AN EFFICIENT APPROXIMATE SOLUTION METHOD FOR PREDICTING THE BUCKLING OF AXIALLY COMPRESSED IMPERFECT CYLINDRICAL SHELLS

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Aeronautical Engineer

California Institute of Technology

Pasadena, California

1975

ACKNOWLEDGMENT

The author wishes to take this opportunity to sincerely thank Dr. J. Arbocz for the patience and guidance he generously extended during the course of this investigation. The author also thanks Drs. C. D. Babcock and E. E. Sechler for their advice and comments and Mrs. Elizabeth Fox for skillfully typing the manuscript.

This study was supported in part by the National Science Foundation under Research Grant GK 16934 and this aid is gratefully acknowledged.

Finally the author is indebted to his loving wife for her patience and encouragement during many days of graduate work.

ABSTRACT

A theoretical investigation of an efficient numerical solution scheme to solve approximately the nonlinear Donnell equations for imperfect isotropic cylindrical shells with edge restraints and under axial compression was carried out.

The nonlinear partial differential equations have been reduced to an equivalent set of nonlinear ordinary differential equations. The resulting two-point boundary value problem was solved, first, by using "Newton's Method of Quasilinearization" to obtain a set of linearized differential equations for the correction terms and, secondly, these differentials were approximated as finite differences to cast the linearized system of equations into the form of a block tridiagonal matrix equation. The Potters' Method solution scheme was used to solve efficiently the block tridiagonal matrix equation. By successive iterations a solution to the set of nonlinear ordinary differential equations was obtained.

The use of this method makes it possible to investigate how the axial load level at the limit point is affected by the following factors: the choice of inplane boundary conditions, the prebuckling growth caused by the radial edge constraint, the orientation and shape of the axisymmetric and asymmetric imperfection components, and the finite length of the shell.

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A_0, A_1	=	axial dependence of the radial imperfection (see eq. 8)
A_i, B_i, C_i	=	coefficient matrices defined by eq. 33
c	=	Poisson's effect ($c = \sqrt{3(1-\nu^2)}$)
$\underset{\sim}{d}_i$	=	error vector defined by eq. 33
E	=	Young's modulus
f_0, f_1, f_2, F	=	Airy stress functions (see eq. 10)
G_1, G_N, H_1, H_N	=	coefficient matrices defined by eqs. 35 and 36
h	=	dimensionless spacing of grid points
i, k	=	number of half waves in the axial direction
L	=	length of shell
n	=	number of full waves in the circumferential direction
N	=	number of grid points
N_x, N_y, N_{xy}	=	stress resultants (lb/in.)
P_i	=	coefficient matrix in the Potters' equation (38)
$\underset{\sim}{q}_i$	=	vector term in the Potters' equation (38)
R	=	radius of shell
t	=	thickness of shell
w_0, w_1, W	=	radial displacement, positive outward (see eq. 9)
$\overline{W}$	=	radial imperfection from perfect circular cylinder (see eq. 8)
x, y	=	axial and circumferential coordinates on middle surface of shell, respectively
$\bar{x}, \bar{y}$	=	nondimensional coordinates ($\bar{x} = x/R$, $\bar{y} = y/R$)
v	=	circumferential displacement

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$\delta \underset{\sim}{Y}_i$	=	correction vector defined by eq. 32
$\underset{\sim}{\epsilon}_1, \underset{\sim}{\epsilon}_N$	=	error vectors defined by eqs. 35 and 36
λ	=	nondimensional loading parameter $\left(\lambda = \sqrt{3(1-\nu^2)}\ \frac{R}{t}\ \frac{\sigma_x}{E}\right)$
ν	=	Poisson's ratio
$\bar{\xi}$	=	initial imperfection amplitude
$\sigma_x, \sigma_y, \sigma_{xy}$	=	normal and shear stresses respectively
$'$	=	$d/d\bar{x}$

I. INTRODUCTION

The stability of circular cylindrical shells under axial compression has been studied extensively both theoretically and experimentally by many investigators. Buckling loads, predicted by linearized small deflection theories, proved much higher than those realized in experiments and the experimental results showed a perplexingly large scatter band. Choice of in-plane boundary conditions and prebuckling deformation caused by edge restraints have been shown to affect the buckling load (Refs. 1, 2, 3, and 4). However, initial geometric imperfections have been accepted as the main cause for poor correlation and wide experimental scatter between the predictions of the linearized small deflection theory and the experimental results.

In earlier works (Refs. 5, 6, 7, 8, and 9) imperfections were assumed to be axisymmetric, having the same form as the classical buckling mode of the shell. In 1969 Arbocz and Babcock (Ref. 10) reported the results of buckling experiments where the actual initial imperfections and the prebuckling growth of electroplated isotropic shells were measured by means of an automated scanning mechanism. The measured imperfections consisted mainly of asymmetric modes that corresponded to buckling loads higher than the classical value. The theoretical analysis used for correlation in this study included asymmetric modes. The comparison of the analytical results with the experimental values showed good agreement. However, the analysis has met with criticism because the double Fourier series expressions used to represent initial imperfections and deflections do not satisfy all of the boundary conditions. An extended theoretical analysis (Ref. (11)) was performed which included one asymmetric mode for the initial imperfection and also satisfied the boundary conditions. The method of numerical solution of the equations required large amounts of computer time, and precluded anything but scant preliminary results. It was the purpose of this study to develop an alternate method of theoretical analysis capable of including asymmetry and satisfying rigorously the experimental boundary conditions which would be less time-consuming.

II. THEORETICAL ANALYSIS

A. Development of the Analysis

Assuming that the radial displacement W is positive outward and that the two-dimensional membrane stress resultants can be obtained from an Airy stress function F as $F,_{yy} = N_x = t\sigma_x$, $F,_{xx} = N_y = t\sigma_y$, and $-F,_{xy} = N_{xy} = t\sigma_{xy}$, then the Donnell equations for an imperfect cylindrical shell (Ref. 12) can be written as

Displacement Compatibility:

$$\frac{1}{Et}\nabla^4 F - \frac{1}{R} W,_{xx} + \frac{1}{2} L(W, W+2\overline{W}) = 0 \qquad (1)$$

Equilibrium:

$$\frac{Et^3}{12(1-\nu^2)} \nabla^4 W + \frac{1}{R} F,_{xx} - L(F, W+\overline{W}) = 0 \qquad (2)$$

where the nonlinear operator L is defined by

$$L(S, T) = S,_{xx}T,_{yy} - 2S,_{xy}T,_{xy} + S,_{yy}T,_{xx} \qquad (3)$$

and ∇^4 is the two-dimensional biharmonic operator.

Arbocz and Babcock (Ref. 10) obtained an approximate solution to the Donnell shell equations by representing the measured initial imperfections as

$$\overline{W} = \overline{\xi}_1 t \cos i \frac{\pi x}{L} + \overline{\xi}_2 t \cos k \frac{\pi x}{L} \cos n y/R$$
$$+ \overline{\xi}_3 t \sin k \frac{\pi x}{L} \cos n y/R \qquad (4)$$

In deriving the nonlinear buckling equations they assumed that any equilibrium state of the axially loaded cylinder could be represented by

$$W = \frac{\nu}{E} \sigma_x R + w = t(\frac{\nu}{c} \lambda) + w \tag{5}$$

$$F = -\frac{1}{2} \sigma_x t y^2 + f = -\frac{1}{2} \frac{Et^2}{cR} \lambda y^2 + f \tag{6}$$

where the terms added to w and f constituted the membrane prebuckling solution for the perfect shell. Further, w was approximated as

$$w = \xi_1 t \cos i \frac{\pi x}{L} + \xi_2 t \cos k \frac{\pi x}{L} \cos n \frac{y}{R} + \xi_3 t \sin k \frac{\pi x}{L} \cos n \frac{y}{R} \tag{7}$$

Approximate solutions of the full nonlinear equations (1) and (2) were obtained as follows. First, the compatibility equation (1) was solved exactly for the stress function F in terms of the assumed radial displacement W and the measured imperfection $\overline{W}$, guaranteeing a kinematically admissible displacement field to be associated with the solution of the second equation. In that analysis, only the effect of initial imperfections on the buckling load was of interest. Therefore the boundary conditions on the finite length shell were neglected. Thus only a particular solution of equation (1) was used. Second, the equation of equilibrium (2) was solved approximately by substituting for F, W, and $\overline{W}$ and applying the Galerkin procedure. The solution of the resulting set of nonlinear algebraic equations yielded the equilibrium configuration of the finite shell as a function of the loading parameter λ. If λ attained a maximum as the compressive axial load was increased, then this value of λ at the limit point was associated with the buckling load of the shell, λ_s. Using this model with experimentally measured imperfection harmonics, Arbocz and Babcock located the "pairs of critical modal components," defined as that combination of one axisymmetric and one asymmetric component that would yield the lowest value for λ_s. The agreement between their analytical predictions and experimental values was good (within 10%).

However encouraging the results were, the analytical model contained several simplifying assumptions that need justification. First of all, as mentioned above, the assumed displacement functions did not satisfy the experimental boundary conditions at the shell edges. It might be argued that since the effect of boundary conditions at moderate load levels is confined to the region next to the shell edges, one is justified in neglecting the boundary conditions if the imperfections have many waves in the axial direction. The "pairs of critical modal components" reported by Arbocz and Babcock turned out to have, in most cases, one wave or one half wave in the axial direction. Secondly, the effect of prebuckling deformations due to edge constraints were neglected. Almroth (Ref. 2) showed that prebuckling deformation caused by edge-constraints would reduce the buckling load predicted by linearized small deflection theory for perfect shells by at most 15%. Finally, it was questionable whether the prebuckling behavior, especially close to the limit point, was adequately represented by the "pair of critical modal components."

1. Formulation in Terms of Ordinary Differential Equations

In order to obtain an approximate solution of the nonlinear Donnell-type equations which will include rigorous satisfaction of the experimental boundary conditions, the form used by Arbocz and Sechler (Ref. 11) is assumed to represent the initial imperfection surface, namely

$$\overline{W}(\bar{x},\bar{y}) = t A_0(\bar{x}) + t A_1(\bar{x}) \cos n\bar{y} \tag{8}$$

where $A_0(\bar{x})$ and $A_1(\bar{x})$ are known functions of $\bar{x}$. Assume also that the deformation and stress state of the axially compressed cylinder is adequately represented by

$$W(\bar{x},\bar{y}) = \frac{t\nu\lambda}{c} + tw_0(\bar{x}) + tw_1(\bar{x}) \cos n\bar{y} \tag{9}$$

$$F(\bar{x},\bar{y}) = \frac{ERt^2}{c}\left\{-\frac{\lambda}{2}\bar{y}^2 + f_0(\bar{x}) + f_1(\bar{x}) \cos n\bar{y} + f_2(\bar{x}) \cos 2n\bar{y}\right\} \tag{10}$$

Assuming the axial dependence of the response to be an unknown function of $\bar{x}$ will reduce the buckling problem to the solution of a set of nonlinear ordinary differential equations, which allows the satisfaction of the experimental boundary conditions.

