ELLIPTICITY AND DEFORMATIONS WITH DISCONTINUOUS GRADIENTS IN FINITE ELASTOSTATICS

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Ποτὸν δὲ αὐτοῖς ἐστιν ἀὴρ συνθλιβόμενος εἰς κύλικα καὶ ὑγρὸν παρέχων ὥσπερ δρόσον.

Λουκιανοῦ Σαμοσατέως Ἀληθὴς Ἱστορία

And a [common] drink among [the inhabitants of the moon] is air which is compressed in a chalice and provides a liquid like dew.

Lucian of Samosata, *A True Story* (250 B.C.)

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ABSTRACT

Loss of ellipticity of the equilibrium equations of finite elastostatics is closely related to the possible emergence of elastostatic shocks, i.e., deformations with discontinuous gradients. In certain situations where constitutive response functions are essentially one-dimentional, such as anti-plane shear or bar theories, *strong ellipticity* is closely related to *convexity* of the elastic potential and *invertibility* of certain constitutive response functions.

The present work addresses the analogous issues within the context of three-dimensional elastostatics of compressible but not necessarily isotropic hyperelastic materials. A certain *direction-dependent resolution* of the deformation gradient is introduced and its existence and uniqueness for a given direction are established. The elastic potential is expressed as a function of kinematic variables arising from this resolution. Strong ellipticity is shown to be equivalent to the positive definiteness of the Hessian matrix of this function, thus sufficing for its strict convexity. The underlying variables are interpretable physically as simple shears and extensions. Their work-conjugates define a *traction response mapping*. It is shown that discontinuous deformation gradients are sustainable if and only if this mapping fails to be invertible. This result is explicit, in the sense that it characterizes the set of all possible piecewise homogeneous deformations given the elastic potential function.

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0. Introduction.

Solid materials exhibiting more than one phase have received increased attention in recent years. Continuum-mechanical theories have emerged which provide models for the type of behavior associated with austenite-martensite transformations in certain alloys, twinning in crystals, or other load-induced phase transformations, such as the ones that occur in shape memory alloys[1]. Macroscopic studies of such phenomena very often involve finite deformation fields with discontinuous deformation gradients. The theory of finite elasticity, generalized in order to encompass such singular fields, predicts that they are necessarily accompanied by a loss of ellipticity of the equilibrium equations. For isotropic hyperelastic materials, conditions on the elastic potential (stored energy function) which are necessary and sufficient for strong and ordinary ellipticity have been obtained in various settings. Within the context of *plane deformations*, the appropriate criteria were furnished by Knowles and Sternberg [1] for compressible and by Abeyaratne [2] for incompressible bodies. The analogous three-dimensional results relevant to incompressible istropic materials were deduced by Zee and Sternberg [3], whereas the ones appropriate for compressible bodies were obtained by Simpson and Spector[4].

There are certain aspects of the interpretation and consequences of the aforementioned conditions that are well understood only in situations where the underlying constitutive response functions are essentially one-dimensional. One such situation is that investigated by Abeyaratne [2]. He shows that for plane deformations of incompressible bodies, the requirement of *strong ellipticity* is only slightly stronger than that of *strict convexity* of the elastic potential con-

[1] For a sample of the literature on the subject the reader is refered to [5,12–17]

sidered as a function of the amount of shear. Moreover, the former suffices for the *invertibility* of the shear stress response function. The implications of the latter property are taken up by Abeyaratne and Knowles in [5]. They conclude that a loss of convexity of the elastic potential is synonymous with the existence of *equilibrium shocks,* i.e., deformations exhibiting discontinuities in their gradient.

On the other hand, the ellipticity conditions appropriate to the higher-dimensional constitutive setting in [1], [3] and [4] do not seemingly allow one to draw conclusions analogous to those in [2] and [5] alluded to above. Indeed, whether or not such an interpretation of ellipticity conditions is feasible within the general framework of three-dimensional elastostatics is a question that has motivated the present work. Our objective is twofold. First, we furnish a set of conditions for strong ellipticity which clarify its connection to a convexity condition on the elastic potential and to the invertibility of a suitably chosen constitutive response function. Additionally we deduce conditions on the constitutive law which are necessary and sufficient for the material to sustain equilibrium shocks. We accomplish the latter task in a way which is constructive, in the sense that it characterizes the set of all possible piecewise homogeneous deformations, given the elastic potential. Our results are pertinent to three-dimensional deformations of compressible, hyperelastic but not necessarily isotropic bodies.

Section 1 recalls some preliminaries from the theory of nonlinear elastostatics as well as the relevant notions of strong and ordinary ellipticity.

In Section 2 we introduce a direction-dependent decomposition for nonsingular linear transformations on a Euclidean space of arbitary dimension. After establishing the existence and uniqueness of this decomposition, we investigate its kinematic significance in a three-dimensional context.

Section 3 is devoted to a derivation of necessary and sufficient conditions for strong and ordinary ellipticity. The elastic potential is reduced to a function of kinematic variables which arise in connection with the directional resolution of the deformation gradient introduced in Section 2. Strong ellipticity is shown to be equivalent to the positive definiteness of the Hessian matrix of this *reduced* elastic potential, and thus it suffices for the strict convexity of this function. Moreover, strong ellipticity implies that the gradient mapping of the latter—expressing Piola tractions as functions of their conjugate resolved shears and extensions—is *globally invertible.* Ordinary ellipticity is analogously related to the local invertibility of this *traction response mapping.* These conclusions offer a mechanical interpretation of the ellipticity conditions. The results are subsequently specialized to the case of isotropy, where they assume a particularly simple form, and are applied to a specific choice of the constitutive law for purposes of illustration.

In Section 4 we set up the problem of existence of equilibrium deformations with piecewise constant gradients. By utilizing the kinematic results of Section 1, we then deduce certain inequalities which restrict the principal stretches of such deformations as a consequence of compatibility. We proceed to derive restrictions on the elastic potential which are necessary and sufficient for such states to be sustainable by the material. These conditions turn out to be equivalent to a loss of invertibility of the traction response mapping introduced in the previous section. Furthermore they provide an explicit representation of all pairs of associated deformation gradients. Hence the question of existence of shocks is answered in a constructive manner. As an application we identify all shocks sustainable by a special class of materials.

Finally, Section 5 establishes a connection between previous work and the present paper by adapting the approach of Sections 2 and 3 to plane deformations. The resulting conditions for ellipticity reproduce the results of Knowles and Sternberg [3].

1. Preliminaries. Finite Elastostatics, Ellipticity.

The symbol E_n, with n a positive integer, is used throughout to represent an n-dimentional real Euclidean space. Accordingly, $\mathcal{L}_+$ is the set of all tensors (linear transformations) on E_n with positive determinant, $\overset{+}{\mathcal{S}}$ is the set of all symmetric, positive-definite tensors, whereas $\overset{+}{\mathcal{S}}_0$ stands for the collection of all symmetric, positive-semidefinite, singular tensors on E_n. Moreover, $\mathcal{O}$ denotes the orthogonal group while $\mathcal{O}_+$ and $\mathcal{O}_-$ are the subgroups of $\mathcal{O}$ consisting of the proper and improper orthogonal tensors, respectively, so that $\mathcal{O}_+ = \mathcal{O} \cap \mathcal{L}_+$, $\mathcal{O}_- = \mathcal{O} - \mathcal{O}_+$. The set of unit vectors in E_n is denoted by $\mathcal{U}$. By a frame is meant a Cartesian coordinate frame X$= \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ with origin O and an orthonormal basis for E_n with vectors $\underset{\sim}{e}_i$, $i = 1, \ldots, n$. Latin indices have the range $\{1, 2, \ldots, n\}$; Greek indices range over $\{1, \ldots, n-1\}$. Summation over the appropriate range of repeated indices is implicit unless otherwise indicated. The matrix of components of a tensor $\underset{\sim}{A}$ in the frame X is denoted by $[\underset{\sim}{A}]^{\mathrm{X}}$. Superscripts -1, T and $-T$ indicate inversion, transposition and inversion of the transpose of a tensor; $\underset{\sim}{1}$ is the idem tensor. The set of strictly positive real numbers is symbolized by $\mathbb{R}_+$.

We consider a body which in some reference configuration occupies a region with interior $\mathcal{R}$ in real, Euclidean three-space E_3. A deformation is an invertible and suitably smooth mapping

$$\hat{\underset{\sim}{y}} : \mathcal{R} \to \mathcal{R}_* \, , \; \hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x}) \, , \; \underset{\sim}{x} \in \mathcal{R}, \tag{1.1}$$

which maps $\mathcal{R}$ onto the interior $\mathcal{R}_*$ of the region occupied by the body in the deformed configuration. Here $\underset{\sim}{x}$ is the reference position vector of a particle, $\hat{\underset{\sim}{y}}(\underset{\sim}{x})$ is its deformation image and $\underset{\sim}{u}(\underset{\sim}{x})$ the displacement. Until further notice, we assume that $\hat{\mathbf{y}}$ is twice continuously differentiable on $\mathcal{R}$. The deformation gradient tensor field,

$$\mathbf{F} = \nabla \hat{\mathbf{y}} \quad \text{on} \quad \mathcal{R}\,, \tag{1.2}$$

is restricted so that the Jacobian determinant of the mapping (1.1) is positive:

$$J = \det \mathbf{F} \ > 0 \quad \text{on} \quad \mathcal{R}\,. \tag{1.3}$$

The right and left Cauchy-Green tensors, $\mathbf{C}$ and $\mathbf{G}$, defined by

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \quad , \quad \mathbf{G} = \mathbf{F}\mathbf{F}^T \quad \text{on} \quad \mathcal{R}\,, \tag{1.4}$$

share the same fundamental scalar invariants. These are known as the deformation invariants associated with (1.1) and are given by

$$\left.\begin{aligned} I_1(\mathbf{C}) &= \operatorname{tr}\mathbf{C} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2\,, \\ I_2(\mathbf{C}) &= \frac{1}{2}[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}(\mathbf{C}^2)] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2\,, \\ I_3(\mathbf{C}) &= \det \mathbf{C} = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2, \end{aligned}\right\} \tag{1.5}$$

where $\lambda_i > 0$ are the principal stretches of the deformation.

The Cauchy stress tensor field $\boldsymbol{\tau}$, defined on $\mathcal{R}_*$, associated with (1.1), is related to the Piola (nominal) stress tensor field $\boldsymbol{\sigma}$, defined on $\mathcal{R}$, via

$$\boldsymbol{\tau}\,(\hat{\mathbf{y}}(\mathbf{x})) = \frac{1}{J(\mathbf{x})}\,\boldsymbol{\sigma}(\mathbf{x})\,\mathbf{F}^T(\mathbf{x})\,, \quad \mathbf{x} \in \mathcal{R}\,. \tag{1.6}$$

The deformation is equilibrated in the absence of body forces if

$$\operatorname{div}\boldsymbol{\sigma} = \mathbf{0} \quad , \quad \boldsymbol{\sigma}\mathbf{F}^T = \mathbf{F}\boldsymbol{\sigma}^T \quad \text{on} \quad \mathcal{R}\,. \tag{1.7}$$

We now specify that the body at hand is hyperelastic and that the reference configuration is homogeneous. Thus the elastic potential W is a scalar function defined and twice continuously differentiable on $\mathcal{L}_+$. The Piola stress is then determined by the constitutive law

$$\underset{\sim}{\sigma} = W_{\underset{\sim}{F}}(\underset{\sim}{F}) \quad \text{or} \quad \sigma_{ij} = \partial W(\underset{\sim}{F})/\partial F_{ij} \,. \tag{1.8}$$

Objectivity requires that

$$W(\underset{\sim}{F}) = W(\underset{\sim}{R}\underset{\sim}{F}) \quad \forall\,(\underset{\sim}{R}, \underset{\sim}{F}) \in \mathcal{O}_+ \times \mathcal{L}_+ \,. \tag{1.9}$$

Upon substitution from (1.9), (1.8) into (1.7) one finds that the second of (1.7), namely balance of moments, is implied by them, whereas the first becomes a system of second order, quasilinear, partial differential equations for the displacement field components:

$$C_{ijk\ell}(\underset{\sim}{1} + \nabla \underset{\sim}{u})u_{k,\ell j} = 0 \quad \text{on} \quad \mathcal{R} \,, \tag{1.10}$$

where

$$C_{ijk\ell}(\underset{\sim}{F}) = C_{k\ell ij}(\underset{\sim}{F}) = \partial^2 W(\underset{\sim}{F})/\partial F_{ij}\partial F_{k\ell} \,, \quad \underset{\sim}{F} \in \mathcal{L}_+ \,, \tag{1.11}$$

are the components of the elasticity four-tensor $\underset{\sim}{C}(\underset{\sim}{F})$. The system (1.10) is known as the displacement equations of equilibrium.

In the event that the body is composed of isotropic material,[1] the elastic potential depends on $\underset{\sim}{F}$ only through the deformation invariants:

$$W(\underset{\sim}{F}) = \hat{W}(I_1(\underset{\sim}{F}^T\underset{\sim}{F}), I_2(\underset{\sim}{F}^T\underset{\sim}{F}), I_3(\underset{\sim}{F}^T\underset{\sim}{F})) \,, \quad \underset{\sim}{F} \in \mathcal{L}_+ \,. \tag{1.12}$$

In the above $\hat{W}$ is a function defined on the Invariant Region

$$\overset{+}{\mathcal{I}} = \{(\xi_1, \xi_2, \xi_3) \,|\, \xi_i = I_i(\underset{\sim}{A}) \,,\, \underset{\sim}{A} \in \overset{+}{\mathcal{S}}\} \,. \tag{1.13}$$

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[1] Here the reference configuration is chosen to be natural for isotropy of the body.

Alternatively, W can be expressed in terms of the principal stretches λ_i:

$$\hat{W}(I_1, I_2, I_3) = \bar{W}(\lambda_1, \lambda_2, \lambda_3)\ , \tag{1.14}$$

where $\bar{W}$, defined on $\mathbb{R}^3_+$ and fully symmetric in its arguments, is obtained from $\hat{W}$ by means of the relations (1.5) connecting the I_i and λ_i.

One defines

$$\sigma_i = \frac{\partial \bar{W}}{\partial \lambda_i} \quad , \quad \tau_i = \frac{\lambda_i}{\lambda_1 \lambda_2 \lambda_3} \sigma_i \quad \text{on } \mathbb{R}^3_+\ , \tag{1.15}$$

as the principal Piola and Cauchy stresses, respectively. These are equal to the principal values of the respective stress tensors in case $\underset{\sim}{F}$ is symmetric with principal values $\lambda_i > 0$, as can be seen from (1.8), (1.6) and (1.14).

Restrictions on mechanical behavior are often imposed by means of certain postulates in the form of constitutive inequalities. In particular, $\bar{W}$ obeys the Baker-Ericksen inequalities whenever

$$(\tau_i - \tau_j)(\lambda_i - \lambda_j) > 0 \quad (\text{no sum}) \quad \lambda_i \neq \lambda_j,\ \ \lambda_i > 0\ . \tag{1.16}$$

On the other hand the Coleman-Noll condition holds whenever $\bar{W}$ is a strictly convex function on $\mathbb{R}^3_+$. A slightly stronger restriction requires that the matrix of second partial derivatives

$$\left[\frac{\partial^2 \bar{W}}{\partial \lambda_i \partial \lambda_j}\right] \quad \text{is positive-definite on } \mathbb{R}^3_+\ . \tag{1.17}$$

We now proceed to the notions of ordinary and strong ellipticity no longer restricting attention to isotropic materials. The acoustic tensor is the tensor with components

$$Q_{ik}(\underset{\sim}{F}, \underset{\sim}{n}) = C_{ijk\ell}(\underset{\sim}{F}) n_j n_\ell \quad \forall\, (\underset{\sim}{F}, \underset{\sim}{n}) \in \mathcal{L}_+ \times \mathcal{U}\ . \tag{1.18}$$

As is seen from (1.11), $\underset{\sim}{Q}$ is symmetric on its domain of definition.

Definition 1.1. *For a given choice of W, ordinary ellipticity holds at $\underset{\sim}{F} \in \mathcal{L}_+$ if and only if*

$$\det \underset{\sim}{Q}(\underset{\sim}{F}, \underset{\sim}{n}) \neq 0 \quad \forall \underset{\sim}{n} \in \mathcal{U} . \tag{1.19}$$

Moreover, strong ellipticity holds at $\underset{\sim}{F}$ if and only if

$$\underset{\sim}{m} \cdot \underset{\sim}{Q}(\underset{\sim}{F}, \underset{\sim}{n})\, \underset{\sim}{m} > 0 \quad \forall\, \underset{\sim}{m}, \underset{\sim}{n} \in \mathcal{U} \quad . \tag{1.20}$$

In the above, ellipticity is regarded as a property of the constitutive function W, whereas the motivation stems from considerations pertaining to the smoothness of solutions of the system (1.10). Following the methodology of Knowles and Sternberg [6], we call a displacement field $\underset{\sim}{u} : \mathcal{R} \to E_3$ a relaxed solution[2] of (1.10) if it has continuous first and piecewise continuous second partial derivatives on $\mathcal{R}$, the latter suffering at most finite jumps across continuously differentiable surfaces in $\mathcal{R}$, and if moreover it satisfies (1.10) at points of continuity of its second derivatives. Accordingly, *the system* (1.10) *is said to be elliptic at a relaxed solution $\underset{\sim}{u}$, and at a point $\underset{\sim}{x} \in \mathcal{R}$, if* (1.19) *holds with* $\underset{\sim}{F} = \underset{\sim}{1} + \nabla \underset{\sim}{u}(\underset{\sim}{x})$. Knowles and Sternberg [6] show that this is *sufficient* for *all* relaxed solutions whose gradient coincides with that of $\underset{\sim}{u}$ at $\underset{\sim}{x}$ to be twice continuously differentiable there. On the other hand, loss of ellipticity does not preclude the existence of twice continuously differentiable solutions of (1.10). This can be demonstrated by choosing a homogeneous deformation whose (constant) gradient is such that (1.19) is violated.

The system (1.10) *is said to be strongly elliptic at a solution $\underset{\sim}{u}$ and at $\underset{\sim}{x} \in \mathcal{R}$ if* (1.20) *holds with* $\underset{\sim}{F} = \underset{\sim}{1} + \nabla \underset{\sim}{u}(\underset{\sim}{x})$. Clearly, strong implies ordinary ellipticity. For an alternative definition, which stems from the elastodynamic motivation of strong ellipticity, we refer the reader to Truesdell and Noll [7].

[2] The term "relaxed solution" was introduced by Zee and Sternberg [3] in the context of a discussion parallel to the one in Knowles and Sternberg [1] but pertaining to *incompressible* bodies.

In general it is desirable to establish the connection between various constitutive inequalities, including (1.19), (1.20). It is known that the Baker-Ericksen inequalities are necessary for strong ellipticity[3]. By means of a special example, Knowles and Sternberg [6] have demonstrated that the Coleman-Noll condition neither implies nor is implied by strong ellipticity.

[3] This result was obtained by Zee and Sternberg [3] for incompressible and Simpson and Spector [4] for compressible isotropic materials.

