REARRANGEMENTS OF MEASURABLE FUNCTIONS

Thesis by
Peter William Day

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ABSTRACT

Let \((X, \Lambda, \mu)\) be a measure space and let \(M(X, \mu)\) denote the set of all extended real valued measurable functions on \(X\). If \((X_1, \Lambda_1, \mu_1)\) is also a measure space and \(f \in M(X, \mu)\) and \(g \in M(X_1, \mu_1)\), then \(f\) and \(g\) are said to be equimeasurable (written \(f \sim g\)) iff \(\mu(f^{-1}[r, s]) = \mu_1(g^{-1}[r, s])\) whenever \([r, s]\) is a bounded interval of the real numbers or \([r, s] = (+\infty)\) or \([-\infty, -\infty]\). Equimeasurability is investigated systematically and in detail.

If \((X, \Lambda, \mu)\) is a finite measure space (i.e. \(\mu(X) < \infty\)) then for each \(f \in M(X, \mu)\) the decreasing rearrangement \(\delta_f\) of \(f\) is defined by

\[
\delta_f(t) = \inf \{s: \mu(\{f > s\}) < t\} \quad 0 \leq t \leq \mu(X).
\]

Then \(\delta_f\) is the unique decreasing right continuous function on \([0, \mu(X)]\) such that \(\delta_f \sim f\). If \((X, \Lambda, \mu)\) is non-atomic, then there is a measure preserving map \(\sigma: X \to [0, \mu(X)]\) such that \(\delta_f(\sigma) = f \mu\text{-a.e.}\).

If \((X, \Lambda, \mu)\) is an arbitrary measure space and \(f \in M(X, \mu)\) then \(f\) is said to have a decreasing rearrangement iff there is an interval \(J\) of the real numbers and a decreasing function \(\delta\) on \(J\) such that \(f \sim \delta\). The set \(D(X, \mu)\) of functions having decreasing rearrangements is characterized, and a particular decreasing rearrangement \(\delta_f\) is defined for each \(f \in D\). If \(\text{ess.} \inf f \leq 0 < \text{ess.} \sup f\), then \(\delta_f\) is obtained as the right inverse of a distribution function of \(f\). If \(\text{ess.} \inf f < 0 < \text{ess.} \sup f\) then formulas relating \((\delta_f)^+\) to \(\delta_{f^+}\), \((\delta_f)^-\) to \(\delta_{f^-}\) and \(\delta_f^+\) to \(\delta_f^-\) are given. If \((X, \Lambda, \mu)\) is non-atomic and \(\sigma\)-finite and \(\delta\) is a decreasing rearrangement of \(f\) on \(J\), then there is a measure preserving map \(\sigma: X \to J\) such that \(\delta(\sigma) = f \mu\text{-a.e.}\).
If \((X, \Lambda, \mu)\) and \((X_1, \Lambda_1, \mu_1)\) are finite measure spaces such that 
\[ a = \mu(X) = \mu_1(X_1), \]
if \(f, g \in M(X, \mu) \cup M(X_1, \mu_1)\), and if 
\[ \int_0^a \delta_{f(t)} \text{ and} \]
\[ \int_0^a \delta_{g(t)} \]
are finite, then \(g \ll f\) means \(\int_0^t \delta_{g(s)} \leq \int_0^t \delta_{f(s)}\) for all \(0 \leq t \leq a\), 
and \(g < f\) means \(g \ll f\) and 
\[ \int_0^a \delta_{f(t)} = \int_0^a \delta_{g(t)}. \]
The preorder relations \(<\) and \(\ll\) are investigated in detail.

If \(f \in L^1(X, \mu)\), let \(\Omega(f) = \{g \in L^1(X, \mu): g < f\}\) and 
\[ \Delta(f) = \{g \in L^1(X, \mu): g \sim f\}. \]
Suppose \(\rho\) is a saturated Fatou norm on 
\(M(X, \mu)\) such that 
\(L^\rho\) is universally rearrangement invariant and 
\(L^\infty \subset L^\rho \subset L^1\). If \(f \in L^\rho\) then \(\Delta(f) \subset L^\rho\) and 
\(\Omega(f)\) is convex and 
\(\sigma(L^\rho, L^\rho')\)-compact. If \(\xi\) is a locally convex topology on \(L^\rho\) in 
which the dual of \(L^\rho\) is \(L^\rho'\), then \(\Omega(f)\) is the \(\xi\)-closed convex hull of 
\(\Delta(f)\) for all \(f \in L^\rho\) iff \((X, \Lambda, \mu)\) is adequate. More generally, if 
\(f \in L^1(X_1, \mu_1)\) let 
\[ \Omega_f(X, \mu) = \{g \in L^1(X, \mu): g < f\}\] 
and 
\[ \Delta_f(X, \mu) = \{g \in L^1(X, \mu): g \sim f\}. \]
Theorems for \(\Omega(f)\) and \(\Delta(f)\) are 
generalized to \(\Omega_f\) and \(\Delta_f\), and a norm \(\rho_1\) on \(M(X_1, \mu_1)\) is given 
such that 
\[ \Omega_f | f | \subset L^\rho\] 
iff \(f \in L^\rho 1\).

A linear map \(T: L^1(X_1, \mu_1) \rightarrow L^1(X_1, \mu_1)\) is said to be doubly stochastic 
iff \(Tf < f\) for all \(f \in L^1(X_1, \mu_1)\). It is shown that \(g < f\) iff there is a 
doubly stochastic \(T\) such that \(g = Tf\).

If \(f \in L^1\) then the members of \(\Delta(f)\) are always extreme in 
\(\Omega(f)\). If \((X, \Lambda, \mu)\) is non-atomic, then \(\Delta(f)\) is the set of extreme points 
and the set of exposed points of \(\Omega(f)\).

A mapping \(\phi: \Lambda_1 \rightarrow \Lambda\) is called a homomorphism if 
(i) \(\mu(\phi(A)) = \mu_1(A)\) for all \(A \in \Lambda_1\); (ii) \(\phi(A \cup B) = \phi(A) \cup \phi(B) [\mu]\) whenever 
\(A \cap B = \emptyset [\mu_1]\); and (iii) \(\phi(A \cap B) = \phi(A) \cap \phi(B) [\mu]\) for all \(A, B \in \Lambda_1\).
where $A = B \mu$ means $C_A = C_B \mu$-a.e. If $\hat{\psi}: \Lambda_1 \to \Lambda$ is a homomorphism, then there is a unique doubly stochastic operator $T_\hat{\psi}: L^1(X_1, \mu_1) \to L^1(X, \mu)$ such that $T_\hat{\psi}C_E = C_\hat{\psi}(E)$ for all $E$. If $T: L^1(X_1, \mu_1) \to L^1(X, \mu)$ is linear then $Tf \sim f$ for all $f \in L^1(X_1, \mu_1)$ iff $T = T_\hat{\psi}$ for some homomorphism $\hat{\psi}$.

Let $X_0$ be the non-atomic part of $X$ and let $A$ be the union of the atoms of $X$. If $f \in L^1(X, \mu)$ then the $\sigma(L^1, L^\infty)$-closure of $A(f)$ is shown to be $\{g \in L^1: \text{there is an } h \sim f \text{ such that } g|_{X_0} < h|_{X_0} \text{ and } g|_A = h|_A\}$ whenever either (i) $X$ consists only of atoms; (ii) $X$ has only finitely many atoms; or (iii) $X$ is separable.
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1. Introduction. The decreasing rearrangement of a non-negative measurable function has, since its treatment in *Inequalities* by Hardy, Littlewood and Polya, played an increasingly important role in analysis, because of its fundamental part in the structure of normed spaces of measurable functions which are rearrangement invariant. Examples of such spaces are the classical $L^p$ spaces, the Orlicz spaces, and the spaces introduced by Halperin [14] and Lorentz [23]. These so-called rearrangement invariant Banach function spaces have been shown by Boyd [2], Shimogaki [45], and Lorentz and Shimogaki [25] to be well suited for studying problems related to Fourier analysis and interpolation of operators.

Recently Luxemburg [28] gave a general account of the theory of rearrangement invariant Banach spaces for measure spaces with finite total measure. Such spaces provide a natural setting in which to generalize a theorem of Hardy, Littlewood and Polya which gives equivalent conditions that two vectors in $\mathbb{R}^n$ be related by a certain preorder relation $\prec$. J. V. Ryff [42] has given the generalization for $L^1[0, 1]$, while Luxemburg [28] has given it in part in the rearrangement invariant Banach space setting. We will complete this generalization in Chapter V.

With an eye to studying these topics when the total measure of the space is not finite, in Chapter I we investigate the concept of equimeasurability for arbitrary measure spaces, though we include as well results which can only be proved in general for finite measure spaces.
In Chapter II we define the decreasing rearrangement for all measurable functions if the measure space is finite. If the measure space is not finite, we characterize the set of measurable functions which have decreasing rearrangements, and define one for each such function. Of importance is the fact that we can prove a theorem relating a function to its decreasing rearrangement by a measure preserving transformation when the measure space is non-atomic and σ-finite.

In Chapter III we introduce the generalization to measurable functions on a finite measure space of the Hardy-Littlewood-Polya preorder relation $<$, and investigate it and some associated inequalities. In particular we give a new and careful proof of a theorem in [28] about the values taken on by certain integrals, and we characterize adequate measures.

For completeness we include as Chapter IV Luxemburg's treatment of rearrangement invariant Banach spaces. For the same reason we include his results on Schur convex functions in Chapter V, where we give finally a complete account of the generalization of the theorem of Hardy, Littlewood and Polya referred to above. We also extend Ryff's generalization of Muirhead's theorem to finite measure spaces. Finally in Chapter VI we settle some extremal and related problems of some sets which arose in Chapter V.

Throughout we use the following abbreviations:

m.s. = measure space; m.p. = measure preserving; pwd = pairwise disjoint.
I. EQUIMEASURABILITY

2. Spectral Measures. Let \((X, \Lambda, \mu)\) be a measure space (m.s.), i.e.,
\(X\) is a non-empty point set, \(\Lambda\) is a \(\sigma\)-algebra of subsets of \(X\), and \(\mu\) is a countably additive measure on \(\Lambda\). Often we will write \(\int f \, d\mu\) for the integral of \(f\) over \(X\) with respect to \(\mu\) when \(X\) is clear from the context.

Also we let \(M = M(X, \mu)\) denote the set of all extended real valued \(\mu\)-measurable functions on \(X\), and if \(E\) is a set, then \(\mathcal{C}_E\) denotes the characteristic function of \(E\).

Let \(\mathbb{R}\) denote the real numbers, let \(\mathbb{R}^\#\) denote the extended real numbers, and let \(S = S(\mathbb{R}^\#)\) denote the Riesz space of all functions \(s\) of the form

\[ s = \sum_{k=1}^{n} \alpha_k \mathcal{C}_{I_k} \]

where each \(\alpha_k \in \mathbb{R}\) and each \(I_k\) is a bounded interval of \(\mathbb{R}^\#\), i.e., \(I_k\) is an interval of \(\mathbb{R}\) of finite length or \(I_k = [+\infty]\) or \(I_k = [-\infty]\). We call the members of \(S\) step functions on \(\mathbb{R}^\#\).

For every \(s \in S\) and \(f \in M\) it is easy to see that \(s(f)\) is a simple measurable function on \(X\). If \(f \in M\), let \(I_f : \mathcal{C} \rightarrow \mathbb{R}^\#\) be defined by

\[ I_f(s) = \int s(f) \, d\mu \]

for all \(s \in S\). Suppose \(f\) has the property that \(\mu(f^{-1}([u, v])) < \infty\) for every bounded interval \([u, v]\) of \(\mathbb{R}^\#\). Then \(I_f(s)\) is finite for every \(s \in S\) and in fact defines a positive linear functional which is continuous in the sense that if \(s_n \downarrow 0\) then \(I_f(s_n) \downarrow 0\). Hence there is a measure \(\mu_f\) on \(\mathbb{R}^\#\) such that for every \(s \in S\) we have

\[ \int s(f) \, d\mu = I_f(s) = \int_{\mathbb{R}^\#} s \, d\mu_f. \]
We call $\mu_f$ the *spectral measure* of $f$, or sometimes the $\mu$-*spectral measure* of $f$. In probability theory $\mu_f$ is known as the distribution measure of $f$.

For many purposes, whether or not $f$ takes the value 0 on a set of finite or infinite measure is of no interest, so our conditions under which $\mu_f$ is defined may seem unduly restrictive. However, we have only to let $X' = \{x \in X : f(x) \neq 0\}$ (called the *carrier* of $f$) $\Lambda' = \Lambda \cap X'$ and $\mu' = \mu|\Lambda'$ to see that if $\mu(f^{-1}([u,v])) < \infty$ for all bounded intervals $[u,v]$ of $R^*$ and thus the $\mu'$-*spectral measure* $\mu'_f|X'$ is defined. We may in a similar manner ignore whether or not $f$ takes the values $+\infty$ or $-\infty$ on sets of finite or infinite measure.

Observe that if $[u,v]$ is a bounded interval of $R^*$, then $\mu_f([u,v]) = \mu(f^{-1}[u,v])$. If $(X, \Lambda, \mu)$ is a *finite m.s.* (i.e. $\mu(X) < \infty$), then $\mu_f$ is defined for every $f \in M$ and can be represented by the distribution function $d_f$ defined for $t \in R$ by

$$d_f(t) = \mu(\{x \in X : f(x) > t\}).$$

Letting $e_f(t) = \mu(\{x \in X : f(x) \leq t\})$ for every $t \in R$ we have that $d_f + e_f = \mu(X)$, $d_f$ is decreasing, $e_f$ is increasing, both $d_f$ and $e_f$ are right continuous, $d_{f_n} \uparrow d_f$ and $e_{f_n} \downarrow e_f$ whenever $f_n \uparrow f$, $\lim_{t \to -\infty} d_f(t) = \mu(\{f = +\infty\})$, $\lim_{t \to -\infty} e_f(t) = \mu(\{f = -\infty\})$, $d_f(t-) = \mu(\{x : f(x) \geq t\})$, and $e_f$ is continuous at $t$ iff $d_f$ is continuous at $t$ iff $\mu(\{f = t\}) = 0$.

(2.1) **PROPOSITION.** If $(X, \Lambda, \mu)$ is a *finite m.s.* and $f_n \to f$ *pointwise* a.e. then $d_{f_n} \to d_f$ at every point of continuity of $d_f$, so $d_{f_n} \to d_f$ pointwise a.e.
PROOF. Let $E = \{ x : f_n(x) \neq f(x) \}$. Then $\mu(E) = 0$. Let $t \in \mathbb{R}$ and let $A_n = \{ t_n > t \}$ and $A = \{ f > t \}$. Then $A \subset E \cup \liminf A_n \subset E \cup \limsup A_n \subset \{ f = t \}$. If $d_f$ is continuous at $t$, then $\mu(\{ f = t \}) = 0$ so $\mu(A) \leq \mu(\liminf A_n) \leq \liminf \mu(A_n) \leq \limsup \mu(A_n) \leq \mu(\limsup A_n) \leq \mu(A)$ and hence $d_f(t) = \mu(A) = \lim \mu(A_n) = \lim d_{f_n}(t)$. Since $d_f$ is decreasing on $\mathbb{R}$, $d_f$ has only countably many discontinuities.

REMARK. If $f = g$ a.e. then $s(f) = s(g)$ a.e. for each $s \in S$. Thus for each $s \in S$ we may define a mapping $T_s : L^1(X, \lambda, \mu) \to L^1(X, \lambda, \mu)$ by $T_s f = s(f)$. Let $S_1 = \{ s \in S : s = C_{\{ u < v \}} \ \text{u \ & \ v \ rational} \}$. Then $\{ T_s : s \in S_1 \}$ separates points of $L^1$.

For if $f \& g \in L^1$ differ on a set of positive $\mu$-measure, then at least one of $\{ f < g \}$ or $\{ g < f \}$ has positive measure. By symmetry we may assume $\{ f < g \}$ has positive measure. If $\{ \beta_i \}_{i=1}^\infty$ is an enumeration of all rationals of $\mathbb{R}$, then $\{ f < g \} = \bigcup_{i=1}^\infty \{ f < \beta_i < g \}$ so there is a rational $v$ such that $\mu(\{ f < v < g \}) > 0$. Since $f \in L^1$, $\mu(\{ f = -\infty \}) = 0$ so there is a rational number $u$ s.t. $\mu(\{ u < f < v < g \}) > 0$. Letting $s = C_{\{ u < f < v \}}$ we have $s(f) = C_{\{ u < f < v \}}$ so $\{ s(f) \# s(g) \} \supset \{ u < f < v < g \}$ and thus $s(f)$ and $s(g)$ differ on a set of positive measure.

3. Spectral Equivalency. Let $(X_1, \lambda_1, \mu_1)$ and $(X_2, \lambda_2, \mu_2)$ be measure spaces (m. s.) with $f_1 \in M(X_1, \mu_1)$ and $f_2 \in M(X_2, \mu_2)$. If both m. s. are finite, then we say that $f_1$ and $f_2$ are spectrally equivalent or equimeasurable iff $\mu_{f_1} = \mu_{f_2}$ and in this case we write $f_1 \sim f_2$. Then $f_1 \sim f_2$ iff $d_{f_1} = d_{f_2}$ iff $\mu_1(\{ f_1^{-1}[u, v] \}) = \mu_2(\{ f_2^{-1}[u, v] \})$ for every bounded interval $[u, v]$ of $\mathbb{R}^\#$. 
In case one of the m.s. is not finite we write $f_1 \sim f_2$ iff
$$\mu_1(f_1^{-1}([u, v])) = \mu_2(f_2^{-1}([u, v]))$$
for every bounded interval $[u, v]$ of $\mathbb{R}^\#$. Then $f_1 \sim f_2$ iff
$$\mu_1(f_1^{-1}([B])) = \mu_2(f_2^{-1}([B]))$$
for every Borel set $B$ of $\mathbb{R}^\#$.

Observe that if there exist $f_1 \in M(X_1, \mu_1)$ and $f_2 \in M(X_2, \mu_2)$ such that $f_1 \sim f_2$ then $\mu_1(X_1) = \mu_2(X_2)$ in the sense that both are infinite or finite and equal.

(3.1) Lemma. If $f \sim g$ and $f$ is a simple function, then $g$ is a simple function. Two simple functions are equimeasurable iff they take the same value on sets of equal measure.

Proof. Let $f = \sum_{i=1}^{n} a_i C_{E_i}$ where $a_1 < \ldots < a_n$ and $\{E_i\}$ partitions $X_1$. Suppose $g \in M(X_2, \mu_2)$ and $g \sim f$. Then $\mu_2([g = a_i]) = \mu_1(E_i)$.

$$\mu_2(\{g \notin \{a_1, \ldots, a_n\}\}) = \mu_2(g^{-1}(-\infty, a_1])$$
$$+ \sum_{i=1}^{n-1} \mu_2(g^{-1}[a_i, a_{i+1}[) + \mu_2(g^{-1}[a_n, +\infty])$$
$$= \mu_1(f^{-1}(-\infty, a_1])$$
$$+ \sum_{i=1}^{n-1} \mu_1(f^{-1}[a_i, a_{i+1}[) + \mu_1(f^{-1}[a_n, +\infty])$$
$$= \mu_1(\{f \notin \{a_1, \ldots, a_n\}\}) = 0.$$
(3.2) PROPOSITION. Let $F$ be a Borel measurable function on an interval $I \subseteq \mathbb{R}^d$. If $G \in M(I)$ and $r > 0$ and $F_1(t) = F(rt)$ and $G_1(t) = G(rt)$ on $\frac{1}{r} I$ then $G \sim F$ iff $G_1 \sim F_1$. In addition, each of the following functions is equimeasurable with $F$.

(i) $H(t) = F(t-r)$ \quad $r \in \mathbb{R}$, $t \in I + r$
(ii) $H(t) = F(-t)$ \quad $t \in - I$
(iii) $H(t) = F(t+)$ \quad if $F$ is monotonic
(iv) $H(t) = F(t-)$ \quad if $F$ is monotonic

PROOF. Let $m$ denote Lebesgue measure, and let $[u, v]$ be a bounded interval of $\mathbb{R}^d$. $m(F_1^{-1}[u,v]) = \frac{1}{r} m(G_1^{-1}[u,v])$

so $m(F_1^{-1}[u,v]) = m(G_1^{-1}[u,v])$ iff $m(F_1^{-1}[u,v]) = m(G_1^{-1}[u,v])$.

(i) $m(H^{-1}[u,v]) = m(F^{-1}[u,v])$ and $m(F^{-1}[u,v]) = m(G^{-1}[u,v])$.
(ii) $m(H^{-1}[u,v]) = m(-F^{-1}[u,v]) = m(-G^{-1}[u,v])$.
(iii) & (iv) $H = F$ a.e.

(3.3) PROPOSITION. The following are true for all measure spaces.

(i) $f \sim g$ implies $s(f) \sim s(g)$ for all $s \in S$
(ii) $f \sim g$ implies $rf \sim rg$ and $f+r \sim g+r$ for all $r \in \mathbb{R}$
(iii) $f \sim g$ implies $|f| \sim |g|$
(iv) $f \sim g$ implies $f^+ \sim g^+$ and $f^- \sim g^-$
(v) If $\sigma: X_1 \rightarrow X_2$ is measure preserving and $f \in M(X_2, \mu_2)$ then $f \circ \sigma \sim f$.
(vi) $f \sim g$ implies ess. inf $f = \text{ess. inf} g$ and ess. sup $f = \text{ess. sup} g$.
(vii) If $f \sim g$ and $f \geq 0$ a.e. then $g \geq 0$ a.e. and there are sequences $\{f_n\}$ and $\{g_n\}$ of simple measurable functions such that $f_n \sim g_n$ and $0 \leq f_n \uparrow f$ and $0 \leq g_n \uparrow g$. In addition, if $f$ is essentially bounded then $g$ is essentially bounded and $f_n \uparrow f$ and $g_n \uparrow g$ uniformly.
(viii) If $f_1 \in M(X_1, \mu_1)$ and $f_2 \in M(X_2, \mu_2)$ and $f_1, f_2 \geq 0$ then $f_1 \sim f_2$ implies $\int f_1 \, d\mu_1 = \int f_2 \, d\mu_2$ in the sense that both are finite and equal, or both are infinite.

(ix) If $f_1 \in L^1(X_1, \mu_1)$ and $f_2 \in M(X_2, \mu_2)$ and $f_2 \sim f_1$ then $f_2 \in L^1(X_2, \mu_2)$ and $\int f_1 \, d\mu_1 = \int f_2 \, d\mu_2$.

(x) If $f \in M(X, \mu)$ and $g \in M(X', \mu')$ and $\{X_i\}_{i=1}^\infty$ and $\{X_i'\}_{i=1}^\infty$ are pairwise disjoint measurable subsets of $X$ and $X'$ respectively such that $\mu(X \cup X_i) = 0 = \mu'(X' \cup X_i')$ then $f|_{X_i} \sim g|_{X_i'}$ for $i = 1, 2, 3, \ldots$. implies $f \sim g$.

In addition the following are true for finite m.s.

(xi) If $f_1 \sim g_1$ and $f_2 \sim g_2$ and $\inf \{ |f_1|, |f_2| \} = 0 = \inf \{ |g_1|, |g_2| \}$ then $f_1 + f_2 \sim g_1 + g_2$.

(xii) $f \sim g$ iff $f^+ \sim g^+$ and $f^- \sim g^-$. 

(xiii) If $f_n \to f$ and $g_n \to g$ a.e. and $f_n \sim g_n$ n = 1, 2, 3, ..., then $f \sim g$.

(xiv) If $f_n, f \in L^1(X_1, \mu_1)$ and $g_n, g \in L^1(X_2, \mu_2)$ and $f_n \to f$ and $g_n \to g$ in $L^1$ norm and $f_n \sim g_n$ n = 1, 2, 3, ..., then $f \sim g$.

(xv) If $\phi$ is Borel measurable on $R^#$ then $f \sim g$ implies $\phi(f) \sim \phi(g)$.

**Proof.** (i) Let $f \sim g$ and $s = \sum_{k=1}^n a_k C_{I_k}$ with $\{I_k\}$ p.wd intervals of $R^#$. Then $s(f) = \sum_{k=1}^n a_k C_{f^{-1}[I_k]} \sim \sum_{k=1}^n a_k C_{g^{-1}[I_k]} = s(g)$ using Lemma (3.1).

(ii) Let $[u, v]$ be a bounded interval of $R^#$. If $r > 0$ then $\mu_1((rf)^{-1}[u, v]) = \mu_1(r^{-1}[u/r, v/r]) = \mu_2(g^{-1}[u/r, v/r]) = \mu_2((rg)^{-1}[u, v]).$

Thus $rf \sim rg$ if $r > 0$ and $f \sim g, srf \sim rg$ for all $r \in R$. Let $r \in R$. $\mu_1((f+r)^{-1}[u, v]) = \mu_1((f^{-1}[u-r, v-r]) = \mu_2((g^{-1}[u-r, v-r]) = \mu_2((g+r)^{-1}[u, v]).$
(iii) Let \( f \sim g \) and let \([u, v]\) be a bounded interval of \( \mathbb{R}^\# \). In case
\[ u \leq 0 \leq v, \quad \mu_1(|f|^{-1}[u, v]) = \mu_1(f^{-1}[-v, v]) = \mu_2(g^{-1}[-v, v]) \]
\[ = \mu_2(|g|^{-1}[u, v]). \] If \( 0 < u \), \( \mu_1(|f|^{-1}[u, v]) = \mu_1(f^{-1}[u, v]) + \mu_1((-f)^{-1}[u, v]) \]
\[ = \mu_2(g^{-1}[u, v]) + \mu_2((-g)^{-1}[u, v]) = \mu_2(|g|^{-1}[u, v]). \]
The case \( v < 0 \) is trivial.

(iv) Let \( f \sim g \) and let \([u, v]\) be a bounded interval of \( \mathbb{R}^\# \). We wish to prove that \( \mu_1(f^{+1}[u, v]) = \mu_2(g^{+1}[u, v]) \) which is clearly true if \( v < 0 \) or \( u > 0 \). Hence suppose \( u \leq 0 \leq v \). Then \( \mu_1(f^{+1}[u, v]) = \mu_1(f^{-1}[-\infty, v]) \)
\[ = \mu_2(g^{-1}[-\infty, v]) = \mu_2(g^{+1}[u, v]). \] For the rest, \( f \sim g \Rightarrow -f \sim -g \Rightarrow f^{-} = (-f)^+ \sim (-g)^+ = g^{-} \).

(v) \( \mu_1((fo \sigma)^{-1}[u, v]) = \mu_1(\sigma^{-1}f^{-1}[u, v]) = \mu_2(f^{-1}[u, v]). \)

(vi) Recall that ess. sup \( f = \inf \{ t : \mu(\{ f > t \}) = 0 \} \) and ess. inf \( f = -\) ess. sup \((-f) \). If \( f \sim g \), then \( t \in \mathbb{R}^\# \Rightarrow \mu_1(\{ f > t \}) = \mu_1(f^{-1}[t, +\infty]) \)
\[ = \mu_2(g^{-1}[t, +\infty]) = \mu_2(\{ g > t \}) \] so ess. sup \( f = \) ess. sup \( g \). \( f \sim g \Rightarrow -f \sim -g \Rightarrow \) ess. inf \( f = -\) ess. sup \((-f) = -\) ess. sup \((-g) = \) ess. inf \( g \).

(vii) Let \( f \sim g \) and \( f \geq 0 \) a.e. Then (vi) above \( \Rightarrow g \geq 0 \) a.e. To construct \( \{ f_n \} \) and \( \{ g_n \} \) let
\[ A_n, i = f^{-1}\left[ \frac{i-1}{2^n}, \frac{i}{2^n} \right], \quad B_n, i = g^{-1}\left[ \frac{i-1}{2^n}, \frac{i}{2^n} \right], \quad 1 \leq i \leq n 2^n \]
\[ A_n = f^{-1}[n, \infty) \quad B_n = g^{-1}[n, \infty) \]
\[ f_n = \sum_{i=1}^{n\frac{2^n}{n}} i-1 \frac{2^n}{2^n} C_{A_n, i} + n C_{A_n} \quad g_n = \sum_{i=1}^{\frac{n\frac{2^n}{n}}{2^n}} i-1 \frac{2^n}{2^n} C_{B_n, i} + n C_{B_n} . \]
Then Lemma (3.1) \( \Rightarrow f_n \sim g_n \) \( \Rightarrow f \sim g \Rightarrow |f| \sim |g| \) so if \( f \) is essentially bounded, then ess. sup \( |g| = \) ess. sup \( |f| < \infty \) so \( g \) is essentially bounded. The rest is well known. See [17, p. 159, (11.35)].
(viii) If \( f \) is a simple function and \( g \sim f \) then Lemma (3.1) \( \Rightarrow \) \( g \) is simple and \( \int f \, d\mu_1 = \int g \, d\mu_2 \). If \( f \geq 0 \) and \( g \sim f \) then \( g \geq 0 \) and there are sequences \( \{f_n\} \) and \( \{g_n\} \) of non-negative simple functions such that \( f_n \sim g_n \) and \( f_n \uparrow f \) and \( g_n \uparrow g \). Then \( \int f \, d\mu_1 = \lim \int f_n \, d\mu_1 = \lim \int g_n \, d\mu_2 = \int g \, d\mu_2 \).

(ix) If \( f \in L^1(X_1, \mu_1) \) and \( g \sim f \) then \( |g| \sim |f| \) so \( \int |g| \, d\mu_2 = \int |f| \, d\mu_1 < \infty \) and thus \( g \in L^1(X_2, \mu_2) \). Finally \( g \sim f \Rightarrow g^+ \sim f^+ \) and \( g^- \sim f^- \) so
\[
\int g \, d\mu_2 = \int g^+ \, d\mu_2 - \int g^- \, d\mu_2 = \int f^+ \, d\mu_1 - \int f^- \, d\mu_1 = \int f \, d\mu_1 .
\]

(x) Letting \( X_0 = X \setminus \bigcup X_i \) and \( X'_0 = X' \setminus \bigcup X'_i \) we have \( f \big| X_i \sim g \big| X_i' \) \( i = 0, 1, 2, \ldots \), and \( \{X_i\}_{i=0}^{\infty} \) and \( \{X'_i\}_{i=0}^{\infty} \) are partitions of \( X \) and \( X' \) respectively. Let \( [r, s] \) be a bounded interval of \( R^\# \). Since
\[
f \big| X_i \sim g \big| X_i \ i=0,1,2,\ldots, \text{we have } \mu(f^{-1}[r, s] \cap X_i) = \mu'(g^{-1}[r, s] \cap X_i')
\]
\( i = 0, 1, 2, \ldots \), so summing from \( i = 0 \) to \( +\infty \) we get \( \mu(f^{-1}[r, s]) = \mu'(g^{-1}[r, s]) \). Hence \( f \sim g \).

(xi) Let \( f_1, f_2, g_1, g_2 \) be as stated. Then \( \mu_1(X_1) = \mu_2(X_2) \). Let
\[
E_i = \{f_i \not\equiv 0\}, \quad F_i = \{g_i \not\equiv 0\}, \quad i = 1, 2, \quad \text{and} \quad E_3 = X_1 - (E_1 \cup E_2),
\]
\( F_3 = X_2 - (F_1 \cup F_2) \). Then \( \mu_1(E_1) = \mu_2(F_1) \) \( i = 1, 2 \) and since the m.s. are finite we may conclude that \( \mu_1(E_3) = \mu_2(F_3) \). Let \( [u, v] \) be a bounded interval of \( R^\# \) and let \( f = f_1 + f_2, \ g = g_1 + g_2 \). Then \( \mu_1(f^{-1}[u, v]) = \sum_{i=1}^{3} \mu_1(E_i \cap f^{-1}[u, v]) = \mu_1(f_1^{-1}[u, v]) + \mu_1(f_2^{-1}[u, v]) + \mu_1(E_4) \)
where \( E_4 = \{E_3 \text{ if } 0 \in [u, v], \ \emptyset \text{ if } 0 \notin [u, v] \} \). Hence if \( F_4 = \{F_3 \text{ if } 0 \in [u, v], \ \emptyset \text{ if } 0 \notin [u, v] \} \), then \( \mu_2(g^{-1}[u, v]) = \mu_2(g_1^{-1}[u, v]) + \mu_2(g_2^{-1}[u, v]) + \mu_2(F_4) \)
\[
= \mu_1(f^{-1}[u, v]) .
\]

(xii) Follows from (iv) and (xi).
Using (2.1) we have
\[ \lim_{n \to \infty} d_n = d \] on \( \mathbb{R} \), so \( f_n \to g \).

(f) \( f_n \to f \) in norm implies \( \{ f_n \} \) has a subsequence \( f_{n_k} \to f \) pointwise a.e. Then \( g_{n_k} \to g \) in norm so \( \{ g_{n_k} \} \) has a subsequence \( g_{n_{m_k}} \to g \) pointwise a.e. But then \( f_{n_{m_k}} \to f \) pointwise a.e. so (3.3)(xiii) implies \( f \sim g \).

From the proof (vii) we see that each of \( \phi^+ \) and \( \phi^- \) is the limit of a sequence of simple functions of the form \( \sum_{i=1}^{n} \alpha_i \chi_{E_i} \) where the sets \( \{ E_i \} \) are Borel sets. Hence \( \phi \) is the limit of a sequence \( \{ s_n \} \) of simple functions of the same form. Hence \( f \sim g = s_n(f) \sim s_n(g) \) and since \( s_n(f) \to \phi(f) \) and \( s_n(g) \to \phi(g) \) we have \( \phi(f) \sim \phi(g) \).

(3.4) LEMMA. (i) Suppose \( f \in M(X, \Lambda, \mu) \), \( g \in M(X, \Lambda, \mu) \), \( E_1 \in \Lambda, \ E_2 \in \Lambda, \ \mu_1(\Lambda_1) = \mu_2(\Lambda_2) < \infty \), and \( f \) and \( g \) have the same constant value on \( E_1 \) and \( E_2 \) respectively. Then \( f \sim g = f|_{X_1-E_1} \sim g|_{X_2-E_2} \).

(ii) Suppose \( f, g \in M(X, \Lambda, \mu) \), \( E \in \Lambda \) has \( \mu(E) < \infty \) and \( f|_E = g|_E \). Then \( f \sim g = f|_{X-E} \sim g|_{X-E} \).

PROOF. (i) Let \( f = f|_{X_1-E_1} \), \( g = g|_{X_2-E_2} \), and let \( \alpha = f|_{E_1} \sim g|_{E_2} \). Let \( [r, s] \) be a bounded interval of \( \mathbb{R}^q \). Then
\[
\mu_1(f^{-1}[r, s] \cap E_1) = \begin{cases} 0 & \text{if } \alpha \notin [r, s] \\ \mu_1(E_1) & \text{if } \alpha \in [r, s] \end{cases}
\]
and similarly for $g$, so $\mu_1(f^{-1}[r, s] \cap E_1) = \mu_2(g^{-1}[r, s] \cap E_2)$. Hence

$$\mu_1([r, s]) = \mu_1([r, s] \cap (X_1 - E_1))$$

$$= \mu_1([r, s]) - \mu_1([r, s] \cap E_1)$$

$$= \mu_1([r, s]) - \mu_2([r, s] \cap E_2) = \mu_2(g^{-1}[r, s])$$

so $f \sim g$.

Let $f_1 = f_{C_{X_1 - E_1}}$ and $g_1 = g_{C_{X_2 - E_2}}$. Then $\mu_1(f_1^{-1}([0, 1])) = \mu_1(E_1 \cup (X_1 - E_1) \cap f^{-1}([0, 1])) = \mu_1(E_1) + \mu_1(f^{-1}([0, 1])) = \mu_2(E_2) + \mu_2(g^{-1}([0, 1])) = \mu_2(g_1^{-1}([0, 1]))$. If $I$ is an interval of $\mathbb{R}^+$ and $0 \notin I$, then $\mu_1(f_1^{-1}(I)) = \mu_1(f^{-1}(I)) = \mu_2(g^{-1}(I)) = \mu_2(g_1^{-1}(I))$. Hence $f_1 \sim g_1$.

(ii) Let $f = f_{|X - E}$ and $g = g_{|X - E}$. Let $[r, s]$ be a bounded interval of $\mathbb{R}^+$. Since $f|E = g|E$, we have $\mu(f^{-1}[r, s] \cap E) = \mu(g^{-1}[r, s] \cap E)$ and the rest is similar to part (i).

(3.5) EXAMPLES. 1. The following functions are equimeasurable with $f(x) = x$ on $[0, 1]$ and are measure preserving maps of $[0, 1] - [0, 1]$.

(i) $g_1(x) = 1 - x \quad 0 \leq x \leq 1$

(ii) $g_2(x) = 2x \mod 1$

(iii) $g_3(x) = \begin{cases} \frac{1}{2} - x & 0 \leq x < \frac{1}{2} \\ \frac{3}{2} - x & \frac{1}{2} \leq x \leq 1 \end{cases}$
2. Let \( f = C_{-\infty, 0}[-C]_2, \infty[ \) and \( g = C_{-\infty, 1}[-C]_2, \infty[ \). Then 
\[
\begin{align*}
    f^+ &= C_{-\infty, 0}[-C]_2, \infty[ \sim C_{-\infty, 1}[-C]_2, \infty[ \sim g^+ \quad \text{and} \\
    f^- &= g^- \quad \text{but} \quad f \not\sim g \quad \text{since} \quad m(f^{-1}(0)) = 2 \\
    \text{while } m(g^{-1}(0)) &= 1.
\end{align*}
\]

3. Let \( f = C_{-\infty, 0}^+ + 2C_{0, \infty}[ \) have the same constant value on \([0, \infty[ \) and \([1, \infty[ \) but \( f|_R - [0, \infty[ \not\sim f|_R - [1, \infty[ \).

4. Let \( \mathcal{F} \) be the collection of all finite subsets \( F \) of \([0, 1] \) and let \( \mathcal{F} \) be directed by \( \subset \). Then \( \{C_F \}_{F \in \mathcal{F}} \) is a net which converges pointwise to \( C_{[0, 1]} \) (since if \( t \in [0, 1] \) then \( F = \{t\} \in \mathcal{F} \) and \( F \subset F \) implies \( C_F(t) = 1 \) but \( C_F \sim 0 \) for all \( F \in \mathcal{F} \).

For \( i = 1, \ldots, k \) let \( (X_i, \Lambda_i, \mu_i) \) and \( (Y_i, \Sigma_i, \nu_i) \) be finite m.s., let \( X = X_1 \times \ldots \times X_k \), \( Y = Y_1 \times \ldots \times Y_k \), \( \mu = \mu_1 \times \ldots \times \mu_k \) and \( \nu = \nu_1 \times \ldots \times \nu_k \).

(3.6) **PROPOSITION.** Let \( f_i \in M(X_i, \mu_i) \) and \( g_i \in M(Y_i, \nu_i) \) \( i = 1, \ldots, k \) and let \( F(x_1, \ldots, x_k) = (f_1(x_1), \ldots, f_k(x_k)) \) for all \( (x_1, \ldots, x_k) \in X \) and \( G(y_1, \ldots, y_k) = (g_1(y_1), \ldots, g_k(y_k)) \) for all \( (y_1, \ldots, y_k) \in Y \). If \( f_i \sim g_i \) \( i = 1, \ldots, k \) then \( F \sim G \).

