Discrete Reduction of Mechanical Systems and Multisymplectic Geometry of Continuum Mechanics

Thesis by
Sergey Pekarsky

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

2000

(Defended May 18, 2000)

To My Family

Acknowledgements

First of all, I would like to thank my advisor, Professor Jerrold E. Marsden, for his encouragement, support, and supervision which all together contributed greatly to the free spirit of research I enjoyed at Caltech. His deep insight and understanding of both the intricate details and the big picture set a truly high standard for me to aspire to. I am also thankful to other faculty, students, and visitors for providing a stimulating work environment and occasional fun and to the the whole department of Control & Dynamical Systems for its support at the moment of a particularly bad luck in my life.

I benefited significantly from collaboration with a number of people. Anthony Blaom has helped me to discovery the mathematical beauty of geometric mechanics. Special thanks to a friend and a colleague Steve Shkoller. A significant portion of the research in this thesis was done while I was visiting him in Los Alamos National Lab. His constant optimism and enthusiasm often helped me out of my energy minima. Many thanks to Matt West for inspiring conversations and collaborations which resulted in Chapter 7 and some other material in Part II of this thesis and other yet unpulished research.

There have been many people around me with whom I have shared the lighter and darker sides of life. I am especially grateful to Alfredo Martinez and Pablo Parrilo, for our joint struggle with endless homeworks during the first years, late dinners followed by watching SNL, sharing opinions about the grad school and life in general, and tolerating my occasional being a pain in the neck during all those years.

During my graduate years here, I have had the pleasure of being a part of a cool gang whose proud members include Federico Spedalieri, Diego Dugatkin, Luz Vela, Monica Giannelli, Martin Basch and others, – we had lots of fun together. Many thanks to Svetlana Lyapina, Matt Perlmutter, Hendra Adiwidjaja, Sameer Jalnakupar, Scott Kelly, Sven Khatri, Alex Fax, Jim Primbs, Couro Kane, Anna Kashina, Leo Eisner and others for numerous conversations about science, business, and life in general. I am grateful to all of my friends here for the fun time I had in

Caltech.

Finally, my deepest gratitude goes to my family, my wife, my brother and my parents, for their support, love, and understanding.

Discrete Reduction of Mechanical Systems and Multisymplectic Geometry of Continuum Mechanics

by

Sergey Pekarsky

In Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

Abstract

This thesis develops discrete reduction techniques for mechanical systems defined on Lie groups and also presents multisymplectic formulation of both compressible and incompressible models of continuum mechanics on general Riemannian manifolds. While the former synthesizes ideas of Euler-Poincaré and Lie-Poisson reduction for mechanical systems with the Veselov type discretization of such systems, the latter sets the stage for multisymplectic reduction and for further development of Veselov type multisymplectic discretizations.

For systems defined on finite dimensional Lie groups G with Lagrangians $L : TG \to \mathbb{R}$ that are G-invariant, the reduced discrete equations provide "reduced" numerical algorithms which manifestly preserve the underlying (symplectic) structure. The manifold $G \times G$ is used as an approximation of TG, and a discrete Lagrangian $\mathbb{L} : G \times G \to \mathbb{R}$ is constructed in such a way that the G-invariance property is preserved. Reduction by G results in new "variational" principle for the reduced Lagrangian $\ell : G \to \mathbb{R}$, and provides the discrete Euler-Poincaré (DEP) equations. The solution of the DEP algorithm immediately leads to a discrete Lie-Poisson (DLP) algorithm.

It is also shown that the reduced Lagrangian $\ell : G \to \mathbb{R}$ defines a Poisson structure on (a subset) of one copy of the Lie group G. This structure governs the corresponding discrete reduced dynamics. The symplectic leaves of this structure

become dynamically invariant manifolds which are manifestly preserved under the structure preserving discrete Euler-Poincaré algorithm.

A variational multisymplectic formulation of non-relativistic continuum mechanics on general Riemannian manifolds is developed. Two main applications of our theory are considered – fluid dynamics and elasticity – each specified by a particular choice of the Lagrangian density. The non-relativistic character of the theory enables applications to such important cases as incompressible hydrodynamics and constrained director models of elastic rods and shells. These are applications of a general formalism developed here for treating non-relativistic first-order multisymplectic field theories with constraints.

The results obtained in this thesis also set the stage for multisymplectic reduction and for the further development of Veselov-type multisymplectic discretizations and numerical algorithms. Combined with the ideas on discretizing systems with symmetries, this approach would results in so called multisymplectic integrators which preserve the discrete analogues of the conservation laws.

Contents

Chapter 1

Introduction

Introductory Remarks. Classical mechanics is a large subject which plays a fundamental role in science. Besides mechanical systems and continuum mechanics, it deals with such fields as electromagnetism, gravity, etc. Mechanics has two main branches, Lagrangian mechanics and Hamiltonian mechanics. The Lagrangian formulation of mechanics can be based on the observation that there are variational principles behind the fundamental laws of force balance as given by Newton's law. This will be described in detail in Chapter 2.

Rarely can one find explicit solutions to the ODEs and PDEs that govern the dynamics of real mechanical systems or continuum mechanics models. Numerical integration is then called upon to approximate solutions, and the question of the compatibility of such approximations with the structure of mechanics arises. Though it can usually be answered on a case by case basis, there are general tools and methods that have been developed in the last 20 years. Such methods are usually based on some basic underlying principles of mechanics.

There are some fundamental properties which are characteristic of problems in classical mechanics. Those include, but are not limited to, conservation of energy and integrals of motion, such as linear and angular momenta, and symplecticity, i.e., preservation of a symplectic structure (see Chapter 2). It then becomes important to analyze how these properties are preserved under the numerical algorithm. This is the subject of geometric integration, which recently has become increasingly more systematic, starting with a seminal work by Moser and Veselov [47]. Their approach is based on discretizing the variational principle that is the cornerstone of Lagrangian mechanics. Several very important and interesting papers followed, and we would like to mention some of them which have direct relationship to this work, [53, 28, 34].

Symmetry has always played an important role in mechanics, from fundamental formulations of basic principles to concrete applications. Various reduction techniques have been developed (see, e.g., [40, 32]) which enable construction of a dynamical system of lower dimension equivalent to the original system. For instance, the Euler-Poincare reduction described in Chapter 2 is a particular example of a general procedure of Lagrangian reduction which is concerned with reducing the variational principle.

Main Achievements. Broadly speaking, Part I of this thesis is a synthesis of ideas and results in discretization and reduction for finite dimensional mechanical systems. We develop a systematic way of constructing reduced structure-preserving integration algorithms for mechanical systems defined on Lie algebras. This formalism can be generalized both to more general configuration manifolds and to continuum mechanics models defined on infinite dimensional spaces. The latter are the subject of Part II of the thesis, which sets the stage for such a theory by developing the so-called multi-symplectic description of continuum mechanics. The main results of this thesis can be summarized as follows:

- Discrete analogues of Euler-Poincaré and Lie-Poisson reduction theory are developed for systems on finite dimensional Lie groups G with Lagrangians $L : TG \to \mathbb{R}$ that are G-invariant. These discrete equations provide "reduced" numerical algorithms which manifestly preserve the symplectic structure,

- A variational and multisymplectic formulation of both compressible and incompressible models of continuum mechanics on general Riemannian manifolds is presented. Two main applications of our theory are considered—fluid dynamics and elasticity—each specified by a particular choice of the Lagrangian density,

- A general formalism is developed for non-relativistic first-order multisymplectic field theories with constraints, such as the incompressibility constraint. Our main example of a constraint in this thesis is the incompressibility constraint in fluids.

The results obtained in this thesis set the stage for multisymplectic reduction and for the further development of Veselov-type multisymplectic discretizations and numerical algorithms.

Outline of the Thesis. The two parts of the thesis deal with quite different objects (which are related in the last chapter) and, hence, have different sets of notations. Rather than compiling an overall vocabulary, we define each notion as it first appears in the exposition unless it is trivial or self-explanatory. We remark on our choice of notations in the end of the introduction chapter. Chapter 10 concludes the thesis by making various connections and outlining directions of the ongoing and future research.

Chapter 2 contains some preliminary standard results in Lagrangian mechanics. Hamilton's variational principle is introduced and the resulting Euler-Lagrange equations are derived for a system defined by some Lagrangian function L on the velocity phase space TG of some Lie group G. One of the key geometric objects – the symplectic structure on the underlying phase space – is obtained both via the Legendre transformations and based on the variational approach. In either case, one demonstrates that this structure is preserved under the evolution flow of the system. Moreover, if the system possesses a symmetry group K, then a corresponding momentum map is defined which provides us with the conserved quantities – integrals of the motion – corresponding to such a symmetry. In the last section, main results of the Euler-Poincaré and Lie-Poisson reduction techniques are outlined.

We continue with an overview of the Veselov discretization method for Lagrangian systems adapted to the case of systems defined on Lie groups. Chapter 3 also contains the first main result of Part I, namely the discrete Euler-Poincaré reduction theorem which mimics the analogous theorem stated in Section 2.2. For a choice of the discrete Lagrangian function ℓ on G, the resulting discrete Euler-Poincaré (DEP) equations provide a structure-preserving algorithm on the Lie group G. We demonstrate how this algorithm can be reconstructed to the full dynamics, resulting in discrete Euler-Lagrange (DEL) equations, and also how it can be used to obtain a discrete Lie-Poisson algorithm which approximates the reduced Hamiltonian dynamics.

In Chapter 4 we analyze various Poisson and symplectic structures arising in our constructions. We establish connections between discretization and reduction on the Lagrangian side and define Legendre type transformations which relate discrete Lagrangian dynamics with continuous Hamiltonian dynamics on both the reduced and unreduced levels. This analysis reveals two intrinsic and absolutely consistent ways of putting a Poisson structure on a (neighborhood of the identity in a) Lie group and points out some advantages of reduced structure preserving algorithms – the numerical trajectories stay on some lower dimensional subspace, called a symplectic leaf, which corresponds to an invariant manifold of the continuous problem under Legendre transformations.

Application of the above results and ideas to the rigid body (RB) dynamics on SO(3) is described in Chapter 5. For a particular choice of the discretization of the RB Lagrangian, the DEP algorithm recovers the well-known Moser-Veselov (completely integrable) scheme [47]. One explanation of the known superior performance of this algorithm lies in the conservation of the underlying discrete reduced Poisson structure on SO(3). We conclude the Chapter by computing the corresponding Casimir function which determines invariant manifolds of the discrete dynamics.

The purpose of Part II of the thesis is to give a variational multisymplectic formulation of continuum mechanics from a point of view that will facilitate the development of a corresponding discrete theory, as in the PDE Veselov formulation due to Marsden, Patrick, and Shkoller [34]. We restrict our attention to non-relativistic theories on general Riemannian manifolds. The relativistic case was considered in [23], where the authors take an alternative approach of inverse fields, effectively exchanging the base and fiber spaces (see also [14]). There are a number of reasons, both functional analytic and geometric for motivating a formulation in terms of *direct particle placement fields* rather than on *inverse fields.* For example, in the infinite dimensional context, this is the setting in which one has the deeper geometric and analytical properties of the Euler equations and related field theories, as in [2, 13, 43]. Moreover, the non-relativistic formalism naturally includes incompressible fluids and incompressible elasticity, which cannot be described within the framework of the relativistic theory.

Two main applications of our theory – fluid dynamics and elasticity – are considered in Chapter 6, each being specified by a particular choice of Lagrangian density. The resulting Euler-Lagrange equations can be written in a well-known form by introducing the pressure function P and the Piola-Kirchhoff stress tensor $\mathcal{P}$ (equations (6.18) and (6.21) below, respectively).

We only consider *ideal,* that is nonviscous, fluid dynamics in this thesis, both compressible and incompressible cases. In the former case, we work out the details for *barotropic* fluids for which the stored energy is a function of the density. These results can be trivially extended to *isentropic* (compressible) fluids, when the stored energy also depends on the entropy. Both the density and the entropy are assumed to be some given functions in material representation, so that our formalism naturally includes *inhomogeneous* ideal fluids with the exception of symmetries and corresponding conservations considered in Chapter 9.

For the theory of elasticity we restrict our attention to *hyperelastic* materials, that is to materials whose constitutive law is derived from a stored energy function. Similarly, we assume that the material density is some given function which describes a *heterogeneous* hyperelastic material and, hence, is non-constant.

Chapter 7 develops a general formalism for treating constrained multisymplectic theories. Often, constraints that are treated in the multisymplectic context are dynamically invariant, as with the constraint $\operatorname{div} \mathbf{E} = 0$ in electromagnetism (see, for example, [16]), or $\operatorname{div} \mathbf{E} = \rho$ for electromagnetism interacting with charged matter. Our main example of a constraint considered in Chapter 8 is the incompressibility constraint in fluids, which, when viewed in the standard *Eulerian, or spatial* view of fluid mechanics is often considered to be a *nonlocal* constraint (because the pressure is determined by an elliptic equation and, correspondingly, the sound speed is infinite), so it is interesting how it is handled in the multisymplectic context, which is, by nature, a local formalism. We restrict our attention to first-order theories, in which both the Lagrangian and the constraints depend only on *first* derivatives of the fields. Moreover, we assume that time derivatives do not enter the constraints, which translate under the space-time split to holonomic constraints on the corresponding infinite-dimensional configuration manifold in material representation.

Symmetries and the corresponding momentum maps and conservation theorems are considered separately in Chapter 9 since they are very different for different models of a continuous media, e.g., homogeneous fluid dynamics has a huge symmetry, namely the particle relabeling symmetry, while standard elasticity (usually assumed to be inhomogeneous) has much smaller symmetry groups, such as rotations and translations in Euclidean case. We emphasize that although the rest of Part II describes general heterogeneous continuous media, the results in Chapter 9 only apply to homogeneous fluid dynamics, where the symmetry group is the full group of volume-preserving diffeomorphisms $\mathcal{D}_\mu$. However, these results can be generalized to inhomogeneous fluids, in which case the symmetry group is a *subgroup* $\mathcal{D}_\mu^\rho \subset \mathcal{D}_\mu$ that preserves the level sets of the material density for barotropic fluids, or a *subgroup* $\mathcal{D}_\mu^{\rho,\mathrm{ent}} \subset \mathcal{D}_\mu$ that preserves the level sets of the material density and entropy for isentropic fluids. This would put us in the realm of a multisymplectic version of the Euler-Poincaré theory – one needs to introduce additional *advected quantities* as basic fields to handle this situation (see discussion in Chapter 10).

Finally, Chapter 10 summarizes the results obtained in this thesis. We relate both parts and outline the main directions of future research. Namely, though neither the structure preserving discretization, nor the reduction of non-trivial symmetries is well-established for multisymplectic theories, we strongly believe that the combination of these techniques a lá Part I will in the long run result in superior new algorithms. One of the ultimate objectives is a derivation of schemes which compute the Eulerian velocity field or the vorticity field of an ideal fluid in a way that preserves all the known Casimirs, e.g., all moments of vorticity in 2D, such as enstrophy.

A note on the notations

The subject of this thesis lies in the intersection of many areas of mathematics. Some of them, such as Lagrangian and Hamiltonian mechanics and reduction theory, have existed for a relatively long time and have a set of well-established notations. We follow them. Others, such as geometric integration, are quite young; their notations vary significantly in the literature and are sometimes inconsistent. We try to develop a standard. Yet others, such as standard hydrodynamics, standard elasticity and multisymplectic field theories, have such different sets of notations that reconciling them was not easy. Here, we make a personal choice. In this case, in particular, we decided to follow the field-theoretical language of sections of various bundles with some established rules, e.g., for base and fiber points and their indices (see, e.g. [16]). Finally we notice that notations in Part I and in Part II are effectively not overlapping, so that there is little danger of confusion.

Part I

Discrete Reduction of Mechanical Systems

Chapter 2

Preliminaries

The purpose of this chapter is to recall some basic results in Lagrangian mechanics, which can be traced back to Euler, Lagrange and Hamilton in the period 1740-1830, and to establish some associated terminology and notations. We shall also review the theory of Euler-Poincaré and Lie-Poisson reduction, originally developed by Poincaré [49] and Lie [29] (see, e.g., [40] for a detailed exposition of the subject and some interesting historic remarks), which in later chapters will be adapted to discrete settings.

2.1 Variational Lagrangian mechanics

Here we closely follow [34]. Let G be an n-dimensional Lie group with its tangent bundle TG. Denote coordinates on G by g^i and those on TG by $(g^i, \dot{g}^i)$. Consider a Lagrangian $L : TG \to \mathbb{R}$. Construct the corresponding action functional S on C^2 curves $g(t)$ in G by integration of L along the tangent to the curve. In coordinate notation, this reads

$$S\big(g(t)\big) \equiv \int_a^b L\left(g^i(t), \frac{dg^i}{dt}(t)\right) dt. \tag{2.1}$$

The action functional depends on a and b, but this is not explicit in the notation for S. ***Hamilton's principle*** seeks the curves $g(t)$ for which the functional S is stationary under variations of $g(t)$ with fixed endpoints; namely, we seek curves $g(t)$

which satisfy

$$dS\big(g(t)\big) \cdot \delta g(t) \equiv \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} S\big(g_\epsilon(t)\big) = 0 \tag{2.2}$$

for all $\delta g(t)$ with $\delta g(a) = \delta g(b) = 0$, where g_ϵ is a smooth family of curves with $g_0 = g$ and $(d/d\epsilon)|_{\epsilon=0} g_\epsilon = \delta g$. Using integration by parts, the calculation for this is simply

$$\begin{aligned} dS\big(g(t)\big) \cdot \delta g(t) &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_a^b L\left(g^i_\epsilon(t), \frac{dg^i_\epsilon}{dt}(t)\right) dt \\ &= \int_a^b \delta g^i \left(\frac{\partial L}{\partial g^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{g}^i}\right) dt + \left.\frac{\partial L}{\partial \dot{g}^i}\delta g^i\right|_a^b . \end{aligned} \tag{2.3}$$

The last term in (2.3) vanishes since $\delta g(a) = \delta g(b) = 0$, so that the requirement (2.2) for S to be stationary yields the ***Euler-Lagrange equations***

$$\frac{\partial L}{\partial g^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{g}^i} = 0. \tag{2.4}$$

Recall that L is called ***regular*** when the symmetric matrix $[\partial^2 L/\partial \dot{g}^i \partial \dot{g}^j]$ is everywhere nonsingular. If L is regular, the Euler-Lagrange equations are second order ordinary differential equations for the required curves.

The standard geometric setting. The action (2.1) is independent of the choice of coordinates, and thus the Euler-Lagrange equations are coordinate independent as well. Consequently, it is natural that the Euler-Lagrange equations may be intrinsically expressed using the language of differential geometry. This intrinsic development of mechanics is now standard, and can be seen, for example, in [3, 1, 40].

The ***canonical*** **1*-form*** θ_0 on the $2n$-dimensional cotangent bundle of G, T^*G is defined by

$$\theta_0(\alpha_g) w_{\alpha_g} \equiv \alpha_g \cdot T\pi_G w_{\alpha_g}, \qquad \alpha_g \in T^*_g G,\ w_{\alpha_g} \in T_{\alpha_g} T^*G,$$

where $\pi_G : T^*G \to G$ is the canonical projection. The Lagrangian L intrinsically defines a fiber preserving bundle map $FL : TG \to T^*G$, the ***Legendre transformation***, by vertical differentiation:

$$FL(v_g)w_g \equiv \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(v_g + \epsilon w_g).$$

We define the ***Lagrange 1-form*** on TG, the Lagrangian side, by pull-back to give $\theta_L \equiv FL^*\theta_0$, and the ***Lagrange 2-form*** by $\omega_L = -d\theta_L$. We then seek a vector field X_E (called the ***Lagrange vector field***) on TG such that $X_E \lrcorner \omega_L = dE$, where the ***energy*** E is defined by $E(v_g) \equiv FL(v_g)v_g - L(v_g)$.

If FL is a local diffeomorphism, then X_E exists and is unique, and its integral curves solve the Euler-Lagrange equations (2.4). In addition, the flow F_t of X_E preserves ω_L; that is, $F_t^*\omega_L = \omega_L$. Such maps are ***symplectic***, and the form ω_L is called a ***symplectic 2-form***. This is an example of a ***symplectic manifold***: a pair (M, ω) where M is a manifold and ω is a closed nondegenerate 2-form.

The variational approach. We next show that one can derive the fundamental differential geometric structures directly from the variational approach in a natural way. This development begins by removing the boundary condition $\delta g(a) = \delta g(b) = 0$ from (2.3). Equation (2.3) becomes

$$dS\big(g(t)\big) \cdot \delta g(t) = \int_a^b \delta g^i \left(\frac{\partial L}{\partial g^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{g}^i} \right) dt + \left.\frac{\partial L}{\partial \dot{g}^i}\delta g^i \right|_a^b, \tag{2.5}$$

where the left side now operates on more general δg, while the last term on the right side does not vanish. That last term of (2.5) is a linear pairing of the function $\partial L/\partial \dot{g}^i$, a function of g^i and $\dot{g}^i$, with the tangent vector δg^i. Thus, one may consider it to be a 1-form on TG; namely the 1-form $(\partial L/\partial \dot{g}^i)dg^i$. This is exactly the Lagrange 1-form, and we can turn this into a formal theorem/definition:

Theorem 2.1.1. *Given a C^k Lagrangian L, $k \geq 2$, there exists a unique C^{k-2} mapping $D_{EL}L : \ddot{G} \to T^*G$, defined on the second order submanifold*

$$\ddot{G} \equiv \left\{ \frac{d^2 g}{dt^2}(0) \;\middle|\; q \text{ a } C^2 \text{ curve in } G \right\}$$

of TTG, and a unique C^{k-1} 1-form θ_L on TG, such that, for all C^2 variations $g_\epsilon(t)$,

$$dS(g(t)) \cdot \delta g(t) = \int_a^b D_{EL}L\left(\frac{d^2g}{dt^2}\right) \cdot \delta g \, dt + \theta_L\left(\frac{dg}{dt}\right) \cdot \hat{\delta g}\Big|_a^b, \tag{2.6}$$

where

$$\delta g(t) \equiv \frac{d}{d\epsilon}\Big|_{\epsilon=0} g_\epsilon(t), \qquad \hat{\delta g}(t) \equiv \frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{d}{dt}\Big|_{t=0} g_\epsilon(t).$$

The 1-form so defined is called the ***Lagrange 1-form.***

Indeed, uniqueness and local existence follow from the calculation (2.3) and the coordinate independence of the action, and then global existence is immediate. This implies that the Lagrange 1-form θ_L is the "boundary part" of the the functional derivative of the action when the boundary is varied. The analogue of the symplectic form is the (negative of) the exterior derivative of θ_L, i.e., $\omega_L \equiv -d\theta_L$.

Lagrangian flows are symplectic. Assuming that L is regular, the variational principle then gives coordinate independent second order ordinary differential equations, as we have noted. We temporarily denote the vector field on TG so obtained by X, and its flow by F_t. Our further development relies on a change of viewpoint: we focus on the restriction of S to the subspace $\mathcal{C}_L$ of solutions of the variational principle. The space $\mathcal{C}_L$ may be identified with the initial conditions, elements of TG, for the flow: to $v_g \in TG$, we associate the integral curve $s \mapsto F_s(v_g)$, $s \in [0,t]$. The value of S on that curve is denoted by S_t, and again called the ***action***. Thus, we define the map $S_t : TG \to \mathbb{R}$ by

$$S_t(v_g) = \int_0^t L(g(s), \dot{g}(s))\, ds, \tag{2.7}$$

where $(g(s), \dot{g}(s)) = F_s(v_g)$. The fundamental equation (2.6) becomes

$$dS_t(v_g)w_{v_g} = \theta_L(F_t(v_g)) \cdot \frac{d}{d\epsilon}\Big|_{\epsilon=0} F_t(v_g^\epsilon) - \theta_L(v_g) \cdot w_{v_g},$$

where $\epsilon \mapsto v_g^\epsilon$ is an arbitrary curve in TG such that $v_g^0 = v_g$ and $(d/d\epsilon)|_0 v_g^\epsilon = w_{v_g}$. We have thus derived the equation

$$dS_t = F_t^* \theta_L - \theta_L. \tag{2.8}$$

Taking the exterior derivative of (2.8) yields the fundamental fact that the flow of X is symplectic:

$$0 = ddS_t = d(F_t^* \theta_L - \theta_L) = -F_t^* \omega_L + \omega_L$$

which is equivalent to

$$F_t^* \omega_L = \omega_L.$$

This leads to the following:

> *Using the variational principle, the fact that the evolution is symplectic is a consequence of the equation $d^2 = 0$, applied to the action restricted to the space of solutions of the variational principle.*

Momentum maps. Suppose that a Lie group K, with Lie algebra $\mathfrak{k}$, acts on G, and hence on curves in G, in such a way that the action S is invariant. Clearly, K leaves the set of solutions of the variational principle invariant, so the action of K restricts to $\mathcal{C}_L$, and the group action commutes with F_t. Denoting the infinitesimal generator of $\xi \in \mathfrak{k}$ on TG by ξ_{TG}, we have by (2.8),

$$0 = \xi_{TG} \lrcorner\, dS_t = \xi_{TG} \lrcorner\, (F_t^* \theta_L - \theta_L) = F_t^* (\xi_{TG} \lrcorner\, \theta_L) - \xi_{TG} \lrcorner\, \theta_L. \tag{2.9}$$

For $\xi \in \mathfrak{k}$, define $J_\xi : TG \to \mathbb{R}$ by $J_\xi \equiv \xi_{TG} \lrcorner\, \theta_L$. Then (2.9) says that J_ξ is an integral of the flow of X. We have arrived at a version of Noether's theorem (rather close to the original derivation of Noether):

> *Using the variational principle, Noether's theorem results from the infinitesimal invariance of the action restricted to the space of solutions of*

the variational principle. The conserved momentum associated to a Lie algebra element ξ is $J_\xi = \xi_{TG} \lrcorner \theta_L$, where θ_L is the Lagrange one-form.

