BENDING-TORSION FLUTTER OF AN AIRFOIL WITH NONLINEAR STRUCTURAL CHARACTERISTICS

Thesis by
Jerry C. Peck

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ABSTRACT

A theoretical analysis of the incompressible bending-torsion flutter of a two-dimensional airfoil with nonlinear structural characteristics is presented. The Method of Slowly Varying Parameters of Kryloff and Bogoliuboff is applied, and the steady-state oscillations are found. The stability of the steady-state oscillation is analyzed through the use of perturbation equations. A quasi-steady aerodynamic approximation is used, and closed-form solutions for the steady-state oscillations are found for the case of elastic torsional nonlinearities. A numerical example of a soft-hard stiffness characteristic is treated, and the steady-state and stability results given.

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TABLE OF CONTENTS

I. INTRODUCTION

Important aspects of the flutter phenomenon are basically nonlinear in nature and thus cannot be explained by the usual (linear) flutter theory. For example, linear theory predicts that once the critical flutter speed is reached any small disturbance will initiate amplitude build-up without limit. However, limited amplitude flutter is a commonly encountered experimental fact. So also is the dependence of the stability upon the size of the initial disturbance, so that stability often persists beyond the linear critical speed until a sufficiently large and sharp disturbance is encountered. These phenomena are typical of nonlinear systems in general, and indicate that the methods of nonlinear mechanics must be invoked to study such effects in detail.

Previous work, both theoretical and experimental, has been done on nonlinear flutter problems. Analog computer studies of the effects of several typical structural nonlinearities on flutter were made by Woolston, et al. (1,2). A theoretical investigation, using the Method of Equivalent Linearization of Kryloff and Bogoliuboff, has been carried out by Shen and Hsu (3,4) and compares favorably with the analog computations.

An aspect of the problem not previously considered is the stability of the various parts of the steady-state amplitude-velocity curve. This is important because nonlinearities resulting in multiple-valued amplitudes are often encountered. A particularly convenient techique for the stability analysis is found in the Method of Slowly Varying Parameters, also by Kryloff and Bogoliuboff. In this method the problem is reduced to the solution of first order ordinary differential equations in the amplitudes and phases, so that the perturbations about the steady state are governed by first order linear ordinary differential equations with constant coefficients. These equations are distinctly easier to study than the Mathieu-Hill equations resulting from the perturbation of the original second order equations for the displacements.

In this study the application of the Method of Slowly Varying Parameters to the nonlinear flutter problem is treated, including both the steady-state and the stability analysis. Specifically, the incompressible bending-torsion flutter of a two-dimensional airfoil with structural nonlinearities is considered. A linear quasi-steady aerodynamic theory is used, allowing a closed-form solution to be obtained. A closed-form solution for the case of uncoupled elastic torsional nonlinearities is presented, and a specific numerical example is worked out.

II. EQUATIONS OF MOTION

1. General Equations of Motion

The equations of motion for the bending-torsion of a two-dimensional airfoil are

$$F_S + m\ddot{h} + m x_\alpha b \ddot{\alpha} + \mathcal{L} = 0 \qquad (1)$$

$$M_S + I_\alpha \ddot{\alpha} + m x_\alpha b \ddot{h} - \mathcal{M} = 0 \qquad (2)$$

where F_s = translational ("bending") structural force, positive upward

M_s = rotational ("torsion") structural moment, positive counter-clockwise

m = mass of the airfoil

I_α = mass moment of inertia about center of rotation

$\mathcal{L}$ = aerodynamic lift, positive upward

$\mathcal{M}$ = aerodynamic moment, positive clockwise;

the remaining symbols are shown in fig. 1.

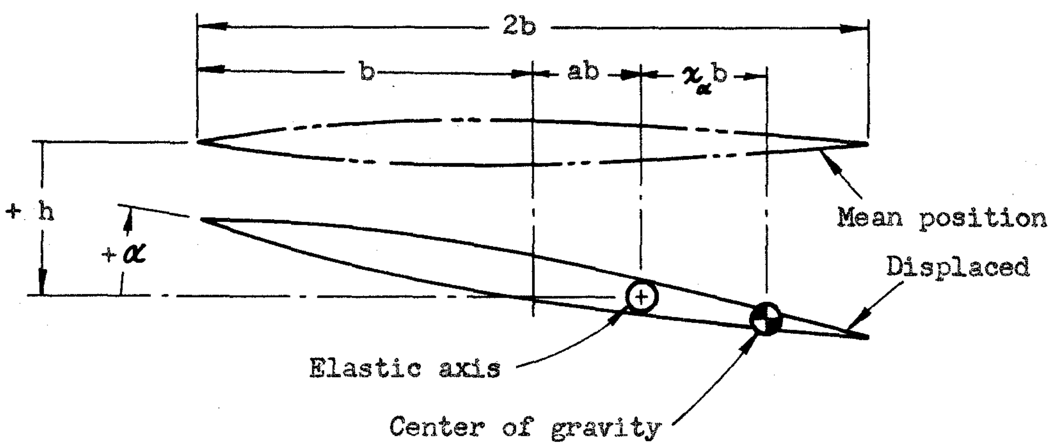

Fig. 1 Description of system

The aerodynamic and structural forces may be linear or nonlinear as eqs. 1 and 2 stand. In the present study structural nonlinearities only are considered, and a linear quasi-steady theory is used for the aerodynamic forces.

2. Nonlinear Structural Forces

Small nonlinearities in the structural forces are considered. These nonlinearities may be dependent on h, α, $\dot{h}$ and/or $\dot{\alpha}$, and thus may be elastic, damping, coupled, or a combination.

$$F_s = \omega_h^2 m \left[h + \epsilon f(h, \alpha, \dot{h}, \dot{\alpha}) \right] \tag{3}$$

$$M_s = \omega_\alpha^2 I_\alpha \left[\alpha + \epsilon g(h, \alpha, \dot{h}, \alpha) \right] \tag{4}$$

The smallness of the nonlinearity is characterized by $\underline{\epsilon}$. The precise statement is

$$\epsilon \ll \frac{h}{f(h, \alpha, \dot{h}, \dot{\alpha})} \quad , \quad \epsilon \ll \frac{\alpha}{g(h, \alpha, \dot{h}, \dot{\alpha})} \tag{5}$$

In the above equations $\omega_h^2 m$ and $\omega_\alpha^2 I_\alpha$ are the spring constants in bending and torsion, respectively.

3. Linear Quasi-Steady Aerodynamic Forces

The air forces used in this study are a quasi-steady approximation to incompressible, unsteady air forces. The approximation is that of Dugundji's, in reference 5. Briefly, the Theodorsen function is set equal to unity and the density ratio, $\underline{\mu}$, is assumed high enough to permit neglecting the apparent mass contribution to the $\underline{hh}$ and $\underline{\alpha\alpha}$ terms. Full details are given in the appendix.

The use of the quasi-steady approximations allows a closed-form solution to be obtained. Of course, the presence of the Theodorsen function precludes this for the complete unsteady formulation.

The quasi-steady lift and moment are given by

$$\mathcal{L} = (2\rho\pi b U^2)\alpha + (2\rho\pi b U)\dot{h} + (\rho\pi b^2 U)2(1-a)\dot{\alpha} - (\rho\pi b^3 a)\ddot{\alpha} \tag{6}$$

$$\mathcal{M} = (2\rho\pi b^2 U^2)(\tfrac{1}{2}+a)\alpha + (2\rho\pi b^2 U)(\tfrac{1}{2}+a)\dot{h} + (2\rho\pi b^3 U a)(\tfrac{1}{2}-a)\dot{\alpha} + (\rho\pi b^3 a)\ddot{h} \tag{7}$$

where ρ is the air density, and U the air speed.

4. Equations of Motion for Quasi-Steady Air Forces and Structural Nonlinearities

Using the above expressions for the forces the equations of motion become

$$\omega_h^2 m[h + \varepsilon f(h,\alpha,\dot{h},\dot{\alpha})] + m\ddot{h} + m x_\alpha b\ddot{\alpha} + 2\rho\pi b U^2\alpha + 2\rho\pi b U\dot{h} + \rho\pi b^2 U 2(1-a)\dot{\alpha} - \rho\pi b^3 a\ddot{\alpha} = 0 \tag{8}$$

$$\omega_\alpha^2 I_\alpha[\alpha + \varepsilon g(h,\alpha,\dot{h},\dot{\alpha})] + I_\alpha\ddot{\alpha} + m x_\alpha\ddot{h} - 2\rho\pi b^2 U^2(\tfrac{1}{2}+a)\alpha - 2\rho\pi b^2 U(\tfrac{1}{2}+a)\dot{h} - 2\rho\pi b^3 U a(\tfrac{1}{2}-a)\dot{\alpha} - \rho\pi b^3 a\ddot{h} = 0 \tag{9}$$

The equations are nondimensionalized in the usual manner for flutter analysis, viz., define a nondimensional time, $\tau = \frac{U}{b}t$, denote derivatives with respect to τ by primes, and divide eq. 8 by $\rho\pi b^3\omega^2$

and eq. 9 by $\rho\pi b^4\omega^2$. Grouping terms, the equations become:

$$\frac{\mu}{k^2}\bar{h}'' + \frac{1}{k^2}(\mu x_\alpha - a)\alpha'' + \frac{2}{k^2}\bar{h}' + \frac{2}{k^2}(1-a)\alpha' + RX\mu\bar{h} + \frac{2}{k^2}\alpha + \varepsilon\mu\bar{f}X = 0 \qquad (10)$$

(1) (2) (3) (4) (5) (6) (7) (8)

$$\frac{1}{k^2}(\mu x_\alpha - a)\bar{h}'' + \frac{1}{k^2}r_\alpha^2\mu\alpha'' - \frac{2}{k^2}\left(\frac{1}{2}+a\right)\bar{h}' - \frac{2}{k^2}\left(\frac{1}{2}-a\right)a\alpha' +$$

(9) (10) (11) (12) (13)

$$\left[X r_\alpha^2\mu - \frac{2}{k^2}\left(\frac{1}{2}+a\right)\right]\alpha + \varepsilon\mu\bar{g}X = 0 \qquad (11)$$

(14) (15) (16)

The new symbols in eqs. 10 and 11 are standard flutter notation, and are defined below.