Substituting the expressions assumed for $\overline{W}$, W and F into the compatibility equation (1), using some trigonometric identities, and equating coefficients of like terms, (see details in Appendix A), results in the following system of 3 nonlinear ordinary differential equations

$$f_0^{iv} - cw_0'' - \frac{c}{2}\frac{t}{R}n^2\left[A_1 w_1'' + A_1'' w_1 + (2A_1' + w_1')w_1' + w_1 w_1''\right] = 0 \tag{11}$$

$$f_1^{iv} - 2n^2 f_1'' + n^4 f_1 - cw_1'' - \frac{ct}{R}n^2\left[A_0'' w_1 + (A_1 + w_1)w_0''\right] = 0 \tag{12}$$

$$f_2^{iv}-2(2n)^2f_2''+(2n)^4f_2-\frac{c}{2}\frac{t}{R}n^2\left[A_1w_1''+A_1''w_1\right.$$

$$\left.-(2A_1'+w_1')w_1'+w_1w_1''\right]=0 \qquad (13)$$

Substituting in turn the expressions assumed for $\overline{W}$, W and F into the equilibrium equation (2) and applying Galerkin's procedure (see Appendix A), gives the following two nonlinear ordinary differential equations:

$$w_0^{iv}+4c(\frac{R}{t})^2\ f_0''+4c\ \frac{R}{t}\lambda(A_0''+w_0'')+2c\ \frac{R}{t}n^2\left[(A_1''+w_1'')f_1\right.$$

$$\left.+(A_1+w_1)f_1''+2(A_1'+w_1')f_1'\right]=0 \qquad (14)$$

$$w_1^{iv}-2n^2w_1''+n^4w_1+4c\,(\frac{R}{t})^2\ f_1''+4c\ \frac{R}{t}\ \lambda(A_1''+w_1'')$$

$$+2c\ \frac{R}{t}\ n^2\ \left[2(A_0''+w_0'')f_1+4(A_1''+w_1'')f_2+(A_1+w_1)f_2''\right.$$

$$\left.+4(A_1'+w_1')f_2'+\ 2(A_1+w_1)f_0''\right]=0 \qquad (15)$$

where $' = \dfrac{d}{d\overline{x}}$

As pointed out by Narasimhan and Hoff (Ref. 13), equation (11) can be integrated twice to yield

$$f_0''=cw_0+\frac{c}{4}\frac{t}{R}n^2(2A_1+w_1)w_1 \qquad (16)$$

In order to satisfy periodicity in the circumferential direction, the constants of integration are set equal to zero (Ref. 14). Substituting equation (16) into equations (12-15) gives the following system of four nonlinear ordinary differential equations.

$$f_1^{iv} - 2n^2 f_1'' + n^4 f_1 - c w_1'' - \frac{ct}{R} n^2 [A_0'' w_1 + (A_1 + w_1) w_0''] = 0 \qquad (17)$$

$$f_2^{iv} - 2(2n)^2 f_2'' + (2n)^4 f_2 - \frac{c}{2}\frac{t}{R} n^2 [A_1 w_1'' + A_1'' w_1$$
$$- (2A_1' + w_1') w_1' + w_1 w_1''] = 0 \qquad (18)$$

$$w_0^{iv} + 4c^2 (\frac{R}{t})^2 w_0 + c^2 \frac{R}{t} n^2 (2A_1 + w_1) w_1 + 4c \frac{R}{t} \lambda (A_0'' + w_0'')$$
$$+ 2c \frac{R}{t} n^2 [(A_1'' + w_1'') f_1 + (A_1 + w_1) f_1'' + 2(A_1' + w_1') f_1'] = 0$$
$$(19)$$

$$w_1^{iv} - 2n^2 w_1'' + n^4 [1 + c^2 (A_1 + w_1)(2A_1 + w_1)] w_1 + 4c^2 \frac{R}{t} n^2 (A_1 + w_1) w_0$$
$$+ 4c (\frac{R}{t})^2 f_1'' + 4c \frac{R}{t} \lambda (A_1'' + w_1'') + 2c \frac{R}{t} n^2 [2(A_0'' + w_0'') f_1$$
$$+ 4(A_1'' + w_1'') f_2 + (A_1 + w_1) f_2'' + 4(A_1' + w_1') f_2'] = 0 \qquad (20)$$

2. Boundary Conditions

Of interest are the SS1, SS3, and C-3 boundary conditions. The C-3 clamped boundary condition corresponds closest to most experimental test conditions. According to Arbocz (Ref. 14) the various boundary conditions may be represented as follows

SS1 ($W = W_{,xx} = N_{,xy} = 0$, $N_{xx} = -N_0$ at $\bar{x} = 0$, L/R) which reduces to

$$\left.\begin{aligned} w_0 &= -\frac{\nu\lambda}{c} \\ w_1 &= w_0'' = w_1'' = f_1 = f_2 = f_1' = f_2' = 0 \end{aligned}\right\} \text{at } \bar{x} = 0,\ L/R \qquad (21)$$

SS3 ($W = W_{,xx} = v = 0$, $N_{xx} = -N_0$ at $\bar{x} = 0$, L/R) which reduces to

$$\left.\begin{aligned} w_0 &= -\nu\lambda/c \\ w_1 &= w_0'' = w_1'' = f_1 = f_2 = f_1'' = f_2'' = 0 \end{aligned}\right\} \text{at } \bar{x} = 0,\ L/R \qquad (22)$$

C-3 ($W = W,_x = v = 0$, $N_{xx} = -N_0$ at $\bar{x} = 0$, L/R) which reduces to

$$\left.\begin{aligned} & w_0 = -\nu\lambda/c \\ & w_1 = w_0' = w_1' = f_1 = f_2 = f_1'' = f_2'' = 0 \end{aligned}\right\} \text{ at } \bar{x} = 0,\ L/R \qquad (23)$$

The system of equations (17-20) and the boundary condition equations (either (21), (22), or (23)) constitute a nonlinear 2-point boundary value problem. The solution of this nonlinear 2-point boundary value problem will locate the limit point of the prebuckling states. The value of the loading parameter λ corresponding to the limit will be, by definition, the theoretical buckling load.

3. Linearization of the Differential Equations

In order to solve the system of nonlinear ordinary differential equations, Newton's Method of Quasilinearization is used. The assumed forms for out-of-plane displacements, and the Airy stress function are written as the sum of initial values plus correction terms. The initial values are considered to be known values, while the correction terms are the unknowns. Writing in detail, the out-of-plane displacement is represented by

$$W = t\left\{\frac{\nu\lambda}{c} + w_0(\bar{x}) + \delta w_0(\bar{x}) + (w_1(\bar{x}) + \delta w_1(\bar{x}))\cos n\bar{y}\right\} \qquad (24)$$

The Airy stress function is written as

$$F = \frac{ERt^2}{c}\left\{-\frac{\lambda}{2}\bar{y}^2 + f_0(\bar{x}) + \delta f_0(\bar{x}) + (f_1(\bar{x}) + \delta f_1(\bar{x}))\cos n\bar{y} + (f_2(\bar{x}) + \delta f_2(\bar{x}))\cos 2n\bar{y}\right\} \qquad (25)$$

Inserting (21) and (22) into the system of differential equations (17), (18), (19), and (20), and dropping terms with products of correction terms (see details in Appendix B) yields the following set of linearized differential equations for the correction terms.

$$n^4\delta f_1-2n^2\delta f_1''+\delta f_1^{iv} - \frac{ct}{R} n^2(w_1+A_1)\delta w_0'' - \frac{ct}{R} n^2(w_0''+A_0'')\delta w_1$$

$$-c\,\delta w_1''= -n^4 f_1+2n^2 f_1''-f_1^{iv} + \frac{ct}{R} n^2\left[(w_1+A_1)w_0''+A_0''w_1\right] +cw_1'' \tag{26}$$

$$(2n)^4\delta f_2-2(2n)^2\delta f_2''+\delta f_2^{iv}-\frac{c}{2}\frac{t}{R} n^2(w_1''+A_1'')\delta w_1+c\frac{t}{R} n^2(w_1'+A_1')\delta w_1'$$

$$-\frac{c}{2}\frac{t}{R} n^2(w_1+A_1)\delta w_1''= -(2n)^4 f_2+2(2n)^2 f_2''-f_2^{iv}$$

$$+\frac{c}{2}\frac{t}{R} n^2\left[(w_1+A_1)w_1''+A_1''w_1-(w_1'+2A_1')w_1'\right] \tag{27}$$

$$2cn^2(w_1''+A_1'')\delta f_1+4cn^2(w_1'+A_1')\delta f_1' + 2cn^2(w_1+A_1)\delta f_1''+4c^2\frac{R}{t}\delta w_0$$

$$+4c\lambda\,\delta w_0''+\frac{t}{R}\delta w_0^{iv} + (2c^2n^2(w_1+A_1)+2cn^2 f_1'')\delta w_1+4cn^2 f_1'\delta w_1'$$

$$+2cn^2 f_1\delta w_1''=-2cn^2((w_1''+A_1'')f_1+2(w_1'+A_1')f_1'+(w_1+A_1)f_1'')$$

$$-4c^2\frac{R}{t}w_0 - \frac{t}{R}w_0^{iv}-4c\lambda(A_0''+w_0'') - c^2n^2(2A_1+w_1)w_1 \tag{28}$$

$$4cn^2(w_0''+A_0'')\delta f_1+4c\left(\frac{R}{t}\right)\delta f_1''+8cn^2(w_1''+A_1'')\delta f_2+8cn^2(w_1'+A_1')\delta f_2'$$

$$+2cn^2(w_1+A_1)\delta f_2''+4c^2n^2(w_1+A_1)\delta w_0+4cn^2 f_1\delta w_0''$$

$$+\left\{n^4\frac{t}{R}+2cn^2 f_2''+4c^2n^2 w_0+n^4\frac{t}{R}c^2\left[3w_1w_1+6w_1A_1+2A_1A_1\right]\right\}\delta w_1$$

$$+8cn^2 f_2'\delta w_1'+\left(-2\frac{t}{R}n^2+4c\lambda+8cn^2 f_2\right)\delta w_1''+\frac{t}{R}\delta w_1^{iv}$$

$$= -4cn^2(w_0''+A_0'')f_1-4c\frac{R}{t}f_1''-8cn^2(w_1''+A_1'')f_2-8cn^2(w_1'+A_1')f_2'$$

$$-2cn^2(w_1+A_1)f_2'' - 4c^2n^2(w_1+A_1)w_0 - n^4\frac{t}{R}w_1$$

$$-n^4\frac{t}{R}c^2\left\{(A_1+w_1)(2A_1+w_1)w_1\right\} + 2\frac{t}{R}n^2w_1'' - 4c\lambda(A_1''+w_1'')$$

$$-\frac{t}{R}w_1^{iv} \qquad (29)$$

Starting with some initial values, equations (26-29) are used in an iterative scheme converging to a solution of the nonlinear equations (17-20) with the enforcement of the appropriate boundary conditions (either (21), (22), or (23)).