2.2 Euler-Poincaré and Lie-Poisson reduction

To describe the Euler-Poincaré and Lie-Poisson reduction procedures in detail and with even a minimal number of examples would probably require writing a separate book (see, e.g., [41]). We therefore restrict ourself to simply stating the necessary results together with the required definitions. We follow here [40, 32] where the proofs of the quoted theorems can be found.

In the setting of the previous section, let us assume that the symmetry group is the configuration space G itself. Then, the dynamics on TG defined by some left-invariant Lagrangian L can be reduced to the dynamics on the Lie algebra $\mathfrak{g}$ defined by $l = L|_{T_eG}$. The resulting system has half the dimension of the original one, but the variational principle and, hence, the equations of motion are, in general, less trivial. The following theorem makes the relation between these systems precise for the right group action, an analogous theorem holds for the left action case.

Theorem 2.2.1. *Let G be a Lie group and let $L : TG \to \mathbb{R}$ be a right invariant Lagrangian. Let $l : \mathfrak{g} \to \mathbb{R}$ be its restriction to the identity. For a curve $g(t) \in G$, let $\xi(t) = \dot{g}(t) \cdot g(t)^{-1}$, i.e., $\xi(t) = T_{g(t)}R_{g(t)^{-1}}\dot{g}(t)$. Then the following are equivalent:*

i *$g(t)$ satisfies the Euler-Lagrange equations for L on G;*

ii *the variational principle*

$$\delta \int L(g(t), \dot{g}(t))dt = 0$$

holds for variations with fixed endpoints;

iii *the Euler-Poincaré equations hold:*

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = -\operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi};$$

iv *the variational principle*

$$\delta \int l(\xi(t))dt = 0$$

holds on $\mathfrak{g}$, *using variations of the form*

$$\delta\xi = \dot{\eta} - \mathrm{ad}_{\xi}\, \eta = \dot{\eta} + [\xi, \eta],$$

where η *vanishes at the endpoints.*

One of our main results for the discrete symmetry reduction, namely Theorem 3.2.1, mimics the above theorem in relating both the variational principles and the equations of motion for the original and for the corresponding reduced systems.

Whereas the Euler-Poincaré procedure is concerned with reducing the variational principle, the Lie-Poisson construction is concentrated on reducing the underlying geometric (Poisson or symplectic) structure. Namely, for a Lie group G, the canonical Poisson bracket on T^*G is related to the Lie-Poisson bracket on $\mathfrak{g}^*$ which completely determines the reduced dynamics for a given Hamiltonian function H. We state two main Lie-Poisson reduction theorems below and remark that both reduction procedures described here can be related by means of Legendre transformations specified by either a right invariant Lagrangian L on TG or a right invariant Hamiltonian H on T^*G.

Theorem 2.2.2. *Identifying the set of functions on* $\mathfrak{g}^*$ *with the set of right invariant functions on* T^*G *endows* $\mathfrak{g}^*$ *with a Poisson structure given by*

$$\{F, H\}_+(\mu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu}\right]\right\rangle, \quad \mu \in \mathfrak{g}^*.$$

Theorem 2.2.3. *Let* G *be a Lie group and let* $H : T^*G \to \mathbb{R}$ *be a right invariant Hamiltonian. Let* $h : \mathfrak{g}^* \to \mathbb{R}$ *be the restriction of* H *to* T_e^*G. *For a curve* $p(t) \in T^*_{g(t)}G$, *let* $\mu(t) = (T^*_{g(t)}R_{g(t)}) \cdot p(t) =: \lambda(p(t))$ *be the induced curve in* $\mathfrak{g}^*$. *Assuming that* $g(t)$ *satisfies the differential equation*

$$\dot{g} = T_e R_g \frac{\delta h}{\delta \mu},$$

where $\mu = p(0)$*, the following are equivalent:*

i $p(t)$ *is an integral curve of Hamilton's equations on* T^*G*;*

ii *for any* $F \in \mathcal{F}(T^*G)$, $\dot{F} = \{F, H\}$*, where* $\{,\}$ *is the canonical bracket on* T^*G*;*

iii $\mu(t)$ *satisfies the Lie-Poisson equations*

$$\frac{d\mu}{dt} = -\operatorname{ad}^*_{\partial h/\partial \mu} \mu,$$

where $\operatorname{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ *is defined by* $\operatorname{ad}_\xi \eta = [\xi, \eta]$ *and* ad^*_ξ *is its dual;*

iv *for any function* $f \in \mathcal{F}(\mathfrak{g}^*)$*, we have*

$$\dot{f} = \{f, h\}_+,$$

where $\{,\}_+$ *is the plus Lie-Poisson bracket.*

Now, let $\mathcal{O} \subset \mathfrak{g}$ be a coadjoint orbit; that is, the orbit of a point under the coadjoint action of G on $\mathfrak{g}^*$. Then $\mathcal{O}$ is a symplectic manifold with unique Kirillov-Kostant forms $\omega_\pm$ as the coadjoint orbit symplectic structures. An important consequence of the momentum conservation and the above theorem is that evolution of the reduced system keeps it on a particular coadjoint orbit specified by the initial conditions. Namely, if $(g(t), p(t))$ is a solution of Hamilton's equations on T^*G with the momentum value $\mu_0 = J(g(t))$, then the dynamics of the reduced system is given by

$$\mu(t) = \operatorname{Ad}^*_{g^{-1}(t)} \mu_0.$$

This formula will be exploited in Chapter 3 for construction of the discrete Lie-Poisson (DLP) algorithm.

Finally, we state here Lemma 14.4.2 from [40] which says that for any $g \in G$, $\operatorname{Ad}^*_{g^{-1}} : \mathcal{O} \to \mathcal{O}$ preserves $\omega_\pm$. We shall appeal to this result in Chapter 3 for the analysis of structure preservation by the DLP scheme.

Chapter 3

Discrete Reduction on Lie Groups

The goal of this chapter is to develop structure preserving numerical integrators on the reduced space of a mechanical system whose configuration space is a Lie group G, and whose Lagrangian $L : TG \rightarrow \mathbb{R}$ is either left or right invariant under the group action. In particular, we shall develop the discrete analogue of Euler-Poincaré theory by following the variational approach introduced by Marsden, Patrick, and Shkoller [34] for the construction of discrete Euler-Lagrange equations that naturally preserve the symplectic structure and the momentum mappings of the Lagrangian system.

The variational approach described below can be used to obtain a symplectic-momentum integrator by discretizing TG and forming a discrete action sum. For every choice of discretization, a unique discrete symplectic structure is obtained, and the algorithm given by the discrete Euler-Lagrange equations is guaranteed to preserve this structure as well as the momentum mappings associated with it. Our goal is to apply the reduction procedure in this discrete setting, restrict the Lagrangian to the reduced space, and derive the algorithm which preserves the induced structure.

Our procedure results in the discrete Euler-Poincaré equation, which defines an algorithm on the reduced space that is shown to be equivalent to the discrete Euler-Lagrange equations in the sense of reconstruction. This reduced algorithm is used together with the coadjoint action to advance points in $\mathfrak{g}^* \cong T^*G/G$ and thus to approximate the Lie-Poisson dynamics.

3.1 Veselov discretization of mechanics

The discrete Lagrangian formalism in Veselov [50, 51] can be applied to the case of the configuration manifold being a Lie group G. In the exposition below we closely follow [53] and [34], substituting a Lie group G for a general configuration manifold Q. One uses $G \times G$ for the discrete version of the tangent bundle of a configuration space G; heuristically, it corresponds to approximating the velocity $\dot{g}$ by $(g_1 - g_0)/\Delta t$ for some a priori choice of time interval Δt. Define a ***discrete Lagrangian*** to be a smooth map $\mathbb{L} : G \times G = \{g_0, g_1\} \to \mathbb{R}$, and the corresponding action to be

$$\mathbb{S} \equiv \sum_{k=1}^{N} \mathbb{L}(g_k, g_{k+1}). \tag{3.1}$$

The discrete Lagrangian can be obtained from the original Lagrangian $L : TG \to \mathbb{R}$ as

$$\mathbb{L}(g_k, g_{k+1}) = L(\kappa(g_k, g_{k+1}), \mathcal{X}(g_k, g_{k+1})),$$

where κ and $\mathcal{X}$ are functions of (g_k, g_{k+1}) which approximate the current configuration $g(t) \in G$ and the corresponding velocity $\dot{g}(t) \in T_gG$, respectively.

Remark 3.1.1. We choose particular discretization schemes so that the discrete Lagrangian $\mathbb{L}$ inherits the symmetries of the original Lagrangian L, i.e., $\mathbb{L}$ is G-invariant on $G \times G$ whenever L is G-invariant on TG. In particular, the induced right (left) lifted action of G onto TG corresponds to the diagonal right (left) action of G on $G \times G$.

The variational principle is to extremize $\mathbb{S}$ for variations holding the endpoints g_0 and g_N fixed. This variational principle determines a "discrete flow" $\mathbb{F} : G \times G \to G \times G$ by $F(g_{k-1}, g_k) = (g_k, g_{k+1})$, where g_{k+1} is found from the ***discrete Euler-Lagrange equations*** (DEL equations):

$$D_2\mathbb{L}(g_{k-1}, g_k) + D_1\mathbb{L}(g_k, g_{k+1}) = 0, \tag{3.2}$$

where D_1 and D_2 denote derivatives with respect to the first and second arguments, respectively.

In the remainder of this section we review the derivation of the basic differential-geometric objects of the discrete mechanics directly from the variational point of view.

Lagrange forms. We begin by calculating $d\mathbb{S}$ for variations that do not fix the endpoints:

$$\begin{aligned} & d\mathbb{S}(g_0, \cdots, g_N) \cdot (\delta g_0, \cdots, \delta g_N) \\ = & \sum_{k=1}^{N} D_2\mathbb{L}(g_k, g_{k+1})\delta g_k + \sum_{k=0}^{N-1} D_1\mathbb{L}(g_{k-1}, g_k)\delta g_k \\ = & \sum_{k=1}^{N-1} \left[D_1\mathbb{L}(g_k, g_{k+1}) + D_2\mathbb{L}(g_{k-1}, g_k)\right] \delta g_k \\ & \quad + D_1\mathbb{L}(g_0, g_1)\delta g_0 + D_2\mathbb{L}(g_{N-1}, g_N)\delta g_N, \end{aligned} \tag{3.3}$$

where we have used the discrete analogue of integration by parts, which simply shifts the sequence $g_k \mapsto g_r$ where $r = k+1$. It is the last two terms that arise from the boundary variations, and so these are the terms amongst which we expect to find the discrete analogue of the Lagrange 1-form. We define *two* 1-forms on $G \times G$:

$$\theta_{\mathbb{L}}^{-}(g_0, g_1) \cdot (\delta g_0, \delta g_1) \equiv D_1\mathbb{L}(g_0, g_1)\delta g_0,$$

and

$$\theta_{\mathbb{L}}^{+}(g_0, g_1) \cdot (\delta g_0, \delta g_1) \equiv D_2\mathbb{L}(g_0, g_1)\delta g_1,$$

which are related by

$$\theta_{\mathbb{L}}^{-} + \theta_{\mathbb{L}}^{+} = d\mathbb{L}, \tag{3.4}$$

so that

$$d\theta_{\mathbb{L}}^{-} + d\theta_{\mathbb{L}}^{+} = 0.$$

Thus, there are *two* generally distinct 1-forms, but (up to sign) only *one* 2-form. We define $\omega_{\mathbb{L}} \equiv d\theta_{\mathbb{L}}^- = -d\theta_{\mathbb{L}}^+$, which in coordinates is given by

$$\omega_{\mathbb{L}}(g_k, g_{k+1}) = d\theta_{\mathbb{L}}^- = \frac{\partial^2 \mathbb{L}}{\partial g_k^i \partial g_{k+1}^j} dg_k^i \wedge dg_{k+1}^j \tag{3.5}$$

and agrees with the discrete symplectic two-form in [50, 51]. Define the discrete Legendre transformation to be the fiber derivative $F\mathbb{L}$ given in the above coordinates by the following expression (see, e.g., [53])

$$F\mathbb{L} : G \times G \to T^*G; \quad (g_k, g_{k+1}) \mapsto (g_k, D_1\mathbb{L}(g_k, g_{k+1})).$$

Notice also that the Lagrange 2-form $\omega_{\mathbb{L}}$ defined by (3.5) coincides with the pull-back under $F\mathbb{L}$ of the canonical 2-form ω_{can} on T^*G (see, e.g., [34, 53]).

Remark 3.1.2. We remark that the discrete symplectic structure $\omega_{\mathbb{L}}$ is not globally defined, but rather is only nondegenerate in a neighborhood Δ of the diagonal in $G \times G$, i.e., whenever g_k and g_{k+1} are nearby. Section 3 of [34] shows that $\omega_{\mathbb{L}}$ arises from the boundary terms of the discrete action sum restricted to the space of solutions of the discrete Euler-Lagrange equations; an implicit function theorem argument relying on the regularity of the discrete Lagrangian $\mathbb{L}$ is required in order to obtain solutions to the discrete Euler-Lagrange equations, and this regularity need only hold in a neighborhood of the diagonal in $G \times G$, which we have denoted as Δ. The local character of the discrete symplectic and Poisson structures is implicitly understood in this Chapter and will be addressed in more detail in Chapter 4.

Symplecticity of the flow. We parameterize the solutions of the variational principle by the initial conditions (g_0, g_1), and restrict $\mathbb{S}$ to that solution space. Then equation (3.3) becomes

$$d\mathbb{S} = \theta_{\mathbb{L}}^- + F^*\theta_{\mathbb{L}}^+. \tag{3.6}$$

Using it we obtain the symplecticity of the flow $\mathbb{F}$ by applying the identity $ddS = 0$:

$$\mathbb{F}^* \omega_{\mathbb{L}} = \mathbb{F}^* (d\theta_{\mathbb{L}}^+) = -d\theta_{\mathbb{L}}^- = \omega_{\mathbb{L}},$$

Noether's Theorem. Suppose a Lie group K with Lie algebra $\mathfrak{k}$ acts on G, and hence diagonally on $G \times G$, and that $\mathbb{L}$ is K-invariant. Clearly, $\mathbb{S}$ is also K-invariant and K sends critical points of $\mathbb{S}$ to critical points. Thus, the action of K restricts to the space of solutions, the map $\mathbb{F}$ is K-equivariant, and from (3.6),

$$0 = \xi_{G\times G} \lrcorner\, d\mathbb{S} = \xi_{G\times G} \lrcorner\, \theta_{\mathbb{L}}^- + \xi_{G\times G} \lrcorner\, (\mathbb{F}^* \theta_{\mathbb{L}}^+),$$

for $\xi \in \mathfrak{k}$ and $\xi_{G\times G}$ being the corresponding infinitesimal generator on $G \times G$, or equivalently, using the equivariance of $\mathbb{F}$,

$$\xi_{G\times G} \lrcorner\, \theta_{\mathbb{L}}^- = -\mathbb{F}^* (\xi_{G\times G} \lrcorner\, \theta_{\mathbb{L}}^+). \tag{3.7}$$

Since $\mathbb{L}$ is K-invariant, (3.4) gives $\xi_{G\times G} \lrcorner\, \theta_{\mathbb{L}}^- = -\xi_{G\times G} \lrcorner\, \theta_{\mathbb{L}}^+$. Defining the ***discrete momentum*** to be

$$\mathbb{J}_\xi \equiv \xi_{G\times G} \lrcorner\, \theta_{\mathbb{L}}^+,$$

we see that (3.7) converts to the conservation of momentum equation

$$\mathbb{J}_\xi = \xi_{G\times G} \lrcorner\, \theta_{\mathbb{L}}^+ = \mathbb{F}^* (\xi_{G\times G} \lrcorner\, \theta_{\mathbb{L}}^+) = \mathbb{F}^* (\mathbb{J}_\xi). \tag{3.8}$$

Remark 3.1.3. For the remainder of this chapter we assume that K coincides with G, i.e., the symmetry group and the configuration space are the same. Another interesting case is when K is a subgroup of G and results in the discrete analogue of the semi-direct product reduction theory, but is outside the scope of this thesis.

3.2 The discrete Euler-Poincaré and Lie-Poisson algorithms

In this section we develop the discrete Euler-Poincaré reduction of a Lagrangian system on TG. The discrete reduction of a right-invariant system proceeds as follows. The induced group action on $G \times G$ is simply right multiplication in each component:

$$\bar{g} : (g_k, g_{k+1}) \mapsto (g_k \bar{g}, g_{k+1} \bar{g}),$$

for all $\bar{g}, g_k, g_{k+1} \in G$. Then the quotient map is given by

$$\pi_d : G \times G \to (G \times G)/G \cong G, \qquad (g_k, g_{k+1}) \mapsto g_k g_{k+1}^{-1}. \tag{3.9}$$

We note that one may alternatively use $g_{k+1} g_k^{-1}$ instead of $g_k g_{k+1}^{-1}$. The projection map (3.9) defines the ***reduced discrete Lagrangian*** $\ell : G \to \mathbb{R}$ for any G-invariant $\mathbb{L}$ by $\ell \circ \pi_d = \mathbb{L}$, so that

$$\ell(g_k g_{k+1}^{-1}) = \mathbb{L}(g_k, g_{k+1}),$$

and the ***reduced action sum*** is given by

$$s = \sum_{k=0}^{N-1} \ell(f_{kk+1}),$$

where $f_{kk+1} \equiv g_k g_{k+1}^{-1}$ denote points in the quotient space. A reduction of the DEL equations results in the discrete Euler-Poincaré (DEP) equations.

Discrete Euler-Poincaré reduction

We state here a general reduction type theorem which mimics Theorem 2.2.1 of Chapter 2. Notice a similarity between the reduced variation of ξ in Theorem 2.2.1 and the variation of f_{kk+1} below, e.g., the ad-action on $\mathfrak{g}$ is replaced by the Ad-action on G.

Theorem 3.2.1. *Let $\mathbb{L}$ be a right invariant Lagrangian on $G \times G$, and let ℓ :*

$(G \times G)/G \cong G \to \mathbb{R}$ be the restriction of $\mathbb{L}$ to G given by $\ell(g_1 g_2^{-1}) = \mathbb{L}(g_1, g_2)$. For any integer $N \geq 3$, let $\{(g_k, g_{k+1})\}_{k=0}^{N-1}$ be a sequence in $G \times G$ and define $f_{kk+1} \equiv g_k g_{k+1}^{-1}$ to be the corresponding sequence in G. Then, the following are equivalent.

(1) The sequence $\{(g_k, g_{k+1})\}_{k=0}^{N-1}$ is an extremum of the action sum $\mathbb{S} : G^{N+1} \to \mathbb{R}$ for arbitrary variations $\delta g_k = (d/d\epsilon)|_0 g_k^\epsilon$ where for each k, $\epsilon \mapsto g_k^\epsilon$ is a smooth curve in G such that $g_k^0 = g_k$; δg_k vanishes at endpoints.

(2) The sequence $\{(g_k, g_{k+1})\}_{k=0}^{N-1}$ satisfies the discrete Euler-Lagrange equations (3.2).

(3) The sequence $\{f_{kk+1}\}_{k=0}^{N-1}$ is an extremum of the reduced action sum $s : G^{N+1} \to \mathbb{R}$ with respect to variations δf_{kk+1} induced by the variations δg_k and given by

$$\delta f_{kk+1} = TR_{f_{kk+1}}(\delta g_k g_k^{-1} - \mathrm{Ad}_{f_{kk+1}} \cdot \delta g_{k+1} g_{k+1}^{-1}).$$

(4) The sequence $\{f_{kk+1}\}_{k=0}^{N-1}$ satisfies the ***discrete Euler-Poincaré*** *(DEP) equations*

$$R^*_{f_{kk+1}} \ell'(f_{kk+1}) - L^*_{f_{k-1k}} \ell'(f_{k-1k}) = 0 \tag{3.10}$$

for $k = 1, ..., N-1$. Here R^*_f and L^*_f are the right and left pull-backs by f, respectively, acting on variations of the form $\vartheta_k = \delta g_k g_k^{-1}$ and $\ell' : G \to T^*G$ is the differential of ℓ defined as follows. Let g^ϵ be a smooth curve in G such that $g^0 = g$ and $(d/d\epsilon)|_{\epsilon=0} g^\epsilon = v$. Then

$$\ell'(g) \cdot v = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \ell(g^\epsilon).$$

Proof. Setting the end-point variations in (3.3) to zero immediately recovers the DEL equations (3.2) and, hence, establishes the equivalence of (1) and (2). To see that (1) is equivalent to (3), notice that since $\mathbb{L} = \ell \circ \pi_d$,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} s(f^{\epsilon}_{kk+1}) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathbb{S}(g^{\epsilon}_k).$$

Now for (3) $\Leftrightarrow$ (4), we compute

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \sum_{k=0}^{N-1} \ell(g^{\epsilon}_k {g^{\epsilon}_{k+1}}^{-1})$$

and find that

$$\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} s(f^{\epsilon}_{kk+1}) &= \sum_{k=0}^{N-1} \ell'(f_{kk+1}) \left[\delta g_k g_{k+1}^{-1} - g_k g_{k+1}^{-1} \delta g_{k+1} g_{k+1}^{-1}\right] \\
&= \sum_{k=1}^{N-1} \ell'(f_{kk+1}) \delta g_k g_k^{-1} g_k g_{k+1}^{-1} - \sum_{r=1}^{N-1} \ell'(f_{r-1r}) g_{r-1} g_r^{-1} \delta g_r g_r^{-1} \\
&= \sum_{k=1}^{N-1} \left(\ell'(f_{kk+1}) TR_{f_{kk+1}} - \ell'(f_{k-1k}) TL_{f_{k-1k}}\right) \delta g_k g_k^{-1} \\
&= \sum_{k=1}^{N-1} \left(R^*_{f_{kk+1}} \ell'(f_{kk+1}) - L^*_{f_{k-1k}} \ell'(f_{k-1k})\right) \vartheta_k
\end{aligned}$$

where $\vartheta_k \equiv \delta g_k g_k^{-1}$ and we have used discrete integration by parts and the fact that $\delta g_0 = \delta g_N = 0$. Using the arbitrariness of ϑ_k, we obtain the discrete Euler-Poincare equations (3.10) for all variations of this form. □

Remark 3.2.1. In the case that $\mathbb{L}$ is left invariant, the discrete Euler-Poincaré equations take the form

$$L^*_{f_{k+1k}} \ell'(f_{k+1k}) - R^*_{f_{kk-1}} \ell'(f_{kk-1}) = 0 \tag{3.11}$$

where $f_{k+1k} \equiv g_{k+1}^{-1} g_k$ is in the left quotient $(G \times G)/G$, and the operators act on variations of the form $\vartheta_k = g_k^{-1} \delta g_k$.

Structure preservation and reconstruction

Below we introduce the issue of structure preservation by reduced algorithms and postpone a more detailed discussion of it until Chapter 4. The exposition here appeals to general Poisson reduction theory, which can be applied both to continuous and to discrete settings, while Chapter 4 is concerned with a hands-on derivation of the reduced structure using Legendre type transformations as well as a groupoid-algebroid formalism. Both descriptions address the symplecticity of the reduced discrete flow and are, in some sense, complementary. Recall also Remark 3.1.2 regarding the local character of the discrete symplectic and Poisson structures.

We may associate to any C^1 function F on $G \times G$ its Hamiltonian vector field X_F satisfying $X_F \lrcorner\, \omega_{\mathbb{L}} = dF$. The symplectic structure $\omega_{\mathbb{L}}$ naturally defines a Poisson structure $\{\cdot,\cdot\}_{G\times G}$ on $G \times G$ by the relation

$$\{F, H\}_{G\times G} = \omega_{\mathbb{L}}(X_F, X_H). \tag{3.12}$$

Theorem 3.2.2. *If the action of G on $G \times G$ is proper, then the algorithm on G defined by the discrete Euler-Poincaré equations (3.10) preserves the induced Poisson structure $\{\cdot,\cdot\}_G$ on G given by*

$$\{f, h\}_G \circ \pi_d = \{f \circ \pi, h \circ \pi\}_{G\times G} \tag{3.13}$$

for any C^1 functions $f, h : (G \times G)/G \cong G \to \mathbb{R}$.

Proof. We saw in the previous section that the DEL algorithm preserves the symplectic structure $\omega_{\mathbb{L}}$ on $G \times G$; hence, by (3.12), the DEL algorithm preserves the Poisson structure on $G \times G$. Since the action of G on $G \times G$ is proper, the general Poisson reduction theorem (see, e.g., [40] and discussion in Chapter 2) states that the projection $\pi_d : G \times G \to G$ is a Poisson map.

