OPTIMAL SIMPLE STRUCTURES WITH BENDING AND MEMBRANE STRESSES

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ABSTRACT

Structural optimization of elastic solid structures with membrane and bending stresses is studied. A necessary and sufficient optimality condition for maximum stiffness is derived using an energy formulation (minimum potential energy). In the case of pure membrane or pure bending stress states, maximum stiffness and maximum strength are governed by the same optimality condition (uniform strength design). For mixed stress states (membrane and bending), this result is not true, and a general method of approach to find a necessary condition for maximum strength is developed by means of the calculus of variation. Two particular statically indeterminate problems with one design variable to be optimized (clamped-clamped beam, clamped-clamped arch) are treated by this method, and, in both examples, maximum strength is achieved by a uniform strength design.

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LIST OF SYMBOLS

A, B, C, D	Constants
E	Young's Modulus
G	Normalized Mass
H	Defined by $\sqrt{1 \pm 24 \dfrac{A[g(\theta)-g(\theta_0)]}{\omega_a[A\,g(\theta)+1]^2}}$
I	Bending Moment of Inertia
K	Curvature
L	Half-length of the Beam
M	Mass
$\mathfrak{M}$	Bending Moment
$\underset{\sim}{M}$	Functional associated to M
N	Inplane Force
P	Potential Energy
$\underset{\sim}{P}$	Functional associated to P
Q	Shear force
Q_i	Discrete Generalized Forces
R	Radius
S	Material capacity (constant)
T	Structure
U	Strain Energy
W	External Work or, Stiffness
$\underset{\sim}{W}$	Functional associated to W
a	Positive Constant such that: at = Membrane Structural Stiffness
b	Constant Width

List of Symbols (Cont'd)

c	Positive Constant such that: ct^3 = Bending Structural Stiffness
d	Constant
$e(q(x))$	Strain Energy per Unit Length (area) per Unit Stiffness
e_M	Strain Energy per Unit Length (area) per Unit Stiffness for Membrane Stress State
e_B	Strain Energy per Unit Length (area) per Unit Stiffness for Bending Stress State
f	Function such that $f \frac{bE}{R} = N_\theta$
g	Function defined by $A(\cos\theta + tg\,\Theta \sin\theta)$
j	Constant
k_1, k_2	Constants
$\ell(\sigma_{ij})$	Comparison Function
m	Intensity of the Load
$p(x)$	Load Distribution
$q(x)$	Generalized Strains
$r(\sigma_{ij})$	Loading Function of the Material
$s(x)$	Structural Stiffness
t	Thickness
$\underset{\sim}{t}$	Normalized Thickness
$u(\underset{\sim}{x})$	Displacement Vector
v	Inplane Displacement
w	Normal Displacement
x_i	Space Variable
dx	Line Element or Area Element
x, y	Principal Axis

List of Symbols (Cont'd)

$\alpha(\underset{\sim}{x}), \beta(\underset{\sim}{x}), \overline{\gamma}(\underset{\sim}{x})$	Switching Functions
γ	Defined by $\gamma = -\frac{\xi^2}{2} + \xi$
δ, ϵ	Small Quantities (Infinitesimal)
ϵ_0	Mid Surface Strain
ζ	Constant
θ	Variable in the Case of the Arch
Θ	Half-Opening Angle of the Arch
λ	Constant Lagrange Multiplier
$\lambda_i(\underset{\sim}{x}), \Lambda(\underset{\sim}{x})$	Variable Lagrange Multipliers
μ, ν	Angles
ξ	Normalized Variable $\xi = \frac{X}{L}$
ρ	Density of the Material
σ	Stress
σ_0	Maximum Allowable Stress
$\Phi(x)$	Slack Function
Ψ	Function Defined in Section 4 of Part VI
ω	Load Parameter
ω_b, ω_a	Load Parameters in the Case of the Beam and the Arch
Ω	Positive Constant such that $M = \Omega \int t \, dx$
*	Index for Optimal Structure (Satisfying Optimality Condition)

I. INTRODUCTION

When considering structural design for minimum weight the problems may be categorized according to their satisfaction of certain restrictive conditions. For example, these may include: (1)

1. Static compliance
2. Dynamic compliance
3. Buckling load limitations
4. Load factor at plastic collapse
5. Multipurpose and/or multiconstraint design
6. Optional layout
7. Reinforcement and/or prestressed conditions.

Excluding the domain of optimal layout, previous work on the optimization of membrane and sandwich structures has usually been restricted to these structures characterized by a linear relationship between the specific structural stiffness and the specific structural weight. Because of this restriction, a general method of design using an energy approach and the calculus of variation can be established. The two fundamental steps in this procedure are: first, find an optimality condition for the displacement field that does not involve design parameters, and then determine the optimal design from the usual differential equations of the structure (2). Conditions of maximum stiffness, maximum fundamental frequency, maximum buckling load, plastic design for maximum safety, and maximum strength are treated as examples in references (2), (3), and (4). More complex problems including additional constraints such as specified minimum cross sections can also be formulated. (5), (6). A slightly different method using the same basic principles (minimum potential energy) is developed in reference (7).

If one is interested in the optimal design of structures, not of the sandwich type, it is difficult to find previous work. Some specific problems such as that of the strongest column (8), (9) and the optimum design of a vibrating bar with minimum cross-section or a specified natural frequency (4), (10), (11) can be found in the literature.

Among all types of problems solved in optimal structural design, the case of maximum strength or maximum allowable stress has received only limited attention. Taylor (3) has shown that for the case of the pure membrane stress state (sandwich cross section) the maximum elastic capacity design corresponds to the maximum stiffness design simultaneously. Huang (7) has shown that for pure bending the optimality condition for maximum stiffness leads to a uniform strength design; however, he did not discuss the relationship between the uniform strength design and the maximum strength design. In both the pure membrane and the pure bending stress state, the derived optimality condition is found to be the Euler equation corresponding to the variation of the functional

$$\underset{\sim}{P} = P - \lambda M \qquad (1.1)$$

with respect to the design variable where

P = potential energy

M = given mass of material

λ = constant Lagrange multiplier

The purpose of this analysis is to extend the existing work done concerning the pure membrane and the pure bending stress state. It will be shown that maximum stiffness leads to maximum strength in the case of pure bending, and that maximum stiffness does not lead to maximum strength for the mixed stress state (membrane and bending). A necessary condition for a stationary value of the mass of material when a maximum allowable stress is specified will be derived for the mixed stress state, and two examples will be treated to illustrate the particular method. In what follows, pure bending is considered to be a stress state in which energies other than bending energies are sufficiently small to be negligible. For example, a long beam under the usual loading systems will be considered to be a pure bending case.

In all of the following, linearity of the elastic isotropic material and small deflections will be assumed. In particular, displacements are proportional to the load. Only static loads independent of the displacement field will be considered, and secondary problems, such as stability considerations will not be treated. To fully illustrate the method, statically indeterminate problems are chosen as examples and for simplicity, the two examples are "one independent variable" problems. Because of the difficulty in finding closed form solutions, the use of a high speed digital computer (IBM: 370-155) was required.

II. MAXIMUM STIFFNESS AND MAXIMUM STRENGTH IN A PURE BENDING STRESS STATE

1. Stiffness definition

The stiffness of a structure can be characterized by the work done by external forces

$$W = m\left\{\int p(x)\, u(x)\, dx + \sum_i Q_i\, u(x_i)\right\} \tag{2.1}$$

where $u(x)$ = displacement field

$p(x)$ = load distribution

Q_i = discrete generalized forces

m = intensity of the load

dx = line element of a one dimensional structure or the area element of a two dimensional structure.

It is assumed for simplicity that the loads act in principal directions and do not depend upon the displacement field. (e.g.: elastic foundations do not fit this assumption).

We shall say that a structure T^* is stiffer than a structure T if: for the same intensity of the load m, and the same load distribution,

$$W(T^*) < W(T) \tag{2.2}$$

or

$$\int p(x)\, u^*(x)\, dx + \sum_i Q_i\, u^*(x_i) < \int p(x)\, u(x) dx + \sum_i Q_i\, u(x_i) \tag{2.3}$$

where

$u^*(x)$ = displacement field corresponding to the structure T^*

$u(x)$ = displacement field corresponding to the structure T

2. Sufficient condition for maximum stiffness

Let one imagine two structures of the same material, T^* and T (T^* in the neighborhood of T) having the same stiffness under the same loading condition.

$$W^* = W \tag{2.4}$$

if the two structures are at "equilibrium conditions", we know that W = 2U where U is the strain energy.

The potential energy P is given by

$$P = U - W \tag{2.5}$$

$$P = - W/2 \tag{2.6}$$

so

$$W^* = W \quad \text{implies} \quad P^* = P \tag{2.7}$$

The strain energy can be represented by

$$U = \int s(x)\, e(q(x))\, dx \tag{2.8}$$

where

$s(x)$ = structural stiffness

$q(x)$ = generalized strains

$e(q(x))$ = strain energy per unit length (or area) per unit stiffness

dx = line element of a one dimensional structure or area element of a two dimensional structure.

In the particular case of pure bending (beam bending without membrane forces - assuming shear strain energies are negligible - long beam case)

$$s(x) \;=\; c\,t^3 = \frac{1}{12}\,E\,t^3$$

$$e(q(x)) = e_B \;=\; \frac{1}{2}\left(\frac{d^2w}{dx^2}\right)^2$$

where t is the design variable (thickness), c is a positive constant, E is Young's Modulus and w is the normal displacement.

With this notation, the potential energy of a structure T can be written

$$P \;=\; \int c\,t^3 e_B dx - m\left\{\int p(x)\,u(x)\,dx + \sum_i Q_i\,u(x_i)\right\} \qquad (2.9)$$

The fact that T^* and T have the same stiffness is now expressed by:

$$\int c\,t^{*3} e_B^* dx - m\left\{\int p(x)u^*(x)dx + \sum_i Q_i\,u^*(x_i)\right\} = \int c\,t^3 e_B dx - m\left\{\int p(x)u(x)dx + \sum_i Q_i\,u(x_i)\right\} \qquad (2.10)$$

where the "*" terms correspond to the T^* structure.

From the principle of Minimum Potential Energy,

$$\int c\,t^3 e_B dx - m\left\{\int p(x)\,u(x)\,dx + \sum_i Q_i\,u(x_i)\right\}$$

$$\le \int c\,t^3 e_B^* dx - m\left\{\int p(x)\,u^*(x)dx + \sum_i Q_i\,u^*(x_i)\right\} \qquad (2.11)$$

Using $P^* = P$ from Eq. (2.7), we obtain

$$\int c\,t^{*3} e_B^* dx - m\left\{\int p(x)\,u^*(x)dx + \sum_i Q_i\,u^*(x_i)\right\}$$

$$\le \int c\,t^3 e_B^* dx - m\left\{p(x)\,u^*(x)dx + \sum_i Q_i\,u^*(x_i)\right\} \qquad (2.12)$$

which reduces to

$$\int (t^3 - t^{*3})\, e_B^* \, dx \geq 0 \qquad (2.13)$$

since c is a positive constant.

Because T^* is in the neighborhood of T, we can write

$$t = t^* + \delta t^* \qquad \text{where} \quad \delta t^* \text{ is small} \qquad (2.14)$$

expanding t^3 and keeping only the first order term, we obtain

$$\int 3\, t^{*2}\, e_B^*\, \delta t^* \, dx \geq 0 \qquad (2.15)$$

What is desired is an optimality condition such that the design corresponding to this optimality condition leads to the structure of minimum mass for a given stiffness W^*. From Eq. (2.14), the masses of T and T^* are related by

$$M = M^* + \delta M^* \qquad (2.16)$$

where $M = \Omega \int t\, dx, \quad M^* = \Omega \int t^* dx, \quad \delta M^* = \Omega \int \delta t^* \, dx \qquad (2.17)$

and Ω is a positive multiplicative constant including the material properties. A necessary and sufficient condition for T^* to be the optimal structure in the previous sense is that

$$M^* \leq M \qquad (2.18)$$

which leads to

$$\delta M^* = \Omega \int \delta t^* dx \geq 0$$

or

$$\int \delta t^* \, dx \geq 0 \tag{2.19}$$

In order to satisfy this condition, it is sufficient in Eq. (2.15) to set

$$t^{*2} e_B^* = d^{*2} \tag{2.20}$$

where d^* is a constant.

Statement 1: A sufficient condition for obtaining the minimum mass design of a structure T^* of prescribed stiffness W^*, and supporting a given load system is:

$$t^{*2} e_B^* = d^{*2}$$

Lemma: If two structures T^* and T (where T^* is in the neighborhood of T) have the same loading and are of the same material but have different stiffnesses, satisfy the conditions that $t^{*2} e_B^* = d^{*2}$ and $t^2 e_B = d^2$, the structure having the greater stiffness will have the greater mass.

Proof: Without loss of generality, assume T^* stiffer than T.

$$W^* = m \int p(x) u^*(x) dx + \sum_i Q_i u^*(x_i) < m \int p(x) u(x) dx + \sum_i Q_i u(x_i) = W \tag{2.21}$$

is T^* and T are at equilibrium

$$P^* = -\frac{W^*}{2} \qquad \text{and} \qquad P = -\frac{W}{2} \tag{2.22}$$

So

$$\int c t^{*3} e_B^* dx - m\left\{\int p(x) u^*(x) dx + \sum_i Q_i u^*(x_i)\right\} > \int c t^3 e_B dx - m\left\{\int p(x) u(x) dx + \sum_i Q_i u(x_i)\right\} \tag{2.23}$$

minimum potential energy gives:

$$\int c\, t^{*3} e_B\, dx - m\left\{\int p(x)u(x)dx + \sum_i Q_i\, u(x_i)\right\} \geq \int c\, t^{*3} e_B^{*} dx - m\left\{\int p(x)u^{*}(x)dx + \sum_i Q_i\, u^{*}(x_i)\right\} \tag{2.24}$$

so

$$\int t^{*3}\, e_B\, dx > \int t^3\, e_B\, dx \tag{2.25}$$

and

$$\int (t^{*3} - t^3)\, e_B\, dx > 0 \tag{2.26}$$

T^* in the neighborhood of T leads to $t^* = t + \delta t$ where δt is small. Keeping the first order term, we obtain

$$\int 3t^2\, e_B\, \delta t\, dx > 0 \tag{2.27}$$

But, by assumption, $t^2\, e_B = d^2$, so

$$\int \delta t\, dx > 0 \tag{2.28}$$

which means

$$M^* > M \tag{2.29}$$

The proof is completed.

Using this result, we can deduce

Statement II: Considering two structures T^* and T of same mass $M^* = M$ and supporting the same load system, it is sufficient for the structure T^* to satisfy the optimality condition $t^{*2}\, e_B^{*} = d^{*2}$ to be as stiff or stiffer than T.

Proof: Let us consider two structures T^* and T of same mass $M^* = M$. The structure T^* of stiffness W^* satisfies $t^{*2}\, e_B^* = d^{*2}$. T is an arbitrary structure of stiffness W, in the neighborhood of T^*. From Statement I, we know that the stiffness W can be achieved by a structure T_0^* satisfying the condition $t_0^{*2}\, e_{B\,0}^* = d_0^{*2}$. T_0^* is such that

$$M_0^* \leq M = M^* \qquad (2.30)$$

using the previous lemma, Eq. (2.30) requires

$$W^* \leq W \qquad (2.31)$$

The structure T^* has the maximum stiffness. We notice that Statement I and Statement II are equivalent.

Conclusion: $t^{*2}\, e_B^* = d^{*2}$ is a sufficient condition for maximum stiffness in a pure bending stress state.

Note: because of the assumption that δt or δt^* are small, this result is only a local optimum.

3. Necessary condition for maximum stiffness

We want to find a necessary condition such that the stiffness W^* of a structure T^* of given mass M^*, under a given load system, is stationary. If the design variable is continuous and differentiable, this problem can be treated by the calculus of variation.

As has been shown in Section 1 of Part II, the stiffness of a structure T^* can be characterized by the external work W^*. Maximum stiffness corresponds to minimum W^*. Thus, an extremum of W^* will correspond to an extremum of the stiffness.

Let us consider the functional

$$\underset{\sim}{W}^* = W^* - \lambda^* \left\{ \Omega \int t^* dx - M^* \right\} \qquad (2.33)$$

The constraint equation characterizing the specified mass M^* is

$$\Omega \int t^* dx - M^* = 0 \qquad (2.34)$$

where λ^* is a constant Lagrange multiplier. A necessary condition for stationary stiffness or external work can be obtained by:

$$\delta_{t^*} (\underset{\sim}{W}^*) = 0 \qquad (2.35)$$

which generates the Euler equation corresponding to the variation of $\underset{\sim}{W}^*$ with respect to the design variable t^*.

The structure is assumed to be in equilibrium, so, for convenience, one can interchange the external work W^* and the strain energy U^* through the relation $W^* = 2\,U^*$. Eq. (2.33) then becomes

$$\underset{\sim}{W}^* = 2\,U^* - \lambda^* \left\{ \Omega \int t^* dx - M^* \right\} \qquad (2.36)$$

or

$$\underset{\sim}{W}^* = 2c \int t^{*3}\, e_B^*\, dx - \lambda^* \left\{ \Omega \int t^* dx - M^* \right\} \qquad (2.37)$$

in the particular case of bending only. The Euler equation associated with the variation with respect to t^* is

$$t^{*2}\, e_B^* = d^{*2} \qquad (2.38)$$

with $\quad d^{*2} = \dfrac{\lambda^* \Omega}{6\,c}$

e_B^*, Ω, c are positive, so, λ^* will be positive.

Eq. (2.38) is a necessary condition for a stationary value of the stiffness W^* of the structure T^* of given mass M^*, in the case of bending only.

Conclusion: A necessary and sufficient condition for maximum stiffness of a structure in the case of pure bending is

$$t^{*2} e_B^* = d^{*2} \tag{2.39}$$

4. Sufficient condition for maximum strength

We previously derived an optimality condition for maximum stiffness in the case of bending. Let us assume that the positive constant d^{*2} involved in this optimality condition can be identified with a specific form of material capacity S^{*2}. Example: For bending of a straight beam, the optimality condition is

$$t^{*2} \left(\frac{d^2u^*}{dx^2}\right)^2 = d^{*2} \tag{2.40}$$

In this particular case, d^* is proportional to the maximum local stress intensity at the location "x" of the beam. To give a constant value to this quantity, means that we fix the value of the maximum local stress intensity in the whole structure.

Consider a structure T^* such that $t^{*2} e_B^* = S^{*2}$ under a load m^*, where S^{*2} is a constant identified as a material capacity. Knowing S^{*2} and t^*, we are able to determine the value m^* of the load intensity. Now consider an arbitrary design T in the neighborhood of T^* such that:

- T and T^* have the same mass
- the maximum local stress intensity $S^2(x)$ in T is less than or equal to the maximum local stress intensity S^{*2} in T^*.

This is equivalent to

- $t^* = t + \delta t$ (2.41)

  with $\int \delta t \, dx = 0$ (2.42)
- $S^2(x) = S^{*2}(1 - \Phi^2(x)) = t^2(x) \, e_B$ (2.43)

where $\Phi(x)$ is an unknown scalar function.

Knowing $S^2(x)$ and $t(x)$, we are able to determine the value m of the load intensity. If $m^* \geq m$, we will say that T^* represents the maximum strength structure and if $m^* < m$, we will say that T^* does not represent the maximum strength structure.

The strain energy for the structure T under the load m is given by

$$U(t, u_m, m) = c \int t^3 \, e_B \, dx = c \int S^2(x) \, t \, dx \qquad (2.44)$$

where u_m represents the displacement field of T under the load of intensity m. Use of Eq. (2.41), (2.43) gives

$$U(t, u_m, m) = c \int S^{*2} \, (1 - \Phi^2(x)) \, (t^* - \delta t) dx \qquad (2.45)$$

After using Eq. (2.42), this reduces to

$$U(t, u_m, m) = U(t^*, u^*_{m^*}, m^*) - c \, S^{*2} \int \Phi^2(x) \, t \, dx \qquad (2.46)$$

where $U(t^*, u^*_{m^*}, m^*) = c \int S^{*2} \, t^* \, dx$

with $u^*_{m^*}$ being the displacement field of T^* under the load m^*. The integral in the right hand side of Eq. (2.46) is positive, so

$$U(t^*, u^*_{m^*}, m^*) \geq U(t, u_m, m) \tag{2.47}$$

T^* and T are in equilibrium so W = 2 U. This implies that

$$m^*\left\{\int p(x)\, u^*_{m^*} dx - \sum_i Q_i\, u^*_{m^*}(x_i)\right\} \geq m\left\{\int p(x) u_m dx - \sum_i Q_i\, u_m(x_i)\right\} \tag{2.48}$$

Let $m^* = \zeta m$

Since we have linearity

$$u^*_{m^*} = \left(u^*_m\right)\cdot\zeta \tag{2.49}$$

where u^*_m is the response of T^* to the load m.

Eq. (2.48) then becomes

$$\zeta^2 m\left\{\int p(x) u^*_m dx - \sum_i Q_i\, u^*_m(x_i)\right\} \geq m\left\{\int p(x) u_m dx - \sum_i Q_i\, u_m(x_i)\right\} \tag{2.50}$$

But, we have seen before that a structure T^* satisfying $t^{*2} e^*_B = S^{*2}$ can be compared to any arbitrary structure T of the same mass under the same load and in its neighborhood, by

$$m\left\{\int p(x)\, u^*_m(x)\, dx - \sum_i Q_i\, u^*_m(x_i)\right\} \leq m\left\{\int p(x) u_m(x) dx - \sum_i Q_i\, u_m(x_i)\right\} \tag{2.51}$$

corresponding to maximum stiffness.

From Eq. (2.50) together with Eq. (2.51) we deduce

$$\zeta^2 \geq 1 \tag{2.52}$$

this implies

$$m^{*} \geq m \tag{2.53}$$

Therefore, a structure satisfying the condition $t^{*2} e_B^{*} = S^{*2}$ is one that, for a given mass, and for a given capacity constraint of the material S^{*2}, the load intensity is maximum among all the other structures of the same material in its neighborhood, providing that the load distribution p(x) is the same. This represents a maximum strength design.

Note:

We have seen that in the case of a beam, the optimality condition derived for maximum stiffness and maximum strength represents the fact that the absolute value of the maximum stress at any location x of the structure is equal to the maximum allowable stress σ_0 (capacity of the material). If we use a linear, isotropic elastic material and if we consider small displacements, the stress distribution in a cross-section is linear and anti-symmetric (pure bending). Therefore, at a location x, the maximum stress will occur in the extreme fibers (external surface of the structure). In the case of a statically determinate problem, where the bending moment does not depend upon the displacement field, the optimality condition for maximum strength can be derived directly.

Assume a structure T of design t such that $|\sigma(x)|_{max} \leq \sigma_0$ ($\sigma_{max}(x)$ is the bending stress occuring in an extreme fiber, at the location x). The mass of T is given by $M = \Omega \int t \, dx$. Without changing the loading diagram (bending moment), it is possible at any location x to reduce

the thickness t to a thickness t^* such that $|\sigma^*_{max}(x)| = \sigma_0$. If we reduce again the thickness, we will get $|\sigma_{max}(x)| > \sigma_0$ and the capacity of the material will be exceeded. So, t^* is the minimum possible thickness at the location x. We can do this at any x so, in the whole interval, $t^* \leq t$. But, the mass of this new structure T^* is given by $M^* = \Omega \int t^* dx$. Therefore, $M^* \leq M$ and T^* has a greater strength than T and the design t^* represents the maximum strength design.
In the case of a statically indeterminate problem, we cannot use the same argument because the loading diagram definitively depends upon the displacement field which is a function of the design and the optimal condition derived in the general case must be used.

III. MAXIMUM STIFFNESS IN COMBINED MEMBRANE-BENDING STRESS STATE

1. Sufficient condition for maximum stiffness

The stiffness definition will remain the same as in the pure bending case.

In the mixed stress state, the potential energy can be expressed by

$$P = \int (a\,t\,e_M + c\,t^3\,e_B)dx - m\left\{\int p(x)\,u(x)dx + \sum_i Q_i\,u(x_i)\right\} \qquad (3.1)$$

with

$\int a\,t\,e_M\,dx$: membrane strain energy

$\int c\,t^3\,e_B\,dx$: bending strain energy

e_M, e_B = functions of generalized strain, defined as strain energies per unit length (or area) per unit stiffness.

$a\,t$ = membrane structural stiffness

$c\,t^3$ = bending structural stiffness

We proceed as in the pure bending stress state case.