2. On the Directional Resolution of a Tensor.

In the present section we introduce the directional resolution of a tensor. To begin with, we establish a theorem which guarantees the existence and uniqueness of such a resolution, given a direction. We then proceed to elucidate the kinematic significance of this result in connection with the deformation gradient.

At this point we introduce a direction-dependent decomposition appropriate for tensors with positive determinant. This depends upon establishing the following

Theorem 2.1. *Let $\underset{\sim}{F} \in \mathcal{L}_+$. Then for each $\underset{\sim}{n} \in \mathcal{U}$, there exist unique $\underset{\sim}{R} \in \mathcal{O}_+$, unique $\underset{\sim}{a} \in E_n$, unique $\underset{\sim}{V} \in \overset{+}{\mathcal{S}}$ with $\underset{\sim}{V}\underset{\sim}{n} = \underset{\sim}{n}$, such that*

$$\underset{\sim}{F} = \underset{\sim}{R}(\underset{\sim}{a} \otimes \underset{\sim}{n} + \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha)\underset{\sim}{V}, \tag{2.1}$$

where $\{\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_{n-1}, \underset{\sim}{n}\}$ is an (orthonormal) principal basis for $\underset{\sim}{V}$.

Our proof of the above rests upon a refinement of the Polar Decomposition Theorem,[1] concerning the extent of nonuniqueness of the orthogonal factor in the right polar decomposition of certain *singular* tensors. The *null space* of a tensor $\underset{\sim}{A}$ is $\mathcal{N}(\underset{\sim}{A}) = \{\underset{\sim}{x} \mid \underset{\sim}{x} \in E_n, \underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{0}\}$.

Lemma 2.1. *Let $\underset{\sim}{B}$ be a tensor on E_n having a one-dimensional null space $\mathcal{N}(\underset{\sim}{B})$. Then there exist unique $\overset{+}{\underset{\sim}{R}} \in \mathcal{O}_+$, $\overset{-}{\underset{\sim}{R}} \in \mathcal{O}_-$ and a unique $\underset{\sim}{U} \in \overset{+}{\mathcal{S}}_0$, such that*

$$\underset{\sim}{B} = \overset{+}{\underset{\sim}{R}}\underset{\sim}{U} = \overset{-}{\underset{\sim}{R}}\underset{\sim}{U}. \tag{2.2}$$

Proof of Lemma 2.1.

[1] see for example Halmos [8].

(a) Existence. The Polar Decomposition Theorem for arbitrary tensors ensures that there are tensors $\underset{\sim}{R} \in \mathcal{O}$ and $\underset{\sim}{U} \in \overset{+}{S}_0$, such that

$$\underset{\sim}{B} = \underset{\sim}{R}\underset{\sim}{U}, \tag{2.3}$$

from which it follows that

$$\mathcal{N}(\underset{\sim}{B}) = \mathcal{N}(\underset{\sim}{U}) \,. \tag{2.4}$$

By hypothesis, there is a unit vector $\underset{\sim}{e}$ such that $\mathcal{N}(\underset{\sim}{B}) \cap \mathcal{U} = \{-\underset{\sim}{e}\,, \underset{\sim}{e}\}$. Define

$$\underset{\sim}{P} = \underset{\sim}{1} - 2\underset{\sim}{e} \otimes \underset{\sim}{e} \,. \tag{2.5}$$

Then

$$\underset{\sim}{P}^2 = \underset{\sim}{1}\,,\ \underset{\sim}{P} = \underset{\sim}{P}^T\,,\ \underset{\sim}{P} \in \mathcal{O} \,. \tag{2.6}$$

Moreover, (2.5) yields

$$\underset{\sim}{P}\underset{\sim}{e} = -\underset{\sim}{e}\,, \quad \underset{\sim}{P}\underset{\sim}{m} = \underset{\sim}{m} \quad \forall\, \underset{\sim}{m} \perp \underset{\sim}{e}\,, \tag{2.7}$$

so that $\underset{\sim}{P}$, which is symmetric, has precisely one principal value equal to -1, corresponding to the principal direction vector $\underset{\sim}{e}$, and $n-1$ repeated principal values equal to 1. This, in view of (2.6), dictates that $\underset{\sim}{P} \in \mathcal{O}_-$. From (2.4), (2.5), noting that $\underset{\sim}{e} \in \mathcal{N}(\underset{\sim}{U})$, one infers that

$$\underset{\sim}{P}\underset{\sim}{U} = \underset{\sim}{U}. \tag{2.8}$$

Now, since (2.3) holds for some $\underset{\sim}{R} \in \mathcal{O}$, then either $\underset{\sim}{R} \in \mathcal{O}_+$ or $\underset{\sim}{R} \in \mathcal{O}_-$. If $\underset{\sim}{R} \in \mathcal{O}_+$, we set $\overset{+}{\underset{\sim}{R}} = \underset{\sim}{R}$, $\overset{-}{\underset{\sim}{R}} = \underset{\sim}{R}\underset{\sim}{P}$. Then clearly $\overset{-}{\underset{\sim}{R}} \in \mathcal{O}_-$, whereas with the aid of (2.3) and (2.8) we conclude that

$$\overset{+}{\underset{\sim}{R}}\underset{\sim}{U} = \overset{+}{\underset{\sim}{R}}\underset{\sim}{P}\underset{\sim}{U} = \overset{-}{\underset{\sim}{R}}\underset{\sim}{U} = \underset{\sim}{B} \quad , \quad \overset{+}{\underset{\sim}{R}} \in \mathcal{O}_+\,,\ \overset{-}{\underset{\sim}{R}} \in \mathcal{O}_-\,,\ \underset{\sim}{U} \in \overset{+}{S}_0 \,. \tag{2.9}$$

On the other hand, if $\underset{\sim}{R} \in \mathcal{O}_-$, setting $\overset{-}{\underset{\sim}{R}} = \underset{\sim}{R}$, $\overset{+}{\underset{\sim}{R}} = \overset{-}{\underset{\sim}{R}}\,\underset{\sim}{P} \in \mathcal{O}_+$, once again validates (2.9), thus demonstrating existence.

(b) Uniqueness. The uniqueness of $\underset{\sim}{U} \in \overset{+}{\mathcal{S}}_0$ satisfying (2.3) is ascertained by the original version of the Polar Decomposition Theorem as stated by Halmos [8]. In particular, (2.3) necessitates that

$$\underset{\sim}{U}^2 = \underset{\sim}{B}^T \underset{\sim}{B} \in \overset{+}{\mathcal{S}}_0 \, , \tag{2.10}$$

so that $\underset{\sim}{U}$ is the unique positive semi-definite square root of $\underset{\sim}{B}^T \underset{\sim}{B}$. Turning now to the uniqueness of each of the orthogonal factors in (2.2), let $\underset{\sim}{R}$, $\underset{\sim}{Q} \in \mathcal{O}$ satisfy

$$\underset{\sim}{R}\underset{\sim}{U} = \underset{\sim}{Q}\underset{\sim}{U} = \underset{\sim}{B} \tag{2.11}$$

and define

$$\underset{\sim}{M} = \underset{\sim}{Q}^T \underset{\sim}{R} \quad , \tag{2.12}$$

so that $\underset{\sim}{M} \in \mathcal{O}$. Because of (2.11), one has

$$\underset{\sim}{M}\underset{\sim}{U} = \underset{\sim}{U}. \tag{2.13}$$

Since $\underset{\sim}{U}$ is symmetric and possesses a one-dimensional null space $\mathcal{N}(\underset{\sim}{U})$, it has an orthonormal principal basis $\{\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_n\}$, in which exactly one of the $\underset{\sim}{e}_i$, say $\underset{\sim}{e}_n$, belongs to $\mathcal{N}(\underset{\sim}{U})$:

$$\underset{\sim}{U}\underset{\sim}{e}_n = \underset{\sim}{0} \, . \tag{2.14}$$

Using (2.13),

$$\underset{\sim}{M}\underset{\sim}{U}\underset{\sim}{e}_i = \underset{\sim}{U}\underset{\sim}{e}_i \, ;$$

hence

$$\underset{\sim}{M}\underset{\sim}{e}_\alpha = \underset{\sim}{e}_\alpha \quad , \quad \alpha = 1, \ldots, n-1 \, . \tag{2.15}$$

Orthogonality of $\underset{\sim}{M}$ forces the $\underset{\sim}{M}\underset{\sim}{e}_i$ $(i = 1, \ldots, n)$ to be orthonormal. In view of (2.15), this leaves two possibilities for $\underset{\sim}{M}\underset{\sim}{e}_n$, namely

$$(i)\ \underset{\sim}{M}\underset{\sim}{e}_n = \underset{\sim}{e}_n\ , \quad \text{or} \quad (ii)\ \underset{\sim}{M}\underset{\sim}{e}_n = -\underset{\sim}{e}_n\ . \tag{2.16}$$

If the first of (2.16) is true, then together with (2.15), it dictates that

$$\underset{\sim}{M} = \underset{\sim}{1}\ . \tag{2.17}$$

In case (ii) of (2.16) holds, once again $\underset{\sim}{M}$ is uniquely determined by (2.15), (2.16): By (2.14), $\underset{\sim}{e}_n \in \{-\underset{\sim}{e}, \underset{\sim}{e}\} \equiv \mathcal{N}(\underset{\sim}{U}) \cap \mathcal{U}$, whereas $\underset{\sim}{P}$ defined by (2.5) satisfies (2.15), (2.16). Hence (2.15), (2.16) ii imply

$$\underset{\sim}{M} = \underset{\sim}{P} \equiv \underset{\sim}{1} - 2\underset{\sim}{e} \otimes \underset{\sim}{e} \quad \in \mathcal{O}_-\ . \tag{2.18}$$

In conclusion, if $\underset{\sim}{R}$ and $\underset{\sim}{Q} \in \mathcal{O}$ satisfy (2.11), then by virtue of (2.12), (2.17), (2.18), they are related by $\underset{\sim}{R} = \underset{\sim}{M}\underset{\sim}{Q}$ with either $\underset{\sim}{M} = \underset{\sim}{1}$, in which case $\underset{\sim}{R} = \underset{\sim}{Q}$, or $\underset{\sim}{M} = \underset{\sim}{1} - 2\underset{\sim}{e} \otimes \underset{\sim}{e}$, where $\underset{\sim}{e}$ is the unit vector belonging to the null space of $\underset{\sim}{B}$ and $\underset{\sim}{U}$ and is unique apart from its sign, which does not affect $\underset{\sim}{M}$. Consequently there are at most two distinct orthogonal factors in the right polar decomposition of $\underset{\sim}{B}$, which cannot both belong to $\mathcal{O}_+$ or $\mathcal{O}_-$, since $\underset{\sim}{M} \in \mathcal{O}_-$. This verifies the uniqueness of each of them, and completes the proof.

Proof of Theorem 2.1.

(a) Existence. Given $\underset{\sim}{F} \in \mathcal{L}_+$ and $\underset{\sim}{n} \in \mathcal{U}$, define

$$\underset{\sim}{B} = \underset{\sim}{F}(\underset{\sim}{1} - \underset{\sim}{n} \otimes \underset{\sim}{n})\ . \tag{2.19}$$

Then clearly,

$$\mathcal{N}(\underset{\sim}{B}) = span\{\underset{\sim}{n}\}\ , \tag{2.20}$$

which has dimension 1, so that $\underset{\sim}{B}$ conforms to the hypotheses of Lemma 2.1. Accordingly there exist (unique) $\underset{\sim}{R} \in \mathcal{O}_+$ and $\underset{\sim}{U} \in \overset{+}{\mathcal{S}}_0$ satisfying

$$\underset{\sim}{B} = \underset{\sim}{R}\underset{\sim}{U}, \quad \mathcal{N}(\underset{\sim}{U}) = \mathcal{N}(\underset{\sim}{B})\ . \tag{2.21}$$

Combining (2.19), (2.21) and rearranging, we obtain

$$\underset{\sim}{F} = \underset{\sim}{R}(\underset{\sim}{R}^T\underset{\sim}{F}\,\underset{\sim}{n}) \otimes \underset{\sim}{n} + \underset{\sim}{R}\underset{\sim}{U}\,. \tag{2.22}$$

From the properties of $\underset{\sim}{U}$ we infer that there are $\underset{\sim}{e}_\alpha \in \mathcal{U}$ such that $\{\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_{n-1}, \underset{\sim}{n}\}$ is an orthonormal principal basis for $\underset{\sim}{U}$. The principal value of $\underset{\sim}{U}$ corresponding to $\underset{\sim}{n}$ is zero, the principal values corresponding to $\underset{\sim}{e}_\alpha$ are strictly positive. Now let

$$\underset{\sim}{V} = \underset{\sim}{U} + \underset{\sim}{n} \otimes \underset{\sim}{n}\,. \tag{2.23}$$

It follows that $\underset{\sim}{V}$ is positive-definite and

$$\underset{\sim}{V}\underset{\sim}{n} = \underset{\sim}{n}\,, \quad \underset{\sim}{V} \in \overset{+}{S}, \quad (\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha)\underset{\sim}{V} = \underset{\sim}{U}\,. \tag{2.24}$$

Substitution of the last of (2.24) into (2.22) and an appeal to the first of (2.24) provide

$$\underset{\sim}{F} = \underset{\sim}{R}[(\underset{\sim}{R}^T\underset{\sim}{F}\underset{\sim}{n}) \otimes \underset{\sim}{n} + \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha]\underset{\sim}{V}. \tag{2.25}$$

Finally, choosing

$$\underset{\sim}{a} = \underset{\sim}{R}^T\underset{\sim}{F}\,\underset{\sim}{n} \tag{2.26}$$

in (2.25), leads to the desired result (2.1).

(b) Uniqueness. Granted the existence of $\underset{\sim}{R}, \underset{\sim}{a}$ and $\underset{\sim}{V}$ satisfying (2.1), we now show them to be unique. Directly from (2.1), we have

$$\underset{\sim}{F}\,\underset{\sim}{n} = \underset{\sim}{R}(\underset{\sim}{a} \otimes \underset{\sim}{n} + \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha)\underset{\sim}{V}\underset{\sim}{n}\,.$$

Since $\underset{\sim}{V}\underset{\sim}{n} = \underset{\sim}{n}$ and the $\underset{\sim}{e}_\alpha$ are orthogonal to $\underset{\sim}{n}$, the above result simplifies to

$$\underset{\sim}{F}\,\underset{\sim}{n} = \underset{\sim}{R}\underset{\sim}{a}\,. \tag{2.27}$$

From the fact that $\{\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_{n-1}, \underset{\sim}{n}\}$ is a principal basis for $\underset{\sim}{V}$, we infer that

$$(\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha)\underset{\sim}{V} = \underset{\sim}{V} - \underset{\sim}{n} \otimes \underset{\sim}{n}\,. \tag{2.28}$$

Upon substitution of (2.27), (2.28) into (2.1) and after some rearrangement, we obtain

$$\underset{\sim}{F} - \underset{\sim}{F}\underset{\sim}{n}\otimes\underset{\sim}{n} = \underset{\sim}{R}(\underset{\sim}{V} - \underset{\sim}{n}\otimes\underset{\sim}{n}) . \tag{2.29}$$

The left hand side of (2.29), completely specified by $\underset{\sim}{F}$ and $\underset{\sim}{n}$, has a one-dimensional null space (spanned by $\underset{\sim}{n}$) and thus conforms to the hypothesis of Lemma 2.1. Also $\underset{\sim}{R} \in \mathcal{O}_+$ whereas $\underset{\sim}{V} - \underset{\sim}{n}\otimes\underset{\sim}{n} \in \overset{+}{\mathcal{S}}_0$, so that—by an appeal to Lemma 2.1 —they are the unique factors of the (proper orthogonal) polar decomposition of the left–hand side (2.29). Thus the uniqueness of $\underset{\sim}{R}$ and $\underset{\sim}{V}$ is established. With this in mind we draw from (2.27) that

$$\underset{\sim}{a} = \underset{\sim}{R}^T\underset{\sim}{F}\underset{\sim}{n}, \tag{2.30}$$

which assures that $\underset{\sim}{a}$ is also uniquely determined by $\underset{\sim}{F}$ and $\underset{\sim}{n}$ and completes the proof.

As suggested by Theorem (2.1), it is in fact possible to express $\underset{\sim}{R}, \underset{\sim}{a}, \underset{\sim}{V}$ in (2.1) directly as functions of $\underset{\sim}{F}$ and $\underset{\sim}{n}$. To that effect, define

$$\left.\begin{aligned} \underset{\sim}{B} = \hat{\underset{\sim}{B}}(\underset{\sim}{F},\underset{\sim}{n}) &= \underset{\sim}{F}(\underset{\sim}{1} - \underset{\sim}{n}\otimes\underset{\sim}{n}) , \\ \overset{*}{\underset{\sim}{n}} = \overset{*}{\underset{\sim}{n}}(\underset{\sim}{F},\underset{\sim}{n}) &= \underset{\sim}{F}^{-T}\underset{\sim}{n}/|\underset{\sim}{F}^{-T}\underset{\sim}{n}| , \\ \underset{\sim}{F} \in \mathcal{L}_+ &, \ \underset{\sim}{n} \in \mathcal{U} , \end{aligned}\right\} \tag{2.31}$$

and let

$$\left.\begin{aligned} &\underset{\sim}{V} = \hat{\underset{\sim}{V}}(\underset{\sim}{F},\underset{\sim}{n}) = (\underset{\sim}{B}^T\underset{\sim}{B} + \underset{\sim}{n}\otimes\underset{\sim}{n})^{1/2}, \\ &\underset{\sim}{R} = \hat{\underset{\sim}{R}}(\underset{\sim}{F},\underset{\sim}{n}) = (\underset{\sim}{B} + \overset{*}{\underset{\sim}{n}}\otimes\underset{\sim}{n})\underset{\sim}{V}^{-1}, \\ &\underset{\sim}{a} = \hat{\underset{\sim}{a}}(\underset{\sim}{F},\underset{\sim}{n}) = \underset{\sim}{R}^T\underset{\sim}{F}\underset{\sim}{n} , \\ &\qquad \underset{\sim}{F} \in \mathcal{L}_+ , \ \underset{\sim}{n} \in \mathcal{U} . \end{aligned}\right\} \tag{2.32}$$

Then one verifies that $\underset{\sim}{R}, \underset{\sim}{a}, \underset{\sim}{V}$ given by (2.32) indeed satisfy (2.1) with $\underset{\sim}{R} \in \mathcal{O}_+$, $\underset{\sim}{V} \in \overset{+}{\mathcal{S}}$, $\underset{\sim}{V}\underset{\sim}{n} = \underset{\sim}{n}$, $\underset{\sim}{a} \in E_3$ and with $\underset{\sim}{e}_\alpha$ as required by the theorem. We remark that a version of (2.1) that does not involve $\underset{\sim}{e}_\alpha$ is easily seen to be

$$\underset{\sim}{F} = \underset{\sim}{R}[(\underset{\sim}{a} - \underset{\sim}{n}) \otimes \underset{\sim}{n} + \underset{\sim}{1}]\underset{\sim}{V}; \tag{2.33}$$

one merely notes that $\underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + \underset{\sim}{n} \otimes \underset{\sim}{n} = \underset{\sim}{1}$. In contrast to the Polar Decomposition of $\underset{\sim}{F}$, the result of Theorem (2.1) is a *direction-dependent* decomposition of $\underset{\sim}{F}$, the "direction" being identified with the unit vector $\underset{\sim}{n}$. This is clarified by the explicit dependence on $\underset{\sim}{n}$ of the items in (2.32). Accordingly, *for given* $\underset{\sim}{F} \in \mathcal{L}_+$ *and* $\underset{\sim}{n} \in \mathcal{U}$ *we call* (2.1)*- with* $\underset{\sim}{R}, \underset{\sim}{a}, \underset{\sim}{V}$ *supplied by* (2.32) *- the Directional Resolution of* $\underset{\sim}{F}$ *with respect to* $\underset{\sim}{n}$.