By Theorem 3.2.1, the projection of the DEL algorithm,

$$\pi_d \circ (g_{k-1}, g_k) \mapsto \pi_d \circ (g_k, g_{k+1})$$

is equivalent to the DEP algorithm on G, $f_{k-1k} \mapsto f_{kk+1}$. Therefore, as the Poisson structure on G is induced by π_d and as π_d is Poisson, we have proven the theorem.

□

As we shall prove in the following theorem, reconstruction of the DEP algorithm (3.10) on G reproduces the DEL algorithm on $G \times G$.

Theorem 3.2.3. *The discrete Euler-Lagrange algorithm governed by $\mathbb{L}$ and the discrete Euler-Poincaré algorithm governed by ℓ are related as follows. The canonical projection of a solution of DEL gives a solution of DEP, while the reconstruction of a solution of the DEP equations results in a solution of the DEL equations.*

Proof. The first assertion follows by construction. For the second assertion, using the definition $f_{kk+1} = g_k g_{k+1}^{-1}$, the DEL algorithm can be reconstructed from DEP algorithm by

$$(g_{k-1}, g_k) \mapsto (g_k, g_{k+1}) = (f_{k-1k}^{-1} \cdot g_{k-1}, f_{kk+1}^{-1} \cdot g_k), \tag{3.14}$$

where f_{kk+1} is the solution of (3.10). Indeed, $f_{kk+1}^{-1} \cdot g_k$ is precisely g_{k+1}. Thus, at each increment, one needs only to compute $f_{kk+1}^{-1} \cdot g_k$ since $g_k = f_{k-1k}^{-1} \cdot g_{k-1}$ is already known.

Similarly, one shows that in the case of a left G action, the reconstruction of the DEP equations (3.11) is given by

$$(g_{k-1}, g_k) \mapsto (g_k, g_{k+1}) = (g_{k-1} \cdot f_{kk-1}^{-1}, g_k \cdot f_{k+1k}^{-1}). \tag{3.15}$$

□

Let us denote by $\overline{\pi}$ the quotient map $\overline{\pi} : TG \to TG/G \cong \mathfrak{g}$ mapping $(g, \dot{g})$ to $\dot{g}g^{-1} \in \mathfrak{g}$. In the limit as the time step $\Delta t \to 0$, the discrete action sum converges to the action integral, and the DEL algorithm converges to the flow of the EL equations.

We denote the reconstruction of the flow of the Euler-Lagrange equations from the flow of the Euler-Poincaré equations by $\mathfrak{R}_{EP}$. Similarly, we denote the reconstruction of the DEL algorithm from the DEP algorithm provided by Theorem 3.2.3 by $\mathfrak{R}_{DEP}$. The following non-commutative diagram shows these relations.

$$
\begin{array}{ccccccc}
G \times G & \overset{\Delta t \to 0}{\longrightarrow} & TG & \qquad & DEL & \overset{\Delta t \to 0}{\longrightarrow} & EL \\
\downarrow{\scriptstyle \pi_d} & & \downarrow{\scriptstyle \overline{\pi}} & & \uparrow{\scriptstyle \mathfrak{R}_{DEP}} & & \uparrow{\scriptstyle \mathfrak{R}_{EP}} \\
G & & \mathfrak{g} & & DEP & & EP
\end{array}
$$

where $G \times G \to TG$ as $\Delta t \to 0$ in the following sense. Locally, $G \times G = F\mathbb{L}^*(T^*G)$ and as $\Delta t \to 0$, $F\mathbb{L} \to FL$ which pulls-back T^*G to TG. Thus, the DEP algorithm approximates the flow of the Euler-Poincaré equations if properly interpreted by means of reconstruction.

Discrete Lie-Poisson algorithm

In addition to reconstructing the dynamics on $G \times G$, we may use the coadjoint action to form a discrete Lie-Poisson algorithm approximating the dynamics on $\mathfrak{g}^*$. Recall (see, e.g., [4]) that in the Lie-Poisson reduction setting, for the momentum $m \in T_g^*G$ corresponding to the velocity vector $\dot{g} \in T_gG$, we define

$$m_c = L_g^* m \in \mathfrak{g}^*, \qquad m_s = R_g^* m \in \mathfrak{g}^*$$

to be the *body* and *spatial* momentum vectors, respectively, with the relation

$$m_s = \mathrm{Ad}^*_{g^{-1}} m_c.$$

For a right invariant system, the first Euler theorem states that $(d/dt)m_c = 0$, so that the body momentum is a constant of the motion. For convenience, we denote the constant m_c by μ_0 and $m_s(t)$ by $\mu(t)$ so that

$$\mu(t) = \mathrm{Ad}^*_{g^{-1}(t)} \cdot \mu_0. \tag{3.16}$$

Now, let $\mathcal{O} \subset \mathfrak{g}$ be a coadjoint orbit; that is, the orbit of a point under the coadjoint action of G on $\mathfrak{g}^*$. Then $\mathcal{O}$ is a symplectic manifold with unique Kirillov-Kostant forms $\omega_\pm$ as the coadjoint orbit symplectic structures.

Recall from Chapter 2 that for any $g \in G$, $\mathrm{Ad}^*_{g^{-1}} : \mathcal{O} \to \mathcal{O}$ preserves $\omega_\pm$. On the other hand, there are natural Lie-Poisson $\{\cdot,\cdot\}_\pm$ structures on $\mathfrak{g}^*$ (coming from Lie-Poisson reduction on T^*G) which induce ($\pm$) symplectic forms on each symplectic leaf in $\mathfrak{g}^*$. These induced symplectic structures coincide with the coadjoint orbit symplectic structures on each coadjoint orbit (see Kostant [24]); hence, the coadjoint action preserves the Lie-Poisson structures.

Using the evolution equation (3.16) along with the sequence $\{f_{kk+1}\}$ obtained by the DEP algorithm, we find that

$$\mu_{k+1} = \mathrm{Ad}^*_{g^{-1}_{k+1}}\, \mu_0 = \mathrm{Ad}^*_{(f^{-1}_{kk+1}\cdot g_k)^{-1}}\, \mu_0 = \mathrm{Ad}^*_{f_{kk+1}} \cdot \mathrm{Ad}^*_{g_k^{-1}}\, \mu_0 = \mathrm{Ad}^*_{f_{kk+1}}\, \mu_k.$$

Thus, we have proven the following

Proposition 3.2.1. *An algorithm, called the* ***discrete Lie-Poisson*** *(DLP) algorithm, on $\mathfrak{g}^*$ defined along the sequence $\{f_{kk+1}\}$ provided by the DEP algorithm on G and given by*

$$\mu_{k+1} = \mathrm{Ad}^*_{f_{kk+1}} \cdot \mu_k \tag{3.17}$$

is Lie-Poisson, i.e., it preserves the (+) Lie-Poisson structure on $\mathfrak{g}^*$.

Remark 3.2.2. The corresponding discrete Lie-Poisson equations for the left invariant system is given by[1]

$$\Pi_{k+1} = \mathrm{Ad}^*_{f^{-1}_{k+1k}} \cdot \Pi_k, \tag{3.18}$$

where $\Pi_k := \mathrm{Ad}^*_{g_k}\, \pi_0$ and the reduced variable $m_c(t)$ is denoted by $\Pi(t)$ and the constant m_s by π_0.

[1]We reserve the notation $\mu \in \mathfrak{g}^*$ for the *right* invariant system and $\Pi \in \mathfrak{g}^*$ for the *left.*

Thus, one can obtain a Lie-Poisson integrator by solving (3.10) for f_{kk+1} and then substituting it into (3.17) to generate the algorithm. This algorithm manifestly preserves the coadjoint orbits and hence the Poisson structure on $\mathfrak{g}^*$. We demonstrate in Section 4.2 of the next chapter that the DEP/DLP algorithms can be thought of as generators of Lie-Poisson Hamilton-Jacobi equations obtained by Ge and Marsden [15]. Also, in Section 5.2, we show that this approach recovers the Moser-Veselov equations for generalized rigid-body dynamics on SO(n).

3.3 Discretization using natural charts

As was stated in the beginning of Section 3.1, discrete Lagrangians $\mathbb{L}$ are typically obtained from the corresponding continuous Lagrangians L by means of some approximation of current configuration and velocity (see (3.1)). In this section, we use the group exponential map at the identity, $\exp_e : \mathfrak{g} \to G$, to construct an appropriate discrete Lagrangian.

For finite dimensional Lie groups G, $\exp_e$ is locally a diffeomorphism and thus provides a natural chart. Namely, there exists an open neighborhood U of $e \in G$ such that $\exp_e^{-1} : U \to \mathfrak{u} \equiv \exp_e^{-1}(U)$ is a C^∞ diffeomorphism (this is not in general true for infinite dimensional groups). Hence, the manifold structure is provided by right translation, so that a chart at $g \in G$ is given by

$$\psi_g = \exp_e^{-1} \circ R_{g^{-1}}. \tag{3.19}$$

We now define the *discrete Lagrangian*, $\mathbb{L} : G \times G \to \mathbb{R}$, by

$$\mathbb{L}(g_1, g_2) = L\left(\psi_g^{-1}\left[\frac{\psi_g(g_1) + \psi_g(g_2)}{2}\right], (\psi_g^{-1})_*\left[\frac{\psi_g(g_2) - \psi_g(g_1)}{\Delta t}\right]\right), \tag{3.20}$$

where $\Delta t \in \mathbb{R}_+$ is the given time step and $g_1, g_2 \in U_g \equiv R_g(U)$.

We shall assume that G has a right invariant Riemannian metric $\langle \cdot, \cdot \rangle$ obtained by right translating a positive bilinear form on $\mathfrak{g}$ over the entire group. We also assume that G has a regular quadratic Lie algebra, as in [15].

For $K \subset G$ a compact set, we define the Riemannian distance function, dist : $K \times K \to \mathbb{R}^+$ by

$$\operatorname{dist}(g_1, g_2) = \int_0^1 \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt,$$

where $\gamma : [0,1] \to G$ is the geodesic with $\gamma(0) = g_1$ and $\gamma(1) = g_2$. It is then clear that $\operatorname{diam}(U) = \operatorname{diam}(U_g)$ for all $g \in G$, so in order for (3.20) to be well defined we require that $\operatorname{dist}(g_1, g_2) < \operatorname{diam}(U)$. In other words, we require that (g_1, g_2) be close to the diagonal in $G \times G$. Our restriction on $\operatorname{dist}(g_1, g_2)$ in turn places a restriction on the timestep Δt.

Next, let

$$\eta = \frac{\psi_g(g_1) + \psi_g(g_2)}{2},$$

with corresponding group element

$$g' = \exp(\eta) \in U.$$

We denote the algebra element approximating the velocity $g^{-1}\dot{g}$ by

$$\zeta = \frac{\psi_g(g_2) - \psi_g(g_1)}{\Delta t}.$$

Using the standard formula for the derivative of the exponential (see, for example, Dragt and Finn [12] or Channel and Scovel [10]) given by

$$T_\eta \exp = T_e R_{g'} \cdot \operatorname{iex}(-\operatorname{ad}_\eta), \quad \eta \in \mathfrak{g}, \quad g' = \exp(\eta) \in U,$$

where iex is the function defined by

$$\operatorname{iex}(w) = \sum_{n=0}^{\infty} \frac{w^n}{(n+1)!}, \tag{3.21}$$

we may evaluate the push-forward of ψ_g^{-1} at η. We obtain the following expression for the discrete Lagrangian

$$\mathbb{L}(g_1, g_2) = L\left(\psi_g^{-1}(\eta), T_{g'}R_g \cdot T_e R_{g'} \cdot \mathrm{iex}(-\operatorname{ad}_\eta)(\zeta)\right).$$

Setting $q \equiv \psi_g^{-1}(\eta) = R_g g'$, the last formula is expressed as

$$\mathbb{L}(g_1, g_2) = L\left(q, T_e R_q \cdot \mathrm{iex}(-\operatorname{ad}_\eta)(\zeta)\right), \tag{3.22}$$

so that locally the Lagrangian is evaluated at the base point $q = \psi_g^{-1}(\eta) \in U_g \subset G$, and the Lie algebra (fiber) element $\mathrm{iex}(-\operatorname{ad}_\eta)(\zeta)$ is right translated to the tangent space at the point q, T_qG; as $\Delta t \to 0$, this fiber element converges to the group velocity $\dot{g} \in T_gG$.

The following lemma establishes that the discrete Lagrangian $\mathbb{L}$ inherits the G-invariance property from the original Lagrangian L, so that the discrete counterpart of the Euler-Poincare reduction is well-defined.

Lemma 3.3.1. *The discrete Lagrangian $\mathbb{L} : G \times G \to \mathbb{R}$ is right (left) invariant under the diagonal action of G on $G \times G$, whenever $L : TG \to \mathbb{R}$ is right (left) invariant.*

Proof. We fix the right action and consider $R_{\bar{g}}^*(\mathbb{L})$ for some $\bar{g} \in G$. By construction, $R_{\bar{g}}g_1, R_{\bar{g}}g_2 \in R_{\bar{g}}(U_g)$, whenever $g_1, g_2 \in U_g \equiv R_g(U)$, so that the chart is given by $\psi_{g\bar{g}} = \exp_e^{-1} \circ R_{(g\bar{g})^{-1}}$.

By definition, both η and ζ are always elements of a neighborhood of $0 \in \mathfrak{g}$, so it is clear that they are right invariant. Hence, using the explicit form of the chart $\psi_{g\bar{g}}$ together with the right invariance of the Lagrangian L, we obtain from (3.20) and (3.22) that

$$\begin{aligned}
R_{\bar{g}}^*\mathbb{L}(g_1, g_2) &= L\left(\psi_{g\bar{g}}^{-1}\left[\frac{\psi_{g\bar{g}}(g_1\bar{g}) + \psi_{g\bar{g}}(g_2\bar{g})}{2}\right], (\psi_{g\bar{g}})_*\left[\frac{\psi_{g\bar{g}}(g_2\bar{g}) - \psi_{g\bar{g}}(g_1\bar{g})}{\Delta t}\right]\right) \\
&= L\left(R_{\bar{g}} \cdot \psi_g^{-1}(\eta), T_qR_{\bar{g}} \cdot T_{g'}R_g \cdot T_eR_{g'} \cdot \mathrm{iex}(-\operatorname{ad}_\eta)(\zeta)\right) \\
&= L\left(R_{\bar{g}} \cdot q,\ T_qR_{\bar{g}} \cdot T_eR_q \cdot \mathrm{iex}(-\operatorname{ad}_\eta)(\zeta)\right) \\
&= \mathbb{L}(g_1, g_2).
\end{aligned}$$

In the case that the group action is on the left, we use $\phi_g = \exp_e^{-1} \circ L_{g^{-1}}$ as the chart, and proceed with the same argument. □

Corollary 3.3.1. *Using the discretization defined by (3.20), the reduced discrete Lagrangian ℓ determined by the projection map (3.9) and given by $\ell(g_1 g_2^{-1}) = \mathbb{L}(g_1, g_2)$ can be expressed in terms of the continuous reduced Lagrangian l by*

$$\ell(g_1 g_2^{-1}) = l(\operatorname{iex}(-\operatorname{ad}_\eta)(\zeta)), \tag{3.23}$$

where $\eta = (\psi_g(g_1) + \psi_g(g_2))/2$, $\zeta = (\psi_g(g_2) - \psi_g(g_1))/\Delta t$, and l can be defined by translation to the identity of the arguments of the right invariant Lagrangian L, i.e., $l(\xi) = L(R_{g^{-1}} g, TR_{g^{-1}} \dot{g}) = L(e, \xi)$, where $\xi = TR_{g^{-1}} \dot{g} \in \mathfrak{g}$.

The proof of this corollary follows from expression (3.22), and the fact that the Lagrangian L is right invariant so that translation by q^{-1} to e gives (3.23).

The expressions (3.22) and (3.23) for the discrete Lagrangian in general require evaluation of the infinite series for the iex function given by (3.21); however, a simplification occurs when g is set to either g_k or g_{k+1}. This is due to the fact that when $g = g_k$ or $g = g_{k+1}$, one may easily verify that $\operatorname{ad}_\zeta \eta := [\zeta, \eta] = 0$, and hence that $\operatorname{iex}(-\operatorname{ad}_\eta)(\zeta) = \zeta$.

For example, with $g = g_{k+1}$, the discrete Lagrangian is simply

$$\mathbb{L}(g_k, g_{k+1}) = L\left(q, T_e R_q(\zeta)\right), \tag{3.24}$$

where

$$\eta = \frac{1}{2} \log(g_k g_{k+1}^{-1}),\ q \equiv \psi_{g_{k+1}}(\eta) = (g_k g_{k+1})^{1/2},\ \zeta = \frac{1}{\Delta t} \log(g_k g_{k+1}^{-1})$$

and $\log \equiv \exp^{-1}$. Consequently, the reduced discrete Lagrangian is given by

$$\ell(f_{kk+1}) = l(\log(f_{kk+1})/\Delta t), \tag{3.25}$$

where $f_{kk+1} = g_k g_{k+1}^{-1}$.

Substituting the discrete Lagrangian (3.25) into the DEP equation (3.10), we obtain the following implicit algorithm on the Lie algebra

$$l'(\xi_{kk+1}/\Delta t) \cdot \chi(\mathrm{ad}_{\xi_{kk+1}}) = l'(\xi_{k-1k}/\Delta t) \cdot \chi(\mathrm{ad}_{\xi_{k-1k}}) \cdot \exp(\mathrm{ad}_{\xi_{k-1k}}), \qquad (3.26)$$

where $\xi_{kk+1} \equiv \log f_{kk+1} \in \mathfrak{g}$ and the function χ is defined to be the inverse of the function iex defined by (3.21), $\chi(\mathrm{ad}_\xi) \cdot \mathrm{iex}(-\mathrm{ad}_\xi) = \mathrm{Id}_\mathfrak{g}$. The function χ in (3.26) arises from taking the derivative of the log function viewed as a map from the Lie group to its algebra. It is interesting to compare the above algorithm with the one obtained by Channel and Scovel [10] using the Hamilton-Jacobi equation (see [36]).

Chapter 4

Poisson Structure and Invariant Manifolds on Lie Groups

Discretization of an Euler-Poincaré system on TG results in a system on $G \times G$ defined by a Lagrangian $\mathbb{L}$. If it is regular, the Legendre transformation (in the sense of Veselov) $F\mathbb{L}$ defines a symplectic form (and, hence, a Poisson structure) on $\mathcal{V} \subset G \times G$ via the pull-back of the canonical form from T^*G. Then, general Poisson reduction applied to this discrete setting defines a Poisson structure on the reduced space $\mathcal{U} = \pi_d(\mathcal{V}) \subset G$. This approach was described in Theorem 3.2.2 of the previous chapter.

Alternatively, without appealing to the reduction procedure, a Poisson structure on a Lie group can be defined using ideas of Weinstein [52] on Lagrangian mechanics on groupoids and their algebroids. The key idea can be summarized in the following statements. A smooth function on a groupoid defines a natural (Legendre type) transformation between the groupoid and the dual of its algebroid. This transformation can be used to pull back a canonical Poisson structure from the dual of the algebroid, provided the regularity conditions are satisfied.

The ideas outlined in this chapter can be easily expressed using the groupoid-algebroid formalism. Such a formalism is suited to the discrete gauge field theory generalization as well as to discrete semidirect product theory; nevertheless, the theory of groupoids and algebroids is not essential for the derivations, but rather contributes nicely to the elegance of the exposition.

4.1 Dynamics on groupoids and algebroids

We briefly summarize results from Weinstein [52] and refer the reader to the original paper for details of proofs and definitions. Let Γ be a groupoid over a set M, with $\alpha, \beta : \Gamma \to M$ being its source and target maps, with a multiplication map $m : \Gamma_2 \to \Gamma$, where $\Gamma_2 \equiv \{(g,h) \in \Gamma \times \Gamma \mid \beta(g) = \alpha(h)\}$. Denote its corresponding algebroid by $\mathcal{A}$.

The Lie groupoids relevant to our exposition are the Cartesian product $G \times G$ of a Lie group G, with multiplication $(g,h)(h,k) = (g,k)$, and the group G itself. The corresponding algebroids are the tangent bundle TG and the Lie algebra $\mathfrak{g}$, respectively. The dual bundle to a Lie algebroid carries a natural Poisson structure. This is the Poisson bracket associated to the canonical symplectic form on T^*G and the Lie-Poisson structure on $\mathfrak{g}^*$, respectively.

Lagrangian mechanics on a groupoid Γ is defined as follows. Let $\mathcal{L}$ be a smooth, real-valued function on Γ, and let $\mathcal{L}_2$ be the restriction to Γ_2 of the function $(g,h) \mapsto \mathcal{L}(g) + \mathcal{L}(h)$.

Definition 4.1.1. *Let $\Sigma_{\mathcal{L}} \subset \Gamma_2$ be the set of critical points of $\mathcal{L}_2$ along the fibers of the multiplication map m; i.e., the points in $\Sigma_{\mathcal{L}}$ are stationary points of the function $\mathcal{L}(g) + \mathcal{L}(h)$ when g and h are restricted to admissible pairs with the constraint that the product gh is fixed [52].*

A ***solution of the Lagrange equations*** for the Lagrangian function $\mathcal{L}$ is a sequence $\ldots, g_{-2}, g_{-1}, g_0, g_1, g_2, \ldots$ of elements of Γ, defined on some "interval" in $\mathbb{Z}$, such that $(g_j, g_{j+1}) \in \Sigma_{\mathcal{L}}$ for each j.

The Hamiltonian formalism for discrete Lagrangian systems is based on the fact that each Lagrangian submanifold of a symplectic groupoid determines a Poisson automorphism on the base Poisson manifold. Recall that the cotangent bundle $T^*\Gamma$ is, in addition to being a symplectic manifold, a groupoid itself, the base being $\mathcal{A}^*$; notice that both manifolds are naturally Poisson. The source and target mappings $\tilde{\alpha}, \tilde{\beta} : T^*\Gamma \to \mathcal{A}^*$ are induced by α and β.

Definition 4.1.2. *Given any smooth function $\mathcal{L}$ on Γ, a Poisson map $\Lambda_{\mathcal{L}}$ from $\mathcal{A}^*$ to itself, which may be said to be generated by $\mathcal{L}$, is defined by the Lagrangian submanifold $d\mathcal{L}(\Gamma)$ (under a suitable hypothesis of nondegeneracy) [52].*

The appropriate "Legendre transformation" $F\mathcal{L}$ in the groupoid context is given by $\tilde{\alpha} \circ d\mathcal{L} : \Gamma \to \mathcal{A}^*$ or $\tilde{\beta} \circ d\mathcal{L} : \Gamma \to \mathcal{A}^*$, depending on whether we consider right or left invariance (through the definition of maps $\tilde{\alpha}$ and $\tilde{\beta}$). The transformation $F\mathcal{L}$ relates the mapping on Γ defined by $\Sigma_{\mathcal{L}}$ with the mapping $\Lambda_{\mathcal{L}}$ on $\mathcal{A}^*$. $F\mathcal{L}$ also pulls back the Poisson structure from $\mathcal{A}^*$ to Γ, which, in general, is defined only *locally* on some neighborhood $\mathcal{U} \subset \Gamma$. In the context of a Lie group, this means that any regular function $\ell : G \to \mathbb{R}$ defines a Poisson structure on $\mathcal{U}$. We shall address this issue in the next section.

4.2 Generating Lie-Poisson dynamics

There are various ways to generate discrete Hamiltonian dynamics, besides the most obvious discretization of the Hamiltonian formalism on T^*G, making use of the rich structure of Lie-Poisson systems. For instance, Ge and Marsden [15] obtained a discrete algorithm using the Lie-Poisson Hamilton-Jacobi equations, while we constructed the DLP algorithm (3.17) using the discrete Legendre transformation. It is instructive to compare such discrete algorithms.

Below we review results obtained in [15] and establish various links to the DEP/DLP algorithms. The groupoid-algebroid formalism described in the previous chapter seems to be the most elegant way of generating a discrete Lie-Poisson dynamics on $\mathfrak{g}^*$ and relating it to the discrete Euler-Poincaré dynamics on G. This relation enables us to put a Poisson structure on the Lie group G and to establish a correspondence between dynamically invariant manifolds of the corresponding continuous and discrete systems. These issues are addressed in Section 4.3.

The Lie-Poisson Hamilton-Jacobi route

We now state results from [15] which were obtained for the *left* action of a group G on itself. Let H be a G-invariant Hamiltonian on T^*G and let H_L be the corresponding left reduced Hamiltonian on $\mathfrak{g}^*$. If a generating function $S : G \times G \to \mathbb{R}$ of canonical transformations is invariant, then there exists a unique function S_L such that $S_L(g^{-1}g_0) = S(g, g_0)$.

