h_1(a_2)} \tag{6.20}$$

$$C_1' = \frac{1}{3} a_1 \frac{\alpha_1}{\alpha_2'} \frac{\left[(1+2\chi)\frac{\alpha_1'}{\alpha_1} G_{B_1'} - 3\chi\right] h_1(a_2) + \chi\left[1 + (1-\delta) G_{B_1} - \frac{\alpha_1'}{\alpha_1} G_{B_1'}\right] a_2 h_0(a_2)}{2(\chi-1) j_1(a_2') h_1(a_2) + a_2' j_0(a_2') h_1(a_2) - \chi a_2 h_0(a_2) j_1(a_2')} \tag{6.21}$$

where $G_{nm}^{B_1}$ and G_{nm} are abbreviations for the following quantities:

$$G_{00}^{B_1} = 1-\epsilon\,, \quad G_{01}^{B_1} = 3(\epsilon-1) - \frac{\epsilon}{2}\beta^2\,, \quad G_{10}^{B_1} = 3(\epsilon-1) + \frac{1}{2}\beta'^2\,, \quad G_{11}^{B_1} = 9(1-\epsilon) + \frac{3}{2}\beta'^2(\delta-1)$$

$$G_{00} = (1-\epsilon)(2+\delta)\,, \quad G_{01} = 9\delta(\epsilon-1) - \frac{\epsilon\beta^2}{2}(\delta+2)\,, \quad G_{10} = 3(\epsilon-1)(\delta+2) + \frac{\beta'^2}{2}(\delta+2)$$

$$G_{11} = 27\delta(1-\epsilon) + \frac{3}{2}\epsilon\beta^2(\delta-1) \tag{6.22}$$

where:

$$\delta = \rho/\rho'$$

It is shown in Appendix VI that these equations reduce to the results of V when $\epsilon = \delta = 0,\ \chi = \infty$, because there we essentially considered an infinitely heavy and infinitely viscous sphere of zero thermal conductivity. This serves as a useful check.

Since we shall not need D_n and D_n' in our later applications, these coefficients were not evaluated.

It can be shown that (6.14)-(6.17) are identical with Epstein's results (3) which is as it should be because these coefficients are independent of the thermal conductivity of either medium (to within our approximation), and it may be safe to say that the same applies to the coefficients in the vector potential. As before, the only essential difference between our results and Epstein's lies in the existence of the second scalar potential, or the conduction wave. But, again, since the conduction wave is strongly damped, we see that at points far away from the obstacle, the acoustic field is essentially unchanged by the inclusion of thermal effects.

VII. ELASTIC SOLID SPHERE

Consider now the case where the diffracting obstacle is a sphere composed of an elastic solid, and let it be embedded in a viscous fluid. Again we shall have to deal with seven different waves, and from II and III the expansions for the potentials assume the following form:

$$\phi_i = \sum_0^\infty i^n (2n+1) j_n(k_1 r) P_n(\cos\theta)$$

$$\phi_1 = \sum_0^\infty i^n (2n+1) B_n h_n(k_1 r) P_n(\cos\theta)$$

$$\phi_2 = \sum_0^\infty i^n (2n+1) C_n h_n(k_2 r) P_n(\cos\theta)$$

$$A_\varphi = \sum_1^\infty i^n (2n+1) D_n h_n(Kr) P_n(\cos\theta)$$

$$\phi_1' = -\frac{1}{i\omega} \sum_0^\infty i^n (2n+1) B_n' j_n(k_1' r) P_n(\cos\theta) \qquad (7.01)$$

$$\phi_2' = -\frac{1}{i\omega} \sum_0^\infty i^n (2n+1) C_n' j_n(k_2' r) P_n(\cos\theta)$$

$$A_\varphi' = -\frac{1}{i\omega} \sum_1^\infty i^n (2n+1) D_n' j_n(Kr) P_n(\cos\theta)$$

where:

$$k_1 = \frac{\omega}{c}, \qquad k_2 = \sqrt{\frac{\omega \gamma}{2\mathcal{H}}}\,(1+i), \qquad K = \sqrt{\frac{\omega}{2\nu}}\,(1+i)$$

$$k_1' = \omega\sqrt{\frac{\rho'}{\lambda+2\mu}}, \quad k_2' = \sqrt{\frac{\omega}{2\mathcal{H}'}}\,(1+i), \quad K' = \omega\sqrt{\frac{\rho'}{\mu}}$$

The reason for introducing $-\frac{1}{i\omega}$ into the primed potentials will become clear later. We must remember that ϕ and A_φ are potential functions for the velocity in the viscous fluid while ϕ' and A'_φ are potential functions for the displacement in the elastic solid.

We must again satisfy the same bounday conditions as before, namely v_r, v_θ, $T, \sigma\frac{\partial T}{\partial r}$, p_{rr}, and $p_{r\theta}$ must be continuous.

As before T has to be expressed in terms of the potentials. This was done in V for a viscous fluid and we must now do the same for the case of an elastic solid.

From (2.24) we obtain:

$$\beta T' = -\rho_0'\omega^2(\phi_1' + \phi_2') - (\lambda+2\mu)\nabla^2(\phi_1' + \phi_2') =$$

$$= \left[-\rho_0'\omega^2 + (\lambda+2\mu)k_1'^2\right]\phi_1' + \left[-\rho_0'\omega^2 + (\lambda+2\mu)k_2'^2\right]\phi_2'$$

and using (3.07) and (3.08):

$$\beta T' = \frac{1}{2} i\omega \frac{\rho_0'^2\omega^2\mathcal{H}'M}{(\lambda+2\mu)^2}\phi_1' + \left[-\rho_0'\omega^2 + (\lambda+2\mu)\frac{i\omega}{\mathcal{H}'}\right]\phi_2' =$$

$$\cong \frac{1}{2} i\omega \frac{\rho_0'^2\omega^2\mathcal{H}'M}{(\lambda+2\mu)^2}\phi_1' + i\omega\frac{\lambda+2\mu}{\mathcal{H}'}\phi_2'$$

since $\frac{\lambda+2\mu}{\mathcal{H}'} \gg \rho_0'\omega$ for the frequency range in which we are interested. Thus, we can write:

$$T' = -i\omega\left(\alpha_1'\phi_1' + \alpha_2'\phi_2'\right) \qquad (7.02)$$

where:

$$\alpha_1' = -\frac{\rho_0'^2\omega^2\mathcal{H}'M}{2\beta(\lambda+2\mu)^2}, \qquad \alpha_2' = -\frac{\lambda+2\mu}{\beta\mathcal{H}'}, \qquad \frac{\alpha_1'}{\alpha_2'} = \frac{\rho_0'^2\omega^2\mathcal{H}'^2M}{2(\lambda+2\mu)^3} \ll 1 \qquad (7.03)$$

Again, the reason for leaving $-iw$ outside will become clear later.

The stresses p_{rr} and $p_{r\theta}$ for an elastic solid differ from the ones for a viscous fluid, (6.02) (6.03), in the following respects: (1) $\underset{\sim}{v}$ is replaced by $\underset{\sim}{s}$, (2) $-p$ in p_{rr} is replaced by $\lambda\nabla\cdot\underset{\sim}{s}$, (3) η is replaced by μ , (4) There appears an extra term $-\beta T$ in p_{rr} due to the thermal stress. We therefore obtain:

$$p_{rr}' = \lambda\left(k_1'^2\phi_1' + k_2'^2\phi_2'\right) + \beta i w\left(\alpha_1'\phi_1' + \alpha_2'\phi_2'\right) + 2\mu\left[-\frac{\partial^2(\phi_1'+\phi_2')}{\partial r^2} + \right.$$

$$\left. + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{1}{r}\frac{\partial A_\varphi'}{\partial r} - \frac{1}{r^2}A_\varphi'\right)\right]\right]$$

But:

$$\lambda k_1'^2 + \beta i w\alpha_1' = \lambda k_1'^2\left[1 - \frac{iw\rho_0'\varkappa' M}{\lambda(\lambda+2\mu)}\right] \cong \lambda k_1'^2$$

$$\lambda k_2'^2 + \beta i w\alpha_2' = \lambda\frac{iw}{\varkappa'} - iw\frac{\lambda+2\mu}{\varkappa'} = -2\mu k_2'^2$$

$$\therefore p_{rr}' = \lambda k_1'^2\phi_1' - 2\mu k_2'^2\phi_2' + 2\mu\left\{-\frac{\partial^2(\phi_1'+\phi_2')}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{1}{r}\frac{\partial A_\varphi'}{\partial r} - \frac{1}{r^2}A_\varphi'\right)\right]\right\} \quad (7.04)$$

If we now define:

$$\chi = \sigma/\sigma', \qquad \varepsilon = -iw\eta/\mu$$

and set up the continuity conditions for v_r, v_θ, T, $\sigma\frac{\partial T}{\partial r}$ and $p_{r\theta}$ (remember that $\underset{\sim}{v}' = -iw\underset{\sim}{s}$) we see that we obtain equations which are identical in form with (6.08)-(6.12). This was the reason for defining the primed potentials and α_1', α_2' in the way that we have done. The continuity condition for p_{rr} becomes:

$$B_n\epsilon\left[b^2\beta_1 h_n(a_1)-2a_1^2h_n''(a_1)\right]-B_n'\left[\frac{\lambda}{\mu}a_1'^2 j_n(a_1')-2a_1'^2 j_n''(a_1')\right]+$$

$$+C_n\epsilon\left[b^2\beta_2 h_n(a_2)-2a_2^2h_n''(a_2)\right]+C_n'\left[2a_2'^2 j_n(a_2')+2a_2'^2 j_n''(a_2')\right]+$$

$$+D_n\epsilon\cdot 2n(n+1)\left[b h_n'(b)-h_n(b)\right]-D_n'\cdot 2n(n+1)\left[b' j_n'(b')-j_n(b')\right]=$$

$$=-\epsilon\left[b^2\beta_1 j_n(a_1)-2a_1^2 j_n''(a_1)\right] \qquad (7.05)$$

The definition of the primed arguments is in the present case:

$$a_1'=R\omega\sqrt{\frac{\rho'}{\lambda+2\mu}}\,,\qquad a_2'=R(1+i)\sqrt{\frac{\omega}{2\mathcal{H}'}}\,,\qquad b'=R\omega\sqrt{\frac{\rho'}{\mu}}$$

The expressions $\sqrt{\frac{\lambda+2\mu}{\rho'}}$ and $\sqrt{\frac{\mu}{\rho'}}$ are simply the longitudinal- and transverse velocities of propagation of sound in an elastic medium. Since they are of the same order or even greater that the longitudinal velocity of sound in a fluid, then if we assume, as before, that a_1 is small, then this implies that a_1' and b' will be small also. We also see that ϵ as defined in this section is an extremely small number (for the frequency range we are concerned with), and, as before, $\left|\frac{\alpha_1}{\alpha_2}\right|$ and $\frac{\alpha_1'}{\alpha_2'}$ are $\ll 1$. The evaluation of the six coefficients therefore proceeds along identical lines as in VI except now we have the additional simplification that not only a_1 and a_1' are small but also b' and ϵ. The solution is indicated in Appendix VII and the results are:

$$B_0 = -\frac{1}{3} i a_1^3 \left[1 - \frac{3\rho c^2}{3\lambda + 2\mu} \right] \tag{7.06}$$

$$B_1 = \frac{1}{3} i a_1^3 (1-\delta) \frac{h_2(b)}{3\delta h_2(b) + 2(\delta - 1) h_0(b)} \tag{7.07}$$

$$B_0' = 3\delta \frac{\lambda + 2\mu}{3\lambda + 2\mu} \tag{7.08}$$

$$B_1' = \frac{a_1}{a_1'} \frac{3\delta h_2(b)}{3\delta h_2(b) + 2(\delta - 1) h_0(b)} \tag{7.09}$$

$$C_0 = \frac{\alpha_1}{\alpha_2} \frac{a_2' j_1(a_2') \left[3\delta \frac{\alpha_1'}{\alpha_1} \frac{\lambda + 2\mu}{3\lambda + 2\mu} - 1 \right]}{a_2' j_1(a_2') h_0(a_2) - \chi a_2 h_1(a_2) j_0(a_2')} \tag{7.10}$$

$$C_0' = \frac{\alpha_1}{\alpha_2'} \frac{\chi a_2 h_1(a_2) \left[3\delta \frac{\alpha_1'}{\alpha_1} \frac{\lambda + 2\mu}{3\lambda + 2\mu} - 1 \right]}{a_2' j_1(a_2') h_0(a_2) - \chi a_2 h_1(a_2) j_0(a_2')} \tag{7.11}$$

C_1 and C_1' are identical in form with (6.19) and (6.21). We see that B_0, B_1, B_0', B_1' are identical with Epstein's result so that here again the only new thing is the presence of the conduction wave.

VIII. ENERGY RELATIONS

Since the primary object of this investigation is to calculate the energy loss, or extinction, due to small spherical obstacles in the path of a sound wave, we must now turn our attention to energy considerations.

(a) Viscous Fluid

Assume at first that the obstacle is not present, so that we have a homogeneous fluid and consider a volume V of the fluid. Let us multiply (2.05) by dV and integrate over the volume of interest. Then:

$$\int_V \frac{d}{dt}(\rho c_v T)\,dV = \int_V \left[\left(Tc_v + \frac{T}{\rho}\frac{\partial \rho}{\partial T}\right)\frac{d\rho}{dt} + \sigma\nabla^2 T + F\right]dV$$

and using Gauss's Theorem and the Equation of continuity, this becomes:

$$\int_V \frac{d}{dt}(\rho c_v T)\,dV = \int_V \left[-\left(\rho c_v + \frac{\partial \rho}{\partial T}\right)T\nabla\cdot\underset{\sim}{v} + F\right]dV + \oint_S \sigma\frac{\partial T}{\partial n}\,dS \qquad (8.01)$$

where S is the surface enclosing V. In II we neglected second order terms. This accuracy was sufficient for the equations of motion but is evidently inadequate when energy relations are to be considered, and we therefore cannot make the same approximations here.

If we now consider a periodic state and average (8.01) over a period, then the conduction term on the right will give no contribution. Writing $\frac{dE}{dt}$ for the average of the left side, we obtain:

$$\frac{dE}{dt} = \left[-\int_V \left(\rho c_v + \frac{\partial p}{\partial T}\right) T \nabla \cdot \underset{\sim}{v}\, dV\right]_{av.} + \left[\int_V F\, dV\right]_{av.} \tag{8.02}$$

We must now turn our attention to the quantity F which represents the rate at which energy is dissipated per unit volume due to viscosity. Multiply (2.01) by $v_i dV$ and (2.04) by $\frac{1}{2}v^2 dV$, add and integrate over V. Then:

$$\int_V \left[\rho \underset{\sim}{v} \cdot \frac{\partial \underset{\sim}{v}}{\partial t} + \rho \underset{\sim}{v} \cdot (\underset{\sim}{v} \cdot \nabla) \underset{\sim}{v} + \frac{1}{2} v^2 \frac{\partial \rho}{\partial t} + \frac{1}{2} v^2 \nabla \cdot (\rho \underset{\sim}{v}) + v \cdot \nabla p - \sum_{i,j} v_i \nabla_j \tau_{ij}\right] dV = 0$$

which can be transformed into:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_V \rho v^2 dV + \int_V \underset{\sim}{v} \cdot \nabla p\, dV - \int_V \sum_{i,j} v_i \nabla_j \tau_{ij}\, dV = -\frac{1}{2}\oint_S \rho v^2 v_n\, dS \tag{8.03}$$

Let us now define the quantity f by:

$$f = \int \frac{dp}{\rho}$$

Then:

$$\nabla p = \rho \nabla f$$

so that:

$$\int_V \underset{\sim}{v} \cdot \nabla p\, dV = \int_V \rho \underset{\sim}{v} \cdot \nabla f\, dV = \int_V \left[\nabla \cdot (\rho \underset{\sim}{v} f) - f \nabla \cdot (\rho \underset{\sim}{v})\right] dV =$$

$$= \oint_S \rho f v_n\, dS + \int_V f \frac{\partial \rho}{\partial t}\, dV \tag{8.04}$$

For the viscous term we obtain:

$$\int_V v_i \nabla_j \tau_{ij}\, dV = \oint_S v_n \tau_{nn}\, dS - \int_V \sum_{i,j} \tau_{ij} \frac{\partial v_i}{\partial x_j}\, dV =$$

$$= \oint_S v_n \tau_{nn}\, dS - \int_V \sum_i \tau_{ii} \frac{\partial v_i}{\partial x_i}\, dV - \frac{1}{2} \int_V {\sum_{i,j}}' \tau_{ij} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) dV$$

$$= \oint_S v_n \tau_{nn}\, dS - \int_V \sum_i \tau_{ii} \frac{\partial v_i}{\partial x_i}\, dV - \frac{1}{2\eta} \int_V {\sum_{i,j}}' \tau_{ij}\, \tau_{ij}\, dV \qquad (8.05)$$

where we have used (2.02). But, again using (2.02):

$$\sum_i \tau_{ii} \frac{\partial v_i}{\partial x_i} = \sum_i \left[2\eta \left(\frac{\partial v_i}{\partial x_i} \right)^2 - \frac{2}{3}\eta (\nabla \cdot \underset{\sim}{v}) \frac{\partial v_i}{\partial x_i} \right] = 2\eta \sum_i \left(\frac{\partial v_i}{\partial x_i} \right)^2 - \frac{2}{3}\eta (\nabla \cdot \underset{\sim}{v})^2$$

$$\sum_i \tau_{ii}\, \tau_{ii} = \sum_i \left[4\eta^2 \left(\frac{\partial v_i}{\partial x_i} \right)^2 - \frac{8}{3}\eta^2 (\nabla \cdot \underset{\sim}{v}) \frac{\partial v_i}{\partial x_i} + \frac{4}{9}\eta^2 (\nabla \cdot \underset{\sim}{v})^2 \right] =$$

$$= 4\eta^2 \sum_i \left(\frac{\partial v_i}{\partial x_i} \right)^2 - \frac{4}{3}\eta^2 (\nabla \cdot \underset{\sim}{v})^2$$

so that:

$$\sum_i \tau_{ii} \frac{\partial v_i}{\partial x_i} = \frac{1}{2\eta} \sum \tau_{ii}\, \tau_{ii}$$

and substituting this into (8.05) we obtain:

$$\int_V \sum_{i,j} v_i \nabla_j \tau_{ij}\, dV = \oint_S v_n \tau_{nn}\, dS - \frac{1}{2\eta} \int_V \sum_{i,j} \tau_{ij}\, \tau_{ij}\, dV \qquad (8.06)$$

Substituting (8.04) and (8.06) into (8.03), there results:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_V \rho v^2 dV + \frac{\partial}{\partial t}\int_V f\rho\, dV + \frac{1}{2\eta}\int_V \sum_{i,j} \tau_{ij}\tau_{ij}\, dV = \oint_S \left(\tau_{nn} - \rho f - \tfrac{1}{2}\rho v^2\right) v_n\, dS$$

Again considering a periodic state and averaging over a period, the first two terms on the left will give no contribution and we are left with:

$$\left[\frac{1}{2\eta}\int_V \sum_{i,j} \tau_{ij}\tau_{ij}\, dV\right]_{av} = \left[\oint_S \left(\tau_{nn} - \rho f - \tfrac{1}{2}\rho v^2\right) v_n\, dS\right]_{av.} \qquad (8.07)$$

This represents the rate at which energy is dissipated due to viscosity, so that we can identify:

$$\frac{1}{2\eta}\sum_{i,j} \tau_{ij}\tau_{ij} = F$$

Substituting, therefore, (8.07) into (8.02) we obtain:

$$\frac{dE}{dt} = \left[\oint_S \left(\tau_{nn} - \rho f - \tfrac{1}{2}\rho v^2\right) v_n\, dS - \int_V \left(\rho c_v + \frac{\partial p}{\partial T}\right) T\, \nabla\cdot \underset{\sim}{v}\, dV\right]_{av.} \qquad (8.08)$$

For the acoustic case, the velocities, condensations, and compressions are so small that their squares may be neglected. Hence:

$$\tau_{nn} - \rho f - \tfrac{1}{2}\rho v^2 \cong \tau_{nn} - p = p_{nn}$$

so that:

$$\frac{dE}{dt} = \left[\oint_S p_{nn} v_n\, dS - \int_V \left(\rho c_v + \frac{\partial p}{\partial T}\right) T\, \nabla\cdot \underset{\sim}{v}\, dV\right]_{av.} \qquad (8.09)$$

This equation represents the energy loss per unit time inside the volume V. If now, the system is inhomogeneous with surfaces of discontinuity between its homogeneous parts, (such as we have for the case of an obstacle) then the surface S will be the outer boundary surface enclosing the whole system since from the boundary conditions, p_{nn} and v_n are continuous. The volume integral must be evaluated over both the inner and the outer medium. We shall see in the next section that if thermal effects had been neglected, the volume integral would vanish, leaving a result identical with Epstein's (Equ. (59) of (3)). The volume integral in (8.09) thus represents the additional absorption due to thermal conduction.