$\mu = \dfrac{m}{\pi\rho b^2}$, the mass ratio

$k = \dfrac{b\omega}{U}$, the reduced frequency, or Strouhal number.

$r_\alpha = \sqrt{\dfrac{I_\alpha}{mb^2}}$, the nondimensional radius of gyration.

$R = \dfrac{\omega_h^2}{\omega_\alpha^2}$

$X = \dfrac{\omega_\alpha^2}{\omega^2}$

$\bar{h} = \dfrac{h}{b}$

$\bar{f} = \dfrac{Rf}{b}$

$\bar{g} = r_\alpha^2\, g$

It is of interest to identify the origin of each of the terms in eqs. 10 and 11. The numbers below each term refer to the following list:

(1) translational inertia due to bending acceleration

(2) translational inertia due to torsional acceleration

(3) apparent mass due to torsional acceleration

(4) lift at the $\frac{1}{4}$ - chord due to $\frac{\dot{h}}{U}$

(5) combination of lift at the $\frac{1}{4}$ - chord due to $\frac{ab\dot{\alpha}}{U}$ and lift at the $\frac{3}{4}$ - chord due to apparent mass arising from $U\dot{\alpha}$

(6) linear structural restoring force in bending

(7) lift at the $\frac{1}{4}$ - chord due to instantaneous angle of attack

(8) nonlinear structural force in bending

(9) rotational inertia due to bending acceleration

(10) apparent moment of inertia due to bending acceleration

(11) rotational inertia due to torsional acceleration

(12) aerodynamic moment arising from term (4) above

(13) aerodynamic moment associated with term (5) above

(14) linear structural restoring moment in torsion

(15) aerodynamic moment arising from term (7) above

(16) nonlinear structural force in torsion

It is necessary that eqs. 10 and 11 be "uncoupled" in the second derivatives for the application of the method of slowly varying parameters. Multiplying eq. 10 by $\frac{k^2}{\mu} r_\alpha^2$ and eq. 11 by $-\frac{k^2}{\mu}\bar{x}_\alpha$, defining $\bar{x}_\alpha = x_\alpha - \frac{a}{\mu}$ and adding results in an equation in h'' only. Multiplying eq. 10 by $-\frac{k^2}{\mu}\bar{x}_\alpha$ and eq. 11 by $\frac{k^2}{\mu}$ and adding results in an equation in α'' only. The two equations are:

$$(r_\alpha^2 - \bar{x}_\alpha^2)\bar{h}'' + \frac{2}{\mu}\left[r_\alpha^2 + \bar{x}_\alpha(\tfrac{1}{2} + a)\right]\bar{h}' +$$

$$\frac{2}{\mu}\left[r_\alpha^2(1-a) + \bar{x}_\alpha(\tfrac{1}{2}-a)a\right]\alpha' + k^2 r_\alpha^2 R X \bar{h} +$$

$$\left\{\frac{2}{\mu}\left[r_\alpha^2 + \bar{x}_\alpha(\tfrac{1}{2}+a)\right] - k^2\bar{x}_\alpha r_\alpha^2 X\right\}\alpha + \epsilon k^2 X(r_\alpha^2\bar{f} - \bar{x}_\alpha\bar{g}) = 0 \qquad (12)$$

$$(r_\alpha^2 - \bar{x}_\alpha^2)\alpha'' - \frac{2}{\mu}(\tfrac{1}{2} + a + \bar{x}_\alpha)\bar{h}' -$$

$$\frac{2}{\mu}\left[a(\tfrac{1}{2}-a) + \bar{x}_\alpha(1-a)\right]\alpha' - k^2\bar{x}_\alpha R X \bar{h} +$$

$$\left[k^2 X r_\alpha^2 - \frac{2}{\mu}(\tfrac{1}{2} + a + \bar{x}_\alpha)\right]\alpha + \epsilon k^2 X(\bar{g} - \bar{x}_\alpha\bar{f}) = 0 \qquad (13)$$

This is the final form of the equations of motion to which the method of slowly varying parameters will be applied.

III. METHOD OF SOLUTION

1. Introduction

The Method of Slowly Varying Parameters is well known in the theory of nonlinear oscillations (see references (6) and (7) for example). It is variously known as the Method of the First Approximation of Kryloff and Bogoliuboff, the Method of Slowly Varying Amplitude and Phase, and by the name used in this study. It should not be confused with the Method of Equivalent Linearization, also by Kryloff and Bogoliuboff.

The S.V.P.* method requires that the equations to be studied be quasi-linear, i.e., that the nonlinearities be small. This makes the harmonic (first) approximation to the motion quite good and leads to the slow variation of the amplitude and phase. Approximations of higher order can be made but for practical purposes the first approximation is almost always sufficient.

While the S.V.P. method is usually applied to single degree of freedom systems, the methods of solution remain essentially the same for multiple degree of freedom systems. First, the displacements corresponding to the n degrees of freedom are replaced by the amplitudes and phase angles. This changes the system of n second order equation to one of $2n$ first order equations, the dependent variables being n amplitudes, $(n - 1)$ phases (since one phase is arbitrary in the free vibration problem) and the frequency. Second, all quantities are averaged over a cycle, which leads to the harmonic motion approximation to the more complex exact motion. This "first approximation"

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* For convenience abbreviate Slowly Varying Parameters to S.V.P.

contains all the significant nonlinear behavior of quasi-linear systems, such as limit cycles and amplitude dependence.

The simplified first order equations resulting from the averaging process are used to study the steady-state oscillations by setting the τ derivatives equal to zero. The resulting set of n algebraic non-linear simultaneous equations is solved for the steady-state amplitudes, phases, and frequency.

The stability of the steady-state is studied by perturbing the steady-state solutions. Since only small perturbations are considered the n first order equations become linear in amplitude and phase perturbations. Furthermore since the dependent variables are amplitude and phase, rather than displacement, the equations have constant coefficients. Thus relatively simple linear theory can be used in the stability analysis.

It will be noted that the oscillations considered are "single-frequency", i.e. only one frequency is associated with the multiple degree of freedom system. In an ordinary vibration problem this might be a considerable restriction, since one would expect to find as many characteristic frequencies as there are degrees of freedom. Fortunately, flutter oscillations are peculiarly suited to a single-frequency analysis. It is a well-documented fact (see references (8), (9), (10)) that bending-torsion flutter systems manifest "frequency coalescence" as the critical flutter speed is approached. Forced vibration studies, both experimental and theoretical, indicate that the two characteristic frequencies approach a common frequency at or near flutter. Thus the requirement of single-frequency oscillations is not restrictive in the present case.

2. Application of Method

Change dependent variables $\bar{h}(\tau)$ and $\alpha(\tau)$ in eqs. 10 and 11 as follows:

$$\bar{h}(\tau) = a_1(\tau)\sin[k\tau + \varphi_1(\tau)] \tag{14}$$

$$\alpha(\tau) = a_2(\tau)\sin[k\tau + \varphi_2(\tau)] \tag{15}$$

Since four new variables have been introduced to replace the original two, four relations must be defined. The remaining two are:

$$\bar{h}'(\tau) = a_1(\tau)\, k\cos[k\tau + \varphi_1(\tau)] \tag{16}$$

$$\alpha'(\tau) = a_2(\tau)\, k\cos[k\tau + \varphi_1(\tau)] \tag{17}$$

Since a_1 and φ_1 are functions of $\underline{\tau}$ eqs. 16 and 17 require

$$a_1'\sin(k\tau + \varphi_1) + \varphi_1' a_1\cos(k\tau + \varphi_1) = 0 \tag{18}$$

$$a_2'\sin(k\tau + \varphi_2) + \varphi_2' a_2\cos(k\tau + \varphi_2) = 0 \tag{19}$$

Differentiating eqs. 16 and 17 to obtain the expressions for the second derivatives, and substituting into eqs. 12 and 13 yields two first order equations. Let $\theta_1 = k\tau + \varphi_1$ and $\theta_2 = k\tau + \varphi_2$.

$$(h_\alpha^2 - \bar{x}_\alpha^2)[a_1' k\cos\theta_1 - a_1 k(k+\varphi_1')\sin\theta_1] + \frac{2}{\mu}[h_\alpha^2 + \bar{x}_\alpha(\tfrac{1}{2}+a)]a_1 k\cos\theta_1 +$$
$$\frac{2}{\mu}[h_\alpha^2(1-a) + \bar{x}_\alpha(\tfrac{1}{2}-a)a]a_2 k\cos\theta_2 + k^2 h_\alpha^2 R X a_1\sin\theta_1 +$$
$$\left\{\frac{2}{\mu}[h_\alpha^2 + \bar{x}_\alpha(\tfrac{1}{2}+a)] - k^2\bar{x}_\alpha h_\alpha^2 X\right\} a_2\sin\theta_2 + \varepsilon k^2 X(h_\alpha^2\bar{\xi} - \bar{x}_\alpha\bar{q}) = 0 \tag{20}$$

$$(\lambda_\alpha^2-\bar{x}_\alpha^2)[a_2' k\cos\theta_2 - a_2 k(k+\varphi_2')\sin\theta_2] - \frac{2}{\mu}(\tfrac{1}{2}+a+\bar{x}_\alpha)a_1 k\cos\theta_1 -$$

$$\frac{2}{\mu}[a(\tfrac{1}{2}-a)+\bar{x}_\alpha(1-a)]a_2 k\cos\theta_2 - k^2\bar{x}_\alpha R X a_1\sin\theta_1 +$$

$$[k^2 X\lambda_\alpha^2 - \frac{2}{\mu}(\tfrac{1}{2}+a+\bar{x}_\alpha)]a_2\sin\theta_2 + \epsilon k^2 X(\bar{g}-\bar{x}_\alpha\bar{f}) = 0 \qquad (21)$$

The above equations and eqs. 18 and 19 form the set of four first order equations. Explicit expressions for a_1', a_2', φ_1', and φ_2' can be obtained by algebraic manipulation.