Consider two structures T^* and T, having the same stiffness W^* and W, under the same load (load intensity = m). If T^* and T are in equilibrium, W = 2 U so,

$$W^* = W \quad \text{implies} \quad P^* = P \qquad (3.2)$$

which can be written as

$$P^* = \int (a\,t^* e_M^* + c\,t^{*3} e_B^*) dx - m\left\{\int p(x) u^*(x) dx + \sum_i Q_i u^*(x_i)\right\}$$

$$= \int (a\,t\,e_M + c\,t^3\,e_B) dx - m\left\{\int p(x) u(x) dx + \sum_i Q_i\,u(x_i)\right\} = P \qquad (3.3)$$

But, the principle of Minimum Potential energy gives us:

$$\int (a\,t\,e_M + c\,t^3\,e_B) dx - m\left\{\int p(x)\,u(x) dx + \sum_i Q_i\,u(x_i)\right\} \qquad (3.4)$$

$$\leq \int (a\,t\,e_M^* + c\,t^3\,e_B^*) dx - m\left\{\int p(x)\,u^*(x) dx + \sum_i Q_i\,u^*(x_i)\right\}$$

Because $P^* = P$ from Eq. (3.2), we obtain

$$\int (a\,t^* e_M^* + c\,t^{*3} e_B^*) dx - m\left\{\int p(x) u^*(x) dx + \sum_i Q_i\,u^*(x_i)\right\}$$

$$\leq \int (a\,t\;e_M^* + c\,t^3\;e_B^*)\,dx - m\left\{\int p(x) u^*(x) dx + \sum_i Q_i\,u^*(x_i)\right\} \qquad (3.5)$$

or

$$\int \left\{a(t-t^*)e_M^* + c(t^3 - t^{*3})e_B^*\right\} dx \geq 0 \qquad (3.6)$$

Assume now the design of T^* in the neighborhood of the design of T.

$$t = t^* + \delta t^* \quad \text{where} \quad \delta t^* \text{ is small} \qquad (3.7)$$

Expansion of t^3 for small values of δt^*, substitution in Eq. (3.6), and retention of first order term only leads to

$$\int (a\,e_M^* + 3\,c\,t^{*2}\,e_B^*)\delta t^*\,dx \geq 0 \qquad (3.8)$$

We want to find an optimality condition such that the design corresponding to this optimality condition leads to the minimum mass structure for a given stiffness W^*.

A necessary and sufficient condition for T^* to be the optimal structure in this sense is:

$$M^* \leq M \tag{3.9}$$

which leads to

$$\int \delta t^* \, dx \geq 0 \tag{3.10}$$

In order to satisfy this condition, it is sufficient in Eq. (3. 8) to set

$$a\, e_M^* + 3\, c\, t^{*2}\, e_B^* = j^{*2} \tag{3.11}$$

where j^* is a constant.

Statement 1: A sufficient condition to obtain the minimum mass design of a structure T^* of prescribed stiffness W^* and supporting a given load system is that

$$a\, e_M^* + 3\, c\, t^{*2}\, e_B^* = j^{*2}$$

Lemma: If two structures of the same material, T^* and T (T^* in the neighborhood of T) of different stiffness W^* and W, under the same load, fulfill the conditions that $a\, e_M^* + 3\, c\, t^{*2}\, e_B^* = j^{*2}$ and $a\, e_M + 3\, c\, t^2\, e_B = j^2$, the one of greater stiffness will have the greater mass.

Proof: Without loss of generality, assume T^* stiffer than T. This can be written

$$W^* = m \int p(x) u^*(x) dx + \sum_i Q_i\, u^*(x_i) < m \int p(x) u(x) dx + \sum_i Q_i\, u(x_i) = W \tag{3.12}$$

T^* and T are in equilibrium, so

$$P^* = -\frac{W^*}{2} \qquad \text{and} \qquad P = \frac{-W}{2} \tag{3.13}$$

Therefore

$$\int (a\,t^*\,e_M^* + c\,t^{*3}\,e_B^*)dx - m\left\{\int p(x)u^*(x)dx + \sum_i Q_i\,u^*(x_i)\right\}$$

$$> \int (a\,t\,e_M + c\,t^3\,e_B)dx - m\left\{\int p(x)u(x)dx + \sum_i Q_i\,u(x_i)\right\} \tag{3.14}$$

Again, minimum potential energy produces:

$$\int (a\,t^*\,e_M + c\,t^{*3}\,e_B)dx - m\left\{\int p(x)u(x)dx + \sum_i Q_i\,u\,(x_i)\right\}$$

$$\geq \int (a\,t^*\,e_M^* + c\,t^{*3}\,e_B^*)dx - m\left\{\int p(x)u^*(x)dx + \sum_i Q_i\,u^*(x_i)\right\} \tag{3.15}$$

Using Eq. (3.14), we find:

$$\int \left\{a(t^* - t)e_M + c(t^{*3} - t^*)e_B\right\} dx > 0 \tag{3.16}$$

T^* is assumed to be in the neighborhood of T, so

$$t^* = t + \delta t \qquad \text{with} \qquad \delta t \text{ small} \tag{3.17}$$

Expansion for t^{*3}, substitution in Eq. (3.16), and retention of first order term only produces:

$$\int (a\,e_M + 3\,c\,t^2\,e_B)\delta t\,dx > 0 \tag{3.18}$$

But, by assumption, $a\,e_M + 3\,c\,t^2\,e_B = j^2$. This implies

$$\int \delta t\,dx > 0 \tag{3.19}$$

which means $M^* > M$ (3.20)

The structure T^* of greater stiffness has a greater mass than the structure T.

Statement 2: Considering two structures T^* and T of the same mass $M^* = M$ and supporting the same load system, it is sufficient for the structure T^* to satisfy the optimality condition $ae^*_M + 3\,c\,t^{*2}\,e^*_B = j^{*2}$ to be as stiff or stiffer than T.

Proof: Consider two structures T^* and T of the same mass M^* and M. T^* satisfies $a\,e^*_M + 3\,c\,t^{*2}\,e^*_B = j^{*2}$ and has stiffness W^*. T is an arbitrary structure of stiffness W in the neighborhood of T^*.

From Statement 1, we know that the stiffness W can be achieved by a structure T^*_0 such that $ae^*_{M_0} + 3\,c\,t^{*2}_0\,e^*_{B_0} = j^{*2}_0$ implying

$$M^*_0 \leq M = M^* \qquad (3.21)$$

Because of this, $W^* \leq W$ (3.22)

from previous lemma, and the proof is complete. Statement 1 and Statement II are thus shown to be equivalent.

Conclusion: $a\,e^*_M + 3\,c\,t^{*2}\,e^*_B = j^{*2}$ is a sufficient condition for maximum stiffness in a mixed stress state (membrane and bending). Again, this is a local optimum.

2. Necessary condition for maximum stiffness

Providing the design variable is continuous, and differentiable, this can be treated by the calculus of variation. We are looking for a necessary condition such that the stiffness W^* of a structure T^* of given mass M^* under a given load (intensity factor = m), is stationary. An extremum of the stiffness is characterized by an extremum of W^*. Consider the functional

$$\underset{\sim}{W}^* = W^* - \lambda^* \left\{ \Omega \int t^* \, dx - M^* \right\} \tag{3.23}$$

The constraint equation characterizing the specified mass M^* is the bracketed term and λ^* is a constant Lagrange multiplier.

A necessary condition for stationary stiffness or external work is obtained from

$$\delta_{t^*} (\underset{\sim}{W}^*) = 0 \tag{3.24}$$

The structure is assumed to be in equilibrium, so again, for convenience, we interchange W^* and U^*. In the particular case of the mixed stress state (membrane and bending) Eq. (3.23) can be written

$$\underset{\sim}{W}^* = 2 \int (a\, t^* e_M^* + c\, t^{*3} e_B^*) dx - \lambda^* \left\{ \Omega \int t^* \, dx - M^* \right\} \tag{3.25}$$

The Euler equation associated with the variation of $\underset{\sim}{W}^*$ with respect to the design variable t^* is:

$$a\, e_M^* + 3\, c\, t^{*2} e_B^* = j^{*2} \tag{3.26}$$

with

$$j^{*2} = \frac{\lambda^* \Omega}{2}$$

Ω is positive, so λ^* will be positive.

Eq. (3.26) is a necessary condition for a stationary value of the stiffness W^* of the structure T^* of given mass M^*, in the case of a membrane-bending stress state.

Conclusion: A necessary and sufficient condition for maximum stiffness of a structure in the case of mixed stress state (membrane and bending) is:

$$a\, e_M^* + 3\, c\, t^{*2}\, e_B^* = j^{*2} \qquad (3.27)$$

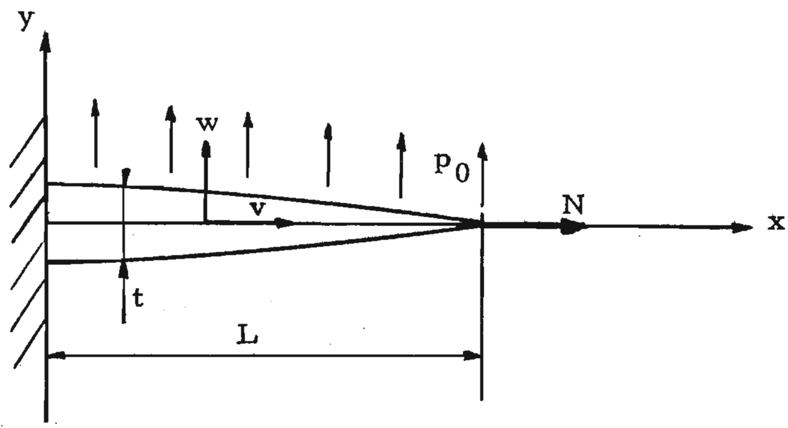
3. Remarks about maximum strength in mixed stress state

We have seen that in pure membrane or pure bending stress state, the optimality condition derived leads to maximum stiffness and maximum strength at the same time. One might be tempted to extend this result to the case of the mixed stress state. The following will show that this first guess is not true.

a) Maximum stiffness design does not lead to maximum strength design.

It will be sufficient if we can find an example where this statement is true to demonstrate this fact.

Consider a cantilever beam subjected to bending moment and traction forces as shown in the figure below where the load and coordinates systems are described.



With these conventions, N, the net traction force acting on the beam, will be positive and p_0, the constant load per unit length, will be negative. v, w are the inplane and normal displacements. The idea of this example is to compare the maximum stress occurring in two structures T^* and T of the same mass M^* and M, but of different design, the load system being the same in both cases.

The structure T^* will obey the design given by the optimality condition for maximum stiffness. The structure T will be designed for a uniform stress σ_0 (yielding stress) in the extreme fiber $(y = +\frac{t}{2})$. The design parameter is the thickness t of the beam. The width b is constant, and E is the Young's modulus.

<u>Structure T^*</u>

The optimality condition and equilibrium equations are derived from the calculus of variation by taking the variation with respect to the design parameter t^* and displacement field v^*, w^*, of the functional

$$\underset{\sim}{P}^* = P^* - \lambda^*\left\{\Omega \int t^* \, dx - M^*\right\} \tag{3.28}$$

where P^* = potential energy

λ^* = a constant Lagrange multiplier

Ω = positive constant equal to ρb

ρ = density of the material

Because the structure is in equilibrium, the potential energy P^* is directly related to the external work W^* by $P^* = -\frac{W^*}{2}$ and the functional $\underset{\sim}{P}^*$, Eq. (3.28) is equivalent to the functional $\underset{\sim}{W}^*$, Eq. (3.23). For this particular example,

$$\underset{\sim}{P}^* = \int_0^L \frac{1}{2}\left\{\frac{Ebt^{*3}}{12}(w^*)''^2 + \frac{bEt^*}{2}(w^*)'^2\right\} dx$$

$$- \int_0^L p_0 \, w^*(x)dx - N\, v^*(L) - \lambda^*\left\{\Omega \int_0^L t^* \, dx - M^*\right\} \tag{3.29}$$

$$(\;)' = \frac{d}{dx}(\;) \; ; \qquad (\;)'' = \frac{d^2}{dx^2}(\;)$$

$\delta \underset{\sim}{P}^* = 0$ with respect to v^*, w^*, t^* will give two equilibrium equations, their associated boundary conditions, and the optimality condition for maximum stiffness

$$\delta_{w^*}(\underset{\sim}{P}^*) = 0 \quad \text{gives} \quad \left(E\frac{bt^{*3}}{12}(w^*)''\right)'' - p_0 = 0 \tag{3.30}$$

$$E \frac{bt^{*3}}{12} (w^*)'' \text{ or } \delta(w^*)' = 0 \text{ at } x = 0, L \tag{3.31}$$

$$\left(E \frac{bt^{*3}}{12} (w^*)''\right)' \text{ or } \delta w^* = 0 \text{ at } x = 0, L \tag{3.32}$$

$\delta_{v^*}(\underset{\sim}{P}^*) = 0$ gives

$$(b\, t^* E(v^*)')' = 0 \tag{3.33}$$

$$(b\, t^* E(v^*)')_{x=L} - N = 0 \tag{3.34}$$

$\delta_{t^*}(\underset{\sim}{P}^*) = 0$ gives

$$\frac{\Omega\lambda^*}{b} = \frac{E}{2}\left[(v^*)'^2 + \frac{t^{*2}}{4}(w^*)''^2\right] \tag{3.35}$$

which is the optimality condition for maximum stiffness.

The boundary conditions are:

$$t^{*3}(w^*)''\Big)_{x=L} = 0 \quad \text{no moment at } x = L \tag{3.36}$$

$$t^{*3}(w^*)''\Big)'_{x=L} = 0 \quad \text{no shear force at } x = L \tag{3.37}$$

$$(w^*)' = w^* = 0 \quad \text{at } x = 0 \quad \text{clamped} \tag{3.38}$$

$$b\, t^* E(v^*)' = N \quad \text{at } x = L \tag{3.39}$$

The solution of these equations is given by

$$t^{*2}_{1,2} = \frac{N^2 \pm \sqrt{N^4 + 72\Omega\lambda^* E b p_0^2 (L-x)^4}}{4\Omega\lambda^* E b} \tag{3.40}$$

The product of the two roots t_1^* and t_2^* is negative. Using a physical argument, we will choose the positive root for t^*.

<u>Structure T</u>

We set the stress in the extreme fiber ($y = +\frac{t}{2}$) equal to σ_0. This stress will be the maximum stress at any location x.

$$\sigma = + \frac{\mathcal{M}}{I}\frac{t}{2} + \frac{N}{bt} = -E\, w'' \frac{t}{2} + \frac{N}{bt} \tag{3.41}$$

where $\mathcal{M}$ = bending moment = $-E\, w'' \frac{bt^3}{12}$, and

I = the moment of Inertia = $\frac{bt^3}{12}$

With the same boundary condition as for T^*, the solution for t is

$$t = \frac{N + \sqrt{N - 12\, b\, \sigma_0 p_0 (L-x)^2}}{2\, b\, \sigma_0} \tag{3.42}$$

Comparison

The following elements of comparison are used:

- $M^* = M$ for T^*, T
- same load, same geometric parameters other than thickness
- we want to compare the absolute maximum generalized stress occurring in T^* and T

First we fix σ_0, p_0, N, L, b. Then calculate t(x) and find by iteration the quantity $\Omega\, \lambda^* E$ such that

$$\int t^* \, dx = \frac{M^*}{\Omega} = \frac{M}{\Omega} = \int t \, dx \tag{3.43}$$

This means that T^* and T have the same mass. $\Omega\, \lambda^* E$ is now known and the equations are solved.

For the structure T^*,

$$\sigma^*_{max} = -\frac{E}{2} (w^*)'' \, t^* + \frac{N}{bt^*} \tag{3.44}$$

Using the optimality condition Eq. (3.35) and the second equilibrium equation, Eq. (3.33), we deduce

$$\sigma^*_{max} = \sqrt{2\,\frac{\Omega\lambda^* E}{b} - \frac{N^2}{b^2\, t^{*2}}} + \frac{N}{b t^*} \tag{3.45}$$

For the structure T, $\sigma_{max} = \sigma_0$ by definition

Numerical results

The following values were chosen:

$L = 40$ in.

$\sigma_0 = 50,000\ lb/in^2$

$p_0 = -6.25\ lb/in$

$N = 12,500$ lbs.

$b = 0.3$ in

$\Omega\lambda^* E$ is found equal to 23.5450×10^7

$\sigma^*_{max} = 56,030\ lb/in^2$ for T^*

$\sigma_{max} = 50,000\ lb/in^2$ for T

so,

$$\sigma_{max} < \sigma^*_{max} \tag{3.46}$$

These results are shown in Fig. 1 for $y = +\frac{t}{2}$. For the same loading conditions, with the same mass of material, the structure T^*, satisfying the optimality condition for maximum stiffness requires a greater yielding stress ($56,030\ lb/in^2$) than T ($50,000\ lb/in^2$) to support the load.

Conclusion: The design using the optimality condition for maximum stiffness does not realize the maximum strength design.

b) Maximum strength for statically determinate problems

If the material is isotropic and linearly elastic, and if we consider small deformations (linearity), the stress distribution in a cross section will be linear but non-antisymmetric with respect to the center line. At a location x, the maximum stress $|\sigma_{max}(x)|$ will occur in one of the surface extreme fibers if we have a mixed stress state. Assume there exists a structure T of design t such that $|\sigma_{max}(x)| \le \sigma_0$ everywhere (σ_0 = the capacity of the material). Because the structure is statically determinate, the loading diagram does not depend upon the design (loading diagram independent of displacement field). Thus, it is always possible to find a thickness t^* such that $t^* \le t$ and $|\sigma^*_{max}(x)| = \sigma_0$. A smaller thickness than t^* will generate $|\sigma_{max}(x)| > \sigma_0$ which is not an acceptable condition. Therefore, t^* is the smallest possible thickness at location x. This is true for all x, so $t^* \le t$ everywhere and $M^* \le M$. Since the structure T^* and T carry the same load, but the structure T^* is lighter than the structure T, T^* can be considered as the maximum strength design. As was stated in the bending only analysis, this argument fails for a statically indeterminate problem, and another approach must be used.

IV. NECESSARY CONDITION FOR STATIONARY VALUE OF THE MASS OF A STRUCTURE SUBJECTED TO A GENERAL STRESS STATE, INCLUDING THE MATERIAL CAPACITY

For the purpose of describing the state of stress at any point in a material, it is convenient to represent each state of stress by a point in a nine dimensional stress space. A basic assumption is made that there exists a scalar function $r(\sigma_{ij})$ which depends on the state of stress and which characterizes the loading of the material. The material capacity can be represented by σ_0, the maximum generalized stress allowed at any point. Generally, σ_0 corresponds to the yielding stress, but it can be any arbitrary value below this yielding stress. The function $\ell(\sigma_{ij}) = r(\sigma_{ij}) - \sigma_0^2$ can be used to compare the loading function to σ_0^2. The equation $\ell(\sigma_{ij}) = 0$ represents a closed surface in the stress space. The function $r(\sigma_{ij})$ is normalized such that as long as $\ell(\sigma_{ij}) \leq 0$ at any point of the structure, we will say that the capacity of the material is not exceeded. $\ell(\sigma_{ij}) > 0$ will mean that the capacity of the material is exceeded and this cannot be an acceptable situation.

The limitation of the material capacity will be expressed by:

$$\ell(\sigma_{ij}) \leq 0 \tag{4.1}$$

or

$$\ell(\sigma_{ij}) + \alpha^2(\underset{\sim}{x}) = 0 \tag{4.2}$$

where $\alpha(\underset{\sim}{x})$ is a scalar function of space. This relation represents one constraint equation.[1]

(1) This description summarizes that given by Fung (Ref. 12).

Another constraint equation comes from the fact that the structure must be in equilibrium. In other words, the design and the displacement field must be compatible so as to satisfy the equilibrium equations. These equilibrium equations can be derived by taking the variation of the Potential Energy with respect to the generalized displacements.

Let us consider the following functional

$$\underset{\sim}{M} = M + \int \left\{ \sum_i \lambda_i(\underset{\sim}{x}) \, \delta_{u_i}(P) \right\} dx + \int \Lambda(\underset{\sim}{x}) \left\{ \ell(\sigma_{ij}) + \alpha^2(\underset{\sim}{x}) \right\} dx \qquad (4.3)$$

where

M	:	mass of the structure given by $\Omega \int t(x)dx$
Ω	:	ρ b, a positive constant
ρ	:	density of the material
b	:	constant width
dx	:	line element for a one dimensional structure or area element for a two dimensional structure
P	:	Potential energy
u_i	:	generalized displacement
$\delta_{u_i}(P)$	:	i^{th} equilibrium equation (variation of potential energy with respect to the generalized displacement u_i)
$\lambda_i(\underset{\sim}{x})$	:	Lagrange multiplier associated with the i^{th} equilibrium equation. It is a scalar function of space $\underset{\sim}{x}$.
σ_{ij}	:	stress
$\ell(\sigma_{ij}) + \alpha^2(\underset{\sim}{x})$	:	material capacity constraint
$\Lambda(\underset{\sim}{x})$	:	Lagrange multiplier associated with this capacity constraint. It is a scalar function of space $\underset{\sim}{x}$.

This functional characterizes the mass M of material under the following constraints

$\delta_{u_i}(P) = 0$	structure is in equilibrium	(4.4)
$\ell(\sigma_{ij})+\alpha^2(\underset{\sim}{x}) = 0$	no generalized stress can exceed the maximum allowable stress σ_0 at any point of the structure.	(4.5)

Providing the variables are continuous and differentiable, we can use the calculus of variation. We take the variation of $\underset{\sim}{M}$ with respect to the variables

t	:	design variable
u_i	:	generalized displacements
$\alpha(\underset{\sim}{x})$	:	unknown switching function
$\lambda_i(\underset{\sim}{x}),\ \Lambda(\underset{\sim}{x})$	:	Lagrange multipliers to restitute the constraint equations (equilibriun equations, stress constraints)

The system of differential equations resulting from $\delta(\underset{\sim}{M}) = 0$ is:

$\delta_{t,u_i,\alpha}(\underset{\sim}{M}) = 0$	Calculus of variation	(4.6)
$\delta_{u_i}(P) = 0$	Equilibrium equations	(4.7)
$\ell(\sigma_{ij})+\alpha^2(\underset{\sim}{x}) = 0$	Stress constraints	(4.8)

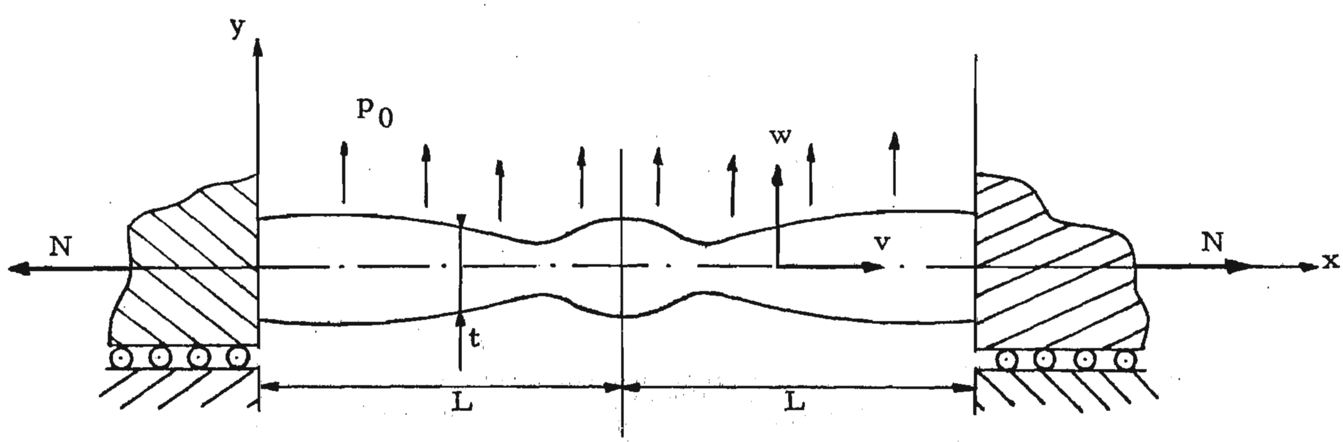
We must add to these equations the appropriate set of boundary conditions to be able to obtain a solution to a particular problem.

The design variable t, coming out of the solution of this system, will provide a structure such that its mass is a stationary value within the considered constraints. Because we assume that the material is homogeneous, the stationary value of the mass of material is equivalent to the stationary value of the volume of the structure.

The solution to this system is a necessary condition for a stationary value of the mass for a given limit of the material capacity. In other words, it is a necessary condition for maximum strength in the case of a general stress state. This general stress state is characterized by the nine components of the stress tensor. and includes the mixed stress state (membrane + bending). Physical examples of such a state are given in the next sections. The first example is the problem of a clamped-clamped beam subjected to traction and bending. The second example is the problem of a clamped-clamped deep arch under a uniform load p_0.

V. EXAMPLE 1: CLAMPED-CLAMPED SOLID BEAM SUBJECTED TO MIXED STRESS STATE

A beam of uniform width b and length 2L is clamped at its two ends. The loading system consists of a constant load per unit length p_0, and of an inplane traction force N. This problem is statically indeterminate. A linear, elastic, isotropic material and small deformations are assumed. The capacity of the material will be defined by its yield stress σ_0. The design variable to be optimized will be the thickness t of the beam. The load and coordinates systems can be described as shown in the following figure



v and w are the inplane and normal displacements. Because the distributed load p_0 is constant, the design will be symmetric with respect to the middle of the beam. (In the example, p_0 is considered negative).

1. Formulation

The usual Bernouilli-Euler beam theory is used. The stress distribution in a cross section is linear. At a location x, the absolute value of the maximum stress will occur in one of the extreme fibers (at $y = +\frac{t}{2}$ or $y = -\frac{t}{2}$).

