REARRANGEMENTS OF MEASURABLE FUNCTIONS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1970

(Submitted April 21, 1970)

ACKNOWLEDGEMENTS

The author thanks Professor W. A. J. Luxemburg for bringing this topic to his attention and for his guidance in its perusal.

The author thanks the National Science Foundation for four years of financial support.

The author also thanks Mrs. Virginia Conner for her efforts in getting this thesis typed quickly.

Finally, the author thanks his wife, Susan, for her patience and understanding during the last three years.

ABSTRACT

Let (X, Λ, μ) be a measure space and let $M(X, \mu)$ denote the set of all extended real valued measurable functions on X. If (X_1, Λ_1, μ_1) is also a measure space and $f \in M(X, \mu)$ and $g \in M(X_1, \mu_1)$, then f and g are said to be equimeasurable (written $f \sim g$) iff $\mu(f^{-1}[r, s]) = \mu_1(g^{-1}[r, s])$ whenever $[r, s]$ is a bounded interval of the real numbers or $[r, s] = \{+\infty\}$ or $= \{-\infty\}$. Equimeasurability is investigated systematically and in detail.

If (X, Λ, μ) is a finite measure space (i. e. $\mu(X) < \infty$) then for each $f \in M(X, \mu)$ the decreasing rearrangement δ_f of f is defined by

$$\delta_f(t) = \inf \{s: \mu(\{f > s\}) \leq t\} \qquad 0 \leq t \leq \mu(X).$$

Then δ_f is the unique decreasing right continuous function on $[0, \mu(X)]$ such that $\delta_f \sim f$. If (X, Λ, μ) is non-atomic, then there is a measure preserving map $\sigma: X \rightarrow [0, \mu(X)]$ such that $\delta_f(\sigma) = f$ μ-a. e.

If (X, Λ, μ) is an arbitrary measure space and $f \in M(X, \mu)$ then f is said to have a decreasing rearrangement iff there is an interval J of the real numbers and a decreasing function δ on J such that $f \sim \delta$. The set $D(X, \mu)$ of functions having decreasing rearrangements is characterized, and a particular decreasing rearrangement δ_f is defined for each $f \in D$. If ess. inf $f \leq 0 <$ ess. sup f, then δ_f is obtained as the right inverse of a distribution function of f. If ess. inf $f < 0 <$ ess. sup f then formulas relating $(\delta_f)^+$ to δ_{f+}, $\delta_f)^-$ to δ_{f-} and δ_{-f} to δ_f are given. If (X, Λ, μ) is non-atomic and σ-finite and δ is a decreasing rearrangement of f on J, then there is a measure preserving map $\sigma: X \rightarrow J$ such that $\delta(\sigma) = f$ μ-a. e.

If (X, Λ, μ) and (X_1, Λ_1, μ_1) are finite measure spaces such that $a = \mu(X) = \mu_1(X_1)$, if $f, g \in M(X, \mu) \cup M(X_1, \mu_1)$, and if $\int_0^a \delta_{|f|}$ and $\int_0^a \delta_{|g|}$ are finite, then $g \ll f$ means $\int_0^t \delta_g \leq \int_0^t \delta_f$ for all $0 \leq t \leq a$, and $g \prec f$ means $g \ll f$ and $\int_0^a \delta_f = \int_0^a \delta_g$. The preorder relations $\prec$ and $\ll$ are investigated in detail.

If $f \in L^1(X, \mu)$, let $\Omega(f) = \{g \in L^1(X, \mu): g \prec f\}$ and $\Delta(f) = \{g \in L^1(X, \mu): g \sim f\}$. Suppose ρ is a saturated Fatou norm on $M(X, \mu)$ such that L^ρ is universally rearrangement invariant and $L^\infty \subset L^\rho \subset L^1$. If $f \in L^\rho$ then $\Omega(f) \subset L^\rho$ and $\Omega(f)$ is convex and $\sigma(L^\rho, L^{\rho'})$-compact. If ξ is a locally convex topology on L^ρ in which the dual of L^ρ is $L^{\rho'}$, then $\Omega(f)$ is the ξ-closed convex hull of $\Delta(f)$ for all $f \in L^\rho$ iff (X, Λ, μ) is adequate. More generally, if $f \in L^1(X_1, \mu_1)$ let $\Omega_f(X, \mu) = \{g \in L^1(X, \mu): g \prec f\}$ and $\Delta_f(X, \mu) = \{g \in L^1(X, \mu): g \sim f\}$. Theorems for $\Omega(f)$ and $\Delta(f)$ are generalized to Ω_f and Δ_f, and a norm ρ_1 on $M(X_1, \mu_1)$ is given such that $\Omega_{|f|} \subset L^\rho$ iff $f \in L^{\rho_1}$.

A linear map $T: L^1(X_1, \mu_1) \to L^1(X, \mu)$ is said to be doubly stochastic iff $Tf \prec f$ for all $f \in L^1(X_1, \mu_1)$. It is shown that $g \prec f$ iff there is a doubly stochastic T such that $g = Tf$.

If $f \in L^1$ then the members of $\Delta(f)$ are always extreme in $\Omega(f)$. If (X, Λ, μ) is non-atomic, then $\Delta(f)$ is the set of extreme points and the set of exposed points of $\Omega(f)$.

A mapping $\Phi: \Lambda_1 \to \Lambda$ is called a homomorphism if (i) $\mu(\Phi(A)) = \mu_1(A)$ for all $A \in \Lambda_1$; (ii) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$ $[\mu]$ whenever $A \cap B = \emptyset$ $[\mu_1]$; and (iii) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ $[\mu]$ for all $A, B \in \Lambda_1$, where $A = B\ [\mu]$ means $C_A = C_B$ μ-a.e. If $\Phi: \Lambda_1 \to \Lambda$ is a homomorphism, then there is a unique doubly stochastic operator $T_\Phi: L^1(X_1, \mu_1) \to L^1(X, \mu)$ such that $T_\Phi C_E = C_{\Phi(E)}$ for all E. If $T: L^1(X_1, \mu_1) \to L^1(X, \mu)$ is linear then $Tf \sim f$ for all $f \in L^1(X_1, \mu_1)$ iff $T = T_\Phi$ for some homomorphism Φ.

Let X_o be the non-atomic part of X and let A be the union of the atoms of X. If $f \in L^1(X, \mu)$ then the $\sigma(L^1, L^\infty)$-closure of $\Delta(f)$ is shown to be $\{g \in L^1:$ there is an $h \sim f$ such that $g|X_o \prec h|X_o$ and $g|A = h|A\}$ whenever either (i) X consists only of atoms; (ii) X has only finitely many atoms; or (iii) X is separable.

TABLE OF CONTENTS

1. Introduction. The decreasing rearrangement of a non-negative measurable function has, since its treatment in Inequalities by Hardy, Littlewood and Polya, played an increasingly important role in analysis, because of its fundamental part in the structure of normed spaces of measurable functions which are rearrangement invariant. Examples of such spaces are the classical L^p spaces, the Orlicz spaces, and the spaces introduced by Halperin [14] and Lorentz [23]. These so-called rearrangement invariant Banach function spaces have been shown by Boyd [2], Shimogaki[45], and Lorentz and Shimogaki [25] to be well suited for studying problems related to Fourier analysis and interpolation of operators.

Recently Luxemburg [28] gave a general account of the theory of rearrangement invariant Banach spaces for measure spaces with finite total measure. Such spaces provide a natural setting in which to generalize a theorem of Hardy, Littlewood and Polya which gives equivalent conditions that two vectors in R^n be related by a certain preorder relation $\prec$. J. V. Ryff [42] has given the generalization for $L^1[0, 1]$, while Luxemburg [28] has given it in part in the rearrangement invariant Banach space setting. We will complete this generalization in Chapter V.

With an eye to studying these topics when the total measure of the space is not finite, in Chapter I we investigate the concept of equimeasurability for arbitrary measure spaces, though we include as well results which can only be proved in general for finite measure spaces.

In Chapter II we define the decreasing rearrangement for all measurable functions if the measure space is finite. If the measure space is not finite, we characterize the set of measurable functions which have decreasing rearrangements, and define one for each such function. Of importance is the fact that we can prove a theorem relating a function to its decreasing rearrangement by a measure preserving transformation when the measure space is non-atomic and σ-finite.

In Chapter III we introduce the generalization to measurable functions on a finite measure space of the Hardy-Littlewood-Polya preorder relation $\prec$, and investigate it and some associated inequalities. In particular we give a new and careful proof of a theorem in [28] about the values taken on by certain integrals, and we characterize adequate measures.

For completeness we include as Chapter IV Luxemburg's treatment of rearrangement invariant Banach spaces. For the same reason we include his results on Schur convex functions in Chapter V, where we give finally a complete account of the generalization of the theorem of Hardy, Littlewood and Polya referred to above. We also extend Ryff's generalization of Muirhead's theorem to finite measure spaces. Finally in Chapter VI we settle some extremal and related problems of some sets which arose in Chapter V.

Throughout we use the following abbreviations:
m. s. = measure space; m.p. = measure preserving; pwd = pairwise disjoint.

I. EQUIMEASURABILITY

2. Spectral Measures. Let (X, Λ, μ) be a measure space (m. s.), i.e., X is a non-empty point set, Λ is a σ-algebra of subsets of X, and μ is a countably additive measure on Λ. Often we will write $\int f\,d\mu$ for the integral of f over X with respect to μ when X is clear from the context. Also we let $M = M(X, \mu)$ denote the set of all extended real valued μ-measurable functions on X, and if E is a set, then C_E denotes the characteristic function of E.

Let R denote the real numbers, let $R^{\#}$ denote the extended real numbers, and let $S = S(R^{\#})$ denote the Riesz space of all functions s of the form

$$s = \sum_{k=1}^{n} \alpha_k C_{I_k}$$

where each $\alpha_k \in R$ and each I_k is a bounded interval of $R^{\#}$, i.e., I_k is an interval of R of finite length or $I_k = \{+\infty\}$ or $I_k = \{-\infty\}$. We call the members of S step functions on $R^{\#}$.

For every $s \in S$ and $f \in M$ it is easy to see that $s(f)$ is a simple measurable function on X. If $f \in M$, let $I_f : S \to R^{\#}$ be defined by

$$I_f(s) = \int s(f)\,d\mu$$

for all $s \in S$. Suppose f has the property that $\mu(f^{-1}[u, v]) < \infty$ for every bounded interval $[u, v]$ of $R^{\#}$. Then $I_f(s)$ is finite for every $s \in S$ and in fact defines a positive linear functional which is continuous in the sense that if $s_n \downarrow 0$ then $I_f(s_n) \downarrow 0$. Hence there is a measure μ_f on $R^{\#}$ such that for every $s \in S$ we have

$$\int s(f)\,d\mu = I_f(s) = \int_{R^{\#}} s\,d\mu_f \,.$$

We call μ_f the spectral measure of f, or sometimes the μ-spectral measure of f. In probability theory μ_f is known as the distribution measure of f.

For many purposes, whether or not f takes the value 0 on a set of finite or infinite measure is of no interest, so our conditions under which μ_f is defined may seem unduly restrictive. However, we have only to let $X' = \{x \in X : f(x) \neq 0\}$ (called the carrier of f) $\Lambda' = \Lambda \cap X'$ and $\mu' = \mu \mid \Lambda'$ to see that if $\mu(f^{-1}[u,v]) < \infty$ for all bounded intervals $[u, v]$ of $R^{\#} - \{0\}$ then $\mu'(f|X'^{-1}[u,v]) < \infty$ for all bounded intervals of $R^{\#}$ and thus the μ'-spectral measure $\mu'_{f|X'}$ is defined. We may in a similar manner ignore whether or not f takes the values $+\infty$ or $-\infty$ on sets of finite or infinite measure.

Observe that if $[u, v]$ is a bounded interval of $R^{\#}$, then $\mu_f([u, v]) = \mu(f^{-1}[u, v])$. If (X, Λ, μ) is a finite m. s. (i. e. $\mu(X) < \infty$), then μ_f is defined for every $f \in M$ and can be represented by the distribution function d_f defined for $t \in R$ by

$$d_f(t) = \mu(\{x \in X : f(x) > t\}) \quad .$$

Letting $e_f(t) = \mu(\{x \in X : f(x) \leq t\})$ for every $t \in R$ we have that $d_f + e_f = \mu(X)$, d_f is decreasing, e_f is increasing, both d_f and e_f are right continuous, $d_{f_n} \uparrow d_f$ and $e_{f_n} \downarrow e_f$ whenever $f_n \uparrow f$, $\lim_{t \to \infty} d_f(t) = \mu(\{f = +\infty\})$, $\lim_{t \to -\infty} e_f(t) = \mu(\{f = -\infty\})$, $d_f(t-) = \mu(\{x : f(x) \geq t\})$, and e_f is continuous at t iff d_f is continuous at t iff $\mu(\{f = t\}) = 0$.

(2.1) PROPOSITION. If (X, Λ, μ) is a finite m. s. and $f_n \to f$ pointwise a. e. then $d_{f_n} \to d_f$ at every point of continuity of d_f, so $d_{f_n} \to d_f$ pointwise a. e.

PROOF. Let $E = \{x: f_n(x) \neq f(x)\}$. Then $\mu(E) = 0$. Let $t \in R$ and let $A_n = \{f_n > t\}$ and $A = \{f > t\}$. Then $A \subset E \cup \liminf A_n \subset E \cup \limsup A_n \subset E \cup A \cup \{f = t\}$. If d_f is continuous at t, then $\mu(\{f=t\}) = 0$ so $\mu(A) \leq \mu(\liminf A_n) \leq \liminf \mu(A_n) \leq \limsup \mu(A_n) \leq \mu(\limsup A_n) \leq \mu(A)$ and hence $d_f(t) = \mu(A) = \lim \mu(A_n) = \lim d_{f_n}(t)$. Since d_f is decreasing on R, d_f has only countably many discontinuities.

REMARK. If $f = g$ a. e. then $s(f) = s(g)$ a. e. for each $s \in S$. Thus for each $s \in S$ we may define a mapping $T_s : L^1(X, \Lambda, \mu) \to L^1(X, \Lambda, \mu)$ by $T_s f = s(f)$. Let $S_1 = \{s \in S : s = C_{]u,v[},\ u \ \&\ v \text{ rational}\}$. Then $\{T_s : s \in S_1\}$ separates points of L^1.

For if f & $g \in L^1$ differ on a set of positive μ-measure, then at least one of $\{f < g\}$ or $\{g < f\}$ has positive measure. By symmetry we may assume $\{f < g\}$ has positive measure. If $\{\beta_i\}_{i=1}^{\infty}$ is an enumeration of all rationals of R, then $\{f < g\} = \bigcup_{i=1}^{\infty} \{f < \beta_i < g\}$ so there is a rational $\underline{v}$ such that $\mu(\{f < v < g\}) > 0$. Since $f \in L^1$, $\mu(\{f = -\infty\}) = 0$ so there is a rational number $\underline{u}$ s.t. $\mu(\{u < f < v < g\}) > 0$. Letting $s = C_{]u,v[}$ we have $s(f) = C_{\{u < f < v\}}$ so $\{s(f) \neq s(g)\} \supset \{u < f < v < g\}$ and thus s(f) and s(g) differ on a set of positive measure.

3. Spectral Equivalency. Let (X_1, Λ_1, μ_1) and (X_2, Λ_2, μ_2) be measure spaces (m. s.) with $f_1 \in M(X_1, \mu_1)$ and $f_2 \in M(X_2, \mu_2)$. If both m. s. are finite, then we say that f_1 and f_2 are spectrally equivalent or equi-measurable iff $\mu_{f_1} = \mu_{f_2}$ and in this case we write $f_1 \sim f_2$. Then $f_1 \sim f_2$ iff $d_{f_1} = d_{f_2}$ iff $\mu_1(f_1^{-1}[u, v]) = \mu_2(f_2^{-1}[u, v])$ for every bounded interval $[u, v]$ of $R^{\#}$.

In case one of the m. s. is not finite we write $f_1 \sim f_2$ iff $\mu_1(f_1^{-1}[u,v]) = \mu_2(f_2^{-1}[u,v])$ for every bounded interval $[u, v]$ of $R^{\#}$. Then $f_1 \sim f_2$ iff $\mu_1(f_1^{-1}[B]) = \mu_2(f_2^{-1}[B])$ for every Borel set B of $R^{\#}$.

Observe that if there exist $f_1 \in M(X_1, \mu_1)$ and $f_2 \in M(X_2, \mu_2)$ such that $f_1 \sim f_2$ then $\mu_1(X_1) = \mu_2(X_2)$ in the sense that both are infinite or finite and equal.

(3. 1) LEMMA. <u>If $f \sim g$ and f is a simple function, then g is a simple function. Two simple functions are equimeasurable iff they take the same value on sets of equal measure.</u>

PROOF. Let $f = \sum_{i=1}^{n} a_i C_{E_i}$ where $a_1 < \ldots < a_n$ and $\{E_i\}$ partitions X_1. Suppose $g \in M(X_2, \mu_2)$ and $g \sim f$. Then $\mu_2(\{g = a_i\}) = \mu_1(E_i)$.

$$\mu_2(\{g \notin \{a_1, \ldots, a_n\}\}) = \mu_2(g^{-1}[-\infty, a_1[) + \sum_{i=1}^{n-1} \mu_2(g^{-1}]a_i, a_{i+1}[) + \mu_2(g^{-1}]a_n, +\infty])$$

$$= \mu_1(f^{-1}[-\infty, a_1[) + \sum_{i=1}^{n-1} \mu_1(f^{-1}]a_i, a_{i+1}[) + \mu_1(f^{-1}]a_n, +\infty])$$

$$= \mu_1(\{f \notin \{a_1, \ldots, a_n\}\}) = 0.$$

Thus if $A_i = \{g = a_i\}$ then $g = \sum_{i=1}^{n} a_i C_{A_i}$ a. e.

Now let f be as above and suppose $g = \sum_{i=1}^{n} a_i C_{A_i}$ where $\{A_i\}$ partitions X_2 and $\mu_2(A_i) = \mu_1(E_i)$, $i = 1, \ldots, n$. If $[u, v]$ is a bounded interval of $R^{\#}$, then $\mu_2(g^{-1}[u,v]) = \mu_2(\cup\{A_i : a_i \in [u,v]\})$

$= \Sigma\{\mu_2(A_i) : a_i \in [u,v]\} = \Sigma\{\mu_1(E_i) : a_i \in [u,v]\}$

$= \mu_1(\cup\{E_i : a_i \in [u,v]\}) = \mu_1(f^{-1}[u,v])$.

(3.2) PROPOSITION. Let F be a Borel measurable function on an interval $I \subset R^{\#}$. If $G \in M(I)$ and $r > 0$ and $F_1(t) = F(rt)$ and $G_1(t) = G(rt)$ on $\frac{1}{r} I$ then $G \sim F$ iff $G_1 \sim F_1$. In addition, each of the following functions is equimeasurable with F.

(i) $H(t) = F(t-r)$ $r \in R$, $t \in I+r$

(ii) $H(t) = F(-t)$ $t \in -I$

(iii) $H(t) = F(t+)$ if F is monotonic

(iv) $H(t) = F(t-)$ if F is monotonic

PROOF. Let m denote Lebesgue measure, and let [u, v] be a bounded interval of $R^{\#}$. $m(F_1^{-1}[u,v]) = \frac{1}{r} m(G^{-1}[u,v])$ so $m(F^{-1}[u,v]) = m(G^{-1}[u,v])$ iff $m(F_1^{-1}[u,v]) = m(G_1^{-1}[u,v])$.

(i) $m(H^{-1}[u,v]) = m(F^{-1}[u,v]+r) = m(F^{-1}[u,v])$.

(ii) $m(H^{-1}[u,v]) = m(-F^{-1}[u,v]) = m(F^{-1}[u,v])$.

(iii) & (iv) $H = F$ a. e.

(3.3) PROPOSITION. The following are true for all measure spaces.

(i) $f \sim g$ implies $s(f) \sim s(g)$ for all $s \in S$

(ii) $f \sim g$ implies $rf \sim rg$ and $f+r \sim g+r$ for all $r \in R$

(iii) $f \sim g$ implies $|f| \sim |g|$

(iv) $f \sim g$ implies $f^+ \sim g^+$ and $f^- \sim g^-$

(v) If $\sigma: X_1 \to X_2$ is measure preserving and $f \in M(X_2, \mu_2)$ then $f \circ \sigma \sim f$.

(vi) $f \sim g$ implies ess. inf f = ess. inf g and ess. sup f = ess. sup g.

(vii) If $f \sim g$ and $f \geq 0$ a. e. then $g \geq 0$ a. e. and there are sequences $\{f_n\}$ and $\{g_n\}$ of simple measurable functions such that $f_n \sim g_n$ and $0 \leq f_n \uparrow f$ and $0 \leq g_n \uparrow g$. In addition, if f is essentially bounded then g is essentially bounded and $f_n \uparrow f$ and $g_n \uparrow g$ uniformly.

(viii) *If* $f_1 \in M(X_1, \mu_1)$ *and* $f_2 \in M(X_2, \mu_2)$ *and* $f_1, f_2 \geq 0$ *then* $f_1 \sim f_2$ *implies* $\int f_1 \, d\mu_1 = \int f_2 \, d\mu_2$ *in the sense that both are finite and equal, or both are infinite.*

(ix) *If* $f_1 \in L^1(X_1, \mu_1)$ *and* $f_2 \in M(X_2, \mu_2)$ *and* $f_2 \sim f_1$ *then* $f_2 \in L^1(X_2, \mu_2)$ *and* $\int f_1 \, d\mu_1 = \int f_2 \, d\mu_2$.

(x) *If* $f \in M(X, \mu)$ *and* $g \in M(X', \mu')$ *and* $\{X_i\}_{i=1}^{\infty}$ and $\{X_i'\}_{i=1}^{\infty}$ *are pairwise disjoint measurable subsets of* X *and* X' *respectively such that* $\mu(X - \cup X_i) = 0 = \mu'(X' - \cup X_i')$ *then* $f|X_i \sim g|X_i'$ $i = 1, 2, 3, \ldots.$ *implies* $f \sim g$.

In addition the following are true for finite m. s.

(xi) *If* $f_1 \sim g_1$ *and* $f_2 \sim g_2$ *and* $\inf\{|f_1|, |f_2|\} = 0 = \inf\{|g_1|, |g_2|\}$ *then* $f_1 + f_2 \sim g_1 + g_2$.

(xii) $f \sim g$ *iff* $f^+ \sim g^+$ *and* $f^- \sim g^-$.

(xiii) *If* $f_n \to f$ *and* $g_n \to g$ *a. e. and* $f_n \sim g_n$ $n = 1, 2, 3, \ldots,$ *then* $f \sim g$.

(xiv) *If* $f_n, f \in L^1(X_1, \mu_1)$ *and* $g_n, g \in L^1(X_2, \mu_2)$ *and* $f_n \to f$ *and* $g_n \to g$ *in* L^1 *norm and* $f_n \sim g_n$ $n = 1, 2, 3, \ldots,$ *then* $f \sim g$.

(xv) *If* φ *is Borel measurable on* $R^{\#}$ *then* $f \sim g$ *implies* $\varphi(f) \sim \varphi(g)$.

PROOF. (i) Let $f \sim g$ and $s = \sum_{k=1}^{n} a_k C_{I_k}$ with $\{I_k\}$ pwd intervals of $R^{\#}$. Then $s(f) = \sum_{k=1}^{n} a_k C_{f^{-1}[I_k]} \sim \sum_{k=1}^{n} a_k C_{g^{-1}[I_k]} = s(g)$ using Lemma (3.1).

(ii) Let $[u, v]$ be a bounded interval of $R^{\#}$. If $r > 0$ then
$\mu_1((rf)^{-1}[u, v]) = \mu_1(f^{-1}[\frac{u}{r}, \frac{v}{r}]) = \mu_2(g^{-1}[\frac{u}{r}, \frac{v}{r}]) = \mu_2((rg)^{-1}[u, v])$.
$\mu_1((-f)^{-1}[u, v]) = \mu_1(f^{-1}[-v, -u]) = \mu_2(g^{-1}[-v, -u]) = \mu_2((-g)^{-1}[u, v])$.
Thus $rf \sim rg$ if $r > 0$ and $-f \sim -g$, so $rf \sim rg$ for all $r \in R$. Let $r \in R$.
$\mu_1((f+r)^{-1}[u, v]) = \mu_1(f^{-1}[u-r, v-r]) = \mu_2(g^{-1}[u-r, v-r]) = \mu_2((g+r)^{-1}[u, v])$.

(iii) Let $f \sim g$ and let $[u, v]$ be a bounded interval of $R^{\#}$. In case $u \leq 0 \leq v$, $\mu_1(|f|^{-1}[u, v]) = \mu_1(f^{-1}[-v, v]) = \mu_2(g^{-1}[-v, v])$ $= \mu_2(|g|^{-1}[u, v])$. If $0 < u$, $\mu_1(|f|^{-1}[u, v]) = \mu_1(f^{-1}[u,v]) + \mu_1((-f)^{-1}[u, v])$ $= \mu_2(g^{-1}[u, v]) + \mu_2((-g)^{-1}[u, v]) = \mu_2(|g|^{-1}[u, v])$.
The case $v < 0$ is trivial.

(iv) Let $f \sim g$ and let $[u, v]$ be a bounded interval of $R^{\#}$. We wish to prove that $\mu_1(f^{+-1}[u, v]) = \mu_2(g^{+-1}[u, v])$ which is clearly true if $v<0$ or $u>0$. Hence suppose $u \leq 0 \leq v$. Then $\mu_1(f^{+-1}[u, v]) = \mu_1(f^{-1}[-\infty, v])$ $= \mu_2(g^{-1}[-\infty, v]) = \mu_2(g^{+-1}[u, v])$. For the rest, $f \sim g \Rightarrow -f \sim -g \Rightarrow f^-$ $= (-f)^+ \sim (-g)^+ = g^-$.

(v) $\mu_1((f \circ \sigma)^{-1}[u, v]) = \mu_1(\sigma^{-1} f^{-1}[u, v]) = \mu_2(f^{-1}[u, v])$.

(vi) Recall that ess. sup $f = \inf\{t : \mu(\{f > t\}) = 0\}$ and ess. inf $f =$ $-$ ess. sup $(-f)$. If $f \sim g$, then $t \in R^{\#} \Rightarrow \mu_1(\{f > t\}) = \mu_1(f^{-1}]t, +\infty])$ $= \mu_2(g^{-1}]t, +\infty]) = \mu_2(\{g > t\})$ so ess. sup $f =$ ess. sup g. $f \sim g \Rightarrow -f \sim -g$ $\Rightarrow$ ess. inf $f = -$ess. sup$(-f) = -$ess. sup$(-g) =$ ess. inf g.

(vii) Let $f \sim g$ and $f \geq 0$ a.e. Then (vi) above $\Rightarrow g \geq 0$ a.e. To construct $\{f_n\}$ and $\{g_n\}$ let

$$A_{n,i} = f^{-1}\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right[\qquad B_{n,i} = g^{-1}\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right[\qquad 1 \leq i \leq n\,2^n$$

$$A_n = f^{-1}[n, \infty] \qquad B_n = g^{-1}[n, \infty]$$

$$f_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} C_{A_{n,i}} + n C_{A_n} \quad g_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} C_{B_{n,i}} + n C_{B_n} .$$

Then Lemma (3.1) $\Rightarrow f_n \sim g_n$. $f \sim g \Rightarrow |f| \sim |g|$ so if f is essentially bounded, then ess. sup $|g| =$ ess. sup $|f| < \infty$ so g is essentially bounded. The rest is well known. See [17, p. 159, (11.35)].

(viii) If f is a simple function and $g \sim f$ then Lemma (3.1) $\Rightarrow$ g is simple and $\int f\,d\mu_1 = \int g\,d\mu_2$. If $f \geq 0$ and $g \sim f$ then $g \geq 0$ and there are sequences $\{f_n\}$ and $\{g_n\}$ of non-negative simple functions s.t. $f_n \sim g_n$ and $f_n \uparrow f$ and $g_n \uparrow g$. Then $\int f\,d\mu_1 = \lim \int f_n d\mu_1 = \lim \int g_n\,d\mu_2$ $= \int g\,d\mu_2$.

(ix) If $f \in L^1(X_1, \mu_1)$ and $g \sim f$ then $|g| \sim |f|$ so $\int |g|\,d\mu_2 = \int |f|\,d\mu_1$ $< \infty$ and thus $g \in L^1(X_2, \mu_2)$. Finally $g \sim f \Rightarrow g^+ \sim f^+$ and $g^- \sim f^-$ so $\int g\,d\mu_2 = \int g^+\,d\mu_2 - \int g^-\,d\mu_2 = \int f^+ d\mu_1 - \int f^- d\mu_1 = \int f\,d\mu_1$.

(x) Letting $X_0 = X - \cup X_i$ and $X_0' = X' - \cup X_i'$ we have $f|X_i \sim g|X_i'$ $i = 0, 1, 2, \ldots$, and $\{X_i\}_{i=0}^{\infty}$ and $\{X_i'\}_{i=0}^{\infty}$ are partitions of X and X' respectively. Let $[r, s]$ be a bounded interval of $R^{\#}$. Since $f|X_i \sim g|X_i$ $i=0,1,2,\cdots$, we have $\mu(f^{-1}[r, s] \cap X_i) = \mu'(g^{-1}[r, s] \cap X_i')$ $i = 0, 1, 2, \cdots$, so summing from $i = 0$ to $+\infty$ we get $\mu(f^{-1}[r, s]) =$ $\mu'(g^{-1}[r, s])$. Hence $f \sim g$.

(xi) Let f_1, f_2, g_1, g_2 be as stated. Then $\mu_1(X_1) = \mu_2(X_2)$. Let $E_i = \{f_i \neq 0\}$, $F_i = \{g_i \neq 0\}$ $i = 1, 2$, and $E_3 = X_1 - (E_1 \cup E_2)$, $F_3 = X_2 - (F_1 \cup F_2)$. Then $\mu_1(E_i) = \mu_2(F_i)$ $i = 1, 2$ and since the m. s. are finite we may conclude that $\mu_1(E_3) = \mu_2(F_3)$. Let $[u, v]$ be a bounded interval of $R^{\#}$ and let $f = f_1 + f_2$, $g = g_1 + g_2$. Then $\mu_1(f^{-1}[u, v])$

$$= \sum_{i=1}^{3} \mu_1(E_i \cap f^{-1}[u, v]) = \mu_1(f_1^{-1}[u, v]) + \mu_1(f_2^{-1}[u, v]) + \mu_1(E_4)$$

where $E_4 = \begin{cases} E_3 & \text{if } 0 \in [u, v] \\ \emptyset & \text{if } 0 \notin [u, v] \end{cases}$. Hence if $F_4 = \begin{cases} F_3 & \text{if } 0 \in [u, v] \\ \emptyset & \text{if } 0 \notin [u, v] \end{cases}$

then
$$\mu_2(g^{-1}[u, v]) = \mu_2(g_1^{-1}[u, v]) + \mu_2(g_2^{-1}[u, v]) + \mu_2(F_4)$$
$$= \mu_1(f^{-1}[u, v]) .$$

(xii) Follows from (iv) and (xi) .

(xiii) Using (2.1) we have

$d_f = \lim d_{f_n} = \lim d_{g_n} = d_g$ on R so $f \sim g$.

(xiv) $f_n \to f$ in norm implies $\{f_n\}$ has a subsequence $f_{n_k} \to f$ pointwise a.e. Then $g_{n_k} \to g$ in norm so $\{g_{n_k}\}$ has a subsequence $g_{n_{m_k}} \to g$ pointwise a.e. But then $f_{n_{m_k}} \to f$ pointwise a.e. so (3.3)(xiii) implies $f \sim g$.

(xv) From the proof (vii) we see that each of φ^+ and φ^- is the limit of a sequence of simple functions of the form $\sum_{i=1}^{n} \alpha_i C_{E_i}$ where the sets $\{E_i\}$ are Borel sets. Hence φ is the limit of a sequence $\{s_n\}$ of simple functions of the same form. Hence $f \sim g \Rightarrow s_n(f) \sim s_n(g)$ and since $s_n(f) \to \varphi(f)$ and $s_n(g) \to \varphi(g)$ we have $\varphi(f) \sim \varphi(g)$.

(3.4) LEMMA. (i) Suppose $f \in M(X_1, \Lambda_1, \mu_1)$, $g \in M(X_2, \Lambda_2, \mu_2)$, $E_1 \in \Lambda_1$, $E_2 \in \Lambda_2$, $\mu_1(E_1) = \mu_2(E_2) < \infty$ and f and g have the same constant value on E_1 and E_2 respectively. Then $f \sim g \Rightarrow f|X_1 - E_1 \sim g|X_2 - E_2$ and $fC_{X_1 - E_1} \sim gC_{X_2 - E_2}$.

(ii) Suppose $f, g \in M(X, \Lambda, \mu)$, $E \in \Lambda$ has $\mu(E) < \infty$ and $f|E = g|E$. Then $f \sim g \Rightarrow f|X-E \sim g|X-E$ and $fC_{X-E} \sim gC_{X-E}$.

PROOF. (i) Let $f| = f|X_1 - E_1$ and $g| = g|X_2 - E_2$, and let $\alpha = f|E_1 = g|E_2$. Let $[r, s]$ be a bounded interval of $R^{\#}$. Then

$$\mu_1(f^{-1}[r, s] \cap E_1) = \begin{cases} 0 & \text{if } \alpha \notin [r, s] \\ \mu_1(E_1) & \text{if } \alpha \in [r, s] \end{cases}$$

and similarly for g, so $\mu_1(f^{-1}[r, s] \cap E_1) = \mu_2(g^{-1}[r, s] \cap E_2)$. Hence

$$\mu_1(f|^{-1}[r, s]) = \mu_1(f^{-1}[r, s] \cap (X_1 - E_1))$$
$$= \mu_1(f^{-1}[r, s]) - \mu_1(f^{-1}[r, s] \cap E_1)$$
$$= \mu_1(g^{-1}[r, s]) - \mu_2(g^{-1}[r, s] \cap E_2) = \mu_2(g|^{-1}[r, s]) ,$$

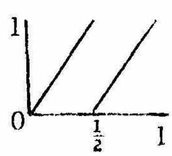
so $f| \sim g|$.

Let $f_1 = f\, C_{X_1 - E_1}$ and $g_1 = g\, C_{X_2 - E_2}$. Then $\mu_1(f_1^{-1}(0)) =$ $\mu_1(E_1 \cup ((X_1 - E_1) \cap f^{-1}(0))) = \mu_1(E_1) + \mu_1(f|^{-1}(0)) = \mu_2(E_2) + \mu_2(g|^{-1}(0)) =$ $\mu_2(g_1^{-1}(0))$. If I is an interval of $R^{\#}$ and $0 \notin I$, then $\mu_1(f_1^{-1}(I)) =$ $\mu_1(f|^{-1}(I)) = \mu_2(g|^{-1}(I)) = \mu_2(g_1^{-1}(I))$. Hence $f_1 \sim g_1$.

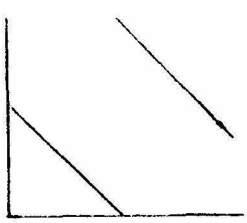
(ii) Let $f| = f|X-E$ and $g| = g|X-E$. Let $[r, s]$ be a bounded interval of $R^{\#}$. Since $f|E = g|E$, we have $\mu(f^{-1}[r, s] \cap E) =$ $\mu(g^{-1}[r, s] \cap E)$ and the rest is similar to part (i).

(3.5) EXAMPLES. 1. The following functions are equimeasurable with $f(x) = x$ on $[0, 1]$ and are measure preserving maps of $[0,1] \to [0,1]$.

(i) $g_1(x) = 1-x \quad 0 \le x \le 1$

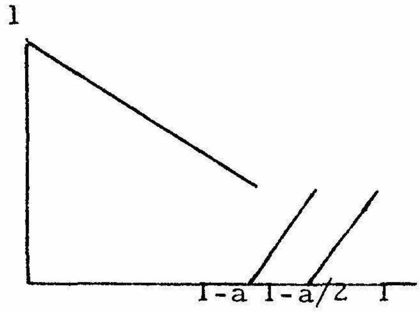
(ii) 

$g_2(x) = 2x \mod 1$

(iii) 

$$g_3(x) = \begin{cases} \frac{1}{2} - x & 0 \le x < \frac{1}{2} \\ \frac{3}{2} - x & \frac{1}{2} \le x \le 1 \end{cases}$$

(iv)



$$g_4(x) = \begin{cases} 1-x & 0 \le x < 1-a \\ 2(x-1+a) & 1-a \le x < 1-\frac{a}{2} \\ 2(x-1+\frac{a}{2}) & 1-\frac{a}{2} \le x \le 1 \end{cases}$$

$$0 \le a \le 1$$

2. Let $f = C_{]-\infty,0[} - C_{]2,\infty[}$ and $g = C_{]-\infty,1[} - C_{]2,\infty[}$. Then $f^+ = C_{]-\infty,0[} \sim C_{]-\infty,1[} = g^+$ and $f^- \sim g^-$ but $f \not\sim g$ since $m(f^{-1}(0)) = 2$ while $m(g^{-1}(0)) = 1$.

3. Let $f = C_{]-\infty,0[} + 2C_{[0,\infty[}$. $m([0,\infty[) = m([1,\infty[)$ and f has the same constant value on $[0,\infty[$ and $[1,\infty[$ but $f|R - [0,\infty[$ $\not\sim f|R - [1,\infty[$.