At this point we specialize the foregoing results to three dimensions and proceed to investigate their kinematic interpretation in the context of three–dimensional deformations. Consider a homogeneous deformation $\hat{\underset{\sim}{y}} : E_3 \to E_3$

$$\hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \underset{\sim}{F}\underset{\sim}{x} \ , \quad \underset{\sim}{x} \in E_3 \ , \quad \underset{\sim}{F} \in \mathcal{L}_+ \ . \tag{2.34}$$

Also fix $\underset{\sim}{n} \in \mathcal{U}$ arbitrarily. Let $\underset{\sim}{R} \in \mathcal{O}_+, \underset{\sim}{a} \in E_3$ and $\underset{\sim}{V} \in \overset{+}{\mathcal{S}}$ supply via (2.1) the directional resolution of the deformation gradient $\underset{\sim}{F}$ of (2.34) with respect to $\underset{\sim}{n}$. In particular, because of the properties of $\underset{\sim}{V}$ there are a frame X= $\{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}, \underset{\sim}{e}_3 = \underset{\sim}{n}$ and two numbers $\beta_\alpha > 0$ such that

$$\underset{\sim}{V} = \sum_{\alpha=1}^{2} \beta_\alpha \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + \underset{\sim}{n} \otimes \underset{\sim}{n}, \tag{2.35}$$

so that $\underset{\sim}{e}_i$ ($\underset{\sim}{e}_3 = \underset{\sim}{n}$) are principal direction-vectors of $\underset{\sim}{V}$ corresponding to principal values $\beta_1, \beta_2, 1$, respectively. In view of (2.35), (2.1) becomes

$$\underset{\sim}{F} = \underset{\sim}{R}\left(\sum_{\alpha=1}^{2} \beta_\alpha \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + a_i \underset{\sim}{e}_i \otimes \underset{\sim}{e}_3\right) \ , \quad a_i = \underset{\sim}{a} \cdot \underset{\sim}{e}_i \ , \quad \underset{\sim}{e}_3 = \underset{\sim}{n} . \tag{2.36}$$

Note that det $\underset{\sim}{F} = \beta_1 \beta_2 a_3 > 0$, so that necessarily $a_3 > 0$. Thus it is a corollary of Theorem 2.1 that for given $\underset{\sim}{n}$, there are a frame X= $\{O; \underset{\sim}{e}_1, \underset{\sim}{e}, \underset{\sim}{e}_3\}$ with $\underset{\sim}{e}_3 = \underset{\sim}{n}$, numbers $\beta_\alpha > 0$ and a_i with $a_3 > 0$, and an $\underset{\sim}{R} \in \mathcal{O}_+$ such that the matrix of components of $\underset{\sim}{F}$ in X, $[\underset{\sim}{F}]^{\mathrm{X}}$, admits the representation

$$[\underset{\sim}{F}]^{\mathrm{X}} = [\underset{\sim}{R}]^{\mathrm{X}} \begin{pmatrix} \beta_1 & 0 & a_1 \\ 0 & \beta_2 & a_2 \\ 0 & 0 & a_3 \end{pmatrix} \quad , \beta_\alpha > 0, \quad a_3 > 0\,. \tag{2.37}$$

In fact, one can resolve (2.37) further into

$$[\underset{\sim}{F}]^{\mathrm{X}} = [\underset{\sim}{R}]^{\mathrm{X}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.38}$$

This provides the desired geometrical interpretation of the resolution (2.1) in terms of particularly simple homogeneous deformations. Consider the arbitrary homogeneous deformation (2.34), and choose an arbitrary direction $\underset{\sim}{n} \in \mathcal{U}$. Then one can find two orthogonal directions $\underset{\sim}{e}_1, \underset{\sim}{e}_2$ in the plane whose normal is $\underset{\sim}{n}$, such that the deformation (2.34) can be decomposed into the following:

(i) An in-plane biaxial strech (in the plane normal to $\underset{\sim}{n}$) in the directions $\underset{\sim}{e}_1, \underset{\sim}{e}_2$, with principal streches $\beta_1 > 0, \beta_2 > 0$, respectively, followed by

(ii) a simple shear parallel to $\underset{\sim}{e}_1$ in the $\underset{\sim}{e}_1, \underset{\sim}{e}_3$ plane, of amount a_1, followed by

(iii) a simple shear parallel to $\underset{\sim}{e}_2$ in the $\underset{\sim}{e}_2, \underset{\sim}{e}_3$ plane, of amount a_2, followed by

(iv) a uniaxial stretch parallel to $\underset{\sim}{e}_3 = \underset{\sim}{n}$, of amount $a_3 > 0$, followed by

(v) a rigid rotation about the origin with rotation tensor $\underset{\sim}{R} \in \mathcal{O}_+$.

A noteworthy property of $\underset{\sim}{R}$ in (2.1), (2.36)–(2.38) is that if Π is a plane normal to $\underset{\sim}{n}$, the unit normal $\overset{*}{\underset{\sim}{n}}$ to its deformation image $\hat{\underset{\sim}{y}}(\Pi)$ under (2.34) is given by

$$\overset{*}{\underset{\sim}{n}} = \underset{\sim}{R}\underset{\sim}{n}\,. \tag{2.39}$$

This can be seen from (2.31), (2.32), or directly from (2.1).

For subsequent use we introduce the following notation. For $\beta_\alpha \in \mathbb{R}_+$, we write $\underset{\sim}{\beta} = (\beta_1, \beta_2) \in \mathcal{D}_1 \equiv \mathbb{R}_+ \times \mathbb{R}_+$, whereas for $a_\alpha \in \mathbb{R}$, $a_3 \in \mathbb{R}_+$, we write

$\underline{a} = (a_1, a_2, a_3) \in \mathcal{D}_2 \equiv \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$. Also we let $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ and denote its elements by $(\underline{\beta}; \underline{a}) \in \mathcal{D}$ where $\underline{\beta} \in \mathcal{D}_1$, $\underline{a} \in \mathcal{D}_2$. For each frame X= $\{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ we define a tensor valued function $\underset{\sim}{A}_{\mathrm{X}}(\,\cdot\,) : \mathcal{D} \rightarrow \mathcal{L}_+$ by

$$\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \underline{a}) = \sum_{\alpha=1}^{2} \beta_\alpha \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + a_i \underset{\sim}{e}_i \otimes \underset{\sim}{e}_3 \quad \forall\, (\underline{\beta}; \underline{a}) \in \mathcal{D} \,. \tag{2.40}$$

That $\underset{\sim}{A}_{\mathrm{X}}(\mathcal{D}) \subset \mathcal{L}_+$ follows directly from (2.4) and the definition of $\mathcal{D}$. According to (2.40), for any frame X and $(\underline{\beta}; \underline{a}) \in \mathcal{D}$, $\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \underline{a})$ is the tensor whose matrix of components *in the same frame* X is

$$[\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \underline{a})]^{\mathrm{X}} = \begin{pmatrix} \beta_1 & 0 & a_1 \\ 0 & \beta_2 & a_2 \\ 0 & 0 & a_3 \end{pmatrix} . \tag{2.41}$$

The connection between $\underset{\sim}{A}_{\mathrm{X}}$ and the directional resolution is established in the following

Lemma 2.2. *Given $\underset{\sim}{n} \in \mathcal{U}$, let $\mathcal{F}_{\underset{\sim}{n}}$ be the set of all frames* $\mathrm{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ *with $\underset{\sim}{e}_3 = \underset{\sim}{n}$. Then*

$$\{\underset{\sim}{F} \,|\, \underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \underline{a}),\ \underset{\sim}{R} \in \mathcal{O}_+,\ \mathrm{X} \in \mathcal{F}_{\underset{\sim}{n}},\ (\underline{\beta}; \underline{a}) \in \mathcal{D}\} = \mathcal{L}_+ \,. \tag{2.42}$$

Proof. Let $\mathcal{A}$ stand for the set of tensors in the set specified in the left–hand side of (2.42). Using (2.41),

$$\det\{\underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \underline{a})\} = \beta_1 \beta_2 a_3 > 0 \quad \forall\, \underset{\sim}{R} \in \mathcal{O}_+ \,,\ \forall\, (\underline{\beta}; \underline{a}) \in \mathcal{D} \,, \tag{2.43}$$

since $(\underline{\beta}; \underline{a}) \in \mathcal{D} \Rightarrow \beta_\alpha > 0$, $a_3 > 0$ and $\det \underset{\sim}{R} = 1$ for $\underset{\sim}{R} \in \mathcal{O}_+$. Hence $\underset{\sim}{F} \in \mathcal{A} \Rightarrow \underset{\sim}{F} \in \mathcal{L}_+$, so that $\mathcal{A} \subset \mathcal{L}_+$. Now choose $\underset{\sim}{F} \in \mathcal{L}_+$ and construct its directional resolution with respect to $\underset{\sim}{n}$. This yields $\underset{\sim}{R} \in \mathcal{O}_+$, $\underset{\sim}{V} \in \overset{+}{\mathcal{S}}$, $\underset{\sim}{a} \in E_3$ satisfying (2.1). Choose β_α and $\underset{\sim}{e}_i$ as in (2.35), and let a_i be the components of $\underset{\sim}{a}$ in $\mathrm{X} \in \mathcal{F}_{\underset{\sim}{n}}$ where X is a principal frame of $\underset{\sim}{V}$ with unit vectors $\underset{\sim}{e}_i$ ($\underset{\sim}{e}_3 = \underset{\sim}{n}$).

Then $\underline{\beta} = (\beta_1, \beta_2)$ and $\underline{a} = (a_1, a_2, a_3)$ form $(\underline{\beta}; \underline{a}) \in \mathcal{D}$ so that (2.36), (2.37) and (2.40) give

$$\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \underline{a}) \tag{2.44}$$

where $\underset{\sim}{R} \in \mathcal{O}_+$, $\mathrm{X} \in \mathcal{F}_{\mathcal{D}}$, $(\underline{\beta}; \underline{a}) \in \mathcal{D}$, so that $\underset{\sim}{F} \in \mathcal{L}_+ \Rightarrow \underset{\sim}{F} \in \mathcal{A}$ because of (2.44) (2.42). Consequently $\mathcal{A} \supset \mathcal{L}_+$ and the proof is complete.

Assume now that $\underset{\sim}{F}$ admits the representation (2.44) for suitable $\underset{\sim}{R} \in \mathcal{O}_+$, frame $\mathrm{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ and $(\underline{\beta}; \underline{a}) \in \mathcal{D}$, so that (2.37) is in force with $(\beta_1, \beta_2) = \underline{\beta}$, $(a_1, a_2, a_3) = \underline{a}$, $(\underline{\beta}; \underline{a}) \in \mathcal{D}$. Motivated by (2.38) and its interpretation, we then call β_1, β_2 the *resolved in-plane components of the directional resolution of $\underset{\sim}{F}$ with respect to $\underset{\sim}{e}_3$*. Moreover, we refer to a_1, a_2 and a_3 as the *resolved out-of-plane components of the directional resolution of $\underset{\sim}{F}$ with respect to $\underset{\sim}{e}_3$*. In particular a_1, a_2 are the *resolved out-of-plane shears* and a_3 is the *resolved out-of-plane stretch of $\underset{\sim}{F}$ with respect to $\underset{\sim}{e}_3$*.

Consider now the right Cauchy-Green tensor $\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F}$ associated with the homogeneous deformation (2.34). Employing (2.44), we obtain

$$\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F} = [\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \underline{a})]^T \underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \underline{a}) \ . \tag{2.45}$$

An appeal to (2.41) in conjunction with (2.45) furnishes an expression for the matrix of components of $\underset{\sim}{C}$ in X in terms of the resolved in- and out-of-plane components of $\underset{\sim}{F}$ with respect to $\underset{\sim}{e}_3$:

$$[\underset{\sim}{C}]^{\mathrm{X}} = \begin{pmatrix} \beta_1^2 & 0 & \beta_1 a_1 \\ 0 & \beta_2^2 & \beta_2 a_2 \\ \beta_1 a_1 & \beta_2 a_2 & a_i a_i \end{pmatrix} \quad . \tag{2.46}$$

In addition, (2.46) facilitates immediate calculation of the corresponding expressions for the deformation invariants (1.5) associated with (2.34). Thus

$$\left.\begin{aligned} &I_1(\underset{\sim}{C}) = \beta_\alpha\beta_\alpha + a_i a_i \ , \\ &I_2(\underset{\sim}{C}) = \beta_1^2(a_2^2 + a_3^2) + \beta_2^2(a_1^2 + a_3^2) + \beta_1^2\beta_2^2 \ , \\ &I_3(\underset{\sim}{C}) = \beta_1^2\beta_2^2 a_3^2 \ . \end{aligned}\right\} \tag{2.47}$$

We may now construct a mapping $\underline{\hat{I}}(\cdot) : \mathcal{D} \to \overset{+}{\mathcal{I}}$, where $\overset{+}{\mathcal{I}}$ is the Invariant Region defined in (1.13), by letting

$$\begin{aligned} \hat{I}_i(\underline{\beta};\underline{a}) &= I_i(\underset{\sim}{C}) \ , \quad \underset{\sim}{C} = [\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})]^T \underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a}) \quad \forall\, (\underline{\beta};\underline{a}) \in \mathcal{D} \ , \\ \underline{\hat{I}} &= (\hat{I}_1, \hat{I}_2, \hat{I}_3) : \mathcal{D} \to \overset{+}{\mathcal{I}} \ , \end{aligned} \tag{2.48}$$

so that $\hat{I}_i(\underline{\beta};\underline{a})$ are given by (2.47). Since $\mathcal{D}$ and $\overset{+}{\mathcal{I}}$ are subregions of $\mathbb{R}^5$ and $\mathbb{R}^3$, respectively, $\underline{\hat{I}}$ cannot be one-to-one on $\mathcal{D}$. On the other hand, Lemma 2.2 allows us to conclude that $\underline{\hat{I}}(\mathcal{D}) = \overset{+}{\mathcal{I}}$ since for any $\underset{\sim}{C} \in \overset{+}{\mathcal{S}}$, one can choose $\underset{\sim}{F} = \underset{\sim}{C}^{1/2} \in \overset{+}{\mathcal{S}} \subset \mathcal{L}_+$ in (2.42) and then employ (2.44) through (2.47). It is of particular interest to determine the set of all $(\underline{\beta};\underline{a}) \in \mathcal{D}$ corresponding through $\underline{\hat{I}}(\cdot)$ to a given $\underline{I} \in \overset{+}{\mathcal{I}}$. This set coincides with the set of all possible values of the resolved in- and out-of-plane components $(\underline{\beta};\underline{a}) \in \mathcal{D}$ of a given $\underset{\sim}{F} \in \mathcal{L}_+$ satisfying $\underline{I}(\underset{\sim}{F}^T\underset{\sim}{F}) = \underline{I}$. A demonstration of this fact, together with a characterization of the set in question, is provided by the following

Lemma 2.3. *Given $\underset{\sim}{F} \in \mathcal{L}_+$, let λ_i^2 be the principal values of $\underset{\sim}{F}^T\underset{\sim}{F} = \underset{\sim}{C}$, ordered so that $0 < \lambda_1 \le \lambda_2 \le \lambda_3$, and set $\underline{I} = \underline{I}(\underset{\sim}{C})$.*[2] *(i) Then $(\underline{\beta};\underline{a}) \in \mathcal{D}$ satisfies $\underline{\hat{I}}(\underline{\beta};\underline{a}) = \underline{I}$ if and only if there is an $\underset{\sim}{R} \in \mathcal{O}_+$ and a frame X, such that $\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})$. (ii) Furthermore, $\underline{\hat{I}}(\underline{\beta};\underline{a}) = \underline{I}$ if and only if $\underline{\beta} = (\beta_1, \beta_2)$ and $\underline{a} = (a_1, a_2, a_3)$ satisfy:*

$$\lambda_1 \le \beta_\alpha \le \lambda_2 \le \beta_\gamma \le \lambda_3 \quad (\alpha \neq \gamma) \ , \tag{2.49}$$

$$a_3 = \sqrt{I_3}/\beta_1\beta_2 \ , \tag{2.50}$$

and in case $\beta_1 \neq \beta_2$,

$$a_\gamma^2 = \frac{P_{\underline{I}}(\beta_\gamma^2)}{\beta_\gamma^2(\beta_\gamma^2 - \beta_\alpha^2)} \ , \ \alpha \neq \gamma \quad (\textit{no sum}) \ , \tag{2.51}$$

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[2] In this case, $\underline{I} = (I_1, I_2, I_3)$ is related to the principal stretches λ_i via (1.5).

while if $\beta_1 = \beta_2$,

$$a_1^2 + a_2^2 = (\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)/\lambda_2^2 \ , \tag{2.52}$$

where in (2.51)

$$P_{\underline{I}}(\xi) = -\xi^3 + I_1\xi^2 - I_2\xi + I_3 \quad \forall\, \xi \in \mathbb{R} \ .$$

Proof. Assume that $(\underline{\beta};\underline{a}) \in \mathcal{D}$, $\underset{\sim}{R} \in \mathcal{O}_+$ and the frame X satisfy (2.44). Then (2.45) holds and thus $\underline{I} = \hat{\underline{I}}(\underline{\beta};\underline{a})$ because of (2.47), (2.48).

Conversely choose any $(\underline{\beta};\underline{a}) \in \mathcal{D}$ such that $\hat{\underline{I}}(\underline{\beta};\underline{a}) = \underline{I}$. Let X$'$ be an arbitrary frame and set $\underset{\sim}{B} = \underset{\sim}{A}_{X'}(\underline{\beta};\underline{a})$, so that $\underset{\sim}{B} \in \mathcal{L}_+$ and

$$\hat{\underline{I}}(\underline{\beta};\underline{a}) = \underline{I}(\underset{\sim}{B}^T\underset{\sim}{B}) = \underline{I}(\underset{\sim}{F}^T\underset{\sim}{F}) \ . \tag{2.54}$$

The above necessitates that for some $\underset{\sim}{Q},\underset{\sim}{P} \in \mathcal{O}_+$, $\underset{\sim}{F} = \underset{\sim}{Q}\underset{\sim}{B}\underset{\sim}{P}$. Letting $\underset{\sim}{R} = \underset{\sim}{Q}\underset{\sim}{P}$ so that $\underset{\sim}{R} \in \mathcal{O}_+$,

$$\underset{\sim}{F} = \underset{\sim}{R}(\underset{\sim}{P}^T\underset{\sim}{B}\underset{\sim}{P}) \ . \tag{2.55}$$

If now $X' = \{O;\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}$, then $X = \{O;\underset{\sim}{P}^T\underset{\sim}{e}_1,\underset{\sim}{P}^T\underset{\sim}{e}_2,\underset{\sim}{P}^T\underset{\sim}{e}_3\}$ is also a frame since $\underset{\sim}{P} \in \mathcal{O}_+$. Moreover it easily follows from (2.40) that

$$\underset{\sim}{P}^T\underset{\sim}{B}\underset{\sim}{P} = \underset{\sim}{P}^T\underset{\sim}{A}_{X'}(\underline{\beta};\underline{a})\underset{\sim}{P} = \underset{\sim}{A}_X(\underline{\beta};\underline{a}) \ . \tag{2.56}$$

Thus (2.55) becomes $\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{A}_X(\underline{\beta};\underline{a})$ and (i) is confirmed.