**PROOF.** Let \( \mathcal{K} \) be the semi-algebra of all measurable rectangles of \( R^k \) and let \( \mathcal{B} \) be the \( \sigma \)-algebra of all Borel subsets of \( R^k \). If \( A \) is a set and \( \mathcal{B} \subset 2^A \) let \( \mathcal{S}(\mathcal{B}) \) denote the \( \sigma \)-algebra generated by \( \mathcal{B} \);
let $C(\mathcal{F})$ denote the monotone class generated by $\mathcal{F}$; and let $R(\mathcal{F})$ be the algebra generated by $\mathcal{F}$. If $\mathcal{F}$ is a semi-algebra we recall that $R(\mathcal{F})$ is the set of all finite pairwise disjoint [pwd] unions of members of $\mathcal{F}$.

Let $\mathcal{G} = \{B \in \mathcal{F}_k : \mu(F^{-1}[B]) = \vee(G^{-1}[B])\}$.

If $B_1 \times \cdots \times B_k \in \mathcal{F}_k$, then

$$
\mu(F^{-1}(B_1 \times \cdots \times B_k)) = \mu(f_1^{-1}[B_1] \times \cdots \times f_k^{-1}[B_k]) = \omega(f_1^{-1}[B_1]) \cdots \omega_k(f_k^{-1}[B_k]) = \vee(g_1^{-1}[B_1]) \cdots \vee_k(g_k^{-1}[B_k]) = \vee(G^{-1}(B_1 \times \cdots \times B_k))
$$

so $\mathcal{F}_k \subseteq \mathcal{G}$. $\mathcal{G}$ is clearly closed under pwd unions so $R(\mathcal{F}_k) \subseteq \mathcal{G}$. $\mathcal{G}$ is easily seen to be a monotone class, so $\mathcal{F}_k = S(\mathcal{F}_k) = S(R(\mathcal{F}_k)) = C(R(\mathcal{F}_k)) \subseteq \mathcal{G}$.

(3.7) **COROLLARY.** For $i = 1, \ldots, k$ let $(X_i, \lambda_i, \omega_i)$ and $(Y_i, \Sigma_i, \upsilon_i)$ be finite m. s., let $f_i \in M(X_i, \lambda_i)$, $g_i \in M(Y_i, \upsilon_i)$, $a_i \in R$, and $n_i \geq 0$ integers. If $f_i \sim g_i$, $i = 1, \ldots, k$ then for each of the following definitions of $F$ and $G$ we have $F \sim G$.

(a) $F(x_1, \ldots, x_k) = \sum_{i=1}^{k} a_i f_i^n(x_i)$

$b_i \frac{n_i}{i}$

$G(y_1, \ldots, y_k) = \sum_{i=1}^{k} a_i g_i^n(y_i)$

(b) $F(x_1, \ldots, x_k) = \prod_{i=1}^{k} f_i^n(x_i)$

$G(y_1, \ldots, y_k) = \prod_{i=1}^{k} g_i^n(y_i)$

**PROOF.** Use Props. (3.6) and (3.3) (xv) with $\varphi(t_1, \ldots, t_k) = \sum_{i=1}^{k} a_i t_i^n$ and $\varphi(t_1, \ldots, t_k) = \prod_{i=1}^{k} t_i^n$. 
REMARKS. (1) For finite m.s. (3.3) (xv) gives another proof of (3.3) (ii), (iii), & (iv) by letting \( \varphi(t) = rt \), \( \varphi(t) = t+r \), \( \varphi(t) = |t| \), and

\[ \varphi(t) = \max \{ 0, t \} \quad \text{and} \quad \varphi(t) = -\min \{ 0, t \} \]

(2) For finite m.s. if \( f \sim af \) for some number \( a \) s.t. \( |a| \neq 1 \), then \( |f| = 0 \) or \( +\infty \). For we have by induction that \( |f| \sim |a|^n |f| \to 0 \) or \( +\infty \) as \( n \to \infty \).
II. DECREASING REARRANGEMENTS

4. The Right Inverse of a Decreasing Function. If \( p \) is a decreasing function defined on an interval \( J \) of \( \mathbb{R} \), we can extend \( p \) to a decreasing function defined on \( \mathbb{R}^\# \) by defining for \( t \notin J \),

\[
p(t) = \begin{cases} +\infty & \text{if } t \leq \inf J \\ -\infty & \text{if } t \geq \sup J \end{cases}
\]

If \( p \) is a decreasing function defined on \( \mathbb{R}^\# \), then its right continuous inverse \( p^* \) is defined by

\[
p^*(t) = \inf \{ u \in \mathbb{R} : p(u) \leq t \}
\]

for each \( t \in \mathbb{R}^\# \), where by \( \inf \emptyset \) we mean \( +\infty \) and \( \inf \mathbb{R} = -\infty \). If \( p \) is 1:1 then \( p^* = p^{-1} \).

It is easy to see that \( p^* \) is decreasing and right continuous, and for every \( t \in \mathbb{R}^\# \), \( p^*(p(t)) \leq t \), \( p^*(p(t+)) \leq t \), \( p^*((p(t)+\varepsilon)-) \leq t \), \( p^*(p(t)-\varepsilon) \geq t \) whenever \( \varepsilon > 0 \), and \( p^*(p(t)-) \geq t \). Furthermore, for all \( t \in \mathbb{R}^\# \), \( p^*(t) = \sup \{ u \in \mathbb{R} : p(u) > t \} = \sup \{ u : p(u) > t \} \) and \( p^*(t-) = \inf \{ u \in \mathbb{R} : p(u) < t \} = \inf \{ u : p(u-) < t \} \).

(4.1) PROPOSITION. If \( p \) is a decreasing function on \( \mathbb{R}^\# \), then for every \( t \in \mathbb{R} \) we have

\[
p^*(t) = (p^*)^*(t) = p(t+).
\]

PROOF. Let \( u \in \mathbb{R} \). Since \( p^*(p(u+)) \leq u \) we have \( p^*(u) = \inf \{ t : p^*(t) \leq u \} \leq p(u+) \). Since \( p^* \) is decreasing and right continuous we have \( p^*(p^*(u)) = p^*(\inf \{ t : p^*(t) \leq u \}) = \sup \{ p^*(t) : p^*(t) \leq u \} \leq u \). Since \( t \rightarrow p(t+) \) is also decreasing and right continuous we have similarly
\[ p(p^+(t)) = \sup \{ p(r^+): p(r) \leq t \} \leq t \] for all \( t \in \mathbb{R} \). Hence
\[ p(u^+) \leq p(p^+(p^+(p^+(u^+)))) \leq p^+(u). \]

(4.2) PROPOSITION. Let \( 0 < a \in \mathbb{R} \) and let \( p \) be a decreasing function defined on \([0, a] \). Then for every \( t \in \mathbb{R} \),
\[ p^+(t) = m([u \in [0, a]: p(u) > t]) = d(t) \]
where \( m \) is Lebesgue measure.

PROOF. Now \( p^+(t) = \sup \{ u: p(u) > t \} \) so \( p(u) > t \Rightarrow u \leq p^+(t) \) and thus \( \{ u \in [0, a]: p(u) > t \} \subset [0, p^+(t)] \).
Again, \( p^+(t) = \inf \{ u: p(u) \leq t \} \) so \( p(u) \leq t \Rightarrow p^+(t) \leq u \), i.e., \( u < p^+(t) \Rightarrow p(u) > t \), so
\[ 0, p^+(t) \subset \{ u \in [0, a]: p(u) > t \} \].

(4.3) PROPOSITION. Let \( p, p_n \) be decreasing functions on \( \mathbb{R} \) for \( n = 1, 2, 3, \ldots \) and suppose \( p_n \to p \) at all but countably many points of \( \mathbb{R} \). Then \( p_n \to p \) at every point of continuity of \( p^+ \).

PROOF. Fix \( t \in \mathbb{R} \). Let \( A_n = \{ u: p_n(u) > t \} \), \( A = \{ u: p(u) > t \} \),
\[ E = \{ u: p_n(u) \neq p(u) \} \] so that \( p_n^+(t) = \sup A_n \), \( p^+(t) = \sup A \) and \( E \) is countable. Then
\[ A \cap E^c \subset \lim \inf A_n \subset \lim \sup A_n \subset \{ u: p(u) \geq t \} \] so
\[ \sup A \cap E^c \leq \sup (\lim \inf A_n) \leq \sup (\lim \sup A_n) \leq \sup \{ u: p(u) \geq t \} \].
Now the functions \( p_n \) are decreasing so the sets \( A_n \) have the form \([-\infty, r[ \) or \([ -\infty, r) \) and thus \( \sup (\lim \inf A_n) = \lim \inf (\sup A_n) = \lim \inf p_n^+(t) \) and similarly \( \sup (\lim \sup A_n) = \lim \sup p_n^+(t) \). Since
E is countable and A is an interval, \( \sup A \cap E^c = \sup A = p^*(t) \). Hence \( p^*(t) \leq \lim \inf p_n^*(t) \leq \lim \sup p_n^*(t) \leq p^*(t-) \). Thus if \( p^* \) is continuous at \( t \), then \( \lim p_n^*(t) = p^*(t) \). Also see \([47, \text{ p. } 508, (18.21)]\).

5. *Decreasing Rearrangements of Functions on Finite m.s.* Let \( f: [0, 1] \to [0, 1] \) be Lebesgue measurable. It is natural to wonder if the values of \( f \) can be rearranged to form a decreasing function \( f^*: [0, 1] \to [0, 1] \) such that \( f^* \sim f \). The affirmative answer is well known \([15]\).

We now generalize this idea for a finite measure space \((X, \Lambda, \mu)\) by showing that if \( f \in M(X, \mu) \) then there is a decreasing right continuous Lebesgue measurable function \( \delta_f \) on \([0, \mu(X)]\) s.t. \( \delta_f \sim f \).

For the rest of this section let \((X, \Lambda, \mu)\) be a finite measure space (m.s.).

(5.1) *DEFINITION.* If \( f \in M(X, \mu) \) we define \( \delta_f \) by \( \delta_f(t) = d_f^*(t) \) if \( 0 \leq t \leq \mu(X) \).

(5.2) *THEOREM.* (i) If \( f \in M(X, \mu) \) then \( \delta_f \) is a decreasing right continuous Lebesgue measurable function on \([0, \mu(X)]\) satisfying \( \delta_f \sim f \).

(ii) Conversely, if \( p \) is a decreasing right continuous Lebesgue measurable function on \([0, \mu(X)]\) satisfying \( p \sim f \), then \( p = \delta_f \).

*PROOF.* Now \( \delta_f \) is by definition decreasing, right continuous and Lebesgue measurable on \([0, \mu(X)]\) and \( \delta_f = d_f^* \). Lemma (4.2) \implies
\[ d_{\delta_f} = \delta_f^* \] and Lemma (4.1) \[ d_{\delta_f}^* = d_f \] so \[ d_{\delta_f} = d_f \] and thus \( \delta_f \sim f \).

Conversely, if \( p \) is right continuous decreasing and Lebesgue measurable on \([0, \mu(X)]\) s.t. \( p \sim f \), then \( d_f = d_p = p^* \) using (4.2) so (4.1) \[ \delta_f = d_f^* = p^* = p. \]

(5.3) PROPPOSITION.

(i) \( f \) is integrable iff \( \delta_f \) is integrable in which case \[ \int_a^b f \, d\mu = \int_0^{a} \delta_f \]

where \( a = \mu(X) \).

(ii) \( s(f) \sim s(\delta_f) \) for all \( s \in \mathcal{S} \).

(iii) \( f_1 \leq f_2 \Rightarrow \delta_{f_1} \leq \delta_{f_2} \).

(iv) If each of \( f \) and \( g \) is a measurable function on a finite m.s. then \( f \sim g \) iff \( \delta_f = \delta_g \).

(v) If \( p \) is increasing on \( \mathbb{R}^+ \) then \( \delta_p(f)(t) = p(\delta_f(t)^-), 0 \leq t \leq \mu(X) \).

(vi) If \( r \geq 0 \) then \( \delta_{rf} = r\delta_f \) while if \( r \) is real, then \( \delta_{f+r} = \delta_f + r \).

(vii) If \( f_n \to f \) a.e. then \( \delta_{f_n} \to \delta_f \) at every point of continuity of \( \delta_f \).

(viii) If \( f_n \to 0 \) in measure then \( \delta_{f_n} \to 0 \) uniformly on every closed subinterval of \( [0, \mu(X)] \).

PROOF. (i) This is an important special case of results of § 3.

Similarly for (ii).

(iii) \( f_1 \leq f_2 \Rightarrow \delta_{f_1} \leq \delta_{f_2} \Rightarrow d_{f_1} = \delta_{f_1}^* \leq d_{f_2}^* = \delta_{f_2}^* \).

(iv) \( f \sim g \Rightarrow \delta_{f} = \delta_{f}^* = \delta_{g} = \delta_{g}^* \). Conversely, \( \delta_{f} = \delta_{g} \Rightarrow d_{f} = d_{g} \Rightarrow \delta_{f}^* = \delta_{g}^* \\
\delta_{g}^* = \delta_{g} \Rightarrow f \sim g. \)

(v) Since \( f \sim \delta_f \) we have \( p(f) \sim p(\delta_f) = p(\delta_f^-) \) a.e. Since \( t \to p(\delta_f(t)^-) \) is decreasing and right continuous, Theorem (5.2) (ii) implies \( p(f) = p(\delta_f^-). \)
(vi) These are important special cases of (v) with \( p(t) = rt \) and \( p(t) = t^r \).

(vii) \( f_n \to f \) a.e. \( \Rightarrow d_{f_n} \to d_f \) at all but countably many points so Lemma (4.3) \( \delta_f = d_f^* = \lim d_{f_n}^* \) at every point of continuity of \( d_f^* = b_f \).

(viii) Let \([u, v] \subset J_0, \mu(X)[\) and let \( \varepsilon > 0 \). Since \( u > 0 \exists N_1 > 0 \) s.t. 
\[ \mu([|f_n| > \varepsilon]) < u \] whenever \( n \geq N_1 \) so \( d_{f_n}(\varepsilon) = \mu([f_n > \varepsilon]) < u \) and thus 
\[ \delta_{f_n}(u) \leq \delta_{f_n}(d_{f_n}(\varepsilon)) = d_{f_n}^*(d_{f_n}(\varepsilon)) \leq \varepsilon \] whenever \( n \geq N_1 \). Since \( v < \mu(X) \) 
\[ \exists N_2 > 0 \text{ s.t. } n \geq N_2 \Rightarrow \mu([|f_n| > \varepsilon/2]) < \mu(X) - v \] so 
\[ v < \mu([|f_n| < \varepsilon/2]) \leq \mu([f_n > -\varepsilon]) = d_f^*(-\varepsilon) = d_{f_n}^*(-\varepsilon) \] so \( \delta_{f_n}(v) = d_{f_n}^*(v) \geq -\varepsilon \) whenever \( n \geq N_2 \). Since \( \delta_{f_n} \) is decreasing on \([u, v]\) we have \[ |\delta_{f_n}(t)| \leq \varepsilon \] for all \( t \in [u, v] \) whenever \( n \geq N_1 + N_2 \).

Recall that if \( f \in M(X, \mu) \), then ess. sup \( f \) = inf \{ t : \mu([f > t]) = 0 \}.

Writing \( E^c = X - E \) if \( E \in \Lambda \) and \( f|E^c \) for \( f \) restricted to \( E^c \) we have:

\[ \text{(5.4) PROPOSITION. } \text{If } f \in M(X, \mu) \text{ and } 0 \leq t \leq \mu(X), \text{ then} \]
\[ \delta_f(t) = \inf \{ \text{ess. sup } (f|E^c) : \mu(E) \leq t \} \]
\[ \delta_f(t-) = \inf \{ \text{ess. sup } (f|E^c) : \mu(E) < t \} \]

where ess. sup \( f \mid \emptyset = -\infty \).

\[ \text{PROOF. } \text{Let } t \in \{0, \mu(X)\}. \text{ If } u \in R^# \text{ and } E_u = [f > u] \text{ so that} \]
\[ d_f(u) = \mu(E_u), \text{ then } d_f(u) \leq t \Rightarrow \mu(E_u) \leq t \text{ and ess. sup } (f|E_u^c) \leq u \text{ and thus} \]
\[ \inf \{ \text{ess. sup } (f|E^c) : \mu(E) \leq t \} \leq \delta_f(t). \]

Conversely, if \( \mu(E) \leq t \), let \( u = \text{ess. sup } (f|E^c) \) so that \( [f > u] \subset E, \)
\[ d_f(u) \leq t, \text{ and thus } \delta_f(t) \leq \inf \{ \text{ess. sup } (f|E^c) : \mu(E) \leq t \} \].
(5.5) THEOREM. If \( f \in M(X, \mu) \) and \( a = \mu(X) \) then

(i) \( (\delta_f)^+ = \delta_f^+ + (\delta_f)^- = -\delta_f^- \)

so that \( \delta_f = \delta_f^+ + \delta_f^- \)

and \( |\delta_f| = \delta_f^+ - \delta_f^- \)

(ii) \( \delta_f(t) = -\delta_f((a-t)-) \quad 0 \leq t \leq a. \)

PROOF. (i) \( \delta_f \sim f = (\delta_f)^+ \sim f^+ \) and \( (\delta_f)^- \sim f^- \). Since \( (\delta_f)^+ \) is decreasing and right continuous, \( (\delta_f)^+ = \delta_f^+ \). \( (\delta_f)^- \sim f^- \Rightarrow -(\delta_f)^- \sim -f^- \)

and \( -(\delta_f)^- \) is decreasing and right continuous, so \( -(\delta_f)^- = \delta_f^- \) and thus \( (\delta_f)^- = -\delta_f^- \).

(ii) \( f \sim \delta_f \sim -f \sim -\delta_f \sim a \) where \( a(t) = -\delta_f((a-t)-) \) using Lemma (3.2)

(iii). Since \( a \) is decreasing and right continuous, \( a = \delta_f^- \).

From now on we will use \( \delta_E \) to denote \( \delta_{C_E} \), where \( E \) is a measurable set. Statement (i) of the following lemma seems to have been first used systematically by F. Riesz ([35], p. 164).

(5.6) LEMMA. (i) If \( f = \sum_{i=1}^{n} f_i \ C_{E_i} \in M(X, \mu) \) and \( 0 < f_1 < \cdots < f_n \),

then \( \delta_f = f_1 \delta_{F_1} + \sum_{i=2}^{n} (f_i-f_{i-1})\delta_{F_i} \) where \( F_i = \bigcup_{k=i}^{n} E_k \)

(ii) If \( f = \sum_{i=1}^{n} f_i \ C_{E_i} \in M(X, \mu) \) with \( f_i > 0 \) and \( E_1 \supset \cdots \supset E_n \), then

\( \delta_f = \sum_{i=1}^{n} f_i \delta_{E_i} \).

PROOF. (i) Let \( F_{n+1} = \emptyset \). Now

\[
d_f(s) = \begin{cases} 
0 & s \geq f_n \\
\mu(F_{i+1}) & f_i \leq s < f_{i+1} \\
\mu(X) & s < f_1
\end{cases} \quad i = 1, \ldots, n-1
\]
so \( \delta_f = \sum_{i=1}^{n} f_i \mathbb{C}[\mu(F_{i+1}), \mu(F_i)] \)

\[ = \sum_{i=1}^{n} f_i \delta_{F_i} - \sum_{i=1}^{n} f_i \delta_{F_{i+1}} \]

\[ = f_1 \delta_{F_1} + \sum_{i=2}^{n} (f_i - f_{i-1}) \delta_{F_i}. \]

(ii) Let \( G_i = E_i - \bigcup_{k=i+1}^{n} E_k \) and \( g_i = \sum_{k=1}^{i} f_k' \) \( g_0 = 0. \)

Then \( f = \sum_{i=1}^{n} g_i \mathbb{C}_{G_i} \) and \( 0 < g_1 < \cdots < g_n \) so

\[ \delta_f = \sum_{i=1}^{n} (g_i - g_{i-1}) \delta_{F_i} = \sum_{i=1}^{n} f \delta_{E_i} \]

since

\( F_i = \bigcup_{k=i}^{n} G_k = E_i \) and \( g_i - g_{i-1} = f_i. \)

(5.7) PROPOSITION. If \( 0 \leq f \in M(X, \mu) \) and \( E \in \Lambda \) then \( \delta_{f \mathbb{C}_E} \leq \delta_f \delta_E. \)

PROOF. First let \( f = C_F \) where \( F \in \Lambda. \) Then \( \delta_{C_E \mathbb{C}_F} = \delta_{E \cap F} = C[0, \mu(E \cap F)] \leq C[0, \min(\mu(E), \mu(F))] = \delta_E \delta_F. \)

Let \( f = \sum_{i=1}^{n} f_i \mathbb{C}_{F_i}, \ F_n \subset \cdots \subset F_1, \ f_i > 0, \) so

\[ \delta_{f \mathbb{C}_E} = \sum_{i=1}^{n} f_i \delta_{E \cap F_i} \leq \sum_{i=1}^{n} f_i \delta_{E \cap F_i} = \delta_{E \cap F} = \delta_{E \cap F}. \]

If \( 0 \leq f \in M(X, \mu) \) then there is a sequence \( 0 \leq f_n \uparrow f \) of simple functions, so \( \delta_{f_n} \uparrow \delta_{f} \) and

\[ \delta_{f_n \mathbb{C}_E} \uparrow \delta_{f \mathbb{C}_E} \]

and hence \( \delta_{f \mathbb{C}_E} = \lim \delta_{f_n \mathbb{C}_E} \leq \lim \delta_{f_n \mathbb{C}_E} \delta_{f_n \mathbb{C}_E} = \delta_{f \mathbb{C}_E}. \)

REMARKS. (1) It follows immediately from Prop. (5.4) that \( \delta_f(0) = \text{ess. sup} \ f \) and \( \delta_f(a-) = \text{ess. inf} \ f \) when \( a = \mu(X). \)

(2) If \( f_n(x) = x^n \) on \([0, 1]\) then \( f_n \rightarrow f = 0 \) in measure but

\( \delta_{f_n}(0) = \text{ess. sup} \ f_n = 1 \) so \( \delta_{f_n}(0) \neq \delta_f(0) = 0. \)
(3) We cannot prove Prop. (5.3)(vii) for nets. Direct the finite subsets of $[0, 1]$ by inclusion and let $f_E = C_E$ if $E$ is a finite subset of $[0, 1]$. Then $t \in [0, 1]$ and $\{t\} \subset E = f_E(t) = C_E(t) = 1$ so the net $\{f_E\}$ converges pointwise everywhere to 1 but $\delta_f^E = 0$ for every $E$ since $f_E = 0$ a.e.

(4) If $g = |\delta_f|$, then $g = \delta_f$ since $f \sim \delta_f = |f| \sim |\delta_f|$.

(5) Let $p$ be decreasing and right continuous on $K^*$, $f \in M(X, \mu)$ and $a = \mu(X)$. $f \sim -\delta_{-f}$ so $p(f) \sim p(-\delta_{-f})$ which is decreasing and right continuous on $[0, a]$, so for $0 \leq t \leq a$, $p(f)(t) = p(\delta_f((a-t)-))$.

(6) Let $\sigma: X \to [0, \mu(X)]$ be m.p and let $a(t) = \mu(X) - t$ $0 \leq t \leq \mu(X)$. Then $\sigma \sim a$ and $a$ is decreasing and right continuous so $\delta_{\sigma}(t) = \mu(X) - t$.

(7) Let $a > 0$. If $f \in M[0, a]$ and $F_1(t) = F(at)$ on $[0, 1]$ then $\delta_{F_1}(t) = \delta_{F}(at)$ on $[0, 1]$. This follows immediately from (3.2).

**Examples.**

1. Let $f(x) = x$ on $[a, b]$. Since $\sigma(x) = b-x$ is a m.p. map of $[0, b-a] \to [a, b]$, $f \circ \sigma \sim f$ (3.3) (v). But $f \circ \sigma(t) = b-t$ is decreasing and right continuous on $[0, b-a]$ so $\delta_f(t) = b-t$ on $[0, b-a]$.

2. Suppose $p$ is increasing on $K^*$, $a > 0$, and $f(x) = p(ax)$ on $[a, b]$. Then $\delta_f(t) = \delta_{p(ax)}(t) = p(\delta_{ax}(t)-) = p(a \delta_x(t)-) = p((ab-at)-)$ on $[0, b-a]$.

3. Let $g(x) = e^{ax}$, $a > 0$, on $[a, b]$. Then $\delta_g(t) = e^{ab-at}$ on $[0, b-a]$ using example 2.

4. Let $g(x) = log ax$, $a > 0$, on $[a, b]$, $a > 0$. $\delta_g(t) = log(ab-at)$ on $[0, b-a]$.

5. Suppose $p$ is increasing on $K^*$, $a < 0$, and $f(x) = p(ax)$ on $[a, b]$. Since $\sigma(x) = x+a$ is a m.p. map of $[0, b-a] \to [a, b]$, $f \circ \sigma \sim f$. Thus $t \to p(a(t+a)-)$ is decreasing and right continuous and equimeasurable
6. Let \( f(x) = \sin x \) on \([0, \pi/2]\). Then \( \delta_f(t) = \cos t \) on \([0, \pi/2]\), because \( \cos x = \sin(x + \pi/2) \) on \([0, \pi/2]\) and thus they are equimeasurable.

7. Let \( F \) be decreasing on \([0, a]\) and extend \( F \) to \( \mathbb{R} \) by periodicity. Let \( f(x) = F(x) \) on \([0, na]\) where \( n > 0 \) is an integer. Then \( \delta_f(t) = F\left(\frac{t}{n}\right) \) on \([0, na]\). To prove this, let \( G(t) = F\left(\frac{t}{n}\right) \) on \([0, na]\), and let \([r, s]\) be a bounded interval of \( \mathbb{R} \). Then \( m(G^{-1}[r, s]) = m((nF^{-1}[r, s]) \cap [0, na]) = nm(F^{-1}[r, s]) \cap [0, a]) = m(f^{-1}[r, s]) \). Thus \( G \sim f \) and since \( G \) is decreasing and right continuous, \( G = \delta_f \).

8. Let \( f(x) = \sin x \) on \([0, \pi]\). Then \( \delta_f(t) = \cos t/2 \) on \([0, \pi]\). For if we let \( g(x) = \cos x \) on \([0, \pi/2]\) and \( g(x) = \cos(x - \pi/2) \) on \([\pi/2, \pi]\) then \( g\mid_{[0, \pi/2]} \sim f\mid_{[0, \pi/2]} \) and \( g\mid_{[\pi/2, \pi]} = f\mid_{[\pi/2, \pi]} \) so \( g \sim f \) by (3.3) (x) and thus \( \delta_f(t) = \delta_g(t) = \cos t/2 \) on \([0, \pi]\) using example 7.

9. Let \( f(x, y) = x + y \) on \([0, 1] \times [0, 1]\) with product Lebesgue measure. Then

\[
d_f(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq 0 \\
1 - \frac{1}{2} t^2 & \text{if } 0 \leq t \leq 1 \\
\frac{1}{2} (2-t)^2 & \text{if } 1 \leq t \leq 2 \\
0 & \text{if } t \geq 2 
\end{cases}
\]

by inverting \( 1 - \frac{1}{2} t^2 \) and \( \frac{1}{2} (2-t)^2 \).

10. Let \( g(x, y) = (1-x) + (1-y) \) on \([0, 1] \times [0, 1]\). Then with \( f \) as in example 9, \( \delta_g = \delta_f \) because \( g \sim f \) by (3.6).

11. If \( g(x, y) = \sqrt{x+y} \) on \([0, 1] \times [0, 1]\) with \( f \) as in example 9, \( \delta_g = \sqrt{\delta_f} \) using (5.3)(v).

12. The Hilbert Transform of a function \( f \) on \( \mathbb{R} \) is defined by
\[ T(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|t-x| > \varepsilon} \frac{f(t)}{t-x} \, dt \]

It is well known that if \( E \) has finite Lebesgue measure, then
\[ d |C_E| (t) = 2 \frac{m(E)}{\sinh \pi t} \quad \text{so} \quad \delta |C_E| (u) = \frac{1}{\pi} \sinh^{-1} \left( \frac{2m(E)}{u} \right). \] [46]

At this point it is natural to wonder if, given a right continuous decreasing function \( \sigma \) on \([0, \mu(X)]\), there is an \( f \in M(X, \mu) \) s.t. \( \delta_f = \sigma \).

(5.7) PROPOSITION. Every right continuous decreasing function on \([0, \mu(X)]\) is the decreasing rearrangement of a measurable function on \((X, \Lambda, \mu)\) iff there is a m.p. map \( \sigma : X \to [0, \mu(X)] \).

PROOF. Let \( \sigma : X \to [0, \mu(X)] \) be m.p. and let \( g \) be a decreasing right continuous function on \([0, \mu(X)]\). Then \( g \circ \sigma \in M(X, \mu) \) and
\[ g \circ \sigma \sim g \quad \text{so} \quad \delta_{g \circ \sigma} = g. \]

Conversely, let \( \sigma \in M(X, \mu) \) s.t \( \delta_{\sigma} (t) = \mu(X) - t, 0 \leq t \leq \mu(X) \). Then \( \delta_{\sigma} \) and hence \( \sigma \) is m.p.

Recall that \( A \in \Lambda \) is said to be an atom of \((X, \Lambda, \mu)\) (sometimes an atom of \( \mu \)) iff \( \mu(A) > 0 \) and \( B \subset A, \ B \in \Lambda \Rightarrow \mu(B) = 0 \) or \( \mu(B) = \mu(A) \).

If \( A \) is an atom of \((X, \Lambda, \mu)\) and \( f \in M(X, \mu) \) then \( \text{ess. sup}(f|A) = \text{ess. inf}(f|A) \), i.e., \( f \) is essentially constant on \( A \). \((X, \Lambda, \mu)\) is said to be non-atomic if it has no atoms. Any Borel subset of \( \mathbb{R} \) with Lebesgue measure is known to be non-atomic. The following is a fundamental result about non-atomic measure spaces.
(5.8) LEMMA. (A. Liapounoff, 1940) If \((X, \Lambda, \mu)\) is a non-atomic finite m. s. then \(\{\mu(E) : E \in \Lambda\} = [0, \mu(X)]\).

For a proof see N. Dinculeanu [5, p. 25].

In order to state Liapounoff's result in another way we make the following definition.

(5.9) DEFINITION. A mapping \(\phi : [0, 1] \to \Lambda\) is said to be a \(\mu\)-resolution of \(A \in \Lambda\) if it has the following three properties.

(i) \(\phi(t) \subseteq A\) for all \(0 \leq t \leq 1\).

(ii) \(0 \leq t_1 \leq t_2\) implies \(\phi(t_1) \subseteq \phi(t_2)\).

(iii) \(\mu(\phi(t)) = t\mu(A)\) for all \(0 \leq t \leq 1\).

Observe that a \(\mu\)-resolution \(\phi\) of a set \(A \in \Lambda\) is continuous in the sense that \(\mu(\phi(u) - \phi(t)) = (u-t)\mu(A) \to 0\) as \(u \downarrow t\) or as \(t \uparrow u\). This is equivalent to saying that \(\phi\) is continuous as a mapping of \([0, 1]\) into the metric space associated with \((X, \Lambda, \mu)\). (Also see [26].)

(5.10) THEOREM. The following four statements are equivalent for the finite m. s. \((X, \Lambda, \mu)\).

(i) \((X, \Lambda, \mu)\) is non-atomic.

(ii) There is a \(\mu\)-resolution of \(X\).

(iii) There is a measure preserving map of \(X\) onto \([0, \mu(X)]\).

(iv) Every right continuous decreasing function on \([0, \mu(X)]\) is the decreasing rearrangement of a measurable function on \((X, \Lambda, \mu)\).
PROOF. (i) = (ii). Let a = \mu(X) and let \( x_0 \in X \). Since \((X, \Lambda, \mu)\) is non-atomic, \( \mu(\{x_0\}) = 0 \). Let \( \phi(0) = \{x_0\} \) and \( \phi(1) = X \). If \( \phi(m/2^n) \) is defined for each \( 0 \leq m \leq 2^n \) so that \( \phi(m/2^n) \subset \phi(m/2^n+1) \) and \( \mu(\phi(m/2^n)) = \frac{m}{2^n} - a \), then there are sets \( \phi(\frac{m}{2^n}) \in \Lambda \) s.t.

\[
\phi(\frac{m-1}{2^n}) \subset \phi(\frac{2m-1}{2^n}) \subset \phi(\frac{m}{2^n}) \quad \text{and} \quad \mu(\phi(\frac{2m-1}{2^n+1})) = \frac{2m-1}{2^n+1} \quad a, \quad 0 \leq m \leq 2^n.
\]

This defines \( \phi \) on \( A = \{ \frac{m}{2^n} : n \geq 0, \ 0 \leq m \leq 2^n \} \subset [0, 1] \) so that for all \( u, v \in A \) we have \( \phi(u) \subset \phi(v) \) when \( u < v \) and \( \mu(\phi(u)) = au \). If \( t \in [0, 1] - A \) we define \( \phi(t) = \cap \{ \phi(u) : u \geq t \} \), which also holds for \( t \in A \).

\[
\mu(\phi(t)) = \inf\{\mu(\phi(u)) : u \geq t \} = t a \quad \text{and if} \quad t_1 < t_2 \quad \text{there is a} \quad v \in B \quad \text{s.t.}
\]

\[
t_1 < v < t_2 \quad \text{and hence} \quad \phi(t_1) = \cap \{\phi(u) : u \geq t_1\} \subset \phi(v) \subset \cap \{\phi(u) : u \geq t_2\} = \phi(t_2).
\]

Thus \( \phi \) is a \( \mu \)-resolution of \( X \).

(ii) \( \Rightarrow \) (iii). Let \( \phi \) be a \( \mu \)-resolution of \( X \). Define

\[
\sigma(x) = a \cdot \inf\{t : x \in \phi(t)\}.
\]

Then \( \{x \in X : \sigma(x) > s\} = \cup\{\phi(t)^c : t > s/a\} \) so

\[
d_\sigma(s) = \mu(\sigma > s) = \lim \mu(\phi(t)^c) = \lim(a-ta) \quad \text{as} \quad t \downarrow s/a
\]

\[=aC_{-\infty}, \quad 0[a-s] \subset [0, a] \quad \text{so} \quad \delta_\sigma(t) = a-t, \quad 0 \leq t \leq a, \quad \text{and hence} \quad \sigma \text{ is m.p.}
\]

Any m.p. map is essentially onto.

(iii) \( \Rightarrow \) (iv) This is (5.7).

(iv) \( \Rightarrow \) (i) Let \( f \in M(X, \mu) \) be s.t. \( \delta_f(t) = a-t, \ 0 \leq t \leq a \). Then \( f \) is not constant on any subset of \( X \) of positive measure so \((X, \Lambda, \mu)\) has no atoms.

EXAMPES. 1. \( \phi(t) = [0, at] \) is an m-resolution of \([0, a]\) with Lebesgue measure \( m \).

2. If \( \sigma : X \to [0, a] \) is m.p. then \( \phi(t) = \sigma^{-1}[0, at] \) is a \( \mu \)-resolution of \( X \), and for all \( x \in X \) we have \( \sigma(x) = \inf\{t : x \in \sigma^{-1}[0, at]\} \).
3. \( \sigma(x) = b-x \) and \( \tau(x) = x-a \) are measure preserving maps of \([a, b]\) onto \([0, b-a]\).

The following theorem, proved by J. V. Ryff [40] for \( M[0, 1] \), shows how each measurable function on a non-atomic m.s. can be related to its decreasing rearrangement. First a lemma.

(5.11) **Lemma.** If the finite m.s. \((X, \Lambda, \mu)\) is non-atomic and \( \mu(X) = b-a \), then there is a m.p. map of \( X \) onto \([a, b]\) and also one of \( X \) onto \([a, b]\).

**Proof.** Let \( \varphi: X \to [0, b-a] \) be m.p. and let \( \psi(t) = t+a, \ t \in \mathbb{R} \). Then \( \sigma = \psi \circ \varphi: X \to [a, b] \) is m.p. and if \( \rho(t) = \begin{cases} t & t \in [a, b], \\ a & t = b \end{cases} \) then \( \sigma_1 = \rho \circ \psi \circ \varphi: X \to [a, b]. \) Any m.p. map is essentially onto.

(5.12) **Theorem** (J. V. Ryff). If the finite m.s. \((X, \Lambda, \mu)\) is non-atomic and \( f \in M(X, \mu) \) then there is a m.p. \( \sigma: X \to [0, \mu(X)] \) s.t.: 
\[ f = \delta_f \circ \sigma \quad \mu - \text{a.e.} \]

**Proof.** Following Ryff [40, p. 96] let \( I_t = \{ s \in [0, \mu(X)]: \delta_f(s) = t \} \) and \( A_t = \{ x \in X: f(x) = t \} \) for each \( t \in \mathbb{R}^f \). Since \( f \sim \delta_f \), there is an \( f' = f \mu\text{-a.e. s.t.} \ f' \) and \( \delta_f \) have the same range \( B \). Denote \( f' \) by \( f \).

Also \( m(I_t) = \mu(A_t) \) for each extended real \( t \), and each \( I_t \) has the form \([a, b]\) or \([a, b]\) since \( \delta_f \) is decreasing and rt-ctn. Hence for each \( t \in B \) there is a m.p. \( \sigma_t: A_t \to I_t \). Define \( \sigma \) by \( \sigma(x) = \sigma_t(x) \) if \( x \in A_t \).

Then clearly \( f = \delta_f \circ \sigma \). Now \( \mu(X) < \infty \Rightarrow m(I_t) > 0 \) for at most countably many \( t \). Let \( F \) be the set of all such \( t \) and let \( I = \bigcup_{t \in F} I_t \), which is
measurable since its complement is, and on which \( \delta_f \) is 1:1. Let 
\( J \subseteq [0, \mu(X)] \) be measurable. Then 
\[ J = \bigcup_{t \in F} J \cap I_t \] 
so 
\[ \mu(\sigma^{-1}(J)) = \sum_{t \in F} \mu(\sigma^{-1}(J \cap I_t)) + \mu(\sigma^{-1}(J \cap I)) \]. 
But 
\[ \mu(\sigma^{-1}(J \cap I)) = \mu(\sigma_t^{-1}(J \cap I_t)) = m(J \cap I_t) \] 
and \( \delta_f \) is 1:1 on \( J \cap I_t \), so \( f = \delta_f \circ \sigma \circ \delta_f^{-1} \circ f = \sigma \) there, and hence 
\[ \mu(\sigma^{-1}(J \cap I)) = \mu(f^{-1}(\delta_f(J \cap I))) = m(\delta_f^{-1}(\delta_f(J \cap I))) = m(J \cap I) \]. Hence \( \sigma \) is m.p.