The constraint equation coming from the finite yield strength of the material will express the fact that, nowhere in the material or on its boundary will exist a stress greater than the yield stress σ_0, in absolute value. Due to the special stress distribution, this is equivalent to $|\sigma_{+t/2}| \leq \sigma_0$ and $|\sigma_{-t/2}| \leq \sigma_0$ everywhere ($\sigma_{+t/2}$, $\sigma_{-t/2}$ are the stresses at $y = \pm t/2$). The capacity constraint of the material Eq. (4.2) of the general form $\ell(\sigma_{ij}) + \alpha^2(\underset{\sim}{x}) = 0$ can be represented in an equivalent form by the two equations:

$$(\sigma_{+t/2})^2 - \sigma_0^2 + \alpha^2(x) = 0 \tag{5.1}$$

and

$$(\sigma_{-t/2})^2 - \sigma_0^2 + \beta^2(x) = 0 \tag{5.2}$$

$\alpha(x)$, $\beta(x)$ are unknown scalar functions of x. The membrane and bending stresses are acting in the same direction, so:

$$\sigma_{+t/2} = \sigma_{B,t/2} + \sigma_M \tag{5.3}$$

$$\sigma_{-t/2} = \sigma_{B,-t/2} + \sigma_M \tag{5.4}$$

where $\sigma_{B,t/2}$, $\sigma_{B,-t/2}$ are the bending stresses at $y = \pm t/2$ and σ_M is the membrane stress (uniform in a cross-section).

$$\sigma_{B,+t/2} = -E\, w'' \frac{t}{2} \; ; \; \sigma_{B,-t/2} = E\, w'' \frac{t}{2} \; ; \; \sigma_M = \frac{N}{b\,t}$$

$$(\;)'' = \frac{d^2}{dx^2}(\;) \tag{5.5}$$

The two constraint equations become:

$$E^2\, w''^2 \frac{t^2}{4} - E\, w'' \frac{N}{b} + \frac{N^2}{b^2 t^2} - \sigma_0^2 + \alpha^2(x) = 0 \text{ (upper surface, } y = \frac{t}{2}) \tag{5.6}$$

$$E^2\, w''^2 \frac{t^2}{4} + E\, w'' \frac{N}{b} + \frac{N^2}{b^2 t^2} - \sigma_0^2 + \beta^2(x) = 0 \text{ (lower surface, } y = -\frac{t}{2}) \tag{5.7}$$

The equilibrium equations can be deduced from the Potential energy P.

$$P = \int_0^{2L} \left[\frac{1}{2} \frac{E\,b\,t^3}{12} w''^2 + \frac{b\,E\,t}{2} v'^2\right] dx - \int_0^{2L} p_0\, w\, dx - N(v\,(2L) - v(0)) \tag{5.8}$$

$$\delta_w P = 0 \quad \text{gives} \quad \left(E \frac{bt^3}{12} w''\right)'' - p_0 = 0 \quad \text{(1st Eq. E)} \tag{5.9}$$

$$\delta_v P = 0 \quad \text{gives} \quad (b\,t\,E\,v')' = 0 \quad \text{(2nd Eq. E)} \tag{5.10}$$

$b\,t\,E\,v'$ represents the inplane force N, so the second equilibrium equation can be integrated directly

$$b\,t\,E\,v' = N \tag{5.11}$$

Because of the symmetry of the structure and of the loading system, it is sufficient to analyze the structure in the interval [0, L]. A stationary value of the mass of material in the interval [0, L] will also be stationary mass in the interval [0, 2L].

Let us consider the functional:

$$\underset{\sim}{M} = \Omega \int_0^L t\,dx$$
$$+ \int_0^L \lambda_1(x)\left[\left(E\frac{bt^3}{12}w''\right)'' - p_0\right]dx + \int_0^L \lambda_2(x)\,[bEtv' - N]dx$$
$$+ \int_0^L \lambda_3(x)\left[E^2 w''^2 \frac{t^2}{4} - Ew''\frac{N}{b} + \frac{N^2}{b^2t^2} - \sigma_0^2 + \alpha^2(x)\right]dx$$
$$+ \int_0^L \lambda_4(x)\left[E^2 w''^2 \frac{t^2}{4} + Ew''\frac{N}{b} + \frac{N^2}{b^2t^2} - \sigma_0^2 + \beta^2(x)\right]dx \qquad (5.12)$$

$\lambda_1(x)$, $\lambda_2(x)$, $\lambda_3(x)$, $\lambda_4(x)$ are unknown Lagrange multipliers which are functions of x, associated with the four constraint equations.
It has been seen before that

$$\delta \underset{\sim}{M} = 0 \qquad (5.13)$$

i.e. the variation of $\underset{\sim}{M}$ with respect to all the variables of the problem (Eq. (4.6), (4.7), 4.8)), will correspond to a stationary value of the mass of material within the above constraints. We take the variation of $\underset{\sim}{M}$ with respect to the variables t, w, v, α, β, λ_1, λ_2, λ_3, λ_4, carry out appropriate integrations by parts, and set the variable coefficients of δt, δw, δv, $\delta\alpha$, $\delta\beta$, $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$, $\delta\lambda_4$ and the terms coming from integration by parts, equal to zero. We then obtain the corresponding Euler's equations, constraint equations (equilibrium and material capacity), and set of boundary conditions governing the solution for: t, w, v, α, β, λ_1, λ_2, λ_3, λ_4.

• Equations:

$$\Omega + \frac{Eb}{12}\left[\lambda_1''(x)\,(w''\ 3t^2)\right] + \lambda_2(x)\, b\, E\, v'$$

$$+ (\lambda_3(x) + \lambda_4(x))\left(\frac{E^2 w''^2}{2}\, t - \frac{2N^2}{b^2 t^3}\right) = 0 \qquad (5.14)$$

$$\left[\frac{Eb}{12}\lambda_1''(x)\, t^3 + (\lambda_3(x) + \lambda_4(x))\frac{E^2 w'' t^2}{2} - (\lambda_3(x) - \lambda_4(x))\frac{EN}{b}\right]'' = 0 \qquad (5.15)$$

$$(\lambda_2(x)\, b\, E\, t)' = 0 \qquad (5.16)$$

$$\lambda_3(x)\,\alpha(x) = 0 \qquad (5.17)$$

$$\lambda_4(x)\,\beta(x) = 0 \qquad (5.18)$$

$$\left(\frac{Eb}{12}\, t^3\, w''\right)'' - p_0 = 0 \qquad (5.19)$$

$$b\, E\, t\, v' - N = 0 \qquad (5.20)$$

$$E^2 w''^2 \frac{t^2}{4} - E\, w'' \frac{N}{b} + \frac{N^2}{b^2 t^2} - \sigma_0^2 + \alpha^2(x) = 0 \qquad (5.21)$$

$$E^2 w''^2 \frac{t^2}{4} + E\, w'' \frac{N}{b} + \frac{N^2}{b^2 t^2} - \sigma_0^2 + \beta^2(x) = 0 \qquad (5.22)$$

• Boundary conditions from calculus of variations:

$$\left\{\lambda_1(x)\,(w''\ 3t^2)' - \lambda_1'(x)\,(w''\ 3t^2)\right\}\delta t\,\Big|_0^L = 0 \qquad (5.23)$$

$$\left\{\lambda_1(x)\,(w''\ 3t^2)\right\}\delta t'\,\Big|_0^L = 0 \qquad (5.24)$$

$$\left[\frac{Eb}{12}(\lambda_1''(x)\, t^3) + (\lambda_3(x) + \lambda_4(x))\left(\frac{E^2 w'' t^2}{2}\right) - \left(\lambda_3(x) - \lambda_4(x)\right)\frac{EN}{b}\right]'\delta w\,\Big|_0^L = 0 \qquad (5.25)$$

$$\left[\frac{Eb}{12}(\lambda_1''(x)\, t^3) + (\lambda_3(x) + \lambda_4(x))\frac{E^2 w'' t^2}{2} - (\lambda_3(x) - \lambda_4(x))\frac{EN}{b}\right]\delta w'\,\Big|_0^L = 0 \qquad (5.26)$$

$$\left\{\lambda_1'(x)\, t^3 - \lambda_1(x)\, t^{3\prime}\right\} \delta w'' \Big|_0^L = 0 \qquad (5.27)$$

$$[\lambda_1(x)\, t^3]\, \delta w''' \Big|_0^L = 0 \qquad (5.28)$$

$$\{\lambda_2(x)\, b\, E\, t\}\, \delta v \Big|_0^L = 0 \qquad (5.29)$$

• Boundary conditions from equilibrium equations

$w'(x = 0)$	=	0	clamped at $x = 0$	(5.30)
$w'(x = L)$	=	0	symmetry at $x = L$	(5.31)
$w\,(x = 0)$	=	0	clamped at $x = 0$	(5.32)
$(w''\, t^3)'_{x=L}$	=	0	symmetry implies shear force = 0 at $x = L$	(5.33)
$v\,(x = L)$	=	0	symmetry	(5.34)

2. Necessary condition for a stationary value of M

It is assumed that there exists at least one solution of this system of equations, leading to a stationary value of M. This system can be greatly simplified, and a physical signification will be given to the condition characterizing this simplification. For convenience, we will specify two different loading systems: $p_0 \neq 0$ and $p_0 = 0$.

$p_0 \neq 0$.

Assume $\alpha^2(x) \neq 0$ and $\beta^2(x) \neq 0$. Equations (5.17) and (5.18) produce

$$\lambda_3(x) \equiv \lambda_4(x) \equiv 0 \qquad (5.35)$$

so $$\lambda_3'(x) \equiv \lambda_4'(x) \equiv 0 \qquad (5.36)$$

and $$\lambda_3''(x) \equiv \lambda_4''(x) \equiv 0 \qquad (5.37)$$

Integration of Eq. (5. 16) gives

$$\lambda_2(x)\, b\, E\, t \; = \; \text{constant} \tag{5.38}$$

but, from Eq. (5. 29),

$$\lambda_2 b\, E\, t)_{x=0} \; = \; 0 \quad \text{because} \quad \delta\, v(x=0) \neq 0 \tag{5.39}$$

We deduce

$$\lambda_2(x)\, b\, E\, t \; \equiv \; 0 \quad \text{in} \quad [0, L] \tag{5.40}$$

$$\text{or } \lambda_2(x) \; \equiv \; 0 \quad \text{because} \quad t \not\equiv 0 \tag{5.41}$$

Eq. (5. 15) reduces to

$$(\lambda_1''(x)\, t^3)'' \; = \; 0 \tag{5.42}$$

after integration,

$$(\lambda_1''(x)\, t^3)' \; = \; k_1 \tag{5.43}$$

but, from Eq. (5. 25)

$$(\lambda_1''\, t^3)'_{x=L} \; = \; 0 \quad \text{because} \quad \delta_w(x=L) \neq 0 \tag{5.44}$$

so, $k_1 = 0$ and

$$(\lambda_1''(x)\, t^3)' \equiv 0 \quad \text{in } [0, L] \tag{5.45}$$

integration gives

$$\lambda_1''\, t^3 \; = \; k_2 \tag{5.46}$$

We substitute this value into Eq. (5. 14) and obtain

$$\frac{\Omega}{b} + \frac{E}{12}\, 3 k_2\, \frac{w''}{t} \; = \; 0 \tag{5.47}$$

or $\frac{w''}{t}$ = constant (5.48)

This is a contradiction because, $w'(x=0) = 0$, $w'(x=L) = 0$ and $w'(x) \neq 0$ for some x belonging to the open interval $]0, L[$. This means that $w'(x)$ has at least one extremum at $x = x_e$. At this extremum, $w''(x_e) = 0$. So, $w''(x)$ must change its sign at least one time in $[0, L]$. But, t has a constant positive sign, so, $\frac{w''}{t}$ = constant shows that $w''(x)$ has a constant sign. The only solution is $w''(x) \equiv 0$. This implies $\Omega = 0$ from Eq. (5.67). This is a contradiction because Ω is different from zero by definition (Eq. (2.17)). Therefore, $\alpha^2(x) \neq 0$ and $\beta^2(x) \neq 0$ are not solutions of this system. At this point, it is convenient to study separately the two loading cases: $N \neq 0$ and $N = 0$.

<u>$N \neq 0$</u> (mixed stress case)

Subtracting Eq. (5.22) from Eq. (5.21), we obtain:

$$- 2 E w'' \frac{N}{b} + (\alpha^2(x) - \beta^2(x)) = 0 \tag{5.49}$$

This equation is nothing else than $\sigma^2_{t/2} - \sigma^2_{-t/2} + (\alpha^2(x) - \beta^2(x)) = 0$. By assumption, $b \neq 0$, $N \neq 0$, $w'' \not\equiv 0$ because $p_o \neq 0$ (Eq. (5.19)). (w'' can be equal to zero only at some discrete point). From Eq. (5.49), we deduce

$$\alpha^2(x) - \beta^2(x) \neq 0 \quad \text{or} \quad \alpha^2(x) \neq \beta^2(x) \tag{5.50}$$

Since $\alpha^2(x) \neq 0$ and $\beta^2(x) \neq 0$ is not a possible solution, the only solution is:

$$\alpha^2(x) = 0 \quad \text{and} \quad \beta^2(x) \neq 0 \tag{5.51}$$

or $\alpha^2(x) \neq 0$ and $\beta^2(x) = 0$ (5.52)

because

$$\alpha^2(x) \neq \beta^2(x)$$

The physical meaning to these equations is that the yielding stress σ_0 must be reached for all x, in the upper extreme fiber at $y = +\frac{t}{2}$, or, in the lower extreme fiber at $y = -\frac{t}{2}$. (At some discrete point where $w'' = 0$, $\sigma = \sigma_0$ everywhere in the cross section). This is the necessary condition for a stationary value of the mass of material for $p_0 \neq 0$ and $N \neq 0$.

N = 0 (pure bending stress state)

Equation (5.49) reduces to

$$\alpha^2(x) - \beta^2(x) \equiv 0 \text{ which implies } \alpha^2(x) \equiv \beta^2(x) \quad (5.53)$$

Because $\alpha^2(x) \neq 0$, $\beta^2(x) \neq 0$ cannot be a solution of the system, the only remaining possibility is:

$$\alpha^2(x) \equiv \beta^2(x) \equiv 0 \quad (5.54)$$

The physical meaning of this is that the yielding stress σ_0 is reached in the two extreme fibers of the structure, at any location x. This is the necessary condition for a stationary value for the mass of the material for $p_0 \neq 0$ and $N = 0$.

$p_0 = 0$ and $N \neq 0$

Assume $\alpha^2(x) \neq 0$ and $\beta^2(x) \neq 0$. Equations (5.17) and (5.18) produce

$$\lambda_3(x) \equiv \lambda_4(x) \equiv 0 \tag{5.55}$$

so $$\lambda_3'(x) \equiv \lambda_4'(x) \equiv 0 \tag{5.56}$$

and $$\lambda_3''(x) \equiv \lambda_4''(x) \equiv 0 \tag{5.57}$$

Again, for exactly the same reasons as in the previous part $(t \neq 0)$,

$$\lambda_2(x) \equiv 0 \qquad \text{(using Eqs. (5.16), (5.29))} \tag{5.58}$$

if $p_0 = 0$, we have no bending moment, and as a consequency

$$w''(x) \equiv 0 \tag{5.59}$$

Substituting $w''(x)$, $\lambda_2(x)$, $\lambda_3(x)$, $\lambda_4(x)$ into (5.14) we obtain

$$\Omega = 0 \tag{5.60}$$

which is a contradiction because $\Omega = \rho b$ is different from zero.

So, $\alpha^2(x) \neq 0$ and $\beta^2(x) \neq 0$ cannot be solutions.

Again, subtract (5.22) from (5.21) and obtain

$$\alpha^2(x) = \beta^2(x) \quad \text{because} \quad w''(x) = 0 \tag{5.61}$$

The only solutions can be $\alpha^2(x) \equiv \beta^2(x) \equiv 0$. The physical meaning of this is that the yield stress σ_0 is equal to the membrane stress for all x. The structure is completely yielded. This is a necessary condition for a stationary value for the mass of material for $p_0 = 0$ and $N \neq 0$.

3. Solution

For normalization purpose, let $\xi = \frac{x}{L}$.

As a consequence, $(\ )' = \frac{d}{dx}(\ ) = \frac{d}{d\xi}(\ )\frac{1}{L}$.

The new interval for the dimensionless variable ξ is $[0, 1]$. The loading system consists of an inplane traction force $N > 0$ and a constant load per unit length $p_0 < 0$.

We have seen before that, because of the particular boundary condition (clamped at $\xi = 0$ and symmetry), $w''(\xi)$ must change its sign at least one time in the interval $[0, 1]$. Multiple changes of sign are not excluded at this point, but, as will be seen in the following, this does not correspond to the physics of the problem.

If $w''(\xi) > 0$, Eq. (5.49) gives $\alpha^2(\xi) > \beta^2(\xi)$. and the condition defined by Eqs. (5.51), (5.52) states that

$$\alpha^2(\xi) \neq 0 \quad \text{and} \quad \beta^2(\xi) \equiv 0 \tag{5.62}$$

If $w''(\xi) < 0$, Eq. (5.49) gives $\alpha^2(\xi) < \beta^2(\xi)$ and the condition defined by Eqs. (5.51), (5.52) is:

$$\alpha^2(\xi) \equiv 0 \quad \text{and} \quad \beta^2(\xi) \neq 0 \tag{5.63}$$

The important location ξ_0 where $w''(\xi_0) = 0$ will be called a "switching point". When we cross the switching point, the structure, previously yielded in its upper surface ($y = +\frac{t}{2}$) will be yielded in its lower surface ($y = -\frac{t}{2}$) and vice versa.

Let us assume there exists only one switching point ξ_0 in the interval $[0, 1]$. This comes from the physical argument that the deflection of the beam has only one inflection point in $[0, 1]$. Since p_0 is negative, the resultant displacement w must be negative. To characterize this type of deflection, it is reasonable to assume

$$w''(\xi) \leq 0 \quad \text{for} \quad 0 \leq \xi \leq \xi_0 \tag{5.64}$$

and

$$w''(\xi) \geq 0 \quad \text{for} \quad \xi_0 \leq \xi \leq 1 \tag{5.65}$$

The location ξ_0, still unknown, must be determined such that the boundary conditions and equations of the problem are satisfied. Using the normalized variable, the two equilibrium equations are transformed into:

$$\left(\frac{E\,b\,t^3}{12\,L^4}\,w''\right)'' - p_0 = 0 \tag{5.66}$$

and

$$\frac{b\,E\,t}{L}\,v' - N = 0 \tag{5.67}$$

Double integration of the first one and utilization of boundary condition Eq. (5.33) gives

$$w'' = \frac{p_0\,12\,L^4}{E\,b\,t^3}\left(\frac{\xi^2}{2} - \xi + \gamma_0\right) \tag{5.68}$$

By assumption, $w''(\xi_0) = 0$ (switching point). So,

$$\gamma_0 = -\frac{\xi_0^2}{2} + \xi_0 \tag{5.69}$$

It is convenient to analyze the solutions separately in the two intervals $[0, \xi_0]$ and $[\xi_0, 1]$. We will use the subscript 1 for variables of the first interval and subscript 2 for variables of the second interval.

<u>$0 \leq \xi \leq \xi_0$</u>

$w_1''(\xi)$ is < 0 by assumption. This leads to $\alpha^2(\xi) \equiv 0$ and $\beta^2(\xi) \neq 0$. The extreme fiber of the upper surface of the beam ($y = +\frac{t_1}{2}$) is yielded in tension.

Eq. (5.21) can be rewritten in terms of the normalized variable:

$$- \frac{E w_1'' t_1}{2 L^2} + \frac{N}{b t_1} = \sigma_0 \tag{5.70}$$

Direct substitution of w_1'' from Eq. (5.68) into Eq. (5.70) gives

$$t_1^2 \sigma_0 - t_1 \frac{N}{b} + 6 \frac{p_0 L^2}{b} \left(\frac{\xi^2}{2} - \xi + \gamma_0\right) = 0 \tag{5.71}$$

The product of two roots is negative, so the design t_1, will be given by the positive root.

$$t_1 = \frac{N}{2 b \sigma_0} \left(1 + \sqrt{1 - 24 \omega_b \left(\frac{\xi^2}{2} - \xi + \gamma_0\right)}\right) \tag{5.72}$$

where $\omega_b = \frac{p_0 \sigma_0 b L^2}{N^2}$ is called the Load parameter.

$\underline{\xi_0 \leq \xi \leq 1}$

In this domain, w_2'' is > 0 by assumption. So, $\alpha^2(\xi) \neq 0$ and $\beta^2(\xi) \equiv 0$. The extreme fiber of the lower surface of the beam ($y = -\frac{t_2}{2}$) is yielded in tension. Eq. (5.22) in terms of normalized variables is:

$$E w_2'' \frac{t_2}{2 L^2} + \frac{N}{b t_2} = \sigma_0 \tag{5.73}$$

Again, substitution of w_2'' from Eq. (5.68) into Eq. (5.73) gives

$$t_2^2 \sigma_0 - t_2 \frac{N}{b} - 6 \frac{p_0 L^2}{b} \left(\frac{\xi^2}{2} - \xi + \gamma_0\right) = 0 \tag{5.74}$$

For the same reason, we choose the positive root

$$t_2 = \frac{N}{2 b \sigma_0} \left(1 + \sqrt{1 + 24 \omega_b \left(\frac{\xi^2}{2} - \xi + \gamma_0\right)}\right) \tag{5.75}$$

with the load parameter $\omega_b = \dfrac{p_0 \sigma_0 b L^2}{N^2}$

<u>Determination of $\xi_0 = \xi_0(\omega_b)$ from boundary conditions</u>

Integration of Eq. (5.68) gives

$$w'(\xi) = \int_0^{\xi} \frac{12 p_0 L^4}{E b t_1^3} \left(\frac{\xi^2}{2} - \xi + \gamma_0\right) d\xi \qquad \text{for } \xi \in [0, \xi_0] \tag{5.76}$$

and

$$w'(\xi) = w'(\xi_0) + \int_{\xi_0}^{\xi} \frac{12 p_0 L^4}{E b t_2^3} \left(\frac{\xi^2}{2} - \xi + \gamma_0\right) d\xi \quad \text{for } \xi \in [\xi_0, 1] \tag{5.77}$$

$w'(0) = 0$ is automatically satisfied.

$w'(1) = 0$ will give the following integral condition to find the unknown constant γ_0.

$$\int_0^{\xi_0} \frac{12 p_0 L^4}{E b t_1^3} \left(\frac{\xi^2}{2} - \xi + \gamma_0\right) d\xi + \int_{\xi_0}^{1} \frac{12 p_0 L^4}{E b t_2^3} \left(\frac{\xi^2}{2} - \xi + \gamma_0\right) d\xi = 0$$

or

$$\int_0^{\xi_0} \left(\frac{\xi^2}{2} - \xi + \gamma_0\right) \frac{d\xi}{\left[1 + \sqrt{1 - 24\,\omega_b\left(\frac{\xi^2}{2} - \xi + \gamma_0\right)}\right]^3} + \int_{\xi_0}^{1} \frac{\left(\frac{\xi^2}{2} - \xi + \gamma_0\right) d\xi}{\left[1 + \sqrt{1 + 24\,\omega_b\left(\xi^2 - \xi + \gamma_0\right)}\right]^3} = 0 \tag{5.78}$$

Closed form integration appeared difficult, so numerical techniques were used to find $\xi_0 = \xi_0(\omega_b)$.

In the case of small or large values of the load parameter ω_b, it is possible to find a closed form solution for ξ_0 or γ_0.

Asymptotic value of ξ_0 for small values of $\omega_b = \dfrac{p_0\,\sigma_0\,b\,L^2}{N^2}$

Let $\omega_b = \epsilon$, small but different from zero.

Because $(\frac{\xi^2}{2} - \xi + \gamma_0)$ is bounded, in [0, 1], it is possible to perform the expansion of the equation (5.78). Keeping only the first order terms, (5.78) is simplified in the following manner:

$$\int_0^{\xi_0} \left\{ (\frac{\xi^2}{2} - \xi + \gamma_0) + (\frac{\xi^2}{2} - \xi + \gamma_0)^2\, 18\epsilon \right\} d\xi$$

$$+ \int_{\xi_0}^{1} \left\{ (\frac{\xi^2}{2} - \xi + \gamma_0) - (\frac{\xi^2}{2} - \xi + \gamma_0)^2\, 18\epsilon \right\} d\xi + 0(\epsilon^2) = 0 \qquad (5.79)$$

Integration can be performed in closed form, and we obtain

$$-\frac{1}{3} - \frac{\xi_0^2}{2} + \xi_0 + 18\epsilon \left(\frac{4}{15}\xi_0^5 - \frac{13}{12}\xi_0^4 + \frac{5}{3}\xi_0^3 - \frac{4}{3}\xi_0^2 + \frac{2}{3}\xi_0 - \frac{2}{15} \right) + 0(\epsilon^2) = 0 \qquad (5.80)$$

When ϵ tends to zero, $\xi_0(0)$ is the solution to the equation

$$-\frac{1}{3} - \frac{\xi_0^2(0)}{2} + \xi_0(0) = 0 \qquad (5.81)$$

giving

$$\xi_0(0) = 1 - \frac{1}{\sqrt{3}} = 0.42\,265 \quad \text{only root in } [0, 1]$$

To know the behavior in the neighborhood of $\xi_0(0)$, let $\xi_0 = \xi_0(0) + \delta$ where δ is small. ξ_0, ξ_0^2, ξ_0^3, ξ_0^4, ξ_0^5 can be approximated. We substitute these values in equation (5.80), use equation (5.81) and we obtain, after keeping only the first order terms

$$\delta(1 - \xi_0(0)) + 18\varepsilon D + 0(\varepsilon\delta) + 0(\delta^2) + 0(\varepsilon^2) = 0 \qquad (5.82)$$

where

$$D = \frac{4}{15}\xi_0^5(0) - \frac{13}{12}\xi_0^4(0) + \frac{5}{3}\xi_0^3(0) - \frac{4}{3}\xi_0^2(0) + \frac{2}{3}\xi_0(0) - \frac{2}{15}$$

Substitution of the numerical value of $\xi_0(0)$ in this equation gives:

$$\delta = -0.159487 \times \varepsilon$$

or $$\varepsilon = -6.270100 \times \delta$$

For small values ε of the load parameter ω_b, the curve $\xi_0(\omega_b)$ can be approximated by

$$\xi_0 = \xi_0(0) - 0.159487 \times \varepsilon \qquad (5.83)$$

This corresponds to the straight line shown in Fig. 2.