The four explicit first order equations are

$$(\lambda_\alpha^2-\bar{x}_\alpha^2)a_1' = -\frac{2}{\mu}[\lambda_\alpha^2+\bar{x}_\alpha(\tfrac{1}{2}+a)]a_1\cos^2\theta_1 - \frac{2}{\mu}[\lambda_\alpha^2(1-a)+\bar{x}_\alpha a(\tfrac{1}{2}-a)]a_2\cos\theta_1\cos\theta_2 -$$

$$k[\lambda_\alpha^2 R X-(\lambda_\alpha^2-\bar{x}_\alpha^2)]a_1\sin\theta_1\cos\theta_1 -$$

$$\left\{\frac{2}{\mu k}[\lambda_\alpha^2+\bar{x}_\alpha(\tfrac{1}{2}+a)]-k\bar{x}_\alpha\lambda_\alpha^2 X\right\}a_2\cos\theta_1\sin\theta_2 - \epsilon k X(\lambda_\alpha^2\bar{f}-\bar{x}_\alpha\bar{g})\cos\theta_1 \qquad (22)$$

$$(\lambda_\alpha^2-\bar{x}_\alpha^2)a_2' = \frac{2}{\mu}(\tfrac{1}{2}+a+\bar{x}_\alpha)a_1\cos\theta_1\cos\theta_2 + \frac{2}{\mu}[a(\tfrac{1}{2}-a)+\bar{x}_\alpha(1-a)]a_2\cos^2\theta_2 +$$

$$k\bar{x}_\alpha R X a_1\sin\theta_1\cos\theta_2 - \left\{k[\lambda_\alpha^2 X-(\lambda_\alpha^2-\bar{x}_\alpha^2)] - \frac{2}{\mu k}(\tfrac{1}{2}+a+\bar{x}_\alpha)\right\}a_2\sin\theta_2\cos\theta_2 -$$

$$\epsilon k X(\bar{g}-\bar{x}_\alpha\bar{f})\cos\theta_2 \qquad (23)$$

$$(\lambda_\alpha^2-\bar{x}_\alpha^2)\varphi_1' = \frac{2}{\mu}[\lambda_\alpha^2+\bar{x}_\alpha(\tfrac{1}{2}+a)]\sin\theta_1\cos\theta_1 + \frac{2}{\mu}[\lambda_\alpha^2(1-a)+\bar{x}_\alpha a(\tfrac{1}{2}-a)]\frac{a_2}{a_1}\sin\theta_1\cos\theta_2 +$$

$$k[\lambda_\alpha^2 R X-(\lambda_\alpha^2-\bar{x}_\alpha^2)]\sin^2\theta_1 + \left\{\frac{2}{\mu k}[\lambda_\alpha^2+\bar{x}_\alpha(\tfrac{1}{2}+a)] - k\bar{x}_\alpha\lambda_\alpha^2 X\right\}\frac{a_2}{a_1}\sin\theta_1\sin\theta_2 + \qquad (24)$$

$$\frac{\epsilon k X}{a_1}(\lambda_\alpha^2\bar{f}-\bar{x}_\alpha\bar{g})\sin\theta_1$$

$$(r_\alpha^2-\bar{x}_\alpha^2)\varphi_2' = -\frac{2}{\mu}\left(\frac{1}{2}+a+\bar{x}_\alpha\right)\frac{a_1}{a_2}\cos\theta_1\sin\theta_2 - \frac{2}{\mu}\left[a\left(\frac{1}{2}-a\right)+\bar{x}_\alpha(1-a)\right]\sin\theta_2\cos\theta_2 -$$

$$k\bar{x}_\alpha R X\frac{a_1}{a_2}\sin\theta_1\sin\theta_2 + \left\{k\left[r_\alpha^2 X-(r_\alpha^2-\bar{x}_\alpha^2)\right]-\frac{2}{\mu k}\left(\frac{1}{2}+a+\bar{x}_\alpha\right)\right\}\sin^2\theta_2 +$$

$$\frac{\epsilon k X}{a_2}(\bar{g}-\bar{x}_\alpha\bar{f})\sin\theta_2 \qquad (25)$$

Note that these equations are exact. The approximation arises in the averaging which follows.

Since the amplitudes and phases are slowly varying*, they may be regarded as constant during a cycle, and eqs. 22 to 25 may be averaged over a cycle. This eliminates explicit dependence on τ, and simplifies the equations considerably. Denote the average value of the nonlinear expressions as follows:

$$\left.\begin{aligned}
N_{1c} &= \frac{\epsilon k X}{2\pi}\int_0^{2\pi}(r_\alpha^2\bar{f}-\bar{x}_\alpha\bar{g})\cos\theta_1\,d\theta_1 \\
N_{2c} &= \frac{\epsilon k X}{2\pi}\int_0^{2\pi}(\bar{g}-\bar{x}_\alpha\bar{f})\cos\theta_2\,d\theta_2 \\
N_{1s} &= \frac{\epsilon k X}{2\pi}\int_0^{2\pi}(r_\alpha^2\bar{f}-\bar{x}_\alpha\bar{g})\sin\theta_1\,d\theta_1 \\
N_{2s} &= \frac{\epsilon k X}{2\pi}\int_0^{2\pi}(\bar{g}-\bar{x}_\alpha\bar{f})\sin\theta_2\,d\theta_2 .
\end{aligned}\right\} \qquad (26)$$

The averaged equations are

$$(r_\alpha^2-\bar{x}_\alpha^2)a_1' = -\frac{1}{\mu}\left[r_\alpha^2+\bar{x}_\alpha\left(\frac{1}{2}+a\right)\right]a_1 - \frac{1}{\mu}\left[r_\alpha^2(1-a)+\bar{x}_\alpha a\left(\frac{1}{2}-a\right)\right]a_2\cos\varphi +$$

$$\left\{\frac{1}{\mu k}\left[r_\alpha^2+\bar{x}_\alpha\left(\frac{1}{2}+a\right)\right]-\frac{1}{2}k\bar{x}_\alpha r_\alpha^2 X\right\}a_2\sin\varphi - N_{1c} \qquad (27)$$

* see p. 19 regarding slow variation assumption

$$(r_\alpha^2-\bar{x}_\alpha^2)a_2' = \frac{1}{\mu}\left(\frac{1}{2}+a+\bar{x}_\alpha\right)a_1\cos\varphi + \frac{1}{\mu}\left[a\left(\frac{1}{2}-a\right)+\bar{x}_\alpha(1-a)\right]a_2 + \frac{1}{2}k\bar{x}_\alpha RXa_1\sin\varphi - N_{2c} \tag{28}$$

$$a_1(r_\alpha^2-\bar{x}_\alpha^2)\varphi_1' = \frac{1}{\mu}\left[r_\alpha^2(1-a)+\bar{x}_\alpha a\left(\frac{1}{2}-a\right)\right]a_2\sin\varphi + \frac{k}{2}\left[r_\alpha^2RX-(r_\alpha^2-\bar{x}_\alpha^2)\right]a_1 + \left\{\frac{1}{\mu k}\left[r_\alpha^2+\bar{x}_\alpha\left(\frac{1}{2}+a\right)\right]-\frac{1}{2}k\bar{x}_\alpha r_\alpha^2X\right\}a_2\cos\varphi + N_{1s} \tag{29}$$

$$a_2(r_\alpha^2-\bar{x}_\alpha^2)\varphi_2' = \frac{1}{\mu}\left(\frac{1}{2}+a+\bar{x}_\alpha\right)a_1\sin\varphi - \frac{1}{2}k\bar{x}_\alpha RXa_1\cos\varphi + \left\{\frac{k}{2}\left[r_\alpha^2X-(r_\alpha^2-\bar{x}_\alpha^2)\right]-\frac{1}{\mu k}\left(\frac{1}{2}+a+\bar{x}_\alpha\right)\right\}a_2 + N_{2s} \tag{30}$$

where $\varphi = \varphi_1 - \varphi_2 = \theta_1 - \theta_2$

The above equations are the basic results of the Method of Slowly Varying Parameters, and are used to study the steady-state oscillations and their stability.