**REMARKS.** (1) In general we cannot always find a measure preserving map \( \phi : [0, a] \to X \) such that \( f \circ \phi = \delta_f \). For example, let 
\( f : [0, 1] \to [0, 1] \) be m.p., suppose \( \phi : [0, 1] \to [0, 1] \) is measure preserving s.t. \( f \circ \phi = \delta_f \) and let \( \sigma : [0, 1] \to [0, 1] \) be m.p. such that \( \delta_f \circ \sigma = f \). Then \( \delta_f(t) = 1 - t = f \circ \phi(t) = 1 - \sigma(\phi(t)) \) so \( \sigma(\phi(t)) = t \) for all \( t \in [0, 1] \) and hence 
\[ t_1 \neq t_2 = \sigma(\phi(t_1)) \neq \sigma(\phi(t_2)) \Rightarrow \phi(t_1) \neq \phi(t_2). \]
Since \( \phi \) is m.p. it is essentially onto and it follows that \( \phi^{-1} = \sigma \), so \( \sigma \) is an invertible m.p. map. Also \( \delta_f \circ \sigma = f \) implies \( f = 1 - \sigma \) so \( f \) is necessarily 1:1. Thus if \( f \) is not 1:1, say \( f(x) = 2x \mod 1 \), then there is no m.p. \( \phi : [0, 1] \to [0, 1] \) such that \( f \circ \phi = \delta_f \) (see §20).

(2) Let \( (X_i, \Lambda_i, \mu_i) \) be finite m.s. with \( \mu_i(X_i) = a_i \), \( i = 1, \ldots, k \), let \( (X, \Lambda, \mu) \) be their product m.s., and let \( J = [0, a_1] \times \cdots \times [0, a_k] \) with product Lebesgue measure. If \( \sigma_i : X_i \to [0, a_i] \) are m.p. \( i = 1, \ldots, k \), and 
\[ \sigma(x_1, \ldots, x_k) = (\sigma_1(x_1), \ldots, \sigma_k(x_k)) \] 
then \( \sigma : X \to J \) is m.p. To prove this take \( f_i = \sigma_i \) and \( g_i(t) = t \) for all \( t \in [0, a_i] \) in Prop. (3.6).

(3) Observe that Prop. (3.6) = \( (f_1, \ldots, f_k) \sim (\delta_{f_1}, \ldots, \delta_{f_k}) \) so as in (3.7) at the end of §3 we may conclude such things as 
\[ |f(x) - g(y)| \sim |f(u) - \delta_{\hat{g}}(v)|. \]
(4) If \( f \in M[0, 1] \), and \( t \in [0, 1] \) let \( \varphi(t) = m(\{ f > \varphi(t) \}) + m(\{ f = \varphi(t) \} \cap [0, t]) \). Ryff \([44]\) has shown that
\( \varphi: [0, 1] \to [0, 1] \) is m.p. and \( f = \delta_f \circ \varphi \).

(5) Actually Liapounoff proved a much stronger result in 1940 than the one stated above: The range of a countably additive finite measure taking values in \( \mathbb{R}^n \) is compact, and if the measure space is non-atomic, then the range is convex as well \([22]\). In 1947 Paul R. Halmos gave a simplified proof of this result \([11]\). In Lemma 7 he shows that a non-atomic m.s. \((X, \Lambda, \mu)\), with \( \mu \) taking values in \( \mathbb{R}^n \), is convex, i.e., for every \( E \in \Lambda \) there is a function \( \phi: E \to [0, 1[ \) s.t. 
\( \mu(\{ \phi < s \}) = s \mu(E) \) for every \( s \in [0, 1] \). We may define a one-dimensional vector valued Lebesgue measure \( \lambda \) on the line segment 
\( [0, \mu(E)[ = \{ s \mu(E): 0 \leq s < 1 \} \) joining the zero vector and the vector \( \mu(E) \) as follows. Let \( m \) be Lebesgue measure on \( [0, 1[ \). If \( B \subset [0, 1[ \) we write \( B \mu(E) \) to denote \( \{ t \mu(E): t \in B \} \), and if \( B \) is Lebesgue measurable we define \( \lambda(B \mu(E)) = m(B) \mu(E) \). Thus \( \lambda \) is a vector valued measure defined on the \( \sigma \)-algebra 
\( S = \{ B \mu(E): B \subset [0, 1[ \) is Lebesgue measurable \} of subsets of \( [0, \mu(E)[ \). Now let \((X, \Lambda, \mu)\) be convex, let \( E \in \Lambda \) and let \( \phi \) be as above. If we define \( \sigma(x) = \phi(x) \mu(E) \) then \( \sigma \) is a m.p. map of \((X, \Lambda, \mu)\) onto \(([0, \mu(E)[, S, \lambda)\).
6. Decreasing Rearrangements on Not Necessarily Finite m.s.

Let \((X, \Lambda, \mu)\) be a measure space (m.s.) and let \(\mu\) denote Lebesgue measure. If \(f \in M(X, \mu)\) and there is an interval \(I \subset \mathbb{R}\) and a decreasing function \(\delta \in M(I, \mu)\) such that \(f \sim \delta\), then we will call \(\delta\) a decreasing rearrangement of \(f\). In this section we will characterize those functions which have a decreasing rearrangement in this sense.

(6.1) DEFINITION. We denote by \(D(X, \mu)\) the set of all \(f \in M(X, \mu)\) which satisfy

(i) \(\mu(f^{-1}[a, b]) < \infty\) whenever \([a, b] \subset \) ess. inf \(f\), ess. sup \(f\)

(ii) \(\mu([a, b]) < \infty\) whenever \(\mu(f^{-1}(\text{ess. sup} f)) > 0\)

and ess. inf \(f < c < \text{ess. sup} f\).

(iii) \(\mu(f^{-1}\text{ess. inf} f, c] < \infty\) whenever \(\mu(f^{-1}(\text{ess. inf} f)) > 0\) and ess. inf \(f < c < \text{ess. sup} f\).

It is our purpose to show that \(D(X, \mu)\) is precisely the set of all those functions which have decreasing rearrangements.

(6.2) LEMMA. If \(I\) is an interval of \(\mathbb{R}\) and \(\delta \in M(I, \mu)\) is monotonic, then \(\delta \in D(I, \mu)\).

PROOF. Assume \(\delta\) is decreasing; the proof when \(\delta\) is increasing is similar. Let \(J = \) ess. inf \(\delta\), ess. sup \(\delta\) \(\subset \), so \(J \subset \) inf \(\delta\), sup \(\delta\) \(\subset \).

(i) Let \([a, b] \subset J\). Then \(b < \sup \delta\) so there is an \(u \in I\) such that \(b < \delta(u)\) and hence \(\delta^{-1}[a, b] \subset ]u, \infty[\). Since inf \(\delta < a\) there is a \(v \in I\) such
that \( \delta(v) < a \) so \( \delta^{-1}[a, b] \subset ]-\infty, v[ \). Hence \( \delta^{-1}[a, b] \subset ]u, v[ \) so \( m(\delta^{-1}[a, b]) < \infty \). (ii) Let \( c \in J \). Then there are \( w, v \in I \) such that \( \delta(v) < c < \delta(w) \). If \( m(\delta^{-1}(\text{ess. sup } \delta)) > 0 \) then there is a \( t \in I \) such that \( \delta(t) = \text{ess. sup } \delta \), so \( \delta^{-1}c, \text{ess. sup } \delta \subset [t, v] \) and thus \( m(\delta^{-1}c, \text{ess. sup } \delta) < \infty \).

The proof when \( m(\delta^{-1}(\text{ess. inf } \delta)) > 0 \) is similar.

(6.3) Lemma. (i) If \( f \in D(X, \mu) \) and \( f_1 \in M(X_1, \mu_1) \) then \( f_1 \sim f \) implies \( f_1 \in D(X_1, \mu_1) \).

(ii) If \( f \in D(X, \mu) \) then \( f + r \in D(X, \mu) \) for all \( r \in R \).

Proof. (i) use (3.3) (vi).

(ii) Let \( f \in D(X, \mu) \) and \( r \in R \). Then \( \text{ess. inf } (f+r) = r + \text{ess. inf } f \) and \( \text{ess. sup } (f+r) = r + \text{ess. sup } f \). If \( [a, b] \subset ]\text{ess. inf } (f+r), \text{ess. sup } (f+r) \[ \) then \( [a-r, b-r] \subset ]\text{ess. inf } f, \text{ess. sup } f \[ \) so \( \mu((f+r)^{-1} [a, b]) = \mu(f^{-1} [a-r, b-r]) < \infty \). The rest of the verifications are similar.

Thus we see that (6.2) and (6.3) (i) imply that \( D(X, \mu) \) contains all measurable functions which have decreasing rearrangements. To prove the converse we have to construct a decreasing rearrangement for each function in \( D(X, \mu) \). It is convenient to do this first for a special subset of \( D(X, \mu) \), which we now define.

(6.4) Definition. \( D'(X, \mu) \) is the set of all functions \( f \in D(X, \mu) \) which satisfy

(i) \( \text{ess. inf } f \leq 0 < \text{ess. sup } f \)

(ii) \( \mu(f^{-1}(0)) > 0 \) if \( \text{ess. inf } f = 0 \).
If $f \in D'(X, \mu)$ we define a distribution function by

$$d_f(t) = \begin{cases} 
\mu(f^{-1}[t, 0]) & \text{if } t < 0 \\
-\mu(f^{-1}[0, t]) & \text{if } t \geq 0 
\end{cases}$$

for all $t \in \mathbb{R}^\#$. Then $d_f$ is decreasing and we define

$$\delta_f(t) = d_f^*(t)$$

for all $t \in I = \mu(f^{-1}[0, \infty]), \mu(f^{-1}[-\infty, 0]).$

(6.5) LEMMA. Let $f \in D'(X, \mu)$ and let

$I = \mu(f^{-1}[0, \infty]), \mu(f^{-1}[-\infty, 0]).$.

(i) $|d_f(t)| < \infty$ if $\text{ess. inf } f < t < \text{ess. sup } f.$

(ii) $d_f$ is right continuous

(iii) $d_f^*(t) = +\infty$ if $t < -\mu(f^{-1}[0, u])$ for all $0 \leq u \in \mathbb{R}$

(iv) $d_f^*(t) = -\infty$ if $t \geq \mu(f^{-1}[-\infty, 0]).$

PROOF. (i) Since $f \in D(X, \mu)$, this is clear when $\text{ess. inf } f \neq 0.$
If $\text{ess. inf } f = 0$ then $\mu(f^{-1}(\text{ess. inf } f)) > 0$ so $\mu(f^{-1}[0, t]) < \infty$ whenever $\text{ess. inf } f = 0 < t < \text{ess. sup } f.$

(ii) Now $d_f$ is clearly right continuous at all $t < 0$, and also at $t \geq 0$ if $|d_f(u)| < \infty$ for some $u > t$. Hence let $t \geq 0$ and suppose

$|d_f(u)| = \infty$ for all $u > t$. For such $u$ we have $u > 0 \geq \text{ess. inf } f$, so

(i) implies $\text{ess. sup } f \leq u$. Hence $\mu(f^{-1}[0, t]) = \mu(f^{-1}[0, \text{ess. sup } f]) = \mu(f^{-1}[0, u]).$

(iii) $d_f^*(t) = +\infty$ iff $\{u: d_f(u) > t\} \supset [0, +\infty[$ and

$\delta_f^*(t) = -\infty$ iff $\{u: d_f(u) \leq t\} \supset ]-\infty, 0[$.
One would hope that \( d_f = d_g \) on \( \mathbb{R}^\# \) implies \( f \sim g \). The following example shows this is not the case. Let \( f(x) = x \) on \( X_1 = [0, +\infty) \) and let

\[
g(x) = \begin{cases} 
  x & \text{if } 0 \leq x \leq +\infty \\
  +\infty & \text{if } -1 \leq x < 0 
\end{cases}
\]
on \( X_2 = [-1, +\infty] \).

Then \( d_f = d_g \) on \( \mathbb{R}^\# \) but \( m(f^{-1} (+\infty)) \neq m(g^{-1} (+\infty)) \). We will be able to prove \( \delta_f = \delta_g \) iff \( f \sim g \), however, by using (6.5) (iii) & (iv) and the following result.

(6.6) **LEMMA.** Let \( f \in D'(X_1, \mu_1) \) and \( g \in D'(X_2, \mu_2) \). If \( d_f = d_g \) on \( \mathbb{R} \) then \( \mu_1(f^{-1} [a, b]) = \mu_2(g^{-1} [a, b]) \) for all intervals \([a, b]\) of \( \mathbb{R} \).

**PROOF.** This is equivalent to proving that \( f|f^{-1}[\mathbb{R}] \sim g|g^{-1}[\mathbb{R}] \) so we may assume (and we do) that \( f \) and \( g \) are essentially finite.

Let \( d_f = d_g \) on \( \mathbb{R} \). Then (6.5) (i) says that \( |d_f(t)| < \alpha \) and \( |d_g(t)| < \alpha \) whenever \( \min \{\text{ess. inf } f, \text{ ess. inf } g\} < t < \max \{\text{ess. sup } f, \text{ ess. sup } g\} \), in which case \( \mu_2(g^{-1} [-\alpha, t]) = \mu_1(f^{-1} [-\alpha, t]) \) and \( \mu_2(g^{-1} [t, +\alpha]) = \mu_1(f^{-1} [t, +\alpha]) \). Then \( \text{ess. inf } f < t < \text{ess. sup } f \) implies \( \mu_2(g^{-1} [t, +\alpha]) > 0 \) and \( \mu_2(g^{-1} [-\alpha, t]) > 0 \) so \( \text{ess. sup } g \geq \text{ess. sup } f \) and \( \text{ess. inf } g \leq \text{ess. inf } f \). The argument is symmetric in \( f \) and \( g \) so we conclude that they have the same ess. inf, say \( u \), and the same ess. sup, say \( v \).

Now \([a, b] \subset [u, v]\) implies \( \mu_1(f^{-1} [a, b]) \) & \( \mu_2(g^{-1} [a, b]) < \alpha \) in which case \( \mu_1(f^{-1} [a, b]) = d_f(a-) - d_f(b) = d_g(a-) - d_g(b) = \mu_2(g^{-1} [a, b]) \). Also if \([a, b] \subset [-\alpha, +\alpha] \) then \( \mu_1(f^{-1} [a, b]) = 0 = \mu_2(g^{-1} [a, b]) \).

Now \( f, g \in D' \) implies \( u = 0 < v \). If \( \mu_1(f^{-1} [0, v]) < \alpha \) then \( \mu_1(f^{-1} (v)) = d_f(v-) - d_f(v) = d_g(v-) - d_g(v) = \mu_2(g^{-1} (v)) \). Otherwise, since \( f, g \in D' \), \( \mu_1(f^{-1} (v)) = 0 = \mu_2(g^{-1} (v)) \). For the rest, if \( \mu_1(f^{-1} [u, 0]) < \alpha \),
then since \( \mu_1(f^{-1}(-\infty)) = 0 = \mu_2(g^{-1}(-\infty)) \), we have \( \mu_1(f^{-1}(u)) = \mu_2(g^{-1}(u)) \).

\[
\begin{align*}
\mu_1(f^{-1}] - x, u[) &= \mu_1(f^{-1}] - \infty, 0[) - \mu_1(f^{-1}] u, 0[) \\
&= \mu_2(g^{-1}] - \infty, 0[) - \mu_2(g^{-1}] u, 0[) = \mu_2(g^{-1}] - \infty, u[) \\
&= \mu_2(g^{-1}(u)).
\end{align*}
\]

Otherwise, \( u < 0 \) and \( \mu_1(f^{-1}(u)) = 0 = \mu_2(g^{-1}(u)) \).

(6.7) **Lemma.** Let \( J \) be an interval of \( R^\# \) such that \( \inf J \leq 0 \leq \sup J \). If \( p \in D'(J, m) \) is decreasing and \( p(t) \leq 0 \) if \( t \geq 0 \), then \( p' = dp \) on \( R^\# \).

**Proof.** From the definition of \( p' \) and the condition on \( p \) it is easy to see that we have:

- if \( t < 0 \), then \([0, p^\ast(t)] \cap J \cap p^{-1}[t, 0] \subset [0, p^\ast(t)]\)
- if \( t \geq 0 \), then \([p^\ast(t), 0] \cap J \cap p^{-1}[0, t] \subset [p^\ast(t), 0]\)

Hence \( p^\ast = dp \).

(6.8) **Definition.** For each \( f \in D(X, \mu) \) define \( b_f = 0 \) if \( f \in D'(X, \mu) \) while if \( f \notin D'(X, \mu) \) define

\[
b_f = \begin{cases}
\frac{1}{2}(\text{ess. inf } f + \text{ess. sup } f) & \text{if } |\text{ess. inf } f| < \infty \text{ and } |\text{ess. sup } f| < \infty \\
-1 + \text{ess. sup } f & \text{if } |\text{ess. sup } f| < \infty \text{ and } |\text{ess. inf } f| = \infty \\
1 + \text{ess. inf } f & \text{if } |\text{ess. inf } f| < \infty \text{ and } |\text{ess. sup } f| = \infty
\end{cases}
\]

(6.9) **Lemma.** If \( f \in D(X, \mu) \) is not essentially constant then \( \text{ess. inf } f < b_f < \text{ess. sup } f \) and \( f - b_f \in D'(X, \mu) \).
(6.10) DEFINITION. For each \( f \in D(X, \mu) \) define
\[
\delta_f = \begin{cases} 
\delta_f - b_f + b_f & \text{if } f \text{ is not essentially constant} \\
\text{ess. sup } f & \text{if } f \text{ is essentially constant}
\end{cases}
on I = \left[-\mu(f^{-1}[b_f, \infty]), \mu(f^{-1}[-\infty, b_f])\right].
\]

(6.11) THEOREM. Let \( f \in D(X, \mu) \) and let
\[
I = \left[-\mu(f^{-1}[b_f, \infty]), \mu(f^{-1}[-\infty, b_f])\right].
\]
(i) Then \( \inf I \leq 0 \leq \sup I \), \( \delta_f \in D(I, m) \) is decreasing and right continuous, \( \delta_f \sim f \), and for each \( t \in I \), \( \delta_f(t) \leq b_f \) iff \( t \geq 0 \).

(ii) Suppose \( J \) is an interval of \( \mathbb{R} \) such that \( \inf J \leq 0 \leq \sup J \), suppose \( p \in D(J, m) \) is decreasing and right continuous, \( p \sim f \), and for each \( t \in J \), \( p(t) \leq b_f \) iff \( t \geq 0 \). Then \( I \subseteq J \) and \( p = \delta_f \) on \( I \).

PROOF. (i) The result is clearly true if \( f \) is essentially constant since \( m(I) = \mu(X) \). If the result is true for \( f \in D'(\text{so } b_f = 0) \) and \( f \in D \) is not essentially constant, then
\[
\delta_f(t) = \delta_f - b_f + b_f \leq b_f \text{ iff } \delta_f - b_f(t) \leq 0 \text{ iff } t \geq 0 \text{ for all } t \in ]u, v[ \text{ where } u = -\mu((f-b_f)^{-1}[0, \infty]), v = -\mu((f-b_f)^{-1}[-\infty, 0]), \mu((f-b_f)^{-1}[-\infty, b_f]) \text{ so }
\]
\[
I = ]u, v[; \text{ and } f - b_f \sim \delta_f - b_f \text{ so } f \sim \delta_f - b_f + b_f = \delta_f.
\]
Hence suppose \( f \in D' \).

If \( 0 \leq t \in \mathbb{R}^\# \), then \( d_f(0) = 0 \leq t \) so \( d_f^*(t) = \inf\{u: d_f(u) \leq t\} \leq 0 \).

Suppose \( 0 > t \in \mathbb{R}^\# \). For all \( 0 < u < \text{ess. sup } f \), \( d_f(u) \) is finite. Hence \( d_f(u) \uparrow 0 \) as \( u \downarrow 0 \) so there is a \( u_0 > 0 \) s.t. \( t < d_f(u_0) \). Then \( d_f^*(t) = \sup\{u: d_f(u) > t\} \geq u_0 > 0 \).
Let \( q = d_f^* \). Then \( d_q = q^* = d_f^{**} = d_f \) on \( \mathbb{R} \). But \( |d_f^*(t)| < \infty \Rightarrow t \in \overline{\mathbb{I}} \).

(Lemma (6.5)) \( m(\delta_f^{-1}[a, b]) = m(q^{-1}[a, b]) = m(q^{-1}[a, b]) = \mu(f^{-1}[a, b]) \)

for all bounded intervals \([a, b]\) of \( \mathbb{R} \). Now (6.5) implies

\[
\delta_f^{-1}(-\infty) = [\mu(f^{-1}] - \infty, 0)], \quad \mu(f^{-1}[-\infty, 0]) = 0 \Rightarrow m(\delta_f^{-1}(-\infty)) = 0. \quad \text{If} \ \mu(f^{-1}(-\infty)) > 0 \ \text{then} \ \mu(f^{-1}] - \infty, 0) < \infty \ \text{and thus}
\]

\[
m(\delta_f^{-1}(-\infty)) = \mu(f^{-1}(-\infty)). \quad \text{Similarly, if} \ \mu(f^{-1}(+\infty)) = 0, \ \text{then}
\]

\[
-\mu(f^{-1}[0, u]) \downarrow -\mu(f^{-1}[0, \infty)) \quad \text{as} \ u \to \infty \ \text{so} \ (6.5) \Rightarrow m(\delta_f^{-1}(+\infty)) = 0.
\]

If \( \mu(f^{-1}(+\infty)) > 0 \), then \( \mu(f^{-1}[0, \infty)) < \infty \) \ \text{so} \ m(\delta_f^{-1}(+\infty)) = \mu(f^{-1}(+\infty)).

(ii) Again this is clearly true if \( p \) is essentially constant since

then \( f \) is, and \( p \sim f \) implies \( m(J) = \mu(X) \). \ If the result is true for all

\( p \in D'(J, m) \) (so \( b_f = 0 \)), and \( p \in D(J, m) \) is not essentially constant,

then \( p - b_f \in D'(J, m) \) satisfies all the conditions in (ii) for

\( f - b_f \in D'(X, \mu) \) so \( I \subset J \) and \( p - b_f = \delta_f - b_f \) i.e. \( p = \delta_f - b_f + b_f = \delta_f \) on \( I \).

Hence suppose \( f \in D' \). Since \( p \sim f \),

\[
d_f = d_p = p^* \quad \text{so} \quad d_f^* = p^{**} = p. \quad \text{Also, because} \ p \sim f,
\]

\[
m(J \cap p^{-1}[0, \infty)] = \mu(f^{-1}[0, \infty)] \quad \text{and}
\]

\[
m(J \cap p^{-1}[-\infty, 0]) = \mu(f^{-1}[-\infty, 0]), \quad \text{and}
\]

\[
\inf J, 0 \subset J \cap p^{-1}[0, \infty] \subset [\inf J, 0] \quad \text{and}
\]

\[
[0, \sup J \subset J \cap p^{-1}[-\infty, 0] \subset [0, \sup J] \quad \text{so}
\]

\[
\inf J = -\mu(f^{-1}[0, \infty)) \quad \text{and} \ \sup J = \mu(f^{-1}[-\infty, 0])
\]

and hence \( I \subset J \), and on \( I \), \( p = d_f^* = \delta_f \).
(6.12) THEOREM. Suppose \( f \in D(x, \mu) \) and ess. \( \inf f < 0 < \text{ess. sup } f \).

Let \( I = \text{]-}\mu(f^{-1}[0, +\infty]), \mu(f^{-1}[-\infty, 0]) \text{]}. \) Then

\[
\begin{align*}
a &= \mu(f^{-1}(0)) < \infty \quad \text{and we have} \\
(i) \quad (\delta_f)^* &= \delta_f^* \quad \text{on } I \\
&\quad (\delta_f)^-(t) = \delta_f^-(a-t-) \quad t \in I \\
(ii) \quad \delta_{f^*}(t) &= -\delta_f^-(a-t-) \quad t \in a - I.
\end{align*}
\]

PROOF. Let \( F(t) = (\delta_f)^-(a-t-) \). We prove that \( \delta_{f^*} = F \) on \( a - I \).

Now \( f \sim \delta_f \) so \( f^- \sim (\delta_f)^- \sim F \). Now \( a = m(\delta_f^{-1}(0)) \), so \( \delta_f^{-1}(0) \subseteq [0, a] \) so \( F(t) \leq 0 \) iff \( \delta_f^-(a-t-) = 0 \) iff \( t \geq 0 \) and hence \( F = \delta_{f^*} \) on \( J = \text{]-}\mu(f^{-1}[0, +\infty)), \mu((f^-)^{-1}[-\infty, 0]) \text{]}. \)

The rest is similar and easier.

We now show how to obtain an analog of Theorem (5.10) for general m.s. Observe that unlike the situation for finite m.s., if \( A \) is an atom and \( f \in M(X, \mu) \), it need not be the case that \( f \) is constant on \( A \), or even on a subset of \( A \) of positive measure. This can only happen, of course, if \( \mu(A) = +\infty \). The situation is nicer if \( f \in D(X, \mu) \).

(6.13) LEMMA. If \( A \) is an atom of \( (X, \Lambda, \mu) \) and \( f \in D(X, \mu) \) then

\[
\mu \{ x \in A : f(x) \notin \{ \text{ess. inf } f, \text{ ess. sup } f \} \} = 0.
\]

Hence \( f \) is constant on a subset of \( A \) of positive measure.
PROOF. Let $J = \text{ess. inf } f$, $\text{ess. sup } f$. If $\mu(A) < \alpha$ then $f$ is essentially constant on $A$, so assume $\mu(A) = +\infty$. Since $f \in D$, 
$$\mu(f^{-1}[r, s]) < \infty$$ whenever $[r, s] \subset J$ so $\mu((f|A)^{-1}[r, s]) = \mu(f^{-1}[r, s] \cap A) < +\infty = \mu(A)$ and thus $\mu(f|A)^{-1}[r, s]) = 0$ for all $[r, s] \subset J$, since $A$ is an atom.

Since $J$ is a countable union of closed intervals, 
$$\mu\{x \in A: f(x) \notin [\text{ess. inf } f, \text{ess. sup } f]\} = \mu(f|A)^{-1}(J)) = 0.$$ 

$0 < \mu(A) \leq \mu(f^{-1}(\text{ess. inf } f)) + \mu(f^{-1}(\text{ess. sup } f))$ shows the rest.

(6.14) LEMMA. If $(X, \lambda, \mu)$ is a m.s. and $I$ is an open interval of $R$ s.t. $\mu(X) = m(I)$ and $\{X_i\}_{i=1}^\infty$ is a partition of $X$ by measurable sets s.t. $\mu(X_i) < \infty$ i = 1, 2, 3, ..., then there are pw intervals 
$$\{a_n, b_n\}_{n=1}^\infty$$ s.t. $\mu(X_n) = b_n - a_n$ n = 1, 2, 3, ..., and 
$$m(I - \bigcup_{n=1}^\infty [a_n, b_n]) = 0.$$

PROOF. Suppose first that $I$ is bounded below, say $a_1 = \inf I > -\infty$.

Let $a_n = a_1 + \sum_{i=1}^{\infty} \mu(X_i)$ and $b_n = a_n + \mu(X_n)$. Then 
$$\bigcup_{n=1}^\infty [a_n, b_n] = I - \{a_n : n = 1, 2, 3, \ldots \}. $$

Similarly if $I$ is bounded above. Hence let $I = R$. Then $\sum_{i=1}^\infty \mu(X_i) = \mu(X) = m(I) = +\infty$ so there is a partition 
$$\{B_k\}_{k=1}^\infty$$ of the positive integers by finite sets $B_k$ such that 
$$\sum_{i=1}^\infty \mu(X_i) \geq 1 \text{ for each } k = 1, 2, 3, \ldots .$$

Let $\{n_1\}_{i=1}^\infty$ and $\{p_i\}_{i=1}^\infty$ enumerate $\bigcup_{k=1}^\infty B_{2k}$ and $\bigcup_{k=1}^\infty B_{2k-1}$ respectively, let $r_0 \in R$, and let 
$$X_1 = \bigcup_{i=1}^\infty X_{n_i}, \quad X_2 = \bigcup_{i=1}^\infty X_{p_i}, \quad I_1 = ]r_0, \infty[, \quad I_2 = ]-\infty, r_0[.$$ 

We get collections of intervals for $I_1$ and $X_1$ and for $I_2$ and $X_2$ as before, and the union of these collections works for $X$. 
THEOREM. Let \((X, \Lambda, \mu)\) be a m.s. and let \(I\) be an open interval of \(R\) with \(0 \in I\). Then the following four statements are equivalent.

(i) There is an \(f \in M(X, \mu)\) s.t. \(\delta_f(t) = -t\) for all \(t \in I\).

(ii) There is a measure preserving map of \(X\) onto \(I\).

(iii) If \(v \in M(I, m)\) is decreasing and right continuous and for each \(t \in I\), \(v(t) \leq b_v\) iff \(t \geq 0\), then there is an \(f \in D(X, \mu)\) s.t. \(\delta_f = v\) on \(I\).

(iv) \((X, \Lambda, \mu)\) is non-atomic and \(\sigma\)-finite and \(\mu(X) = m(I)\).

PROOF (i) \Rightarrow (ii) Let \(\sigma = -f\). Then \(-\sigma = f \sim \delta_f\) so \(\sigma \sim -\delta_f\). Then \(\sigma\) is m.p. and has the same essential range as \(-\delta_f\), namely \(I\).

(ii) \Rightarrow (iii). Now ess. inf \(\sigma = \inf I\) and ess. sup \(\sigma = \sup I\). Let \(f = v \circ \sigma\). Then \(f \sim v\) so \(b_f = b_v\).

\[\mu(f^{-1}[a, \infty]) = m(v^{-1}[b_v, \infty]) = -\inf I\]

\[\mu(f^{-1}(-\infty, b_v]) = m(v^{-1}(-\infty, b_v]) = \sup I\] Hence Theorem (6.11) implies \(\delta_f = v\) on \(I\).

(iii) \Rightarrow (iv) Let \(f \in M(X, \mu)\) s.t. \(\delta_f(t) = -t\) for all \(t \in I\). Then \(X = \bigcup f^{-1}[i, i+1[\) s.t. \(\mu(f^{-1}[i, i+1[) = m(\delta_f^{-1}[i, i+1[) \leq 1\)
so \(X\) is \(\sigma\)-finite. Since \(\mu(f^{-1}(r)) = m(\delta_f^{-1}(r)) = 0\) for all \(r \in R\), \(f\) is not constant on any set of positive measure, so \(X\) has no atoms. \(m(I) = m(\delta_f^{-1}(R)) = \mu(f^{-1}(R)) = \mu(X)\).

(iv) \Rightarrow (i) Let \(\{X_i\}_{i=1}^\infty\) be a partition of \(X\) by sets of finite measure.
Since \(X\) is non-atomic, so is each \(X_i\). Since \(m(I) = \mu(X)\) there are pairwise disjoint intervals \(\{a_i, b_i\}_{i=1}^\infty\) s.t. \(m(I - \bigcup_{i=1}^\infty a_i, b_i[) = 0\), and \(b_i - a_i = \mu(X_i)\).
Let \( t(t) = t \) for all \( t \in \mathbb{R} \). Now \( \alpha_n(t) = -(t+a_n) \) for \( 0 \leq t < b_n - a_n \) is decreasing and right continuous on \([0, \mu(X_n)]\) and hence there is an \( f_n \in M(X_n, \mu) \) s.t. \( f_n \sim \alpha_n \). Now for \( v_n(t) = \alpha_n(t-a_n) = -t \) \((a_n \leq t < b_n)\) we have \( f_n \sim \alpha_n \sim v_n \sim -t \mid [a_n, b_n] \). Let \( f(x) = f_n(x) \) if \( x \in X_n \). Then \( f \in M(X, \mu) \) and \( f|X_n = f_n \sim f \mid [a_n, b_n] \) so \( f \sim -t \mid I \). Then \( f \in D'(X, \mu) \) and Theorem (6.11) implies \( \delta_f = -t \) on \( I \).

(6.16) THEOREM. If \((X, \Lambda, \mu)\) is non-atomic and \( \sigma \)-finite, if \( f \in D(X, \mu) \), and if \( \delta \in M(\Lambda, m) \) is a decreasing rearrangement of \( f \), then there is a measure preserving map \( \sigma : X \to I \) such that \( \delta \circ \sigma = f \) \( \mu \)-a.e.

PROOF. Let \( J_t = \{ s \in I : \delta(s) = t \} \) for each \( t \in \mathbb{R}^\# \). Each \( J_t \) is an interval, and since the topology of \( \mathbb{R} \) has a countable base, we have \( m(J_t) > 0 \) for at most countably many \( t \). The rest of the proof is like that of (5.12).
REMARKS. 1. If \((X, \Lambda, \mu)\) is a m. s. and \(0 < f \in M(X, \mu)\) and 
\(\mu(\{f \geq \xi\}) < \infty\) for some \(\xi\), then we may define \([f]\) on \([0, \infty[\) to be the right continuous inverse of \(d_f(t) = \mu(\{f > t\})\) \([8]\). This is almost the same as the definition of \(\bar{f}\) given by Hardy, Littlewood and Polya \([15]\) for \(X\) an infinite interval of \(\mathbb{R}\). Although 
\(m(\{[f] > \xi\}) = \mu(\{f > \xi\})\) for all \(\xi \in \mathbb{R}\), in general \([f]\) need not be 
equimeasurable with \(f\). If \(f \in D(X, \mu)\) then 
\(a = \mu(f^{*-1}[b_f, \infty]) < \infty\) and for all \(0 \leq u < \infty\) we have 
\([f](u) = \delta_f(u-a)\).

2. If \((X, \Lambda, \mu)\) is a finite m. s. and \(f \in M(X, \mu)\) then the decreasing 
rearrangement of \(f\) defined in this section is a translation of the 
decreasing rearrangement defined in §5 for finite m. s. Letting 
\(f^*\) denote the one defined in §5, we have 
\[\delta_f(t) = f^*(t+a)\]
for all \(t \in I = [-\mu(f^{*-1}[b_f, \infty]), \mu(f^{*-1}[\infty, b_f])\]
where \(a = \mu(f^{*-1}[b_f, \infty])\).

3. Suppose \((X, \Lambda, \mu)\) is a finite m. s. and \(X\) is the union of a 
finite number of atoms of equal measure, say \(X = A_1 \cup \cdots \cup A_n\).

Then \(M(X, \mu)\) is isomorphic with \(\mathbb{R}^n\) under the correspondence 
\(f \mapsto a = (a_1, \ldots, a_n) \in \mathbb{R}^n\) where \(a_i = f|A_i\). For each \(f \in M(X, \mu)\) let 
\(\delta_f\) denote the decreasing rearrangement defined in §5. If \(f \in M(X, \mu)\) 
then \(\delta_f\) is constant on \([a_{k-1}^*, a_k^*]\) where 
\(a_k^* = \sum_{i=1}^{k} \mu(A_i)\) \(k=1, \ldots, n\), and 
if \(a \mapsto f\) and \(a^*_k = \delta_f(a_{k-1}^*, a_k^*]\) then \(a^*_k\) is the point of \(\mathbb{R}^n\) whose 
components are the components of \(a\) arranged in decreasing order.
4. G. F. D. Duff [6] has defined a generally multiple valued decreasing rearrangement for each member of \( M(I, m) \) where \( I \) is an infinite interval of \( \mathbb{R} \) and \( m \) is Lebesgue measure. The conditions (6.1) on \( f \in M(I, m) \) are necessary and sufficient that this decreasing rearrangement be single valued.
III. INEQUALITIES AND REARRANGEMENTS

7. A Theorem of Hardy. The following theorem of G. H. Hardy will be used often in what follows.

(7.1) THEOREM. (Hardy) (i) Suppose \( f_1, f_2 \in M[a, b], f_1, f_2 \in L^1[a, t], \) for all \( a \leq t < b \) and \( \int_a^t f_1 < \int_a^t f_2 \) for all \( a \leq t < b \). If \( g \) is a non-negative decreasing function on \([a, b]\) s.t. \( f_i g \in L^1[a, t] \) (\( i = 1, 2 \)) for all \( a \leq t < b \), then \( \int_a^t f_1 g \leq \int_a^t f_2 g \) for all \( a \leq t < b \).

(ii) Suppose \( f_1, f_2 \in L^1[a, b], \) \( \int_a^t f_1 \leq \int_a^t f_2 \) for all \( a \leq t \leq b \), \( \int_a^b f_1 = \int_a^b f_2 \), and \( g \in M[a, b] \) s.t. \( f_i g \in L^1[a, b] \) (\( i = 1, 2 \)). If \( g \) is decreasing then \( \int_a^b f_1 g \leq \int_a^b f_2 g \), while if \( g \) is increasing then \( \int_a^b f_1 g \geq \int_a^b f_2 g \).

PROOF. (i) Let \( F_i(t) = \int_a^t f_i \) (\( i = 1, 2 \)) if \( a < t < b \). Then for all \( a \leq t \leq b \),

\[
\int_a^t f_2 g - f_1 g = \int_a^t g (F_2 - F_1) = g(t) (F_2 - F_1)(t) - \int_a^t (F_2 - F_1) dg \geq 0.
\]

(ii) is similar.

(7.2) THEOREM. If \( f \in L^1[0, a] \) is decreasing, then \( \frac{1}{t} \int_0^t f \) is a decreasing function of \( t \), while if \( f \) is increasing, then \( \frac{1}{t} \int_0^t f \) is an increasing function of \( t \), \( 0 < t \leq a \).

PROOF. Let \( 0 < t_1 < t_2 \leq a \). Then \( t_1 t \leq t_2 t \) for all \( 0 \leq t \leq a \)
so \( \min \{ t_1 t_1, t_1 t_2 \} \leq \min \{ t_2 t_1, t_1 t_2 \} \) and hence

\[
\int_0^t \frac{1}{t_2} C[0, t_2] = \frac{1}{t_2} \min \{ t, t_2 \} \leq \frac{1}{t_1} \min \{ t, t_1 \} = \int_0^t \frac{1}{t_1} C[0, t_1]
\]

for all \( 0 \leq t \leq a \). Thus (7.1) (ii) implies

\[
\frac{1}{t_2} \int_0^t f = \int_0^a \frac{1}{t_2} C[0, t_2] f \leq \int_0^a \frac{1}{t_1} C[0, t_1] f = \frac{1}{t_1} \int_0^t f \text{ if } f \text{ is decreasing},
\]
and the reverse inequality if \( f \) is increasing.

8. A Preorder Relation. In this and in subsequent sections we will use \( \delta_f \) to denote the decreasing rearrangement defined in \( \S 5 \), since the measure spaces will always be finite. In addition we will use \( \delta_E \) to denote the decreasing rearrangement of \( CE \).

In [15] Hardy, Littlewood and Polya introduced for the first time an important preorder relation for \( n \)-tuples of real numbers and later for integrable functions on a finite interval. We present this relation for finite m. s. as follows.

\[(8.1)\) DEFINITION. Suppose \((X_1, \Lambda_1, \mu_1)\) and \((X, \Lambda, \mu)\) are finite m. s. with \( a = \mu_1(X_1) = \mu_1(X) \) and \( f, g \in M(X, \mu) \cup M(X_1, \mu_1) \) and \( \int_0^a \delta_f \) and \( \int_0^a \delta_g \) are finite.