Asymptotic value of ξ_0 for large values of ω_b

ω_b becomes large when the bending effect is much larger than the membrane effect. As a limit case, let's consider the case of pure bending, when N = 0. The design variable t reduces to

$$t_1 = \sqrt{-6\frac{p_0 L^2}{\sigma_0 b}\left(\frac{\xi^2}{2} - \xi + \gamma_0\right)} \quad \text{in } 0 \le \xi \le \xi_0 \qquad (5.84)$$

and

$$t_2 = \sqrt{6\frac{p_0 L^2}{\sigma_0 b}\left(\frac{\xi^2}{2} - \xi + \gamma_0\right)} \quad \text{in } \xi_0 \le \xi \le 1 \qquad (5.85)$$

The integral condition (5.78) to find ξ_0 is given by boundary conditions $w'(0) = 0$ and $w'(1) = 0$. It reduces to

$$\int_0^{\xi_0} \frac{d\xi}{\left[\frac{\xi^2}{2} - \xi + \gamma_0\right]^{\frac{1}{2}}} - \int_{\xi_0}^{1} \frac{d\xi}{\left[-(\frac{\xi^2}{2} - \xi + \gamma_0)\right]^{\frac{1}{2}}} = 0 \qquad (5.86)$$

There is a singularity of the integrand at $\xi = \xi_0$, but we know that a singularity of order $\frac{1}{2}$ is integrable. So, we will not run into trouble. After integration, Eq. (5.96) becomes

$$\mathrm{Log}\left\{\frac{\xi_0 - 1}{\sqrt{2(-\frac{\xi_0^2}{2} + \xi_0) - 1}}\right\} = \frac{\pi}{2} \qquad (5.87)$$

which can be written:

$$\xi_0 - 1 - e^{\pi/2}\left\{\sqrt{2\left(-\frac{\xi_0^2}{2} + \xi_0\right) - 1}\right\} = 0 \qquad (5.88)$$

The solution of this equation in the interval [0, 1] is

$$\xi_0 = 0.6015$$

This is the vertical asymptote of Figs. 3, 4. The values of ξ_0 displayed in Figs. 2, 3, 4 were found by numerically evaluating Eq. 5.78. As can be seen from the figures the numerical results agree nicely with the asymptotic value previously found. It is interesting to note that the switching point is always located in the rather narrow region $0.42 \le \xi_0 \le 0.60$.

4. <u>Comparison with the equivalent uniform thickness beam</u>

The equivalent beam of uniform thickness (subscript u) will be the uniform thickness beam of minimum mass able to support the load system without exceeding the yield stress σ_0.

Uniform thickness design

Triple integration of the first equilibrium equation (5.6 6), use of boundary conditions $w_u'(0) = 0$, $w_u'(1) = 0$ (clamped at $\xi = 0$ and symmetry) and $w_u'''(1) = 0$ (no shear force at the center for symmetry reasons) gives:

$$\frac{E b t_u^3}{12} \quad w_u' = p_0 L^4 \left(\frac{\xi^3}{6} - \frac{\xi^2}{2} + \frac{\xi}{3} \right) \tag{5.89}$$

The stress in a fiber is

$$\sigma(y, \xi) = - \frac{E w_u''}{L^2} y + \frac{N}{b t_u} \tag{5.90}$$

$w_u''(\xi)$ can be found from Eq. (5.89). It can be checked that the maximum stress in absolute value σ_{max}, occurs at $\xi = 0$ and $y = + \frac{t_u}{2}$.

Let $\sigma_{max} = \sigma_0$; we obtain

$$\sigma_{max} = \sigma_0 = \sigma_{+\frac{t_u}{2}}(0) = - \frac{2 p_0 L^2}{b t_u^2} + \frac{N}{b t_u} \tag{5.91}$$

which is equivalent to:

$$t_u^2 \sigma_0 - t_u \frac{N}{b} + 2 \frac{p_0 L^2}{b} = 0 \tag{5.92}$$

The product of the two roots t_{u_1}, t_{u_2} is negative, so, for physical reasons, we choose the positive root:

$$t_u = \frac{N}{2 b \sigma_0} \left(1 + \sqrt{1 - 8 \omega_b} \right) \tag{5.93}$$

where $\omega_b = \dfrac{p_0 \sigma_0 b L^2}{N^2}$

This is the thickness of the minimum mass uniform thickness beam able to support the load system p_0 and N.

Normalization:

The design which corresponds to an extremum of M (uniform strength), can be normalized by the design of the uniform thickness beam

$$\underset{\sim}{t}_1 = \frac{1 + \sqrt{1 - 24\,\omega_b \left(\frac{\xi^2}{2} - \xi + \gamma_0\right)}}{1 + \sqrt{1 - 8\,\omega_b}} \qquad 0 \le \xi \le \xi_0 \qquad (5.94)$$

$$\underset{\sim}{t}_2 = \frac{1 + \sqrt{1 + 24\,\omega_b \left(\frac{\xi^2}{2} - \xi + \gamma_0\right)}}{1 + \sqrt{1 - 8\,\omega_b}} \qquad \xi_0 \le \xi \le 1 \qquad (5.95)$$

Figure 5 represents $\underset{\sim}{t} = \underset{\sim}{t}(\omega_b, \xi)$. In order to quantify the gain in mass realized by using the design of uniform strength, let us form the quantity

$$G(\omega_b) = \frac{M_{\text{uniform strength}}}{M_{\text{uniform thickness}}} = \int_0^{\xi_0} \underset{\sim}{t}_1 \, d\xi + \int_{\xi_0}^{1} \underset{\sim}{t}_2 \, d\xi \qquad (5.96)$$

It is seen in Fig. 6 that an appreciable gain is achieved for values of ω_b less than -4. The advantage of using the uniform strength design instead of the uniform thickness design is evident as soon as the bending due to the load system is not negligible. Maximum gain occurs for full bending ($\omega_b \to -\infty$).

For large values of $|\omega_b|$,

$$\underset{\sim}{t}_1 \sim \sqrt{3}\sqrt{\left(\frac{\xi^2}{2} - \xi + \gamma_0\right)} \qquad 0 \le \xi \le \xi_0 \qquad (5.97)$$

$$t_2 \sim \sqrt{3}\sqrt{-\left(\frac{\xi^2}{2} - \xi + \gamma_0\right)} \qquad \xi_0 \le \xi \le 1 \qquad (5.98)$$

So,

$$\underset{\omega_b \to -\infty}{G(\omega_b)} = \int_0^{\xi_0} \underset{\sim}{t}_1 d\xi + \int_{\xi_0}^{1} \underset{\sim}{t}_2 \, d\xi = \frac{\sqrt{3}}{2}\sqrt{\gamma_0} \quad \text{with } \gamma_0 = -\frac{\xi_0^2}{2} + \xi_0 \qquad (5.99)$$

For large values of $|\omega_b|$, $\xi_o \sim 0.6015$ (found in previous part),

so $$\lim_{\omega_b \to -\infty} G(\omega_b) = 0.561327$$

This asymptote is shown in Fig. 6

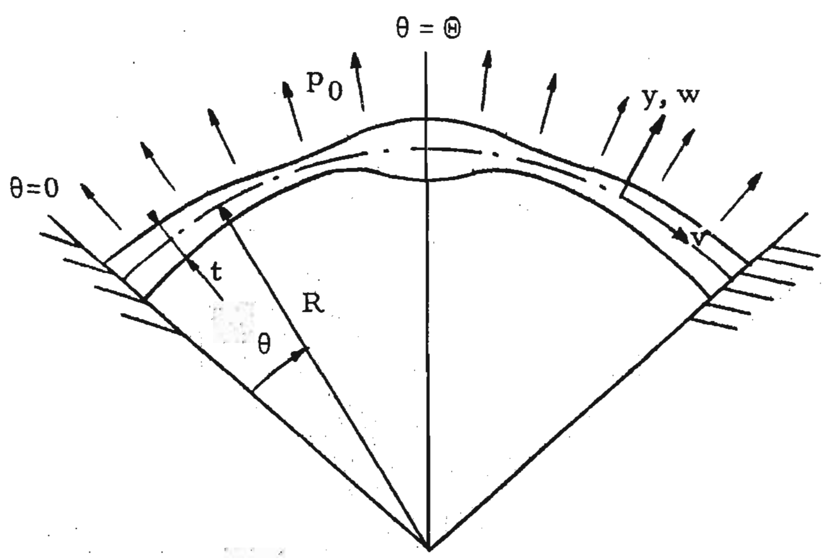
5. Concluding remarks

It should be remembered that the above analysis neglects the shear stresses due to the normal shear force. This assumption becomes invalid when the thickness is small, and in particular for the predominant bending case, near the switching point where the shear stress goes to infinity. Shear stresses can be considered if included in the evaluation of the maximum local generalized stress which is compared to σ_0. Another approach would be to give a minimum allowable thickness. This condition would generate a constraint of the form $t - t_{minimum} - \bar{\gamma}^2(\underset{\sim}{x}) = 0$ where $\bar{\gamma}(\underset{\sim}{x})$, an unknown scalar function of space $\underset{\sim}{x}$. The assumption of only one switching point in the interval $[0, 1]$ permits us to find at least one solution. It is the more realistic physical consideration. Compared to the uniform thickness beam, a saving of 40% or more can be achieved for values of $|\omega_b| \geq 16$ by using a proper design. This corresponds to a greater contribution of bending in the total stress: $\sigma = \sigma_B + \sigma_M$.

VI. EXAMPLE 2: CLAMPED-CLAMPED CIRCULAR ARCH

A solid circular arch of radius R, uniform width b and total opening angle of 2 Θ, has its two ends rigidly clamped. A uniform load per unit length p_0 constitutes the loading. Because of the arch effect, this load will generate bending and membrane stresses. One of the main differences from the beam problem is that the membrane stress and bending stress are coupled. A linear elastic isotropic material and small deformations are assumed. The yielding stress σ_0 characterizes the capacity of the material. The width b is assumed constant, so, the design variable will be the thickness t.

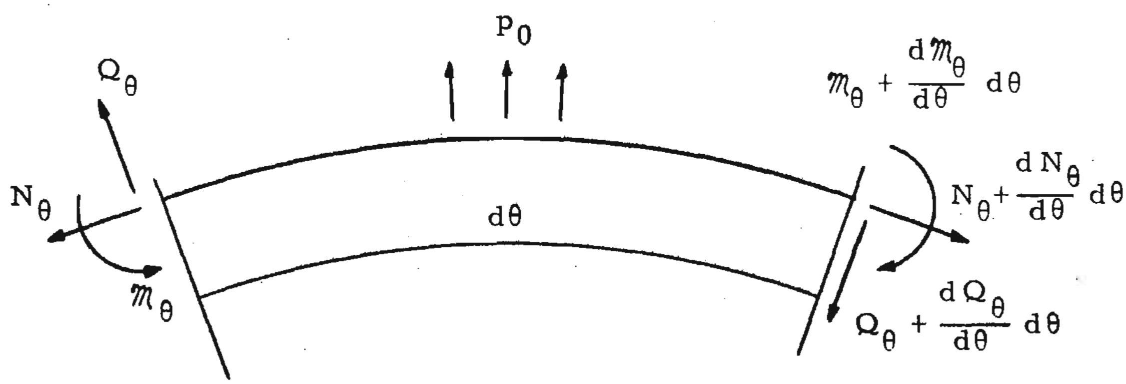
This problem is statically indeterminate and an analysis similar to the beam analysis will be used to derive the necessary condition for maximum strength. The coordinates systems and the loading are shown in the following figure.



w and v are the normal and inplane displacements. The load per unit length p_0 is acting along the y axis and will be negative, so that there will be a compressive force generated in the arch. The design is symmetric with respect to $\theta = \Theta$. No consideration is given to stability, which could be the design constraint for a physical problem.

1. Preliminary

The equilibrium equations can be deduced directly from the diagram:



they are:

$$\frac{dN_\theta}{d\theta} - Q_\theta = 0 \tag{6.1}$$

$$-\frac{dQ_\theta}{d\theta} - N_\theta + p_0 R = 0 \tag{6.2}$$

$$-\frac{d\mathcal{M}_\theta}{d\theta} - Q_\theta R = 0 \tag{6.3}$$

with N_θ = inplane force

Q_θ = normal shear force

$\mathcal{M}_\theta$ = bending moment

From geometrical considerations,

$$\text{curvature} = K = -\frac{1}{R^2}(w'' + v') \tag{6.4}$$

$$\text{mid surface strain} = \epsilon_0 = \frac{1}{R}(v' + w) \quad \text{where } (\ \ )' = \frac{d}{d\theta} \tag{6.5}$$

The stress is given by $\sigma(y, \theta) = E(\epsilon_0 + yK)$ (6.6)

$$\sigma(y, \theta) = E\left[\frac{1}{R}(v'+w) - y\frac{1}{R^2}(w'' + v')\right] \tag{6.7}$$

so

$$N_\theta = b\int_{-t/2}^{+t/2} \sigma(y, \theta)dy = btE\frac{1}{R}(v' + w) \tag{6.8}$$

$$\mathcal{M}_\theta = b\int_{-t/2}^{t/2} \sigma(y,\theta)\, y\, dy = -bE\frac{t^3}{12R^2}(w''+v') \tag{6.9}$$

After substitution of N_θ, $\mathcal{M}_\theta$, the equilibrium equations reduce to

$$[t(v'+w)]' - \frac{1}{12R^2}[t^3(w''+v')]' = 0 \tag{6.10}$$

$$-\frac{E}{R^2}t(v'+w) - \frac{E}{12R^4}[t^3(w''+v')]'' + \frac{p_0}{b} = 0 \tag{6.11}$$

2. Formulation of the Problem

The shear stresses are assumed negligible in this analysis. The stress distribution in a cross section is linear but non-antisymmetric. (It would be antisymmetric if $(v'+w)$ were identical to zero. But this would lead to $p_0 = 0$ which is not compatible with the original assumption $p_0 \neq 0$). For this reason, the maximum stress in absolute value $|\sigma_{max}|$ will occur in one of the extreme fibers ($y = +\frac{t}{2}$ or $y = -\frac{t}{2}$). The constraint equation coming from the limited capacity of the material can be formulated exactly the same way as for the beam, namely,

$$|\sigma_{+t/2}| \leq \sigma_0$$

$$|\sigma_{-t/2}| \leq \sigma_0$$

which is equivalent to the two equations:

$$E^2\left\{\frac{1}{R^2}(v'+w)^2 - \frac{t}{R^3}(v'+w)(w''+v') + \frac{t^2}{4R^4}(w''+v')^2\right\} - \sigma_0^2 + \alpha^2(\theta) = 0 \quad (6.12)$$

$$E^2\left\{\frac{1}{R^2}(v'+w)^2 + \frac{t}{R^3}(v'+w)(w''+v') + \frac{t^2}{4R^4}(w''+v')^2\right\} - \sigma_0^2 + \beta^2(\theta) = 0 \quad (6.13)$$

$\alpha(\theta)$ and $\beta(\theta)$ are two switching functions unspecified at this point. The load p_0 is symmetric with respect to $\theta = \Theta$. As a consequence, the design will also be symmetric with respect to this point, and it is sufficient to analyze the structure in the interval $[0, \Theta]$ only.

Let us consider the functional

$$\begin{aligned}
\underset{\sim}{M} = & \int_\mu^\nu \Omega \, t \, d\theta \\
& + \int_\mu^\nu \lambda_1(\theta)\left[t(v'+w) - \frac{1}{12R^2}t^3(w''+v')\right]' d\theta \\
& + \int_\mu^\nu \lambda_2(\theta)\left[t(v'+w) + \{t(v'+w)\}'' - \frac{p_0 R^2}{b \cdot E}\right] d\theta \\
& + \int_\mu^\nu \lambda_3(\theta)\left\{E^2\left[\frac{1}{R^2}(v'+w)^2 - \frac{t}{R^3}(v'+w)(w''+v') + \frac{t^2}{4R^4}(w''+v')^2\right]\right. \\
& \left. - \sigma_0^2 + \alpha^2(\theta)\right\} d\theta \\
& + \int_\mu^\nu \lambda_4(\theta)\left\{E^2\left[\frac{1}{R^2}(v'+w)^2 + \frac{t}{R^3}(v'+w)(w''+v') + \frac{t^2}{4R^4}(w''+v')^2\right.\right. \\
& \left.\left. - \sigma_0^2 + \beta^2(\theta)\right]\right\} d\theta \qquad (6.14)
\end{aligned}$$

$\lambda_1(\theta)$, $\lambda_2(\theta)$, $\lambda_3(\theta)$, $\lambda_4(\theta)$ are unknown Lagrange multipliers.

$\int_\mu^\nu \Omega \, t \, d\theta$ represents the mass to be optimized in the interval $[\mu, \nu]$ within the specified constraints.

The second and third integrals indicate that the two equilibrium equations have to be satisfied, and act as constraint equations (the quantity $t(v'+w) + \{t(v'+w)\}'' - \frac{p_0 R^2}{b E}$ is the second equilibrium equation after substitution of $[t^3(w''+v')]''$ by using the first equilibrium equation (6.10)). The fourth and fifth integrals represent the constraints on the capacity of the material.

It has been seen before (Eq. (4.6), (4.7), (4.8) that

$$\delta(\underset{\sim}{M}) = 0 \tag{6.15}$$

(variation of $\underset{\sim}{M}$ with respect to all the variables involved) will produce the set of equations for $\underset{\sim}{M}$ stationary, or $M = \int_\mu^\nu \Omega \, t \, d\theta$ is stationary, with the four constraints equations satisfied. We perform this variation, carry out the appropriate integrations by parts, set the variable coefficients of δt, δw, δv, $\delta\alpha$, $\delta\beta$, $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$, $\delta\lambda_4$, and the terms coming from integration by parts equal to zero. We thus obtain the system of differential equations and the associated boundary conditions governing the solution for t, w, v, α, β, λ_1, λ_2, λ_3, λ_4.

• Equations

$$\Omega - \lambda_1'(\theta)\left[(v'+w) - \frac{3t^2}{12R^2}(w''+v)\right] + \lambda_2(\theta)(v'+w) + \lambda_2''(\theta)(v'+w)$$
$$+ (\lambda_3(\theta)+\lambda_4(\theta))\frac{E^2 t}{2R^4}(w''+v')^2 - (\lambda_3(\theta)-\lambda_4(\theta))\frac{E^2}{R^3}(v'+w)(w''+v') = 0 \tag{6.16}$$

$$-\lambda_1'(\theta)t + \left(\frac{\lambda_1'(\theta)t^3}{12R^2}\right)'' + \lambda_2(\theta)t + \lambda_2''(\theta)t$$
$$+ (\lambda_3(\theta)+\lambda_4(\theta))\frac{E^2 2}{R^2}(v'+w) + \left[(\lambda_3(\theta)+\lambda_4(\theta))\frac{t^2E^2}{4R^4}2(w''+v')\right]''$$
$$- (\lambda_3(\theta)-\lambda_4(\theta))\frac{E^2 t}{R^3}(w''+v') - \left[(\lambda_3(\theta)-\lambda_4(\theta))\frac{tE^2}{R^3}(v'+w)\right]'' = 0 \tag{6.17}$$

$$\Big\{-\lambda_1'(\theta)\left(t-\frac{t^3}{12R^2}\right)+\lambda_2(\theta)\,t+\lambda_2''(\theta)\,t$$

$$+(\lambda_3(\theta)+\lambda_4(\theta))\left(\frac{E^2}{R^2}\,2(v'+w)+\frac{E^2t^2}{4R^4}\,2(v'+w'')\right)$$

$$-(\lambda_3(\theta)-\lambda_4(\theta))\,\frac{E^2t}{R^3}\,[2v'+(w+w'')]\Big\}'=0 \qquad (6.18)$$

$$\lambda_3(\theta)\,\alpha(\theta)=0 \qquad (6.19)$$

$$\lambda_4(\theta)\,\beta(\theta)=0 \qquad (6.20)$$

$$\left[t(v'+w)-\frac{1}{12R^2}\,t^3(w''+v')\right]'=0 \qquad (6.21)$$

$$t(v'+w)+[t(v'+w)]''-\frac{p_0R^2}{bE}=0 \qquad (6.22)$$

$$E^2\left\{\frac{1}{R^2}(v'+w)^2-\frac{t}{R^3}(v'+w)(w''+v')+\frac{t^2}{4R^4}(w''+v')^2\right\}-\sigma_0^2+\alpha^2(\theta)=0 \qquad (6.23)$$

$$E^2\left\{\frac{1}{R^2}(v'+w)^2+\frac{t}{R^3}(v'+w)(w''+v')+\frac{t^2}{4R^4}(w''+v')^2\right\}-\sigma_0^2+\beta^2(\theta)=0 \qquad (6.24)$$

<u>Note</u>: The four last equations are the constraint equations.

• Boundary conditions from calculus of variations.

$$\left\{\lambda_1(\theta)\left[(v'+w)-\frac{3t^2}{12R^2}(w''+v')\right]+\lambda_2(\theta)\,(v'+w)'-\lambda_2'(\theta)(v'+w)\right\}\delta t\,\Big|_\mu^\nu=0 \qquad (6.25)$$

$$\lambda_2(\theta)\,(v'+w)\,\delta t'\,\Big|_\mu^\nu=0 \qquad (6.26)$$

$$\Big\{\lambda_1(\theta)t-\left(\frac{\lambda_1'(\theta)\,t^3}{12R^2}\right)'+\lambda_2(\theta)t'-\lambda_2'(\theta)t+\left[(\lambda_3(\theta)-\lambda_4(\theta))\,\frac{E^2t}{R^3}(v'+w)\right]'$$

$$-\left[(\lambda_3(\theta)+\lambda_4(\theta))\,\frac{E^2t^2}{2R^4}\,(w''+v')\right]'\Big\}\;\delta w\,\Big|_\mu^\nu=0 \qquad (6.27)$$

$$\left\{\frac{\lambda_1'(\theta)t^3}{12R^2} + \lambda_2(\theta)t - (\lambda_3(\theta)-\lambda_4(\theta))\frac{E^2 t}{R^3}(v'+w) + (\lambda_3(\theta)+\lambda_4(\theta))\frac{t^2E^2}{4R^4}2(w''+v')\right\}\delta w'\Big|_\mu^\nu = 0 \qquad (6.28)$$

$$\left\{-\frac{\lambda_1(\theta)t^3}{12R^2}\right\}\delta w''\Big|_\mu^\nu = 0 \qquad (6.29)$$

$$\left\{-\lambda_1'(\theta)\left(t - \frac{t^3}{12R^2}\right) + \lambda_2(\theta)t + \lambda_2''(\theta)t - (\lambda_3(\theta)-\lambda_4(\theta))\frac{E^2 t}{R^3}(2v'+w+w'') + (\lambda_3(\theta)+\lambda_4(\theta))E^2 2\left(\left(\frac{v'+w)}{R^2} + \frac{t^2}{4R^4}(v'+w'')\right)\right\}\delta v\Big|_\mu^\nu = 0 \quad (6.30)$$

$$\left\{\lambda_1(\theta)t - \lambda_1(\theta)\frac{t^3}{12R^2} + \lambda_2(\theta)t' - \lambda_2'(\theta)t\right\}\delta v'\Big|_\mu^\nu = 0 \qquad (6.31)$$

$$\left\{\lambda_2(\theta)t\right\}\delta v''\Big|_\mu^\nu = 0 \qquad (6.32)$$

Boundary conditions from equilibrium equations

$w'(\theta=0) = 0$		clamped at $\theta = 0$	(6.33)
$w'(\theta=\Theta) = 0$		symmetry	(6.34)
$w(\theta=0) = 0$		fixed at $\theta = 0$	(6.35)
$(t^3(w''+v'))'_{\theta=\Theta} = 0$	symmetry	→ shear force = 0 at $\theta = \Theta$	(6.36)
$v(\theta=0) = 0$		fixed at $\theta = 0$	(6.37)
$v(\theta=\Theta) = 0$		symmetry	(6.38)

3. Necessary condition for stationary value of M

As for the beam, a discussion of this system of equations, together with the boundary conditions, can bring some special properties to the switching functions $\alpha(\theta)$ and $\beta(\theta)$.