The left reduced Hamilton-Jacobi equation for the function $S_L : G \to \mathbb{R}$ is given by

$$\frac{\partial S_L}{\partial t} + H_L(-TR_g^* \cdot dS_L(g)) = 0, \tag{4.1}$$

and is called the ***Lie-Poisson Hamilton-Jacobi*** equation. The Lie-Poisson flow of the Hamiltonian H_L is generated by its solution S_L; in particular, the flow $t \mapsto F_t$ of S_L taking initial data Π_0 to $\Pi(t)$ is Poisson for each t in the domain of definition. Next, one defines $g \in G$ as the solution of

$$\Pi_0 = -TL_g^* \cdot D_g S_L \tag{4.2}$$

and then sets

$$\Pi = \mathrm{Ad}^*_{g^{-1}} \Pi_0. \tag{4.3}$$

Thus, one obtains a Lie-Poisson integrator by approximately solving (4.1), and then using (4.2) and (4.3) to generate the algorithm.

Note that (4.1) is the analogue of the usual Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}\right) = 0$$

and that (4.2) and (4.3) are the analogues of the corresponding canonical transformations generated by a solution S which in a local chart are given by

$$p_{0i} = -\frac{\partial S}{\partial q_0^i} \qquad p_i = \frac{\partial S}{\partial q^i}.$$

It is interesting to compare the approach using the Lie-Poisson Hamilton-Jacobi equation (4.1) with that using the discrete Euler-Lagrange equations. The choice of discrete Lagrangian ℓ may be viewed as a choice of approximate solution to the Hamilton-Jacobi equation. Then the steps of solving (4.2) and (4.3) are parallel to the solution of equations (3.10) and (3.17). Namely, the DLP equation provides a time evolution map $\mu_k \mapsto \mu_{k+1}$ on $\mathfrak{g}^*$ using a *known* solution f_{kk+1}, while (4.3) advances the initial value Π_0 along the coadjoint orbit and requires at each time step the solution g of (4.2) that approximates the current "position" $g(t)$.

The groupoid route

A Lie group G is the simplest example of a groupoid, with the base being just a point. Its algebroid is the corresponding Lie algebra $\mathfrak{g}$, with the dual being $\mathfrak{g}^*$. Consider left invariance and let a general function $\mathcal{L}$ on the group be specified by the discrete reduced Lagrangian $\ell : G \to \mathbb{R}$. Then, the Legendre transform $F\mathcal{L}$ defined above is given by

$$F\ell = L_g^* \circ d\ell \ : G \to \mathfrak{g}^*,$$

where $d\ell : G \to T^*G$. Using these transformations we define

$$\Pi_{k-1} \equiv F\ell(f_{kk-1}) = L^*_{f_{kk-1}} \circ d\ell(f_{kk-1}).$$

Recall the DEP equation (3.11) for left-invariant systems:

$$L^*_{f_{k+1k}} d\ell(f_{k+1k}) - R^*_{f_{kk-1}} d\ell(f_{kk-1}) = 0,$$

where we have identified the notations ℓ' and $d\ell$. The latter equation can be rewritten as a system

$$\begin{cases} \Pi_k = L_f^* \circ d\ell(f), \\ \Pi_{k+1} = R_f^* \circ d\ell(f), \end{cases} \tag{4.4}$$

where the first equation is to be solved for f (which stands for f_{k+1k}) which then is substituted into the second equation to compute Π_{k+1}.

This system is precisely the Lie-Poisson Hamilton-Jacobi system with the reduced discrete Lagrangian ℓ playing the role of the generating function. This means that there is no need to find an approximate solution of the reduced Hamilton-Jacobi equation (4.1). Notice also that the DLP equation (3.18) is a direct consequence of the system (4.4):

$$\Pi_{k+1} = \mathrm{Ad}^*_{f^{-1}_{k+1k}} \cdot \Pi_k.$$

The following diagrams relate the dynamics on G and on $\mathfrak{g}^*$:

$$\begin{array}{ccc} G & \xrightarrow{\Sigma_\ell} & G \\ \downarrow{\scriptstyle F\ell} & & \downarrow{\scriptstyle F\ell} \\ \mathfrak{g}^* & \xrightarrow{\Lambda_\ell} & \mathfrak{g}^* \end{array} \qquad \begin{array}{ccc} f_{kk-1} & \xrightarrow{\Sigma_\ell} & f_{k+1k} \\ \downarrow{\scriptstyle F\ell} & & \downarrow{\scriptstyle F\ell} \\ \Pi_{k-1} & \xrightarrow{\Lambda_\ell} & \Pi_k, \end{array} \tag{4.5}$$

where Σ_ℓ and Λ_ℓ are given in Definitions 4.1.1 and 4.1.2.

4.3 Some advantages of structure-preserving integrators

As we mentioned above, the "Legendre transform" $F\ell$ allows us to put a Poisson structure on the Lie group G, which, of course, depends on the discrete Lagrangian $\mathbb{L}$ on $G \times G$, and hence on the original Lagrangian L on TG (if we consider this from the discrete reduction point of view). It follows that the reduction of the discrete Euler-Lagrange dynamics on $G \times G$ is necessarily restricted to the symplectic leaves of this Poisson structure, so that these leaves are invariant manifolds, and correspond (under $F\ell$) to the symplectic leaves (coadjoint orbits) of the continuous reduced system on $\mathfrak{g}^*$.

These ideas are the content of the following theorems.

Theorem 4.3.1. *Let L be a right invariant Lagrangian on TG and let $\mathbb{L}$ be the Lagrangian of the corresponding discrete system on $\mathcal{V} \subset G \times G$. Assume that $\mathbb{L}$ is regular, in the sense that the Legendre transformation $F\mathbb{L} : \mathcal{V} \to F\mathbb{L}(\mathcal{V}) \subset T^*G$ is a local diffeomorphism, and let the quotient maps be given by*

$$\pi_d : G \times G \to (G \times G)/G \cong G \quad \textit{and} \quad \pi : T^*G \to (T^*G)/G \cong \mathfrak{g}^*.$$

Let ℓ be the reduced Lagrangian on G defined by

$$\mathbb{L} = \ell \circ \pi_d,$$

and let

$$F\ell : \mathcal{U} \subset G \to \mathfrak{g}^*$$

be the corresponding Legendre transform. Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{V} \subset G \times G & \xrightarrow{F\mathbb{L}} & T^*G \\ \downarrow{\scriptstyle \pi_d} & & \downarrow{\scriptstyle \pi} \\ \mathcal{U} \subset G & \xrightarrow{F\ell} & \mathfrak{g}^*. \end{array} \tag{4.6}$$

Proof. First, we choose coordinate systems on each space. Let $(g_k, g_{k+1}) \in G \times G$ and $(g, p) \in T^*G$, so that the discrete quotient map (3.9) is given by $\pi_d : (g_k, g_{k+1}) \mapsto f_{kk+1} = g_k g_{k+1}^{-1}$, and the continuous quotient map by $\pi : (g, p) \mapsto \mu = R_g^* p$. Recall that the fiber derivative $F\mathbb{L}$ in these coordinates has the following form

$$F\mathbb{L} : G \times G \to T^*G; \quad (g_k, g_{k+1}) \mapsto (g_k, D_1\mathbb{L}(g_k, g_{k+1})).$$

Then the above diagram is given by:

$$\begin{array}{ccc} (g_k, g_{k+1}) & \xrightarrow{F\mathbb{L}} & (g_k, p_k = \dfrac{\partial \mathbb{L}}{\partial g_k}) \\ \Big\downarrow{\scriptstyle \pi_d} & & \Big\downarrow{\scriptstyle \pi} \\ f = R_{g_{k+1}^{-1}} g_k & & \mu = R^*_{g_k} p_k, \end{array} \tag{4.7}$$

where f stands for $f_{kk+1} = g_k g_{k+1}^{-1}$. To close this diagram and to verify the arrow determined by $F\ell$, compute the derivative of $\mathbb{L}$ using the chain rule to obtain

$$\mu = R^*_{g_k} p_k = R^*_{g_k} \frac{\partial(\ell \circ \pi)}{\partial g_k} = R^*_{g_k} \left(R^*_{g_{k+1}^{-1}} \frac{\partial \ell}{\partial f} \right) = R^*_f \frac{\partial \ell}{\partial f} = R^*_f \circ \ell'(f), \tag{4.8}$$

where we have used that, according to the definition of f, the partial derivative $\dfrac{\partial f}{\partial g_k}$ is given by the linear operator $TR_{g_{k=1}^{-1}}$. Equation (4.8) is precisely the Legendre transformation $F\ell$ for a right invariant system (see the previous section). □

Corollary 4.3.1. *Reconstruction of the discrete Lie-Poisson (DLP) dynamics on $\mathfrak{g}^*$ by π^{-1} corresponds to the image of the discrete Euler-Lagrange (DEL) dynamics on $G \times G$ under the Legendre transformation $F\mathbb{L}$ and results in an algorithm on T^*G approximating the continuous flow of the corresponding Hamiltonian system.*

Proof. The proof follows from the results of the previous section. In particular, diagram (4.5) relates the DLP dynamics on $\mathfrak{g}^*$ with the DEP dynamics on $\mathcal{U} \subset G$ which, in turn, is related to the DEL dynamics on $\mathcal{V} \subset G \times G$ via the reconstruction (3.14). □

An important remark on this corollary which follows from the results in [20] (see also [21]) is that, in general, to obtain a corresponding algorithm on the Hamiltonian side which is consistent with the corresponding continuous Hamiltonian system on T^*G, one must use the time step dependent Legendre transform given by the map

$$(g_k, g_{k+1}) \mapsto (g_k, \Delta t \cdot D_1 \mathbb{L}(g_k, g_{k+1})).$$

The results here are not effected, however, as we assume Δt to be constant and so we would simply add a constant multiplier to the corresponding symplectic and Poisson structures. For variable time-stepping algorithms, this remark is crucial and must be taken into account.

Theorem 4.3.2. *The Poisson structure on the Lie group G obtained by reduction of the Lagrange symplectic form $\omega_{\mathbb{L}}$ on $\mathcal{V} \subset G \times G$ via π_d coincides with the Poisson structure on $\mathcal{U} \subset G$ obtained by the pull-back of the Lie-Poisson structure ω_μ on $\mathfrak{g}^*$ by the Legendre transformation $F\ell$ (see diagram (4.6) above).*

Proof. The proof is based on the commutativity of diagram (4.6) and the G invariance of the unreduced symplectic forms. Notice that G and $\mathfrak{g}^*$ in (4.6) are Poisson manifolds, each being foliated by symplectic leaves, which we denote Σ_f and $\mathcal{O}_\mu$ for $f \in G$ and $\mu \in \mathfrak{g}^*$, respectively. Denote by ω_f and ω_μ the corresponding symplectic forms on these leaves. Below we shall prove the compatibility of these structures under the diagram (4.6). Repeating this proof leaf by leaf then establishes the equivalence of the Poisson structures and proves the theorem.

Recall that the Lagrange 2-form $\omega_{\mathbb{L}}$ on $\mathcal{V} \subset G \times G$ derived from the variational principle coincides with the pull-back of the canonical 2-form ω_{can} on T^*G. Recall also that for a right-invariant system, reduction of T^*G to $\mathfrak{g}^*$ is given by right translation to the identity $e \in G$, i.e., any $p \in T^*_g G$ is mapped to $\mu = R^*_g p \in \mathfrak{g}^* \cong T^*_e G$. Thus, for any $g \in \pi^{-1}(\mu)$, where $\mu \in \mathfrak{g}^*$,

$$\pi^{-1}|_{T^*G} = R^*_{g^{-1}} : \mathfrak{g}^* \to T^*_g G,$$

so that $(\pi^{-1})^* = (R^*_{g^{-1}})^*$ pulls back ω_{can} to ω_μ. Henceforth, π^{-1} shall denote the inverse map of π restricted to $T_g G^*$.

Let us write down, using the above notations, how the symplectic forms are mapped under the transformations in diagram (4.6); we see that

$$\begin{array}{ccc} \mathcal{V} \subset G \times G & \xrightarrow{F\mathbb{L}} & T^*G \\ \big\downarrow \pi_d & & \big\downarrow \pi \\ \mathcal{U} \subset G & \xrightarrow{F\ell} & \mathfrak{g}^* \end{array} \qquad \begin{array}{ccc} \omega_{\mathbb{L}} & \xleftarrow{F\mathbb{L}^*} & \omega_{\text{can}} \\ & & \big\uparrow (\pi^{-1})^* \\ \omega_f & \xleftarrow{F\ell^*} & \omega_\mu . \end{array} \tag{4.9}$$

Then, using the coordinate notations of diagram (4.7), $\forall\, f \in G$ and $u, v \in T_f\Sigma_f$,

$$\omega_f(f)(u,v) \equiv \omega_\mu(\mu)(TF\ell(u), TF\ell(v)), \tag{4.10}$$

where $\mu = F\ell(f) \in \mathfrak{g}^*$. Continuing this equation using diagram (4.9), we have that

$$\begin{aligned} \omega_f(f)(u,v) &= \omega_{\text{can}}((g_k, p_k))(T\pi^{-1} \circ TF\ell(u), T\pi^{-1} \circ TF\ell(v)) \\ &= \omega_{\mathbb{L}}((g_k, g_{k+1}))(TF\mathbb{L}^{-1} \circ T\pi^{-1} \circ TF\ell(u), TF\mathbb{L}^{-1} \circ T\pi^{-1} \circ TF\ell(v)), \end{aligned} \tag{4.11}$$

where $(g_k, p_k) \in \pi^{-1}(\mu)$ and $T\pi^{-1}$ denotes $TR^*_{g^{-1}}$.

Using (4.6), it follows that

$$F\ell \circ \pi_d = \pi \circ F\mathbb{L}$$

and, hence, for the tangent maps, we have that

$$TF\ell \circ T\pi_d = T\pi \circ TF\mathbb{L}.$$

So, if u, v in (4.10) are images of some G invariant vector fields U, V on $\mathcal{V} \subset G \times G$, i.e., $u = T\pi_d(U), v = T\pi_d(V)$, then from (4.11) it follows that

$$\omega_f(f)(u,v) = \omega_{\mathbb{L}}((g_k, g_{k+1}))(U, V),$$

where $(g_k, g_{k+1}) = \pi_d^{-1}(f)$ and $U, V \in T_{(g_k, g_{k+1})}G \times G$. The last equation precisely means that ω_f is the discretely reduced symplectic form, i.e., the image of $\omega_{\mathbb{L}}$ under the quotient map π_d. □

Analogous theorems hold for the case of left invariant systems.

Chapter 5

Example: Generalized Rigid Body

As an example of applications of the ideas in the previous chapters we consider the dynamics of a generalized rigid body with the configuration space being the group $\mathrm{SO}(n)$, as well as its associated reduction and discretization. In the last section we shall obtain an explicit form of the reduced discrete Poisson structure on $\mathrm{SO}(3)$ and analyze its symplectic leaves.

The Basic Set Up.

The configuration space of the system is $\mathrm{SO}(n)$. The corresponding Lagrangian is determined by a symmetric positive definite operator $J : \mathfrak{so}(n) \to \mathfrak{so}(n)$, defined by $J(\xi) = \Lambda\xi + \xi\Lambda$, where $\xi \in \mathfrak{so}(n)$ and Λ is a diagonal matrix satisfying $\Lambda_i + \Lambda_j > 0$ for all $i \neq j$. The left invariant metric on $\mathrm{SO}(n)$ is obtained by left translating the bilinear form at $e \in \mathrm{SO}(n)$ given by

$$(\xi, \xi) = \frac{1}{4}\operatorname{Tr}\left(\xi^T J(\xi)\right).$$

The operator J, viewed as a mapping $\mathcal{J} : \mathfrak{so}(n) \to \mathfrak{so}(n)^*$, has the usual interpretation of the inertia tensor, and the Λ_i correspond to the sums of certain principal moments of inertia.

The rigid body Lagrangian is the kinetic energy of the system

$$L(g, \dot{g}) = \frac{1}{4}\langle g^{-1}\dot{g}, \mathcal{J}(g^{-1}\dot{g})\rangle = \frac{1}{4}\langle \xi, \mathcal{J}\xi)\rangle = l(\xi), \tag{5.1}$$

where $\xi = g^{-1}\dot{g} \in \mathfrak{so}(n)$ and $\langle \cdot, \cdot \rangle$ is the pairing between the Lie group and its dual.

5.1 Natural charts discretization

We apply our DEP algorithm to the generalized rigid body problem. We discretize $T\,\mathrm{SO}(n)$ by $\mathrm{SO}(n) \times \mathrm{SO}(n)$ and construct the discrete Lagrangian following (3.24) as

$$\mathbb{L}(g_k, g_{k+1}) = L\left(q_{k+1k}, T_e L_{q_{k+1k}}(\zeta_{k+1k})\right),$$

where $q_{k+1k} = g_{k+1}(g_{k+1}^{-1} g_k)^{1/2}$ and $\zeta_{k+1k} = \frac{1}{h}\log(g_{k+1}^{-1} g_k)$. Using the left invariance of the metric, we may express the discrete rigid body Lagrangian as

$$\mathbb{L}(g_k, g_{k+1}) = \frac{1}{2}(\zeta_{k+1k}, \zeta_{k+1k}) = \frac{1}{2}\langle \zeta_{k+1k}, \mathcal{J}(\zeta_{k+1k})\rangle. \tag{5.2}$$

The Lagrangian for the reduced system on $(\mathrm{SO}(n) \times \mathrm{SO}(n))/\,\mathrm{SO}(n) \cong \mathrm{SO}(n)$ is then given by

$$\ell(f_{k+1k}) = \mathbb{L}(g_k, g_{k+1}) = \frac{1}{2h^2}\langle \log f_{k+1k}, \mathcal{J}(\log f_{k+1k})\rangle, \tag{5.3}$$

where $f_{k+1k} \equiv g_{k+1}^{-1} g_k \in \mathrm{SO}(n)$ is an element of the reduced space and h is the time step.

The DEP equation (3.11) has the following implicit form

$$\zeta_{k+1k} = \mathcal{J}^{-1}\left(\mathrm{iex}(-\operatorname{ad}^*_{h\zeta_{k+1k}}) \cdot \chi(\operatorname{ad}^*_{h\zeta_{kk-1}}) \cdot \operatorname{Ad}^*_{\exp(-h\zeta_{kk-1})} \mathcal{J}(\zeta_{kk-1})\right). \tag{5.4}$$

5.2 Moser-Veselov discretization

An alternative discretization approach may be taken if we first embed our group G into a linear space; for finite dimensional matrix groups, the linear ambient space is $\mathfrak{gl}(n)$. Then, summation of the group elements becomes a legitimate operation provided we project the result back onto the group G by using Lagrange multipliers.

Using the definition of J we rewrite the Lagrangian (5.1) in the following form:

$$L = \frac{1}{4}\operatorname{Tr}\left(\xi^T J(\xi)\right) = \frac{1}{2}\operatorname{Tr}\left(\xi^T \Lambda \xi\right).$$

We now discretize the Lie algebra elements $\xi = g^{-1}\dot{g}$ by

$$\xi \approx \frac{1}{h} g_{k+1}^T (g_{k+1} - g_k), \tag{5.5}$$

where h is the time step. Substituting (5.5) into the Lagrangian L (and using properties of the trace), we obtain the following expression for the discrete Lagrangian (modulo $\mathbb{R}$):

$$\mathbb{L}(g_k, g_{k+1}) = -\frac{1}{h^2}\operatorname{Tr}\left(g_k \Lambda g_{k+1}^T\right).$$

We remark that exactly the same expression is obtained if we instead discretize ξ by $\frac{1}{h} g_k^T (g_{k+1} - g_k)$. Notice that up to a multiplier of $-1/h^2$, this is precisely the Lagrangian used by Moser and Veselov [47].

We scale the above Lagrangian and introduce matrix Lagrange multipliers λ_k, imposing the constraint $\Phi_k(g_k) = g_k g_k^T - \mathrm{Id} = 0$. By decomposing λ_k into symmetric and skew components, we see that the skew component of λ_k does not contribute to the action because the constraint Φ_k is symmetric; thus, we find that $\lambda_k = \lambda_k^T$. The action sum then takes the form

$$S = \sum_k \operatorname{Tr}\left(g_k \Lambda g_{k+1}^T\right) - \frac{1}{2}\sum_k \operatorname{Tr}\left(\lambda_k (g_k g_k^T - \mathrm{Id})\right) \tag{5.6}$$

Notice that the discrete Lagrangian $\mathbb{L}$ is left invariant and can be reduced to a Lagrangian $\ell : G \to \mathbb{R}$ using the canonical projection $\pi_d : (g_k, g_{k+1}) \mapsto f_{k+1k} = g_{k+1}^{-1} g_k$ so that

$$\ell(f_{k+1k}) = \operatorname{Tr}(f_{k+1k}\Lambda).$$

Because the constraint, ensuring that each $g_k \in G$, is G-invariant, there exists a Lagrange multiplier $\bar{\lambda}_k$ in the conjugacy class of λ_k, i.e., $\bar{\lambda}_k = g^T \lambda_k g$ for all $g \in G$, so that $\bar{\lambda}_k = \bar{\lambda}_k^T$. Hence, computing the discrete variation of (5.6) with respect to

g_k, we obtain the operator equation

$$-\ell'(f_{kk-1})TR_{f_{kk-1}} + \ell'(f_{k+1k})\,\mathrm{Ad}_{f_{k+1k}}TR_{f_{k+1k}} = \bar{\lambda}_k,$$

where the operators act on the variations $\vartheta_k = g_k^T \delta g_k$. Using the expression for the reduced Lagrangian ℓ, the DEP equation can then be written as

$$f_{k+1k}^T \Lambda + f_{kk-1}\Lambda = \bar{\lambda}_k.$$

Using the fact that $\bar{\lambda}_k^T = \bar{\lambda}_k$, we obtain the DEP algorithm on $\mathrm{SO}(n)$ as

$$f_{k+1k}^T \Lambda - \Lambda f_{k+1k} = \Lambda f_{kk-1}^T - f_{kk-1}\Lambda. \tag{5.7}$$

This is an implicit scheme to be solved for f_{k+1k} using the current value f_{kk-1}. The solution of (5.7) generates the *explicit* DLP algorithm on $\mathfrak{so}(n)^*$ given by

$$\Pi_{k+1} = \mathrm{Ad}_{f_{k+1k}^{-1}}\, \Pi_k = f_{k+1k}\Pi_k f_{k+1k}^T. \tag{5.8}$$

Finally, reconstruction of the DEP algorithm recovers the DEL algorithm on $G \times G$ which, according to (3.15), is given by

$$(g_{k-1}, g_k) \mapsto (g_k, g_{k+1}) = (g_k, g_k \cdot f_{k+1k}^{-1}).$$

Theorem 5.2.1. *The above DEP and DLP algorithms given by (5.7) and (5.8), respectively, are equivalent to the Moser-Veselov equations*

$$\begin{cases} M_{k+1} \equiv \omega_{k-1} M_k \omega_{k-1}^{-1} \\ M_k = \omega_k^T \Lambda - \Lambda \omega_k, \qquad \omega_k \in \mathrm{SO}(n), \end{cases} \tag{5.9}$$

where (using the notation of [47]) $\omega_k = g_k^T g_{k-1} \in \mathrm{SO}(n)$ is the discrete angular velocity, $M_k = g_{k-1}^T m_k g_{k-1} = \omega_k^T \Lambda - \Lambda \omega_k \in \mathrm{so}(n)$ is the discrete body angular momentum, and $m_k = m_0$ is the constant discrete spatial angular momentum.

Proof. Comparing the definitions of $f_{kk-1} = g_k^T g_{k-1}$ and $\omega_k = g_k^T g_{k-1}$, we see that

$f_{kk-1} \equiv \omega_k$. Similarly, comparing the definitions of $\Pi_k = \mathrm{Ad}^*_{g_k} \pi_0$ and

$$M_k = g_{k-1}^T m_k g_{k-1} = g_{k-1}^T m_o g_{k-1} = \mathrm{Ad}^*_{g_{k-1}} m_0,$$

we conclude that $\Pi_{k-1} \equiv M_k$ and $\pi_0 \equiv m_0$. Hence, the first equation in (5.9) is precisely the DLP algorithm (5.8).

Substituting the second equation of (5.9) into the first results in the following expression:

$$\omega_{k+1}^T \Lambda - \Lambda \omega_{k+1} = \Lambda \omega_k^T - \omega_k \Lambda,$$

which is precisely the DEP equation (5.7) when the above identifications are invoked.

$\square$

The Moser-Veselov algorithm (5.9) has an an obvious geometric mechanical interpretation. The first equation can be viewed as a discretization of the left Lie-Poisson equation

$$M_k = g_{k-1}^T m_0 g_{k-1} = \mathrm{Ad}^*_{g_{k-1}} m_0,$$

rewritten in terms of the ω_k and this corresponds to the DLP algorithm (5.8). The second equation is a discrete version of the relation between the angular momentum and angular velocity, as it is obtained by substitution of (5.5) into $M = J(\xi) = \Lambda\xi + \xi\Lambda$.

The DEP algorithm (5.7) provides an equivalent alternative to the Moser-Veselov scheme (5.9), the difference being that the former is an algorithm on G only, while the latter is a combined algorithm on G and $\mathfrak{g}^*$ and schematically can be represented by the mappings $\mathfrak{g}^* \mapsto G \mapsto \mathfrak{g}^* \mapsto G; \quad M_k \mapsto \omega_k \mapsto M_{k+1} \mapsto \omega_{k+1}$.