(i) \( g << f \) means \( \int_0^t \delta_g \leq \int_0^t \delta_f \) for all \( 0 \leq t \leq a \);

(ii) \( g < f \) means \( g << f \) and \( \int_0^a \delta_g = \int_0^a \delta_f \).

It is obvious that

(i) \( f \sim g \iff g << f \) and \( f << g \iff g < f \) and \( f < g \);

(ii) \( g << f \iff \delta_g << \delta_f \)

(iii) \( g < f \iff \delta_g < \delta_f \)

(iv) \( << \) and \( < \) are reflexive and transitive.

The reader should compare the conditions in Definition (8.1) with the hypotheses of Theorem (7.1).
The following are some other useful but less obvious properties of these preorder relations.

(8.2) **LEMMA.** (i) If $g << f$ then $g + r << f + r$ for all $r \in \mathbb{R}$ and $rg << rf$ if $r \geq 0$.

(ii) If $g < f$ then $g + r < f + r$ for all $r \in \mathbb{R}$ and $rg < rf$ for all $0 \leq r \in \mathbb{R}$.

If in addition $f \in L^1$ then $-g < -f$.

(iii) If $r \leq s$ where $r, s \in \mathbb{R}^+$ and $g < f$ then $r \leq g \leq s$.

(iv) If $f \in L^1(X_1, \mu_1)$ then $\left( \frac{1}{\mu_1(X_1)} \int_{X_1} f \, d\mu_1 \right) C_X < f$.

(v) If $f \in L^1(X_1, \mu_1)$ and $E \in \Lambda$ then $f < C_E$ iff $0 < f \leq 1$ and $\int_{X_1} f \, d\mu_1 = \mu(E)$. In particular, if $f \in C_X$ then $f = C_{X_1} \cup_1$-a.e.

(vi) If $g^+ < f^+$ and $g^- < f^-$ then $g < f$.

(vii) If $g << f$ then $g^+ << f^+$. If $g < f \in L^1$ then in addition $g^- << f^-.

(viii) If $0 \leq f_n, 0 \leq g_n \uparrow g$, and $g_n << f_n$, $n = 1, 2, 3, \ldots$, then $g < f$.

**PROOF.** (i) & (ii) If $g << f$ and $r \in \mathbb{R}$ then

$$\int_0^t g + r = \int_0^t (g + r) \leq \int_0^t (f + r) = \int_0^t f + r$$

for all $0 \leq t \leq a$ using (5.3) (vi).

If $g < f$ then in addition we have equality when $t = a$. Suppose $r \geq 0$.

If $g << f$ then $\int_0^t \delta r g = \int_0^t r \, \delta g \leq \int_0^t r \, \delta f = \int_0^t \delta f$ for all $0 \leq t \leq a$. If $g < f$ then we also have equality when $t = a$. Now assume $g < f$. To prove the rest it suffices to show that $-g < -f$.

$$\int_0^t \delta (u) du = \int_0^t - \delta (a-u) du$$

for all $0 \leq t \leq a$ with equality when $t = a$ using (5.5) (ii).

(iii) Let $g < f$ and suppose first that $f \geq 0$. If $g$ is negative on a set of positive measure then there is a $0 < u < a$ s.t. $\delta_f (u) < 0$. Then $\delta_f (t) < 0$ for all $u \leq t \leq a$ so $\int_0^a \delta_f = \int_0^a \delta g \leq \int_0^a \delta_f \leq \int_0^a \delta f$ since
\[ \delta_t \geq 0, \text{ a contrd. Hence } g \geq 0. \text{ Now suppose } r \in \mathbb{R} \text{ and } r \leq f. \text{ Then } \\
\delta_t \geq 0 \text{ and } g-r < f-r \text{ so } g-r \geq 0, \text{ i.e. } r \leq g. \text{ Suppose } s \in \mathbb{R} \text{ and } \\
f \leq s. \text{ Then } f + s \geq 0 \text{ and } -g+s < -f+s \text{ so } -g+s \geq 0, \text{ i.e. } g \leq s. \]

(iv) Let \( g = \left( \frac{1}{\mu_1(X_1)} \int_{X_1} f \, d\mu_1 \right) c_1 \) so we have to show \( g < f \).

\[
\int_0^t \delta_g = t \frac{1}{\mu_1(X_1)} \int_{X_1} f \, d\mu_1 = t \frac{1}{a} \int_0^a \delta_f \leq \int_0^t \delta_f \quad \text{for all } 0 \leq t \leq a,
\]

by Theorem (7.2), and we have equality when \( t = a \).

(v) Suppose \( f \in \mathcal{C}_E \). Then (iii) \( 0 \leq f \leq 1 \), and \( \int f \, d\mu_1 = \\
\int_{X_1} C_E \, d\mu = \mu(E) \). Assume \( 0 \leq f \leq 1 \) and \( \int_{X_1} f \, d\mu_1 = \mu(E) \). Then

\[
\delta_f \leq \leq 1 \text{ so } \int_0^t \delta_f \leq t, \text{ and } 0 \leq \delta_f \text{ so } \int_0^t \delta_f \leq \int_0^a \delta_f = \mu(E). \text{ Hence}
\]

\[
\int_0^t \delta_f \leq \min\{t, \mu(E)\} = \int_0^t \delta_E \text{ for all } 0 \leq t \leq a \text{ and we have equality when } t = a.
\]

(vi) Now \( g^- < f^- \Rightarrow -g^- < -f^- \) so \( \int_0^t \delta_g \leq \int_0^t \delta_f \). \( \int_0^t \delta_f \leq \int_0^t \delta_f + \int_0^t \delta_f = t \)

\[= \int_0^t \delta_f \text{ for all } 0 \leq t \leq a \text{ with equality when } t = a, \text{ using (5.5)(i)}. \]

(vii) Assume \( g \in \mathcal{M}(X, \mu) \& f \in \mathcal{M}(X_1, \mu_1) \). Let \( b = \mu(\{g \geq 0\}) = \mu(\{g \geq 0\}) \)

and \( b_1 = \mu_1(\{f \geq 0\}) \). Then \( \int_0^a \delta_g = \int_0^a \delta_f \leq \int_0^b \delta_f \leq \int_0^b \delta_f = \int_0^a \delta_f \). If \( g < f \in \mathcal{L}_1 \)

then \( -g < -f \) so \( \delta_g = (-g)^+ \ll (-f)^+ = f^- \).

(viii) Use monotone convergence and (5.3).

(8.3) LEMMA. Suppose \( f, g \in \mathcal{L}_1[0, a] \), \( f \) and \( g \) are decreasing, \( r, s \leq r \leq u, v \leq [0, a] = 1 \), \( s-r = v-u \), and \( f \leq \int r g. \) If \( g < f \)
then \( g \ll [1-r]u, v \ll [f \ll [1-r]]r, s \).

PROOF. The proof is essentially due to Ryff [43, p. 433].

Let \( g \ll [1-r]u, v \ll [f \ll [1-r]]r, s \) and let \( h = s-r = v-u \). Except at most at countably many points of \( 1 \),
\[
\delta_g(x) = \begin{cases} 
g(x) & 0 \leq x \leq u 
g(x+h) & u < x < a-h 
\end{cases}
\]

and similarly for \( f \). We consider four cases.

**Case 1.** \( t < \min \{u, r\} \).

\[
\begin{align*}
\int_t^u g & = \int_t^u g < \int f = \int_t^s \delta_f \\
\int_0^0 & = \int_0^0 g < \int f = \int_0^0 \delta_f.
\end{align*}
\]

**Case 2.** \( t > \max \{u, r\} \). Now

\[
\begin{align*}
\int_t^u g & = \int_t^u g + \int g(x+h)dx = \int_v^u g + \int_v^u g \\
\int_0^0 & = \int_0^0 g < \int f = \int_0^0 \delta_f + \int f.
\end{align*}
\]

and similarly for \( f \). Hence

\[
\begin{align*}
\int_v^s g & = \int_v^s g - \int f \leq \int_s^f \delta_f \\
\int_0^0 & = \int_0^0 g < \int f \leq \int_0^0 \delta_f.
\end{align*}
\]

Since \( \int g - \int f \geq 0 \) we obtain

\[
\begin{align*}
\int_t^u g & \leq \int_t^u \delta_g + \int f = \int_t^u \delta_g + \int f \\
\int_0^0 & = \int_0^0 g < \int f \leq \int_0^0 \delta_f.
\end{align*}
\]

**Case 3.** \( r \leq t \leq u \). Since \( g \) is decreasing, \( \int_v^s g \leq \int_t^{t+h} g \). Then

\[
\begin{align*}
\int_v^s g & = \int_v^s g - \int f = \int_s^f \delta_f \\
\int_0^0 & = \int_0^0 g < \int f \leq \int_0^0 \delta_f.
\end{align*}
\]

Thus \( \int v^s g < \int f \).

**Case 4.** \( u \leq t \leq r \).

\[
\begin{align*}
\int_t^{t+h} g & = \int_t^{t+h} g - \int f = \int_t^{t+h} \delta_f \\
\int_0^0 & = \int_0^0 g < \int f \leq \int_0^0 \delta_f.
\end{align*}
\]

Thus \( g | < | f \).
LEMMA. Suppose \( f, g \in L^1[0, a], \) \( s[ \& ]u, v[ \subset ] 0, a = 1. \) Then
\[ gC_{I-}u, v[ + \alpha C]u, v[ \ll f C_{I-}r, s[ + \alpha C]r, s[ \quad \text{for all } \alpha \in \mathbb{R}. \]

PROOF. Let \( g|_r = g|_{I-}u, v[ \text{ and } f|_r = f|_{I-}r, s[ , \text{ so } g|_r \ll f|_r, \)
let \( \alpha \in \mathbb{R}, \) let \( g_1 = g C_{I-}u, v[ + \alpha C]u, v[ , \)
let \( f_1 = f C_{I-}r, s[ + \alpha C]r, s[ , \) and let \( w, z[ \subset \delta^{-1}_1(\alpha) \) \& \;
\]
\( ]p, q[ \subset \delta^{-1}_1(\alpha) \) s.t. \( z-w = q-p = h = s-r = v-u \).
Then \( \delta g|_r \ll \alpha \) on \([0, w], \ \delta g|_1 \ll \alpha \) on \([z, a-h], \) and
except at most at countably many points of \( I,
\[
\delta g_1(x) = \begin{cases} 
\delta g|(x) & 0 \leq x \leq w \\
\alpha & w < x \leq z \\
\delta g|(x-h) & z < x \leq a 
\end{cases}
\]
and similarly for \( f. \) The inequality \( \int t g_1 \ll \int t f_1 \) follows by
examining cases. Rather than go through them all we give the details
for three representative ones.

Case: \( w \leq p \leq t \leq \min \{q, z\} \). Now \( \alpha \leq \delta f|_1 \) on \([w, p] \)
so
\[
\int t g_1 = \int t g + \int t \alpha = \int t g + \int t \alpha + \int t \alpha \\
o g_1 o g | w w w w p p p
\leq \int t g_1 + \int t \alpha = \int t \alpha + \int t \alpha = \int t f_1
\]

Case: \( p \leq w \leq t \leq \min \{q, z\} \). Now \( \delta f|_1 \ll \alpha \) on \([p, w] \) so
\[
\int t g_1 = \int t g + \int t \alpha \leq \int t g + \int t \alpha \leq \int t f_1 + \int t \alpha + \int t \alpha \\
o g_1 o g | w w w w p p p
= \int t f_1 + \int t \alpha = \int t \delta f_1.
\]
Case: \( z \leq t \leq q \). \[
\int^t_0 \delta_t = \int^w_0 \delta_t + \int^z_0 \alpha + \int^t_0 \delta_t (x-h) dx
\]

\[
= \int^w_0 \delta_t + \alpha h + \int^w_0 \delta_t = \int^t_0 \delta_t + \int^t_0 \alpha
\]

If \( t \leq p \),
\[
\int^t_0 \delta_t = \int^t_0 \delta_t + \int^t_0 \alpha \leq \int^t_0 \delta_t + \int^t_0 \delta_f = \int^t_0 \delta_t = \int^t_0 \delta_f
\]

since \( \alpha \leq \delta_f \) on \([t-h, t]\).

If \( p < t \), then \( t < q \Rightarrow t-h \leq p \& \alpha \leq \delta_f \) on \([t-h, p]\) so
\[
\int^t_0 \delta_t = \int^t_0 \delta_t + \alpha \leq \int^t_0 \delta_f + \int^t_0 \delta_f + \int^t_0 \alpha = \int^t_0 \delta_f
\]

(8.5) LEMMA. Suppose \( f, g \in L^1 [0, a] \), \( N \), \( s[-] u, v[-] \subseteq [0, a] = 1 \)

and \( s-r = v-u \).

(i) [J. V. Ryff] If \( f \) and \( g \) are decreasing and \( \int^s_r f = \int^v_u g \) and \( g < f \text{ then } g|_{I-} u, v ] < f|_{I-} r, s ] \).

(ii) If \( g|_{I-} u, v ] < f|_{I-} r, s [ \text{ then for all } \alpha \in R \),
\( g C_{I-} u, v[-] + \alpha C_{I-} r, s [-] + \alpha C ] r, s [ \).

PROOF. (i) Let \( f, g \) be as in the proof of (8.4). Then
\[
\int^a_{a-h} \delta_t = \int^a_{a-h} \delta_t = \int^a_{a-h} \delta_t = \int^a_{a-h} \delta_t = \int^a_{a-h} \delta_t
\]

where \( h = s-r = v-u \). Since we already have \( g|_{I-} f \) \( \text{ from (8.3) we conclude that } g|_{I-} f \).

(ii) Let \( g_1 \) and \( f_1 \) be as in proof of (8.4). Then
\[
\int^a_{a-h} g_1 = \int^a_{a-h} g + \alpha h = \int^a_{a-h} g + \alpha h = \int^a_{a-h} g_1
\]
(8.6) **Lemma.** Suppose \( f \in L^1(X_1, \mu_1) \), \( g \in L^1(X, \mu) \), \( f \) is constant on \( E_1 \in \Lambda_1 \), \( g \) is constant on \( E \in \Lambda \), \( f|E_1 \leq g|E \) and \( \mu_1(E_1) = \mu(E) \). If \( g < f \), then

(i) \( g|X-E < f|X-E_1 \) and

(ii) \( g C_{X-E} + \alpha C_E < f C_{X-E_1} + \alpha C_{E_1} \) for all \( \alpha \in \mathbb{R} \).

**Proof.** Let \( \beta = f|E_1 \), \( \gamma = g|E \), \( a = \mu(X) = \mu_1(X_1) \),
\[
I = [0, a], \quad s \subseteq \delta_f^{-1}(\beta) \quad \text{s.t.} \quad s - r = \mu_1(E_1), \quad \text{and}
\]
\[
j \subseteq \delta_f^{-1}(\gamma) \quad \text{s.t.} \quad v - u = \mu(E). \]
Then \( s - r = v - u \) and \( \int_r^s \delta_f \leq \int_v^u \delta_f \).

If \( g < f \) then \( \delta_f \) is defined as \( g|X-E \leq \delta_f \) so \( f|X-E \sim \delta_f \).

\[
\begin{align*}
gC_{X-E} + \alpha C_E &< f C_{X-E_1} + \alpha C_{E_1} \\
\text{for all } \alpha &\in \mathbb{R}
\end{align*}
\]

for all \( \alpha \in \mathbb{R} \) using Lemmas (3.3) (xi), (3.4) (i), (8.3), and (8.4).

(8.7) **Lemma.** Suppose \( f \in L^1(X_1, \mu_1) \), \( g \in L^1(X, \mu) \) and suposse \( f \) and \( g \) have the same constant value on \( E_1 \in \Lambda_1 \) and \( E \in \Lambda \) respectively, where \( \mu_1(E_1) = \mu(E) \). If \( g < f \) then

(i) \( g|X-E < f|X-E_1 \) and

(ii) \( g C_{X-E} + \alpha C_E < f C_{X-E_1} + \alpha C_{E_1} \) for all \( \alpha \in \mathbb{R} \).

**Remark.** Let \( g \not< f \) mean \( g < f \) and \( g \) and \( f \) are not equimeasurable, and define \( g \not> f \) similarly. Let \((X, \Lambda, \mu)\) be a non-atomic m.s. and let \( f, g \in L^1(X, \mu) \). Then \( g \not< f \) means there is an \( h \in L^1(X, \mu) \) s.t. \( g \not< h \not< f \), and \( g \not> f \) means there is an \( h \in L^1(X, \mu) \) s.t. \( g \not> h \not> f \) for by Theorem (5.10) there is an \( h \in M(X, \mu) \) s.t. \( \delta_h = \frac{1}{2} [\delta_f + \delta_g] \).
We now pause to show how a finite m.s. can be embedded in a non-atomic m.s. We will use this device to show which results about non-atomic m.s. carry over the general m.s.

9. Embedding of a Finite m.s. in a Non-Atomic Finite m.s. Let $(X, \Lambda, \mu)$ be a finite m.s. We recall that a measurable set $A \in \Lambda$ is called a $\mu$-atom or an atom of $(X, \Lambda, \mu)$ whenever $\mu(A) > 0$ and $B \subset A$ implies $\mu(B) = 0$ or $\mu(B) = \mu(A)$. If $A$ is a $\mu$-atom and $f \in M(X, \mu)$ then $f$ is essentially constant on $A$. If $A$ and $B$ are $\mu$-atoms, they are equal or disjoint a.e., i.e., $\mu(A \Delta B) = 0$, where $A \Delta B = A - B \cup B - A$, or $\mu(A \cap B) = 0$.

A finite (or $\sigma$-finite) m.s. can have at most countably many atoms. Thus $X = X_0 \cup \bigcup_{n \in \mathbb{P}} A_n$ where $\mathbb{P}$ is $\{1, \ldots, k\}$ for some natural number $k$ or $\mathbb{P}$ is $\{1, 2, 3, \ldots\}$; each $A_n$ is a $\mu$-atom; and $(X_0, \Lambda \cap X_0, \mu)$ is non-atomic.

We may embed $(X, \Lambda, \mu)$ in a non-atomic m.s. $(X^\#, \Lambda^\#, \mu^\#)$ as follows. Let $X^\# = X_0 \cup \bigcup_{n \in \mathbb{P}} I[a_n, b_n]$ where the $I[a_n, b_n]$ are disjoint intervals of $\mathbb{R}$ with end-points $a_n$ & $b_n$ s.t. $b_n - a_n = \mu(A_n)$. Then $(X^\#, \Lambda^\#, \mu^\#)$ is the direct sum of $(X_0, \Lambda \cap X_0, \mu)$ and the Lebesgue m.s. $(I[a_n, b_n], m)$, i.e., $E \in \Lambda^\#$ iff $E = E_0 \cup \bigcup_{n \in \mathbb{P}} E_n$ where $E_0 \in \Lambda, E_0 \subset X_0$, and for each $n \in \mathbb{P}$, $E_n$ is a Lebesgue measurable subset of $I[a_n, b_n]$, and in this case $\mu^\#(E) = \mu(E_0) + \sum_{n \in \mathbb{P}} m(E_n)$.

Clearly $(X^\#, \Lambda^\#, \mu^\#)$ is a non-atomic finite m.s. with $\mu^\#(X^\#) = \mu(X)$.

Each $f \in M(X, \Lambda, \mu)$ is identified with $f^\# = f C_{X_0} \sum_{n \in \mathbb{P}} (f|_{A_n}) C_{I[a_n, b_n]}$ in $M(X^\#, \Lambda^\#, \mu^\#)$ and it is easy to see that $f \sim f^\#$ so $\delta_f = \delta_{f^\#}$. Of more
importance is the fact that $L^1(X, \mu)$ is a retract of $L^1(X^#, \mu^#)$ under
the following mapping.

(9.1) DEFINITION. If $f \in L^1(X^#, \mu^#) \cup M^+(X^#, \mu^#)$ let
\[
T_\mu f = f \mathbf{1}_X + \sum_{n \in P} \frac{1}{b_n - a_n} \left( \int_{a_n}^{b_n} f \right) \mathbf{1}_{A_n}.
\]

(9.2) THEOREM. (i) $T_\mu : L^1(X^#, \mu^#) \to L^1(X, \mu)$.
(ii) $T_\mu f = f$ for all $f \in L^1(X, \mu) \cup M^+(X, \mu)$.
(iii) $T_\mu f \geq 0$ for all $0 \leq f \in M(X, \mu)$.
(iv) $\int T_\mu f \, \mu = \int f \, \mu^#$ for all $f \in (L^1 \cup M^+)(X^#, \mu^#)$
and $\|T_\mu f\|_1 \leq \|f\|_1$ for all $f \in L^1(X^#, \mu^#)$.
(v) $\|T_\mu f\|_\infty \leq \|f\|_\infty$ for all $f \in L^\infty(X^#, \mu^#)$.
(vi) For all $f \in (L^1 \cup M^+)(X^#, \mu^#)$ and $g \in M(X, \mu)$ s.t.
$fg \in (L^1 \cup M^+)(X^#, \mu^#)$ we have $T_\mu (fg) = (T_\mu f) g$.

The reader who is familiar with the concept of a conditional
expectation will recognize that $T_\mu f$ is the conditional expectation of $f$
with respect to the $\sigma$-ring generated by $X_0 \cap \Lambda$ and the intervals
$[a_n, b_n], n \in P$.

At the end of §13 we will prove that $T_\mu f < f$ for all
$f \in (L^1 \cup M^+)(X^#, \mu^#)$. We will investigate transformations of this
type in detail in §17.
10. An Inequality of Hardy and Littlewood. In this section we prove an inequality which was originally proved by Hardy and Littlewood for non-negative functions ([15], Theorem 378). This inequality will be fundamental for what is to follow.

\[(10.1) \text{THEOREM (Hardy & Littlewood).} \text{ If } f, g \in M(X, \mu), a = \mu(X) < \infty, \text{ and if } \delta_{f} \delta_{g} \in L^{1}[0, a], \text{ then } f g \in L^{1}(X, \mu) \text{ and} \]
\[
\int_{0}^{a} \delta_{f}(a-t) \delta_{g}(t) dt = \int_{0}^{a} \delta_{f}(t) \delta_{g}(a-t) dt \leq \int f g d\mu \leq \int_{0}^{a} \delta_{f} \delta_{g} .
\]

PROOF. First suppose \( f = C_{E} \) and \( g = C_{F} \) when \( E, F \in \Lambda \).

Then
\[
\int_{0}^{a} \delta_{f}(t) \delta_{g}(a-t) dt = (\mu(E) + \mu(F) - a)^{+} = (\mu(E \cup F) + \mu(E \cap F) - a)^{+} \leq \mu(E \cap F) = \int f g d\mu \leq \min \{\mu(E), \mu(F)\} = \int_{0}^{a} \delta_{f} \delta_{g} .
\]

Next suppose \( f \) and \( g \) are non-negative simple functions. Then \( f \) and \( g \) can be written in the form

\[
f = \sum_{i=1}^{n} f_{i} C_{E_{i}} \quad f_{i} > 0 \quad i = 1, \ldots, n \quad \text{and} \quad E_{1} \supset \ldots \supset E_{n} \]

\[
g = \sum_{j=1}^{m} g_{j} C_{F_{j}} \quad g_{j} > 0 \quad j = 1, \ldots, m \quad \text{and} \quad F_{1} \supset \ldots \supset F_{m} .
\]

Then
\[
\int_{0}^{a} \delta_{f}(t) \delta_{g}(a-t) dt = \sum_{i,j} f_{i} g_{j} \int_{E_{i}}^{a} \delta_{E_{i}}(t) \delta_{F_{j}}(a-t) dt \]
\[
\leq \sum_{i,j} f_{i} g_{j} \int C_{E_{i}} C_{F_{j}} d\mu = \int f g d\mu
\]
\[
\leq \sum_{i,j} f_{i} g_{j} \int_{E_{i}}^{a} \delta_{E_{i}} \delta_{F_{j}} = \int_{0}^{a} \delta_{f} \delta_{g} \quad \text{using (5.6) (ii).}
\]
If \( 0 \leq f, g \in M(X, \mu) \) then there are sequences \( \{f_n\} \) and \( \{g_n\} \) of non-negative simple functions such that \( f_n \uparrow f \) and \( g_n \uparrow g \) everywhere. Then
\[
\frac{1}{a} \int_0^a \delta_f(t) \delta_g(a-t) \, dt = \lim_{n \to \infty} \frac{1}{a} \int_0^a \delta_{f_n}(t) \delta_{g_n}(a-t) \, dt
\]
\[
\leq \lim_{n \to \infty} \frac{1}{a} \int_0^a f_n g_n \, d\mu = \frac{1}{a} \int f g \, d\mu
\]
\[
\leq \lim_{n \to \infty} \frac{1}{a} \int \delta_f \delta_g = \frac{1}{a} \int \delta_f \delta_g
\]

Let \( f, g \in M(X, \mu) \) be arbitrary s.t. \( \delta|f\| \delta|g| \in L^1[0, a] \). Then by what we have already proved \( \int |fg| \, d\mu \leq \int\left|\frac{a}{0} \delta|f\| \delta|g|\right| \) so \( fg \in L^1(X, \mu) \). Then using Theorem (5.5),
\[
\int_0^a \delta_{f_+(t)} \delta_{g_+(a-t)} \, dt = \int_0^a \delta_{f_-(t)} \delta_{g_+(a-t)} \, dt - \int_0^a \delta_{f_-(t)} \delta_{g_-(a-t)} \, dt - \int_0^a \delta_{f_+(t)} \delta_{g_-(a-t)} \, dt
\]
\[
+ \int_0^a \delta_{f_-(a-t)} \delta_{g_-(t)} \, dt \leq \int f^+ \, d\mu - \int f^- \, d\mu + \int g^+ \, d\mu - \int g^- \, d\mu
\]
\[
= \int f g \, d\mu \leq \int \left( \int_0^a \delta_{f_+(t)} \delta_{g_+(a-t)} \, dt - \int_0^a \delta_{f_-(t)} \delta_{g_+(a-t)} \, dt - \int_0^a \delta_{f_+(t)} \delta_{g_-(a-t)} \, dt
\]
\[
+ \int_0^a \delta_{f_-(a-t)} \delta_{g_-(t)} \, dt \right) \delta_f \delta_g
\]

REMARK. Observe that the inequalities in (10.1) are true for all \( 0 \leq f, g \in M(X, \mu) \) even if \( \delta_f, \delta_g \notin L^1[0, a] \).

(10.2) PROPOSITION. If \( F \) and \( G \) are measurable functions on \([0, a]\) such that \( \delta|F| \delta|G| \in L^1[0, a] \) and \( G \) is decreasing, then the convolution
\[
F \ast G(u) = \int_0^u F(t) G(u-t) \, dt
\]
is a continuous function of \( u \) on \([0, a]\).
PROOF. First we show that $|F|*|G|(u)$ is finite for every $u \in [0, a]$. Let $u \in [0, a]$ and let

$$H(t) = \begin{cases} G(u-t) & \text{on } [0, u] \\ G(t) & \text{on } ]u, a] \end{cases}$$

Then $H[u, u] \sim G[u, u]$ and $H[u, a] \sim G[u, a]$ so $H \sim G$ and thus $|H| \sim |G|$ so

$$|F|*|G|(u) = \int_0^u |F|(t) |G|(u-t)dt \leq \int_0^u |F|(t) |G|(u-t)dt + \int_u^a |F| |G|$$

$$= \int_0^a |F| |H| \leq \int_0^a \delta |F| \delta |H| = \int_0^a \delta |F| \delta |G| < \infty .$$

Let $0 \leq u < u_0 \leq a$, so

$$|F*G(u_0) - F*G(u)| \leq \int_0^{u_0} |F(t)G(u_0-t)|dt + \int_0^u |(G(u-t) - G(u_0-t))F(t)|dt.$$ 

Let $H(t) = G(u_0-t)$ on $[u, u_0]$ and 0 elsewhere. Then $H \sim G C[0, u_0-u]$ so $|H| \sim |G| C[0, u_0-u]$ and thus

$$|H| = \delta |G| C[0, u_0-u] \leq \delta |G| \delta [0, u_0-u] = \delta |G| C[0, u_0-u].$$

Hence

$$0 \leq \int_0^{u_0} |F(t)| |G(u_0-t)|dt = \int_0^a |F| |H| \leq \int_0^a \delta |F| \delta |H| \leq \int_0^{u_0-u} \delta |F| \delta |G|^{-0}$$

as $u \uparrow u_0$.

For the rest, let $u_n \uparrow u_0$, $n = 1, 2, 3, \ldots$, and let $H_n(t) = G(u_n-t)F(t)$ on $[0, u_n]$ and 0 elsewhere and $H(t) = G(u_0-t)F(t)$ on $[0, u]$ and 0 elsewhere. Then $H_n \uparrow H$ pointwise a.e. since the discontinuities of $G$ form a set of measure 0, and

$$\int_0^a |H_1| = |F|*|G|(u_1) < \infty ,$$

so
\[
\int_0^u |(G(u_n-t)-G(u_0-t))F(t)| \, dt = \int_0^a |H_n-H| \to 0 \text{ as } n \to \infty \text{ by Dominated convergence.}
\]

The case \(0 \leq u_0 < u \leq a\) is similar.

11. The Values of an Integral. Let \(f, g \in M(X, \mu)\). If \(\delta |f| \delta |g| \in L^1[0,a]\)
where \(a = \mu(X)\) and \(f' \sim f\) and \(g' \sim g\) then \(|f'| \sim |f|\) and \(|g'| \sim |g|\) and we
know that \(\int |f'g'| \, d\mu \leq \int_0^a \delta |f'| \delta |g'| = \int_0^a \delta |f| \delta |g|\) and
\(\int_0^a \delta f(t) \delta g(a-t) \, dt \leq \int f'g' \, d\mu \leq \int_0^a \delta f \delta g\).

(11.1) THEOREM. Let \(f, g \in M(X, \mu)\) and \(\delta |f| \delta |g| \in L^1[0,a]\),
where \(a = \mu(X)\). If \((X, \Lambda, \mu)\) is non-atomic, then
\[
\{\int f'g' \, d\mu : g' \sim g\} = \left[\int_0^a \delta f(t) \delta g(a-t) \, dt, \int_0^a \delta f \delta g\right].
\]

PROOF. We already know that
\[
\{\int f'g' \, d\mu : g' \sim g\} \subset \left[\int_0^a \delta f(t) \delta g(a-t) \, dt, \int_0^a \delta f \delta g\right],
\]
so it remains to show that all the values are taken on.

Let \(\delta |f| \delta |g| \in L^1[0,a]\), and for \(0 \leq u \leq a\) let
\[
\gamma(u) = \int_0^u \delta f(t) \delta g(u-t) \, dt + \int_u^a \delta f \delta g.
\]
Then \(\gamma\) is a continuous function of \(u\) with \(\gamma(0) = \int_0^a \delta f \delta g\) and \(\gamma(a) = \int_0^a \delta f(t) \delta g(a-t) \, dt\), so it suffices to
prove that for each \(0 \leq u \leq a\) there is a \(g' \sim g\) s.t. \(\int g' \, d\mu = \gamma(u)\).

Fix \(u \in [0,a]\). We prove that there are sets \(X_1 \in \Lambda\) and \(X_2 = X-X_1\)
s.t. \(\mu(X_1) = u\), \(\delta_f|X_1 = \delta_f\) on \([0,u]\) and \(\delta_f|X_2(t) = \delta_f(t+u)\) on \(0 \leq t < a-u\).
To this end, if \( t \in \mathbb{R}^+ \) let \( F_t = \{ x \in X : f(x) > t \} \) and 
\[ F_{t-} = \{ x \in X : f(x) > t \} \] so \( \mu(F_{t-}) = \lim_{s \uparrow t} \mu(F_s) \). Now \( \mu(F_{+x}) = 0 \) if \( s \uparrow t \) so let \( t_0 = \inf \{ t \in \mathbb{R}^+ : \mu(F_t) \leq u \} \). Then \( \mu(F_{t_0}) \leq u \) and 
\[ s < t_0 \Rightarrow \mu(F_s) < u, \text{ so } \mu(F_{t_0}) \leq u \leq \mu(F_{t_0}) \] Since \((X, \Lambda, \mu)\) is non-atomic, there is an \( X_1 \in \Lambda \) s.t. \( F_{t_0} \subseteq X_1 \subseteq F_{t_0} \) and \( \mu(X_1) = u \).

Now \( s < t_0 \Rightarrow d_f(s) = \mu(F_s) > u \) so \( d_f(s) < u \Rightarrow s \geq t_0 \Rightarrow F_s \subseteq F_{t_0} \subseteq X_1 \Rightarrow d_f|X_1 (s) = \mu(F_s \cap X_1) = \mu(F_s) = d_f(s) \). Conversely, \( s < t_0 \Rightarrow \)
\[ X_1 \subseteq F_{t_0} \subseteq F_s \Rightarrow d_f|X_1 (s) = \mu(F_s \cap X_1) = \mu(X_1) = u \] so 
\[ d_f|X_1 (s) < u \Rightarrow s \geq t_0 \Rightarrow d_f(s) = d_f|X_1 \text{ as before. Hence if } 0 \leq t < u \]
we have \( \{ s : d_f(s) \leq t \} = \{ s : d_f|X_1 (s) \leq t \} \) so \( \delta_f(t) = \delta_f|X_1 (t) \).

Now let \( v(t) = \delta_f(t+u) \) on \( 0 \leq t < a-u \). Then for each bounded interval \([r, s]\) of \( \mathbb{R}^+ \) we have 
\[ \mu(f^{-1}[r, s] \cap X_1) = \mu((f|X_1)^{-1}[r, s]) = m(\delta_f^{-1}[r, s]) \] 
\[ = m(\delta_f^{-1}[r, s]) \cap [0, u[). \]

Subtracting this equation from \( \mu(f^{-1}[r, s] \cap X) = m(\delta_f^{-1}[r, s]) \cap [0, u[) \)
we get \( \mu(f^{-1}[r, s] \cap (X-X_1)) = m(\delta_f^{-1}[r, s]) \cap [u, a[), \text{ i.e.,} \)
\[ \mu((f|X_2)^{-1}[r, s]) = m(v^{-1}[r, s] \cap [0, a-u[) + u) \] 
\[ = m(v^{-1}[r, s]) \cap [0, a-u[). \]

Hence \( \delta_f|X_2 (t) = \delta_f(t+u) \) on \( 0 \leq t < a-u \).

Recall that if \( T : (X, \Lambda, \mu) \rightarrow (Y, \Sigma, \nu) \) is a measurable transformation and \( g \in M(Y, \Sigma) \) then \( \int_Y g(d\mu T^{-1}) = \int_X g \circ T d\mu \) in the sense that either both integrals exist and are equal or neither exist \([12, \text{ p. 163, Theorem C}]\) where \( (\mu T^{-1})(E) = \mu(T^{-1}(E)) \) for all \( E \in \Sigma \).

If \( T \) is measure preserving, then \( \mu T^{-1} = \nu \).
Let \( \sigma_1: X_1 \to [0, u] \) and \( \sigma_2: X_2 \to [0, a-u] \) be m.p. s.t. \( \delta_f|X_1 \circ \sigma_1 = f|X_1 \) and \( \delta_f|X_2 \circ \sigma_2 = f|X_2 \).

Also let \( G_1(t) = \delta_g(u-t) \) on \( 0 \leq t \leq u \) so \( G_1 \sim \delta_g\) \([0, u]\) and let \( G_2(t) = \delta_g(t+u) \) on \( 0 \leq t \leq a-u \) so \( G_2 \sim \delta_g\) \([u, a]\).

Then \( \int_0^u \mathfrak{f}(t)\delta_g(u-t)dt = \int_0^u \mathfrak{f}|X_1 G_1 dm = \int \mathfrak{f}|X_1 \circ \sigma_1 \delta_f|X_1 \circ \sigma_1 \) \( \delta_f|X_1 \circ \sigma_1 \) \( \delta_g\) \([0, u]\).

Also \( \int_0^a \mathfrak{f}\delta_g = \int_0^{a-u} \mathfrak{f}|X_2 G_2 dm = \int \mathfrak{f}|X_2 \circ \sigma_2 \delta_f|X_2 \circ \sigma_2 \) \( \delta_f|X_2 \circ \sigma_2 \) \( \delta_g\) \([u, a]\).

and \( \delta_f|X_2 \circ \sigma_2 \sim \delta_g\) \([u, a]\). Hence if we let
\[
g'(x) = \begin{cases} 
\sigma_1(x) & \text{if } x \in X_1 \\
\sigma_2(x) & \text{if } x \in X_2
\end{cases}
\]
then \( g' \sim \delta_g \sim g \)

and \( \gamma(u) = \int f g' \) \( \mu \).

(11.2) COROLLARY If \((X, \Lambda, \mu)\) is non-atomic then for all \(0 \leq f, g \in M(X, \mu)\) we have
\[
\max \{ \int f' g' \) \( \mu : f' \sim f, g' \sim g \} = \max \{ \int f g' \) \( \mu : g' \sim g \} = \int_0^a \delta_f\delta_g ,
\]
in the sense that both are infinite or are finite and equal.

PROOF. As we observed, the inequalities of (10.1) hold for all \(0 \leq f, g \in M(X, \mu)\) even if \( \delta_f\delta_g \notin L^1\) \([0, a]\). To show that the sup is attained, let \( \sigma: X \to [0, a]\) be measure preserving s.t. \( \delta_f \circ \sigma = f \). Then
\[
\int_0^a \delta_f\delta_g = \int (\delta_f \circ \sigma) (\delta_g \circ \sigma) \) \( \mu = \int f g' \) \( \mu \) where \( g' = \delta_g \circ \sigma \sim g \).
Many investigators have used $g < f$ to mean $|g| \ll |f|$ ([25] for example). Hence the following theorem is of interest. It will also be very useful.

(11.3) **THEOREM.** Let $(X_1, \Lambda_1, \mu_1)$ and $(X, \Lambda, \mu)$ be finite m.s. such that $\mu(X) = \mu_1(X_1) = a$, and let $f \in L^1(X_1, \mu_1)$ and $g \in M(X, \mu)$. Then $g < f$ implies $g \in L^1(X, \mu)$ and $|g| \ll |f|$.

**PROOF.** (8.2)(vii) $g \in L^1$. From §9 it follows that (11.2) implies

$$\int_0^t \delta |g| = \int |g| C_E d\mu^#$$

for some $E \in \Lambda^#$ with $\mu^#(E) = t$

$$= \int g \sgn(g) C_E d\mu^#$$

$$\leq \int_0^t g \delta h$$

where $h = \sgn(g) C_E$

$$\leq \int_0^t \delta f \delta h$$

since $g < f$

$$= \int f h' d\mu_1^#$$

for some $h' \in L^1(X_1^#, \mu_1^#)$ s.t. $h' \sim h$

$$\leq \int |f| |h'| d\mu_1^#$$

and $|h'| \sim |h| = C_E$ so $|h'| = C_F$

$$= \int |f| C_F d\mu_1^#$$

$$\leq \int_0^t \delta |f|$$

for all $0 \leq t \leq a$.