(b) Elastic Solid

From (2.18) and (2.20) the equation of motion is:

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \nabla_j v_i = \nabla_j \tau_{ij} - \beta \nabla_i T$$

Multiply this by $v_i dV$ and the equation of continuity by $\frac{1}{2}v^2 dV$, add and integrate. Then:

$$\int_V \Big[\rho \underset{\sim}{v} \cdot \frac{\partial \underset{\sim}{v}}{\partial t} + \rho \underset{\sim}{v} \cdot (\underset{\sim}{v} \cdot \nabla) \underset{\sim}{v} + \frac{1}{2} v^2 \frac{\partial \rho}{\partial t} + \frac{1}{2} v^2 \nabla \cdot (\rho \underset{\sim}{v}) - \\ - \sum_{i,j} v_i \nabla_j \tau_{ij} + \beta \underset{\sim}{v} \cdot \nabla T \Big] dV = 0$$

which we can bring into the form:

$$\frac{1}{2} \frac{\partial}{\partial t} \int_V \rho v^2 dV - \int_V \Big(\sum_{i,j} v_i \nabla_j \tau_{ij} - \beta \underset{\sim}{v} \cdot \nabla T \Big) dV = -\frac{1}{2} \oint_S \rho v^2 v_n \, dS \qquad (8.10)$$

Now, using Gauss's Theorem, we can write:

$$\int_V \sum_{i,j} v_i \nabla_j \tau_{ij}\, dV = \oint_S v_n \tau_{nn}\, dS - \int_V \sum_{i,j} \tau_{ij} \nabla_j v_i\, dV$$

Let us define the following quantities:

$$e_{ii} = \frac{\partial s_i}{\partial x_i} \quad , \quad e_{ij} = \frac{\partial s_i}{\partial x_j} + \frac{\partial s_j}{\partial x_i} \tag{8.11}$$

Then, since τ_{ij} is symmetric, we have:

$$\sum_{i,j} \tau_{ij} \nabla_j v_i = \sum_{i,j} \tau_{ij} \frac{\partial e_{ij}}{\partial t}$$

Consider now the function W, defined by:

$$W = \tfrac{1}{2}\lambda\left(e_{xx} + e_{yy} + e_{zz}\right)^2 + \mu\left(e_{xx}^2 + e_{yy}^2 + e_{zz}^2\right) + \tfrac{1}{2}\mu\left(e_{xy}^2 + e_{yz}^2 + e_{xz}^2\right)$$

Then, we easily see from (8.11) and (2.18) that:

$$\frac{\partial W}{\partial e_{ij}} = \tau_{ij}$$

so that:

$$\sum_{i,j} \tau_{ij} \nabla_j v_i = \sum_{i,j} \frac{\partial W}{\partial e_{ij}} \frac{\partial e_{ij}}{\partial t} = \frac{\partial W}{\partial t}$$

Hence:

$$\int_V \sum_{i,j} v_i \nabla_j \tau_{ij}\, dV = \oint_S v_n \tau_{nn}\, dS - \frac{\partial}{\partial t}\int W\, dV$$

so that (8.10) becomes:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_V \rho v^2\, dV + \frac{\partial}{\partial t}\int_V W\, dV - \int_V \beta T \nabla\cdot \underset{\sim}{v}\, dV = \oint_S \left(\tau_{nn} - \beta T - \tfrac{1}{2}\rho v^2\right) v_n\, dS$$

Considering, as before, a periodic state and averaging over a period, the first two terms on the left will not contribute. Since, for the acoustic case the velocities will be small we obtain:

$$\left[-\int_V \beta T \nabla\cdot \underset{\sim}{v}\, dV\right]_{av.} = \left[\oint_S (\tau_{nn}-\beta T) v_n\, dS\right]_{av.} \tag{8.12}$$

This equation states that the average rate at which work is done on the boundary (right term) is equal to the average rate of production of heat inside. If we had not taken temperature effects into account, then the left side of (8.12) would have vanished and the term $\beta T v_n$ would not have appeared on the right, giving us the result that no energy is dissipated.

We can identify (8.12) with $\left[\int F\, dV\right]_{av.}$ of (8.02) so that (8.02) becomes:

$$\frac{dE}{dt} = \left[\oint_S (\tau_{nn}-\beta T) v_n\, dS - \int_V \left(\rho c_v + \frac{\partial \rho}{\partial T}\right) T \nabla\cdot \underset{\sim}{v}\, dV\right]_{av.} \tag{8.13}$$

This equation represents the energy loss per unit time inside the volume V. Let the system now be inhomogeneous and consider the case of an elastic solid sphere embedded in a viscous fluid. The energy loss inside the obstacle will then be given by (8.13) where the surface integral is taken over the surface of the obstacle. The energy loss in the fluid will be given by (8.09) which will now, however, involve two surface integrals: one integral over the surface of the obstacle and a second integral over a surface surrounding the whole system. The total energy loss will be given by the sum of (8.09) and (8.13). But since, from the boundary conditions, $\tau'_{nn} - \beta T' = p_{nn}$ on the boundary, the two integrals over the surface of the obstacle will cancel and we are left with

$$\frac{dE}{dt} = \left[\oint_S p_{nn} v_n \, dS - \int_V \left(\rho c_v + \frac{\partial \rho}{\partial T} \right) T \nabla \cdot \underset{\sim}{v} \, dV \right]_{av.} \tag{8.14}$$

where the surface integral is taken over the boundary of the whole system and the volume integral is evaluated over both the inner and the outer medium. We see that (8.14) is identical in form with (8.09). The various coefficients involved, however, are of course quite different.

Equs. (8.09) and (8.14) are the expressions we desired to obtain, since they will enable us to calculate the extinction of the incident sound wave.

IX. CALCULATION OF THE EXTINCTION

(a) Viscous fluid sphere

We must keep in mind that p_{nn}, T, and v_n in (8.09) stand for complex quantities, supplied with the time factor e^{-iwt}. In actually evaluating (8.09) we must replace these terms by their real parts.

Consider now two functions, f and g, both of which having an exponential time dependence. Then we can write:

$$\left[Re\, f \cdot Re\, g \right]_{av.} = \tfrac{1}{2} Re\,(f g^*)$$

Applying this to (8.09) we have:

$$\frac{dE}{dt} = \tfrac{1}{2} Re \oint_S p_{nn} v_n^* \, dS - \tfrac{1}{2} Re \int_V \left[\rho c_v + \left(\frac{\partial p}{\partial T} \right)_V \right] T \nabla \cdot \underset{\sim}{v}^* \, dV \tag{9.01}$$

We now take as our surface a sphere of radius large compared to the obstacle, and concentric with it so that the normal components become the radial components. We saw in VI that both the transverse wave (described by the vector potential $\underset{\sim}{A}$) and the conduction wave (described by the scalar potential ϕ_2) are absorbed very quickly. Hence, since the radius of our surface is large, we can in the surface integral of (9.01) neglect any terms which are due to either $\underset{\sim}{A}$ or ϕ_2 . Then, from (6.02) and (6.05):

$$p_{rr} \Big|_S \cong -p - 2\eta \frac{\partial^2 \phi_1}{\partial r^2} = K^2 \eta \beta_1 \phi_1 - 2\eta \frac{\partial^2 \phi_1}{\partial r^2} \cong i\omega \rho_0 \phi_1$$

since the second term is negligible compared to the first. So that:

$$\oint_S p_{nn} v_n^* dS = -i\omega\rho_0 \oint_S \phi_1 \frac{\partial \phi_1^*}{\partial n} dS \tag{9.02}$$

For the volume integral, we obtain from (5.04) and (5.03):

$$\mathrm{Re}\int_{\text{outside}} \left[\rho c_v + \left(\frac{\partial \rho}{\partial T}\right)_v\right] T \nabla\cdot \underset{\sim}{v}^* dV = -\left(\frac{\rho_0 c^2 \alpha_v}{\gamma} + \rho_0 c_v\right) \mathrm{Re}\int (\alpha_1\phi_1 + \alpha_2\phi_2)(\nabla^2\phi_1^* + \nabla^2\phi_2^*) dV =$$

$$= \left(\frac{\rho_0 c^2 \alpha_v}{\gamma} + \rho_0 c_v\right) \mathrm{Re}\int \left(\alpha_1 k_1^{*2}\phi_1\phi_1^* + \alpha_2 k_1^{*2}\phi_2\phi_1^* + \alpha_1 k_2^{*2}\phi_1\phi_2^* + \alpha_2 k_2^{*2}\phi_2\phi_2^*\right) dV$$

But α_1 and k_2^2 are imaginary while α_2 and k_1^2 are real. Thus, the first and last term are imaginary and will not contribute. Hence:

$$\mathrm{Re}\int \left[\rho c_v + \left(\frac{\partial \rho}{\partial T}\right)_v\right] T \nabla\cdot \underset{\sim}{v}\, dV = \left(\frac{\rho_0 c^2 \alpha_v}{\gamma} + \rho_0 c_v\right)\left(\alpha_2 k_1^2 + \alpha_1 k_2^{*2}\right) \mathrm{Re}\int \phi_1 \phi_2^* dV$$

But:

$$\alpha_2 k_1^2 + \alpha_1 k_2^{*2} = -\frac{\omega^2 \gamma^2}{\alpha_v \mathcal{H} c^2}$$

so that:

$$\mathrm{Re}\int_{\text{outside}} \left[\rho c_v + \left(\frac{\partial \rho}{\partial T}\right)_v\right] T \nabla\cdot \underset{\sim}{v}\, dV = -\frac{\rho_0 \omega^2 \gamma}{\mathcal{H}}\left(1 + \frac{\gamma c_v}{\alpha_v c^2}\right) \mathrm{Re}\int \phi_1 \phi_2^* dV \tag{9.03}$$

Similarly:

$$\mathrm{Re}\int_{\text{inside}} \left[\rho c_v + \left(\frac{\partial \rho}{\partial T}\right)_v\right] T \nabla\cdot \underset{\sim}{v}\, dV = -\frac{\rho_0' \omega^2 \gamma'}{\mathcal{H}'}\left(1 + \frac{\gamma' c_v'}{\alpha_v' c'^2}\right) \mathrm{Re}\int \phi_1' \phi_2'^* dV \tag{9.04}$$

Substituting (9.02) (9.03) and (9.04) into (9.01) we obtain:

$$\frac{dE}{dt} = -\frac{1}{2}\rho_0 \omega \, Re \, i \oint_S (\phi_i + \phi_1) \frac{\partial}{\partial r}(\phi_i^* + \phi_1^*) dS + \frac{\rho' \omega^2 \gamma'}{2\mathcal{H}'}\left(1 + \frac{\gamma' C_v'}{\alpha_V' C'^2}\right) Re \int_{inside} \phi_1' \phi_2'^* dV +$$

$$+ \frac{\rho \omega^2 \gamma}{2\mathcal{H}}\left(1 + \frac{\gamma C_v}{\alpha_V C^2}\right) Re \int_{outside} (\phi_i + \phi_1)\phi_2^* dV \qquad (9.05)$$

since outside the obstacle, we must identify ϕ_1 with $\phi_i + \phi_1$

Let n be the number of obstacles per unit volume. Then the total energy loss per unit time per unit volume will be $n\frac{dE}{dt}$, if the obstacles take up only a small fraction of the total volume. The relative energy loss or the extinction per unit length of path will be $\frac{n}{E_0}\frac{dE}{dt}$ where E_0 is the average energy which the incident wave carries across unit area per unit time.

From (9.01) we obtain:

$$E_0 = \frac{1}{2} Re \, p v_n^* = \frac{1}{2} Re \, i\omega\rho_0 \phi_i \frac{\partial \phi_i^*}{\partial x} = \frac{1}{2}\omega\rho_0 k_1$$

Denoting the extinction by α we then obtain from (9.05):

$$\alpha = \frac{n}{k_1}\left[-I_s + \frac{\rho' \omega \gamma'}{\rho \mathcal{H}'}\left(1 + \frac{\gamma' C_v'}{\alpha_V' C'^2}\right) I_{V_1} + \frac{\omega \gamma}{\mathcal{H}}\left(1 + \frac{\gamma C_v}{\alpha_V C^2}\right) I_{V_2}\right] \qquad (9.06)$$

where:

$$I_s = Re \, i \oint_S (\phi_i + \phi_1) \frac{\partial}{\partial r}(\phi_i^* + \phi_1^*) dS \, , \quad I_{V_1} = Re \int \phi_1' \phi_2'^* dV , \quad I_{V_2} = Re \int (\phi_i + \phi_1)\phi_2^* dV \qquad (9.07)$$

These integrals are evaluated in Appendix VIII. It is also shown there that the result for the first approximation to the extinction (9.06), neglecting terms of order a_1^2 and $a_1'^2$ compared to unity is as follows:

$$\alpha = \frac{4\pi n R^3}{k_1}\left\{-\frac{1}{R^3 k_1}\,\mathrm{Re}\,(3B_1) + \frac{\rho' w \gamma'}{\rho \mathcal{H}'}\left(1+\frac{\gamma' C_v'}{\alpha_v' c'^2}\right)\mathrm{Re}\,\frac{1}{a_2'}B_0' C_0' j_1(a_2') - \right.$$

$$\left. - \frac{\gamma w}{\mathcal{H}}\left(1+\frac{\gamma C_v}{\alpha_v c^2}\right)\mathrm{Re}\,\frac{1}{a_2}C_0 h_1(a_2)\right\} \qquad (9.08)$$

Since we have neglected the absorption of ϕ_i and ϕ_1 , α then represents the additional extinction due to the presence of the obstacles. The first term in (9.08) is identical with Epstein's result and represents the extinction due to viscosity alone. The remaining two terms come from the additional absorption due to thermal conduction.

We shall apply (9.08) to a few special cases in the next section.

(b) Elastic solid sphere

The only difference between this case and the one just considered lies in the volume integral over the obstacle, as can be seen from (8.14). Using (2.27) and (7.02) we obtain, omitting the primes:

$$\mathrm{Re}\int_{\text{inside}} T\nabla\cdot \underset{\sim}{v}^* dV = \mathrm{Re}\int T i\omega \nabla\cdot \underset{\sim}{s}^* dV =$$

$$= \omega^2\,\mathrm{Re}\int\left(\alpha_1 k_1^{*2}\phi_1\phi_1^* + \alpha_2 k_1^{*2}\phi_2\phi_1^* + \alpha_1 k_2^{*2}\phi_1\phi_2^* + \alpha_2 k_2^{*2}\phi_2\phi_2^*\right)dV$$

Since α_1, α_2 and k_1 are real while k_2^2 is imaginary, the last term will give no contribution, so that:

$$\mathrm{Re}\int T\nabla\cdot \underset{\sim}{v}^* dV = \omega^2 \mathrm{Re}\int \left[\alpha_1 k_1^2 \phi_1\phi_1^* + (\alpha_2 k_1^2 + \alpha_1 k_2^{*2})\phi_1\phi_2^*\right] dV$$

But:

$$\alpha_2 k_1^2 + \alpha_1 k_2^{*2} \cong -\frac{\rho_0 \omega^2}{\beta \mathcal{H}}$$

since for the frequency range we are interested in, the second term is very much smaller than the first. Also:

$$\alpha_1 k_1^2 = -\frac{\alpha_1}{\alpha_2}\frac{\rho_0\omega^2}{\beta\mathcal{H}}$$

so that:

$$\mathrm{Re}\int_{inside} T\nabla\cdot \underset{\sim}{v}^* dV = -\frac{\rho_0\omega^4}{\beta\mathcal{H}}\left[\frac{\alpha_1}{\alpha_2}\int \phi_1\phi_1^* dV + \mathrm{Re}\int \phi_1\phi_2^* dV\right]$$

and hence we obtain by similarity with (9.05):

$$\frac{dE}{dt} = -\tfrac{1}{2}\rho_0\omega\, \mathrm{Re}\, i \oint_S (\phi_i+\phi_1)\frac{\partial}{\partial n}(\phi_i^* + \phi_1^*) dS + \frac{\rho_0\omega^2\gamma}{2\mathcal{H}}\left(1+\frac{\gamma C_v}{\alpha_v c^2}\right)\mathrm{Re}\int (\phi_i+\phi_1)\phi_2^* dV +$$

$$+ \frac{\rho_0'\omega^4}{2\beta\mathcal{H}'}(\rho_0' C_v' + \alpha_v' \bar{K})\left[\frac{\alpha_1'}{\alpha_2'}\int \phi_1'\phi_1'^* dV + \mathrm{Re}\int \phi_1'\phi_2'^* dV\right] \qquad (9.09$$

where we have written $\alpha_v' \bar{K}$ for $\left(\frac{\partial p}{\partial T}\right)_v$, $\bar{K}$ being the isothermal bulk modulus.

Defining the extinction in the same manner as before, there results:

$$\alpha = \frac{n}{k_1}\left[-I_s + \frac{\omega\gamma}{\mathcal{H}}\left(1+\frac{\gamma C_v}{\alpha_\gamma C^2}\right)I_{V_2} + \frac{\rho'\omega^3}{\rho\beta\mathcal{H}'}\left(\rho' C_v' + \alpha_\gamma' K\right)\left(\frac{\alpha_1'}{\alpha_2'}I_{V_3} + I_{V_1}\right)\right] \tag{9.10}$$

where I_s, I_{V_1}, and I_{V_2} have been defined by (9.07) and:

$$I_{V_3} = \int \phi_1' \phi_1'^{*} dV \tag{9.11}$$

It is shown in Appendix IX that the first approximation to the extinction, neglecting terms of order a and a compared to unity is as follows:

$$\alpha = \frac{4\pi n R^3}{k_1}\left\{-\frac{1}{R^3 k_1} Re\,(3B_1) + \frac{\rho'\omega}{\rho\beta\mathcal{H}'}\left(\rho' C_v' + \alpha_\gamma' K\right)\left(\frac{1}{3}\frac{\alpha_1'}{\alpha_2'}B_0'^2 + Re\,\frac{1}{a_2'}B_0' C_0' j_1(a_2')\right) - \right.$$

$$\left. - \frac{\gamma\omega}{\mathcal{H}}\left(1+\frac{\gamma C_v}{\alpha C^2}\right) Re\,\frac{1}{a_2} C_0 h_1(a_2)\right\} \tag{9.12}$$

Since we have again neglected the absorption of ϕ_i and ϕ_1, α will represent the additional extinction due to the presence of the obstacles.