Subtract Eq. (6.24) from Eq. (6.23)

$$\frac{-2t}{R^3}(v'+w)(w''+v') + \alpha^2(\theta) - \beta^2(\theta) = 0 \qquad (6.39)$$

The quantity $(v'+w)(w''+v')$ is equivalent to the product $N_\theta \mathcal{M}_\theta$. But, N_θ or $\mathcal{M}_0$ are different from zero except at all but some possible discrete points. So,

$$\alpha^2(\theta) - \beta^2(\theta) \neq 0 \text{ implies } \alpha^2(\theta) \neq \beta^2(\theta) \qquad (6.40)$$

Assuming $\alpha^2(\theta) \neq 0$ and $\beta^2(\theta) \neq 0$

From Eq. (6.19) and Eq. (6.20) we deduce

$$\lambda_3(\theta) \equiv \lambda_4(\theta) \equiv 0 \qquad (6.41)$$

$$\lambda_3'(\theta) \equiv \lambda_4'(\theta) \equiv 0 \qquad (6.42)$$

$$\lambda_3''(\theta) \equiv \lambda_4''(\theta) \equiv 0 \qquad (6.43)$$

and Eq. (6.16), (6.17), (6.18) reduce to

$$\Omega - \lambda_1'(\theta)\left[(v'+w) - \frac{3t^2}{12R^2}(w''+v)\right] + \lambda_2(\theta)(v'+w) + \lambda_2''(\theta)(v'+w) = 0 \qquad (6.44)$$

$$-\lambda_1'(\theta)t + \left(\frac{\lambda_1'(\theta)t^3}{12R^2}\right)'' + \lambda_2(\theta)t + \lambda_2''(\theta)t = 0 \qquad (6.45)$$

$$\left(-\lambda_1'(\theta)\left(t - \frac{t^3}{12R^2}\right) + \lambda_2(\theta)t + \lambda_2''(\theta)t\right)' = 0 \qquad (6.46)$$

The initial arch defined in the interval $[0, \Theta]$ can be decomposed in two elementary arches n°1 and n°2, by a cut at an arbitrary location $\theta = \theta_a$ and matching of the boundary conditions. These two arches are defined in the interval $[0, \theta_a]$ and $[\theta_a, \Theta]$. We assume there exists a location θ_a such that

$$w', w'', v, v'' \neq 0 \qquad \text{at} \qquad \theta = \theta_a$$

The purpose of the following is to find a necessary condition to make the mass of these two elementary arches stationary. This will correspond to a stationary value of the mass of the complete arch because the location θ_a is arbitrary.

Arch No. 1: This concerns the interval $[0, \theta_a]$, so $\mu = 0$, $\nu = \theta_a$.

From Eq. (6.29) $\lambda_1(\theta = \theta_a) = 0$ because $\delta w''(\theta = \theta_a) \neq 0$.

From Eq. (6.32) $\lambda_2(\theta = \theta_a) = 0$ because $\delta v''(\theta = \theta_a) \neq 0$

Using these results, Eq. (6.28) reduces to

$$\lambda_1'(\theta = \theta_a) = 0 \text{ because } \delta w'(\theta = \theta_a) \neq 0 \quad .$$

In the same manner, Eq. (6.30) reduces to

$$\lambda_2''(\theta = \theta_a) = 0 \text{ because } \delta v\,(\theta = \theta_a) \neq 0 \quad .$$

Substitution of these values in Eq. (6.16) gives $\Omega(\theta = \theta_a) = 0$. But, Ω is assumed to be constant so,

$$\Omega \equiv 0 \qquad \text{in} \qquad [0, \theta_a] \tag{6.47}$$

This is a contradiction.

Arch No. 2: This concerns the interval $[\theta_a, \Theta]$, so, $\mu = \theta_a$, $\nu = \Theta$.

Exactly the same method can be used to find the same contradiction

$$\Omega \equiv 0 \qquad \text{in} \qquad [\theta_a, \Theta] \tag{6.48}$$

This can be done for any arbitrary θ_a meeting the conditions w', w'', v, $v'' \neq 0$. Because of these contradictions, the assumption that $\alpha^2(\theta) \neq 0$ and $\beta^2(\theta) \neq 0$, cannot be true.

The only possibility is

$$\alpha^2(\theta) = 0 \qquad \text{and} \qquad \beta^2(\theta) \neq 0 \tag{6.49}$$

or

$$\alpha^2(\theta) \neq 0 \qquad \text{and} \qquad \beta^2(\theta) = 0 \tag{6.50}$$

This set of equations constitutes the necessary condition for a stationary value of the mass of the arch. As for the beam, the physical meaning of this condition is that the yielding stress σ_0 must be reached for any x (except at some discrete point), in the upper or lower extreme fiber at $y = \pm \frac{t}{2}$.

4. Solution

According to our sign convention, the load per unit length p_0 will be < 0.

Eq. (6.22) can be integrated directly to give

$$t(v'+w) = \frac{p_0 R^2}{b E} (A \cos\theta + B \sin\theta + 1) \tag{6.51}$$

Assume $\Theta < \frac{\pi}{2}$; by use of Eq. (6.21) and boundary condition (6.36), this can be written

$$t(v'+w) = \frac{p_0 R^2}{b E} [A(\cos\theta + \operatorname{tg}\Theta \sin\theta) + 1] = f(A, \theta) = f \tag{6.52}$$

Up to this point, A is an unknown constant, but its interval of definition can be estimated. One can show that f(A, θ) is monotonic and that the sign of $df/d\theta$ = - sign A, providing that $p_0 < 0$. But sign of $df/d\theta$ = sign of $dN_\theta/d\theta$ which is > 0 because $dN_\theta/d\theta = Q_\theta$, and Q_θ is > 0 at $\theta = 0$. So, A must be negative. On the other hand, the sign of f = sign of N_θ. But, the inplane force N_θ can only be of negative sign (compression). $f \leq 0$

will require $-\cos\Theta \leq A$. So, A can be bounded by the interval $[-\cos\Theta,\ 0]$.

Because of the particular boundary conditions, (clamped at $\theta = 0$, symmetry at $\theta = \Theta$), the curvature $K = -\frac{1}{R^2}(w'' + v')$ must change its sign in $[0, \Theta]$ at least one time.

If $K > 0$, Eq. (6.39) together with $N_\theta < 0$ requires

$$\alpha^2(\theta) \quad > \quad \beta^2(\theta) \qquad (6.53)$$

and from Eq. (6.49), (6.50)

$$\alpha^2(\theta) \not\equiv 0 \quad \text{and} \quad \beta^2(\theta) \equiv 0 \qquad (6.54)$$

If $K < 0$, Eq. (6.39) together with $N_\theta < 0$ requires

$$\alpha^2(\theta) \quad < \quad \beta^2(\theta) \qquad (6.55)$$

and from Eq. (6.49), (6.50)

$$\alpha^2(\theta) \equiv 0 \quad \text{and} \quad \beta^2(\theta) \not\equiv 0 \qquad (6.56)$$

As for the beam, the location $\theta = \theta_0$ where $K = 0$ is called the switching point and characterizes the same properties.

Using the same argumentation as for the beam, it is reasonable to think that there is only one switching point in $[0, \Theta]$ and that:

$$K \geq 0 \qquad \text{for} \qquad 0 \leq \theta \leq \theta_0 \qquad (6.57)$$

and

$$K \leq 0 \qquad \text{for} \qquad \theta_0 \leq \theta \leq \Theta \qquad (6.58)$$

(6.57) and (6.58) means that the deflection w is negative.

Eq. (6.21) can be integrated to give

$$t(v'+w) - \frac{1}{12R^2} t^3(w''+v') = f_0 \text{ (constant)} \tag{6.59}$$

At the switching point θ_0, the curvature $K = 0$.

$\therefore f_0$ is found to be

$$f_0 = t(v'+w)_{\theta=\theta_0} = \frac{p_0 R^2}{b E} [A(\cos\theta_0 + tg\,\Theta \sin\theta_0)+1] = f(A, \theta_0) \tag{6.60}$$

and

$$K = -(f-f_0) \frac{12}{t^3} \tag{6.61}$$

<u>$0 \le \theta \le \theta_0$ (Subscript:1)</u>

$K \ge 0$ by assumption requires $\alpha^2(\theta) \ne 0$ and $\beta^2(\theta) \equiv 0$. The extreme fiber in the lower surface ($y = -\frac{t}{2}$) is yielded in compression (yielding stress in compression is σ_0, negative). Therefore

$$\sigma_{-\frac{t}{2}} = E \left[\frac{(v_1'+w_1)}{R} - \frac{t_1}{2} K\right] = \sigma_0 \tag{6.62}$$

We substitute for $v_1' + w_1$, K and solve for f. We identify this expression of f with the one of Eq. (6.52) and obtain:

$$t_1^2 \frac{R\sigma_0}{E} - t_1 f + 6R[f_0 - f] = 0 \tag{6.63}$$

The product of two real roots is negative. So, we will chose the positive root by physical considerations.

It can be written:

$$t_1 = \omega_a \frac{R}{2}(A\,g(\theta)+1)\left\{1+\sqrt{1+24\,\frac{A(g(\Theta)-g(\theta_0))}{\omega_a(A\,g(\theta)+1)^2}}\right\} \qquad (6.64)$$

with $\omega_a = \dfrac{p_0}{\sigma_0 b}$ being the load parameter

$$g(\theta) = \cos\theta + tg\,\Theta\,\sin\theta$$

$$g(\theta_0) = \cos\theta_0 + tg\,\Theta\,\sin\theta_0$$

$\theta_0 \le \theta \le \Theta$ (subscript:2)

This time, $K \le 0$ by assumption. So, $\alpha^2(\theta) \equiv 0$ and $\beta^2(\theta) \neq 0$ and the extreme fiber in the upper surface ($y = +\frac{t}{2}$) is yielded (σ_0, negative)

$$\sigma_{+t/2} = E\left[\frac{(v_2' + w_2)}{R} + \frac{t_2}{2}K\right] = \sigma_0 \qquad (6.65)$$

Again, we solve for f after substitution, using Eq. (6.52) and obtain:

$$t_2^2\,\frac{R\,\sigma_0}{E} - t_2 f - 6R\,(f_0 - f) = 0 \qquad (6.66)$$

the physical root is:

$$t_2 = \omega_a \frac{R}{2}(A\,g(\theta)+1)\left\{1+\sqrt{1-24\,\frac{A(g(\Theta)-g(\theta_0))}{\omega_a(A\,g(\theta)+1)^2}}\right\} \qquad (6.67)$$

where ω_a, $g(\theta)$, $g(\theta_0)$ have the same significance as for t_1. In the interval $[0, \Theta]$, the design t is defined by

$$t(\theta) = t_1(\theta) \qquad \text{for } 0 \le \theta \le \theta_0 \qquad (6.68)$$

$$t(\theta) = t_2(\theta) \qquad \text{for } \theta_0 \le \theta \le \Theta \qquad (6.69)$$

t depends upon two constants A, θ_0 which have to be found from the boundary conditions.

Determination of A and θ_0

From the first equilibrium equation after integration (Eq. (6.21)), we can find

$$w'' + v' = \frac{(f-f_0)}{t^3} \; 12\, R^2 \tag{6.70}$$

Let us integrate on $[0, \Theta]$

$$(w' + v)_{\theta=\Theta} - (w'+v)_{\theta=0} = \int_0^{\Theta} \frac{(f-f_0)}{t^3} \, 12\, R^2 \, d\theta \tag{6.71}$$

but, from boundary conditions,

$w'(0)$	=	0	clamped
$w'(\Theta)$	=	0	symmetry
$v\ (0)$	=	0	fixed support
$v\ (\Theta)$	=	0	symmetry

So, $$\int_0^{\Theta} \frac{(f-f_0)}{t^3} \, d\theta = 0 \tag{6.72}$$

which can be rewritten

$$\int_0^{\theta_0} \frac{(f-f_0)}{t_1^3} \, d\theta + \int_{\theta_0}^{\Theta} \frac{(f-f_0)}{t_2^3} \, d\theta = 0 \tag{6.73}$$

This integral condition gives a relation between A and θ_0. If we fix θ_0, we can find A such that this equation is satisfied.

Now, let us find the second relation between A and θ_0. v' can be substituted in Eq. (6.70), by its value from Eq. (6.52) giving

$$w'' - w = \frac{(f-f_0)}{t^3} 12 R^2 - \frac{f}{t} = \Psi(A, \theta_0, \theta) \tag{6.74}$$

This differential equation is linear in w. Using the boundary conditions: $w(0) = w'(0) = 0$, we find the solution:

$$w(\theta) = \int_0^{\theta} \sinh(\theta - \tau) \Psi(A, \theta_0, \tau) d\tau \tag{6.75}$$

$$w'(\theta) = \int_0^{\theta} \cosh(\theta - \tau) \Psi(A, \theta_0, \tau) d\tau \tag{6.76}$$

But, $w'(\Theta) = 0$, so

$$\int_0^{\Theta} \cosh(\Theta - \tau) \Psi(A, \theta_0, \tau) d\tau = 0 \tag{6.77}$$

with

$$\Psi(A, \theta_0, \tau) = \frac{(f-f_0)}{t_1^3} 12 R^2 - \frac{f}{t_1} \quad \text{for } 0 \le \theta \le \theta_0 \tag{6.78}$$

and

$$\Psi(A, \theta_0, \tau) = \frac{(f-f_0)}{t_2^3} 12 R^2 - \frac{f}{t_2} \quad \text{for } \theta_0 \le \theta \le \Theta \tag{6.79}$$

This is a second integral relation which, together with Eq. (6.73), is sufficient to find A and θ_0.

After substitution of t_1, t_2, f, f_0 in Eq. (6.73) and (6.77) we obtain:

$$\int_0^{\Theta} \frac{g(\theta) - g(\theta_0)}{\{(A\, g(\theta)+1)(1+H)\}^3} d\theta = 0 \tag{6.80}$$

and

$$\int_0^{\Theta} \cos h(\Theta - \theta) \left[\frac{48\, A[g(\theta)-g(\theta_0)]}{\{(A\, g(\theta)+1)(1+H)\}^3 \; \omega_a^{\,2}} - \frac{1}{1+H} \right] d\theta = 0 \qquad (6.81)$$

where

$$H = \sqrt{1 \pm 24 \; \frac{A[g(\theta) - g(\theta_0)]}{\omega_a (Ag(\theta) + 1)^2}}$$

with + sign if $\theta \in [0, \theta_0]$

and - sign if $\theta \in [\theta_0, \Theta]$

It should be noticed that, for a given opening angle, A and θ_0 are only functions of the load parameter ω_a, because Eq. (6.80) and (6.81) are only functions of Θ and ω_a. It was difficult to find a closed form solution to Eq. (6.80) and (6.81) so a numerical procedure was used.

Remarks about numerical procedures

For a given opening angle and load parameter, the numerical scheme is to set θ_0, and to deduce A_1 and A_2, from Eq. (6.80) and (6.81). We know that A_1 and A_2 must belong to the interval $[-\cos\Theta, 0]$. We will have a solution when A_1 equals A_2. The first difficulty arising is that the solution of Eq (6.80) or Eq. (6.81) is not always single valued, (generally, two roots for each). Nevertheless, if we consider values of $\frac{p_0}{\sigma_0 b} = \omega_a$ less than 0.2, there is no major difficulty and, numerically, we have found only one couple θ_0, A satisfying Eq. (6.80) and (6.81) for given values of Θ and ω_a. From a physical point of view, values of ω_a greater than .2 are unreasonable. For values of ω_a between 0.08 and 0.2, double precision must be used because the quantity $A_1 - A_2$, which must be zero, tends to be insensitive to θ_0. For greater values of ω_a (greater than 0.2), numerical difficulties for some values of the opening angle ($10^{\circ} < \Theta < 30^{\circ}$) might be encountered. The uniqueness of the solution (θ_0, A) is not well established numerically in this case, and it is difficult to know if this is a problem due to numerical techniques, or, if really the solution of the system of Equations (6.80) and (6.81) is not unique.

An automatic numerical procedure was developed for small values of ω_a (less than 10^{-2}), but for greater values, the lack of characteristic properties of Eqs. (6.80) and (6.81) made easier the use of a more basic technique. (The quantity $A_1 - A_2$ is oscillatory if we vary θ_0). The two unknown constants $\theta_0 = \theta_0(\Theta, \omega_a)$ and $A = A(\Theta, \omega_a)$ are represented on Figs. 7 and 8 for some values of the $\frac{1}{2}$ opening angle Θ. For small values of ω_a, one can predict the location θ_0 of the switching point.

• Asymptotic value of θ_0 for small values of the load parameter ω_a

Let $\omega_a = \epsilon$, small but different from zero.

The quantities mainly involved in Eqs. (6.80) and (6.81) are A and $A g(\theta) + 1$. A priori, we do not know the behavior of these quantities for small values of ω_a. There exists only three possible combinations,

a) $A = 0(1)$ and $A g + 1 = 0(1)$

b) $A = 0(1)$ and $A g + 1 = 0(\epsilon^m)$ $m > 0$

c) $A = 0(\epsilon^n)$ and $A g + 1 = 0(1)$ $n > 0$

because A belongs to $[-\cos\Theta, 0]$ and that $g(\theta)$ is bounded in the interval $\theta \in [0, \Theta]$.

Using the physical argument that, for small values of ω_a, the forces in the structure are small, the thickness will be small, and,

the bending stress $\sigma_B = -E\, y \frac{1}{R^2} (w'' + v')$ will be small everywhere.
After substitution of $w'' + v'$ from Eq. (6.70)

$$\left|\sigma_{B\,max}(\theta)\right| = \left| 6\, p_0\, R^2\, A(g-g_0) \frac{1}{t^2 b} \right| \qquad (6.82)$$

Except at the switching point θ_0, the quantity $g(\theta) - g(\theta_0)$ is finite and different from zero. The thickness t is defined by Eqs. (6.68), (6.69)

in case a), $\sigma_B = 0(1)$ (6.83)

in case b), $\sigma_B = 0(1)$ (6.84)

in case c), $\sigma_B = 0(1)$ if $0 < n \leq 1$ (6.85)

$\sigma_B = 0(\varepsilon^{n-1}) = 0(\varepsilon^{p>0})$ if $1 < n$ (6.86)

From this result, we can deduct that the only possible configuration is

$$A = 0(\varepsilon^q), \quad q > 1, \quad A\, g+1 = 0(1) \qquad (6.87)$$

(this corresponds to case c)).

As a consequence, Eq. (6.80) reduces to:

$$\int_0^{\Theta} [g(\theta) - g(\theta_0)]\, d\theta = 0 \qquad (6.88)$$

After integration, θ_0 is found to be the solution of

$$\Theta \cos\theta_0 + \Theta\, tg\, \Theta \sin\theta_0 - tg\, \Theta = 0 \qquad (6.89)$$

This equation can be solved numerically to give the results shown in Fig. 9.

5. Comparison with the equivalent uniform thickness arch

The equivalent arch of uniform thickness (subscript u) will be the uniform thickness arch of minimum mass able to support the load without exceeding the yielding stress σ_0. The uniform thickness design is found in the appendix, Eq. (A.17)

$$t_u = \omega_a \frac{R}{2} [C+1] \left[1 + \sqrt{1 + 24 \frac{1}{\omega_a} \frac{C}{[C+1]^2} \left[1 - \frac{tg\, \Theta}{\Theta} \right]} \right] \quad (6.90)$$

where C is an unknown constant which can be determined from the boundary conditions. We can use t_u to normalize the design which gives an extremum of M. The normalized thickness $\underset{\sim}{t}$ is defined by

$$\underset{\sim}{t} = \underset{\sim}{t}_1 = \frac{t_1}{t_u} \qquad \text{for} \qquad 0 \le \theta \le \theta_0 \quad (6.91)$$

and

$$\underset{\sim}{t} = \underset{\sim}{t}_2 = \frac{t_2}{t_u} \qquad \text{for} \qquad \theta_0 \le \theta \le \Theta \quad (6.92)$$

$\underset{\sim}{t}$ is only a function of ω_a and Θ.

Figures 10 through 17 represent $\underset{\sim}{t}$ for opening angles from 10^o to 80^o and various values of ω_a.

It can be noticed from these figures that:

$\underset{\sim}{t}$ at $\theta = \Theta$ is decreasing when ω_a increases

$\underset{\sim}{t}$ at $\theta = \theta_0$ is decreasing when ω_a increases

$\underset{\sim}{t}$ at $\theta = 0$ decreases and then increases when ω_a increases.

It is also convenient to quantify the gain in mass by looking at the quantity

$$G(\omega_a, \Theta) \doteq \frac{M_{\text{uniform strength}}}{M_{\text{uniform thickness}}} \quad (6.93)$$

$G(\omega_a, \Theta)$ is shown in Fig. 18. In engineering applications, ($\omega_a \sim 10^{-2}$ to 10^{-1}), a noticeable saving of material can be achieved by using the uniform strength design.

6. Shallow arch

If the assumption of a shallow arch is made, the opening angle 2Θ has to be small. The equations of equilibrium (6.10) and (6.11) reduce to

$$[t(v' + w)]' = 0 \tag{6.94}$$

and

$$-\frac{E}{R^2} t(v'+w) - \frac{E}{12 R^4} (t^3 w'')'' + \frac{p_0}{b} = 0 \tag{6.95}$$

because the projection of Q_θ on the centroidal axis is assumed negligible compared to N_θ, and v' is small compared to w'' in the expression for the curvature. Therefore, the inplane force N_θ is constant. But, for the general arch, the inplane force is given by

$$N_\theta = \frac{bE}{R} t(v'+w) = p_0 R[A\, g(\theta) + 1] \tag{6.96}$$

So, $\quad A\, g(\theta) + 1 = \dfrac{N_1}{p_0 R}$ (constant) $\qquad$ (6.96)

but $\quad A\, g(\theta) + 1 = A + 1 + 0(\Theta^2)$ for small values of Θ

So $\quad A + 1 \sim \dfrac{N_1}{p_0 R} \qquad$ (6.98)

Expanding $g(\theta) - g(\theta_0)$ for small Θ gives,

$$g(\theta) - g(\theta_0) = -\Theta^2 \left[\frac{\xi^2}{2} - \xi + \gamma_0 \right] \tag{6.99}$$

where $\xi = \dfrac{\theta}{\Theta}$

and

$$\gamma_0 = -\frac{\xi_0^2}{2} + \xi_0$$

After using these results, the integral relation (6.80) which relates the two unknown constants A and ξ_0 can be written

$$\int_0^{\xi_0} \frac{\left[\frac{\xi^2}{2} - \xi + \gamma_0\right] d\xi}{\left[1+\sqrt{1+24\,\bar{\omega}\left(\frac{\xi^2}{2} - \xi + \gamma_0\right)}\right]^3} + \int_{\xi_0}^{1} \frac{\left[\frac{\xi^2}{2} - \xi + \gamma_0\right] d\xi}{\left[1+\sqrt{1-24\,\bar{\omega}\left(\frac{\xi^2}{2} - \xi + \gamma_0\right)}\right]^3} = 0 \quad (6.100)$$

with

$$\bar{\omega} = + \frac{\sigma_0 b}{p_0} \frac{A\Theta^2}{(A+1)^2} \quad \text{load parameter of the shallow arch}$$

This equation is identical to the equation (5.88) used in the problem of the beam (ω_b is substituted by $\bar{\omega}$). The solution is represented by Figs. 2, 3, 4 and this can greatly simplify the numerical solution of the shallow arch. Only the equation (6.81) has to be solved by trying different couples A and θ_0 satisfying (6.80), the solution of which we already know.

7. Concluding remarks

The restriction concerning the shear stresses mentioned in the concluding remarks of the beam analysis is still present in the analysis of the deep arch. It would be possible also to incorporate a constraint equation on a minimum allowable thickness. With the assumption of only one switching point, we are able to find a solution in the interval considered for the load parameter. Uniqueness of the solution in this interval seems established from a numerical point of view, but, for greater values of the load parameter ($\omega_a > 0.2$), it is difficult to draw such a conclusion. For some range of ω_a, the saving in mass of material is appreciable when we use the "optimum design" (uniform strength). The efficiency of the uniform thickness design increases when we increase the opening angle. This means that, for large values of opening angle, the stress state tends to be dominately membrane.

VII. CONCLUSION

Maximum stiffness: Assuming a direct equivalence between stiffness and external work, a necessary and sufficient condition for maximum stiffness is derived for all types of structures which have membrane and bending stresses only. This optimality condition which characterizes an energy approach to the optimal structural design can be summarized by:

$$\frac{\partial W}{\partial t} = d^2 \qquad (7.1)$$

with t, the design variable, W, external work, and d a constant. It must be remembered that this is only a local optimality condition and that the uniqueness of the solution of the resulting system of nonlinear differential equations is not established.

Maximum strength: Statically determinate problems can be treated by a direct optimization of the design variable at each point of the structure, as long as the capacity of the material is not exceeded. Statically indeterminate problems cannot be treated in this way, and other techniques must be used.

In case of pure membrane or pure bending stress states, the optimality condition for maximum strength is shown to be identical to the optimality condition for maximum stiffness. Eq. (7.1) which is, from this fact, the optimality condition for maximum strength, is identified as a maximum generalized stress criterion and is directly related to the capacity of the material. In case of a mixed stress case, this is not true. In this case one must choose a suitable stress criterion such as maximum allowable stress or yielding criterion, to characterize the capacity of the material through an inequality constraint. With the help of the calculus of variation, a necessary condition is derived for a stationary value of the mass of a structure supporting a given load.

For the two examples considered, the optimality condition derived is a uniform strength design. This means that the two inequality constraints become equalities in the intervals defined by the two switching functions. This method can be extended without major difficulty, to the case of an axisymmetric shell of revolution of prescribed geometry, thickness excepted. The material capacity constraint can be specified by any yielding criterion which must not be violated anywhere in the structure. In the case of the pure bending of a beam, a substantial saving in material of 44% can be achieved by using the optimal design instead of the equivalent uniform thickness design.