B. Numerical Analysis

1. Formulation in Terms of Central Difference Matrix Equations

Due to the nonlinear nature of the 2-point boundary value problem, anything but a numerical solution is precluded. Trying to devise an efficient method of numerical solution it was decided to use the "Potters' Method" (Ref. 15) which lends itself particularly well to the solution of differential equations. The linearized differential equations for the correction terms (26-29) are first converted to finite difference equations by using central difference formulae for derivatives of certain correction term variables. The central difference equations are then written as matrix equations. Observe that first, second, and fourth derivatives of correction terms appear in the equations (26-29). The central difference formulae for first and second derivatives involve values of the variables at 3 different points. The central difference formula for fourth derivatives, however, involves values at more than 3 points. In order to use the Potters' Method, the matrix central difference equations are written to include values at three points. It is therefore necessary to carry along second derivatives of variables in the solution vectors of the central difference matrix equations. Fourth derivatives are then computed as second derivatives of the second derivative variables. Choosing a grid of N points spaced at a distance h apart, the first and second derivatives at point i of a variable g are written (Ref. 16)

$$g_i' = (g_{i+1} - g_{i-1})/2h \tag{30}$$

$$g_i'' = (g_{i-1} - 2g_i + g_{i+1})/h^2 \tag{31}$$

The solution vector of the matrix central difference equation is constructed at point i

$$\delta \underset{\sim}{Y}_i = (\delta f_1, \delta f_1'', \delta f_2, \delta f_2'', \delta w_0, \delta w_0'', \delta w_1, \delta w_1'')^T \tag{32}$$

The four differential equations (26-29) yield four central difference equations (see details in Appendix B). The solution vector is constructed with eight unknowns, therefore four additional equations are needed. These additional equations are identities between the second derivative variables in the solution vector, and the second derivatives of the corresponding variables constructed by the central difference formula (31). The eight central difference equations are written in matrix form (see Appendix B)

$$A_i \, \delta \underset{\sim}{Y}_{i-1} + B_i \, \delta \underset{\sim}{Y}_i + C_i \, \delta \underset{\sim}{Y}_{i+1} = \underset{\sim}{d}_i \tag{33}$$

where A_i, B_i, and C_i are (8 x 8) matrices and the $\delta \underset{\sim}{Y}_i$ and $\underset{\sim}{d}_i$ are 8 dimensional vectors.

2. Enforcement of the Boundary Conditions

The matrix central difference equations (33) are applied to points 2 to N-1 yielding a system of N-2 matrix equations and N vector unknowns. It remains to construct 2 matrix boundary condition equations using the boundary condition equations (either (21), (22), or (23)) with the help of the forward (or backward) difference formula

$$g_i' = (g_{i+1} - g_i)/h \tag{34}$$

The boundary condition equations written in matrix form are (see Appendix B)

$$H_1\, \delta \underset{\sim}{Y}_1 + G_1\, \delta \underset{\sim}{Y}_2 = \underset{\sim}{\epsilon}_1 \tag{35}$$

$$H_N\, \delta \underset{\sim}{Y}_{N-1} + G_N\, \delta \underset{\sim}{Y}_N = \underset{\sim}{\epsilon}_N \tag{36}$$

where the G and H are (8 x 8) matrices, and the $\underset{\sim}{\epsilon}$ are 8 dimensional vectors. The resulting system of N matrix equations and N vector unknowns may be written in terms of a single block tridiagonal matrix equation.

$$\begin{bmatrix} H_1\ G_1 & & & & & 0 \\ A_2\ B_2\ C_2 & & & & & \\ & A_3\ B_3\ C_3 & & & & \\ & & \ddots & & & \\ & & & A_{N-2}\ B_{N-2}\ C_{N-2} & & \\ 0 & & & & A_{N-1}\ B_{N-1}\ C_{N-1} & \\ & & & & H_N\ \ G_N & \end{bmatrix} \begin{Bmatrix} \delta \underset{\sim}{Y}_1 \\ \delta \underset{\sim}{Y}_2 \\ \delta \underset{\sim}{Y}_3 \\ \vdots \\ \delta \underset{\sim}{Y}_{N-2} \\ \delta \underset{\sim}{Y}_{N-1} \\ \delta \underset{\sim}{Y}_N \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{\epsilon}_1 \\ \underset{\sim}{d}_2 \\ \underset{\sim}{d}_3 \\ \vdots \\ \underset{\sim}{d}_{N-2} \\ \underset{\sim}{d}_{N-1} \\ \underset{\sim}{\epsilon}_N \end{Bmatrix} \tag{37}$$

3. Solution Via the Potters' Method

The block tridiagonal system (37) is solved using the Potters' method (Ref. 15) assuming that the solution vector at the i^{th} point $\delta \underset{\sim}{Y}_i$ can be written in terms of the corresponding solution vector at the $i+1^{th}$ point $\delta \underset{\sim}{Y}_{i+1}$ by

$$\delta \underset{\sim}{Y}_i = P_i \, \delta \underset{\sim}{Y}_{i+1} + \underset{\sim}{q}_i \tag{38}$$

where P_i is an (8 x 8) matrix and $\underset{\sim}{q}_i$ is an 8 dimensional vector.

Substituting the Potters' equation (38) into the matrix central difference equation (33) yields

$$A_i[P_{i-1} \, \delta \underset{\sim}{Y}_i + \underset{\sim}{q}_{i-1}] + B_i \, \delta \underset{\sim}{Y}_i + C_i \, \delta \underset{\sim}{Y}_{i+1} = \underset{\sim}{d}_i \tag{39}$$

which gives

$$\delta \underset{\sim}{Y}_i = -[A_i P_{i-1} + B_i]^{-1} C_i \, \delta \underset{\sim}{Y}_{i+1} + [A_i P_{i-1} + B_i]^{-1} \{\underset{\sim}{d}_i - A_i \underset{\sim}{q}_{i-1}\} \tag{40}$$

providing that the inverse exists.

Comparison of the equations (40) and (38) yields the recurrence relations.

$$P_i = -[A_i P_{i-1} + B_i]^{-1} C_i \tag{41}$$

$$\underset{\sim}{q}_i = [A_i P_{i-1} + B_i]^{-1} \{\underset{\sim}{d}_i - A_i \underset{\sim}{q}_{i-1}\} \tag{42}$$

To start the solution of the recurrence relations it is necessary to determine starting P and $\underset{\sim}{q}$ values. Normally the first equation in the system (37) would be solved for $\delta \underset{\sim}{Y}_1$ in terms of $\delta \underset{\sim}{Y}_2$, giving values for P_1 and $\underset{\sim}{q}_1$. The matrix H_1 in this formulation may however be singular. Accordingly, the solution is started by the simultaneous solution for $\delta \underset{\sim}{Y}_2$ and $\delta \underset{\sim}{Y}_3$ in terms of $\delta \underset{\sim}{Y}_1$ from the first two equations of the system (37)

$$H_1 \, \delta \underset{\sim}{Y}_1 + G_1 \, \delta \underset{\sim}{Y}_2 = \underset{\sim}{\epsilon}_1 \tag{43}$$

$$A_2 \, \delta \underset{\sim}{Y}_1 + B_2 \, \delta \underset{\sim}{Y}_2 + C_2 \, \delta \underset{\sim}{Y}_3 = \underset{\sim}{d}_2 \tag{44}$$

Then

$$\delta \underset{\sim}{Y}_1 = A_2^{-1} [\underset{\sim}{d}_2 - B_2 \, \delta \underset{\sim}{Y}_2 - C_2 \, \delta \underset{\sim}{Y}_3] \tag{45}$$

providing that the inverse of A_2 exists, also

$$[G_1 - H_1 A_2^{-1} B_2] \, \delta \underset{\sim}{Y}_2 - H_1 A_2^{-1} C_2 \, \delta \underset{\sim}{Y}_3 = \underset{\sim}{\epsilon}_1 - H_1 A_2^{-1} \underset{\sim}{d}_2 \tag{46}$$

$$\delta \underset{\sim}{Y}_2 = [G_1 - H_1 A_2^{-1} B_2]^{-1} H_1 A_2^{-1} C_2 \, \delta \underset{\sim}{Y}_3 + [G_1 - H_1 A_2^{-1} B_2]^{-1} \{\underset{\sim}{\epsilon}_1 - H_1 A_2^{-1} \underset{\sim}{d}_2\} \tag{47}$$

providing that the inverse exists. Comparing (47) with the Potters' equation (38) yields

$$P_2 = [G_1 - H_1 A_2^{-1} B_2]^{-1} H_1 A_2^{-1} C_2 \tag{48}$$

$$\underset{\sim}{g}_2 = [G_1 - H_1 A_2^{-1} B_2]^{-1} \{\underset{\sim}{\epsilon}_1 - H_1 A_2^{-1} \underset{\sim}{d}_2\} \tag{49}$$

Starting with the values (48) and (49) $(P_3, \ldots\ldots, P_{N-1})$ and $(\underset{\sim}{g}_3, \ldots\ldots, \underset{\sim}{g}_{N-1})$ are computed via the recurrence relations (41) and (42). Using the last of the matrix equations in the system (37)

$$H_N \, \delta \underset{\sim}{Y}_{N-1} + G_N \, \delta \underset{\sim}{Y}_N = \underset{\sim}{\epsilon}_N \tag{50}$$

and writing $\delta \underset{\sim}{Y}_{N-1}$ in terms of $\delta \underset{\sim}{Y}_N$ using the Potters' equation (38)

$$\delta \underset{\sim}{Y}_{N-1} = P_{N-1}\, \delta \underset{\sim}{Y}_N + \underset{\sim}{g}_{N-1} \tag{51}$$

Substituting (51) into (50)

$$[H_N P_{N-1} + G_N]\ \delta \underset{\sim}{Y}_N = \{\underset{\sim}{\epsilon}_N - H_N \underset{\sim}{g}_{N-1}\} \tag{52}$$

equation (52) is solved for $\delta \underset{\sim}{Y}_N$ providing that the inverse of $[H_N P_N + G_N]$ exists. Next the solution vectors $(\delta \underset{\sim}{Y}_{N-1}, \ldots, \delta \underset{\sim}{Y}_2)$ are found using the Potters' equation (38). Finally the solution vector $\delta \underset{\sim}{Y}_1$ is found using equation (45).

