NON-STATIONARY MOTION OF PURELY SUPERSONIC WINGS

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Jack E. Froehlich

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ABSTRACT

A general theory is presented for the calculation of the total forces acting on purely supersonic wings. The method applies to wings having an arbitrary downwash distribution (stationary or non-stationary) and is valid whenever all of the wing edges are supersonic. The general three-dimensional non-stationary problem is reduced to an equivalent two-dimensional problem. In the case of harmonic oscillations the aerodynamic coefficients are expressed in terms of known or tabulated functions. The specific example of an oscillating delta wing is considered and values of the aerodynamic coefficients for plunging, pitching, and rolling oscillations are calculated for two Mach numbers.

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I. INTRODUCTION

With the advent of high performance aircraft and the possibility of supersonic flight, the problem of flutter and aerodynamic stability of airfoils moving at supersonic speeds has become of increasing importance. Consequently, much attention has been directed to the problem of an oscillating airfoil in a supersonic stream.

As in the case of the subsonic theory developed by Glauert (Ref. 1), von Kármán and Sears (Ref. 2), and others, the approach to the supersonic non-stationary problem has generally been made through the use of the linearized equations of motion. It is well known that if one considers a non-viscous, non-heat-conducting fluid with the assumptions of irrotationality and small disturbances, the equations which govern the disturbed motion of the fluid may be reduced to the wave equation. Under these conditions it is possible to find general solutions of the wave equation which may be superimposed to satisfy the boundary condition of tangential flow on the wing surface.

Possio (Ref. 3) and Borbely (Ref. 4) have both treated the two-dimensional airfoil in this manner. These authors have obtained solutions for the pressures on an airfoil due to an arbitrary chordwise downwash distribution which exhibits a harmonic time dependence. These solutions are expressed as integrals which cannot be evaluated in terms of known functions. However, Schwarz (Ref. 5) has calculated these integrals by numerical methods for a sufficient range of the parameters to make the use of this theory practical.

Garrick and Rubinow (Ref. 6), and recently Miles (Ref. 7), following the early work of Possio and Borbely, have treated the two-dimensional theory as applied specifically to the flutter problem. These authors have obtained analytical expressions for the flutter derivatives in terms of integrals tabulated by Schwarz.

The work of Garrick and Rubinow, which includes some numerical results, indicates that for a certain range of Mach numbers the aerodynamic damping due to torsional oscillations becomes negative; hence the motion is unstable. This point is of paramount interest to the designer, and the question naturally arises as to what further instabilities may arise in the case of a finite wing.

Miles (Ref. 8) has carried out an approximate solution of the problem of an oscillating delta wing, including only terms of first order in frequency. This analysis indicates that the instabilities found by Garrick and Rubinow may also occur for wings of finite span.

Recently Stewart and Li (Ref. 9) have obtained the solution for an oscillating rectangular wing. They have found that there is a marked change in the nature of the damping of certain modes due to the finite aspect ratio.

The general solution for an oscillating three-dimensional wing with supersonic edges* may easily be expressed in terms of elementary solutions

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* It is common practice to refer to a wing boundary as being either subsonic or supersonic according to whether the normal component of the free stream Mach number is greater or less than 1.

of the wave equation; however, even for relatively simple planforms, the calculation of the pressures involves integrals much too difficult to evaluate. Fortunately, knowledge of pressure distributions is not of prime importance in the application of the theory. On the contrary, interest is centered on the nature of the total forces acting on the wing.

Miles (Ref. 10) has pointed out that it is possible to reduce the problem of calculating the forces acting on an oscillating delta wing to an equivalent two-dimensional problem. It is the purpose of this paper to present a general solution for the arbitrary motion of any three-dimensional wing having supersonic edges in terms of an equivalent two-dimensional problem. In the case of harmonic oscillations the solution is expressed in terms of the familiar two-dimensional functions tabulated by Schwarz. This reduction is possible only if the trailing edge of the wing is straight and perpendicular to the direction of the free stream velocity.

The specific example of an oscillating delta wing is considered. The forces acting on the wing due to rolling, plunging, and pitching oscillations are calculated for two Mach numbers as a function of frequency.

II. GENERAL THEORY

Consider a wing contained in the $\bar{x}$, $\bar{y}$ plane which is extremely thin in the $\bar{z}$ direction and is moving in the direction of the negative $\bar{x}$ axis with a supersonic velocity U (Fig. 1). Let the wing perform any motion of small amplitude and small velocity about the plane $\bar{z} = 0$.

If the flow is assumed to be irrotational, a velocity potential may be introduced such that

$$\begin{aligned} u &= \frac{\partial \varphi}{\partial \bar{x}} \\ v &= \frac{\partial \varphi}{\partial \bar{y}} \\ w &= \frac{\partial \varphi}{\partial \bar{z}} \end{aligned} \tag{1}$$

where u, v, and w are the components of the velocity of the fluid in the $\bar{x}$, $\bar{y}$, and $\bar{z}$ directions, respectively.

Since the disturbances are assumed to be small, the equations of motion for an inviscid non-heat-conducting fluid may be linearized to give

$$\frac{\partial^2 \varphi}{\partial \bar{x}^2} + \frac{\partial^2 \varphi}{\partial \bar{y}^2} + \frac{\partial^2 \varphi}{\partial \bar{z}^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial \bar{t}^2} = 0 \tag{2}$$

$$\begin{aligned} \frac{\partial \varphi}{\partial \bar{t}} &= \frac{P_0 - P}{\rho_0} \\ c^2 &= \left(\frac{dP}{d\rho}\right)_0 \end{aligned} \tag{3}$$

where P_0 and ρ_0 are the pressure and density far from the wing, and c is the acoustic velocity.

The boundary condition on φ is expressed in terms of the downwash necessary to produce tangential flow over the wing. If $\bar{z}_w = \bar{z}(\bar{x}, \bar{y}, \bar{t})$ describes the altitude of points on the wing at the time $\bar{t}$, then the vertical velocity of the fluid adjacent to the wing will be given by

$$\frac{\partial \varphi}{\partial \bar{z}}(\bar{x}, \bar{y}, 0, \bar{t}) = -W(\bar{x}, \bar{y}, \bar{t}) \tag{4}$$

where

$$W(\bar{x}, \bar{y}, \bar{t}) = -\frac{\partial \bar{z}_w}{\partial \bar{t}} \tag{5}$$

If the edges of the wing are all supersonic no disturbances will occur in the plane of the wing off the wing except in its wake. Furthermore, if the trailing edge is supersonic, the pressures on the wing are independent of conditions in its wake. Thus the pressures on the wing are completely specified by Eqs. (2) through (5).

The problem represented by this system of equations can, in principle, be solved. However, in certain cases (harmonic oscillations, for example), the integrals encountered in the calculation of the pressures are so complicated that they have thus far resisted evaluation.

If only the total forces acting on the wing need be calculated, considerable simplification can be obtained. It has already been noted that due to the hyperbolic nature of the differential equation and the assumption of supersonic edges, the two sides of the wing may be treated independently.* Furthermore, since the wing is essentially a lamina, the

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* This is equivalent to the statement that there are no disturbances in plane of the wing off the wing.

downwash on the wing will be an even function of $\bar{z}$ and hence the over pressure will be an odd function of $\bar{z}$. Thus it can readily be seen that the total lift acting on the wing will be given by

$$L = 2\rho_0 \iint_w \frac{p_0 - p}{\rho_0} \, dS$$

where the integral is evaluated over the upper surface of the wing. Introducing Eq. (3)

$$L = 2\rho_0 \int_{-U\bar{t}}^{-U\bar{t}+2b} \int_{\bar{y}_p(\bar{x}+U\bar{t})}^{\bar{y}_s(\bar{x}+U\bar{t})} \frac{\partial\varphi}{\partial\bar{t}}(\bar{x},\bar{y},0,\bar{t}) \, d\bar{y} \, d\bar{x} \tag{6}$$

where $\bar{y}_p$ and $\bar{y}_s$ are the extreme left and right hand coordinates, respectively, of the leading ledge, 2b is the chord of the wing, and

$$\bar{y}_p(0) = \bar{y}_s(0) = 0$$

In order to perform the integration as indicated in Eq. (6) it is necessary to assume that the trailing edge is straight and perpendicular to the $\bar{x}$ axis. In an analogous manner the pitching moment about the point $\bar{x} = -U\bar{t}$ and the rolling moment about the $\bar{x}$ axis can be written as

$$\mathfrak{M} = -2\rho_0 \int_{-U\bar{t}}^{-U\bar{t}+2b} (\bar{x}+U\bar{t}) \int_{\bar{y}_p}^{\bar{y}_s} \frac{\partial\varphi}{\partial\bar{t}}(\bar{x},\bar{y},0,\bar{t}) \, d\bar{y} \, d\bar{x} \tag{7}$$

$$R = -2\rho_0 \int_{-U\bar{t}}^{-U\bar{t}+2b} \int_{\bar{y}_p}^{\bar{y}_s} \bar{y} \, \frac{\partial\varphi}{\partial\bar{t}}(\bar{x},\bar{y},0,\bar{t}) \, d\bar{y} \, d\bar{x} \tag{8}$$

Since it is known that $\frac{\partial \varphi}{\partial \bar{t}}(\bar{x},\bar{y},0,\bar{t})$ is zero for

$$\bar{y} < \bar{y}_P \quad ; \quad \bar{y} > \bar{y}_S$$

it is possible to extend the intervals of integration in the $\bar{y}$ direction without affecting the results. Thus Eqs. (6), (7), and (8) may be written as

$$L = 2\rho_0 \int_{-U\bar{t}}^{-U\bar{t}+2b} \int_{-\infty}^{\infty} \frac{\partial \varphi}{\partial \bar{t}}(\bar{x},\bar{y},0,\bar{t})\, d\bar{y}\, d\bar{x} \tag{9}$$

$$\mathfrak{M} = -2\rho_0 \int_{-U\bar{t}}^{-U\bar{t}+2b} (\bar{x}+U\bar{t}) \int_{-\infty}^{\infty} \frac{\partial \varphi}{\partial \bar{t}}(\bar{x},\bar{y},0,\bar{t})\, d\bar{y}\, d\bar{x} \tag{10}$$

$$R = -2\rho_0 \int_{-U\bar{t}}^{-U\bar{t}+2b} \int_{-\infty}^{\infty} \bar{y}\, \frac{\partial \varphi}{\partial \bar{t}}(\bar{x},\bar{y},0,\bar{t})\, d\bar{y}\, d\bar{x} \tag{11}$$

The above equations indicate that if the following two-dimensional functions are introduced,

$$\psi_1(\bar{x},\bar{z},\bar{t}) = \int_{-\infty}^{\infty} \varphi(\bar{x},\bar{y},\bar{z},\bar{t})\, d\bar{y} \tag{12}$$

$$\psi_2(\bar{x},\bar{z},\bar{t}) = \int_{-\infty}^{\infty} \bar{y}\, \varphi(\bar{x},\bar{y},\bar{z},\bar{t})\, d\bar{y} \tag{13}$$

the total forces and moments acting on the wing may be calculated without explicit knowledge of the solution for φ , provided that the functions ψ_1 and ψ_2 can be evaluated.* For if Eqs. (12) and (13) are differentiated with respect to $\bar{t}$ and substituted into Eqs. (9), (10), and (11), the expressions for the forces become

$$L = 2\rho_0 \int_{-U\bar{t}}^{-U\bar{t}+2b} \frac{\partial \psi_1}{\partial \bar{t}}(\bar{x},0,\bar{t})\, d\bar{x} \tag{14}$$

$$\mathfrak{M} = -2\rho_0 \int_{-U\bar{t}}^{-Ut+2b} (\bar{x}+U\bar{t})\, \frac{\partial \psi_1}{\partial \bar{t}}(\bar{x},0,\bar{t})\, d\bar{x} \tag{15}$$

$$R = -2\rho_0 \int_{-U\bar{t}}^{-U\bar{t}+2b} \frac{\partial \psi_2}{\partial \bar{t}}(\bar{x},0,\bar{t})\, d\bar{x} \tag{16}$$

The functions ψ_1 and ψ_2 have an important physical interpretation. $\frac{\partial \psi_1}{\partial \bar{t}}$ represents the lift of a spanwise element of the wing and $\frac{\partial \psi_2}{\partial \bar{t}}$ represents the rolling moment contributed by a spanwise element of the wing.