We shall apply (9.12) to a few special cases in the next section.

X. APPLICATIONS

Since equs. (9.08) and (9.12) are still rather complicated formulas when the expressions for the coefficients are substituted, it is best to consider a few special cases which are of practical importance, and for which some further simplifications can be made.

(a) Water drops in air

This case is evidently of great importance in connection with the absorption of sound in fogs. Since the average radius of the water drops in fogs is about 10^{-3} cm (10), we see that our assumption of $a_1 = \frac{wR}{c} \ll 1$ will permit us to go up to $w \sim 10^6 \text{sec}^{-1}$ without violating any of our approximations.

The primed quantities will pertain to water and the unprimed ones to air. Since water is much more viscous, much heavier and a much better thermal conductor than air, the parameters ϵ, δ and χ will be very small. The actual values are:

$$\epsilon = .0167\,, \quad \delta = .00117\,, \quad \chi = .044$$

Under these conditions we obtain:

$$B_1 \cong \frac{1}{3} i a_1^3 G_{B_1}\,, \quad B_0' = \delta\,, \quad C_0 \cong -\frac{\alpha_1}{\alpha_2} \frac{a_2' j_1(a_2')}{a_2' j_1(a_2') h_0(a_2) - \chi a_2 j_0(a_2') h_1(a_2)}$$

$$C_0' \cong -\frac{\alpha_1}{\alpha_2} \frac{\alpha_r' \gamma c_v}{\alpha_r \gamma' c_v'} \delta \frac{a_2 h_1(a_2)}{a_2' j_1(a_2') h_0(a_2) - \chi a_2 j_0(a_2') h_1(a_2)} \tag{10.01}$$

Even though χ is small, we cannot neglect the second term in the denominator of C_0 and C_0' because for very low frequencies, when a_2 and a_2' are also small, we can see from Appendix III that the first term will be proportional to $\sqrt{\omega}$ while the second term will be proportional to $1/\sqrt{\omega}$ so that as $w \to 0$, the second term will eventually predominate even though $\chi \ll 1$.

Substituting (10.01) into (9.08) we obtain, after some slight rearranging:

$$\alpha = \frac{4\pi n R^3}{k_1}\left\{-\frac{a_1^3}{k_1 R^3}\,\mathrm{Re}\left(iG_{B_1}\right) - \left[\frac{\gamma'\omega}{\mathcal{H}'}\left(1+\frac{\gamma' C_v'}{\alpha_v' C'^2}\right)\frac{\alpha_v'\gamma C_v}{\alpha_v\gamma' C_v'}\,\delta\,\frac{a_2^2}{a_2'^2} - \right.\right.$$

$$\left.\left. - \frac{\gamma\omega}{\mathcal{H}}\left(1+\frac{\gamma C_v}{\alpha C^2}\right)\right]\mathrm{Re}\cdot\frac{\alpha_1}{a_2\alpha_2}\,\frac{a_2' h_1(a_2) j_1(a_2')}{a_2' j_1(a_2') h_0(a_2) - \chi a_2 j_0(a_2') h_1(a_2)}\right\}$$

But:

$$\frac{\gamma' C_v'}{\alpha_v' C'^2} \sim 200 \gg 1, \quad \frac{\gamma C_v}{\alpha_v C^2} \sim 3$$

Hence, we can approximate the bracketed factor in the second term by:

$$\frac{\gamma\omega}{\mathcal{H}}\left[\frac{\gamma C_v}{\alpha_v C^2}\,\delta\left(\frac{c}{c'}\right)^2 - \left(1+\frac{\gamma C_v}{\alpha_v C^2}\right)\right] \cong -\frac{\gamma\omega}{\mathcal{H}}\left(1+\frac{\gamma C_v}{\alpha_v C^2}\right)$$

since $\delta\left(\frac{c}{c'}\right)^2 \ll 1$, in our case. This essentially means that we could have neglected the term arising from the integration inside the drop. Using (5.07) we then obtain:

$$\alpha = \frac{4\pi n R^3 \omega}{C} \Big\{ - Re\,(i\,G_{B_1}) +$$

$$+ (\gamma - 1)\left(1 + \frac{\gamma C_v}{\alpha_v c^2}\right) Re \cdot \frac{i}{a_2} \frac{a_2' j_1(a_2') h_1(a_2)}{a_2' j_1(a_2') h_0(a_2) - \chi a_2 j_0(a_2') h_1(a_2)} \Big\} \qquad (10.02)$$

We must now investigate what happens to G_{B_1} if ϵ and δ are small. Under these conditions, (6.22) becomes:

$$G_{00}^{B_1} = 1 \;,\quad G_{01}^{B_1} = -3 \;,\quad G_{10}^{B_1} = -\tfrac{1}{2}(6 - b'^2) \;,\quad G_{10}^{B_1} = \tfrac{3}{2}(6 - b'^2)$$

$$G_{00} = 2 \;,\quad G_{01} = -9\delta \;,\quad G_{10} = b'^2 - 6 \;,\quad G_{11} = 27\delta$$

Here, again, we must keep G_{01} and G_{11} even though $\delta \ll 1$, because for very low frequencies, b and b' will both be small, so that the first and third term in the denominator of (6.15) will be proportional to $\sqrt{\omega}$ while the second and fourth term (those involving G_{01} and G_{11}) will be proportional to $1/\sqrt{\omega}$, so that as $w \to 0$, the G_{01} and G_{11} terms will eventually predominate even though $\delta \ll 1$. Using Appendix III and the calculations of Appendix VI we obtain, for the case that ϵ and δ are small:

$$G_{B_1} = -\frac{h_2(b)}{2h_0(b)} \frac{1}{1 + \frac{9}{2}\delta \frac{h_1(b)}{b h_0(b)} \frac{j_2(b')}{\frac{1}{2} b' j_1(b') - j_2(b')}} \qquad (10.03)$$

The difference between this and the result of Appendix VI consists of the fact that here δ is small whereas in Appendix VII it was identically zero since we wanted to go back to the case considered in V.

Substituting (10.03) into (10.02) we obtain the rather complicated expression:

$$\alpha = \frac{4\pi n R^3 \omega}{c}\left\{\frac{1}{2}\,Re\cdot i\,\frac{h_2(b)}{h_0(b)}\;\frac{1}{1+\frac{9}{2}\delta\,\frac{h_1(b)}{b\,h_0(b)}\;\frac{j_2(b')}{\frac{1}{2}b' j_1(b')-j_2(b')}}\;+\right.$$

$$\left.+(\gamma-1)\left(1+\frac{\chi c_v}{\alpha_v c^2}\right)Re\cdot i\,\frac{h_1(a_2)}{a_2 h_0(a_2)}\;\frac{1}{1-\chi\,\frac{j_0(a_2')}{a_2' j_1(a_2')}\cdot\frac{a_2 h_1(a_2)}{h_0(a_2)}}\right\} \qquad (10.04)$$

It would be very cumbersome to evaluate these real parts exactly, but fortunately this is not necessary in our case. Due to the smallness of δ and χ we see that the second term in each denominator will be of importance only when b^2 is smaller or of the same order as δ , and when $a_2'^2$ is smaller or of the same order as χ. We can therefore replace the second term in the denominators by their assymptotic expressions for the case of b, b', a_2, a_2' small. This will of course not hold for intermediate or high frequencies, but then the whole second term is so small anyway, that it is immaterial what form we assume for it. Subject to these approximations, the real parts in (10.04) can then be easily evaluated and the calculations are indicated in

Appendix X. The resulting expression for (10.04) is:

$$\alpha = \frac{4\pi n R^2}{c}\left\{ \frac{3/2}{1+\left(\frac{3\delta\nu}{R^2\omega}\right)^2}\left[\frac{\nu}{R}+\sqrt{\frac{\nu\omega}{2}}\right] + \frac{(\gamma-1)\left(1+\frac{\gamma c_v}{\alpha_\gamma c^2}\right)}{1+\left(\frac{3\chi\mathcal{H}'}{R^2\omega\gamma'}\right)^2}\left[\frac{\mathcal{H}}{R\gamma}+\sqrt{\frac{\mathcal{H}\omega}{2\gamma}}\right]\right\} \qquad (10.05)$$

The first term represents the extinction due to viscosity and the second term is the additional extinction due to thermal conduction. As $\mathcal{H}\rightarrow 0$, only the first term remains and our result becomes identical with Epstein's*. If in addition, the frequency is high enough so that $\left(\frac{3\delta\nu}{R^2\omega}\right)^2 \ll 1$, then we are left with Sewell's result. We see that (10.05) gives zero extinction for zero frequency which eliminates a bothersome point in Sewell's theory. Since in Sewell's case, δ was not small, but identically zero, the denominator of the viscous term was unity and hence Sewell came out with a finite attenuation at zero frequency. This difference in behavior, as pointed out by Epstein, is due to the oscillations of the droplets, which were not taken into account by Sewell. These oscillations will become more important at the low frequency end, and less important the heavier the droplets are compared to the surrounding medium. These conclusions are borne out by (10.05). Since $\chi\mathcal{H}' = \mathcal{H}\delta\frac{c_v}{c_v'}$ we see that the deonominator of the conduction term in (10.05) is also due to the finite density of drops and is therefore connected with their oscillations rather than with thermal conduction.

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*Epstein considered frequencies that were high enough so that $(3\delta\nu/R^2\omega)^2 \ll 1$, in obtaining his equ. (69) from (60), and thus came out with Sewell's result for the special case of fogs.

We note that α/n has the dimensions of an area, and it represents the absorption cross-section per drop. Denoting it by σ_a we obtain from (10.05):

$$\sigma_a = \frac{4\pi R}{c}\left[\frac{3/2}{1+\left(\frac{3\delta\nu}{R^2\omega}\right)^2}\left(\nu+\sqrt{\frac{R^2\omega\nu}{2}}\right)+\frac{(\gamma-1)\left(1+\frac{\gamma c_v}{\alpha_v c^2}\right)}{1+\left(\frac{3\chi\varkappa'}{R^2\omega\gamma'}\right)^2}\left(\frac{\varkappa}{\gamma}+\sqrt{\frac{R^2\omega\varkappa}{2\gamma}}\right)\right] \quad (10.06)$$

If we now plot $\sigma_a \frac{c}{4\pi R}$ against R^2w we obtain a universal plot for all R, which is quite an advantage. This is done on Graph I, where we have also plotted Sewell's and Epstein's results for comparison. We see that for $R^2w \gtrsim 10^{-3}\text{cm}^2\text{sec}^{-1}$, the inclusion of thermal effects just about doubles the extinction that would have been obtained if viscosity alone had been taken into account. We also observe that for $R^2w \lesssim 10^{-2}$ (which corresponds to $w \lesssim 10^4\text{sec}^{-1}$ if $R \sim 10^{-3}$cm) the absorption is much more strongly frequency dependent than would appear from Sewell's result.

We shall compare (10.05) with experimental findings in the next section.

(b) Air bubbles in water

A great deal of experimental work was done during the war in connection with the absorption of sound in water due to air bubbles because of its bearing on submarine detection. Recently, a paper by Loye and Arndt appeared on the acoustical insulation of a submarine repair dock by an air bubble screen across the open end of the dock (11), and it was found to be extremely effective. It would therefore be of interest to apply our equations to this particular case.

Here we have the opposite situation to the one just considered. The primed quantities refer to air and the unprimed ones to water. If ϵ, δ and χ become large then, from (6.22):

$$G_{00}^{B_1} = -\epsilon \,, \quad G_{01}^{B_1} = \frac{\epsilon}{2}(6 - b^2) \,, \quad G_{10}^{B_1} = 3\epsilon \,, \quad G_{11}^{B_1} = -\frac{3\epsilon}{2}(6 - b^2)$$

$$G_{00} = -\epsilon\delta \,, \quad G_{01} = \delta\epsilon\left(9 - \frac{b^2}{2}\right) \,, \quad G_{10} = 3\epsilon\delta \,, \quad G_{11} = -3\delta\epsilon\left(9 - \frac{b^2}{2}\right)$$

Substituting this into (6.15) and using the relations of Appendix III, we obtain:

$$B_1 \cong -\frac{1}{3} i a_1^3 \frac{\frac{1}{2} b^2 h_1(b) - b h_2(b)}{(\frac{1}{2} b^2 - 6) h_1(b) - b h_2(b)} \tag{10.07}$$

Also from (6.16) (6.18) and (6.20) we obtain, if δ and χ are large:

$$B_0' = \delta \,, \quad C_0' \cong -\frac{\alpha_1'}{\alpha_2'} \frac{\delta}{j_0(a_2')} \,, \qquad C_0 \cong -\frac{\alpha_1'}{\alpha_2'} \frac{\alpha_\gamma \gamma' c_v'}{\alpha_\gamma' \gamma c_v} \frac{a_2' j_1(a_2')}{a_2 j_0(a_2') h_1(a_2)} \tag{10.8}$$

If we now substitute (10.07) and (10.08) into (9.08) and make use of the fact that $\delta \gg \frac{\gamma'}{\gamma}\left(\frac{C'}{C}\right)^2$, it can be shown that the term in (9.08) arising from the volume integral over the outer region is negligible. We then obtain:

$$\alpha = \frac{4\pi n R^3 w}{c}\left[Re \cdot i \frac{\frac{1}{2} b^2 h_1(b) - b h_2(b)}{(\frac{1}{2} b^2 - 6) h_1(b) - b h_2(b)} - \delta\left(\frac{c}{c'}\right)^2 (\gamma' - 1)\left(1 + \frac{\gamma' c_v'}{\alpha_\gamma' c'^2}\right) Re \frac{i j_1(a_2')}{a_2' j_0(a_2')}\right]$$

Closer analysis reveals that for the case we are considering, the first term is negligible compared to the second, since $\delta\left(\frac{c}{c'}\right)^2 \gg 1$. This essentially means that the main energy loss occurs inside the bubble and is due to thermal conduction. We then obtain:

$$\alpha = -\frac{4\pi n R^3 \omega}{c}\,\delta\left(\frac{c}{c'}\right)^2(\gamma'-1)\left(1+\frac{\gamma' c_0'}{\alpha_V' c'^2}\right)\mathrm{Re}\,\frac{i j_1(a_2')}{a_2' j_0(a_2')} \qquad (10.09)$$

The real part is evaluated in Appendix XI and the final result is:

$$\alpha = \frac{2\pi n R \varkappa'}{c}\,\delta\left(\frac{c}{c'}\right)^2\left(1-\frac{1}{\gamma'}\right)\left(1+\frac{\gamma' c_0'}{\alpha_V' c'^2}\right)\left(z\,\frac{\sinh 2z+\sin 2z}{\cosh^2 z-\cos^2 z}-2\right) \qquad (10.10)$$

where:

$$z = R\sqrt{\frac{\omega\gamma'}{2\varkappa'}}$$

Thus, the absorption cross-section per bubble turns out to be:

$$\sigma_a = \frac{2\pi R}{c}\cdot A\cdot F(z) \qquad (10.11)$$

where:

$$A = \varkappa'\delta\left(\frac{c}{c'}\right)^2\left(1-\frac{1}{\gamma'}\right)\left(1+\frac{\gamma' c_0'}{\alpha_V' c'^2}\right) \qquad (10.12)$$

$$F(z) = z\,\frac{\sinh 2z+\sin 2z}{\cosh^2 z-\cos^2 z}-2 \qquad (10.13)$$

so that if we plot $\sigma_a \frac{c}{2\pi R A}$ against z we obtain again a universal curve for all R. This is done on Graph II. It can be shown that for $z \lesssim \frac{1}{2}$, F(z) is proportional to w^2 while for $z \gtrsim 3$, F(z) is proportional to $\sqrt{\omega}$. Thus, as in the previous case, we have a strong frequency dependence for low frequencies.

Whereas in the case of fogs, the extinction due to viscosity and thermal conduction are of the same order of magnitude, for the case of air bubbles in water the extinction due to thermal conduction is the dominant term, which is a very interesting result.

Assuming reasonable values for R, we see that the extinction calculated by (10.10) will be very large, indicating that an air bubble screen for purposes of acoustic insulation would be very effective.

We shall compare (10.10) with experimental measurements in the next section.

(c) Elastic solid obstacles suspended in air.

In this case δ will be evidently small, and since most solids are much better thermal conductors than air, χ will also be small. By analogy with part (a) of this section, the second term in (9.12) (arising from the integral over the volume of the obstacle) will be negligible, so that we come out with a result identical with (10.05). Hence, to our approximation, the influence of elasticity is imperceptible.

XI. COMPARISON WITH EXPERIMENT

The absorption of sound in fogs has been determined by many observers, but in most cases an analysis of the water drops as to distribution in size and number was not attempted. This, of course, makes it very difficult to compare the measurements with theory. Recently, however, Knudsen (5) made some new measurements, using the reverberation chamber technique and he also determined the size-number relationship for the droplets used.

If V_s represents the total volume occupied by the obstacles per unit volume (V_s is thus dimensionless) then (10.05) becomes:

$$\alpha = \frac{3V_s}{cR^2}\left[\frac{3/2}{1+\left(\frac{3\delta\nu}{R^2\omega}\right)^2}\left(\nu+\sqrt{\frac{R^2\omega\nu}{2}}\right)+\frac{(\gamma-1)\left(1+\frac{\gamma c_v}{\alpha_1 c^2}\right)}{1+\left(\frac{3\chi\mathcal{H}'}{R^2\omega\gamma'}\right)^2}\left(\frac{\mathcal{H}}{\gamma}+\sqrt{\frac{R^2\omega\mathcal{H}}{2\gamma}}\right)\right] \qquad (11.01)$$

Now, Knudsen split his drops into five groups according to size and determined the number of drops in each group. The quantity V_s(pertaining to all the drops) was also determined. We can then calculate the extinction for each of the five radii with the same V_s for each and then take a weighted average according to volume. The resulting curve is shown on Graph III where we have plotted the extinction (in db/m) against frequency (in cycles per second). Knudsen's experimental values are also indicated. We see that the agreement is rather good at the higher frequencies, but the points for frequencies of 500 and 1000 cps fall slightly below the calculated curve. This may be due to the fact that at the low frequency end the errors in measurement tend to increase. The dotted curve on Graph III indicates the extinction we would have obtained if thermal effects had been neglected, which would give values too small by about a factor of two. Thus, the inclusion of thermal conduction seems definitely to be a step in the right direction toward a better understanding of the absorption of sound in fogs.