**REMARKS.** 1. If $(X, \Lambda, \mu)$ has atoms, then Theorem (11.1) may be false. Let $X = X_0 \cup A_1$ where $X_0$ is non-atomic, $A_1$ is an atom, and $\mu(A_1) > \mu(X_0) > 0$. Let $f = C_{X_0}$ and $g = C_{A_1}$. Then $\int_0^a \delta f \delta g = \mu(X_0)$ but $g' \sim g$ implies $g' = C_{A_1}$ since $\mu(A_1) > \mu(X_0)$, so

$$\int f g' d\mu = \int C_{X_0} C_{A_1} d\mu = 0.$$
2. If \((X, \lambda, \mu)\) is a finite m.s. consisting only of atoms of equal measure, then it is said to be **discrete**, and the set of values of the corresponding sums do not fill up the whole interval, although the endpoints are attained. (See [15], Theorem 368).

(11.4) DEFINITION. A finite m.s. \((X, \lambda, \mu)\) is called adequate if

for all \(0 \leq f, g \in M(X, \mu)\) we have

\[
\max \{ \int f g' \, d\mu : g' \sim g \} = \int_a^a \delta f \delta g', \quad a = \mu(X).
\]

(11.5) THEOREM. The following are equivalents for the finite m.s. \((X, \lambda, \mu)\).

(i) \((X, \lambda, \mu)\) is adequate.

(ii) \((X, \lambda, \mu)\) is non-atomic or consists only of atoms of equal measure.

(iii) For all \(A, B \in \Lambda\) we have

\[
\sup \{ \int C_A C_E \, d\mu : C_E \sim C_B \} = \int_a^a \delta_A \delta_B.
\]

**PROOF.** Now we know that (ii) \(\Rightarrow\) (i) \(\Rightarrow\) (iii). Hence it remains to prove (iii) \(\Rightarrow\) (ii). Suppose (ii) is not true. Then either \(X\) has at least two atoms \(A, B\) with \(0 < \mu(B) < \mu(A)\), or \(X\) has an atom \(A\) and a non-atomic part \(X_0\) of positive measure, in which case (5.8) implies there is a \(B \subset X_0\) such that \(0 < \mu(B) < \mu(A)\). In either case, for all \(E \in \Lambda\) with \(C_E \sim C_B\) we have \(\mu(E) = \mu(B)\) and hence

\[
\mu(A \cap E) \leq \mu(E) = \mu(B) < \mu(A) \quad \text{so} \quad \mu(A \cap E) = 0.
\]

Thus \(\int C_A C_E \, d\mu = 0\) for all \(E \in \Lambda\) with \(C_E \sim C_B\) but \(\int_a^a \delta_A \delta_B = \mu(B) > 0\).
12. **The Decreasing Rearrangements of Sums and Products.** If \( f, g \in L^1(X, \mu) \) then in general \( \delta_{f+g} \neq \delta_f + \delta_g \). For example, let \( E \) and \( F \) be disjoint sets of equal positive measure. Then
\[
\delta_E = \delta_F = C[0, \mu(E)] \quad \text{while} \quad \delta_{E \cup F} = C[0, 2\mu(E)] \quad \text{so} \quad \delta_{E \cup F} \neq \delta_E + \delta_F.
\]
We can, however, prove the following.

(12.1) **THEOREM.** If \( f_1, \ldots, f_n \in L^1(X, \mu) \), then
\[
\int_0^t \delta_{f_1 + f_2} + \sum_{j=1}^{t} \int_0^j \delta_{f_1} (a-u) du < \int_0^t \delta_{f_1} + \cdots + \int_0^t \delta_{f_n} \quad i = 1, \ldots, n
\]
for all \( 0 \leq t < a \) with equality on both sides when \( t = a \).

**PROOF.** From §9 it follows that Theorem (11.1) implies
\[
\int_0^t \delta_{f_1 + f_2} = \max \{ \int_E (f_1 + f_2) d\mu^\#: E \in \mathcal{A} \& \mu^\#(E) = t \} \leq \int_0^t \delta_{f_1} + \int_0^t \delta_{f_2}
\]
and
\[
\int_0^t \delta_{f_1 + f_2} = \max \{ \int_E (f_1 + f_2) d\mu^\#: \mu^\#(E) = t \}
\]
\[= \max \{ \int_E f_1 d\mu^\#: \mu^\#(E) = t \} + \int_0^t \delta_{f_2} (a-u) du = \int_0^t \delta_{f_1} + \int_0^t \delta_{f_2} (a-u) du \] is the general case follows by induction.

(12.2) **COROLLARY.** If \( f \in L^1(X, \mu) \) then
\[
\int_0^t |\delta_f| < \int_0^t \delta |f| \quad \text{for all} \quad 0 \leq t < a \quad \text{with equality when} \quad t = a.
\]

**PROOF.**
\[
\int_0^t |\delta_f| = \int_0^t \delta_{f^+} + \int_0^t \delta_{f^-} (a-u) du \leq \int_0^t \delta_{f^+ + f^-} = \int_0^t \delta |f|
\]
using Theorems (5.5) and (12.1). We have equality when \( t = a \) because \( |f| \sim |\delta_f| \).

Along with (ii) of (7.1), (12.1) also yields the following.
(12.3) Theorem. If \((X, \Lambda, \mu)\) is a finite m.s. with \(a = \mu(X)\) and \(g\) is a bounded decreasing function on \([0, a]\), then the function \(p(f) = \int_0^a f(u)g(u)du\) is sublinear on \(L^1(X, \mu)\).

(12.4) Lemma. If \(f, g \in L^1(X, \mu)\) then
\[
\left|\int_0^t f - \int_0^t g\right| \leq \int_0^t |f-g| \quad \text{for all} \quad 0 \leq t \leq a.
\]

Proof. For all \(0 \leq t \leq a\),
\[
\int_0^t f \leq \int_0^t g + \int_0^t f - g \\
\leq \int_0^t g + \int_0^t |f-g| \\
\leq \int_0^t f + \int_0^t g - f + \int_0^t |f-g|
\]
so \(-\int_0^t f \leq \int_0^t f - \int_0^t g \leq \int_0^t |f-g|\).

(12.5) Corollary. Suppose \(f, f_n \in L^1(X,\mu)\) and \(g, g_n \in L^1(X,\mu)\) and \(g_n < f_n\) \(n = 1, 2, 3 \ldots\). If \(\|f_n - f\|_1 \to 0\) and \(\|g_n - g\|_1 \to 0\) then \(g < f\).

Proof. \(\left|\int_0^t f_n - \int_0^t f\right| \leq \int_0^t |f_n - f| \leq \int_0^a |f_n - f| = \|f_n - f\|_1\) for all \(0 \leq t \leq a\), and similarly for \(g\).

As in the case of sums, if \(f, g \in L^1(X, \mu)\) then in general,
\[
\delta_{fg} \neq \delta f \cdot \delta g \quad \text{even if} \quad 0 \leq f, g.
\]
For example, let \(E\) and \(F\) be disjoint sets of equal positive measure and let \(f = \chi_E + 2\chi_F\) and \(g = 2\chi_E + \chi_F\). Then
\[
f g = 2 C_{E \cup F} \quad \text{and} \quad \delta f = \delta g = 2 C_{[0, \mu(E)]} + C_{\mu(E)}, 2\mu(E)\]
so \(\delta f \delta g = 4C_{[0, \mu(E)]} + C_{\mu(E)}, 2\mu(E)\)
\[
\delta f C_E \leq \delta f \delta E. \quad \text{More generally, we have the following.}
\]
(12.6) **THEOREM.** (i) If $0 \leq f_1, \ldots, f_n \in M(X, \mu)$ then
\[
\int_0^t \delta_{f_1} \ldots \delta_{f_n} dt \leq \int_0^t \delta_{f_1} \ldots \delta_{f_n} dt, \quad \text{for all } 0 \leq t \leq a, \text{ so in particular,}
\]
\[
\int f_1 \ldots f_n d\mu \leq \int_0^a \delta_{f_1} \ldots \delta_{f_n}. \]
(ii) If $f_1, \ldots, f_n \in M(X, \mu)$ and $\delta |f_1| \ldots \delta |f_n| \in L^1[0, a]$ where $a = \mu(X)$, then for all $0 \leq t \leq a$ we have
\[
\int_0^t \delta_{f_1} \ldots f_n dt \leq \int_0^t \delta_{f_1} \ldots f_n dt \leq \int_0^t \delta_{f_1} \ldots \delta_{f_n}. \]

**PROOF.** (i) There is an $E \in \Lambda^\#$ with $\mu^\#(E) = t$ s.t.
\[
\int_0^t \delta_{f_1} f_2 = \int E \delta_{f_1} f_2 d\mu^\# = \int (f_1 C_E)(f_2 C_E) d\mu^\# \leq \int_0^a \delta_{f_1} C_E \delta_{f_2} C_E
\]
\[
\leq \int_0^a \delta_{f_1} \delta_{f_2} = \int_0^t \delta_{f_1} \delta_{f_2}. \quad \text{The general case follows by induction}
\]
using (7.1) (i). (ii) follows from (i) and (12.2).

13. **T** $\mu f < f$ and Some Consequences.

(13.1) **THEOREM.** Let $(X, \Lambda, \mu)$ be a finite m.s. Then for all $f \in L^1(X^\#, \mu^\#) \cup M^+(X^\#, \mu^\#)$ we have $T\mu f < f$.

**PROOF.** Let $E \in \Lambda^\#$. \[
\int T\mu C_E d\mu = \int C_E d\mu^\# = \mu^\#(E) \text{ by (9.2) (iv)}
\]
and $T\mu C_E = C_{X_0 \cap E} + \sum_{n \in \mathbb{P}} \frac{m(E \cap I[a_n, b_n])}{m(I[a_n, b_n])}$ so $0 \leq T\mu C_E \leq 1$ and hence $T\mu C_E < C_E$ by (8.2) (v).

Actually, for all $f \in L^1(X^\#, \mu^\#)$ we have already that
\[
\int_0^a \delta_{T\mu f} = \int T\mu f d\mu = \int f d\mu^\# = \int_0^a \delta_f \text{ using (9.2)(iv)}.
\]

Let $f$ be a non-negative simple measurable function. Then $f$ can be written in the form $f = \sum_{i=1}^n f_i C_{E_i}$ with $f_i > 0$ $i = 1, \ldots, n$.
and $E_1 \supset \cdots \supset E_n$. For all $0 \leq t \leq a$,
\[
\int_0^t \delta_{T^*} f \leq \sum_{i=1}^n f_i \int_0^t \delta_{T^* C E_i} \leq \sum_{i=1}^n f_i \int_0^t \delta_{E_i} = \int_0^t \delta_f, \quad \text{so } T^* f < f.
\]

Let $0 \leq f \in L^1(X^#, \mu^#)$ be arbitrary. Then there is a sequence
\[
\{f_n\}
\]
of non-negative simple functions s.t. $f_n \uparrow f$. From the definition
of $T^*$ we see that $0 \leq T^* f_n \uparrow T^* f$. Since we have $T^* f_n < f_n$ it follows
from (8.2)(viii) that $T^* f < f$.

If $f \in L^1(X^#, \mu^#)$ then $T^* f^+ < f^+$ and $T^* f^- < f^-$ so $-T^* f^- < -f^-$. For
all $0 \leq t \leq a$ we then have
\[
\int_0^t \delta_{T^*} f \leq \int_0^t \delta_{T^* f^+} + \int_0^t \delta_{T^* f^-} = \int_0^t \delta_f
\]
so $T^* f < f$.

(13.2) THEOREM. Let $(X, \mu)$ and $(X_1, \mu_1)$ be finite m.s. such
that $a = \mu(X) = \mu_1(X_1)$, and let $f \in M(X, \mu)$ and $g \in M(X_1, \mu_1)$.

(i) If $\delta \| f \| \delta \| g \| \in L^1[0, a]$ then
\[
\{\int g' \mathop{d}\mu : g' \in M(X, \mu) \text{ and } g' < g\} = \left[ \int_0^a \delta_f (u) \delta_g (a-u) \mathop{d}u, \int_0^a \delta_f \delta_g \right].
\]

(ii) If $0 \leq f, g$ then $\max\{\int g' \mathop{d}\mu : g' < g\} = \int_0^a \delta_f \delta_g$ in the
sense that both are infinite or are finite and equal.

PROOF. (i) According to Theorem (5.10) there is a
$g_o \in M(X^#, \mu^#)$ such that $\delta_{g_o} = \delta_g$. Let $I$ be the interval in question.
Then $I \ni \{\int g' \mathop{d}\mu : g' < g\} \ni \{\int \mathop{T^* g'} \mathop{d}\mu : g' \sim g, \ g' \in M(X^#, \mu^#)\}$
$= \{\int g' \mathop{d}\mu : g' \sim g_o\} = I$ using (7.1), (13.1), (9.2) and (11.1).

(ii) Follows from (11.2) in a manner similar to (i).
EXAMPLE. If \( n \geq 0 \) is an integer and \( \lambda \geq 1 \), find

\[
\sup\left\{ \int_0^1 x^n f(x) \, dx : 0 \leq f \leq \lambda \text{ and } \int_0^1 f(x) \, dx = 1 \right\}.
\]

Now \( 0 \leq f \leq \lambda \) and \( \int_0^1 f = 1 \) iff \( 0 \leq \frac{1}{\lambda} f \leq 1 \) and \( \int_0^1 \frac{1}{\lambda} f = \frac{1}{\lambda} \)

iff \( \frac{1}{\lambda} f \in C \left[ 0, \frac{1}{\lambda} \right] \), using (8.2)(v). Thus the sup is the same as

\[
\sup\left\{ \int_0^1 x^n f(x) \, dx : f \in C \left[ 0, \frac{1}{\lambda} \right] \right\} = \int_0^1 \delta^n \lambda \, C \left[ 0, \frac{1}{\lambda} \right] = \lambda \int_0^1 (1-x)^n \, dx,
\]

using (13.2) and (5.3)(v).
IV. REARRANGEMENT INVARIANT BANACH FUNCTION SPACES

14. Rearrangement Invariant Banach Function Spaces. Extensive studies have been made of Banach function spaces and more generally normed Riesz spaces (see [27] and [29]). In this chapter we review some rearrangement invariant properties of such spaces when the underlying measure space is finite. The treatment closely parallels Luxemburg in [28]. Rearrangement invariant Banach spaces will be the setting in which we will work in future sections.

Definition. Let $\mu$ be a finite m. s. and let $M^+ = M^+(X, \mu)$ denote the set of all non-negative extended real valued measurable functions on $(X, \lambda, \mu)$. A mapping $\rho: M^+ \to \mathbb{R}^+$ is called a function norm if it has the following two properties.

(i) $0 \leq \rho(f) \leq +\infty$ for all $f \in M^+$ and $\rho(f) = 0$ iff $f = 0$ $\mu$-a. e.

(ii) $\rho(f+g) \leq \rho(f) + \rho(g)$ for all $f, g \in M^+$

$\rho(\lambda f) = \lambda \rho(f)$ for all $f \in M^+$ and $\lambda \geq 0$.

$f \leq g = \rho(f) \leq \rho(g)$ for all $f, g \in M^+$.

In addition, $\rho$ is said to have the sequential Fatou property and is called a Fatou norm if it also satisfies.

(iii) $0 \leq f_n \uparrow f$ pointwise everywhere implies $\rho(f_n) \uparrow \rho(f)$.

We extend the domain of definition of a function norm $\rho$ by defining $\rho(f) = \rho(|f|)$ for all $f \in M(X, \mu)$, and we denote by $L^p = L^p(X, \mu)$ the set of all $f \in M(X, \mu)$ such that $\rho(f) < \infty$. If $\mu$-a. e equal members
of $L^p$ are identified as usual, then $L^p$ is a normed linear space with the norm $\|f\| = \|f\|_p = \rho(f)$. These spaces are clearly generalizations of the classical Lebesgue and Orlicz spaces.

Unfortunately the hypotheses we have placed on $\rho$ so far do not preclude the existence of a $\rho$-purely infinite set, i.e., a set $A$ for which $\rho(C_B) = +\infty$ for all $B \subset A$ such that $\mu(B) > 0$. Of course, if $f \in M(X, \mu)$ and $A$ is $\rho$-purely infinite, then $f \mathbb{C}_A = 0 \mu$ - a.e. [29, Note IV, p. 251, Lemma 8.2]. Fortunately it can be shown that $X$ has a maximal $\rho$-purely infinite set $X_\infty$, i.e., $X_\infty$ is $\rho$-purely infinite and $X - X_\infty$ has no $\rho$-purely infinite subsets [Note IV, Theorem 8.3]. We will assume in this and in subsequent sections that $X_\infty$ has been removed from $X$ so that our $m.s.$ has no $\rho$-purely infinite subsets. In this case $\rho$ is said to be saturated.

Given a function norm $\rho$ we define the first and second associates of $\rho$ as follows.

(14.2) DEFINITION. For all $f \in M^+$

$$\rho'(f) = \sup \{ \int |fg| \, d\mu : \rho(g) \leq 1 \}$$

$$\rho''(f) = \sup \{ \int |fg| \, d\mu : \rho'(g) \leq 1 \}$$

It is not hard to show that $\rho'$ and $\rho''$ have the sequential Fatou property [Note IV, p. 254, Theorem 9.2], and that $\rho$ is saturated implies $\rho'$ is a norm [Note IV, Theorem 9.7]. It is true but harder to show that in addition, $\rho'$ is saturated [Note IV, p. 261, Corollary 11.6]. Hence $\rho''$ is also a saturated Fatou norm. The spaces $L^\rho'$ and $L^\rho''$ are called the first and second associate spaces of $L^\rho$,
respectively.

We have the following Hölder type inequality [Note IV, p. 261, Corollary 11.7].

\[(14.3) \text{THEOREM.} \quad \text{If } f \in L^p \text{ and } g \in L^{p'} \text{ then} \]

\[| \int f g \, d\mu | \leq \int |fg| \, d\mu \leq \rho''(f)\rho'(g) \leq \rho(f)\rho'(g). \]

Note that \(\rho'' \leq \rho\). In the other direction the following has been shown. [Note IV, p. 259, Lemma 11.3].

\[(14.4) \text{THEOREM.} \quad \text{(Lorentz and Luxemburg)} \rho'' = \rho \text{ if and only if} \rho \text{ has the sequential Fatou property.} \]

The following converse of the Hölder inequality also holds. [Note V, p. 499, Corollary 14.2].

\[(14.5) \text{THEOREM.} \quad \text{(Lorentz and Wertheim). Suppose } L^p \text{ is complete and } g \in M(X, \mu). \text{ Then } g \in L^{p'} \text{ iff } \int |fg| \, d\mu < \infty \text{ for all } f \in L^p. \]

It has been shown that \(L^p\) is complete iff \(\rho\) has the following property called the Riesz-Fischer Property: \(\sum \rho(f_n) < \infty\) implies \(\rho(\sum |f_n|) < \infty\). [Note I, p. 143, Theorem 4.8]. In particular, \(L^p\) is complete whenever \(\rho\) is Fatou. [Note II, p. 149, Theorem 5.3]. Thus \(L^{p'}\) and \(L^{p''}\) are complete.
If $L^p = L^p$, $1 \leq p < \infty$, then $L^{p'} = L^q$ where $p^{-1} + q^{-1} = 1$.

In general $L^{p'}$ is a closed normal subspace of the Banach Dual $(L^p)^*$ [30, p. 153]. We have $L^{p'} = (L^p)^*$ iff $\rho(f_n) \downarrow 0$ whenever $f_n \downarrow 0$ everywhere. $G \in (L^p)^*$ is in $L^{p'}$ iff $G(f_n) \downarrow 0$ whenever $f_n \downarrow 0$, $f_n \in L^p$.

Since the m.s. is finite we will, in accordance with the Lebesgue and Orlicz space situation, assume in this and in subsequent sections that $L^\infty \subset L^p$, $L^{p'} \subset L^1$. This is easily seen to be equivalent to $\rho(C \chi) < \infty$ and $\rho'(C \chi) < \infty$. If $\rho$ is Fatou, then $L^\infty \subset L^p \subset L^1$ iff $\rho(C \chi) < \infty$ and $\rho'(C \chi) < \infty$, in which case $L^\infty \subset L^{p'} \subset L^1$ also.

(14.6) DEFINITION. A function norm $\rho$ is called rearrangement invariant (r.i.) if $f_1 \sim f_2$ implies $\rho(f_1) = \rho(f_2)$.

$L^p$ is called rearrangement invariant if $f_1 \sim f_2 \in L^p$ implies $f_1 \in L^p$.

If $\rho$ is rearrangement invariant then so is $L^p$, but the converse is not true. For example, let $\rho(f) = \int_0^1 |f(t)| \, dt + 2 \int_1^2 |f(t)| \, dt$ if $f \in M[0,2]$. $L^p$ is r.i. since $g \sim f \in L^p$ implies $\int_0^1 |g| + 2 \int_1^2 |g| \leq \int_0^1 |g| + 2 \int_0^1 |g|$ = $3 \int_0^1 \delta |f| \leq 3 \int_0^1 \delta |f| = 3 \int_0^1 |f| < \infty$. However $\rho(C[0,1]) = 1$ and $\rho(C[1,2]) = 2$ even though $C[0,1] \sim C[1,2]$.

The following lemma is fundamental to what follows.
LEMMA. Suppose \((X, \lambda, \mu)\) is adequate (i.e. discrete or non-atomic) and \(\rho\) is a Fatou norm and let \(a = \mu(X)\). Then \(L^\rho\) is rearrangement invariant iff \(\delta_f \delta_g \in L^1[0, a]\) for all \(0 \leq f \in L^\rho\) and \(0 \leq g \in L^{\rho'}\), in which case \(L^{\rho'}\) and \(L^{\rho''}\) are rearrangement invariant iff \(L^\rho\) is.

PROOF. Suppose \(L^\rho\) is rearrangement invariant. Let \(0 \leq f \in L^\rho\) and \(0 \leq g \in L^{\rho'}\). If \(f' \sim f\) then \(f' \in L^\rho\) so \(\int f' \, g \, d\mu \leq \rho(f') \rho'(g) < \infty\). Hence \(\delta_f \delta_g \in L^1[0, a]\) since \(\mu\) is adequate.

Suppose \(\delta_f \delta_g \in L^1[0, a]\) for all \(0 \leq f \in L^\rho\) and \(0 \leq g \in L^{\rho'}\). If \(0 \leq f' \sim f \in L^\rho\), \(\int f' \, g \, d\mu \leq \int_0^a \delta_f \delta_g < \infty\) so \(f' \in L^{\rho''} = L^\rho\) using (13.4) and (13.5) since \(\rho\) is Fatou.

The rest follows by replacing \(\rho\) by \(\rho'\) and \(L^{\rho''}\) by \(L^\rho\).

THEOREM. Suppose \((X, \lambda, \mu)\) is adequate.

(i) If \(\rho\) is a Fatou norm and if \(L^\rho\) is rearrangement invariant then \(0 \leq g \ll f \in L^\rho\) implies \(g \in L^\rho\) and hence \(g < f \in L^\rho\) implies \(g \in L^\rho\).

(ii) If \(L^\rho\) is rearrangement invariant, then \(0 \leq g \ll f \in L^{\rho'}\) implies \(g \in L^{\rho'}\) and \(g < f \in L^{\rho'}\) implies \(g \in L^\rho\).

PROOF. (i) Suppose \(0 \leq g \ll f \in L^\rho\). Then for all \(0 \leq h \in L^{\rho'}\) we have \(\int g \, h \, d\mu \leq \int_0^a \delta_g \delta_h \leq \int_0^a \delta_f \delta_h < \infty\) using (10.1) and (i) of (7.1), so \(g \in L^{\rho''} = L^\rho\). If \(g < f \in L^\rho \subset L^1\) then (11.3) implies \(|g| \ll |f| \in L^\rho\) so \(|g| \in L^\rho\).

(ii) Follows from (i) since \(\rho'\) and \(\rho''\) are Fatou norms, and \(L^\rho\) (r.i.) implies \(L^{\rho'}\) and \(L^{\rho''}\) are (r.i.).
(14.9) DEFINITION. A function norm \( \rho \) is called universally rearrangement invariant (u. r. i.) if \( f' \sim f \geq 0 \) implies \( \rho(T_{\mu} f') \leq \rho(f) \) for all \( f' \in M(X^\#, \mu^#) \).

\( L^\rho \) is called universally rearrangement invariant (u. r. i.) if \( f' \sim f \in L^\rho \) implies \( T_{\mu} f' \in L^\rho \) for all \( f' \in M(X^\#, \mu^#) \).

If \( (X, \Lambda, \mu) \) is adequate (i.e. discrete or non-atomic) then we will see that \( \rho \) is (u. r. i.) iff it is (r. i.) and \( L^\rho \) is (u. r. i.) iff it is (r. i.). Our previous results generalize as follows.

(14.10) THEOREM. (i) If \( L^\rho \) is (u. r. i.) then \( \delta_f \delta_g \in L^1[0, a] \) for all \( 0 \leq f \in L^\rho \) and \( 0 \leq g \in L^\rho' \) where \( a = \mu(X) \). If \( \rho \) is a Fatou norm, then the converse holds, and in that case \( L^\rho' \) and \( L^\rho'' \) are (u. r. i.) iff \( L^\rho \) is (u. r. i.)

(ii) If \( \rho \) is a Fatou norm and \( L^\rho \) is (u. r. i.) then \( 0 \leq g \ll f \in L^\rho \) implies \( g \in L^\rho \) and \( g < f \in L^\rho \) implies \( g \in L^\rho \).

(iii) If \( L^\rho \) is (u. r. i.) then \( 0 \leq g \ll f \in L^\rho' \) implies \( g \in L^\rho' \) and \( g < f \in L^\rho' \) implies \( g \in L^\rho' \). Similarly for \( L^\rho'' \).

PROOF. (i) For all \( f' \sim f, f' \in M(X^\#, \mu^#) \), we have

\[
\int f' g \ d\mu^# = \int (T_{\mu} f') g \ d\mu \leq \rho(T_{\mu} f') \rho'(g) < \infty, \quad \text{so} \quad \delta_f \delta_g \in L^1[0, a]
\]

since \( (X^\#, \mu^#) \) is non-atomic.

Suppose \( \rho \) is Fatou and let \( 0 \leq f' \sim f \in L^\rho, f' \in M(X^\#, \mu^#) \). Then

\[
\int (T_{\mu} f') g \ d\mu = \int f' g \ d\mu^# \leq \int_0^a \delta_f \delta_g \ < \infty \text{ for all } 0 \leq g \in L^\rho' \text{ so } T_{\mu} f' \in L^\rho'' = L^\rho.
\]
The rest follows by replacing \( L^\rho \) by \( L^\rho' \) and \( L^\rho'' \) by \( L^\rho \).
(ii) and (iii) Follow as in (14.8).

We now investigate rearrangement invariant (and (u. r. i)) norms.

(14.11) THEOREM. Suppose \((X, \Lambda, \mu)\) is adequate and \(\rho\) is a rearrangement invariant norm and let \(a = \mu(X)\). Then for all \(0 \leq f \in M(X, \mu)\) we have

\[
\rho'(f) = \sup \left\{ \int_0^a \delta_f \delta |g| : \rho(g) \leq 1 \right\}
\]

\[
\rho''(f) = \sup \left\{ \int_0^a \delta_f \delta |g| : \rho'(g) \leq 1 \right\}
\]

so \(\rho'\) and \(\rho''\) are rearrangement invariant.

In addition, if \(\rho\) is a Fatou norm, then

\[
\rho(f) = \sup \left\{ \int_0^a \delta_f \delta |g| : \rho'(g) \leq 1 \right\}
\]

for all \(0 \leq f \in M(X, \mu)\).

PROOF. Since \((X, \Lambda, \mu)\) is adequate, for each \(g\) with \(\rho(g) \leq 1\) there is a \(g' \sim g\) such that \(\int_0^a \delta_f \delta |g| = \int f |g'| d\mu\). \(g' \sim g\) implies \(\rho(g') = \rho(g) \leq 1\) since \(\rho\) is (r. i.) so

\[
\left\{ \int_0^a \delta_f \delta |g| : \rho(g) \leq 1 \right\} \subseteq \left\{ \int f |g| d\mu : \rho(g) \leq 1 \right\}.
\]

But \(\int f |g| d\mu \leq \int_0^a \delta_f \delta |g|\) so

\[
\rho'(f) = \sup \left\{ \int f |g| d\mu : \rho(g) \leq 1 \right\} = \sup \left\{ \int_0^a \delta_f \delta |g| : \rho(g) \leq 1 \right\}.
\]

We get the formula for \(\rho''\) by replacing \(\rho\) by \(\rho'\). If \(\rho\) is Fatou, then \(\rho = \rho''\), giving the last formula.
The following is a kind of converse which will be useful later.

(14.12) **Lemma.** Let \((X, \Lambda, \mu)\) and \((X', \Lambda', \mu')\) be finite m.s. with 
\[\mu(X) = \mu'(X') = a.\] If \(A \subset M(X', \mu')\) with \(r C_{\chi', e} \in A\) for some \(r \neq 0\), then the mapping \(0 \leq f - \rho(f)\) defined on \(M^+(X, \mu)\) by

\[\rho(f) = \sup \{ \int_0^a \delta_f \delta |g| : g \in A \}\]

is a Fatou norm which is universally rearrangement invariant.

**Proof.** \(f = 0\) \(\mu\) - a.e. implies \(\rho(f) = 0\) is clear. If \(\rho(f) = 0\) we take \(g = r C_{\chi'}\), and obtain \(\int_0^a \delta_f = 0\), so \(\delta_f\) and hence \(f = 0\) a.e.

since \(f \geq 0\). If \(u \geq 0\), then \(\delta_{uf} = u \delta_f\) so \(\rho(uf) = u \rho(f)\). If \(0 \leq f_1 \leq f_2\) then \(\delta_{f_1} \leq \delta_{f_2}\) so \(\rho(f_1) \leq \rho(f_2)\). To prove the triangle inequality we have from (12.1) that \(\int_0^t \delta_{f_1+f_2} \leq \int_0^t \delta_{f_1+f_2}\) for all \(0 \leq t \leq a\), so

if \(\rho(f_1) + \rho(f_2) < \infty\) then (7.1) implies that for each \(g \in A\) we have

\[\int_0^a \delta_{f_1+f_2} |g| \leq \int_0^a \delta_{f_1} |g| + \int_0^a \delta_{f_2} |g|\]

so \(\rho(f_1+f_2) \leq \rho(f_1)+\rho(f_2)\).

Since \(0 \leq f_n \uparrow f\) everywhere implies \(\int_0^a \delta_{f_n} \delta |g| \uparrow \int_0^a \delta_f \delta |g|\) for each \(g \in A\), we have \(\rho(f_n) \uparrow \rho(f)\) as in [29, Note II, p. 149, Theorem 5.4]. Hence \(\rho\) is a Fatou norm. To prove that \(\rho\) is (u. r. i.), let

\(0 \leq f \in M(X, \mu)\) and let \(f' \sim f\) where \(f' \in M(X^#, \mu^#)\). Then \(0 \leq T_{\mu} f' \ll f\) so \(\int_0^a \delta_{T_{\mu} f'} |g| \leq \int_0^a \delta_f |g|\) for each \(g \in A\) and hence

\[\rho(T_{\mu} f') \leq \rho(f).\]
Suppose $(X, \lambda, \mu)$ is adequate and $\rho$ is a rearrangement invariant norm. If $\rho$ is also Fatou, then

(i) $0 \leq g \ll f$ implies $\rho(g) \leq \rho(f)$.

(ii) $g < f \in L^1(X, \mu)$ implies $\rho(g) \leq \rho(f)$.

Whether $\rho$ is Fatou or not, $\rho'$ and $\rho''$ satisfy (i) and (ii).

If $(X, \lambda, \mu)$ is not adequate, then Theorem (14.13) may not be true. For example, suppose $X = X_0 \cup A$ where $A$ is an atom, $X_0$ is non-atomic, $0 < \mu(X_0) < \mu(A)$, and $\mu(A) > 1$. Let

$\rho(h) = \int_{X_0} |h| d\mu + |h C_A|$. Pick $\varepsilon > 0$ such that $0 < 1 - \varepsilon \leq \mu(X_0) \leq 1 - \varepsilon \mu(A)$, and let $f = C_A$ and $g = C_{X_0} + \varepsilon C_A$. Then $\rho$ is a Fatou rearrangement invariant norm and $0 \leq g \ll f$, but $\rho(f) = 1 < \mu(X_0) + \varepsilon = \rho(g)$.

For general m. s. our previous results take the following form.

(14.14) THEOREM. If $\rho$ is a universally rearrangement invariant norm, then the following four statements are true.

(i) For all $0 \leq f \in M(X, \mu)$ we have

$$\rho'(f) = \sup \left\{ \int_0^a \delta_f \delta |g| : \rho(g) \leq 1 \right\}$$

and

$$\rho''(f) = \sup \left\{ \int_0^a \delta_f \delta |g| : \rho'(g) \leq 1 \right\},$$

where $a = \mu(X)$. If $\rho$ is in addition Fatou, then

$$\rho(f) = \sup \left\{ \int_0^a \delta_f \delta |g| : \rho'(g) \leq 1 \right\}$$

for all $0 \leq f \in M(X, \mu)$.
(ii) \(\rho'\) and \(\rho''\) are universally rearrangement invariant.

(iii) \(0 \leq g < f\) implies \(\rho'(g) \leq \rho'(f)\) and \(\rho''(g) \leq \rho''(f)\), and similarly for \(\rho\) if \(\rho\) is Fatou.

(iv) \(g < f \in L^1(X, \mu)\) implies \(\rho'(g) \leq \rho'(f)\) and \(\rho''(g) \leq \rho''(f)\), and similarly for \(\rho\) if \(\rho\) is Fatou.

**Proof.** (i) For each \(g\) with \(\rho(g) \leq 1\) there is a \(g' \in M(X^+, \mu^+)\) such that \(g' \sim g\) and \(\int_0^a \delta_f \delta \mid g \mid = \int f \mid g' \mid d\mu^+ = \int f T_\mu \mid g' \mid d\mu\).

\(g' \sim g\) implies \(\mid g' \mid \sim \mid g \mid\) so \(\rho(T_\mu \mid g' \mid) \leq \rho(\mid g \mid) \leq 1\) and thus \(\{\int_0^a \delta_f \delta \mid g \mid: \rho(g) \leq 1\} \subset \{\int f \mid g \mid d\mu: \rho(g) \leq 1\}\). Then

\[\rho'(f) = \sup\{\int f \mid g \mid d\mu: \rho(g) \leq 1\} = \sup\{\int_0^a \delta_f \delta \mid g \mid: \rho(g) \leq 1\}\]

as in (14.11).

(ii) This follows from (i) since \(0 \leq f \sim f' \in M(X^+, \mu^+)\) implies \(T_\mu f' < f\) implies \(T_\mu f' \geq 0\) (8.2), and \(\int_0^a \delta T_\mu f' \delta \mid g \mid \leq \int_0^a \delta_f \delta \mid g \mid\).

(iii) & (iv) are immediate using the representations in (i).

15. A Representation Theorem. The rearrangement invariant spaces such as the classical Lebesgue spaces, the Orlicz spaces, and the spaces introduced by Halperin and Lorentz (see [14], [23], [24], Boyd [2] and Shimogaki [45] are all of the following kind.

Let \((X, \Lambda, \mu)\) be a finite m. s. with \(a = \mu(X)\) and let \(M^+ [0, a]\) denote the set of all non-negative extended real valued Lebesgue measurable functions on \([0, a]\).
(15.1) **Lemma.** If $\lambda$ is a Fatou rearrangement invariant norm defined on $M^+[0, a]$ such that $L^\infty \subset L^\lambda \subset L^1$, then the mapping $0 \leq f \to \rho(f) = \lambda(\delta_f)$ is a Fatou norm which is universally rearrangement invariant, and $\rho'(f) = \lambda'(\delta_f)$ for all $0 \leq f \in M(X, m)$. Furthermore $L^\infty \subset L^p$, $L^p \subset L^1$.

**Proof.** Since Lebesgue measure is non-atomic, (14.11) implies that for all $0 \leq F \in M[0, a]$ we have

$$\lambda(F) = \sup \{ \int_0^a \delta_F \delta |G| : \lambda'(G) \leq 1 \}$$

so

$$\rho(f) = \sup \{ \int_0^a \delta_f \delta |G| : \lambda'(G) \leq 1 \}.$$ 

To show that $\rho$ is a (u.r.i.) Fatou norm we have by (14.12) only to show that $\lambda'(r C[0, a]) \leq 1$ for some $r \neq 0$. If $\lambda'(C[0, a]) = 0$, take $r = 1$. Otherwise, let $r = 1/\lambda'(C[0, a])$.

Finally we show that $\rho'(f) = \lambda'(\delta_f)$. For each $G$ with $\lambda(G) \leq 1$ there is a $g' \in M(X, m)$ such that $g' \sim |G|$ and $\int_0^a \delta_f \delta |G| = \int f g' d\mu'$. But $\int f g' d\mu' = \int f T_\mu' g' d\mu$ and $\rho(T_\mu' g') = \lambda(\delta_{T_\mu' g'}) \leq \lambda(|G|) \leq 1$ since $0 \leq T_\mu' g' \ll |G|$, so

$$\{ \int_0^a \delta_f \delta |G| : \lambda(G) \leq 1 \} \subset \{ \int |g| d\mu : \rho(g) \leq 1 \}$$

and thus $\lambda'(\delta_f) \leq \rho'(f)$.

For the rest, if $\rho(g) \leq 1$, then $\int |g| d\mu \leq \int_0^a \delta_f \delta |g|$ and

$$\lambda(\delta |g|) = \rho(g) \leq 1 \quad \text{so} \quad \rho'(f) \leq \lambda'(\delta_f). \quad \rho(C_X) = \lambda(C[0, a]) \&$$

$\rho'(C_X) = \lambda'(C[0, a])$ shows the rest.
(15.2) **Theorem.** If \((X, \Lambda, \mu)\) is a finite m. s. and \(\rho\) is a Fatou norm, then \(\rho\) is universally rearrangement invariant (rearrangement invariant if \((X, \Lambda, \mu)\) is adequate) if and only if there is a Fatou rearrangement invariant norm \(\lambda\) on \(M^+[0, a]\), where \(a = \mu(X)\), such that \(\rho(f) = \lambda(\delta_f)\) for all \(0 \leq f \in M(X, \mu)\).

**Proof.** It only remains to prove the existence of \(\lambda\). By \((14. 14)\) we have that for all \(0 \leq f \in M(X, \mu)\) \(\rho(f) = \sup\{ \int_0^a \delta_f \delta |g| : \rho'(g) \leq 1\}\).

For every \(0 \leq F \in M[0, a]\) we define \(\lambda(F) = \sup\{ \int_0^a \delta_F \delta |g| : \rho'(g) \leq 1\}\).