It is now necessary to determine, if possible, the differential equations which these functions satisfy, and the appropriate boundary conditions that must be imposed upon them. This can best be accomplished

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* This possibility was suggested by Dr. P. A. Lagerstrom.

by examining the derivatives of ψ_1 and ψ_2. Consider ψ_1,

$$\frac{\partial^2 \psi_1}{\partial \bar{x}^2} = \int_{-\infty}^{\infty} \frac{\partial^2 \varphi}{\partial \bar{x}^2} \, d\bar{y}$$

$$\frac{\partial^2 \psi_1}{\partial \bar{z}^2} = \int_{-\infty}^{\infty} \frac{\partial^2 \varphi}{\partial \bar{z}^2} \, d\bar{y} \qquad (17)$$

$$\frac{\partial^2 \psi_1}{\partial \bar{t}^2} = \int_{-\infty}^{\infty} \frac{\partial^2 \varphi}{\partial \bar{t}^2} \, d\bar{y}$$

Since from physical considerations infinite velocities and pressures in the flow field are not allowed, φ is at least a piece-wise continuous function of the independent variables. Therefore the integral of Eq. (12) exists in the ordinary sense. However, φ may have discontinuities in its first derivatives. Therefore the integrals of Eq. (17) must be considered as Stieltjes integrals. The derivatives defined by Eq. (17) will thus have meaning whenever the Stieltjes integrals exist.

For values of $\bar{x}$, $\bar{z}$, $\bar{t}$, such that the second derivatives of φ exist, ψ_1 satisfies the equation

$$\frac{\partial^2 \psi_1}{\partial \bar{x}^2} + \frac{\partial^2 \psi_1}{\partial \bar{z}^2} - \frac{1}{c^2} \frac{\partial^2 \psi_1}{\partial \bar{t}^2} = \int_{-\infty}^{\infty} \left\{ \frac{\partial^2 \varphi}{\partial \bar{x}^2} + \frac{\partial^2 \varphi}{\partial \bar{z}^2} - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial \bar{t}^2} \right\} d\bar{y}$$

or, from Eq. (2),

$$\frac{\partial^2 \psi_1}{\partial \bar{x}^2} + \frac{\partial^2 \psi_1}{\partial \bar{z}^2} - \frac{1}{c^2} \frac{\partial^2 \psi_1}{\partial \bar{t}^2} = - \int_{-\infty}^{\infty} \frac{\partial^2 \varphi}{\partial \bar{y}^2} \, d\bar{y} \qquad (18)$$

Since the side wash must vanish at infinity, the right hand side of Eq. (18)

may be integrated to give

$$\frac{\partial^2 \Psi_1}{\partial \bar{x}^2} + \frac{\partial^2 \Psi_1}{\partial \bar{z}^2} - \frac{1}{c^2}\frac{\partial^2 \Psi_1}{\partial \bar{t}^2} = 0 \tag{19}$$

For certain values of $\bar{x}$, $\bar{z}$, and $\bar{t}$ terms of Eq. (19) may not be defined. This is to be expected since Eq. (19) is a hyperbolic differential equation. The boundary condition on Ψ_1 may be obtained in a similar manner. Consider

$$\frac{\partial \Psi_1}{\partial \bar{z}} = \int_{-\infty}^{\infty} \frac{\partial \varphi}{\partial \bar{z}}\, d\bar{y}$$

From Eq. (4)

$$\frac{\partial \Psi_1}{\partial \bar{z}}(\bar{x},0,\bar{t}) = -\int_{\bar{y}_P}^{\bar{y}_S} W(\bar{x},\bar{y},\bar{t})\, d\bar{y} = -W_1(\bar{x},\bar{t}) \tag{20}$$

$$0 < \bar{x} + U\bar{t} < 2b$$

where W $(\bar{x}, \bar{y}, \bar{t})$ is the downwash on the wing. Also

$$\frac{\partial \Psi_1}{\partial \bar{z}}(\bar{x},0,\bar{t}) = 0 \qquad \bar{x} + U\bar{t} < 0$$

The solution to Eqs. (19) and (20) is well known. For points on the wing the solution is

$$\Psi_1(\bar{x},0,\bar{t}) = \frac{c}{\pi}\int_0^{\bar{t}} \int_{\bar{x}-c(\bar{t}-\tau)}^{\bar{x}+c(\bar{t}-\tau)} \frac{W_1(\xi,\tau)\, d\xi\, d\tau}{\sqrt{c^2(\bar{t}-\tau)^2 - (\bar{x}-\xi)^2}} \tag{21}$$

The region of integration in the $\bar{x}$, $\bar{t}$ plane is illustrated in Fig. (2).

Now consider ψ_2 .

$$\frac{\partial^2 \psi_2}{\partial \bar{x}^2} = \int_{-\infty}^{\infty} \bar{y}\, \frac{\partial^2 \varphi}{\partial \bar{x}^2}\, d\bar{y}$$

$$\frac{\partial^2 \psi_2}{\partial \bar{z}^2} = \int_{-\infty}^{\infty} \bar{y}\, \frac{\partial^2 \varphi}{\partial \bar{z}^2}\, d\bar{y} \tag{22}$$

$$\frac{\partial^2 \psi_2}{\partial \bar{t}^2} = \int_{-\infty}^{\infty} \bar{y}\, \frac{\partial^2 \varphi}{\partial \bar{t}^2}\, d\bar{y}$$

Here the presence of $\bar{y}$ in the integrand introduces no convergence difficulties because φ and all of its derivatives are zero outside of a finite $\bar{y}$ interval. From Eq. (2)

$$\frac{\partial^2 \psi_2}{\partial \bar{x}^2} + \frac{\partial^2 \psi_2}{\partial \bar{z}^2} - \frac{1}{c^2}\frac{\partial^2 \psi_2}{\partial \bar{t}^2} = -\int_{-\infty}^{\infty} \bar{y}\, \frac{\partial^2 \varphi}{\partial \bar{y}^2}\, d\bar{y} \tag{23}$$

and

$$\varphi = \frac{\partial \varphi}{\partial \bar{y}} = \frac{\partial^2 \varphi}{\partial \bar{y}^2} = 0 \qquad |\bar{y}| \geq L(\bar{x},\bar{z},\bar{t})$$

The right hand side of Eq. (23) may thus be written

$$\int_{-\infty}^{\infty} \bar{y}\, \frac{\partial^2 \varphi}{\partial \bar{y}^2}\, d\bar{y} = \int_{-L}^{L} \bar{y}\, \frac{\partial^2 \varphi}{\partial \bar{y}^2}\, d\bar{y} = \bar{y}\,\frac{\partial \varphi}{\partial \bar{y}}\Bigg|_{-L}^{L} - \int_{-L}^{L} \frac{\partial \varphi}{\partial \bar{y}}\, d\bar{y} = 0$$

Therefore

$$\frac{\partial^2 \psi_2}{\partial \bar{x}^2} + \frac{\partial^2 \psi_2}{\partial \bar{z}^2} - \frac{1}{c^2}\frac{\partial^2 \psi_2}{\partial \bar{t}^2} = 0 \tag{24}$$

The boundary condition on Ψ_2 becomes

$$\frac{\partial \Psi_2}{\partial \bar{z}}(\bar{x},0,\bar{t}) = -\int_{\bar{y}_p}^{\bar{y}_s} \bar{y}\, W(\bar{x},\bar{y},\bar{t})\, d\bar{y} = -W_2(\bar{x},\bar{t})$$

$$0 < \bar{x}+U\bar{t} < 2b \tag{25}$$

$$\frac{\partial \Psi_2}{\partial \bar{z}}(\bar{x},0,\bar{t}) = 0 \qquad\qquad \bar{x}+U\bar{t} < 0$$

The solution for Ψ_2 for points on the wing may be written down immediately as

$$\Psi_2(\bar{x},0,\bar{t}) = \frac{c}{\pi}\int_0^{\bar{t}}\int_{\bar{x}-c(\bar{t}-\tau)}^{\bar{x}+c(\bar{t}-\tau)} \frac{W_2(\xi,\tau)\, d\xi\, d\tau}{\sqrt{c^2(\bar{t}-\tau)^2-(\bar{x}-\xi)^2}} \tag{26}$$

The original three-dimensional problem has now been reduced to an equivalent two-dimensional problem. The methods previously used for two-dimensional wings can be applied to obtain the forces acting on the wing.

The solutions represented by Eqs. (21) and (26) are quite general and apply to any downwash distribution, such as gust loading, unit step loading, or accelerating flight. The special case that will be of interest here is the downwash distribution which, at any point on the wing, has a harmonic time dependence. This condition is

$$\frac{\partial \varphi}{\partial \bar{z}}(\bar{x},\bar{y},0,\bar{t}) = -W(\bar{x}+U\bar{t},\bar{y})\, e^{j\omega\bar{t}}$$

or, from Eq. (20),

$$\frac{\partial \Psi_1}{\partial \bar{z}}(\bar{x},0,\bar{t}) = -\int_{\bar{y}_p(\bar{x}+U\bar{t})}^{\bar{y}_s(\bar{x}+U\bar{t})} W(\bar{x}+U\bar{t},\bar{y})\, e^{j\omega\bar{t}}\, d\bar{y} = -W_1(\bar{x}+U\bar{t})\, e^{j\omega\bar{t}} \tag{27}$$

and from Eq. (25)

$$\frac{\partial \psi_2}{\partial \bar{z}}(\bar{x},0,\bar{t}) = -\int_{\bar{y}_p(\bar{x}+U\bar{t})}^{\bar{y}_s(\bar{x}+U\bar{t})} \bar{y}\, W(\bar{x}+U\bar{t},\bar{y})\, e^{j\omega\bar{t}}\, d\bar{y} = -W_2(\bar{x}+U\bar{t})\, e^{j\omega\bar{t}} \tag{28}$$

With these boundary conditions the expressions for ψ_1 and ψ_2 can be simplified. Consider Eq. (21). Using Eq. (27)

$$\psi_1(\bar{x},0,\bar{t}) = \frac{c}{\pi}\int_0^{\bar{t}}\int_{\bar{x}-c(\bar{t}-\tau)}^{\bar{x}+c(\bar{t}-\tau)} \frac{W_1(\xi+U\tau)\, e^{j\omega\tau}\, d\xi\, d\tau}{\sqrt{c^2(\bar{t}-\tau)^2-(\bar{x}-\xi)^2}} \tag{29}$$

Introducing the Gallilean transformation

$$x = \bar{x}+U\bar{t}$$
$$y = \bar{y}$$
$$z = \bar{z}$$
$$t = \bar{t}$$

and integrating first along $\xi+U\tau$ = constant Eq. (29) yields the well known integral relation (see, for example, Ref. 4 or 7)

$$\psi_1(x,0,t) = \frac{e^{j\omega t}}{\beta}\int_0^x W_1(s)\, J_0\left[\frac{k(x-s)}{b}\right] e^{\frac{-jkM(x-s)}{b}}\, ds \tag{30}$$

where

$$M = \frac{U}{c}$$

$$\beta^2 = M^2-1$$

$$k = \frac{\omega b}{c\,\beta^2}$$

and J_0 is the Bessel function of the first kind of zero order.

Similarly

$$\psi_2(x,0,t) = \frac{e^{j\omega t}}{\beta} \int_0^x w_2(s)\, J_0\left[\frac{k(x-s)}{b}\right] e^{\frac{-jkM(x-s)}{b}} ds \tag{31}$$

The calculation of ψ_1 and ψ_2 has been reduced to the evaluation of a single integral. The calculation of the forces and moments requires another integration. In many cases of interest $W_1(x)$ and $W_2(x)$ are polynomials in x . When this is true it will be domonstrated that the expressions for the forces and moments can be reduced to the single integrals tabulated by Schwarz. The special case of an oscillating delta wing will now be considered.

III. THE DELTA WING

The functions ψ_1 and ψ_2 will now be used to calculate the aerodynamic forces acting on an oscillating delta wing. The planform of the wing is an isosceles triangle with a vertex angle of $\pi - 2\sigma$ (Fig. 3). The vertex is placed at the origin of the x, y, z coordinate system. The maximum chord of the wing is 2b. Three types of motion will be considered: plunging, pitching, and rolling oscillations.

<u>Case a:</u> Plunging Oscillation.