For future convenience, the equations may be written in simpler form with the definition of some new symbols. Let the equations be

$$Qa_1' = Ba_1 + (-C\sin\varphi + D\cos\varphi)a_2 - N_{1c} \tag{31}$$

$$Qa_2' = (E\sin\varphi + F\cos\varphi)a_1 + Ha_2 - N_{2c} \tag{32}$$

$$a_1Q\varphi_1' = -Aa_1 - (D\sin\varphi + C\cos\varphi)a_2 + N_{1s} \tag{33}$$

$$a_2Q\varphi_2' = (F\sin\varphi - E\cos\varphi)a_1 - Ga_2 + N_{2s} \tag{34}$$

where

$$\left.\begin{aligned}
A &= -\frac{k}{2}[r_\alpha^2 RX - (r_\alpha^2 - \bar{x}_\alpha^2)] \\
B &= -\frac{1}{\mu}[r_\alpha^2 + \bar{x}_\alpha(\tfrac{1}{2}+a)] \\
C &= -\left\{\frac{1}{\mu k}[r_\alpha^2 + \bar{x}_\alpha(\tfrac{1}{2}+a)] - \tfrac{1}{2}k\bar{x}_\alpha r_\alpha^2 X\right\} \\
D &= -\frac{1}{\mu}[r_\alpha^2(1-a) + \bar{x}_\alpha a(\tfrac{1}{2}-a)] \\
E &= \tfrac{1}{2}k\bar{x}_\alpha RX \\
F &= \frac{1}{\mu}(\tfrac{1}{2}+a+\bar{x}_\alpha) \\
G &= -\left\{\frac{k}{2}[r_\alpha^2 X - (r_\alpha^2 - \bar{x}_\alpha^2)] - \frac{1}{\mu k}(\tfrac{1}{2}+a+\bar{x}_\alpha)\right\} \\
H &= \frac{1}{\mu}[a(\tfrac{1}{2}-a) + \bar{x}_\alpha(1-a)] \; ; \qquad Q = r_\alpha^2 - \bar{x}_\alpha^2
\end{aligned}\right\} \quad (35)$$

Note that this form exhibits variables a_1, a_2 and φ but not variables k and X. This turns out to be convenient for later use.

3. Steady-State Oscillations

The steady-state oscillations, i.e., at constant amplitudes and phases, are the nonlinear equivalent of the flutter condition. Setting a_1', a_2', φ_1', and φ_2' all equal to zero in eqs. 27 to 30 results in four algebraic simultaneous equations in the five variables a_1, a_2, φ, k and X. Note that in the free vibration problem one phase is arbitrary and thus only the phase difference is significant.

The equations can be solved to give velocity as a function of amplitude, i.e., a closed-form solution, for the case of elastic nonlinearities in one of the degrees of freedom, and such a solution is carried out in Parts IV and V. For more complicated cases, considerable algebraic reduction can still be achieved, however, some iteration may be necessary to obtain numerical results.

4. Stability of Steady-State Oscillations

The procedure in the preceding paragraphs generates relations between amplitudes, phase, frequency, and velocity which may exist at steady-state, but makes no statement regarding the stability of small perturbations from the steady state. If, in fact, parts of the steady-state relations are unstable in the above sense they will not be encountered in practice, since they are perhaps more accurately described as critical energy levels, which separate motions converging on different amplitudes, or limit cycles. Unstable portions of the amplitude--velocity curve arise frequently due to soft-hard or similar types of stiffnesses, which result in multiple-valued amplitudes for certain ranges of velocity. Thus an investigation of the stability is seen to be an important part of the nonlinear flutter problem.

Assume the steady-state equations have been solved and denote the steady-state values by a_{1o}, a_{2o}, φ_o, k_o and X_o. The perturbations are defined as follows:

$$\left.\begin{aligned} a_1 &= a_{1_0} + \xi_1 \\ a_2 &= a_{2_0} + \xi_2 \\ &\left.\begin{aligned} \varphi_1 &= \varphi_{1_0} + \eta_1 \\ \varphi_2 &= \varphi_{2_0} + \eta_2 \end{aligned}\right\} \varphi = \varphi_0 + \eta \;;\; i.e.,\; \eta = \eta_1 - \eta_2 \end{aligned}\right\} \quad (36)$$

The perturbations are assumed to occur at constant velocity and the frequency perturbations are accounted for by $\underline{\eta}$, so that $\underline{k}$

and X do not require perturbation.

Substituting eqs. 36 into eqs. 31 to 34 yields

$$Q\xi_1' = B(a_{1_0}+\xi_1) - C(a_{2_0}+\xi_2)\sin(\varphi_0+\eta) + D(a_{2_0}+\xi_2)\cos(\varphi_0+\eta) -$$

$$\left[N_{1c}(a_{1_0},a_{2_0},\varphi_0) + \frac{\partial N_{2c}}{\partial a_1}\Big|_0 \xi_1 + \frac{\partial N_{1c}}{\partial a_2}\Big|_0 \xi_2 + \frac{\partial N_{1c}}{\partial \varphi}\Big|_0 \eta + \cdots\right] \qquad (37)$$

$$Q\xi_2' = E(a_{1_0}+\xi_1)\sin(\varphi_0+\eta) + F(a_{1_0}+\xi_1)\cos(\varphi_0+\eta) + H(a_{2_0}+\xi_2) -$$

$$\left[N_{2c}(a_{1_0},a_{2_0},\varphi_0) + \frac{\partial N_{2c}}{\partial a_1}\Big|_0 \xi_1 + \frac{\partial N_{2c}}{\partial a_2}\Big|_0 \xi_2 + \frac{\partial N_{2c}}{\partial \varphi}\Big|_0 \eta + \cdots\right] \qquad (38)$$

$$(a_{1_0}+\xi_1)Q\eta_1' = -A(a_{1_0}+\xi_1) - D(a_{2_0}+\xi_2)\sin(\varphi_0+\eta) - C(a_{2_0}+\xi_2)\cos(\varphi_0+\eta) +$$

$$\left[N_{1s}(a_{1_0},a_{2_0},\varphi_0) + \frac{\partial N_{1s}}{\partial a_1}\Big|_0 \xi_1 + \frac{\partial N_{1s}}{\partial a_2}\Big|_0 \xi_2 + \frac{\partial N_{1s}}{\partial \varphi}\Big|_0 \eta + \cdots\right] \qquad (39)$$

$$(a_{2_0}+\xi_2)Q\eta_2' = F(a_{1_0}+\xi_1)\sin(\varphi_0+\eta) - E(a_{1_0}+\xi_1)\cos(\varphi_0+\eta) - G(a_{2_0}+\xi_2) +$$

$$\left[N_{2s}(a_{1_0},a_{2_0},\varphi_0) + \frac{\partial N_{2s}}{\partial a_1}\Big|_0 \xi_1 + \frac{\partial N_{2s}}{\partial a_2}\Big|_0 \xi_2 + \frac{\partial N_{2s}}{\partial \varphi}\Big|_0 \eta + \cdots\right]. \qquad (40)$$

To first order in the perturbations, and with the subtraction of sums equal to zero by virtue of the fact that a_{10}, a_{20}, φ_0, k_0, and X_0 are solutions of the steady-state equations, the perturbation equations become

$$Q\xi_1' = \left(B - \frac{\partial N_{1c}}{\partial a_1}\Big|_0\right)\xi_1 + \left(-C\sin\varphi_0 + D\cos\varphi_0 - \frac{\partial N_{1c}}{\partial a_2}\Big|_0\right)\xi_2 +$$

$$\left(-Ca_{2_0}\cos\varphi_0 - Da_{2_0}\sin\varphi_0 - \frac{\partial N_{1c}}{\partial \varphi}\Big|_0\right)\eta \qquad (41)$$

$$Q\xi_2' = \left(E\sin\varphi_0 + F\cos\varphi_0 - \frac{\partial N_{2c}}{\partial a_1}\Big|_0\right)\xi_1 + \left(H - \frac{\partial N_{2c}}{\partial a_2}\Big|_0\right)\xi_2 +$$

$$\left(E a_{1_0}\cos\varphi_0 - F a_{1_0}\sin\varphi_0 - \frac{\partial N_{2c}}{\partial\varphi}\Big|_0\right)\eta \qquad (42)$$

$$Q\eta' = \left(-\frac{A}{a_{1_0}} - \frac{F}{a_{2_0}}\sin\varphi_0 + \frac{E}{a_{2_0}}\cos\varphi_0 + \frac{1}{a_{1_0}}\frac{\partial N_{1s}}{\partial a_1}\Big|_0 - \frac{1}{a_{2_0}}\frac{\partial N_{2s}}{\partial a_1}\right)\xi_1 +$$

$$\left(-\frac{D}{a_{1_0}}\sin\varphi_0 - \frac{C}{a_{1_0}}\cos\varphi_0 + \frac{G}{a_{2_0}} + \frac{1}{a_{1_0}}\frac{\partial N_{1s}}{\partial a_2}\Big|_0 - \frac{1}{a_{2_0}}\frac{\partial N_{2s}}{\partial a_2}\Big|_0\right)\xi_2 +$$

$$\left(-\frac{D a_{2_0}}{a_{1_0}}\cos\varphi_0 + \frac{C a_{2_0}}{a_{1_0}}\sin\varphi_0 - \frac{F a_{1_0}}{a_{2_0}}\cos\varphi_0 - \frac{E a_{1_0}}{a_{2_0}}\sin\varphi_0 + \frac{1}{a_{1_0}}\frac{\partial N_{1s}}{\partial\varphi}\Big|_0 - \frac{1}{a_{2_0}}\frac{\partial N_{2s}}{\partial\varphi}\Big|_0\right)\eta. \qquad (43)$$

Note that, again, since only $\eta = \eta_1 - \eta_2$ is significant there are only three equations.