\((14. 12)\) implies that \(\lambda\) is a (r. i.) Fatou norm. Clearly \(\rho(f) = \lambda(\delta_f)\) for all \(0 \leq f \in M(X, \mu)\).
16. **Schur Convex Functions.**

If \((X, \lambda, \mu)\) is a finite m.s. consisting of \(n\) atoms of equal measure, then each member of \(M(X, \mu)\) may be identified with a point 
\[
\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n,
\]
in which case its decreasing rearrangement, denoted by \(\bar{x}^*\), is the point obtained by rearranging the components of \(\bar{x}\) in decreasing order. If \(\bar{x}, \bar{y} \in \mathbb{R}^n\) then the definition of \(\bar{y} < \bar{x}\) assumes the form
\[
\sum_{i=1}^{k} y_i^* \leq \sum_{i=1}^{k} x_i^* \quad 1 \leq k \leq n
\]
with equality when \(k = n\). It was in this form that the relation \(<\) was first introduced by Hardy, Littlewood, and Pólya.

Let \(S_n\) denote the symmetric group of all permutations of \([1, \ldots, n]\). For each \(\bar{x} \in \mathbb{R}^n\) let \(\Delta(\bar{x}) = \{y \in \mathbb{R}^n : y \sim \bar{x}\} = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \sigma \in S_n\}\). Recall that an \(n \times n\) matrix \(A = [a_{ij}]\) is said to be doubly stochastic (d.s.) if \(a_{ij} \geq 0\) and
\[
\sum_{i=1}^{n} a_{ij} = 1 = \sum_{j=1}^{n} a_{ij} \quad \text{for all } i, j = 1, \ldots, n.
\]

**THEOREM.** (Hardy, Littlewood and Pólya). The following are equivalent for \(\bar{x}, \bar{y} \in \mathbb{R}^n\).

(i) \(\bar{y} < \bar{x}\)

(ii) \(\sum_{i=1}^{n} \varphi(y_i) \leq \sum_{i=1}^{n} \varphi(x_i)\) for all convex functions \(\varphi\) on \(\mathbb{R}\).

(iii) \(\bar{y}\) is in the convex hull of \(\Delta(\bar{x})\).
(iv) There is a doubly stochastic matrix $A$ such that $\vec{y} = \vec{A}\vec{x}$.

An interesting discussion of this result and of the theory of doubly stochastic transformations can be found in a paper by L. Mirsky [31].

(16.2) THEOREM. (R. Muirhead) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then $\vec{y} < \vec{x}$ iff for all positive $\vec{a} \in \mathbb{R}^n$ we have

$$\sum_{\sigma \in S_n} a_{\sigma(1)} \ldots a_{\sigma(n)} \leq \sum_{\sigma \in S_n} a_{\sigma(1)} \ldots a_{\sigma(n)}.$$

Equality holds iff $\vec{a}$ is constant or $\vec{y} \sim \vec{x}$.

It will be our purpose in the next few sections to show in what sense these two theorems can be extended to functions in $L^p(X, \Lambda, \mu)$, where $(X, \Lambda, \mu)$ is a finite m. s., $L^\infty \subset L^p, L^0 \subset L^1$, $p$ is a saturated Fatou norm, and $L^p$ is universally rearrangement invariant. The generalization to $L^1[0, 1]$ of the equivalence of (i), (iii) and (iv) and Theorem (16.2) was given by Ryff ([39], [40], [42]). The generalization of the equivalence of (i) $\&$ (ii), was given for $L^\infty(X, \Lambda, \mu)$ by Grothendieck [10]. The generalization of the equivalence of (i), (ii) and (iii) was given for $L^p(X, \mu)$ independently by Luxemburg [28].
(16.3) DEFINITION. A mapping \( \hat{\psi} : L^p \to R^\# \) which satisfies the following properties is called a Schur Convex function.

(i) \(-\infty < \hat{\psi}(f) \leq +\infty \) for all \( f \in L^p \), and \( \hat{\psi}(f) < +\infty \) for some \( f \in L^p \).

(ii) \( \hat{\psi} \) is convex, i.e., \( \hat{\psi}(rf_1 + (1-r)f_2) \leq r \hat{\psi}(f_1) + (1-r)\hat{\psi}(f_2) \) for all \( f_1, f_2 \in L^p \) and \( 0 \leq r \leq 1 \).

(iii) \( \hat{\psi} \) is \( \sigma(L^p, L^p') \) - lower semi-continuous, i.e.,

\[ \{ f : \hat{\psi}(f) \leq r \} \text{ is } \sigma(L^p, L^p') - \text{closed for all real } r. \]

(iv) \( \hat{\psi} \) is rearrangement invariant, i.e., \( f_1 \sim f_2 \) implies \( \hat{\psi}(f_1) = \hat{\psi}(f_2) \) for all \( f_1, f_2 \in L^p \).

If \( \hat{\psi} \) is convex, then for all real \( r \), the set \( \{ f : \hat{\psi}(f) \leq r \} \) is convex. Hence it is \( \sigma(L^p, L^p') \) closed iff it is closed in each locally convex topology on \( L^p \) in which \( L^p' \) is the dual of \( L^p \). In particular the topology \( \sigma(L^p, L^p') \) generated by the seminorms

\[ P_g(f) = \int |f_g| \, d\mu, \quad f \in L^p, \quad g \in L^p' \]

is a locally convex topology on \( L^p \) in which \( L^p' \) is the dual of \( L^p \), and thus a \( \hat{\psi} \) satisfying (i) and (ii) satisfies (iii) iff:

\[ \int |f_n - f| \, |g| \, d\mu \to 0 \text{ as } n \to \infty \]

for all \( g \in L^p' \) implies

\[ \hat{\psi}(f) \leq \lim \inf \hat{\psi}(f_n). \]

In particular, such a \( \hat{\psi} \) satisfies (iii) whenever:

\[ |f_n| \leq f_0 \in L^p \text{ and } f_n \to f \text{ } \mu\text{-a.e. implies } \hat{\psi}(f) \leq \lim \inf \hat{\psi}(f_n). \]

We would like to prove that \( g < f \) iff \( \hat{\psi}(g) \leq \hat{\psi}(f) \) for all Schur Convex functions \( \hat{\psi} \). However it is clear that we cannot prove this for general m.s. because any rearrangement invariant Fatou norm \( \rho_1 \) is Schur convex, and we have seen that unless \( (X, \lambda, \mu) \) is adequate, we cannot guarantee that \( g < f \) implies \( \rho_1(g) \leq \rho_1(f) \).
In order to alleviate this problem we introduced the concept of a universally rearrangement invariant norm. The same idea works here also.

(16.4) DEFINITION. A mapping $\hat{\varphi} : L^p \to \mathbb{R}$ is said to be a universal Schurconvex function (u. s. c.) if it satisfies properties (i), (ii) and (iii) of (16.3) and (iv): $\hat{\varphi}$ is universally rearrangement invariant, i.e., $\hat{\varphi}(T f') \leq \hat{\varphi}(f)$ whenever $f \in L^p$, $f' \in L^p(X^*, \mu')$ and $f' \sim f$.

If $\hat{\varphi}$ is universal Schur convex then it is Schur convex.

The next lemma will imply (among other things) that if $(X, \Lambda, \mu)$ is adequate, then $\hat{\varphi}$ is u.s.c. whenever it is Schur convex.

EXAMPLES. (i) If $\varphi$ is a real convex function on $\mathbb{R}$ then $\hat{\varphi}(f) = \int \varphi(f) \, d\mu$ is Schur convex on $L^\infty$. If in addition $\varphi$ is increasing then $\hat{\varphi}_t(f) = \int_0^t \varphi(\delta_t f) = \int_0^t \varphi(f)$ is, for each $0 \leq t \leq \mu(X)$, Schur convex on $L^\infty$. If $\lim \inf_{u \to -\infty} \varphi(u)/u$ is finite, then $\hat{\varphi}$ and $\hat{\varphi}_t$ are Schur convex on $L^p$.

(ii) $\hat{\varphi}_1(f) = \int f \, d\mu$ and $\hat{\varphi}_2(f) = \int (-f) \, d\mu$ are u.s.c. on $L^p$.

(iii) For each $0 \leq t \leq \mu(X)$, $\hat{\varphi}_t(f) = \int_0^t \delta_t f$ is u.s.c.

(iv) If $g \in L^p$ and $a = \mu(X)$ then $\hat{\varphi}_g(f) = \int_0^a \delta_f \delta_g + b$ is universal Schur convex on $L^p$ where $b \in \mathbb{R}$.

(v) [Ryff] $\hat{\varphi}(f) = \int_a^t \int_0^s \delta_f(s) \, ds \, dt = \int_a^t \delta_f(s)(a-s) \, ds$ is u.s.c. on $L^p$ and $\hat{\varphi}(f) = \hat{\varphi}(g)$ iff $f \sim g$. 
If \( \Phi \) is a Fatou rearrangement invariant function norm, then \( \Phi \) is Schur convex on \( L^p \). (see [30], Theorem 3.1, p. 162).

(vii) The supremum of a family of Schur convex [Universal Schur Convex] functions is Schur Convex [Universal Schur Convex].

(16.5) LEMMA. If \( \Phi \) is universal Schur convex, or if \( (X, \Lambda, \mu) \) is adequate and \( \Phi \) is Schur convex, then there are functions \( g_i \in L^{p'} \) and real numbers \( b_i \) such that

\[
\Phi(f) = \sup_i \left\{ \int_0^a \delta f \delta g_i + b_i \right\} \text{ for all } f \in L^p.
\]

If in addition \( \Phi \) is increasing, then

\[
\Phi(f) = \sup_i \left\{ \int_0^a \delta f \delta |g_i| + b_i \right\} \text{ for all } f \in L^p.
\]

PROOF. It is well known that if \( \Phi \) is convex and lower semi-continuous, then there are continuous linear functionals \( L_i \) and real numbers \( b_i \) such that \( \Phi(f) = \sup_i \{ L_i(f) + b_i \} \), the \( L_i \) being non-negative if \( \Phi \) is increasing [1]. For each \( i \), there is a \( g_i \in L^{p'} \) s.t. \( L_i(f) = \int f g_i d\mu \) for all \( f \in L^p \), and \( g_i \geq 0 \) if \( L_i \) is non-negative, so

\[
\Phi(f) = \sup_i \{ \int f g_i d\mu + b_i \} \leq \left\{ \int_0^a \delta f \delta g_i + b_i \right\}.
\]

Suppose \( \Phi \) is u.s.c. and let \( f \in L^p \). For each \( i \) there is an \( f' \sim f \) with \( f' \in L^p(X, \# \mu, \# \mu) \) such that \( \int f' g_i d\mu = \int_0^a \delta f \delta g_i \). Then

\[
\Phi(f) \geq \Phi(T_{f'}^i) \geq \int g_i T_{f'} d\mu + b_i = \int f' g_i d\mu + b_i = \int_0^a \delta f \delta g_i + b_i.
\]

and this holds for all \( i \) so
The proof when \((X, \Lambda, \mu)\) is adequate and \(\Phi\) is Schur Convex is similar.

**REMARK.** Luxemburg has observed that if we let

\[ Y(g) = \sup \{ \int f g \, d\mu - \Phi(f) : f \in L^\rho \} \]

then the conclusion of the lemma assumes the form

\[ \Phi(f) = \sup \{ \int_0^a \delta_f g - Y(g) : g \in L^\rho \} \].

(16.6) **THEOREM.** (i) Suppose \((X, \Lambda, \mu)\) is any finite m. s. and \(f_1, f_2 \in L^\rho(X, \mu)\).

1. \(f_1 \ll f_2 \iff \Phi(f_1) \leq \Phi(f_2)\) for all increasing universal Schur convex functions \(\Phi\) on \(L^\rho\).

2. \(f_1 < f_2 \iff \Phi(f_1) < \Phi(f_2)\) for all u. s. c. functions \(\Phi\) on \(L^\rho\).

(ii) Suppose \((X, \Lambda, \mu)\) is adequate and \(f_1, f_2 \in L^\rho\).

1. \(f_1 \ll f_2 \iff \Phi(f_1) \leq \Phi(f_2)\) for all increasing Schur convex functions \(\Phi\) on \(L^\rho\).

2. \(f_1 < f_2 \iff \Phi(f_1) < \Phi(f_2)\) for all Schur convex functions \(\Phi\) on \(L^\rho\).

**PROOF.** Lemma (16.5) in conjunction with the Hardy inequalities (7.1) shows that \(\Phi(f_1) \leq \Phi(f_2)\) whenever \(f_1 < f_2\), or whenever \(f_1 \ll f_2\) and \(\Phi\) is increasing.

If \(\Phi(f_1) \leq \Phi(f_2)\) for all increasing Schur convex \(\Phi\), then since \(f - \int_0^t \delta_f\) is increasing and u. s. c. for each \(0 \leq t \leq \mu(X)\), we have
$f_1 \ll f_2$. If $\mathcal{S}(f_1) \leq \mathcal{S}(f_2)$ for all Schur convex $\mathcal{S}$, then since $f \to \int (-f)\,d\mu$ is universal Schur convex we have $f_1 \ll f_2$.

Recall that if $\rho$ is a Fatou u.r.i. norm, then there is a Fatou r.i. norm $\lambda$ on $M^+[0, a]$ such that $\rho(f) = \lambda(\delta_f)$.

(16.7) THEOREM. Suppose $\rho(f) = \lambda(\delta_f)$ where $\lambda$ is a Fatou rearrangement invariant norm. Then $\mathcal{S}$ is a universal Schur convex function on $L^\rho$ iff there is a Schur convex function $\mathcal{S}_o$ on $L^\lambda[0, a]$ such that $\mathcal{S}(f) = \mathcal{S}_o(\delta_f)$.

PROOF. If $\mathcal{S}$ is u.s.c. on $L^\rho$ then there are $g_i \in L^{\rho'}$ and $b_i \in \mathbb{R}$ such that for all $f \in L^\rho$, $\mathcal{S}(f) = \sup \{ \int_0^a \delta_f \delta_{g_i} + b_i \}$. Since $\rho'(g) = \lambda'(\delta_g)$ for all $g \in L^{\rho'}$ we see that $\delta_{g_i} \in L^{\lambda'}[0, a]$ for each $i$. Hence define $\mathcal{S}_o(F) = \sup \{ \int_0^a \delta_F \delta_{g_i} + b_i \}$ for $F \in L^\lambda[0, a]$. Then $\mathcal{S}_o$ is Schur convex and $\mathcal{S}_o(\delta_f) = \mathcal{S}(f)$.

Conversely, if $\mathcal{S}_o$ is Schur convex on $L^\lambda[0, a]$ then there are $G_i \in L^{\lambda'}[0, a]$ and $b_i \in \mathbb{R}$ s.t. $\mathcal{S}_o(F) = \sup \{ \int_0^a \delta_F \delta_{G_i} + b_i \}$ for all $F \in L^\lambda[0, a]$. But $f \in L^\rho$ iff $\delta_f \in L^{\lambda'}[0, a]$ so for all $f \in L^\rho$, $\mathcal{S}(f) = \mathcal{S}_o(\delta_f) = \sup \{ \int_0^a \delta_f \delta_{G_i} + b_i \}$ and thus $\mathcal{S}$ is universal Schur convex.

17. The Sets $\Omega(f) = \{ f' : f' \ll f \}$. Recall that we are assuming that $\rho$ is a saturated Fatou norm on $M^+(X, \mu)$, $L^\rho(X, \mu)$ is u.r.i., $L^\infty(\mu) \subset L^\rho(\mu)$, $L^{\rho'}(\mu) \subset L^1(\mu)$, and $a = \mu(X)$.  

The equivalence \( \vec{y} \preceq \vec{x} \) iff \( \vec{y} \in \text{the convex hull of } \Delta(\vec{x}) \) can be reformulated:

\[ \{ \vec{y} : \vec{y} \preceq \vec{x} \} = \text{the closed convex hull of } \{ \vec{y} : \vec{y} \sim \vec{x} \} \]

If \( f \in L^p(X, \mu) \) we let \( \Omega(f) = \{ f' \in L^1(X, \mu) : f' \preceq f \} \), and \( \Delta(f) = \{ f' \in L^1(X, \mu) : f' \sim f \} \). Since \( L^p \) is u. r. i., \( f' \preceq f \in L^p \) implies \( f' \in L^p \) and hence both \( \Omega(f) \& \Delta(f) \subseteq L^p \).

\( \Omega(f) \) has a smallest element \( \overline{f} = \left( \frac{1}{\mu(X)} \int f \, d\mu \right) \chi_X \) in the sense that \( h < f' \) for all \( f' \in \Omega(f) \) iff \( h = \overline{f} \). This follows from (8.2)(vi). Also \( \Omega(f) \) is contained in the hyperplane \( \{ f' \in L^1 : \int f' \, d\mu = \int f \, d\mu \} \), and \( \overline{f} \) is equidistant (in the \( L^1 \) norm) from the members of \( \Delta(f) \).

(17.1) PROPOSITION. Any \( \sigma(L^p, L^p') \)-bounded set \( A \subseteq L^p \) is \( \rho \)-bounded.

PROOF. Since \( A \) is \( \sigma(L^p, L^p') \)-bounded, for each \( g \in L^p' \) we have

\[ \sup \{ \int f' g \, d\mu : f' \in A \} < \infty. \]

For each \( f' \in A \), \( L_{f'}(g) = \int f' g \, d\mu \) defines a \( \rho' \)-continuous linear functional on \( L^p' \) with \( \| L_{f'} \| = \rho''(f') = \rho(f') \)

[29, Note IV, p. 257]. Then for each \( g \in L^p' \), \( \sup \{ \| L_{f'}(g) \| : f' \in A \} < \infty \), and since \( L^p' \) is complete, the Banach Steinhaus Theorem implies

\[ \sup \{ \| L_{f'} \| : f' \in A \} = \sup \{ \rho(f') : f' \in A \} < \infty. \]

(17.2) COROLLARY. If \( \{ f_{\alpha} \} \) is a net in \( L^p \) which is \( \sigma(L^p, L^p') \)

convergent to \( f_0 \in L^p \) then \( \{ f_{\alpha} \} \) is \( \rho \)-bounded.

(17.3) THEOREM. For all \( f \in L^p \), \( \Omega(f) \) is \( \rho \)-bounded.

PROOF. We have only to show that \( \Omega(f) \) is \( \sigma(L^p, L^p') \) bounded.

This follows at once since for each \( g \in L^p' \) we have

\[ \sup \{ \| f' g \| : f' \in \Omega(f) \} \leq \sup \{ \| f' g' \| : f' \in \Omega(f) \} \leq f_0 \delta ||f|| \delta ||g|| < \infty. \]
(17.4) **THEOREM.** If $f \in L^p$ then $\Omega(f)$ is a convex and $\sigma(L^p, L^{p'})$ compact subset of $L^p$.

**PROOF.** Let $f \in L^p$, let $f_1, f_2 \in \Omega(f)$ and let $0 \leq r \leq 1$. Then (12.1) and (5.3)(vi) imply that for all $0 \leq t \leq a$, 

$$
\int_0^t r \delta f_1 + \int_0^t (1-r) \delta f_2 \leq \int_0^t \delta f \quad \text{so} \quad rf_1 + (1-r)f_2 \in \Omega(f).
$$

Hence $\Omega(f)$ is convex.

Let $\{f_\alpha\}$ be a net in $\Omega(f)$. For each $\alpha$, $F_\alpha (g) = \int f_\alpha g \, d\mu$ defines a $\rho'$-continuous linear functional on $L^{p'}$ with $\|F_\alpha\| = \rho''(f_\alpha) = \rho(f_\alpha)$ [29, Note IV, p. 257]. Since $\Omega(f)$ is $\rho$-bounded, there is a number $M > 0$ such that $\rho(f') \leq M$ for all $f' \in \Omega(f)$, and thus $\|F_\alpha\| = \rho(f_\alpha) \leq M$ for all $\alpha$. Hence $\{F_\alpha\}$ is a net in a $\sigma(L^{p'}, L^{p'})$-compact set [Alaoglu's Theorem, 7, p. 424], so it has a convergent subnet $F_\beta \rightarrow F_0$ say. To show that $F_0 \in L^p = L^{p''}$ it is necessary and sufficient to show that $F_0 (g_n) \rightarrow 0$ whenever $g_n \rightarrow 0$ [30, p. 155].

Now for each $g \in L^{p'}$ and for each $\alpha$, 

$$
|F_\alpha (g)| = |\int f_\alpha g \, d\mu| \leq \|f_\alpha g\| \, d\mu \leq \int_0^a \delta |f| \delta |g| \, d\mu \quad \text{because} \quad f_\alpha < f \in L^1 \quad \text{implies} \quad |f_\alpha| < |f| \cdot
$$

Since the bound is independent of $\alpha$ we have $|F_\alpha (g)| \leq \int_0^a \delta |f| \delta |g| \, d\mu$ and this $\rightarrow 0$ as $|g| \rightarrow 0$. Hence $F_\alpha \in L^p$, i.e., there is an $f_\alpha \in L^p$ such that $F_\alpha (g) = \int f_\alpha g \, d\mu$ for all $g \in L^{p'}$. Since $F_\beta \rightarrow F_0$ we have 

$$
\int f_\beta g \, d\mu = F_\beta (g) \rightarrow F_0 (g) = \int f_\alpha g \, d\mu \quad \text{for all} \quad g \in L^{p'}, \quad \text{i.e.,} \quad \{f_\beta\} \quad \text{is} \quad \sigma(L^p, L^{p'}) \quad \text{convergent to} \quad f_\alpha. \quad \text{Thus} \quad \Omega(f) \quad \text{is} \quad \sigma(L^p, L^{p'}) \quad \text{compact.}
$$

Let $V$ be a locally convex linear topological space and let $V^*$ denote the dual of $V$, i.e. $V^*$ is the collection of all continuous linear
functionals $F$ on $V$. For each $A \subset V$ and for each linear functional $F$ on $V$ let $F[A] = \{F(v) : v \in A\}$ and let $\text{cov}(A)$ denote the convex hull of $A$.

As a corollary of the Hahn-Banach Separation Theorem we have the following characterization of closed convex sets.

(17.5) PROPOSITION. Let $K$ be a closed convex subset of $V$. Then for each $v \in V$, $v \in K$ iff $F(v) \leq \sup F[K]$ for all $F \in V^*$.

PROOF. If $v \in K$, then clearly $F(v) \leq \sup F[K]$ for all $F \in V^*$.

Conversely, if $v \notin K$, then the Hahn-Banach Separation Theorem [36, p. 30, Cor 1] implies there is an $F \in V^*$ such that $F(v) \in F[K] \supseteq [\inf F[K], \sup F[K]]$ so either $\sup F[K] < F(v)$ or $\sup -F[K] < -F(v)$.

(17.6) THEOREM. Let $f \in L^p$. Then $f_1 \in \Omega(f)$ iff $\int f_1 g d\mu \leq \int_0^a \delta_f \delta_g$ for all $g \in L^{p'}$.

PROOF. The continuous linear functionals on $L^p$ with the $\sigma(L^p, L^{p'})$ topology are $F_g(f) = \int f g d\mu$ where $f \in L^p$, $g \in L^{p'}$. For each $g \in L^{p'}$ it follows from (13.2) that $\sup F_g[\Omega(f)] = \int_0^a \delta_f \delta_g$. Now use (17.5).

We may similarly give a criterion for deciding when a closed convex set is the closed convex hull of another set.
PROPOSITION. Suppose \( K \) is a closed convex subset of \( V \) and \( D \subseteq K \). Then \( K \) is the closed convex hull of \( D \) iff
\[
\sup F[D] = \sup F[K] \text{ for all } F \in V^*.
\]

PROOF. Suppose the criterion holds. Then
\[
\sup F[K] = \sup F[D] \leq \sup F[\overline{\operatorname{cov}(D)}] \text{ for all } F \in V^*.
\]
Since \( \overline{\operatorname{cov}(D)} \) is a closed convex set, (17.5) implies \( K \subseteq \overline{\operatorname{cov}(D)} \). But \( \overline{\operatorname{cov}(D)} \subseteq K \) since \( D \subseteq K \) and \( K \) is closed and convex. Hence \( K = \overline{\operatorname{cov}(D)} \).

Suppose now \( K = \overline{\operatorname{cov}(D)} \). Let \( v \in K \). Then there is a net \( \{w_\alpha\} \) in \( \operatorname{cov}(D) \) such that \( w_\alpha \to v \), so for each \( F \in V^* \) we have \( F(w_\alpha) \to F(v) \) and thus \( \sup F[D] = \sup F[\overline{\operatorname{cov}(D)}] \geq \sup F(w_\alpha) \geq F(v) \) since \( F[D] = F[\overline{\operatorname{cov}(D)}] \).

In view of (17.7) we have immediately:

(17.8) THEOREM. Let \( f \in L^p \) and let \( \xi \) be a topology on \( L^p \) in which \( L^p' \) is the dual of \( L^p \). Then \( \Omega(f) \) is the \( \xi \)-closed convex hull of a set \( D \subseteq \Omega(f) \) iff
\[
\sup \{ \int f' \ g \ d\mu : f' \in D \} = \int_0^a \delta \ f \ g \text{ for all } g \in L^p'.
\]

(17.9) THEOREM. Let \( \xi \) be a topology on \( L^p \) in which \( L^p' \) is the dual of \( L^p \). In particular \( \xi = \sigma(L^p, L^p') \) or if \( \rho(f_n) \downarrow 0 \) whenever \( f_n \downarrow 0 \) then \( \xi = \rho \)-topology are such topologies.

(i) \( \Omega(f) \) is the \( \xi \)-closed convex hull of \( \Delta(f) = \{ f' \in L^1 : f' \sim f \} \) for all \( f \in L^p \) iff \( (X, \Lambda, \mu) \) is adequate.

(ii) For any finite m.s. \( (X, \Lambda, \mu) \) and \( f \in L^p \), \( \Omega(f) \) is the \( \xi \)-closed convex hull of \( \{ T_{f'} f : f' \in L^1(X^#, \mu^#) \text{ and } f' \sim f \} \).
PROOF. (i) If \((X, \Lambda, \mu)\) is adequate then the result follows from (17.8) and (11.1). If \((X, \Lambda, \mu)\) is not adequate, then (11.5) shows that the condition in (17.8) fails. (ii) Follows from (17.8) and (9.1) (iv) \& (vi).

As Luxemburg has pointed out, Theorem (17.9) answers the following question asked recently by Z.Nehari [32]: Let \((X, \Lambda, \mu)\) be a non-atomic finite m.s. and let \(E \in \Lambda\) have positive measure. What is the smallest closed convex set \(A \subset L^1(X, \mu)\) which contains all the functions \(C_F\) such that \(\mu(F) = \mu(E)\)? Obviously \(A\) is the closed convex hull of \(\Delta(C_E)\) so \(A = \Omega(C_E) = \{f \in L^1(X, \mu): 0 \leq f \leq 1\} \text{ and } \int f \, d\mu = \mu(E)\} \text{ using (8.2)(v).}

We can define the sets \(\Omega\) in a slightly more general way. Let \((X_1, \Lambda_1, \mu_1)\) and \((X, \Lambda, \mu)\) be finite m.s. such that \(\mu_1(X_1) = \mu(X) = a\).

If \(f \in L^1(X_1, \mu_1)\) let \(\Omega_f(X, \mu) = \{h \in L^1(X, \mu): h < f\}\) and let \(\Delta_f(X, \mu) = \{h \in L^1(X, \mu): h \sim f\}.\) Observe that \(\Omega_f\) is never empty because it contains \(\bar{f} = \left(\frac{1}{\mu_1(X_1)} \int f \, d\mu_1\right)C_X\), but it may happen that \(\Delta_f = \emptyset\). This is the only interesting case, because if \(\Delta_f \neq \emptyset\), then we are doing nothing new, since for each \(f_0 \in \Delta_f\) we have \(\Omega_f = \Omega(f_0)\). Theorem (5.10) implies that \(\Delta_f(X^+, \mu^+) \neq \emptyset\) for all \(f \in L^1(X_1, \mu_1)\), so if \(f_0 \in \Delta_f(\mu^+)\) we then have \(\Omega_f(X, \mu) \subset \Omega(f_0)\).

Now let \(\rho\) be a saturated Fatou norm on \(M^+(X, \mu)\) such that \(L^\infty(\mu) \subset L^\rho, L^\rho \subset L^1(\mu)\) and \(L^\rho\) is u.r.i. The question arises: For what \(f \in L^1(X_1, \mu_1)\) is \(\Omega_f \subset L^\rho(X, \mu)?\)
PROPOSITION. Let \( f \in L^1(X, \mu) \). Then \( \Omega |f| \subseteq L^\rho \) iff \( \delta |f| \, \delta |g| \in L^1[0, a] \) for all \( g \in L^\rho \), in which case \( \Omega f \subseteq L^\rho \).

PROOF. If \( \Omega |f| \subseteq L^\rho \) then \( f' < |f| \), \( f' \in M(\mu) \) implies \( 0 < f' \in L^1(X, \mu) \) so \( f' \in \Omega |f| \subseteq L^\rho \) and thus for all \( g \in L^\rho \) we have
\[
\int f' \, |g| \, d\mu \text{ is finite, and hence (13.2) says } \delta |f| \, \delta |g| \in L^1[0, a].
\]
Conversely, if \( \delta |f| \, \delta |g| \in L^1[0, a] \) for all \( g \in L^\rho \) then \( f' < f \) implies \( |f'| < |f| \) so \( \delta |f'| \, \delta |g| \in L^1[0, a] \) for all \( g \in L^\rho \) and hence \( f' \in L^\rho \) by (13.2) & (14.5).

Observe that Theorems (17.3), (17.4), (17.6), (17.8) and (17.9) are true for \( \Omega f \) under the hypothesis that \( \Omega |f| \subseteq L^\rho \), because in view of (17.10), the proofs are practically the same.

In view of the condition \( \Omega |f| \subseteq L^\rho \) it is natural to wonder: Is there a norm \( \rho_1 \) on \( M^+(X, \mu) \) such that \( f \in L^1 \) implies \( \Omega |f| \subseteq L^\rho \)?

(17.11) THEOREM. Define the norm \( \rho_1 \) on \( M(X, \mu) \) by
\[
\rho_1(f) = \sup \{ \int_0^a \delta |f| \, \delta |g| : \rho'(g) \leq 1 \}.
\]
Then \( \rho_1 \) is a Fatou u. r. i. norm satisfying
\begin{enumerate}
\item \( L^\infty(\mu) \subseteq L^{\rho_1}, L^{\rho_1} \subseteq L^1(\mu) \)
\item \( f' < f \in L^{\rho_1} \) implies \( \rho(f') \leq \rho_1(f) \)
\item \( \Omega |f| \subseteq L^\rho \) iff \( f \in L^{\rho_1} \).
\end{enumerate}
PROOF. (i) $\rho_1(C_X) = \sup \{ \int_0^a \delta |g| : \rho'(g) \leq 1 \}
= \sup \{ \int |g| d\mu : \rho'(g) \leq 1 \} = \rho(C_X) < \infty$. Since $0 < \rho'(C_X) < \infty$ let $g_1 = \frac{1}{\rho'(C_X)} C_X$. Then $\frac{1}{\rho'(C_X)} \int |f| d\mu_1 = \int_0^a \delta |f| |g_1|$
\leq \sup \{ \int_0^a \delta |f| |g| : \rho'(g) \leq 1 \} = \rho_1(f)$ so $f \in L^{\rho_1}$ implies $f \in L^1(\mu_1)$.
Since $\rho_1$ is Fatou, $L^\infty(\mu_1) \subset L^{\rho_1} \subset L^1(\mu_1)$ also.

(ii) In order to apply the theorem of Hardy (7.1) we need to know that $\delta |f'| |g| \in L^1[0,a]$ for all $g \in L^{\rho_1}$ with $\rho'(g) \leq 1$. Now $f' < f \in L^{\rho_1} \subset L^1(\mu_1)$ implies $\delta_{f'} < \delta_f \in L^1[0,a]$ so
$$\delta |f'| |g| \leq \sup \{ \int_0^a \delta |f| |g| : \rho'(g) \leq 1 \} = \rho_1(f).$$
In fact $\delta |f'| |g| \in L^1[0,a]$ for all $g \in L^\infty(\mu)$, so $\int_0^a \delta |f| |g| \leq \int_0^a \delta |f'| |g|$
for all $g \in L^\infty(\mu)$.
Let $g \in L^{\rho'}(\mu)$. Then there is a sequence $\{g_n\} \subset L^\infty(\mu)$ such that $0 \leq g_n \uparrow |g|$ so $\delta g_n \uparrow \delta |g|$ and hence the monotone convergence theorem implies
$$\int_0^a \delta |f'| |g| = \lim \int_0^a \delta |f'| |g_n| \leq \lim \int_0^a \delta |f| |g_n| = \int_0^a \delta |f| |g| < \infty$$
so $\delta |f'| |g| \in L^1[0,a]$. Hence $\rho(f') = \sup \{ \int |f'| g |d\mu : \rho'(g) \leq 1 \}$
\leq \sup \{ \int_0^a \delta |f| |g| : \rho'(g) \leq 1 \} = \rho_1(f).

(iii) Part (ii) shows that $f \in L^{\rho_1}$ implies $\Omega |f| \subset L^\rho$. Suppose $\Omega |f| \subset L^\rho$. Then $\delta |f| |g| \in L^1[0,a]$ for all $g \in L^{\rho'}$
so \( \max \{ \int f' |g| \, d\mu : f' < |f| \} = \int_0^a \delta f' |g| \, \delta g < \alpha \) for all \( g \in L^{p'} \) and hence \( \rho_1(f) = \sup \{ \max \{ \int f' |g| \, d\mu : f' < |f| \} : \rho'(g) < 1 \} \)

\[ = \sup \{ \sup \{ \int f' |g| \, d\mu : \rho'(g) < 1 \} : f' < |f| \} \]

\[ = \sup \{ \rho(f') : f' \in \Omega_f \} < \alpha \) because \( \Omega_f \in \mathbb{L}^p \)

implies \( \Omega_f \) is \( \rho \)-bounded.

If (as in §15) \( \lambda \) is a Fatou r. i. norm on \( M^+[0,a] \) such that \( L^{\infty} \subset L^{\lambda} \), \( L^{\lambda'} \subset L^1 \) and \( \rho(f) = \lambda(\delta_f) \) for \( f \in M^+(X,\mu) \) then the natural norm to choose on \( M^+(X_1,\mu_1) \) is \( \rho_2(f) = \lambda(\delta_f) \). In this connection we have the following.

(17.12) PROPOSITION. Let \( \lambda, \rho, \rho_1, \rho_2 \) be as above.

(i) \( L^{\infty}(\mu_1) \subset L^{\rho_2}, L^{\rho_2'} \subset L^1(\mu_1) \)

(ii) \( \rho_1 \leq \rho_2 \)

(iii) \( f' < f \in L^{\rho_2} \) implies \( \rho(f') \leq \rho_2(f) \)

PROOF. (i) See (15.1).

(ii) \( \rho_1(f) = \sup \{ \int_0^a \delta f' |h| : \lambda'(\delta_h) \leq 1, h \in M(X,\mu) \} \)

\[ \leq \sup \{ \int_0^a \delta f' |G| : \lambda'(G) \leq 1 \} = \rho_2(f). \]

(iii) Use (17.12) (ii).
18. **Doubly Stochastic Transformations.** Observe that (16.1)
implies that an \( n \times n \) matrix \( A \) is doubly stochastic iff \( A \vec{x} \leq \vec{x} \) for
all \( \vec{x} \in \mathbb{R}^n \). Let \( (X_1, \lambda_1, \mu_1) \) and \( (X, \lambda, \mu) \) be finite measure spaces
with \( \mu_1(X_1) = \mu(X) = a \).

(18.1) **DEFINITION.** A linear mapping \( T: L^1(X_1, \mu_1) \to L^1(X, \mu) \)
is called doubly stochastic (d.s.) iff \( Tf < f \) for all \( f \in L^1(X_1, \mu_1) \).

**EXAMPLES.** 1. If \( (X, \lambda, \mu) \) is a non-atomic m.s. and
\( \sigma: X \to [0, a] \) is measure preserving (m.p.) let \( T_\sigma f = f \circ \sigma \). Then
\( T_\sigma: L^1([0, a]) \to L^1(X, \mu) \) is d.s.

2. \( T_\mu: L^1(X^#, \mu^#) \to L^1(X, \mu) \) is d.s.

3. \( Tf = (\frac{1}{\mu_1(X_1)}) \int f \, d\mu_1)C_X \) defines a d.s. \( T: L^1(X_1, \mu_1) \to L^1(X, \mu) \).

(18.2) **PROPOSITION.** Let \( T: L^1(\mu_1) \to L^1(\mu) \) be d.s. Then

(i) \( T \) is non-negative, i.e., \( f \geq 0 \) implies \( Tf \geq 0 \).

(ii) \( TC_{X_1} = C_X \)

(iii) \( (Tf)^+ \leq Tf^+ \), \( (Tf)^- \leq Tf^- \), \( |Tf| = T|f| \) for all \( f \in L^1(\mu_1) \)

(iv) \( T: L^\infty(X_1, \mu_1) \to L^\infty(X, \mu) \)

(v) \( f \in L^1(\mu_1) \) implies \( \|Tf\|_1 \leq \|f\|_1 \)

\( f \in L^\infty(\mu_1) \) implies \( \|Tf\|_\infty \leq \|f\|_\infty \)
PROOF. (i) Follows from (8.2)(iii).

(ii) Follows from (8.2)(v).

(iii) Since \( Tf = Tf^+ - Tf^- \) and \( Tf^+ \geq 0 \) we have \( (Tf)^+ = Tf^+ \) and \( (Tf)^- \leq (Tf)^- \). Hence \( |Tf| = (Tf)^+ + (Tf)^- \leq Tf^+ + Tf^- = T|f| \).

(iv) & (v) Follow from \( |Tf| \leq T|f| < |f| \).

(18.3) LEMMA. Suppose \( T : L^1(\mu_1) \rightarrow L^1(\mu) \) is linear.

(i) If \( T f^+ < f^+ \) and \( T f^- < f^- \) then \( T f < f \).

(ii) \( T \) is d. s. iff \( T f < f \) for all \( 0 \leq f \in L^1(\mu_1) \).

PROOF. (i) \( \int_0^t \delta_{Tf} = \int_0^t \delta_{Tf^+} - \int_0^t \delta_{Tf^-} \leq \int_0^t \delta_{Tf^+} + \int_0^t \delta_{-Tf^-} \)

\[ \leq \int_0^t \delta_{f^+} + \int_0^t \delta_{-f^-} = \int_0^t \delta_f \]

for all \( 0 \leq t \leq a \) with equality when \( t = a \) since \( T f^- < f^- \) implies \( -T f^- < -f^- \).

(ii) \( T f < f \) for all \( f \geq 0 \) implies \( T f^+ < f^+ \) and \( T f^- < f^- \) so \( T f < f \) for all \( f \).

The following Theorem, first proved by J. V. Ryff [39] for \( L^1[0,1] \), is fundamental.

(18.4) THEOREM [J. V. Ryff]. A linear transformation \( T \) mapping the simple functions of \( (X_1, \Lambda_1, \mu_1) \) into \( L^1(X, \mu) \) has a unique extension to a doubly stochastic transformation of \( L^1(\mu_1) \rightarrow L^1(\mu) \) iff for all \( E \in \Lambda_1 \), we have \( 0 \leq T C_E \leq C_X \) and \( \int_X T C_E \, d\mu = \mu_1(E) \).