This motion is characterized by a downwash which is the same for each point on the wing. Analytically this is expressed by

$$\frac{\partial \varphi}{\partial z}(x,y,0,t) = -w_0 \, e^{j\omega t}$$

or

$$w(x,y) = w_0 = \text{constant}$$

The leading edge of the wing is described by the equations

$$\begin{aligned} y_p &= -\frac{x}{\tan\sigma} \\ y_s &= \frac{x}{\tan\sigma} \end{aligned} \tag{32}$$

Now from Eq. (27)

$$w_1(x) = \int_{\frac{-x}{\tan\sigma}}^{\frac{x}{\tan\sigma}} w_0 \, dy = \frac{2 w_0 x}{\tan\sigma} \tag{33}$$

Substituting Eq. (33) into Eq. (30)

$$\psi_1(x,0,t) = \frac{2 w_0 e^{j\omega t}}{\beta \tan\sigma} \int_0^x s \, G(x-s) \, ds \tag{34}$$

where

$$G(x-s) = J_0\left[\frac{k(x-s)}{b}\right] e^{\frac{-jkM(x-s)}{b}} \tag{35}$$

It will often be convenient to express integrals of the type appearing in Eq. (34) in slightly different form. A change of the variable of integration establishes the following relation

$$\int_0^x f(s)\,h(x-s)\,ds = \int_0^x f(x-s)\,h(s)\,ds \tag{36}$$

With the aid of the Galillean transformation introduced in the previous section, Eq. (14) becomes

$$L = 2\rho_0 \int_0^{2b} \left[\frac{\partial\psi_1}{\partial t} + U\frac{\partial\psi_1}{\partial x}\right] dx \tag{37}$$

Substituting Eq. (34) and using the relation of Eq. (36)

$$L = 2\rho_0 \int_0^{2b} \left[\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right] \frac{2W_0 e^{j\omega t}}{\beta\tan\sigma} \int_0^x (x-s)\,G(s)\,ds\,dx$$

or

$$L = \frac{4\rho_0 W_0 e^{j\omega t}}{\beta\tan\sigma}\left\{ j\omega \int_0^{2b}\int_0^x (x-s)\,G(s)\,ds\,dx + U\int_0^{2b}\int_0^x G(s)\,ds\,dx \right\}$$

Since the area of the wing is

$$S = \frac{4b^2}{\tan\sigma}$$

and the dynamic pressure is

$$q = \frac{1}{2}\rho_0 U^2$$

the lift coefficient becomes

$$C_L = \frac{2 W_0 e^{j\omega t}}{\beta U^2 b^2} \left\{ j\omega \int_0^{2b} \int_0^{x} (x-s)\, G(s)\, ds\, dx + U \int_0^{2b} \int_0^{x} G(s)\, ds\, dx \right\} \qquad (38)$$

It is interesting to note that the C_L is independent of the sweep angle σ . It will be found that this is true for all of the coefficients considered in this analysis.

Introducing the quasi-steady value for the lift coefficient

$$C_{L_0} = \frac{4 W_0 e^{j\omega t}}{\beta U}$$

Eq. (38) becomes*

$$\left(\frac{C_L}{C_{L_0}}\right)_a = \frac{j\omega}{2 U b^2} \int_0^{2b} \int_0^{x} (x-s)\, G(s)\, ds\, dx + \frac{1}{2 b^2} \int_0^{2b} \int_0^{x} G(s)\, ds\, dx \qquad (39)$$

Now if the variables of integration are changed by

$$x = 2b\xi$$

$$s = 2b\eta$$

then

$$\left(\frac{C_L}{C_{L_0}}\right)_a = 4j\nu \int_0^{1} \int_0^{\xi} (\xi-\eta)\, G(2b\eta)\, d\eta\, d\xi + 2 \int_0^{1} \int_0^{\xi} G(2b\eta)\, d\eta\, d\xi \qquad (40)$$

where

$$\nu = \frac{\omega b}{U}$$

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* Arabic letters are used as subscripts to denote the mode of oscillation (Cf. Table of Notation).

Since integrals of this type will occur throughout this analysis, it will be convenient to obtain a special reduction formula. Consider

$$I = \int_0^1 \xi^p \int_0^\xi \eta^n \, G(2b\eta)\, d\eta\, d\xi$$

Integrating by parts

$$I = \frac{1}{p+1} \int_0^1 \xi^n G(2b\xi)\, d\xi - \frac{1}{p+1} \int_0^1 \xi^{p+n+1} G(2b\xi)\, d\xi$$

or

$$I = \frac{T_n}{(p+1)\, 2^{n+1}} - \frac{T_{p+n+1}}{(p+1)\, 2^{p+n+2}} \tag{41}$$

where

$$T_n = 2^{n+1} \int_0^1 s^n \, G(2bs)\, ds \tag{42}$$

Eq. (40) can now be written

$$\left(\frac{C_L}{C_{L_0}}\right)_a = j\nu\left(T_0 - T_1 + \tfrac{1}{4} T_2\right) + T_0 - \tfrac{1}{2} T_1 \tag{43}$$

The stalling moment about the vertex of the wing is obtained from Eq. (15)

$$\mathfrak{M} = -\frac{4\rho_0 W_0 e^{j\omega t}}{\beta \tan\sigma} \left\{ j\omega \int_0^{2b} x \int_0^x (x-s)\, G(s)\, ds\, dx + U \int_0^{2b} x \int_0^x G(s)\, ds\, dx \right\}$$

Now

$$C_M = \frac{\mathfrak{M}}{q S 2b}$$

and the quasi-steady value of C_M is

$$C_{M_0} = -\frac{8 W_0 e^{j\omega t}}{3 \beta U}$$

Therefore, using Eq. (41) the final expression becomes

$$\left(\frac{C_M}{C_{M_0}}\right)_a = j\nu\left(T_0 - \tfrac{3}{4}T_1 + \tfrac{1}{16}T_3\right) + \tfrac{3}{4}T_0 - \tfrac{3}{16}T_2 \tag{44}$$

Since the wing is symmetrical about the x axis and the downwash is an even function of y for this mode of oscillation, the total rolling moment about the x-axis is zero.

<u>Case b:</u> Pitching Oscillation.

In this case the wing is assumed to be pitching about an axis a distance $2\mu b$ downstream of the vertex. The instantaneous angle of attack is given by

$$\alpha = \alpha_0 e^{j\omega t} \tag{45}$$

where

$$\alpha_0 = \text{constant}$$

Therefore the z coordinate of any point on the wing is

$$z_w = -\alpha(x - 2\mu b)$$

and the downwash necessary to produce tangential flow over the wing is

$$\frac{\partial\varphi}{\partial z} = \frac{\partial z_w}{\partial t} + U\frac{\partial z_w}{\partial x} = -\alpha_0 e^{j\omega t}\left[j\omega(x - 2\mu b) + U\right]$$

or

$$w(x,y) = (1 - 2j\nu\mu)U\alpha_0 + j\nu\alpha_0 U\frac{x}{b} \tag{46}$$

The first term on the right hand side of Eq. (46) corresponds to the downwash used in Case a. Therefore it is only necessary to obtain a solution for the parabolic downwash

$$w_p = \frac{\alpha_0 U x}{b}$$

From Eq. (27)

$$W_1(x) = \int_{\frac{-x}{\tan\sigma}}^{\frac{x}{\tan\sigma}} \frac{\alpha_0 U x}{b}\, dy = \frac{2\alpha_0 U x^2}{b\tan\sigma}$$

and from Eq. (30)

$$\psi_1(x,0,t) = \frac{2\alpha_0 U e^{j\omega t}}{\beta b\tan\sigma}\int_0^x (x-s)^2 G(s)\, ds \tag{47}$$

Introducing Eq. (47) into Eq. (14)

$$L_p = \frac{4\rho_0\alpha_0 U e^{j\omega t}}{\beta b\tan\sigma}\left\{ j\omega\int_0^{2b}\int_0^x (x-s)^2 G(s)\, ds\, dx + 2U\int_0^{2b}\int_0^x (x-s)\, G(s)\, ds\, dx \right\}$$

Changing variable as before and dividing by $q\,S$

$$C_{L_P} = \frac{2\alpha_0 e^{j\omega t}}{\beta b^3 U}\left\{ 16 j\omega b^4 \int_0^1\int_0^\xi (\xi-\eta)^2 G(2b\eta)\, d\eta\, d\xi + 16 U b^3 \int_0^1\int_0^\xi (\xi-\eta)\, G(2b\eta)\, d\eta\, d\xi \right\}$$

or

$$C_{L_P} = \frac{32\alpha_0 e^{j\omega t}}{\beta}\left\{ j\nu \int_0^1\int_0^\xi (\xi-\eta)^2 G(2b\eta)\, d\eta\, d\xi + \int_0^1\int_0^\xi (\xi-\eta)\, G(2b\eta)\, d\eta\, d\xi \right\}$$

Using the reduction formula of Eq. (41)

$$C_{L_P} = \frac{16\alpha_0 e^{j\omega t}}{3\beta}\left\{ j\nu\left(T_0 - \tfrac{3}{2}T_1 + \tfrac{3}{4}T_2 - \tfrac{1}{8}T_3\right) + \tfrac{3}{2}T_0 - \tfrac{3}{2}T_1 + \tfrac{3}{8}T_2 \right\}$$

A comparison of this expression with Eqs. (43) and (44) leads to the following result

$$C_{L_P} = \frac{16\alpha_0 e^{j\omega t}}{3\beta}\left\{ 3\left(\frac{C_L}{C_{L_0}}\right)_a - 2\left(\frac{C_M}{C_{M_0}}\right)_a \right\} \tag{48}$$

Since the differential equations which are being solved are linear it is permissible to superimpose solutions. Therefore, from Eq. (46)

$$C_{Lb} = (1-2j\nu\mu)\frac{U\alpha_o}{W_o}C_{La} + j\nu C_{LP}$$

For this case the quasi-steady C_L is given by

$$(C_{Lo})_b = \frac{4\alpha_o}{\beta}e^{j\omega t} = \frac{\alpha_o U}{W_o}(C_{Lo})_a$$

so that

$$\left(\frac{C_L}{C_{Lo}}\right)_b = (1-2j\nu\mu + 4j\nu)\left(\frac{C_L}{C_{Lo}}\right)_a - \frac{8j\nu}{3}\left(\frac{C_M}{C_{Mo}}\right)_a \tag{49}$$

It is now necessary to calculate the moment about the vertex due to the parabolic downwash. From Eqs. (15) and (47)

$$\mathcal{M}_P = -\frac{4\rho_o\alpha_o U e^{j\omega t}}{\beta\, b\tan\sigma}\left\{j\omega\int_0^{2b}x\int_0^x(x-s)^2G(s)\,ds\,dx + 2U\int_0^{2b}x\int_0^x(x-s)\,G(s)\,ds\,dx\right\}$$

By a change of variable and integration by parts

$$C_{MP} = -\frac{4\alpha_o e^{j\omega t}}{\beta}\left\{j\nu\left(T_o - \frac{4}{3}T_1 + \frac{1}{2}T_2 - \frac{1}{48}T_4\right) + \frac{4}{3}T_o - T_1 + \frac{1}{12}T_3\right\}$$

Since

$$(C_{Mb})_{x=o} = (1-2j\nu\mu)\frac{U\alpha_o}{W_o}C_{Ma} + j\nu C_{MP}$$

the total moment coefficient about the vertex is

$$(C_{Mb})_{x=o} = -\frac{8\alpha_o e^{j\omega t}}{3\beta}(1-2j\nu\mu)\left(\frac{C_M}{C_{Mo}}\right)_a + j\nu C_{MP}$$

In the case of pitching motions it is generally desirable to calculate the pitching moment about the axis of rotation. If $(C_{Mb})_\mu$ is the pitching moment coefficient about the axis of rotation, then

$$(C_{Mb})_\mu = (C_{Mb})_{x=0} + \mu C_{Lb}$$

Since the quasi-steady value for $(C_{Mb})_\mu$ is

$$(C_{Mo})_\mu = \frac{4\alpha_o(3\mu - 2)}{3\beta} e^{j\omega t}$$

then

$$\left(\frac{C_M}{C_{Mo}}\right)_{b\mu} = \frac{1}{2-3\mu}\left\{2(1-2j\nu\mu)\left(\frac{C_M}{C_{Mo}}\right)_a - 3\mu\left(\frac{C_L}{C_{Lo}}\right)_b + j\nu\left(\frac{C_M}{C_{Mo}}\right)_p\right\} \tag{50}$$

where

$$\left(\frac{C_M}{C_{Mo}}\right)_p = j\nu\left(3T_o - 4T_1 + \frac{3}{2}T_2 - \frac{1}{16}T_4\right) + 4T_o - 3T_1 + \frac{1}{4}T_3 \tag{51}$$

In this case the downwash is again symmetrical about the x-axis and the rolling moment is therefore zero.

<u>Case c:</u> Rolling Oscillation.

In this case the motion of the wing, and hence the downwash, will be specified by the rate of roll $\dot{\phi}$. Thus if

$$\dot{\phi} = P e^{j\omega t}$$

then

$$W(x,y) = P y$$

Since the downwash is antisymmetrical with respect to the x-axis the lift and pitching moment will be zero. It is therefore only necessary to calculate the rolling moment.

From Eq. (28)

$$W_2(x) = \int_{\frac{-x}{\tan\sigma}}^{\frac{x}{\tan\sigma}} P y^2 \, dy = \frac{2Px^3}{3\tan^3\sigma} \tag{52}$$

Introducing this into Eq. (31)

$$\psi_2(x,0,t) = \frac{2Pe^{j\omega t}}{3\beta\tan^3\sigma} \int_0^x (x-s)^3 G(s)\, ds \tag{53}$$

The rolling moment is obtained from Eq. (16)

$$R = -\frac{4\rho_0 P e^{j\omega t}}{3\beta\tan^3\sigma}\left\{ j\omega \int_0^{2b}\int_0^x (x-s)^3 G(s)\, ds\, dx + 3U \int_0^{2b}\int_0^x (x-s)^2 G(s)\, ds\, dx \right\}$$

If the rolling moment coefficient is defined as

$$C_\ell = \frac{R}{qS\ell}$$

where

$$\ell = \frac{4b}{\tan\sigma} = \text{span of wing}$$

Then

$$C_\ell = -\frac{2\ell P e^{j\omega t}}{3\beta U}\left\{ 2j\nu \int_0^1\int_0^\xi (\xi-\eta)^3 G(2b\eta)\, d\eta\, d\xi + 3\int_0^1\int_0^\xi (\xi-\eta)^2 G(2b\eta)\, d\eta\, d\xi \right\}$$

Since the quasi-steady value for the C_ℓ is

$$C_{\ell_0} = -\frac{\ell P e^{j\omega t}}{6\beta U} \tag{54}$$

and using Eqs. (41) and (54) together with the results of Case a and Case b the final result becomes

$$\left(\frac{C_\ell}{C_{\ell_0}}\right)_c = 12\left(\frac{C_L}{C_{L_0}}\right)_a - 8\left(\frac{C_M}{C_{M_0}}\right)_a - \left(\frac{C_M}{C_{M_0}}\right)_p \tag{55}$$

Note that for a given span the rolling moment coefficient is independent of the sweep angle.