4. Completion of the Solution of Correction Terms

The solution vectors $\delta \underset{\sim}{Y}_i$ as in equation (32) are composed of correction term variables and second derivatives of the correction term variables. In order to provide the complete solution of the differential equations for the correction terms (26-29), first and fourth derivatives of the correction term variables are computed from the terms in the solution vectors using the formulae (30) and (31). Special forward and backward difference formulae are used in computing first and second derivatives on the boundaries. The special forward and backward difference formulae are (Ref. 16)

At point 1 (Forward Gregory Newton)

$$g_1' = (g_2 - g_1)/h \tag{53}$$

$$g_1'' = (g_3 - 2g_2 + g_1)/h^2 \tag{54}$$

At point N (Backward Gregory Newton)

$$g_N' = (g_N - g_{N-1})/h \tag{55}$$

$$g_N'' = (g_N - 2g_{N-1} + g_{N-2})/h^2 \tag{56}$$

III. PREPARATION AND DEBUGGING OF THE COMPUTER PROGRAM

In coding the solution for the IBM 370/158 a modular approach was used. The various tasks of computation were divided into several subroutines, which were tested and debugged separately before attempting to run the program as an entire integrated unit. In order to check the completed program, two special cases were run which have closed form solutions available for comparison. The special cases studied are 1) the case of axisymmetric imperfections only, with enforcement of the C-3 boundary conditions, and 2) the case of asymmetric imperfections only, at low load level, with enforcement of the SS3 boundary conditions. For the first case the program was run by setting the asymmetric initial imperfection amplitudes on the order of 10^{-7}. Similarly the second case was run by setting the axisymmetric initial imperfection amplitude on the order of 10^{-7}, setting Poisson's ratio equal to 10^{-5}, and setting the load level equal to 10^{-1}.

A. Case of Axisymmetric Imperfections Only

If the initial imperfection is axisymmetric, then it can be represented by

$$\overline{W}(\overline{x}) = t\, A_0(\overline{x}) \tag{57}$$

The prebuckling stress and deformation state is also axisymmetric since the uniform axial compression loading and the boundary conditions are axisymmetric. Thus

$$W(\bar{x}) = \frac{\nu t \lambda}{c} + t\, w_0(\bar{x}) \tag{58}$$

$$F(\bar{x}, \bar{y}) = \frac{ERt^2}{c} \left\{ - \frac{\lambda}{2}\, \bar{y}^2 + f_0(\bar{x}) \right\} \tag{59}$$

As shown by Arbocz (Ref. 14) upon substituting eqs. (57) and (58) into the Donnell type shell equations (1) and (2), an inhomogeneous linear ordinary differential equation with constant coefficients for w_0 is obtained.

$$w_0^{iv} + 4c\, \frac{R}{t} \lambda\, w_0'' + 4c^2\, \frac{R^2}{t^2}\, w_0 = -4c\, \frac{R}{t}\, \lambda\, A_0'' \tag{60}$$

where $' = d/d\bar{x}$

Assuming that the initial imperfection function has the form

$$A_0(\bar{x}) = \bar{\xi} \cos i\pi\, \frac{R}{L}\, \bar{x} \tag{61}$$

the general solution for the axisymmetric deflection response obtained by Arbocz is

$$\begin{aligned} \frac{\overline{W(\bar{x})}}{t} &= \frac{\nu\lambda}{c} + e^{\alpha R\bar{x}} (C_1 \sin \beta R\bar{x} + C_2 \cos \beta R\bar{x}) \\ &+ e^{-\alpha R\bar{x}} (C_3 \sin \beta R\bar{x} + C_4 \cos \beta R\bar{x}) \\ &+ \frac{4\mu^2 \bar{\xi} \lambda}{4\mu^4 + 1 - 4\mu^2 \lambda} \cos i\pi\, \frac{R\bar{x}}{L} \end{aligned} \tag{62}$$

where

$$\alpha = \sqrt{1-\lambda}\, \sqrt{\frac{c}{Rt}}$$

$$\beta = \sqrt{1+\lambda}\, \sqrt{\frac{c}{Rt}} \tag{63}$$

$$\mu^2 = \frac{1}{4c}\ \frac{t}{R}\ (\frac{i\pi R}{L})^2$$

and for the C-3 boundary condition:

$$C_1 = e^{-\alpha L}\ (\frac{\alpha}{\beta} \cos \beta L - \sin \beta L)(A \cos i\pi + \frac{\nu}{c}\lambda)$$

$$C_2 = -e^{-\alpha L}\ (\frac{\alpha}{\beta} \sin \beta L + \cos \beta L)\ (A \cos i\pi + \frac{\nu}{c}\lambda)$$

$$C_3 = -\frac{\alpha}{\beta}\ (A + \frac{\nu\lambda}{c})$$

$$C_4 = -(A + \frac{\nu}{c}\lambda) \tag{64}$$

where

$$A = \frac{4\mu^2\bar{\xi}\lambda}{4\mu^4+1-4\mu^2\lambda}$$

Figure 1 shows the results of the computer program compared to the analytical solution (62) for the case when $\bar{\xi} = 0.1$, $i = 2$, $\nu = 0.3$, $\lambda = 0.2$, $R = 4$, $L = 4$, and $t = 0.004$.

B. Case of Asymmetric Imperfections Only

If the initial imperfection is purely asymmetric, then it can be represented by

$$\overline{W} = tA_1(x) \cos n\bar{y} \tag{65}$$

At low load level with a zero Poisson's ratio the Donnell-type equations (1) and (2) will admit a solution of the form

$$W = t\ w_1(x) \cos n\bar{y} \tag{66}$$

$$F = \frac{ERt^2}{c}\{-\frac{\lambda}{2}\bar{y}^2 + f_1(\bar{x}) \cos n\bar{y}\} \tag{67}$$

Substituting the forms (65), (66) and (67) into the Donnell equations (1) and (2), and dropping the highest order terms gives

$$(f_1^{iv} - 2n^2 f_1'' + n^4 f_1) - cw_1'' = 0 \tag{68}$$

$$\frac{t}{R}(w_1^{iv} - 2n^2 w_1'' + n^4 w_1) + \frac{4cR}{t} f_1'' + 4c\lambda w_1'' = -4c\lambda A_1'' \tag{69}$$

where $' = d/d\bar{x}$.

Imposing the SS3 boundary conditions - see equations (22) - a sine function may be used for the axial dependence of the initial imperfection and the deflection and stress response functions.

Let

$$A_1(\bar{x}) = \bar{\xi} \sin \pi\bar{x} \tag{70}$$

The response will be

$$f_1(\bar{x}) = a \sin \pi\bar{x} \tag{71}$$

$$w_1(\bar{x}) = b \sin \pi\bar{x} \tag{72}$$

Substituting the forms (70), (71), and (72) into the governing equations (68) and (69) the coefficients a and b are found to be

$$b = \frac{-\bar{\xi}\lambda}{\left(\lambda - \frac{c\pi^2}{\alpha}\frac{R}{t} - \frac{\alpha t}{4c\pi^2 R}\right)} \tag{73}$$

$$a = \frac{-c\pi^3}{\alpha} b \tag{74}$$

where $\alpha = (\pi^4 + 2\pi^2 n^2 + n^4)$

Figure 2 shows the results of the computer program solution for the case when $\xi = -0.05$, $n = 13$, $R = 4$, $t = 0.004$, $L = 4$, and $\lambda = 0.1$. The analytic solution for this particular case is

$$w_1(\bar{x}) \doteq -5.54198 \times 10^{-3} \sin \pi\bar{x}$$

$$A_1(\bar{x}) \doteq 2.961179 \times 10^{-6} \sin \pi\bar{x}$$

IV. NUMERICAL RESULTS

A. Convergence Study

In order to study the convergence of the numerical solution, i.e., the number of grid points required and the computation time, a general case is chosen entailing both axisymmetric and asymmetric initial imperfection modes. The cylindrical shell under study has a length of 4 inches, a radius of 4 inches, and a thickness of 0.004 inches. The C-3 boundary conditions are enforced. The initial imperfection functions are

$$A_0 = 0.5 \cos(2\pi\bar{x})$$

$$A_1 = -0.05 \sin(\pi\bar{x})$$

An axial load level of $\lambda = 0.4$ is applied. The circumferential wave number for the asymmetric response is $n = 13$.

The numerical solution scheme requires the computer program to have an internal convergence check, to determine whether or not a satisfactory solution has been obtained. The check is made by forming the ratio of a correction term variable to its associated variable, for all variables at each grid point. The maximum in absolute value of ratios thus obtained is compared to an input acceptable error value. If it is smaller than this acceptable value, then the solution is considered to be satisfactory. For the cases run in the convergence study, the acceptable error value was set equal to 10^{-5}.

To start, a solution was obtained using 26 grid points. The program was then run repeatedly, each time doubling the number of grid points.

Table I lists results of the convergence check, giving the number of grid points used, the number of iterations required for the numerical scheme to bring the solution from an initial guess to a satisfactory solution, and the total computation time for the run. Figures 3, 4, 5, and 6 show the convergence of the functions w_0, w_1, f_1, and f_2.

V. CONCLUSIONS

The aim of this study was to devise an efficient numerical method for finding an approximate solution to the Donnell-type non-linear shell equations. By incrementing the axial load level, one may ascertain the limit point which is, by definition, the theoretical buckling load of the shell.