4. Let $\mathfrak{F}$ be the collection of all finite subsets F of $[0,1]$ and let $\mathfrak{F}$ be directed by $\subset$. Then $\{C_F\}_{F\in\mathfrak{F}}$ is a net which converges pointwise to $C_{[0,1]}$ (since if $t \in [0,1]$ then $F_0 = \{t\} \in \mathfrak{F}$ and $F_0 \subset F$ implies $C_F(t) = 1$) but $C_F \sim 0$ for all $F \in \mathfrak{F}$.

For $i = 1, \ldots, k$ let (X_i, Λ_i, μ_i) and (Y_i, Σ_i, ν_i) be finite m. s., let $X = X_1 \times \ldots \times X_k$, $Y = Y_1 \times \ldots \times Y_k$, $\mu = \mu_1 \times \ldots \times \mu_k$ and $\nu = \nu_1 \times \ldots \times \nu_k$.

(3.6) PROPOSITION. Let $f_i \in M(X_i, \mu_i)$ and $g_i \in M(Y_i, \nu_i)$ $i = 1, \ldots, k$ and let $F(x_1, \ldots, x_k) = (f_1(x_1), \ldots, f_k(x_k))$ for all $(x_1, \ldots, x_k) \in X$ and $G(y_1, \ldots, y_k) = (g_1(y_1), \ldots, g_k(y_k))$ for all $(y_1, \ldots, y_k) \in Y$. If $f_i \sim g_i$ $i = 1, \ldots, k$ then $F \sim G$.

PROOF. Let $\mathcal{A}_k$ be the semi-algebra of all measurable rectangles of R^k and let $\mathcal{B}_k$ be the σ-algebra of all Borel subsets of R^k. If A is a set and $\mathcal{D} \subset 2^A$ let $S(\mathcal{D})$ denote the σ-algebra generated by $\mathcal{D}$;

let $C(\mathcal{D})$ denote the monotone class generated by $\mathcal{D}$; and let $R(\mathcal{D})$ be the algebra generated by $\mathcal{D}$. If $\mathcal{D}$ is a semi-algebra we recall that $R(\mathcal{D})$ is the set of all finite pairwise disjount [pwd] unions of members of $\mathcal{D}$.

Let $\mathcal{A} = \{B \in \mathcal{B}_k : \mu(F^{-1}[B]) = \nu(G^{-1}[B])\}$.

If $B_1 \times \cdots \times B_k \in \mathcal{N}_k$, then

$$\begin{aligned}\mu(F^{-1}(B_1 \times \ldots \times B_k)) &= \mu(f_1^{-1}[B_1] \times \cdots \times f_k^{-1}[B_k]) \\ &= \mu_1(f^{-1}[B_1]) \cdots \mu_k(f_k^{-1}[B_k]) \\ &= \nu_1(g_1^{-1}[B_1]) \cdots \nu_k(g_k^{-1}[B_k]) \\ &= \nu\,(G^{-1}(B_1 \times \cdots \times B_k))\end{aligned}$$

so $\mathcal{N}_k \subset \mathcal{A}$. $\mathcal{A}$ is clearly closed under pwd unions so $R(\mathcal{N}_i) \subset \mathcal{A}$. $\mathcal{A}$ is easily seen to be a monotone class, so $\mathcal{B}_k = S(\mathcal{N}_k) = S(R(\mathcal{N}_k)) = C(R(\mathcal{N}_k)) \subset \mathcal{A}$.

(3.7) COROLLARY. <u>For</u> $i = 1, \ldots, k$ <u>let</u> (X_i, Λ_i, μ_i) <u>and</u> (Y_i, Σ_i, ν_i) <u>be finite m. s., let</u> $f_i \in M(X_i, \mu_i)$, $g_i \in M(Y_i, \nu_i)$, $a_i \in R$, <u>and</u> $n_i \geq 0$ integers. <u>If</u> $f_i \sim g_i$, $i=1, \ldots, k$ <u>then for each of the following definitions</u> of F <u>and</u> G <u>we have</u> $F \sim G$.

(a) $F(x_1, \ldots, x_k) = \sum_{i=1}^{k} a_i f_i^{n_i}(x_i)$

$G(y_1, \ldots, y_k) = \sum_{i=1}^{k} a_i g_i^{n_i}(y_i)$

(b) $F(x_1, \ldots, x_k) = \prod_{i=1}^{k} f_i^{n_i}(x_i)$

$G(y_1, \ldots, y_k) = \prod_{i=1}^{k} g_i^{n_i}(y_i)$

PROOF. Use Props. (3.6) and (3.3) (xv) with $\varphi(t_1, \ldots, t_k) = \sum_1^k a_i t_i^{n_i}$ and $\varphi(t_1, \ldots, t_k) = \prod_1^k t_i^{n_i}$.

REMARKS. (1) For finite m. s. (3.3) (xv) gives another proof of (3.3) (ii), (iii), & (iv) by letting $\varphi(t) = rt$, $\varphi(t) = t+r$, $\varphi(t) = |t|$, and $\varphi(t) = \max\{0, t\}$, $\varphi(t) = -\min\{0, t\}$.

(2) For finite m. s. if $f \sim a f$ for some number a s. t. $|a| \neq 1$, then $|f| = 0$ or $+\infty$. For we have by induction that $|f| \sim |a|^n |f| \to 0$ or $+\infty$ as $n \to \infty$.

II. DECREASING REARRANGEMENTS

4. <u>The Right Inverse of a Decreasing Function.</u> If p is a decreasing function defined on an interval J of R, we can extend p to a decreasing function defined on $R^{\#}$ by defining for $t \notin J$,

$$p(t) = \begin{cases} +\infty & \text{if } t \le \inf J \\ -\infty & \text{if } t \ge \sup J \end{cases} .$$

If p is a decreasing function defined on $R^{\#}$, then its <u>right continuous inverse</u> $p^{\bullet}$ is defined by

$$p^{\bullet}(t) = \inf\{u \in R: p(u) \le t\}$$

for each $t \in R^{\#}$, where by $\inf \emptyset$ we mean $+\infty$ and $\inf R = -\infty$. If p is 1:1 then $p^{\bullet} = p^{-1}$.

It is easy to see that $p^{\bullet}$ is decreasing and right continuous, and for every $t \in R^{\#}$, $p^{\bullet}(p(t)) \le t$, $p^{\bullet}(p(t+)) \le t$, $p^{\bullet}((p(t)+\epsilon)-) \le t$, $p^{\bullet}(p(t)-\epsilon) \ge t$ whenever $\epsilon > 0$, and $p^{\bullet}(p(t)-) \ge t$. Furthermore, for all $t \in R^{\#}$, $p^{\bullet}(t) = \sup\{u \in R: p(u) > t\} = \sup\{u: p(u-) > t\}$ and $p^{\bullet}(t-) = \inf\{u \in R: p(u) < t\} = \inf\{u: p(u-) < t\}$.

(4.1) PROPOSITION. <u>If</u> p <u>is a decreasing function on</u> $R^{\#}$, <u>then for every</u> $t \in R$ <u>we have</u>

$$p^{\bullet\bullet}(t) = (p^{\bullet})^{\bullet}(t) = p(t+).$$

PROOF. Let $u \in R$. Since $p^{\bullet}(p(u+)) \le u$ we have $p^{\bullet\bullet}(u) = \inf\{t: p^{\bullet}(t) \le u\} \le p(u+)$. Since $p^{\bullet}$ is decreasing and right continuous we have $p^{\bullet}(p^{\bullet\bullet}(u)) = p^{\bullet}(\inf\{t: p^{\bullet}(t) \le u\}) = \sup\{p^{\bullet}(t): p^{\bullet}(t) \le u\} \le u$. Since $t \to p(t+)$ is also decreasing and right continuous we have similarly

$p(p^{\bullet}(t)+) = \sup\{p(r+): p(r) \leq t\} \leq t$ for all $t \in R^{\#}$. Hence

$p(u+) \leq p(p^{\bullet}(p^{\bullet\bullet}(u))+) \leq p^{\bullet\bullet}(u)$.

(4.2) PROPOSITION. Let $0 < a \in R$ and let p be a decreasing function defined on $[0, a]$. Then for every $t \in R$,

$$p^{\bullet}(t) = m(\{u \in [0, a]: p(u) > t\}) = d_p(t)$$

where m is Lebesgue measure.

PROOF. Now $p^{\bullet}(t) = \sup\{u: p(u) > t\}$ so $p(u) > t \Rightarrow u \leq p^{\bullet}(t)$ and thus $\{u \in [0, a]: p(u) > t\} \subset [0, p^{\bullet}(t)]$.
Again, $p^{\bullet}(t) = \inf\{u: p(u) \leq t\}$ so $p(u) \leq t \Rightarrow p^{\bullet}(t) \leq u$, i.e., $u < p^{\bullet}(t) \Rightarrow p(u) > t$, so
$]0, p^{\bullet}(t)[\subset \{u \in [0, a]: p(u) > t\}$.

(4.3) PROPOSITION. Let p, p_n be decreasing functions on $R^{\#}$ for $n = 1, 2, 3, \ldots$ and suppose $p_n \to p$ at all but countably many points of $R^{\#}$. Then $p_n^{\bullet} \to p^{\bullet}$ at every point of continuity of $p^{\bullet}$.

PROOF. Fix $t \in R$. Let $A_n = \{u: p_n(u) > t\}$, $A = \{u: p(u) > t\}$, $E = \{u: p_n(u) \not\to p(u)\}$ so that $p_n^{\bullet}(t) = \sup A_n$, $p^{\bullet}(t) = \sup A$ and E is countable. Then
$A \cap E^c \subset \liminf A_n \subset \limsup A_n \subset \{u: p(u) \geq t\}$ so
$\sup A \cap E^c \leq \sup(\liminf A_n) \leq \sup(\limsup A_n) \leq \sup\{u: p(u) \geq t\}$.
Now the functions p_n are decreasing so the sets A_n have the form $[-\infty, r[$ or $[-\infty, r]$ and thus $\sup(\liminf A_n) = \liminf(\sup A_n) = \liminf p_n^{\bullet}(t)$ and similarly $\sup(\limsup A_n) = \limsup p_n^{\bullet}(t)$. Since

E is countable and A is an interval, $\sup A \cap E^c = \sup A = p^\bullet(t)$. Hence $p^\bullet(t) \leq \liminf p_n^\bullet(t) \leq \limsup p_n^\bullet(t) \leq p^\bullet(t-)$. Thus if $p^\bullet$ is continuous at t, then $\lim p_n^\bullet(t) = p^\bullet(t)$. Also see [47, p. 508, (18.21)].

5. Decreasing Rearrangements of Functions on Finite m. s. Let $f:[0, 1] \to [0, 1]$ be Lebesgue measurable. It is natural to wonder if the values of f can be rearranged to form a decreasing function $f^*:[0, 1] \to [0, 1]$ such that $f^* \sim f$. The affirmative answer is well known [15].

We now generalize this idea for a finite measure space (X, Λ, μ) by showing that if $f \in M(X, \mu)$ then there is a decreasing right continuous Lebesgue measurable function δ_f on $[0, \mu(X)]$ s.t. $\delta_f \sim f$.

For the rest of this section let (X, Λ, μ) be a finite measure space (m. s.).

(5.1) DEFINITION. If $f \in M(X, \mu)$ we define δ_f by $\delta_f(t) = d_f^\bullet(t)$ if $0 \leq t \leq \mu(X)$.

(5.2) THEOREM. (i) If $f \in M(X, \mu)$ then δ_f is a decreasing right continuous Lebesgue measurable function on $[0, \mu(X)]$ satisfying $\delta_f \sim f$.

(ii) Conversely, if p is a decreasing right continuous Lebesgue measurable function on $[0, \mu(X)]$ satisfying $p \sim f$, then $p = \delta_f$.

PROOF. Now δ_f is by definition decreasing, right continuous and Lebesgue measurable on $[0, \mu(X)]$ and $\delta_f = d_f^\bullet$. Lemma (4.2) $\Rightarrow$

$d_{\delta_f} = \delta_f^{\bullet}$ and Lemma (4.1) $\Rightarrow d_f^{\bullet\bullet} = d_f$ so $d_{\delta_f} = d_f$ and thus $\delta_f \sim f$.

Conversely, if p is right continuous decreasing and Lebesgue measurable on $[0, \mu(X)]$ s.t. $p \sim f$, then $d_f = d_p = p^{\bullet}$ using (4.2) so (4.1) $\Rightarrow \delta_f = d_f^{\bullet} = p^{\bullet\bullet} = p$.

(5.3) PROPOSITION.

(i) f <u>is integrable iff</u> δ_f <u>is integrable in which case</u> $\int f\,d\mu = \int_0^a \delta_f$ <u>where</u> $a = \mu(X)$.

(ii) $s(f) \sim s(\delta_f)$ <u>for all</u> $s \in S$.

(iii) $f_1 \leq f_2 \Rightarrow \delta_{f_1} \leq \delta_{f_2}$.

(iv) <u>If each of</u> f <u>and</u> g <u>is a measurable function on a finite m.s. then</u> $f \sim g$ <u>iff</u> $\delta_f = \delta_g$.

(v) <u>If</u> p <u>is increasing on</u> $R^{\#}$ <u>then</u> $\delta_{p(f)}(t) = p(\delta_f(t)-)$ $0 \leq t \leq \mu(X)$.

(vi) <u>If</u> $r \geq 0$ <u>then</u> $\delta_{rf} = r\delta_f$, <u>while if</u> r <u>is real, then</u> $\delta_{f+r} = \delta_f + r$.

(vii) <u>If</u> $f_n \to f$ <u>a.e. then</u> $\delta_{f_n} \to \delta_f$ <u>at every point of continuity of</u> δ_f.

(viii) <u>If</u> $f_n \to 0$ <u>in measure then</u> $\delta_{f_n} \to 0$ <u>uniformly on every closed subinterval of</u> $]0, \mu(X)[$.

PROOF. (i) This is an important special case of results of § 3. Similarly for (ii).

(iii) $f_1 \leq f_2 \Rightarrow d_{f_1} \leq d_{f_2} \Rightarrow \delta_{f_1} = d_{f_1}^{\bullet} \leq d_{f_2}^{\bullet} = \delta_{f_2}$.

(iv) $f \sim g \Rightarrow \delta_f = d_f^{\bullet} = d_g^{\bullet} = \delta_g$. Conversely, $\delta_f = \delta_g \Rightarrow d_f = \delta_f^{\bullet} = \delta_g^{\bullet} = d_g \Rightarrow f \sim g$.

(v) Since $f \sim \delta_f$ we have $p(f) \sim p(\delta_f) = p(\delta_f -)$ a.e. Since $t \to p(\delta_f(t)-)$ is decreasing and right continuous, Theorem (5.2) (ii) implies $\delta_{p(f)} = p(\delta_f -)$.

(vi) These are important special cases of (v) with $p(t) = rt$ and $p(t) = t+r$.

(vii) $f_n \to f$ a.e. $\Rightarrow d_{f_n} \to d_f$ at all but countably many points so Lemma (4.3) $\Rightarrow \delta_f = d_f^{\bullet} = \lim d_{f_n}^{\bullet}$ at every point of continuity of $d_f^{\bullet} = \delta_f$.

(viii) Let $[u, v] \subset]0, \mu(X)[$ and let $\varepsilon > 0$. Since $u > 0 \ \exists\ N_1 > 0$ s.t. $\mu(\{|f_n| > \varepsilon\}) < u$ whenever $n \geq N_1$ so $d_{f_n}(\varepsilon) = \mu(\{f_n > \varepsilon\}) < u$ and thus $\delta_{f_n}(u) \leq \delta_{f_n}(d_{f_n}(\varepsilon)) = d_{f_n}^{\bullet}(d_{f_n}(\varepsilon)) \leq \varepsilon$ whenever $n \geq N_1$. Since $v < \mu(X)$ $\exists N_2 > 0$ s.t. $n \geq N_2 \Rightarrow \mu(\{|f_n| > \varepsilon/2\}) < \mu(X) - v$ so $v < \mu(\{|f_n| \leq \varepsilon/2\}) \leq \mu(\{f_n > -\varepsilon\}) = d_{f_n}(-\varepsilon) = d_{f_n}^{\bullet\bullet}(-\varepsilon)$ so $\delta_{f_n}(v) = d_{f_n}^{\bullet}(v) \geq -\varepsilon$ whenever $n \geq N_2$. Since δf_n is decreasing on $[u, v]$ we have $|\delta_{f_n}(t)| \leq \varepsilon$ for all $t \in [u, v]$ whenever $n \geq N_1 + N_2$.

Recall that if $f \in M(X, \mu)$, then ess. sup $f = \inf\{t: \mu(\{f>t\}) = 0\}$. Writing $E^c = X-E$ if $E \in \Lambda$ and $f|E^c$ for f restricted to E^c we have:

(5.4) PROPOSITION. If $f \in M(X, \mu)$ and $0 \leq t \leq \mu(X)$, then

$$\delta_f(t) = \inf\{\text{ess. sup } (f|E^c): \mu(E) \leq t\}$$

$$\delta_f(t-) = \inf\{\text{ess. sup } (f|E^c): \mu(E) < t\}$$

where ess. sup $f|\emptyset = -\infty$.

PROOF. Let $t \in [0, \mu(X)]$. If $u \in R^{\#}$ and $E_u = \{f > u\}$ so that $d_f(u) = \mu(E_u)$, then $d_f(u) \leq t \Rightarrow \mu(E_u) \leq t$ and ess. sup $(f|E_u^c) \leq u$ and thus $\inf\{\text{ess. sup } (f|E^c): \mu(E) \leq t\} \leq \delta_f(t)$.

Conversely, if $\mu(E) \leq t$, let $u = \text{ess. sup } (f|E^c)$ so that $\{f > u\} \subset E$, $d_f(u) \leq t$, and thus $\delta_f(t) \leq \inf\{\text{ess. sup } (f|E^c): \mu(E) \leq t\}$.

(5.5) THEOREM. If $f \in M(X,\mu)$ and $a = \mu(X)$ then

(i) $(\delta_f)^+ = \delta_{f^+}$ and $(\delta_f)^- = -\delta_{-f^-}$

so that $\delta_f = \delta_{f^+} + \delta_{-f^-}$

and $|\delta_f| = \delta_{f^+} - \delta_{-f^-}$

(ii) $\delta_{-f}(t) = -\delta_f((a-t)-) \qquad 0 \le t \le a.$

PROOF. (i) $\delta_f \sim f \Rightarrow (\delta_f)^+ \sim f^+$ and $(\delta_f)^- \sim f^-$. Since $(\delta_f)^+$ is decreasing and right continuous, $(\delta_f)^+ = \delta_{f^+}$. $(\delta_f)^- \sim f^- \Rightarrow -(\delta_f)^- \sim -f^-$ and $-(\delta_f)^-$ is decreasing and right continuous, so $-(\delta_f)^- = \delta_{-f^-}$ and thus $(\delta_f)^- = -\delta_{-f^-}$.

(ii) $f \sim \delta_f \Rightarrow -f \sim -\delta_f \sim \alpha$ where $\alpha(t) = -\delta_f((a-t)-)$ using Lemma (3.2) (iii). Since α is decreasing and right continuous, $\alpha = \delta_{-f}$.

From now on we will use δ_E to denote δ_{C_E}, where E is a measurable set. Statement (i) of the following lemma seems to have been first used systematically by F. Riesz ([35], p. 164).

(5.6) LEMMA. (i) If $f = \sum_{i=1}^{n} f_i C_{E_i} \in M(X,\mu)$ and $0 < f_1 < \cdots < f_n$, then $\delta_f = f_1 \delta_{F_1} + \sum_{i=2}^{n} (f_i - f_{i-1})\delta_{F_i}$ where $F_i = \bigcup_{k=i}^{n} E_k$

(ii) If $f = \sum_{i=1}^{n} f_i C_{E_i} \in M(X,\mu)$ with $f_i > 0$ and $E_1 \supset \cdots \supset E_n$, then $\delta_f = \sum_{i=1}^{n} f_i \delta_{E_i}$.

PROOF. (i) Let $F_{n+1} = \emptyset$. Now

$$d_f(s) = \begin{cases} 0 & s \ge f_n \\ \mu(F_{i+1}) & f_i \le s < f_{i+1} \qquad i = 1,\dots,n-1 \\ \mu(X) & s < f_1 \end{cases}$$

$$\text{so } \delta_f = \sum_{i=1}^{n} f_i \, C_{[\mu(F_{i+1}), \mu(F_i)[}$$

$$= \sum_{i=1}^{n} f_i \, \delta_{F_i} - \sum_{i=1}^{n} f_i \, \delta_{F_{i+1}}$$

$$= f_1 \, \delta_{F_1} + \sum_{i=2}^{n} (f_i - f_{i-1}) \delta_{F_i} .$$

(ii) Let $G_i = E_i - \bigcup_{k=i+1}^{n} E_k$ and $g_i = \sum_{k=1}^{i} f_k$, $g_0 = 0$.

Then $f = \sum_{i=1}^{n} g_i \, C_{G_i}$ and $0 < g_1 < \cdots < g_n$ so

$$\delta_f = \sum_{i=1}^{n} (g_i - g_{i-1}) \delta_{F_i} = \sum_{i=1}^{n} f_i \, \delta_{E_i} \qquad \text{since}$$

$$F_i = \bigcup_{k=i}^{n} G_k = E_i \text{ and } g_i - g_{i-1} = f_i \; .$$

(5.7) PROPOSITION. _If_ $0 \le f \in M(X, \mu)$ _and_ $E \in \Lambda$ _then_ $\delta_{fC_E} \le \delta_f \, \delta_E$.

PROOF. First let $f = C_F$ where $F \in \Lambda$. Then $\delta_{C_E C_F} = \delta_{E \cap F} = C_{[0, \mu(E \cap F)[} \le C_{[0, \min\{\mu(E), \mu(F)\}[} = \delta_E \delta_F$.

Let $f = \sum_{i=1}^{n} f_i \, C_{F_i}$, $F_n \subset \cdots \subset F_1$, $f_i > 0$, so

$\delta_{fC_E} = \sum f_i \, \delta_{E \cap F_i} \le \sum f_i \, \delta_E \, \delta_{F_i} = \delta_E \, \delta_f$. If $0 \le f \in M(X, \mu)$ then there is a sequence $0 \le f_n \uparrow f$ of simple functions, so $\delta_{f_n} \uparrow \delta_f$ and $\delta_{f_n C_E} \uparrow \delta_{fC_E}$ and hence $\delta_{fC_E} = \lim \delta_{f_n C_E} \le \lim \delta_{f_n} \delta_E = \delta_f \delta_E$.

REMARKS. (1) It follows immediately from Prop. (5.4) that $\delta_f(0) = \text{ess. sup} f$ and $\delta_f(a-) = \text{ess. inf } f$ when $a = \mu(X)$.

(2) If $f_n(x) = x^n$ on $[0, 1]$ then $f_n \to f = 0$ in measure but $\delta_{f_n}(0) = \text{ess. sup } f_n = 1$ so $\delta_{f_n}(0) \not\to \delta_f(0) = 0$.

(3) We cannot prove Prop. (5.3)(vii) for nets. Direct the finite subsets of $[0, 1]$ by inclusion and let $f_E = C_E$ if E is a finite subset of $[0, 1]$. Then $t \in [0, 1]$ and $\{t\} \subset E \Rightarrow f_E(t) = C_E(t) = 1$ so the net $\{f_E\}$ converges pointwise everywhere to 1 but $\delta_{f_E} = 0$ for every E since $f_E = 0$ a.e.

(4) If $g = |\delta_f|$, then $\delta_g = \delta_{|f|}$ since $f \sim \delta_f \Rightarrow |f| \sim |\delta_f|$.

(5) Let p be decreasing and right continuous on $R^{\#}$, $f \in M(X, \mu)$ and $a = \mu(X)$. $f \sim -\delta_{-f}$ so $p(f) \sim p(-\delta_{-f})$ which is decreasing and right continuous on $[0, a]$, so for $0 \le t \le a$, $\delta_{p(f)}(t) = p(\delta_f((a-t)-))$.

(6) Let $\sigma: X \to [0, \mu(X)]$ be m.p and let $\alpha(t) = \mu(X)-t$ $0 \le t \le \mu(X)$. Then $\sigma \sim \alpha$ and α is decreasing and right continuous so $\delta_\sigma(t) = \mu(X)-t$.

(7) Let $a > 0$. If $F \in M[0, a]$ and $F_1(t) = F(at)$ on $[0, 1]$ then $\delta_{F_1}(t) = \delta_F(at)$ on $[0, 1]$. This follows immediately from (3.2).

EXAMPLES. 1. Let $f(x) = x$ on $[a, b]$. Since $\sigma(x) = b-x$ is a m.p. map of $[0, b-a] \to [a, b]$, $f \circ \sigma \sim f$ (3.3)(v). But $f \circ \sigma(t) = b-t$ is decreasing and right continuous on $[0, b-a]$ so $\delta_f(t) = b-t$ on $[0, b-a]$.

2. Suppose p is increasing on R, $\alpha > 0$, and $f(x) = p(\alpha x)$ on $[a, b]$. Then $\delta_f(t) = \delta_{p(\alpha x)}(t) = p(\delta_{\alpha x}(t)-) = p(\alpha\, \delta_x(t)-) = p((\alpha b-\alpha t)-)$ on $[0, b-a]$.

3. Let $g(x) = e^{\alpha x}$, $\alpha > 0$, on $[a, b]$. Then $\delta_g(t) = e^{\alpha b-\alpha t}$ on $[0, b-a]$ using example 2.

4. Let $g(x) = \log \alpha x$, $\alpha > 0$, on $[a, b]$, $a > 0$. $\delta_g(t) = \log(\alpha b-\alpha t)$ on $[0, b-a]$.

5. Suppose p is increasing on R, $\alpha < 0$, and $f(x) = p(\alpha x)$ on $[a, b]$. Since $\sigma(x) = x+a$ is a m.p. map of $[0, b-a] \to [a, b]$, $f \circ \sigma \sim f$. Thus $t \to p(\alpha(t+a)-)$ is decreasing and right continuous and equimeasurable

with f so $\delta_f(t) = p((\alpha t + \alpha a)-)$.

6. Let $f(x) = \sin x$ on $[0, \pi/2]$. Then $\delta_f(t) = \cos t$ on $[0, \pi/2]$, because $\cos x = \sin(x+\pi/2)$ on $[0, \pi/2]$ and thus they are equimeasurable.

7. Let F be decreasing on $[0, a[$ and extend F to R by periodicity. Let $f(x) = F(x)$ on $[0, na]$ where $n > 0$ is an integer. Then $\delta_f(t) = F(\frac{t}{n}+)$ on $[0, na]$. To prove this, let $G(t) = F(\frac{t}{n}+)$ on $[0, na]$, and let $[r, s]$ be a bounded interval of $R^{\#}$. Then $m(G^{-1}[r, s]) = m(nF^{-1}[r, s] \cap [0, na])$ $= n\,m(F^{-1}[r, s] \cap [0, a]) = m(f^{-1}[r, s])$. Thus $G \sim f$ and since G is decreasing and right continuous, $G = \delta_f$.

8. Let $f(x) = \sin x$ on $[0, \pi]$. Then $\delta_f(t) = \cos t/2$ on $[0, \pi]$. For if we let $g(x) = \cos x$ on $[0, \pi/2[$ and $g(x) = \cos(x-\pi/2)$ on $[\pi/2, \pi]$ then $g|[0, \pi/2[\sim f|[0, \pi/2[$ and $g|[\pi/2, \pi] = f|[\pi/2, \pi]$ so $g \sim f$ by (3.3) (x) and thus $\delta_f(t) = \delta_g(t) = \cos t/2$ on $[0, \pi]$ using example 7.

9. Let $f(x, y) = x+y$ on $[0, 1] \times [0, 1]$ with product Lebesgue measure. Then

$$d_f(t) = \begin{cases} 1 & t \le 0 \\ 1 - \frac{1}{2}t^2 & 0 \le t \le 1 \\ \frac{1}{2}(2-t)^2 & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases} \qquad \delta_f(u) = \begin{cases} 2 - \sqrt{2u} & 0 \le u \le \frac{1}{2} \\ \sqrt{2(1-u)} & \frac{1}{2} \le u \le 1 \end{cases}$$

by inverting $1 - \frac{1}{2}t^2$ and $\frac{1}{2}(2-t)^2$.

10. Let $g(x, y) = (1-x) + (1-y)$ on $[0, 1] \times [0, 1]$. Then with f as in example 9, $\delta_g = \delta_f$ because $g \sim f$ by (3.6).

11. If $g(x, y) = \sqrt{x+y}$ on $[0, 1] \times [0, 1]$ with f as in example 9, $\delta_g = \sqrt{\delta_f}$ using (5.3)(v).

12. The Hilbert Transform of a function f on R is defined by

$$\tilde{f}(x) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{|t-x|>\epsilon} \frac{f(t)}{t-x}\, dt \quad .$$

It is well known that if E has finite Lebesgue measure, then

$d_{|\tilde{\chi}_E|}(t) = 2\, m(E)/\sinh \pi t$ so $\delta_{|\tilde{\chi}_E|}(u) = \frac{1}{\pi} \sinh^{-1}(2m(E)/u)$. [46]

At this point it is natural to wonder if, given a right continuous decreasing function α on $[0, \mu(X)]$, there is an $f \in M(X, \mu)$ s.t. $\delta_f = \alpha$.

(5.7) PROPOSITION. <u>Every right continuous decreasing function on $[0, \mu(X)]$ is the decreasing rearrangement of a measurable function on (X, Λ, μ) iff there is a m.p. map</u> $\sigma: X \to [0, \mu(X)]$).

PROOF. Let $\sigma: X \to [0, \mu(X)]$ be m.p. and let g be a decreasing right continuous function on $[0, \mu(X)]$. Then $g \circ \sigma \in M(X, \mu)$ and $g \circ \sigma \sim g$ so $\delta_{g \circ \sigma} = g$.

Conversely, let $\sigma \in M(X, \mu)$ s.t $\delta_\sigma(t) = \mu(X) - t, 0 \le t \le \mu(X)$. Then δ_σ and hence σ is m.p.

Recall that $A \in \Lambda$ is said to be <u>an atom of</u> (X, Λ, μ) (sometimes an <u>atom of μ</u>) iff $\mu(A) > 0$ and $B \subset A$, $B \in \Lambda \Rightarrow \mu(B) = 0$ or $\mu(B) = \mu(A)$. If A is an atom of (X, Λ, μ) and $f \in M(X, \mu)$ then $\text{ess.sup}(f|A) = \text{ess.inf}(f|A)$, i.e., f is essentially constant on A. (X, Λ, μ) is said to be <u>non-atomic</u> if it has no atoms. Any Borel subset of R with Lebesgue measure is known to be non-atomic. The following is a fundamental result about non-atomic measure spaces.

(5.8) LEMMA. (A. Liapounoff, 1940) If (X, Λ, μ) is a non-atomic finite m.s. then $\{\mu(E): E \in \Lambda\} = [0, \mu(X)]$.

For a proof see N. Dinculeanu [5, p. 25].

In order to state Liapounoff's result in another way we make the following definition.

(5.9) DEFINITION. A mapping $\phi : [0, 1] \to \Lambda$ is said to be a μ-resolution of $A \in \Lambda$ if it has the following three properties.

(i) $\phi(t) \subset A$ for all $0 \leq t \leq 1$.

(ii) $0 \leq t_1 \leq t_2$ implies $\phi(t_1) \subset \phi(t_2)$.

(iii) $\mu(\phi(t)) = t\mu(A)$ for all $0 \leq t \leq 1$.

Observe that a μ-resolution ϕ of a set $A \in \Lambda$ is continuous in the sense that $\mu(\phi(u) - \phi(t)) = (u-t)\mu(A) \to 0$ as $u \downarrow t$ or as $t \uparrow u$. This is equivalent to saying that ϕ is continuous as a mapping of $[0, 1]$ into the metric space associated with (X, Λ, μ). (Also see [26].)

(5.10) THEOREM. The following four statements are equivalent for the finite m.s. (X, Λ, μ).

(i) (X, Λ, μ) is non-atomic.

(ii) There is a μ-resolution of X.

(iii) There is a measure preserving map of X onto $[0, \mu(X)]$.

(iv) Every right continuous decreasing function on $[0, \mu(X)]$ is the decreasing rearrangement of a measurable function on (X, Λ, μ).

PROOF. (i) ⇒ (ii). Let $a = \mu(X)$ and let $x_0 \in X$. Since (X, Λ, μ) is non-atomic, $\mu(\{x_0\}) = 0$. Let $\phi(0) = \{x_0\}$ and $\phi(1) = X$. If $\phi(m/2^n)$ is defined for each $0 \leq m \leq 2^n$ so that $\phi(\frac{m-1}{2^n}) \subset \phi(\frac{m}{2^n})$ and $\mu(\phi(\frac{m}{2^n})) = \frac{m}{2^n}a$, then there are sets $\phi(\frac{2m-1}{2^{n+1}}) \in \Lambda$ s.t. $\phi(\frac{m-1}{2^n}) \subset \phi(\frac{2m-1}{2^{n+1}}) \subset \phi(\frac{m}{2^n})$ and $\mu(\phi(\frac{2m-1}{2^{n+1}})) = \frac{2m-1}{2^{n+1}}a$, $0 \leq m \leq 2^n$. This defines ϕ on $A = \{\frac{m}{2^n} : n \geq 0,\ 0 \leq m \leq 2^n\} \subset [0, 1]$ so that for all $u, v \in A$ we have $\phi(u) \subset \phi(v)$ when $u < v$ and $\mu(\phi(u)) = au$. If $t \in [0, 1] - A$ we define $\phi(t) = \cap\{\phi(u): u \geq t\}$, which also holds for $t \in A$. $\mu(\phi(t)) = \inf\{\mu(\phi(u)): u \geq t\} = t\,a$ and if $t_1 < t_2$ there is a $v \in B$ s.t. $t_1 < v < t_2$ and hence $\phi(t_1) = \cap\{\phi(u): u \geq t_1\} \subset \phi(v) \subset \cap\{\phi(u): u \geq t_2\} = \phi(t_2)$. Thus ϕ is a μ-resolution of X.

(ii) ⇒ (iii). Let ϕ be a μ - resolution of X. Define $\sigma(x) = a \cdot \inf\{t: x \in \phi(t)\}$. Then $\{x \in X: \sigma(x) > s\} = \cup\{\phi(t)^c : t > s/a\}$ so $d_\sigma(s) = \mu\{\sigma > s\} = \lim \mu(\phi(t)^c) = \lim(a - ta)$ as $t \downarrow s/a$ $= aC_{]-\infty, 0[} + (a-s)\,C_{[0,a]}$ so $\delta_\sigma(t) = a - t$, $0 \leq t \leq a$, and hence σ is m.p. Any m.p. map is essentially onto.

(iii) ⇒ (iv) This is (5.7).

(iv) ⇒ (i) Let $f \in M(X, \mu)$ be s.t. $\delta_f(t) = a - t$, $0 \leq t \leq a$. Then f is not constant on any subset of X of positive measure so (X, Λ, μ) has no atoms.

EXAMPLES. 1. $\phi(t) = [0, at]$ is an m-resolution of $[0, a]$ with Lebesgue measure m.

2. If $\sigma: X \to [0, a]$ is m.p. then $\phi(t) = \sigma^{-1}[0, at]$ is a μ-resolution of X, and for all $x \in X$ we have $\sigma(x) = \inf\{t: x \in \sigma^{-1}[0, at]\}$.

3. $\sigma(x) = b-x$ and $\tau(x) = x-a$ are measure preserving maps of $[a, b]$ onto $[0, b-a]$.

The following theorem, proved by J. V. Ryff [40] for $M[0, 1]$, shows how each measurable function on a non-atomic m. s. can be related to its decreasing rearrangement. First a lemma.

(5.11) LEMMA. If the finite m. s. (X, Λ, μ) is non-atomic and $\mu(X) = b-a$, then there is a m. p. map of X onto $[a, b]$ and also one of X onto $[a, b[$.

PROOF. Let $\varphi: X \to [0,b-a]$ be m.p. and let $\psi(t) = t+a$, $t \in R$. Then $\sigma = \psi \circ \varphi: X \to [a, b]$ is m. p. and if $p(t) = \begin{cases} t & t \in [a, b[\\ a & t = b \end{cases}$ then $\sigma_1 = p \circ \psi \circ \varphi: X \to [a, b[$. Any m.p. map is essentially onto.

(5.12) THEOREM (J. V. Ryff). If the finite m. s. (X, Λ, μ) is non-atomic and $f \in M(X, \mu)$ then there is a m. p. $\sigma: X \to [0, \mu(X)]$ s.t. $f = \delta_f \circ \sigma$ μ - a. e.

PROOF. Following Ryff [40, p. 96] let $I_t = \{s \in [0, \mu(X)]: \delta_f(s)=t\}$ and $A_t = \{x \in X: f(x) = t\}$ for each $t \in R^{\#}$. Since $f \sim \delta_f$, there is an $f' = f$ μ-a. e. s. t. f' and δ_f have the same range B. Denote f' by f. Also $m(I_t) = \mu(A_t)$ for each extended real t, and each I_t has the form $[a, b]$ or $[a, b[$ since δ_f is decreasing and rt-ctn. Hence for each $t \in B$ there is a m.p. $\sigma_t: A_t \xrightarrow{\text{onto}} I_t$. Define σ by $\sigma(x) = \sigma_t(x)$ if $x \in A_t$. Then clearly $f = \delta_f \circ \sigma$. Now $\mu(X) < \infty \Rightarrow m(I_t) > 0$ for at most countably many t. Let F be the set of all such t and let $I = \bigcup_{t \in F} I_t$, which is measurable since its complement is, and on which δ_f is 1:1. Let $J \subset [0, \mu(X)]$ be measurable. Then $J = \bigcup_{t \in F} J \cap I_t \cup J \cap I$ so $\mu(\sigma^{-1}(J)) = \sum_{t \in F} \mu(\sigma^{-1}(J \cap I_t)) + \mu(\sigma^{-1}(J \cap I))$. But $\mu(\sigma^{-1}(J \cap I_t)) = \mu(\sigma_t^{-1}(J \cap I_t)) = m(J \cap I_t)$ and δ_f is 1:1 on $J \cap I$, so $f = \delta_f \circ \sigma \Rightarrow \delta_f^{-1} \circ f = \sigma$ there, and hence $\mu(\sigma^{-1}(J \cap I)) = \mu(f^{-1}(\delta_f(J \cap I))) = m(\delta_f^{-1}(\delta_f(J \cap I))) = m(J \cap I)$. Hence σ is m.p.