To prove (ii) we let

$$c_\alpha = \beta_\alpha^2 \ , \ d_i = a_i^2 \ , \tag{2.57}$$

so that (2.47) becomes

$$\left.\begin{aligned} d_1 + d_2 + d_3 &= I_1 - (c_1 + c_2) \ , \\ c_2d_1 + c_1d_2 + (c_1 + c_2)d_3 &= I_2 - c_1c_2 \ , \\ c_1c_2d_3 &= I_3 \ . \end{aligned}\right\} \tag{2.58}$$

In view of (2.57) we need to determine all possible values of $c_\alpha > 0$, $d_\alpha \geq 0$ and $d_3 > 0$ such that (2.58) holds. For fixed c_α (2.58) is a linear system for d_i, the determinant of the coefficient matrix of which vanishes if and only if $c_1 = c_2$.

Assume first that $c_1 \neq c_2$. Then the solution of (2.58) is

$$d_\alpha = \frac{P_{\underline{I}}(c_\alpha)}{c_\alpha(c_\alpha - c_\gamma)} \,, \ \alpha \neq \gamma \quad (\text{no sum}) \,, \quad d_3 = I_3/c_1 c_2 \tag{2.59}$$

where $P_{\underline{I}}(\cdot)$ is the characteristic polynomial (2.53) of $\underset{\sim}{C} = \underset{\sim}{F}^T \underset{\sim}{F}$ and has roots λ_i^2. Note first that $c_\alpha > 0 \Rightarrow d_3 > 0$, whereas $d_\alpha \geq 0$ if and only if

$$P_{\underline{I}}(c_\alpha) \geq 0 \geq P_{\underline{I}}(c_\gamma) \quad \text{for} \quad c_\alpha > c_\gamma \quad (\alpha \neq \gamma) \,. \tag{2.60}$$

Requiring c_α to be positive, (2.60) are equivalent to

$$\lambda_1^2 \leq c_\alpha \leq \lambda_2^2 \leq c_\gamma \leq \lambda_3^2 \quad (\alpha \neq \gamma) \,, \qquad (c_1, c_2) \neq (\lambda_2^2, \lambda_2^2) \,. \tag{2.61}$$

In view of (2.57), (2.59) and (2.61) confirm (2.49), (2.50), (2.51).

Finally assume $c_1 = c_2 = c > 0$. Then the first and second of (2.58) reduce to

$$\left.\begin{aligned} d_1 + d_2 &= I_1 - 2c - I_3/c^2 \,, \\ d_1 + d_2 &= I_2/c - c - 2I_3/c^2 \,. \end{aligned}\right\} \tag{2.62}$$

Substracting the first of (2.62) from the second yields

$$\frac{1}{c^2} P_{\underline{I}}(c) = 0$$

whence $c \in \{\lambda_1^2, \lambda_2^2, \lambda_3^2\}$. Expressing I_i in terms of λ_j and choosing $c = \lambda_j^2$ in (2.62) gives

$$d_1 + d_2 = (\lambda_i^2 - \lambda_j^2)(\lambda_j^2 - \lambda_k^2)/\lambda_j^2 \,, \quad i \neq j \neq k \quad (\text{no sum}). \tag{2.63}$$

Now $d_\alpha \geq 0$ necessitates that

$$c = \lambda_2^2 \,, \quad d_1 + d_2 = (\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)/\lambda_2^2 \tag{2.64}$$

with the understanding that whenever two or more of λ_i coincide they equal λ_2. The sufficiency of (2.64) is confirmed directly by substitution into (2.62). This completes the proof.

3. Ellipticity Conditions and their Interpretation for Hyperelastic Materials.

This section aims at establishing conditions necessary and sufficient for strong and ordinary ellipticity based on the kinematic developments of the previous section.

Our discussion concerns elastic potential functions $W : \mathcal{L}_+ \to \mathbb{R}$, restricted to be *twice continuously differentiable on* $\mathcal{L}_+$ *and subject to the requirement of objectivity* (1.9). We *do not* restrict our attention to isotropic materials until later on, where we eventually specialize our main results to the case of isotropy.

Let $\hat{\sigma}(\cdot) = W_F(\cdot)$ on $\mathcal{L}_+$ denote the nominal stress response function and recall the definition (1.18) of the acoustic tensor Q. It is a direct consequence of objectivity (1.9) that

$$\hat{\sigma}(RF) = R\hat{\sigma}(F) \quad \forall\, (R, F) \in \mathcal{O}_+ \times \mathcal{L}_+ \,, \tag{3.1}$$

whereas the acoustic tensor obeys

$$Q(RF, n) = RQ(F, n)R^T \quad \forall\, (R\,, F\,, n) \in \mathcal{O}_+ \times \mathcal{L}_+ \times \mathcal{U} \,. \tag{3.2}$$

This identity is obtained by differentiating (1.9) twice with respect to F and using (1.18).

We now consider the class of homogeneous deformations $\hat{y} : E_3 \to E_3$

$$\hat{y}(x) = Fx \,,\; x \in E_3 \,,\; F = A_{\mathrm{X}}(\underline{\beta}; \underline{a}) \tag{3.3}$$

where X is a fixed frame, A_{X} is as in (2.40) and $(\underline{\beta}; \underline{a}) \in \mathcal{D}$. Since $A_{\mathrm{X}}(\underline{\beta}; \underline{a})$ is fully determined by X and $(\underline{\beta}; \underline{a}) \in \mathcal{D}$, the elastic potential associated with (3.3) is expressible – for a given choice of frame X – as a function only of $(\underline{\beta};\underline{a})$. Thus for each frame X we define $\mathcal{W}^{\mathrm{X}}:\mathcal{D}\to\mathbb{R}$ by

$$\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a})=W\big(\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})\big)\quad\forall\,(\underline{\beta};\underline{a})\in\mathcal{D}\;. \tag{3.4}$$

One recognizes from (3.4) that $\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a})$ coincides with the stored energy per unit reference volume due to a homogeneous deformation (3.3) with (constant) gradient whose matrix of components in the frame X is given by (2.41). The next result assembles some properties of $\mathcal{W}^{\mathrm{X}}$ important to our analysis.

Lemma 3.1. *For each frame* $\mathrm{X}=\{O;\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}$, $\mathcal{W}^{\mathrm{X}}:\mathcal{D}\to\mathbb{R}$ *defined by (3.4) is twice continuously differentiable on* $\mathcal{D}$. *Moreover,*

$$\frac{\partial}{\partial a_i}\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a})\underset{\sim}{e}_i=\hat{\underset{\sim}{\sigma}}\big(\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})\big)\underset{\sim}{e}_3\quad\forall\,(\underline{\beta};\underline{a})\in\mathcal{D}\;; \tag{3.5}$$

also

$$\frac{\partial^2}{\partial a_i\partial a_j}\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a})\underset{\sim}{e}_i\otimes\underset{\sim}{e}_j=\underset{\sim}{Q}\big(\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a}),\underset{\sim}{e}_3\big)\quad\forall\,(\underline{\beta};\underline{a})\in\mathcal{D}\;. \tag{3.6}$$

Proof. For each frame X, and by definition, the function $\mathcal{W}^{\mathrm{X}}$ is the composition of $W:\mathcal{L}_+\to\mathbb{R}$ and $\underset{\sim}{A}_{\mathrm{X}}:\mathcal{D}\to\mathcal{L}_+$. The latter is seen from (2.40) to be a linear—hence infinitely smooth—function on $\mathcal{D}$. Hence the regularity of $\mathcal{W}^{\mathrm{X}}$ on $\mathcal{D}$ follows from that of W. Differentiating (2.40) once and twice with respect to a_i furnishes

$$\frac{\partial}{\partial a_i}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})=\underset{\sim}{e}_i\otimes\underset{\sim}{e}_3\;,\;\frac{\partial^2}{\partial a_i\partial a_j}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})=\underset{\sim}{0}\quad\forall\,(\underline{\beta};\underline{a})\in\mathcal{D}\;. \tag{3.7}$$

In view of the smoothness of $\mathcal{W}^{\mathrm{X}}$, the chain rule and the first of (3.7), we may extract from (3.4)

$$\frac{\partial}{\partial a_i}\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a})=\hat{\underset{\sim}{\sigma}}\big(\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})\big)\cdot(\underset{\sim}{e}_i\otimes\underset{\sim}{e}_3)\quad\forall\,(\underline{\beta};\underline{a})\in\mathcal{D}\;. \tag{3.8}$$

This leads directly to (3.5). (In the above the dot indicates the scalar product $\underset{\sim}{A}\cdot\underset{\sim}{B} = \mathrm{tr}(\underset{\sim}{A}^T\underset{\sim}{B})$ of two tensors $\underset{\sim}{A}, \underset{\sim}{B}$.) Differentiating (3.8) once more with respect to a_j and observing the second of (3.7), we arrive at

$$\frac{\partial^2}{\partial a_i \partial a_j}\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a}) = (\underset{\sim}{e}_j \otimes \underset{\sim}{e}_3)\cdot\left[\underset{\sim}{C}\left(\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})\right)(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_3)\right],\ (\underline{\beta};\underline{a}) \in \mathcal{D} \qquad (3.9)$$

where the elasticity tensor $\underset{\sim}{C}(\cdot)$ is the second tensor gradient of W on $\mathcal{L}_+$; see (1.11). By virtue of (1.18), for every $\underset{\sim}{A} \in \mathcal{L}_+$,

$$(\underset{\sim}{e}_j \otimes \underset{\sim}{e}_3)\cdot[\underset{\sim}{C}(\underset{\sim}{A})(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_3)] = \underset{\sim}{Q}(\underset{\sim}{A},\underset{\sim}{e}_3)\cdot(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)$$

which, combined with (3.9), validates (3.5) and completes the proof.

We will refer to $\mathcal{W}^{\mathrm{X}}$ in (3.4) as *the reduced elastic potential associated with the frame X.*

The results of the foregoing Lemma suggest the following interpretation of the first and second partial derivatives of $\mathcal{W}^{\mathrm{X}}$ with respect to the out-of-plane kinematic variables a_i. The nominal stress tensor $\underset{\sim}{\sigma}$ associated with (3.3) is given by $\underset{\sim}{\sigma} = \hat{\underset{\sim}{\sigma}}(\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a}))$, with components in X given by $\sigma_{ij}^{\mathrm{X}} = \underset{\sim}{e}_i\cdot\underset{\sim}{\sigma}\underset{\sim}{e}_j$. Then, by (3.5), the first partial derivatives $\frac{\partial}{\partial a_i}\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a})$ are equal to the components σ_{i3}^{X} of the nominal traction $\underset{\sim}{\sigma}\underset{\sim}{e}_3$ acting on a reference plane with normal $\underset{\sim}{e}_3$. Bearing in mind the kinematic interpretation of a_i following (2.38), we remark that the shear tractions $\sigma_{\alpha 3}^{\mathrm{X}}$ are conjugate to the out-of-plane shears a_α with respect to $\mathcal{W}^{\mathrm{X}}(\beta;\cdot)$, whereas the normal traction component σ_{33}^{X} is conjugate to the out-of-plane stretch a_3. At this point we define a function $\underline{g}^{\mathrm{X}} : \mathcal{D} \to \mathbb{R}^3$ by letting

$$g_i^{\mathrm{X}}(\underline{\beta};\underline{a}) = \frac{\partial}{\partial a_i}\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a}) \quad \forall\ (\underline{\beta};\underline{a}) \in \mathcal{D}\ . \qquad (3.10)$$

In view of our previous observations, for each $\underline{\beta} = (\beta_1,\beta_2) \in \mathcal{D}_1, \underline{g}^{\mathrm{X}}(\underline{\beta};\cdot) : \mathcal{D}_2 \to \mathbb{R}^3$ constitutes a *traction response mapping* of the triplet of out-of-plane components $(a_1,a_2,a_3) = \underline{a} \in \mathcal{D}_2$ to the triplet of their conjugate nominal tractions $(\sigma_{13}^{\mathrm{X}}, \sigma_{23}^{\mathrm{X}}, \sigma_{33}^{\mathrm{X}})$. In this context, the in-plane components β_α play the role of parameters. The Jacobian matrix of $\underline{g}^{\mathrm{X}}(\underline{\beta};\,\cdot)$ coincides with the Hessian matrix of the reduced elastic potential $\mathcal{W}^{\mathrm{X}}(\underline{\beta};\,\cdot)$ on the set $\mathcal{D}_2$ of triplets $\underline{a}$ of out-of-plane components. The result (3.6) of Lemma (3.1) thus establishes that the Jacobian matrix of the traction response mapping $\underline{g}^{\mathrm{X}}(\underline{\beta};\,\cdot)$ associated with the frame X, evaluated at $\underline{a} \in \mathcal{D}_2$, coincides with the matrix of components in X of the acoustic tensor $\underset{\sim}{Q}$ evaluated at $\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})$ and direction $\underset{\sim}{e}_3$.

Now let $\underset{\sim}{F} \in \mathcal{L}_+$ be the gradient tensor of an arbitrary homogeneous deformation, and choose any $\underset{\sim}{n} \in \mathcal{U}$. According to Lemma 2.3, there exist $\underset{\sim}{R} \in \mathcal{O}_+$, a frame $\mathrm{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3 = \underset{\sim}{n}\}$ and $(\underline{\beta};\underline{a}) \in \mathcal{D}$, such that $\underset{\sim}{F}$ admits the representation $\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})$. The objectivity requirement (1.9) together with (3.4) implies

$$W(\underset{\sim}{F}) = W\big(\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})\big) = \mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a})\ . \tag{3.11}$$

Similarly, from (3.1), (3.2) and an appeal to (3.5), (3.6) of Lemma 3.1, we infer that the Piola traction $\hat{\underset{\sim}{\sigma}}(\underset{\sim}{F})\underset{\sim}{n}$ and the acoustic tensor $\underset{\sim}{Q}(\underset{\sim}{F},\underset{\sim}{n})$ are expressible as

$$\begin{aligned} \hat{\underset{\sim}{\sigma}}(\underset{\sim}{F})\underset{\sim}{n} &= g_i^{\mathrm{X}}(\underline{\beta};\underline{a})\underset{\sim}{R}\underset{\sim}{e}_i\ , \\ \underset{\sim}{Q}(\underset{\sim}{F},\underset{\sim}{n}) &= \frac{\partial^2}{\partial a_i \partial a_j}\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a})\underset{\sim}{R}\underset{\sim}{e}_i \otimes \underset{\sim}{R}\underset{\sim}{e}_j\ , \end{aligned} \tag{3.12}$$

where

$$\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})\ ,\ \mathrm{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3 = \underset{\sim}{n}\}\ ,\ \underset{\sim}{R} \in \mathcal{O}_+\ ,\ (\underline{\beta};\underline{a}) \in \mathcal{D}\ .$$

Thus the stored energy density, the Piola traction and the acoustic tensor associated with any direction $\underset{\sim}{n}$ are expressible in terms of the rotation $\underset{\sim}{R}$, the frame X and the resolved in– and out–of–plane components of the directional resolution of $\underset{\sim}{F}$ with respect to $\underset{\sim}{n}$. This involves the reduced elastic potential $\mathcal{W}^{\mathrm{X}}$ (associated with X) and its derivatives (with respect to the out–of–plane components) in the manner specified by (3.12). In particular, the components of the Piola traction and the acoustic tensor *in a frame rotated with respect to* X *by* $\underset{\sim}{R}$, i.e., with unit vectors $\underset{\sim}{R}\underset{\sim}{e}_i$, again coincide with those of the traction response mapping and the Hessian matrix of $\mathcal{W}^{X}$, respectively.

The relevance of the above observations to the notions of ordinary and strong ellipticity spelled out in Definition 1.1 is exhibited by the following theorem, which constitutes the main result of this section.

Theorem 3.1. *Ordinary ellipticity fails at* $\underset{\sim}{F} \in \mathcal{L}_+$ *if and only if there is* $(\underline{\beta};\underline{a}) \in \mathcal{D}$ *and a frame* X *at which the Hessian matrix* $[\partial^2\mathcal{W}^{X}(\underline{\beta};\underline{a})/\partial a_i \partial a_j]$ *is singular, such that* $\underset{\sim}{F}$ *admits the representation*

$$\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{A}_{X}(\underline{\beta};\underline{a})$$

for some $\underset{\sim}{R} \in \mathcal{O}_+$. *Moreover, the above remains valid if "ordinary" and "singular" are replaced by "strong" and "not positive-definite" respectively.*

Proof. Assume first that ordinary ellipticity fails to hold at $\underset{\sim}{F} \in \mathcal{L}_+$. Then by Definition 1.1, there is $\underset{\sim}{n} \in \mathcal{U}$ such that the acoustic tensor $\underset{\sim}{Q}(\underset{\sim}{F},\underset{\sim}{n})$ is singular. By constructing the directional resolution of $\underset{\sim}{F}$ with respect to $\underset{\sim}{n}$, in view of Lemma 2.3, we can find $\underset{\sim}{R} \in \mathcal{O}_+$, a frame $X = \{O;\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3 = \underset{\sim}{n}\}$ and $(\underline{\beta};\underline{a}) \in \mathcal{D}$ such that $\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{A}_{X}(\underline{\beta};\underline{a})$. Moreover $\underset{\sim}{Q}(\underset{\sim}{F},\underset{\sim}{n})$ now satisfies the second of(3.12). Hence $[\partial^2\mathcal{W}^{X}(\underline{\beta};\underline{a})/\partial a_i \partial a_j]$ is singular , since it equals the matrix of components of $\underset{\sim}{Q}(\underset{\sim}{F},\underset{\sim}{n})$ in a frame with unit vectors $\underset{\sim}{R}\underset{\sim}{e}_i$.

To show the converse, assume that a frame $X = \{O;\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}$ and $(\underline{\beta};\underline{a}) \in \mathcal{D}$ exist, such that $[\partial^2\mathcal{W}^{X}(\underline{\beta};\underline{a})/\partial a_i \partial a_j]$ is not positive-definite. Choose any $\underset{\sim}{R} \in \mathcal{O}_+$ and set $\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{A}_{X}(\underline{\beta};\underline{a})$ which belongs to $\mathcal{L}_+$ because of Lemma 2.3. The acoustic tensor $\underset{\sim}{Q}(\underset{\sim}{F},\underset{\sim}{e}_3)$ now satisfies (3.12), the right hand side of which is singular by hypothesis. Hence, ordinary ellipticity fails at $\underset{\sim}{F}$ since $\underset{\sim}{Q}(\underset{\sim}{F},\underset{\sim}{n})$ is singular for $\underset{\sim}{n} = \underset{\sim}{e}_3$. The analogous result for strong ellipticity is obtained by substituting "ordinary" and "singular" by "strong" and "not positive-definite" respectively in the above. This completes the proof.