No proof, concerning absolute maximum strength and uniqueness of the solution is given. But, it must be noticed that in the two examples studied, the numerical results tend to indicate uniqueness of the solution. However, if we assume that there exists a solution to our problem, and if the necessary condition derived leads to a unique solution, this solution characterizes an absolute minimum or maximum. The uniform thickness design has a smaller strength than the uniform strength design. Therefore, from a numerical point of view, the uniform strength design represents the absolute maximum strength design.

Multivariable optimization: This analysis, as the majority of the references, is restricted to problems where only one design variable has to be optimized. Nevertheless, multivariable optimization can be set up by using the same techniques. In the case of the arch, for example, two geometric design variables can be optimized: radius, thickness. Additional equations will come from the variation of the mass functional $\underset{\sim}{M}$ with respect to the additional design variables. The complexity of the resulting nonlinear system of differential equations will be increased considerably, but, mathematical derivation of the equations is straightforward.

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APPENDIX

Equivalent uniform thickness deep arch

This appendix determines the constant thickness t_u of the arch of minimum mass able to support the load p_0. The same notation as for the uniform strength arch is used. The two equilibrium equations are

$$\left[t_u(v_u'+w_u) - \frac{1}{12R^2} t_u^3 (w_u''+v_u')\right]' = 0 \qquad (A.1)$$

$$t_u(v_u'+w_u) + [t_u(v_u'+w_u)]'' - \frac{p_0 R^2}{bE} = 0 \qquad (A.2)$$

Integration of Eq. (A.2) gives

$$t_u(v_u'+w_u) = \frac{p_0 R^2}{b\,E} \left[C(\cos\theta + \mathrm{tg}\,\Theta \sin\theta)+1\right] = f(C, \theta) \qquad (A.3)$$

Integration of Eq. (A.1) gives

$$(w_u''+v_u') = \frac{f(C, \theta) - f(C, \theta_1)}{t_u^3} 12\,R^2 \qquad (A.4)$$

where θ_1 is the location where the bending moment $\mathcal{M} = 0$.

The stress in the lower extreme fiber is given by

$$\sigma_{-t_u/2} = E\left[\frac{1}{R}(v_u'+w_u) + \frac{t_u}{2R^2}(w_u''+v_u')\right] \qquad (A.5)$$

which reduces to

$$\sigma_{-t_u/2} = E\left[\frac{f(C, \theta)}{t_u R} + \frac{6}{t_u^2}\left[f(C, \theta) - f(C, \theta_1)\right]\right] \qquad (A.6)$$

It can be checked that the maximum stress occurs in compression at the location $\theta = 0$ in the lower extreme fiber $y = -\frac{t_u}{2}$. This will determine the design through the relation

$$\sigma_0 = E\left[\frac{f(C,0)}{t_u R} + \frac{6}{t_u^2}[f(C,0) - f(C,\theta_1)]\right] = \sigma_{-t_u/2} (\theta = 0) \quad (A.7)$$

or

$$t_u^2 \frac{\sigma_0}{E} - t_u \frac{f(C,0)}{R} - 6[f(C,0) - f(C,\theta_1)] = 0 \quad (A.8)$$

Where σ_0 is the yielding stress in compression (negative).

The two roots t_{u_1} and t_{u_2} of this equation are of different sign, so, using a physical argument, the positive one is chosen

$$t_u = \frac{f(C,0)}{2\sigma_0 R} E\left[1 + \sqrt{1 + 24 \frac{R^2 \sigma_0}{f(C,0)^2 E}[f(C,0)-f(C,\theta_1)]}\right] \quad (A.9)$$

This is the thickness of the equivalent arch. The two unknown constants C, θ_1 must still be calculated.

It is easy to calculate $f(C, \theta_1)$ by using the boundary conditions

$w_u'(0)$	$= 0$	clamped at $\theta = 0$
$w_u'(\Theta)$	$= 0$	symmetry
$v_u(0)$	$= 0$	rigid support
$v_u(\Theta)$	$= 0$	symmetry

After integration of Eq. (A.4) and use of the above boundary conditions, the following expression is obtained.

$$\int_0^{\Theta} [f(C,\theta) - f(C,\theta_1)] d\theta = 0 \quad (A.10)$$

which gives

$$f(C,\theta_1) = \frac{p_0 R^2}{b E}\left[\frac{C \operatorname{tg}\Theta}{\Theta} + 1\right] \quad (A.11)$$

v_u' can be eliminated from Eq. (A. 3) and Eq. (A. 4) and produces the linear differential equation

$$w_u - w_u'' = \frac{f(C,\theta)}{t_u} - \frac{f(C,\theta) - f(C,\theta_1)}{t_u^3} 12 R^2 \qquad (A.12)$$

using the boundary conditions $w_u(0) = 0$, $w_u'(0) = 0$, the solution is:

$$w_u(\theta) = \int_0^\theta \sinh(\theta - \tau) \left[- \frac{f(C,\tau)}{t_u} + \frac{[f(C,\tau) - f(C,\theta_1)]}{t_u^3} 12 R^2 \right] d\tau \qquad (A.13)$$

$$w_u'(\theta) = \int_0^\theta \cosh(\theta - \tau) \left[- \frac{f(C,\tau)}{t_u} + \frac{[f(C,\tau) - f(C,\theta_1)]}{t_u^3} 12 R^2 \right] d\tau \qquad (A.14)$$

The last boundary condition to be used is $w_u'(\Theta) = 0$. This condition will give the only remaining unknown constant C. Eq. (A. 14) for $\theta = \Theta$ can be rewritten in the form

$$\omega_a^2 (C+1)^2 \left\{ 1 + \sqrt{1 + 24 \frac{C}{\omega_a (C+1)^2} \left[1 - \frac{tg\,\Theta}{\Theta} \right]} \right\}^2$$

$$- 48 \left\{ \frac{\frac{C}{2}\left(1 + tg\,\Theta . \frac{\cosh\Theta}{\sinh\Theta}\right) - C\frac{tg\,\Theta}{\Theta}}{\frac{C}{2}\left(1 + tg\,\Theta . \frac{\cosh\Theta}{\sinh\Theta}\right) + 1} \right\} = 0 \qquad (A.15)$$

This equation can be solved numerically without any difficulty to give C. The uniform thickness t_u is completely defined by the expression

$$t_u = \omega_a \frac{R}{2} [C+1] \left[1 + \sqrt{1 + 24 \frac{1}{\omega_a} \frac{C}{[C+1]^2} \left[1 - \frac{tg\,\Theta}{\Theta} \right]} \right] \qquad (A.16)$$

As can be seen, the uniform thickness design depends only upon ω_a and Θ.

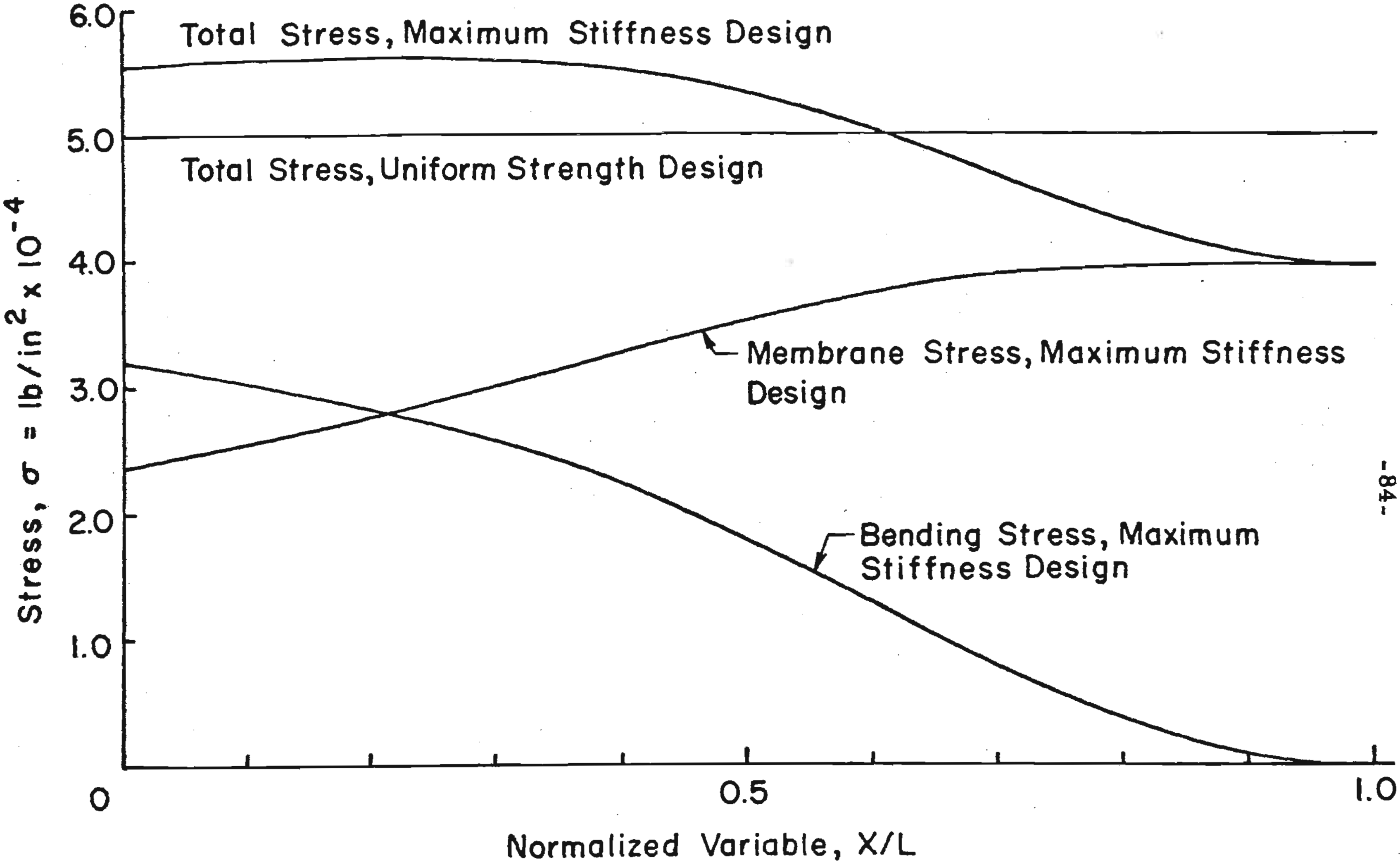

FIG. I STRESS COMPARISON BETWEEN A UNIFORM STRENGTH DESIGN AND A MAXIMUM STIFFNESS DESIGN

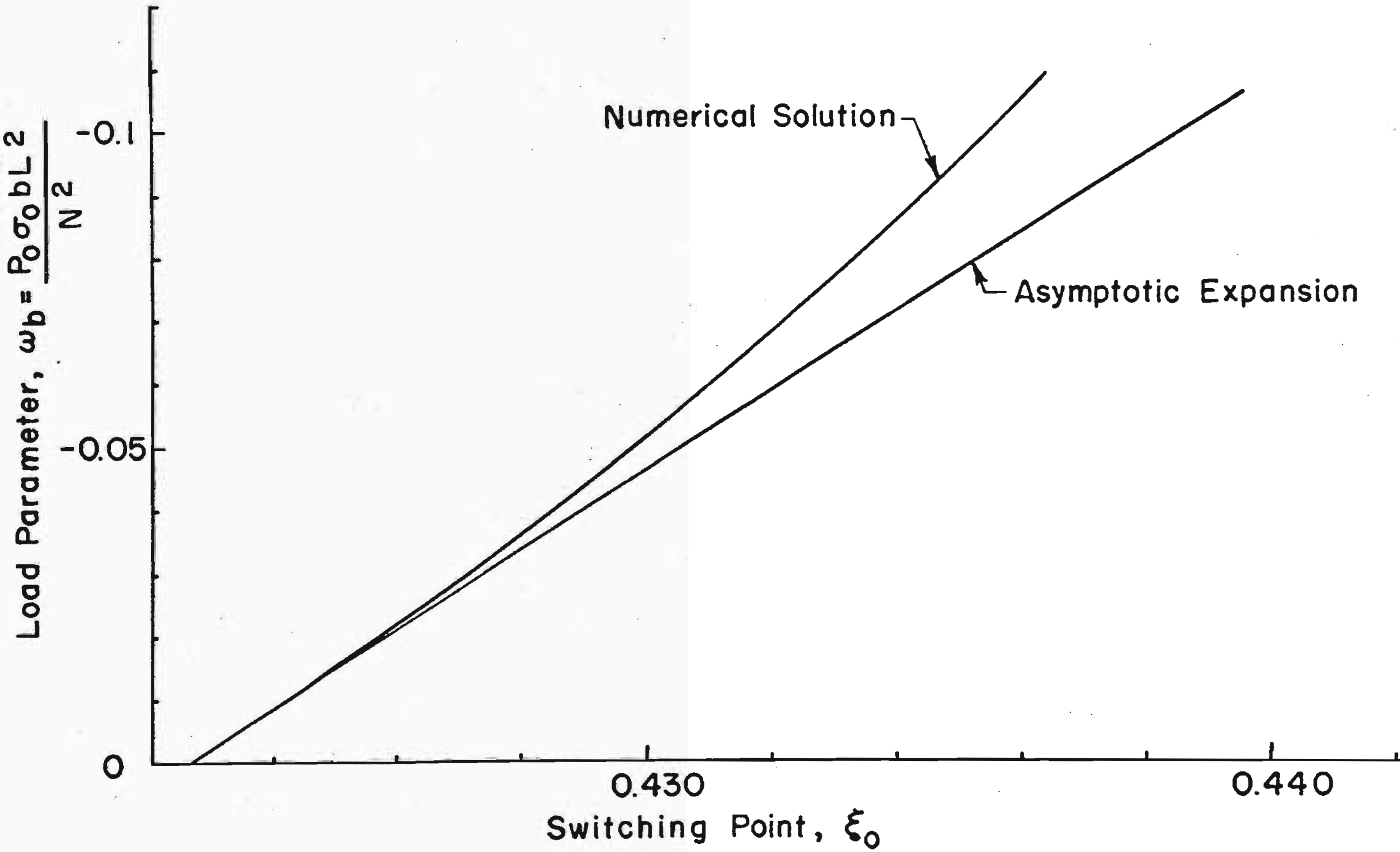

FIG. 2 SWITCHING POINT VS. SMALL VALUES OF LOAD PARAMETER (BEAM)

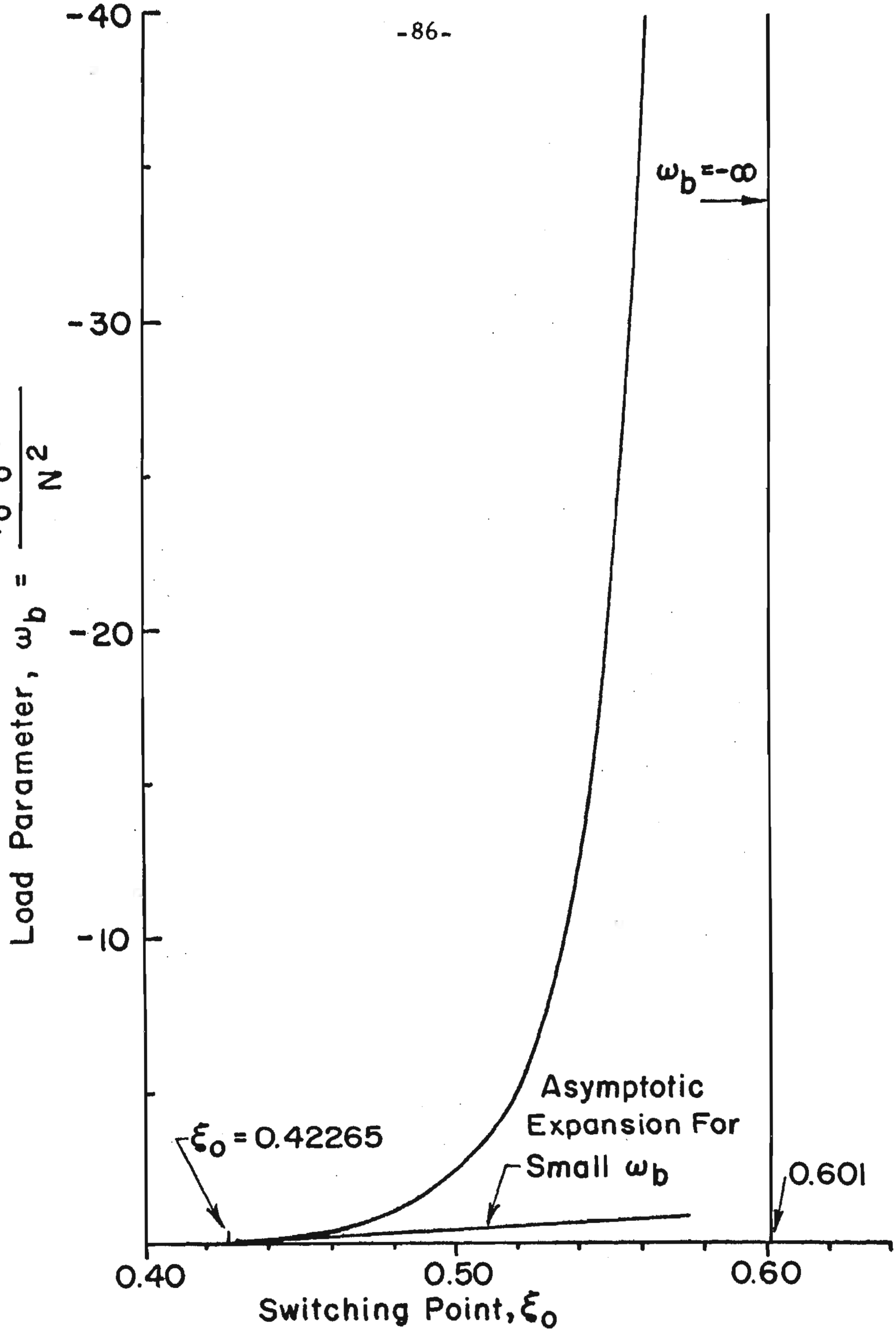

FIG. 3 SWITCHING POINT VERSUS LOAD PARAMETER (BEAM)

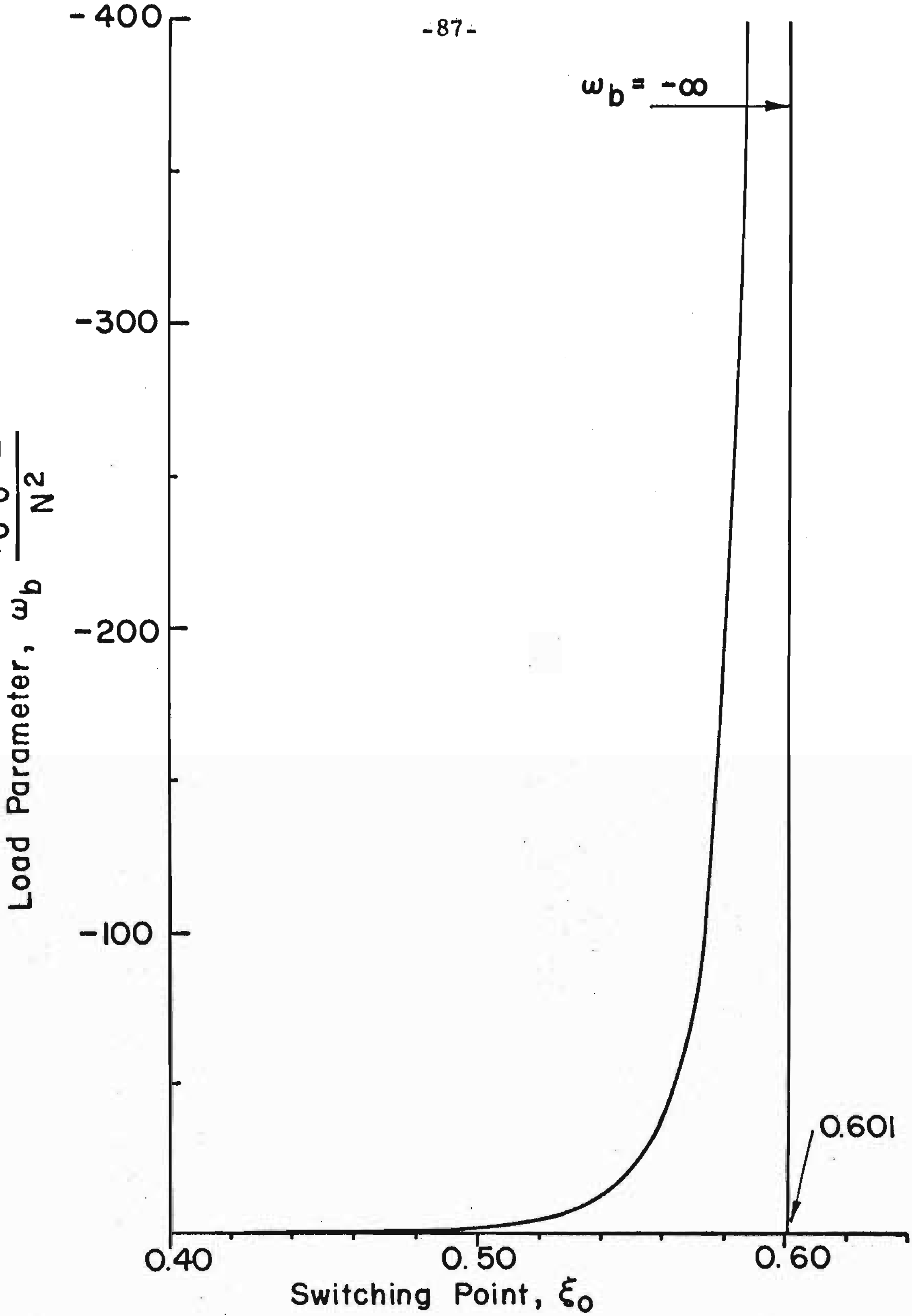

FIG. 4 SWITCHING POINT VS. LARGE VALUES OF LOAD PARAMETER (BEAM)

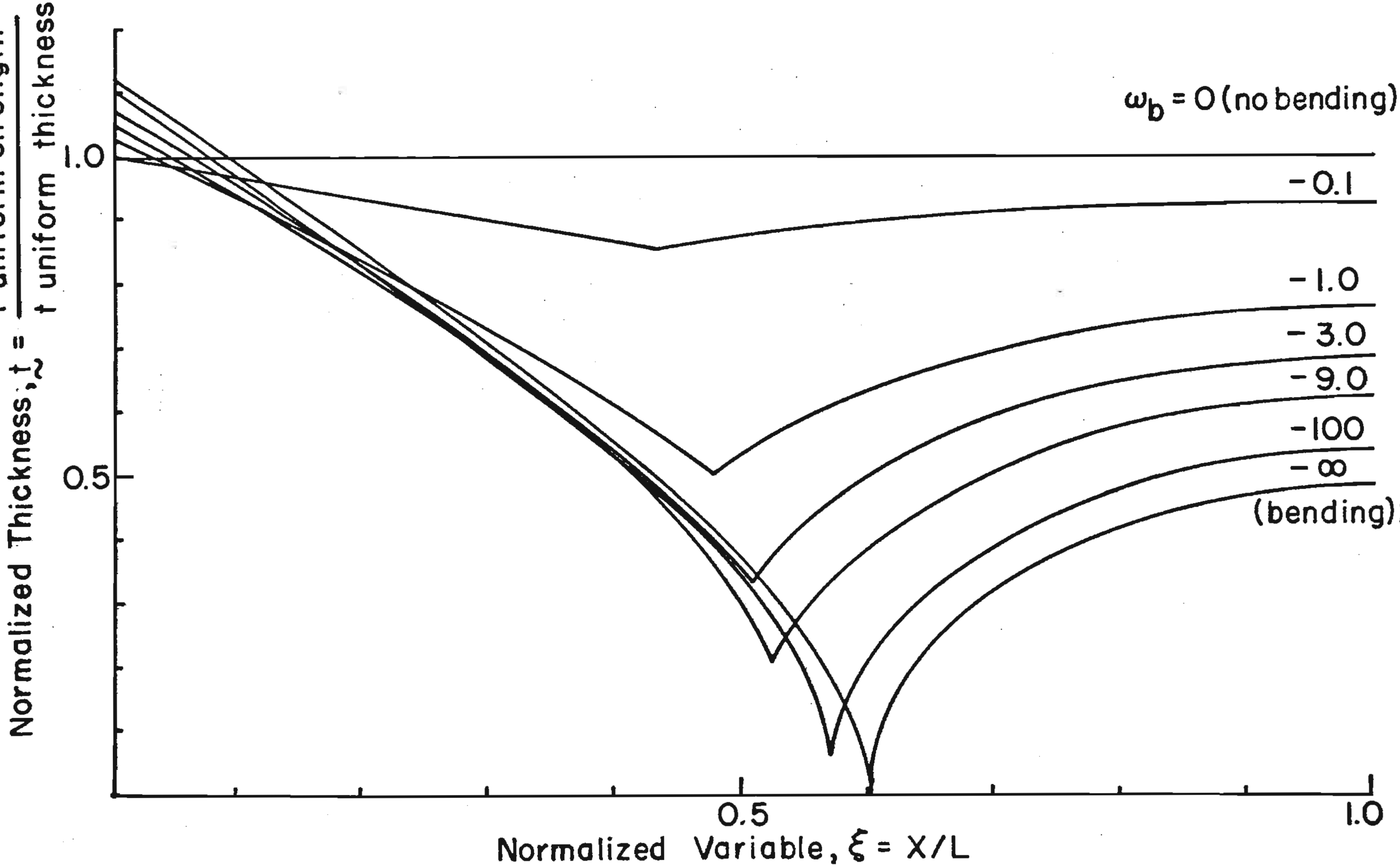

FIG. 5 NORMALIZED THICKNESS FOR SEVERAL VALUES OF LOAD PARAMETER (BEAM)