REMARKS. (1) In general we cannot always find a measure preserving map $\phi : [0, a] \to X$ such that $f \circ \phi = \delta_f$. For example, let $f: [0, 1] \to [0, 1]$ be m.p., suppose $\phi : [0, 1] \to [0, 1]$ is measure preserving s.t. $f \circ \phi = \delta_f$, and let $\sigma: [0, 1] \to [0, 1]$ be m.p. such that $\delta_f \circ \sigma = f$. Then $\delta_f(t) = 1-t = f \circ \phi(t) = 1 - \sigma(\phi(t))$ so $\sigma(\phi(t)) = t$ for all $t \in [0, 1]$ and hence $t_1 \neq t_2 \Rightarrow \sigma(\phi(t_1)) \neq \sigma(\phi(t_2)) \Rightarrow \phi(t_1) \neq \phi(t_2)$. Since ϕ is m.p. it is essentially onto and it follows that $\phi^{-1} = \sigma$, so σ is an invertible m.p. map. Also $\delta_f \circ \sigma = f$ implies $f = 1 - \sigma$ so f is necessarily 1:1. Thus if f is not 1:1, say $f(x) = 2x \bmod 1$, then there is no m.p. $\phi : [0, 1] \to [0, 1]$ such that $f \circ \phi = \delta_f$. (see §20).

(2) Let (X_i, Λ_i, μ_i) be finite m.s. with $a_i = \mu_i(X_i)$, $i=1, \ldots, k$, let (X, Λ, μ) be their product m.s., and let $J = [0, a_1] \times \cdots \times [0, a_k]$ with product Lebesgue measure. If $\sigma_i : X_i \to [0, a_i]$ are m.p. $i=1, \ldots, k$, and $\sigma(x_1, \ldots, x_k) = (\sigma_1(x_1), \ldots \sigma_k(x_k))$ then $\sigma: X \to J$ is m.p. To prove this take $f_i = \sigma_i$ and $g_i(t) = t$ for all $t \in [0, a_i]$ in Prop. (3.6).

(3) Observe that Prop. (3.6) $\Rightarrow (f_1, \ldots, f_k) \sim (\delta_{f_1}, \ldots, \delta_{f_k})$ so as in (3.7) at the end of § 3 we may conclude such things as $|f(x) - g(y)| \sim |\delta_f(u) - \delta_g(v)|$.

(4) If $f \in M[0, 1]$, and $t \in [0, 1]$ let $\varphi(t) = m(\{f > f(t)\}) + m(\{f = f(t)\} \cap [0, t])$. Ryff [44] has shown that $\varphi: [0, 1] \to [0, 1]$ is m.p. and $f = \delta_f \circ \varphi$.

(5) Actually Liapounoff proved a much stronger result in 1940 than the one stated above: The range of a countably additive finite measure taking values in R^n is compact, and if the measure space is non-atomic, then the range is convex as well [22]. In 1947 Paul R. Halmos gave a simplified proof of this result [11]. In Lemma 7 he shows that a non-atomic m. s. (X, Λ, μ), with μ taking values in R^n, is convex, i.e., for every $E \in \Lambda$ there is a function $\phi : E \to [0, 1[$ s.t. $\mu(\{\phi < s\}) = s\,\mu(E)$ for every $s \in [0, 1]$. We may define a one-dimensional vector valued Lebesgue measure λ on the line segment $[0, \mu(E)[= \{s\,\mu(E): 0 \le s < 1\}$ joining the zero vector and the vector $\mu(E)$ as follows. Let m be Lebesgue measure on $[0, 1[$. If $B \subset [0, 1[$ we write $B\,\mu(E)$ to denote $\{t\mu(E): t \in B\}$, and if B is Lebesgue measurable we define $\lambda(B\mu(E)) = m(B)\mu(E)$. Thus λ is a vector valued measure defined on the σ-algebra $S = \{B\mu(E): B \subset [0, 1[$ is Lebesgue measurable$\}$ of subsets of $[0, \mu(E)[$. Now let (X, Λ, μ) be convex, let $E \in \Lambda$ and let ϕ be as above. If we define $\sigma(x) = \phi(x)\,\mu(E)$ then σ is a m.p. map of (X, Λ, μ) onto $([0, \mu(E)[, S, \lambda)$.

6. Decreasing Rearrangements on Not Necessarily Finite m. s.

Let (X, Λ, μ) be a measure space (m. s.) and let m denote Lebesgue measure. If $f \in M(X, \mu)$ and there is an interval $I \subset R$ and a decreasing function $\delta \in M(I, m)$ such that $f \sim \delta$, then we will call δ *a decreasing rearrangement of* f. In this section we will characterize those functions which have a decreasing rearrangement in this sense.

(6.1) DEFINITION. *We denote by* $D(X, \mu)$ *the set of all* $f \in M(X, \mu)$ *which satisfy*

(i) $\mu(f^{-1}[a, b]) < \infty$ *whenever* $[a, b] \subset]\text{ess. inf } f, \text{ ess. sup } f[$;

(ii) $\mu(f^{-1}]c, \text{ ess. sup } f[) < \infty$ *whenever* $\mu(f^{-1}(\text{ess. sup } f)) > 0$ *and* $\text{ess. inf } f < c < \text{ess. sup } f$.

(iii) $\mu(f^{-1}]\text{ess. inf } f, c[) < \infty$ *whenever* $\mu(f^{-1}(\text{ess. inf } f)) > 0$ *and* $\text{ess. inf } f < c < \text{ess. sup } f$.

It is our purpose to show that $D(X, \mu)$ is precisely the set of all those functions which have decreasing rearrangements.

(6.2) LEMMA. *If* I *is an interval of* R *and* $\delta \in M(I, m)$ *is monotonic, then* $\delta \in D(I, m)$.

PROOF. Assume δ is decreasing; the proof when δ is increasing is similar. Let $J =]\text{ess. inf } \delta, \text{ ess. sup } \delta[$, so $J \subset]\inf \delta, \sup \delta[$.

(i) Let $[a, b] \subset J$. Then $b < \sup \delta$ so there is a $u \in I$ such that $b < \delta(u)$ and hence $\delta^{-1}[a, b] \subset]u, \infty[$. Since $\inf \delta < a$ there is a $v \in I$ such that $\delta(v) < a$ so $\delta^{-1}[a, b] \subset]-\infty, v[$. Hence $\delta^{-1}[a, b] \subset]u, v[$ so $m(\delta^{-1}[a, b]) < \infty$. (ii) Let $c \in J$. Then there are $w, v \in I$ such that $\delta(v) < c < \delta(w)$. If $m(\delta^{-1}(\text{ess. sup } \delta)) > 0$ then there is a $t \in I$ such that $\delta(t) = \text{ess. sup } \delta$, so $\delta^{-1}]c, \text{ess. sup } \delta[\subset [t, v]$ and thus $m(\delta^{-1}]c, \text{ess. sup } \delta[) < \infty$. The proof when $m(\delta^{-1}(\text{ess. inf } \delta)) > 0$ is similar.

(6.3) LEMMA. (i) <u>If</u> $f \in D(X, \mu)$ <u>and</u> $f_1 \in M(X_1, \mu_1)$ <u>then</u> $f_1 \sim f$ <u>implies</u> $f_1 \in D(X_1, \mu_1)$.

(ii) <u>If</u> $f \in D(X, \mu)$ <u>then</u> $f + r \in D(X, \mu)$ <u>for all</u> $r \in R$.

PROOF. (i) use (3.3) (vi).

(ii) Let $f \in D(X, \mu)$ and $r \in R$. Then $\text{ess. inf } (f+r) = r + \text{ess. inf } f$ and $\text{ess. sup}(f+r) = r + \text{ess. sup } f$. If $[a, b] \subset]\text{ess. inf}(f+r), \text{ess. sup}(f+r)[$ then $[a-r, b-r] \subset]\text{ess. inf } f, \text{ess. sup } f[$ so $\mu((f+r)^{-1}[a, b]) = \mu(f^{-1}[a-r, b-r]) < \infty$. The rest of the verifications are similar.

Thus we see that (6.2) and (6.3) (i) imply that $D(X, \mu)$ contains all measurable functions which have decreasing rearrangements. To prove the converse we have to construct a decreasing rearrangement for each function in $D(X, \mu)$. It is convenient to do this first for a special subset of $D(X, \mu)$, which we now define.

(6.4) DEFINITION. $D'(X, \mu)$ <u>is the set of all functions</u> $f \in D(X, \mu)$ <u>which satisfy</u>

(i) $\text{ess. inf } f \leq 0 < \text{ess. sup } f$

(ii) $\mu(f^{-1}(0)) > 0$ <u>if</u> $\text{ess. inf } f = 0$.

If $f \in D'(X,\mu)$ *we define a distribution function by*

$$d_f(t) = \begin{cases} \mu(f^{-1}]t,0]) & \text{if } t < 0 \\ -\mu(f^{-1}]0,t]) & \text{if } t \geq 0 \end{cases}$$

for all $t \in R^{\#}$. *Then* d_f *is decreasing and we define*

$$\delta_f(t) = d_f^{\bullet}(t)$$

for all $t \in I =]-\mu(f^{-1}]0,\infty]), \mu(f^{-1}[-\infty,0])[$

(6.5) LEMMA. *Let* $f \in D'(X,\mu)$ *and let* $I =]-\mu(f^{-1}]0,\infty]), \mu(f^{-1}[-\infty,0])[$.

(i) $|d_f(t)| < \infty$ *if* ess. inf $f < t <$ ess. sup f.

(ii) d_f *is right continuous*

(iii) $d_f^{\bullet}(t) = +\infty$ *iff* $t < -\mu(f^{-1}]0,u])$ *for all* $0 \leq u \in R$

(iv) $d_f^{\bullet}(t) = -\infty$ *iff* $t \geq \mu(f^{-1}]-\infty,0])$.

PROOF. (i) Since $f \in D(X,\mu)$, this is clear when ess. inf $f \neq 0$. If ess. inf $f = 0$ then $\mu(f^{-1}(\text{ess. inf } f)) > 0$ so $\mu(f^{-1}]0,t]) < \infty$ whenever ess. inf $f = 0 < t <$ ess. sup f.

(ii) Now d_f is clearly right continuous at all $t < 0$, and also at $t \geq 0$ if $|d_f(u)| < \infty$ for some $u > t$. Hence let $t \geq 0$ and suppose $|d_f(u)| = \infty$ for all $u > t$. For such u we have $u > 0 \geq$ ess. inf f, so (i) implies ess. sup $f \leq u$. Hence $\mu(f^{-1}]0,t]) = \mu(f^{-1}]0, \text{ess. sup } f]) = \mu(f^{-1}]0,u])$.

(iii) $d_f^{\bullet}(t) = +\infty$ iff $\{u: d_f(u) > t\} \supset [0,+\infty[$ and $d_f^{\bullet}(t) = -\infty$ iff $\{u: d_f(u) \leq t\} \supset]-\infty, 0[$.

One would hope that $d_f = d_g$ on $R^{\#}$ implies $f \sim g$. The following example shows this is not the case. Let $f(x) = x$ on $X_1 = [0, +\infty]$ and let

$$g(x) = \begin{cases} x & \text{if } 0 \le x \le +\infty \\ +\infty & \text{if } -1 \le x < 0 \end{cases} \qquad \text{on } X_2 = [-1, +\infty].$$

Then $d_f = d_g$ on $R^{\#}$ but $m(f^{-1}(+\infty)) \neq m(g^{-1}(+\infty))$. We will be able to prove $\delta_f = \delta_g$ iff $f \sim g$, however, by using (6.5) (iii) & (iv) and the following result.

(6.6) LEMMA. <u>Let</u> $f \in D'(X_1, \mu_1)$ <u>and</u> $g \in D'(X_2, \mu_2)$. <u>If</u> $d_f = d_g$ <u>on</u> R <u>then</u> $\mu_1(f^{-1}[a, b]) = \mu_2(g^{-1}[a, b])$ <u>for all intervals</u> $[a, b]$ of R.

PROOF. This is equivalent to proving that $f|f^{-1}[R] \sim g|g^{-1}[R]$ so we may assume (and we do) that f and g are essentially finite. Let $d_f = d_g$ on R. Then (6.5) (i) says that $|d_f(t)| < \infty$ and $|d_g(t)| < \infty$ whenever $\min\{\text{ess.inf } f, \text{ ess.inf } g\} < t < \max\{\text{ess.sup } f, \text{ ess.sup } g\}$, in which case $\mu_2(g^{-1}]-\infty, t]) = \mu_1(f^{-1}]-\infty, t])$ and $\mu_2(g^{-1}]t, \infty[) = \mu_1(f^{-1}]t, \infty[)$. Then ess.inf $f < t <$ ess.sup f implies $\mu_2(g^{-1}]t, \infty]) > 0$ and $\mu_2(g^{-1}]-\infty, t]) > 0$ so ess.sup $g \geq$ ess.sup f and ess.inf $g \leq$ ess.inf f. The argument is symmetric in f and g so we conclude that they have the same ess.inf, say u, and the same ess.sup, say v.

Now $[a, b] \subset]u, v[$ implies $\mu_1(f^{-1}[a, b])$ & $\mu_2(g^{-1}[a, b]) < \infty$ in which case $\mu_1(f^{-1}[a, b]) = d_f(a-) - d_f(b) = d_g(a-) - d_g(b) = \mu_2(g^{-1}[a, b])$. Also if $[a, b] \subset]-\infty, u[\cup]v, +\infty[$ then $\mu_1(f^{-1}[a, b]) = 0 = \mu_2(g^{-1}[a, b])$.

Now f, $g \in D'$ implies $u \leq 0 < v$. If $\mu_1(f^{-1}]0, v[) < \infty$ then $\mu_1(f^{-1}(v)) = d_f(v-) - d_f(v) = d_g(v-) - d_g(v) = \mu_2(g^{-1}(v))$. Otherwise, since f, $g \in D'$, $\mu_1(f^{-1}(v)) = 0 = \mu_2(g-1(v))$. For the rest, if $\mu_1(f^{-1}]u, 0]) < \infty$, then since $\mu_1(f^{-1}(-\infty)) = 0 = \mu_2(g^{-1}(-\infty))$, we have $\mu_1(f^{-1}(u)) =$

$\mu_1(f^{-1}]-\infty, u]) = \mu_1(f^{-1}]-\infty, 0]) - \mu_1(f^{-1}]u, 0])$

$$= \mu_2(g^{-1}]-\infty, 0]) - \mu_2(g^{-1}]u, 0]) = \mu_2(g^{-1}]-\infty, u])$$

$$= \mu_2(g^{-1}(u)) .$$

Otherwise, $u < 0$ and $\mu_1(f^{-1}(u)) = 0 = \mu_2(g^{-1}(u))$.

(6.7) LEMMA. Let J be an interval of $R^{\#}$ such that $\inf J \leq 0 \leq \sup J$. If $p \in D'(J, m)$ is decreasing and $p(t) \leq 0$ iff $t \geq 0$, then $p^{\bullet} = d_p$ on $R^{\#}$.

PROOF. From the definition of $p^{\bullet}$ and the condition on p it is easy to see that we have:

if $t < 0$, then $[0, p^{\bullet}(t)[\subset J \cap p^{-1}]t, 0] \subset [0, p^{\bullet}(t)]$

if $t \geq 0$, then $]p^{\bullet}(t), 0[\subset J \cap p^{-1}]0, t] \subset [p^{\bullet}(t), 0[$.

Hence $p^{\bullet} = d_p$.

(6.8) DEFINITION. For each $f \in D(X, \mu)$ define $b_f = 0$ if $f \in D'(X, \mu)$ while if $f \notin D'(X, \mu)$ define

$$b_f = \begin{cases} \frac{1}{2}(\text{ess. inf } f + \text{ess. sup} f) & \text{if } |\text{ess. inf } f| \,\&\, |\text{ess. sup } f| < \infty \\ -1 + \text{ess. sup } f & \text{if } |\text{ess.sup } f| < \infty \,\&\, |\text{ess. inf } f| = \infty \\ 1 + \text{ess. inf } f & \text{if } |\text{ess. inf } f| < \infty \,\&\, |\text{ess.sup } f| = +\infty \end{cases}$$

(6.9) LEMMA. If $f \in D(X, \mu)$ is not essentially constant then $\text{ess. inf } f < b_f < \text{ess. sup } f$ and $f - b_f \in D'(X, \mu)$.

(6.10) DEFINITION. <u>For each</u> $f \in D(X,\mu)$ <u>define</u>

$$\delta_f = \begin{cases} \delta_{f-b_f} + b_f & \underline{\text{if}}\ f\ \underline{\text{is not essentially constant}} \\ \text{ess. sup } f & \underline{\text{if}}\ f\ \underline{\text{if essentially constant}} \end{cases}$$

<u>on</u> $I =]-\mu(f^{-1}]b_f, +\infty]), \mu(f^{-1}[-\infty, b_f])[$.

(6.11) THEOREM. <u>Let</u> $f \in D(X,\mu)$ <u>and let</u>
$I =]-\mu(f^{-1}]b_f, \infty]), \mu(f^{-1}[-\infty, b_f])[$.

(i) <u>Then</u> $\inf I \le 0 \le \sup I$, $\delta_f \in D(I, m)$ <u>is decreasing and right continuous</u>, $\delta_f \sim f$, <u>and for each</u> $t \in I$, $\delta_f(t) \le b_f$ <u>iff</u> $t \ge 0$.

(ii) <u>Suppose</u> J <u>is an interval of</u> R <u>such that</u> $\inf J \le 0 \le \sup J$, <u>suppose</u> $p \in D(J, m)$ <u>is decreasing and right continuous</u>, $p \sim f$, <u>and for each</u> $t \in J$, $p(t) \le b_f$ <u>iff</u> $t \ge 0$. <u>Then</u> $I \subset J$ <u>and</u> $p = \delta_f$ on I.

PROOF. (i) The result is clearly true if f is essentially constant since $m(I) = \mu(X)$. If the result is true for $f \in D'$ (so $b_f = 0$) and $f \in D$ is not essentially constant, then
$\delta_f(t) = \delta_{f-b_f}(t) + b_f \le b_f$ iff $\delta_{f-b_f}(t) \le 0$ iff $t \ge 0$ for all $t \in]u, v[$ where
$u = -\mu((f-b_f)^{-1}]0, \infty]) = -\mu(f^{-1}]b_f, \infty])$ and
$v = \mu((f-b_f)^{-1}[-\infty, 0]) = \mu(f^{-1}[-\infty, b_f])$ so
$I =]u, v[$; and $f - b_f \sim \delta_{f-b_f}$ so $f \sim \delta_{f-b_f} + b_f = \delta_f$.
Hence suppose $f \in D'$.

If $0 \le t \in R^{\#}$, then $d_f(0) = 0 \le t$ so $d_f^{\bullet}(t) = \inf\{u: d_f(u) \le t\} \le 0$. Suppose $0 > t \in R^{\#}$. For all $0 < u <$ ess. sup f, $d_f(u)$ is finite. Hence $d_f(u) \uparrow 0$ as $u \downarrow 0$ so there is a $u_0 > 0$ s.t. $t < d_f(u_0)$. Then $d_f^{\bullet}(t) = \sup\{u: d_f(u) > t\} \ge u_0 > 0$.

Let $q = d_f^{\bullet}$. Then $d_q = q^{\bullet} = d_f^{\bullet\bullet} = d_f$ on R. But $|d_f^{\bullet}(t)| < \infty \Rightarrow t \in \bar{I}$ (Lemma (6.5)) $\Rightarrow m(\delta_f^{-1}[a,b]) = m(q|I^{-1}[a,b]) = m(q^{-1}[a,b]) = \mu(f^{-1}[a,b])$ for all bounded intervals $[a,b]$ of R. Now (6.5) implies $\delta_f^{-1}(-\infty) = [\mu(f^{-1}]-\infty, 0]), \mu(f^{-1}[-\infty, 0])[$ so $\mu(f^{-1}(-\infty)) = 0 \Rightarrow$ $m(\delta_f^{-1}(-\infty)) = 0$. If $\mu(f^{-1}(-\infty)) > 0$ then $\mu(f^{-1}]-\infty, 0]) < \infty$ and thus $m(\delta_f^{-1}(-\infty)) = \mu(f^{-1}(-\infty))$. Similarly, if $\mu(f^{-1}(+\infty)) = 0$, then $-\mu(f^{-1}]0,u]) \downarrow -\mu(f^{-1}]0,\infty])$ as $u \to \infty$ so (6.5) $\Rightarrow m(\delta_f^{-1}(+\infty)) = 0$. If $\mu(f^{-1}(+\infty)) > 0$, then $\mu(f^{-1}]0,\infty[) < \infty$ so $m(\delta_f^{-1}(+\infty)) = \mu(f^{-1}(+\infty))$.

(ii) Again this is clearly true if p is essentially constant since then f is, and $p \sim f$ implies $m(J) = \mu(X)$. If the result is true for all $p \in D'(J, m)$ (so $b_f = 0$), and $p \in D(J, m)$ is not essentially constant, then $p - b_f \in D'(J, m)$ satisfies all the conditions in (ii) for $f - b_f \in D'(X, \mu)$ so $I \subset J$ and $p - b_f = \delta_{f-b_f}$ i.e. $p = \delta_{f-b_f} + b_f = \delta_f$ on I.

Hence suppose $f \in D'$. Since $p \sim f$,

$d_f = d_p = p^{\bullet}$ so $d_f^{\bullet} = p^{\bullet\bullet} = p$. Also, because $p \sim f$,

$m(J \cap p^{-1}]0,\infty]) = \mu(f^{-1}]0,\infty])$ and

$m(J \cap p^{-1}[-\infty, 0]) = \mu(f^{-1}[-\infty, 0])$, and

$]\inf J, 0[\subset J \cap p^{-1}]0,\infty] \subset [\inf J, 0[$ and

$[0, \sup J[\subset J \cap p^{-1}[-\infty, 0] \subset [0, \sup J]$ so

$\inf J = -\mu(f^{-1}]0,\infty])$ and $\sup J = \mu(f^{-1}[-\infty, 0])$

and hence $I \subset J$, and on I, $p = d_f^{\bullet} = \delta_f$.

(6.12) THEOREM. Suppose $f \in \mathbf{D}(x, \mu)$ and ess. inf $f < 0 <$ ess. sup f. Let $I =]-\mu(f^{-1}]0, +\infty]), \ \mu(f^{-1}[-\infty, 0])[$. Then

$a = \mu(f^{-1}(0)) < \infty$ and we have

(i) $(\delta_f)^{-1} = \delta_{f^{-1}}$ on I

$(\delta_f)^{-}(t) = \delta_{f^-}((a-t)-) \qquad t \in I$

(ii) $\delta_{-f}(t) = -\delta_f((a-t)-) \qquad t \in a - I$.

PROOF. Let $F(t) = (\delta_f)^{-}((a-t)-)$. We prove that $\delta_{f^-} = F$ on $a - I$. Now $f \sim \delta_f$ so $f^- \sim (\delta_f)^- \sim F$. Now $a = m(\delta_f^{-1}(0))$, & (6.11) (i) implies $\delta_f^{-1}(0) \subset [0, a]$ so $F(t) \leq 0$ iff $\delta_f((a-t)-) = 0$ iff $t \geq 0$ and hence $F = \delta_{f^-}$ on $J =] - \mu((f^-)^{-1}]0, \infty]), \ \mu((f^-)^{-1}[-\infty, 0])[= a - I$.
The rest is similar and easier.

We now show how to obtain an analog of Theorem (5.10) for general m. s. Observe that unlike the situation for finite m. s., if A is an atom and $f \in M(X, \mu)$, it need not be the case that f is constant on A, or even on a subset of A of positive measure. This can only happen, of course, if $\mu(A) = +\infty$. The situation is nicer if $f \in D(X, \mu)$.

(6.13) LEMMA. If A is an atom of (X, Λ, μ) and $f \in D(X, \mu)$ then $\mu\{x \in A: f(x) \notin \{\text{ess. inf } f, \text{ ess. sup } f\}\} = 0$. Hence f is constant on a subset of A of positive measure.

PROOF. Let $J =]\text{ess.inf } f, \text{ ess.sup } f[$. If $\mu(A) < \infty$ then f is essentially constant on A, so assume $\mu(A) = +\infty$. Since $f \in D$, $\mu(f^{-1}[r,s]) < \infty$ whenever $[r,s] \subset J$ so $\mu((f|A)^{-1}[r,s]) = \mu(f^{-1}[r,s] \cap A) < +\infty = \mu(A)$ and thus $\mu(f|A^{-1}[r,s]) = 0$ for all $[r,s] \subset J$, since A is an atom. Since J is a countable union of closed intervals, $\mu\{x \in A: f(x) \notin \{\text{ess.inf } f, \text{ ess.sup } f\}\} = \mu(f|A^{-1}(J)) = 0$. $0 < \mu(A) \le \mu(f^{-1}(\text{ess.inf } f)) + \mu(f^{-1}(\text{ess.sup } f))$ shows the rest.

(6.14) LEMMA. <u>If</u> (X, Λ, μ) <u>is a m.s. and</u> I <u>is an open interval of</u> R <u>s.t.</u> $\mu(X) = m(I)$ <u>and</u> $\{X_i\}_{i=1}^{\infty}$ <u>is a partition of</u> X <u>by measurable sets s.t.</u> $\mu(X_i) < \infty$ $i = 1, 2, 3, \ldots,$ <u>then there are pwd open intervals</u> $\{]a_n, b_n[\}_{n=1}^{\infty}$ <u>s.t.</u> $\mu(X_n) = b_n - a_n$ $n = 1, 2, 3, \ldots,$ <u>and</u> $m(I - \bigcup_{n=1}^{\infty}]a_n, b_n[) = 0$.

PROOF. Suppose first that I is bounded below, say $a_1 = \inf I > -\infty$. Let $a_n = a_1 + \sum_{i=1}^{n-1} \mu(X_i)$ and $b_n = a_n + \mu(X_n)$. Then $\bigcup_{n=1}^{\infty}]a_n, b_n[= I - \{a_n : n = 1, 2, 3, \ldots\}$. Similarly if I is bounded above. Hence let $I = R$. Then $\sum_{i=1}^{\infty} \mu(X_i) = \mu(X) = m(I) = +\infty$ so there is a partition $\{B_k\}_{k=1}^{\infty}$ of the positive integers by finite sets B_k such that $\sum \{\mu(X_i): i \in B_k\} \ge 1$ for each $k = 1, 2, 3, \ldots$. Let $\{n_i\}_{i=1}^{\infty}$ and $\{p_i\}_{i=1}^{\infty}$ enumerate $\bigcup_{k=1}^{\infty} B_{2k}$ and $\bigcup_{k=1}^{\infty} B_{2k-1}$ respectively, let $r_0 \in R$, and let $X_1 = \bigcup_{i=1}^{\infty} X_{n_i}$, $X_2 = \bigcup_{i=1}^{\infty} X_{p_i}$, $I_1 =]r_0, \infty[$ and $I_2 =]-\infty, r_0[$. We get collections of intervals for I_1 and X_1 and for I_2 and X_2 as before, and the union of these collections works for X.

(6.15) THEOREM. Let (X, Λ, μ) be a m.s. and let I be an open interval of R with $0 \in I$. Then the following four statements are equivalent.

(i) There is an $f \in M(X, \mu)$ s.t. $\delta_f(t) = -t$ for all $t \in I$.

(ii) There is a measure preserving map of X onto I.

(iii) If $v \in M(I, m)$ is decreasing and right continuous and for each $t \in I$, $v(t) \leq b_v$ iff $t \geq 0$, then there is an $f \in D(X, \mu)$ s.t. $\delta_f = v$ on I.

(iv) (X, Λ, μ) is non-atomic and σ-finite and $\mu(X) = m(I)$.

PROOF (i) $\Rightarrow$ (ii) Let $\sigma = -f$. Then $-\sigma = f \sim \delta_f$ so $\sigma \sim -\delta_f$. Then σ is m.p. and has the same essential range as $-\delta_f$, namely I.

(ii) $\Rightarrow$ (iii). Now ess. inf σ = inf I and ess. sup σ = sup I. Let $f = v \circ \sigma$. Then $f \sim v$ so $b_f = b_v$.

$\mu(f^{-1}]b_f, \infty]) = m(v^{-1}]b_v, \infty]) = -\inf I$

$\mu(f^{-1}[-\infty, b_f]) = m(v^{-1}[-\infty, b_v]) = \sup I$. Hence Theorem (6.11) implies $\delta_f = v$ on I.

(iii) $\Rightarrow$ (iv) Let $f \in M(X, \mu)$ s.t. $\delta_f(t) = -t$ for all $t \in I$. Then $X = \cup\{f^{-1}[i, i+1[: i \text{ is an integer}\}$ and $\mu(f^{-1}[i, i+1[) = m(\delta_f^{-1}[i, i+1[) \leq 1$ so X is σ-finite. Since $\mu(f^{-1}(r)) = m(\delta_f^{-1}(r)) = 0$ for all $r \in R$, f is not constant on any set of positive measure, so X has no atoms. $m(I) = m(\delta_f^{-1}(R)) = \mu(f^{-1}(R)) = \mu(X)$.

(iv) $\Rightarrow$ (i) Let $\{X_i\}_{i=1}^{\infty}$ be a partition of X by sets of finite measure. Since X is non-atomic, so is each X_i. Since $m(I) = \mu(X)$ there are pairwise disjoint intervals $\{]a_i, b_i[\}_{i=1}^{\infty}$ s.t. $m(I - \bigcup_{i=1}^{\infty}]a_i, b_i[) = 0$, and $b_i - a_i = \mu(X_i)$.

Let $\iota(t) = t$ for all $t \in R$. Now $\alpha_n(t) = -(t+a_n)$ for $0 \leq t \leq b_n - a_n$ is decreasing and right continuous on $[0, \mu(X_n)]$ and hence there is an $f_n \in M(X_n, \mu)$ s.t. $f_n \sim \alpha_n$. Now for $v_n(t) = \alpha_n(t-a_n) = -t$ $(a_n \leq t < b_n)$ we have $f_n \sim \alpha_n \sim v_n \sim -\iota | [a_n, b_n[$. Let $f(x) = f_n(x)$ if $x \in X_n$. Then $f \in M(X, \mu)$ and $f | X_n = f_n \sim \iota | [a_n, b_n[$ so $f \sim -\iota | I$. Then $f \in D'(X, \mu)$ and Theorem (6.11) implies $\delta_f = -\iota$ on I.

(6.16) THEOREM. If (X, Λ, μ) is non-atomic and σ-finite, if $f \in D(X, \mu)$, and if $\delta \in M(I, m)$ is a decreasing rearrangement of f, then there is a measure preserving map $\sigma: X \to I$ such that $\delta \circ \sigma = f$ μ-a.e.

PROOF. Let $J_t = \{s \in I: \delta(s) = t\}$ for each $t \in R^{\#}$. Each J_t is an interval, and since the topology of R has a countable base, we have $m(J_t) > 0$ for at most countably many t. The rest of the proof is like that of (5.12).

REMARKS. 1. If (X, Λ, μ) is a m. s. and $0 \leq f \in M(X, \mu)$ and $\mu(\{f \geq \xi\}) < \infty$ for some ξ, then we may define $[f]$ on $[0, \infty[$ to be the right continuous inverse of $d_f(t) = \mu(\{f > t\})$ [8] . This is almost the same as the definition of $\bar{f}$ given by Hardy, Littlewood and Polya [15] for X an infinite interval of R. Although $m(\{[f] > \xi\}) = \mu(\{f > \xi\})$ for all $\xi \in R$, in general $[f]$ need not be equimeasurable with f. If $f \in D(X, \mu)$ then $a = \mu(f^{-1}]b_f, \infty]) < \infty$ and for all $0 \leq u < \infty$ we have

$$[f](u) = \delta_f(u-a) .$$

2. If (X, Λ, μ) is a finite m. s. and $f \in M(X, \mu)$ then the decreasing rearrangement of f defined in this section is a translation of the decreasing rearrangement defined in § 5 for finite m. s. Letting f^* denote the one defined in § 5, we have

$$\delta_f(t) = f^*(t+a)$$

for all $t \in I =]-\mu(f^{-1}]b_f, \infty]), \mu(f^{-1}[-\infty, b_f])[$

where $a = \mu(f^{-1}]b_f, \infty]) .$

3. Suppose (X, Λ, μ) is a finite m. s. and X is the union of a finite number of atoms of equal measure, say $X = A_1 \cup \cdots \cup A_n$. Then $M(X, \mu)$ is isomorphic with R^n under the correspondence $f \leftrightarrow a = (a_1, \ldots, a_n) \in R^n$ where $a_i = f|A_i$. For each $f \in M(X, \mu)$ let δ_f denote the decreasing rearrangement defined in § 5. If $f \in M(X, \mu)$ then δ_f is constant on $[\alpha_{k-1}, \alpha_k[$ where $\alpha_k = \sum_{i=1}^{k} \mu(A_i)$ $k=1, \ldots, n$, and if $a \leftrightarrow f$ and $a_k^* = \delta_f|[\alpha_{k-1}, \alpha_k[$ then a^* is the point of R^n whose components are the components of $\underline{a}$ arranged in decreasing order.

4. G. F. D. Duff [6] has defined a generally multiple valued decreasing rearrangement for each member of M(I, m) where I is an infinite interval of R and m is Lebesgue measure. The conditions (6.1) on $f \in M(I, m)$ are necessary and sufficient that this decreasing rearrangement be single valued.

III. INEQUALITIES AND REARRANGEMENTS

7. A Theorem of Hardy. The following theorem of G. H. Hardy will be used often in what follows.

(7.1) THEOREM. (Hardy) (i) Suppose $f_1, f_2 \in M[a, b]$, $f_1, f_2 \in L^1[a,t]$, for all $a \le t < b$ and $\int_a^t f_1 \le \int_a^t f_2$ for all $a \le t \le b$. If g is a non-negative decreasing function on $[a, b]$ s.t. $f_i g \in L^1[a, t]$ $(i = 1, 2)$ for all $a \le t < b$, then $\int_a^t f_1 g \le \int_a^t f_2 g$ for all $a \le t \le b$.

(ii) Suppose $f_1, f_2 \in L^1[a, b]$, $\int_a^t f_1 \le \int_a^t f_2$ for all $a \le t \le b$, $\int_a^b f_1 = \int_a^b f_2$, and $g \in M[a, b]$ s.t. $f_i g \in L^1[a, b]$ $(i=1, 2)$. If g is decreasing then $\int_a^b f_1 g \le \int_a^b f_2 g$, while if g is increasing then $\int_a^b f_1 g \ge \int_a^b f_2 g$.

PROOF. (i) Let $F_i(t) = \int_a^t f_i$ $(i=1, 2)$ if $a < t \le b$. Then for all $a \le t \le b$, $\int_a^t f_2 g - f_1 g = \int_a^t g \, d(F_2 - F_1) = g(t)(F_2 - F_1)(t) - \int_a^t (F_2 - F_1) dg \ge 0$. (ii) is similar.

(7.2) THEOREM. If $f \in L^1[0, a]$ is decreasing, then $\frac{1}{t}\int_0^t f$ is a decreasing function of t, while if f is increasing, then $\frac{1}{t}\int_0^t f$ is an increasing function of t, $0 < t \le a$.

PROOF. Let $0 < t_1 < t_2 \le a$. Then $t_1 t \le t_2 t$ for all $0 \le t \le a$ so $\min\{t_1 t, t_1 t_2\} \le \min\{t_2 t, t_1 t_2\}$ and hence

$$\int_0^t \frac{1}{t_2} C_{[0, t_2]} = \frac{1}{t_2} \min\{t, t_2\} \le \frac{1}{t_1} \min\{t, t_1\} = \int_0^t \frac{1}{t_1} C_{[0, t_1]}$$

for all $0 \le t \le a$. Thus (7.1) (ii) implies

$$\frac{1}{t_2}\int_0^{t_2} f = \int_0^a \frac{1}{t_2} C_{[0, t_2]} f \le \int_0^a \frac{1}{t_1} C_{[0, t_1]} f = \frac{1}{t_1}\int_0^{t_1} f \text{ if } f \text{ is decreasing,}$$

and the reverse inequality if f is increasing.

8. A Preorder Relation. In this and in subsequent sections we will use δ_f to denote the decreasing rearrangement defined in § 5, since the measure spaces will always be finite. In addition we will use δ_E to denote the decreasing rearrangement of C_E.

In [15] Hardy, Littlewood and Polya introduced for the first time an important preorder relation for n-tuples of real numbers and later for integrable functions on a finite interval. We present this relation for finite m. s. as follows.

(8.1) DEFINITION. Suppose (X_1, Λ_1, μ_1) and (X, Λ, μ) are finite m. s. with $a = \mu_1(X_1) = \mu_1(X_1)$ and $f, g \in M(X,\mu) \cup M(X_1,\mu_1)$ and $\int_0^a \delta_{f^+}$ and $\int_0^a \delta_{g^+}$ are finite.

(i) $g << f$ means $\int_0^t \delta_g \le \int_0^t \delta_f$ for all $0 \le t \le a$;

(ii) $g < f$ means $g << f$ and $\int_0^a \delta_g = \int_0^a \delta_f$.