The data taken by Loye and Arndt (11) does not lend itself readily to comparison with our theory since no analysis of the bubbles as to size and number was made. The bubbles were produced by compressed air being passed through a hollow pipe with holes drilled in it at regular intervals. If the airflow is kept fixed then n will be roughly inversely proportional to R^3 so that the factor in front of $F(z)$ in (10.10) will be inversely proportional to R^2. Now if $z \sim 3$ or greater, $F(z)$ is proportional to R so that the extinction will decrease inversely as R. It would thus be advantageous for purposes of acoustic insulation to use a larger number of small holes, rather than a smaller number of big ones. This effect was actually observed, but undoubtedly it was also partly due to the fact that the bubbles made less noise upon emergence when they were small. On the other hand, when $z \sim \frac{1}{2}$ or less, then $F(z)$ is proportional to R^4 so that the extinction will be proportional to R^2. It therefore appears that for a given air flow there exists an optimum size for the holes for maximum absorption.

Hartmann and Focke (4) measured the extinction of supersonics in aqueous suspensions of lycopodium particles of density $\rho' = 1.00$ gr/cm^2. In this case $\delta = 1$ so that from (7.07) $B_1 = 0$ and thus the first order extinction due to viscosity will vanish, but the extinction due to thermal conductivity will not. Since in Epstein's case the additional term due to thermal conductivity did not appear, he calculated the second order terms in the extinction due to viscosity and added the additional energy loss due to scattering but came out with a result that was still about 10 times too small. Since in our case the first order extinction due to thermal conduction still remains, it may be that this additional term will increase the calculated values by about the right amount. Unfortunately, however, certain constants of lycopodium which we must know for numerical computation have not been determined, so that it is not possible to compare our results with these particular measurements.

PART II

This part contains the various mathematical appendices referred to in part I. We shall here adopt the same system for numbering equations. If reference is made to an equation of part II, the number will be followed by the letter A (such as 3.04A). If the letter A does not appear, it is to be understood that the reference is made to an equation in part I.

APPENDIX I

The solution of (3.02) is:

$$k^2 = \frac{\frac{4}{3}i\omega\nu + i\omega\mathcal{H} - c^2 \pm \left[\left(\frac{4}{3}i\omega\nu + i\omega\mathcal{H} - c^2\right)^2 + 4\mathcal{H}\omega^2\left(\frac{4}{3}\nu + \frac{ic^2}{\omega\gamma}\right)\right]^{1/2}}{2\mathcal{H}\left(\frac{4}{3}\nu + \frac{ic^2}{\omega\gamma}\right)} \tag{1.01}$$

Let:

$$A = \frac{4}{3}\omega\nu + \omega\mathcal{H} \quad , \quad B = c^2$$
$$C = \frac{16}{3}\mathcal{H}\omega^2\nu \quad , \quad D = \frac{4\mathcal{H}\omega c^2}{\gamma} \tag{1.02}$$

Then we can write (1.02A) as follows:

$$k^2 = \frac{2\omega^2(iA-B)}{C+iD}\left[1 \pm \left(1 + \frac{C+iD}{(iA-B)^2}\right)^{1/2}\right]$$

It turns out that as long as $w \leqslant 10^7 \sec^{-1}$, then we can write with sufficient accuracy:

$$\left|\frac{C+iD}{(iA-B)^2}\right| \cong \frac{D}{B^2} \ll 1$$

since $A \ll B$, $C \ll D$. (See Table I for actual values in the case of air and water). Since in our later applications we shall not go to frequencies beyond $w = 10^6 \sec^{-1}$, we can expand the square root and, taking the lower sign, we obtain:

$$k_1^2 \cong -\frac{\omega^2}{iA-B} + \frac{\omega^2(C+iD)}{4(iA-B)^3} \cong \frac{\omega^2}{B}\left(1 + i\frac{A}{B} - i\frac{D}{4B^2}\right) \tag{1.03}$$

using (1.02A) this becomes:

$$k_1^2 \cong \frac{\omega^2}{c^2}\left[1 + \frac{i\omega}{c^2}\left(\frac{4}{3}\nu + \mathcal{H} - \frac{\mathcal{H}}{\gamma}\right)\right]\cdot$$

Since the imaginary term is small, we have:

$$k_1 \cong \frac{\omega}{c}\left[1+\frac{i\omega}{c^2}\left(\frac{2}{3}\nu+\frac{\mathcal{H}}{2}\left(1-\frac{1}{\gamma}\right)\right)\right] \qquad (1.04)$$

Combining (1.03A) with (3.01) there results

$$k_2^2 \cong -\frac{\omega^2}{\frac{1}{4\omega^2}(C+iD)\frac{\omega^2}{B}\left(1+i\frac{A}{B}-i\frac{D}{4B^2}\right)}$$

$$\cong \frac{4i\omega^2 B}{D}\left[1+i\left(\frac{D}{4B^2}+\frac{C}{D}-\frac{A}{B}\right)\right]$$

In (1.03A) we had to keep terms of order A/B and D/B^2 since they were the only imaginary terms we had (indicating absorption) but for k_2^2 this is not necessary. Hence:

$$k_2^2 \cong \frac{4i\omega^2 B}{D} = \frac{i\omega\gamma}{\mathcal{H}}$$

so that:

$$k_2 = (1+i)\sqrt{\frac{\omega\gamma}{2\mathcal{H}}} \qquad (1.05)$$

(1.04A) and (1.05A) are the desired equations.

APPENDIX II

The solution of (3.06) is:

$$k^2 = \frac{\mathcal{H}\rho_0 w^2 + iw(\lambda+2\mu+M) \pm \left\{\left[\mathcal{H}\rho_0 w^2 + iw(\lambda+2\mu+M)\right]^2 - 4\mathcal{H}i\rho_0 w^3(\lambda+2\mu)\right\}^{1/2}}{2\mathcal{H}(\lambda+2\mu)}$$

This can be written as:

$$k^2 = \frac{\mathcal{H}\rho_0 w^2 + iw(\lambda+2\mu+M) \pm iw(\lambda+2\mu+M)\left\{1 + \frac{2iw\mathcal{H}\rho_0(\lambda+2\mu)}{(\lambda+2\mu+M)^2} - \frac{2iw\mathcal{H}\rho_0 M}{(\lambda+2\mu+M)^2} - \frac{\mathcal{H}^2\rho_0^2 w^2}{(\lambda+2\mu+M)^2}\right\}^{1/2}}{2\mathcal{H}(\lambda+2\mu)} \tag{2.01}$$

We can estimate the magnitude of M in the following way: we know that:

$$\left(\frac{\partial p}{\partial T}\right)_V = -\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \cdot V\left(\frac{\partial p}{\partial V}\right)_T = \alpha_V \mathbb{K}$$

where $\mathbb{K}$ is the isothermal bulk modulus. For most solids:

$\alpha_V \sim 10^{-5}\ {}^\circ C^{-1}$, $\mathbb{K} \sim 10^{12}$ dynes-cm^{-2}, so that $\left(\frac{\partial p}{\partial T}\right)_V \sim 10^7$

Since for most solids λ and μ are of order 10^{11}, and C_V is of order 10^8 ergs - C^{-1} - gr^{-1} we see from the definition of M that M is of order 10^7, so that $M \ll \lambda$ or μ. Now consider the terms inside the square root. Since $\mathcal{H}$ and ρ are of order 1 for most solids, then at $w = 10^6$ the terms inside the square root are of order 10^{-5}, 10^{-9}, 10^{-10} respectively, so that we can expand the square root. Then, (2.01A) becomes:

$$k^2 = \left[2\mathcal{H}(\lambda+2\mu)\right]^{-1}\left[\mathcal{H}\rho_0 w^2 + iw(\lambda+2\mu+M) \pm iw(\lambda+2\mu+M) \mp \frac{w^2\mathcal{H}\rho_0(\lambda+2\mu)}{\lambda+2\mu+M} \pm \right.$$

$$\left. \pm \frac{w^2\mathcal{H}\rho_0 M}{\lambda+2\mu+M} \mp \frac{iw^3\mathcal{H}^2\rho_0^2}{2(\lambda+2\mu+M)} \pm \frac{iw^3\mathcal{H}^2\rho_0^2(\lambda+2\mu)^2}{2(\lambda+2\mu+M)^3}\right] \tag{2.02}$$

Taking the lower sign, this becomes:

$$k_1^{\,2} \cong \left[2\mathcal{H}(\lambda+2\mu)\right]^{-1}\left[\mathcal{H}\rho_0\omega^2\left(1+\frac{\lambda+2\mu-M}{\lambda+2\mu+M}\right)+\frac{i\omega^3\mathcal{H}^2\rho_0^{\,2}}{2(\lambda+2\mu+M)}\left(1-\frac{(\lambda+2\mu)^2}{(\lambda+2\mu+M)^2}\right)\right]$$

Since $M \ll \lambda$ or μ, we have:

$$1+\frac{\lambda+2\mu-M}{\lambda+2\mu+M} \cong 2\,, \quad 1-\frac{(\lambda+2\mu)^2}{(\lambda+2\mu+M)^2} \cong \frac{2M}{\lambda+2\mu}$$

so that:

$$k_1^{\,2} \cong \frac{\rho_0\omega^2+\frac{1}{2}\,\frac{i\omega^3\mathcal{H}\rho_0^{\,2}M}{(\lambda+2\mu)^2}}{\lambda+2\mu}$$

Since the imaginary term is small, we obtain:

$$k_1 \cong \omega\sqrt{\frac{\rho_0}{\lambda+2\mu}}\left(1+\frac{i\omega\mathcal{H}\rho_0 M}{4(\lambda+2\mu)^2}\right)$$

which is (3.07). Taking the upper sign in (2.02A), we get:

$$k_2^2 \cong \left[2\mathcal{H}(\lambda+2\mu)\right]^{-1}\left[2i\omega(\lambda+2\mu+M)+\mathcal{H}\rho_0\omega^2\left(1-\frac{\lambda+2\mu-M}{\lambda+2\mu+M}\right)-\frac{i\omega^3\mathcal{H}^2\rho_0^{\,2}}{2(\lambda+2\mu+M)}\left(1-\frac{(\lambda+2\mu)^2}{(\lambda+2\mu+M)^2}\right)\right]=$$

$$\cong \left[2\mathcal{H}(\lambda+2\mu)\right]^{-1}\left[2i\omega(\lambda+2\mu+M)+\frac{2\mathcal{H}\rho_0\omega^2 M}{\lambda+2\mu}-\frac{i\omega^3\mathcal{H}^2\rho_0^{\,2}M}{(\lambda+2\mu)^2}\right]$$

Since the second and third terms are negligible compared to the first, we obtain:

$$k_2 \cong (1+i)\sqrt{\frac{\omega}{2\mathcal{H}}}$$

which is (3.08).

APPENDIX III

We shall find the following properties of the spherical Bessel and Hankel Functions very useful:

$$j_n(x) = \frac{x^n}{1\cdot 3\cdots(2n+1)}\left(1 - \frac{x^2}{2(2n+3)} + \frac{x^4}{2\cdot 4\cdot(2n+3)(2n+5)} - \cdots\right) \tag{3.01}$$

$$h_n(x) = i^{-n-1}\frac{e^{ix}}{x}\left(1 + \frac{(n+1)!\,i}{(n-1)!\,1!\,2x} + \frac{(n+2)!\,i^2}{(n-2)!\,2!\,(2x)^2} + \cdots + \frac{(2n)!\,i^n}{n!\,(2x)^n}\right) =$$

$$= i^{-n-1}\frac{e^{ix}}{x}\sum_{\lambda=0}^{n}\frac{(n+\lambda)!\,i^\lambda}{(n-\lambda)!\,\lambda!\,(2x)^\lambda} \tag{3.02}$$

Also:

$$x j_n'(x) = n j_n(x) - x j_{n+1}(x) \tag{3.03}$$

$$x j_n'(x) = -(n+1) j_n(x) + x j_{n-1}(x) \tag{3.04}$$

$$(2n+1) j_n(x) = x j_{n+1}(x) + x j_{n-1}(x) \tag{3.05}$$

$$x^2 j_n''(x) - 2x j_{n+1}(x) = -x^2 j_n(x) - n(n-1) j_n(x) \tag{3.06}$$

and identical expressions in which $j_n(x)$ is replaced by $h_n(x)$ If x is small, so that we can neglect x^2 compared to unity, then:

$$x j_n'(x) \cong n j_n(x) \qquad (n \neq 0) \tag{3.07}$$

$$x h_n'(x) \cong -(n+1) h_n(x) \tag{3.08}$$

$$j_n(x) \cong \frac{2^n n!}{(2n+1)!} x^n \tag{3.09}$$

$$h_n(x) \cong \frac{(2n)!}{i n! 2^n x^{n+1}} \qquad (x \ll 1) \tag{3.10}$$

In the following appendices we shall find occasion to use the two relations above very frequently.

APPENDIX IV

The coefficient B_n is given by:

$$B_n = \frac{\Delta_{B_n}}{\Delta}$$

where, from (5.09):

$$\Delta_{B_n} = \begin{vmatrix} -a_1 j_n'(a_1) & a_2 h_n'(a_2) & -n(n+1) h_n(b) \\ -j_n(a_1) & h_n(a_2) & -b h_n'(b) - h_n(b) \\ \alpha_1 a_1 j_n'(a_1) & \alpha_2 a_2 h_n'(a_2) & 0 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a_1 h_n'(a_1) & a_2 h_n'(a_2) & -n(n+1) h_n(b) \\ h_n(a_1) & h_n(a_2) & -b h_n'(b) - h_n(b) \\ \alpha_1 a_1 h_n'(a_1) & \alpha_2 a_2 h_n'(a_2) & 0 \end{vmatrix}$$

Hence:

$$\Delta_{B_n} = n(n+1) h_n(b) \left[\alpha_2 a_2 j_n(a_1) h_n'(a_2) - \alpha_1 a_1 j_n'(a_1) h_n(a_2) \right] -$$

$$- (\alpha_2 - \alpha_1) a_1 a_2 j_n'(a_1) h_n'(a_2) \left[b h_n'(b) + h_n(b) \right]$$

and using (3.07A) and (3.08A) this becomes:

$$\Delta_{B_n} \cong n(n+1) h_n(b) \left[\alpha_2 a_2 j_n(a_1) h_n'(a_2) - \alpha_1 n j_n(a_1) h_n(a_2) \right] -$$

$$- (\alpha_2 - \alpha_1) a_2 n j_n(a_1) h_n'(a_2) \left[b h_n'(b) + h_n(b) \right]$$

The two terms in the first bracket are of the same order except for α_1 and α_2. But since $\left|\frac{\alpha_1}{\alpha_2}\right| \ll 1$, we have very nearly:

$$\Delta_{B_n} \cong \alpha_2 a_2 j_n(a_1) h_n(b) h_n'(a_2) \left(n(n+1) - n \frac{b h_n'(b)}{h_n(b)} - n \right)$$

and using (3.03A) this becomes:

$$\Delta_{B_n} \cong \alpha_2 a_2 b n j_n(a_1) h_n'(a_2) h_{n+1}(b) \tag{4.01}$$

For Δ we obtain:

$$\Delta = -n(n+1) h_n(b) \left[\alpha_2 a_2 h_n(a_1) h_n'(a_2) - \alpha_1 a_1 h_n'(a_1) h_n(a_2) \right] +$$
$$+ (\alpha_2 - \alpha_1) a_1 a_2 h_n'(a_1) h_n'(a_2) \left[b h_n'(b) + h_n(b) \right]$$

Again neglecting α_1 compared to α_2 and using (3.08A), we get:

$$\Delta \cong -\alpha_2 a_2 h_n'(a_2) h_n(b) \left\{ n(n+1) h_n(a_1) + (n+1) h_n(a_1) \left[1 + \frac{b h_n'(b)}{h_n(b)} \right] \right\}$$

and using (3.04A):

$$\Delta \cong -\alpha_2 a_2 b (n+1) h_n'(a_2) h_n(a_1) h_{n-1}(b) \tag{4.02}$$

Combining (4.01A) and (4.02A), B_n becomes:

$$B_n \cong -\frac{n j_n(a_1) h_{n+1}(b)}{(n+1) h_n(a_1) h_{n-1}(b)}$$

and from (3.09A) (3.10A):

$$B_n \cong -i \frac{2^{2n} n! n!}{(2n+2)!(2n-1)!} \frac{h_{n+1}(b)}{h_{n-1}(b)} a_1^{2n+1} \tag{4.03}$$

For C_n we have:

$$C_n = \frac{\Delta_{C_n}}{\Delta}$$

where:

$$\Delta_{C_n} = \begin{vmatrix} a_1 h_n'(a_1) & -a_1 j_n'(a_1) & -n(n+1) h_n(b) \\ h_n(a_1) & -j_n(a_1) & -b h_n'(b) - h_n(b) \\ \alpha_1 a_1 h_n'(a_1) & -\alpha_1 a_1 j_n'(a_1) & 0 \end{vmatrix} =$$

$$= n(n+1)\alpha_1 h_n(b)\left[a_1 j_n'(a_1) h_n(a_1) - a_1 h_n'(a_1) j_n(a_1)\right]$$

and using (3.03A):

$$\Delta_{C_n} \cong n(n+1)\,\alpha_1 a_1 h_n(b)\left[\,j_n(a_1) h_{n+1}(a_1) - j_{n+1}(a_1) h_n(a_1)\right]$$

But from (3.09A) and (3.10A) we see that the first term is of order $1/a_1^2$ and the second of order unity so that we can retain only the first term. Hence:

$$C_n \cong -\frac{\alpha_1 h_n(b)}{\alpha_2 a_2 b h_n'(a_2) h_{n-1}(b)} \cdot \frac{n a_1 j_n(a_1) h_{n+1}(a_1)}{h_n(a_1)}$$

Using (3.09A), (3.10A) and (3.04A) this becomes:

$$C_n = \frac{\alpha_1}{\alpha_2}\,\frac{2^n n\cdot n!}{(2n)!}\,\frac{h_n(b)\, a_1^n}{\left[(n+1) h_n(a_2) - a_2 h_{n-1}(a_2)\right] b\, h_{n-1}(b)} \tag{4.04}$$

Also:

$$\Delta_{D_n} = \begin{vmatrix} a_1 h_n'(a_1) & a_2 h_n'(a_2) & -a_1 j_n'(a_1) \\ h_n(a_1) & h_n(a_2) & -j_n(a_1) \\ \alpha_1 h_n'(a_1) a_1 & \alpha_2 a_2 h_n'(a_2) & -\alpha_1 a_1 j_n'(a_1) \end{vmatrix}$$

making the same approximations as before; we obtain:

$$\Delta_{D_n} \cong -\alpha_2 a_1 a_2 h_n'(a_2) j_n(a_1) h_{n+1}(a_1)$$

Hence;

$$D_n = \frac{\Delta_{D_n}}{\Delta} \cong \frac{a_1 j_n(a_1) h_{n+1}(a_1)}{h_n(a_1)} \frac{1}{b h_{n-1}(b)} \cong \frac{2^n n!}{(2n)!} \frac{a_1^n}{b h_{n-1}(b)} \qquad (4.05)$$

When n = 0 we must solve the following set of equations:

$$a_1 h_0'(a_1) B_0 + a_2 h_0'(a_2) C_0 = -a_1 j_0'(a_1)$$

$$\alpha_1 a_1 h_0'(a_1) B_0 + \alpha_2 a_2 h_0'(a_2) C_0 = -\alpha_1 a_1 j_0'(a_1)$$

from which:

$$\Delta = (\alpha_2 - \alpha_1) a_1 a_2 h_0'(a_1) h_0'(a_2)$$

$$\Delta_{B_0} = -(\alpha_2 - \alpha_1) a_1 a_2 j_0'(a_1) h_0'(a_2)$$

$$\Delta_{C_0} = 0$$

so that:

$$B_0 \cong -\frac{1}{3} i a_1^3 \qquad (4.06)$$

$$C_0 = 0 \qquad (4.07)$$

Equs. (4.03A)-(4.07A) are the desired solutions.