Case d: Oscillating Flap.

The final case investigated is that of a flap oscillating about its hinge axis. The flap considered consists of the aft portion of a delta wing (Fig. 4). The hinge line is perpendicular to the x-axis. The chord of the flap is $(1-\mu)2b$. The portion of the wing forward of the hinge line remains at zero angle of attack.

Under the above conditions the expression for the downwash is

$$w(x,y) = 0 \qquad x < 2\mu b$$

$$w(x,y) = \alpha_0 U\left(1 - 2j\nu\mu + \frac{j\nu x}{b}\right) \qquad 2\mu b < x < 2b$$

From Eq. (27)

$$w_1(x) = 0 \qquad x < 2\mu b$$

$$w_1(x) = \frac{2\alpha_0 U x}{\tan\sigma}\left(1 - 2j\nu\mu + \frac{j\nu x}{b}\right) \qquad 2\mu b < x < 2b$$

and from Eq. (30)

$$\psi_1(x,0,t) = 0 \qquad x < 2\mu b$$

$$\psi_1(x,0,t) = \frac{2\alpha_0 U e^{j\omega t}}{\beta\tan\sigma}\int_{2\mu b}^{x}\left(1 - 2j\nu\mu + \frac{j\nu s}{b}\right) s\, G(x-s)\, ds \qquad 2\mu b < x < 2b \tag{56}$$

but the lift is given by

$$L = 2\rho_0 \int_{2\mu b}^{2b} \left(\frac{\partial \psi_1}{\partial t} + U \frac{\partial \psi_1}{\partial x} \right) dx$$

and

$$\frac{\partial \psi_1}{\partial t} = j\omega \psi_1$$

Integrating the second term of the equation for lift,

$$L = 2\rho_0 \left\{ j\omega \int_{2\mu b}^{2b} \psi_1 \, dx + U \psi_1 (2b, 0, t) \right\}$$

which becomes, on substituting Eq. (56),

$$L = \frac{4\rho_0 \alpha_0 U e^{j\omega t}}{\beta \tan\sigma} \left\{ j\omega \int_{2\mu b}^{2b} \int_{2\mu b}^{x} \left(1 - 2j\nu\mu + \frac{j\nu s}{b}\right) s \, G(x-s) \, ds \, dx + U \int_{2\mu b}^{2b} \left(1 - 2j\nu\mu + \frac{j\nu s}{b}\right) s \, G(2b-s) \, ds \right\}$$

The area of the flap is

$$S_f = \frac{4b^2 (1-\mu^2)}{\tan\sigma}$$

Introducing the new variables

$$s = 2b\eta$$

$$x = 2b\xi$$

The C_L based on the area of the flap becomes

$$C_L = \frac{2\alpha_0 e^{j\omega t}}{\beta U b^2 (1-\mu^2)} \left\{ 8 j\omega b^3 \int_{\mu}^{1} \int_{\mu}^{\xi} (1 - 2j\nu\mu + 2j\nu\eta) \eta \, G[2b(\xi-\eta)] \, d\eta \, d\xi + 4b^2 U \int_{\mu}^{1} (1 - 2j\nu\mu + 2j\nu\eta) \eta \, G[2b(1-\eta)] \, d\eta \right\}$$

Let

$$\eta' = \eta - \mu$$

$$\xi' = \xi - \mu$$

then

$$C_L = \frac{8\alpha_\circ e^{j\omega t}}{\beta(1-\mu^2)} \left\{ 2j\nu \int_0^{1-\mu} \int_0^{\xi'} (1 + 2j\nu\eta')(\eta'+\mu)\, G\left[2b(\xi'-\eta')\right] d\eta' d\xi' \right.$$

$$\left. + \int_0^{1-\mu} (1+2j\nu\eta')(\eta'+\mu)\, G\left[2b(1-\mu-\eta')\right] d\eta' \right\}$$

Finally, let

$$\eta' = (1-\mu)s$$

$$\xi' = (1-\mu)x$$

then

$$C_L = \frac{8\alpha_\circ e^{j\omega t}}{\beta(1-\mu^2)} \left\{ 2j\nu(1-\mu)^2 \int_0^1 \int_0^x \left[1+2j\nu(1-\mu)s\right]\left[(1-\mu)s+\mu\right] G\left[2b(1-\mu)(x-s)\right] ds\, dx \right.$$

$$\left. + (1-\mu) \int_0^1 \left[1+2j\nu(1-\mu)s\right]\left[(1-\mu)s+\mu\right] G\left[2b(1-\mu)(1-s)\right] ds \right\}$$

or

$$C_L = \frac{8\alpha_\circ e^{j\omega t}}{\beta(1+\mu)} \left\{ 2j\nu(1-\mu) \int_0^1 \int_0^x \left[1+2j\nu(1-\mu)(x-s)\right]\left[(1-\mu)(x-s)+\mu\right] G\left[2b(1-\mu)s\right] ds\, dx \right.$$

$$\left. + \int_0^1 \left[1+2j\nu(1-\mu)(1-s)\right]\left[(1-\mu)(1-s)+\mu\right] G\left[2b(1-\mu)s\right] ds \right\}$$

It is now convenient to adopt the following convention. If

$$F = F(b)$$

Then

$$F' = F(b')$$

where

$$b' = (1-\mu)\, b$$

This convention makes it possible to use expressions previously obtained, but with the reduced frequency ν' based on the chord of the flap.

The expression for the C_L can now be reduced in the usual manner using Eq. (41). The result, in terms of expressions already obtained, is

$$\left(\frac{C_L}{C_{L_0}}\right)_d = \frac{2}{1+\mu}\left\{\mu\left[\frac{1}{2}T_0' + j\nu'\left(T_0' - \frac{T_1'}{2}\right) + \frac{1}{2}\left(1-\mu+4j\nu'-2j\nu'\mu\right)\left(\frac{C_L}{C_{L_0}}\right)_a' - \frac{4j\nu'(1-\mu)}{3}\left(\frac{C_M}{C_{M_0}}\right)_a' \right.\right. \tag{57}$$

where the quasi-steady lift coefficient of the flap is given by

$$C_{L_0} = \frac{4\alpha_0 e^{j\omega t}}{\beta}$$

The pitching moment about the hinge axis is

$$\mathcal{M} = -2\rho_0 \int_{2\mu b}^{2b} \left[\frac{\partial \Psi_1}{\partial t} + U\frac{\partial \Psi_1}{\partial x}\right](x - 2\mu b)\, dx$$

The second term can be integrated by parts to give

$$\mathcal{M} = -2\rho_0 \left\{ j\omega \int_{2\mu b}^{2b} (x-2\mu b)\Psi_1\, dx + U 2b(1-\mu)\Psi_1(2b,0,t) - U\int_{2\mu b}^{2b} \Psi_1\, dx \right\}$$

Using Eq. (56)

$$\mathcal{M} = -\frac{4\rho_0 \alpha_0 U e^{j\omega t}}{\beta \tan\sigma} \Big\{ j\omega \int_{2\mu b}^{2b} (x-2\mu b) \int_{2\mu b}^{2b} \left(1-2j\nu\mu + j\frac{\nu s}{b}\right) s\, G(x-s)\, ds\, dx$$

$$+ 2b(1-\mu)U \int_{2\mu b}^{2b} \left(1-2j\nu\mu + \frac{j\nu s}{b}\right) s\, G(2b-s)\, ds$$

$$- \int_{2\mu b}^{2b} \int_{2\mu b}^{x} \left(1-2j\nu\mu + \frac{j\nu s}{b}\right) s\, G(x-s)\, ds\, dx \Big\}$$

Proceeding exactly as in the case of the C_L the final result is

$$\left(\frac{C_M}{C_{M_0}}\right)_d = \frac{6}{2+\mu} \left\{ \mu\left[\frac{1}{4}T_1' + j\nu'\left(\frac{1}{2}T_0' - \frac{1}{8}T_2'\right)\right] + \frac{(1-\mu+2j\nu'\mu)}{3}\left(\frac{C_M}{C_{M_0}}\right)_a' + \frac{j\nu'(1-\mu)}{6}\left(\frac{C_M}{C_{M_0}}\right)_p' \right\} \quad (58)$$

where

$$C_{M_0} = -\frac{4\alpha_0(2+\mu)e^{j\omega t}}{3\beta(1+\mu)}$$

Here the C_M is based on the chord of the flap.

IV. CALCULATIONS AND RESULTS

The aerodynamic coefficients obtained in the previous section have all been expressed in terms of the functions $T_n(k,M)$. A recurrence formula may be obtained for T_n by integrating Eq. (42) by parts. This formula was first given by Borbely (Ref. 4); however, the R. T. P. Translation of the paper by Borbely is in error. The correct formula is

$$\nu T_n = 2^n \left[j J_0(2k) - \frac{1}{M} J_1(2k) + \frac{(1-n)}{2kM} J_0(2k) \right] e^{-2jkM} + j(1-2n) T_{n-1} + \frac{(1-n)^2}{kM} T_{n-2} \qquad (59)$$

where J_0 and J_1 are Bessel functions of the first kind of zero and first order, respectively. With this formula it is possible to express all of the results for the delta wing in terms of the single function T_0 . Schwarz has tabulated the values of two functions, $J_s(\lambda, x)$, and $J_c(\lambda, x)$, which are related to T_0 by the equation

$$T_0(k,M) = \frac{1}{kM} \left[J_c\left(\frac{1}{M}, 2kM\right) - j J_s\left(\frac{1}{M}, 2kM\right) \right] \qquad (60)$$

The results of the previous section have been evaluated by means of Eqs. (59) and (60) for two values of Mach number, 1.25 and 2. The results are plotted as polar diagrams in Figures 5 through 11. In order to interpret the polar diagrams in terms of the aerodynamic damping it is convenient to calculate the work done on the wing per cycle. This will be done for each of the four cases.

Case a:

The work done on the wing in one cycle is given by

$$\mathcal{W}_a = \int_0^{\frac{2\pi}{\omega}} L_a \frac{\partial Z_w}{\partial t} dt$$

For real oscillations

$$\frac{\partial Z_w}{\partial t} = -W_o \cos \omega t$$

But $\left(\frac{C_L}{C_{Lo}}\right)_a$ was of the form

$$\left(\frac{C_L}{C_{Lo}}\right)_a = A e^{j\epsilon}$$

thus for real oscillations

$$L_a = B \cos(\omega t + \epsilon) \qquad B > 0$$

Therefore

$$\mathcal{W}_a = -\frac{B W_o}{\omega} \int_0^{2\pi} \cos \omega t \cos(\omega t + \epsilon)\, d(\omega t) = -\frac{\pi B W_o}{\omega} \cos \epsilon \tag{61}$$

Case b:

In this case the work per cycle is given by

$$\mathcal{W}_b = \int_0^{2\pi} \mathcal{M}_b \frac{d\alpha}{d(\omega t)} d(\omega t)$$

The pitching moment may be expressed as

$$\left(\frac{C_M}{C_{M_o}}\right)_b = A e^{j\epsilon}$$

and the angle of attack is

$$\alpha = \alpha_o e^{j\omega t}$$

Thus for real oscillations

$$\mathfrak{m} = B\,(3\mu - 2)\cos(\omega t + \epsilon) \qquad B > 0$$

$$\alpha = \alpha_0 \cos \omega t$$

Therefore

$$\mathcal{W}_b = -B\alpha_0(3\mu - 2)\int_0^{2\pi} \cos(\omega t + \epsilon)\sin\omega t\, d(\omega t) = B\alpha_0 \pi (3\mu - 2)\sin\epsilon \tag{62}$$

<u>Case c</u>:

Here the work per cycle is given by

$$\mathcal{W}_c = \int_0^{\frac{2\pi}{\omega}} R\,\dot{\phi}\, dt$$

Since the rolling moment is of the form

$$\left(\frac{C_\ell}{C_{\ell_0}}\right)_c = A\, e^{j\epsilon}$$

and the rate of roll is

$$\dot{\phi} = p\, e^{j\omega t}$$

then for real oscillations

$$R = -B\cos(\omega t + \epsilon) \qquad B > 0$$

$$\dot{\phi} = p \cos \omega t$$

thus

$$\mathcal{W}_c = -Bp\int_0^{\frac{2\pi}{\omega}} \cos(\omega t + \epsilon)\cos\omega t\, dt = -\frac{Bp\pi}{\omega}\cos\epsilon \tag{63}$$

Case d:

This case is the same as Case b except that for real oscillations

$$m_d = -B \cos(\omega t + \epsilon) \qquad B > 0$$

Therefore

$$w_d = -B \alpha_0 \pi \sin \epsilon \tag{64}$$

In Figure 8 at M = 1.25 the phase angle is negative for small values of the reduced frequency. Since this curve is plotted for μ = .5 Eq. (62) indicates that the damping in pitch is negative, i.e., that work is being done on the wing. In Figure 11 the oscillating flap is also shown to be unstable at M = 1.25 for small values of the frequency.