It is obvious that

(i) $f \sim g$ iff $g << f$ and $f << g$ iff $g < f$ and $f < g$;

(ii) $g << f$ iff $\delta_g << \delta_f$

(iii) $g < f$ iff $\delta_g < \delta_f$

(iv) $<<$ and $<$ are reflexive and transitive .

The reader should compare the conditions in Definition (8.1) with the hypotheses of Theorem (7.1).

The following are some other useful but less obvious properties of these preorder relations.

(8.2) LEMMA. (i) If $g \prec\prec f$ then $g+r \prec\prec f+r$ for all $r \in R$ and $rg \prec\prec rf$ if $r \geq 0$.

(ii) If $g \prec f$ then $g+r \prec f+r$ for all $r \in R$ and $rg \prec rf$ for all $0 \leq r \in R$. If in addition $f \in L^1$ then $-g \prec -f$.

(iii) If $r \leq f \leq s$ where $r, s \in R^{\#}$ and $g \prec f$ then $r \leq g \leq s$.

(iv) If $f \in L^1(X_1, \mu_1)$ then $\left(\frac{1}{\mu_1(X_1)} \int_{X_1} f \, d\mu_1\right) C_X \prec f$.

(v) If $f \in L^1(X_1, \mu_1)$ and $E \in \Lambda$ then $f \prec C_E$ iff $0 \leq f \leq 1$ and $\int_{X_1} f \, d\mu_1 = \mu(E)$. In particular, if $f \prec C_X$ then $f = C_{X_1}$ μ_1-a.e.

(vi) If $g^+ \prec f^+$ and $g^- \prec f^-$ then $g \prec f$.

(vii) If $g \prec\prec f$ then $g^+ \prec\prec f^+$. If $g \prec f \in L^1$ then in addition $g^- \prec\prec f^-$.

(viii) If $0 \leq f_n \uparrow f$, $0 \leq g_n \uparrow g$, and $g_n \prec f_n$, $n = 1, 2, 3, \ldots$, then $g \prec f$.

PROOF. (i) & (ii) If $g \prec\prec f$ and $r \in R$ then $\int_0^t \delta_{g+r} = \int_0^t (\delta_g + r) \leq \int_0^t (\delta_f + r) = \int_0^t \delta_{f+r}$ for all $0 \leq t \leq a$ using (5.3)(vi). If $g \prec f$ then in addition we have equality when $t = a$. Suppose $r \geq 0$. If $g \prec\prec f$ then $\int_0^t \delta_{rg} = \int_0^t r\,\delta_g \leq \int_0^t r\,\delta_f = \int_0^t \delta_{rf}$ for all $0 \leq t \leq a$. If $g \prec f$ then we also have equality when $t = a$. Now assume $g \prec f$. To prove the rest it suffices to show that $-g \prec -f$. $\int_0^t \delta_{-g}(u)du = \int_0^t -\delta_g(a-u)\,du$ $= \int_a^{a-t} \delta_g = \int_0^{a-t} \delta_g - \int_0^a \delta_g \leq \int_0^{a-t} \delta_f - \int_0^a \delta_f = \int_0^t -\delta_f(a-u)du = \int_0^t \delta_{-f}(u)du$ for all $0 \leq t \leq a$ with equality when $t = a$ using (5.5)(ii).

(iii) Let $g \prec f$ and suppose first that $f \geq 0$. If g is negative on a set of positive measure then there is a $0 < u < a$ s.t. $\delta_g(u) < 0$. Then $\delta_g(t) < 0$ for all $u \leq t \leq a$ so $\int_0^a \delta_f = \int_0^a \delta_g < \int_0^u \delta_g \leq \int_0^u \delta_f \leq \int_0^a \delta_f$ since

$\delta_f \geq 0$, a contrd. Hence $g \geq 0$. Now suppose $r \in R$ and $r \leq f$. Then $f - r \geq 0$ and $g - r \prec f - r$ so $g - r \geq 0$, i.e. $r \leq g$. Suppose $s \in R$ and $f \leq s$. Then $-f + s \geq 0$ and $-g + s \prec -f + s$ so $-g + s \geq 0$, i.e. $g \leq s$.

(iv) Let $g = \left(\frac{1}{\mu_1(X_1)} \int_{X_1} f \, d\mu_1\right) C_X$ so we have to show $g \prec f$.

$$\int_o^t \delta_g = t \frac{1}{\mu_1(X_1)} \int_{X_1} f \, d\mu_1 = t \frac{1}{a} \int_o^a \delta_f \leq \int_o^t \delta_f \qquad \text{for all } 0 \leq t \leq a$$

by Theorem (7.2), and we have equality when $t = a$.

(v) Suppose $f \prec C_E$. Then (iii) $\Rightarrow 0 \leq f \leq 1$, and $\int_{X_1} f \, d\mu_1 = \int_X C_E \, d\mu = \mu(E)$. Assume $0 \leq f \leq 1$ and $\int_{X_1} f \, d\mu_1 = \mu(E)$. Then $\delta_f \leq 1$ so $\int_o^t \delta_f \leq t$, and $0 \leq \delta_f$ so $\int_o^t \delta_f \leq \int_o^a \delta_f = \mu(E)$. Hence $\int_o^t \delta_f \leq \min\{t, \mu(E)\} = \int_o^t \delta_E$ for all $0 \leq t \leq a$ and we have equality when $t = a$.

(vi) Now $g^- \prec f^- \Rightarrow -g^- \prec -f^-$ so $\int_o^t \delta_g = \int_o^t \delta_{g^+} + \int_o^t \delta_{-g^-} \leq \int_o^t \delta_{f^+} + \int_o^t \delta_{-f^-} = \int_o^t \delta_f$ for all $0 \leq t \leq a$ with equality when $t = a$, using (5.5)(i).

(vii) Assume $g \in M(X,\mu)$ & $f \in M(X_1, \mu_1)$. Let $b = \mu(\{g \geq 0\}) = m(\{\delta_g \geq 0\})$ and $b_1 = \mu_1(\{f \geq 0\})$. Then $\int_o^a \delta_{g^+} = \int_o^b \delta_g \leq \int_o^b \delta_f \leq \int_o^{b_1} \delta_f = \int_o^a \delta_{f^+}$. If $g \prec f \in L^1$ then $-g \prec -f$ so $g^- = (-g)^+ \prec\prec (-f)^+ = f^-$.

(viii) Use monotone convergence and (5.3).

(8.3) LEMMA. <u>Suppose</u> $f, g \in L^1[0, a]$, f <u>and</u> g <u>are decreasing</u>, $]r, s[\ \&\ ]u, v[\ \subset [0, a] = I$, $s - r = v - u$, <u>and</u> $\int_r^s f \leq \int_u^v g$. <u>If</u> $g \prec\prec f$ <u>then</u> $g|I-]u, v[\ \prec\prec f|I-]r, s[$.

PROOF. The proof is essentially due to Ryff [43 , p. 433] . Let $g| = g|I -]u, v[$, $f| = f|I-]r, s[$ and let $h = s - r = v - u$. Except at most at countably many points of I,

$$\delta_{g|}(x) = \begin{cases} g(x) & 0 \le x \le u \\ g(x+h) & u < x < a-h \end{cases}$$

and similarly for f. We consider four cases.

Case 1. $t < \min\{u, r\}$.

$$\int_o^t \delta_{g|} = \int_o^t g \le \int_o^t f = \int_o^t \delta_{f|} .$$

Case 2. $t > \max\{u, r\}$. Now

$$\int_o^t \delta_{g|} = \int_o^u g + \int_u^t g(x+h)dx = \int_o^u g + \int_v^{t+h} g$$

and similarly for f. Hence

$$\int_o^t \delta_{g|} + \int_u^v g = \int_o^{t+h} g \le \int_o^{t+h} f = \int_o^t \delta_{f|} + \int_r^s f .$$

Since $\int_u^v g - \int_r^s f \ge 0$ we obtain

$$\int_o^t \delta_{g|} \le \int_o^t \delta_{g|} + \int_u^v g - \int_r^s f \le \int_o^t \delta_{f|} .$$

Case 3. $r \le t \le u$. Since g is decreasing, $\int_u^v g \le \int_t^{t+h} g$. Then

$$\int_o^t \delta_{g|} + \int_u^v g = \int_o^t g + \int_u^v g \le \int_o^t g + \int_t^{t+h} g = \int_o^{t+h} g$$

$\le \int_o^{t+h} f = \int_o^t \delta_{f|} + \int_r^s f$ so $\int_o^t \delta_{g|} \le \int_o^t \delta_{f|}$ as before.

Case 4. $u \le t \le r$.

$$\int_o^t \delta_{g|} = \int_o^u g + \int_v^{t+h} g \le \int_o^u g + \int_u^t g = \int_o^t g \le \int_o^t f = \int_o^t \delta_{f|} .$$

Thus $g| << f|$.

(8.4) LEMMA. Suppose $f, g \in L^1[0,a]$, $]r,s[$ & $]u,v[\subset [0,a] = I$, $s-r = v-u$, and $g|I -]u,v[\ll f|I -]r,s[$. Then

$g\,C_{I-]u,v[} + \alpha\, C_{]u,v[} \ll f\,C_{I-]r,s[} + \alpha\, C_{]r,s[}$ for all $\alpha \in R$.

PROOF. Let $g| = g|I -]u,v[$ and $f| = f|I-]r,s[$, so $g| \ll f|$, let $\alpha \in R$, let $g_1 = g\,C_{I-]u,v[} + \alpha\, C_{]u,v[}$, let

$f_1 = f\,C_{I-]r,s[} + \alpha\, C_{]r,s[}$, and let $]w,z[\subset \delta_{g_1}^{-1}(\alpha)$ &

$]p,q[\subset \delta_{f_1}^{-1}(\alpha)$ s.t. $z-w = q-p = h = s-r = v-u$.

Then $\delta_{g|} \geq \alpha$ on $[0,w]$, $\delta_{g|} \leq \alpha$ on $[z, a-h]$, and except at most at countably many points of I,

$$\delta_{g_1}(x) = \begin{cases} \delta_{g|}(x) & 0 \leq x \leq w \\ \alpha & w < x \leq z \\ \delta_{g|}(x-h) & z < x \leq a \end{cases}$$

and similarly for f. The inequality $\int_0^t \delta_{g_1} \leq \int_0^t \delta_{f_1}$ follows by examining cases. Rather than go through them all we give the details for three representative ones.

Case: $w \leq p \leq t \leq \min\{q,z\}$. Now $\alpha \leq \delta_{f|}$ on $[w,p]$

$$\text{so } \int_0^t \delta_{g_1} = \int_0^w \delta_{g|} + \int_w^t \alpha = \int_0^w \delta_{g|} + \int_w^p \alpha + \int_p^t \alpha$$

$$\leq \int_0^w \delta_{f|} + \int_w^p \delta_{f|} + \int_p^t \alpha = \int_0^p \delta_{f|} + \int_p^t \alpha = \int_0^t \delta_{f_1}$$

Case: $p \leq w \leq t \leq \min\{q,z\}$. Now $\delta_{f|} \leq \alpha$ on $[p,w]$ so

$$\int_0^t \delta_{g_1} = \int_0^w \delta_{g|} + \int_w^t \alpha \leq \int_0^w \delta_{f|} + \int_w^t \alpha \leq \int_0^p \delta_{f|} + \int_p^w \alpha + \int_w^t \alpha$$

$$= \int_0^p \delta_{f|} + \int_p^t \alpha = \int_0^t \delta_{f_1}.$$

Case: $z < t \leq q$. $\int_o^t \delta_{g_1} = \int_o^w \delta_{g|} + \int_w^z \alpha + \int_z^t \delta_{g|}(x-h)dx$

$$= \int_o^w \delta_{g|} + \alpha h + \int_w^{t-h} \delta_{g|} = \int_o^{t-h} \delta_{g|} + \int_{t-h}^t \alpha .$$

If $t \leq p$, $\int_o^t \delta_{g_1} = \int_o^{t-h} \delta_{g|} + \int_{t-h}^t \alpha \leq \int_o^{t-h} \delta_{f|} + \int_{t-h}^t \delta_{f|} = \int_o^t \delta_{f|} = \int_o^t \delta_{f_1}$

since $\alpha \leq \delta_{f|}$ on $[t-h, t]$.

If $p < t$, then $t < q \Rightarrow t-h \leq p$ & $\alpha \leq \delta_{f|}$ on $[t-h, p]$ so

$$\int_o^t \delta_{g_1} = \int_o^{t-h} \delta_{g|} + \int_{t-h}^p \alpha + \int_p^t \alpha \leq \int_o^{t-h} \delta_{f|} + \int_{t-h}^p \delta_{f|} + \int_p^t \alpha = \int_o^t \delta_{f_1} .$$

(8.5) LEMMA. Suppose $f, g \in L^1[0, a]$, $]r, s[$ & $]u, v[\subset [0, a] = I$ and $s-r = v-u$.

(i) [J. V. Ryff] If f and g are decreasing and $\int_r^s f = \int_u^v g$ and $g < f$ then $g| I-]u, v] < f| I-]r, s[$.

(ii) If $g| I-]u, v[< f| I-]r, s[$ then for all $\alpha \in R$,

$$g\, C_{I-]u,v[} + \alpha\, C_{]u,v[} < f\, C_{I-]r,s[} + \alpha\, C_{]r,s[} .$$

PROOF. (i) Let $f|, g|$ be as in the proof of (8.4). Then

$$\int_o^{a-h} \delta_{g|} = \int_{I-]u,v[} g = \int_o^a g - \int_u^v g = \int_o^a f - \int_r^s f = \int_o^{a-h} \delta_{f|}$$

where $h = s-r = v-u$. Since we already have $g| << f|$ from (8.3) we conclude that $g| < f|$.

(ii) Let g_1 and f_1 be as in proof of (8.4). Then

$$\int_o^a g_1 = \int_o^a g - \int_u^v g + \alpha h = \int_o^a f - \int_r^s f + \alpha h = \int_o^a f_1 .$$

(8.6) LEMMA Suppose $f \in L^1(X_1, \mu_1)$, $g \in L^1(X, \mu)$, f is constant on $E_1 \in \Lambda_1$, g is constant on $E \in \Lambda$, $f|E_1 \leq g|E$ and $\mu_1(E_1) = \mu(E)$. If $g \prec\prec f$, then

(i) $g|X-E \prec\prec f|X-E_1$ and

(ii) $g\,C_{X-E} + \alpha\, C_E \prec\prec f\,C_{X_1-E_1} + \alpha\, C_{E_1}$ for all $\alpha \in R$.

PROOF. Let $\beta = f|E_1$, $\gamma = g|E$, $a = \mu(X) = \mu_1(X_1)$, $I = [0, a]$, $]r, s[\subset \delta_f^{-1}(\beta)$ s.t. $s-r = \mu_1(E_1)$, and $]u, v[\subset \delta_g^{-1}(\gamma)$ s.t. $v-u = \mu(E)$. Then $s-r = v-u$ and $\int_r^s \delta_f \leq \int_u^v \delta_g$. If $g \prec\prec f$ then $\delta_g \prec\prec \delta_f$ so $g|X-E \sim \delta_g|I-]u, v[\prec\prec \delta_f|I-]r, s[\sim f|X_1-E_1$ and $gC_{X-E}+\alpha C_E \sim \delta_g C_{I-]u,v[}+\alpha C_{]u,v[} \prec\prec \delta_f C_{I-]r,s[}+\alpha C_{]r,s[} \sim fC_{X_1-E_1}+\alpha C_{E_1}$ for all $\alpha \in R$ using Lemmas (3.3) (xi), (3.4) (i), (8.3), and (8.4).

(8.7) LEMMA. Suppose $f \in L^1(X_1, \mu_1)$, $g \in L^1(X, \mu)$ and suppose f and g have the same constant value on $E_1 \in \Lambda_1$ and $E \in \Lambda$ respectively, where $\mu_1(E_1) = \mu(E)$. If $g \prec f$ then

(i) $g|X-E \prec f|X_1-E_1$ and

(ii) $gC_{X-E} + \alpha\, C_E \prec f\,C_{X_1-E_1} + \alpha\, C_{E_1}$ for all $\alpha \in R$.

REMARK. Let $g \underset{\neq}{\prec} f$ mean $g \prec f$ and g and f are not equimeasurable, and define $g \underset{\neq}{\prec\prec} f$ similarly. Let (X, Λ, μ) be a non-atomic m. s. and let $f, g \in L^1(X, \mu)$. Then $g \underset{\neq}{\prec} f \Rightarrow$ there is an $h \in L^1(X, \mu)$ s.t. $g \underset{\neq}{\prec} h \underset{\neq}{\prec} f$, and $g \underset{\neq}{\prec\prec} f \Rightarrow$ there is an $h \in L^1(X, \mu)$ s.t. $g \underset{\neq}{\prec\prec} h \underset{\neq}{\prec\prec} f$ for by Theorem (5.10) there is an $h \in M(X, \mu)$ s.t. $\delta_h = \frac{1}{2}[\delta_f + \delta_g]$.

We now pause to show how a finite m. s. can be embedded in a non-atomic m. s. We will use this device to show which results about non-atomic m. s. carry over the general m. s.

9. Embedding of a Finite m. s. in a Non-Atomic Finite m. s. Let (X, Λ, μ) be a finite m. s. We recall that a measurable set $A \in \Lambda$ is called a μ-atom or an atom of (X, Λ, μ) whenever $\mu(A) > 0$ and $B \subset A$ implies $\mu(B) = 0$ or $\mu(B) = \mu(A)$. If A is a μ-atom and $f \in M(X, \mu)$ then f is essentially constant on A. If A and B are μ - atoms, they are equal or disjoint a. e., i. e., $\mu(A \Delta B) = 0$, where $A \Delta B = A-B \cup B-A$, or $\mu(A \cap B) = 0$.

A finite (or σ-finite) m. s. can have at most countably many atoms. Thus $X = X_0 \cup \bigcup_{n \in P} A_n$ where P is $\{1, \ldots, k\}$ for some natural number k or P is $\{1, 2, 3, \ldots\}$; each A_n is a μ-atom; and $(X_0, \Lambda \cap X_0, \mu)$ is non-atomic.

We may embed (X, Λ, μ) in a non-atomic m. s. $(X^\#, \Lambda^\#, \mu^\#)$ as follows. Let $X^\# = X_0 \cup \bigcup_{n \in P} I[a_n, b_n]$ where the $I[a_n, b_n]$ are disjoint intervals of R with end-points a_n & b_n s. t. $b_n - a_n = \mu(A_n)$. Then $(X^\#, \Lambda^\#, \mu^\#)$ is the direct sum of $(X_0, \Lambda \cap X_0, \mu)$ and the Lebesgue m. s. $(I[a_n, b_n], m)$, i. e., $E \in \Lambda^\#$ iff $E = E_0 \cup \bigcup_{n \in P} E_n$ where $E_0 \in \Lambda$, $E_0 \subset X_0$, and for each $n \in P$, E_n is a Lebesgue measurable subset of $I[a_n, b_n]$, and in this case $\mu^\#(E) = \mu(E_0) + \sum_{n \in P} m(E_n)$.

Clearly $(X^\#, \Lambda^\#, \mu^\#)$ is a non-atomic finite m. s. with $\mu^\#(X^\#) = \mu(X)$. Each $f \in M(X, \Lambda, \mu)$ is identified with $f^\# = f\, C_{X_0} + \sum_{n \in P} (f|A_n)\, C_{I[a_n, b_n]}$ in $M(X^\#, \Lambda^\#, \mu^\#)$ and it is easy to see that $f \sim f^\#$ so $\delta_f = \delta_{f^\#}$. Of more importance is the fact that $L^1(X, \mu)$ is a retract of $L^1(X^{\#}, \mu^{\#})$ under the following mapping.

(9.1) DEFINITION. If $f \in L^1(X^{\#}, \mu^{\#}) \cup M^+(X^{\#}, \mu^{\#})$ let

$$T_\mu f = f\, C_{X_0} + \sum_{n \in P} \frac{1}{b_n - a_n} \left(\int_{a_n}^{b_n} f \right) C_{A_n} .$$

(9.2) THEOREM. (i) $T_\mu : L^1(X^{\#}, \mu^{\#}) \to L^1(X, \mu)$.

(ii) $T_\mu f = f$ for all $f \in L^1(X, \mu) \cup M^+(X, \mu)$.

(iii) $T_\mu f \geq 0$ for all $0 \leq f \in M(X, \mu)$.

(iv) $\int T_\mu f \, d\mu = \int f \, d\mu^{\#}$ for all $f \in (L^1 \cup M^+)(X^{\#}, \mu^{\#})$ and $\|T_\mu f\|_1 \leq \|f\|_1$ for all $f \in L^1(X^{\#}, \mu^{\#})$.

(v) $\|T_\mu f\|_\infty \leq \|f\|_\infty$ for all $f \in L^\infty(X^{\#}, \mu^{\#})$.

(vi) For all $f \in (L^1 \cup M^+)(X^{\#}, \mu^{\#})$ and $g \in M(X, \mu)$ s.t. $fg \in (L^1 \cup M^+)(X^{\#}, \mu^{\#})$ we have $T_\mu(fg) = (T_\mu f) g$.

The reader who is familiar with the concept of a conditional expectation will recognize that $T_\mu f$ is the conditional expectation of f with respect to the σ-ring generated by $X_0 \cap \Lambda$ and the intervals $I[a_n, b_n]$, $n \in P$.

At the end of § 13 we will prove that $T_\mu f \prec f$ for all $f \in (L^1 \cup M^+)(X^{\#}, \mu^{\#})$. We will investigate transformations of this type in detail in § 17.

10. An Inequality of Hardy and Littlewood. In this section we prove an inequality which was originally proved by Hardy and Littlewood for non-negative functions ([15], Theorem 378). This inequality will be fundamental for what is to follow.

(10.1) THEOREM (Hardy & Littlewood). If $f, g \in M(X, \mu)$, $a = \mu(X) < \infty$, and if $\delta_{|f|}\delta_{|g|} \in L^1[0, a]$, then $f g \in L^1(X, \mu)$ and

$$\int_0^a \delta_f(a-t)\delta_g(t)dt = \int_0^a \delta_f(t)\delta_g(a-t)dt \leq \int_X f g \, d\mu \leq \int_0^a \delta_f\delta_g .$$

PROOF. First suppose $f = C_E$ and $g = C_F$ when $E, F \in \Lambda$. Then $\int_0^a \delta_f(t)\,\delta_g(a-t)dt = (\mu(E) + \mu(F) - a)^+ = (\mu(E \cup F) + \mu(E \cap F) - a)^+ \leq \mu(E \cap F) = \int f g \, d\mu \leq \min\{\mu(E), \mu(F)\} = \int_0^a \delta_f\,\delta_g$.

Next suppose f and g are non-negative simple functions. Then f and g can be written in the form

$$f = \sum_{i=1}^{n} f_i C_{E_i} \qquad f_i > 0 \quad i = 1, \ldots, n \quad \text{and} \quad E_1 \supset \cdots \supset E_n$$

$$g = \sum_{j=1}^{m} g_j C_{F_j} \qquad g_j > 0 \quad j = 1, \ldots, m \quad \text{and} \quad F_1 \supset \cdots \supset F_m.$$

$$\text{Then } \int_0^a \delta_f(t)\delta_g(a-t)dt = \sum_{i,j} f_i\, g_j \int_0^a \delta_{E_i}(t)\,\delta_{F_j}(a-t)dt$$

$$\leq \sum_{i,j} f_i\, g_i \int C_{E_i} C_{F_j}\, d\mu = \int f g \, d\mu$$

$$\leq \sum_{i,j} f_i\, g_i \int_0^a \delta_{E_i}\,\delta_{F_j} = \int_0^a \delta_f\,\delta_g \quad \text{using (5.6) (ii).}$$

If $0 \le f,\ g \in M(X,\mu)$ then there are sequences $\{f_n\}$ and $\{g_n\}$ of non-negative simple functions such that $f_n \uparrow f$ and $g_n \uparrow g$ everywhere. Then $\int_0^a \delta_f(t)\,\delta_g(a-t)dt = \lim_{n\to\infty} \int_0^a \delta_{f_n}(t)\,\delta_{g_n}(a-t)dt$

$$\le \lim_{n\to\infty} \int f_n g_n \, d_\mu = \int f g \, d_\mu$$

$$\le \lim_{n\to\infty} \int_0^a \delta_{f_n}\,\delta_{g_n} = \int_0^a \delta_f\,\delta_g$$

Let $f,\ g \in M(\mathbf{X},\mu)$ be arbitrary s.t. $\delta_{|f|}\,\delta_{|g|} \in L^1[0,a]$. Then by what we have already proved $\int |f g|\,d\mu \le \int_0^a \delta_{|f|}\,\delta_{|g|}$ so $f g \in L^1(X,\mu)$. Then using Theorem (5.5)

$$\int_0^a \delta_f(t)\,\delta_g(a-t)dt = \int_0^a \delta_{f^+}(t)\,\delta_{g^+}(a-t)dt - \int_0^a \delta_{f^-}\,\delta_{g^+} - \int_0^a \delta_{f^+}\,\delta_{g^-}$$

$$+ \int_0^a \delta_{f^-}(a-t)\,\delta_{g^-}(t)dt \le \int f^+g^+\,d\mu - \int f^-g^+\,d\mu + \int f^-g^-\,d\mu - \int f^+g^-\,d\mu$$

$$= \int f g\,d\mu \le \int_0^a \delta_{f^+}\,\delta_{g^+} - \int_0^a \delta_{f^-}(a-t)\,\delta_{g^+}(t)dt - \int_0^a \delta_{f^+}(t)\,\delta_{g^-}(a-t)dt$$

$$+ \int_0^a \delta_{f^-}\,\delta_{g^-} = \int_0^a \delta_f\,\delta_g \ .$$

REMARK. Observe that the inequalities in (10.1) are true for all $0 \le f,\ g \in M(X,\mu)$ even if $\delta_f\,\delta_g \notin L^1[0,a]$.

(10.2) PROPOSITION. <u>If</u> F <u>and</u> G <u>are measurable functions on</u> $[0,a]$ <u>such that</u> $\delta_{|F|}\,\delta_{|G|} \in L^1[0,a]$ <u>and</u> G <u>is decreasing, then the convolution</u>

$$F*G(u) = \int_0^u F(t)\,G(u-t)dt$$

<u>is a continuous function of</u> u <u>on</u> $[0,a]$.

PROOF. First we show that $|F|*|G|(u)$ is finite for every $u \in [0, a]$. Let $u \in [0, a]$ and let

$$H(t) = \begin{cases} G(u-t) & \text{on } [0, u] \\ G(t) & \text{on }]u, a] \end{cases}$$

Then $H|[0, u] \sim G|[0, u]$ and $H|\,]u, a] \sim G|\,]u, a]$ so $H \sim G$ and thus $|H| \sim |G|$ so

$$|F|*|G|(u) = \int_0^u |F|(t)\,|G|(u-t)dt \le \int_0^u |F|(t)\,|G|(u-t)dt + \int_u^a |F|\,|G|$$

$$= \int_0^a |F|\,|H| \le \int_0^a \delta|F|\,\delta|H| = \int_0^a \delta|F|\,\delta|G| < \infty .$$

Let $0 \le u < u_0 \le a$, so

$$|F*G(u_0) - F*G(u)| \le \int_u^{u_0} |F(t)G(u_0-t)|dt + \int_0^u |(G(u-t)-G(u_0-t))F(t)|dt.$$

Let $H(t) = G(u_0-t)$ on $[u, u_0]$ and 0 elsewhere. Then $H \sim G\,C_{[0,\, u_0-u]}$ so $|H| \sim |G|\,C_{[0,u_0-u]}$ and thus

$$\delta|H| = \delta|G|C_{[0,\,u_0-u]} \le \delta|G|\,\delta_{[0,u_0-u]} = \delta|G|\,C_{[0,\,u_0-u[} .$$ Hence

$$0 \le \int_u^{u_0} |F(t)|\,|G(u_0-t)|dt = \int_0^a |F|\,|H| \le \int_0^a \delta|F|\,\delta|H| \le \int_0^{u_0-u} \delta|F|\,\delta|G| \to 0$$

as $u \uparrow u_0$.

For the rest, let $u_n \uparrow u_0$ $n = 1, 2, 3, \ldots,$ and let $H_n(t) = G(u_n-t)F(t)$ on $[0, u_n]$ and 0 elsewhere and $H(t) = G(u_0-t)F(t)$ on $[0, u]$ and 0 elsewhere. Then $H_n \downarrow H$ pointwise a.e. since the discontinuities of G form a set of measure 0, and $\int_0^a |H_1| = |F|*|G|(u_1) < \infty$, so

$\int_0^{u_n} |(G(u_n - t) - G(u_0 - t)) F(t)| \, dt = \int_0^a |H_n - H| \to 0$ as $n \to \infty$ by Dominated convergence.

The case $0 \le u_0 < u \le a$ is similar.

11. The Values of an Integral. Let f, g $\in M(X, \mu)$. If $\delta_{|f|} \delta_{|g|} \in L^1[0, a]$ where $a = \mu(X)$ and $f' \sim f$ and $g' \sim g$ then $|f'| \sim |f|$ and $|g'| \sim |g|$ and we know that $\int |f'g'| \, d\mu \le \int_0^a \delta_{|f'|} \delta_{|g'|} = \int_0^a \delta_{|f|} \delta_{|g|}$ and $\int_0^a \delta_f(t) \delta_g(a-t) dt \le \int f'g' \, d\mu \le \int_0^a \delta_f \delta_g$.

(11.1) THEOREM. Let f, g $\in M(X, \mu)$ and $\delta_{|f|} \delta_{|g|} \in L^1[0, a]$, where $a = \mu(X)$. If (X, Λ, μ) is non-atomic, then

$$\{\int f g' \, d\mu : g' \sim g\} = \left[\int_0^a \delta_f(t) \delta_g(a-t) dt, \int_0^a \delta_f \delta_g \right].$$

PROOF. We already know that

$$\{\int f g' \, d\mu : g' \sim g\} \subset \left[\int_0^a \delta_f(t) \delta_g(a-t) dt, \int_0^a \delta_f \delta_g \right], \quad \text{so}$$

it remains to show that all the values are taken on.

Let $\delta_{|f|} \delta_{|g|} \in L^1[0, a]$, and for $0 \le u \le a$ let

$\gamma(u) = \int_0^u \delta_f(t) \delta_g(u-t) dt + \int_u^a \delta_f \delta_g$. Then γ is a continuous function of u with $\gamma(0) = \int_0^a \delta_f \delta_g$ and $\gamma(a) = \int_0^a \delta_f(t) \delta_g(a-t) dt$, so it suffices to prove that for each $0 \le u \le a$ there is a $g' \sim g$ s.t. $\int f g' \, d\mu = \gamma(u)$.

Fix $u \in [0, a]$. We prove that there are sets $X_1 \in \Lambda$ and $X_2 = X - X_1$ s.t. $\mu(X_1) = u$, $\delta_{f|X_1} = \delta_f$ on $[0, u[$ and $\delta_{f|X_2}(t) = \delta_f(t+u)$ on $0 \le t < a-u$.

To this end, if $t \in R^{\#}$ let $F_t = \{x \in X: f(x) > t\}$ and $F_{t-} = \{x \in X: f(x) \geq t\}$ so $\mu(F_{t-}) = \lim_{s \uparrow t} \mu(F_s)$. Now $\mu(F_{+\infty}) = 0 \leq u$ so let $t_o = \inf\{t \in R^{\#}: \mu(F_t) \leq u\}$. Then $\mu(F_{t_o}) \leq u$, and $s < t_o \Rightarrow \mu(F_s) > u$, so $\mu(F_{t_o}) \leq u \leq \mu(F_{t_o -})$. Since (X, Λ, μ) is non-atomic, there is an $X_1 \in \Lambda$ s.t. $F_{t_o} \subset X_1 \subset F_{t_o -}$ and $\mu(X_1) = u$. Now $s < t_o \Rightarrow d_f(s) = \mu(F_s) > u$ so $d_f(s) < u \Rightarrow s \geq t_o \Rightarrow F_s \subset F_{t_o} \subset X_1 \Rightarrow d_{f|X_1}(s) = \mu(F_s \cap X_1) = \mu(F_s) = d_f(s)$. Conversely, $s < t_o \Rightarrow X_1 \subset F_{t_o -} \subset F_s \Rightarrow d_{f|X_1}(s) = \mu(F_s \cap X_1) = \mu(X_1) = u$ so $d_{f|X_1}(s) < u \Rightarrow s \geq t_o \Rightarrow d_f(s) = d_{f|X_1}$ as before. Hence if $0 \leq t < u$ we have $\{s: d_f(s) \leq t\} = \{s: d_{f|X_1}(s) \leq t\}$ so $\delta_f(t) = \delta_{f|X_1}(t)$.

Now let $v(t) = \delta_f(t+u)$ on $0 \leq t < a-u$. Then for each bounded interval $[r, s]$ of $R^{\#}$ we have

$$\mu(f^{-1}[r, s] \cap X_1) = \mu((f|X_1)^{-1}[r, s]) = m(\delta_{f|X_1}^{-1}\ [r, s])$$
$$= m(\delta_f^{-1}[r, s] \cap [0, u[).$$

Subtracting this equation from $\mu(f^{-1}[r, s] \cap X) = m(\delta_f^{-1}[r, s] \cap [0, a[)$ we get $\mu(f^{-1}[r, s] \cap (X - X_1)) = m(\delta_f^{-1}[r, s] \cap [u, a[)$, i.e.,

$$\mu((f|X_2)^{-1}[r, s]) = m((v^{-1}[r, s] \cap [0, a-u[) + u)$$
$$= m(v^{-1}[r, s] \cap [0, a-u[) .$$

Hence $\delta_{f|X_2}(t) = \delta_f(t+u)$ on $0 \leq t < a-u$.

Recall that if $T: (X, \Lambda, \mu) \to (Y, \Sigma, \nu)$ is a measurable transformation and $g \in M(Y, \Sigma)$ then $\int_Y g(d\mu T^{-1}) = \int_X g \circ T \, d\mu$ in the sense that either both integrals exist and are equal or neither exist [12, p. 163, Theorem C] where $(\mu T^{-1})(E) = \mu(T^{-1}(E))$ for all $E \in \Sigma$. If T is measure preserving, then $\mu T^{-1} = \nu$.

Let $\sigma_1 : X_1 \to [0, u[$ and $\sigma_2 : X_2 \to [0, a-u[$ be m.p. s.t. $\delta_{f|X_1} \circ \sigma_1 = f|X_1$ and $\delta_{f|X_2} \circ \sigma_2 = f|X_2$.

Also let $G_1(t) = \delta_g(u-t)$ on $0 \le t < u$ so $G_1 \sim \delta_g|[0, u[$

and let $G_2(t) = \delta_g(t+u)$ on $0 \le t < a-u$ so $G_2 \sim \delta_g|[u, a[$.

Then $\int_0^u \delta_f(t)\delta_g(u-t)dt = \int_0^u \delta_{f|X_1} G_1 \, dm = \int_{X_1} (\delta_{f|X_1} \circ \sigma_1)(G_1 \circ \sigma_1) \, d\mu$

$= \int_{X_1} f \, G_1 \circ \sigma_1 \, d\mu$ and $G_1 \circ \sigma_1 \sim G_1 \sim \delta_g|[0, u[$. Also

$$\int_u^a \delta_f \delta_g = \int_0^{a-u} \delta_f(t+u)\delta_g(t+u)dt = \int_0^{a-u} \delta_{f|X_2} G_2 \, dm$$

$$= \int_{X_2} (\delta_{f|X_2} \circ \sigma_2)(G_2 \circ \sigma_2)d\mu = \int_{X_2} f \, (G_2 \circ \sigma_2) \, d\mu$$

and $G_2 \circ \sigma_2 \sim G_2 \sim \delta_g|[u, a[$. Hence if we let

$$g'(x) = \begin{cases} G_1 \circ \sigma_1(x) & \text{if } x \in X_1 \\ G_2 \circ \sigma_2(x) & \text{if } x \in X_2 \end{cases} \qquad \text{then } g' \sim \delta_g \sim g$$

and $\gamma(u) = \int f \, g' \, d\mu$.

(11.2) COROLLARY <u>If</u> (X, Λ, μ) <u>is non-atomic then for all</u> $0 \le f, \; g \in M(X, \mu)$ <u>we have</u>

$$\max\{ \int f' g' \, d\mu : f' \sim f, \; g' \sim g\} = \max\{\int f \, g' \, d\mu : g' \sim g\} = \int_0^a \delta_f \delta_g ,$$

<u>in the sense that both are infinite or are finite and equal.</u>

PROOF. As we observed, the inequalities of (10.1) hold for all $0 \le f, \; g \in M(X, \mu)$ even if $\delta_f \delta_g \notin L^1[0, a]$. To show that the sup is attained, let $\sigma : X \to [0, a]$ be measure preserving s.t. $\delta_f \circ \sigma = f$. Then

$$\int_0^a \delta_f \delta_g = \int (\delta_f \circ \sigma)(\delta_g \circ \sigma)d\mu = \int f \, g' \, d\mu \text{ where } g' = \delta_g \circ \sigma \sim g.$$

Many investigators have used $g \prec f$ to mean $|g| \ll |f|$ ([25] for example). Hence the following theorem is of interest. It will also be very useful.

(11.3) THEOREM. Let (X_1, Λ_1, μ_1) and (X, Λ, μ) be finite m. s. such that $\mu(X) = \mu_1(X_1) = a$, and let $f \in L^1(X_1, \mu_1)$ and $g \in M(X, \mu)$. Then $g \prec f$ implies $g \in L^1(X, \mu)$ and $|g| \ll |f|$.