APPENDIX V

Consider at first the case when $n = 1$. We shall have to evaluate determinants of the form:

$$\begin{vmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\ C_1 & C_2 & C_3 & C_4 & 0 & 0 \\ D_1 & D_2 & D_3 & D_4 & 0 & 0 \\ E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \\ F_1 & F_2 & F_3 & F_4 & F_5 & F_6 \end{vmatrix}$$

(This notation will cause no confusion with the coefficients in the expansions of the potentials).

Let us now investigate the determinant of the coefficients of the unknown in equs. (6.08)-(6.13). Since a_1 and a_1' are small we obtain, using the relations of Appendix III:

$$A_1 \cong -2h_1(a_1), \quad A_2 \cong -j_1(a_1'), \quad A_3 = a_2 h_1'(a_2), \quad A_4 = -a_2' j_1(a_2')$$

$$A_5 = -2h_1(b), \quad A_6 = 2j_1(b')$$

$$B_1 = h_1(a_2), \quad B_2 = -j_1(a_1), \quad B_3 = h_1(a_1), \quad B_4 = -j_1(a_2')$$

$$B_5 = -b h_1'(b) - h_1(b), \quad B_6 = b' j_1'(b') + j_1(b')$$

$$C_1 = \alpha_1 h_1(a_1), \quad C_2 = -\alpha_1' j_1(a_1'), \quad C_3 = \alpha_2 h_1(a_2), \quad C_4 = -\alpha_2' j_1(a_2')$$

$$D_1 \cong -2\chi\alpha_1 h_1(a_1), \quad D_2 \cong -\alpha_1' j_1(a_1'), \quad D_3 = \chi\alpha_2 a_2 h_1'(a_2), \quad D_4 = -\alpha_2' a_2' j_1'(a_2')$$

$$E_1 \cong -3\epsilon h_1(a_1), \quad E_2 \cong 0, \quad E_3 = -\epsilon a_2 h_2(a_2), \quad E_4 = a_2' j_2(a_2')$$

$$E_5 = -\frac{\epsilon}{2} b^2 h_1''(b), \quad E_6 = \frac{1}{2} b'^2 j_1''(b')$$

$$F_1 \cong \epsilon h_1(a_1)(b^2-12)\,, \quad F_2 \cong -b'^2 j_1(a_1')\,, \quad F_3 = \epsilon\left[b^2\beta_2 h_1(a_2) - 2a_2^2 h_1''(a_2)\right]$$

$$F_4 = -b'^2\beta_2' j_1(a_2') + 2a_2'^2 j_1''(a_2')\,, \quad F_5 = 4\epsilon\left[b h_1'(b) - h_1(b)\right], \quad F_6 = -4\left[b' j_1'(b') - j_1(b')\right]$$

Since $\left|\frac{\alpha_1}{\alpha_2}\right| \ll 1$ and $\left|\frac{\alpha_1'}{\alpha_2'}\right| \ll 1$ we can, in view of the results of Appendix IV, neglect all terms involving α_1 or α_1' in the determinant. This means that we can set $C_1 = C_2 = D_1 = D_2 = 0$. Under these conditions the determinant becomes:

$$\begin{aligned}\Delta = (C_3D_4 - C_4D_3)\Big[&(B_6A_5 - B_5A_6)E_1F_2 + (A_6E_5 - A_5E_6)(B_1F_2 - B_2F_1) - \\ &-(A_5F_6 - A_6F_5)B_2E_1 + (B_5E_6 - B_6E_5)(A_1F_2 - A_2F_1) - (B_6F_5 - B_5F_6)A_2E_1 + \\ &+(E_5F_6 - E_6F_5)(A_1B_2 - A_2B_1)\Big]\end{aligned} \qquad (5.01)$$

Now:

$$E_1F_2 = 3\epsilon b'^2 j_1(a_1')h_1(a_1)\,, \qquad B_1F_2 - B_2F_1 = \left[\epsilon(b^2-12) - b'^2\right] j_1(a_1')h_1(a_1)$$

$$B_2E_1 = 3\epsilon j_1(a_1')h_1(a_1)\,, \qquad A_1F_2 - A_2F_1 = \left[\epsilon(b^2-12) + 2b'^2\right] j_1(a_1')h_1(a_1)$$

$$A_2E_1 = 3\epsilon j_1(a_1')h_1(a_1)\,, \qquad A_1B_2 - A_2B_1 = 3 j_1(a_1')h_1(a_1)$$

$$B_6A_5 - B_5A_6 = 2\left[j_1(b')\,b\,h_0(b) - b' j_0(b')h_1(b)\right]$$

$$A_6E_5 - A_5E_6 = \left[6 - b'^2 - \epsilon(6-b^2)\right] h_1(b)j_1(b') + 2\left[\epsilon b j_1(b')h_0(b) - b'h_1(b)j_0(b')\right]$$

$$A_5F_6 - A_6F_5 = 8\left[3(\epsilon-1)j_1(b')h_1(b) + j_0(b')b'h_1(b) - \epsilon j_1(b')\,b\,h_0(b)\right]$$

$$B_5E_6 - B_6E_5 = \tfrac{1}{2}\left[6 - b'^2 - \epsilon(6 - b^2)\right] j_1(b')h_1(b) + (1-\epsilon)bb'h_0(b)j_1(b') +$$
$$+ \left[\tfrac{\epsilon}{2}(6 - b'^2) - 1\right] h_1(b)b'j_0(b') + \left[\epsilon - \tfrac{1}{2}(6 - b'^2)\right] j_1(b')bh_0(b)$$

$$B_6F_5 - B_5F_6 = 4\Big[(\epsilon-1)3j_1(b')h_1(b) + (\epsilon-1)b'j_0(b')bh_0(b) + (1-3\epsilon)b'j_0(b')h_1(b) +$$
$$+ (3-\epsilon)bh_0(b)j_1(b')\Big]$$

$$E_5F_6 - E_6F_5 = 2\epsilon\left\{3(b^2 - b'^2)h_1(b)j_1(b') + bb'\left[b'h_0(b)j_1(b') - bj_0(b')h_1(b)\right]\right\}$$

Substituting these expressions into (5.01A) and collecting the different combinations of Bessel functions, we obtain:

$$\Delta = j_1(a_1')h_1(a_1)(C_3D_4 - C_4D_3)\Big[\Delta_{00}\, bb'j_0(b')h_0(b) + \Delta_{01}\, b'j_0(b')h_1(b) +$$
$$+ \Delta_{10}\, bj_1(b')h_0(b) + \Delta_{11}\, j_1(b')h_1(b)\Big] \qquad (5.02)$$

where:

$$\left.\begin{aligned}
\Delta_{00} &= (1-\epsilon)(\epsilon b^2 + 2b'^2) \\
\Delta_{01} &= \epsilon b^2\left[9(\epsilon-1) - \tfrac{1}{2}b^2\epsilon - b'^2\right] \\
\Delta_{10} &= \epsilon b^2\left[3(\epsilon-1) + \tfrac{1}{2}b'^2\right] + b'^2(b'^2 - 6) + 6\epsilon b'^2 \\
\Delta_{11} &= 3\epsilon b^2\left[9(1-\epsilon) + \tfrac{1}{2}(\epsilon b^2 - b'^2)\right]
\end{aligned}\right\} \qquad (5.03)$$

The coefficient B_1 is given by $\frac{\Delta_{B_1}}{\Delta}$ where Δ_{B_1} is the determinant obtained by replacing the first column of Δ by the terms on the right sides of (6.08)-(6.13). Hence, now:

$$A_1 \cong -j_1(a_1)\,, \quad B_1 = -j_1(a_1)\,, \quad E_1 = C_1 = D_1 \cong 0\,, \quad F_1 \cong -\epsilon b^2 j_1(a_1)$$

and all other elements are the same, so that (5.01) becomes:

$$\Delta_{B_1} = (C_3 D_4 - C_4 D_3)\left(A_6 E_5 - A_5 E_6 + B_5 E_6 - B_6 E_5\right)(A_1 F_2 - A_2 F_1)$$

where we have made use of the fact that $A_1 = B_1$, $A_2 = B_2$. Again collecting the different combinations of Bessel functions, we obtain:

$$\Delta_{B_1} = j_1(a_1) j_1(a_1')(C_3 D_4 - C_4 D_3)(b'^2 - \epsilon b^2)\Big[\Delta_{00}^{B_1} b b' j_0(b') h_0(b) + \Delta_{01}^{B_1} b' j_0(b') h_1(b) +$$
$$+ \Delta_{10}^{B_1} b j_1(b') h_0(b) + \Delta_{11}^{B_1} j_1(b') h_1(b)\Big] \qquad (5.04)$$

where:

$$\Delta_{00}^{B_1} = 1 - \epsilon \;, \quad \Delta_{01}^{B_1} = \frac{\epsilon}{2}(6 - b^2) - 3 \;, \qquad \Delta_{10}^{B_1} = 3\epsilon - \frac{1}{2}(6 - b'^2)$$
$$\Delta_{11}^{B_1} = \frac{3}{2}\left[6 - b'^2 - \epsilon(6 - b^2)\right] \qquad (5.05)$$

Dividing (5.04A) by (5.02A), dividing numerator and denominator by b'^2, and making use of the fact that $\delta b'^2 = \epsilon b^2$ and $j_1(a_1)/h_1(a_1) \cong \frac{1}{3} i a_1^3$ we obtain (6.15) and (6.22).

To solve for B_1' we must evaluate $\Delta_{B_1'}$ which is the determinant obtained by replacing the second column of Δ with the right hand terms of (6.08)-(6.13). Comparing $\Delta_{B_1'}$ with Δ we see that to our approximation they differ only in so far as a_1' in Δ is replaced by a_1 in $\Delta_{B_1'}$, and that the b'^2 in the F_2 element of Δ is replaced by ϵb^2 in the F_2 element of $\Delta_{B_1'}$. Hence, we obtain:

$$E_1 F_2 = 3\epsilon^2 b^2 j_1(a_1) h_1(a_1) \;, \qquad B_1 F_2 - B_2 F_1 = -12\epsilon j_1(a_1) h_1(a_1)$$
$$B_2 E_1 = 3\epsilon j_1(a_1) h_1(a_1) \;, \qquad A_1 F_2 - A_2 F_1 = 3\epsilon(b^2 - 4) j_1(a_1) h_1(a_1)$$
$$A_2 E_1 = 3\epsilon j_1(a_1) h_1(a_1) \;, \qquad A_1 B_2 - A_2 B_1 = 3 j_1(a_1) h_1(a_1)$$

Otherwise everything is the same as in the calculation of Δ and we obtain:

$$\Delta_{B_1'} = j_1(a_1)h_1(a_1)(C_3D_4 - C_4D_3)\left[\Delta_{00}^{B_1'} bb' j_0(b')h_0(b) + \Delta_{01}^{B_1'} b' j_0(b')h_1(b) + \right.$$
$$\left. + \Delta_{10}^{B_1'} b j_1(b')h_0(b) + \Delta_{11}^{B_1'} j_1(b')h_1(b)\right] \tag{5.06}$$

where:

$$\Delta_{00}^{B_1'} = 3\epsilon b^2(1-\epsilon), \quad \Delta_{01}^{B_1'} = \epsilon b^2\left[9(\epsilon-1) - \tfrac{3}{2}\epsilon b^2\right], \quad \Delta_{10}^{B_1'} = \epsilon b^2\left[9(\epsilon-1) + \tfrac{3}{2}\epsilon b^2\right]$$
$$\Delta_{11}^{B_1'} = \epsilon b^2\left[27(1-\epsilon) + \tfrac{3}{2}(\epsilon b^2 - b'^2)\right] \tag{5.07}$$

Dividing (5.06A) by (5.02A), factoring out a b'^2 in the numerator and denominator, and making use of the fact that $\delta b'^2 = \epsilon b^2$ and that $j_1(a_1)/j_1(a_1') \cong a_1/a_1'$ we obtain (6.17).

Having found B_1 and B_1', we can determine C_1 and C_1' from (6.10) and 6.11) since D_1 and D_1' do not appear in those equations. We can write (6.10) and (6.11) in the form:

$$C_1\alpha_2 h_1(a_2) - C_1'\alpha_2' j_1(a_2') = P_1 \tag{5.08}$$

$$C_1\alpha_2 \chi a_2 h_1'(a_2) - C_1'\alpha_2' a_2' j_1(a_2') = Q_1 \tag{5.09}$$

where:

$$P_1 = B_1'\alpha_1' j_1(a_1') - \alpha_1 j_1(a_1) - B_1\alpha_1 h_1(a_1) \tag{5.10}$$

$$Q_1 = B_1'\alpha_1' j_1(a_1') - \chi\alpha_1 j_1(a_1) + 2B_1\chi\alpha_1 h_1(a_1) \tag{5.11}$$

Making use of the fact that the arguments are small, we obtain, using (6.15) and (6.17):

$$P_1 \cong \tfrac{1}{3}a_1\alpha_1\left[\frac{\alpha_1'}{\alpha_1}G_{B_1'} - 1 - (1-\delta)G_{B_1}\right]$$

$$Q \cong \frac{1}{3} a_1 \alpha_1 \left[\frac{\alpha_1'}{\alpha_1} G_{B_1'} - \chi + 2\chi(1-\delta) G_{B_1} \right]$$

Hence, making use of (3.04A) we obtain:

$$\Delta = -\alpha_2 \alpha_2' a_2' j_1'(a_2') h_1(a_2) + \alpha_2 \alpha_2' \chi a_2 j_1(a_2') h_1'(a_2) =$$

$$= -\alpha_2 \alpha_2' \left[2(\chi - 1) j_1(a_2') h_1(a_2) + a_2' j_0(a_2') h_1(a_2) - \chi a_2 h_0(a_2) j_1(a_2') \right] \tag{5.12}$$

$$\Delta_{C_1} = -P_1 \alpha_2' a_2' j_1'(a_2') + Q_1 \alpha_2' j_1(a_2') =$$

$$= -\frac{1}{3} \alpha_1 \alpha_2' a_1 \left\{ j_1(a_2') \left[2 + \chi - 3\frac{\alpha_1'}{\alpha_1} G_{B_1'} + 2(1-\chi)(1-\delta) G_{B_1} \right] + a_2' j_0(a_2') \left[\frac{\alpha_1'}{\alpha_1} G_{B_1'} - 1 - (1-\delta) G_{B_1} \right] \right\} \tag{5.13}$$

$$\Delta_{C_1'} = Q_1 \alpha_2 h_1(a_2) - P_1 \chi \alpha_2 a_2 h_1'(a_2) =$$

$$= -\frac{1}{3} \alpha_1 \alpha_2 a_1 \left\{ h_1(a_2) \left[(1+2\chi) \frac{\alpha_1'}{\alpha_1} G_{B_1'} - 3\chi \right] + a_2 h_0(a_2) \chi \left[1 + (1-\delta) G_{B_1} - \frac{\alpha_1'}{\alpha_1} G_{B_1'} \right] \right\} \tag{5.14}$$

Combining (5.12A) (5.13A) and (5.14A), we obtain (6.19) and (6.21). Since we are not interested in D_1 and D_1', this completes the case for n = 1.

When n = 0, we must solve the following set of equations (where we have already simplified the terms, due to the smallness of a_1 and a_1'):

$$-B_0 a_1 h_1(a_1) + B_0' a_1' j_1(a_1') - C_0 a_2 h_1(a_2) + C_0' a_2' j_1(a_2') = a_1 j_1(a_1) \tag{5.15}$$

$$B_0 \alpha_1 h_0(a_1) - B_0' \alpha_1' j_0(a_1') + C_0 \alpha_2 h_0(a_2) - C_0' \alpha_2' j_0(a_2') = -\alpha_1 j_0(a_1) \tag{5.16}$$

$$-B_0 \chi \alpha_1 a_1 h_1(a_1) + B_0' \alpha_1' a_1' j_1(a_1') - C_0 \chi \alpha_2 a_2 h_1(a_2) + C_0' \alpha_2' a_2' j_1(a_2') = \chi \alpha_1 a_1 j_1(a_1) \tag{5.17}$$

$$B_0 \epsilon \left[b^2 h_0(a_1) - 4 a_1 h_1(a_1) \right] - B_0' b'^2 j_0(a_1') + C_0 \epsilon \left[b^2 \beta_2 h_0(a_2) - 2 a_2^2 h_0''(a_2) \right] -$$

$$- C_0' \left[b'^2 \beta_2' j_0(a_2') - 2 a_2'^2 j_0''(a_2') \right] = -\epsilon b^2 j_0(a_1) \tag{5.18}$$

Labeling the elements of the determinant as before, we obtain (since $\left|\frac{\alpha_1}{\alpha_2}\right| \ll 1$, $\left|\frac{\alpha_1'}{\alpha_2'}\right| \ll 1$):

$$\Delta \cong (A_1 D_2 - A_2 D_1)(B_3 C_4 - B_4 C_3) =$$

$$= \left[a_1 h_1(a_1)\, b'^2 j_0(a_1') - a_1' j_1(a_1')\epsilon\left(b^2 h_0(a_1) - 4a_1 h_1(a_1)\right)\right](B_3 C_4 - B_4 C_3) =$$

$$= b'^2 (B_3 C_4 - B_4 C_3)\left[a_1 h_1(a_1) j_0(a_1') - \delta a_1' h_0(a_1) j_1(a_1')\right] =$$

$$= b'^2 (B_3 C_4 - B_4 C_3)\, a_1 h_1(a_1) j_0(a_1') \tag{5.19}$$

since the second term is of order $a_1'^2$ compared to the first. Making the same approximations in Δ_{B_0} and $\Delta_{B_0'}$ we obtain:

$$\Delta_{B_0} \cong - b'^2 \left[a_1 j_1(a_1) j_0(a_1') - \delta a_1' j_0(a_1) j_1(a_1')\right](B_3 C_4 - B_4 C_3) \tag{5.20}$$

$$\Delta_{B_0'} \cong a_1 \epsilon b^2 h_1(a_1) j_0(a_1)\,(B_3 C_4 - B_4 C_3) \tag{5.21}$$

so that:

$$B_0 \cong -\tfrac{1}{3} i a_1^3 \left(1 - \delta \frac{a_1'^2}{a_1^2}\right) \tag{5.22}$$

$$B_0' = \delta \tag{5.23}$$

which are equs. (6.14) and (6.16) respectively.