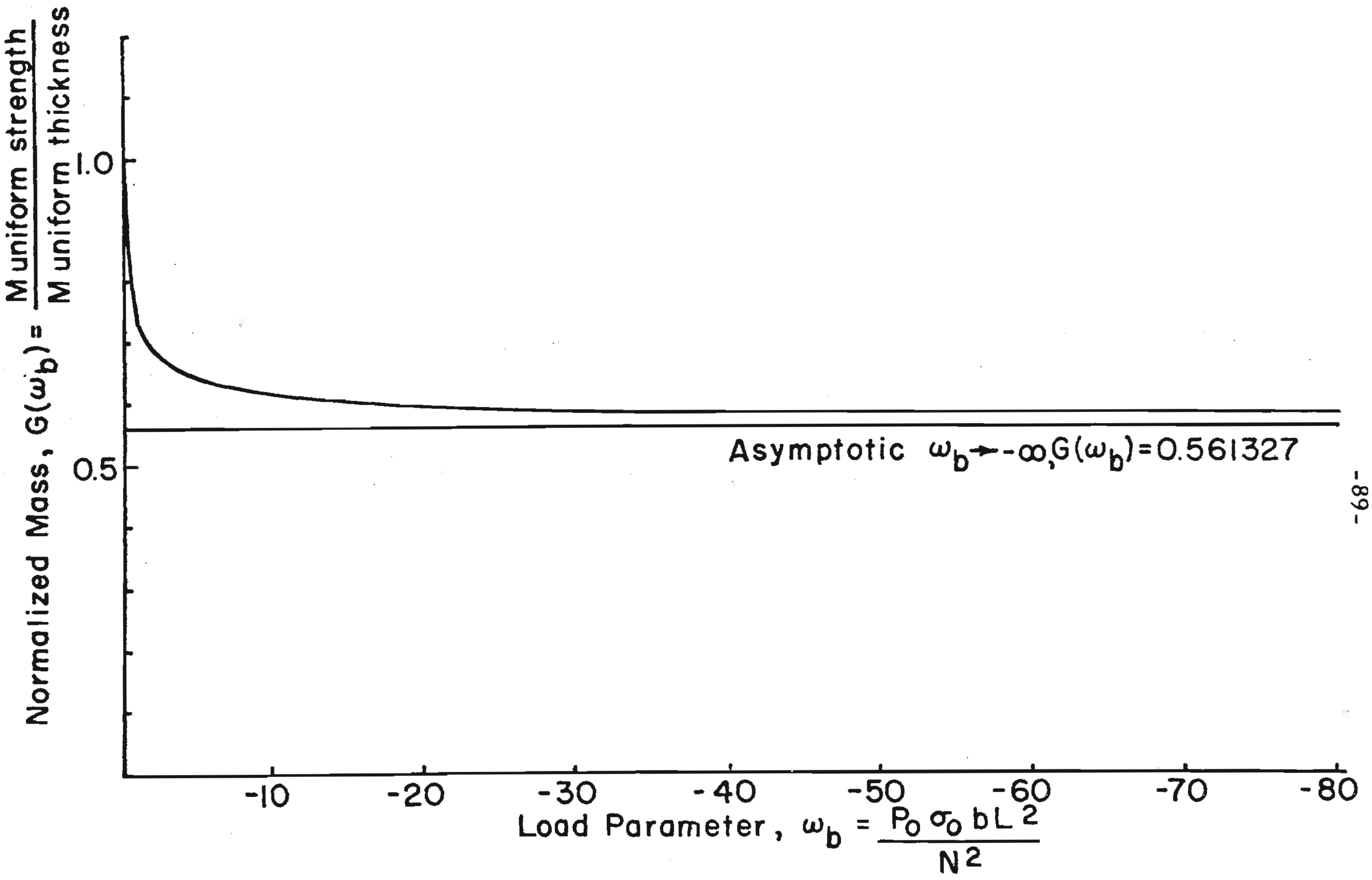

FIG. 6 NORMALIZED MASS VS. LOAD PARAMETER (BEAM)

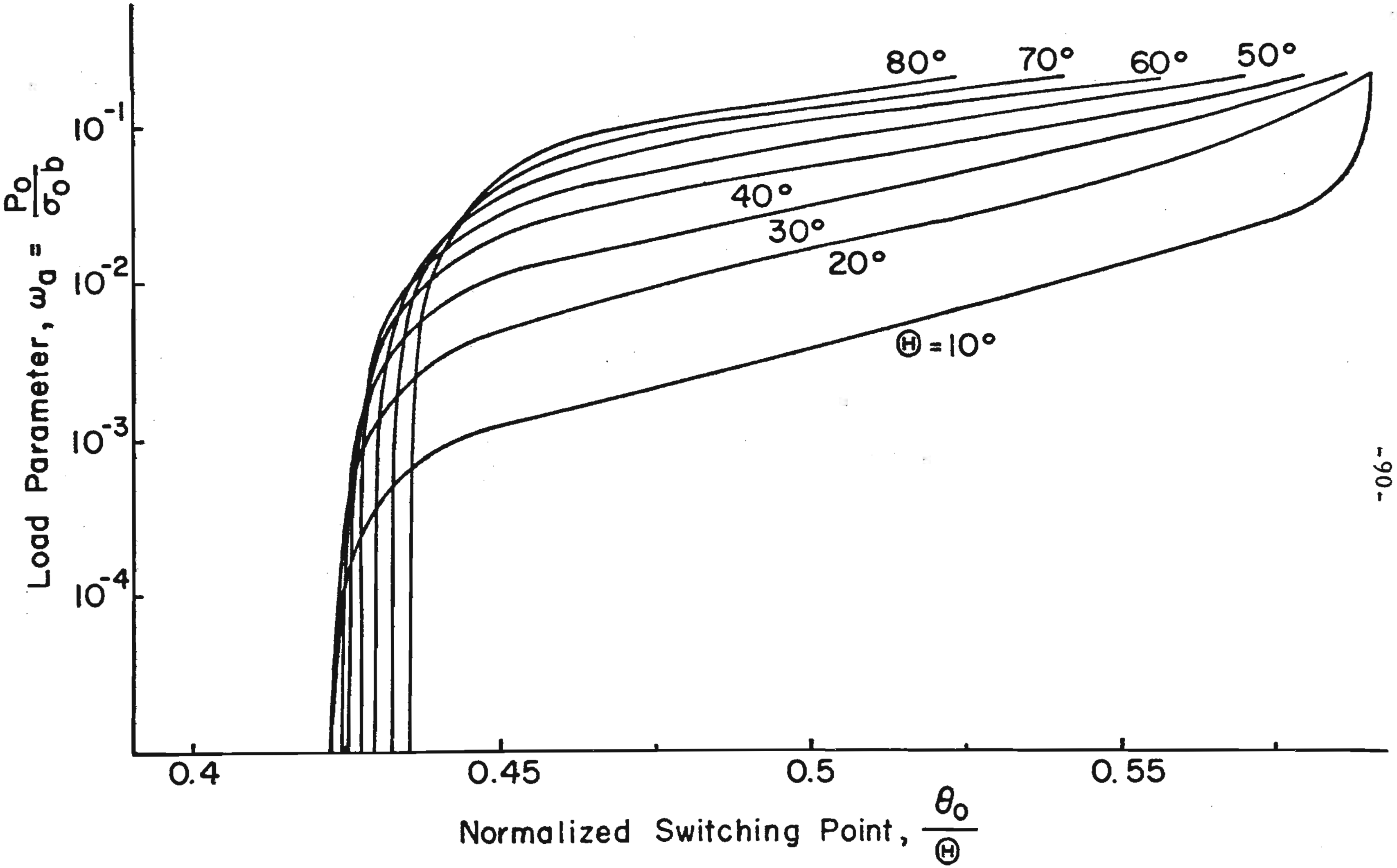

FIG. 7 SWITCHING POINT VS LOAD PARAMETER FOR DIFFERENT OPENING ANGLES (ARCH)

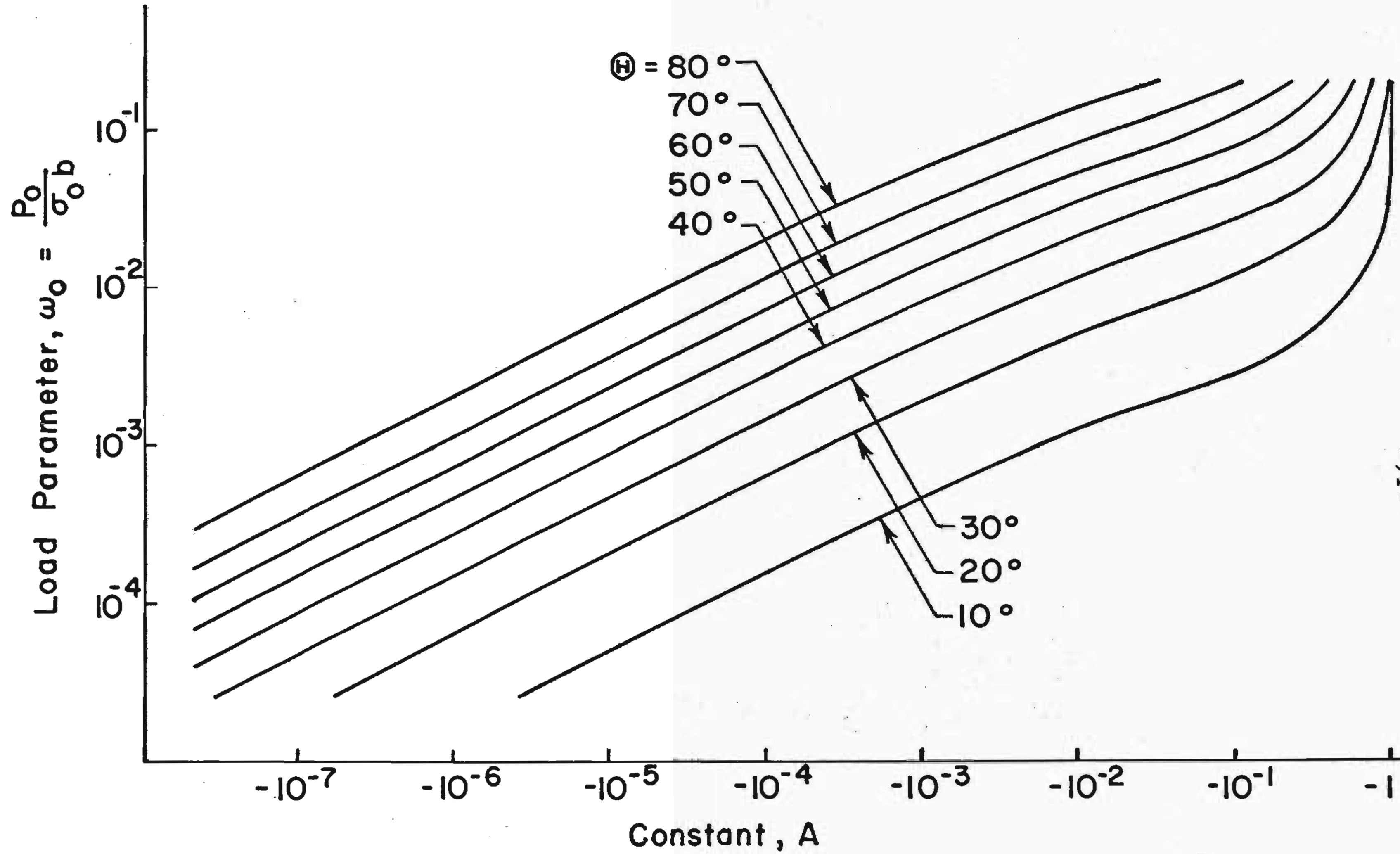

FIG. 8 A VS. LOAD PARAMETER FOR DIFFERENT OPENING ANGLES (ARCH)

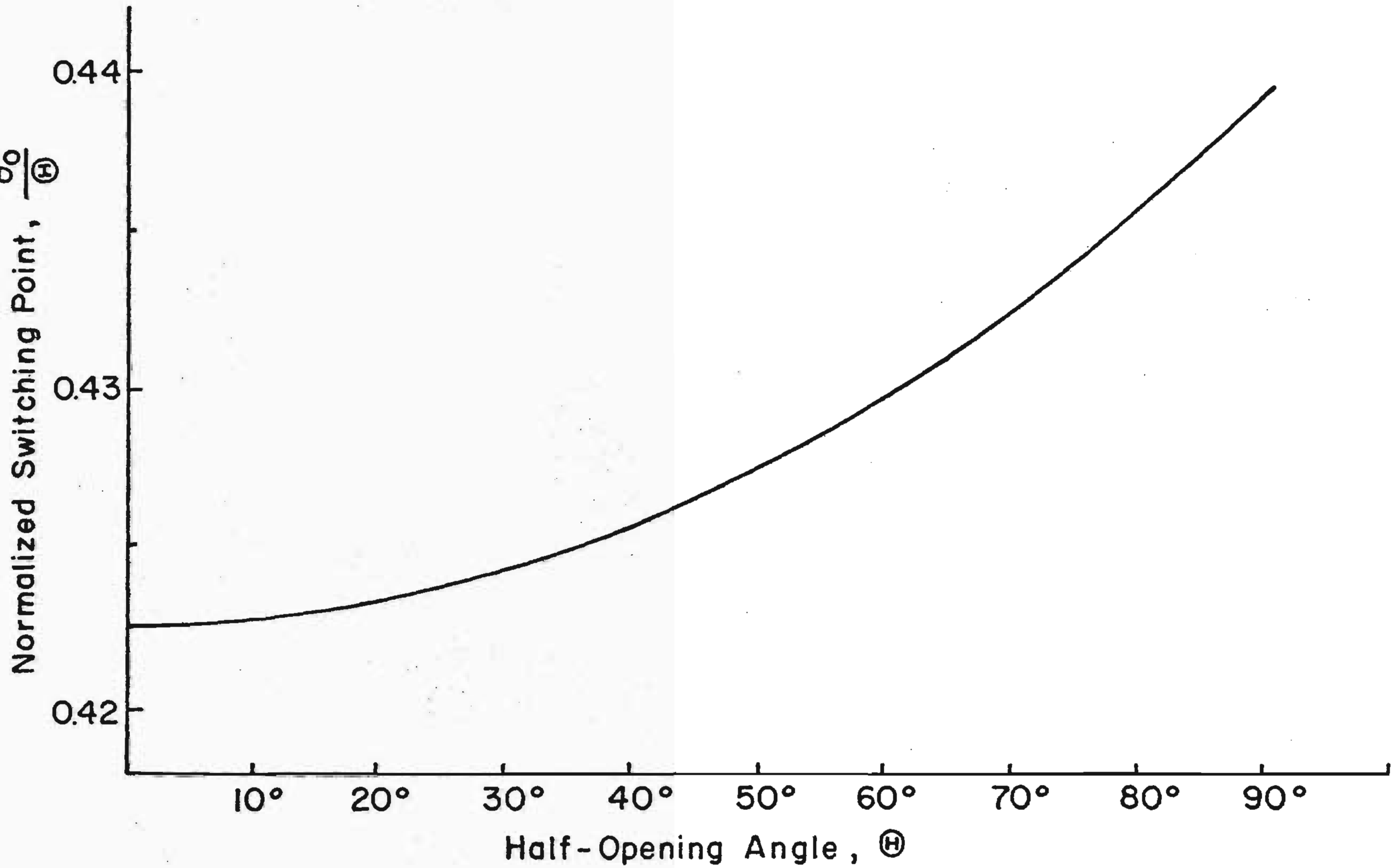

FIG. 9 SWITCHING POINT FOR SMALL VALUES OF LOAD PARAMETER (ARCH)

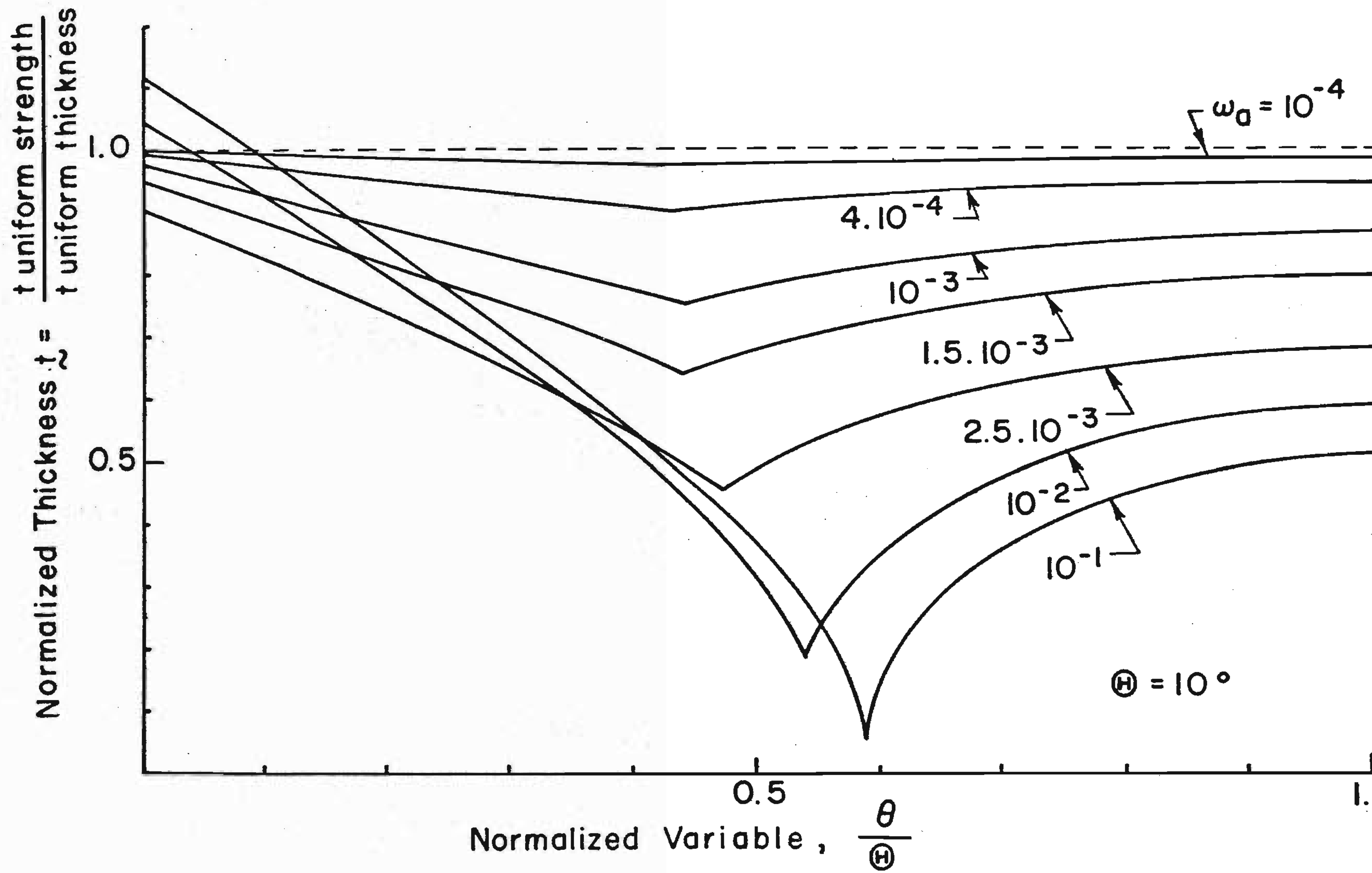

FIG. 10 NORMALIZED THICKNESS VS. LOAD PARAMETER (ARCH)

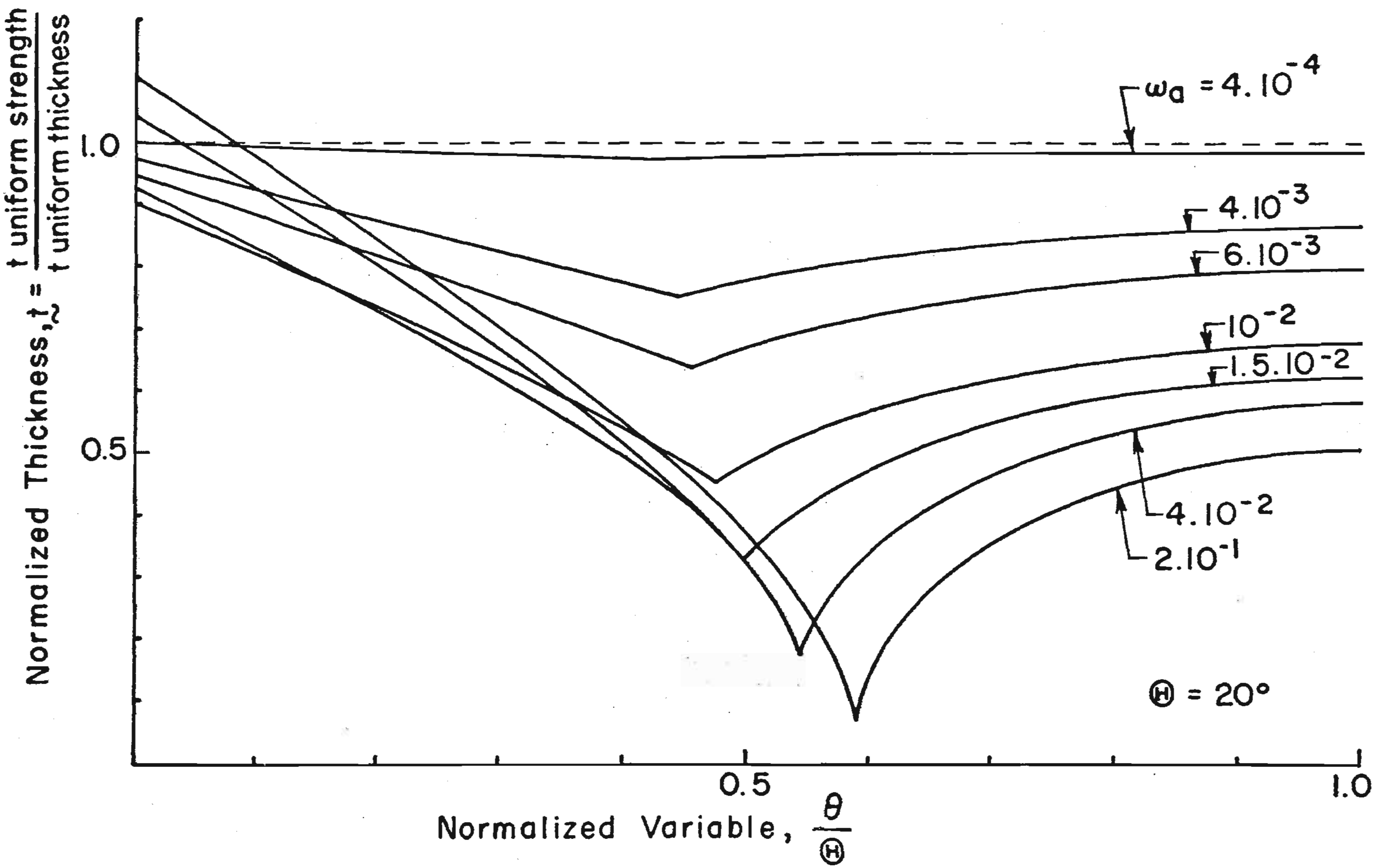

FIG. 11 NORMALIZED THICKNESS VS LOAD PARAMETER (ARCH)