It is of interest to determine the Mach number at which the pitching oscillations first become unstable for small values of frequency. This may be done by obtaining a Taylor expansion of the T_n. Thus, neglecting terms of order k^2

$$T_n = \frac{2^{n+1}}{n+1} - j\,\frac{2^{n+2}}{n+2}\,kM \tag{65}$$

By means of Eq. (65) the expression for $\left(\frac{C_M}{C_{M_0}}\right)_{b\mu}$ becomes, for small values of frequency,

$$\left(\frac{C_M}{C_{M_0}}\right)_{b\mu} = 1 + \frac{jk}{M(3\mu-2)}\left(8\beta^2\mu - 6\beta^2\mu^2 - 3\beta^2 + \frac{3}{2} - 2\mu\right) \tag{66}$$

The condition that the damping is positive is

$$8\beta^2\mu - 6\beta^2\mu^2 - 3\beta^2 + \frac{3}{2} - 2\mu < 0$$

In Figure 12 the Mach number at which the damping reverses sign is plotted as a function of μ. A similar region of instability was found in the two-dimensional case calculated by Garrick and Rubinow (Ref. 6). Figure 12 shows that the maximum Mach number for instability is $\sqrt{2}$ and that all instability vanishes for rotations about axes aft of the three quarters chord. These results agree with those obtained by Miles (Ref. 8).

Figure 11 indicates that the damping in the case of the flap also becomes negative at M = 1.25. Using Eq. (65) the expression for becomes

$$\left(\frac{C_M}{C_{M_0}}\right)_d = 1 + \frac{jk'}{M(2+\mu)}\left\{\mu\left[M^2 - \frac{7}{2}\right] + 3M^2 - \frac{9}{2}\right\} \tag{67}$$

Thus the condition for positive damping is

$$\mu\left(M^2 - \frac{7}{2}\right) + 3M^2 - \frac{9}{2} > 0$$

The Mach number for zero damping is plotted as a function of μ in Figure 13. It is seen that for a flap of infinite aspect ratio (μ = 1) the Mach number for zero damping is $\sqrt{2}$. This is the two-dimensional result obtained by Garrick and Rubinow.

V. CONCLUSIONS

Using the linearized equations for a compressible isentropic flow, one can reduce any three-dimensional wing problem to an equivalent two-dimensional problem, provided that the edges of the wing are supersonic and that the trailing edge is normal to the flow. For the usual types of planforms and downwash distributions the forces acting on the wing can be expressed in terms of tabulated functions.

As an example, the forces acting on a delta wing have been calculated for the case in which the downwash exhibits a harmonic time dependence. All of the aerodynamic coefficients were found to be independent of the sweep angle. Since any non-stationary motion of a rigid delta wing can be obtained by harmonic analysis from the three modes of oscillation considered here, it is proved that the force coefficients for such an arbitrary motion are independent of the sweep angle. It is also found that the pitching oscillations of a delta wing exhibit a characteristic instability for small values of the frequency. This is in agreement with the two-dimensional result obtained previously.

Finally, it should be noted that while it is correct to calculate the total loads acting on the wing by strip theory, this procedure does not necessarily give the correct spanwise loading. It is interesting, however, that although the load distribution is disturbed by the three-dimensional effect, the two-dimensional lift of a chordwise strip is preserved. This result is in accord with the theorem on the preservation of lift for steadily lifting wings given by Dr. P. A. Lagerstrom (Ref. 11).

TABLE OF NOTATION

$\bar{x}, \bar{y}, \bar{z}$	Cartesian space coordinates
$\bar{t}, t$	Time
φ	Perturbation velocity potential
P	Pressure
ρ	Density
c	Acoustic velocity
L	Lift
m	Pitching moment
R	Rolling moment
y_P, y_S	Coordinates of the port and starboard leading edges
b	Semichord
ω	Frequency
x, y, z	Wing coordinates
$M = \frac{U}{c}$	Mach number
U	Free stream velocity
$\nu = \frac{\omega b}{U}$	Reduced frequency

TABLE OF NOTATION
(continued)

$K = \frac{\nu M}{\beta^2}$	
$\beta^2 = M^2 - 1$	
σ	Sweep angle
S	Wing area
$q = \frac{1}{2}\rho_0 U^2$	Dynamic pressure
$C_L = \frac{L}{qS}$	Lift coefficient
$C_M = \frac{m}{2bqS}$	Pitching moment coefficient
ℓ	Span
$C_\ell = \frac{R}{qS\ell}$	Rolling moment coefficient
α	Angle of attack
p	Rate of roll
$\mathcal{W}$	Work per cycle
ϵ	Phase angle
$(\ )_a$	Case a; plunging oscillation
$(\ )_b$	Case b; pitching oscillation
$(\ )_c$	Case c; rolling oscillation
$(\ )_d$	Case d; pitching oscillation of a flap

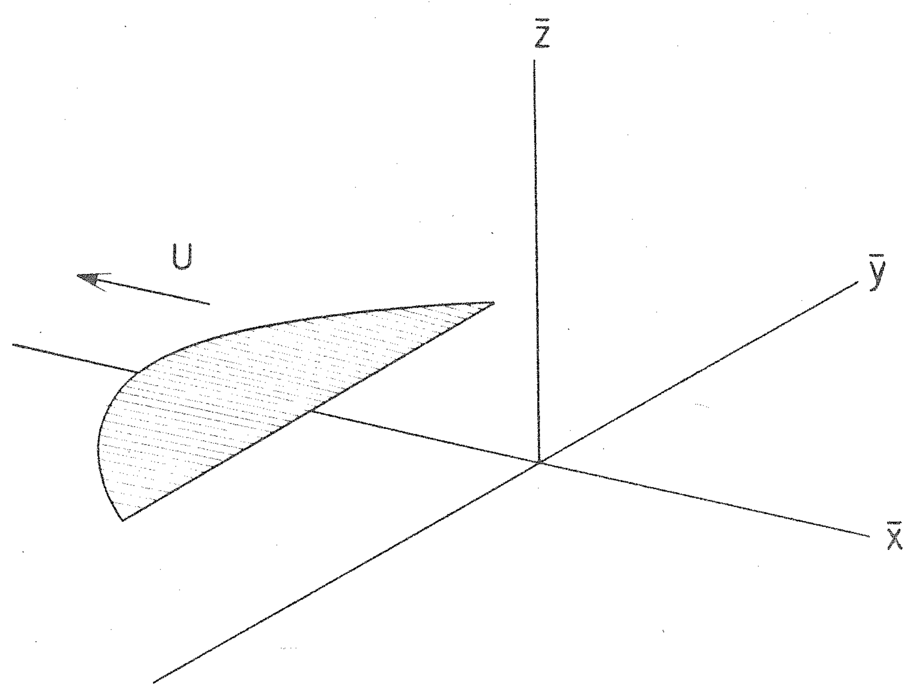

Figure 1.

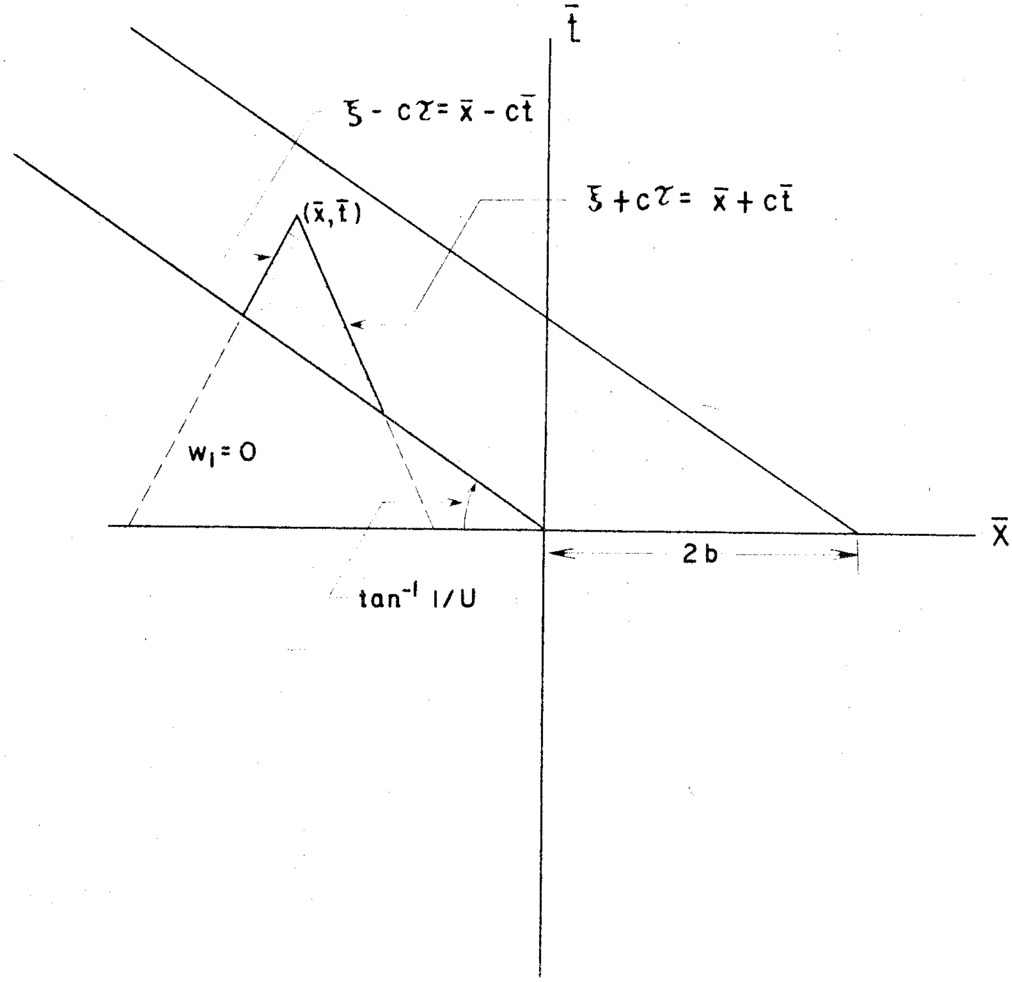
$\bar{t}$
$\xi - c\tau = \bar{x} - c\bar{t}$
$\xi + c\tau = \bar{x} + c\bar{t}$
$(\bar{x}, \bar{t})$
$w_1 = 0$
$\bar{X}$
2 b
$\tan^{-1} 1/U$

Figure 2.

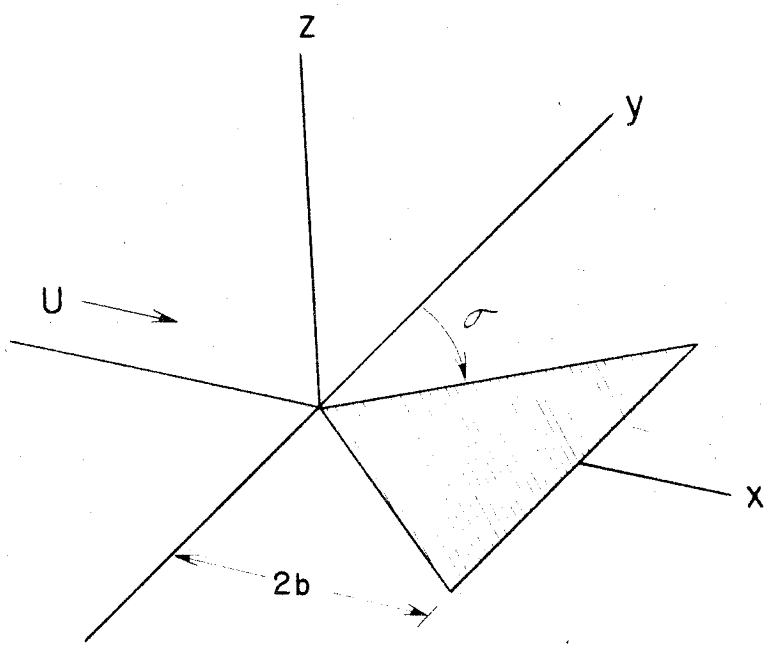

Figure 3.