PROOF. (8.2)(vii) $\Rightarrow g \in L^1$. From §9 it follows that (11.2) implies

$\int_0^t \delta_{|g|} = \int |g| C_E \, d\mu^\#$ for some $E \in \Lambda^\#$ with $\mu^\#(E) = t$

$= \int g \operatorname{sgn}(g) C_E \, d\mu^\#$

$\leq \int_0^a \delta_g \delta_h$ where $h = \operatorname{sgn}(g) C_E$

$\leq \int_0^a \delta_f \delta_h$ since $g \prec f$

$= \int f h' \, d\mu_1^\#$ for some $h' \in L^1(X_1^\#, \mu_1^\#)$ s.t. $h' \sim h$

$\leq \int |f| \, |h'| \, d\mu_1^\#$ and $|h'| \sim |h| = C_E$ so $|h'| = C_F$ for some $F \in \Lambda_1^\#$ s.t. $\mu_1^\#(F) = \mu_1^\#(E) = t$

$= \int |f| C_F \, d\mu_1^\#$

$\leq \int_0^t \delta_{|f|}$ for all $0 \leq t \leq a$.

REMARKS. 1. If (X, Λ, μ) has atoms, then Theorem (11.1) may be false. Let $X = X_0 \cup A_1$ where X_0 is non-atomic, A_1 is an atom, and $\mu(A_1) > \mu(X_0) > 0$. Let $f = C_{X_0}$ and $g = C_{A_1}$. Then $\int_0^a \delta_f \delta_g = \mu(X_0)$ but $g' \sim g$ implies $g' = C_{A_1}$ since $\mu(A_1) > \mu(X_0)$, so

$\int f g' \, d\mu = \int C_{X_0} C_{A_1} \, d\mu = 0.$

2. If (X, Λ, μ) is a finite m. s. consisting only of atoms of equal measure, then it is said to be discrete, and the set of values of the corresponding sums do not fill up the whole interval, although the endpoints are attained. (See [15], Theorem 368).

(11.4) DEFINITION. A finite m. s. (X, Λ, μ) is called adequate if for all $0 \le f, g \in M(X, \mu)$ we have

$$\max\{ \int f g' \, d\mu : g' \sim g \} = \int_o^a \delta_f \delta_g , \quad a = \mu(X).$$

(11.5) THEOREM. The following are equivalents for the finite m. s. (X, Λ, μ) .

(i) (X, Λ, μ) is adequate.

(ii) (X, Λ, μ) is non-atomic or consists only of atoms of equal measure.

(iii) For all $A, B \in \Lambda$ we have

$$\sup \{ \int C_A C_E \, d\mu : C_E \sim C_B \} = \int_o^a \delta_A \delta_B .$$

PROOF. Now we know that (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). Hence it remains to prove (iii) $\Rightarrow$ (ii). Suppose (ii) is not true. Then either X has at least two atoms A, B with $0 < \mu(B) < \mu(A)$, or X has an atom A and a non-atomic part X_0 of positive measure, in which case (5. 8) implies there is a $B \subset X_0$ such that $0 < \mu(B) < \mu(A)$. In either case, for all $E \in \Lambda$ with $C_E \sim C_B$ we have $\mu(E) = \mu(B)$ and hence $\mu(A \cap E) \le \mu(E) = \mu(B) < \mu(A)$ so $\mu(A \cap E) = 0$. Thus $\int C_A C_E \, d\mu = 0$ for all $E \in \Lambda$ with $C_E \sim C_B$ but $\int_o^a \delta_A \delta_B = \mu(B) > 0$.

12. The Decreasing Rearrangements of Sums and Products. If $f, g \in L^1(X, \mu)$ then in general $\delta_{f+g} \neq \delta_f + \delta_g$. For example, let E and F be disjoint sets of equal positive measure. Then $\delta_E = \delta_F = C_{[0, \mu(E)[}$ while $\delta_{E \cup F} = C_{[0, 2\mu(E)[}$ so $\delta_{E\cup F} \neq \delta_E + \delta_F$. We can, however, prove the following.

(12.1) THEOREM. If $f_1, \ldots, f_n \in L^1(X, \mu)$, then

$$\int_0^t \delta_{f_i} + \sum_{j \neq i} \int_0^t \delta_{f_j}(a-u)du \le \int_0^t \delta_{f_1+\cdots+f_n} \le \sum_{j=1}^n \int_0^t \delta_{f_i} \quad i = 1, \ldots, n$$

for all $0 \le t \le a$ with equality on both sides when $t = a$.

PROOF. From § 9 it follows that Theorem (11.1) implies

$$\int_0^t \delta_{f_1+f_2} = \max\{ \int_E (f_1+f_2)\, d\mu^{\#} : E \in \Lambda^{\#} \ \&\ \mu^{\#}(E) = t\} \le \int_0^t \delta_{f_1} + \int_0^t \delta_{f_2}$$

and $\int_0^t \delta_{f_1+f_2} = \max\{ \int_E (f_1+f_2)\, d\mu^{\#} : \mu^{\#}(E) = t\}$

$$\ge \max\{ \int_E f_1 \, d\mu^{\#} : \mu^{\#}(E) = t\} + \int_0^t \delta_{f_2}(a-u)du = \int_0^t \delta_{f_1} + \int_0^t \delta_{f_2}(a-u)du .$$

The general case follows by induction.

(12.2) COROLLARY. If $f \in L^1(X, \mu)$ then $\int_0^t |\delta_f| \le \int_0^t \delta_{|f|}$ for all $0 \le t \le a$ with equality when $t = a$.

PROOF. $\int_0^t |\delta_f| = \int_0^t \delta_{f^+} + \int_0^t \delta_{f^-}(a-u)du \le \int_0^t \delta_{f^+ + f^-} = \int_0^t \delta_{|f|}$

using Theorems (5.5) and (12.1). We have equality when $t = a$ because $|f| \sim |\delta_f|$.

Along with (ii) of (7.1), (12.1) also yields the following.

(12.3) THEOREM. If (X, Λ, μ) is a finite m.s. with $a = \mu(X)$ and g is a bounded decreasing function on $[0, a]$, then the function $p(f) = \int_0^a \delta_f(u)g(u)du$ is sublinear on $L^1(X, \mu)$.

(12.4) LEMMA. If $f, g \in L^1(X, \mu)$ then

$$\left| \int_0^t \delta_f - \int_0^t \delta_g \right| \leq \int_0^t \delta_{|f-g|} \quad \text{for all} \quad 0 \leq t \leq a .$$

PROOF. For all $0 \leq t \leq a$, $\int_0^t \delta_f \leq \int_0^t \delta_g + \int_0^t \delta_{f-g} \leq \int_0^t \delta_g + \int_0^t \delta_{|f-g|}$ and $\int_0^t \delta_g \leq \int_0^t \delta_f + \int_0^t \delta_{g-f} \leq \int_0^t \delta_f + \int_0^t \delta_{|f-g|}$ so $-\int_0^t \delta_{|f-g|} \leq \int_0^t \delta_f - \int_0^t \delta_g \leq \int_0^t \delta_{|f-g|}$.

(12.5) COROLLARY. Suppose $f, f_n \in L^1(X_1, \mu_1)$ and $g, g_n \in L^1(X, \mu)$ and $g_n \prec f_n$, $n = 1, 2, 3 \ldots\ldots$ If $\|f_n - f\|_1 \to 0$ and $\|g_n - g\|_1 \to 0$ then $g \prec f$.

PROOF. $\left| \int_0^t \delta_{f_n} - \int_0^t \delta_f \right| \leq \int_0^t \delta_{|f_n - f|} \leq \int_0^a \delta_{|f_n - f|} = \|f_n - f\|_1$ for all $0 \leq t \leq a$, and similarly for g.

As in the case of sums, if $f, g \in L^1(X, \mu)$ then in general, $\delta_{fg} \nleq \delta_f \delta_g$ even if $0 \leq f, g$. For example, let E and F be disjoint sets of equal positive measure and let $f = C_E + 2C_F$ and $g = 2C_E + C_F$. Then $fg = 2\, C_{E \cup F}$, and $\delta_f = \delta_g = 2\, C_{[0, \mu(E)[} + C_{[\mu(E), 2\mu(E)[}$ so $\delta_f \delta_g = 4C_{[0, \mu(E)[} + C_{[\mu(E), 2\mu(E)[} \ngeq 2\, C_{[0, 2\mu(E)[} = \delta_{fg}$.

Recall, however, that if $0 \leq f \in M(X, \mu)$ and $E \in \Lambda$ then $\delta_{fC_E} \leq \delta_f \delta_E$. More generally, we have the following.

(12.6) THEOREM. (i) If $0 \le f_1, \ldots, f_n \in M(X,\mu)$ then $\int_o^t \delta_{f_1 \cdots f_n} \le \int_o^t \delta_{f_1} \cdots \delta_{f_n}$ for all $0 \le t \le a$, so in particular, $\int f_1 \cdots f_n \, d\mu \le \int_o^a \delta_{f_1} \cdots \delta_{f_n}$.

(ii) If $f_1, \ldots, f_n \in M(X,\mu)$ and $\delta_{|f_1|} \cdots \delta_{|f_n|} \in L^1[0,a]$ where $a = \mu(X)$, then for all $0 \le t \le a$ we have

$$\int_o^t |\delta_{f_1 \cdots f_n}| \le \int_o^t \delta_{|f_1 \cdots f_n|} \le \int_o^t \delta_{|f_1|} \cdots \delta_{|f_n|}.$$

PROOF. (i) There is an $E \in \Lambda^{\#}$ with $\mu^{\#}(E) = t$ s. t.

$$\int_o^t \delta_{f_1 f_2} = \int_E f_1 f_2 \, d\mu^{\#} = \int_{X^{\#}} (f_1 C_E)(f_2 C_E) \, d\mu^{\#} \le \int_o^a \delta_{f_1 C_E} \delta_{f_2 C_E}$$

$\le \int_o^a \delta_{f_1} \delta_E \delta_{f_2} \delta_E = \int_o^t \delta_{f_1} \delta_{f_2}$. The general case follows by induction using (7.1) (i). (ii) follows from (i) and (12.2).

13. $T_\mu f \prec f$ and Some Consequences.

(13.1) THEOREM. Let (X, Λ, μ) be a finite m. s. Then for all $f \in L^1(X^{\#}, \mu^{\#}) \cup M^+(X^{\#}, \mu^{\#})$ we have $T_\mu f \prec f$.

PROOF. Let $E \in \Lambda^{\#}$. $\int T_\mu C_E \, d\mu = \int C_E \, d\mu^{\#} = \mu^{\#}(E)$ by (9.2) (iv) and

$$T_\mu C_E = C_{X_0 \cap E} + \sum_{n \in P} \frac{m(E \cap I[a_n, b_n])}{m(I[a_n, b_n])}$$

so $0 \le T_\mu C_E \le 1$ and hence $T_\mu C_E \prec C_E$ by (8.2) (v).

Actually, for all $f \in L^1(X^{\#}, \mu^{\#})$ we have already that

$$\int_o^a \delta_{T_\mu f} = \int T_\mu f \, d\mu = \int f \, d\mu^{\#} = \int_o^a \delta_f \quad \text{using (9.2)(iv).}$$

Let f be a non-negative simple measurable function. Then f can be written in the form $f = \sum_{i=1}^{n} f_i C_{E_i}$ with $f_i > 0$ $i = 1, \ldots, n$

and $E_1 \supset \cdots \supset E_n$. For all $0 \le t \le a$,

$$\int_0^t \delta_{T_\mu f} \le \sum_{i=1}^n f_i \int_0^t \delta_{T_\mu C_{E_i}} \le \sum_{i=1}^n f_i \int_0^t \delta_{E_i} = \int_0^t \delta_f , \quad \text{so } T_\mu f \prec f.$$

Let $0 \le f \in L^1(X^\#, \mu^\#)$ be arbitrary. Then there is a sequence $\{f_n\}$ of non-negative simple functions s. t. $f_n \uparrow f$. From the definition of T_μ we see that $0 \le T_\mu f_n \uparrow T_\mu f$. Since we have $T_\mu f_n \prec f_n$ it follows from (8. 2)(viii) that $T_\mu f \prec f$.

If $f \in L^1(X^\#, \mu^\#)$ then $T_\mu f^+ \prec f^+$ and $T_\mu f^- \prec f^-$ so $-T_\mu f^- \prec -f^-$. For all $0 \le t \le a$ we then have

$$\int_0^t \delta_{T_\mu f} \le \int_0^t \delta_{T_\mu f^+} + \int_0^t \delta_{-T_\mu f^-} \le \int_0^t \delta_{f^+} + \int_0^t \delta_{-f^-} = \int_0^t \delta_f$$

so $T_\mu f \prec f$.

(13. 2) THEOREM. Let $(X, \Lambda\, \mu)$ and (X_1, Λ_1, μ_1) be finite m. s. such that $a = \mu(X) = \mu_1(X_1)$, and let $f \in M(X, \mu)$ and $g \in M(X_1, \mu_1)$.

(i) If $\delta_{|f|}\, \delta_{|g|} \in L^1[0, a]$ then

$$\{\int f g'\, d\mu : g' \in M(X,\mu) \;\&\; g' \prec g\} = \left[\int_0^a \delta_f(u)\delta_g(a-u)du, \int_0^a \delta_f \delta_g \right].$$

(ii) If $0 \le f, g$ then $\max\{\int f g'\, d\mu : g' \prec g\} = \int_0^a \delta_f\, \delta_g$ in the sense that both are infinite or are finite and equal.

PROOF. (i) According to Theorem (5. 10) there is a $g_0 \in M(X^\#, \mu^\#)$ such that $\delta_{g_0} = \delta_g$. Let I be the interval in question. Then $I \supset \{\int f g'\, d\mu : g' \prec g\} \supset \{\int f\, T_\mu g'\, d\mu : g' \sim g,\ g' \in M(X^\#, \mu^\#)\}$ $= \{\int f g'\, d\mu^\# : g' \sim g_0\} = I$ using (7. 1), (13. 1), (9. 2) and (11. 1).

(ii) Follows from (11. 2) in a manner similar to (i).

EXAMPLE. If $n \geq 0$ is an integer and $\lambda \geq 1$, find

$$\sup\{\int_0^1 x^n f(x)dx: \ 0 \leq f \leq \lambda \ \text{and} \ \int_0^1 f(x)dx = 1\}.$$

Now $0 \leq f \leq \lambda$ and $\int_0^1 f = 1$ iff $0 \leq \frac{1}{\lambda} f \leq 1$ and $\int_0^1 \frac{1}{\lambda} f = \frac{1}{\lambda}$ iff $\frac{1}{\lambda} f < C_{[0,\frac{1}{\lambda}]}$, using (8.2) (v). Thus the sup is the same as

$$\sup\{\int_0^1 x^n f(x)dx: \ f < C_{[0,\frac{1}{\lambda}]}\} = \int_0^1 \delta_{x^n} \, \delta_\lambda \, C_{[0,\frac{1}{\lambda}]} = \lambda \int_0^{\frac{1}{2}} (1-x)^n \, dx,$$

using (13.2) and (5.3) (v).

IV. REARRANGEMENT INVARIANT BANACH FUNCTION SPACES

14. Rearrangement Invariant Banach Function Spaces. Extensive studies have been made of Banach function spaces and more generally normed Riesz spaces (see [27] and [29]). In this chapter we review some rearrangement invariant properties of such spaces when the underlying measure space is finite. The treatment closely parallels Luxemburg in [28]. Rearrangement invariant Banach spaces will be the setting in which we will work in future sections.

(14.1) DEFINITION. Let (X, Λ, μ) be a finite m. s. and let $M^+=M^+(X,\mu)$ denote the set of all non-negative extended real valued measurable functions on (X, Λ, μ). A mapping $\rho: M^+ \to R^{\#}$ is called a function norm if it has the following two properties.

(i) $0 \leq \rho(f) \leq +\infty$ for all $f \in M^+$ and $\rho(f) = 0$ iff $f = 0$ μ-a.e.

(ii) $\rho(f+g) \leq \rho(f) + \rho(g)$ for all $f, g \in M^+$

$\rho(af) = a\,\rho(f)$ for all $f \in M^+$ and $a \geq 0$.

$f \leq g \Rightarrow \rho(f) \leq \rho(g)$ for all $f, g \in M^+$.

In addition, ρ is said to have the sequential Fatou property and is called a Fatou norm if it also satisfies.

(iii) $0 \leq f_n \uparrow f$ pointwise everywhere implies $\rho(f_n) \uparrow \rho(f)$.

We extend the domain of definition of a function norm ρ by defining $\rho(f) = \rho(|f|)$ for all $f \in M(X, \mu)$, and we denote by $L^\rho = L^\rho(X, \mu)$ the set of all $f \in M(X, \mu)$ such that $\rho(f) < \infty$. If μ-a.e equal members of L^ρ are identified as usual, then L^ρ is a normed linear space with the norm $\|f\| = \|f\|_\rho = \rho(f)$. These spaces are clearly generalizations of the classical Lebesgue and Orlicz spaces.

Unfortunately the hypotheses we have placed on ρ so far do not preclude the existence of a ρ-purely infinite set, i. e., a set A for which $\rho(C_B) = +\infty$ for all $B \subset A$ such that $\mu(B) > 0$. Of course, if $f \in M(X, \mu)$ and A is ρ-purely infinite, then $f\, C_A = 0$ μ - a. e. [29, Note IV, p. 251, Lemma 8. 2]. Fortunately it can be shown that X has a maximal ρ-purely infinite set X_∞, i. e., X_∞ is ρ-purely infinite and $X - X_\infty$ has no ρ-purely infinite subsets [Note IV, Theorem 8. 3]. We will assume in this and in subsequent sections that X_∞ has been removed from X so that our m. s. has no ρ-purely infinite subsets. In this case ρ is said to be <u>saturated.</u>

Given a function norm ρ we define the first and second associates of ρ as follows.

(14. 2) DEFINITION. <u>For all</u> $f \in M^+$

$$\rho'(f) = \sup\{ \int |fg|\, d\mu : \rho(g) \le 1\}$$

$$\rho''(f) = \sup\{ \int |fg|\, d\mu : \rho'(g) \le 1\}$$

It is not hard to show that ρ' and ρ'' have the sequential Fatou property [Note IV, p. 254, Theorem 9. 2], and that ρ is saturated implies ρ' is a norm [Note IV, Theorem 9. 7]. It is true but harder to show that in addition, ρ' is saturated [Note IV, p. 261, Corollary 11. 6]. Hence ρ'' is also a saturated Fatou norm. The spaces $L^{\rho'}$ and $L^{\rho''}$ are called the first and second associate spaces of L^ρ, respectively.

We have the following Hölder type inequality [Note IV, p. 261, Corollary 11.7].

(14.3) THEOREM. If $f \in L^{\rho}$ and $g \in L^{\rho'}$ then

$$\left| \int f g \, d\mu \right| \le \int |fg| \, d\mu \le \rho''(f)\rho'(g) \le \rho(f)\rho'(g) .$$

Note that $\rho'' \le \rho$. In the other direction the following has been shown. [Note IV, p. 259, Lemma 11.3].

(14.4) THEOREM. (Lorentz and Luxemburg) $\rho'' = \rho$ if and only if ρ has the sequential Fatou property.

The following converse of the Hölder inequality also holds. [Note V, p. 499, Corollary 14.2].

(14.5) THEOREM. (Lorentz and Wertheim). Suppose L^{ρ} is complete and $g \in M(X, \mu)$. Then $g \in L^{\rho'}$ iff $\int |fg| \, d\mu < \infty$ for all $f \in L^{\rho}$.

It has been shown that L^{ρ} is complete iff ρ has the following property called the Riesz-Fischer Property: $\sum \rho(f_n) < \infty$ implies $\rho(\sum |f_n|) < \infty$. [Note I, p. 143, Theorem 4.8]. In particular, L^{ρ} is complete whenever ρ is Fatou. [Note II, p. 149, Theorem 5.3]. Thus $L^{\rho'}$ and $L^{\rho''}$ are complete.

If $L^\rho = L^p$, $1 \le p < \infty$, then $L^{\rho'} = L^q$ where $p^{-1} + q^{-1} = 1$. In general $L^{\rho'}$ is a closed normal subspace of the Banach Dual $(L^\rho)^*$ [30, p. 153]. We have $L^{\rho'} = (L^\rho)^*$ iff $\rho(f_n) \downarrow 0$ whenever $f_n \downarrow 0$ everywhere. $G \in (L^\rho)^*$ is in $L^{\rho'}$ iff $G(f_n) \downarrow 0$ whenever $f_n \downarrow 0$, $f_n \in L^\rho$.

Since the m. s. is finite we will, in accordance with the Lebesgue and Orlicz space situation, assume in this and in subsequent sections that $L^\infty \subset L^\rho$, $L^{\rho'} \subset L^1$. This is easily seen to be equivalent to $\rho(C_X) < \infty$ and $\rho'(C_X) < \infty$. If ρ is Fatou, then $L^\infty \subset L^\rho \subset L^1$ iff $\rho(C_X) < \infty$ and $\rho'(C_X) < \infty$, in which case $L^\infty \subset L^{\rho'} \subset L^1$ also.

(14.6) DEFINITION. <u>A function norm</u> ρ <u>is called rearrangement invariant</u> (r. i.) <u>if</u> $f_1 \sim f_2$ <u>implies</u> $\rho(f_1) = \rho(f_2)$.

L^ρ <u>is called rearrangement invariant if</u> $f_1 \sim f_2 \in L^\rho$ <u>implies</u> $f_1 \in L^\rho$.

If ρ is rearrangement invariant then so is L^ρ, but the converse is not true. For example, let $\rho(f) = \int_0^1 |f(t)|dt + 2\int_1^2 |f(t)|dt$ if $f \in M[0,2]$. L^ρ is r. i. since $g \sim f \in L^\rho$ implies $\int_0^1 |g| + 2\int_1^2 |g| \le \int_0^1 \delta_{|g|} + 2\int_0^1 \delta_{|g|} = 3\int_0^1 \delta_{|f|} \le 3\int_0^2 \delta_{|f|} = 3\int_0^2 |f| < \infty$. However $\rho(C_{[0,1]}) = 1$ and $\rho(C_{[1,2]}) = 2$ even though $C_{[0,1]} \sim C_{[1,2]}$.

The following lemma is fundamental to what follows.

(14.7) LEMMA. Suppose (X, Λ, μ) is adequate (i.e. discrete or non-atomic) and ρ is a Fatou norm and let $a = \mu(X)$. Then L^ρ is rearrangement invariant iff $\delta_f\delta_g \in L^1[0, a]$ for all $0 \le f \in L^\rho$ and $0 \le g \in L^{\rho'}$, in which case $L^{\rho'}$ and $L^{\rho''}$ are rearrangement invariant iff L^ρ is.

PROOF. Suppose L^ρ is rearrangement invariant. Let $0 \le f \in L^\rho$ and $0 \le g \in L^{\rho'}$. If $f' \sim f$ then $f' \in L^\rho$ so $\int f' g \, d\mu \le \rho(f')\rho'(g) < \infty$. Hence $\delta_f\delta_g \in L^1[0, a]$ since μ is adequate.

Suppose $\delta_f\delta_g \in L^1[0, a]$ for all $0 \le f \in L^\rho$ and $0 \le g \in L^{\rho'}$. If $0 \le f' \sim f \in L^\rho$, $\int f' g \, d\mu \le \int_0^a \delta_f\delta_g < \infty$ so $f' \in L^{\rho''} = L^\rho$ using (13.4) and (13.5) since ρ is Fatou.

The rest follows by replacing ρ by ρ' and $L^{\rho''}$ by L^ρ.

(14.8) THEOREM. Suppose (X, Λ, μ) is adequate.

(i) If ρ is a Fatou norm and if L^ρ is rearrangement invariant then $0 \le g \ll f \in L^\rho$ implies $g \in L^\rho$ and hence $g < f \in L^\rho$ implies $g \in L^\rho$.

(ii) If L^ρ is rearrangement invariant, then $0 \le g \ll f \in L^{\rho'}$ implies $g \in L^{\rho'}$ and $g < f \in L^{\rho'}$ implies $g \in L^{\rho'}$.

PROOF. (i) Suppose $0 \le g \ll f \in L^\rho$. Then for all $0 \le h \in L^{\rho'}$ we have $\int g h \, d\mu \le \int_0^a \delta_g\delta_h \le \int_0^a \delta_f\delta_h < \infty$ using (10.1) and (i) of (7.1), so $g \in L^{\rho''} = L^\rho$. If $g < f \in L^\rho \subset L^1$ then (11.3) implies $|g| \ll |f| \in L^\rho$ so $|g| \in L^\rho$.

(ii) Follows from (i) since ρ' and ρ'' are Fatou norms, and L^ρ (r.i.) implies $L^{\rho'}$ and $L^{\rho''}$ are (r.i.).

(14. 9) DEFINITION. A function norm ρ is called universally rearrangement invariant (u. r. i.) if $f' \sim f \geq 0$ implies $\rho(T_\mu f') \leq \rho(f)$ for all $f' \in M(X^\#, \mu^\#)$.

L^ρ is called universally rearrangement invariant (u. r. i.) if $f' \sim f \in L^\rho$ implies $T_\mu f' \in L^\rho$ for all $f' \in M(X^\#, \mu^\#)$.

If (X, Λ, μ) is adequate (i. e. discrete or non-atomic) then we will see that ρ is (u. r. i.) iff it is (r. i.) and L^ρ is (u. r. i.) iff it is (r. i.). Our previous results generalize as follows.

(14. 10) THEOREM. (i) If L^ρ is (u. r. i.) then $\delta_f \delta_g \in L^1[0, a]$ for all $0 \leq f \in L^\rho$ and $0 \leq g \in L^{\rho'}$ where $a = \mu(X)$. If ρ is a Fatou norm, then the converse holds, and in that case $L^{\rho'}$ and $L^{\rho''}$ are (u. r. i.). iff L^ρ is (u. r. i.)

(ii) If ρ is a Fatou norm and L^ρ is (u. r. i.) then $0 \leq g \ll f \in L^\rho$ implies $g \in L^\rho$ and $g < f \in L^\rho$ implies $g \in L^\rho$.

(iii) If L^ρ is (u. r. i.) then $0 \leq g \ll f \in L^{\rho'}$ implies $g \in L^{\rho'}$ and $g < f \in L^{\rho'}$ implies $g \in L^{\rho'}$. Similarly for $L^{\rho''}$.

PROOF. (i) For all $f' \sim f$, $f' \in M(X^\#, \mu^\#)$, we have $\int f' g \, d\mu^\# = \int (T_\mu f') g \, d\mu \leq \rho(T_\mu f')\rho'(g) < \infty$, so $\delta_f \delta_g \in L^1[0, a]$ since $(X^\#, \mu^\#)$ is non-atomic.

Suppose ρ is Fatou and let $0 \leq f' \sim f \in L^\rho$, $f' \in M(X^\#, \mu^\#)$. Then $\int (T_\mu f') g \, d\mu = \int f' g \, d\mu^\# \leq \int_0^a \delta_f \delta_g < \infty$ for all $0 \leq g \in L^{\rho'}$ so $T_\mu f' \in L^{\rho''} = L^\rho$. The rest follows by replacing L^ρ by $L^{\rho'}$ and $L^{\rho''}$ by L^ρ.

(ii) and (iii) Follow as in (14. 8).

We now investigate rearrangement invariant (and (u. r. i)) norms.

(14. 11) THEOREM. Suppose (X, Λ, μ) is adequate and ρ is a rearrangement invariant norm and let $a = \mu(X)$. Then for all $0 \le f \in M(X, \mu)$ we have

$$\rho'(f) = \sup\{ \int_0^a \delta_f \, \delta_{|g|} : \rho(g) \le 1\}$$

$$\rho''(f) = \sup\{ \int_0^a \delta_f \, \delta_{|g|} : \rho'(g) \le 1\}$$

so ρ' and ρ'' are rearrangement invariant.

In addition, if ρ is a Fatou norm, then

$$\rho(f) = \sup\{ \int_0^a \delta_f \delta_{|g|} : \rho'(g) \le 1\}$$

for all $0 \le f \in M(X, \mu)$.

PROOF. Since (X, Λ, μ) is adequate, for each g with $\rho(g) \le 1$ there is a $g' \sim g$ such that $\int_0^a \delta_f \delta_{|g|} = \int f|g'|d\mu$. $g' \sim g$ implies $\rho(g') = \rho(g) \le 1$ since ρ is (r. i.) so

$$\{ \int_0^a \delta_f \delta_{|g|} : \rho(g) \le 1\} \subset \{ \int f|g|d\mu : \rho(g) \le 1\}.$$

But $\int f|g|d\mu \le \int_0^a \delta_f \, \delta_{|g|}$ so

$$\rho'(f) = \sup\{\int f|g|d\mu : \rho(g) \le 1\} = \sup\{ \int_0^a \delta_f \delta_{|g|} : \rho(g) \le 1 \}.$$

We get the formula for ρ'' by replacing ρ by ρ'. If ρ is Fatou, then $\rho = \rho''$, giving the last formula.

The following is a kind of converse which will be useful later.

(14.12) LEMMA. Let (X, Λ, μ) and (X', Λ', μ') be finite m. s. with $\mu(X) = \mu'(X') = a$. If $A \subset M(X', \mu')$ with $rC_{X'} \in A$ for some $r \neq 0$, then the mapping $0 \leq f \to \rho(f)$ defined on $M^+(X, \mu)$ by

$$\rho(f) = \sup\{ \int_o^a \delta_f \delta_{|g|} : g \in A \}$$

is a Fatou norm which is universally rearrangement invariant.

PROOF. $f = 0$ μ - a. e. implies $\rho(f) = 0$ is clear. If $\rho(f) = 0$ we take $g = r C_{X'}$ and obtain $\int_o^a \delta_f = 0$, so δ_f and hence $f = 0$ a. e. since $f \geq 0$. If $u \geq 0$, then $\delta_{uf} = u\, \delta_f$ so $\rho(uf) = u\, \rho(f)$. If $0 \leq f_1 \leq f_2$ then $\delta_{f_1} \leq \delta_{f_2}$ so $\rho(f_1) \leq \rho(f_2)$. To prove the triangle inequality we have from (12.1) that $\int_o^t \delta_{f_1+f_2} \leq \int_o^t (\delta_{f_1} + \delta_{f_2})$ for all $0 \leq t \leq a$, so if $\rho(f_1)$ & $\rho(f_2) < \infty$ then (7.1) implies that for each $g \in A$ we have $\int_o^a \delta_{f_1+f_2} \delta_{|g|} \leq \int_o^a \delta_{f_1} \delta_{|g|} + \int_o^a \delta_{f_2} \delta_{|g|}$ so $\rho(f_1+f_2) \leq \rho(f_1) + \rho(f_2)$. Since $0 \leq f_n \uparrow f$ everywhere implies $\int_o^a \delta_{f_n} \delta_{|g|} \uparrow \int_o^a \delta_f \delta_{|g|}$ for each $g \in A$, we have $\rho(f_n) \uparrow \rho(f)$ as in [29, Note II, p. 149, Theorem 5.4]. Hence ρ is a Fatou norm. To prove that ρ is (u. r. i.), let $0 \leq f \in M(X, \mu)$ and let $f' \sim f$ where $f' \in M(X^\#, \mu^\#)$. Then $0 \leq T_\mu f' << f$ so $\int_o^a \delta_{T_\mu f'} \delta_{|g|} \leq \int_o^a \delta_f \delta_{|g|}$ for each $g \in A$ and hence $\rho(T_\mu f') \leq \rho(f)$.

(14.13) THEOREM. <u>Suppose</u> (X, Λ, μ) <u>is adequate and</u> ρ <u>is a rearrangement invariant norm.</u> <u>If</u> ρ <u>is also Fatou, then</u>

(i) $0 \leq g \ll f$ <u>implies</u> $\rho(g) \leq \rho(f)$.

(ii) $g \prec f \in L^1(X, \mu)$ <u>implies</u> $\rho(g) \leq \rho(f)$.

<u>Whether</u> ρ <u>is Fatou or not,</u> ρ' <u>and</u> ρ'' <u>satisfy (i) and (ii).</u>

If (X, Λ, μ) is not adequate, then Theorem (14.13) may not be true. For example, suppose $X = X_0 \cup A$ where A is an atom, X_0 is non-atomic, $0 < \mu(X_0) < \mu(A)$, and $\mu(A) > 1$. Let $\rho(h) = \int_{X_0} |h| d\mu + |h C_A|$. Pick $\varepsilon > 0$ such that $0 < 1-\varepsilon < \mu(X_0) \leq (1-\varepsilon)\mu(A)$, and let $f = C_A$ and $g = C_{X_0} + \varepsilon C_A$. Then ρ is a Fatou rearrangement invariant norm and $0 \leq g \ll f$, but $\rho(f) = 1 < \mu(X_0) + \varepsilon = \rho(g)$.

For general m.s. our previous results take the following form.

(14.14) THEOREM. <u>If</u> ρ <u>is a universally rearrangement invariant norm, then the following four statements are true.</u>

(i) <u>For all</u> $0 \leq f \in M(X, \mu)$ <u>we have</u>

$$\rho'(f) = \sup \{ \int_0^a \delta_f \delta_{|g|} : \rho(g) \leq 1 \} \quad \text{and}$$

$$\rho''(f) = \sup \{ \int_0^a \delta_f \delta_{|g|} : \rho'(g) \leq 1 \}, \quad \text{where}$$

$a = \mu(X)$. <u>If</u> ρ <u>is in addition Fatou, then</u>

$$\rho(f) = \sup \{ \int_0^a \delta_f \delta_{|g|} : \rho'(g) \leq 1 \}$$

<u>for all</u> $0 \leq f \in M(X, \mu)$

(ii) ρ' and ρ'' are universally rearrangement invariant.

(iii) $0 \le g << f$ implies $\rho'(g) \le \rho'(f)$ and $\rho''(g) \le \rho''(f)$, and similarly for ρ if ρ is Fatou.

(iv) $g < f \in L^1(X, \mu)$ implies $\rho'(g) \le \rho'(f)$ and $\rho''(g) \le \rho''(f)$, and similarly for ρ if ρ is Fatou.

PROOF. (i) For each g with $\rho(g) \le 1$ there is a $g' \in M(X^{\#}, \mu^{\#})$ such that $g' \sim g$ and $\int_0^a \delta_f \delta_{|g|} = \int f|g'|\, d\mu^{\#} = \int f\, T_\mu |g'| d\mu$. $g' \sim g$ implies $|g'| \sim |g|$ so $\rho(T_\mu |g'|) \le \rho(|g|) \le 1$ and thus

$\{\int_0^a \delta_f \delta_{|g|} : \rho(g) \le 1\} \subset \{ \int f|g|d\mu : \rho(g) \le 1 \}$. Then

$$\rho'(f) = \sup\{ \int f|g|d\mu : \rho(g) \le 1\} = \sup\{ \int_0^a \delta_f \delta_{|g|} : \rho(g) \le 1\}$$

as in (14.11).

(ii) This follows from (i) since $0 \le f \sim f' \in M(X^{\#}, \mu^{\#})$ implies $T_\mu f' < f$ implies $T_\mu f' \ge 0$ (8.2), and $\int_0^a \delta_{T_\mu f'} \delta_{|g|} \le \int_0^a \delta_f \delta_{|g|}$.

(iii) & (iv) are immediate using the representations in (i).

15. A Representation Theorem. The rearrangement invariant spaces such as the classical Lebesgue spaces, the Orlicz spaces, and the spaces introduced by Halperin and Lorentz (see [14], [23], [24]), Boyd [2] and Shimogaki [45] are all of the following kind.

Let (X, Λ, μ) be a finite m. s. with $a = \mu(X)$ and let $M^+[0, a]$ denote the set of all non-negative extended real valued Lebesgue measurable functions on $[0, a]$.

(15.1) LEMMA. If λ is a Fatou rearrangement invariant norm defined on $M^+[0,a]$ such that $L^\infty \subset L^\lambda \subset L^1$, then the mapping $0 \le f \to \rho(f) = \lambda(\delta_f)$ is a Fatou norm which is universally rearrangement invariant, and $\rho'(f) = \lambda'(\delta_f)$ for all $0 \le f \in M(X,\mu)$. Furthermore $L^\infty \subset L^\rho$, $L^{\rho'} \subset L^1$.

PROOF. Since Lebesgue measure is non-atomic, (14.11) implies that for all $0 \le F \in M[0,a]$ we have

$$\lambda(F) = \sup\{ \int_0^a \delta_F \, \delta_{|G|} : \lambda'(G) \le 1\} \qquad \text{so}$$

$$\rho(f) = \sup\{ \int_0^a \delta_f \, \delta_{|G|} \quad \lambda'(G) \le 1\} \; .$$

To show that ρ is a (u.r.i.) Fatou norm we have by (14.12) only to show that $\lambda'(r\, C_{[0,a]}) \le 1$ for some $r \neq 0$. If $\lambda'(C_{[0,a]}) = 0$, take $r = 1$. Otherwise, let $r = 1/\lambda'(C_{[0,a]})$.

Finally we show that $\rho'(f) = \lambda'(\delta_f)$. For each G with $\lambda(G) \le 1$ there is a $g' \in M(X^\#, \mu^\#)$ such that $g' \sim |G|$ and $\int_0^a \delta_f \delta_{|G|} = \int f g' \, d\mu^\#$. But $\int f g' \, d\mu^\# = \int f \, T_\mu g' \, d\mu$ and $\rho(T_\mu g') = \lambda(\delta_{T_\mu g'}) \le \lambda(|G|) \le 1$ since $0 \le T_\mu g' << |G|$, so

$\{ \int_0^a \delta_f \, \delta_{|G|} : \lambda(G) \le 1\} \subset \{ \int f|g|d\mu : \rho(g) \le 1\}$ and thus $\lambda'(\delta_f) \le \rho'(f)$.