Referring to figures 3 through 6, the convergence of the numerical solution scheme appears to be quadratic with respect to the spacing of grid points. As the mesh size is halved upon successive iterations, the mean distance between solution curves seems to be reduced by about one fourth. For the case used in the convergence check, the solution computed with 51 grid points appears to have sufficient accuracy to describe the response behavior of the shell. An important consideration is that the boundary layer has sufficient grid points to insure convergence in that region. Within the characteristic wavelength of $\sqrt{Rt}$, there should apparently be at least 7 or 8 grid points. As shown in table I, the total computation time for a solution with 51 grid points is about 20 seconds. Even if 101 grid points were necessary to insure sufficient accuracy, the solution at a single load level would be obtained in approximately 30 seconds. The "shooting method" numerical solution scheme used by Arbocz and Sechler (Ref. 11) required approximately 10 minutes to obtain a solution at a single load level for the same degree of accuracy.

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APPENDIX A

The approximate solution of Donnell's equations for an imperfect cylindrical shell assumes that the initial imperfection surface is represented by

$$\overline{W} = t A_0(\bar{x}) + t A_1(\bar{x}) \cos n\bar{y} \tag{1}$$

The equilibrium state of the axially loaded cylinder is approximated as:

$$W(\bar{x}, \bar{y}) = \frac{t\nu\lambda}{c} + t\, w_0(\bar{x}) + t w_1(\bar{x}) \cos n\bar{y} \tag{2}$$

$$F(\bar{x}, \bar{y}) = \frac{ERt^2}{c} \{ - \frac{\lambda}{2} \bar{y}^2 + f_0(\bar{x}) + f_1(\bar{x}) \cos n\bar{y} + f_2(\bar{x}) \cos 2n\bar{y} \} \tag{3}$$

In order to facilitate the substitution of (1), (2) and (3) into the Donnell-type equations, write the spacial derivatives in the Donnell equations in nondimensional form by using

$$\frac{1}{R} \frac{d}{dx} = \frac{d}{d\bar{x}} , \quad \frac{1}{R} \frac{d}{dy} = \frac{d}{d\bar{y}} \tag{4}$$

The Donnell shell equations are then

$$\frac{1}{R^4 Et} \bar{\nabla}^4 F - \frac{1}{R^3} W,_{\bar{x}\bar{x}} + \frac{1}{2R^4} L_{NL}(W, W+2\overline{W}) = 0 \tag{5}$$

$$\frac{Et^3}{12R^4(1-\nu^2)} \bar{\nabla}^4 w + \frac{1}{R^3} F,_{\bar{x}\bar{x}} - \frac{1}{R^4} L_{NL}(F, W+\overline{W}) = 0 \tag{6}$$

where the nonlinear operator L_{NL} is defined by

$$L_{NL}(S, T) = S,_{\overline{x}\overline{x}}\, T,_{\overline{y}\overline{y}} - 2S,_{\overline{x}\overline{y}}\, T,_{\overline{x}\overline{y}} + S,_{\overline{y}\overline{y}}\, T,_{\overline{x}\overline{x}} \tag{7}$$

and the two-dimensional biharmonic operator $\overline{\nabla}^4$ is

$$\overline{\nabla}^4 = \frac{\partial^4}{\partial \overline{x}^4} + 2\frac{\partial^4}{\partial \overline{x}^2 \partial \overline{y}^2} + \frac{\partial^4}{\partial \overline{y}^4} \tag{8}$$

Performing the various operations on $\overline{W}$, W, and F in equations (5) and (6), using the assumed forms (1), (2) and (3)

$$\overline{\nabla}^4 F = \frac{ERt^2}{c}\{f_0^{iv} + f_1^{iv}\cos n\overline{y} + f_2^{iv}\cos 2n\overline{y} - 2n^2 f_1'' \cos n\overline{y}$$

$$-8n^2 f_2'' \cos 2n\overline{y} + n^4 f_1 \cos n\overline{y} + 16\, n^4 f_2 \cos 2n\overline{y}\} \tag{9}$$

$$W,_{\overline{x}\overline{x}} = t\{w_0'' + w_1'' \cos n\overline{y}\} \tag{10}$$

$$L_{NL}(W, W+2\overline{W}) = -t^2 n^2 (w_0''+w_1'' \cos n\overline{y})(w_1+2A_1)\cos n\overline{y}$$

$$-2t^2n^2(w_1'(w_1'+2A_1'))\sin^2 n\overline{y}$$

$$-t^2n^2\{w_1(w_0''+2A_0'')\cos n\overline{y} + w_1(w_1''+2A_1'')\cos^2 n\overline{y}\}$$

$$\tag{11}$$

$$\overline{\nabla}^4 W = t\{w_0^{iv} + w_1^{iv}\cos n\overline{y} - 2n^2 w_1'' \cos n\overline{y} + n^4 w_1 \cos n\overline{y}\} \tag{12}$$

$$F,_{\overline{x}\overline{x}} = \frac{ERt^2}{c}\{f_0'' + f_1'' \cos n\overline{y} + f_2'' \cos 2n\overline{y}\} \tag{13}$$

$$L_{NL}(F, W+\overline{W}) = -\frac{ERt^3}{c} n^2 (w_1+A_1)(f_0'' \cos n\overline{y} + f_1'' \cos^2 n\overline{y}$$

$$+ f_2'' \cos n\overline{y} \cos 2n\overline{y})$$

$$-2\frac{ERt^3}{3} n^2 (w_1'+A_1')(f_1' \sin^2 n\overline{y} + 2f_2' \sin n\overline{y} \sin 2n\overline{y})$$

$$-\frac{ERt^3}{c}((w_0''+A_0'')+(w_1''+A_1'')\cos n\overline{y})(\lambda + n^2 f_1 \cos n\overline{y} + 4n^2 f_2 \cos 2n\overline{y})$$

$$\tag{14}$$

where $' = d/d\overline{x}$.

Substituting the trigonometric identities

$$\sin^2 a = \frac{1}{2}(1-\cos 2a) \tag{15}$$

$$\cos^2 a = \frac{1}{2}(1+\cos 2a) \tag{16}$$

into equation (11) yields

$$\begin{aligned} L_{NL}(W, W+2\overline{W}) = & -t^2n^2\left(\frac{w_1''}{2}(w_1+2A_1)+w_1'(w_1'+2A_1')+\frac{w_1}{2}(w_1''+2A_1'')\right) \\ & -t^2n^2(w_0''(w_1+2A_1)+w_1(w_0''+2A_0''))\cos n\overline{y} \\ & -t^2n^2\left(\frac{w_1''}{2}(w_1+2A_1)-w_1'(w_1'+2A_1')+\frac{w_1}{2}(w_1''+2A_1'')\right)\cos 2n\overline{y} \end{aligned} \tag{17}$$

Substituting the terms (9), (10), and (17) into the compatibility equation (5) yields

$$\begin{aligned} & \left\{\frac{t}{R^3c}f_0^{iv} - \frac{t}{R^3}w_0'' - \frac{t^2n^2}{2R^4}\left(\frac{w_1''}{2}(w_1+2A_1)+w_1'(w_1'+2A_1')+\frac{w_1}{2}(w_1''+2A_1'')\right)\right\} \\ & +\left\{\frac{t}{R^3c}(f_1^{iv}-2n^2f_1''+n^4f_1) - \frac{t}{R^3}w_1'' - \frac{t^2n^2}{2R^4}(w_0''(w_1+2A_1) \right. \\ & \qquad \left. +w_1(w_0''+2A_0''))\right\}\cos n\overline{y} \\ & +\left\{\frac{t}{R^3c}(f_2^{iv}-8n^2f_2''+16n^4f_2) - \frac{t^2n^2}{2R^4}\left(\frac{w_1''}{2}(w_1+2A_1) \right.\right. \\ & \qquad \left.\left. -w_1'(w_1'+2A_1')+\frac{w_1}{2}(w_1''+2A_1'')\right)\right\}\cos 2n\overline{y} \\ & = 0 \end{aligned} \tag{18}$$

Multiplying through by $\frac{R^3c}{t}$ and equating coefficients of like terms yields the following 3 equations

$$f_0^{iv} - cw_0'' - \frac{c}{2}\frac{t}{R}n^2[A_1w_1''+A_1''w_1+(2A_1'+w_1')w_1'+w_1w_1''] = 0 \qquad (19)$$

$$f_1^{iv} - 2n^2f_1''+n^4f_1-cw_1'' - \frac{ct}{R}n^2[A_0''w_1+(A_1+w_1)w_0''] = 0 \qquad (20)$$

$$f_2^{iv}-2(2n)^2f_2''+(2n)^4f_2 - \frac{c}{2}\frac{t}{R}n^2[A_1w_1''+A_1''w_1$$

$$-(2A_1'+w_1')w_1'+w_1w_1''] = 0 \qquad (21)$$

Substituting the trigonometric identities (15) and (16) into equation (14) yields

$$L_{NL}(F, W+\overline{W}) = -\frac{ERt^3}{c}n^2\{\frac{1}{2}(w_1+A_1)f_1''+(w_1'+A_1')f_1'$$

$$+(w_0''+A_0'')\frac{\lambda}{n^2}+\frac{1}{2}(w_1''+A_1'')f_1\}$$

$$-\frac{ERt^3}{c}n^2\{(w_1+A_1)f_0''+(w_0''+A_0'')f_1 + (w_1''+A_1'')\frac{\lambda}{n^2}\} \cos n\overline{y}$$

$$-\frac{ERt^3}{c}n^2\{\frac{1}{2}(w_1+A_1)f_1''-(w_1'+A_1')f_1'+4(w_0''+A_0'')f_2$$

$$+\frac{1}{2}(w_1''+A_1'')f_1\} \cos 2n\overline{y}$$

$$-\frac{ERt^3}{c}n^2\{(w_1+A_1)f_2''+4(w_1''+A_1'')f_2\} \cos n\overline{y} \cos 2n\overline{y}$$

$$-\frac{ERt^3}{c}n^2\{4(w_1'+A_1')f_2'\} \sin n\overline{y} \sin 2n\overline{y}$$

$$(22)$$

Substituting the terms (12), (13), and (22) into the equilibrium equation (6) yields

$$\left\{\frac{Et^4}{12R^4(1-\nu^2)} w_0^{iv} + \frac{Et^2}{cR^2} f_0'' + \frac{E}{c}\frac{t^3}{R^3} n^2 \left\{\frac{1}{2}(w_1+A_1)f_1'' + (w_1'+A_1')f_1' + (w_0''+A_0'')\frac{\lambda}{n^2} + \frac{1}{2}(w_1''+A_1'')f_1\right\}\right\}$$

$$+\left\{\frac{Et^4}{12R^4(1-\nu^2)}(w_1^{iv}-2n^2w_1''+n^4w_1) + \frac{E}{c}\frac{t^2}{R^2} f_1'' + \frac{E}{c}\frac{t^3}{R^3} n^2\left\{(w_1+A_1)f_0''+(w_0''+A_0'') f_1 + (w_1''+A_1'')\frac{\lambda}{n^2}\right\}\right\} \cos n\overline{y}$$

$$+\left\{\frac{E}{c}\frac{t^2}{R^2} f_2'' + \frac{E}{c}\frac{t^3}{R^3} n^2 \left\{\frac{1}{2}(w_1+A_1)f_1''-(w_1'+A_1')f_1'+4(w_0''+A_0'') f_2 + \frac{1}{2}(w_1''+A_1'')f_1\right\}\right\} \cos 2n\overline{y}$$

$$+\frac{E}{c}\frac{t^3}{R^3} n^2 \left\{(w_1+A_1)f_2'' + 4(w_1''+A_1'')f_2\right\} \cos n\overline{y} \cos 2n\overline{y}$$

$$+\frac{E}{c}\frac{t^3}{R^3} n^2\left\{2(w_1'+A_1')\, f_2'\right\} \left\{\sin n\overline{y} \sin 2n\overline{y}\right\} = \epsilon \qquad (23)$$