The above equations determine the stability of small perturbations about the steady state. Since they are linear differential equations with constant coefficients they can be solved by standard techniques. Specifically, it can be assumed that

$$\xi_1 = \bar{\xi}_1 e^{\lambda\tau}, \quad \xi_2 = \bar{\xi}_2 e^{\lambda\tau}, \quad \eta = \bar{\eta} e^{\lambda\tau}.$$

Then eqs. 41 to 43 become

$$\lambda\bar{\xi}_1 = \alpha_1\bar{\xi}_1 + \beta_1\bar{\xi}_2 + \gamma_1\bar{\eta}$$

$$\lambda\bar{\xi}_2 = \alpha_2\bar{\xi}_1 + \beta_2\bar{\xi}_2 + \gamma_2\bar{\eta}$$

$$\lambda\bar{\eta} = \alpha_3\bar{\xi}_1 + \beta_3\bar{\xi}_2 + \gamma_3\bar{\eta}$$

where α_i, β_i and γ_i are defined in an obvious manner. For non-trivial solutions the determinant of the coefficients must be zero.

The determinant generates a cubic polynomial in λ. The stability, i.e., real part of λ greater than or less than zero, may then be studied with the aid of Routh's criteria for cubic polynomials, or by actually solving for the roots.

In addition to stability studies, the perturbation equations provide an *a posteriori* check on the assumption of slow variations. Since λ characterizes the rate of change of the perturbations, its magnitude is the critical factor, and the roots of the cubic in λ give this information.

It is not always necessary that all the roots be small, however, since some may be associated with initial transients which do not affect the significant part of the perturbation motion. The example in Part V illustrates this point. There the three roots consist of one real, small root which determines the stability, and two rather large, complex conjugate roots (negative real part) which correspond to a highly damped oscillatory mode which will be in evidence only for a short time after the perturbation is initiated.

Since the solutions obtained are approximate, the question arises as to whether some undesirable large roots may have been precluded by the averaging approximation itself. Such questions generally accompany *a posteriori* justifications of this kind, and are usually answered on physical grounds. For the present case of quasi-linear systems there appears to be little danger that such values of λ have been precluded.

Further analysis of the stability in these general terms is beyond the scope of this study. In any specific numerical example, however, the procedure is straightforward. In Part V such a calculation will be carried out.

IV. CLOSED-FORM SOLUTION FOR ELASTIC TORSIONAL NONLINEARITY

Solution of the set of four simultaneous nonlinear algebraic equations for the steady-state oscillations (eqs. 27 to 30 or 31 to 34) appears to be a formidable task, however, there is a relatively simple procedure which yields closed-form results in some cases, and almost closed-form results in other cases. The closed-form solutions are possible for the important case of nonlinearities in one degree of freedom, which are dependent only on the amplitude of that degree of freedom, i.e., for single, uncoupled, elastic nonlinearities. The procedure will be shown for an elastic, uncoupled nonlinearity in torsion.

For an elastic torsional nonlinearity the functions N_{1c}, N_{2c}, N_{1s}, and N_{2s}, defined in eqs. 26, take on a simple form.

$$N_{1c} = -\frac{\epsilon k X \bar{x}_\alpha}{2\pi}\int_0^{2\pi} \bar{g}\cos\theta_1\, d\theta_1 = -\frac{\epsilon k X \bar{x}_\alpha r_\alpha^2}{2\pi}\int_0^{2\pi} g(a_2\sin\theta_2)\cos\theta_1\, d\theta_1$$

$$N_{1c} = -\frac{\epsilon k X \bar{x}_\alpha r_\alpha^2}{2\pi}\int_{-\varphi}^{2\pi-\varphi} g(a_2\sin\theta_2)\left[\cos\theta_2\cos\varphi - \sin\theta_2\sin\varphi\right] d\theta_2$$

$$N_{1c} = \bar{x}_\alpha k\sigma \sin\varphi\, a_2 \tag{44}$$

Similarly,

$$N_{2c} = 0 \tag{45}$$

$$N_{1s} = -\bar{x}_\alpha k\sigma\cos\varphi\, a_2 \tag{46}$$

$$N_{2s} = k\sigma a_2 \tag{47}$$

where

$$\sigma = \frac{\epsilon X h_\alpha^2}{2\pi a_2}\int_{-\varphi}^{2\pi-\varphi} g(a_2 \sin\theta_2)\sin\theta_2 \, d\theta_2 \tag{48}$$

Note that σ is a function of a_2, torsional amplitude.

With the definitions of eqs. 44 to 47 the steady-state equations corresponding to eqs. 31 to 34 become

$$Ba_1 + \left[-(C+\bar{x}_\alpha k\sigma)\sin\varphi + D\cos\varphi\right]a_2 = 0 \tag{49}$$

$$\left[E\sin\varphi + F\cos\varphi\right]a_1 + Ha_2 = 0 \tag{50}$$

$$Aa_1 + \left[D\sin\varphi + (C+\bar{x}_\alpha k\sigma)\cos\varphi\right]a_2 = 0 \tag{51}$$

$$\left[F\sin\varphi - E\cos\varphi\right]a_1 - (G-k\sigma)a_2 = 0 \tag{52}$$

Working with eqs. 49 and 51 leads to the system

$$(-B\sin\varphi + A\cos\varphi)a_1 + (C+\bar{x}_\alpha k\sigma)a_2 = 0 \tag{53}$$

$$(B\cos\varphi + A\sin\varphi)a_1 + Da_2 = 0 \tag{54}$$

$$(E\sin\varphi + F\cos\varphi)a_1 + Ha_2 = 0 \tag{55}$$

$$(E\cos\varphi - F\sin\varphi)a_1 + (G-k\sigma)a_2 = 0 \tag{56}$$

From eqs. 54 and 55 an expression for the amplitude ratio, in terms of φ only, is obtained.

$$\frac{a_1}{a_2} = -\frac{D}{B\cos\varphi + A\sin\varphi} = -\frac{H}{E\sin\varphi + F\cos\varphi} \tag{57}$$

From eqn. 57

$$\left.\begin{aligned}
\tan\varphi &= \frac{DF-BH}{AH-DE} \\
\sin\varphi &= \frac{DF-BH}{\sqrt{(DF-BH)^2+(AH-DE)^2}} \\
\cos\varphi &= \frac{AH-DE}{\sqrt{(DF-BH)^2+(AH-DE)^2}} \\
\frac{a_1}{a_2} &= \frac{\sqrt{(DF-BH)^2+(AH-DE)^2}}{BE-AF}
\end{aligned}\right\} \tag{58}$$

Substituting eqs. 58 into eqs. 53 and 56 yields

$$\left.\begin{aligned}
-B(DF-BH)+A(AH-DE)-(AF-BE)(C+\bar{x}_\alpha k\sigma) &= 0 \\
E(AH-DE)-F(DF-BH)-(AF-BE)(G-k\sigma) &= 0
\end{aligned}\right\} \tag{59}$$

Working with eqs. 59 yields finally

$$BH-DF+(C+\bar{x}_\alpha k\sigma)E-(G-k\sigma)A=0 \tag{60}$$

$$DE-HA+(C+\bar{x}_\alpha k\sigma)F-(G-k\sigma)B=0 \tag{61}$$

When the definitions from eqs. 35 are substituted into eqs. 60 and 61 only collection of terms is necessary to obtain relations between flutter frequency, flutter velocity, and torsional amplitude. Equation 61 yields

$$\frac{1}{X} = \frac{\omega^2}{\omega_\alpha^2} = \frac{r_\alpha^2 + R(a-\frac{1}{2})a + 2r_\alpha^2\delta}{r_\alpha^2 + 2\bar{x}_\alpha a + a^2 - \frac{1}{2}(\bar{x}_\alpha + a)} \qquad (62)$$

Equation 60 yields

$$\frac{1}{k^2} = \frac{r_\alpha^2 - \bar{x}_\alpha^2 - r_\alpha^2 X[1 + R(1-X) + 2(1-RX)\delta]}{\frac{2}{\mu}[RX(\frac{1}{2}+a) - (\frac{1}{2}+a+\bar{x}_\alpha - \frac{1}{\mu})]} \qquad (63)$$

In the above

$$\delta = \frac{\sigma}{X r_\alpha^2} = \frac{\epsilon}{2\pi a_2}\int_{-\varphi}^{2\pi-\varphi} g(a_2\sin\theta_2)\sin\theta_2 d\theta_2 = \text{fnc}(a_2 \text{ only}) \qquad (64)$$

From eqs. 62 and 63 one obtains

$$\left(\frac{U}{b\omega_\alpha}\right)^2 = \left(\frac{U}{b\omega}\right)^2\left(\frac{\omega}{\omega_\alpha}\right)^2 = \frac{1}{k^2X}$$

$$\left(\frac{U}{b\omega_\alpha}\right)^2 = \frac{\mu}{2}\left\{\frac{\frac{1}{X}[r_\alpha^2 - \frac{1}{X}(r_\alpha^2 - \bar{x}_\alpha^2)] - Rr_\alpha^2(1-\frac{1}{X}) + 2r_\alpha^2(\frac{1}{X}-R)\delta}{\frac{1}{X}(\frac{1}{2}+a+\bar{x}_\alpha - \frac{1}{\mu}) - R(a+\frac{1}{2})}\right\} \qquad (65)$$

Since $\underline{\delta}$ is a given function of a_2, $\underline{X}$, and hence $\underline{U}$, can be found directly from eqs. 62 and 65, thus forming a closed-form solution.