5.3 Poisson structures of the rigid body

The Lie algebra dual $\mathfrak{so}(3)^*$ has a well-known Lie-Poisson structure with a Casimir $C_{\mathfrak{so}(3)^*}(\mu) = \mathrm{Tr}(\mu^2)$, where $\mu \in \mathfrak{so}(3)^*$. Upon identification with $\mathbb{R}^3$, its generic symplectic leaves become concentric spheres with Kirilov-Kostant symplectic form being proportional to the area form. If y denotes coordinates on $\mathbb{R}^3 \cong \mathfrak{so}(3)^*$, then the above Casimir function is given by $C_{\mathfrak{so}(3)^*}(y) = \|y\|^2$.

Recall that the reduced form of the Moser-Veselov Lagrangian on the group SO(3) is given by

$$\ell(f) = \mathrm{Tr}(f\Lambda),$$

where $f \in \mathrm{SO}(3)$ and SO(3) is embedded into the linear space $\mathfrak{gl}(3)$. Then, the Legendre transform $F\ell$ takes the form

$$F\ell(f) = L_f^* \circ d\ell(f) = \mathrm{skew}(f\Lambda) = f\Lambda - \Lambda f^T \ : \mathrm{SO}(3) \to \mathfrak{so}(3)^*,$$

where the constraint that f be in SO(3) has been enforced. The pull-back of $C_{\mathfrak{so}(3)^*}$ under $F\ell^*$ defines a Casimir function *on the group*, which up to a constant term and a sign, is given by

$$C_{\mathrm{SO}(3)}(f) = \mathrm{Tr}(f\Lambda f\Lambda) \qquad f \in \mathrm{SO}(3). \tag{5.10}$$

Its symplectic leaves constitute the invariant manifolds of the reduced discrete dynamics corresponding to the Lagrangian (5.1).

Analogously, one can define a Poisson structure on the Lie algebra $\mathfrak{so}(3)$ using the duality between Lie-Poisson and Euler-Poincaré reduced systems on $\mathfrak{so}(3)^*$ and $\mathfrak{so}(3)$, respectively. The Lagrangian (5.1) defines the Legendre transformations Fl from $\mathfrak{so}(3)$ to $\mathfrak{so}(3)^*$ given by $\mu = \dfrac{\partial l}{\partial \xi} = \mathcal{J}(\xi)$. Then, the pull-back by Fl^* defines a Casimir on $\mathfrak{so}(3)$:

$$C_{\mathfrak{so}(3)}(\xi) = Fl^* \cdot C_{\mathfrak{so}(3)^*}(\xi) = \langle\langle \mathcal{J}(\xi), \mathcal{J}(\xi)\rangle\rangle,$$

where the metric on the dual is induced by the one on the algebra, i.e., by the symmetric positive definite operator J. If x denotes coordinates on $\mathbb{R}^3 \cong \mathfrak{so}(3)$, then the above Casimir function is given by $C_{\mathfrak{so}(3)}(x) = \|\mathcal{J}(x)\|^2$. Thus, the corresponding symplectic leaves are ellipsoids of $\mathcal{J}^2$. They *do not* coincide with adjoint orbits, which are spheres in $\mathbb{R}^3$. The dynamic orbits are obtained by intersecting these ellipsoids, determined by $\mathcal{J}^2$, with the energy ellipsoids, determined by $\mathcal{J}$.

Part II

Multisymplectic Geometry of Continuum Mechanics

Chapter 6

Compressible Continuum Mechanics

To describe the multisymplectic framework of continuum mechanics, we must first specify the covariant configuration and phase spaces. Once we obtain a better understanding of the geometry of these manifolds we can consider the dynamics determined by a particular covariant Lagrangian.

6.1 Configuration and Phase Spaces

The Jet Bundle. We shall set up a formalism in which a continuous medium is described using sections of a fiber bundle Y over X; here X is the base manifold and Y consists of fibers Y_x at each point $x \in X$. Sections of the bundle $\pi_{XY} : Y \to X$ represent ***configurations***, e.g., particle placement fields or deformations.

Let (B, G) be a smooth n-dimensional compact oriented Riemannian manifold with a smooth boundary and let (M, g) be a smooth N-dimensional compact oriented Riemannian manifold. For the non-relativistic case, the base manifold can be chosen to be a spacetime manifold represented by the product $X = B \times \mathbb{R}$ of the manifold B together with time; $(x, t) \in X$. Let us set $x^0 = t$, so that $x^\mu = (x^i, x^0) = (x^i, t)$, with $\mu = 0, \ldots, n$, $i = 1, \ldots, n$, denote coordinates on the $(n+1)$-dimensional manifold X. Construct a trivial bundle Y over X with M being a fiber at each point; that is, $Y = X \times M \ni (x, t, y)$ with $y \in M$ — the fiber coordinate. This bundle,

$$\pi_{XY} : Y \to X; \quad (x, t, y) \mapsto (x, t),$$

with π_{XY} being the projection on the first factor, is the covariant configuration manifold for our theory. Let $\mathcal{C} \equiv C^\infty(Y)$ be the set of smooth sections of Y. Then, a section ϕ of $\mathcal{C}$ represents a time dependent configuration.

Let y^a, $i = 1, \ldots, N$ denote fiber coordinates so that a section ϕ has a coordinate representation $\phi(x) = (x^\mu, \phi^a(x)) = (x^\mu, y^a)$. The first jet bundle J^1Y is the affine bundle over Y whose fiber above $y \in Y_x$ consists of those linear maps $\gamma : T_xX \to T_yY$ satisfying $T\pi_{XY} \circ \gamma = Id_{T_xX}$. Coordinates on J^1Y are denoted $\gamma = (x^\mu, y^a, v^a{}_\mu)$. For a section ϕ, its tangent map at $x \in X$, denoted $T_x\phi$, is an element of $J^1Y_{\phi(x)}$. Thus, the map $x \mapsto T_x\phi$ is a local section of J^1Y regarded as a bundle over X. This section is denoted $j^1\phi$ and is called the first jet extension of ϕ. In coordinates, $j^1\phi$ is given by $(x^\mu, \phi^a(x), \partial_\mu\phi^a)$, where $\partial_0\phi^a = \partial_t\phi^a$ and $\partial_k\phi^a = \partial_{x^k}\phi^a$.

Notice that we have introduced *two different* Riemannian structures on the configuration bundle. The internal metric on the spatial part B of the base manifold X is denoted by G and the fiber, or field, metric on M is denoted by g. There are two main cases, which we consider:

(i) fluid dynamics on a fixed background with fixed boundaries, when B and M are the same and the fiber metric g coincides with the base metric G; a special case of this is fluid dynamics on a region in Euclidean space;

(ii) elasticity on a fixed background, when the metric spaces (B, G) and (M, g) are essentially different.

Both approaches result in *background theories.* The case of relativistic fluid and elasticity was considered by Kijowski (see, e.g., [23]).

Define the following function on the first jet bundle:

$$J(x, t, y, v) = \det[v]\sqrt{\frac{\det[g(y)]}{\det[G(x)]}} : J^1Y \to \mathbb{R}. \tag{6.1}$$

We shall see later that its pull-back by a section ϕ has the interpretation of the Jacobian of the linear transformation $D\phi_t$.

A very important remark here is that even though in fluid dynamics metrics g and G coincide, i.e., on each fiber Y_x, g is a copy of G, there is no cancellation because the metric tensors are evaluated at different points. For instance, in (6.1) $g(y)$ does not coincide with $G(x)$ unless $y = x$ or both metrics are constant. Hence, only for fluid dynamics in Euclidean spaces, can one trivially raise and lower indices and drop all metric determinants and derivatives in the expressions below.

The Dual Jet Bundle. Recall that the dual jet bundle J^1Y^* is a vector bundle over Y whose fiber at $y \in Y_x$ is the set of affine maps from J^1Y to $\Lambda^{n+1}X_x$, where $\Lambda^{n+1}X$ denotes the bundle of $(n+1)$-forms on X. A smooth section of J^1Y^* is an affine bundle map of J^1Y to $\Lambda^{n+1}X$ covering π_{XY}. Fiber coordinates on J^1Y^* are $(\Pi, p_a{}^\mu)$, which correspond to the affine map given in coordinates by $v^a{}_\mu \mapsto (\Pi + p_a{}^\mu v^a{}_\mu)d^{n+1}x$.

To define canonical forms on J^1Y^*, another description of the dual bundle is convenient. Let $\Lambda = \Lambda^{n+1}Y$ denote the bundle of $(n+1)$-forms on Y, with fiber over $y \in Y$ denoted by Λ_y and with projection $\pi_{Y\Lambda} : \Lambda \to Y$. Let Z be its "vertically invariant" subbundle whose fiber is given by

$$Z_y = \{z \in \Lambda_y \mid v \lrcorner w \lrcorner z = 0 \text{ for all } v, w \in V_yY\},$$

where $V_yY = \{v \in T_yY \mid T\pi_{XY} \cdot v = 0\}$ is a vertical subbundle. Elements of Z can be written uniquely as

$$z = \Pi d^{n+1}x + p_a{}^\mu dy^a \wedge d^n x_\mu$$

where $d^n x_\mu = \partial_\mu d^{n+1}x$, so that $(x^\mu, y^a, \Pi, p_a{}^\mu)$ give coordinates on Z. Equating the coordinates $(x^\mu, y^a, \Pi, p_a{}^\mu)$ of Z and of J^1Y^* defines a vector bundle isomorphism $Z \leftrightarrow J^1Y^*$. This isomorphism can also be defined intrinsically (see [16]).

Define the canonical $(n+1)$-form Θ_Λ on Λ by $\Theta_\Lambda(z) = (\pi^*_{Y\Lambda}z)$, where $z \in \Lambda$. The canonical $(n+2)$-form is given by $\Omega_\Lambda = -d\Theta_\Lambda$. If $i_{\Lambda Z} : Z \to \Lambda$ denotes the inclusion, the corresponding canonical forms on Z are given by $\Theta = i^*_{\Lambda Z}\Theta_\Lambda$ and $\Omega = -d\Theta = i^*_{\Lambda Z}\Omega_\Lambda$. In coordinates they have the following representation

$$\Theta = p_a{}^\mu dy^a \wedge d^n x_\mu + \Pi d^{n+1}x, \quad \Omega = dy^a \wedge dp_a{}^\mu \wedge d^n x_\mu - d\Pi \wedge d^{n+1}x.$$

Ideal Fluid

We now recall the classical material and spatial descriptions of ideal (i.e., nonviscous) fluids moving in a fixed region, i.e., with fixed boundary conditions (see, e.g., [4]). We set $B = M$ and call it the ***reference fluid container***. A fluid flow is given by a family of diffeomorphisms $\eta_t : M \to M$ with $\eta_0 = \mathrm{Id}$, where $\eta_t(M)$ is the fluid configuration at some later time t. Let $x \in M$ denote the original position of a fluid particle, then $y \equiv \eta_t(x) \in M$ is its position at time t; x and y are called ***material*** and ***spatial*** points, respectively. The ***material velocity*** is defined by $V(x,t) = (\partial/\partial t)\eta_t(x)$. The same velocity viewed as a function of (y,t) is called the ***spatial velocity***. It is denoted by u; that is, $u(y,t) = V(x(y),t)$, where $x = \eta_t^{-1}(y)$, so that $u = V \circ \eta_t^{-1} = \dot{\eta} \circ \eta_t^{-1}$.

Thus, in the bundle picture above, the spatial part of the base manifold $B \subset X$ has the interpretation of the reference configuration, while an extra dimension of X corresponds to the time evolution. All later configurations of the fluid are captured by a section ϕ of the bundle Y, which gets the interpretation of a particle placement field. Pointwise this implies that x in the base point (x,t) represents the material point, while $y \in Y_{(x,t)}$ represents the spatial point and corresponds to a position $y = \phi(x,t) = \eta_t(x)$ of the fluid particle x at time t.

Elasticity

For the theory of elasticity (as well as for fluids with a free boundary), the base and fiber spaces are generally different; (B,G) is traditionally called the ***reference configuration***, while (M,g) denotes the ambient space. For classical 2 or 3-dimensional elasticity, M and B have the same dimension, while for rods and shells models the dimension of the reference configuration B is less than that of the ambient space.

For a fixed time t, sections of the bundle Y, denoted by ϕ_t, play the role of ***deformations***, they map reference configuration B into spatial configuration M. Upon restriction to the space of first jets, the fiber coordinates v of $\gamma = (x, y, v) \in J^1Y$ become partial derivatives $\partial\phi^a/\partial x^\mu$; they consist of the time derivative of the deformation $\dot{\phi}^a$ and the ***deformation gradient***, $F^a{}_i = \partial\phi^a/\partial x^i$. The first jet of a section ϕ then has the following local representation $j^1\phi = ((x,t), \phi(x,t), \dot{\phi}(x,t), F(x,t)) : X \to J^1Y$.

Using the map ϕ, one can pull back and push forward metrics on the base and fiber manifolds. In particular, a pull-back of the field metric g on M to $B \subset X$ defines the ***Green deformation tensor*** (also called the right Cauchy-Green tensor) C by $C^\flat = \phi_t^*(g)$, while a push-forward of the base metric G on $B \subset X$ to M defines the inverse of the ***Finger deformation tensor*** b (also called the left Cauchy-Green tensor): $c = b^{-1} = (\phi_t)_*(G)$. In coordinates,

$$C_{ij}(x,t) = g_{ab}F^a{}_i F^b{}_j(x,t), \qquad c_{ab}(y) = G_{ij}(F^{-1})^i{}_a(F^{-1})^j{}_b(y), \tag{6.2}$$

where F^{-1} is thought of as a function of y. We remark that C is defined whether or not the deformation is regular, while b and c rely on the regularity of ϕ_t. Another important remark is that operations flat $\flat$ and sharp $\sharp$ are taken with respect to the corresponding metrics on the space, so that, e.g., $(\phi_t^* g)^\sharp \neq \phi_t^*(g^\sharp)$.

Notice that J restricted to the first jets of sections is the ***Jacobian*** of $D\phi_t$, that is, the determinant of the linear transformation $D\phi_t$; it is given in coordinates by

$$J(j^1\phi) = \det[F]\sqrt{\frac{\det[g]}{\det[G]}}(j^1\phi) \;:\; X \to \mathbb{R}.$$

It is a scalar function of x and t, invariant under coordinate transformations. Notice also that $J(x,t) > 0$ for regular deformations with $\phi(x,0) = x, F(x,0) = \mathrm{Id}$ because $J(x,0) = 1$.

6.2 Lagrangian Dynamics.

To obtain the Euler-Lagrange equations for a particular model of a continuous medium, one needs to specify a Lagrangian density $\mathcal{L}$. Naturally, it should contain terms corresponding to the kinetic energy and to the potential energy of the medium. Such terms depend on material properties such as mass density ρ as well as on the constitutive relation. The latter is determined by the form of the potential energy of the material. We remark that such an approach excludes from our consideration non-hyperelastic materials whose constitutive laws cannot be obtained from a potential energy function.

Lagrangian Density. Let the mass density $\rho : B \to \mathbb{R}$ be given for a particular model of continuum mechanics. The Lagrangian density $\mathcal{L} : J^1Y \to \Lambda^{n+1}X$ for a multisymplectic model of continuum mechanics is a smooth bundle map

$$\begin{aligned}\mathcal{L}(\gamma) = L(\gamma)d^{n+1}x = \mathbb{K} - \mathbb{P} = \frac{1}{2}\sqrt{\det[G]}\rho(x)g_{ab}v^a{}_0v^b{}_0d^{n+1}x \\ - \sqrt{\det[G]}\rho(x)W(x, G(x), g(y), v^a{}_j)d^{n+1}x,\end{aligned} \quad (6.3)$$

where $\gamma \in J^1Y$ and W is the ***stored energy function***. The first term in (6.3) corresponds to the kinetic energy of the matter when restricted to first jet extensions, as $v^a{}_0$ becomes the time derivative $\partial_t\phi^a$ of the section ϕ. The second term reflects the potential energy and depends on the spatial derivatives of the fields (upon restriction to first jet extensions), i.e., on the deformation gradient F.

A choice of the stored energy function specifies a particular model of a continuous medium. While different general functional forms distinguish various broad classes of materials (elastic, fluid, etc.), the specific functional forms determine specific materials. Typically, for elasticity, W depends on the field's partial derivatives through the (Green) deformation tensor C, while for Newtonian fluid dynamics, W is only a function of the Jacobian J (6.1).

Legendre Transformations. The Lagrangian density (6.3) determines the Legendre transformation $\mathbb{F}\mathcal{L} : J^1Y \to J^1Y^*$. The conjugate momenta are given by the following expressions:

$$p_a{}^0 = \frac{\partial L}{\partial v^a{}_0} = \rho g_{ab} v^b{}_0 \sqrt{\det[G]}, \qquad p_a{}^j = \frac{\partial L}{\partial v^a{}_j} = -\rho \frac{\partial W}{\partial v^a{}_j} \sqrt{\det[G]}, \tag{6.4}$$

$$\Pi = L - \frac{\partial L}{\partial v^a{}_\mu} v^a{}_\mu = \left[-\frac{1}{2} g_{ab} v^a{}_0 v^b{}_0 - W + \frac{\partial W}{\partial v^a{}_j} v^a{}_j \right] \rho \sqrt{\det[G]}$$

Define the ***energy density*** e by

$$e = p_a{}^0 v^a{}_0 - L \quad \text{or, equivalently} \quad e d^{n+1}x = \mathbb{K} + \mathbb{P}, \tag{6.5}$$

then

$$\Pi = -p_a{}^j v^a{}_j - \sqrt{\det[G]}\, e.$$

The Cartan Form. Using the Legendre transformation (6.4), we can pull-back the canonical $(n+1)$-form from the dual bundle. The resulting form on J^1Y is called the Cartan form and is given by

$$\begin{aligned} \Theta_{\mathcal{L}} = \rho g_{ab} v^b{}_0 \sqrt{\det[G]} dy^a \wedge d^n x_0 - \rho \frac{\partial W}{\partial v^a{}_j} \sqrt{\det[G]} dy^a \wedge d^n x_j \\ + \left[-\frac{1}{2} g_{ab} v^a{}_0 v^b{}_0 - W + \frac{\partial W}{\partial v^a{}_j} v^a{}_j \right] \rho \sqrt{\det[G]} d^{n+1}x. \end{aligned} \tag{6.6}$$

We set $\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}}$.

Theorem 6.2.1 below provides a nicer method for obtaining the Cartan form via the Calculus of Variations and remains entirely on the Lagrangian bundle J^1Y. Moreover, the variational approach is essential for the Veselov type discretization of our multisymplectic theory. We present it here for the benefit of the reader, but remark that it is not essential for our current exposition (see [34] for details).

Variational Approach. To make the variational derivation of the equations of motion rigorous as well as that of the geometric objects, such as the multisymplectic form and the Noether current, we need to introduce some new notations (see [34]). These are generalizations of the notations used in the rest of Part II of the thesis. They only apply to the variational derivation described here and later in Section 9.1 and do not influence the formalism and results in the rest of the thesis. The reason for such generalizations is very important yet subtle: one should allow for *arbitrary* and not only *vertical* variations of the sections.

Vertical variations are confined to the vertical subbundle $VY \subset TY$, $V_yY = \{\mathcal{V} \in T_yY | T\pi_{XY} \cdot \mathcal{V} = 0\}$; this allows only for *fiber-preserving* variations, i.e., if $\phi(X) \in Y_x$ and $\tilde{\phi}$ is a new section, then $\tilde{\phi} \in Y_x$. In general, one should allow for arbitrary variations in TY, when $\tilde{\phi} \in Y_{\tilde{x}}$ for some $\tilde{x} \neq x$. Introducing a splitting of the tangent bundle into a vertical and a horizontal parts, $T_yY = V_yY \oplus H_yY$ (H_yY is not uniquely defined), one can decompose a general variation into a vertical and horizontal components, respectively.

Explicit calculations show (see [39]) that while both vertical and arbitrary variations result in the same Euler-Lagrange equations, the Cartan form obtained from the vertical variations only is missing one term (corresponding to the $d^{n+1}x$ from on X); the horizontal variations account precisely for this extra term and make the Cartan form complete.

One can account for general variations either by introducing new "tilted sections," or by introducing some true new sections that compensate for the horizontal variation. The latter can be implemented in the following way. Let $U \subset X$ be a smooth manifold with smooth closed boundary. Define the set of smooth maps

$$\mathcal{C} = \{\varphi : U \to Y | \pi_{XY} \circ \varphi : U \to X \text{is an embedding}\}.$$

For each $\varphi \in \mathcal{C}$, set $\varphi_X = \pi_{XY} \circ \varphi$ and $U_X = \pi_{XY} \circ \varphi(U)$, so that $\varphi_X : U \to U_X$ is a diffeomorphism and $\varphi \circ \varphi_X^{-1}$ is a section of Y. The ***tangent space*** to the manifold $\mathcal{C}$ at a point φ is the set $T_\varphi\mathcal{C}$ defined by

$$\{\mathcal{V} \in C^\infty(X, TY) | \pi_{Y,TY} \circ \mathcal{V} = \varphi \ \& \ T\pi_{XY} \circ \mathcal{V} = \mathcal{V}_X, \text{a vector field on} X\}.$$

Arbitrary (i.e., including both vertical and horizontal) variations of sections of Y can be induced by a family of maps φ defined through the action of some Lie group. Let $\mathcal{G}$ be a Lie group of π_{XY} bundle automorphisms η_Y covering diffeomorphisms η_X. Define the ***action*** of $\mathcal{G}$ on $\mathcal{C}$ by composition: $\eta_Y \cdot \varphi = \eta_Y \circ \varphi$. Hence, while $\varphi \circ \varphi_X^{-1}$ is a section of $\pi_{U_X,Y}$, $\eta_Y \cdot \varphi$ induces a section $\eta_Y \circ (\varphi \cdot \varphi_X^{-1}) \circ \eta_X^{-1}$ of $\pi_{\eta_X(U_X),Y}$.

A one parameter family of variations can be obtained in the following way. Let $\varepsilon \mapsto \eta_Y^\varepsilon$ be an arbitrary smooth path in $\mathcal{G}$ with $\eta_Y^0 = e$, and let $\mathcal{V} \in T_\varphi\mathcal{C}$ be given by

$$\mathcal{V} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \eta_Y^\varepsilon \cdot \varphi.$$

Define the ***action function***

$$S(\varphi) = \int_{U_X} \mathcal{L}(j^1(\varphi \circ \varphi_X^{-1})) : \mathcal{C} \to \mathbb{R},$$

and call φ a ***critical point (extremum)*** of S if

$$dS(\varphi) \cdot \mathcal{V} \equiv \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\eta_Y^\varepsilon \cdot \varphi) = 0.$$

The Euler-Lagrange equations and the Cartan form can be obtained by analyzing this condition. We summarize the results in the following theorem from [34] which illustrates the application of the variational principle to multisymplectic field theory.

Theorem 6.2.1. *Given a smooth Lagrangian density $\mathcal{L} : J^1Y \to \Lambda^{n+1}(X)$, there exist a unique smooth section $D_{EL}\mathcal{L} \in C^\infty(Y'', \Lambda^{n+1}(X) \otimes T^*Y))$ (Y'' being the space of second jets of sections) and a unique differential form $\Theta_\mathcal{L} \in \Lambda^{n+1}(J^1Y)$ such that for any $\mathcal{V} \in T_\phi\mathcal{C}$, and any open subset U_X such that $\overline{U}_X \cap \partial X = \emptyset$,*

$$dS(\varphi) \cdot \mathcal{V} = \int_{U_X} D_{EL}\mathcal{L}(j^2(\varphi \circ \varphi_X^{-1})) \cdot \mathcal{V} + \int_{\partial U_X} j^1(\varphi \circ \varphi_X^{-1})^*[j^1(\mathcal{V}) \lrcorner \Theta_\mathcal{L}]. \quad (6.7)$$

Furthermore,

$$D_{EL}\mathcal{L}(j^2(\varphi \circ \varphi_X^{-1})) \cdot \mathcal{V} = j^1(\varphi \circ \varphi_X^{-1})^*[j^1(\mathcal{V}) \lrcorner \Omega_\mathcal{L}] \quad \textit{in } U_X. \quad (6.8)$$

In coordinates, the action of the Euler-Lagrange derivative $D_{EL}\mathcal{L}$ on Y'' is given by

$$\begin{aligned} D_{EL}\mathcal{L}(j^2(\varphi\circ\varphi_X^{-1})) = \Bigg[&\frac{\partial L}{\partial y^a}(j^1(\varphi\circ\varphi_X^{-1})) - \frac{\partial^2 L}{\partial x^\mu \partial v^a{}_\mu}(j^1(\varphi\circ\varphi_X^{-1})) \\ &- \frac{\partial^2 L}{\partial y^b \partial v^a{}_\mu}(j^1(\varphi\circ\varphi_X^{-1}))\cdot(\varphi\circ\varphi_X^{-1})^b_{,\mu} \\ &- \frac{\partial^2 L}{\partial v^b{}_\nu \partial v^a{}_\mu}(j^1(\varphi\circ\varphi_X^{-1}))\cdot(\varphi\circ\varphi_X^{-1})^b_{,\mu\nu}\Bigg]\, dy^a\wedge d^{n+1}x, \end{aligned} \tag{6.9}$$

while the form $\Theta_\mathcal{L}$ matches the definition of the Cartan form obtained via Legendre transformation and has the coordinate expression

$$\Theta_\mathcal{L} = \frac{\partial L}{\partial v^a{}_\mu} dy^a \wedge d^n x_\mu + \left(L - \frac{\partial L}{\partial v^a{}_\mu} v^a{}_\mu\right) d^{n+1}x. \tag{6.10}$$

Corollary 6.2.1. *The $(n+1)$-form $\Theta_\mathcal{L}$ defined by the variational principle satisfies the relationship*

$$\mathcal{L}(\mathfrak{z}) = \mathfrak{z}^*\Theta_\mathcal{L}$$

for all holonomic sections $\mathfrak{z}\in C^\infty(\pi_{X,J^1Y})$.