The right-hand side of the equation is nonzero because the equilibrium equation is not necessarily satisfied exactly by the assumed form of the solution. Applying the Galerkin procedure, the following integrals are evaluated.

$$\int_0^{2\pi} \epsilon \, d\overline{y} = 0 \qquad (24)$$

$$\int_0^{2\pi} \epsilon \cos(n\overline{y})\, d\overline{y} = 0 \qquad (25)$$

Using equation (23) to evaluate the integral (24) yields

$$w_0^{iv} + 4c\,\frac{R^2}{t^2}\,f_0'' + 4c\,\frac{R}{t}\,\lambda\,(A_0''+w_0'')$$

$$+ \frac{2cR}{t}\,n^2\left\{(A_1''+w_1'')f_1+(A_1+w_1)f_1''+2(A_1'+w_1')f_1'\right\} = 0 \qquad (25)$$

Again using equation (23) to evaluate the integral (25) yields

$$w_1^{iv} - 2n^2 w_1'' + n^4 w_1 + 4c\,\frac{R^2}{t^2}\,f_1'' + 4c\,\frac{R}{t}\,\lambda\,(A_1''+w_1'')$$

$$+ \frac{2cR}{t}\,n^2\left[2(A_0''+w_0'')f_1+4(A_1''+w_1'')f_2\right.$$

$$\left.+ (A_1+w_1)f_2''+4(A_1'+w_1')f_2'+2(A_1+w_1)f_0''\right] = 0 \qquad (26)$$

APPENDIX B

The Donnell-type shell equations are written using a special form for the imperfection surface

$$\overline{W} = tA_0(\overline{x}) + tA_1(\overline{x})\cos n\overline{y} \qquad (1)$$

and the corresponding form for the out-of-plane displacements, and Airy stress function.

$$W = t\left(\frac{\nu\lambda}{c} + w_0(\overline{x}) + w_1(\overline{x})\cos n\overline{y}\right) \qquad (2)$$

$$F = \frac{ERt^2}{c}\left\{-\frac{\lambda}{2}\overline{y}^2 + f_0(\overline{x}) + f_1(\overline{x})\cos n\overline{y} + f_2(\overline{x})\cos 2n\overline{y}\right\} \qquad (3)$$

The resulting system of 4 nonlinear ordinary differential equations are, after integrating the equation involving f_0 and substituting into the remaining equations,

$$f_1^{iv} - 2n^2 f_1'' + n^4 f_1 - cw_1'' - \frac{ct}{R} n^2 \left[A_0'' w_1 + (A_1 + w_1) w_0''\right] = 0 \qquad (4)$$

$$f_2^{iv} - 2(2n)^2 f_2'' + (2n)^4 f_2 - \frac{c}{2}\frac{t}{R} n^2 \left[A_1 w_1'' + A_1'' w_1 - (2A_1' + w_1') w_1' + w_1 w_1''\right] = 0 \qquad (5)$$

$$w_0^{iv} + 4c^2\left(\frac{R}{t}\right)^2 w_0 + c^2 \frac{R}{t} n^2 (2A_1 + w_1) w_1 + 4c \frac{R}{t}\lambda (A_0'' + w_0'') + 2c\frac{R}{t} n^2 \left[(A_1'' + w_1'') f_1 + (A_1 + w_1) f_1'' + 2(A_1' + w_1') f_1'\right] = 0 \qquad (6)$$

$$w_1^{iv} - 2n^2 w_1'' + n^4\left\{1 + c^2 (A_1 + w_1)(2A_1 + w_1)\right\} w_1 + 4c^2 \frac{R}{t} n^2 (A_1 + w_1) w_0 + 4c\left(\frac{R}{t}\right)^2 f_1'' + 4c\frac{R}{t}\lambda (A_1'' + w_1'') + 2c\frac{R}{t} n^2 \left[2(A_0'' + w_0'') f_1 + 4(A_1'' + w_1'') f_2 + (A_1 + w_1) f_2'' + 4(A_1' + w_1') f_2'\right] = 0 \qquad (7)$$

Applying Newton's Method of Quasilinearization to linearize the system of differential equations, the out-of-plane displacement and Airy stress function (2) and (3), are rewritten in terms of initial values, and corrections to these values.

$$W = t\left(\frac{\nu\lambda}{c} + w_0(\bar{x}) + \delta w_0(\bar{x}) + (w_1(\bar{x}) + \delta w_1(\bar{x}))\cos n\bar{y}\right) \tag{8}$$

$$F = \frac{ERt^2}{c}\left\{-\frac{\lambda}{2}\bar{y}^2 + f_0(\bar{x}) + \delta f_0(\bar{x}) + (f_1(\bar{x}) + \delta f_1(\bar{x}))\cos n\bar{y} + (f_2(\bar{x}) + \delta f_2(\bar{x}))\cos 2n\bar{y}\right\} \tag{9}$$

The initial values are considered to be known values, while the correction terms are the unknowns. Rewriting equations (4), (5), (6) and (7), using the form of (8) and (9) gives

$$(f_1^{iv} + \delta f_1^{iv}) - 2n^2(f_1'' + \delta f_1'') + n^4(f_1 + \delta f_1) - c(w_1'' + \delta w_1'') - \frac{ct}{R}n^2\left[A_0''(w_1 + \delta w_1) + (A_1 + (w_1 + \delta w_1))(w_0'' + \delta w_0'')\right] = 0 \tag{10}$$

$$(f_2^{iv} + \delta f_2^{iv}) - 2(2n)^2(f_2'' + \delta f_2'') + (2n)^4(f_2 + \delta f_2) - \frac{c}{2}\frac{t}{R}n^2\left[A_1(w_1'' + \delta w_1'') + A_1''(w_1 + \delta w_1) - (2A_1' + (w_1' + \delta w_1')(w_1' + \delta w_1') + (w_1 + \delta w_1)(w_1'' + \delta w_1'')\right] = 0 \tag{11}$$

$$(w_0^{iv} + \delta w_0^{iv}) + 4c^2\left(\frac{R}{t}\right)^2(w_0 + \delta w_0) + c^2\left(\frac{R}{t}\right)n^2(2A_1 + (w_1 + \delta w_1))(w_1 + \delta w_1) + 4c\frac{R}{t}\lambda(A_0'' + (w_0'' + \delta w_0'')) + 2c\frac{R}{t}n^2\left[(A_1'' + (w_1'' + \delta w_1'')(f_1 + \delta f_1) + (A_1 + (w_1 + \delta w_1))(f_1'' + \delta f_1'') + 2(A_1' + (w_1' + \delta w_1'))(f_1' + \delta f_1')\right] = 0 \tag{12}$$

$$(w_1^{iv}+\delta w_1^{iv})-2n^2(w_1''+\delta w_1'')+n^4\{1+c^2(A_1+(w_1+\delta w_1))(2A_1+(w_1+\delta w_1))\}(w_1+\delta w_1)$$

$$+ 4c^2 \frac{R}{t} n^2(A_1+(w_1+\delta w_1))(w_0+\delta w_0) + 4c(\frac{R}{t})^2(f_1''+\delta f_1'')$$

$$+ 4c \frac{R}{t}\lambda (A_1''+(w_1''+\delta w_1''))+2c \frac{R}{t} n^2 \left[2(A_0''+(w_0''+\delta w_0''))(f_1+\delta f_1)\right.$$

$$+ 4(A_1''+(w_1''+\delta w_1''))(f_2+\delta f_2)+(A_1+(w_1+\delta w_1))(f_2''+\delta f_2'')$$

$$\left.+ 4(A_1'+(w_1'+\delta w_1'))(f_2'+\delta f_2')\right] = 0 \qquad (13)$$

Rearranging terms and dropping products of correction terms,

$$n^4\delta f_1-2n^2\delta f_1''+\delta f_1^{iv} - \frac{ct}{R} n^2(w_1+A_1)\delta w_0'' - \frac{ct}{R} n^2(w_0''+A_0'')\delta w_1$$

$$-c\delta w_1'' = -n^4 f_1+2n^2 f_1''-f_1^{iv} + \frac{ct}{R} n^2 \left[(w_1+A_1)w_0''+A_0''w_1\right] +cw_1'' \qquad (14)$$

$$(2n)^4\delta f_2-2(2n)^2\delta f_2''+\delta f_2^{iv}- \frac{c}{2} \frac{t}{R} n^2(w_1''+A_1'')\delta w_1$$

$$+c \frac{t}{R} n^2(w_1'+A_1')\delta w_1' - \frac{c}{2} \frac{t}{R} n^2(w_1+A_1)\delta w_1'' =$$

$$-(2n)^4 f_2+2(2n)^2 f_2'' - f_2^{iv} + \frac{c}{2} \frac{t}{R} n^2 \left[(w_1+A_1)w_1''+A_1''w_1-(w_1'+2A_1')w_1'\right]$$

$$(15)$$

$$2cn^2(w_1''+A_1'')\delta f_1+4cn^2(w_1'+A_1')\delta f_1'+2cn^2(w_1+A_1)\delta f_1''+4c^2 \frac{R}{t}\delta w_0$$

$$+4c\lambda\, \delta w_0'' + \frac{t}{R} \delta w_0^{iv}+(2c^2n^2(w_1+A_1)+2cn^2 f_1'')\delta w_1+4^2 f_1'\delta w_1'$$