In carrying out a calculation the procedure would be as follows. 1) Assume a value of a_2, thus determining $\underline{\delta}$. 2) Using $\underline{\delta}$, compute $\underline{X}$. 3) Using $\underline{X}$, compute $U/b\omega_\alpha$ and $\underline{k}$. 4) Phase difference, $\underline{\varphi}$, and amplitude ratio, $\frac{a_1}{a_2}$, can then be computed from the following formulas, derived from eqs. 58.

$$\tan\varphi = -\frac{1}{\mu k\left[\bar{x}_\alpha(1-a) + (1-RX)\left(\frac{a}{2}-a^2\right)\right]} \tag{66}$$

$$\frac{a_1}{a_2} = \frac{r_\alpha^2(1-a) + \bar{x}_\alpha a\left(\frac{1}{2}-a\right)}{\frac{\mu k}{2}\left[r_\alpha^2 - \bar{x}_\alpha^2 - RXr_\alpha^2\right]\sin\varphi - \left[r_\alpha^2 + \bar{x}_\alpha\left(\frac{1}{2}+a\right)\right]\cos\varphi} \tag{67}$$

Equations 62, 65, 66, and 67 constitute a complete solution for the steady-state oscillations in the nonlinear flutter problem for an elastic torsional nonlinearity*. A numerical example is worked out in Part V.

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* Note that when $\delta = 0$ (the linear case) eqs. 62 and 65 reduce to eqs. 3 and 4 of reference (5).

V. NUMERICAL EXAMPLE: SOFT-HARD QUINTIC CHARACTERISTIC

1. Specification of Example

The example chosen for this study is particularly interesting from the point of view of stability, since it manifests multiple-valued steady-state amplitudes over part of the velocity range. The perturbation analysis shows that the limit cycles corresponding to these amplitudes vary alternately in their stability, a result one might intuitively expect from single degree of freedom theory. Such multiple-valuedness arises from the fact that the relative size of the nonlinearity, $\frac{\varepsilon g}{\alpha}$, is not a monotonic function of the amplitude. Examples of stiffness characteristics of this type are shown in fig. 2, together with the relative size of the nonlinearity.

The specific characteristic chosen is a soft-hard quintic, given by

$$M_S = \omega_\alpha^2 I_\alpha(\alpha - 4\alpha^3 + 32\alpha^5) \qquad (68)$$

Hence

$$\varepsilon g = -4\alpha^3 + 32\alpha^5 \qquad (69)$$

It is plotted in fig. 3.

The system parameters are

$$s_\alpha^2 = 0.25, \quad \frac{\omega_h}{\omega_\alpha} = 0.2, \quad a = -0.4, \quad x_\alpha = 0.1, \quad \mu = 10. \qquad (70)$$

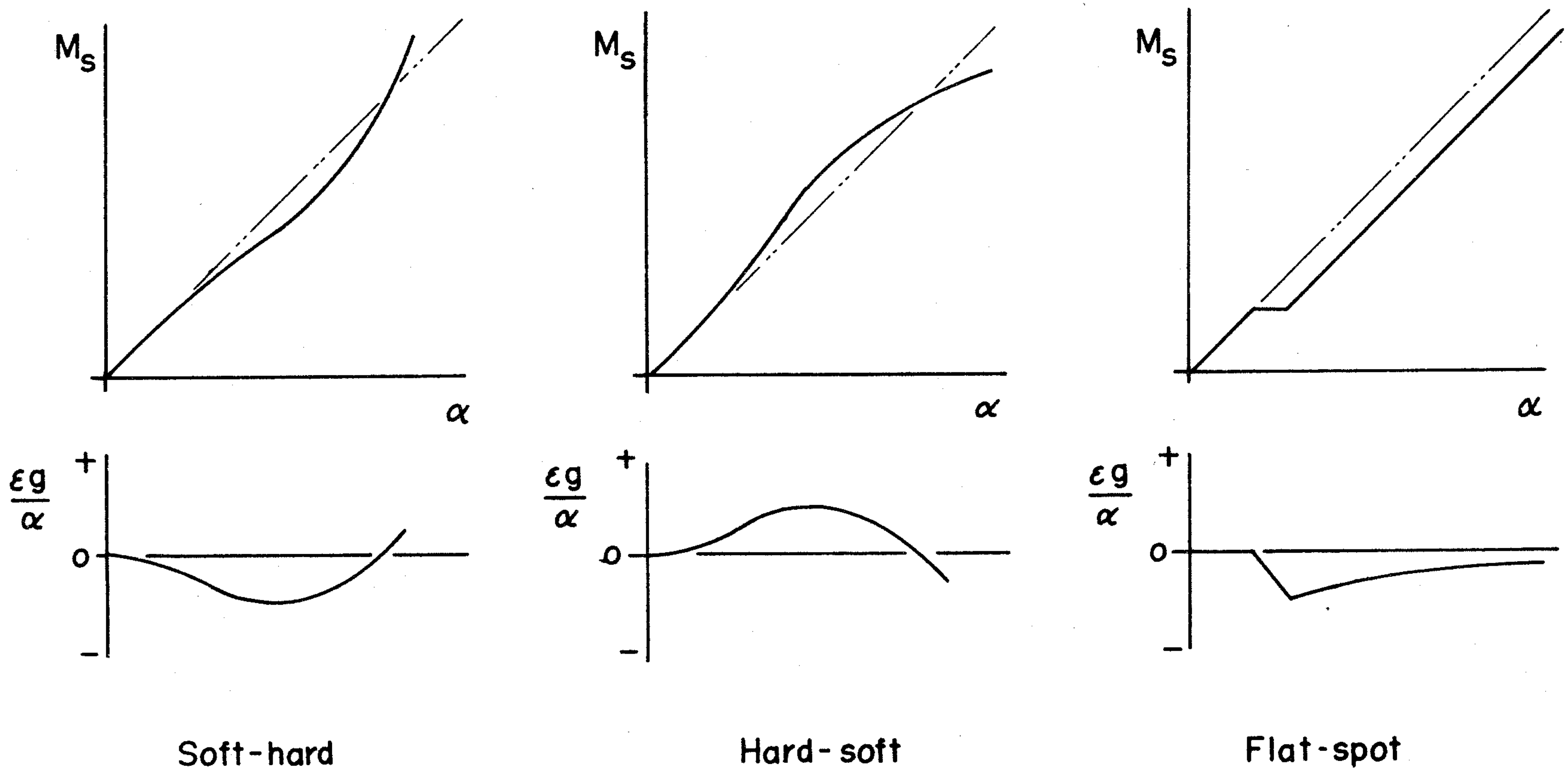

Fig. 2. Examples of stiffness characteristics and relative sizes of nonlinearities which result in multiple-valued amplitudes.