A corollary of the above theorem is that strong ellipticity holds *globally* (that is, for *every* $\underset{\sim}{F} \in \mathcal{L}_+$) if and only if the reduced elastic potential function $\mathcal{W}^{\mathrm{X}}(\underline{\beta};\cdot) : \mathcal{D}_2 \to \mathbb{R}$ has positive-definite Hessian matrix on $\mathcal{D}_2 = \{(a_1, a_2, a_3) | a_i \in \mathbb{R}, a_3 > 0\}$ for every frame X and $\underline{\beta} \in \mathcal{D}_1$. Bernstein and Toupin [9] have obtained a characterization of strictly convex functions in terms of properties of their Hessian matrix. They show that a twice continuously differentiable scalar function defined on a convex region in $\mathbb{R}^n$ and having positive-definite Hessian matrix there has the following two properties. It is strictly convex on its domain of definition and the mapping defined by its gradient is globally invertible there. These results, asserted respectively in Theorems I and IV in [9], are readily applicable to the reduced elastic potential $\mathcal{W}^{\mathrm{X}}(\underline{\beta};\cdot)$, *considered as a function of the out-of-plane component triplet* $\underline{a}$ on $\mathcal{D}_2$, and yield the following

Theorem 3.2. *Assume that strong ellipticity holds globally on* $\mathcal{L}_+$. *Then for every frame* X *and* $\underline{\beta} = (\beta_1, \beta_2) \in \mathcal{D}_1$,

(i) $\mathcal{W}^{\mathrm{X}}(\underline{\beta};\cdot)$ *is strictly convex on* $\mathcal{D}_2$, *i.e.,*

$$\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a}') - \mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a}) - g_i^{\mathrm{X}}(\underline{\beta};\underline{a})(a_i' - a_i) > 0 \quad \forall \underline{a}\,,\, \underline{a}' \in \mathcal{D}_2\,,\, \underline{a} \neq \underline{a}'\,. \tag{3.13}$$

(ii) the traction response mapping $\underline{g}^{\mathrm{X}}(\underline{\beta};\cdot) : \mathcal{D}_2 \to \mathbb{R}^3$ *satisfies*

$$\left(g_i^{\mathrm{X}}(\underline{\beta};\underline{a}') - g_i^{\mathrm{X}}(\underline{\beta};\underline{a})\right)(a_i' - a_i) > 0 \quad \forall\, \underline{a}', \underline{a} \in \mathcal{D}_2\,,\, \underline{a}' \neq \underline{a}\,, \tag{3.14}$$

so that it is globally invertible on $\mathcal{D}_2$.

Proof. By hypothesis and Theorem 3.1, for each frame X and $\underline{\beta} \in \mathcal{D}_1$ the Hessian matrix of $\mathcal{W}^{\mathrm{X}}(\underline{\beta};\cdot)$ is positive-definite on $\mathcal{D}_2 = \{(a_1, a_2, a_3) | a_i \in \mathbb{R}, a_3 > 0\}$ which is a half-space and thus a convex subregion of $\mathbb{R}^3$. Now $\underline{g}^{\mathrm{X}}(\underline{\beta};\cdot)$ is the gradient of $\mathcal{W}^{\mathrm{X}}(\underline{\beta};\cdot)$ on $\mathcal{D}_2$ by definition (3.10). Assertion (i) then follows from Theorem II of Bernstein and Toupin [9] applied to $\mathcal{W}^{\mathrm{X}}(\beta;\cdot)$, whereas (3.14) is deduced from an appeal to (i) and Theorem IV of [9]. In particular, the global invertibility of $\underline{g}^{\mathrm{X}}(\underline{\beta};\,\cdot)$ on $\mathcal{D}_2$ follows from the *strict monotonicity inequality* (3.14). This completes the proof.

The invertibility property of the traction response mapping for a globally strongly elliptic potential admits the following constitutive interpretation. Consider two homogenous deformations of the same body, with gradients whose directional resolutions with respect to some direction $\underset{\sim}{n}$ possess common rotation, frame and in-plane components but *distinct out-of-plane components.* Then the respective nominal tractions acting on a reference plane with normal $\underset{\sim}{n}$ are necessarily distinct.

On the other hand, if ordinary ellipticity holds globally on $\mathcal{L}_+$ then Theorem 3.1 demands that the Jacobian determinant of the traction response mapping never vanish, so that $\underline{g}^{\mathrm{X}}(\underline{\beta};\,\cdot)$ is merely *locally invertible* at every point in $\mathcal{D}_2$

Our next objective is a specialization of the foregoing results to the case of isotropy. Thus in the present circumstances we assume that the elastic potential is expressible by means of (1.12) as a function $\hat{W}$ of the deformation invariants. Certain properties of the reduced elastic potential which pertain to the class of isotropic hyperelastic solids are established below.

Lemma 3.2. *Assume W satisfies (1.12) and define $\mathcal{W} : \mathcal{D} \to \mathbb{R}$ by*

$$\mathcal{W}(\underline{\beta};\underline{a}) = \hat{W}\left(\hat{I}_1(\underline{\beta};\underline{a}), \hat{I}_2(\underline{\beta};\underline{a}), \hat{I}_3(\underline{\beta};\underline{a})\right) \quad \forall\, (\underline{\beta};\underline{a}) \in \mathcal{D} \tag{3.15}$$

where $\hat{I}_i(\underline{\beta};\underline{a})$ are given by (2.47), (2.48). Then for any frame X, *the reduced elastic potential $\mathcal{W}^{\mathrm{X}}$ obeys*

$$\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a}) = \mathcal{W}(\underline{\beta};\underline{a}) \quad \forall\, (\underline{\beta};\underline{a}) \in \mathcal{D}\,. \tag{3.16}$$

Moreover, $\mathcal{W}$ determines W completely in the sense that given $\underset{\sim}{F} \in \mathcal{L}_+$, there is $(\underline{\beta};\underline{a}) \in \mathcal{D}$ (with $\hat{\underline{I}}(\underline{\beta};\underline{a}) = \underline{I}(\underset{\sim}{F}^T\underset{\sim}{F})$,) such that

$$W(\underset{\sim}{F}) = \mathcal{W}(\underline{\beta};\underline{a})\,. \tag{3.17}$$

Proof. By the definition (3.4) of $\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a})$,

$$\mathcal{W}^{\mathrm{X}}(\underline{\beta};\underline{a}) = W\big(\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})\big) .$$

The deformation invariants corresponding to $\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})$ are given by (2.47), (2.48). This, together with (1.12), demonstrates that the left–hand side of (3.4) is equal to the right–hand side of (3.15). Hence (3.16) holds. Now given $\underset{\sim}{F} \in \mathcal{L}_+$, we construct its directional resolution with respect to an arbitrary direction to obtain (2.44). Then $\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\underline{a})$ so that $\underline{I}(\underset{\sim}{F}^T\underset{\sim}{F}) = \hat{\underline{I}}(\underline{\beta};\underline{a})$ because of (2.45), (2.47), (2.48) and since $\underset{\sim}{R} \in \mathcal{O}_+$. Also, (3.11) is now in force, which together with (3.16) validates (3.17) and completes the proof.

As asserted by (3.16), it is a special feature of isotropic materials that the reduced elastic potential is independent of the frame X in (3.4). Once $\hat{W}(I_1, I_2, I_3)$ is specified, it is a simple matter to construct $\mathcal{W}(\underline{\beta};\underline{a})$ in the way suggested by (3.15). One merely has to make the substitution

$$\left.\begin{aligned} &\mathcal{W}(\underline{\beta};\underline{a}) = \hat{W}(I_1, I_2, I_3)\ , \quad \text{where} \\ &\qquad I_1 = \hat{I}_1(\underline{\beta};\underline{a}) = \beta_\alpha\beta_\alpha + a_i a_i\ , \\ &\qquad I_2 = \hat{I}_2(\underline{\beta};\underline{a}) = \beta_1^2(a_2^2 + a_3^2) + \beta_1^2(a_1^2 + a_2^2) + \beta_1^2\beta_2^2\ , \\ &\qquad I_3 = \hat{I}_3(\underline{\beta};\underline{a}) = (\beta_1\beta_2 a_3)^2\ , \quad (\underline{\beta};\underline{a}) \in \mathcal{D} \end{aligned}\right\} \tag{3.18}$$

using (2.47) (2.48). Furthermore, the second assertion of Lemma 3.2 gives rise to the conclusion that the single function $\mathcal{W}$ defined on $\mathcal{D}$ determines the elastic potential W completely on $\mathcal{L}_+$. Given $\mathcal{W}(\underline{\beta};\underline{a})$ one substitutes for the a_i from the corresponding expression (2.50), (2.51), so that the result only involves the invariants I_i and the in-plane components β_α. However, since substitution from (2.50), (2.51) into (2.47) satisfies the latter identically, and because of (3.15), the resulting expression for the potential is independent of the β_α; it involves merely the invariants and coincides with $\hat{W}(I_1, I_2, I_3)$.

The following version of Theorem 3.1 is relevant in the case of isotropy.

Theorem 3.3. *Assume W satisfies (1.12), and define*

$$H_{ij}(\underline{\beta};\underline{a}) = \frac{\partial^2}{\partial a_i \partial a_j} \mathcal{W}(\underline{\beta};\underline{a}) \, , \ (\underline{\beta};\underline{a}) \in \mathcal{D} \, . \tag{3.19}$$

Let $\mathcal{D}_- = \{ (\underline{\beta};\underline{a}) \mid (\underline{\beta};\underline{a}) \in \mathcal{D}, \ [H_{ij}(\underline{\beta};\underline{a})]$ *not positive definite* $\}$. *Then strong ellipticity fails at* $\underset{\sim}{F} \in \mathcal{L}_+$ *with* $I_i = I_i(\underset{\sim}{F}^T\underset{\sim}{F})$ *if and only if*

$$(I_1, I_2, I_3) \in \hat{\underline{I}}(\mathcal{D}_-) \, , \tag{3.20}$$

where $\hat{\underline{I}} : \mathcal{D} \to \overset{+}{\mathcal{I}}$ *is defined by (2.47), (2.48). The above remains valid if "strong" and "not positive definite" are replaced by "ordinary" and "singular," respectively.*

Proof. By virtue of Theorem 3.1 and Lemma 3.2, if strong ellipticity fails at $\underset{\sim}{F}$, then $\underset{\sim}{F}$ admits the representation (2.44) with $(\underline{\beta};\underline{a}) \in \mathcal{D}_-$. From (i) of Lemma 2.3 we draw that $\underline{I} = (I_1, I_2, I_3) = \hat{\underline{I}}(\underline{\beta};\underline{a})$ so that (3.20) holds true. Conversely if (3.20) holds, then there exists $(\underline{\beta};\underline{a}) \in \mathcal{D}_-$ with $\underline{I} = \hat{\underline{I}}(\underline{\beta};\underline{a})$. With the aid of Lemma 3.2 we infer that there is an $\underset{\sim}{R} \in \mathcal{O}_+$ and a frame X such that $\underset{\sim}{F}$ conforms to (2.44) with $(\underline{\beta};\underline{a}) \in \mathcal{D}_-$. Then strong ellipticity is guaranteed to fail by Theorems 3.1 and 3.2. This completes the proof.

With a view toward illustrating some of the above ideas, we turn to an example involving a class of elastic potential introduced by Hadamard [10] and discussed extensively by John [11]. These have the form

$$\hat{W}(I_1, I_2, I_3) = \frac{c}{2} I_1 + \frac{d}{2} I_2 + \Phi(\sqrt{I_3}) \, , \tag{3.21}$$

where c and d are material constants and Φ is a function twice continuously differentiable on $(0, \infty)$. Upon using (3.18), we immediately obtain

$$\begin{aligned} \mathcal{W}(\underline{\beta};\underline{a}) = \frac{1}{2}\big[&(c + d\beta_2^2)a_1^2 + (c + d\beta_1^2)a_2^2 + (c + d\beta_\alpha\beta_\alpha)a_3^2 + c\beta_\alpha\beta_\alpha + d\beta_1^2\beta_2^2\big] \\ &+ \Phi(\beta_1\beta_2 a_3) \, , \ \beta_\gamma \in \mathbb{R}_+ \, , \ a_\gamma \in \mathbb{R} \, , \ a_3 \in \mathbb{R}_+ \end{aligned} \tag{3.22}$$

This is the reduced elastic potential generated by $\hat{W}$ in (3.21). Our next task is to derive necessary and sufficient conditions for ellipticity appropriate to the Hadamard materials.

Proposition 3.1. *Strong ellipticity holds at $\underset{\sim}{F} \in \mathcal{L}_+$ for the potential (3.21) if and only if*

$$c + d\lambda_i^2 > 0\ ,\ c + d(I_1 - \lambda_i^2) + \frac{I_3}{\lambda_i^2}\,\Phi''(\sqrt{I_3}) > 0 \tag{3.23}$$

where[1] $\lambda_i > 0$ *are the principal stretches of* $\underset{\sim}{F}$ *and* $I_1 = \lambda_j\lambda_j, I_3 = (\lambda_1\lambda_2\lambda_3)^2$. *Moreover, strong ellipticity holds globally on* $\mathcal{L}_+$ *if and only if*

$$c \geq 0\ ,\ d \geq 0\ ,\ c + d > 0\ ,\ \Phi''(J) \geq 0\ \ \forall\, J > 0\ . \tag{3.24}$$

Proof. For the Hadamard potential we calculate the Hessian components (3.19) from (3.22) to obtain

$$\begin{aligned} &H_{11}(\underline{\beta};\underline{a}) = c + d\beta_2^2\ ,\ H_{22}(\underline{\beta};\underline{a}) = c + d\beta_1^2\ ,\\ &H_{33}(\underline{\beta};\underline{a}) = c + d\beta_\alpha\beta_\alpha + \beta_1^2\beta_2^2\Phi''(\beta_1\beta_2 a_3)\ ,\\ &H_{ij}(\underline{\beta};\underline{a}) = 0\ ,\ i \neq j \quad (\underline{\beta};\underline{a}) \in \mathcal{D}\ . \end{aligned} \tag{3.25}$$

In the notation of Theorem 3.3, $(I_1, I_2, I_3) \notin \mathcal{D}_-$ if and only if

$$\underline{\hat{I}}(\underline{\beta};\underline{a}) = (I_1, I_2, I_3) \Rightarrow H_{ii}(\underline{\beta};\underline{a}) > 0 \quad \text{(no sum)}. \tag{3.26}$$

We appeal to (ii) of Lemma 2.3 and observe that (3.25) only involves β_1, β_2, a_3. Taking note of (2.49), we make use of (2.50) in (3.25). In view of (3.26) and Theorem 3.3, strong ellipticity at $\underset{\sim}{F}$ is equivalent to

$$\begin{aligned} &c + d\beta_\gamma^2 > 0\ ,\ B(\beta_1,\beta_2) \equiv c + d\beta_\alpha\beta_\alpha + \beta_1^2\beta_2^2\Phi''(J) > 0\\ &\qquad\qquad \forall\ (\beta_1,\beta_2) \in \Lambda = [\lambda_1,\lambda_2] \times [\lambda_2,\lambda_3] \end{aligned} \tag{3.27}$$

where $J = \det \underset{\sim}{F} = \sqrt{I_3}$. The first of (3.27) is equivalent to the first of (3.23). In view of (1.5), the second of (3.23) becomes

$$B(\lambda_i,\lambda_j) > 0, \quad i < j \tag{3.28}$$

[1] Primes indicate differentiation with respect to the argument.

so that it is necessary for the second of (3.27). To show it is also sufficient, we first draw from (3.27) that $B(\cdot, \beta_2)$ is linear in β_1^2 so that the extrema of $B(\cdot, \cdot)$ in Λ may only occur at the four vertices $(\beta_1, \beta_2) = (\lambda_i, \lambda_j)$ for $i < j$ or $(i, j) = (2, 2)$. Thus we need only show that (3.28) suffices for $B(\lambda_2, \lambda_2) > 0$. From the definition of $B(\beta_1, \beta_2)$,

$$\left[B(\lambda_1, \lambda_2) - B(\lambda_2, \lambda_2)\right]\left[B(\lambda_2, \lambda_3) - B(\lambda_2, \lambda_2)\right] = \\ \left[d + \lambda_2^2 \Phi''(J)\right]^2 (\lambda_1^2 - \lambda_2^2)(\lambda_3^2 - \lambda_2^2) \leq 0 \, , \tag{3.29}$$

the inequality following because of the ordering of the λ_i. With the aid of (3.28) we then deduce from (3.29) that $B(\lambda_2, \lambda_2) > 0$, so that B is positive on all four vertices and hence throughout Λ if the second of (3.23) holds.

Turning now to the global inequalities (3.24), we first observe that strong ellipticity holds globally if and only if (3.23) hold for all $\lambda_i > 0$. Clearly, (3.24) are sufficient for (3.23). To show their necessity, assume first that $c + d \leq 0$. Then, choosing $\lambda_i = 1$ contradicts the first of (3.23). If $c < 0$ or $d < 0$, again $(3.23)_1$ is violated for some $\lambda_i > 0$. This proves that the first three of (3.23) are necessary for global strong ellipticity. It remains to consider the case where these hold but $\Phi''(J_0) < 0$ for some $J_0 > 0$. We may pick $\lambda_1 = \lambda_2 = \lambda > 0$, $\lambda_3 = J_0/\lambda^2$ so that $\lambda_1\lambda_2\lambda_3 = J_0$ for each $\lambda > 0$. Then $(3.23)_2$ with $i = 3$ demands

$$h(\lambda^2) \equiv c + 2d\lambda^2 + \Phi''(J_0)\lambda^4 > 0 \quad \forall\, \lambda > 0 \tag{3.30}$$

for global strong ellipticity. However since $c \geq 0, d \geq 0, c + d > 0, \Phi''(J_0) < 0$, (3.30) is contradicted by choosing

$$\lambda^2 = -\left[d + \left(d^2 - c\Phi''(J_0)\right)^{1/2}\right]/\Phi''(J_0) \, ,$$

which is a real and positive root of $h(\lambda^2) = 0$ in (3.30). This completes the proof.

It is worth remarking that the first three of (3.24) are equivalent to the Baker-Ericksen inequalities (1.16). Also, the last of (3.24) is the condition for convexity of the function Φ.

The local strong ellipticity conditions (3.23) may alternatively be obtained by means of the criteria deduced by Simpson and Spector [4]. Although our approach and theirs should lead to identical conclusions for special materials, a *direct* proof of the equivalence of the two methods *in general* has so far elluded our efforts. On the other hand for another special potential, whose ellipticity was investigated by Knowles and Sternberg in [6], we have derived necessary and sufficient conditions for ellipticity which are in complete agreement with both the results reported in [6] and an application of the conditions presented by Simpson and Spector in [4].