$$+2cn^2 f_1\delta w_1'' = -2cn^2((w_1''+A_1'')f_1+2(w_1'+A_1')f_1'+(w_1+A_1)f_1'')$$

$$-4c^2 \frac{R}{t} w_0 - \frac{t}{R} w_0^{iv}-4c\lambda (A_0''+w_0'')-c^2n^2(2A_1+w_1)w_1 \qquad (16)$$

$$4cn^2(w_0''+A_0'')\delta f_1+4c(\frac{R}{t})\delta f_1''+8cn^2(w_1''+A_1'')\delta f_2+8cn^2(w_1'+A_1')\delta f_2'$$

$$+2cn^2(w_1+A_1)\delta f_2''+4c^2n^2(w_1+A_1)\delta w_0+4cn^2 f_1\delta w_0''$$

$$+\{n^4\frac{t}{R}+2cn^2f_2''+4c^2n^2w_0+n^4\frac{t}{R}c^2[3w_1w_1+6w_1A_1+2A_1A_1)]\}\delta w_1$$

$$+8cn^2f_2'\delta w_1'+(-2\frac{t}{R}n^2+4c\lambda+8cn^2f_2)\delta w_1''+\frac{t}{R}\delta w_1^{iv}$$

$$= -4cn^2(w_0''+A_0'')f_1-4c\frac{R}{t}f_1''-8cn^2(w_1''+A_1'')f_2-8cn^2(w_1'+A_1')f_2'$$

$$-2cn^2(w_1+A_1)\,f_2''-4c^2n^2(w_1+A_1)W_0-n^4\frac{t}{R}w_1$$

$$-n^4\frac{t}{R}c^2\{(A_1+w_1)(2A_1+w_1)w_1\}+2\frac{t}{R}n^2w_1''-4c\lambda(A_1''+w_1'')$$

$$-\frac{t}{R}w_1^{iv} \qquad (17)$$

Equations (14-17) are the four linearized differential equations for the correction terms. Equations (14-17) are converted to central difference equations by using the central difference formulae for first and second derivatives. Choosing a grid of N points, spaced at a distance h apart, the central difference formulae for the first and second derivatives of a variable g at point i are

$$g_i' = (g_{i+1} - g_{i-1})/2h \qquad (18)$$

$$g_i'' = (g_{i-1} - 2g_i + g_{i+1})/h^2 \qquad (19)$$

Using the formulae (18) and (19) on first and fourth derivatives of the correction term variables in equations (14)-(17) yields

$$n^4\,\delta f_{1_i} - 2n^2\delta f''_{1_i} + \frac{1}{h^2}(\delta f''_{1_{i-1}} - 2\delta f''_{1_i} + \delta f''_{1_{i+1}}) - \frac{ct}{R}\,n^2(w_1+A_1)\delta w''_{0_i}$$
$$-(ct/R)n^2(w''_0+A''_0)\delta w_{1_i} - c\delta w''_{1_i} = -n^4 f_1 + 2n^2 f''_1 - f_1^{iv}$$
$$+\frac{ct}{R}\,n^2\left[(w_1+A_1)w''_0 + A''_0 w_1\right] + cw''_1 \qquad (20)$$

$$(2n)^4\delta f_{2_i} - 2(2n)^2\delta f''_{2_i} + \frac{1}{h^2}(\delta f''_{2_{i-1}} - 2\delta f''_{2_i} + \delta f''_{2_{i+1}}) - \frac{ct}{2R}\,n^2(w''_1+A''_1)\delta w_{1_i}$$
$$+\frac{c}{2h}\,\frac{t}{R}\,n^2(w'_1+A'_1)(\delta w_{1_{i+1}} - \delta w_{1_{i-1}}) - \frac{c}{2}\,\frac{t}{R}\,n^2(w_1+A_1)\delta w''_{1_i} = -(2n)^4 f_2 + 2(2n)^2 f''_2$$
$$-f_2^{iv} + \frac{c}{2}\,\frac{t}{R}\,n^2\left[(w_1+A_1)w''_1 + A''_1 w_1 - (w'_1+2A'_1)w'_1\right] \qquad (21)$$

$$2cn^2(w''_1+A''_1)\delta f_{1_i} + \frac{2cn^2}{h}(w'_1+A'_1)(\delta f_{1_{i+1}} - \delta f_{1_{i-1}}) + 2cn^2(w_1+A_1)\delta f''_{1_i}$$
$$+4c^2\,\frac{R}{t}\,\delta w_{0_i} + 4c\lambda\,\delta w''_{o_i} + \frac{t}{h^2 R}(\delta w''_{0_{i-1}} - 2\delta w''_{0_i} + \delta w''_{0_{i+1}})$$
$$+(2c^2n^2(w_1+A_1) + 2cn^2 f''_1)\delta w_{1_i} + \frac{2cn^2}{h}\,f'_1(\delta w_{1_{i+1}} - \delta w_{1_{i-1}})$$
$$+2cn^2 f_1\delta w''_{1_i} = -2cn^2((w''_1+A''_1)f_1 + 2(w'_1+A'_1)f'_1 + (w_1+A_1)f''_1)$$
$$-4c^2\,\frac{R}{t}\,w_0 - \frac{t}{R}\,w_0^{iv} - 4c\lambda\,(A''_0+w''_0) - c^2n^2(2A_1+w_1)w_1 \qquad (22)$$

$$4cn^2(w''_0+A''_0)\delta f_{1_i} + 4c(\tfrac{R}{t})\,\delta f''_{1_i} + 8cn^2(w''_1+A''_1)\,\delta f_{2_i}$$
$$+\frac{4cn^2}{h}(w'_1+A'_1)(\delta f_{2_{i+1}} - \delta f_{2_{i-1}}) + 2cn^2(w_1+A_1)\delta f''_{2_i}$$
$$+4c^2n^2(w_1+A_1)\delta w_{0_i} + 4cn^2 f_1\delta w''_{0_i} + \Big\{n^4\,\frac{t}{R} + 2cn^2 f''_2 + 4c^2n^2 w_0$$
$$+n^4\,\frac{t}{R}\,c^2\left[3w_1w_1 + 6w_1A_1 + 2A_1A_1\right]\Big\}\delta w_{1_i} + \frac{4cn^2}{h}\,f'_2(\delta w_{1_{i+1}} - \delta w_{1_{i-1}})$$
$$+(-\,2\frac{t}{R}\,n^2 + 4c\lambda + 8cn^2 f_2)\delta w''_{1_i} + \frac{t}{h^2 R}(\delta w''_{1_{i-1}} - 2\delta w''_{1_i} + \delta w''_{1_{i+1}})$$

$$= -4cn^2(w_0''+A_0'')f_1 - 4c\frac{R}{t}f_1'' - 8cn^2(w_1''+A_1'')f_2 - 8cn^2(w_1'+A_1')f_2'$$

$$-2cn^2(w_1+A_1)f_2'' - 4c^2n^2(w_1+A_1)w_0 - n^4\frac{t}{R}w_1$$

$$-n^4\frac{t}{R}c^2\{(A_1+w_1)(2A_1+w_1)w_1\} + 2\frac{t}{R}n^2w_1'' - 4c\lambda(A_1''+w_1'') - \frac{t}{R}w_1^{iv} \qquad (23)$$

where all non-correction term variables are evaluated at point i. Since correction term second derivatives are carried along as unknowns, four additional equations are needed to relate the second derivatives to the corresponding variables. Using the central difference formula (19) the additional equations are

$$\delta f_{1_i}'' - \frac{1}{h^2}\{\delta f_{1_{i-1}} - 2\delta f_{1_i} + \delta f_{1_{i+1}}\} = -f_{1_i}'' + \frac{1}{h^2}\{f_{1_{i-1}} - 2f_{1_i} + f_{1_{i+1}}\} \qquad (24)$$

$$\delta f_{2_i}'' - \frac{1}{h^2}\{\delta f_{2_{i-1}} - 2\delta f_{2_i} + \delta f_{2_{i+1}}\} = -f_{2_i}'' + \frac{1}{h^2}\{f_{2_{i-1}} - 2f_{2_i} + f_{2_{i+1}}\} \qquad (25)$$

$$\delta w_{0_i}'' - \frac{1}{h^2}\{\delta w_{0_{i-1}} - 2\delta w_{0_i} + \delta w_{0_{i+1}}\} = -w_{0_i}'' + \frac{1}{h^2}\{w_{0_{i-1}} - 2w_{0_i} + w_{0_{i+1}}\} \qquad (26)$$

$$\delta w_{1_i}'' - \frac{1}{h^2}\{\delta w_{1_{i-1}} - 2\delta w_{1_i} + \delta w_{1_{i+1}}\} = -w_{1_i}'' + \frac{1}{h^2}\{w_{1_{i-1}} - 2w_{1_i} + w_{1_{i+1}}\} \qquad (27)$$

The system of central difference equations (20)-(27) is written as a matrix central difference equation

$$A_i\,\delta \underset{\sim}{Y}_{i-1} + B_i\,\delta \underset{\sim}{Y}_i + C_i\,\delta \underset{\sim}{Y}_{i+1} = \underset{\sim}{d}_i \qquad (28)$$

The terms in equation (28) are written in detail

$$\delta \underset{\sim}{Y}_i = \begin{Bmatrix} \delta f_1 \\ \delta f_1'' \\ \delta f_2 \\ \delta f_2'' \\ \delta w_0 \\ \delta w_0'' \\ \delta w_1 \\ \delta w_1'' \end{Bmatrix}_i \tag{29}$$