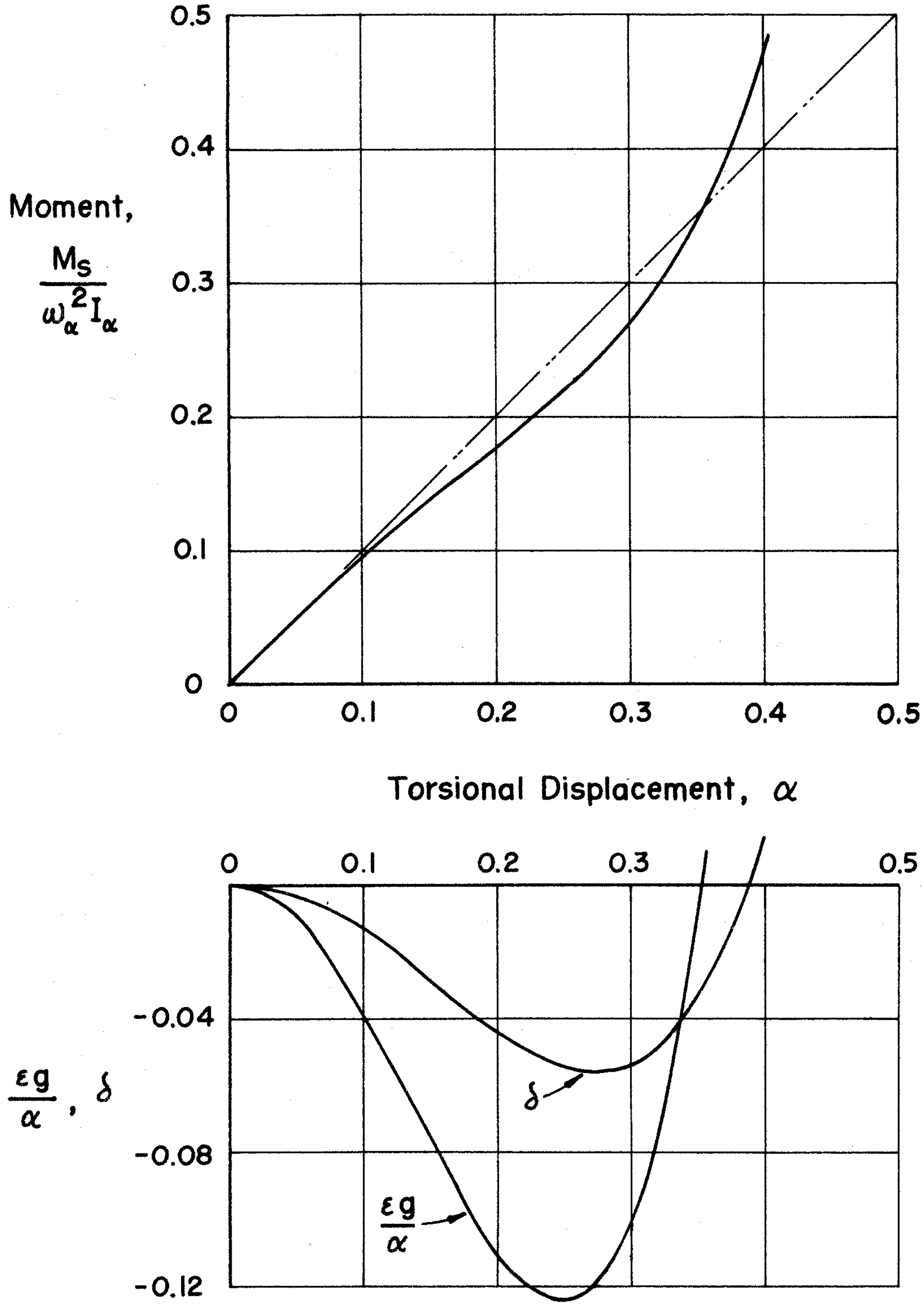

Fig. 3. Soft-hard quintic nonlinearity: $M_S = \left[\alpha - 4\alpha^3 + 32\alpha^5\right]\omega_\alpha^2 I_\alpha$

They are taken from reference (11), fig. I-A (m), where the linear case is worked out for the full unsteady incompressible aerodynamics. The linear ($a_2 = 0$) case worked here indicates that the quasi-steady approximation results in a reduction in the critical velocity of about 25%. For the present purposes this error is reasonable, since it should have no direct bearing on the effect of the non-linearity.

2. Steady-State Solution

Using the values given in eqs. 69 and 70 the governing formulas (eqs. 62 to 67) become as follows

$$\delta = a_2^2(-1.5 + 10a_2^2) \tag{71}$$

$$\frac{1}{X} = 0.617 + 1.17\delta \tag{72}$$

$$\left(\frac{U}{b\omega_\alpha}\right)^2 = \frac{\frac{1}{X}(1.30 - 1.152\frac{1}{X}) - 0.05 + 2.5(\frac{1}{X} - 0.04)\delta}{0.14\frac{1}{X} - 0.004} \tag{73}$$

$$k^2 = \frac{1}{X}\left(\frac{b\omega_\alpha}{U}\right)^2 \tag{74}$$

$$\tan\varphi = \frac{1}{k(1.64 - 0.144X)} \tag{75}$$

$$\frac{a_1}{a_2} = \frac{0.2996}{k(1.152 - 0.05X)\sin\varphi - 0.264\cos\varphi} \tag{76}$$

The calculations for each point are begun by choosing a value of a_2, then $\underline{\delta}$ and all other quantities follow in succesion. The results are plotted in fig. 4.

3. Stability of the Steady-State

Once the steady state solution has been obtained the coefficients of the perturbation equations, eqs. 41 to 43, are determined for any given amplitude. The amplitudes $a_{2_0} = 0.1658$ and $a_{2_0} = 0.3499$ are chosen as representative; they both correspond to a value of U/U_{linear} of 0.963, and their corresponding coefficients differ only where $\underline{\delta}$, a_{1_0} and a_{2_0} are involved. The two sets of perturbation equations, characteristic equations, and roots are as follows

$a_{2_0} = 0.1658$:

Perturbation equations.

$$\left.\begin{aligned} 0.2304\,\xi_1' &= -0.0264\,\xi_1 + 0.0332\,\xi_2 + 0.00930\,\eta \\ 0.2304\,\xi_2' &= 0.0135\,\xi_1 - 0.0164\,\xi_2 - 0.00404\,\eta \\ 0.2304\,\eta' &= -0.334\,\xi_1 + 0.434\,\xi_2 - 0.0482\,\eta \end{aligned}\right\} \quad (77)$$

Assuming $\xi_1 = \bar{\xi}_1 e^{\lambda\tau}$ etc. and forming the determinant of the coefficients yields the characteristic equation

$$\lambda^3 + 0.395\lambda^2 + 0.1245\lambda - 0.00013 = 0. \quad (78)$$

The roots are:

$$\lambda_1 = 0.0010\,; \quad \lambda_{2,3} = -0.198 \pm 0.29i. \quad (79)$$

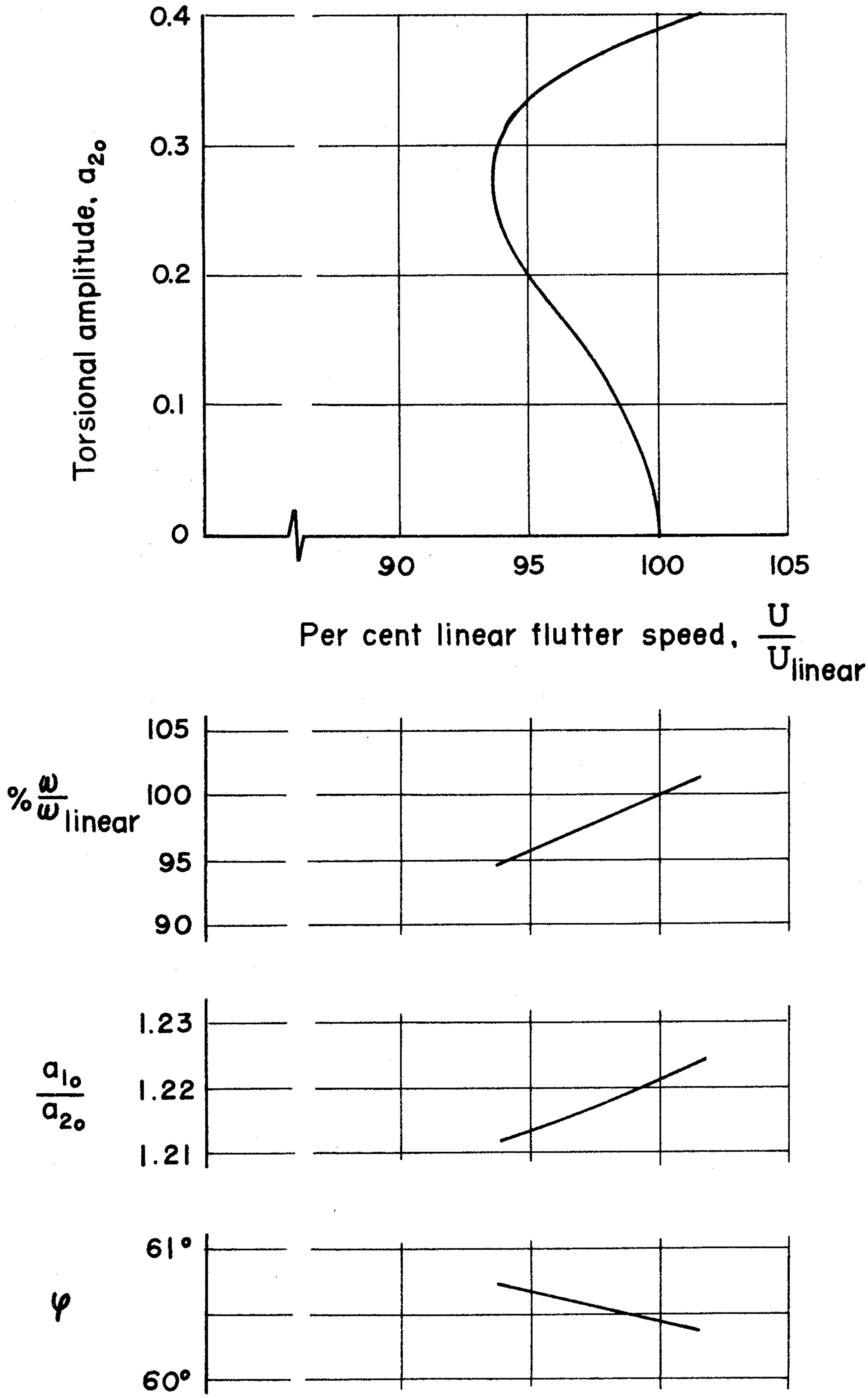

Fig. 4. Steady-state solution for soft-hard quintic nonlinearity.

$\underline{a_{20} = 0.3499}$

Perturbation equations:

$$\left.\begin{aligned} 0.2304\,\xi_1' &= -0.0264\,\xi_1 + 0.0271\,\xi_2 + 0.0192\,\eta \\ 0.2304\,\xi_2' &= 0.0135\,\xi_1 - 0.0164\,\xi_2 - 0.00852\,\eta \\ 0.2304\,\eta' &= -0.158\,\xi_1 + 0.0565\,\xi_2 - 0.0482\,\eta \end{aligned}\right\} \quad (80)$$

Characteristic equations:

$$\lambda^3 + 0.395\lambda^2 + 0.1075\lambda + 0.00128 = 0 \quad (81)$$

Roots:

$$\lambda_1 = -0.0125; \quad \lambda_{2,3} = -0.191 \pm 0.26i \quad (82)$$

4. Discussion of Results

The steady-state amplitude--velocity curve is double-valued in amplitude at speeds $0.937 < (U/U_{linear}) \leq 1$. That this should be the case is clear from fig. 3, where it is seen that $\underline{\alpha}$ is double-valued in $\frac{\varepsilon g}{\alpha}$ and $\underline{\delta}$. Since $\underline{U}$ is dependent on $\underline{\delta}$ only, it follows that $\underline{\alpha}$ will be double-valued in $\underline{U}$.