Another important general theorem, which we quote from [34], is the so-called ***multisymplectic form formula***

Theorem 6.2.2. *If ϕ is a solution of the Euler-Lagrange equation (6.9), then*

$$\int_{\partial U_X} (j^1(\varphi\circ\varphi_X^{-1}))^*\left[j^1\mathcal{V}\,\lrcorner\, j^1\mathcal{W}\,\lrcorner\,\Omega_\mathcal{L}\right] = 0 \tag{6.11}$$

for any $\mathcal{V},\mathcal{W}$ which solve the first variation equations of the Euler-Lagrange equations, i.e., any tangent vectors to the space of solutions of (6.9).

This result is the multisymplectic analog of the fact that the time t map of a mechanical system consists of canonical transformations. See [34] for the proofs.

Finally we remark that in order to obtain vertical variations we can require φ_X (and, hence, φ_X^{-1}) to be the identity map on X. Then, $\phi=\varphi\circ\varphi_X^{-1}$ becomes a true section of the bundle Y.

Euler-Lagrange Equations. Treating $(J^1Y, \Omega_{\mathcal{L}})$ as a multisymplectic manifold, the Euler-Lagrange equations can be derived from the following condition on a section ϕ of the bundle Y:

$$(j^1\phi)^*(\mathcal{W} \lrcorner \Omega_{\mathcal{L}}) = 0,$$

for any vector field $\mathcal{W}$ on J^1Y (see [16] for the proof). This translates to the following familiar expression in coordinates

$$\frac{\partial L}{\partial y^a}(j^1\phi) - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^a{}_\mu}(j^1\phi)\right) = 0, \tag{6.12}$$

which is equivalent to equation (6.9). Substituting the Lagrangian density (6.3) into equation (6.12) we obtain the following Euler-Lagrange equation for a continuous medium:

$$\rho g_{ab}\left(\frac{D_g\dot{\phi}}{Dt}\right)^b - \frac{1}{\sqrt{\det[G]}}\frac{\partial}{\partial x^k}\left(\rho\frac{\partial W}{\partial v^a{}_k}(j^1\phi)\sqrt{\det[G]}\right) = -\rho\frac{\partial W}{\partial g_{bc}}\frac{\partial g_{bc}}{\partial y^a}(j^1\phi), \tag{6.13}$$

where

$$\left(\frac{D_g\dot{\phi}}{Dt}\right)^b \equiv \frac{\partial\dot{\phi}^b}{\partial t} + \gamma^a_{bc}\,\dot{\phi}^b\dot{\phi}^c$$

is the covariant time derivative, which corresponds to ***material acceleration***, with

$$\gamma^c_{ab} = \frac{1}{2}g^{cd}\left(\frac{\partial g_{ad}}{\partial y^b} + \frac{\partial g_{bd}}{\partial y^a} - \frac{\partial g_{ab}}{\partial y^d}\right)$$

being the Christoffel symbols associated with the 'field' metric g. We remark that all terms in this equation are functions of x and t and hence have the interpretation of material quantities.

Equation (6.13) is a PDE to be solved for a section $\phi(x,t)$ for a given type of potential energy W. As the gravity here is treated parametrically, the term on the right-hand side of (6.13) can be thought of as a derivative with respect to a parameter, and we can define a multisymplectic analogue of the Cauchy stress tensor σ as follows:

$$\sigma^{ab} = \frac{2\rho}{J}\frac{\partial W}{\partial g_{ab}}(j^1\phi) \; : \; X \to \mathbb{R}, \tag{6.14}$$

where $J = \det[F]\sqrt{\det[g]/\det[G]}$ is the Jacobian. Equation (6.14) is known in the elasticity literature as the Doyle-Ericksen formula (recall that our ρ corresponds to ρ_{Ref}, so that the Jacobian J in the denominator disappears).

Another important remark is that the balance of moment of momentum

$$\sigma^T = \sigma$$

follows from definition (6.14) and the symmetry of the metric tensor g.

Finally, in the case of Euclidean manifolds with constant metrics g and G, equation (6.13) simplifies to

$$\rho\frac{\partial^2 \phi_a}{\partial t^2} = \frac{\partial}{\partial x^k}\left(\rho\frac{\partial W}{\partial v^a{}_k}(j^1\phi)\right). \tag{6.15}$$

Barotropic Fluid

For standard models of barotropic fluids, the potential energy of a fluid depends only on the Jacobian of the fluid's "deformation," so that $W = W(J(g, G, v))$. For a general inhomogeneous barotropic fluid, the material density is a given function $\rho(x)$. In material representation, this formalism also includes the case of isentropic fluids in which there is a possible dependence on entropy. Since, in that case, entropy is advected, this dependency in the material representation is subsumed by the dependency of the stored energy function on the deformation gradient. [1]

The Legendre transformation can be thought of as defining the pressure function.

[1] In spatial representation, of course one has to introduce the entropy as an independent variable, but this naturally happens via reduction. See [18] for related results from the point of view of the Euler-Poincaré theory with advected quantities.

Notice that

$$p_a{}^i = -\rho\frac{\partial W}{\partial v^a{}_i}\sqrt{\det[G]} = -\rho\frac{\partial W}{\partial J}\frac{\partial J}{\partial v^a{}_i}\sqrt{\det[G]} = -\rho\frac{\partial W}{\partial J}J(v^{-1})_a{}^i\sqrt{\det[G]}$$

and define the pressure function to be

$$P(\phi, x) = -\rho(x)\frac{\partial W}{\partial J}(j^1\phi(x)) \;:\; \mathcal{C}\times X \to \mathbb{R}. \tag{6.16}$$

Then for a given section ϕ, $P(\phi) : X \to \mathbb{R}$ has the interpretation of the ***material pressure*** which is a function of the material density. In this case, the Cauchy stress tensor defined by (6.14) is proportional to the metric with the coefficient being minus the pressure itself:

$$\sigma^{ab}(x) = \frac{2\rho}{J}\frac{\partial W}{\partial J}\frac{\partial J}{\partial g_{ab}}(j^1\phi) = -\frac{2P}{J}J\frac{1}{2}g^{ab}(j^1\phi) = -P(x)\;g^{ab}(y(x)).$$

We remark that this can be a defining equation for the pressure from which (6.16) would follow. With this notation the left-hand side of the Euler-Lagrange equations (6.13) becomes

$$\rho g_{ab}\left(\frac{D_g\dot\phi}{Dt}\right)^b - \frac{1}{\sqrt{\det[G]}}\frac{\partial}{\partial x^k}\left(-PJ\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k\sqrt{\det[G]}\right) =$$

$$\rho g_{ab}\left(\frac{D_g\dot\phi}{Dt}\right)^b + \frac{\partial P}{\partial x^k}J\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k + \frac{P\det\left(\frac{\partial\phi}{\partial x}\right)}{\sqrt{\det[G]}}\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k\frac{\partial\det[g]}{\partial x^k}$$

$$+ (I) + (II) = \rho g_{ab}\left(\frac{D_g\dot\phi}{Dt}\right)^b + \frac{\partial P}{\partial x^k}J\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k + \frac{P}{2}Jg^{bc}\frac{\partial g_{bc}}{\partial y^a}, \tag{6.17}$$

where terms (I) and (II) arise from differentiating $\det[v]$ and $(v^{-1})_a{}^k$ and cancel each other. The right-hand side of (6.13) is given by

$$-\rho\frac{\partial W}{\partial g_{bc}}\frac{\partial g_{bc}}{\partial y^a} = -\rho\frac{\partial W}{\partial J}\frac{\partial J}{\partial g_{bc}}\frac{\partial g_{bc}}{\partial y^a} = \frac{P}{2}Jg^{bc}\frac{\partial g_{bc}}{\partial y^a}.$$

Notice that the last term in (6.17) and in the equation above coincide, so that the

Euler-Lagrange equations for the barotropic fluid have the following form

$$\rho g_{ab}\left(\frac{D_g\dot{\phi}}{Dt}\right)^b = -\frac{\partial P}{\partial x^k}J\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)^k_a, \tag{6.18}$$

where the pressure depends on the section ϕ and the density ρ and is defined by (6.16). Both the metric g_{ab} and the Christoffel symbols γ^c_{ab} in the covariant derivative are evaluated at $y = \phi(x,t)$.

One can re-write (6.18) introducing the ***spatial density*** $\rho_{\mathrm{sp}} = \rho/J$ and defining the ***spatial pressure*** $p(y)$ by the relation $P(x) = p(y(x)) = p(\phi_t(x)$. This yields

$$\frac{D_gV}{Dt}(x,t) = -\frac{1}{\rho_{\mathrm{sp}}}\,\mathrm{grad}\,p\circ\phi\;(x,t),$$

where $V = \dot{\phi}$. Compare this to the equations for incompressible ideal hydrodynamics in Chapter 8.

Elasticity

The Legendre transformation defines the first Piola-Kirchhoff stress tensor $\mathcal{P}_a{}^i$. It is given, up to the multiple of $-1/\sqrt{\det[G]}$, by the matrix of the partial derivatives of the Lagrangian with respect to the deformation gradient:

$$\mathcal{P}_a{}^i(\phi,x) = \rho(x)\frac{\partial W}{\partial v^a{}_i}(j^1\phi(x)), \tag{6.19}$$

and for a given section ϕ, $\mathcal{P}_a{}^i$ is a stress tensor defined on X.

Notice that the first Piola-Kirchhoff stress tensor is proportional to the spatial momenta, $\mathcal{P}_a{}^i = -p_a{}^i/\sqrt{\det[G]}$. The coefficient $\sqrt{\det[G]}$ arises from the difference in the volume forms used in standard and multisymplectic elasticity. In the former, the Lagrangian density is integrated over a space area using the volume form $\mu_G = \sqrt{\det[G]}d^nx$ associated with the metric G, while in the latter, the integration is done over the space-time using $d^{n+1}x = dt\wedge d^nx$. We also remark that though traditionally the first Piola-Kirchhoff stress tensor is normally taken with both indices up, our choice is more natural in the sense that it arises from the Lagrange transformation (6.19) which relates ${\mathcal{P}_a}^i$ with the spatial momenta.

Using definitions (6.14) and (6.19), we can re-write equation (6.13) in the following form

$$\rho g_{ab}\left(\frac{D_g\dot{\phi}}{Dt}\right)^b = {\mathcal{P}_a}^i{}_{|i} + \gamma^b_{ac}({\mathcal{P}_b}^j {F^c}_j - J g_{bd}\sigma^{dc}), \tag{6.20}$$

where we have introduced a ***covariant divergence*** according to

$$ {\mathcal{P}_a}^i{}_{|i} = \mathrm{DIV}\,\mathcal{P} = \frac{\partial {\mathcal{P}_a}^i}{\partial x^i} + {\mathcal{P}_a}^j \Gamma^k_{jk} - {\mathcal{P}_b}^i \gamma^b_{ac} {F^c}_i.$$

Here Γ^i_{jk} are the Christoffel symbols corresponding to the base metric G on $B \subset X$ (see, e.g., [33] for an exposition on covariant derivatives of two-point tensors).

We emphasize that in (6.20) there is no a-priori relationship between the first Piola-Kirchhoff stress tensor and the Cauchy stress tensor, that is, W has the most general form $W(x, G, g, v)$. Such a relationship can, however, be derived from the fact that for standard models of elasticity the stored energy function W depends on the deformation gradient F (i.e., on v) and on the field metric g only via the Green deformation tensor C given by (6.2), that is $W = W(C(v, g))$. Thus, the partial derivatives of W with respect to g and v are related, and the following equation

$${\mathcal{P}_a}^i = J{(\sigma F^{-1})_a}^i$$

follows from definitions (6.14) and (6.19). This relation immediately follows from the form of the stored energy function; it recovers the Piola transformation law, which in conventional elasticity relates the first Piola-Kirchhoff stress tensor and the Cauchy stress tensor. Substituting this relation in (6.20) one easily notices that the last term on the right-hand side cancels, so that the Euler-Lagrange equation for the standard elasticity model can be written in the following covariant form

$$\rho\frac{D_g V}{Dt} = \mathrm{DIV}\,\mathcal{P}, \tag{6.21}$$

where $V = \dot{\phi}$. For elasticity in a Euclidean space, this equation simplifies and takes a well-known form:

$$\rho \frac{\partial V^a}{\partial t} = \frac{\partial \mathcal{P}^{ai}}{\partial x^i}.$$

Chapter 7

Constrained Multisymplectic Field Theories

Multisymplectic field theory is a formalism for the construction of Lagrangian field theories. This is to be contrasted with the formalism in which one takes the view of infinite dimensional manifolds of fields as configuration spaces. The multisymplectic view makes explicit use of the fact that many Lagrangian field theories are local theories, that is, the Lagrangian depends only pointwise on the values of the fields and their derivatives. In formulating a constrained multisymplectic theory, we will therefore only be concerned with the imposition of pointwise constraints $\Phi(\gamma)$, $\gamma \in J^1Y$, depending on point values of the fields and their derivatives. In the current work we also restrict our attention to first-order theories, in which only first derivatives of the fields are considered.

Despite the pointwise nature of the Lagrangian $\mathcal{L}(\gamma)$, $\gamma \in J^1Y$, the variational principle assumes variations of local sections over some region $U \subset X$, that is, it is the action $S(\phi) = \int_U \mathcal{L}(j^1\phi)$ as a function of sections that is being minimized. In order to use the theory of Lagrange multipliers to impose the constraints, it is therefore necessary to form a function $\Psi(\phi)$ of local sections which is defined through point values of the constraint $\Phi(j^1\phi)$ evaluated at the first jets of sections. It is then possible, however, to use the pointwise nature of the Lagrangian and the constraint function to derive a purely local condition, the Euler-Lagrange equations, for the constrained field variables. We will make these ideas precise in Section 7.2.

For holonomic constraints it is well known that Hamilton's principle constrained to the space of allowable configurations gives the correct equations of motion. Hamilton's principle can be naturally extended by either extremizing over the space of motions satisfying the constraints (so-called vakonomic mechanics), which is appropriate for optimal control, but not for dynamics, or by requiring stationarity of the action with respect to variations which satisfy the constraints (the Lagrange-d'Alembert or virtual work principle). The equations of motion derived in each case are, however, different.

Derivations from balance laws ([19]), evidence from experiments ([27]) and comparison to Gauss' Principle of Least Constraint and the Gibbs-Appell equations ([26]) indicates that it is the Lagrange-d'Alembert principle which gives the correct equations of motion; see [5] for further discussion and references.

While the subject of linear and affine non-holonomic constraints is relatively well-understood (see [7]), it is less clear how to proceed for non-linear non-holonomic constraints. Part of the problem lies in the lack of examples for which the correct equations are clear from physical grounds. In the context of constrained field theories, however, there are many cases where nonlinear constraints involving spatial derivatives of the fields need to be applied, such as incompressibility in fluid mechanics, and it is clear what the physically correct equations should be. Here we deliberately avoid the use of the term non-holonomic, to avoid confusion with its standard meaning in the ODE context, where it applies only to time derivatives. Other examples of nonlinearly constrained field theories include constrained director models of elastic rods and shells.

The fact that the constraints involve only spatial and not time derivatives means that imposing the constraints is equivalent to restricting the infinite-dimensional configuration manifold used to formulate the theory as a traditional Hamiltonian or Lagrangian field theory. In this case, the constraint is simply a holonomic or configuration constraint and it is known that restricting Hamilton's principle to the constraint submanifold gives the correct equations for the system.

7.1 Lagrange Multipliers

The Lagrange multiplier theorem naturally makes use of the dual of the space of constraints. In a finite-dimensional setting this is a well defined object, with all definitions being equivalent. When considering infinite-dimensional constraint spaces, however, the issue of what is being used as the dual becomes less clear and more important.

We shall consider constrained multisymplectic field theories for which the constraint space is the space of smooth sections of a particular vector bundle. In the case of the incompressibility constraint, the vector space is one-dimensional and the constraint bundle is, effectively, the space of real valued functions on the base space X. A dual of the constraint space is then defined with respect to an inner product structure on the vector bundle. This is made explicit in the following statement of the Lagrange multiplier theorem where we assume that fields and Lagrange multipliers are sufficiently regular (see [31]).

Theorem 7.1.1 (Lagrange multiplier theorem). *Let $\pi_{\mathcal{M},\mathcal{E}} : \mathcal{E} \to \mathcal{M}$ be an inner product bundle over a smooth manifold $\mathcal{M}$, Ψ a smooth section of $\pi_{\mathcal{M},\mathcal{E}}$, and $h : \mathcal{M} \to \mathbb{R}$ a smooth function. Setting $\mathcal{N} = \Psi^{-1}(0)$, the following are equivalent:*

1. *$\varphi \in \mathcal{N}$ is an extremum of $h|_{\mathcal{N}}$*

2. *there exists an extremum $\bar{\varphi} \in \mathcal{E}$ of $\bar{h} : \mathcal{E} \to \mathbb{R}$ such that $\pi_{\mathcal{M},\mathcal{E}}(\bar{\varphi}) = \varphi$*

where $\bar{h}(\bar{\varphi}) = h(\pi_{\mathcal{M},\mathcal{E}}(\bar{\varphi})) - \langle \bar{\varphi}, \Psi(\pi_{\mathcal{M},\mathcal{E}}(\bar{\varphi}))\rangle_{\mathcal{E}}$.

If $\mathcal{E}$ is a trivial bundle over $\mathcal{M}$, then in coordinates of the trivialization we have $\bar{\varphi} = (\varphi, \lambda)$, where $\lambda : \mathcal{M} \to \mathcal{E}/\mathcal{M}$ is a Lagrange multiplier function.

In the next section we shall use this theorem to relate the constrained Hamilton's principle with the extremum of the augmented action integral which contains the constraint paired with a Lagrange multiplier. Both of them result in constrained Euler-Lagrange equations. We shall furthermore demonstrate that, using the trivialization coordinates, these equations can be equivalently obtained from a Lagrangian defined on an extended configuration bundle. In this picture, the Lagrange multiplier corresponds to a new field, which extends the dimension of the fiber space, and the augmented Lagrangian contains an additional part corresponding to the pairing of this field with the constraint. The Euler-Lagrange equations of motion then follow from *unconstrained* Hamilton's principle in a standard way.

7.2 Multisymplectic Field Theories

In the setting above, the *configuration bundle* is a fiber bundle $\pi_{X,Y} : Y \to X$ and $\pi_{Y,J^1Y} : J^1Y \to Y$ is the corresponding first jet bundle with x^μ and y^a being a local coordinate system on X and Y respectively, and $v^a{}_\mu$ the fiber coordinates on J^1Y.

Choose the *configuration manifold* $\mathcal{M}$ to be the space $\mathcal{C}$ of smooth sections ϕ of $\pi_{X,Y}$. Recall that for a Lagrangian density $\mathcal{L} : J^1Y \to \Lambda^{n+1}X$, a section $\phi \in \mathcal{M}$ is said to satisfy Hamilton's principle if ϕ is an extremum of the action function $S(\phi) = \int_X \mathcal{L}(j^1\phi) : \mathcal{M} \to \mathbb{R}$. Choose the h above to be the action function S and use $\bar{S}$ instead of $\bar{h}$.

To apply the Lagrange multiplier theorem we need to define constraints as a section of some bundle $\mathcal{E} \to \mathcal{M}$ (below called the constraint bundle). As mentioned above, we restrict our attention to constraints Φ which depend only on point values of the fields and their derivatives. Using such constraints we can construct induced constraints Ψ according to (7.1). This is made precise below. We point out, however, that our treatment excludes inherently global constraints, such as those on the inverse Laplacian of the field, which cannot be derived from pointwise values. On the other hand, we also exclude from the consideration a (simple) case when the constrained subbundle of J^1Y can be trivially realized as the first jet of some subbundle of Y.

Define an inner product vector bundle $\pi_{X,\mathcal{V}} : \mathcal{V} \to X$ with the inner product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ whose fibers are isomorphic to $\mathbb{R}^n$. Let $\mathcal{C}^\infty(\mathcal{V})$ be the inner product space of smooth sections of $\pi_{X,\mathcal{V}}$ with the inner product given by

$$\langle a, b\rangle = \int_X \langle a(x), b(x)\rangle_{\mathcal{V}} d^{n+1}x.$$

The constraint function is an $\mathbb{R}^n$-valued function on J^1Y:

$$\Phi : J^1Y \to \mathbb{R}^n.$$

We say that a point $\gamma \in J^1Y$ satisfies the constraint if $\Phi(\gamma) = 0$. By restricting Φ to the space of first jets of sections ϕ of Y, we can define the ***induced constraint function*** Ψ from Φ by setting

$$\Psi(\phi)(x) = \Phi((j^1\phi)(x)) \tag{7.1}$$

for all $\phi \in \mathcal{M}$ and $x \in X$. By construction, Ψ is a map from the space $\mathcal{M}$ of sections of Y to the space $\mathcal{C}^\infty(\mathcal{V})$ of sections of $\mathcal{V}$, hence it can be thought of as a smooth section $\Psi : \mathcal{M} \to \mathcal{E}$ of the ***constraint bundle*** $\mathcal{E}$. This bundle is the trivial inner product bundle given by $\mathcal{M} \times \mathcal{C}^\infty(\mathcal{V})$ over $\mathcal{M}$. Then, a configuration $\phi \in \mathcal{M}$ is said to *satisfy the constraints* if $\Phi((j^1\phi)(x)) = 0$ for all $x \in X$, that is, the section $\Psi(\phi)$ must be a zero function on X. This implies that the space of configurations which satisfy the constraints is given by $\mathcal{N} = \Psi^{-1}(0)$.

The *constrained Hamilton's principle* now seeks a $\phi \in \mathcal{N}$ which is an extremum of $S|_{\mathcal{N}}$. The Lagrange multiplier theorem given in the previous subsection can be applied to conclude that this is equivalent to the existence of $\bar{\phi} \in \mathcal{E}$ with $\pi_{\mathcal{M},\mathcal{E}}(\bar{\phi}) = \phi$ which is an extremum of $\bar{S}$. Using the coordinates of the trivialization of $\mathcal{E}$ we can write $\bar{\phi} = (\phi, \lambda)$, where $\phi = \pi_{\mathcal{M},\mathcal{E}}(\bar{\phi})$ is the base point, i.e., section ϕ of Y, and λ is thought of as a section of the bundle $\pi_{X,\mathcal{V}}$, i.e., an $\mathbb{R}^n$-valued function on X. Then

$\bar{S} : \mathcal{E} \to \mathbb{R}$ is given by

$$\begin{aligned}\bar{S}(\bar{\phi}) &= S(\phi) - \langle \lambda, \Psi(\phi) \rangle_{\mathcal{E}} \\ &= \int_X L((j^1\phi)(x)) d^{n+1}x - \int_X \langle \lambda(x), \Phi((j^1\phi)(x)) \rangle_{\mathcal{V}} d^{n+1}x \\ &= \int_X \left[L((j^1\phi)(x)) - \langle \lambda(x), \Phi((j^1\phi)(x)) \rangle_{\mathcal{V}} \right] d^{n+1}x.\end{aligned}$$

In the next chapter we demonstrate these constructions for the incompressibility constraint for continuum theories. The requirement that $\bar{S}$ be stationary with respect to variations in λ at the point $\bar{\phi}$ implies that

$$0 = \frac{\delta \bar{S}}{\delta \lambda}(\bar{\phi}) \cdot \delta \lambda = \int_X \left[-\langle \delta\lambda(x), \Phi((j^1\phi)(x)) \rangle_{\mathcal{V}} \right] d^{n+1}x$$

for all variations $\delta\lambda$, and thus that $\Phi((j^1\phi)(x)) = 0$ for all $x \in X$. This therefore recovers the condition that ϕ must satisfy the constraints.

Stationarity of $\bar{S}$ with respect to variations in ϕ can be used to derive the *constrained Euler-Lagrange equations*, which have the form

$$\begin{aligned}&\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^a{}_\mu}((j^1\phi)(x)) \right) - \frac{\partial L}{\partial y^a}((j^1\phi)(x)) \\ &\qquad + \left\langle \lambda(x), \frac{\partial \Phi}{\partial y^a}((j^1\phi)(x)) \right\rangle - \frac{\partial}{\partial x^\mu} \left\langle \lambda(x), \frac{\partial \Phi}{\partial v^a{}_\mu}((j^1\phi)(x)) \right\rangle = 0. \quad (7.2)\end{aligned}$$

Alternatively, one can handle the constraints by introducing another bundle, denoted by E, which is a product bundle over X with fibers diffeomorphic to $Y_x \times \mathcal{V}_x$. One can think of E as a configuration bundle of the corresponding *unconstrained* system whose Lagrangian contains an additional part corresponding to the pairing of the constraint with the Lagrange multiplier:

$$L_\Phi = L + \langle \lambda, \Phi \rangle_{\mathcal{V}}.$$

The Euler-Lagrange equations of motion then follow from *unconstrained* Hamilton's principle in a standard way and coincide with (7.2).