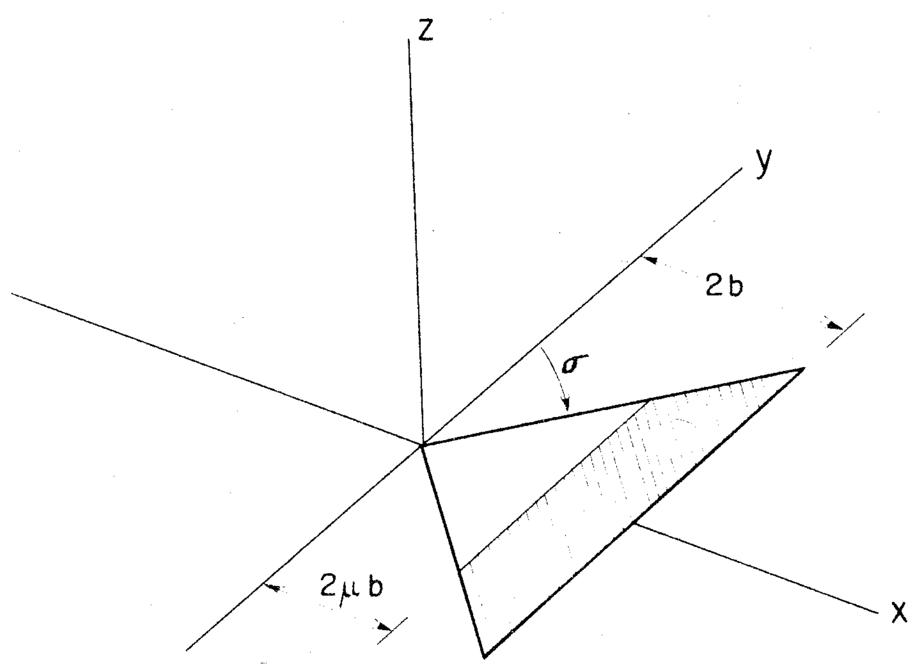

Figure 4.

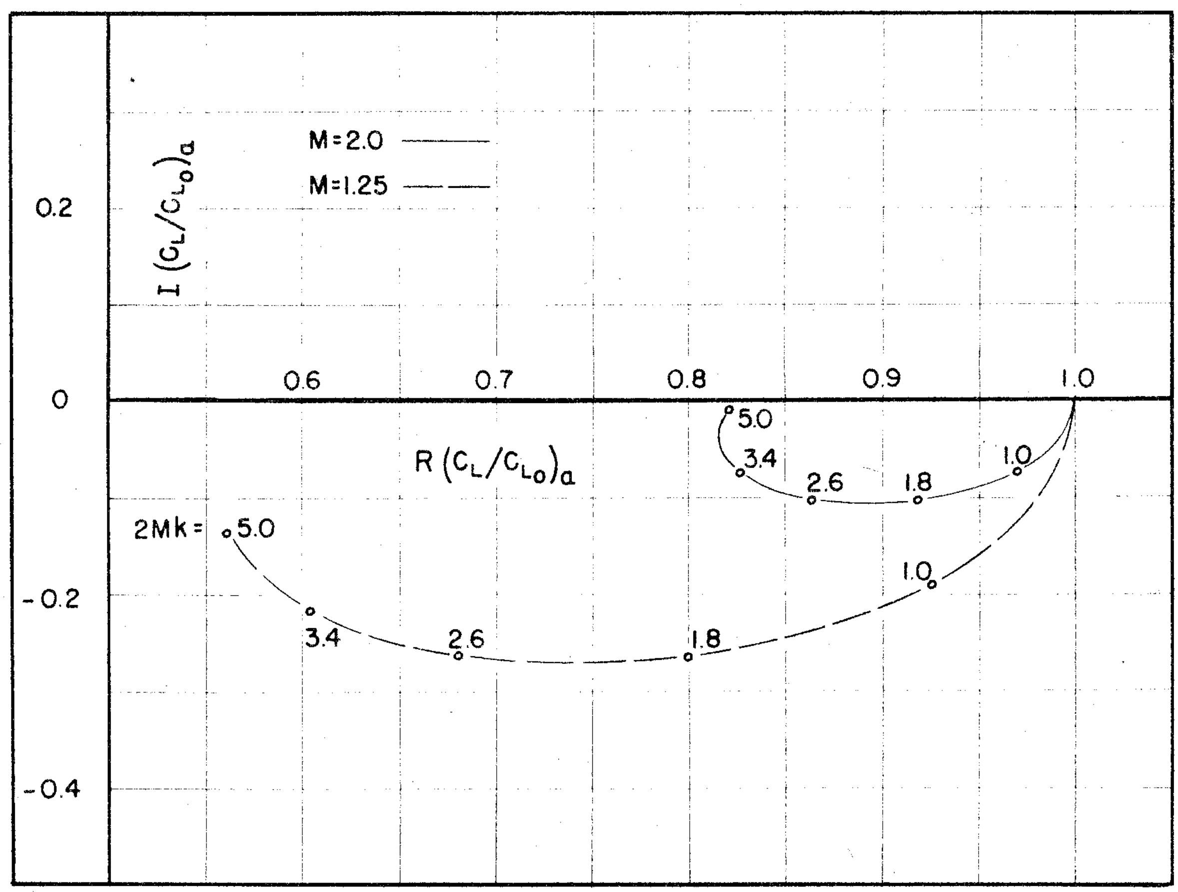

Figure 5. Lift Coefficient, Case a.

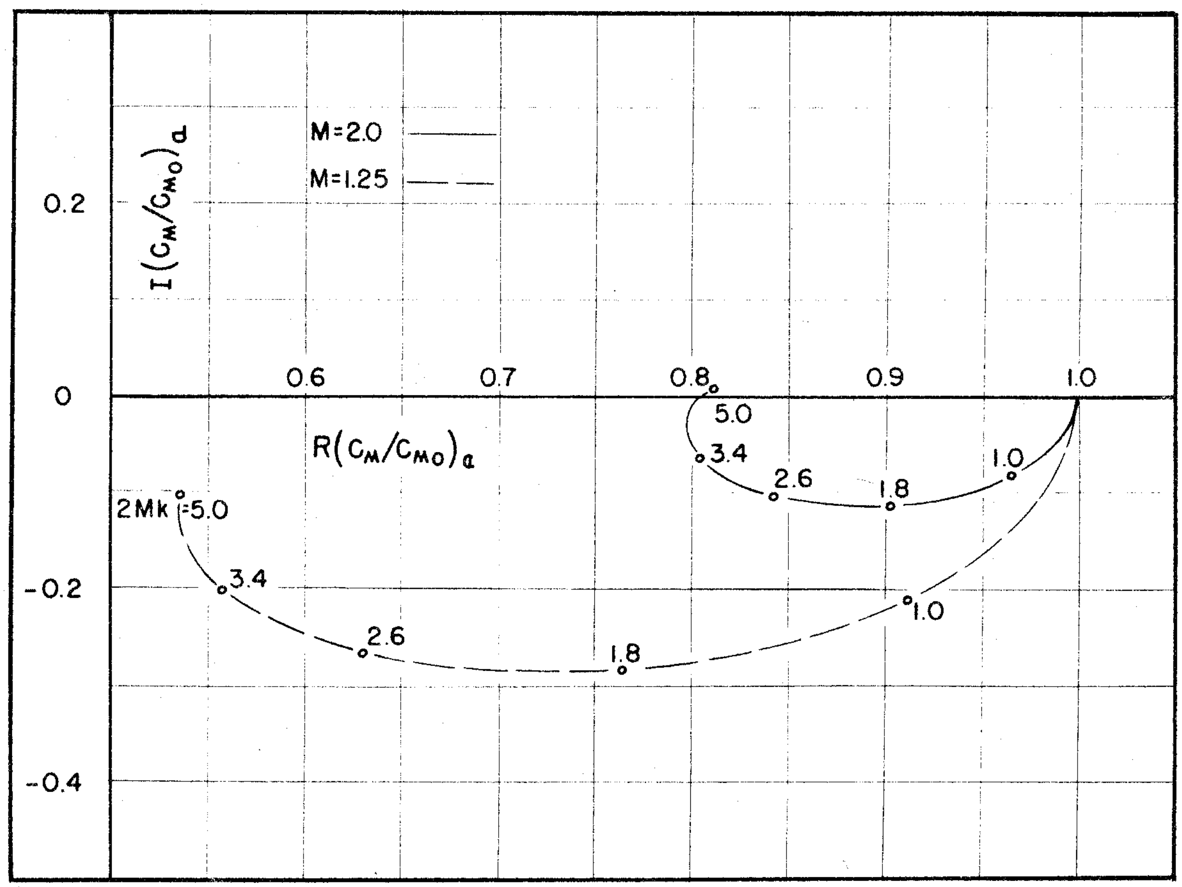

Figure 6. Pitching Moment Coefficient, Case a.

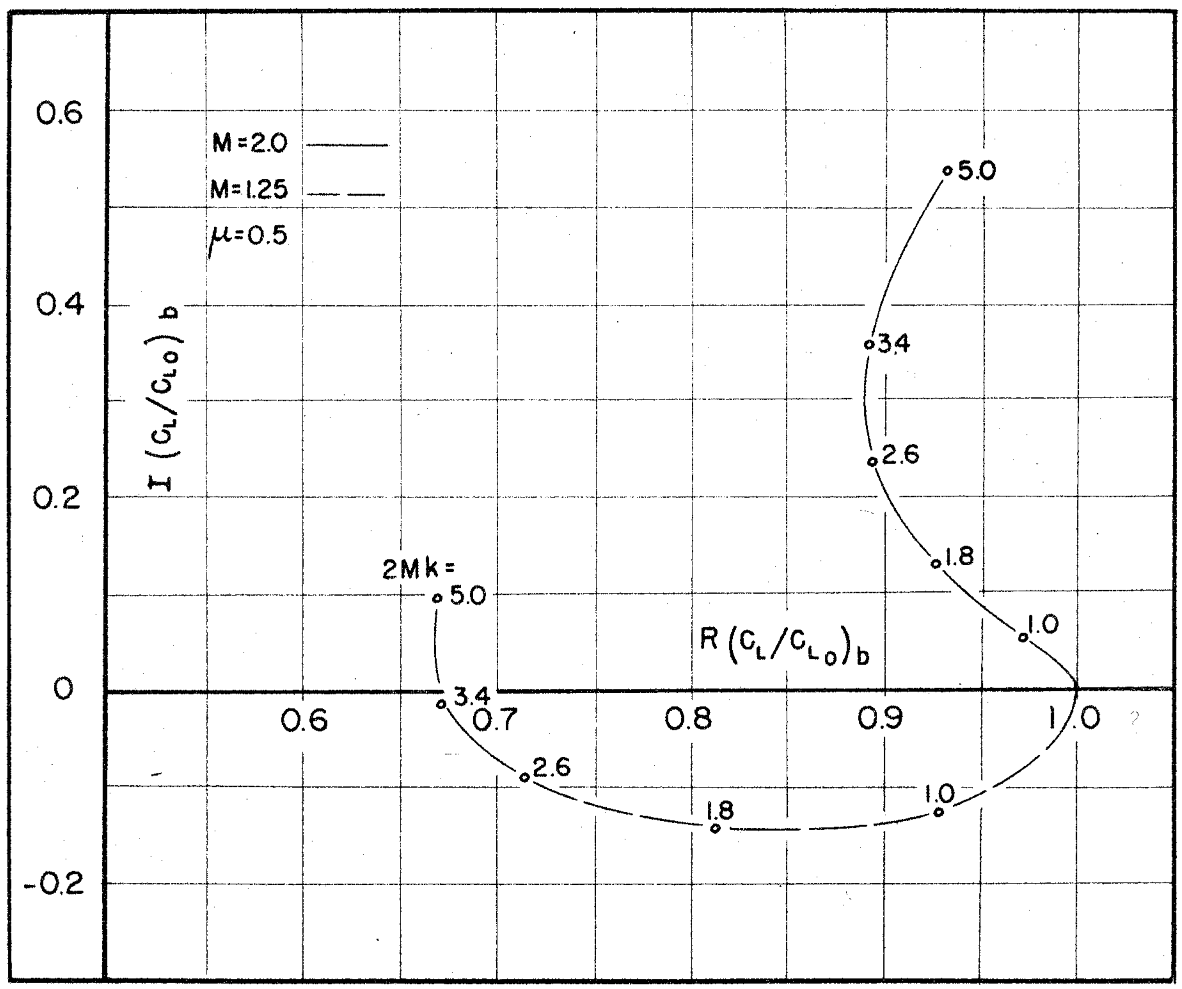

Figure 7. Lift Coefficient, Case b.