The stability of the steady-state is determined by the signs of the roots of the characteristic equations corresponding to the perturbation equations at each amplitude. For each amplitude studied a real root and two complex conjugate roots are found. For $a_2 = 0.3499$ the real parts are all negative, and the perturbations decay exponentially, so that the steady-state oscillation is stable. For $a_2 = 0.1658$ the real root is positive hence the perturbations diverge and the steady-state oscillation is unstable. By computing and plotting the coefficients (or the roots) of the characteristic equation, the stability can be established for any range of amplitudes of interest.

The _a posteriori_ check on the slowness of the variation of the amplitudes and phase has received previous comment in Part III-4. There it was stated that, although in general the roots $\underline{\lambda}$ must be small, large values will not disqualify the assumption of slow variation if they are associated with highly damped initial transients. The present example is a good illustration. The real root at both amplitudes is the critical stability root, and is small, so that the number of cycles (of basic motion) required for double (or half) amplitude to be attained are

$a_{2o} = 0.1658$: 44.5 cycles to double amplitude,

$a_{2o} = 0.3499$: 3.56 cycles to half amplitude.

Considering the least favorable of the above, $a_{2o} = 0.3499$, it is seen that the perturbation changes amplitude at a rate of about 15% of its initial amplitude per cycle (during the initial part of the motion). Assuming the perturbations are limited to 10% of the basic amplitude, then there will be a maximum variation of about 1½% per cycle in the basic amplitude, which admits with little error the averaging approximation. Thus these perturbation modes are slowly varying.

The complex conjugate roots are rather large, and thus correspond to a mode in which the perturbations vary rapidly. However, the real part is negative, so that after only one cycle this mode has damped to 6% of its initial amplitude, and hence has essentially disappeared after a short time. The subsequent perturbation motion consists of essentially only the slowly-varying mode, and it is this part of the motion which is of major concern. Thus the original assumption is justified.

VI. GENERAL DISCUSSION AND CONCLUSIONS

1. Applicability and Advantages of the S.V.P. Method

It has been seen that the Method of Slowly Varying Parameters yields rather concise closed-form solutions for the nonlinear bending-torsion flutter problem using the simplified aerodynamics of the quasi-steady approximation. In addition, the steady-state equations are in a form that makes the perturbation equations exceptionally easy to analyze, thus providing a straightforward stability analysis.

The results of the stability analysis confirm what other nonlinear flutter investigations have indicated, and what intuition leads one to expect. In references (3) and (4) bending-torsion flutter with free play in the torsional stiffness was analyzed by the Method of Equivalent Linearization. Multiple-valued amplitudes were found, but the question of stability of the limit cycles did not enter since essentially linear theory was used. In this theory the linear flutter velocity is found for the equivalent stiffnesses corresponding to the amplitude under consideration. Thus the region to the right of the steady-state curve is termed unstable, as shown in fig. 5.

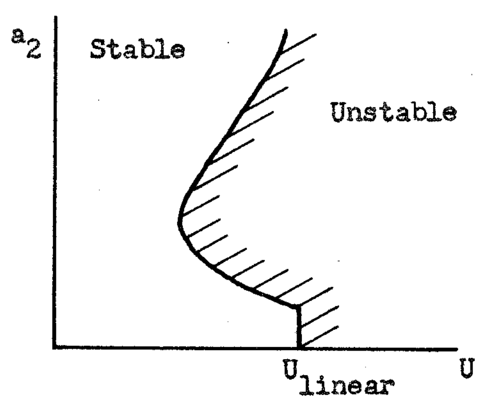

Fig. 5 Equivalent linearization stability result

If such regions of "instability" are regarded as ranges of amplitudes, for a given velocity, where the oscillations are converging on a limit cycle of higher amplitude, then the results of both analyses are identical. However, besides the difference in point of view, the present analysis is the more rigorous. The rigor is not without cost, however. As the member of degrees of freedom considered increases, and if multiple nonlinearities are to be considered, and if nonsteady aerodynamics theory is used, then the solution of the steady-state equations becomes much more difficult. For this reason, it may well be that for complicated cases the method of equivalent linearization would be preferable, at least from the standpoint of the practicing flutter engineer.

The applicability of the S.V.P. method is of course not limited to elastic nonlinearities, or, for that matter, to structural nonlinearities. A fruitful field for future research would be the investigation of the effects of various types of damping and aerodynamic nonlinearities.

2. General Features of Results

The steady-state amplitude--velocity curve is of course directly dependent on the type of nonlinearity under consideration, and there will be as many type of curves as there are nonlinearities. However, it is seen from eqs. 62 and 65 that $\underline{U}$ can conveniently be regarded as a function of $\underline{\delta}$, the averaged relative size of the nonlinearity, and for a given system $\underline{U}$ can be plotted against $\underline{\delta}$. Then from the amplitude-- $\underline{\delta}$ curves corresponding to the nonlinearity under consideration rapid cross-plotting will yield the amplitude--velocity curves. Since for moderate ranges of $\underline{\delta}$ the $\underline{U}$--$\underline{\delta}$ relation will be approximately linear, the shape of the amplitude---velocity curves can be qualitatively or quantitatively determined once the amplitude--- δ relation is given. Thus the steady-state results are relatively easily found and interpreted.

The stability analysis does not yield so easily to rigorous generalization. However, intuition predicts that the limit cycles will alternate in their stability, and the specific example treated lends further support to such expectations.

The *a posteriori* justification of the assumption of slow variations introduces an interesting concept. The perturbation motion was seen to be composed of a slowly varying "stability" mode and a highly damped oscillatory mode, which affected neither the stability nor the assumption of slow variations. Such secondary modes are not uncommon in multiple degree of freedom nonlinear systems, and their study may well lead to fuller understanding of such systems.

VII. REFERENCES

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VIII. APPENDIX: QUASI-STEADY AERODYNAMICS APPROXIMATION

The approximation to the aerodynamic forces used in this study is due to Dugundji, and is described in reference (5). In this reference the details are not too clear, however, since approximation is made after the air forces have been put into the equations of motion. Therefore a derivation will be shown here.

The approximation differs from the usual quasi-steady theory, as given in reference (8), in that one starts with the full unsteady expressions and reduces them to a much simpler, essentially quasi-steady, form. The unsteady expressions for harmonic motions are given by eqs. 9, Section 6.9 of reference 8, and are repeated below

$$\frac{\mathcal{L}(\tau)\,e^{-ik\tau}}{\rho\pi b^2} = -k^2\Big(\overbrace{\frac{h_o}{b}}^{(1)} - a\alpha_o\Big) + ik\alpha_o + 2\,C(k)\Big[\alpha_o + \frac{i}{b}kh_o + \big(\tfrac{1}{2}-a\big)ik\alpha_o\Big] \qquad \text{(A1)}$$

$$\frac{\mathcal{M}(\tau)\,e^{-ik\tau}}{\rho\pi b^2} = \big(\tfrac{1}{2}+a\big)\,2\,C(k)\Big[\alpha_o + \frac{i}{b}kh_o + \big(\tfrac{1}{2}-a\big)ik\alpha_o\Big] - k^2a\Big(\underbrace{\frac{h_o}{b} - a\alpha_o}_{(2)}\Big) - \big(\tfrac{1}{2}-a\big)ik\alpha_o + \underbrace{\frac{k^2}{8}\alpha_o}_{(3)} \qquad \text{(A2)}$$

In the above, $C(k)$ is the Theodorsen function, and h_o and α_o are complex amplitudes in bending and torsion, respectively.

Essentially two approximations are made. First, the Theodorsen function is set equal to unity, its limiting value as $\underline{k}$ approaches zero. Second, the contribution of bending acceleration to the apparent mass, term (1), and the contribution of torsional acceleration to the apparent moment of inertia, terms (2) and (3), are neglected. This approximation will be good if $\frac{1}{\mu} \ll 1$.

With the above approximations, and reverting to general time derivatives where

$$\frac{d}{dt} = \frac{U}{b}\frac{d}{d\tau} \rightarrow \frac{U}{b} ik(\;)_o,$$

eqs. A1 and A2 become

$$\frac{\mathcal{L}(t)}{\rho\pi b U^2} = -a\frac{b^2}{U^2}\ddot{\alpha} + \frac{b}{U}\dot{\alpha} + 2\left[\alpha + \frac{\dot{h}}{U} + \left(\frac{1}{2}-a\right)\frac{b}{U}\dot{\alpha}\right] \tag{A3}$$

$$\frac{\mathcal{M}(t)}{\rho\pi b^2 U^2} = 2\left(\frac{1}{2}+a\right)\left[\alpha + \frac{\dot{h}}{U} + \left(\frac{1}{2}-a\right)\frac{b}{U}\dot{\alpha}\right] + \frac{ab}{U^2}\ddot{h} - \left(\frac{1}{2}-a\right)\frac{b}{U}\dot{\alpha}. \tag{A4}$$

Collecting terms yields eqs. 8 and 9 of Part II.