From (5.16A) and (5.17A) we obtain:

$$C_0 \alpha_2 h_0(a_2) - C_0' \alpha_2' j_0(a_2') = P_0 \tag{5.24}$$

$$-C_0 \chi \alpha_2 a_2 h_1(a_2) + C_0' \alpha_2' a_2' j_1(a_2') = Q_0 \tag{5.25}$$

where:

$$P_0 = -\alpha_1 j_0(a_1) - B_0\alpha_1 h_0(a_1) + \alpha_1' B_0' j_0(a_1')$$

$$Q_0 = \chi\alpha_1 a_1 j_1(a_1) + B_0\chi\alpha_1 a_1 h_1(a_1) - B_0'\alpha_1' a_1' j_1(a_1')$$

Making use of the fact that the arguments are small, and using (5.22A) and (5.23A) we obtain:

$$P_0 \cong \alpha_1\left(\delta\frac{\alpha_1'}{\alpha_1} - 1\right) \quad , \quad Q_0 \cong \tfrac{1}{3}\alpha_1 a_1^2 \delta\left(\chi - \frac{\alpha_1'}{\alpha_1}\right)$$

so that, from (5.24A) and (5.25A):

$$\Delta = \alpha_2\alpha_2'\left[a_2' j_1(a_2') h_0(a_2) - \chi a_2 j_0(a_2') h_1(a_2)\right] \tag{5.26}$$

$$\Delta_{C_0} \cong \alpha_2'\alpha_1 a_2' j_1(a_2')\left(\delta\frac{\alpha_1'}{\alpha_1} - 1\right) \tag{5.27}$$

$$\Delta_{C_0'} \cong \alpha_1\alpha_2\chi a_2 h_1(a_2)\left(\delta\frac{\alpha_1'}{\alpha_1} - 1\right) \tag{5.28}$$

Dividing (5.27A) and (5.28A) by (5.26A) we obtain (6.18) and (6.20) respectively.

This completes the calculations for the viscous fluid sphere.

APPENDIX VI

To compare the results of VI with those of V we must consider the case when $\epsilon = \delta = 0,\ \chi = \infty$.

When $\delta = 0$, we have from (6.14):

$$\mathcal{B}_0 \xrightarrow[\delta \to 0]{} -\tfrac{1}{3} i a_1^3$$

which is identical with (5.10).

Also, when $\epsilon = \delta = 0$ we get from (6.22):

$$G_{00}^{\mathcal{B}_1} = 1, \quad G_{01}^{\mathcal{B}_1} = -3, \quad G_{10}^{\mathcal{B}_1} = -\tfrac{1}{2}(6 - b'^2), \quad G_{11}^{\mathcal{B}_1} = \tfrac{3}{2}(6 - b'^2)$$

$$G_{00} = 2, \quad G_{01} = 0, \quad G_{10} = b'^2 - 6, \quad G_{11} = 0$$

so that, from (6.15):

$$G_{\mathcal{B}_1} \xrightarrow[\epsilon,\delta \to 0]{} \frac{b b' j_0(b') h_0(b) - 3 b' j_0(b') h_1(b) + \tfrac{1}{2}(6 - b'^2)\left[-b j_1(b') h_0(b) + 3 j_1(b') h_1(b)\right]}{2 b b' j_0(b') h_0(b) + (b'^2 - 6) b j_1(b') h_0(b)} =$$

$$= \frac{b' j_0(b')\left[b h_0(b) - 3 h_1(b)\right] + \tfrac{1}{2}(6 - b'^2) j_1(b')\left[-b h_0(b) + 3 h_1(b)\right]}{2 b h_0(b)\left[b' j_0(b') - 3 j_1(b')\right] + b'^2 b j_1(b') h_0(b)} =$$

$$= \frac{b h_2(b)\left[-b' j_0(b') + 3 j_1(b') - \tfrac{1}{2} b'^2 j_1(b')\right]}{2 b h_0(b)\left[-b' j_2(b') + \tfrac{1}{2} b'^2 j_1(b')\right]} =$$

$$= \frac{h_2(b)\left[b' j_2(b') - \tfrac{1}{2} b'^2 j_1(b')\right]}{2 h_0(b)\left[-b' j_2(b') + \tfrac{1}{2} b'^2 j_1(b')\right]} = -\frac{h_2(b)}{2 h_0(b)}$$

so that:

$$\mathcal{B}_1 \xrightarrow[\epsilon,\delta \to 0]{} -\tfrac{1}{6} i a_1^3 \frac{h_2(b)}{h_0(b)}$$

which is identical with (5.10) specialized for the case of n = 1.

We see immediately from (6.16) and (6.17) that if $\delta = 0$, then $B_0' = 0$ and also $B_1' = 0$, since $G_{B_1'} = 3\delta G_{B_1}$ and G_{B_1} remains finite.

It is also evident from (6.18), that if $\delta = 0, \chi = \infty$, then $C_0 = 0$ because of the presence of χ in the denominator. For C_1 we obtain:

$$C_1 \xrightarrow[\substack{\epsilon,\delta\to 0 \\ \chi\to\infty}]{} \frac{1}{3} a_1 \frac{\alpha_1}{\alpha_2} \frac{j_1(a_2')\left(\chi + \chi \frac{h_2(b)}{h_0(b)}\right)}{\left(2\chi h_1(a_2) - \chi a_2 h_0(a_2)\right) j_1(a_2')} = \frac{1}{3} a_1 \frac{\alpha_1}{\alpha_2} \frac{h_0(b) + h_2(b)}{\left(2h_1(a_2) - a_2 h_0(a_2)\right) h_0(b)} =$$

$$= a_1 \frac{\alpha_1}{\alpha_2} \frac{h_1(b)}{\left(2h_1(a_2) - a_2 h_0(a_2)\right) \, b h_0(b)}$$

where we have used (3.05A). This is identical with (5.11) specialized for the case of n = 1.

Also, from (6.20) and (6.21):

$$C_0' \xrightarrow[\substack{\epsilon,\delta\to 0 \\ \chi\to\infty}]{} \frac{\alpha_1}{\alpha_2} \frac{-\chi a_2 h_1(a_2)}{-\chi a_2 j_0(a_2') h_1(a_2)} = \frac{\alpha_1}{\alpha_2'} \frac{1}{j_0(a_2')}$$

$$C_1' \xrightarrow[\substack{\epsilon,\delta\to 0 \\ \chi\to\infty}]{} \frac{1}{3} a_1 \frac{\alpha_1}{\alpha_2'} \frac{-3h_1(a_2) + a_2 h_0(a_2)\left[1 - \frac{h_2(b)}{2h_0(b)}\right]}{j_1(a_2')\left[2h_1(a_2) - a_2 h_0(a_2)\right]}$$

But from (5.06) we have:

$$\frac{\alpha_1}{\alpha_2'} = \frac{\alpha_1}{\alpha_2} \frac{\alpha_2}{\alpha_2'} = \frac{\alpha_1}{\alpha_2} \frac{\alpha_V' \gamma C_V}{\alpha_V \gamma' C_V'} \frac{\delta}{\chi}$$

so that both C_0' and C_1' will go to zero as $\delta \to 0$ and $\chi \to 0$. This agrees with V where the acoustic field inside the obstacle vanished.

APPENDIX VII

From (7.05), the coefficient of B_1' can be put into the following form:

$$\frac{\lambda}{\mu} a_1'^2 j_1(a_1') - 2a_1'^2 j_1''(a_1') = \frac{\lambda}{\mu} a_1'^2 j_1(a_1') - 4a_1' j_2(a_1') + 2a_1'^2 j_1(a_1) =$$

$$= \frac{\lambda+2\mu}{\mu} a_1'^2 j_1(a_1') - 4a_1' j_2(a_1') =$$

$$= b'^2 j_1(a_1') - 4a_1' j_2(a_1') \qquad (7.01)$$

If we now again set up the six equations, then, as was pointed out in VII, the first five equations are identical with the ones of VI. The sixth equation (continuity of p_{rr}) differs only in the coefficients of B_1' and C_1'. But to our approximation the coefficient of C_1' (which is the F_4 element in the determinants) does not appear when the determinants are evaluated, so that this difference is immaterial. We see from (7.01A) that as far as the present coefficient of B_1' is concerned, it differs from the one in VI by the additional term-- $-4a_1' j_2(a_1')$. But since this term is of order $a_1'^3$, it will, as can be seen from (5.01A), give us additional terms of order $a_1'^3$ while the other terms will be of order a_1' and we can thus neglect it. It is true that the first term in (7.01A) is of the same order as the second (since b' is of the same order as a'), but we shall retain it temporarily because then, in order to evaluate B_1 and B_1', all that we have to do is to take over the results for B_1 and B_1' from Appendix V and let ϵ and b'^2 become small.

Hence, neglecting terms involving ϵ and b'^2 we obtain from (6.22):

$$G^{B_1}_{00} = 1, \quad G^{B_1}_{01} = -3, \quad G^{B_1}_{10} = -3, \quad G^{B_1}_{11} = 9, \quad G_{00} = \delta+2, \quad G_{01} = -9\delta, \quad G_{10} = -3(\delta+2), \quad G_{11} = 27\delta$$

so that:

$$G_{B_1} = \frac{bb'j_0(b')h_0(b) - 3b'j_0(b')h_1(b) - 3bj_1(b')h_0(b) + 9j_1(b')h_1(b)}{(\delta+2)bh_0(b)\left[b'j_0(b') - 3j_1(b')\right] - 9\delta \cdot h_1(b)\left[b'j_0(b') - 3j_1(b')\right]}$$

$$= \frac{b'j_0(b')\left[bh_0(b) - 3h_1(b)\right] - 3j_1(b')\left[bh_0(b) - 3h_1(b)\right]}{(\delta+2)bh_0(b)\left[b'j_0(b') - 3j_1(b')\right] - 9\delta h_1(b)\left[b'j_0(b') - 3j_1(b')\right]}$$

$$= \frac{-b'j_0(b')bh_2(b) + 3j_1(b')bh_2(b)}{-(\delta+2)bh_0(b)b'j_2(b') + 9\delta h_1(b)b'j_2(b')} = \frac{bh_2(b)}{9\delta h_1(b) - (\delta+2)bh_0(b)}$$

$$= \frac{h_2(b)}{3\delta h_2(b) + 2(\delta-1)h_0(b)}$$

so that:

$$B_1 = \tfrac{1}{3} i a_1^3 (1-\delta) \frac{h_2(b)}{3\delta h_2(b) + 2(\delta-1)h_0(b)} \tag{7.02}$$

which is (7.07). Since: $G_{B_1'} = 3\delta G_{B_1}$, we also obtain (7.09).

We cannot use the same method for B_0, B_0', C_0, and C_0' since ϵ and b'^2 do not appear in the equations of VI for these quantities.

If n = 0, then in view of what was said before, we must solve the following four equations:

$$-B_0\alpha_1 h_1(a_1) + B_0' a_1' j_1(a_1') - C_0 a_2 h_1(a_2) + C_0' a_2' j_1(a_2') = a_1 j_1(a_1) \tag{7.03}$$

$$B_0\alpha_1 h_0(a_1) - B_0'\alpha_1' j_0(a_1') + C_0\alpha_2 h_0(a_2) - C_0'\alpha_2' j_0(a_2') = -\alpha_1 j_0(a_1) \tag{7.04}$$

$$-B_0\chi\alpha_1 a_1 h_1(a_1) + B_0'\alpha_1' a_1' j_1(a_1') - C_0\chi\alpha_2 a_2 h_1(a_2) + C_0'\alpha_2' a_2' j_1(a_2') = \qquad (7.05)$$
$$= \chi\alpha_1 a_1 j_1(a_1)$$

$$B_0\epsilon\left(b^2 h_0(a_1) - 4a_1 h_1(a_1)\right) - B_0'\left(b'^2 j_0(a_1') - 4a_1' j_1(a_1')\right) + C_0\epsilon\left(b^2\beta_2 h_0(a_2) - 2a_2^2 h_0''(a_2)\right) +$$

$$+ C_0'\left(2a_2'^2 j_0(a_2') + 2a_2'^2 j_0''(a_2')\right) = -\epsilon b^2 j_0(a_1) \qquad (7.06)$$

Label~~l~~ing the elements in the determinant as before, then since $\left|\frac{\alpha_1}{\alpha_2}\right| \ll 1,\ \frac{\alpha_1'}{\alpha_2'} \ll 1$, we obtain:

$$\Delta \cong \left(A_1 D_2 - A_2 D_1\right)\left(B_3 C_4 - B_4 C_3\right) =$$

$$= \left(B_3 C_4 - B_4 C_3\right)\left[a_1 h_1(a_1)\left(b'^2 j_0(a_1') - 4a_1' j_1(a_1')\right) - a_1' j_1(a_1')\epsilon\left(b^2 h_0(a_1) - 4a_1 h_1(a_1)\right)\right] =$$

$$\cong -\left(B_3 C_4 - B_4 C_3\right)\frac{i b'^2}{a_1}\left(1 - \frac{4}{3}\frac{a_1'^2}{b'^2}\right)$$

since a_1, a_1', b' and ϵ are small. Making the same approximations for Δ_{B_0} and $\Delta_{B_0'}$, there results:

$$\Delta_{B_0} = \left(B_3 C_4 - B_4 C_3\right)\left[-a_1 j_1(a_1)\left(b'^2 j_0(a_1') - 4a_1' j_1(a_1')\right) + a_1' j_1(a_1')\epsilon b^2 j_0(a_1)\right] =$$

$$\cong -\left(B_3 C_4 - B_4 C_3\right)\frac{1}{3}a_1^2 b'^2\left(1 - \frac{4}{3}\frac{a_1'^2}{b'^2} - \delta\frac{a_1'^2}{a_1^2}\right)$$

$$\Delta_{B_0'} = \left(B_3 C_4 - B_4 C_3\right)\left[a_1 h_1(a_1)\epsilon b^2 j_0(a_1) - a_1 j_1(a_1)\epsilon\left(b^2 h_0(a_1) - 4a_1 h_1(a_1)\right)\right] =$$

$$\cong -\left(B_3 C_4 - B_4 C_3\right)\frac{i}{a_1}\delta b'^2$$

so that:

$$B_0 = \frac{\Delta_{B_0}}{\Delta} \cong -\frac{1}{3}i a_1^3\left(1 - \frac{\frac{a_1'^2}{a_1^2}\delta}{1 - \frac{4}{3}\frac{a_1'^2}{b'^2}}\right)$$

$$\mathcal{B}_0' \cong \frac{\Delta_{B_0'}}{\Delta} \cong \frac{\delta -}{1 - \frac{4}{3}\frac{a_1'^2}{b'^2}}$$

But:

$$\frac{a_1'^2}{b'^2} = \frac{\mu}{\lambda+2\mu} \quad , \quad \delta\frac{a_1'^2}{a_1^2} = \frac{\rho c^2}{\lambda+2\mu}$$

and hence:

$$\mathcal{B}_0 \cong -\tfrac{1}{3} i a_1^3 \left(1 - \frac{3\rho c^2}{3\lambda+2\mu}\right) \tag{7.07}$$

$$\mathcal{B}_0' \cong 3\delta\,\frac{\lambda+2\mu}{3\lambda+2\mu} \tag{7.08}$$

From (7.04A) and (7.05A) we obtain:

$$C_0 \alpha_2 h_0(a_2) - C_0' \alpha_2' j_0(a_2') = P_0$$

$$-C_0 \chi \alpha_2 a_2 h_1(a_2) + C_0' \alpha_2' a_2' j_1(a_2') = Q_0$$

where:

$$P_0 = -\alpha_1 j_1(a_1) - \mathcal{B}_0 \alpha_1 h_0(a_1) + \mathcal{B}_0' \alpha_1' j_0(a_1')$$

$$Q_0 = \chi \alpha_1 a_1 j_1(a_1) + \mathcal{B}_0 \chi \alpha_1 a_1 h_1(a_1) - \mathcal{B}_0' \alpha_1' a_1' j_1(a_1')$$

Using (7.07A) and (7.08A), we easily obtain (since a_1, a_1' are small):

$$P_0 \cong \alpha_1\left(3\delta\frac{\alpha_1'}{\alpha_1}\,\frac{\lambda+2\mu}{3\lambda+2\mu} - 1\right) \quad , \quad Q_0 \cong 0$$

so that:

$$\Delta = \alpha_2\alpha_2'\left(a_2' j_1(a_2') h_0(a_2) - \chi a_2 h_1(a_2) j_0(a_2')\right)$$

$$\Delta_{C_0} \cong \alpha_1 \alpha_2' a_2' j_1(a_2')\left(3\delta\frac{\alpha_1'}{\alpha_1}\,\frac{\lambda+2\mu}{3\lambda+2\mu} - 1\right)$$

$$\Delta_{C_0'} \cong \chi \alpha_2 \alpha_1 a_2 h_1(a_2)\left(3\delta\frac{\alpha_1'}{\alpha_1}\,\frac{\lambda+2\mu}{3\lambda+2\mu} - 1\right)$$

Consequently:

$$C_0 \cong \frac{\alpha_1}{\alpha_2} \frac{a_2' j_1(a_2')\left(3\delta \frac{\alpha_1'}{\alpha_1} \frac{\lambda+2\mu}{3\lambda+2\mu} - 1\right)}{a_2' j_1(a_2') h_0(a_2) - \chi a_2 h_1(a_2) j_0(a_2')} \tag{7.09}$$

$$C_0' \cong \frac{\alpha_1}{\alpha_2'} \frac{\chi a_2 h_1(a_2)\left(3\delta \frac{\alpha_1'}{\alpha_1} \frac{\lambda+2\mu}{3\lambda+2\mu} - 1\right)}{a_2' j_1(a_2') h_0(a_2) - \chi a_2 h_1(a_2) j_0(a_2')} \tag{7.10}$$

which are (7.10) and (7.11) of part I.