PROOF. (8.2)(v) shows that these conditions are equivalent to \( T C_E < C_E \) for all \( E \in \Lambda_1 \), and are thus necessary. Hence suppose \( T \) is given satisfying the above conditions. Let \( 0 \leq f \in M(X_1, \mu_1) \) be
simple. Then \( f = \sum_{i=1}^{n} f_i \cdot C_{E_i} \) where \( E_n \subset \cdots \subset E_1 \) and \( f_i > 0 \), so
\[
\delta_f = \sum_{i=1}^{n} f_i \cdot \delta_{E_i}.
\]
Then (12.1) implies for all \( 0 \leq t \leq a, \)
\[
\int_{0}^{t} \delta_{Tf} \cdot \sum_{i=1}^{n} f_i \cdot \int_{0}^{t} \delta_{T C_{E_i}} \leq \sum_{i=1}^{n} f_i \cdot \int_{0}^{t} \delta_{E_i} = \int_{0}^{t} \delta_f
\]
with equality when \( t = a \), so \( Tf < f \) for all non-negative simple functions and hence for all simple functions. Thus \( T \) is a contraction in the \( L^1 \) and \( L^\infty \) norms on the simple functions, so it extends uniquely to \( L^1 \). To show that \( T \) is d.s. let \( 0 \leq f \in L^1(\mu_1) \). Then there is a sequence \( \{f_n\} \) of simple functions s.t. \( 0 \leq f_n \uparrow f \). Then \( Tf_n \) is increasing so \( Tf_n \uparrow Tf \) and since \( Tf_n < f_n \), (8.2)(vii) implies \( Tf < f \). Hence (18.3)(ii) implies \( T \) is d.s.

If \( T: L^1(\mu_1) \to L^1(\mu) \) is linear, let \( T^* \) denote the adjoint of \( T \), defined by \( \int g \cdot Tf \, d\mu = \int f \cdot T^*g \, d\mu_1 \) for all \( f \in L^1(\mu_1) \), \( g \in L^\infty(\mu) \). It follows that \( T^*: L^\infty(X,\mu) \to L^\infty(X_1,\mu_1) \) and hence \( T \) is weakly continuous under the topologies \( \sigma(L^1(\mu_1), L^\infty(\mu_1)) \) and \( \sigma(L^1(\mu), L^\infty(\mu)) \)

([36, p.38, Prop. 12] or use nets and the defining equation.)

(18.5) THEOREM. If \( T: L^1(\mu_1) \to L^1(\mu) \) is d.s. then \( T^* \) has a unique extension to a d.s. map of \( L^1(\mu) \to L^1(\mu_1) \).

PROOF. This follows from (18.4) as follows. Let \( E \in \Lambda \).

Then \( \int T^*C_E \, d\mu_1 = \int C_E \cdot T C_{X_1} \, d\mu = \int C_E \, d\mu = \mu(E) \). Also for all \( A \in \Lambda_1 \), \( \int_A T^*C_E \, d\mu_1 = \int C_E \cdot T C_A \, d\mu \) and \( 0 \leq \int C_E \cdot T C_A \, d\mu \leq \int T C_A \, d\mu_1 \)
\( = \mu_1(A) = \int_A C_{X_1} \, d\mu_1 \) so \( 0 \leq T^*C_E \leq C_{X_1} \).
(18.6) PROPOSITION. A linear mapping \( T: L^1(\mu_1) \to L^1(\mu) \) is doubly stochastic iff

\[
\begin{align*}
(i) & \quad T \geq 0 \\
(ii) & \quad TC_{X_1} = C_X \\
(iii) & \quad T^*C_X = C_{X_1}
\end{align*}
\]

PROOF. This follows from (18.4) since for all \( E \in \Lambda_1 \)

\[
\int TC_E \, d\mu = \int C_E T^*C_X \, d\mu_1 = \int C_E \, d\mu_1 = \mu_1(E) \quad \text{and} \quad 0 \leq C_E \leq C_X_1
\]

implies \( 0 \leq TC_E \leq TC_{X_1} = C_X. \)

Observe that if \((X_2, \Lambda_2, \mu_2)\) is also a finite m. s. with \( \mu_2(X_2) = a, \) and \( T_1: L^1(\mu_1) \to L^1(\mu) \) and \( T_2: L^1(\mu_2) \to L^1(\mu_1) \) are both d. s. then \( T_1 T_2: L^1(\mu_2) \to L^1(\mu) \) is d. s., since for all \( f \in L^1(\mu_2) \) we have \( T_1 T_2 f < T_2 f < f. \)

(18.7) LEMMA. Let \( f \in L^1(X, \mu) \) and let \( \sigma: X^\# \to [0, a] \) be measure preserving s. t. \( T_\sigma \delta_f = \delta_f \circ \sigma = f \mu^\# \)-a. e. Then \( T_\sigma^* f = \delta_f \)-a. e.

PROOF. For all measurable \( J \subset [0, a] \) we have

\[
\begin{align*}
\frac{a}{\sigma(C_J)} T_\sigma^* f \, dm = \int f T_\sigma C_J \, d\mu^\# = \int (\delta_f \circ \sigma)(C_J \circ \sigma) \, d\mu^\# = \frac{a}{\sigma(C_J)} \delta_f C_J \, dm.
\end{align*}
\]

Let \( 6(X_1, X) = \{ T: L^1(X_1, \mu_1) \to L^1(X, \mu) \} \) such that \( T \) is doubly stochastic. If \( f \in L^1(\mu_1) \) let \( 6_f(X_1, X) = \{ Tf: T \in 6(X_1, X) \} \) and we recall that \( \Omega_f(X, \mu) = \{ g \in L^1(X, \mu): g < f \} \) and
\[ \Delta_f(X, \mu) = \{ g \in L^1(X, \mu) : g \sim f \} . \]

As we indicated in §17, if \( \Delta_f \neq \emptyset \), then \( \Omega_f = \Omega(f_o) \) for all \( f_o \in \Delta_f \). Note that \( g_f \subset \Omega_f \) for all \( f \in L^1(\mu_1) \).

(18.8) THEOREM. \( \mathcal{G}(X_1, X) \) is convex and compact in the weak operator topology determined by the linear functionals
\[
T \rightarrow \int f \ T g \ d\mu \quad f \in L^1(\mu), \ g \in L^\infty(\mu_1).
\]

PROOF. To show \( \mathcal{G} \) is convex, let \( T_1, T_2 \in \mathcal{G} \) and \( 0 \leq r \leq 1 \). Then for all \( f \in L^1(\mu_1) \) we have \( T_1 f, T_2 f \in \Omega_f \) so \( r T_1 f + (1-r)T_2 f \in \Omega_f \), i.e., \( (r T_1 f + (1-r)T_2 f) = r T_1 f + (1-r)T_2 f < f \), so \( r T_1 + (1-r)T_2 \in \mathcal{G} \).

Let \( S_1 \) be the unit ball of \( L^\infty(\mu_1) \) and let \( S \) be the unit ball in \( L^\infty(\mu) \). Since \( S \) is \( \sigma(L^\infty(\mu), L^1(\mu)) \) compact, the set of all linear operators which map \( S_1 \rightarrow S \) is compact [18]. Hence it only remains to show that \( \mathcal{G} \) is closed. Let \( T_a \rightarrow T \). For each \( E \in \Lambda \) we have \( T_a C_E \in \Omega C_E \) and \( T_a C_E \rightarrow TC_E \) weakly. Since \( \Omega C_E \) is \( \sigma(L^1, L^\infty) \) closed we get \( TC_E \in \Omega C_E \). Hence (18.4) implies \( T \in \mathcal{G} \).

(18.9) THEOREM. Let \( f \in L^1(X_1, \mu_1) \). If \( g \in M(X, \mu) \) then \( g < f \) iff there is a doubly stochastic \( T : L^1(\mu_1) \rightarrow L^1(\mu) \) such that \( g = Tf \).

PROOF. We have to show that \( \mathcal{G}_f = \Omega_f \). It suffices to show that \( T_\mu \Delta_f(\mu^\#) \subset \mathcal{G}_f \) (where, of course, \( T_\mu \Delta_f(\mu^\#) = \{ T_\mu f' : f' \in L^1(X^\#, \mu^\#) \land f' \sim f \} \) and that \( \mathcal{G}_f \) is \( \sigma(L^1(\mu), L^\infty(\mu)) \) closed and convex, because we know from (17.11) that \( \Omega_f \) is the closed convex hull of \( T_\mu \Delta_f(\mu^\#) \) in this topology. \( \mathcal{G}_f \) is convex and closed in this topology, because it is the
image of the convex, compact (in the weak operator topology described in (18.8)) set $\mathcal{D}(X, X_1)$ under the continuous linear map $T \mapsto T^\ast f$. For the rest, let $f' \in \Delta_f^\ast(\mu^\#)$. There are measure preserving transformations $\sigma: X_1^\# \to [0, a]$ and $\phi: X_1^\# \to [0, a]$ such that $T_\sigma \delta_{f'} = f'$ and $T_\phi \delta_{f'} = f$. Since $f' \sim f$, $\delta_{f'} = \delta_f$ so $T_\mu f' = T_\mu T_\sigma T_\phi f \in \mathcal{D}_f$.

(18.10) THEOREM. If $f_1, f_2 \in L^1(X_1, \mu_1)$ and $g \in M(X, \mu)$ and $g < f_1 + f_2$ then there are $g_1, g_2 \in L^1(X, \mu)$ such that $g = g_1 + g_2$ and $g_1 < f_1$ and $g_2 < f_2$.

PROOF. There is a $T \in \mathcal{D}(X_1, X)$ s.t. $g = T(f_1 + f_2)$, so let $g_i = T f_i$, $i = 1, 2$.

A good example of a class of doubly stochastic operators is provided by conditional expectations. Let $\Lambda'$ be a $\sigma$-subalgebra of $\Lambda$ and let $f \in L^1(X, \mu)$. It follows from the Radon-Nikodym theorem that there is a unique $\Lambda'$-measurable function $T f$ such that $\int_E f \, d\mu = \int_E T f \, d\mu$ for all $E \in \Lambda'$. Using (18.4) it is clear that $T: L^1(X, \Lambda, \mu) \to L^1(X, \Lambda', \mu)$ is doubly stochastic. $T$ is called the $\Lambda'$ conditional expectation.

As a special case of this process let $X = X_0 \cup \bigcup_{i \in P} X_i$ be the union of an at most countable number of sets of positive measure and let $\Lambda'$ be the $\sigma$-sub-algebra generated by the sets $\Lambda \cap X_0$ and $\{X_i\}_{i \in P}$. Then the $\Lambda'$ conditional expectation has the form

$$T f = f_{X_0} C_{X_0} + \sum_{i \in P} \frac{1}{\mu(X_i)} (\int_{X_i} f \, d\mu) C_{X_i}.$$
Of course $T$ is of this type.

Now let $\rho$ be a saturated Fatou norm on $M(X, \mu)$ such that $L^p(\mu)$ is u.r.i. and $L^\infty(\mu) \subset L^p$, $L^{p'} \subset L^1(\mu)$, and define the u.r.i. Fatou norm $\rho_1$ on $M(X_1, \mu_1)$ by

$$\rho_1(f) = \sup \{ \int_0^a \delta |f| \delta |g| : \rho'(g) \leq 1 \}.$$

(18.11) **THEOREM.** If $T : L^{p_1}(\mu_1) \rightarrow L^1(\mu)$ is linear and $Tf < f$ for all $f \in L^{p_1}(\mu_1)$ then

(i) $T : L^{p_1}(\mu_1) \rightarrow L^1(\mu)$ and $\rho(Tf) \leq \rho_1(f)$ for all $f \in L^{p_1}$.

(ii) $T$ has a unique extension to a doubly stochastic $T : L^1(\mu_1) \rightarrow L^1(\mu)$.

(iii) $T^* : L^{p_1}(\mu) \rightarrow L^{p_1}(\mu_1)$ and $\rho_1^*(T^*g) \leq \rho'(g)$ for all $g \in L^{p_1}$.

**PROOF.** (i) We already know that $Tf < f \in L^{p_1}(\mu_1)$ implies $\rho(Tf) < \rho_1(f)$ so $Tf \in L^0(\mu)$.

(ii) This follows from (18.4) since $L^\infty(\mu_1) \subset L^{p_1}(\mu_1)$.

(iii) Let $g \in L^{p_1}(\mu)$. Since we have $\int f T^* g d\mu_1 = \int g T f d\mu$ is finite for all $f \in L^{p_1}(\mu_1)$, (14.15) implies $T^* g \in L^{p_1}(\mu_1)$. Let $g \in L^{p_1}(\mu)$.

$$\rho_1^*(T^*g) = \sup \{ \int |f| T^* g d\mu_1 : \rho_1(f) \leq 1 \}$$

$$\leq \sup \{ \int |f| T^* g d\mu_1 : \rho_1(f) \leq 1 \}$$

$$= \sup \{ \int |g| T f d\mu : \rho_1(f) \leq 1 \}$$

$$\leq \sup \{ \int |g| |f'| d\mu : \rho(f') \leq 1 \} = \rho'(g).$$
COROLLARY. Let $g_1 \in L^1(X_1, \mu_1)$. If $g_1 < g \in L^{p'}(\mu)$ then $g_1 \in L^{p'_1}$ and $\rho'_1(g_1) \leq \rho'(g)$.

PROOF. Let $g_1 < g \in L^{p'}(\mu)$. There is a d. s. $T: L^1(\mu) \to L^1(\mu_1)$ s. t. $g_1 = Tg$. Then $T^*$ extends uniquely to a d. s. map of $L^1(\mu_1) \to L^1(\mu)$ so applying (18.11) to $T^*$ we have $g_1 = (T^*)^* g \in L^{p_1}(\mu_1)$ and $\rho'_1(g_1) = \rho'_1((T^*)^* g) \leq \rho'(g)$.

REMARKS. (i) If $(X_1, \Lambda_1, \mu_1) = (X, \Lambda, \mu)$ and $\rho$ is u. r. i. then $\rho_1 = \rho$.

(ii) If $\lambda$ is a r. i. Fatou norm on $M^+[0, a]$ and $\rho(f) = \lambda(\delta_f)$ for $f \in M^+(X, \mu)$ as in §15, and $\rho_2(f) = \lambda(\delta_f)$ for $f \in M^+(X_1, \mu_1)$ then (18.11) holds with $\rho_1$ replaced by $\rho_2$.

19. Muirhead's Theorem. J. V. Ryff has given a generalization of Muirhead's Theorem for bounded measurable functions on $[0, 1]$ (see [42]). In this section we will show when this generalization is valid for arbitrary finite measure spaces.

Let $(X, \Lambda, \mu)$ and $(X_1, \Lambda_1, \mu_1)$ and $(X_2, \Lambda_2, \mu_2)$ be finite m. s. such that $\mu(X) = \mu_1(X_1) = \mu_2(X_2) = a$. If $u \in M(X_2, \mu_2)$ is positive (i. e. $u(x) > 0$ $\mu_2$-a. e.) and $f \in M(X, \mu)$ let

$$[f; u] = \int_X \log \left( \int_{X_2} u(x)^{f(y)} \, d\mu_2(x) \right) \, d\mu(y)$$

and similarly for $g \in M(X_1, \mu_1)$. 
(19.1) LEMMA. If $u \in M(X_2, \mu_2)$ is positive and $f \in M(X, u)$ then 

$$[f; u] = [\delta_f; \delta_u]$$

in the sense that both are finite and equal or both are infinite with the same sign.

PROOF. Let $p \in \mathbb{R}$ and let $\psi(t) = t^p$ for all $t \in \mathbb{R}$. Since $u \sim \delta_u$ and the measure spaces involved are finite, (3.3)(xv) says $\psi(u) \sim \psi(\delta_u)$ so 

$$\int_{X_2} \psi(u) \mu_2 = \int_{\mathcal{A}} \psi(\delta_u),$$

i.e. 

$$\int_{X_2} u^p \mu_2 = \int_{\mathcal{A}} (\delta_u)^p.$$ Let 

$$\varphi(p) = \log(\int_{X_2} u_2^p \mu_2) = \log(\int_{\mathcal{A}} (\delta_u)^p).$$

Again, since $f \sim \delta_f$ we have $\varphi(f) \sim \varphi(\delta_f)$ so 

$$[f; u] = \int_{X_2} \varphi(f) \mu_2 = \int_{\mathcal{A}} \varphi(\delta_f) = [\delta_f; \delta_u].$$

(19.2) THEOREM. Let $f \in L^\infty(X, u)$ and $g \in L^\infty(X_1, \mu_1)$.

(i) If $g < f$ then $[g; u] \leq [f; u]$ and both are finite for all positive $u \in M(X_2, \mu_2)$ such that $u^p \in L^1(X_2, \mu_2)$ whenever 

$$\text{ess. inf } (\min(f, g)) \leq p \leq \text{ess. sup } (\max(f, g))$$

[we call such $u$ admissible for $f$ and $g$.]

(ii) If $[g; u] = [f; u]$ and $u$ is admissible for $f$ and $g$ then $f \sim g$ or $u$ is constant $\mu_2$-a.e.

(iii) If $(X_2, \lambda_2, \mu_2)$ is non-atomic and $[g; u] \leq [f; u]$ for all positive $u \in M(X_2, \mu_2)$ such that $u^p \in L^1(X_2, \mu_2)$ whenever $p \in \mathbb{R}$ [we call such $u$ admissible for $L^\infty$], then $g < f$.

PROOF. Ryff has shown that this theorem is true when $X = X_1 = X_2 = [0, 1]$ with Lebesgue measure ([42, p. 596] and [43, p. 436]).

Our first step is to show that the theorem is true when $X = X_1 = X_2 = [0, a]$ with Lebesgue measure. Let $F, G \in M[0, a]$ and let $u \in M[0, a]$ be admissible for $F$ and $G$. Then
\[ [F;u] = \int_0^a \log \left( \int_0^a u(s)F(t) \, ds \right) \, dt \]
\[ = a \int_0^1 \log \left( \int_0^1 u(as)F(at) \, ds \right) \, dt + a \log a \]
\[ = a [F_1;u_1] + a \log a \]

where \( F_1(t) = F(at) \) and \( u_1(s) = u(as) \) on \([0, 1]\). Similarly
\[ [G;u] = a [G_1;u_1] + a \log a \] where \( G_1(t) = G(at) \) on \([0, 1]\). Since \( a > 0 \), (3.2) implies \( G_1 < F_1 \) iff \( G < F \), and thus it is easy to see that the theorem is true when \( X = X_1 = X_2 = [0, a] \).

Now let \( f \in L^\infty(X, \mu) \) and \( g \in L^\infty(X_1, \mu_1) \).

(i) Suppose \( g < f \) and let \( u \in M(X_2, \mu_2) \) be admissible for \( f \) and \( g \). Then \( \delta_g < \delta_f \) and it is easy to see that \( \delta_g \) is admissible for \( \delta_f \) and \( \delta_g \) so
\[ [g;u] = [\delta_g ; \delta_u] < [\delta_f ; \delta_u] = [f;u]. \]

(ii) Suppose \( [f;u] = [g;u] \). Then \( [\delta_f ; \delta_u] = [\delta_g ; \delta_u] \) so either \( \delta_f \sim \delta_g \) (so \( f \sim g \)) or \( \delta_u \) is essentially constant (so \( u \) is also).

(iii) Suppose \( (X_2, \Lambda_2, \mu_2) \) is non-atomic and \( [g;u] \leq [f;u] \) for all \( u \in M(X_2, \mu_2) \) admissible for \( L^\infty \). Let \( v \in M[0, a] \) be admissible for \( L^\infty[0, a] \). Then there is a \( u \in M(X_2, \mu_2) \) such that \( \delta_u = \delta_v \), in which case \( u \) is admissible for \( L^\infty \) and \( [\delta_g ; v] = [\delta_g ; u] \leq [f;u] = [\delta_f ; v] \), so \( [\delta_g ; v] \leq [\delta_f ; v] \) for all admissible \( v \) and hence \( \delta_g < \delta_f \) so \( g < f \).

REMARKS. I. Ryff has also shown that (iii) is not in general true if we interchange the order of integration in the definition of \( [f;u] \).

2. (iii) may also fail if \( (X_2, \Lambda_2, \mu_2) \) has atoms and \( (X, \mu) \) and \( (X_1, \mu_1) \) are more complex than \( (X_2, \mu_2) \). For example, if \( X_2 \) is an atom and both \( X \) and \( X_1 \) are not atoms, then (iii) fails. The assumption that \( (X_2, \Lambda_2, \mu_2) \)
is non-atomic is sufficient to insure that there are enough admissible $u \in M(X_2, \mu_2)$ to distinguish when $g \prec f$, no matter how complex $(X, \mu)$ and $(X_1, \mu_1)$ are.
VI. EXTREMAL AND RELATED PROBLEMS

20. Extreme, Exposed and Support Points of \( \Omega_f \). Let \( V \) be a locally convex topological vector space and let \( K \) be a convex subset of \( V \). A point \( v \in K \) is said to be an extreme point of \( K \) if \( v = \frac{1}{2}v_1 + \frac{1}{2}v_2 \) and \( v_1, v_2 \in K \) implies \( v_1 = v_2 \). \( v \) is said to be an exposed point of \( K \) if there is an \( F \in V^* \) (the continuous linear functionals on \( V \)) such that \( F(w) < F(v) \) whenever \( w \in K \), \( w \neq v \). If \( K \) is contained in a hyperplane, then a point \( v \in K \) is called a support point of \( K \) if there is an \( F \in V^* \) such that \( F(w) < F(v) \) for all \( w \in K \) and \( F(w) < F(v) \) for some \( w \in K \). It is clear that every exposed point is both an extreme point and a support point.

It would be desirable to characterize the extreme points of \( K \) in terms of the sets \( F[K] \), \( F \in V^* \). For example, it is clear that if \( F(v) \) is extreme in \( F[K] \) for all \( F \in V^* \), then \( v \) is extreme in \( K \). The converse is not true as can be seen by considering the closed unit disk. However, it is true that if \( v \) is extreme in \( K \), then there is an \( F \in V^* \) such that \( F(v) \) is extreme in \( F[K] \). For if \( v \) is extreme in \( K \), then \( v \) is a boundary point of \( K \), and since the interior of \( K \) is convex, the Hahn-Banach Separation theorem [36, p. 29] gives the required \( F \).

It is well known that \( v \in K \) is not extreme iff there is a \( 0 \neq u \in V \) such that \( v + u \) and \( v - u \in K \). Suppose \( K \) is closed and convex. It is easy to see using (17.5) that this condition becomes: \( v \in K \) is not extreme iff there is a \( 0 \neq u \in V \) such that both

\[
F(u) \leq \sup F[K] - F(v) \\
F(u) \leq F(v) - \inf F[K]
\]
for all $F \in V$. This condition has been given for $\Omega(f)$ by Luxemburg [28, p. 141].

It does not appear likely that a useful characterization of this type is possible. Let $K$ be an ice-cream cone in the plane formed by intersecting tangents to a circle. The points of tangency are extreme but not exposed, and there seems to be no way to distinguish them using closed hyperplanes from the points on the sides of the cone.

Let $(X, \Lambda, \mu)$ and $(X_1, \Lambda_1, \mu_1)$ be finite measure spaces such that $\mu(X) = \mu_1(X_1) = a$. Recall that if $f \in L^1(X_1, \mu_1)$ we let

$$\Omega_f(X, \mu) = \{ g \in L^1(X, \mu) : g < f \}$$

$$\Delta_f(X, \mu) = \{ g \in L^1(X, \mu) : g \sim f \}.$$ The reader is referred to § 17 for a detailed discussion of these sets.

The extreme, exposed and support points of $\Omega_f$ have been determined by J. V. Ryff when $X = X_1 = [0, 1]$ (see [41]). The proof that the functions in $\Delta_f$ are all extreme in $\Omega_f$ is due to J. L. Doob [41]. We present it in the following way. The proof of (20.2) is different from Doob's.

\begin{equation}
(20.1) \text{LEMMA. (J. L. Doob). (i) If } \frac{1}{2}(f_1 + f_2) \sim f_1 \sim f_2 \text{ then } f_1 f_2 \geq 0 \text{ } \mu\text{-a.e.}
\end{equation}

(ii) Let $g \in \Omega_f$. If $g = \frac{1}{2}f_1 + \frac{1}{2}f_2$ & $f_1, f_2 \in \Omega_f$ implies $f_1 \sim f_2 \sim g$ then $g$ is extreme.

\begin{proof}
(i) $f_1 \sim f_2$ implies $|f_1| \sim |f_2|$ so $\int |f_1| \, d\mu = \int |f_2| \, d\mu$. Also $|f_1 + f_2| \sim 2|f_1|$ so $\int |f_1 + f_2| \, d\mu = \int (|f_1| + |f_2|) \, d\mu$ so the triangle inequality implies $|f_1 + f_2| = |f_1| + |f_2| \mu\text{-a.e.}$ and thus $f_1 f_2 = |f_1 f_2| \geq 0 \text{ } \mu\text{-a.e.}$
\end{proof}
(ii) Suppose \( g, f_1, f_2 \in \Omega_f \) and \( g = \frac{1}{3}f_1 + \frac{2}{3}f_2 \). Then \( g \sim f_1 \sim f_2 \) and we have to show \( f_1 = f_2 \). By symmetry it suffices to show that
\[
\{ x : f_1(x) < f_2(x) \} \text{ has measure zero.}
\]
For all \( r \in \mathbb{R} \),
\[
(f_1 - r) \sim (f_2 - r) \sim (g - r) = \frac{1}{3}(f_1 - r) + \frac{2}{3}(f_2 - r)
\]
so \( (f_1 - r)(f_2 - r) \geq 0 \) \( \mu \)-a.e.

Let \( \{ r_n \} \) be an enumeration of all the rationals of \( \mathbb{R} \). Then
\[
\{ x : f_1(x) < f_2(x) \} = \bigcup_{n=1}^{\infty} \{ x : f_1(x) < r_n < f_2(x) \}
\]
\[
\subseteq \bigcup_{n=1}^{\infty} \{ x : (f_1(x) - r_n)(f_2(x) - r_n) < 0 \}
\]
and each of these sets has measure zero.

(20.2) **THEOREM.** If \( g \in \Omega_f \) and \( g \sim f \) then \( g \) is extreme.

**PROOF.** Let \( \Phi(h) = \int_0^a \int_0^t \delta_h = \int_0^a \delta_h(s)(a-s)ds \) if
\[
h \in L^1(X, \mathcal{L}) \cup L^1(X_1, \mathcal{L}_1). \quad \Phi \text{ is Shur convex and } \Phi(h_1) = \Phi(h_2) \text{ iff } h_1 \sim h_2.
\]
Suppose \( g \in L^1(X, \mathcal{L}) \) and \( g \sim f \in L^1(X_1, \mathcal{L}_1) \). If \( g = \frac{1}{3}f_1 + \frac{2}{3}f_2 \) where \( f_1, f_2 \in \Omega_f \) then \( \Phi(f) = \Phi(g) = \frac{1}{3} \Phi(f_1) + \frac{2}{3} \Phi(f_2) \leq \frac{1}{3} \Phi(f_1) + \frac{2}{3} \Phi(f) \) so
\[
\Phi(f) \leq \Phi(f_1) \leq \Phi(f)
\]
and thus \( \Phi(f) = \Phi(f_1) \) so \( f \sim f_1 \). Similarly \( f \sim f_2 \).

Hence \( g \sim f_1 \sim f_2 \) so \( g \) is extreme.

Using Theorems (5.10), (5.12) and (10.1) it is easy to see that Ryff's proofs are valid when \( (X, \Lambda, \mu) \) is non-atomic. Hence we have the following.

(20.3) **THEOREM.** If \( (X, \Lambda, \mu) \) is non-atomic and \( f \in L^1(X_1, \mathcal{L}_1) \) then the set of extreme points and the set of exposed points of \( \Omega_f \) are identical with the set \( \Delta_f \). A function \( g \in \Omega_f \) is a support point of
If \((X, \lambda, \mu)\) is not adequate, then there will be a function \(f \in L^1(X, \mu)\) such that \(\Omega(f) = \{g \in L^1(\mu): g < f\}\) is not the closed convex hull of \(\Delta(f) = \{g \in L^1(\mu): g \sim f\}\). If \(f_0 \in L^1(X, \mu_1)\) such that \(f_0 \sim f\) then \(\Omega_{f_0}\) will not be the closed convex hull of \(\Delta_{f_0}\), and hence \(\Delta_{f_0}\) cannot contain all the extreme points of \(\Omega_{f_0}\).

It is a good conjecture that every extreme point of \(\Omega_f\) has the form \(T_{\mu} f'\) where \(f' \in L^1(X, \mu')\) and \(f' \sim f\), and that every function in \(\Omega_f\) equimeasurable with an extreme point is an extreme point. For example, if \(f\) is extreme and either (i) \(g \circ \sigma = f\) where \(\sigma: X \to X\) is measure preserving or (ii) \(g \upharpoonright X_0 = f \upharpoonright X_0\) and \(g \upharpoonright A \sim f \upharpoonright A\) (where \(X_0\) is the non-atomic part and \(A\) is the union of the atoms of \(X\)) then \(g\) is extreme.

The following example shows that not every function \(T_{\mu} f', f' \in L^1(X, \mu')\), \(f' \sim f\) is an extreme point. Let \(X\) be the union of two atoms \(A\) and \(B\) with \(\mu(A) < \mu(B)\). Then every \(g \in M(X, \mu)\) has the form \(g = xC_A + yC_B\). Let \(f = 2C_A + C_B\). Then \(g \in \Omega(f)\) iff \(x \leq 2, y \mu(B) \leq 2\mu(A) + (\mu(B) - \mu(A)) = \mu(A) + \mu(B)\) and \(x\mu(A) + y\mu(B) = 2\mu(A) + \mu(B)\). If we define \(\varphi: M(X, \mu) \to \mathbb{R}^2\) by \(\varphi(xC_A + yC_B) = (x, y)\) then \(\varphi\) is linear, 1:1, and onto the line segment joining the points \((2, 1)\) and \((1, 1 + \frac{\mu(A)}{\mu(B)} - 1)\). Hence the extreme points of \(\Omega(f)\) are \(f = 2C_A + C_B\) and \(g = C_A + (1 + \frac{\mu(A)}{\mu(B)}) C_B\). Observe that \(\Delta(f) = \{f\}, \Delta(g) = \{g\}\), and \(g = T_{\mu} f'\) where \(f' = C[0, \mu(B)][+2C[-\mu(B), \mu(X)]\sim f\). It is clear in this case that \(\{T_{\mu} f': f' \sim f, f' \in L^1(X, \mu')\}\) = \(\Omega(f)\).
The following example shows how extreme points not in \( \Delta(f) \) may arise when \( X \) has an atom \( A \) and a non-atomic part \( X_o \) of positive measure. Let \( B \subset X_o \) be s. t. \( 0 < \mu(B) \leq \mu(A) \). Using (5.10) there is an \( f \) on \( B \) s. t. \( \delta_f|_B(t) = \mu(B) - t \) on \([0, \mu(B)]\). Define \( f \) to be 0 on the rest of \( X \). Let \( I[\alpha, \beta] \) be the interval of \((X^#, \Lambda^#, \mu^#)\) corresponding to \( A \) as in §9. Define \( g^f \in L^1(X^#, \mu^#) \) by \( g^f(t) = \alpha + \mu(B) - t \) if \( \alpha < t < \alpha + \mu(B) \) and \( g^f = 0 \) elsewhere. Then \( g^f \sim f \) so \( g = \frac{1}{2} \mu(B)C_A = T_\mu g^f < f \) and \( g \not\leq f \). To show that \( g \) is extreme, suppose \( g \pm u < f \) (here \( \pm \) means + and -) where \( u \in M(X, \mu) \).

Then \( f \pm u \) d\( \mu = 0 \) and since \( f \geq 0 \), we have \( g \pm u \geq 0 \). Thus

\[ 0 \leq g \pm u \| A^c = \pm u \| A^c \quad \text{so} \quad u \| A^c = 0. \]

Then \( 0 = f \pm u \) d\( \mu = (u \| A) \mu(A) \)

implies \( u \| A = 0 \) so \( u = 0 \). Hence \( g \) is extreme.

In each of the above examples extreme points were obtained in the following way. Write \( X = X_o \cup \bigcup X \) where \( P = \{1, \ldots, n\} \) or \( P = \{1, 2, 3, \ldots \} \), \( X_o \) is the non-atomic part of \( X \), and \( X, i \in P \), are the atoms of \( X \). For the intervals \( I[a_i, b_i] \) defining \( X^# \) take

\[ I[a_i, b_i] = [\mu(X_o) + \sum_{k=1}^{i-1} \mu(X_k), \mu(X_o) + \sum_{k=1}^{i} \mu(X_k) \] .

Let \( f \in L^1(X, \mu) \). For each partition \( \pi = \{[\alpha_i, \beta_i] \}_{i \in P} \) of the interval \([0, \mu(X)] \) such that \( \beta_i - \alpha_i = \mu(X_i) \) define

\[ f''_\pi = \delta_f|_{X_o} (t-\alpha_o) C_{[\alpha_o, \beta_o]} + \sum_{i \in P} (f|_{X_i}) C_{[\alpha_i, \beta_i]} \]

Now for each measure preserving \( \sigma X_o \to [0, \mu(X_o)] \) let

\[ f'_{\pi, \sigma} = (f''_\pi \circ \sigma) C_{X_o} + f''_\pi C_{X-X_o} . \]
Then $f'_{\pi, \sigma} \in L^1(X^\#, \mu^\#)$, $f'_{\pi, \sigma} \sim f$, and it is a good conjecture that $T \mu f'_{\pi, \sigma}$ is extreme in $\Omega_f$.

21. **Permutator Transformations.** Because of the importance of the sets $\Delta(f) = \{ h : h \sim f \}$ it seems natural to investigate the necessarily doubly stochastic operators $T$ such that $Tf \sim f$ for all $f$. Such an operator is called a permutator, because for discrete measures they correspond to the permutation matrices.

Let $(X, \lambda, \mu)$ be a finite m.s. with $\mu(X) = a$. If $A, B \in \Lambda$ we write $A = B [\mu]$ to mean $C_A = C_B \mu$-a.e., i.e., $\mu(A \Delta B) = 0$ where $A \Delta B = A - B \cup B - A$. We also write $A \subset B [\mu]$ to mean $C_A \leq C_B \mu$ -a.e., i.e., $\mu(A - B) = 0$. Note that $A \subset B [\mu]$ implies $\mu(A) \leq \mu(B)$ and $A = B [\mu]$ implies $\mu(A) = \mu(B)$. If $A \subset B [\mu]$, then $A = B [\mu]$ iff $\mu(A) = \mu(B)$.

Now let $(X_1, \lambda_1, \mu_1)$ be a finite m.s. with $\mu_1(X_1) = \mu(X) = a$.

(21.1) **DEFINITION.** A mapping $\phi: \Lambda_1 \rightarrow \Lambda$ is said to be a homomorphism of $\Lambda_1$ into $\Lambda$ if it satisfies

(i) $\mu(\phi(A)) = \mu_1(A)$ for all $A \in \Lambda_1$

(ii) $\phi(A \cap B) = \phi(A) \cap \phi(B) [\mu]$ for all $A, B \in \Lambda_1$

(iii) $\phi(A \cup B) = \phi(A) \cup \phi(B) [\mu]$ whenever $A \cap B = \emptyset [\mu_1]$.

We call $\phi$ an isomorphism if in addition its range is $\Lambda [\mu]$, i.e., for every $E \in \Lambda$ there is an $A \in \Lambda_1$ such that $\phi(A) = E [\mu]$. In this case there is an isomorphism $\phi^{-1}: \Lambda \rightarrow \Lambda_1$ such that $\phi \circ \phi^{-1}$ is the identity $[\mu]$ on $\Lambda$. 
(21.2) PROPOSITION. If \( \varphi : \Lambda \to \Lambda \) is a homomorphism, then

(i) \( \varphi \) preserves disjoint sets, i.e., \( A \cap B = \emptyset [\mu] \) implies

\[ \varphi(A) \cap \varphi(B) = \emptyset [\mu]. \]

(ii) \( \varphi \) is monotone, i.e., \( A \subseteq B [\mu] \) implies \( \varphi(A) \subseteq \varphi(B) [\mu] \).

(iii) \( \varphi \) preserves differences, i.e., \( \varphi(B - A) = \varphi(B) - \varphi(A) [\mu] \) whenever

\( A \subseteq B [\mu] \).

(iv) \( \varphi \) is countably additive, i.e., if \( \{ A_i \}_{i=1}^{\infty} \subseteq \Lambda \) are \( \text{pwd} [\mu] \) then

\[ \varphi(\bigcup A_i) = \bigcup \varphi(A_i) [\mu]. \]

(v) \( \varphi \) is \( 1:1 \), i.e., \( \varphi(A) = \varphi(B) [\mu] \) implies \( A = B [\mu] \).

PROOF. (i) \( A \cap B = \emptyset [\mu] \) iff \( \mu_1(A \cap B) = 0 \) so

\[ \mu(\varphi(A) \cap \varphi(B)) = \mu(\varphi(A \cap B)) = \mu_1(A \cap B) = 0 \] implies \( \varphi(A) \cap \varphi(B) = \emptyset [\mu] \).

(ii) & (iii) If \( A \subseteq B [\mu] \) then \( B = B - A \cup A [\mu] \) is a pairwise disjoint union so \( \varphi(B) = \varphi(B - A) \cup \varphi(A) [\mu] \) is a \( \text{pwd} \) union \( [\mu] \) and hence \( \varphi(A) \subseteq \varphi(B) [\mu] \) and \( \varphi(B) - \varphi(A) = \varphi(B - A) [\mu] \)

(iv) Let \( \{ A_i \} \subseteq \Lambda \) be \( \text{pwd} [\mu] \). Now \( A_j \subseteq \bigcup A_j \) for all \( j \) implies

\( \varphi(A_j) \subseteq \varphi(\bigcup A_j) \) for all \( j \) and hence \( \varphi(A_j) \subseteq \varphi(\bigcup A_j) \). Since

\[ \mu(\varphi(\bigcup A_j)) = \mu_1(\bigcup A_j) = \lim_{n \to \infty} \mu_1(\bigcup A_j) = \lim_{n \to \infty} \mu(\varphi(\bigcup A_j)) \]

\[ = \lim_{n \to \infty} \mu(\bigcup \varphi(A_j)) = \mu(\bigcup \varphi(A_j)) \] we have \( \varphi(\bigcup A_j) = \bigcup \varphi(A_j) [\mu] \).

(v) Suppose \( \varphi(A) = \varphi(B) [\mu] \). Let \( E = A \) or \( B \).

\[ \mu_1(E - A \cap B) = \mu(\varphi(E - A \cap B)) = \mu(\varphi(E) - \varphi(A \cap B)) = \mu(\varphi(E) - \varphi(A) \cap \varphi(B)) = 0 \]

so \( A = B [\mu] \).