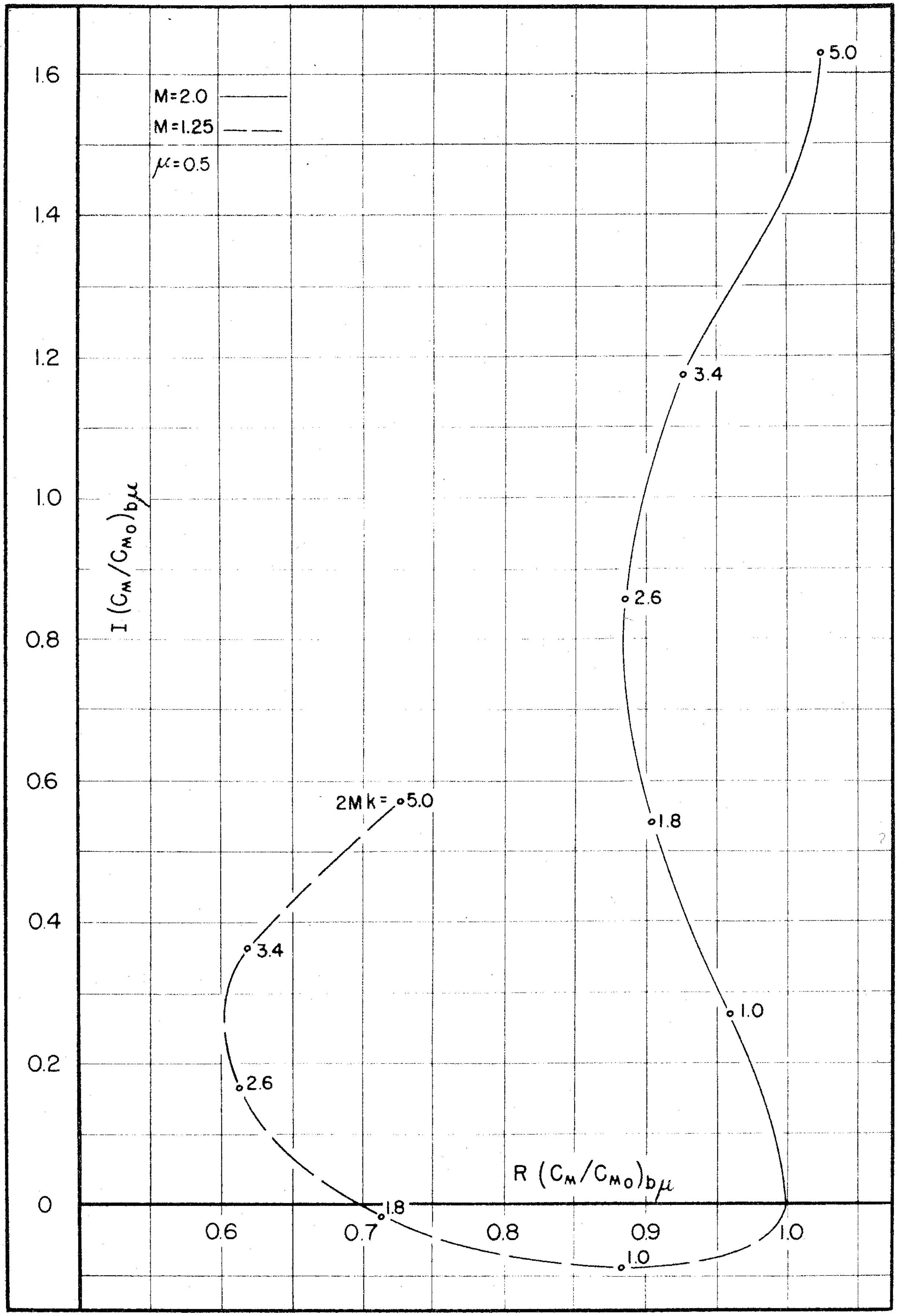

Figure 8. Pitching Moment Coefficient, Case b.

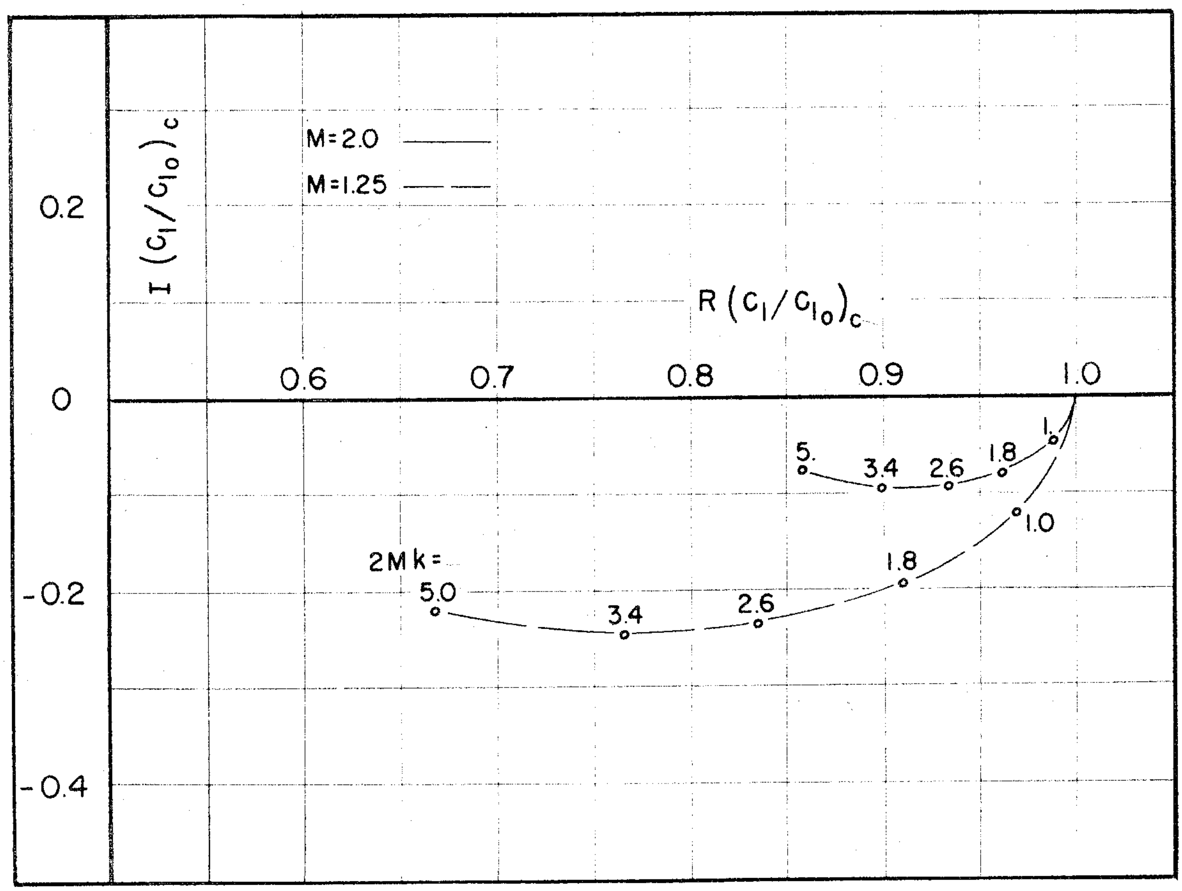

Figure 9. Rolling Moment Coefficient, Case c.

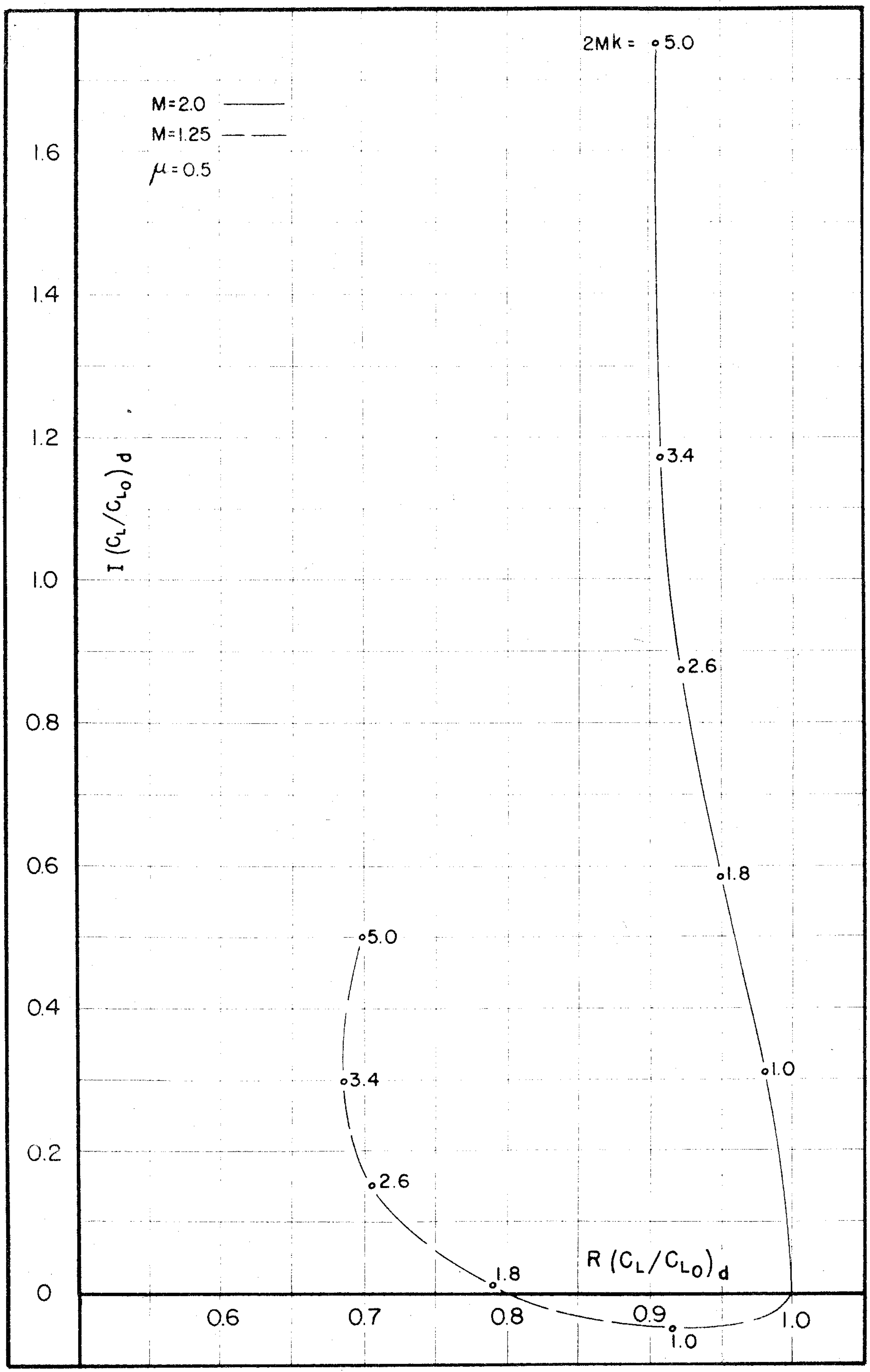

Figure 10. Lift Coefficient, Case d.

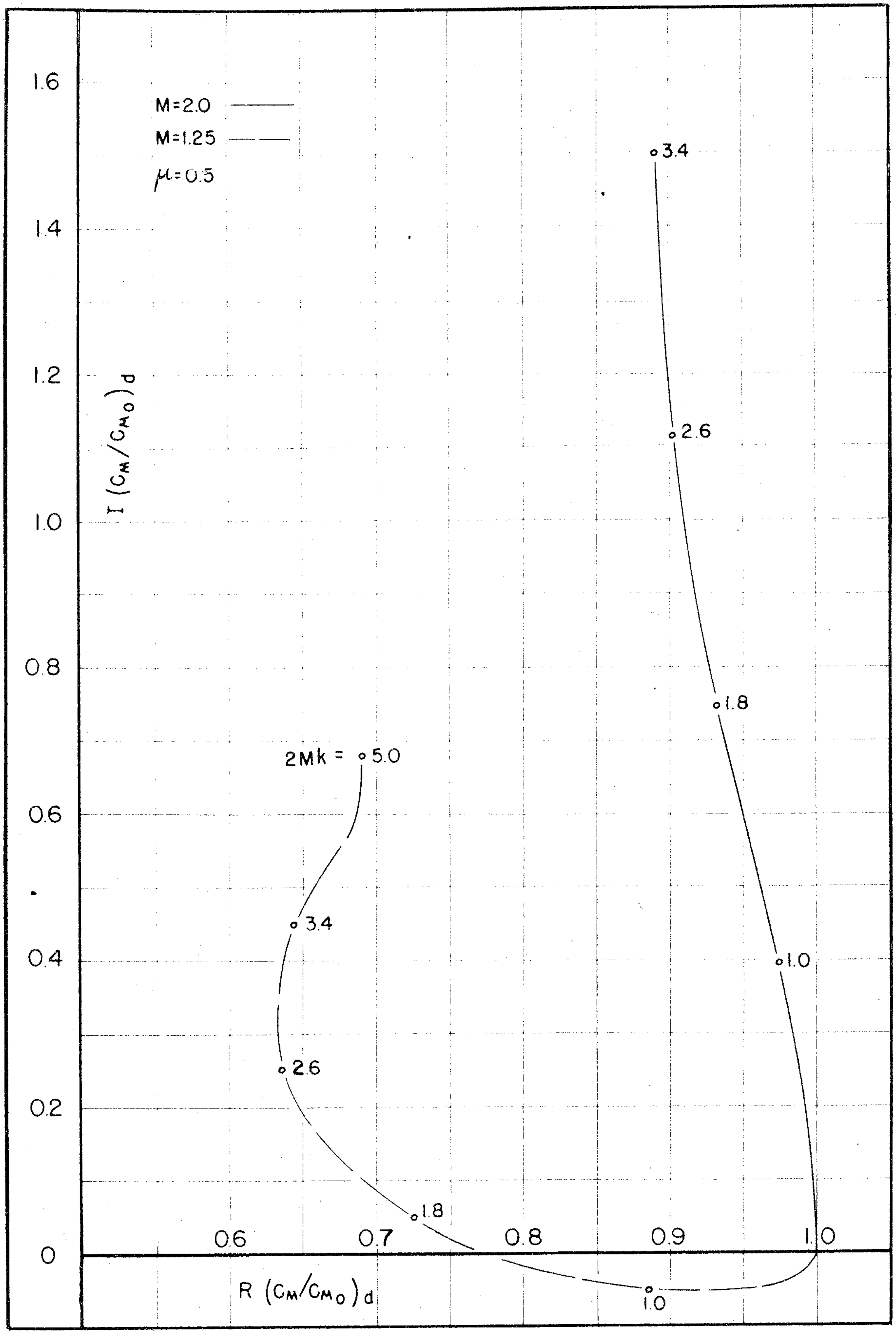

Figure 11. Pitching Moment Coefficient, Case d.