APPENDIX VIII

From (9.07), we have:

$$I_s = Re\, i \int_S (\phi_i + \phi_1) \frac{\partial}{\partial r}(\phi_i^* + \phi_1^*) dS = Re\, i \int_S \left(\phi_i \frac{\partial \phi_i^*}{\partial r} + \phi_1 \frac{\partial \phi_i^*}{\partial r} + \phi_i \frac{\partial \phi_1^*}{\partial r} + \phi_1 \frac{\partial \phi_1^*}{\partial r}\right) dS$$

Since k_1 is real, the first term will give no contribution. For large r, we get:

$$j_n(x) \longrightarrow \frac{1}{x} \cos\left[x - (n+1)\frac{\pi}{2}\right]$$

$$h_n(x) \longrightarrow \frac{1}{x} e^{i\left[x - (n+1)\frac{\pi}{2}\right]}$$

Then, since the radius of our sphere is large, we obtain from (6.01):

$$I_s = \sum_{n,m} Re\, i \int_0^\pi \Big\{ i^{n-m}(2n+1)(2m+1) P_n P_m \Big[-\frac{B_n}{k_1 r^2} e^{i(k_1 r - \delta_n)} \sin(k_1 r - \delta_m) -$$

$$- \frac{i B_n^*}{k_1 r^2} e^{-i(k_1 r - \delta_n)} \cos(k_1 r - \delta_m) - \frac{i B_n B_m^*}{k_1 r^2} \Big] \Big\} \cdot 2\pi r^2 \sin\theta \, d\theta$$

But:

$$\int_0^\pi P_n P_m \cdot \sin\theta \, d\theta = \frac{2\delta_{nm}}{2n+1}$$

so that:

$$I_s = \frac{4\pi}{k_1} \sum_n (2n+1) Re \left[-i B_n e^{i(k_1 r - \delta_n)} \sin(k_1 r - \delta_n) + B_n^* e^{-i(k_1 r - \delta_n)} \cos(k_1 r - \delta_n) + B_n B_n^* \right] =$$

$$= \frac{2\pi}{k_1} \sum_n (2n+1) \left[- \sin(k_1 r - \delta_n) \left(i B_n e^{i(k_1 r - \delta_n)} - i B_n^* e^{-i(k_1 r - \delta_n)} \right) + \right.$$

$$+\cos(k_1 r-\delta_n)\left(B_n^* e^{-i(k_1 r-\delta_n)}+B_n e^{i(k_1 r-\delta_n)}\right)+2B_nB_n^*\Big]=$$

$$=\frac{2\pi}{k_1}\sum_n(2n+1)\Big[B_n e^{i(k_1 r-\delta_n)}\left(-i\sin(k_1 r-\delta_n)+\cos(k_1 r-\delta_n)\right)+$$

$$+B_n^* e^{-i(k_1 r-\delta_n)}\left(i\sin(k_1 r-\delta_n)+\cos(k_1 r-\delta_n)\right)+2B_nB_n^*\Big]=$$

$$=\frac{2\pi}{k_1}\sum_n(2n+1)\left(B_n+B_n^*+2B_nB_n^*\right)$$

so that:

$$I_S=\frac{4\pi}{k_1}\,Re\sum_n(2n+1)\left(B_n+B_nB_n^*\right)$$

Keeping only the term for n = 0 and 1, we get:

$$I_S\cong\frac{4\pi}{k_1}\,Re\left(B_0+B_0B_0^*+3B_1+3B_1B_1^*\right)$$

But B_0 is imaginary and $B_0B_0^*$, $B_1B_1^*$ are negligible compared to B_1. Hence :

$$I_S\cong\frac{4\pi}{k_1}\,Re\,(3B_1) \tag{8.01}$$

Also:

$$I_{\gamma_1}=Re\int_0^R\!\!\int_0^\pi \phi_1'\phi_2'^*\cdot 2\pi r^2\sin\theta\,dr\,d\theta=$$

$$=Re\sum_{n,m} i^{n-m}(2n+1)(2m+1)B_n'C_m'^*\int_0^R\!\!\int_0^\pi j_n(k_1'r)j_m^*(k_2'r)P_nP_m\cdot 2\pi r^2\sin\theta\,dr\,d\theta=$$

$$=4\pi\,Re\sum_n(2n+1)B_n'C_n'^*\int_0^R j_n(k_1'r)j_n^*(k_2'r)\,r^2\,dr$$

Since:

$$j_n^*(k_2'r) = j_n(k_2'^* r)$$

we can write:

$$I_{V_1} = Re \frac{2\pi^2}{(k_1' k_2'^*)^{1/2}} \sum_n (2n+1) B_n' C_n'^* \int_0^R J_{n+\frac{1}{2}}(k_1' r) J_{n+\frac{1}{2}}(k_2'^* r)\, r\, dr \qquad (8.02)$$

For I_{V_2} we obtain:

$$I_{V_2} = 4\pi\, Re \sum_n (2n+1) \int_R^\infty \left[j_n(k_1 r) + B_n h_n(k_1 r) \right] C_n^* h_n^*(k_2 r) r^2 dr =$$

$$= Re \frac{2\pi^2}{(k_1 k_2^*)^{1/2}} \sum_n (2n+1) \int_R^\infty \left[J_{n+\frac{1}{2}}(k_1 r) + B_n H_{n+\frac{1}{2}}(k_1 r) \right] C_n^* H_{n+\frac{1}{2}}^*(k_2 r)\, r\, dr =$$

$$= Re \frac{2\pi^2}{(k_1 k_2^*)^{1/2}} \sum_n (2n+1) \int_R^\infty \left[(1+B_n) J_{n+\frac{1}{2}}(k_1 r) + B_n i Y_{n+\frac{1}{2}}(k_1 r) \right] C_n^* \left[J_{n+\frac{1}{2}}(k_2^* r) - i Y_{n+\frac{1}{2}}(k_2^* r) \right] r\, dr$$

(8.03)

In order to evaluate these integrals, consider the following: let u and v be two solutions of Bessel's equation such that:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) + \left(k_u^2 - \frac{n^2}{r^2}\right) u = 0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \left(k_v^2 - \frac{n^2}{r^2}\right) v = 0$$

Multiply the first equation by v and the second by u, subtract and integrate from, say a to b:

$$(k_u^2 - k_v^2) \int_a^b r u v\, dr = - \int_a^b \left[v \frac{d}{dr}\left(r \frac{du}{dr}\right) - u \frac{d}{dr}\left(r \frac{dv}{dr}\right) \right] dr$$

Integrating the right side by parts we obtain:

$$\int_a^b ruv\,dr = \frac{1}{k_u^2 - k_v^2}\left[ru\frac{dv}{dr} - rv\frac{du}{dr}\right]_a^b \qquad (8.04)$$

This does not make use of the form of u and v. They can be any solutions of Bessel's equations, in particular u and v can be linear combinations of J_n and Y_n.

Applying (8.04A) to (8.02A), we get:

$$\int_0^R J_{n+\frac{1}{2}}(k_1'r)J_{n+\frac{1}{2}}(k_2'^*r)\,r\,dr = \frac{1}{k_1'^2 - k_2'^{*2}}\left(a_2'^* J_{n+\frac{1}{2}}(a_1')J'_{n+\frac{1}{2}}(a_2'^*) - a_1' J_{n+\frac{1}{2}}(a_2'^*)J'_{n+\frac{1}{2}}(a_1')\right) =$$

$$= \frac{1}{k_1'^2 - k_2'^{*2}}\left(a_1' J_{n+\frac{3}{2}}(a_1')J_{n+\frac{1}{2}}(a_2'^*) - a_2'^* J_{n+\frac{3}{2}}(a_2'^*)J_{n+\frac{1}{2}}(a_1')\right)$$

where we have used (3.03A). Substituting this into (8.02A) we get:

$$I_{V_1} = 2\pi^2 R^3 \,\mathrm{Re} \sum_n \frac{(2n+1)B_n' C_n'^*}{(a_1' a_2'^*)^{1/2}(a_1'^2 - a_2'^{*2})}\left(a_1' J_{n+\frac{3}{2}}(a_1')J_{n+\frac{1}{2}}(a_2'^*) - a_2'^* J_{n+\frac{3}{2}}(a_2'^*)J_{n+\frac{1}{2}}(a_1')\right) =$$

$$= 4\pi R^3 \,\mathrm{Re} \sum_n \frac{(2n+1)B_n' C_n'^*}{a_1'^2 - a_2'^{*2}}\left(a_1' j_{n+1}(a_1')j_n(a_2'^*) - a_2'^* j_{n+1}(a_2'^*)j_n(a_1')\right)$$

It will be sufficient for our purposes to maintain only the n = 0 term. Then:

$$I_{V_1} \cong 4\pi R^3 \,\mathrm{Re} \frac{B_0' C_0'^*}{a_1'^2 - a_2'^{*2}}\left(a_1' j_1(a_1')j_0(a_2'^*) - a_2'^* j_1(a_2'^*)j_0(a_1')\right)$$

The first term is negligible compared to the second so that:

$$I_{\gamma_1} \cong 4\pi R^3 \, Re \, \frac{B_0' C_2'^*}{a_2'^*} j_1(a_2'^*) j_0(a_1')$$

But since $j_0(a_1') \cong 1$ and B_0' is real, we have:

$$I_{\gamma_1} \cong 4\pi R^3 \, Re \, \frac{B_0' C_0'}{a_2'} j_1(a_2') \tag{8.05}$$

Using (8.04A) in (8.03A), we obtain:

$$I_{\gamma_2} = -Re \, \frac{2\pi^2}{(k_1 k_2^*)^{1/2}(k_1^2 - k_2^{*2})} \sum_n (2n+1) C_n^* \Big\{ a_2 \Big[(1+B_n) J_{n+\frac{1}{2}}(a_1) + i B_n Y_{n+\frac{1}{2}}(a_1) \Big] \times$$

$$\times \Big[J'_{n+\frac{1}{2}}(a_2^*) - i Y'_{n+\frac{1}{2}}(a_2^*) \Big] - a_1 \Big[J_{n+\frac{1}{2}}(a_2^*) - i Y_{n+\frac{1}{2}}(a_2^*) \Big] \Big[(1+B_n) J'_{n+\frac{1}{2}}(a_1) + i B_n Y'_{n+\frac{1}{2}}(a_1) \Big] \Big\} =$$

$$= -4\pi R^3 \, Re \sum_n \frac{(2n+1) C_n^*}{a_1^2 - a_2^{*2}} \Big\{ a_2^* h_n'^*(a_2) \Big[j_n(a_1) + B_n h_n(a_1) \Big] -$$

$$- a_1 h_n^*(a_2) \Big[j_n'(a_1) + B_n h_n'(a_1) \Big] \Big\}$$

Using (3.03A) and neglecting a_1^2 compared to a_2^{*2} in the denominator, there results:

$$I_{\gamma_2} \cong 4\pi R^3 \, Re \sum_n \frac{(2n+1) C_n^*}{a_2^{*2}} \Big\{ a_1 h_n^*(a_2) \Big[j_{n+1}(a_1) + B_n h_{n+1}(a_1) \Big] -$$

$$- a_2^* h_{n+1}^*(a_2) \Big[j_n(a_1) + B_n h_n(a_1) \Big] \Big\}$$

Again, retaining only the n = 0 term, we get:

$$I_{\gamma_2} \cong 4\pi R^3 \, Re \, \frac{C_0^*}{a_2^{*2}} \Big\{ a_1 h_0^*(a_2) \Big[j_1(a_1) + B_0 h_1(a_1) \Big] -$$

$$- a_2^* h_1^*(a_2) \Big[j_0(a_1) + B_0 h_0(a_1) \Big] \Big\}$$

The first, second and fourth term are of order a_1^2 compared to the third and can hence be discarded. What remains, yields:

$$I_{\gamma_2} \cong -4\pi R^3 \, Re \, \frac{C_0}{a_2} h_1(a_2) \tag{8.06}$$

Substituting (8.01A), (8.05A) and (8.06A) into (9.06) we obtain:

$$\alpha \cong \frac{4\pi n R^3}{k_1}\Big[-\frac{1}{R^3 k_1} Re(3B_1) + \frac{\rho' \omega \gamma'}{\rho \mathcal{K}'}\Big(1+\frac{\gamma' C_v'}{\alpha_\gamma' c'^2}\Big) Re \, \frac{B_0' C_0'}{a_2'} j_1(a_2') -$$

$$- \frac{\gamma \omega}{\mathcal{K}}\Big(1+\frac{\gamma C_v}{\alpha c^2}\Big) Re \, \frac{C_0}{a_2} h_1(a_2)\Big]$$

which is equ. (9.08).

APPENDIX IX

In this case I_s and I_{V_2} are identical with (8.01A) and (8.06A). In view of the definition of the primed potentials, we see from (7.01) and (8.05A) that now:

$$I_{V_1} = \frac{4\pi R^3}{\omega^2} \, Re \, \frac{B_0' C_0'}{a_2'} j_1(a_2') \tag{9.01}$$

and furthermore:

$$I_{V_3} = \frac{1}{\omega^2} \sum_{n,m} i^{n-m} (2n+1)(2m+1) B_n' B_m'^{*} \int_0^R \int_0^\pi j_n(k_1' r) j_m(k_1'^{*} r) P_n P_m \cdot 2\pi r^2 \sin\theta \, dr \, d\theta =$$

$$= \frac{4\pi}{\omega^2} \sum_n (2n+1) |B_n'|^2 \int_0^R [j_n(k_1' r)]^2 r^2 dr$$

Neglecting terms of order $a_1'^2$ compared to one, this becomes:

$$I_{V_3} \cong \frac{4\pi}{\omega^2} |B_0'|^2 \int_0^R r^2 dr = \frac{4\pi R^3}{3\omega^2} |B_0'|^2 \tag{9.02}$$

Substituting (8.01A), (8.06A), (9.01A) and (9.02A) into (9.10), there results:

$$\alpha \cong \frac{4\pi n R^3}{k_1} \left\{ -\frac{1}{R^3 k_1} Re(3B_1) + \frac{\rho' \omega}{\rho \beta \mathcal{H}'} (\rho' C_v' + \alpha_1' K) \left(\frac{1}{3} \frac{\alpha_1'}{\alpha_2'} |B_0'|^2 + Re \frac{B_0' C_0'}{a_2'} j_1(a_2') \right) - \right.$$

$$\left. - \frac{\rho \omega}{\mathcal{H}} \left(1 + \frac{\gamma C_v}{\alpha c^2} \right) Re \, \frac{C_0}{a_2} h_1(a_2) \right\}$$

which is (9.12).

We were able to replace $[j_0(k_1' r)]^2$ in (9.02A) by unity since the argument of the Bessel function was small over the entire range of integration.

APPENDIX X

In view of the discussion in X we can in the second term of the denominator of the first real part consider b and b' to be small. Then:

$$\frac{j_2(b')}{\frac{1}{2}b'j_1(b') - j_2(b')} \cong \frac{\frac{b'^2}{15}}{\frac{1}{2}b' \cdot \frac{b'}{3} - \frac{b'^2}{15}} = \frac{2}{3}$$

$$\frac{h_1(b)}{b\,h_0(b)} = -\frac{i}{b}\left(1 + \frac{i}{b}\right) \cong \frac{1}{b^2}$$

so that:

$$Re \cdot i\frac{h_2(b)}{h_0(b)} \frac{1}{1 + \frac{9}{2}\delta \frac{h_1(b)}{b\,h_0(b)} \frac{j_2(b')}{\frac{1}{2}b'j_1(b') - j_2(b')}} \cong Re \cdot i\frac{h_2(b)}{h_0(b)} \frac{1}{1 + \frac{3\delta}{b^2}}$$

Now, let:

$$b = y(1+i) = R\sqrt{\frac{w}{2\nu}}\,(1+i)$$

Then:

$$i\frac{h_2(b)}{h_0(b)} = -i + \frac{3}{b} + \frac{3i}{b^2} = -i + \frac{3}{2y}(1-i) + \frac{3}{2y^2} = \frac{3}{2y}\left(1 + \frac{1}{y}\right) - i\left(1 + \frac{3}{2y}\right) =$$

$$= \frac{3}{2y}\left(1 + \frac{1}{y}\right)\left(1 - i\,\frac{1 + \frac{2y}{3}}{1 + \frac{1}{y}}\right)$$

and hence:

$$Re \cdot i\frac{h_2(b)}{h_0(b)} \frac{1}{1 + \frac{3\delta}{b^2}} = \frac{3}{2y}\left(1 + \frac{1}{y}\right) Re\, \frac{1 - i\,\frac{1 + \frac{2y}{3}}{1 + \frac{1}{y}}}{1 - \frac{3i\delta}{2y^2}}$$

The real part differs appreciably from unity only when $y \ll 1$, so that we can write:

$$Re \cdot i \frac{h_2(\beta)}{h_0(\beta)} \frac{1}{1+\frac{3\delta}{\beta^2}} \cong \frac{3}{2y}\left(1+\frac{1}{y}\right) Re \frac{1-iy}{1-\frac{3i\delta}{2y^2}} \cong \frac{3}{2y}\left(1+\frac{1}{y}\right) Re \frac{y^2}{y^2-\frac{3i\delta}{2}} =$$

$$= \frac{3}{2y}\left(1+\frac{1}{y}\right)\frac{y^4}{y^4+\frac{9}{4}\delta^2} = \frac{3}{2}\left(\frac{2\nu}{R^2\omega}+\frac{1}{R}\sqrt{\frac{2\nu}{\omega}}\right)\frac{\left(\frac{R^2\omega}{2\nu}\right)^2}{\left(\frac{R^2\omega}{2\nu}\right)^2+\frac{9}{4}\delta^2} =$$

$$= \frac{3}{R}\left(\frac{\nu}{R\omega}+\sqrt{\frac{\nu}{2\omega}}\right)\frac{1}{1+\left(\frac{3\delta\nu}{R^2\omega}\right)^2} \qquad (10.01)$$

Similarly, in the second term of the denominator of the second real part we can consider a_2' and a_2 to be small. So that:

$$\frac{j_0(a_2')}{a_2' j_1(a_2')} \cdot \frac{a_2 h_1(a_2)}{h_0(a_2)} \cong -\frac{3}{a_2'^2} a_2 i \left(1+\frac{i}{a_2}\right) \cong \frac{3}{a_2'^2}$$

Hence:

$$Re \cdot i \frac{h_1(a_2)}{a_2 h_0(a_2)} \frac{1}{1-\chi \frac{j_0(a_2')}{a_2' j_1(a_2')} \cdot \frac{a_2 h_1(a_2)}{h_0(a_2)}} \cong Re \cdot i \frac{h_1(a_2)}{a_2 h_0(a_2)} \frac{1}{1-\frac{3\chi}{a_2'^2}}$$

Let:

$$a_2 = z(1+i) = R\sqrt{\frac{\omega\gamma}{2\mathcal{H}}}\,(1+i)$$

$$a_2' = z'(1+i) = R\sqrt{\frac{\omega\gamma'}{2\mathcal{H}'}}\,(1+i)$$

Then:

$$i\frac{h_1(a_2)}{a_2 h_0(a_2)} = \frac{1}{a_2}+\frac{i}{a_2^2} = \frac{1}{2z}(1-i)+\frac{1}{2z^2} = \frac{1}{2z}\left(1+\frac{1}{z}\right)\left(1-\frac{iz}{1+z}\right)$$

and hence:

$$\mathrm{Re}\cdot i\,\frac{h_1(a_2)}{a_2 h_0(a_2)}\,\frac{1}{1-\frac{3\chi}{a_2'^2}} \cong \frac{1}{2z}\left(1+\frac{1}{z}\right)\mathrm{Re}\,\frac{1-\frac{iz}{1+z}}{1+\frac{3i\chi}{2z'^2}}$$

Now, when z' is small, z is even smaller, so that since the real part differs appreciably from unity only when z' is small, we obtain:

$$\mathrm{Re}\cdot i\,\frac{h_1(a_2)}{a_2 h_0(a_2)}\,\frac{1}{1-\frac{3\chi}{a_2'^2}} \cong \frac{1}{2z}\left(1+\frac{1}{z}\right)\mathrm{Re}\,\frac{z'^4}{z'^4+\frac{9}{4}\chi^2} = \frac{1}{2}\left(\frac{2\mathcal{H}}{R^3\omega\gamma}+\frac{1}{R}\sqrt{\frac{2\mathcal{H}}{\omega\gamma}}\right)\frac{\left(\frac{R^2\omega\gamma'}{2\mathcal{H}'}\right)^2}{\left(\frac{R^2\omega\gamma'}{2\mathcal{H}'}\right)^2+\frac{9}{4}\chi^2} =$$

$$= \frac{1}{R}\left(\frac{\mathcal{H}}{R\omega\gamma}+\sqrt{\frac{\mathcal{H}}{2\omega\gamma}}\right)\frac{1}{1+\left(\frac{3\chi\mathcal{H}'}{R^2\omega\gamma'}\right)^2} \qquad (10.02)$$

Substituting (10.01A) and (10.02A) into (10.04), we get:

$$\alpha = \frac{4\pi n R^2}{c}\left[\frac{3/2}{1+\left(\frac{3\delta\nu}{R^2\omega}\right)^2}\left(\frac{\nu}{R}+\sqrt{\frac{\nu\omega}{2}}\right)+\frac{(\gamma-1)\left(1+\frac{\gamma c_v}{\alpha_v c^2}\right)}{1+\left(\frac{3\chi\mathcal{H}'}{R^2\omega\gamma'}\right)^2}\left(\frac{\mathcal{H}}{R\gamma}+\sqrt{\frac{\mathcal{H}\omega}{2\gamma}}\right)\right]$$

which is (10.05).