If \( (X, \Lambda, \mu) \) is a product of a possibly uncountable number of copies of \([0, 1]\) with Lebesgue measure, then every homomorphism
$\phi: \Lambda \to \Lambda$ is induced by a measure preserving $\sigma: X \to X$ in the sense that $\phi(E) = \sigma^{-1}(E)$ for all $E \in \Lambda$. If $\sigma: [0, a] \to [0, a]$ is measure preserving then $\phi(E) = \sigma^{-1}(E)$ is a homomorphism of the Borel subsets of $[0, a]$ into $\Lambda$. Recall that in this case $Tf = f \circ \sigma$ defines a doubly stochastic transformation of $L^1([0, a])$ into $L^1(X, \mu)$.

(21.3) **PROPOSITION.** To each homomorphism $\phi: \Lambda \to \Lambda$ there corresponds a unique linear transformation $T_\phi: L^1(X_1, \mu_1) \to L^1(X, \mu)$ such that $T_\phi C_E = C_{\phi(E)}$ for all $E \in \Lambda$. In addition we have the following:

(i) $T_\phi f \sim f$ for all $f \in L^1(X_1, \mu_1)$

(ii) $T_\phi(fh) = (T_\phi f)(T_\phi h)$ whenever $f, h \in L^1(X_1, \mu_1)$

(iii) If $\phi': \Lambda \to \Lambda$ and $T_{\phi'} = T_\phi$, then $\phi = \phi' [\mu]$.

**PROOF.** If $f = \sum_{i=1}^{n} \alpha_i C_{A_i}$ is a simple function with $\{A_i\}$ a partition of $X_1 [\mu_1]$ we define

$$T_\phi f = \sum_{i=1}^{n} \alpha_i C_{\phi(A_i)}.$$  

It follows from (21.2) that $\{\phi(A_i)\}_{i=1}^{n}$ is a partition of $X$, so $Tf$ and $f$ are simple functions taking the same values on sets of equal measure and hence $Tf \sim f$.

To see that $T_\phi f$ is well defined, suppose also $f = \sum_{i=1}^{m} \beta_j C_{B_j}$, where $\{B_j\}_{j=1}^{m}$ is a partition of $X_1$. Then

$$f = \sum_{i} \sum_{j} \alpha_i C_{A_i \cap B_j} = \sum_{i} \sum_{j} \beta_j C_{A_i \cap B_j}$$

so $\alpha_i = \beta_j$ whenever $A_i \cap B_j \neq \emptyset [\mu_1]$ and thus
\[ \alpha_i = \beta_j \text{ whenever } \emptyset \neq (A_i \cap B_j), \quad \text{Hence} \]

\[
\sum_i \sum_j \alpha_i \ C_\phi(A_i \cap B_j) = \sum_i \sum_j \beta_j \ C_\phi(A_i \cap B_j)
\]

\[
\sum_i \sum_j \alpha_i \ C_\psi(A_i) \cap \emptyset(B_j) = \sum_i \sum_j \beta_j \ C_\psi(A_i) \cap \emptyset(B_j)
\]

\[
\sum_i \alpha_i \ C_\psi(A_i) = \sum_i \beta_i \ C_\psi(B_j)
\]

It is easy to see that \( T_\phi(rf) = rT_\phi f \) for all \( r \in \mathbb{R} \). To show that \( T_\phi \) is linear, let \( g = \sum_{j=1}^m \beta_j C_{B_j} \) where \( \{B_j\} \) is a partition of \( X_1 \). Then

\[
f+g = \sum_{j=1}^m (\alpha_j + \beta_j) C_{A_j \cap B_j} \quad \text{and} \quad \{A_j \cap B_j\} \text{ is a partition of } X_1
\]

so

\[
T_\phi(f+g) = \sum \sum (\alpha_i + \beta_j) C_\phi(A_i \cap B_j) = T_\phi f + T_\phi g
\]

since \( \{\emptyset(A_i) \cap \emptyset(B_j)\} \) is a partition of \( X [\mu] \). Similarly, \( T_\phi(fh) = (T_\phi f)(T_\phi h) \) for all simple functions \( f, h \).

It follows from (18.4) that \( T_\phi \) extends uniquely to a doubly stochastic transformation of \( L^1(\mu) \rightarrow L^1(\mu) \). Since the simple functions are dense in \( L^1(\mu) \) and \( T_\phi \) is continuous in the \( L^\infty \) and \( L^1 \) norms, it follows from (3.3) (xiii) that \( T_\phi f \sim f \) for all \( f \in L^1(\mu) \). Similarly,

\[ T_\phi(fh) = (T_\phi f)(T_\phi h) \text{ whenever } f, h, fh \in L^1(\mu) \).

If \( T: L^1(\mu) \rightarrow L^1(\mu) \) is also linear such that \( TC_{E_i} = C_{T \phi}(E) \) for all \( E \in \Lambda_1 \) and \( f = \sum_{i=1}^n \alpha_i C_{A_i} \) where \( \{A_i\} \) is a partition of \( X_1 \) then

\[ Tf = \sum \alpha_i TC_{A_i} = \sum \alpha_i C_{T \phi}(A_i) = T_\phi f. \]

It follows that \( T \) extends uniquely to a d. s. transformation of \( L^1(\mu) \rightarrow L^1(\mu) \). Since \( T = T_\phi \) on the simple functions, \( T = T_\phi \) on \( L^1(\mu) \).

Finally, if \( \phi': \Lambda_1 \rightarrow \Lambda \) is a homomorphism for which \( T_{\phi'} = T_\psi \) then

\[ C_{\phi'}(E) = T_{\phi'} C_{E} = T_\psi C_{E} = C_\phi(E) \text{ for all } E \in \Lambda_1 \text{ so } \phi' = \phi [\mu]. \]
It is easy to see that if \((X_2, \Lambda_2, \mu_2)\) is a finite m.s. with 
\[ \mu_2(X_2) = \mu(X) = a \quad \text{and} \quad \Psi: \Lambda_2 \rightarrow \Lambda_1 \text{ is a homomorphism}, \]
then \(\hat{\Phi} \circ \Psi: \Lambda_2 \rightarrow \Lambda\) is a homomorphism and \(T_\hat{\Phi} \circ \Psi = T_\Psi \circ T_\hat{\Phi}\).

Let \(T: L^2(\mu_1) \rightarrow L^2(\mu)\) be linear. \(T\) is called multiplicative if 
\[ T(fh) = (Tf)(Th) \quad \text{whenever} \quad f, h \in L^2(\mu_1). \]
\(T\) is called isometric if 
\[ \int Tf \, d\mu = \int f \, d\mu_1 \quad \text{whenever} \quad f, h \in L^2(\mu_1). \]
\(T\) is called unitary if both \(T\) and \(T^*\) are isometric.

Observe that if \(T\) is multiplicative and isometric then (18.4) implies that \(T\) has a unique extension to a d.s. \(T: L^1(\mu_1) \rightarrow L^1(\mu)\).

If \(T: L^1(\mu_1) \rightarrow L^1(\mu)\) is d.s., then (17.13) (or Remark (ii) after (18.12)) implies that \(T: L^2(\mu_1) \rightarrow L^2(\mu)\). It is easy to see that it is multiplicative iff \(T(fh) = (Tf)(Th)\) whenever \(f, h, fh \in L^1(\mu_1)\) iff 
\[ T(fh) = (Tf)(Th) \quad \text{whenever} \quad f, h \in L^\infty(\mu_1); \]
similarly for \(T\) isometric.

(21.4) THEOREM. Let \(T: L^1(\mu_1) \rightarrow L^1(\mu)\) be linear. Then the following are equivalent:

(i) \(Tf \sim f \quad \text{for all} \quad f \in L^1(\mu_1)\).

(ii) \(T\) is induced by a homomorphism \(\hat{\Phi}: \Lambda_1 \rightarrow \Lambda\).

(iii) \(T\) is d.s. and multiplicative.

(iv) \(T\) is d.s. and \(T^*T\) is the identity function on \(L^1(\mu_1)\).

(v) \(T\) is d.s. and isometric.

PROOF. We have already proved (ii) \(\Rightarrow\) (i) in (21.3). (i) \(\Rightarrow\) (ii):

If \(E \in \Lambda_1\) then \(TC_E \sim C_E\) so there is a \(\hat{\Phi}(E) \in \Lambda\) such that 
\[ TC_E = C_{\hat{\Phi}(E)} \]
and \(\mu(\hat{\Phi}(E)) = \mu_1(E)\). Let \(A, B \in \Lambda_1\) and 
\[ f = C_A + C_B = C_{A-A \cap B} + C_{B-A \cap B} + 2 \, C_{A \cap B} \].
Then \(Tf = C_{\hat{\Phi}(A)} + C_{\hat{\Phi}(B)} = C_{\hat{\Phi}(A-A \cap B)} + C_{\hat{\Phi}(B-A \cap B)} + 2 \, C_{\hat{\Phi}(A \cap B)}\).
so \( \Phi(A \cap B) \subset \Phi(A) \cap \Phi(B) \) and thus to show equality it suffices to show they have equal measure. Now \( \Phi(A) \cap \Phi(B) = \{ T f = 2 \} \) and \( A \cap B = \{ f = 2 \} \) so since \( T f \sim f \), \( \mu(\Phi(A \cap B)) = \mu_1(A \cap B) = \mu(\Phi(A) \cap \Phi(B)) \) and thus
\[
\Phi(A) \cap \Phi(B) = \Phi(A) \cap \Phi(B) \{ \mu \}.
\]

If \( A \cap B = \emptyset \{ \mu_1 \} \) then \( C_{A \cup B} = C_A + C_B \) \( \mu \)-a.e. so applying \( T \) we get \( C_{\Phi(A \cup B)} = C_{\Phi(A)} + C_{\Phi(B)} = C_{\Phi(A) \cup \Phi(B)} \) since \( \mu(\Phi(A) \cap \Phi(B)) = \mu_1(A \cap B) = 0 \).

Hence \( \Phi(A \cup B) = \Phi(A) \cup \Phi(B) \).

(ii) \( \Rightarrow \) (iii). This is (21.3).

(iii) \( \Rightarrow \) (iv). Let \( f \in L^1(\mu_1) \). For all \( g \in L^\infty(\mu_1) \) we have \( fg \in L^1(\mu_1) \) so
\[
\int fg \, d\mu_1 = \int T(\Phi_g) \, d\mu = \int T f T_g \, d\mu = \int g T^* T f \, d\mu_1
\]
and thus \( f = T^* T f \).

(iv) \( \Rightarrow \) (v). Let \( f, h, fh \in L^1(\mu_1) \). Then \( \int fh \, d\mu_1 = \int h T^* T f \, d\mu_1 = \int T f \, T h \, d\mu \).

(v) \( \Rightarrow \) (ii) [3, p. 22]. Let \( A, B \in \Lambda_1 \). \( \int T C_A T C_B \, d\mu = \int C_A C_B \, d\mu_1 = \int T C_{A \cap B} \, d\mu_1 \) since \( T \) is d.s., \( 0 \leq T C_A \leq 1 \) so
\[
0 \leq (T C_A)^2 \leq T C_A \leq 1,
\]
and for \( A = B \), \( \int T C_A - (T C_A)^2 \, d\mu = 0 \) so \( (T C_A)^2 = T C_A \) and thus \( T C_A \) takes only the values 0 and 1. Letting \( \Phi(A) = \{ T C_A = 1 \} \) we have \( T C_A = C_\Phi(A) \). Now \( T C_{A \cap B} \leq \min\{ T C_A, T C_B \} \) so
\[
0 \leq T C_{A \cap B} = (T C_{A \cap B})^2 \leq (T C_A)(T C_B) \text{ and since } \int T C_{A \cap B} \, d\mu = \int T C_A T C_B \, d\mu \text{ we have } T C_{A \cap B} = T C_A T C_B \text{ so } \Phi(A \cap B) = \Phi(A) \cap \Phi(B).
\]

When \( A \cap B = \emptyset \), apply \( T \) to \( C_{A \cup B} = C_A + C_B \) to deduce \( \Phi(A \cup B) = \Phi(A) \cup \Phi(B) \).

Finally, \( \mu(\Phi(A)) = \int T C_A \, d\mu = \int C_A \, d\mu_1 = \mu_1(A) \) since \( T \) is d.s.
(21.5) COROLLARY. Let $\Phi: \Lambda_1 \rightarrow \Lambda$ be a homorphism. Then

(i) $T_\Phi$ is 1:1.

(ii) $T_\Phi^*$ maps $L^\infty(\mu)$ onto $L^\infty(\mu_1)$ and $L^1(\mu)$ onto $L^1(\mu_1)$.

PROOF. $T_\Phi^* T_\Phi$ is the identity mapping on $L^1(\mu_1)$ and $T_\Phi: L^\infty(\mu_1) \rightarrow L^\infty(\mu)$.

(21.6) THEOREM. Let $\Phi: \Lambda_1 \rightarrow \Lambda$ be a homorphism. Then the following are equivalent.

(i) $T_\Phi^*$ is 1:1.

(ii) $T_\Phi T_\Phi^*$ is the identity mapping on $L^1(\mu)$.

(iii) The range of $T_\Phi$ is all of $L^1(\mu)$.

(iv) $\Phi$ is an isomorphism.

(v) $T_\Phi^*$ is induced by a homorphism of $\Lambda \rightarrow \Lambda_1$.

(vi) $T_\Phi$ is unitary.

PROOF. (i) $\Rightarrow$ (ii). Let $f \in L^1(\mu_1)$. Then $T_\Phi^* (T_\Phi T_\Phi^* f) = T_\Phi^* f$ so $T_\Phi T_\Phi^* f = f$.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (iv). Let $A \in \Lambda$. Then there is an $f_A \in L^1(\mu_1)$ such that $T_\Phi f_A = C_A$. For all $g \in L^\infty(\mu)$ we have

$$\int_A f_A^2 T_\Phi^* g \, d\mu_1 = \int g T_\Phi f_A^2 \, d\mu$$

$$= \int g(T_\Phi f_A) (T_\Phi f_A) \, d\mu = \int g C_A \, d\mu = \int g T_\Phi f_A \, d\mu = \int f_A T_\Phi^* g \, d\mu_1 .$$

Since $T_\Phi^*$ maps $L^\infty(\mu)$ onto $L^\infty(\mu_1)$ (this is implied by $(21.5)(ii)$) we have $f_A^2 = f_A$ so $f_A = C_B$ where $B = \{ f_A = 1 \}$. Then $C_{\Phi(B)} = T_\Phi f_A = C_A$ so $\Phi(B) = A \{ \mu \}$. 
(iv) = (v). $\psi^{-1} : \Lambda \rightarrow \Lambda_1$ is a homomorphism, and 

$$T_{\psi}^* = T_{\psi}(T_{\phi}^* T_{\phi}^{-1}) = (T_{\phi}^* T_{\phi}) T_{\phi}^{-1} = T_{\phi}^{-1}.$$ 

(v) = (vi). If $T_{\psi}^* = T_{\psi}$ then $T_{\psi}^*$ is isometric so $T_{\phi}$ is unitary.

(vi) = (i). If $T_{\psi}$ is unitary, then $T_{\psi}^*$ is isometric and hence induced by a homomorphism, and it follows from (21.5) that it is 1:1.

REMARKS. 1. Let $\sigma : X \rightarrow X_1$ be measure preserving and let $T_\sigma$ be defined on $M(X_1, \mu_1)$ by $T_\sigma f = f(\sigma)$. Then $T_\sigma C_E = C_\sigma^{-1}(E)$ for all $E \in \Lambda_1$ so $T_\sigma = T_{\varphi}$ where $\varphi(E) = \sigma^{-1}(E)$.

2. The relationship between multiplicative unitary operators (which are necessarily d.s.) and the operators $T_\sigma$ where $\sigma : X \rightarrow X$ is measure preserving, was studied as early as 1932 by von Neumann [33, p. 618], who assumed that $X$ is a complete separable metric space with a finite Borel measure $\mu$ such that spheres have positive measure and every measurable set is contained in a $G_6$ with the same measure.

3. With $(X, \Lambda, \mu)$ and $\sigma$ as in 2, Paul R. Halmos has used the operators $T_\sigma$ to find necessary and sufficient conditions for the existence of a square root of $\sigma$ [13].

4. It follows from (20.2) that the operators $T_{\psi}$ are extreme points of $\mathfrak{g}(X_1, X)$ (another proof can be gleaned from [34, p. 269, Theorem 1.4]). It is easy to see that $T \in \mathfrak{g}(X_1, X)$ is extreme iff $T^*$ is extreme in $\mathfrak{g}(X, X_1)$. Hence the operators $T_{\psi}^*$ such that $\psi : \Lambda \rightarrow \Lambda_1$ is a homomorphism are also extreme in $\mathfrak{g}(X_1, X)$. A complete characterization of the extreme points of $\mathfrak{g}(X_1, X)$ does not seem to be known.
22. The Weak Closure of \( \{g : g \sim f\} \). As always let \((X, \Lambda, \mu)\) and 
\((X_1, \Lambda_1, \mu_1)\) be finite measure spaces with \(\mu(X) = \mu_1(X_1) = a\). Recall 
that if \(f \in L^1(X_1, \mu_1)\) then \(\Omega_f(X, \mu) = \{g \in L^1(X, \mu): g < f\}\) and 
\(\Delta_f(X, \mu) = \{g \in L^1(\mu): g \sim f\}\). The problem is to determine the \(\sigma(L^1, L^\infty)\) closure of \(\Delta_f\). The case \(\Delta_f = \emptyset\) is easy, while if \(\Delta_f \neq \emptyset\) and \(f_0 \in \Delta_f\) 
then \(\Omega_f = \Omega(f_0)\) and \(\Delta_f = \Delta(f_0)\). Hence from the beginning we work only 
with the sets \(\Omega(f)\) and \(\Delta(f)\) where \(f \in L^1(X, \mu)\).

J. V. Ryff has shown the following result which we state as a 
lemma. For the proof see [43, p. 432].

(22.1) LEMMA. If \(F \in L^1[0, 1]\) then \(\Omega(F) = \{G \in L^1[0, 1]: G < F\}\) 
is the \(\sigma(L^1, L^\infty)\) closure of \(\Delta(F) = \{G \in L^1[0, 1]: G \sim F\}\).

(22.2) THEOREM. \(\Omega(f)\) is the \(\sigma(L^1, L^\infty)\) closure of \(\Delta(f)\) for all 
f \(\in L^1(X, \mu)\) iff \((X, \Lambda, \mu)\) is non-atomic or \(X\) is an atom.

PROOF. Suppose \((X, \Lambda, \mu)\) is non-atomic. Let \(g \in \Omega(f)\) and let 
\(\sigma: X \rightarrow [0, a]\) be measure preserving such that \(\delta_g \circ \sigma = g\). Let 
\(F(t) = \delta_f(at)\) and \(G(t) = \delta_g(at)\) on \([0, 1]\). Then \(G < F\) so there is a 
net \(\{H_\alpha\} \subset \Delta(F)\) such that \(H_\alpha \rightarrow G\) in the \(\sigma(L^1[0, 1], L^\infty[0, 1])\) topology.

Letting \(h_\alpha(t) = H_\alpha(t/a)\) on \([0, a]\) we have \(h_\alpha \sim \delta_g\), and for all \(v \in L^\infty[0, a]\),
\[
\int_0^a h_\alpha v dt = a \int_0^1 H_\alpha(t) v(at) dt \rightarrow a \int_0^1 G(t) v(at) dt = a \int_0^a \delta_g v dt, \text{ i.e.,}
\]
\(h_\alpha \rightarrow \delta_g\) in the \(\sigma(L^1[0, a], L^\infty[0, a])\) topology. Since \(T_\sigma\) is weakly 
continuous, \(T_\sigma h_\alpha \rightarrow T_\sigma \delta_g = g\) and \(T_\sigma h_\alpha \in \Delta(f)\). Hence \(\Delta(f)\) is weakly 
dense in \(\Omega(f)\).
If \( X \) is an atom, then \( \Delta(f) = \{ f \} = \Omega(f) \).

For the converse, suppose \( X \) is not an atom and \( (X, \Lambda, \mu) \) is not non-atomic. Then \( X = A_1 \cup A_2 \) where \( A_1 \) and \( A_2 \) are disjoint sets of positive measure, and \( X \) has an atom \( B \). Let \( f = 2C_{A_1} + C_{A_2} \), let \( g_0 = \frac{1}{\lambda} \int f \, d\mu = 1 + \mu(A_1)/\mu(X) \) so \( 1 < g_0 < 2 \), and let \( g = g_0 C_X \), so \( g \in \Omega(f) \). If \( h \in \Delta(f) \) then \( h = 2C_{B_1} + C_{B_2} \) where \( \mu(B_i) = \mu(A_i) i = 1, 2 \), so \( \int (g-h)C_B \, d\mu = g_0 \mu(B) - 2\mu(B_1 \cap B) - \mu(B_2 \cap B) \) and \( \mu(B \cap B_i) = 0 \) or \( \mu(B) \) since \( B \) is an atom. Hence for \( \varepsilon = \mu(B) \min(g_0-1, 2-g_0) \) there is no \( h \in \Delta(f) \) such that \( \int (g-h)C_B \, d\mu < \varepsilon \) so \( \Delta(f) \) is not dense in \( \Omega(f) \).

(22.3) COROLLARY. For any finite m. s. \( (X, \Lambda, \mu) \) and \( f \in L^1(X, \mu) \) we have that \( \Omega(f) \) is the \( \sigma(L^1, L^\infty) \) closure of \( \{ T_{\mu}f': f' \in L^1(X, \mu^\#) \) and \( f' \sim f \} \).

PROOF. Let \( g \in \Omega(f) \). Since \( (X^\#, \mu^\#) \) is non-atomic, there is a net \( \{ f'_\alpha \} \subset L^1(X^\#, \mu^\#) \) such that \( f'_\alpha \to g \) in \( \sigma(L^1(\mu^\#), L^\infty(\mu^\#)) \). Since \( T_{\mu} \) is weakly continuous (see \S 17 or use (9.2)), \( T_{\mu}f'_\alpha \to T_{\mu}g = g \) in \( \sigma(L^1(\mu), L^\infty(\mu)) \).

If \( (X, \Lambda, \mu) \) consists only of atoms of equal measure then \( \Delta(f) \) is finite and hence weakly closed. We will now determine the weak closure of \( \Delta(f) \) when \( X \) consists only of atoms, or \( X \) has only finitely many atoms, or when \( X \) is separable.

Let \( X_0 \) be the non-atomic part of \( X \), let \( \{ A_n \}_{n \in \mathbb{P}} \) be the atoms of \( X \) and let \( A = \bigcup_{n \in \mathbb{P}} A_n \); also let \( a_0 = \mu(X_0) \) (see \S 9).
(22.4) DEFINITION. If \( f \in L^1(X, \mu) \) let \( Z(f) \) be the set to which \( g \) belongs iff there is an \( h \sim f \) such that \( g|_{X_0} < h|_{X_0} \) and \( g|A = h|A \).

(22.5) LEMMA. For all \( f \in L^1 \) we have \( \Delta(f) \subset Z(f) \subset \overline{\Delta(f)} \).

PROOF. \( \Delta \subset Z \) is easy. For the other inclusion, let \( g \in Z \) so there is an \( h \sim f \) such that \( g|_{X_0} < h|_{X_0} \) and \( g|A = h|A \). Since \( X_0 \) is non-atomic there is a net \( \{h_\alpha\} \subset L^1(X_0, \mu) \) with \( h_\alpha \sim h|_{X_0} \) and \( h_\alpha \rightarrow g|_{X_0} \) weakly. Extend each \( h_\alpha \) to \( X \) by \( h_\alpha|_A = h|A \). Then \( h_\alpha \sim h \sim f \) (see (3.3)(x)) and for each \( v \in L^\infty(X, \mu) \) we have \( v|_{X_0} \in L^\infty(X_0, \mu) \) so \( \int_{X_0} h_\alpha v \, d\mu \rightarrow \int_{X_0} g v \, d\mu \), but since \( \int_A h_\alpha v \, d\mu = \int_A g v \, d\mu \) we have finally that \( \int_X h_\alpha v \, d\mu \rightarrow \int_X g v \, d\mu \), i.e., \( h_\alpha \rightarrow g \) weakly. Hence \( g \in \overline{\Delta} \).

(22.6) THEOREM. If \( (X, \Lambda, \mu) \) consists only of atoms, then \( \overline{\Delta(f)} = Z(f) \) for all \( f \in L^1(X, \mu) \).

PROOF. We have only to show that \( \overline{\Delta(f)} \subset Z(f) \). Let \( g \in \overline{\Delta(f)} \).

Then there is a net \( \{h_\alpha\} \subset \Delta(f) \) with \( h_\alpha \rightarrow g \) weakly. Let \( B \) be an atom. Now \( \mu(B) > \mu(h_\alpha^{-1}(h_\beta|B)) = \mu(f^{-1}(h_\alpha|B)) \) and \( f^{-1}(h_\alpha|B) \cap f^{-1}(h_\beta|B) = \emptyset \) whenever \( h_\alpha|B \neq h_\beta|B \) so \( \mu(X) < \infty \) implies there are only finitely many different values \( h_\alpha|B \). But \( h_\alpha|B \rightarrow g|B \), so for some \( \alpha_0 \), \( \alpha \geq \alpha_0 \) implies \( h_\alpha|B = g|B \). Hence there is an increasing sequence \( \alpha_n \) such that \( \alpha \geq \alpha_n \) implies \( h_\alpha|A_k = g|A_k \), \( k = 1, \ldots, n \). Then \( \|h_\alpha - g\|_1 \rightarrow 0 \) so \( g \sim f \) and hence \( g \in \Delta(f) = Z(f) \) in this case.

I can now prove \( \overline{\Delta(f)} = Z(f) \) in general.
(22.7) **THEOREM.** If \((X, \lambda, \mu)\) has only a finite number of atoms, then \(\Delta(f) = Z(f)\).

**PROOF.** We have only to show that \(\Delta(f) \subset Z(f)\). It is easy to see that the condition \(h_1|_A = h_2|_A\) defines an equivalence relation on \(\Delta(f)\). Since there are only finitely many atoms, there are only finitely many equivalence classes, \(H_1, \ldots, H_n\) say. \(\Delta(f) = H_1 \cup \cdots \cup H_n\) so \(\Delta(f) = H_1 \cup \cdots \cup H_m\). Let \(g \in \Delta(f)\). Then \(g \in H_k\) for some \(1 \leq k \leq m\), so there is an \(h_o \in H_k\) and a net \(\{h_\alpha\} \subset H_k\) with \(h_\alpha \to g\) weakly. Since \(h_\alpha, h_o \in H_k\), \(h_\alpha|_A = h_o|_A\) for all \(\alpha\). Let \(B\) be an atom of \((X, \lambda, \mu)\) (so \(B \subset A\)). \(g\) is constant on \(B\) so \((g|B)\mu(B) = \int g C_B \, d\mu = \lim_\alpha \int h_\alpha C_B \, d\mu = (h_o|B)\mu(B)\) and thus \(g|B = h_o|B\) since \(\mu(B) > 0\). This holds for all atoms \(B\), so \(g|A = h_o|A\). Let \(v \in L^\infty(X_o, \mu)\) and extend \(v\) to all of \(X\) by \(v|A = 0\), so \(v \in L^\infty(X, \mu)\). Then \(\int_X g \, v \, d\mu = \int_X g \, v \, d\mu = \lim_\alpha \int_X h_\alpha \, v \, d\mu\) so \(h_\alpha|X_o \to g|X_o\) weakly. But \(h_o|A = h_o|A\) and \(h_\alpha \sim h_o\) so \((3.4)(ii)\) implies \(h_\alpha|X_o \sim h_o|X_o\). Hence \(g|X_o \in \Delta(h_o|X_o) = \Omega(h_o|X_o)\), i.e., \(g|X_o < h_o|X_o\).

(22.8) **PROPOSITION.** If \(f, g \in L^1(X, \mu)\) and \(g < f\) and \(g|A = f|A\) then \(g|X_o < f|X_o\).

**PROOF.** Define \(G_n, F_n\) inductively by \(G_1 = g, F_1 = f, G_{n+1} = G_n C_{X-A_n}, F_{n+1} = F_n C_{X-A_n}\). Since \(g < f\) we have \(G_1 < F_1\).

Since \(f|A = g|A\) we have by induction using (8.7) that \(G_n < F_n, n \in \mathbb{P} \).
If \( X \) has only a finite number of atoms, then for some \( n \in \mathbb{P} \),

\[
G_n \downarrow g_{C_{X_0}} \text{ and } F_n \downarrow f_{C_{X_0}} \text{ and } \left\| F_n - f_{C_{X_0}} \right\| = \int F_n - f_{C_{X_0}} \, d\mu
\]

\[
= \int f_{C_{\bigcup_{i=n}^{\infty} A_i}} \, d\mu \to 0 \text{ as } n \to \infty \text{ since } f \in L^1.
\]

Similarly, \( \left\| G_n - g_{C_{X_0}} \right\| \to 0. \)

Hence (12.5) implies \( g_{C_{X_0}} \sim f_{C_{X_0}} \) so \( g_{X_0} \sim f_{X_0} \).

**REMARK.** If \( f \& g \notin L^1 \) then (22.8) may fail since we could have

\[
\int A \, f \, d\mu = \int A \, g \, d\mu = +\infty \text{ with } \int_0^a \delta \, g_{X_0} < \int_0^a \delta \, f_{X_0}
\]

(22.9) **LEMMA.** If \( f \in L^1(X, \mu) \) then \( Z(f) \) is \( \sigma(L^1, L^\infty) \) sequentially closed.

**PROOF.** Let the sequence \( \{g_n\} \subset Z(f) \) with \( g_n \rightharpoonup g \) weakly. Then

\[
g_n \big| A \to g \big| A \text{ pointwise, and there are functions } h_n \sim f \text{ such that}
\]

\[
g_n \big| X_0 \to h_n \big| X_0 \text{ and } g_n \big| A = h_n \big| A, \ n = 1, 2, 3, \ldots
\]

Let \( \{[a_n, b_n]\}_{n \in \mathbb{P}} \) be pairwise disjoint intervals such that \( \bigcup_n [a_n, b_n] = [a_0, a] \) and \( b_n - a_n = \mu(A_n), \ n \in \mathbb{P} \) (see §9). For each \( n = 1, 2, 3, \ldots \) let

\[
H_n = \delta_{h_n} \big| X_o \big| [0, a_0] + \sum_{k \in \mathbb{P}} (h_n \big| A_k) \big| C_{[0, a_0]} b_k
\]

Then \( \{H_n\} \subset [-\infty, \infty] [0, a] \) which is compact in the product topology, so there is a subsequence \( \{H_{n_k}\}_{k=1}^{\infty} \) which converges pointwise everywhere to a function \( H \). \( H \big| [0, a_0] \) is the limit of a sequence of decreasing functions so it is decreasing, and \( X_0 \) is non-atomic, so there is an \( h \in M(X_o, \mu) \) such that \( \delta_h \big| X_0 = H \big| [0, a_0] \) a.e. Extend \( h \) to \( X \) by \( h \big| A = g \big| A \). Now \( H_n \sim h_{n_k} \sim f \) and \( H_{n_k} \to H \) pointwise so
H \sim f. Also \( H\big|[0, a_0] = \delta_h |X_0 \sim h|X_0 \), and since
\[
\begin{aligned}
H_n &\big|[a_0, a] = g_n |A = g|A \\
H &\big|[a_0, a] \quad g|A = h|A
\end{aligned}
\]
pointwise, we have \( H\big|[a_0, a] \sim h|A \). Hence \( h \sim H \sim f \in L^1 \) and thus \( h \in L^1 \). Since \( \Omega(f) \) is weakly closed, \( g \in \Omega(f) \) so \( g < f \sim h \) and hence \( g < h \). Since \( g|A = h|A \), we have using (22.8) that \( g|X_0 < h|X_0 \).

Thus \( g \in Z(f) \).

Recall that the metric space associated with a finite m.s. 
\((X, \Lambda, \mu)\) is \((\Lambda(\mu), d)\) where \( \Lambda(\mu) \) is \( \Lambda \) modulo the sets of measure zero and \( d(A, B) = \mu(A-B) + \mu(B-A) \). \( \Lambda(\mu) \) will be viewed as \( \Lambda \) with the equality \( A = B[\mu] \) iff \( C_A = C_B \) \( \mu \)-a.e. A finite m.s. is said to be separable if its associated metric space is separable.

Note that Lebesgue measure on bounded subsets of \( \mathbb{R}^k \) and Stieltjes-Lebesgue measure on bounded subsets of \( \mathbb{R} \) are separable [47, p. 69].

(22.10) PROPOSITION. (i) \((X_0, \Lambda \cap X_0, \mu)\) is separable iff \((X, \Lambda, \mu)\) is separable.

(ii) If \((X, \Lambda, \mu)\) is separable then for each \( f \in L^1(X, \mu) \) the relative \( \sigma(L^1, L^\infty) \) topology on \( \Omega(f) \) is metrizable.

PROOF. (i) \((\Lambda \cap X_0(\mu), d)\) is a subspace of \((\Lambda(\mu), d)\) so \((\Lambda(\mu), d)\) separable implies \((\Lambda \cap X_0(\mu), d)\) is separable. Conversely, if \((\Lambda \cap X_0(\mu), d)\) is separable, then the union of the atoms of \( X \) and a countable dense subset of \((\Lambda \cap X_0(\mu), d)\) is countable and dense in
(ii) Let \( \mathcal{G} \) be a countable dense subset of \((\Lambda(\mu), d)\) and let \( f \in L^1 \). Then \( \Omega(f) \) is weakly compact, so according to [20, p. 143, Theorem 16.7] we have only to show there is a countable subset of \( L^\infty \) which separates points of \( L^1 \). Let \( \mathcal{G} = \{ C_E : E \in \mathcal{G} \} \), so \( \mathcal{G} \) is countable. To show that \( \mathcal{G} \) separates points of \( L^1 \) let \( g, h \in L^1 \). If \( \int (g-h)C_E d\mu = 0 \) for all \( C_E \in \mathcal{G} \) then \( \int_E (g-h) d\mu = 0 \) on a dense subset of \((\Lambda(\mu), d)\). Since \( g-h \in L^1 \), \( E \rightarrow \int_E (g-h) d\mu \) is continuous on \((\Lambda(\mu), d)\) and hence we conclude that \( \int_E (g-h) d\mu = 0 \) for all \( E \in \Lambda(\mu) \) so \( g = h \).

(22.11) THEOREM. If \((X, \Lambda \cap X', \mu)\) is separable then for every \( f \in L^1(X, \mu) \) we have \( \Delta(f) = \overline{Z(f)} \).

PROOF. Now the weak topology on \( \Omega(f) \) is metrizable so \( \overline{Z(f)} \) is closed and thus \( \Delta(f) \subseteq \overline{Z(f)} \subseteq \Delta(\overline{f}) \) implies \( \Delta(f) = \overline{Z(f)} \).

Now suppose \( \rho \) is a saturated Fatou norm on \( M(X, \mu) \) such that \( L^\infty \subseteq L^\rho \), \( L^\rho' \subseteq L^1 \) and \( L^\rho \) is u. r. i. If \( f \in L^\rho \) then \( \Delta(f) \subseteq L^\rho \) and the problem is to determine the \( \sigma(L^\rho, L^\rho') \)-closure of \( \Delta(f) \). If \( A \subseteq L^\rho \) we denote its \( \sigma(L^\rho, L^\rho') \) closure by \( \overline{L^\rho A} \). Since \( \sigma(L^\rho, L^\infty) \subseteq \sigma(L^\rho, L^\rho') \) we see that \( \overline{L^\rho A} \subseteq \overline{A} \).

By examining the proof of (22.10) we see that if \( f \in L^\rho \) and \((X, \Lambda \cap X', \mu)\) is separable, then the relative \( \sigma(L^\rho, L^\rho') \) topology on \( \Omega(f) \) is metrizable.
(22.12) PROPOSITION. If the simple functions are dense in \( L^p \) then for every \( f \in L^p \) and \( A \subseteq \Omega(f) \), the \( \sigma(L^p, L^p') \) closure of \( A \) equals the \( \sigma(L^1, L^\infty) \) closure of \( A \).

PROOF. Now \( \Omega(f) \) is \( \rho \)-bounded so there is an \( M > 0 \) such that \( \rho(g) < M \) for all \( g \in \Omega(f) \). We have only to show that \( A \subseteq \rho(A) \). Now \( \rho(A) \subseteq \Omega(f) \) by (17. 4). Let \( g_o \in A \). Then there is a net \( \{ g_{\alpha} \} \subseteq A \) with \( g_{\alpha} \to g_o \) in \( \sigma(L^1, L^\infty) \). If \( h \in L^p', \) \( F_h(g) = \int g \, h \, d\mu \) defines a continuous linear functional on \( L^p \). It suffices to show that \( F_h(g_{\alpha}) \to F_h(g_o) \) for all \( h \in L^p' \). Hence let \( h \in L^p' \) and let \( \epsilon > 0 \). Then there is a simple function \( v \) such that \( \rho'(h-v) < \epsilon \), so for all \( g \in \Omega(f) \),

\[
|F_h(g) - F_v(g)| = |\int (h-v)g \, d\mu| \leq \rho(g) \rho'(h-v) \leq M \epsilon.
\]

Now there is an \( \alpha_o \) such that \( \alpha \geq \alpha_o \) implies \( |F_v(g_{\alpha}) - F_v(g_o)| < \epsilon \). Hence for \( \alpha \geq \alpha_o \),

\[
|F_h(g_{\alpha}) - F_h(g_o)| \leq |F_h(g_{\alpha}) - F_v(g_{\alpha})| + |F_v(g_{\alpha}) - F_v(g_o)| + |F_v(g_o) - F_h(g_o)| \leq M \epsilon + \epsilon + M \epsilon.
\]

Thus \( F_h(g_{\alpha}) \to F_h(g_o) \).

(22.13) THEOREM. If the simple functions are dense in \( L^p' \) and \( f \in L^p \) then the \( \sigma(L^p, L^p') \) closure of \( \Delta(f) \) equals the \( \sigma(L^1, L^\infty) \) closure of \( \Delta(f) \).

REMARK. The intuitive idea behind the definition of \( Z \) is that every member of \( \Delta(f) \) can be reached by a net in which eventually the rearrangements of \( f \) are formed by rearrangements on \( X_o \) and rearrangements on \( A \). This means that if \( g \in \Delta(f) \) there is a net \( \{ h_{\alpha} \} \subseteq \Delta(f) \) and an index \( \alpha_o \) such that for \( \alpha, \beta \geq \alpha_o \) we have
\( h_\alpha |X_0 \sim h_\beta |X_0 \) and \( h_\alpha |A \sim h_\beta |A \). In this case \( h_\alpha |X_0 \sim g |X_0 \) implies \( g |X_0 < h_\alpha |X_0 \) and \( h_\alpha |A \sim g |A \) implies \( g |A \sim h_\alpha |A \).

Hence defining \( h \) by \( h|X_0 = h_\alpha |X_0 \) and \( h|A = g |A \) we have \( h \sim f, g |X_0 < h |X_0 \) and \( g |A = h |A \).
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