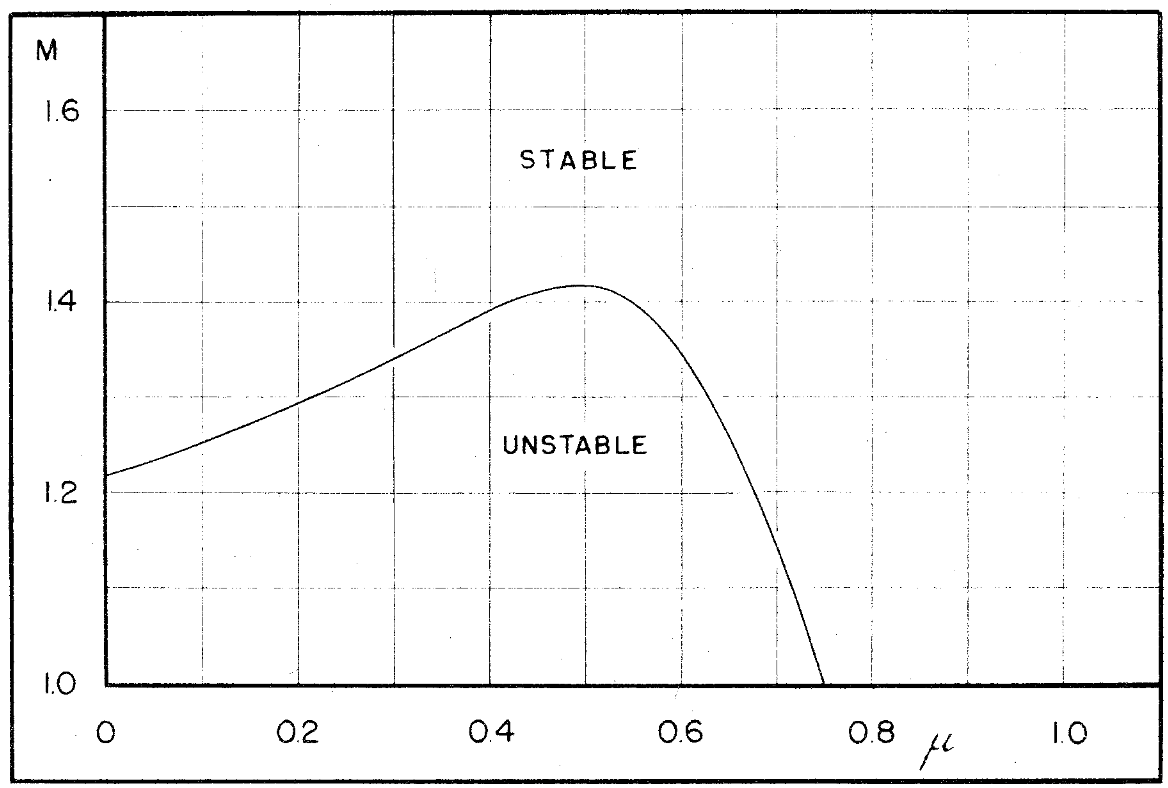

Figure 12. Stability Regions for Pitching Oscillation.

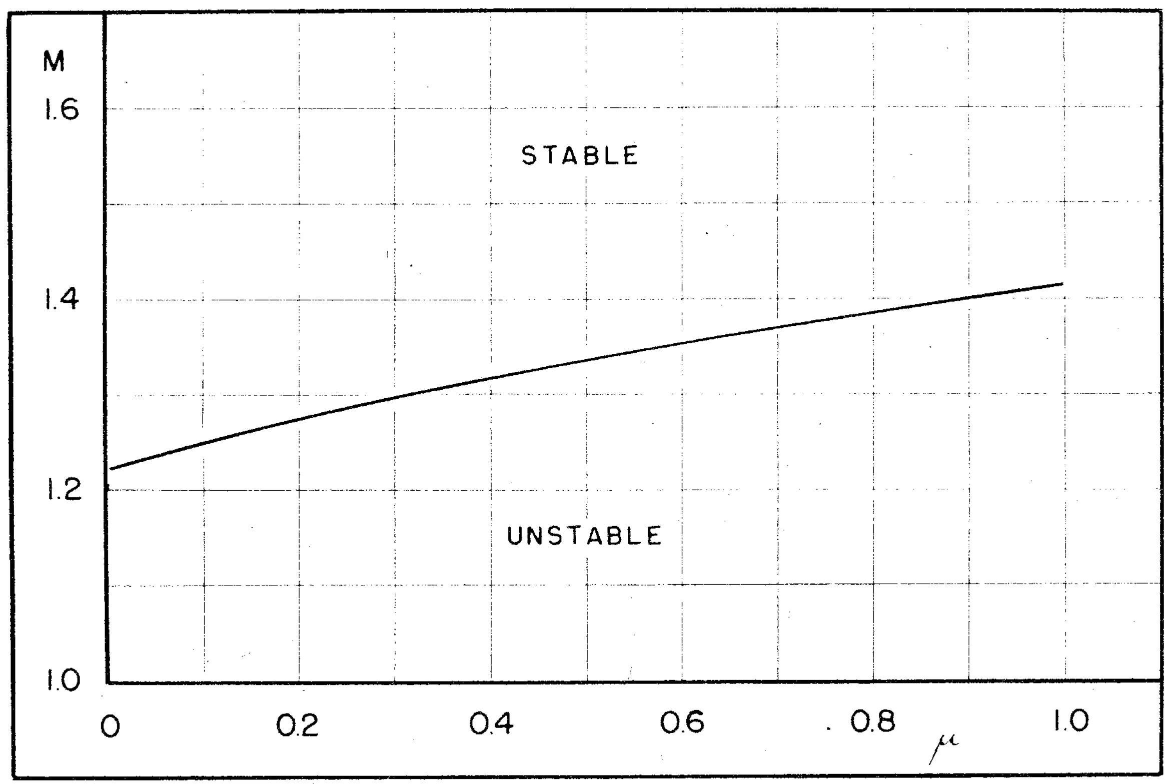

Figure 13. Stability Regions for Oscillating Flap.

TABLE I

M = 1.25

Cases b and d; μ = .5

2kM	$R\left(\frac{C_L}{C_{L_0}}\right)_a$	$I\left(\frac{C_L}{C_{L_0}}\right)_a$	$R\left(\frac{C_M}{C_{M_0}}\right)_a$	$I\left(\frac{C_M}{C_{M_0}}\right)_a$	$R\left(\frac{C_L}{C_{L_0}}\right)_b$	$I\left(\frac{C_L}{C_{L_0}}\right)_b$	$R\left(\frac{C_M}{C_{M_0}}\right)_{b\mu}$	$I\left(\frac{C_M}{C_{M_0}}\right)_{b\mu}$
.6	.9720	-.0996	.9665	-.1375	.9722	-.0862	.9545	-.0727
1.0	.9263	-.1572	.9120	-.2110	.9276	-.1277	.8832	-.0919
1.4	.8662	-.2035	.8413	-.2611	.8710	-.1489	.7970	-.0742
1.8	.8001	-.2364	.7647	-.2851	.8124	-.1460	.7136	-.0206
2.2	.7359	-.2552	.6922	-.2847	.7581	-.1283	.6500	.0659
2.6	.6798	-.2612	.6314	-.2654	.7161	-.0952	.6138	.1670
3.4	.6044	-.2444	.5576	-.1997	.6727	-.0152	.6210	.3611
4.2	.5726	-.2119	.5364	-.1400	.6670	.0510	.6827	.4918
5.0	.5613	-.1793	.5356	-.1052	.6686	.0968	.7271	.5703

2kM	$R\left(\frac{C_\ell}{C_{\ell_0}}\right)_c$	$I\left(\frac{C_\ell}{C_{\ell_0}}\right)_c$	$R\left(\frac{C_L}{C_{L_0}}\right)_d$	$I\left(\frac{C_L}{C_{L_0}}\right)_d$	$R\left(\frac{C_M}{C_{M_0}}\right)_d$	$I\left(\frac{C_M}{C_{M_0}}\right)_d$
.6	.9867	-.0743	.9673	-.0431	.9556	-.0491
1.0	.9694	-.1212	.9158	-.0510	.8866	-.0523
1.4	.9424	-.1612	.8526	-.0343	.8039	-.0200
1.8	.9097	-.1936	.7903	.0086	.7265	.0479
2.2	.8738	-.2180	.7391	.0722	.6680	.1426
2.6	.8367	-.2343	.7051	.1476	.6364	.2499
3.4	.7656	-.2460	.6869	.2970	.6442	.4479
4.2	.7096	-.2373	.6878	.4117	.6841	.5821
5.0	.6675	-.2195	.6998	.5016	.6927	.6800

TABLE II

$M = 2.00$

Cases b and d; $\mu = .5$

$2kM$	$R\left(\frac{C_L}{C_{L_0}}\right)_a$	$I\left(\frac{C_L}{C_{L_0}}\right)_a$	$R\left(\frac{C_M}{C_{M_0}}\right)_a$	$I\left(\frac{C_M}{C_{M_0}}\right)_a$	$R\left(\frac{C_L}{C_{L_0}}\right)_b$	$I\left(\frac{C_L}{C_{L_0}}\right)_b$	$R\left(\frac{C_M}{C_{M_0}}\right)_{b\mu}$	$I\left(\frac{C_M}{C_{M_0}}\right)_{b\mu}$
.6	.9890	-.0389	.9868	-.0539	.9892	.0274	.9839	.1542
1.0	.9707	-.0618	.9651	-.0832	.9718	.0521	.9591	.2685
1.4	.9461	-.0804	.9360	-.1036	.9499	.0853	.9300	.3971
1.8	.9181	-.0937	.9033	-.1135	.9274	.1278	.9040	.5404
2.2	.8897	-.1013	.8708	-.1127	.9081	.1785	.8880	.6954
2.6	.8637	-.1032	.8420	-.1025	.8951	.2352	.8863	.8560
3.4	.8269	-.0927	.8049	-.0638	.8919	.3525	.9274	1.1739
4.2	.8144	-.0708	.7985	-.0205	.9117	.4555	.9990	1.4240
5.0	.8203	-.0468	.8135	.0098	.9342	.5356	1.0508	1.6285

$2kM$	$R\left(\frac{C_l}{C_{l_0}}\right)_c$	$I\left(\frac{C_l}{C_{l_0}}\right)_c$	$R\left(\frac{C_L}{C_{L_0}}\right)_d$	$I\left(\frac{C_L}{C_{L_0}}\right)_d$	$R\left(\frac{C_M}{C_{M_0}}\right)_d$	$I\left(\frac{C_M}{C_{M_0}}\right)_d$
.6	.9955	-.0296	.9927	.1848	.9902	.2346
1.0	.9879	-.0476	.9813	.3119	.9745	.3966
1.4	.9771	-.0634	.9657	.4442	.9546	.5659
1.8	.9638	-.0764	.9493	.5822	.9338	.7431
2.2	.9488	-.0863	.9340	.7254	.9155	.9273
2.6	.9330	-.0928	.9215	.8726	.9018	1.1163
3.4	.8987	-.0961	.9078	1.1716	.8915	1.4985
4.2	.8758	-.0892	.9052	1.4661	.8959	1.8684
5.0	.8572	-.0765	.9035	1.7523	.8980	2.2249

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