Chapter 8

Incompressible Continuum Mechanics

In this chapter we shall consider the incompressibility constraint using the multisymplectic description of continuum mechanics. The main issue is a proper interpretation of the constraint using the Lagrange multiplier formalism developed in the previous chapter.

8.1 Configuration and Phase Spaces

Here we briefly summarize the results. See the analogous parts of Chapter 6 for more details.

Extended Covariant Configuration Bundle. The fibers of $\mathcal{V}$ in this case are one-dimensional and sections $\bar{\phi} = (\phi, \lambda)$ of E contain both the deformation field and the Lagrange multiplier, i.e., E denotes a bundle over X whose fibers are diffeomorphic to the product manifold $M \times \mathbb{R}$ with the projection map

$$\pi_{XE} : E \to X; \quad (x, t, y, \lambda) \mapsto (x, t).$$

Here λ is a section of the trivial bundle $X \times \mathbb{R}$ over X, which can be thought of as a function $\lambda(x,t)$ on X. The phase space is then the first jet bundle J^1E with coordinates $\bar{\gamma} = (x^\mu, y^a, \lambda, v^a{}_\mu, \beta_\mu)$; the first jet extension of a section $\bar{\phi} = (\phi, \lambda)$ has the following coordinate representation $(x^\mu, y^a, \lambda, \partial_\mu \phi^a, \partial_\mu \lambda)$.

The Dual Jet Bundle. We can consider the affine dual bundle J^1E^* as a "vertically invariant" subbundle Z of the bundle $\Lambda = \Lambda^{n+1}E$ of all $(n+1)$-forms on E. Elements of Z can be written uniquely as

$$z = \Pi d^{n+1}x + p_a{}^{\mu} dy^a \wedge d^n x_\mu + \pi^\mu d\lambda \wedge d^n x_\mu,$$

where $d^n x_\mu = \partial_\mu d^{n+1}x$, so that $(x^\mu, y^a, \lambda, \Pi, p_a{}^\mu, \pi^\mu)$ give coordinates on Z.

The canonical $(n+1)$-form is constructed in a standard manner and in the above coordinates has the following representation

$$\Theta = p_a{}^\mu dy^a \wedge d^n x_\mu + \pi^\mu d\lambda \wedge d^n x_\mu + \Pi d^{n+1}x.$$

We set $\Omega = -d\Theta$.

The *primary constraint manifold* $\mathfrak{C}$ is a subbundle of the dual jet bundle and corresponds to the incompressibility constraint. The pull-back of the inclusion map $i_{\mathfrak{C}} : \mathfrak{C} \to J^1E^*$ defines a degenerate $(n+2)$-form $\Omega_{\mathfrak{C}}$ on $\mathfrak{C}$. We shall discuss the explicit form of the constraint in the next subsection.

Incompressibility Constraint. Recall that such a constraint in, for example, incompressible fluid dynamics, is a reflection of the divergence-free property of the Eulerian fluid velocity and, hence, has a pointwise character. The divergence-free character of the velocity field arises from the requirement that the particle placement map be volume preserving at each instant of time. Then, according to the general theory of constrained multisymplectic fields outlined above, it can be obtained from a pointwise constraint Φ defined on the first jet bundle J^1Y.

For $\gamma \equiv (x^\mu, y^a, v^a{}_\mu) \in J^1Y$ we impose the constraint $\Phi(\gamma) = 0$ on the Jacobian of the deformation, where

$$\Phi : J^1Y \to \mathbb{R}; \quad \gamma \mapsto J(\gamma) - 1, \quad J(\gamma) = \det[v]\sqrt{\frac{\det[g(y)]}{\det[G(x)]}}, \tag{8.1}$$

where we have used the definition of J given in (6.1). Restricting Φ to the first jet of a section ϕ results in a constraint on the matrix of spatial partial derivatives $\partial_j \phi^a$.

For the Lagrange multiplier itself, we choose the following Ansatz

$$\lambda(x) = \sqrt{\det[G]} P(x) \ : \ X \to \mathbb{R}, \tag{8.2}$$

where P will be shown later to have the interpretation of the *material* pressure. Equation (8.2) guarantees that λ transforms like a density under the transformations of the base manifold X, so that the pairing of λ and Φ, defined by integrating over X, has the correct transformation law.

8.2 Lagrangian Dynamics

As we have already mentioned, the main distinguishing feature of incompressible models of continuum mechanics is the presence of the constraint (8.1). We shall now explain how this modification to the Lagrangian alters the Legendre transform as well as the Euler-Lagrange equations.

The Lagrangian Density. The Lagrangian density $\mathcal{L} : \ J^1E \to \Lambda^{n+1}X$ for the multisymplectic model of incompressible continuum mechanics is a smooth bundle map defined by

$$\mathcal{L}_\Phi(\bar{\gamma}) = (L(\gamma) + \lambda \cdot \Phi(\gamma)) \ d^{n+1}x = \mathbb{K} - \mathbb{P} + \lambda \cdot \Phi \ d^{n+1}x, \tag{8.3}$$

where L (i.e., $\mathbb{K}$ and $\mathbb{P}$) is given by (6.3) and depends on the choice of the stored energy function W.

The Legendre Transformation. For the above choice of the Lagrangian, the Legendre transform thought of as a fiber preserving bundle map $\mathbb{F}\mathcal{L}_\Phi : J^1E \to J^1E^*$ over E is degenerate due to the constrained character of the dynamics. Indeed, the Lagrange multiplier λ is a cyclic variable as the Lagrangian (8.3) does not depend on its derivatives, β_μ. Hence, the conjugate momentum to λ is identically zero: $\pi^\mu \equiv \partial L_\Phi / \partial \beta_\mu = 0$. The set $\{\pi^\mu = 0\}$ defines the primary constraint set as a subset of the dual bundle J^1E^* to which we restrict the Legendre transformation to make it non-degenerate. The rest of the momenta are given by the following expressions:

$$p_a{}^0 = \frac{\partial L_\Phi}{\partial v^a{}_0} = \rho g_{ab} v^b{}_0 \sqrt{\det[G]},$$

$$p_a{}^j = \frac{\partial L_\Phi}{\partial v^a{}_j} = \left(PJ(v^{-1})_a{}^j - \rho \frac{\partial W}{\partial v^a{}_j} \right) \sqrt{\det[G]}, \tag{8.4}$$

$$\Pi = \left[-\rho \frac{1}{2} g_{ab} v^a{}_0 v^b{}_0 + \rho \frac{\partial W}{\partial v^a{}_j} v^a{}_j - \rho W - P\left(J(n-1)+1\right) \right] \sqrt{\det[G]}.$$

Euler-Lagrange Equations. Using the trivialization (ϕ, λ), we now consider the Euler-Lagrange equations for a section $\bar{\phi}$ of E, both with respect to the deformation ϕ and with respect to the Lagrange multiplier λ. The former can be written in coordinates as follows:

$$\frac{\partial L_\Phi}{\partial y^a}(j^1\bar{\phi}) - \frac{\partial}{\partial x^\mu} \left(\frac{\partial L_\Phi}{\partial v^a{}_\mu}(j^1\bar{\phi}) \right) = 0. \tag{8.5}$$

The Euler-Lagrange equation for λ trivially recovers the constraint $\Phi = 0$ itself restricted to the first jet:

$$\frac{\partial L_\Phi}{\partial \lambda}(j^1\bar{\phi}) - \frac{\partial}{\partial x^\mu} \left(\frac{\partial L_\Phi}{\partial \beta_\mu}(j^1\bar{\phi}) \right) = \Phi(j^1\phi) d^{n+1}x = (J(j^1\phi) - 1) d^{n+1}x = 0. \tag{8.6}$$

These two equations are to be solved for the Lagrange multiplier λ (equivalently, for the pressure P) and for the section ϕ.

Substituting Lagrangian (8.3) into (8.5), we obtain the Euler-Lagrange equation (6.13) modified by the pressure term:

$$\rho g_{ab} \left(\frac{D_g \dot{\phi}}{Dt} \right)^b - \frac{1}{\sqrt{\det[G]}} \frac{\partial}{\partial x^k} \left(\rho \frac{\partial W}{\partial v^a{}_k}(j^1\phi) \sqrt{\det[G]} \right) =$$
$$- \rho \frac{\partial W}{\partial g_{bc}} \frac{\partial g_{bc}}{\partial y^a}(j^1\phi) - \frac{\partial P}{\partial x^k} (v^{-1})_a{}^k \, J(j^1\phi). \tag{8.7}$$

Notice that in the case of parameterized *non-constant* metrics, the extra pressure term in (8.4) gives rise to the term

$$\frac{\partial}{\partial y^b}\left(\left(P\ J(v^{-1})_a{}^j\sqrt{\det[G]}\right)(j^1\phi)\right)\frac{\partial y^b}{\partial x^j},$$

which follows from the chain rule applied to $\partial_{x^j} g(y(x))$. This term exactly cancels another term coming from differentiating the constraint with respect to y due again to the composition $g = g(y)$:

$$\lambda\frac{\partial\Phi}{\partial y^a} = \frac{P}{2}Jg^{bc}\frac{\partial g_{bc}}{\partial y^a}\sqrt{\det[G]}$$

and other cancellations occur as in equation (6.17).

In the case of Euclidean manifolds with constant metrics g and G, the Euler-Lagrange equations simplify to

$$\rho\frac{\partial^2\phi_a}{\partial t^2} = \frac{\partial}{\partial x^k}\left(\rho\frac{\partial W}{\partial v^a{}_k}(j^1\phi)\right) - \frac{\partial P}{\partial x^k}(v^{-1})_a{}^k\ J(j^1\phi) \tag{8.8}$$

together with the constraint (8.6).

8.3 Incompressible Ideal Hydrodynamics

For fluid dynamics, the stored energy term in the Lagrangian is a constant function precisely because of the incompressibility constraint. Indeed, as we have mentioned above, W in ideal fluid models is a function of the Jacobian J, but the latter is constrained to be 1. For simplicity, consider an ideal homogeneous incompressible fluid, so that the material density ρ is constant, and we can set $\rho = 1$ (for inhomogeneous fluids the dependence of material density on the point x is implicit in the pressure function P).

The Lagrangian is given by (8.3) with $\mathbb{P} = \text{const}$. Hence, two terms in equation (8.7) which correspond to the derivatives of W vanish, so that the dynamics of an incompressible ideal fluid is described by

$$g_{ab}\left(\frac{D_g\dot{\phi}}{Dt}\right)^b = -\frac{\partial P}{\partial x^k}J\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)^k_a, \tag{8.9}$$

together with the constraint

$$J(j^1\phi) = \left(\frac{\sqrt{\det[G\circ\phi]}}{\sqrt{\det[G]}}\det\left(\frac{\partial\phi}{\partial x}\right)\right)(x,t) = 1, \tag{8.10}$$

where we have used the fact that $g = G$.

Compare (8.9) with (6.18) and notice that the incompressibility constraint $J = 1$ implies that the ***spatial density*** $\rho_{\mathrm{sp}} = \rho/J$ is constant, e.g., 1. Introducing the ***spatial pressure*** $p = P\circ\phi_t^{-1}$, the above equation can be written as

$$\frac{D_g\dot{\phi}}{Dt}(x,t) = -\operatorname{grad} p\circ\phi(x,t), \tag{8.11}$$

where we have set $\rho_{\mathrm{sp}} = 1$. We remark again that the covariant derivative is evaluated at $y = \phi(x,t)$.

A New Look at the Pressure. Here we shall demonstrate that the same equations of motion are obtained if the potential energy in the Lagrangian (8.3) is not set to a constant, but rather is treated as a function of the Jacobian, $W = W(J)$. This will also clarify the relation between the two definitions of pressure that we have thus far examined.

Recall the definition of the pressure function for barotropic fluids given by (6.16) as a partial derivative of the stored energy function W with respect to the Jacobian J. Compare this to the definition (8.2) of the pressure as a Lagrange multiplier corresponding to the incompressibility constraint (8.1) (modulo a $\sqrt{\det[G]}$ term). In this subsection we shall denote these objects by P_W and P_λ, respectively:

$$P_W = -\rho\frac{\partial W}{\partial J}, \qquad P_\lambda = \frac{1}{\sqrt{\det[G]}}\lambda.$$

The resulting Euler-Lagrange equations can be obtained by combining (6.18) with (8.9) and are given by:

$$g_{ab}\left(\frac{D_g\dot{\phi}^b}{Dt}\right)^b = -\frac{\partial(P_W+P_\lambda)}{\partial x^k}J\left(\left(\frac{\partial\phi}{\partial x}\right)^{-1}\right)_a^k$$

together with the constraint (8.10). We can define a new pressure function

$$P = P_W + P_\lambda. \tag{8.12}$$

Notice that when the constraint $J = 1$ is enforced by the Euler-Lagrange equation (8.6), $P_W(J)$ = const, so that $P = P_\lambda$ + const. This is equivalent to a redefinition of the Lagrange multiplier λ. At the same time, the above Euler-Lagrange equation coincides with (8.9) because $\partial_k P = \partial_k P_\lambda$.

Relation to Standard Ideal Hydrodynamics. Recall the Lie-Poisson description of fluid dynamics as a right-invariant system on the group $\mathcal{D}_\mu(M)$ of volume-preserving diffeomorphisms of a Riemannian manifold (M, G). Here we follow [40] and [4], using our notations. The Lie algebra of $\mathcal{D}_\mu(M)$ is the algebra of divergence-free vector fields on M tangential to the boundary with minus the Jacobi-Lie bracket. The L^2 inner-product on this algebra is given by

$$\langle u, v\rangle_{L^2} = \int_M \langle u(x), v(x)\rangle_G \mu,$$

where μ is the Riemannian volume form on M.

We extend this metric by right invariance to the entire group. The resulting Riemannian manifold with right invariant L^2 metric, denoted by $(\mathcal{D}_\mu(M), L^2)$, is the configuration space for the Lie-Poisson or Euler-Poincaré model of ideal hydrodynamics. Its tangent bundle is the phase space, so that $(\eta_t, \dot{\eta}_t)$ are the basic "coordinates"; here $\eta_t \in \mathcal{D}_\mu(M)$ is a diffeomorphism that transforms the reference fluid configuration to its configuration at time t. Then, using the kinetic energy of fluid particles as a Lagrangian, one obtains the following covariant equations of motion:

$$\frac{D\dot{\eta}}{Dt}(x) = -\operatorname{grad} p \circ \eta(x), \tag{8.13}$$

where

$$\frac{D\dot{\eta}}{Dt} = \ddot{\eta} + \Gamma_\eta(\dot{\eta}, \dot{\eta})$$

denotes covariant material time derivative with respect to the metric (Γ_η denotes the connection associated to the metric) and p is the *spatial pressure.* Notice that covariant derivative is evaluated at $\eta(x)$.

Now define $\eta_t(x) = \eta(t, x)$ to be the flow of the time-dependent vector field $u(t, x)$, so that $\partial_t \eta(t, x) = u(t, \eta(t, x))$. Then composing (8.13) on the right with η^{-1} gives the classical Eulerian description of incompressible ideal fluids:

$$\partial_t u(t, x) + (u \cdot \nabla) u = -\operatorname{grad} p, \qquad \operatorname{div} u = 0.$$

Taking the divergence of both sides of this expression yields the equation for the pressure

$$\Delta p = -\operatorname{div}\left((u \cdot \nabla) u\right). \tag{8.14}$$

One readily notices that equations (8.11) and (8.13) coincide provided $\eta_t(x) = \phi(x, t)$. Upon this identification, the Euler-Lagrange equations for the multisymplectic model of incompressible ideal hydrodynamics recover the well-known evolution of fluid diffeomorphisms (8.13). Similarly, taking the divergence of both sides of (8.11) results in the Poisson equation on the pressure (8.14).

8.4 Incompressible Elasticity

In a manner similar to the previous section, we modify the elasticity Lagrangian by the constraint and extend the phase space to include the Lagrange multiplier. Recall that the stored energy is a function of the Green deformation tensor $W = W(C)$

and use the definition of the first Piola-Kirchhoff stress tensor $\mathcal{P}_a{}^i$ (6.19) to write down the equations of motion:

$$\rho g_{ab} \left(\frac{D_g \dot{\phi}}{Dt} \right)^b = \mathcal{P}_a{}^i{}_{|i} - \frac{\partial P}{\partial x^k} J \left(\left(\frac{\partial \phi}{\partial x} \right)^{-1} \right)^k_a ,$$

together with the constraint (8.6). The above equation can be written in a fully covariant form

$$\rho \frac{D_g V}{Dt} = \operatorname{DIV} \mathcal{P} - \operatorname{grad} p \circ \phi,$$

where $V = \dot{\phi}$ is the velocity vector field, $\mathcal{P}$ is the first Piola-Kirchhoff stress tensor, and p is the spatial pressure.

Chapter 9

Symmetries, Momentum Maps and Noether Theorem

We already mentioned in the introduction that homogeneous fluid dynamics has a huge symmetry, namely the particle relabeling symmetry, while standard elasticity (usually assumed to be inhomogeneous) has much smaller symmetry groups, such as rotations and translations in the Euclidean case. While inhomogeneous fluids (especially the compressible ones) are of great interest, the results worked out in Section 9.1 only apply to homogeneous fluid dynamics, when the symmetry group is the full group of volume-preserving diffeomorphisms $\mathcal{D}_\mu$. However, these results can be generalized to inhomogeneous fluids, in which case the symmetry group is a *subgroup* $\mathcal{D}_\mu^\rho \subset \mathcal{D}_\mu$ that preserves the level sets of the material density for barotropic fluids, or a *subgroup* $\mathcal{D}_\mu^{\rho,\mathrm{ent}} \subset \mathcal{D}_\mu$ that preserves the level sets of the material density and entropy for isentropic fluids.

A general model of continuum mechanics will have the metric g isometry as its symmetry. In particular, the group of rotations and translations is a symmetry for models of fluid dynamics and elasticity in Euclidean spaces (see [39] for an overall emphasis on continuum mechanics in Euclidean spaces).

The only symmetry which is universal for *non-relativistic* continuum mechanics is the time translation invariance. This is due to the fact that the base manifold is a tensor product of the spatial part and the time direction, rather than a space-time, so that all material quantities, such as density ρ, metric G, etc., depend only on

$x \in B \subset X$. In this chapter we shall treat these symmetries separately. We start with the particle relabeling symmetry, introducing the necessary notations.

9.1 Relabeling Symmetry of Ideal Homogeneous Hydrodynamics

In this section we shall consider both the barotropic model and the incompressible model of ideal homogeneous fluids with fixed boundaries at the same time. Their corresponding Lagrangians differ only by the constraint term and both are equivariant with respect to the action of the group of volume-preserving diffeomorphisms.

The Group Action. The action of the diffeomorphism group $\mathcal{D}_\mu(B)$ on the (spatial part of the) base manifold $B \subset X$ captures precisely the meaning of particle relabeling. For any $\eta \in \mathcal{D}_\mu(B)$, denote this action by $\eta_X : (x,t) \mapsto (\eta(x),t)$. The lifts of this action to the bundles Y and E are given by $\eta_Y : (x,t,y) \mapsto (\eta(x),t,y)$ and $\eta_E : (x,t,y,\lambda) \mapsto (\eta(x),t,y,\lambda)$, respectively. Both lifts are fiber-preserving and act on the fibers themselves by the identity transformation. The coordinate expressions have the following form:

$$\eta_X^0 = \mathrm{Id}\cdot t, \ \eta_X^i = \eta^i(x), \quad \eta_Y^a = \delta_b^a y^b, \quad \eta_E^a = (\delta_b^a y^b, \mathrm{Id}\cdot\lambda). \tag{9.1}$$

Jet Prolongations. The jet prolongations are natural lifts of automorphisms of Y to automorphisms of its first jet J^1Y and can be viewed as covariant analogues of the tangent maps (see [16]).

Let γ be an element of J^1Y and $\bar{\gamma}$ be a corresponding element of the extended phase phase space J^1E, in coordinates $\gamma = (x^\mu, y^a, v_\mu^a)$ and $\bar{\gamma} = (x^\mu, y^a, \lambda, v_\mu^a, \beta_\mu)$. The prolongation of η_Y is defined by

$$\eta_{J^1Y}(\gamma) = T\eta_Y \circ \gamma \circ T\eta_X^{-1}, \qquad \eta_{J^1E}(\bar{\gamma}) = T\eta_E \circ \bar{\gamma} \circ T\eta_X^{-1}. \tag{9.2}$$

We shall henceforth consider η_{J^1E}, since it includes η_{J^1Y} as a special case. In coordinates, we have

$$\eta_{J^1E}(\bar{\gamma}) = \left(\eta^k(x), t; y^b, \lambda; v^a{}_0, v^a{}_m \left((\frac{\partial \eta}{\partial x})^{-1} \right)^m_j ; \beta_0, \beta_m \left((\frac{\partial \eta}{\partial x})^{-1} \right)^m_j \right).$$

If ξ is a vector field on E whose flow is η_ϵ, then its prolongation $j^1\xi$ is the vector field on J^1E whose flow is $j^1(\eta_\epsilon)$, that is $j^1\xi \circ j^1(\eta_\epsilon) = (d/d\epsilon) j^1(\eta_\epsilon)$. In particular, the vector field ξ corresponding to η_E given by (9.1) has coordinates $(\xi^i, 0, 0, 0)$ and is divergence-free; its prolongation $j^1\xi$, which corresponds to the prolongation η_{J^1E} of η_E, has the following coordinate expression:

$$j^1\xi = \left(\xi^i, 0; 0, 0; 0, -v^a{}_m \frac{\partial \xi^m}{\partial x^j}; 0, -\beta_m \frac{\partial \xi^m}{\partial x^j}\right). \tag{9.3}$$

Noether's Theorem. Suppose the Lie group $\mathcal{G}$ acts on $\mathcal{C}$ and leaves the action S invariant. This is equivalent to the Lagrangian density $\mathcal{L}$ being ***equivariant*** with respect to $\mathcal{G}$, that is, for all $\eta \in \mathcal{G}$ and $\gamma \in J^1Y$,

$$\mathcal{L}(\eta_{J^1Y}(\gamma)) = (\eta_X^{-1})^* \mathcal{L}(\gamma),$$

where $(\eta_X^{-1})^* \mathcal{L}(\gamma) = (\eta_X)_* \mathcal{L}(\gamma)$ is a push-forward; this equality means equality of $(n+1)$-forms at $\eta(x)$. Denote the ***covariant momentum map*** on J^1Y by $J_\mathcal{L} \in L(\mathfrak{g}, \Lambda^n(J^1Y))$. It is defined by the following expression

$$j^1(\xi) \lrcorner \Omega_\mathcal{L} = dJ_\mathcal{L}(\xi) \tag{9.4}$$

and can be thought of as a Lie algebra valued n-form on J^1Y.

Recall that ϕ is a solution of the Euler-Lagrange equations if and only if

$$(j^1\phi)^*(\mathcal{W} \lrcorner \Omega_\mathcal{L}) = 0$$

for any vector field $\mathcal{W}$ on J^1Y. In particular, setting $\mathcal{W} = j^1(\xi)$ and applying $(j^1\phi)^*$ to (9.4), we obtain the following basic Noether conservation law:

Theorem 9.1.1. *Assume that group $\mathcal{G}$ acts on Y by π_{XY}-bundle automorphisms* *and that the Lagrangian density $\mathcal{L}$ is equivariant with respect to this action for any $\gamma \in J^1Y$. Then, for each $\xi \in \mathfrak{g}$*

$$\mathbf{d}\left((j^1\phi)^* J_{\mathcal{L}}(\xi)\right) = 0 \tag{9.5}$$

for any section ϕ of π_{XY} satisfying the Euler-Lagrange equations. The quantity $(j^1\phi)^* J_{\mathcal{L}}(\xi)$ is called the ***Noether current****.*

See [16] for a proof.

The Variational Route to Noether's Theorem. The variational route to the covariant Noether's theorem on J^1Y was first presented in [34]. We shall briefly describe this formulation now.

Recall the notations of the maps $\varphi : U \to Y$ and the corresponding induced local sections $\varphi \circ \varphi_X^{-1}$ of Y from Section 6.2. Here again it is important to allow for both vertical and horizontal variations of the sections. Vertical variations alone capture only fiber preserving symmetries (i.e., spatial symmetries), while taking arbitrary variations allows for both material and spatial symmetries to be included.