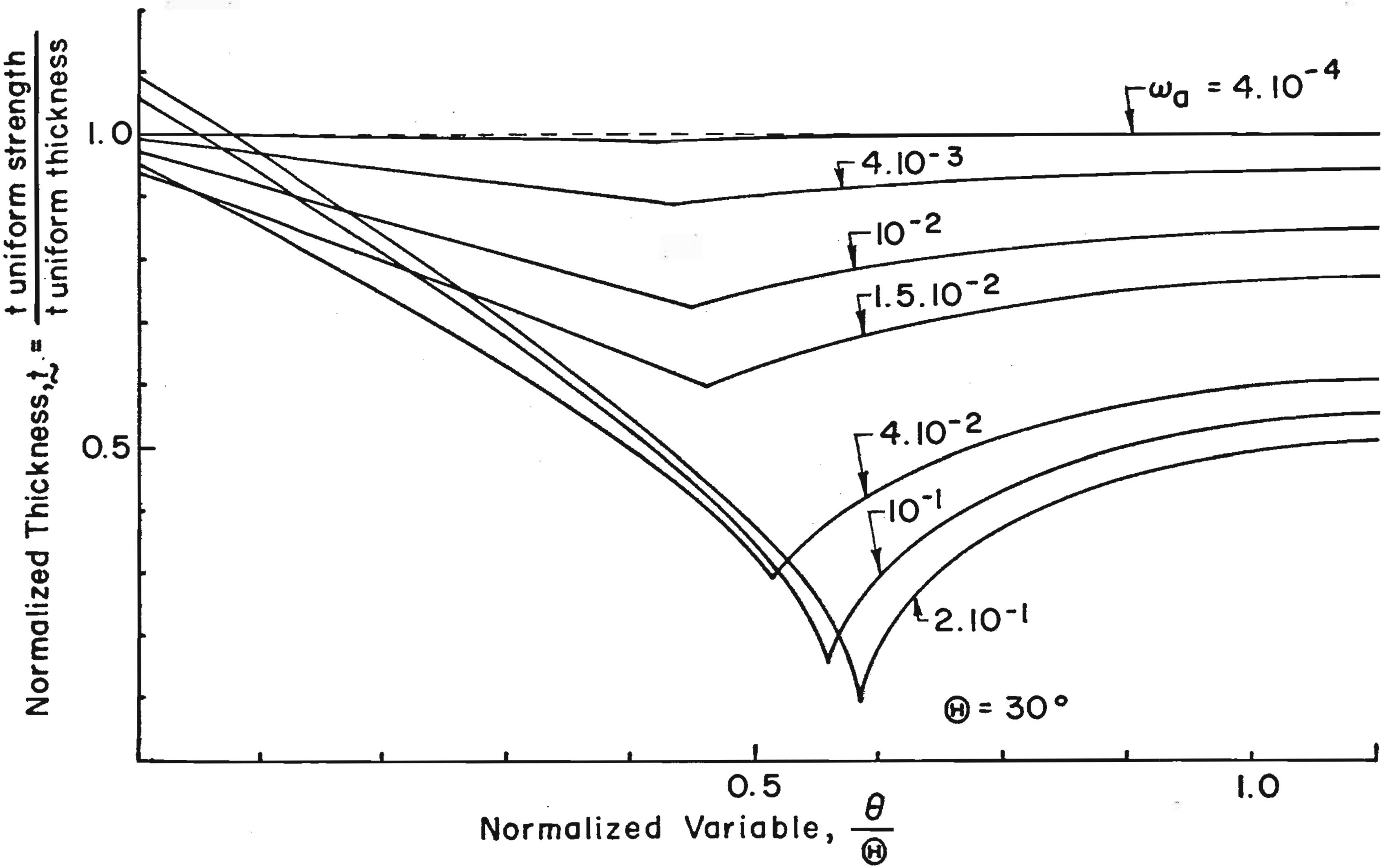

FIG. 12 NORMALIZED THICKNESS VS LOAD PARAMETER (ARCH)

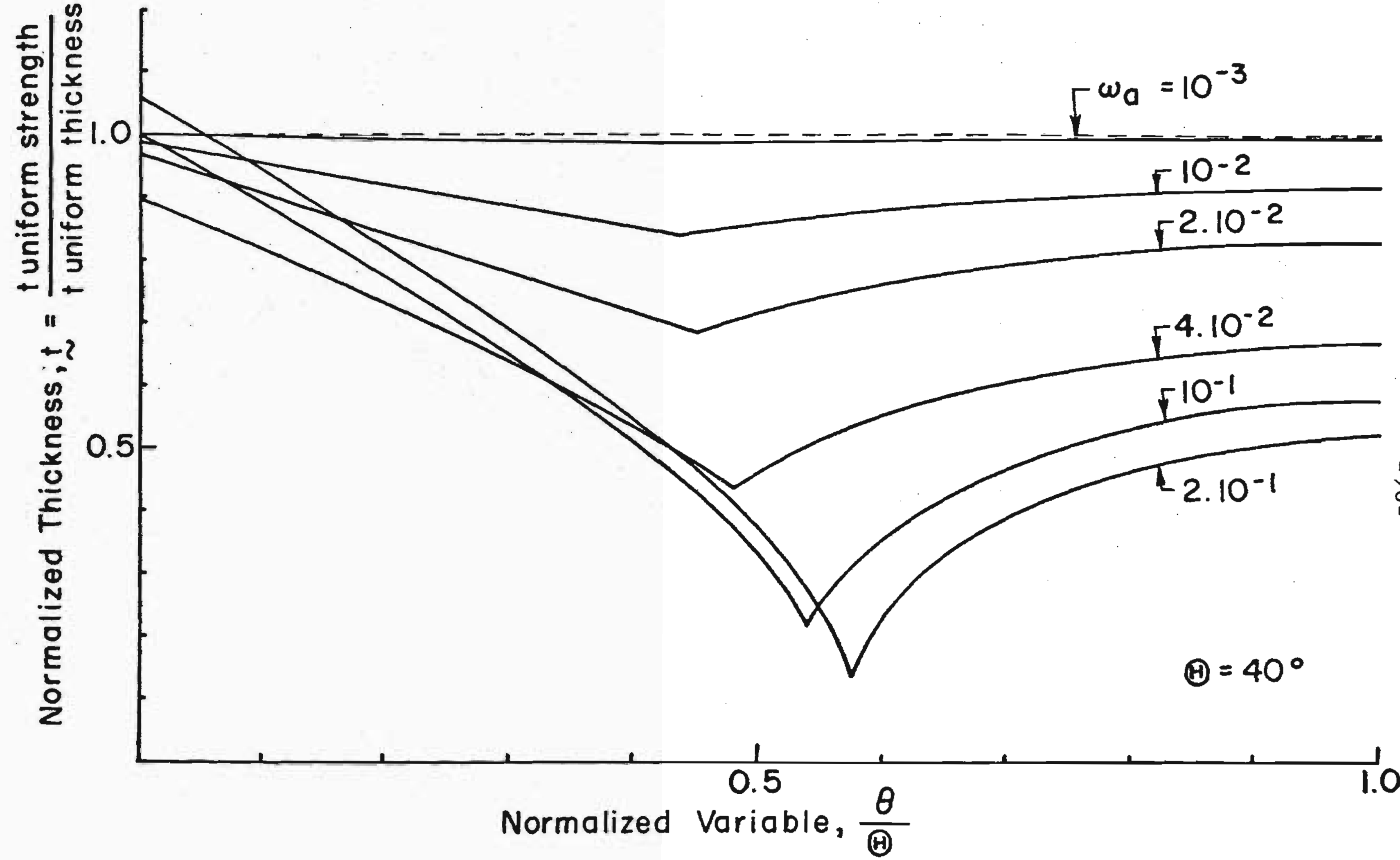

FIG. 13 NORMALIZED THICKNESS VS LOAD PARAMETER (ARCH)

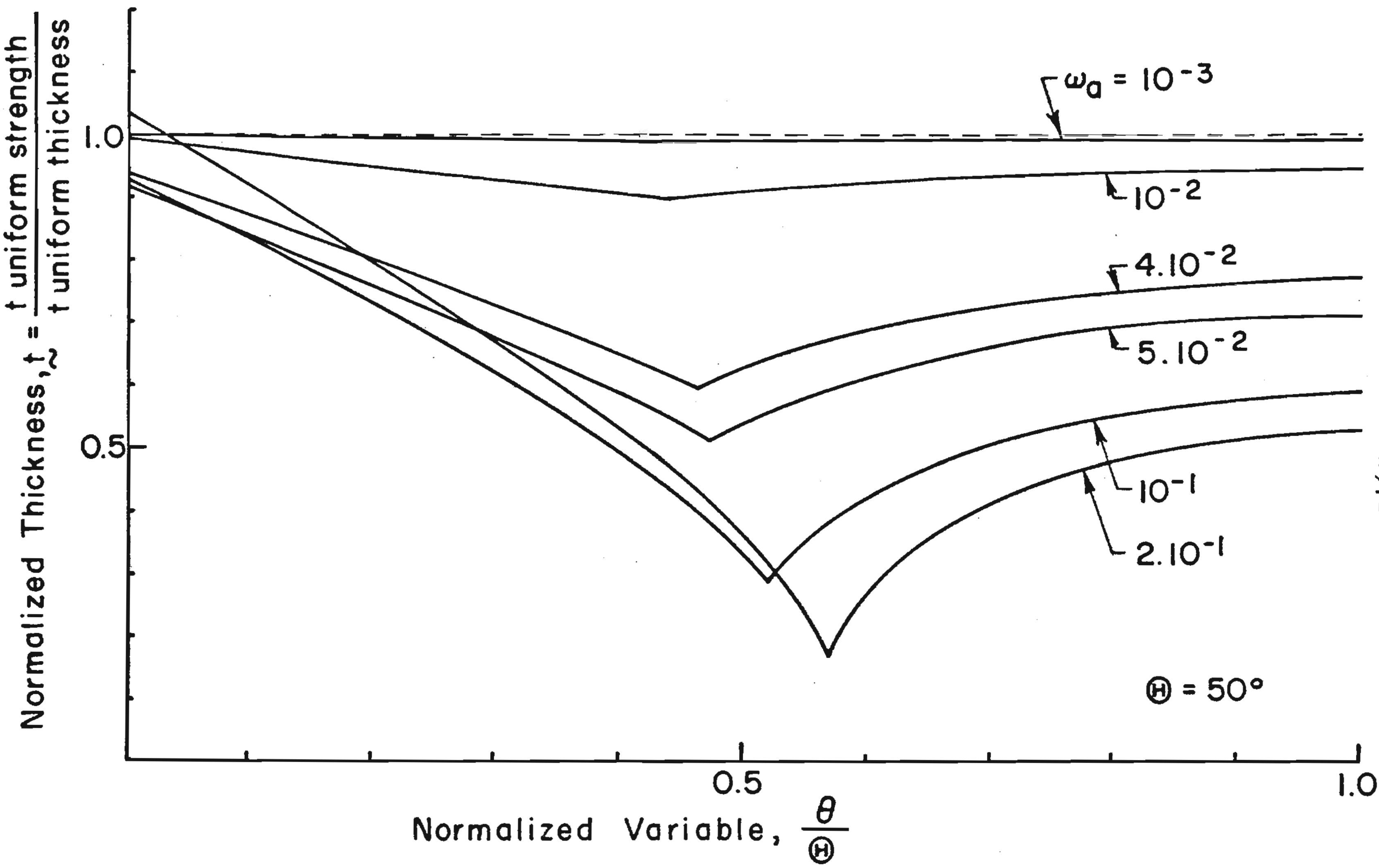

FIG. 14 NORMALIZED THICKNESS VS LOAD PARAMETER (ARCH)

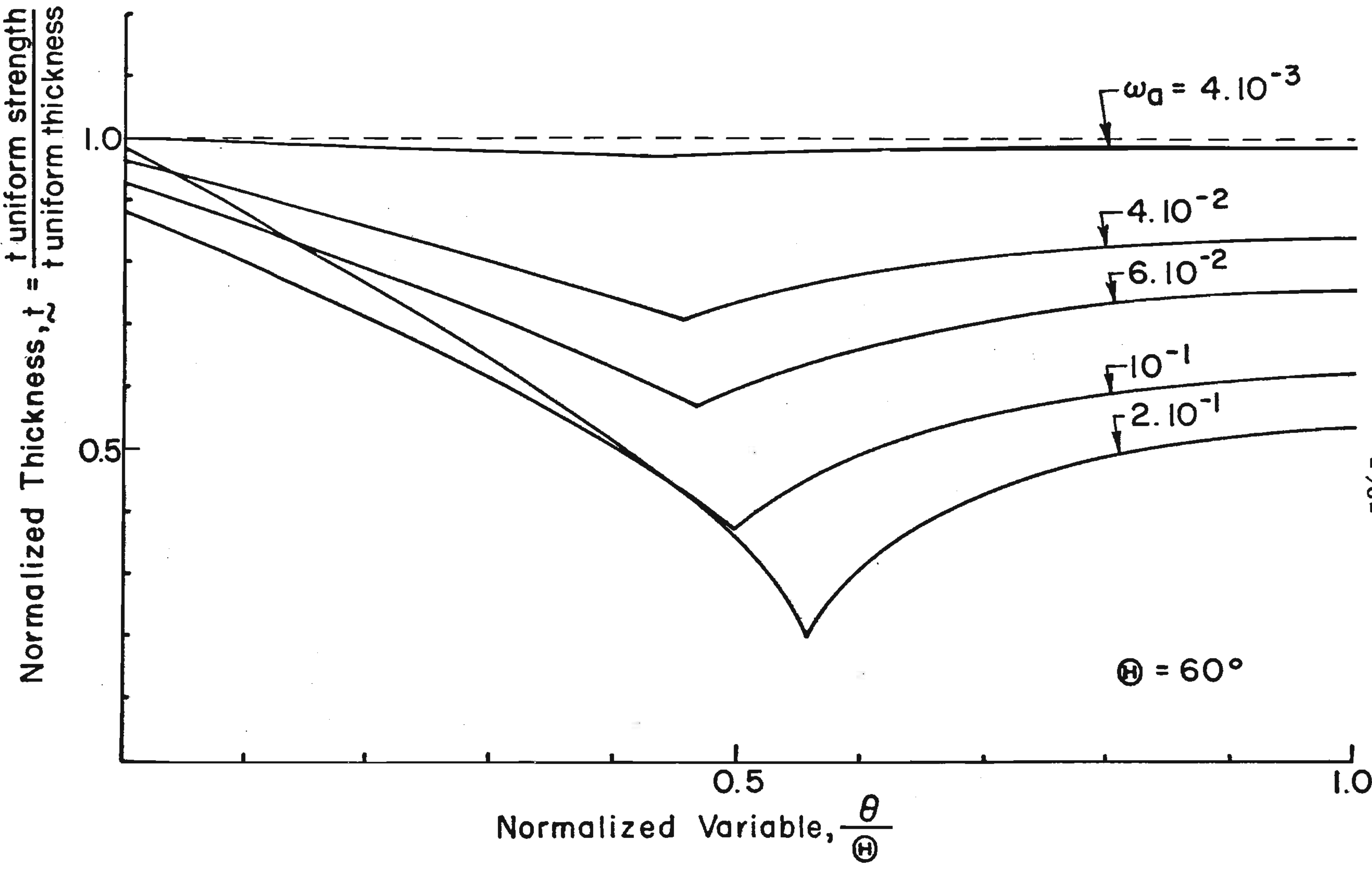

FIG. 15 NORMALIZED THICKNESS VS LOAD PARAMETER (ARCH)

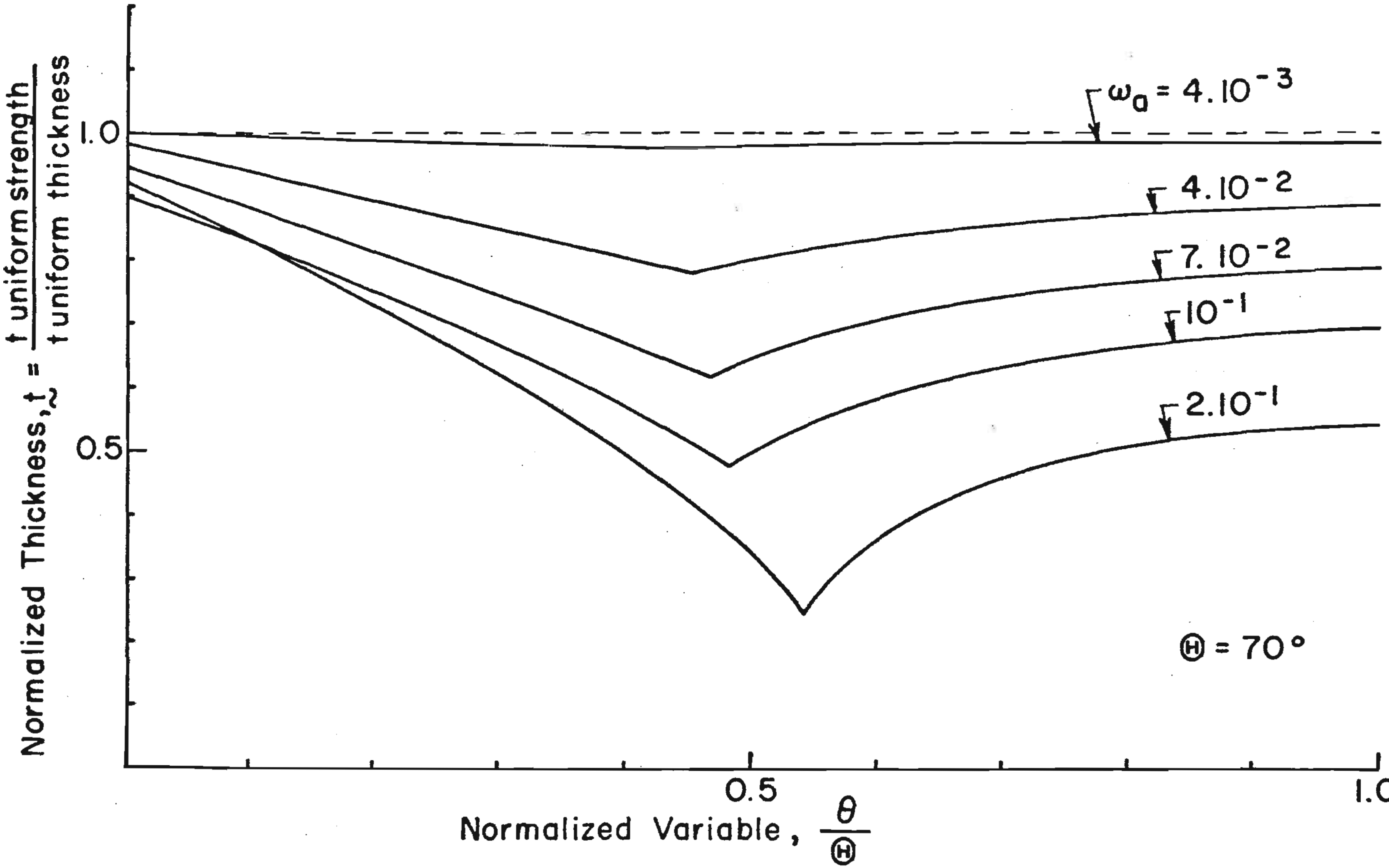

FIG. 16 NORMALIZED THICKNESS VS LOAD PARAMETER (ARCH)

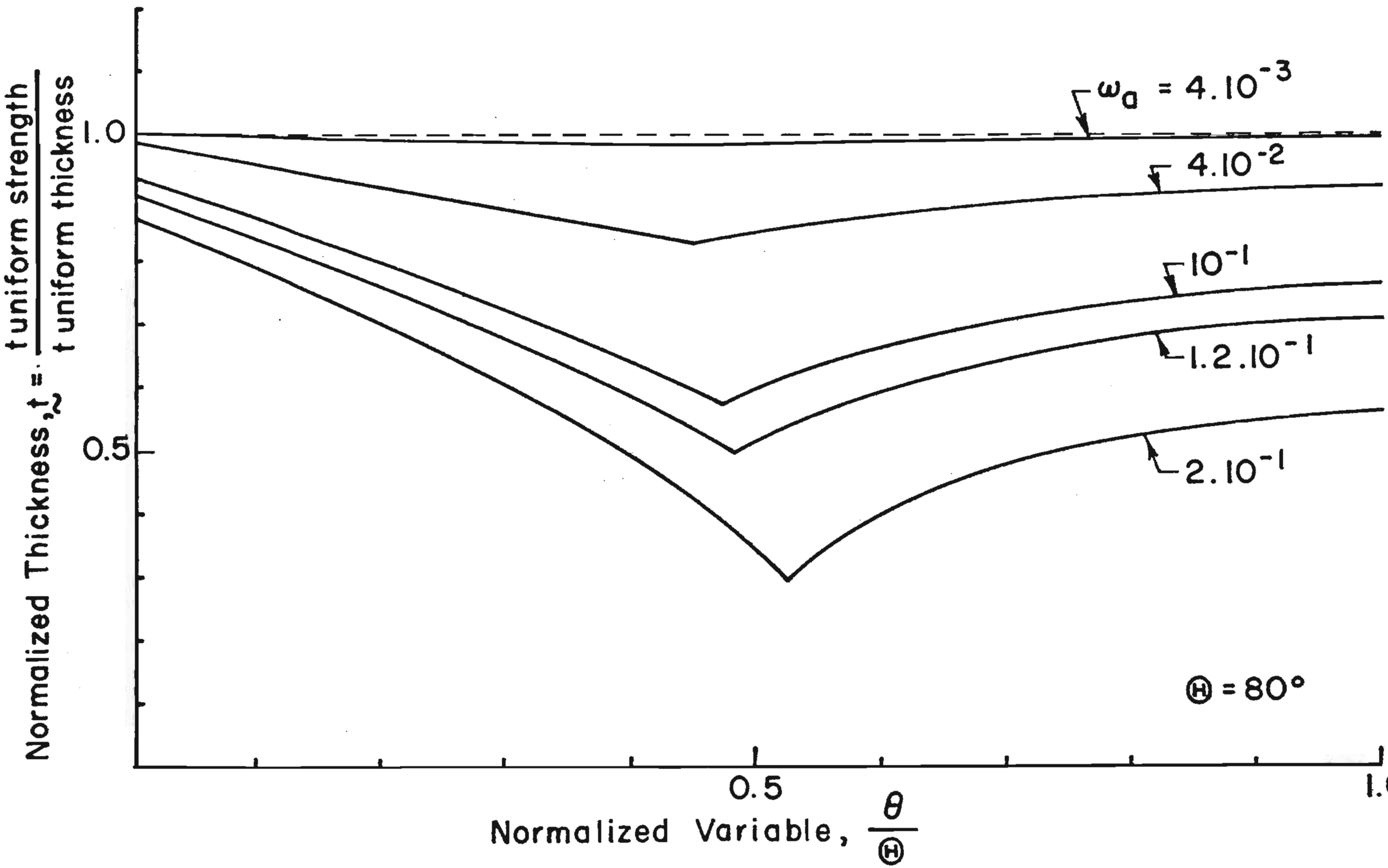

FIG. 17 NORMALIZED THICKNESS VS LOAD PARAMETER (ARCH)

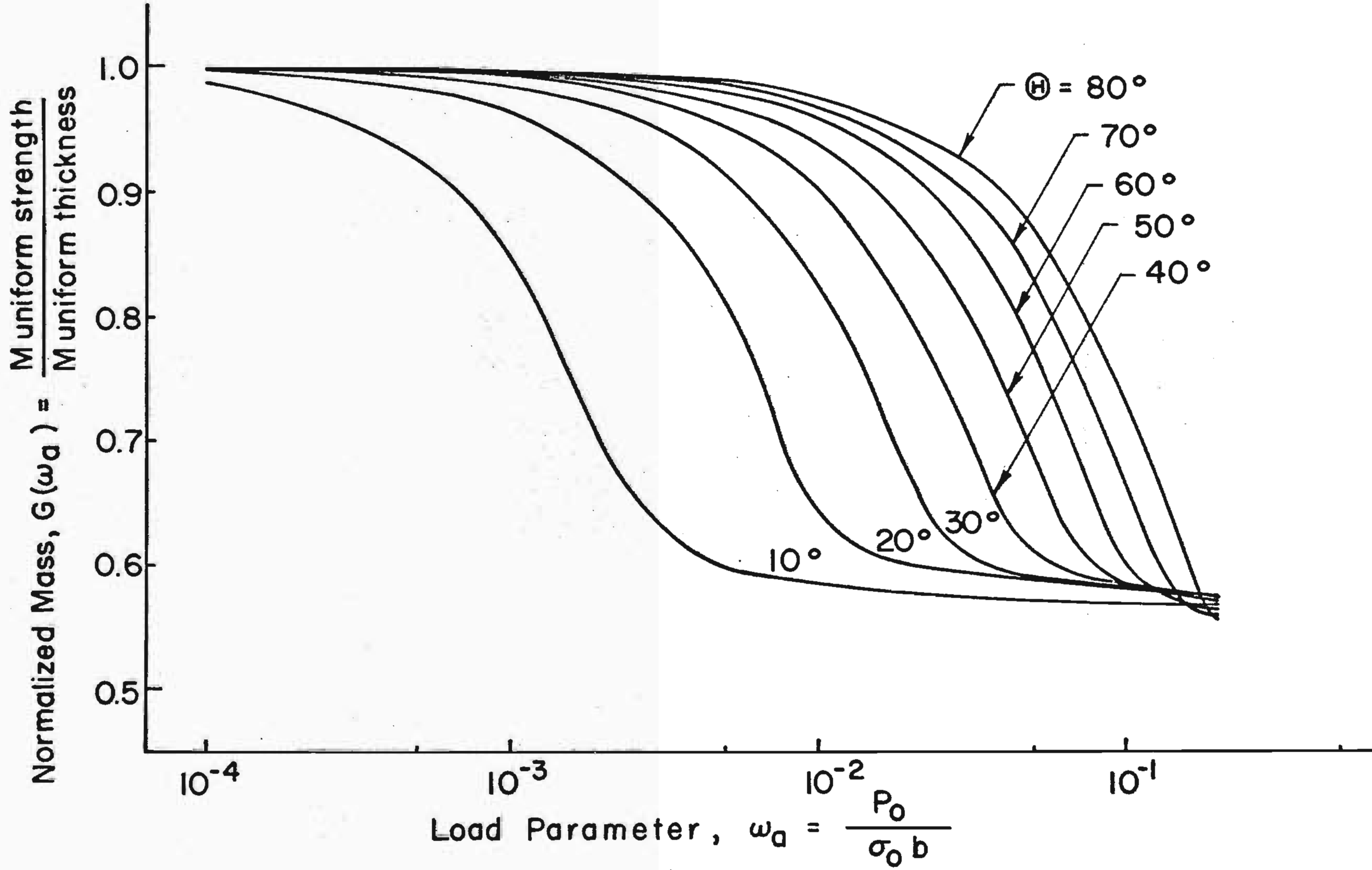

FIG. 18 NORMALIZED MASS VS LOAD PARAMETER FOR DIFFERENT OPENING ANGLES (ARCH)