4. Deformations with Discontinuous Gradients.

In the event that strong ellipticity fails for some deformations, the possibility arises that weak solutions of the equilibrium equations might exist that do not abide by the smoothness requirements imposed in Section 1. We presently envisage a situation where the deformation $\hat{\underset{\sim}{y}}$ remains continuous on the reference region $\mathcal{R}$, but there is a continuously differentiable surface $\Sigma \subset \mathcal{R}$ such that $\hat{\underset{\sim}{y}}$ is twice continuously differentiable on $\mathcal{R} - \Sigma$, whereas $\underset{\sim}{F} = \nabla \hat{\underset{\sim}{y}}$ suffers jump discontinuities across Σ.[1] It is a well known consequence of the continuity of the deformation under the present circumstances that there exists a vector field $\underset{\sim}{b}: \Sigma \to E_3$, such that

$$[\![\underset{\sim}{F}]\!] = \underset{\sim}{b} \otimes \underset{\sim}{n} \quad \text{on} \quad \Sigma \,, \tag{4.1}$$

where $[\![\cdot]\!]$ denotes the jump of a function across Σ and $\underset{\sim}{n} : \Sigma \to \mathcal{U}$ is a continuous field of unit normals on Σ.[2] In view of the constitutive law (1.8), the smoothness properties of $\underset{\sim}{F}$ are shared by the associated nominal stress field $\underset{\sim}{\sigma}$. The deformation under consideration is *equilibrated* if the integral of the nominal traction over the boundary of any bounded regular subregion of $\mathcal{R}$ vanishes. This global balance law reduces to the local equations (1.10) on $\mathcal{R} - \Sigma$. Moreover it necessitates that the nominal traction acting on Σ be balanced in the sense that

$$[\![\underset{\sim}{\sigma}]\!]\underset{\sim}{n} = \underset{\sim}{0} \quad \text{on} \quad \Sigma \,. \tag{4.2}$$

---

[1] That is, if $\mathcal{P} \subset \mathcal{R}$ is separated by Σ into two complementary subregions with nonempty interiors $\overset{\pm}{\mathcal{P}}$, then $\underset{\sim}{F}$ coincides on $\overset{+}{\mathcal{P}}$ (respectively $\overset{-}{\mathcal{P}}$) with a function continuous on the closure of $\overset{+}{\mathcal{P}}$ (respectively $\overset{-}{\mathcal{P}}$.)

[2] See for example Gurtin [12] and James [13].

The additional conditions (4.1), (4.2) pose restrictions on the class of elastic potentials capable of sustaining equilibrated deformations with discontinuous gradients. Rather transparent conclusions on the existence of such deformations may be arrived at by studying a specialized class of them, namely the *piecewise homogeneous ones.* We let $\mathcal{R}$ be the whole space E_3, agree that the deformation gradient takes two *distinct* values $\overset{+}{\underset{\sim}{F}}$ and $\overset{-}{\underset{\sim}{F}} \in \mathcal{L}_+$ and that there is a single surface of discontinuity Σ. It then follows from (4.1) that Σ is a plane. We choose a unit normal to Σ, $\underset{\sim}{n} \in \mathcal{U}$, so that the deformation $\hat{\underset{\sim}{y}}$ is given by

$$\hat{\underset{\sim}{y}}(\underset{\sim}{x}) = \begin{cases} \overset{+}{\underset{\sim}{F}}\underset{\sim}{x}\,, & \underset{\sim}{x} \in E_3\,, \quad \underset{\sim}{x}\cdot\underset{\sim}{n} > 0 \\ \overset{-}{\underset{\sim}{F}}\underset{\sim}{x}\,, & \underset{\sim}{x} \in E_3\,, \quad \underset{\sim}{x}\cdot\underset{\sim}{n} < 0 \end{cases} \tag{4.3}$$

Continuity of $\hat{\underset{\sim}{y}}$ on E_3 implies through (4.1) that there is a nonzero constant vector $\underset{\sim}{b}$, such that

$$\overset{+}{\underset{\sim}{F}} - \overset{-}{\underset{\sim}{F}} = \underset{\sim}{b} \otimes \underset{\sim}{n}\,. \tag{4.4}$$

In view of (4.2) the deformation (4.3) is equilibrated if and only if

$$\hat{\underset{\sim}{\sigma}}(\overset{+}{\underset{\sim}{F}})\underset{\sim}{n} = \hat{\underset{\sim}{\sigma}}(\overset{-}{\underset{\sim}{F}})\underset{\sim}{n}, \tag{4.5}$$

equilibrium being trivially satisfied on $E_3 - \Sigma$. Whether or not a given piecewise homogeneous deformation – one that satisfies (4.3), (4.4) – is *equilibrated* depends on the elastic potential because of the appearance of the stress response function $\hat{\underset{\sim}{\sigma}} = W_{\underset{\sim}{F}}$ in (4.5).

Definition 4.1. *A pair of tensors* $(\overset{+}{\underset{\sim}{F}}, \overset{-}{\underset{\sim}{F}}) \in \mathcal{L}_+ \times \mathcal{L}_+$ *is a shock for the potential* W *if* $\overset{+}{\underset{\sim}{F}} \neq \overset{-}{\underset{\sim}{F}}$ *and (4.4), (4.5) hold for some* $\underset{\sim}{b} \in E_3$ *and* $\underset{\sim}{n} \in \mathcal{U}$. *In the event that such a shock exists we say that* W *sustains shocks.*

Our intention is to deduce conditions on the elastic potential which are necessary and sufficient for it to sustain shocks. To begin with we concentrate on the kinematic jump condition (4.4). *Two tensors are said to be rank-one apart if their difference is a tensor of rank one*, i.e., a tensor product $\underset{\sim}{c} \otimes \underset{\sim}{d}$ of two *non-null* vectors $\underset{\sim}{c}$ and $\underset{\sim}{d} \in E_3$. The next result utilizes the directional resolution to characterize such pairs of tensors.

Lemma 4.1. *Two tensors* $\overset{+}{\underset{\sim}{F}}, \overset{-}{\underset{\sim}{F}} \in \mathcal{L}_+$ *are rank-one apart if and only if there is an* $\underset{\sim}{R} \in \mathcal{O}_+$, *a frame* $\mathrm{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$, $\underline{\beta} = (\beta_1, \beta_2) \in \mathcal{D}_1$ *and distinct* $\overset{+}{\underline{a}}$, $\overset{-}{\underline{a}} \in \mathcal{D}_2$, $\overset{\pm}{\underline{a}} = (\overset{\pm}{a}_1, \overset{\pm}{a}_2, \overset{\pm}{a}_3)$, *such that*

$$\overset{\pm}{\underset{\sim}{F}} = \underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \overset{\pm}{\underline{a}}) , \tag{4.6}$$

in which case

$$\overset{+}{\underset{\sim}{F}} - \overset{-}{\underset{\sim}{F}} = \underset{\sim}{b} \otimes \underset{\sim}{n}, \quad \underset{\sim}{b} = \underset{\sim}{R}(\overset{+}{a}_i - \overset{-}{a}_i)\underset{\sim}{e}_i , \quad \underset{\sim}{n} = \underset{\sim}{e}_3 . \tag{4.7}$$

Proof. Assume first that (4.6) is true. Recalling (2.40) we have

$$\overset{\pm}{\underset{\sim}{F}} = \underset{\sim}{R}\left(\sum_{\alpha=1}^{2} \beta_\alpha \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + \overset{\pm}{a}_i \underset{\sim}{e}_i \otimes \underset{\sim}{e}_3 \right) , \tag{4.8}$$

whence

$$\overset{+}{\underset{\sim}{F}} - \overset{-}{\underset{\sim}{F}} = \underset{\sim}{R}(\overset{+}{a}_i - \overset{-}{a}_i)\underset{\sim}{e}_i \otimes \underset{\sim}{e}_3 . \tag{4.9}$$

Upon setting $\underset{\sim}{b} = \underset{\sim}{R}(\overset{+}{a}_i - \overset{-}{a}_i)\underset{\sim}{e}_i$, $\underset{\sim}{n} = \underset{\sim}{e}_3$, we recover (4.7) and note that by hypothesis $\underset{\sim}{b} \neq \underset{\sim}{0}$, $\underset{\sim}{n} \in \mathcal{U}$. Hence $\overset{+}{\underset{\sim}{F}}$ and $\overset{-}{\underset{\sim}{F}}$ are rank-one apart.

To show the converse, assume that $\overset{+}{\underset{\sim}{F}} - \overset{-}{\underset{\sim}{F}} = \underset{\sim}{c} \otimes \underset{\sim}{d} \neq \underset{\sim}{0}$ and set $\underset{\sim}{n} = \underset{\sim}{d}/|\underset{\sim}{d}| \in \mathcal{U}$, $\underset{\sim}{b} = |\underset{\sim}{d}|\underset{\sim}{c}$, so that

$$\overset{-}{\underset{\sim}{F}} = \overset{+}{\underset{\sim}{F}} - \underset{\sim}{b} \otimes \underset{\sim}{n}, \quad \underset{\sim}{b} \in E_3 , \quad \underset{\sim}{n} \in \mathcal{U} . \tag{4.10}$$

By constructing the directional resolution of $\overset{+}{\underset{\sim}{F}}$ *with respect to* $\underset{\sim}{n}$, we obtain $\underset{\sim}{R} \in \mathcal{O}_+$, a frame $\mathrm{X} = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ with $\underset{\sim}{e}_3 = \underset{\sim}{n}$ and $(\underline{\beta}; \overset{+}{\underline{a}}) \in \mathcal{D}$, such that

$$\overset{+}{\underset{\sim}{F}} = \underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \overset{+}{\underline{a}}) = \underset{\sim}{R}\left(\sum_{\alpha=1}^{2} \beta_\alpha \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + \overset{+}{a}_i \underset{\sim}{e}_i \otimes \underset{\sim}{e}_3 \right) , \tag{4.11}$$

Substituting (4.11) into (4.10) and recalling that $\underset{\sim}{n} = \underset{\sim}{e}_3$, we find that

$$\bar{\underset{\sim}{F}} = \underset{\sim}{R}\left(\sum_{\alpha=1}^{2} \beta_\alpha \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha\right) + \underset{\sim}{R}(\overset{+}{a}_i \underset{\sim}{e}_i - \underset{\sim}{R}^T \underset{\sim}{b}) \otimes \underset{\sim}{e}_3 \,. \tag{4.12}$$

Letting

$$\bar{a}_i = \overset{+}{a}_i - (\underset{\sim}{R}^T \underset{\sim}{b}) \cdot \underset{\sim}{e}_i \,, \quad \bar{\underline{a}} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) \,, \tag{4.13}$$

we draw from (4.12)

$$\bar{\underset{\sim}{F}} = \underset{\sim}{R}\left(\sum_{\alpha=1}^{2} \beta_\alpha \underset{\sim}{e}_\alpha \otimes \underset{\sim}{e}_\alpha + \bar{a}_i \underset{\sim}{e}_i \otimes \underset{\sim}{e}_3\right) = \underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \bar{\underline{a}}) \,. \tag{4.14}$$

That $\bar{\underline{a}}$ defined in (4.13) also belongs to $\mathcal{D}_2$ follows from (4.14), since by hypothesis det $\bar{\underset{\sim}{F}} > 0$. Hence (4.14), (4.11) confirm (4.6), whereas (4.7) follows from (4.10), (4.13). This completes the proof.

It becomes evident that by choosing an arbitrary rotation tensor $\underset{\sim}{R} \in \mathcal{O}_+$, frame X= $\{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$, $\underline{\beta} = (\beta_1, \beta_2) \in \mathcal{D}_1$ and $\overset{\pm}{\underline{a}} = (\overset{\pm}{a}_1, \overset{\pm}{a}_2, \overset{\pm}{a}_3) \in \mathcal{D}_2$ with $\overset{+}{\underline{a}} \neq \bar{\underline{a}}$, one may construct the most general piecewise homogeneous deformation. This is done by assigning to $\overset{+}{\underset{\sim}{F}}$, $\bar{\underset{\sim}{F}}$ the values (4.6) and defining $\hat{\underset{\sim}{y}}$ via (4.3) with $\underset{\sim}{n} = \underset{\sim}{e}_3$. The plane of discontinuity Σ is specified by its normal $\underset{\sim}{n}$ and (4.4) is satisfied provided $\underset{\sim}{b}$ is chosen according to (4.7). In other words, *the directional resolutions of the two values of the deformation gradient taken in the direction normal to the surface of discontinuity possess common rotation* $\underset{\sim}{R}$, *frame* X *and in-plane components* β_α , *but at least one of the out-of-plane-components* a_i *jumps across* Σ. Thus there is a frame X with its third unit vector normal to the plane of discontinuity, in which

$$[\overset{\pm}{\underset{\sim}{F}}]^{\mathrm{X}} = \underset{\sim}{R} \begin{pmatrix} \beta_1 & 0 & \overset{\pm}{a}_1 \\ 0 & \beta_2 & \overset{\pm}{a}_2 \\ 0 & 0 & \overset{\pm}{a}_3 \end{pmatrix} , \tag{4.15}$$

where $\underline{R}$ is a proper orthogonal matrix.

The next result shows that the principal stretches associated with piecewise homogeneous deformations are restricted to obey certain inequalities because of the kinematic jump condition(4.4).

Theorem 4.1. *Let* $\overset{+}{\lambda}_i > 0$ *be the principal stretches of* $\overset{+}{\underset{\sim}{F}} \in \mathcal{L}_+$ *, with* $\overset{+}{\lambda}_1 \leq \overset{+}{\lambda}_2 \leq \overset{+}{\lambda}_3$ *. Then there exists* $\overset{-}{\underset{\sim}{F}} \in \mathcal{L}_+$ *with principal stretches* $\overset{-}{\lambda}_i > 0$ $(\overset{-}{\lambda}_1 \leq \overset{-}{\lambda}_2 \leq \overset{-}{\lambda}_3)$ *, such that* $\overset{+}{\underset{\sim}{F}}$ *and* $\overset{-}{\underset{\sim}{F}}$ *are rank-one apart if and only if*

$$\overset{-}{\lambda}_1 \leq \overset{+}{\lambda}_2 \ , \quad \overset{+}{\lambda}_1 \leq \overset{-}{\lambda}_2 \leq \overset{+}{\lambda}_3 \ , \quad \overset{+}{\lambda}_2 \leq \overset{-}{\lambda}_3 \ , \tag{4.16}$$

with the first or the last of the above inequalities or both being strict in case $\overset{+}{\lambda}_i = \lambda > 0 \,, i = 1,2,3$.

Proof. First we assume that $\overset{+}{\underset{\sim}{F}}$ and $\overset{-}{\underset{\sim}{F}}$ are rank-one apart. Then they satisfy (4.10) for some $\underset{\sim}{b} \in E_3$ and $\underset{\sim}{n} \in \mathcal{U}$. Moreover, their directional resolutions with respect to $\underset{\sim}{n}$ possess the same in-plane component pair $\underline{\beta} = (\beta_1, \beta_2) \in \mathcal{D}_1$, as guaranteed by Lemma 4.1. We may assume $\beta_2 \geq \beta_1$. Invoking (ii) of Lemma 2.3 and applying it to $\overset{+}{\underset{\sim}{F}}$ and $\overset{-}{\underset{\sim}{F}}$, we draw from (2.49)

$$\overset{\pm}{\lambda}_1 \leq \beta_1 \leq \overset{\pm}{\lambda}_2 \leq \beta_2 \leq \overset{\pm}{\lambda}_3 \tag{4.17}$$

which immediately yields (4.16). To demonstrate the assertion following (4.16), suppose that $\overset{+}{\lambda}_1 = \overset{+}{\lambda}_2 = \overset{+}{\lambda}_3 \equiv \lambda > 0$. We now assume that equality holds in the first and last of the inequalities in (4.16), and show that this leads to a contradiction. It follows that $\overset{-}{\lambda}_1 = \overset{-}{\lambda}_2 = \overset{-}{\lambda}_3 = \lambda$. Then $\overset{+}{\underset{\sim}{F}}$ and $\overset{-}{\underset{\sim}{F}}$ are similarity transformations ; $\overset{+}{\underset{\sim}{F}} = \lambda \overset{+}{\underset{\sim}{R}}$, $\overset{-}{\underset{\sim}{F}} = \lambda \overset{-}{\underset{\sim}{R}}$, $\overset{\pm}{\underset{\sim}{R}} \in \mathcal{O}_+$. Lemma 4.1 demands that (4.6) be in force. However,

$$\overset{+}{\underset{\sim}{F}} = \lambda \overset{+}{\underset{\sim}{R}} = \underset{\sim}{R} \underset{\sim}{A} \mathrm{x}(\underline{\beta}; \overset{+}{\underline{a}})$$

can be satisfied by setting $\underset{\sim}{R} = \overset{+}{\underset{\sim}{R}}$, $\beta_\alpha = \overset{+}{\alpha}_3 = \lambda$, $\overset{+}{\alpha}_\gamma = 0$. By uniqueness of the directional resolution asserted in Theorem 2.1, $\underset{\sim}{R} = \overset{+}{\underset{\sim}{R}}$, $\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \overset{+}{\underline{a}}) = \lambda \underset{\sim}{1}$. The same argument applied to $\overset{-}{\underset{\sim}{F}}$ gives $\underset{\sim}{R} = \overset{-}{\underset{\sim}{R}}$. Then by (4.6), $\overset{+}{\underset{\sim}{R}} = \overset{-}{\underset{\sim}{R}}$, hence $\overset{+}{\underset{\sim}{F}} = \overset{-}{\underset{\sim}{F}}$ and they are not rank-one apart.