The invariance of the action $S = \int_{U_X} \mathcal{L}$ under the Lie group action is formally represented by the following expression:

$$S(\eta_Y \cdot \varphi) = S(\varphi) \quad \text{for all} \quad \eta_Y \in \mathcal{G}. \tag{9.6}$$

Equation (9.6) implies that for each $\eta_Y \in \mathcal{G}$, $\eta_Y \cdot \varphi$ is a solution of the Euler-Lagrange equations, whenever φ is a solution. We restrict the action of $\mathcal{G}$ to the space of solutions, and let $\xi_{\mathcal{C}}$ be the corresponding infinitesimal generator on $\mathcal{C}$ restricted to the space of solutions; then

$$\begin{aligned} 0 = (\xi_{\mathcal{C}} \lrcorner \, dS)(\varphi) &= \int_{\partial U_X} j^1(\varphi \circ \varphi_X^{-1})^* [j^1(\xi) \lrcorner \, \Theta_{\mathcal{L}}] \\ &= \int_{U_X} j^1(\varphi \circ \varphi_X^{-1})^* [j^1(\xi) \lrcorner \, \Omega_{\mathcal{L}}], \end{aligned}$$

since the Lie derivative $\mathfrak{L}_{j^1(\xi)} \Theta_{\mathcal{L}} = 0$ by (9.6) and Corollary 6.2.1.

Using (9.4), we find that $\int_{U_X} d[j^1(\varphi \circ \varphi_X^{-1})^* J_{\mathcal{L}}(\xi)] = 0$, and since this holds for arbitrary regions U_X, the integrand must also vanish. Recall that $\phi = \varphi \circ \varphi_X^{-1}$ is a true section of the bundle Y, so that this is precisely a restatement of the Noether's Theorem 9.1.1.

Covariant Canonical Transformations. The computations of the momentum map from definition (9.4) can be simplified significantly in some special cases which we discuss here. A π_{XJ^1Y}-bundle map $\eta_{J^1Y} : J^1Y \to J^1Y$ covering the diffeomorphism $\eta_X : X \to X$ is called a ***covariant canonical transformation*** if $\eta^*_{J^1Y}\Omega_{\mathcal{L}} = \Omega_{\mathcal{L}}$. It is called a ***special covariant canonical transformation*** if $\eta^*_{J^1Y}\Theta_{\mathcal{L}} = \Theta_{\mathcal{L}}$. Recall that forms $\Omega_{\mathcal{L}}$ and $\Theta_{\mathcal{L}}$ can be obtained either by variational arguments or by pulling back canonical forms Ω and Θ from the dual bundle using the Legendre transformation $\mathbb{F}\mathcal{L}$.

From [16], any η_{J^1Y} which is obtained by lifting some action η_Y on Y to J^1Y, is automatically a special canonical transformation. In this case the momentum mapping is given by

$$J_{\mathcal{L}}(\xi) = j^1\xi \,\lrcorner\, \Theta_{\mathcal{L}}. \tag{9.7}$$

We remark that the validity of this expression does not rely on the way in which the Cartan form was derived, i.e., for simplicity of the computations in concrete examples, one can forgo the issues of vertical vs. arbitrary variations in the variational derivation and obtain the Cartan form directly from the dual bundle by means of Legendre transformations. Then, evaluating this form on the prolongation of a vector of an infinitesimal generator gives the momentum n-form.

Equivariance of the Lagrangian. To apply Theorem 9.1.1 to our case we need to establish equivariance of the fluid Lagrangians:

Proposition 9.1.1. *The Lagrangian of an ideal homogeneous barotropic fluid (6.3) and the Lagrangian of an ideal homogeneous incompressible fluid (8.3) are equivari-*

ant with respect to the $\mathcal{D}_\mu(B)$ action (9.1):

$$\mathcal{L}(\eta_{J^1E}(\bar{\gamma})) = (\eta_X^{-1})^* \mathcal{L}(\bar{\gamma}),$$

for all $\bar{\gamma} \in J^1E$.

Proof. First observe that the material density of an ideal homogeneous (compressible or incompressible) fluid is constant. Notice also that Lagrangians (6.3) and (8.3) differ only in the potential energy terms. Both these terms are functions of the Jacobian, which is equivariant with respect to the action of volume preserving diffeomorphisms given by (9.3). Indeed,

$$\begin{aligned} f\left(J(\eta_{J^1E}(\bar{\gamma}))\right) d^{n+1}x &= f\left(\frac{\sqrt{\det[g]}}{\sqrt{\det[G]}} \det(v) \det\left(\frac{\partial \eta}{\partial x}\right)^{-1}\right) d^{n+1}x \\ &= (\eta_X^{-1})^* \left(f(J(\bar{\gamma})) d^{n+1}x\right) \end{aligned}$$

due to the fact that $\det \partial_i \eta^j = 1$ for a volume preserving diffeomorphism η; here f can be any function, e.g., the stored energy W or the constraint Φ.

For the same reason, and the fact that (9.3) acts trivially on $v^a{}_0$, the kinetic part of both Lagrangians is also equivariant. □

Proposition 9.1.1 enables us to use (9.7) for explicit computations of the momentum maps for the relabeling symmetry of homogeneous hydrodynamics. We shall consider barotropic and incompressible ideal fluids separately because their Lagrangians and, hence, their momentum mappings are different.

Barotropic Fluid

Using (9.7) we can compute the Noether current corresponding to the relabeling symmetry of the Lagrangian (6.3) to be

$$\begin{aligned} j^1(\phi)^* J_{\mathcal{L}}(\xi) = \left(\frac{1}{2}\rho g_{ab}\dot{\phi}^a\dot{\phi}^b - \rho W - PJ\right) \sqrt{\det[G]}\xi^k d^n x_k - \\ \left(g_{ab}\dot{\phi}^b\phi^a_{,k}\right) \rho\sqrt{\det[G]}\xi^k d^n x_0, \quad (9.8) \end{aligned}$$

where $j^1\xi$ is the prolongation of the vector field ξ and is given by (9.3).

The differential of this quantity restricted to the solutions of the Euler-Lagrange equation is identically zero according to Theorem 9.1.1. Conversely, requiring the differential of (9.8) to be zero for arbitrary sections ϕ recovers the Euler-Lagrange equation. Indeed, computing the exterior derivative and taking into account that the vector field ξ is divergence free, we obtain:

$$g_{ab}\left(\frac{D_g\dot{\phi}}{Dt}\right)^b = -\frac{\partial P}{\partial x^k} J \left(\left(\frac{\partial \phi}{\partial x}\right)^{-1}\right)^k_a,$$

which coincides with the Euler-Lagrange equation (6.18).

Incompressible Ideal Fluid

Similar computations using Lagrangian (8.3) with the potential energy set to a constant give the following expression for the Noether current corresponding to the relabeling symmetry:

$$j^1(\phi)^* J_{\mathcal{L}}(\xi) = \left(\frac{1}{2}\rho g_{ab}\dot{\phi}^a\dot{\phi}^b - P\right)\sqrt{\det[G]}\xi^k d^n x_k - \left(g_{ab}\dot{\phi}^b\phi^a_{,k}\right)\rho\sqrt{\det[G]}\xi^k d^n x_0. \quad (9.9)$$

The assumptions of Theorem 9.1.1 are satisfied; hence the exterior differential of this Noether current $d\left(j^1(\phi)^* J_{\mathcal{L}}(j^1\xi)\right)$ is equal to zero for all section ϕ which are solutions of the Euler-Lagrange equations.

Now consider the inverse statement. That is, let us analyze whether the Noether conservation law implies the Euler-Lagrange equations for incompressible ideal fluids. Computing the exterior differential of (9.9) for an arbitrary section $\bar{\phi} = (\phi, \lambda)$ we obtain:

$$g_{ab}\left(\frac{D_g\dot{\phi}}{Dt}\right)^b = -\frac{\partial P}{\partial x^k}\left(\left(\frac{\partial \phi}{\partial x}\right)^{-1}\right)^k_a.$$

Here, we have used the fact that ξ is a divergence free vector field on X. This is precisely the Euler-Lagrange equation (8.9) with the constraint $J = 1$ substituted in it. We point out that the above equation is not equivalent to the Euler-Lagrange equations, i.e., the constraint cannot be recovered from the Noether current. Indeed, the symmetry group is the same both for homogeneous compressible barotropic fluids and for homogeneous incompressible fluids. Notice also that the Noether currents (9.8) and (9.9) are different due to the difference in the corresponding Lagrangians.

9.2 Time Translation Invariance

Lagrangian densities (6.3) and (8.3) are equivariant with respect to the group $\mathbb{R}$ action on Y, given by $\tau_Y : (x, t, y) \mapsto (x, t + \tau, y)$ for any $\tau \in \mathbb{R}$. This is because the Lagrangians are explicitly time independent. One can readily compute the jet prolongation of the corresponding infinitesimal generator vector field $\zeta_Y = (0, \zeta, 0)$, where $\tau = \exp \zeta$. Then, the pull-back by $j^1\phi$ of the covariant momentum map corresponding to this symmetry, which we denote by $J^t_{\mathcal{L}}$ to distinguish it from expressions in the previous section, is given by the following n-form on X:

$$
\begin{aligned}
(j^1\phi)^* J^t_{\mathcal{L}}(\zeta) = \zeta \left(L(j^1\phi) d^n x_0 - p_a{}^{\mu}(j^1\phi)\dot{\phi}^a d^n x_\mu \right) = \\
- \zeta \left(e(j^1\phi) d^n x_0 + p_a{}^{j}(j^1\phi)\dot{\phi}^a d^n x_j \right) (j^1\phi),
\end{aligned}
$$

where, in the last equality, we have used the definition of the energy density e given by (6.5).

Noether's Theorem 9.1.1 implies that the exterior derivative of this expression will be zero along solutions of the Euler-Lagrange equations. Computing this divergence for an arbitrary ζ recovers the energy continuity equation. For a barotropic fluid, it is given by

$$
\dot{e} = -\sqrt{\det[G]}\,\mathrm{DIV}\left(PJ \left(\left(\frac{\partial \phi}{\partial x} \right)^{-1} \right)^j_a \dot{\phi}^a \right),
$$

while for standard elasticity the equation has the form:

$$\dot{e} = \sqrt{\det[G]}\,\mathrm{DIV}(\mathcal{P}_a{}^j \dot{\phi}^a).$$

The expressions for an incompressible fluid and elastic medium are similar.

Alternatively, one can consider the inverse statement and *require* that $d\left(J^t_{\mathcal{L}}(\zeta)\right) = 0$. This forces the energy continuity equation to be satisfied for some arbitrary section ϕ.

Chapter 10

Conclusions and Future Directions

In the last chapter we would like to comment on the work in progress and point out general future directions of the multisymplectic discretization program. Some of the aspects discussed will be addressed in [39].

Brief Summary. This thesis develops discrete reduction techniques for mechanical systems defined on Lie groups and also presents multisymplectic formulation of both compressible and incompressible models of continuum mechanics on general Riemannian manifolds. While the former synthesizes ideas of Euler-Poincaré and Lie-Poisson reduction for mechanical systems with the Veselov type discretization of such systems, the latter sets the stage for multisymplectic reduction and for further development of Veselov type multisymplectic discretizations.

More General Configuration Spaces for Mechanical Systems. Besides generalizations to infinite dimensional systems, the ideas of Part I carry over to the integration of systems defined on a general (finite dimensional) configuration space M with some symmetry group G. In this case, the reduced discrete space $(M \times M)/G$ inherits a Poisson structure from the one defined on $M \times M$ (analogously to (3.13)). Its symplectic leaves again become dynamically invariant manifolds for structure-preserving integrators and can be viewed as images of the symplectic leaves of the reduced Poisson manifold T^*M/G under appropriately defined "Legendre transformations." This is a topic of ongoing research that builds on recent progress in

Lagrangian reduction theory; see [42].

Other Models of Continuum Mechanics. The formalism set up in Part II naturally includes other models of three-dimensional linear and non-linear elasticity and fluid dynamics, as well as rod and shell models. For elasticity, the choice of the stored energy W determines a particular model with the corresponding Euler-Lagrange equation given by (6.13); this is a PDE to be solved for the deformation field ϕ. Introducing the first Piola-Kirchhoff stress tensor $\mathcal{P}$, the same equation can be written in a compact fully covariant form (6.21). An explicit form of the Euler-Lagrange equations and conservation laws for rod and shell models are not included here but can be easily derived by following the steps outlined above. The constrained director models which are common in such models are handled well by the formulation of constraints that we use in Chapter 7.

Constrained Multisymplectic Theories. The issue of holonomic vs. non-holonomic constraints in classical mechanics has a long history in the literature. Though there are still many open questions, the subject of linear and affine non-holonomic constraints is relatively well-understood (see, e.g., [7]). We already mentioned in Chapter 7 that this topic is wide open for multisymplectic field theories, partly due to the fact that there is simply no well-defined notion of a non-holonomic constraint for such theories – it appears that one needs to distinguish between time and space partial derivatives.

As all of the examples under present consideration are non-relativistic and do not have constraints involving time derivatives, we used the restriction of Hamilton's principle to the space of allowed configurations to derive the equations of motion. We note, however, that this reduces to vakonomic mechanics in the case of an ODE system with non-holonomic constraints, and is thus incorrect. Whether the approach taken here is correct for constrained fully covariant relativistic field theories is left as an open question, awaiting a reasonable test example.

Also, constraints involving higher than first-order derivatives are beyond the current exposition and should be treated in the context of higher-order multisymplectic field theories defined on $J^kY,\ k > 1$ (see, e.g. [25]).

Multisymplectic Form Formula and Conservation Laws. A very important aspect of any multisymplectic field theory is the existence of the multisymplectic form formula (6.11) which is the covariant analogue of the fact that the flow of conservative systems consists of symplectic maps. The implications of this formula to the multisymplectic continuum mechanics in a special case of Euclidean spaces in considered in [39]. Preliminary results indicate, however, that applications of the multisymplectic form formula not only can be linked to some known principles in elasticity (such as the Betti reciprocity principle), but also can produce some new interesting relations which depend on the space-time direction of the first variations $\mathcal{V}, \mathcal{W}$ in (6.11). An accurate and consistent discretization of the model then results in so called ***multisymplectic integrators*** which preserve the discrete analogues of the multisymplectic form and the conservation laws.

Discretization. A structure preserving discretization is one of the key aspects of the multisymplectic project and is currently under investigation. It is demonstrated in [39]) that the finite element method for static elasticity is a multisymplectic integrator. Moreover, based on the result in [44], it is shown that the finite elements time-stepping with the Newmark algorithm is a multisymplectic discretization.

As we mentioned in the previous paragraph, a consistent discretization based on the variational principle would preserve the discrete multisymplectic form together with the discrete multi-momentum maps corresponding to the symmetries of a particular system. Then, the global (integral) form of the discrete Noether's theorem would imply that a sum of the values of the discrete momentum map over some set of nodes is zero. One implication of this statement for incompressible fluid dynamics is a discrete version of the vorticity preservation. Such discrete conservations are among the hot topics of the ongoing research.

Symmetry Reduction. In the previous chapter we discussed at length the particle relabeling symmetry of ideal homogeneous hydrodynamics and its multisymplectic realization. Reduction by this symmetry takes us from the Lagrangian description in terms of *material* positions and velocities to the Eulerian description in terms of *spatial* velocities. In the compressible case one only reduces by the subgroup of the particle relabeling group that leaves the stored energy function invariant; for example, if the stored energy function depends on the deformation only through the density and entropy, then this means that one introduces them as dynamic fields in the reduction process, as in Euler-Poincaré theory (see [18]).

In the unconstrained (i.e., defined on the extended jet bundle J^1E) multisymplectic description of ideal incompressible fluids, the multisymplectic reduced space is realized as a fiber bundle Υ over X whose fiber coordinates include the Eulerian velocity u and some extra field corresponding to compressibility. Then, the reduced Lagrangian density determines, by means of a constrained variational principle, the Euler-Lagrange equations which give the evolution of the spatial velocity field $u(x) \in \Upsilon_x$ together with a condition of u being divergence-free. A general Euler-Poincaré type theorem relates this equation with equation (8.11) by relating the corresponding variational principles.

Such a description is a particular example of a general procedure of multisymplectic reduction. The case of a finite-dimensional vertical group action was first considered in [8]. More general cases of an infinite-dimensional group action such as that for incompressible ideal hydrodynamics, electro-magnetic fields and symmetries in complex fluids is planned for a future publication (the reader is also referred to a related work by [14]). Applications of structure preserving discretization methods to the reduced hydrodynamic systems would result in schemes which compute the Eulerian velocity field or the vorticity field of an ideal fluid in a way that preserves all the known first integrals.

Vortex Methods. One of our ultimate objectives is to further develop, using the multisymplectic approach, some methods and techniques which were derived in the infinite-dimensional framework and which proved to be very useful. One of them is the vortex blob method developed by Chorin [11], which recently has been linked to the so-called averaged Euler equations of ideal fluid (see citeOS).

Higher Order Theories. Constraints involving higher than first-order derivatives are beyond the current exposition and should be treated in the context of higher-order multisymplectic field theories defined on $J^kY,\ k > 1$.

The averaged Euler equations (see [18] and [42] and references therein) provide an interesting example of a higher order fluid theory with constraints (depending only on first derivatives of the field) to which the multisymplectic methods can presumably be applied by using the techniques of [25]. It would be interesting to carry this out in detail. In the long run, this promises to be an important computational model, so that its formulation as a multisymplectic field theory and the multisymplectic discretization of this theory is of considerable interest.

Bibliography

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics.* Second Edition, Addison-Wesley, 1978.

[2] V.I. Arnold, Sur la géométrie differentielle des groupes de Lie de dimenson infinie et ses applications à l'hydrodynamique des fluids parfaits. *Ann. Inst. Fourier, Grenoble* **16**, (1966), 319–361.

[3] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, 1978. Second Edition, 1989.

[4] V.I. Arnold and B. Khesin, *Topological Methods in Hydrodynamics*, Springer Verlag, New York, 1998.

[5] A. M. Bloch and P.E. Crouch, Optimal control, optimization and analytical mechanics. *Mathematical control theory*, 268–321, Springer, New York (1999).

[6] A. Bloch, P. Crouch, J.E. Marsden and T.S. Ratiu, Optimal Control and Discrete Rigid Body Equations, *Proc. CDC*, **38**, (1998).

[7] A.M. Bloch, P.S. Krishnaprasad and J.E. Marsden and R.M. Murray, Nonholonomic mechanical systems with symmetry, *Arch. Rat. Mech. An.*, **136**, (1996), 21–99.

[8] M. Castrillón-López, T.S. Ratiu and S. Shkoller, Reduction in Principal Fiber Bundles: Covariant Euler-Poincaré Equations, *Proc. Am. Math. Soc.*, (2000), to appear.

[9] H. Cendra, J.E. Marsden and T.S. Ratiu, *Lagrangian reduction by stages*, (1998), preprint.

[10] P.J. Channell and J.C. Scovel, *Integrators for Lie-Poisson dynamical systems*, Physica D, **50**, (1991), 80–88.

[11] A. Chorin, Numerical study of slightly viscous flow, *J. Fluid Mech.* **57**, (1973), 785-796.

[12] A.J. Dragt and J.M. Finn, Lie series and invariant functions for analytic symplectic maps, *J. Math. Phys.*, **17**, 2215–2227 (1976).

[13] D.G. Ebin and J.E. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. Math.* **92**, (1970), 102–163.

[14] A. Fernández, P.L. García and C. Rodrigo, Stress-energy-momentum tensors in higher order variational calculus, *J. Geom. Phys.*, (1999), to appear.

[15] Z. Ge and J.E. Marsden, *Lie-poisson hamilton-jacobi theory and lie-poisson integrators*, Phys. Lett A, **133**, (1988), 134–139.

[16] M.J. Gotay, J.A. Isenberg and J.E. Marsden, *Momentum Maps and the Hamiltonian Structure of Classical Relativistic Field Theories, I and II*, (1997), preprints.

[17] M.J. Gotay, and J.E. Marsden, Stress-energy-momentum tensors and the Belifante-Resenfeld formula. *Cont. Math. AMS.* **132**, (1992), 367–392.

[18] D.D. Holm, J. E. Marsden and T. S. Ratiu, The Euler–Poincaré equations and semidirect products with applications to continuum theories, *Adv. in Math.*, **137**, (1998), 1-81.

[19] S.M. Jalnapurkar, Modeling of Constrained Systems, (1994), *unpublished.*

[20] C. Kane, J.E. Marsden and M. Ortiz, Symplectic-energy-momentum preserving variational integrators, *J. Math. Phys.*, **40**, (1999), 3353–3371.

[21] C. Kane, J.E. Marsden, M. Ortiz and M. West, Variational Integrators and the Newmark Algorithm for Conservative and Dissipative Mechanical Systems (*submitted for publication*).

[22] J. Kijowski and G. Magil, Relativistic elastomechanics as a Lagrangian field theory. *J. Geo. Phys.* **9**, (1992), 207-223.

[23] J. Kijowski and W. Tulczyjew, *A Symplectic Framework for Field Theories.* Springer Lect. Notes in Physics **107** (1979).

[24] B. Kostant, *Orbits, symplectic structures and representation theory*, Proc. US-Japan Seminar in Differential Geometry, Nippon Hyoronsha, Tokyo (1966).

[25] S. Kouranbaeva and S. Shkoller, A variational approach to second-order multisymplectic field theory, *J. Geom. Phys.*, (1999) to appear.

[26] A.D. Lewis, The geometry of the Gibbs-Appell equations and Gauss's Principle of Least Constraint, *Reports on Mathematical Physics*, **38** (1), (1996), 11-28.

[27] A.D. Lewis and R.M. Murray, Variational principles for constrained systems: theory and experiment, *The International Journal of Nonlinear Mechanics*, **30**, (6) (1995), 793–815.

[28] D. Lewis and J.C. Simo, *Conserving algorithms for the N-dimensional rigid body*, Fields. Inst. Comm., **10**, (1996), 121–139.

[29] S. Lie, *Theorie der Transformationsgruppen.* Zweiter Abschnitt. Teubner (1890).

[30] J. Lu and P. Papadopoulos, A covariant constitutive approach to finite plasticity, to appear in *Proc. 8th International Symposium on Plasticity* (1999).

[31] D. G. Luenberger, *Optimization by Vector Space Methods.* John Wiley, New York (1969).

[32] J.E. Marsden, *Lectures on Mechanics*, Cambridge University Press, 1992.

[33] J.E. Marsden and T.J.R. Hughes, *Mathematical Foundations of Elasticity.* Prentice Hall, reprinted by Dover Publications, N.Y., 1994.

[34] J.E. Marsden, G. Patrick and S. Shkoller, *Multisymplectic geometry, variational integrators and nonlinear PDEs*, Comm. Math. Phys., **199**, (1998), 351–395.

[35] J.E. Marsden and S. Pekarsky, Multisymplectic reduction with applications to electro-magnetic field and ideal hydrodynamics, *preprint* (2000).

[36] J.E. Marsden, S. Pekarsky and S. Shkoller, Discrete Euler-Poincaré and Lie-Poisson Equations, *Nonlinearity*, **12**, (1999), 1647-1662.

[37] J.E. Marsden, S. Pekarsky and S. Shkoller, Symmetry reduction of discrete Lagrangian mechanics on Lie groups, to appear in *J. Geom. Phys.* (1999).

[38] J. E. Marsden, S. Pekarsky, S. Shkoller and M. West, Variational methods, multisymplectic geometry and continuum mechanics, submitted to *J. Geom. Phys.* (2000).

[39] J. E. Marsden, S. Pekarsky, S. Shkoller and M. West, Multisymplectic continuum mechanics in Euclidean spaces, *preprint* (2000).

[40] J.E. Marsden and T.S. Ratiu, *Introduction to Mechanics and Symmetry*, Springer-Verlag, 1994. Second Edition, 1999.

[41] J.E. Marsden and T.S. Ratiu, *Mechanics and Symmetry: Reduction Theory*, in preparation.

[42] J.E. Marsden, T.S. Ratiu and J. Scheurle, Reduction theory and the Lagrange-Routh equations, *J. Math. Phys.* (to appear) (1999).

[43] J.E. Marsden, T.S. Ratiu and S. Shkoller, The geometry and analysis of the averaged Euler equations and a new diffeomorphism group, *Geom. and Funct. An.* (to appear) (2000).

[44] J. E. Marsden and M. West, The multisymplectic nature of classical elasticity and its finite element discretization, *preprint* (2000).

[45] R. McLachlan, *Explicit Lie-Poisson entegration and the euler equations*, Phys. Rev. Lett., **71**, (1993), 3043–3046.

[46] R. McLachlan and C. Scovel, *A survey of open problems in symplectic integration*, Fields Inst. Comm., **10**, (1996), 151–180.

[47] J. Moser and A.P. Veselov, *Discrete versions of some classical integrable systems and factorization of matrix polynomials*, Comm. Math. Phys., **139**, (1991), 217–243.

[48] M. Oliver and S. Shkoller, The vortex blob method as a second-grade non-Newtonian fluid, *preprint* (1999).

[49] H. Poincaré, Sur une forme nouvelle des équations de la mécanique. *C.R. Acad. Sci.*, **132**, (1901), 369-371.

[50] A.P. Veselov, *Integrable discrete-time systems and difference operators*, Funk. Anal. Prilozhen., **22**, (1988), 1–13.

[51] A.P. Veselov, *Integrable lagrangian correspondences and the factorization of matrix polynomials*, Funk. Anal. Prilozhen., **25**, (1991), 38–49.

[52] A. Weinstein, Lagrangian mechanics and groupoids, *Fields. Inst. Comm.*, **7**, (1996), 207–231.

[53] J.M. Wendlandt and J.E. Marsden, Mechanical integrators derived from a discrete variational principle, *Physica D*, **106**, (1997), 223–246.