For the rest, if $\rho(g) \le 1$, then $\int f|g|d\mu \le \int_0^a \delta_f \, \delta_{|g|}$ and

$\lambda(\delta_{|g|}) = \rho(g) \le 1$ so $\rho'(f) \le \lambda'(\delta_f)$. $\rho(C_X) = \lambda(C_{[0,a]})$ &

$\rho'(C_X) = \lambda'(C_{[0,a]})$ shows the rest.

(15.2) THEOREM. If (X, Λ, μ) is a finite m. s. and ρ is a Fatou norm, then ρ is universally rearrangement invariant (rearrangement invariant if (X, Λ, μ) is adequate) if and only if there is a Fatou rearrangement invariant norm λ on $M^+[0, a]$, where $a = \mu(X)$, such that $\rho(f) = \lambda(\delta_f)$ for all $0 \leq f \in M(X,\mu)$.

PROOF. It only remains to prove the existence of λ. By (14.14) we have that for all $0 \leq f \in M(X, \mu)$ $\rho(f) = \sup\{ \int_0^a \delta_f \delta_{|g|} : \rho'(g) \leq 1\}$. For every $0 \leq F \in M[0, a]$ we define $\lambda(F) = \sup\{ \int_0^a \delta_F \delta_{|g|} : \rho'(g) \leq 1\}$. (14.12) implies that λ is a (r.i.) Fatou norm. Clearly $\rho(f) = \lambda(\delta_f)$ for all $0 \leq f \in M(X, \mu)$.

V. INEQUALITIES OF HARDY, LITTLEWOOD, PÓLYA AND MUIRHEAD

16. Schur Convex Functions. If (X, Λ, μ) is a finite m. s. consisting of n atoms of equal measure, then each member of $M(X, \mu)$ may be identified with a point $\vec{x} = (x_1, \ldots, x_n)$ of R^n, in which case its decreasing rearrangement, denoted by $\vec{x}^*$, is the point obtained by rearranging the components of $\vec{x}$ in decreasing order. If $\vec{x}, \vec{y} \in R^n$ then the definition of $\vec{y} \prec \vec{x}$ assumes the form

$$\sum_{i=1}^{k} y_i^* \leq \sum_{i=1}^{k} x_i^* \qquad 1 \leq k \leq n$$

with equality when k = n. It was in this form that the relation $\prec$ was first introduced by Hardy, Littlewood, and Pólya.

Let S_n denote the symmetric group of all permutations of $\{1, \ldots, n\}$. For each $\vec{x} \in R^n$ let $\Delta(x) = \{y \in R^n : y \sim x\} = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \sigma \in S_n\}$. Recall that an n×n matrix $A = [a_{ij}]$ is said to be doubly stochastic (d. s.) if $a_{ij} \geq 0$ and

$$\sum_{i=1}^{n} a_{ij} = 1 = \sum_{j=1}^{n} a_{ij} \text{ for all } i, j = 1, \ldots, n.$$

(16.1) THEOREM. (Hardy, Littlewood and Pólya). The following are equivalent for $\vec{x}, \vec{y} \in R^n$.

(i) $\vec{y} \prec \vec{x}$

(ii) $\sum_{i=1}^{n} \varphi(y_i) \leq \sum_{i=1}^{n} \varphi(x_i)$ for all convex functions φ on R.

(iii) $\vec{y}$ is in the convex hull of $\Delta(\vec{x})$.

(iv) There is a doubly stochastic matrix A such that $\vec{y} = A\vec{x}$.

An interesting discussion of this result and of the theory of doubly stochastic transformations can be found in a paper by L. Mirsky [31].

(16.2) THEOREM. (R. Muirhead) Let $\vec{x}, \vec{y} \in R^n$. Then $\vec{y} \prec \vec{x}$ iff for all positive $\vec{a} \in R^n$ we have

$$\sum_{\sigma \in S_n} a_{\sigma(1)}^{y(1)} \cdots a_{\sigma(n)}^{y(n)} \leq \sum_{\sigma \in S_n} a_{\sigma(1)}^{x(1)} \cdots a_{\sigma(n)}^{x(n)} .$$

Equality holds iff $\vec{a}$ is constant or $\vec{y} \sim \vec{x}$.

It will be our purpose in the next few sections to show in what sense these two theorems can be extended to functions in $L^\rho(X, \Lambda, \mu)$, where (X, Λ, μ) is a finite m. s., $L^\infty \subset L^\rho$, $L^{\rho'} \subset L^1$, ρ is a saturated Fatou norm, and L^ρ is universally rearrangement invariant. The generalization to $L^1[0, 1]$ of the equivalence of (i), (iii) and (iv) and Theorem (16.2) was given by Ryff ([39], [40], [42]). The generalization of the equivalence of (i) & (ii), was given for $L^\infty(X, \Lambda, \mu)$ by Grothendieck [10]. The generalization of the equivalence of (i), (ii) and (iii) was given for $L^\rho(X, \mu)$ independently by Luxemburg [28].

(16.3) DEFINITION. A mapping $\Phi: L^\rho \to R^\#$ which satisfies the following properties is called a Schur Convex function.

(i) $-\infty < \Phi(f) \leq +\infty$ for all $f \in L^\rho$, and $\Phi(f) < +\infty$ for some $f \in L^\rho$.

(ii) Φ is convex, i.e., $\Phi(rf_1+(1-r)f_2) \leq r\,\Phi(f_1)+(1-r)\Phi(f_2)$ for all $f_1, f_2 \in L^\rho$ and $0 \leq r \leq 1$.

(iii) Φ is $\sigma(L^\rho, L^{\rho'})$ - lower semi-continuous, i.e., $\{f: \Phi(f) \leq r\}$ is $\sigma(L^\rho, L^{\rho'})$ - closed for all real r.

(iv) Φ is rearrangement invariant, i.e., $f_1 \sim f_2$ implies $\Phi(f_1) = \Phi(f_2)$ for all $f_1, f_2 \in L^\rho$.

If Φ is convex, then for all real r, the set $\{f: \Phi(f) \leq r\}$ is convex. Hence it is $\sigma(L^\rho, L^{\rho'})$ closed iff it is closed in each locally convex topology on L^ρ in which $L^{\rho'}$ is the dual of L^ρ. In particular the topology $|\sigma|(L^\rho, L^{\rho'})$ generated by the seminorms $P_g(f) = \int |fg|\,d\mu$, $f \in L^\rho$, $g \in L^{\rho'}$ is a locally convex topology on L^ρ in which $L^{\rho'}$ is the dual of L^ρ, and thus a Φ satisfying (i) and (ii) satisfies (iii) iff: $\int |f_n - f|\,|g|\,d\mu \to 0$ as $n \to \infty$ for all $g \in L^{\rho'}$ implies $\Phi(f) \leq \liminf_{n \to \infty} \Phi(f_n)$. In particular, such a Φ satisfies (iii) whenever: $|f_n| \leq f_o \in L^\rho$ and $f_n \to f$ μ-a.e. implies $\Phi(f) \leq \liminf_{n \to \infty} \Phi(f_n)$.

We would like to prove that $g \prec f$ iff $\Phi(g) \leq \Phi(f)$ for all Schur Convex functions Φ. However it is clear that we cannot prove this for general m.s. because any rearrangement invariant Fatou norm ρ_1 is Schur convex, and we have seen that unless (X, Λ, μ) is adequate, we cannot guarantee that $g \prec f$ implies $\rho_1(g) \leq \rho_1(f)$.

In order to alleviate this problem we introduced the concept of a universally rearrangement invariant norm. The same idea works here also.

(16.4) DEFINITION. A mapping $\Phi: L^\rho \to R^\#$ is said to be a universal Schurconvex function (u.s.c.) if it satisfies properties (i), (ii) and (iii) of (16.3) and (iv)' Φ is universally rearrangement invariant, i.e., $\Phi(T_\mu f') \leq \Phi(f)$ whenever $f \in L^\rho$, $f' \in L^\rho(X^\#, \mu^\#)$ and $f' \sim f$.

If Φ is universal Schur convex then it is Schur convex. The next lemma will imply (among other things) that if (X, Λ, μ) is adequate, then Φ is u.s.c. whenever it is Schur convex.

EXAMPLES. (i) If φ is a real convex function on R then $\Phi(f) = \int \varphi(f)\, d\mu$ is Schur convex on L^∞. If in addition φ is increasing then $\Phi_t(f) = \int_o^t \varphi(\delta_f) = \int_o^t \delta_{\varphi(f)}$ is, for each $0 \leq t \leq \mu(X)$, Schur convex on L^∞. If $\liminf_{u \to -\infty} \varphi(u)/u$ is finite, then Φ and Φ_t are Schur convex on L^ρ.

(ii) $\Phi_1(f) = \int f\, d\mu$ and $\Phi_2(f) = \int(-f) d\mu$ are u.s.c. on L^ρ.

(iii) For each $0 \leq t \leq \mu(X)$, $\Phi_t(f) = \int_o^t \delta_f$ is u.s.c.

(iv) If $g \in L^{\rho'}$ and $a = \mu(X)$ then $\Phi_g(f) = \int_o^a \delta_f \delta_g + b$ is universal Schur convex on L^ρ where $b \in R$.

(v) [Ryff] $\Phi(f) = \int_o^a \int_o^t \delta_f(s) ds dt = \int_o^a \delta_f(s)(a-s) ds$ is u.s.c. on L^ρ and $\Phi(f) = \Phi(g)$ iff $f \sim g$.

(vi) If Φ is a Fatou rearrangement invariant function norm, then Φ is Schur convex on L^{ρ}. (see [30], Theorem 3.1, p. 162).

(vii) The supremum of a family of Schur convex [Universal Schur Convex] functions is Schur Convex [Universal Schur Convex].

(16.5) LEMMA. If Φ is universal Schur convex, or if (X, Λ, μ) is adequate and Φ is Schur convex, then there are functions $g_i \in L^{\rho'}$ and real numbers b_i such that

$$\Phi(f) = \sup_i \{ \int_0^a \delta_f \delta_{g_i} + b_i \} \text{ for all } f \in L^{\rho}.$$

If in addition Φ is increasing, then

$$\Phi(f) = \sup_i \{ \int_0^a \delta_f \delta_{|g_i|} + b_i \} \text{ for all } f \in L^{\rho}.$$

PROOF. It is well known that if Φ is convex and lower semi-continuous, then there are continuous linear functionals L_i and real numbers b_i such that $\Phi(f) = \sup_i \{L_i(f) + b_i\}$, the L_i being non-negative if Φ is increasing [1]. For each i, there is a $g_i \in L^{\rho'}$ s.t. $L_i(f) = \int f g_i \, d\mu$ for all $f \in L^{\rho}$, and $g_i \geq 0$ if L_i is non-negative, so

$$\Phi(f) = \sup_i \{\int f g_i \, d\mu + b_i\} \leq \{ \int_0^a \delta_f \delta_{g_i} + b_i \} .$$

Suppose Φ is u.s.c. and let $f \in L^{\rho}$. For each i there is an $f' \sim f$ with $f' \in L^{\rho}(X^{\#}, \mu^{\#})$ such that $\int f' g_i \, d\mu^{\#} = \int_0^a \delta_f \delta_{g_i}$. Then

$$\begin{aligned} \Phi(f) \geq \Phi(T_{\mu} f') &\geq \int g_i T_{\mu} f' \, d\mu + b_i = \int f' g_i \, d\mu^{\#} + b_i \\ &= \int_0^a \delta_f \delta_{g_i} + b_i \quad \text{and this holds for all i so} \end{aligned}$$

$\Phi(f) \geq \sup\{\int_0^a \delta_f \delta_{g_i} + b_i\}$.

The proof when (X, Λ, μ) is adequate and Φ is Schur Convex is similar.

REMARK. Luxemburg has observed that if we let $Y(g) = \sup\{ \int fg\, d\mu - \Phi(f): f \in L^\rho\}$ then the conclusion of the lemma assumes the form

$$\Phi(f) = \sup\{ \int_0^a \delta_f \delta_g - Y(g): g \in L^{\rho'}\} .$$

(16.6) THEOREM. (i) Suppose (X, Λ, μ) is any finite m. s. and $f_1, f_2 \in L^\rho(X, \mu)$.

1. $f_1 \ll f_2$ iff $\Phi(f_1) \leq \Phi(f_2)$ for all increasing universal Schur convex functions Φ on L^ρ.

2. $f_1 \prec f_2$ iff $\Phi(f_1) \leq \Phi(f_2)$ for all u. s. c. functions Φ on L^ρ.

(ii) Suppose (X, Λ, μ) is adequate and $f_1, f_2 \in L^\rho$.

1. $f_1 \ll f_2$ iff $\Phi(f_1) \leq \Phi(f_2)$ for all increasing Schur convex functions Φ on L^ρ.

2. $f_1 \prec f_2$ iff $\Phi(f_1) \leq \Phi(f_2)$ for all Schur convex functions Φ on L^ρ.

PROOF. Lemma (16.5) in conjunction with the Hardy inequalities (7.1) shows that $\Phi(f_1) \leq \Phi(f_2)$ whenever $f_1 \prec f_2$, or whenever $f_1 \ll f_2$ and Φ is increasing.

If $\Phi(f_i) \leq \Phi(f_2)$ for all increasing Schur convex Φ, then since $f \to \int_0^t \delta_f$ is increasing and u. s. c. for each $0 \leq t \leq \mu(X)$, we have

$f_1 << f_2$. If $\Phi(f_1) \leq \Phi(f_2)$ for all Schur convex Φ, then since $f \to \int(-f)d\mu$ is universal Schur convex we have $f_1 < f_2$.

Recall that if ρ is a Fatou u. r. i. norm, then there is a Fatou r. i. norm λ on $M^+[0, a]$ such that $\rho(f) = \lambda(\delta_f)$.

(16.7) THEOREM. Suppose $\rho(f) = \lambda(\delta_f)$ where λ is a Fatou rearrangement invariant norm. Then Φ is a universal Schur convex function on L^ρ iff there is a Schur convex function Φ_o on $L^\lambda[0, a]$ such that $\Phi(f) = \Phi_o(\delta_f)$.

PROOF. If Φ is u. s. c. on L^ρ then there are $g_i \in L^{\rho'}$ and $b_i \in R$ such that for all $f \in L^\rho$, $\Phi(f) = \sup_i \{ \int_o^a \delta_f \delta_{g_i} + b_i \}$. Since $\rho'(g) = \lambda'(\delta_g)$ for all $g \in L^{\rho'}$ we see that $\delta_{g_i} \in L^{\lambda'}[0, a]$ for each i. Hence define $\Phi_o(F) = \sup_i \{ \int_o^a \delta_F \delta_{g_i} + b_i \}$ for $F \in L^\lambda[0, a]$. Then Φ_o is Schur convex and $\Phi_o(\delta_f) = \Phi(f)$.

Conversely, if Φ_o is Schur convex on $L^\lambda[0, a]$ then there are $G_i \in L^{\lambda'}[0, a]$ and $b_i \in R$ s. t. $\Phi_o(F) = \sup_i \{ \int_o^a \delta_F \delta_{G_i} + b_i \}$ for all $F \in L^\lambda[0, a]$. But $f \in L^\rho$ iff $\delta_f \in L^\lambda[0, a]$ so for all $f \in L^\rho$, $\Phi(f) = \Phi_o(\delta_f) = \sup_i \{ \int_o^a \delta_f \delta_{G_i} + b_i \}$ and thus Φ is universal Schur convex.

17. The Sets $\Omega(f) = \{f' : f' < f\}$. Recall that we are assuming that ρ is a saturated Fatou norm on $M^+(X, \mu)$, $L^\rho(X, \mu)$ is u. r. i., $L^\infty(\mu) \subset L^\rho(\mu)$, $L^{\rho'}(\mu) \subset L^1(\mu)$, and $a = \mu(X)$.

The equivalence $\vec{y} < \vec{x}$ iff $\vec{y} \in$ the convex hull of $\Delta(\vec{x})$ can be reformulated: $\{\vec{y}: \vec{y} < \vec{x}\}$ = the closed convex hull of $\{\vec{y}: \vec{y} \sim \vec{x}\}$. If $f \in L^{\rho}(X, \mu)$ we let $\Omega(f) = \{f' \in L^1(X,\mu): f' < f\}$, and $\Delta(f) = \{f' \in L^1(X, \mu): f' \sim f\}$. Since L^{ρ} is u. r. i., $f' < f \in L^{\rho}$ implies $f' \in L^{\rho}$ and hence both $\Omega(f)$ & $\Delta(f) \subset L^{\rho}$. $\Omega(f)$ has a smallest element $\overline{f} = (\frac{1}{\mu(X)} \int f\, d\mu)\, C_X$ in the sense that $h < f'$ for all $f' \in \Omega(f)$ iff $h = \overline{f}$. This follows from (8.2)(vi). Also $\Omega(f)$ is contained in the hyperplane $\{f' \in L^1: \int f'\, d\mu = \int f\, d\mu\}$, and $\overline{f}$ is equidistant (in the L^1 norm) from the members of $\Delta(f)$.

(17.1) PROPOSITION. Any $\sigma(L^{\rho}, L^{\rho'})$-bounded set $A \subset L^{\rho}$ is ρ-bounded.

PROOF. Since A is $\sigma(L^{\rho}, L^{\rho'})$-bounded, for each $g \in L^{\rho'}$ we have $\sup\{|\int f' g\, d\mu|: f' \in A\} < \infty$. For each $f' \in A$, $L_{f'}(g) = \int f' g\, d\mu$ defines a ρ'-continuous linear functional on $L^{\rho'}$ with $\|L_{f'}\| = \rho''(f') = \rho(f')$ [29, Note IV, p. 257]. Then for each $g \in L^{\rho'}$, $\sup\{|L_{f'}(g)|: f' \in A\} < \infty$, and since $L^{\rho'}$ is complete, the Banach Steinhaus Theorem implies $\sup\{\|L_{f'}\|: f' \in A\} = \sup\{\rho(f'): f' \in A\} < \infty$.

(17.2) COROLLARY. If $\{f_{\alpha}\}$ is a net in L^{ρ} which is $\sigma(L^{\rho}, L^{\rho'})$ convergent to $f_o \in L^{\rho}$ then $\{f_{\alpha}\}$ is ρ-bounded.

(17.3) THEOREM. For all $f \in L^{\rho}$, $\Omega(f)$ is ρ-bounded.

PROOF. We have only to show that $\Omega(f)$ is $\sigma(L^{\rho}, L^{\rho'})$ bounded. This follows at once since for each $g \in L^{\rho'}$ we have

$$\sup\{|\int f' g\, d\mu|: f' \in \Omega(f)\} \le \sup\{\int |f' g|\, d\mu: f' \in \Omega(f)\} \le \int_o^a \delta_{|f|}\, \delta_{|g|} < \infty .$$

(17.4) THEOREM. If $f \in L^{\rho}$ then $\Omega(f)$ is a convex and $\sigma(L^{\rho}, L^{\rho'})$ compact subset of L^{ρ}.

PROOF. Let $f \in L^{\rho}$, let $f_1, f_2 \in \Omega(f)$ and let $0 \le r \le 1$. Then (12.1) and (5.3)(vi) imply that for all $0 \le t \le a$, $\int_0^t \delta_{rf_1+(1-r)f_2} \le \int_0^t r\,\delta_{f_1} + \int_0^t (1-r)\delta_{f_2} \le \int_0^t \delta_f$ so $rf_1 + (1-r)f_2 \in \Omega(f)$. Hence $\Omega(f)$ is convex.

Let $\{f_\alpha\}$ be a net in $\Omega(f)$. For each α, $F_\alpha(g) = \int f_\alpha\, g\, d\mu$ defines a ρ'-continuous linear functional on $L^{\rho'}$ with $\|F_\alpha\| = \rho''(f_\alpha) = \rho(f_\alpha)$ [29, Note IV, p. 257]. Since $\Omega(f)$ is ρ-bounded, there is a number $M > 0$ such that $\rho(f') \le M$ for all $f' \in \Omega(f)$, and thus $\|F_\alpha\| = \rho(f_\alpha) \le M$ for all α. Hence $\{F_\alpha\}$ is a net in a $\sigma((L^{\rho'})^*, L^{\rho'})$-compact set [Alaoglu's Theorem, 7, p. 424], so it has a convergent subnet $F_\beta \to F_0$ say. To show that $F_0 \in L^{\rho} = L^{\rho''}$ it is necessary and sufficient to show that $F_0(g_n) \downarrow 0$ whenever $g_n \downarrow 0$ [30, p. 155]. Now for each $g \in L^{\rho'}$ and for each α, $|F_\alpha(g)| = |\int f_\alpha\, g\, d\mu| \le \int |f_\alpha g|\, d\mu \le \int_0^a \delta_{|f|}\, \delta_{|g|}$ because $f_\alpha \prec f \in L^1$ implies $|f_\alpha| \prec\prec |f|$. Since the bound is independent of α we have $|F_0(g)| \le \int_0^a \delta_{|f|}\, \delta_{|g|}$ and this $\downarrow 0$ as $|g| \downarrow 0$. Hence $F_0 \in L^{\rho}$, i.e., there is an $f_0 \in L^{\rho}$ such that $F_0(g) = \int f_0\, g\, d\mu$ for all $g \in L^{\rho'}$. Since $F_\beta \to F_0$ we have $\int f_\beta\, g\, d\mu = F_\beta(g) \to F_0(g) = \int f_0\, g\, d\mu$ for all $g \in L^{\rho'}$, i.e., $\{f_\beta\}$ is $\sigma(L^{\rho}, L^{\rho'})$ convergent to f_0. Thus $\Omega(f)$ is $\sigma(L^{\rho}, L^{\rho'})$ compact.

Let V be a locally convex linear topological space and let V^* denote the dual of V, i.e. V^* is the collection of all continuous linear functionals F on V. For each $A \subset V$ and for each linear functional F on V let $F[A] = \{F(v): v \in A\}$ and let cov(A) denote the convex hull of A.

As a corollary of the Hahn-Banach Separation Theorem we have the following characterization of closed convex sets.

(17.5) PROPOSITION. Let K be a closed convex subset of V. Then for each $v \in V$, $v \in K$ iff $F(v) \le \sup F[K]$ for all $F \in V^*$.

PROOF. If $v \in K$, then clearly $F(v) \le \sup F[K]$ for all $F \in V^*$. Conversely, if $v \notin K$, then the Hahn-Banach Separation Theorem [36, p. 30, Cor 1] implies there is an $F \in V^*$ such that $F(v) \notin \overline{F[K]} \supset [\inf F[K], \sup F[K]]$ so either $\sup F[K] < F(v)$ or $\sup -F[K] < -F(v)$.

(17.6) THEOREM. Let $f \in L^\rho$. Then $f_1 \in \Omega(f)$ iff $\int f_1 g \, d\mu \le \int_0^a \delta_f \delta_g$ for all $g \in L^{\rho'}$.

PROOF. The continuous linear functionals on L^ρ with the $\sigma(L^\rho, L^{\rho'})$ topology are $F_g(f) = \int f g \, d\mu$ where $f \in L^\rho$, $g \in L^{\rho'}$. For each $g \in L^{\rho'}$ it follows from (13.2) that $\sup F_g [\Omega(f)] = \int_0^a \delta_f \delta_g$. Now use (17.5).

We may similarly give a criterion for deciding when a closed convex set is the closed convex hull of another set.

(17.7) PROPOSITION. Suppose K is a closed convex subset of V and $D \subset K$. Then K is the closed convex hull of D iff $\sup F[D] = \sup F[K]$ for all $F \in V^*$.

PROOF. Suppose the criterion holds. Then $\sup F[K] = \sup F[D] \leq \sup F[\overline{\text{cov}(D)}]$ for all $F \in V^*$. Since $\overline{\text{cov}(D)}$ is a closed convex set, (17.5) implies $K \subset \overline{\text{cov}(D)}$. But $\overline{\text{cov}(D)} \subset K$ since $D \subset K$ and K is closed and convex. Hence $K = \overline{\text{cov}(D)}$.

Suppose now $K = \overline{\text{cov}(D)}$. Let $v \in K$. Then there is a net $\{w_\alpha\}$ in $\text{cov}(D)$ such that $w_\alpha \to v$, so for each $F \in V^*$ we have $F(w_\alpha) \to F(v)$ and thus $\sup F[D] = \sup F[\text{cov}(D)] \geq \sup_\alpha F(w_\alpha) \geq F(v)$ since $F[D] = F[\text{cov}(D)]$.

In view of (17.7) we have immediately:

(17.8) THEOREM. Let $f \in L^\rho$ and let ξ be a topology on L^ρ in which $L^{\rho'}$ is the dual of L^ρ. Then $\Omega(f)$ is the ξ-closed convex hull of a set $D \subset \Omega(f)$ iff $\sup\{\int f' g \, d\mu : f' \in D\} = \int_0^a \delta_f \delta_g$ for all $g \in L^{\rho'}$.

(17.9) THEOREM. Let ξ be a topology on L^ρ in which $L^{\rho'}$ is the dual of L^ρ. In particular $\xi = \sigma(L^\rho, L^{\rho'})$ or if $\rho(f_n) \downarrow 0$ whenever $f_n \downarrow 0$ then $\xi =$ the ρ-topology are such topologies.

(i) $\Omega(f)$ is the ξ-closed convex hull of $\Delta(f) = \{f' \in L^1 : f' \sim f\}$ for all $f \in L^\rho$ iff (X, Λ, μ) is adequate.

(ii) For any finite m.s. (X, Λ, μ) and $f \in L^\rho$, $\Omega(f)$ is the ξ-closed convex hull of $\{T_\mu f' : f' \in L^1(X^\#, \mu^\#)$ and $f' \sim f\}$.

PROOF. (i) If (X, Λ, μ) is adequate then the result follows from (17.8) and (11.1). If (X, Λ, μ) is not adequate, then (11.5) shows that the condition in (17.8) fails. (ii) Follows from (17.8) and (9.1) (iv) & (vi).

As Luxemburg has pointed out, Theorem (17.9) answers the following question asked recently by Z. Nehari [32]: Let (X, Λ, μ) be a non-atomic finite m. s. and let $E \in \Lambda$ have positive measure. What is the smallest closed convex set $A \subset L^1(X, \mu)$ which contains all the functions C_F such that $\mu(F) = \mu(E)$? Obviously A is the closed convex hull of $\Delta(C_E)$ so $A = \Omega(C_E) = \{f \in L^1(X, \mu): 0 \leq f \leq 1$ and $\int f\, d\mu = \mu(E)\}$ using (8.2)(v).

We can define the sets Ω in a slightly more general way. Let (X_1, Λ_1, μ_1) and (X, Λ, μ) be finite m. s. such that $\mu_1(X_1) = \mu(X) = a$. If $f \in L^1(X_1, \mu_1)$ let $\Omega_f(X, \mu) = \{h \in L^1(X, \mu): h \prec f\}$ and let $\Delta_f(X, \mu) = \{h \in L^1(X, \mu): h \sim f\}$. Observe that Ω_f is never empty because it contains $\bar{f} = (\frac{1}{\mu_1(X_1)} \int f\, d\mu_1) C_X$, but it may happen that $\Delta_f = \emptyset$. This is the only interesting case, because if $\Delta_f \neq \emptyset$, then we are doing nothing new, since for each $f_o \in \Delta_f$ we have $\Omega_f = \Omega(f_o)$. Theorem (5.10) implies that $\Delta_f(X^{\#}, \mu^{\#}) \neq \emptyset$ for all $f \in L^1(X_1, \mu_1)$, so if $f_o \in \Delta_f(\mu^{\#})$ we then have $\Omega_f(X, \mu) \subset \Omega(f_o)$.

Now let ρ be a saturated Fatou norm on $M^+(X, \mu)$ such that $L^\infty(\mu) \subset L^\rho$, $L^{\rho'} \subset L^1(\mu)$ and L^ρ is u. r. i. The question arises: For what $f \in L^1(X_1, \mu_1)$ is $\Omega_f \subset L^\rho(X, \mu)$?

(17.10) PROPOSITION. Let $f \in L^1(X_1, \mu_1)$. Then $\Omega_{|f|} \subset L^\rho$ iff $\delta_{|f|}\,\delta_{|g|} \in L^1[0, a]$ for all $g \in L^{\rho'}$, in which case $\Omega_f \subset L^\rho$.

PROOF. If $\Omega_{|f|} \subset L^\rho$ then $f' < |f|$, $f' \in M(X, \mu)$ implies $0 \le f' \in L^1(X, \mu)$ so $f' \in \Omega_{|f|} \subset L^\rho$ and thus for all $g \in L^{\rho'}$ we have that $\int f' \, |g| \, d\mu$ is finite, and hence (13.2) says $\delta_{|f|}\,\delta_{|g|} \in L^1[0, a]$. Conversely, if $\delta_{|f|}\,\delta_{|g|} \in L^1[0, a]$ for all $g \in L^{\rho'}$ then $f' < f$ implies $|f'| << |f|$ so $\delta_{|f'|}\delta_{|g|} \in L^1[0, a]$ for all $g \in L^{\rho'}$ and hence $f' \in L^\rho$ by (13.2) & (14.5).

Observe that Theorems (17.3), (17.4), (17.6), (17.8) and (17.9) are true for Ω_f under the hypothesis that $\Omega_{|f|} \subset L^\rho$, because in view of (17.10), the proofs are practically the same.

In view of the condition $\Omega_{|f|} \subset L^\rho$ it is natural to wonder: Is there a norm ρ_1 on $M^+(X_1, \mu_1)$ such that $f \in L^{\rho_1}$ implies $\Omega_{|f|} \subset L^\rho$?

(17.11) THEOREM. Define the norm ρ_1 on $M(X_1, \mu_1)$ by

$$\rho_1(f) = \sup\{ \int_0^a \delta_{|f|}\delta_{|g|} : \rho'(g) \le 1 \}.$$

Then ρ_1 is a Fatou u.r.i. norm satisfying

(i) $L^\infty(\mu_1) \subset L^{\rho_1}$, $L^{\rho_1'} \subset L^1(\mu_1)$

(ii) $f' < f \in L^{\rho_1}$ implies $\rho(f') \le \rho_1(f)$

(iii) $\Omega_{|f|} \subset L^\rho$ iff $f \in L^{\rho_1}$.

PROOF. (i) $\rho_1(C_{X_1}) = \sup\{\int_o^a \delta_{|g|} : \rho'(g) \le 1\}$ $= \sup\{\int |g| d\mu : \rho'(g) \le 1\} = \rho(C_X) < \infty$. Since $0 < \rho'(C_X) < \infty$ let $g_1 = \frac{1}{\rho'(C_X)} C_X$. Then $\frac{1}{\rho'(C_X)} \int |f| d\mu_1 = \int_o^a \delta_{|f|}\delta_{|g_1|}$ $\le \sup\{\int_o^a \delta_{|f|}\delta_{|g|} : \rho'(g) \le 1\} = \rho_1(f)$ so $f \in L^{\rho_1}$ implies $f \in L^1(\mu_1)$. Since ρ_1 is Fatou, $L^\infty(\mu_1) \subset L^{\rho_1'} \subset L^1(\mu_1)$ also.

(ii) In order to apply the theorem of Hardy (7.1) we need to know that $\delta_{|f'|}\delta_{|g|} \in L^1[0,a]$ for all $g \in L^{\rho'}$ with $\rho'(g) \le 1$. Now $f' \prec f \in L^{\rho_1} \subset L^1(\mu_1)$ implies $\delta_{f'} \prec \delta_f \in L^1[0,a]$ so $\delta_{|f'|} \sim |\delta_{f'}| \ll |\delta_f| \in L^1[0,a]$ so $\delta_{|f'|} \in L^1[0,a]$ and hence $\delta_{|f'|}\delta_{|g|} \in L^1[0,a]$ for all $g \in L^\infty(\mu)$, so $\int_o^a \delta_{|f'|}\delta_{|g|} \le \int_o^a \delta_{|f|}\delta_{|g|}$ for all $g \in L^\infty(\mu)$.

Let $g \in L^{\rho'}(\mu)$. Then there is a sequence $\{g_n\} \subset L^\infty(\mu)$ such that $0 \le g_n \uparrow |g|$ so $\delta_{g_n} \uparrow \delta_{|g|}$ and hence the monotone convergence theorem implies

$$\int_o^a \delta_{|f'|}\delta_{|g|} = \lim \int_o^a \delta_{|f'|}\,\delta_{g_n} \le \lim \int_o^a \delta_{|f|}\,\delta_{g_n} = \int_o^a \delta_{|f|}\delta_{|g|} < \infty$$

so $\delta_{|f'|}\delta_{|g|} \in L^1[0,a]$. Hence $\rho(f') = \sup\{\int |f'g| d\mu : \rho'(g) \le 1\}$ $\le \sup\{\int_o^a \delta_{|f|}\delta_{|g|} : \rho'(g) \le 1\} = \rho_1(f)$.

(iii) Part (ii) shows that $f \in L^{\rho_1}$ implies $\Omega_{|f|} \subset L^\rho$. Suppose $\Omega_{|f|} \subset L^\rho$. Then $\delta_{|f|}\delta_{|g|} \in L^1[0,a]$ for all $g \in L^{\rho'}$

so $\max\{ \int f'|g|d\mu : f' \prec |f| \} = \int_0^a \delta_{|f|}\delta_{|g|} < \infty$ for all $g \in L^{\rho'}$ and

hence $\rho_1(f) = \sup\{\max\{ \int f'|g|d\mu : f' \prec |f|\} : \rho'(g) \le 1\}$

$= \sup\{\sup\{ \int f'|g|d\mu : \rho'(g) \le 1\} : f' \prec |f| \}$

$= \sup\{\rho(f') : f' \in \Omega_f \} < \infty$ because $\Omega_{|f|} \subset L^{\rho}$

implies Ω_f is ρ-bounded.

If (as in §15) λ is a Fatou r.i. norm on $M^+[0, a]$ such that $L^{\infty} \subset L^{\lambda}$, $L^{\lambda'} \subset L^1$ and $\rho(f) = \lambda(\delta_f)$ for $f \in M^+(X, \mu)$ then the natural norm to choose on $M^+(X_1, \mu_1)$ is $\rho_2(f) = \lambda(\delta_f)$. In this connection we have the following.

(17.12) PROPOSITION. Let λ, ρ, ρ_1, ρ_2 be as above.

(i) $L^{\infty}(\mu_1) \subset L^{\rho_2}$, $L^{\rho_2'} \subset L^1(\mu_1)$

(ii) $\rho_1 \le \rho_2$

(iii) $f' \prec f \in L^{\rho_2}$ implies $\rho(f') \le \rho_2(f)$

PROOF. (i) See (15.1).

(ii) $\rho_1(f) = \sup\{ \int_0^a \delta_{|f|}\delta_{|h|} : \lambda'(\delta_h) \le 1,\ h \in M(X, \mu) \}$

$\le \sup\{ \int_0^a \delta_{|f|}\delta_{|G|} : \lambda'(G) \le 1 \} = \rho_2(f)$.

(iii) Use (17.12)(ii).

18. Doubly Stochastic Transformations. Observe that (16.1) implies that an nXn matrix A is doubly stochastic iff $A\vec{x} \prec \vec{x}$ for all $\vec{x} \in R^n$. Let (X_1, Λ_1, μ_1) and (X, Λ, μ) be finite measure spaces with $\mu_1(X_1) = \mu(X) = a$.

(18.1) DEFINITION. A linear mapping $T: L^1(X_1, \mu_1) \to L^1(X, \mu)$ is called doubly stochastic (d. s.) iff $Tf \prec f$ for all $f \in L^1(X_1, \mu_1)$.

EXAMPLES. 1. If (X, Λ, μ) is a non-atomic m. s. and $\sigma: X \to [0, a]$ is measure preserving (m.p.) let $T_\sigma f = f \circ \sigma$. Then $T_\sigma: L^1[0, a] \to L^1(X, \mu)$ is d. s.

2. $T_\mu: L^1(X^\#, \mu^\#) \to L^1(X, \mu)$ is d. s.

3. $Tf = (\frac{1}{\mu_1(X_1)} \int f \, d\mu_1) C_X$ defines a d. s. $T: L^1(X_1, \mu_1) \to L^1(X, \mu)$.

(18.2) PROPOSITION. Let $T: L^1(\mu_1) \to L^1(\mu)$ be d. s. Then

(i) T is non-negative, i. e., $f \geq 0$ implies $Tf \geq 0$.

(ii) $TC_{X_1} = C_X$

(iii) $(Tf)^+ \leq Tf^+$, $(Tf)^- \leq Tf^-$, $|Tf| \leq T|f|$ for all $f \in L^1(\mu_1)$

(iv) $T: L^\infty(X_1, \mu_1) \to L^\infty(X, \mu)$

(v) $f \in L^1(\mu_1)$ implies $\|Tf\|_1 \leq \|f\|_1$

$f \in L^\infty(\mu_1)$ implies $\|Tf\|_\infty \leq \|f\|_\infty$

PROOF. (i) Follows from (8. 2)(iii).

(ii) Follows from (8. 2)(v).

(iii) Since $Tf = Tf^+ - Tf^-$ and Tf^+, $Tf^- \geq 0$ we have $(Tf)^+ \leq Tf^+$ and $(Tf)^- \leq (Tf)^-$. Hence $|Tf| = (Tf)^+ + (Tf)^- \leq Tf^+ + Tf^- = T|f|$.

(iv) & (v) Follow from $|Tf| \leq T|f| < |f|$.

(18. 3) LEMMA. <u>Suppose</u> $T: L^1(\mu_1) \to L^1(\mu)$ <u>is linear</u>.

(i) <u>If</u> $Tf^+ < f^+$ <u>and</u> $Tf^- < f^-$ <u>then</u> $Tf < f$.