APPENDIX XI

We have:

$$j_0(a_2') = \frac{\sin a_2'}{a_2'} \quad , \quad j_1(a_2') = \frac{\sin a_2'}{a_2'^2} - \frac{\cos a_2'}{a_2'}$$

so that:

$$\frac{i j_1(a_2')}{a_2' j_0(a_2')} = \frac{i}{a_2'^2} - \frac{i \cot a_2'}{a_2'}$$

Let:

$$a_2' = z(1+i) = R\sqrt{\frac{\omega\gamma'}{2\kappa'}}\,(1+i)$$

Then:

$$\frac{i j_1(a_2')}{a_2' j_0(a_2')} = \frac{1}{2z^2} - \frac{i}{z(1+i)}\,\frac{\cos z(1+i)}{\sin z(1+i)} = \frac{1}{2z^2} - \frac{1+i}{2z}\,\frac{\cos z\cosh z - i\sin z\sinh z}{\sin z\cosh z + i\cos z\sinh z}$$

and taking the real part we obtain:

$$Re\cdot\frac{i j_1(a_2')}{a_2' j_0(a_2')} = \frac{1}{2z^2} - \frac{1}{2z}\,Re\cdot(1+i)\,\frac{(\cos z\cosh z - i\sin z\sinh z)(\sin z\cosh z - i\cos z\sinh z)}{\sin^2 z\cosh^2 z + \cos^2 z\sinh^2 z} =$$

$$= \frac{1}{2z^2} - \frac{1}{2z}\,\frac{\sin z\cos z\cosh^2 z - \sin z\cos z\sinh^2 z + \sin^2 z\sinh z\cosh z + \cos^2 z\sinh z\cosh z}{\sin^2 z\cosh^2 z + (1-\sin^2 z)(\cosh^2 z - 1)} =$$

$$= \frac{1}{2z^2} - \frac{1}{4z}\,\frac{\sinh 2z + \sin 2z}{\cosh^2 z - \cos^2 z} = -\frac{1}{4z^2}\left(z\,\frac{\sinh 2z + \sin 2z}{\cosh^2 z - \cos^2 z} - 2\right) \qquad (11.01)$$

Substituting this into (10.09) we obtain:

$$\alpha = \frac{2\pi n R\kappa'}{c}\,\delta\left(\frac{c}{c'}\right)^2\left(1 - \frac{1}{\gamma'}\right)\left(1 + \frac{\gamma' c_v'}{\alpha_v' c'^2}\right)\left(z\,\frac{\sinh 2z + \sin 2z}{\cosh^2 z - \cos^2 z} - 2\right)$$

which is (10.10).

If $z \ll 1$, then we can write:

$$\frac{\sinh 2z + \sin 2z}{\cosh^2 z - \cos^2 z} \longrightarrow \frac{2z + \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} + 2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!}}{\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!}\right)^2 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!}\right)^2} =$$

$$= \frac{2}{z} \, \frac{1 + \frac{2}{15} z^4}{1 + \frac{2}{45} z^4} \cong \frac{2}{z}\left(1 + \frac{4}{45} z^4\right)$$

so that:

$$F(z) \xrightarrow[z \to 0]{} \frac{8}{45} z^4 \tag{11.02}$$

If $z \gg 1$, then:

$$\frac{\sinh 2z + \sin 2z}{\cosh^2 z - \cos^2 z} \longrightarrow \frac{\frac{1}{2} e^{2z}}{\frac{1}{4} e^{2z}} = 2$$

so that:

$$F(z) \xrightarrow[z \to \infty]{} 2(z-1) \tag{11.03}$$

We see from Graph II that (11.02) is actually valid up to $z \sim \frac{1}{2}$ and (11.03) begins to hold beyond $z \sim 3$.

REFERENCES

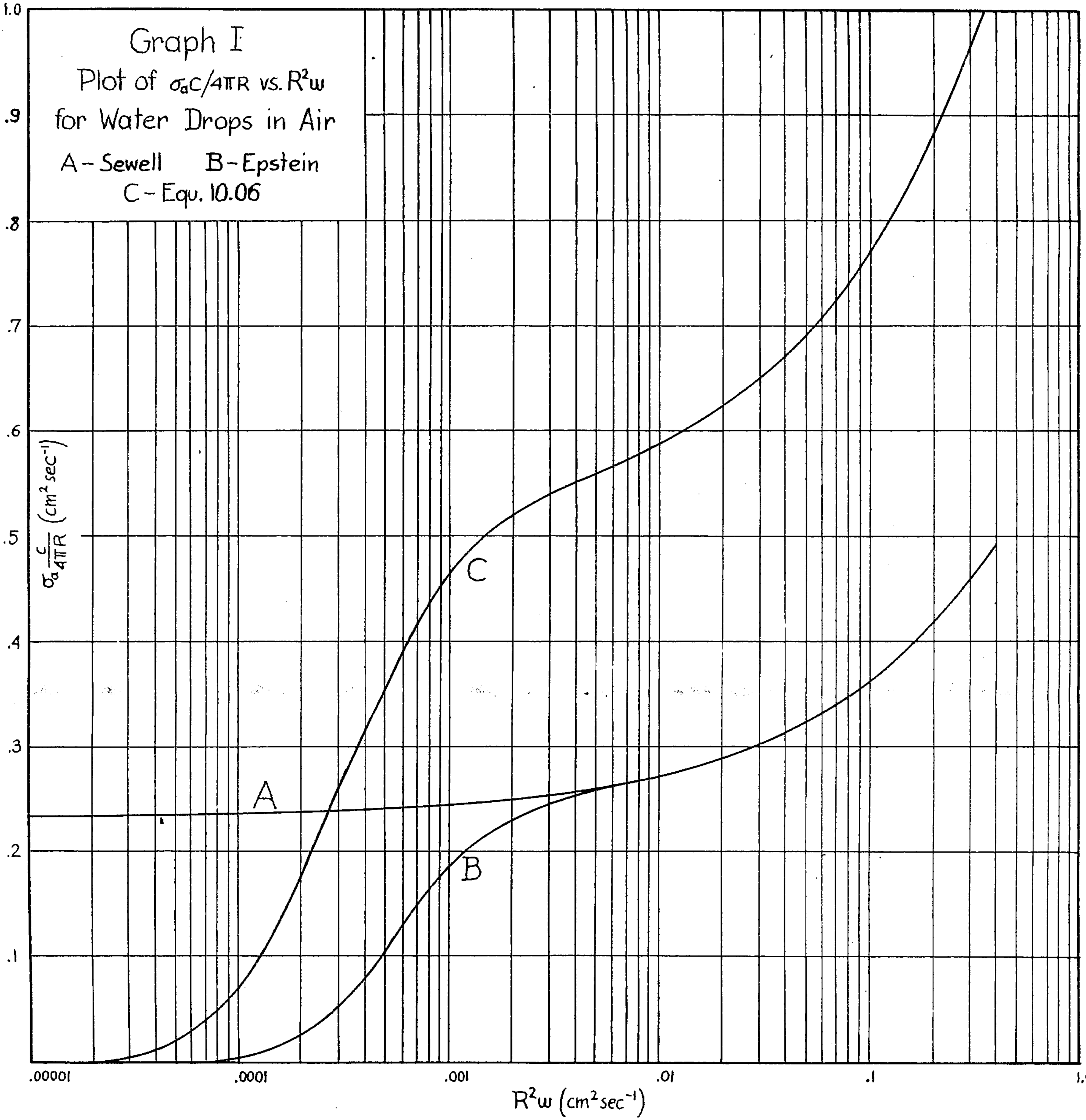
1. Lord Rayleigh: Theory of Sound.
2. C.J.T. Sewell: The extinction of Sound in a Viscous Atmosphere by Small Obstacles of Cylindrical and Spherical Form; Phil. Trans. Roy. Soc. Lond. (1910), (A)210, p. 239.
3. P.S. Epstein: On the Absorption of Sound Waves in Suspensions and Emulsions; Theodore von Karman Anniversary volume (1941).
4. G.K. Hartmann & A.B. Focke: Absorption of Supersonic Waves in Water and Aqueous Suspensions; Phys. Rev., (1940), 57, p. 221.
5. V.O. Knudson: To be published shortly in Jour. Acoust. Soc.
6. G. Kirchhoff: Vorlesungen über Mathematische Physik, vol.IV.
7. S. Chapman & T.G. Cowling: The Mathematical Theory of Non-Uniform Gases; pp. 265-269.
8. A.E.H. Love: A Treatise on the Mathematical Theory of Elasticity; 4th ed. p. 108.
9. W.R. Smythe: Static & Dynamic Electricity; 1st ed. p.394,456.
10. Hann-Suring: Lehrbuch der Meteorologie; 5th ed., vol.1,p.385.
11. D.P. Loye & W.F. Arndt: A Sheet of Air Bubbles as an Acoustic Screen for Underwater Noise; Jour. Acoust. Soc. (1948) 20, p. 143.

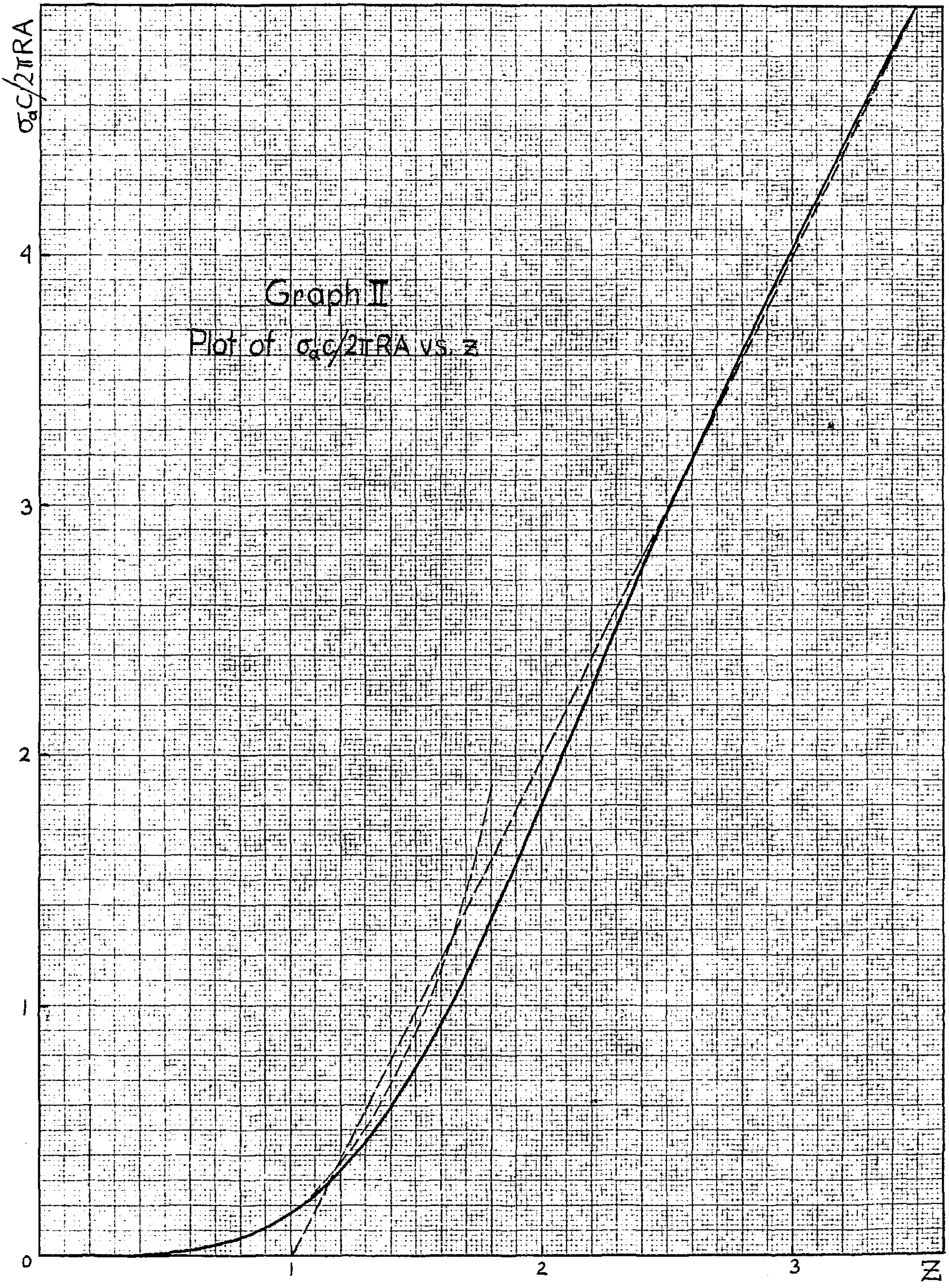
TABLE I

	Air			Water		
w sec^{-1}	10^{3}	10^{5}	10^{7}	10^{3}	10^{5}	10^{7}
C cm sec^{-1}		$3.30 10^{4}$			$1.45 10^{5}$	
ρ cm^{-3}		$1.29 10^{-3}$			1.00	
η g cm^{-1} sec^{-1}		$1.82 10^{-4}$			$1.10 10^{-2}$	
ν cm^{2}sec^{-1}		$.141$			$.011$	
σ cal cm^{-1} sec^{-1}deg^{-1}		$5.80 10^{-5}$			$1.43 10^{-3}$	
$\mathcal{K}$ cm^{2}sec^{-1}		$.204$			$1.43 10^{-3}$	
A cm^{2}sec^{-1}	$3.92 10^{2}$	$3.92 10^{4}$	$3.92 10^{6}$	16.1	$1.61 10^{3}$	$1.61 10^{5}$
B cm^{2}sec^{-2}	$1.09 10^{9}$	$1.09 10^{9}$	$2.10 10^{10}$	$2.10 10^{10}$	$2.10 10^{10}$	$2.10 10^{10}$
$\mathcal{C}$ cm^{4}sec^{-4}	$1.53 10^{5}$	$1.53 10^{9}$	$1.53 10^{13}$	83.8	$8.38 10^{5}$	$8.38 10^{9}$
D cm^{4}sec^{-4}	$6.36 10^{11}$	$6.36 10^{13}$	$6.36 10^{15}$	$1.20 10^{11}$	$1.20 10^{13}$	$1.20 10^{15}$
D/B^{2}	$5.37 10^{-7}$	$5.37 10^{-5}$	$5.37 10^{-3}$	$2.72 10^{-10}$	$2.72 10^{-8}$	$2.72 10^{-6}$

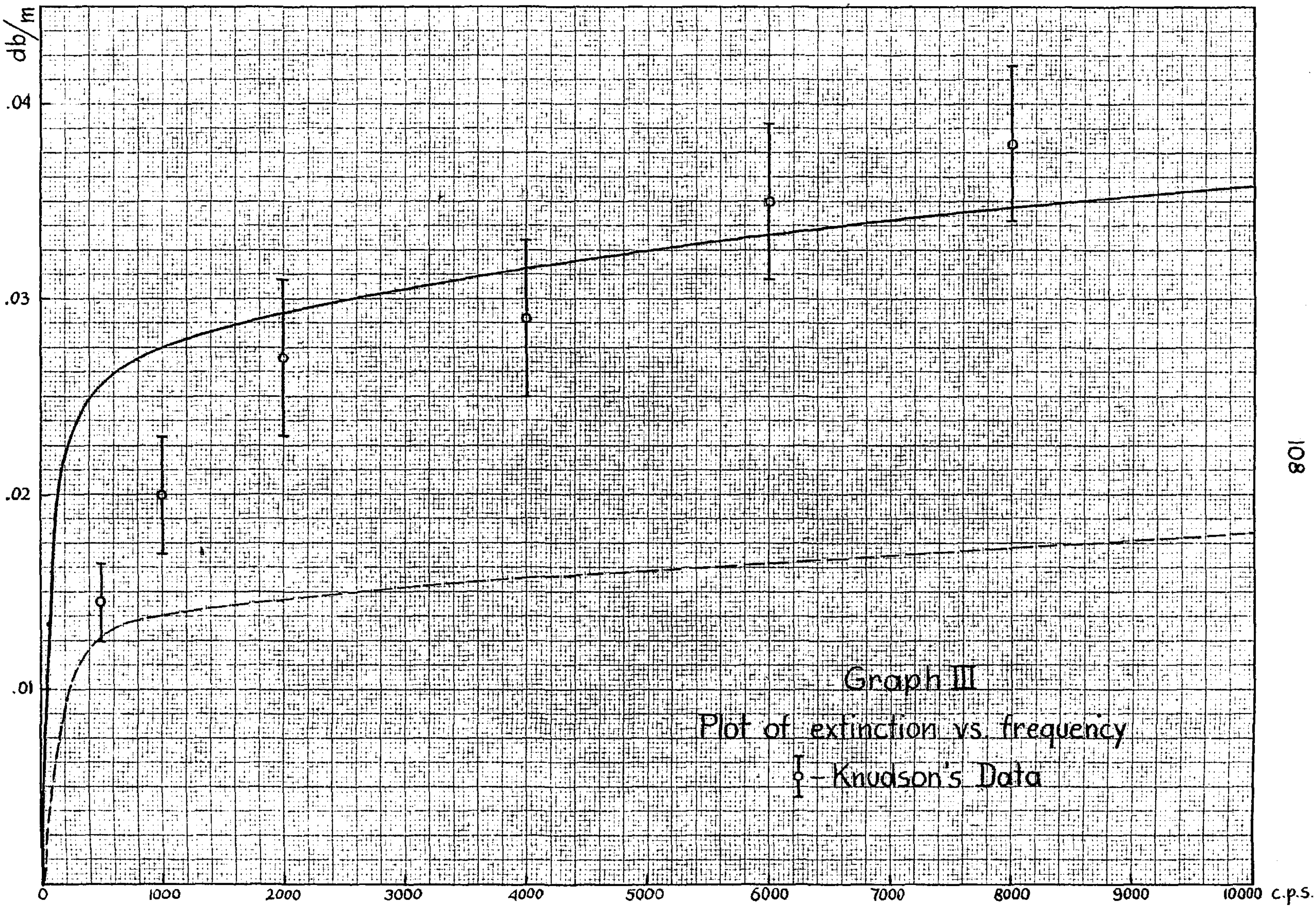
TABLE II

	Air			Water		
w sec^{-1}	10^{3}	10^{5}	10^{7}	10^{3}	10^{5}	10^{7}
l_{L_1} cm	$2.41\,10^{8}$	$2.41\,10^{4}$	2.41	$4.15\,10^{11}$	$4.15\,10^{7}$	$4.15\,10^{3}$
l_{L_2} cm	$1.71\,10^{-2}$	$1.71\,10^{-3}$	$1.71\,10^{-4}$	$1.69\,10^{-3}$	$1.69\,10^{-4}$	$1.69\,10^{-5}$
l_{T} cm	$1.68\,10^{-2}$	$1.68\,10^{-3}$	$1.68\,10^{-4}$	$4.69\,10^{-3}$	$4.69\,10^{-4}$	$4.69\,10^{-5}$


Graph I
Plot of $\sigma_a C/4\pi R$ vs. $R^2 w$
for Water Drops in Air
A - Sewell B - Epstein
C - Equ. 10.06
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
$\sigma_a \frac{c}{4\pi R}$ $(cm^2 sec^{-1})$
A
B
C
.00001
.0001
.001
.01
.1
1.0
$R^2 w$ $(cm^2 sec^{-1})$



$\sigma_a c/2\pi RA$
4
3
2
1
0
1
2
3
Z
Graph II
Plot of $\sigma_a c/2\pi RA$ vs. z


db/m
.04
.03
.02
.01
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000 c.p.s.
Graph III
Plot of extinction vs. frequency
- Knudson's Data