To show the converse, assume the $\overset{-}{\lambda}_i$ conform to (4.16) and its subsequent qualification. Because of the ordering of the $\overset{\pm}{\lambda}_i$, it follows from (4.16) that there are β_γ with $\beta_2 \geq \beta_1 > 0$, such that

$$\overset{\pm}{\lambda}_1 \leq \beta_1 \leq \overset{\pm}{\lambda}_2 \leq \beta_2 \leq \overset{\pm}{\lambda}_3 \, . \tag{4.17}$$

In particular, (2.49) now holds with $\lambda_i = \overset{+}{\lambda}_i$, so that (ii) of Lemma 2.3 with $\underset{\sim}{F} = \overset{+}{\underset{\sim}{F}}$ can be used to define $\overset{+}{\underline{a}} \in \mathcal{D}_2$ by means of (2.49) through (2.52). Then $(\underline{\beta}; \overset{+}{\underline{a}}) \in \mathcal{D}$ satisfies the hypotheses of Lemma 2.3 (i) with $\underset{\sim}{F} = \overset{+}{\underset{\sim}{F}}$. Hence there is a frame X and an $\underset{\sim}{R} \in \mathcal{O}_+$, such that

$$\overset{+}{\underset{\sim}{F}} = \underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \overset{+}{\underline{a}}) \, , \tag{4.18}$$

with $\underline{\beta}$ abiding by (4.17). For such $\underline{\beta}$, (2.49) is valid with $\lambda_i = \overset{-}{\lambda}_i$. Then, obtaining $I_i = \overset{-}{I}_i$ by setting $\lambda_i = \overset{-}{\lambda}_i$ in (1.5), we may construct $\overset{-}{\underline{a}}$ such that (2.49) through (2.52) are satisfied by $(\underline{\beta}; \overset{-}{\underline{a}}) \in \mathcal{D}$. We then choose

$$\overset{-}{\underset{\sim}{F}} = \underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta}; \overset{-}{\underline{a}}) \, , \tag{4.19}$$

with $\underset{\sim}{R}$, X as in (4.18). By construction, $\overset{-}{\underset{\sim}{F}}$ has principal stretches $\overset{-}{\lambda}_i$, as is guaranteed by Lemma 2.3. In view of Lemma 4.1 it remains to be shown that $\underline{\beta}$ satisfying (4.17) can always be chosen so that $\overset{+}{\underline{a}} \neq \overset{-}{\underline{a}}$ in (4.19), (4.18). In case $(\overset{+}{\lambda}_1, \overset{+}{\lambda}_2, \overset{+}{\lambda}_3) \neq (\overset{-}{\lambda}_1, \overset{-}{\lambda}_2, \overset{-}{\lambda}_3)$ we have $\overset{+}{\underline{I}} \neq \overset{-}{\underline{I}}$, since the mapping $(\lambda_1, \lambda_2, \lambda_3) \mapsto (I_1, I_2, I_3)$ is invertible for $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$. This suffices for $\overset{+}{\underline{a}}$ and $\overset{-}{\underline{a}}$ to be distinct because of (2.47). If $\overset{+}{\lambda}_i = \overset{-}{\lambda}_i = \lambda_i$, then by hypothesis not all three of the λ_i coincide, and (2.47) permits us to choose $(\beta_1, \beta_2) \neq (\lambda_i, \lambda_j)$, where $i \neq j$. Examination of (2.50) then reveals that $\overset{+}{\alpha}_\gamma \overset{+}{\alpha}_\gamma > 0$. Hence choosing $\overset{-}{\alpha}_\gamma = -\overset{+}{\alpha}_\gamma$ yields the requisite $\overset{+}{\underset{\sim}{a}} \neq \overset{-}{\underset{\sim}{a}}$, so that $(\underline{\beta}; \overset{+}{\underset{\sim}{a}})$, $(\underline{\beta}; \overset{-}{\underset{\sim}{a}})$ satisfy (2.49) through (2.52). In view of (4.18), (4.19), an appeal to Lemma 4.1 now completes the proof.

In the special case when $\overset{+}{\underset{\sim}{F}} = \underset{\sim}{1}$ the appropriate version of (4.16) is in agreement with a result of Ball and James [14][3]. Moreover, Theorem 4.1 dictates that given a (non-trivial) piecewise homogeneous deformation, its restrictions to either side of Σ cannot be related to each other by a similarity transformation. That is, if

$$\overset{-}{\underset{\sim}{F}} = \varpi \underset{\sim}{R} \overset{+}{\underset{\sim}{F}} \quad , \quad \overset{+}{\underset{\sim}{F}} - \overset{-}{\underset{\sim}{F}} = \underset{\sim}{b} \otimes \underset{\sim}{n} \tag{4.20}$$

for some $\varpi > 0$, $\underset{\sim}{R} \in \mathcal{O}_+$, $\underset{\sim}{b} \in E_3$ and $\underset{\sim}{n} \in \mathcal{U}$, then necessarily $\underset{\sim}{b} = \underset{\sim}{0}$ and $\overset{+}{\underset{\sim}{F}} = \overset{-}{\underset{\sim}{F}}$. To see this, we note that (4.20) dictates that

$$\underset{\sim}{1} - \varpi \underset{\sim}{R} = \underset{\sim}{c} \otimes \underset{\sim}{m} \tag{4.21}$$

where $\underset{\sim}{c} = |\overset{+}{\underset{\sim}{F}}{}^{-T} \underset{\sim}{n}| \underset{\sim}{b}$, $\underset{\sim}{m} = \overset{+}{\underset{\sim}{F}}{}^{-T} \underset{\sim}{n} / |\overset{+}{\underset{\sim}{F}}{}^{-T} \underset{\sim}{n}|$. Applying Theorem 4.1 to $\underset{\sim}{1}$ and $\varpi \underset{\sim}{R}$ we have $\overset{+}{\lambda}_i = 1$, $\overset{-}{\lambda}_i = \varpi$. Then (4.16) demands that $\varpi = 1$, which is prohibited by its subsequent qualification. Hence $\underset{\sim}{c} = \underset{\sim}{0}$ in (4.21) so that $\underset{\sim}{b} = \underset{\sim}{0}$ in (4.20) and $\overset{+}{\underset{\sim}{F}} = \overset{-}{\underset{\sim}{F}}$. In particular, $\overset{+}{\underset{\sim}{F}}$ and $\overset{-}{\underset{\sim}{F}}$ cannot both be similarity transformations if they are rank-one apart.

In our study of equilibrated piecewise homogeneous deformations, it remains to consider the traction continuity condition (4.5). This we propose to do by utilizing the properties of the traction response mapping introduced in Section

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[3] See, in particular, the first assertion of Proposition 4 in [14], where the λ_i stand for the squares of the principal stretches.

3, within the framework laid down by Lemma 4.1. In what follows, the traction response mapping generated by the elastic potential W in the frame X is the function $\underset{\sim}{g}^{\mathrm{X}}(\underline{\beta}\,;\cdot):\mathcal{D}_2\rightarrow\mathbb{R}^3$ defined by (3.10), (3.4). We now state the main result of this section.

Theorem 4.2. *Necessary and sufficient for the potential W to sustain shocks is that there is a frame X and $\underline{\beta}\in\mathcal{D}_1$ such that the traction response mapping $\underset{\sim}{g}^{\mathrm{X}}(\underline{\beta}\,,\cdot):\mathcal{D}_2\rightarrow\mathbb{R}^3$ generated by W in X is not invertible on $\mathcal{D}_2$. Moreover every shock $(\overset{+}{\underset{\sim}{F}},\overset{-}{\underset{\sim}{F}})$ sustained by W admits the representation*

$$[\overset{\pm}{\underset{\sim}{F}}]^{\mathrm{X}}=[\underset{\sim}{R}]^{\mathrm{X}}\begin{pmatrix}\beta_1 & 0 & \overset{\pm}{a}_1\\ 0 & \beta_2 & \overset{\pm}{a}_2\\ 0 & 0 & \overset{\pm}{a}_3\end{pmatrix}, \tag{4.22}$$

where $\underset{\sim}{R}\in\mathcal{O}_+$, whereas X, $\underline{\beta}=(\beta_1,\beta_2)\in\mathcal{D}_1$ and $\overset{\pm}{\underline{a}}=(\overset{\pm}{a}_1,\overset{\pm}{a}_2,\overset{\pm}{a}_3)\in\mathcal{D}_2$ are such that $\overset{+}{\underline{a}}\neq\overset{-}{\underline{a}}$ and

$$g_i^{\mathrm{X}}(\underline{\beta};\overset{+}{\underline{a}})=g_i^{\mathrm{X}}(\underline{\beta};\overset{-}{\underline{a}})\,. \tag{4.23}$$

Proof. We recall $(3.12)_1$ and apply it to two tensors $\underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\overset{\pm}{\underline{a}})$. By subtracting the resulting expressions, we obtain

$$\left[\hat{\underset{\sim}{\sigma}}\left(\underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\overset{+}{\underline{a}})\right)-\hat{\underset{\sim}{\sigma}}\left(\underset{\sim}{R}\underset{\sim}{A}_{\mathrm{X}}(\underline{\beta};\overset{-}{\underline{a}})\right)\right]\underset{\sim}{e}_3=\left[g_i^{\mathrm{X}}(\underline{\beta};\overset{+}{\underline{a}})-g_i^{\mathrm{X}}(\underline{\beta};\overset{-}{\underline{a}})\right]\underset{\sim}{R}\underset{\sim}{e}_i \tag{4.24}$$

for any $\underset{\sim}{R}\in\mathcal{O}_+$, frame X$=\{O;\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}$ and arbitrary $(\underline{\beta};\overset{\pm}{\underline{a}})\in\mathcal{D}$. Assume now that $(\overset{+}{\underset{\sim}{F}},\overset{-}{\underset{\sim}{F}})$ is a shock for W. Then by Definition 4.1, (4.4) is in force for suitable $\underset{\sim}{b},\underset{\sim}{n}$. We infer from Lemma 4.1 that $\overset{+}{\underset{\sim}{F}},\overset{-}{\underset{\sim}{F}}$ are expressible via (4.6). Since by hypothesis (4.5) also holds, we observe that the left hand side of (4.24) vanishes for those choices of $\underset{\sim}{R}$, X, $\underline{\beta}$, $\overset{+}{\underline{a}}$ and $\overset{-}{\underline{a}}$ that correspond to $\overset{\pm}{\underset{\sim}{F}}$ through (4.6) and (4.7). Moreover, $\overset{+}{\underline{a}}\neq\overset{-}{\underline{a}}$ and, since $\underset{\sim}{R}$ is orthogonal, we recover (4.23), whereas (4.6) is equivalent to (4.24) in view of (2.41). We have shown that if W

sustains shocks, then every shock admitted by W conforms to the representation (4.22), and a suitably chosen $\underline{g}^{X}(\underline{\beta};\cdot)$ fails to be invertible because of (4.23). It remains to show that if there exist $\underline{\beta} \in \mathcal{D}_1$ and a frame X for which $\underline{g}^{X}(\underline{\beta};\cdot)$ is not one-to-one, then W sustains shocks. By this hypothesis, (4.23) must hold for some X, $\underline{\beta}$ and distinct $\overset{+}{\underline{a}}, \overset{-}{\underline{a}} \in \mathcal{D}_2$. In terms of these and an arbitrary $\underset{\sim}{R} \in \mathcal{O}_+$ define $\overset{\pm}{\underset{\sim}{F}}$ through (4.22). Then in view of (2.41), $\overset{+}{\underset{\sim}{F}}, \overset{-}{\underset{\sim}{F}}$ satisfy (4.6) of Lemma 4.1 and hence (4.4) with $\underset{\sim}{b}$, $\underset{\sim}{n}$ given by (4.7). Thus $\overset{+}{\underset{\sim}{F}} \neq \overset{-}{\underset{\sim}{F}}$. Moreover, the right hand side of (4.24) vanishes due to (4.23). With the aid of (4.6) and the last of (4.7), we confirm that $\overset{+}{\underset{\sim}{F}}, \overset{-}{\underset{\sim}{F}}$ also satisfy (4.5), and so by definition, $(\overset{+}{\underset{\sim}{F}}, \overset{-}{\underset{\sim}{F}})$ is a shock for W. This completes the proof.

Theorem (4.2) provides a complete characterization of the class of hyperelastic materials that sustain equilibrated piecewise homogeneous deformations. To obtain a mechanical interpretation of the theorem, we recall that the traction response mapping is a constitutive response function, expressing components of Piola traction in terms of their respective conjugate resolved out-of-plane components of the deformation gradient. Whenever this mapping fails to be invertible, one may find deformation gradients with common in-plane but distinct out-of-plane components associated with a given direction, but giving rise to identical tractions. Such deformation gradients are the successful candidates for a shock. In principle one may construct all the shocks sustainable by a given material. This is done as follows. For each frame X and in-plane component pair $\underline{\beta}$ one finds all pairs of out-of-plane component triplets $\overset{\pm}{\underline{a}}$ satisfying (4.23). The deformation gradients $\overset{\pm}{\underset{\sim}{F}}$ of the associated piecewise homogeneous deformation (4.3) are then given by (4.22). The normal $\underset{\sim}{n}$ to the reference surface Σ of discontinuity is related to the frame X through the last of (4.7). It is possible that for some *anisotropic* materials the traction response mapping might lose invertibility only for certain restricted choices of the frame X. If this is the case, then shocks are sustainable by such materials only for special orientations of the discontinuity surface.

Combining the conclusions of Theorem 3.2(ii) and Theorem 4.1, one readily draws that *a loss of strong ellipticity at some deformation is a necessary condition for an elastic potential to sustain shocks.* Thus we arrive at a result established by Knowles and Sternberg [15].

In the event that the material at hand is isotropic, the preceeding discussion may be simplified by an appeal to Lemma 3.2. In these circumstances, the reduced elastic potential is obtained by means of (3.15), (3.16), so that the traction response mapping is independent of the frame X in (3.10). In particular, if we define

$$g_i(\underline{\beta};\underline{a}) = \frac{\partial}{\partial a_i}\mathcal{W}(\underline{\beta};\underline{a}) \ , \quad (\underline{\beta};\underline{a}) \in \mathcal{D} \ , \tag{4.25}$$

with $\mathcal{W}$ supplied by (3.15), then for any choise of the frame X,

$$\underline{g}^{\mathrm{X}}(\underline{\beta};\underline{a}) \ = \ \underline{g}(\underline{\beta};\underline{a}) \quad \forall\, (\underline{\beta};\underline{a}) \in \mathcal{D} \ . \tag{4.26}$$

Thus in attempting to find the shocks sustained by an *isotropic* material, we only need examine the invertibility of a single family of mappings $\underline{g}(\underline{\beta}\,;\,\cdot\,) : \mathcal{D}_2 \to \mathbb{R}^3$, depending on the pair of scalar parameters $\underline{\beta}$. Once this is done, the frame X in the representation (4.22) is seen to be arbitrary. Evidently, if an isotropic material sustains shocks, it does so for every possible orientation of the plane of discontinuity.

In order to illustrate the above results, we apply them to the special case of the elastic potential (3.21) for the Hadamard material, the strong ellipticity of which was investigated in Proposition 3.1.

Proposition 4.1. *The Hadamard potential (3.21) with $c \geq 0$, $d \geq 0$, $c+d \geq 0$, sustains shocks if and only if*

$$[\Phi'(\overset{+}{J}) - \Phi'(\overset{-}{J})]\,[\,\overset{+}{J} - \overset{-}{J}\,] < 0 \tag{4.29}$$

for some distinct $\overset{+}{J}$, $\overset{-}{J} > 0$. *Moreover, if* $\underset{\sim}{F} \in \mathcal{O}_+$ *has principal sretches* λ_i $(0 < \lambda_1 \le \lambda_2 \le \lambda_3)$, *then there is a shock* $(\underset{\sim}{F}, \overset{-}{\underset{\sim}{F}})$ *with* $\det \underset{\sim}{F} = \overset{-}{J}$ *if and only if*

$$c + d(I_1 - \lambda_i^2) + \frac{I_3}{\lambda_i^2}\left(\frac{\Phi'(J) - \Phi'(\bar{J})}{J - \bar{J}}\right) \begin{array}{l} \ge 0 \ \ \text{for } i = 3, \\ \le 0 \ \ \text{for } i = 1, \end{array} \tag{4.30}$$

where $I_1 = \lambda_i \lambda_i$, $J = \lambda_1 \lambda_2 \lambda_3$.

Proof. By means of (3.22), used in conjunction with (3.21), the components of the traction response mapping specialize for the material under consideration to

$$\begin{aligned} g_\gamma(\underline{\beta}; \underline{a}) &= \left[c + (\beta_1\beta_2/\beta_\gamma)^2 d\right] a_\gamma \, , \quad \text{(no sum)} \\ g_3(\underline{\beta}; \underline{a}) &= \left[c + \left(\beta_1^2 + \beta_2^2\right) d\right] a_3 + \beta_1\beta_2\Phi'(\beta_1\beta_2 a_3) \, , \end{aligned} \tag{4.31}$$

where $(\underline{\beta}; \underline{a}) \in \mathcal{D}$. In view of Theorem 4.2 and the preceeding remarks pertaining to isotropy, we now set out to show that (4.29) is necessary and sufficient for the function $\underline{g}(\underline{\beta}; \cdot)$ to lose invertibility on $\mathcal{D}_2$. From (4.31) and the hypotheses on c, d, we draw

$$g_\gamma(\underline{\beta}; \overset{+}{\underline{a}}) - g_\gamma(\underline{\beta}; \overset{-}{\underline{a}}) = 0 \iff \overset{+}{a}_\gamma = \overset{-}{a}_\gamma \ . \tag{4.32}$$

Hence the resolved out-of-plane shear components a_1, a_2 may not jump. Setting $\overset{\pm}{J} = \beta_1\beta_2 \overset{\pm}{a}_3$ we conclude from (4.32) that for a nontrivial shock $\overset{+}{a}_3 \ne \overset{-}{a}_3$, which is equivalent to $\overset{+}{J} \ne \overset{-}{J}$. The last of (4.31) gives

$$g_3(\underline{\beta}; \overset{+}{\underline{a}}) - g_3(\underline{\beta}; \overset{-}{\underline{a}}) = 0 \iff \Delta = 0 \, ,$$

where

$$\Delta \equiv (c + d\beta_\alpha\beta_\alpha)/\beta_1^2\beta_2^2 + \frac{\Phi'(\overset{+}{J}) - \Phi'(\overset{-}{J})}{\overset{+}{J} - \overset{-}{J}} \, . \tag{4.33}$$

If a shock exists, then Δ vanishes in (4.33) for some $\beta_\alpha > 0$ and distinct $\overset{\pm}{J} > 0$. Because of our assumptions on c and d the first term in the right-hand side in (4.33) is positive. This confirms the necessity of (4.29). Now suppose (4.29) holds true. Then, assigning any fixed positive value to β_1, we may confirm easily that $\beta_2 > 0$ can always be chosen so that Δ vanishes. For such $\underline{\beta}$ we may then choose $\overset{\pm}{a}_3 = \overset{\pm}{J} / \beta_1\beta_2$, choose arbitrary a_α, set $\overset{\pm}{a}_\alpha = a_\alpha$ and construct $(\overset{+}{\underset{\sim}{F}}, \overset{-}{\underset{\sim}{F}})$ by way of (4.22) with $\underset{\sim}{R}$ and X also arbitrary. Then (4.33), (4.32) confirm that $(\overset{+}{\underset{\sim}{F}}, \overset{-}{\underset{\sim}{F}})$ is a shock for the Hadamard potential.

Turning now to a proof of (4.30), we recall the results of Lemma 2.3. If $\underset{\sim}{F}$ has principal stretches λ_i then there is a shock $(\underset{\sim}{F}, \overset{-}{\underset{\sim}{F}})$ if and only if the first of (4.32) holds and Δ vanishes with $(\underline{\beta}; \overset{+}{\underline{a}})$ satisfying (2.49) through (2.52). Clearly we only need confirm that (4.30) are equivalent to (4.33) with $\overset{+}{J} = \det \underset{\sim}{F}$, for some

$$(\beta_1, \beta_2) \in \Lambda \equiv \{ (\beta_1, \beta_2) \mid \lambda_1 \leq \beta_1 \leq \lambda_2 \leq \beta_2 \leq \lambda_3 \}$$

and for some positive $\overset{+}{J} \neq \overset{-}{J}$. Now Δ in (4.33) depends continuously on (β_1, β_2) and is easily seen to achieve its maximum and minimum for $(\beta_1, \beta_2) = (\lambda_1, \lambda_2)$ and (λ_2, λ_3), respectively. Moreover, Δ attains all intermediate values on Λ, which is connected. Hence it vanishes for some $(\beta_1, \beta_2) \in \Lambda$ if and only if it is nonnegative at (λ_1, λ_2) and nonpositive at (λ_2, λ_3). These conditions are equivalent to (4.30) if one sets $I_1 = \lambda_i\lambda_i$, $I_3 = \lambda_1\lambda_2\lambda_3 = J^2$, $\overset{+}{J} = J$. This completes the proof.