(ii) T <u>is d. s. iff</u> $Tf < f$ <u>for all</u> $0 \leq f \in L^1(\mu_1)$.

PROOF. (i) $\int_o^t \delta_{Tf} = \int_o^t \delta_{Tf^+ - Tf^-} \leq \int_o^t \delta_{Tf^+} + \int_o^t \delta_{-Tf^-}$

$$\leq \int_o^t \delta_{f^+} + \int_o^t \delta_{-f^-} = \int_o^t \delta_f$$

for all $0 \leq t \leq a$ with equality when $t = a$ since $Tf^- < f^-$ implies $-Tf^- < -f^-$.

(ii) $Tf < f$ for all $f \geq 0$ implies $Tf^+ < f^+$ and $Tf^- < f^-$ so $Tf < f$ for all f.

The following Theorem, first proved by J. V. Ryff [39] for $L^1[0, 1]$, is fundamental.

(18. 4) THEOREM [J. V. Ryff]. <u>A linear transformation</u> T <u>mapping the simple functions of</u> (X_1, Λ_1, μ_1) <u>into</u> $L^1(X, \mu)$ <u>has a unique extension to a doubly stochastic transformation of</u> $L^1(\mu_1) \to L^1(\mu)$ <u>iff for all</u> $E \in \Lambda_1$, <u>we have</u> $0 \leq T C_E \leq C_X$ <u>and</u> $\int_X T C_E \, d\mu = \mu_1(E)$.

PROOF. (8. 2)(v) shows that these conditions are equivalent to $T C_E < C_E$ for all $E \in \Lambda_1$, and are thus necessary. Hence suppose T is given satisfying the above conditions. Let $0 \leq f \in M(X_1, \mu_1)$ be simple. Then $f = \sum_{i=1}^{n} f_i C_{E_i}$ where $E_n \subset \cdots \subset E_1$ and $f_i > 0$, so $\delta_f = \sum_{i=1}^{n} f_i \delta_{E_i}$. Then (12. 1) implies for all $0 \le t < a$,

$\int_0^t \delta_{Tf} < \sum_{i=1}^{n} f_i \int_0^t \delta_{TC_{E_i}} < \sum_{i=1}^{n} f_i \int_0^t \delta_{E_i} = \int_0^t \delta_f$ with equality when $t = a$, so $Tf \prec f$ for all non-negative simple functions and hence for all simple functions. Thus T is a contraction in the L^1 and L^∞ norms on the simple functions, so it extends uniquely to L^1. To show that T is d. s. let $0 \le f \in L^1(\mu_1)$. Then there is a sequence $\{f_n\}$ of simple functions s. t. $0 \le f_n \uparrow f$. Then Tf_n is increasing so $Tf_n \uparrow Tf$ and since $Tf_n \prec f_n$, (8. 2)(vii) implies $Tf \prec f$. Hence (18. 3)(ii) implies T is d. s.

If $T: L^1(\mu_1) \to L^1(\mu)$ is linear, let T^* denote the adjoint of T, defined by $\int_X g\, Tf\, d\mu = \int_{X_1} f\, T^* g\, d\mu_1$ for all $f \in L^1(\mu_1)$, $g \in L^\infty(\mu)$. It follows that $T^*: L^\infty(X, \mu) \to L^\infty(X_1, \mu_1)$ and hence T is weakly continuous under the topologies $\sigma(L^1(\mu_1), L^\infty(\mu_1))$ and $\sigma(L^1(\mu), L^\infty(\mu))$ ([36, p. 38, Prop. 12] or use nets and the defining equation.)

(18. 5) THEOREM. <u>If</u> $T: L^1(\mu_1) \to L^1(\mu)$ <u>is d. s. then</u> T^* <u>has a unique extension to a d. s. map of</u> $L^1(\mu) \to L^1(\mu_1)$.

PROOF. This follows from (18. 4) as follows. Let $E \in \Lambda$. Then $\int T^* C_E\, d\mu_1 = \int C_E\, TC_{X_1}\, d\mu = \int C_E\, d\mu = \mu(E)$. Also for all $A \in \Lambda_1$, $\int_A T^* C_E\, d\mu_1 = \int C_E\, TC_A\, d\mu$ and $0 \le \int C_E TC_A\, d\mu \le \int TC_A\, d\mu = \mu_1(A) = \int_A C_{X_1}\, d\mu_1$ so $0 \le T^* C_E \le C_{X_1}$.

(18.6) PROPOSITION. <u>A linear mapping</u> $T: L^1(\mu_1) \to L^1(\mu)$ <u>is doubly stochastic iff</u>

(i) $T \geq 0$

(ii) $TC_{X_1} = C_X$

(iii) $T^* C_X = C_{X_1}$

PROOF. This follows from (18.4) since for all $E \in \Lambda_1$ $\int TC_E \, d\mu = \int C_E \, T^* C_X \, d\mu_1 = \int C_E \, d\mu_1 = \mu_1(E)$ and $0 \leq C_E \leq C_{X_1}$ implies $0 \leq TC_E \leq TC_{X_1} = C_X$.

Observe that if (X_2, Λ_2, μ_2) is also a finite m. s. with $\mu_2(X_2) = a$, and $T_1: L^1(\mu_1) \to L^1(\mu)$ and $T_2: L^1(\mu_2) \to L^1(\mu_1)$ are both d. s. then $T_1 T_2: L^1(\mu_2) \to L^1(\mu)$ is d. s., since for all $f \in L^1(\mu_2)$ we have $T_1 T_2 f \prec T_2 f \prec f$.

(18.7) LEMMA. <u>Let</u> $f \in L^1(X, \mu)$ <u>and let</u> $\sigma: X^\# \to [0, a]$ <u>be measure preserving s.t.</u> $T_\sigma \delta_f = \delta_f \circ \sigma = f \ \mu^\#$-a. e. <u>Then</u> $T_\sigma^* f = \delta_f$ <u>a. e.</u>

PROOF. For all measurable $J \subset [0, a]$ we have

$$\int_0^a C_J \, T_\sigma^* f \, dm = \int f \, T_\sigma C_J \, d\mu^\# = \int (\delta_f \circ \sigma)(C_J \circ \sigma) d\mu^\# = \int_0^a \delta_f \, C_J \, dm .$$

Let $\mathcal{D}(X_1, X) = \{T: L^1(X_1, \mu_1) \to L^1(X, \mu)$ such that T is doubly stochastic$\}$. If $f \in L^1(\mu_1)$ let $\mathcal{D}_f(X_1, X) = \{Tf: T \in \mathcal{D}(X_1, X)\}$ and we recall that $\Omega_f(X, \mu) = \{g \in L^1(X, \mu): g \prec f\}$ and

$$\Delta_f(X, \mu) = \{g \in L^1(X, \mu): g \sim f\}.$$

As we indicated in §17, if $\Delta_f \neq \emptyset$, then $\Omega_f = \Omega(f_o)$ for all $f_o \in \Delta_f$. Note that $\mathcal{D}_f \subset \Omega_f$ for all $f \in L^1(\mu_1)$.

(18.8) THEOREM. $\mathcal{D}(X_1, X)$ is convex and compact in the weak operator topology determined by the linear functionals $T \to \int f\, Tg\, d\mu \quad f \in L^1(\mu),\ g \in L^\infty(\mu_1)$.

PROOF. To show $\mathcal{D}$ is convex, let $T_1, T_2 \in \mathcal{D}$ and $0 \le r \le 1$. Then for all $f \in L^1(\mu_1)$ we have $T_1 f, T_2 f \in \Omega_f$ so $r\, T_1 f + (1-r)T_2 f \in \Omega_f$, i.e., $(r\, T_1 + (1-r)T_2)f = r\, T_1 f + (1-r)T_2 f \prec f$, so $r\, T_1 + (1-r)\, T_2 \in \mathcal{D}$.

Let S_1 be the unit ball of $L^\infty(\mu_1)$ and let S be the unit ball in $L^\infty(\mu)$. Since S is $\sigma(L^\infty(\mu), L^1(\mu))$ compact, the set of all linear operators which map $S_1 \to S$ is compact [18]. Hence it only remains to show that $\mathcal{D}$ is closed. Let $T_\alpha \to T$. For each $E \in \Lambda_1$ we have $T_\alpha C_E \in \Omega_{C_E}$ and $T_\alpha C_E \to TC_E$ weakly. Since Ω_{C_E} is $\sigma(L^1, L^\infty)$ closed we get $TC_E \in \Omega_{C_E}$. Hence (18.4) implies $T \in \mathcal{D}$.

(18.9) THEOREM. Let $f \in L^1(X_1, \mu_1)$. If $g \in M(X, \mu)$ then $g \prec f$ iff there is a doubly stochastic $T: L^1(\mu_1) \to L^1(\mu)$ such that $g = Tf$.

PROOF. We have to show that $\mathcal{D}_f = \Omega_f$. It suffices to show that $T_\mu\, \Delta_f(\mu^\#) \subset \mathcal{D}_f$ (where, of course, $T_\mu\, \Delta_f(\mu^\#) = \{T_\mu f' : f' \in L^1(X^\#, \mu^\#)\ \&\ f' \sim f\}$) and that $\mathcal{D}_f$ is $\sigma(L^1(\mu), L^\infty(\mu))$ closed and convex, because we know from (17.11) that Ω_f is the closed convex hull of $T_\mu\, \Delta_f(\mu^\#)$ in this topology. $\mathcal{D}_f$ is convex and closed in this topology, because it is the image of the convex, compact (in the weak operator topology described in (18.8)) set $\mathfrak{D}(X, X_1)$ under the continuous linear map $T \to T^* f$. For the rest, let $f' \in \Delta_f(\mu^{\#})$. There are measure preserving transformations $\sigma: X^{\#} \to [0,a]$ and $\phi: X_1^{\#} \to [0,a]$ such that $T_\sigma \delta_{f'} = f'$ and $T_\phi \delta_f = f$. Since $f' \sim f$, $\delta_{f'} = \delta_f$ so $T_\mu f' = T_\mu T_\sigma T_\phi^* f \in \mathfrak{D}_f$.

(18.10) THEOREM. <u>If</u> $f_1, f_2 \in L^1(X_1, \mu_1)$ <u>and</u> $g \in M(X, \mu)$ <u>and</u> $g \prec f_1 + f_2$ <u>then there are</u> $g_1, g_2 \in L^1(X, \mu)$ <u>such that</u> $g = g_1 + g_2$ <u>and</u> $g_1 \prec f_1$ <u>and</u> $g_2 \prec f_2$.

PROOF. There is a $T \in \mathfrak{D}(X_1, X)$ s.t. $g = T(f_1 + f_2)$, so let $g_i = Tf_i$, $i = 1, 2$.

A good example of a class of doubly stochastic operators is provided by <u>conditional expectations.</u> Let Λ' be a σ-subalgebra of Λ and let $f \in L^1(X, \mu)$. It follows from the Radon-Nikodyn theorem that there is a unique Λ'-measurable function Tf such that $\int_E f\, d\mu = \int_E Tf\, d\mu$ for all $E \in \Lambda'$. Using (18.4) it is clear that $T: L^1(X, \Lambda, \mu) \to L^1(X, \Lambda', \mu)$ is doubly stochastic. T is called the Λ' conditional expectation.

As a special case of this process let $X = X_o \cup \bigcup_{i \in P} X_i$ be the union of an at most countable number of sets of positive measure and let Λ' be the σ-sub-algebra generated by the sets $\Lambda \cap X_o$ and $\{X_i\}_{i \in P}$. Then the Λ' conditional expectation has the form

$$Tf = f\, C_{X_o} + \sum_{i \in P} \frac{1}{\mu(X_i)} \left(\int_{X_i} f\, d\mu \right) C_{X_i} .$$

Of course T_μ is of this type.

Now let ρ be a saturated Fatou norm on $M(X,\mu)$ such that $L^\rho(\mu)$ is u.r.i. and $L^\infty(\mu) \subset L^\rho$, $L^{\rho'} \subset L^1(\mu)$, and define the u.r.i. Fatou norm ρ_1 on $M(X_1, \mu_1)$ by

$$\rho_1(f) = \sup\{ \int_0^a \delta_{|f|} \delta_{|g|} : \rho'(g) \le 1\}.$$

(18.11) THEOREM. <u>If</u> $T: L^{\rho_1}(\mu_1) \to L^1(\mu)$ <u>is linear and</u> $Tf \prec f$ <u>for all</u> $f \in L^{\rho_1}(\mu_1)$ <u>then</u>

(i) $T: L^{\rho_1}(\mu_1) \to L^\rho(\mu)$ <u>and</u> $\rho(Tf) \le \rho_1(f)$ <u>for all</u> $f \in L^{\rho_1}$.

(ii) T <u>has a unique extension to a doubly stochastic</u> $T: L^1(\mu_1) \to L^1(\mu)$.

(iii) $T^*: L^{\rho'}(\mu) \to L^{\rho_1'}(\mu_1)$ <u>and</u> $\rho_1'(T^*g) \le \rho'(g)$ for all $g \in L^{\rho'}$.

PROOF. (i) We already know that $Tf \prec f \in L^{\rho_1}(\mu_1)$ implies $\rho(Tf) \le \rho_1(f)$ so $Tf \in L^\rho(\mu)$.

(ii) This follows from (18.4) since $L^\infty(\mu_1) \subset L^{\rho_1}(\mu_1)$.

(iii) Let $g \in L^{\rho'}(\mu)$. Since we have $\int f\, T^* g\, d\mu_1 = \int gTf\, d\mu$ is finite for all $f \in L^{\rho_1}(\mu_1)$, (14.15) implies $T^*g \in L^{\rho_1'}(\mu_1)$. Let $g \in L^{\rho'}(\mu)$.

$$\rho_1'(T^*g) = \sup\{ \int |f\ T^*g|\, d\mu_1 : \rho_1(f) \le 1\}$$

$$\le \sup\{ \int |f|\ T^* |g|\, d\mu_1 : \rho_1(f) \le 1\}$$

$$= \sup\{ \int |g|\ T\,|f|\, d\mu : \rho_1(f) \le 1 \}$$

$$\le \sup\{ \int |g|\ |f'|\, d\mu : \rho(f') \le 1 \} = \rho'(g).$$

(18.12) COROLLARY. Let $g_1 \in L^1(X_1, \mu_1)$. If $g_1 \prec g \in L^{\rho'}(\mu)$ then $g_1 \in L^{\rho'_1}$ and $\rho'_1(g_1) \le \rho'(g)$.

PROOF. Let $g_1 \prec g \in L^{\rho'}(\mu)$. There is a d. s. $T: L^1(\mu) \to L^1(\mu_1)$ s.t. $g_1 = Tg$. Then T^* extends uniquely to a d. s. map of $L^1(\mu_1) \to L^1(\mu)$ so applying (18.11) to T^* we have $g_1 = (T^*)^* g \in L^{\rho_1}(\mu_1)$ and $\rho'_1(g_1) = \rho'_1((T^*)^* g) \le \rho'(g)$.

REMARKS. (i) If $(X_1, \Lambda_1, \mu_1) = (X, \Lambda, \mu)$ and ρ is u. r. i. then $\rho_1 = \rho$.

(ii) If λ is a r. i. Fatou norm on $M^+[0, a]$ and $\rho(f) = \lambda(\delta_f)$ for $f \in M^+(X, \mu)$ as in §15, and $\rho_2(f) = \lambda(\delta_f)$ for $f \in M^+(X_1, \mu_1)$ then (18.11) holds with ρ_1 replaced by ρ_2.

19. Muirhead's Theorem. J. V. Ryff has given a generalization of Muirhead's Theorem for bounded measurable functions on [0, 1] (see [42]). In this section we will show when this generalization is valid for arbitrary finite measure spaces.

Let (X, Λ, μ) and (X_1, Λ_1, μ_1) and (X_2, Λ_2, μ_2) be finite m. s. such that $\mu(X) = \mu_1(X_1) = \mu_2(X_2) = a$. If $u \in M(X_2, \mu_2)$ is positive (i. e. $u(x) > 0$ μ_2-a. e.) and $f \in M(X, \mu)$ let

$$[f;u] = \int_X \log\left(\int_{X_2} u(x)^{f(y)} \, d\mu_2(x) \right) d\mu(y)$$

and similarly for $g \in M(X_1, \mu_1)$.

(19.1) LEMMA. If $u \in M(X_2, \mu_2)$ is positive and $f \in M(X, \mu)$ then $[f;u] = [\delta_f;\delta_u]$ in the sense that both are finite and equal or both are infinite with the same sign.

PROOF. Let $p \in R$ and let $\psi(t) = t^p$ for all $t \in R$. Since $u \sim \delta_u$ and the measure spaces involved are finite, (3.3)(xv) says $\psi(u) \sim \psi(\delta_u)$ so $\int_{X_2} \psi(u) d\mu_2 = \int_0^a \psi(\delta_u)$, i.e. $\int_{X_2} u^p \, d\mu_2 = \int_0^a (\delta_u)^p$. Let $\varphi(p) = \log(\int_{X_2} u^p d\mu_2) = \log(\int_0^a (\delta_u)^p)$. Again, since $f \sim \delta_f$ we have $\varphi(f) \sim \varphi(\delta_f)$ so $[f;u] = \int_X \varphi(f) d\mu = \int_0^a \varphi(\delta_f) = [\delta_f;\delta_u]$.

(19.2) THEOREM. Let $f \in L^\infty(X, \mu)$ and $g \in L^\infty(X_1, \mu_1)$.

(i) If $g \prec f$ then $[g;u] \leq [f;u]$ and both are finite for all positive $u \in M(X_2, \mu_2)$ such that $u^p \in L^1(X_2, \mu_2)$ whenever ess. inf $(\min(f, g)) \leq p \leq$ ess. sup $(\max(f, g))$ [we call such u admissible for f and g.]

(ii) If $[g;u] = [f;u]$ and u is admissible for f and g then $f \sim g$ or u is constant μ_2-a.e.

(iii) If (X_2, Λ_2, μ_2) is non-atomic and $[g;u] \leq [f;u]$ for all positive $u \in M(X_2, \mu_2)$ such that $u^p \in L^1(X_2, \mu_2)$ whenever $p \in R$ [we call such u admissible for L^∞], then $g \prec f$.

PROOF. Ryff has shown that this theorem is true when $X = X_1 = X_2 = [0, 1]$ with Lebesgue measure ([42 , p. 596] and [43 , p. 436]).

Our first step is to show that the theorem is true when $X = X_1 = X_2 = [0, a]$ with Lebesgue measure. Let $F, G \in M[0, a]$ and let $u \in M[0, a]$ be admissible for F and G. Then

$$[F;u] = \int_0^a \log\left(\int_0^a u(s)^{F(t)}\, ds\right)dt$$

$$= a\int_0^1 \log\left(\int_0^1 u(as)^{F(at)}\, ds\right)dt + a\log a$$

$$= a\,[F_1;u_1] + a\log a$$

where $F_1(t) = F(at)$ and $u_1(s) = u(as)$ on $[0, 1]$. Similarly $[G;u] = a[G_1;u_1] + a\log a$ where $G_1(t) = G(at)$ on $[0, 1]$. Since $a > 0$, (3.2) implies $G_1 \prec F_1$ iff $G \prec F$, and thus it is easy to see that the theorem is true when $X = X_1 = X_2 = [0, a]$.

Now let $f \in L^\infty(X, \mu)$ and $g \in L^\infty(X_1, \mu_1)$.

(i) Suppose $g \prec f$ and let $u \in M(X_2, \mu_2)$ be admissible for f and g. Then $\delta_g \prec \delta_f$ and it is easy to see that δ_u is admissible for δ_f and δ_g so $[g;u] = [\delta_g;\delta_u] \leq [\delta_f;\delta_u] = [f;u]$.

(ii) Suppose $[f;u] = [g;u]$. Then $[\delta_f;\delta_u] = [\delta_g;\delta_u]$ so either $\delta_f \sim \delta_g$ (so $f \sim g$) or δ_u is essentially constant (so u is also).

(iii) Suppose (X_2, Λ_2, μ_2) is non-atomic and $[g;u] \leq [f;u]$ for all $u \in M(X_2, \mu_2)$ admissible for L^∞. Let $v \in M[0, a]$ be admissible for $L^\infty[0, a]$. Then there is a $u \in M(X_2, \mu_2)$ such that $\delta_u = \delta_v$, in which case u is admissible for L^∞ and $[\delta_g;v] = [\delta_g;\delta_u] = [g;u] \leq [f;u] = [\delta_f;v]$, so $[\delta_g;v] \leq [\delta_f;v]$ for all admissible v and hence $\delta_g \prec \delta_f$ so $g \prec f$

REMARKS. 1. Ryff has also shown that (iii) is not in general true if we interchange the order of integration in the definition of $[f;u]$.

2. (iii) may also fail if (X_2, Λ_2, μ_2) has atoms and (X, μ) and (X_1, μ_1) are more complex than (X_2, μ_2). For example, if X_2 is an atom and both X and X_1 are not atoms, then (iii) fails. The assumption that (X_2, Λ_2, μ_2)

is non-atomic is sufficient to insure that there are enough admissible $u \in M(X_2, \mu_2)$ to distinguish when $g \prec f$, no matter how complex (X, μ) and (X_1, μ_1) are.

VI. EXTREMAL AND RELATED PROBLEMS

20. Extreme, Exposed and Support Points of Ω_f. Let V be a locally convex topological vector space and let K be a convex subset of V. A point $v \in K$ is said to be an extreme point of K if $v = \frac{1}{2}v_1 + \frac{1}{2}v_2$ & $v_1, v_2 \in K$ implies $v_1 = v_2$. v is said to be an exposed point of K if there is an $F \in V^*$ (the continuous linear functionals on V) such that $F(w) < F(v)$ whenever $w \in K$, $w \neq v$. If K is contained in a hyperplane, then a point $v \in K$ is called a support point of K if there is an $F \in V^*$ such that $F(w) \leq F(v)$ for all $w \in K$ and $F(w) < F(v)$ for some $w \in K$. It is clear that every exposed point is both an extreme point and a support point.

It would be desirable to characterize the extreme points of K in terms of the sets $F[K]$, $F \in V^*$. For example, it is clear that if $F(v)$ is extreme in $F[K]$ for all $F \in V^*$, then v is extreme in K. The converse is not true as can be seen by considering the closed unit disk. However, it is true that if v is extreme in K, then there is an $F \in V^*$ such that $F(v)$ is extreme in $F[K]$. For if v is extreme in K, then v is a boundary point of K, and since the interior of K is convex, the Hahn-Banach Separation theorem [36, p. 29] gives the required F.

It is well known that $v \in K$ is not extreme iff there is a $0 \neq u \in V$ such that $v+u$ and $v-u \in K$. Suppose K is closed and convex. It is easy to see using (17.5) that this condition becomes: $v \in K$ is not extreme iff there is a $0 \neq u \in V$ such that both

$$F(u) \leq \sup F[K] - F(v)$$

$$F(u) \leq F(v) - \inf F[K]$$

for all $F \in V^*$. This condition has been given for $\Omega(f)$ by Luxemburg [28, p. 141].

It does not appear likely that a useful characterization of this type is possible. Let K be an ice-cream cone in the plane formed by intersecting tangents to a circle. The points of tangency are extreme but not exposed, and there seems to be no way to distinguish them using closed hyperplanes from the points on the sides of the cone.

Let (X, Λ, μ) and (X_1, Λ_1, μ_1) be finite measure spaces such that $\mu(X) = \mu_1(X_1) = a$. Recall that if $f \in L^1(X_1, \mu_1)$ we let

$\Omega_f(X,\mu) = \{g \in L^1(X,\mu): g \prec f\}$

$\Delta_f(X,\mu) = \{g \in L^1(X,\mu): g \sim f\}$. The reader is referred to § 17 for a detailed discussion of these sets.

The extreme, exposed and support points of Ω_f have been determined by J. V. Ryff when $X = X_1 = [0,1]$ (see [41]). The proof that the functions in Δ_f are all extreme in Ω_f is due to J. L. Doob [41]. We present it in the following way. The proof of (20.2) is different from Doob's.

(20.1) LEMMA. (J. L. Doob). (i) <u>If</u> $\frac{1}{2}(f_1+f_2) \sim f_1 \sim f_2$ <u>then</u> $f_1 f_2 \geq 0$ μ-a.e.

(ii) <u>Let</u> $g \in \Omega_f$. <u>If</u> $g = \frac{1}{2}f_1+\frac{1}{2}f_2$ & $f_1, f_2 \in \Omega_f$ <u>implies</u> $f_1 \sim f_2 \sim g$ <u>then</u> g <u>is extreme</u>.

PROOF. (i) $f_1 \sim f_2$ implies $|f_1| \sim |f_2|$ so $\int |f_1|\, d\mu = \int |f_2|\, d\mu$. Also $|f_1 + f_2| \sim 2|f_1|$ so $\int |f_1 + f_2| d\mu = \int (|f_1| + |f_2|) d\mu$ so the triangle inequality implies $|f_1 + f_2| = |f_1| + |f_2|$ μ-a.e. and thus $f_1 f_2 = |f_1 f_2| \geq 0$ μ-a.e.

(ii) Suppose $g, f_1, f_2 \in \Omega_f$ and $g = \frac{1}{2}f_1 + \frac{1}{2}f_2$. Then $g \sim f_1 \sim f_2$ and we have to show $f_1 = f_2$. By symmetry it suffices to show that $\{x: f_1(x) < f_2(x)\}$ has measure zero. For all $r \in R$, $(f_1 - r) \sim (f_2 - r) \sim (g - r) = \frac{1}{2}(f_1 - r) + \frac{1}{2}(f_2 - r)$ so $(f_1 - r)(f_2 - r) \geq 0$ μ-a.e. Let $\{r_n\}$ be an enumeration of all the rationals of R. Then

$$\{x: f_1(x) < f_2(x)\} = \bigcup_{n=1}^{\infty} \{x: f_1(x) < r_n < f_2(x)\}$$

$\subset \bigcup_{n=1}^{\infty} \{x: (f_1(x) - r_n)(f_2(x) - r_n) < 0\}$ and each of these sets has measure zero.

(20.2) THEOREM. <u>If</u> $g \in \Omega_f$ <u>and</u> $g \sim f$ <u>then</u> g <u>is extreme.</u>

PROOF. Let $\Phi(h) = \int_0^a \int_0^t \delta_h = \int_0^a \delta_h(s)(a-s)ds$ if $h \in L^1(X, \mu) \cup L^1(X_1, \mu_1)$. Φ is Shur convex and $\Phi(h_1) = \Phi(h_2)$ iff $h_1 \sim h_2$. Suppose $g \in L^1(X, \mu)$ and $g \sim f \in L^1(X_1, \mu_1)$. If $g = \frac{1}{2}f_1 + \frac{1}{2}f_2$ where $f_1, f_2 \in \Omega_f$ then $\Phi(f) = \Phi(g) = \frac{1}{2}\Phi(f_1) + \frac{1}{2}\Phi(f_2) \leq \frac{1}{2}\Phi(f_1) + \frac{1}{2}\Phi(f)$ so $\Phi(f) \leq \Phi(f_1) \leq \Phi(f)$ and thus $\Phi(f) = \Phi(f_1)$ so $f \sim f_1$. Similarly $f \sim f_2$. Hence $g \sim f_1 \sim f_2$ so g is extreme.

Using Theorems (5.10), (5.12) and (10.1) it is easy to see that Ryff's proofs are valid when (X, Λ, μ) is non-atomic. Hence we have the following.

(20.3) THEOREM. <u>If</u> (X, Λ, μ) <u>is non-atomic and</u> $f \in L^1(X_1, \mu_1)$ <u>then the set of extreme points and the set of exposed points of</u> Ω_f <u>are identical with the set</u> Δ_f. <u>A function</u> $g \in \Omega_f$ <u>is a support point of</u>

Ω_f iff there is a $0 < t < a$ such that $\int_0^t \delta_g = \int_0^t \delta_f$.

If (X, Λ, μ) is not adequate, then there will be a function $f \in L^1(X, \mu)$ such that $\Omega(f) = \{g \in L^1(\mu): g \prec f\}$ is not the closed convex hull of $\Delta(f) = \{g \in L^1(\mu): g \sim f\}$. If $f_o \in L^1(X_1, \mu_1)$ such that $f_o \sim f$ then Ω_{f_o} will not be the closed convex hull of Δ_{f_o}, and hence Δ_{f_o} cannot contain all the extreme points of Ω_{f_o}.

It is a good conjecture that every extreme point of Ω_f has the form $T_\mu f'$ where $f' \in L^1(X^\#, \mu^\#)$ and $f' \sim f$, and that every function in Ω_f equimeasurable with an extreme point is an extreme point. For example, if f is extreme and either (i) $g \circ \sigma = f$ where $\sigma: X \to X$ is measure preserving or (ii) $g|X_o = f|X_o$ and $g|A \sim f|A$ (where X_o is the non-atomic part and A is the union of the atoms of X) then g is extreme.

The following example shows that not every function $T_\mu f'$, $f' \in L^1(X^\#, \mu^\#)$, $f' \sim f$ is an extreme point. Let X be the union of two atoms A and B with $\mu(A) < \mu(B)$. Then every $g \in M(X, \mu)$ has the form $g = xC_A + yC_B$. Let $f = 2C_A + C_B$. Then $g \in \Omega(f)$ iff $x \leq 2$, $y\,\mu(B) \leq 2\mu(A) + (\mu(B)-\mu(A)) = \mu(A) + \mu(B)$ and $x\mu(A) + y\,\mu(B) = 2\mu(A) + \mu(B)$. If we define $\varphi: M(X, \mu) \to R^2$ by $\varphi(xC_A + yC_B) = (x, y)$ then φ is linear, 1:1, and onto the line segment joining the points $(2, 1)$ and $(1, 1 + \frac{\mu(A)}{\mu(B)})$. Hence the extreme points of $\Omega(f)$ are $f = 2C_A + C_B$ and $g = C_A + (1 + \frac{\mu(A)}{\mu(B)})\, C_B$. Observe that $\Delta(f) = \{f\}$, $\Delta(g) = \{g\}$, and $g = T_\mu f'$ where $f' = C_{[0, \mu(B)[} + 2C_{[\mu(B), \mu(X)[}$ $\sim f$. It is clear in this case that $\{T_\mu f': f' \sim f,\ f' \in L^1(X^\#, \mu^\#)\} = \Omega(f)$.

The following example shows how extreme points not in $\Delta(f)$ may arise when X has an atom A and a non-atomic part X_o of positive measure. Let $B \subset X_o$ be s.t. $0 < \mu(B) \leq \mu(A)$. Using (5.10) there is an f on B s.t. $\delta_{f|B}(t) = \mu(B)-t$ on $[0, \mu(B)[$. Define f to be 0 on the rest of X. Let $I[\alpha, \beta]$ be the interval of $(X^{\#}, \Lambda^{\#}, \mu^{\#})$ corresponding to A as in §9. Define $g' \in L^1(X^{\#}, \mu^{\#})$ by $g'(t) = \alpha + \mu(B)-t$ if $\alpha < t < \alpha + \mu(B)$ and $g' = 0$ elsewhere. Then $g' \sim f$ so $g = \frac{1}{2}\mu(B)^2 C_A = T_\mu g' \prec f$ and $g \not\sim f$. To show that g is extreme, suppose $g \pm u \prec f$ (here $\pm$means + and -) where $u \in M(X, \mu)$. Then $\int u \, d\mu = 0$ and since $f \geq 0$, we have $g \pm u \geq 0$. Thus $0 \leq g \pm u|A^c = \pm u|A^c$ so $u|A^c = 0$. Then $0 = \int u \, d\mu = (u|A)\mu(A)$ implies $u|A = 0$ so $u = 0$. Hence g is extreme.

In each of the above examples extreme points were obtained in the following way. Write $X = X_o \cup \bigcup_{i \in P} X_i$ where $P = \{1, \ldots, n\}$ or $P = \{1,2,3, \ldots\}$, X_o is the non-atomic part of X, and X_i, $i \in P$, are the atoms of X. For the intervals $I[a_i, b_i]$ defining $X^{\#}$ take

$$I[a_i, b_i] = [\mu(X_o) + \sum_{k=1}^{i-1} \mu(X_k), \ \mu(X_o) + \sum_{k=1}^{i} \mu(X_k)[\ .$$

Let $f \in L^1(X, \mu)$. For each partition $\pi = \{[\alpha_i, \beta_i[\}_{i \in P}$ of the interval $[0, \mu(X)[$ such that $\beta_i - \alpha_i = \mu(X_i)$ define

$$f''_\pi = \delta_{f|X_o}(t-\alpha_o) C_{[\alpha_o, \beta_o[} + \sum_{i \in P} (f|X_i) C_{[\alpha_i, \beta_i[} \ .$$

Now for each measure preserving $\sigma: X_o \to [0, \mu(X_o)[$ let

$$f'_{\pi, \sigma} = (f''_\pi \circ \sigma) C_{X_o} + f''_\pi C_{X - X_o} \ .$$

Then $f'_{\pi,\sigma} \in L^1(X^\#, \mu^\#)$, $f'_{\pi,\sigma} \sim f$, and it is a good conjecture that $T_\mu f'_{\pi,\sigma}$ is extreme in Ω_f.

21. Permutator Transformations. Because of the importance of the sets $\Delta(f) = \{h: h \sim f\}$ it seems natural to investigate the necessarily doubly stochastic operators T such that $Tf \sim f$ for all f. Such an operator is called a permutator, because for discrete measures they correspond to the permutation matrices.

Let (X, Λ, μ) be a finite m. s. with $\mu(X) = a$. If $A, B \in \Lambda$ we write $A = B\ [\mu]$ to mean $C_A = C_B$ μ-a. e., i. e., $\mu(A \Delta B) = 0$ where $A \Delta B = A-B \cup B-A$. We also write $A \subset B\ [\mu]$ to mean $C_A \leq C_B$ μ-a. e., i. e., $\mu(A-B) = 0$. Note that $A \subset B\ [\mu]$ implies $\mu(A) \leq \mu(B)$ and $A = B\ [\mu]$ implies $\mu(A) = \mu(B)$. If $A \subset B\ [\mu]$, then $A = B\ [\mu]$ iff $\mu(A) = \mu(B)$.

Now let (X_1, Λ_1, μ_1) be a finite m. s. with $\mu_1(X_1) = \mu(X) = a$.

(21. 1) DEFINITION. A mapping $\Phi: \Lambda_1 \to \Lambda$ is said to be a homomorphism of Λ_1 into Λ if it satisfies

(i) $\mu(\Phi(A)) = \mu_1(A)$ for all $A \in \Lambda_1$

(ii) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)\ [\mu]$ for all $A, B \in \Lambda_1$

(iii) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)\ [\mu]$ whenever $A \cap B = \emptyset\ [\mu_1]$.

We call Φ an isomorphism if in addition its range is $\Lambda\ [\mu]$, i. e., for every $E \in \Lambda$ there is an $A \in \Lambda_1$ such that $\Phi(A) = E\ [\mu]$. In this case there is an isomorphism $\Phi^{-1}: \Lambda \to \Lambda_1$ such that $\Phi \circ \Phi^{-1}$ is the identity $[\mu]$ on Λ.

(21.2) PROPOSITION. *If $\Phi: \Lambda_1 \to \Lambda$ is a homorphism, then*

(i) Φ *preserves disjoint sets*, *i.e.*, $A \cap B = \emptyset\ [\mu]$ *implies* $\Phi(A) \cap \Phi(B) = \emptyset\ [\mu]$.

(ii) Φ *is monotone*, *i.e.*, $A \subset B\ [\mu_1]$ *implies* $\Phi(A) \subset \Phi(B)\ [\mu]$.

(iii) Φ *preserves differences*, *i.e.*, $\Phi(B-A) = \Phi(B) - \Phi(A)[\mu]$ *whenever* $A \subset B\ [\mu_1]$.

(iv) Φ *is countably additive*, *i.e.*, *if* $\{A_i\}_{i=1}^{\infty} \subset \Lambda_1$ *are pwd* $[\mu_1]$ *then* $\Phi(\cup A_i) = \cup \Phi(A_i)\ [\mu]$.

(v) Φ *is* 1:1, *i.e.*, $\Phi(A) = \Phi(B)\ [\mu]$ *implies* $A = B\ [\mu_1]$.

PROOF. (i) $A \cap B = \emptyset\ [\mu_1]$ iff $\mu_1(A \cap B) = 0$ so $\mu(\Phi(A) \cap \Phi(B)) = \mu(\Phi(A \cap B)) = \mu_1(A \cap B) = 0$ implies $\Phi(A) \cap (B) = \emptyset\ [\mu]$.

(ii) & (iii) If $A \subset B\ [\mu_1]$ then $B = B - A \cup A\ [\mu_1]$ is a pairwise disjoint union so $\Phi(B) = \Phi(B-A) \cup \Phi(A)\ [\mu]$ is a pwd union $[\mu]$ and hence $\Phi(A) \subset \Phi(B)\ [\mu]$ and $\Phi(B) - \Phi(A) = \Phi(B-A)\ [\mu]$

(iv) Let $\{A_i\} \subset \Lambda_1$ be pwd $[\mu_1]$. Now $A_j \subset \cup A_i$ for all j implies $\Phi(A_j) \subset \Phi(\cup A_i)$ for all j and hence $\cup\Phi(A_i) \subset \Phi(\cup A_i)$. Since

$$\mu(\Phi(\bigcup_{i=1}^{\infty} A_i)) = \mu_1(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu_1(\bigcup_{i=1}^{n} A_i) = \lim_{n\to\infty} \mu(\Phi(\bigcup_{i=1}^{n} A_i))$$

$$= \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} \Phi(A_i)) = \mu(\bigcup_{i=1}^{\infty} \Phi(A_i))$$ we have $\Phi(\cup A_i) = \cup \Phi(A_i)\ [\mu]$.