The restrictions on the material constants c and d imposed in the foregoing proposition are equivalent to the Baker-Ericksen inequalities (1.16). In case these hold, one readily infers from (3.29) that the potential under consideration sustains shocks if and only if the last of the global strong ellipticity conditions (3.24) fails to hold, i.e., in case $\Phi''(J) < 0$ for some $J > 0$. Hence for the Hadamard materials that satisfy the Baker-Ericksen inequalities, existence of shocks is *equivalent* to loss of strong ellipticity at some deformation.

5. Ellipticity for Plane Deformations of Isotropic Materials

Conditions for strong and ordinary ellipticity of the equilibrium equations appropriate for *plane* deformations of compressible isotropic materials have been established by Knowles and Sternberg [1]. This section aims at arriving at these results via an alternative route: the kinematic results of Section 2 are specialized to two dimensions, and the developments of Section 3 are modified to make them relevant to plane deformations. Our intention is not only to test the methods developed in this work against previously known results, but also to provide an example of the applicability of our approach in a setting simpler than, but similar, to the three–dimensional one.

We identify E_2 with a plane in E_3 containing the origin O and spanned by two unit vectors $\underset{\sim}{e}_\alpha$ of a fixed frame $\{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$. Consider a body that, in the reference configuration, occupies a cylindrical region with open middle cross-section $\mathcal{R} \subset E_2$ and generators along $\underset{\sim}{e}_3$. A deformation of the cylindrical body is *plane* if it maps the particle at $\underset{\sim}{x} = x_i \underset{\sim}{e}_i$ to $\underset{\sim}{y} = y_i \underset{\sim}{e}_i$, so that

$$\underset{\sim}{y} = \hat{\underset{\sim}{y}}\,(x_\alpha \underset{\sim}{e}_\alpha) + x_3 \underset{\sim}{e}_3 , \quad x_\alpha \underset{\sim}{e}_\alpha \in \mathcal{R} , \tag{5.1}$$

where $\hat{\underset{\sim}{y}} : \mathcal{R} \to E_2$ is a two-dimensional mapping defined on the cross-section $\mathcal{R}$ of the cylinder. Greek indices are understood to take the values 1,2 throughout this Section. If we refer to $\hat{\underset{\sim}{y}}$ in (5.1) as the *(plane) deformation of* $\mathcal{R}$, we may reinterpret the exposition in Section 1 in a two-dimensional setting. The sets $\mathcal{U}$, $\mathcal{L}_+$, $\mathcal{O}_+$ and $\overset{+}{\mathcal{S}}$ now take on two-dimensional meaning. Let X stand for the *two-dimensional* frame $\{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2\}$. We write $\underset{\sim}{x} = x_\alpha \underset{\sim}{e}_\alpha$ for the position vector of a particle in $\mathcal{R}$, $\underset{\sim}{F}(\underset{\sim}{x}) \in \mathcal{L}_+$ for the two-dimensional gradient of the mapping $\hat{\underset{\sim}{y}}$, and $\underset{\sim}{C}(\underset{\sim}{x}) = \underset{\sim}{F}^T(\underset{\sim}{x})\underset{\sim}{F}(\underset{\sim}{x})$, for the two-dimensional version of the right Cauchy-Green tensor. From the three-dimensional nominal and Cauchy stress tensors associated with the deformation, we construct two-dimensional tensors $\underset{\sim}{\sigma}$ and $\underset{\sim}{\tau}$ by specifying that their components in X are given by $\sigma_{\alpha\beta}$ and $\tau_{\alpha\beta}$, respectively. (Note that σ_{i3} , σ_{3i} and $\tau_{i3} = \tau_{3i}$ need not vanish). We speak of $\underset{\sim}{\sigma}$ and $\underset{\sim}{\tau}$ as the *in-plane* nominal and Cauchy Stress tensors, respectively.

For a more detailed account of the theory of plane deformations, we refer the reader to Knowles and Sternberg [1], [15]. The three deformation invariants in (1.5) are fully determined for plane deformations by the two *plane* invariants

$$\begin{aligned} I(\underset{\sim}{C}) &= \operatorname{tr} \underset{\sim}{C} = \lambda_1^2 + \lambda_2^2 , \\ J(\underset{\sim}{C}) &= \sqrt{\det \underset{\sim}{C}} = \det \underset{\sim}{F} = \lambda_1 \lambda_2 , \end{aligned} \tag{5.2}$$

where $\lambda_\alpha > 0$ are the principal stretches.

The in-plane mechanical response of an isotropic body is characterized by its in-plane elastic potential

$$W(\underset{\sim}{F}) = \hat{W}(I, J) , \quad I = \operatorname{tr} \underset{\sim}{F}^T \underset{\sim}{F}, \ J = \det \underset{\sim}{F}, \ \underset{\sim}{F} \in \mathcal{L}_+ \qquad (5.3$$

where $\hat{W}(\cdot, \cdot)$ is defined and twice continuously differentiable on the two-dimensional invariant region $\overset{+}{I} = \{(I, J) | I \geq 2J > 0\}$. In view of (5.2) we may define

$$\bar{W}(\lambda_1, \lambda_2) = \hat{W}(\lambda_1^2 + \lambda_2^2, \lambda_1 \lambda_2) , \quad \lambda_\alpha > 0 , \tag{5.4}$$

thus expressing the potential in terms of principal stretches.

The equilibrium equations are now given by the two-dimensional counterpart of (1.10) which is to be satisfied by the two components of the displacement field $\underset{\sim}{u} : \mathcal{R} \to E_2$. Analogous remarks apply to the acoustic tensor (1.18), so that the ordinary and strong ellipticity conditions for plane deformations are supplied by the two-dimensional version of Definition 1.1.

Theorem 2.1 can be directly applied to tensors in E_2, for which it asserts the following. Let $\underset{\sim}{n} \in \mathcal{U}$ and $\underset{\sim}{F} \in \mathcal{L}_+$ be given. Let $\underset{\sim}{e} \in \mathcal{U}$ be orthogonal to $\underset{\sim}{n}$.

Then $\underset{\sim}{F}$ admits the representation

$$\underset{\sim}{F} = \underset{\sim}{R}(\underset{\sim}{a} \otimes \underset{\sim}{n} + \underset{\sim}{e} \otimes \underset{\sim}{e})\underset{\sim}{V}, \tag{5.5}$$

where $\underset{\sim}{R} \in \mathcal{O}^+$, $\underset{\sim}{a} \in E_2$ and $\underset{\sim}{V} \in \mathcal{S}^+$ are uniquely determined by $\underset{\sim}{n}$ and $\underset{\sim}{F}$, and $\underset{\sim}{V}\underset{\sim}{n} = \underset{\sim}{n}$. Clearly X$= \{O; \underset{\sim}{e}, \underset{\sim}{n}\}$ is a principal frame for $\underset{\sim}{V}$. If $\beta > 0$ is the principal value of $\underset{\sim}{V}$ corresponding to $\underset{\sim}{e}$ and $a_1 = \underset{\sim}{a} \cdot \underset{\sim}{e}$, $a_2 = \underset{\sim}{a} \cdot \underset{\sim}{n}$, the matrix of components of $\underset{\sim}{F}$ in the frame X$= \{O; \underset{\sim}{e}, \underset{\sim}{n}\}$ is given by

$$[\underset{\sim}{F}]^{\mathrm{X}} = [\underset{\sim}{R}]^{\mathrm{X}} \begin{pmatrix} \beta & a_1 \\ 0 & a_2 \end{pmatrix} . \tag{5.6}$$

From (5.6) we draw that $a_2 > 0$, since det $\underset{\sim}{F} = \beta a_2 > 0$. Equation (5.6) is the two-dimensional analogue of (2.37). In the current circumstances there are only one in-plane component $\beta > 0$ and two out-of-plane components a_1 and $a_2 > 0$ associated with the directional resolution of $\underset{\sim}{F}$ with respect to $\underset{\sim}{n}$. Using (5.6) and (5.2), we obtain

$$\begin{aligned} I &= \mathrm{tr}\underset{\sim}{F}^T\underset{\sim}{F} = \beta^2 + a_1^2 + a_2^2 , \\ J &= \det \underset{\sim}{F} = \beta a_2 . \end{aligned} \tag{5.7}$$

Utilizing (5.5)–(5.7) and proceeding in a fashion parallel to the proof of Lemma 2.3, we arrive at its two-dimensional counterpart.

Lemma 5.1. *A two-dimensional tensor $\underset{\sim}{F} \in \mathcal{L}_+$ with principal stretches λ_1, λ_2 ($\lambda_2 \geq \lambda_1 > 0$) admits the representation (5.6) for some $\underset{\sim}{R} \in \mathcal{O}^+$, frame* X *and $(\beta; a_1, a_2) \in \mathcal{D} = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ if and only if*

$$\lambda_1 \leq \beta \leq \lambda_2 , \; a_1^2 = I - (J/\beta)^2 - \beta^2 , \; a_2 = J/\beta , \tag{5.8}$$

where $I = \lambda_1^2 + \lambda_2^2$, $J + \lambda_1\lambda_2$.

The proof is omitted, since it mimics that of Lemma 2.3. Instead, we remark that in view of the restrictions $\beta > 0$, $a_2 > 0$, (5.8) is equivalent to (5.7).

With the above at our disposal we may retrace the construction of the results of Section 3 to obtain their plane counterparts. Keeping in mind our present confinement to isotropic materials, we define the *plane reduced elastic potential* by means of

$$\mathcal{W}(\beta; a_1, a_2) = \hat{W}(\beta^2 + a_1^2 + a_2^2, \beta a_2), \quad (\beta; a_1, a_2) \in \mathcal{D}. \tag{5.9}$$

This is justified by the observation that if $(\beta; a_1, a_2) \in \mathcal{D}$ is related to $\underset{\sim}{F} \in \mathcal{L}_+$ through (5.6), then (5.7), (5.3) imply that

$$W(\underset{\sim}{F}) = \mathcal{W}(\beta; a_1, a_2). \tag{5.10}$$

With the aid of (3.2)—reinterpreted in the present context—and the chain rule, we recover an expression for the acoustic tensor analogous to the second of (3.12).

$$\underset{\sim}{Q}(\underset{\sim}{F}, \underset{\sim}{n}) = \frac{\partial^2}{\partial a_\alpha \partial a_\beta} \mathcal{W}(\beta; a_1, a_2) \underset{\sim}{R}\underset{\sim}{e}_\alpha \otimes \underset{\sim}{R}\underset{\sim}{e}_\beta \tag{5.11}$$

where $X = \{O; \underset{\sim}{e}_1, \underset{\sim}{e}_2\}$, $\underset{\sim}{e}_2 = \underset{\sim}{n}$, $\underset{\sim}{R} \in \mathcal{O}_+$ and $(\beta; a_1, a_2)$ are related to $\underset{\sim}{F}$ through (5.6).

The foregoing results allow us to obtain a result parallel to Theorem 3.3.

Lemma 5.2. *Ordinary Ellipticity holds at* $\underset{\sim}{F} \in \mathcal{L}_+$ *with principal stretches* $\lambda_2 \geq \lambda_1$ $(\lambda_\alpha > 0)$ *if an only if*

$$\det \underline{H}(\beta; (I - J^2/\beta^2 - \beta^2)^{1/2}, J/\beta) \neq 0, \quad \lambda_1 \leq \beta \leq \lambda_2, \tag{5.12}$$

where $H_{\gamma\delta}(\beta; a_1, a_2) = \partial^2 \mathcal{W}(\beta; a_1, a_2)/\partial a_\gamma \partial a_\delta$ *and* $I = \lambda_1^2 + \lambda_2^2$, $J = \lambda_1 \lambda_2$.

Proof. In view of (5.11) and Definition 1.1, ordinary ellipticity holds at $\underset{\sim}{F}$ if and only if the Hessian matrix of $\mathcal{W}(\beta; a_1, a_2)$ is nonsingular for every choise of $(\beta; a_1, a_2) \in \mathcal{D}$ that corresponds to *some* directional resolution of $\underset{\sim}{F}$ through (5.6). Lemma 5.1 asserts that such $(\beta; a_1, a_2)$ are precisely the ones satisfying (5.8). This completes the proof of (5.12).

At this point, we obtain conditions for ordinary ellipticity at $\underset{\sim}{F} \in \mathcal{L}_+$ involving only the invariants I, J and the partial derivatives of $\hat{W}$ with respect to them. With this in mind, we introduce the following notation

$$W_I = \frac{\partial \hat{W}}{\partial I}\,,\; W_{II} = \frac{\partial^2 \hat{W}}{\partial I^2} \quad \text{on } \overset{+}{I} \tag{5.13}$$

and so on. Employing the chain rule in (5.9) and invoking the smoothness of $\hat{W}$, we derive the following expressions for the components $H_{\gamma\delta}(\beta; a_1, a_2) = \partial^2 \mathcal{W}(\beta; a_1, a_2)/\partial a_\gamma \partial a_\delta$ of the Hessian matrix of $\mathcal{W}$, where we omit arguments for brevity

$$\left.\begin{aligned} &H_{11} = 2W_I + 4W_{II}a_1^2\,,\\ &H_{22} = 2W_I + 4W_{II}a_2^2 + 4W_{IJ}\beta a_2 + W_{JJ}\beta^2\,,\\ &H_{12} = H_{21} = 4W_{II}a_1a_2 + 2W_{IJ}\beta a_1\,. \end{aligned}\right\} \tag{5.14}$$

Observing that $\det \underline{H} = H_{11}H_{22} - H_{12}^2$ and assigning to a_1 and a_2 the values indicated in (5.8), we obtain from (5.14) that

$$\begin{aligned} \det \underline{H}\big(\beta\,; (I - J^2/\beta^2 - \beta^2)^{1/2}, J/\beta\big) &\equiv \Delta(\beta^2) = \\ &= -4D\beta^4 + [2W_I(W_{JJ} - 4W_{II}) + 4DI]\beta^2 + \\ &\quad +4W_I^2 + 8W_I(W_{II}I + W_{IJ}J) - 4DJ^2\,, \end{aligned}$$

$$\lambda_1 \le \beta \le \lambda_2\,, \quad I + \lambda_1^2 + \lambda_2^2\,, J = \lambda_1\lambda_2\,, \quad \lambda_2 \ge \lambda_1 > 0\,, \tag{5.15}$$

where we have set

$$D = W_{II}W_{JJ} - W_{IJ}^2\,. \tag{5.16}$$

With (5.15) at our disposal, we may arrive at the requisite ellipticity conditions.

Theorem 5.1. *Given* $I = \lambda_1^2 + \lambda_2^2\,,\ J = \lambda_1\lambda_2\ \ (\lambda_2 \ge \lambda_1 > 0)$, *define*

$$\begin{aligned} &E_{\alpha\alpha} = 2W_I(2W_I + 4W_{II}J^2/\lambda_\alpha^2 + 4W_{IJ}J + W_{JJ}\lambda_\alpha^2) \quad \textit{(no sum)}\\ &E_{12} = E_{21} = 2D(I^2 - 4J^2) + (E_{11} + E_{22})/2 \quad . \end{aligned} \tag{5.17}$$

Then ordinary ellipticity holds at $\underset{\sim}{F} \in \mathcal{L}_+$ *with principal stretches* λ_α *if and only if*

$$E_{11}E_{22} > 0\,,\ \eta E_{12}/\sqrt{E_{11}E_{22}} > -1 \tag{5.18}$$

where $\eta = sgn\ E_{11}$.

Proof. Lemma 5.2 asserts that necessary and sufficient for ordinary ellipticity at $\underset{\sim}{F}$ is that

$$\Delta(\beta^2) \neq 0\ ,\ \lambda_1 \leq \beta \leq \lambda_2\ , \tag{5.18}$$

where $\Delta(\beta^2)$ is defined by (5.15), with the partial derivatives (5.13) of $\hat{W}$ evaluated at the invariants of $\underset{\sim}{C}$. Assume first that $\lambda_1 < \lambda_2$, and define ξ via

$$(\lambda_2^2 - \lambda_1^2)\xi/2 + (\lambda_1^2 + \lambda_2^2)/2 = \beta^2\ . \tag{5.19}$$

One easily sees that for $\beta > 0$, (5.19) ensures

$$\lambda_1 \leq \beta \leq \lambda_2 \iff -1 \leq \xi \leq 1\ . \tag{5.20}$$

Thus we may set $\xi = \cos 2\theta\ ,\ \theta \in [0, \pi/2]$ in (5.19) and substitute the result into (5.15) to express $\Delta(\beta^2)$ as a quadratic in $\cos 2\theta$, for $0 \leq \theta \leq \pi/2$. By means of standard trigonometric formulae and some manipulation of (5.15), one finds that

$$\Delta(\beta^2) = E_{\alpha\beta}\varsigma_\alpha(\theta)\varsigma_\beta(\theta)\ , \tag{5.21}$$

where $\varsigma_1(\theta) = \sin^2\theta\ ,\ \varsigma_2(\theta) = \cos^2\theta\ ,\ \theta \in [0, \pi/2]\ ,\ E_{\alpha\beta}$ are given by (5.17), and β^2 obeys (5.19) with $\xi = \cos 2\theta$. In view of (5.21), (5.18) becomes equivalent to

$$E_{\alpha\beta}z_\alpha z_\beta \neq 0 \quad \text{for} \quad z_\alpha \geq 0\ ,\ z_1 + z_2 > 0\ . \tag{5.22}$$

Knowles and Sternberg have shown in Section 2 of [1] that (5.22) is true if and only if (5.18) holds. It remains to consider the case where $\lambda_1 = \lambda_2 = \lambda > 0$. In this instance (5.18) reduces to

$$\delta(\lambda^2) \neq 0\ . \tag{5.23}$$

Moreover, since $I = 2J = 2\lambda^2$, inspection of (5.17), (5.15) reveals that

$$E_{11} = E_{22} = E_{12} = \delta(\lambda^2)\ . \tag{5.24}$$

Thus in the present circumstances ordinary ellipticity reduces to $E_{11} \neq 0$, which in view of (5.24), remains equivalent to (5.18). This completes the proof.

The above is an alternative derivation of the ellipticity conditions established by Knowles and Sternberg [1]. Here we have arrived at a version of these conditions involving the derivatives of the function $\hat{W}(I,J)$, whereas their result is given in terms of the plane elastic potential $\bar{W}(\lambda_1, \lambda_2)$ expressed as a function of the principal stretches.

The strong ellipticity conditions are easily obtained in a similar fashion. They are equivalent to the requirement that the matrix with components $H_{\alpha\beta}(\beta\,;\,(I - J^2/\beta^2 - \beta^2)^{1/2},\, J/\beta)$ be positive-definite for $\beta \in [\lambda_1, \lambda_2]$. Since the $H_{\alpha\beta}$ are continuous in β, $\underline{H}$ is positive-definite on $[\lambda_1, \lambda_2]$ if and only if it is nonsingular there and positive-definite for $\beta = \lambda_1$. The latter condition becomes

$$W_I > 0, \quad E_{11} > 0\,, \tag{5.25}$$

as inspection of (5.14), (5.17) reveals. Hence *the conditions for strong ellipticity are given by* (5.25) *together with* (5.18), *in which now* $\eta = +1$ *is appropriate* because of the second of (5.25). The above is once again in complete agreement with the conclusions of [1].

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