(v) Suppose $\Phi(A) = \Phi(B)\ [\mu]$. Let $E = A$ or B.

$$\mu_1(E - A \cap B) = \mu(\Phi(E - A \cap B)) = \mu(\Phi(E) - \Phi(A \cap B)) = \mu(\Phi(E) - \Phi(A) \cap \Phi(B)) = 0$$

so $A = B\ [\mu]$.

If (X, Λ, μ) is a product of a possibly uncountable number of copies of $[0, 1]$ with Lebesgue measure, then every homomorphism

$\Phi: \Lambda \to \Lambda$ is induced by a measure preserving $\sigma: X \to X$ in the sense that $\Phi(E) = \sigma^{-1}(E)$ for all $E \in \Lambda$. If $\sigma: X \to [0, a]$ is measure preserving then $\Phi(E) = \sigma^{-1}(E)$ is a homorphism of the Borel subsets of $[0, a]$ into Λ. Recall that in this case $T_\sigma f = f \circ \sigma$ defines a doubly stochastic transformation of $L^1[0, a]$ into $L^1(X, \mu)$.

(21. 3) PROPOSITION. <u>To each homomorphism</u> $\Phi: \Lambda_1 \to \Lambda$ <u>there corresponds a unique linear transformation</u> $T_\Phi : L^1(X_1, \mu_1) \to L^1(X, \mu)$ <u>such that</u> $T_\Phi C_E = C_{\Phi(E)}$ <u>for all</u> $E \in \Lambda_1$. <u>In addition we have the following:</u>

(i) $T_\Phi f \sim f$ <u>for all</u> $f \in L^1(X_1, \mu_1)$

(ii) $T_\Phi(fh) = (T_\Phi f)(T_\Phi h)$ <u>whenever</u> $f, h, fh \in L^1(X_1, \mu_1)$

(iii) <u>If</u> $\Phi': \Lambda_1 \to \Lambda$ <u>and</u> $T_\Phi = T_{\Phi'}$ <u>then</u> $\Phi = \Phi' [\mu]$.

PROOF. If $f = \sum_{i=1}^{n} \alpha_i C_{A_i}$ is a simple function with $\{A_i\}$ a partition of X_1 $[\mu_1]$ we define

$$T_\Phi f = \sum_{i=1}^{n} \alpha_i C_{\Phi(A_i)} \quad .$$

It follows from (21. 2) that $\{\Phi(A_i)\}_{i=1}^{n}$ is a partition of X, so Tf and f are simple functions taking the same values on sets of equal measure and hence $Tf \sim f$.

To see that $T_\Phi f$ is well defined, suppose also $f = \sum_{i=1}^{m} \beta_j C_{B_j}$, where $\{B_j\}_{j=1}^{m}$ is a partition of X_1. Then

$$f = \sum_i \sum_j \alpha_i C_{A_i \cap B_j} = \sum_i \sum_j \beta_j C_{A_i \cap B_j}$$

so $\alpha_i = \beta_j$ whenever $A_i \cap B_j \neq \emptyset$ $[\mu_1]$ and thus

$\alpha_i = \beta_j$ whenever $\Phi(A_i \cap B_j) \neq \emptyset$ $[\mu]$. Hence

$$\sum_i \sum_j \alpha_i C_{\Phi(A_i \cap B_j)} = \sum_i \sum_j \beta_j C_{\Phi(A_i \cap B_j)}$$

$$\sum_i \sum_j \alpha_i C_{\Phi(A_i) \cap \Phi(B_j)} = \sum_i \sum_j \beta_j C_{\Phi(A_i) \cap \Phi(B_j)}$$

$$\sum_i \alpha_i C_{\Phi(A_i)} = \sum_i \beta_i C_{\Phi(B_j)} .$$

It is easy to see that $T_\Phi(rf) = rT_\Phi f$ for all $r \in R$. To show that T_Φ is linear, let $g = \sum_{j=1}^{m} \beta_j C_{B_j}$ where $\{B_j\}$ is a partition of X_1. Then $f+g = \sum_i \sum_j (\alpha_i + \beta_j) C_{A_i \cap B_j}$ and $\{A_i \cap B_j\}$ is a partition of X_1 so

$$T_\Phi(f+g) = \sum\sum(\alpha_i+\beta_j) C_{\Phi(A_i \cap B_j)} = \sum\sum(\alpha_i+\beta_j) C_{\Phi(A_i)\cap\Phi(B_j)} = T_\Phi f + T_\Phi g$$

since $\{\Phi(A_i) \cap \Phi(B_j)\}$ is a partition of X $[\mu]$. Similarly, $T_\Phi(fh) = (T_\Phi f)(T_\Phi h)$ for all simple functions f, h.

It follows from (18.4) that T_Φ extends uniquely to a doubly stochastic transformation of $L^1(\mu_1) \to L^1(\mu)$. Since the simple functions are dense in $L^1(\mu_1)$ and T_Φ is continuous in the L^∞ and L^1 norms, it follows from (3.3) (xiii) that $T_\Phi f \sim f$ for all $f \in L^1(\mu_1)$. Similarly, $T_\Phi(fh) = (T_\Phi f)(T_\Phi h)$ whenever $f, h, fh \in L^1(\mu_1)$.

If $T: L^1(\mu_1) \to L^1(\mu)$ is also linear such that $TC_E = C_{\Phi(E)}$ for all $E \in \Lambda_1$ and $f = \sum_{i=1}^{n} \alpha_i C_{A_i}$ where $\{A_i\}$ is a partition of X_1 then $Tf = \sum \alpha_i TC_{A_i} = \sum \alpha_i C_{\Phi(A_i)} = T_\Phi f$. It follows that T extends uniquely to a d. s. transformation of $L^1(\mu_1) \to L^1(\mu)$. Since $T = T_\Phi$ on the simple functions, $T = T_\Phi$ on $L^1(\mu_1)$.

Finally, if $\Phi': \Lambda_1 \to \Lambda$ is a homomorphism for which $T_{\Phi'} = T_\Phi$ then $C_{\Phi'(E)} = T_{\Phi'} C_E = T_\Phi C_E = C_{\Phi(E)}$ for all $E \in \Lambda_1$ so $\Phi' = \Phi$ $[\mu]$.

It is easy to see that if (X_2, Λ_2, μ_2) is a finite m. s. with $\mu_2(X_2) = \mu(X) = a$ and $\Psi: \Lambda_2 \to \Lambda_1$ is a homomorphism, then $\Phi \circ \Psi: \Lambda_2 \to \Lambda$ is a homomorphism and $T_{\Phi \circ \Psi} = T_\Phi \circ T_\Psi$.

Let $T: L^2(\mu_1) \to L^2(\mu)$ be linear. T is called <u>multiplicative</u> if $T(fh) = (Tf)(Th)$ whenever $f, h \in L^2(\mu_1)$. T is called <u>isometric</u> if $\int Tf\, Th\, d\mu = \int f\, h\, d\mu_1$ whenever $f, h \in L^2(\mu_1)$. T is called <u>unitary</u> if both T and T^* are isometric.

Observe that if T is multiplicative and isometric then (18.4) implies that T has a unique extension to a d. s. $T: L^1(\mu_1) \to L^1(\mu)$.

If $T: L^1(\mu_1) \to L^1(\mu)$ is d. s., then (17.13) (or Remark (ii) after (18.12)) implies that $T: L^2(\mu_1) \to L^2(\mu)$. It is easy to see that it is multiplicative iff $T(fh) = (Tf)(Th)$ whenever $f, h, fh \in L^1(\mu_1)$ iff $T(fh) = (Tf)(Th)$ whenever $f, h \in L^\infty(\mu_1)$; similarly for T isometric.

(21.4) THEOREM. <u>Let</u> $T: L^1(\mu_1) \to L^1(\mu)$ <u>be linear. Then the following are equivalent:</u>

(i) $Tf \sim f$ <u>for all</u> $f \in L^1(\mu_1)$.

(ii) T <u>is induced by a homomorphism</u> $\Phi: \Lambda_1 \to \Lambda$.

(iii) T <u>is d. s. and multiplicative.</u>

(iv) T <u>is d. s. and</u> T^*T <u>is the identity function on</u> $L^1(\mu_1)$.

(v) T <u>is d. s. and isometric.</u>

PROOF. We have already proved (ii) ⇒ (i) in (21.3). (i) ⇒ (ii): If $E \in \Lambda_1$ then $TC_E \sim C_E$ so there is a $\Phi(E) \in \Lambda$ such that $TC_E = C_{\Phi(E)}$ and $\mu(\Phi(E)) = \mu_1(E)$. Let $A, B \in \Lambda_1$ and $f = C_A + C_B = C_{A-A\cap B} + C_{B-A\cap B} + 2C_{A\cap B}$. Then $Tf = C_{\Phi(A)} + C_{\Phi(B)} = C_{\Phi(A-A\cap B)} + C_{\Phi(B-A\cap B)} + 2C_{\Phi(A\cap B)}$

so $\Phi(A\cap B) \subset \Phi(A)\cap\Phi(B)$ and thus to show equality it suffices to show they have equal measure. Now $\Phi(A)\cap\Phi(B) = \{Tf = 2\}$ and $A\cap B = \{f = 2\}$ so since $Tf \sim f$, $\mu(\Phi(A\cap B)) = \mu_1(A\cap B) = \mu(\Phi(A)\cap\Phi(B))$ and thus $\Phi(A)\cap\Phi(B) = \Phi(A\cap B)$ $[\mu]$.

If $A\cap B = \emptyset$ $[\mu_1]$ then $C_{A\cup B} = C_A + C_B$ μ-a. e. so applying T we get $C_{\Phi(A\cup B)} = C_{\Phi(A)} + C_{\Phi(B)} = C_{\Phi(A)\cup\Phi(B)}$ since $\mu(\Phi(A)\cap\Phi(B)) = \mu_1(A\cap B) = 0$. Hence $\Phi(A\cup B) = \Phi(A)\cup\Phi(B)$.

(ii) $\Rightarrow$ (iii). This is (21. 3).

(iii) $\Rightarrow$ (iv). Let $f \in L^1(\mu_1)$. For all $g \in L^\infty(\mu_1)$ we have $fg \in L^1(\mu_1)$ so $\int fg\, d\mu_1 = \int T(fg)\, d\mu = \int Tf\, Tg\, d\mu = \int g\, T^*Tf\, d\mu_1$ and thus $f = T^*Tf$.

(iv) $\Rightarrow$ (v). Let $f, h, fh \in L^1(\mu_1)$. Then $\int fh\, d\mu_1 = \int h\, T^*Tf\, d\mu_1$ $= \int Tf\, Th\, d\mu$.

(v) $\Rightarrow$ (ii) [3, p. 22]. Let $A, B \in \Lambda_1$. $\int TC_A\, TC_B\, d\mu = \int C_A C_B\, d\mu_1$ $= \int C_{A\cap B}\, d\mu_1 = \int TC_{A\cap B}\, d\mu$. Since T is d. s., $0 \le TC_A \le 1$ so $0 \le (TC_A)^2 \le TC_A \le 1$, and for $A = B$, $\int TC_A - (TC_A)^2\, d\mu = 0$ so $(TC_A)^2 = TC_A$ and thus TC_A takes only the values 0 and 1. Letting $\Phi(A) = \{TC_A = 1\}$ we have $TC_A = C_{\Phi(A)}$. Now $TC_{A\cap B} \le \min\{TC_A, TC_B\}$ so $0 \le TC_{A\cap B} = (TC_{A\cap B})^2 \le (TC_A)(TC_B)$ and since $\int TC_{A\cap B} d\mu =$ $\int TC_A\, TC_B\, d\mu$ we have $TC_{A\cap B} = TC_A\, TC_B$ so $\Phi(A\cap B) = \Phi(A)\cap\Phi(B)$. When $A\cap B = \emptyset$, apply T to $C_{A\cup B} = C_A + C_B$ to deduce $\Phi(A\cup B) = \Phi(A)\cup\Phi(B)$. Finally, $\mu(\Phi(A)) = \int TC_A\, d\mu = \int C_A\, d\mu_1 = \mu_1(A)$ since T is d. s.

(21.5) COROLLARY. Let $\Phi : \Lambda_1 \to \Lambda$ be a homorphism. Then

(i) T_Φ is 1:1.

(ii) T_Φ^* maps $L^\infty(\mu)$ onto $L^\infty(\mu_1)$ and $L^1(\mu)$ onto $L^1(\mu_1)$.

PROOF. $T_\Phi^* T_\Phi$ = the identity mapping on $L^1(\mu_1)$ and $T_\Phi : L^\infty(\mu_1) \to L^\infty(\mu)$.

(21.6) THEOREM. Let $\Phi : \Lambda_1 \to \Lambda$ be a homorphism. Then the following are equivalent.

(i) T_Φ^* is 1 : 1.

(ii) $T_\Phi T_\Phi^*$ is the identity mapping on $L^1(\mu)$.

(iii) The range of T_Φ is all of $L^1(\mu)$.

(iv) Φ is an isomorphism .

(v) T_Φ^* is induced by a homorphism of $\Lambda \to \Lambda_1$.

(vi) T_Φ is unitary.

PROOF. (i) $\Rightarrow$ (ii). Let $f \in L^1(\mu_1)$. Then $T_\Phi^*(T_\Phi T_\Phi^* f) = T_\Phi^* f$ so $T_\Phi T_\Phi^* f = f$.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (iv). Let $A \in \Lambda$. Then there is an $f_A \in L^1(\mu_1)$ such that $T_\Phi f_A = C_A$. For all $g \in L^\infty(\mu)$ we have $\int f_A^2 T_\Phi^* g \, d\mu_1 = \int g \, T_\Phi f_A^2 \, d\mu$

$$= \int g(T_\Phi f_A)(T_\Phi f_A) d\mu = \int g \, C_A \, d\mu = \int g \, T_\Phi f_A \, d\mu = \int f_A \, T_\Phi^* g \, d\mu_1 \quad .$$

Since T_Φ^* maps $L^\infty(\mu)$ onto $L^\infty(\mu_1)$ (this is implied by (21.5)(ii)) we have $f_A^2 = f_A$ so $f_A = C_B$ where $B = \{f_A = 1\}$. Then $C_{\Phi(B)} = T_\Phi f_A = C_A$ so $\Phi(B) = A \ [\mu]$.

(iv) ⇒ (v). $\Phi^{-1}: \Lambda \rightarrow \Lambda_1$ is a homorphism, and

$T_\Phi^* = T_\Phi^*(T_\Phi T_{\Phi^{-1}}) = (T_\Phi^* T_\Phi)T_{\Phi^{-1}} = T_{\Phi^{-1}}$.

(v) ⇒ (vi). If $T_\Phi^* = T_\Psi$ then T_Φ^* is isometric so T_Φ is unitary.

(vi) ⇒ (i). If T_Φ is unitary, then T_Φ^* is isometric and hence induced by a homomorphism, and it follows from (21.5) that it is 1:1.

REMARKS. 1. Let $\sigma: X \rightarrow X_1$ be measure preserving and let T_σ be defined on $M(X_1, \mu_1)$ by $T_\sigma f = f(\sigma)$. Then $T_\sigma C_E = C_{\sigma^{-1}(E)}$ for all $E \in \Lambda_1$ so $T_\sigma = T_\Phi$ where $\Phi(E) = \sigma^{-1}(E)$.

2. The relationship between multiplicative unitary operators (which are necessarily d. s.) and the operators T_σ where $\sigma: X \rightarrow X$ is measure preserving, was studied as early as 1932 by von Neumann [33, p. 618], who assumed that X is a complete separable metric space with a finite Borel measure μ such that spheres have positive measure and every measurable set is contained in a G_δ with the same measure.

3. With (X, Λ, μ) and σ as in 2, Paul R. Halmos has used the operators T_σ to find necessary and sufficient conditions for the existence of a square root of σ. [13]

4. It follows from (20. 2) that the operators T_Φ are extreme points of $\mathfrak{D}(X_1, X)$ (another proof can be gleaned from [34, p. 269, Theorem 1.4]). It is easy to see that $T \in \mathfrak{D}(X_1, X)$ is extreme iff T^* is extreme in $\mathfrak{D}(X, X_1)$. Hence the operators T_Φ^* such that $\Phi: \Lambda \rightarrow \Lambda_1$ is a homomorphism are also extreme in $\mathfrak{D}(X_1, X)$. A complete characterization of the extreme points of $\mathfrak{D}(X_1, X)$ does not seem to be known.

22. The Weak Closure of $\{g: g \sim f\}$. As always let (X, Λ, μ) and (X_1, Λ_1, μ_1) be finite measure spaces with $\mu(X) = \mu_1(X_1) = a$. Recall that if $f \in L^1(X_1, \mu_1)$ then $\Omega_f(X, \mu) = \{g \in L^1(X, \mu): g \prec f\}$ and $\Delta_f(X, \mu) = \{g \in L^1(\mu): g \sim f\}$. The problem is to determine the $\sigma(L^1, L^\infty)$ closure of Δ_f. The case $\Delta_f = \emptyset$ is easy, while if $\Delta_f \neq \emptyset$ and $f_o \in \Delta_f$ then $\Omega_f = \Omega(f_o)$ and $\Delta_f = \Delta(f_o)$. Hence from the beginning we work only with the sets $\Omega(f)$ and $\Delta(f)$ where $f \in L^1(X, \mu)$.

J. V. Ryff has shown the following result which we state as a lemma. For the proof see [43, p. 432].

(22.1) LEMMA. If $F \in L^1[0, 1]$ then $\Omega(F) = \{G \in L^1[0, 1]: G \prec F\}$ is the $\sigma(L^1, L^\infty)$ closure of $\Delta(F) = \{G \in L^1[0, 1]: G \sim F\}$.

(22.2) THEOREM. $\Omega(f)$ is the $\sigma(L^1, L^\infty)$ closure of $\Delta(f)$ for all $f \in L^1(X, \mu)$ iff (X, Λ, μ) is non-atomic or X is an atom.

PROOF. Suppose (X, Λ, μ) is non-atomic. Let $g \in \Omega(f)$ and let $\sigma: X \to [0, a]$ be measure preserving such that $\delta_g \circ \sigma = g$. Let $F(t) = \delta_f(at)$ and $G(t) = \delta_g(at)$ on $[0, 1]$. Then $G \prec F$ so there is a net $\{H_\alpha\} \subset \Delta(F)$ such that $H_\alpha \to G$ in the $\sigma(L^1[0, 1], L^\infty[0, 1])$ topology. Letting $h_\alpha(t) = H_\alpha(t/a)$ on $[0, a]$ we have $h_\alpha \sim \delta_f$, and for all $v \in L^\infty[0, a]$, $\int_0^a h_\alpha\, v\, dt = a \int_0^1 H_\alpha(t) v(at)dt \to a \int_0^1 G(t) v(at)dt = \int_0^a \delta_g\, v\, dt$, i.e., $h_\alpha \to \delta_g$ in the $\sigma(L^1[0, a], L^\infty[0, a])$ topology. Since T_σ is weakly continuous, $T_\sigma h_\alpha \to T_\sigma \delta_g = g$ and $T_\sigma h_\alpha \in \Delta(f)$. Hence $\Delta(f)$ is weakly dense in $\Omega(f)$.

If X is an atom, then $\Delta(f) = \{f\} = \Omega(f)$.

For the converse, suppose X is not an atom and (X, Λ, μ) is not non-atomic. Then $X = A_1 \cup A_2$ where A_1 and A_2 are disjoint sets of positive measure, and X has an atom B. Let $f = 2C_{A_1} + C_{A_2}$, let $g_o = \frac{1}{a} \int f\, d\mu = 1 + \mu(A_1)/\mu(X)$ so $1 < g_o < 2$, and let $g = g_o C_X$, so $g \in \Omega(f)$. If $h \in \Delta(f)$ then $h = 2C_{B_1} + C_{B_2}$ where $\mu(B_i) = \mu(A_i)$ $i = 1, 2$, so $\int (g-h)C_B\, d\mu = g_o\, \mu(B) - 2\mu(B_1 \cap B) - \mu(B_2 \cap B)$ and $\mu(B \cap B_i) = 0$ or $\mu(B)$ since B is an atom. Hence for $\epsilon = \mu(B) \min(g_o - 1, 2 - g_o)$ there is no $h \in \Delta(f)$ such that $\left| \int (g-h)C_B\, d\mu \right| < \epsilon$ so $\Delta(f)$ is not dense in $\Omega(f)$.

(22.3) COROLLARY. <u>For any finite m. s.</u> (X, Λ, μ) <u>and</u> $f \in L^1(X, \mu)$ <u>we have that</u> $\Omega(f)$ <u>is the</u> $\sigma(L^1, L^\infty)$ <u>closure of</u> $\{T_\mu f' : f' \in L^1(X^\#, \mu^\#)$ and $f' \sim f\}$.

PROOF. Let $g \in \Omega(f)$. Since $(X^\#, \mu^\#)$ is non-atomic, there is a net $\{f'_\alpha\} \subset L^1(X^\#, \mu^\#)$ such that $f'_\alpha \to g$ in $\sigma(L^1(\mu^\#), L^\infty(\mu^\#))$. Since T_μ is weakly continuous (see §17 or use (9.2)), $T_\mu f'_\alpha \to T_\mu g = g$ in $\sigma(L^1(\mu), L^\infty(\mu))$.

If (X, Λ, μ) consists only of atoms of equal measure then $\Delta(f)$ is finite and hence weakly closed. We will now determine the weak closure of $\Delta(f)$ when X consists only of atoms, or X has only finitely many atoms, or when X is separable.

Let X_o be the non-atomic part of X, let $\{A_n\}_{n \in P}$ be the atoms of X and let $A = \bigcup_{n \in P} A_n$; also let $a_o = \mu(X_o)$ (see § 9).

(22.4) DEFINITION. If $f \in L^1(X,\mu)$ let $Z(f)$ be the set to which g belongs iff there is an $h \sim f$ such that $g|X_o \prec h|X_o$ and $g|A = h|A$.

(22.5) LEMMA. For all $f \in L^1$ we have $\Delta(f) \subset Z(f) \subset \overline{\Delta(f)}$.

PROOF. $\Delta \subset Z$ is easy. For the other inclusion, let $g \in Z$ so there is an $h \sim f$ such that $g|X_o \prec h|X_o$ and $g|A = h|A$. Since X_o is non-atomic there is a net $\{h_\alpha\} \subset L^1(X_o, \mu)$ with $h_\alpha \sim h|X_o$ and $h_\alpha \to g|X_o$ weakly. Extend each h_α to X by $h_\alpha|A = h|A$. Then $h_\alpha \sim h \sim f$ (see (3.3)(x)) and for each $v \in L^\infty(X,\mu)$ we have $v|X_o \in L^\infty(X_o, \mu)$ so $\int_{X_o} h_\alpha v \, d\mu \to \int_{X_o} g v \, d\mu$, but since $\int_A h_\alpha v \, d\mu = \int_A g v \, d\mu$ we have finally that $\int_X h_\alpha v \, d\mu \to \int_X g v \, d\mu$, i.e., $h_\alpha \to g$ weakly. Hence $g \in \overline{\Delta}$.

(22.6) THEOREM. If (X, Λ, μ) consists only of atoms, then $\overline{\Delta(f)} = Z(f)$ for all $f \in L^1(X,\mu)$.

PROOF. We have only to show that $\overline{\Delta(f)} \subset Z(f)$. Let $g \in \overline{\Delta(f)}$. Then there is a net $\{h_\alpha\} \subset \Delta(f)$ with $h_\alpha \to g$ weakly. Let B be an atom. Now $\mu(B) \le \mu(h_\alpha^{-1}(h_\alpha|B)) = \mu(f^{-1}(h_\alpha|B))$ and $f^{-1}(h_\alpha|B)) \cap f^{-1}(h_\beta|B) = \emptyset$ whenever $h_\alpha|B \neq h_\beta|B$ so $\mu(X) < \infty$ implies there are only finitely many different values $h_\alpha|B$. But $h_\alpha|B \to g|B$, so for some α_0, $\alpha \ge \alpha_0$ implies $h_\alpha|B = g|B$. Hence there is an increasing sequence α_n such that $\alpha \ge \alpha_n$ implies $h_\alpha|A_k = g|A_k$, $k = 1, \ldots, n$. Then $\|h_{\alpha_n} - g\|_1 \to 0$ so $g \sim f$ and hence $g \in \Delta(f) = Z(f)$ in this case.

I can now prove $\overline{\Delta(f)} = Z(f)$ in general.

(22.7) THEOREM. <u>If</u> (X, Λ, μ) <u>has only a finite number of atoms, then</u> $\overline{\Delta(f)} = Z(f)$.

PROOF. We have only to show that $\overline{\Delta(f)} \subset Z(f)$. It is easy to see that the condition $h_1|A = h_2|A$ defines an equivalence relation on $\Delta(f)$. Since there are only finitely many atoms, there are only finitely many equivalence classes, $H_1, \cdots, H_n$ say. $\Delta(f) = H_1 \cup \cdots \cup H_n$ so $\overline{\Delta(f)} = \overline{H_1} \cup \cdots \cup \overline{H_m}$. Let $g \in \overline{\Delta(f)}$. Then $g \in \overline{H_k}$ for some $1 \leq k \leq m$, so there is an $h_o \in H_k$ and a net $\{h_\alpha\} \subset H_k$ with $h_\alpha \to g$ weakly. Since $h_\alpha, h_o \in H_k$, $h_\alpha|A = h_o|A$ for all α. Let B be an atom of (X, Λ, μ) (so $B \subset A$). g is constant on B so $(g|B)\mu(B) = \int g\, C_B\, d\mu = \lim_\alpha \int h_\alpha C_B d\mu = (h_o|B)\mu(B)$ and thus $g|B = h_o|B$ since $\mu(B) > 0$. This holds for all atoms B, so $g|A = h_o|A$. Let $v \in L^\infty(X_o, \mu)$ and extend v to all of X by $v|A = 0$, so $v \in L^\infty(X, \mu)$. Then $\int_{X_o} g\, v\, d\mu = \int_X g\, v\, d\mu = \lim_\alpha \int_X h_\alpha\, v\, d\mu = \lim_\alpha \int_{X_o} h_\alpha v\, d\mu$ so $h_\alpha|X_o \to g|X_o$ weakly. But $h_\alpha|A = h_o|A$ and $h_\alpha \sim f \sim h_o$ so (3.4)(ii) implies $h_\alpha|X_o \sim h_o|X_o$. Hence $g|X_o \in \overline{\Delta(h_o|X_o)} = \Omega(h_o|X_o)$, i.e., $g|X_o \prec h_o|X_o$.

(22.8) PROPOSITION. <u>If</u> $f, g \in L^1(X, \mu)$ <u>and</u> $g \prec f$ <u>and</u> $g|A = f|A$ <u>then</u> $g|X_o \prec f|X_o$.

PROOF. Define F_n, G_n inductively by $G_1 = g$, $F_1 = f$, $G_{n+1} = G_n C_{X-A_n}$, $F_{n+1} = F_n C_{X-A_n}$. Since $g \prec f$ we have $G_1 \prec F_1$. Since $f|A = g|A$ we have by induction using (8.7) that $G_n \prec F_n$, $n \in P$.

If X has only a finite number of atoms, then for some $n \in P$, $g\,C_{X_o} = G_n \prec F_n = f\,C_{X_o}$ and (8.7) implies $g|X_o \prec f|X_o$. Otherwise, $G_n \downarrow g\,C_{X_o}$ and $F_n \downarrow f\,C_{X_o}$ and $\|F_n - f\,C_{X_o}\|_1 = \int F_n - f\,C_{X_o}\,d\mu$ $= \int f\,C_{\bigcup_{i=n}^{\infty} A_i}\,d\mu \to 0$ as $n \to \infty$ since $f \in L^1$. Similarly, $\|G_n - g\,C_{X_o}\|_1 \to 0$. Hence (12.5) implies $g\,C_{X_o} \prec f\,C_{X_o}$ so $g|X_o \prec f|X_o$.

REMARK. If $f \,\&\, g \notin L^1$ then (22.8) may fail since we could have $\int_A f\,d\mu = \int_A g\,d\mu = +\infty$ with $\int_0^{a_0} \delta_{g|X_o} < \int_0^{a_0} \delta_{f|X_o}$

(22.9) LEMMA. <u>If</u> $f \in L^1(X,\mu)$ <u>then</u> $Z(f)$ <u>is</u> $\sigma(L^1, L^\infty)$ <u>sequentially closed</u>.

PROOF. Let the sequence $\{g_n\} \subset Z(f)$ with $g_n \to g$ weakly. Then $g_n|A \to g|A$ pointwise, and there are functions $h_n \sim f$ such that $g_n|X_o \prec h_n|X_o$ and $g_n|A = h_n|A$, $n = 1, 2, 3, \ldots$ Let $\{[a_n, b_n[\}_{n\in P}$ be pairwise disjoint intervals such that $\bigcup_{n \in P} [a_n, b_n[= [a_0, a[$ and $b_n - a_n = \mu(A_n)$, $n \in P$ (see §9). For each $n = 1, 2, 3, \ldots$ let

$$H_n = \delta_{h_n|X_o} C_{[0, a_0[} + \sum_{k \in P} (h_n|A_k) C_{[a_k, b_k[} .$$

Then $\{H_n\} \subset [-\infty, \infty]^{[0, a[}$ which is compact in the product topology, so there is a subsequence $\{H_{n_k}\}_{k=1}^{\infty}$ which converges pointwise everywhere to a function H. $H|[0, a_0[$ is the limit of a sequence of decreasing functions so it is decreasing, and X_o is non-atomic, so there is an $h \in M(X_o, \mu)$ such that $\delta_{h|X_o} = H|[0, a_0[$ a.e.. Extend h to X by $h|A = g|A$. Now $H_n \sim h_n \sim f$ and $H_{n_k} \to H$ pointwise so

$H \sim f$. Also $H | [0, a_0[= \delta_{h|X_o} \sim h|X_0$, and since

$$\begin{array}{ccc} H_{n_k} | [a_0, a[\sim h_{n_k} |A = g_{n_k} |A \\ \downarrow \qquad\qquad\qquad \downarrow \\ H| [a_0, a[\qquad\qquad g|A = h|A \end{array}$$

pointwise, we have $H|[a_0, a[\sim h|A$. Hence $h \sim H \sim f \in L^1$ and thus $h \in L^1$. Since $\Omega(f)$ is weakly closed, $g \in \Omega(f)$ so $g \prec f \sim h$ and hence $g \prec h$. Since $g|A = h|A$, we have using (22.8) that $g|X_o \prec h|X_o$. Thus $g \in Z(f)$.

Recall that the metric space associated with a finite m. s. (X, Λ, μ) is $(\Lambda(\mu), d)$ where $\Lambda(\mu)$ is Λ modulo the sets of measure zero and $d(A, B) = \mu(A-B) + \mu(B-A)$. $\Lambda(\mu)$ will be viewed as Λ with the equality $A = B[\mu]$ iff $C_A = C_B$ μ-a.e. A finite m. s. is said to be separable if its associated metric space is separable. Note that Lebesgue measure on bounded subsets of R^k and Stieltjes-Lebesgue measure on bounded subsets of R are separable [47, p. 69].

(22.10) PROPOSITION. (i) $(X_o, \Lambda \cap X_o, \mu)$ *is separable iff* (X, Λ, μ) *is separable.*

(ii) *If* (X, Λ, μ) *is separable then for each* $f \in L^1(X, \mu)$ *the relative* $\sigma(L^1, L^\infty)$ *topology on* $\Omega(f)$ *is metrizable.*

PROOF. (i) $(\Lambda \cap X_o(\mu), d)$ is a subspace of $(\Lambda(\mu), d)$ so $(\Lambda(\mu), d)$ separable implies $(\Lambda \cap X_o(\mu), d)$ is separable. Conversely, if $(\Lambda \cap X_o(\mu), d)$ is separable, then the union of the atoms of X and a countable dense subset of $(\Lambda \cap X_o(\mu), d)$ is countable and dense in

$(\Lambda(\mu), d)$.

(ii) Let $\mathscr{B}$ be a countable dense subset of $(\Lambda(\mu), d)$ and let $f \in L^1$. Then $\Omega(f)$ is weakly compact, so according to [20, p. 143, Theorem 16.7] we have only to show there is a countable subset of L^∞ which separates points of L^1. Let $\mathscr{D} = \{C_E: E \in \mathscr{B}\}$, so $\mathscr{D}$ is countable. To show that $\mathscr{D}$ separates points of L^1 let $g, h \in L^1$. If $\int (g-h)C_E \, d\mu = 0$ for all $C_E \in \mathscr{D}$ then $\int_E (g-h)d\mu = 0$ on a dense subset of $(\Lambda(\mu), d)$. Since $g-h \in L^1$, $E \to \int_E (g-h)d\mu$ is continuous on $(\Lambda(\mu), d)$ and hence we conclude that $\int_E (g-h)d\mu = 0$ for all $E \in \Lambda(\mu)$ so $g = h$.

(22.11) THEOREM. <u>If</u> $(X_0, \Lambda \cap X_0, \mu)$ <u>is separable then for every</u> $f \in L^1(X, \mu)$ <u>we have</u> $\overline{\Delta(f)} = Z(f)$.

PROOF. Now the weak topology on $\Omega(f)$ is metrizable so $Z(f)$ is closed and thus $\Delta(f) \subset Z(f) \subset \overline{\Delta(f)}$ implies $\overline{\Delta(f)} = Z(f)$.

Now suppose ρ is a saturated Fatou norm on $M(X, \mu)$ such that $L^\infty \subset L^\rho$, $L^{\rho'} \subset L^1$ and L^ρ is u.r.i. If $f \in L^\rho$ then $\Delta(f) \subset L^\rho$ and the problem is to determine the $\sigma(L^\rho, L^{\rho'})$-closure of $\Delta(f)$. If $A \subset L^\rho$ we denote its $\sigma(L^\rho, L^{\rho'})$ closure by ${}^\rho\overline{A}$. Since $\sigma(L^\rho, L^\infty) \subset \sigma(L^\rho, L^{\rho'})$ we see that ${}^\rho\overline{A} \subset \overline{A}$.

By examining the proof of (22.10) we see that if $f \in L^\rho$ and $(X_0, \Lambda \cap X_0, \mu)$ is separable, then the relative $\sigma(L^\rho, L^{\rho'})$ topology on $\Omega(f)$ is metrizable.

(22.12) PROPOSITION. <u>If the simple functions are dense in</u> $L^{\rho'}$ <u>then for every</u> $f \in L^{\rho}$ <u>and</u> $A \subset \Omega(f)$, <u>the</u> $\sigma(L^{\rho}, L^{\rho'})$ <u>closure of</u> A <u>equals the</u> $\sigma(L^1, L^{\infty})$ <u>closure of</u> A.

PROOF. Now $\Omega(f)$ is ρ-bounded so there is an $M > 0$ such that $\rho(g) \leq M$ for all $g \in \Omega(f)$. We have only to show that $\overline{A} \subset {}^{\rho}\overline{A}$. Now ${}^{\rho}\overline{A} \subset \Omega(f)$ by (17.4). Let $g_o \in \overline{A}$. Then there is a net $\{g_{\alpha}\} \subset A$ with $g_{\alpha} \to g_o$ in $\sigma(L^1, L^{\infty})$. If $h \in L^{\rho'}$, $F_h(g) = \int g\, h\, d\mu$ defines a continuous linear functional on L^{ρ}. It suffices to show that $F_h(g_{\alpha}) \to F_h(g_o)$ for all $h \in L^{\rho'}$. Hence let $h \in L^{\rho'}$ and let $\varepsilon > 0$. Then there is a simple function v such that $\rho'(h-v) < \varepsilon$, so for all $g \in \Omega(f)$, $|F_h(g) - F_v(g)| = |\int (h-v) g\, d\mu| \leq \rho(g)\rho'(h-v) \leq M\varepsilon$. Now there is an α_o such that $\alpha \geq \alpha_o$ implies $|F_v(g_{\alpha}) - F_v(g_o)| < \varepsilon$. Hence for $\alpha \geq \alpha_o$, $|F_h(g_{\alpha}) - F_h(g_o)| \leq |F_h(g_{\alpha}) - F_v(g_{\alpha})| + |F_v(g_{\alpha}) - F_v(g_o)| + |F_v(g_o) - F_h(g_o)| \leq M\varepsilon + \varepsilon + M\varepsilon$. Thus $F_h(g_{\alpha}) \to F_h(g_o)$.

(22.13) THEOREM. <u>If the simple functions are dense in</u> $L^{\rho'}$ <u>and</u> $f \in L^{\rho}$ <u>then the</u> $\sigma(L^{\rho}, L^{\rho'})$ <u>closure of</u> $\Delta(f)$ <u>equals the</u> $\sigma(L^1, L^{\infty})$ <u>closure of</u> $\Delta(f)$.

REMARK. The intuitive idea behind the definition of Z is that every member of $\overline{\Delta(f)}$ can be reached by a net in which eventually the rearrangements of f are formed by rearrangements on X_o and rearrangements on A. This means that if $g \in \overline{\Delta(f)}$ there is a net $\{h_{\alpha}\} \subset \Delta(f)$ and an index α_o such that for $\alpha, \beta \geq \alpha_o$ we have

$h_\alpha | X_o \sim h_\beta | X_o$ and $h_\alpha | A \sim h_\beta | A$. In this case $h_\alpha | X_o \to g | X_o$ implies $g | X_o < h_{\alpha_o} | X_o$ and $h_\alpha | A \to g | A$ implies $g | A \sim h_{\alpha_o} | A$. Hence defining h by $h | X_o = h_{\alpha_o} | X_o$ and $h | A = g | A$ we have $h \sim f$, $g | X_o < h | X_o$ and $g | A = h | A$.

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