Multiplying equation (28) through by h^2 the nonzero components of A_i are

$$A(1,2) = 1$$

$$A(2,4) = 1$$

$$A(2,7) = -\frac{ch}{2}\ \frac{t}{R}\, n^2(w_1' + A_1')$$

$$A(3,1) = -2hcn^2(w_1' + A_1')$$

$$A(3,6) = \frac{t}{R}$$

$$A(3,7) = -2hcn^2 f_1'$$

$$A(4,3) = -4hcn^2$$

$$A(4,7) = -4hcn^2 f_2'$$

$$A(4,8) = \frac{t}{R}$$

$$A(5,1) = -1$$

$$A(6,3) = -1$$

$$A(7,5) = -1$$

$$A(8,7) = -1$$

The nonzero components of B_i are

$$B(1,1) = h^2 n^4$$

$$B(1,2) = -2h^2 n^2 - 2$$

$$B(1,6) = -\frac{ct}{R} h^2 n^2 (w_1 + A_1)$$

$$B(1,7) = -\frac{ct}{R} h^2 n^2 (w_0'' + A_0'')$$

$$B(1,8) = -ch^2$$

$$B(2,3) = (2n)^4 h^4$$

$$B(2,4) = -2(2n)^2 h^2 - 2$$

$$B(2,7) = -\frac{ct}{2R} h^2 n^2 (w_1'' + A_1'')$$

$$B(2,8) = -\frac{c}{2} \frac{t}{R} h^2 n^2 (w_1 + A_1)$$

$$B(3,1) = 2ch^2 n^2 (w_1'' + A_1'')$$

$$B(3,2) = 2ch^2 n^2 (w_1 + A_1)$$

$$B(3,5) = 4c^2 h^2 \frac{R}{t}$$

$$B(3,6) = 4ch^2 \lambda - 2 \frac{t}{R}$$

$$B(3,7) = 2c^2 h^2 n^2 (w_1 + A_1) + 2ch^2 n^2 f_1''$$

$$B(3,8) = 2ch^2 n^2 f_1$$

$$B(4,1) = 4ch^2 n^2 (w_0'' + A_0'')$$

$B(4,2) = 4ch^2 (\frac{R}{t})$

$B(4,3) = 8ch^2n^2 \; (w_1'' + A_1'')$

$B(4,4) = 2ch^2n^2 \; (w_1 + A_1)$

$B(4,5) = 4c^2h^2n^2 \; (w_1 + A_1)$

$B(4,6) = 4ch^2n^2 f_1$

$$B(4,7) = n^4h^2 \frac{t}{R} + 2cH^2n^2 f_2'' + 4c^2h^2n^2 w_0$$
$$+ n^4h^2 \frac{t}{R} c^2 [3w_1w_1 + 6w_1A_1 + 2A_1A_1]$$

$B(4,8) = -2h^2 \frac{t}{R} n^2 + 4h^2 c\lambda + 8h^2cn^2 f_2 - 2 \frac{t}{R}$

$B(5,1) = 2$

$B(5,2) = h^2$

$B(6,3) = 2$

$B(6,4) = h^2$

$B(7,5) = 2$

$B(7,6) = h^2$

$B(8,7) = 2$

$B(8,8) = h^2$

The nonzero components of C_i are

$$C(1,2) = 1$$

$$C(2,4) = 1$$

$$C(2,7) = \frac{ch}{2} \frac{t}{R} n^2 (w_1' + A_1')$$

$$C(3,1) = 2hcn^2 (w_1' + A_1')$$

$$C(3,6) = \frac{t}{R}$$

$$C(3,7) = 2hcn^2 f_1'$$

$$C(4,3) = 4hcn^2 (w_1' + A_1')$$

$$C(4,7) = 4hcn^2 f_2'$$

$$C(4,8) = \frac{t}{R}$$

$$C(5,1) = -1$$

$$C(6,3) = -1$$

$$C(7,5) = -1$$

$$C(8,7) = -1$$

The components of the vector $\underset{\sim}{d}_i$ are

$$d(1) = -h^2 n^4 f_1 + 2h^2 n^2 f_1'' - h^2 f_1^{iv} + \frac{ct}{R} h^2 n^2 [(w_1 + A_1) w_0'' + A_0'' w_1] + ch^2 w_1''$$

$$d(2) = -(2n)^4h^2f_2 + 2(2n)^2h^2f_2'' - h^2f_2^{iv}$$

$$+ \frac{c}{2}\frac{t}{R}h^2n^2\left[(w_1+A_1)w_1''+A_1''w_1 -(w_1'+2A_1')w_1'\right]$$

$$d(3) = -2ch^2n^2\left[(w_1''+A_1'')f_1+2(w_1'+A_1')f_1' + (w_1+A_1)f_1''\right] -4h^2c^2\frac{R}{t}w_0$$

$$-\frac{t}{R}h^2w_0^{iv} - 4ch^2\lambda(A_0''+w_0'')-c^2h^2n^2(2A_1+w_1)w_1$$

$$d(4) = -4ch^2n^2(w_0''+A_0'')f_1 - 4c\frac{R}{t}h^2f_1''-8ch^2n^2(w_1''+A_1'')f_2$$

$$-8ch^2n^2(w_1'+A_1')f_2'-2ch^2n^2(w_1+A_1)f_2''$$

$$-4c^2h^2n^2(w_1+A_1)w_0$$

$$-n^4h^2\frac{t}{R}w_1-n^4h^2\frac{t}{R}c^2\{(A_1+w_1)(2A_1+w_1)w_1\}$$

$$+2\frac{t}{R}h^2n^2w_1''-4ch^2\lambda(A_1''+w_1'')-\frac{t}{R}h^2w_1^{iv}$$

$$d(5) = -h^2f_{1_i}''+f_{1_{i-1}}-2f_{1_i}+f_{1_{i+1}}$$

$$d(6) = -h^2f_{2_i}'' + f_{2_{i-1}}-2f_{2_i} + f_{2_{i+1}}$$

$$d(7) = -h^2w_{0_i}''+w_{0_{i-1}}-2w_{0_i}+w_{0_{i+1}}$$

$$d(8) = -h^2w_{1_i}'' + w_{1_{i-1}} - 2w_{1_i}+w_{1_{i+1}}$$

Matrix Boundary Condition Equations

The boundary condition equations are written in matrix form with the help of the forward difference formula

$$g_i' = (g_{i+1}-g_i)/h \tag{30}$$

$$H_1 \delta \underset{\sim}{Y}_1 + G_1 \delta \underset{\sim}{Y}_2 = \underset{\sim}{\epsilon}_1 \qquad (31)$$

$$H_N \delta \underset{\sim}{Y}_{N-1} + G_N \delta \underset{\sim}{Y}_N = \underset{\sim}{\epsilon}_N \qquad (32)$$

where the G and H are (8 x 8) matrices, and the $\underset{\sim}{\epsilon}$ are 8 dimensional vectors. The components of the matrices for the various boundary conditions are

(i) <u>SS1 Boundary Conditions</u>

$$w_0 = -\frac{\nu\lambda}{c}, \; w_1 = w_0'' = w_1'' = f_1 = f_2 = f_1' = f_2' = 0$$

$$\text{at } \bar{x} = 0, \; L/R$$

The nonzero components of the G_1 matrix are

$G_1(2,1) = 1$

$G_1(4,3) = 1$

The nonzero components of H_1 are

$H_1(1,1) = 1$

$H_1(2,1) = -1$

$H_1(3,3) = 1$

$H_1(4,3) = -1$

$H_1(5,5) = 1$

$H_1(6,6) = 1$

$H_1(7,7) = 1$

$H_1(8,8) = 1$

The error vectors $\underset{\sim}{\epsilon}_1$ and $\underset{\sim}{\epsilon}_N$ have the components evaluated at points 1, and N respectively.

$\epsilon(1) = -f_1$

$\epsilon(2) = -hf_1'$

$\epsilon(3) = -f_2$

$\epsilon(4) = -hf_2'$

$\epsilon(5) = -(w_0 + \frac{\nu\lambda}{c})$

$\epsilon(6) = -w_0''$

$\epsilon(7) = -w_1$

$\epsilon(8) = -w_1''$

The nonzero components of the G_N matrix are

$G_N(1,1) = 1$

$G_N(2,1) = 1$

$G_N(3,3) = 1$

$G_N(4,3) = 1$

$G_N(5, 5) = 1$

$G_N(6, 6) = 1$

$G_N(7, 7) = 1$

$G_N(8, 8) = 1$

The nonzero components of H_N are

$H_N(2, 1) = -1$

$H_N(4, 3) = -1$

(ii) SS3 Boundary Conditions

$$w_0 = -\frac{\nu\lambda}{c}, \quad w_0'' = w_1 = w_1'' = f_1 = f_1'' = f_2 = f_2'' = 0$$

$$\text{at } \overline{x} = 0, \ L/R$$

All of the components of matrix G_1 are equal to zero.

H_1 is equal to the identity matrix.

G_N is equal to the identity matrix.

All of the components of matrix H_N are equal to zero.

The error vectors $\underset{\sim}{\epsilon}_1$ and $\underset{\sim}{\epsilon}_N$ have the components at points 1 and N respectively.

$\epsilon(1) = -f_1$

$\epsilon(2) = -f_1''$

$\epsilon(3) = -f_2$

$\epsilon(4) = -f_2''$

$\epsilon(5) = -(w_0 + \nu\lambda/c)$

$\epsilon(6) = -w_0''$

$\epsilon(7) = -w_1$

$\epsilon(8) = -w_1''$

(iii) C-3 Boundary Conditions

$$w_0 = -\nu\lambda/c, \ w_0' = w_1 = w_1' = f_1 = f_1'' = f_2 = f_2'' = 0$$

$$\text{at } \overline{x} = 0, L/R$$

The nonzero components of the G_1 matrix are

$G_1(6,5) = 1$

$G_1(8,7) = 1$

The nonzero components of the H_1 matrix are

$H_1(1,1) = 1$

$H_1(2,2) = 1$

$H_1(3,3) = 1$

$H_1(4,4) = 1$

$H_1(5,5) = 1$

$H_1(6,5) = -1$

$H_1(7,7) = 1$

$H_1(8,7) = -1$

The nonzero components of the G_N matrix are

$G_N(1,1) = 1$

$G_N(2,2) = 1$

$G_N(3,3) = 1$

$G_N(4,4) = 1$

$G_N(5,5) = 1$

$G_N(6,5) = 1$

$G_N(7,7) = 1$

$G_N(8,7) = 1$

The nonzero components of the H_N matrix are

$H_N(6,5) = -1$

$H_N(8,7) = -1$

The error vectors $\underset{\sim}{\epsilon}_1$ and $\underset{\sim}{\epsilon}_N$ have the components evaluated at points 1 and N respectively

$\epsilon(1) = -f_1$

$\epsilon(2) = -f_1''$

$\epsilon(3) = -f_2$

$\epsilon(4) = -f_2''$

$\epsilon(5) = -(w_0 + \nu\lambda/c)$

$\epsilon(6) = -hw_0'$

$\epsilon(7) = -w_1$

$\epsilon(8) = -hw_1'$

TABLE I

RESULTS OF CONVERGENCE CHECK

$R = L = 4 \quad t = .004 \quad \lambda = .4 \quad n = 13$

$A_0 = 0.5 \cos(2\pi\bar{x}) \quad A_1 = -0.05 \sin(\pi\bar{x})$

The C-3 boundary conditions are enforced

Number of Grid Points	Number of Iterations Required	Total Computation Time (sec.)
26	4	9.19
51	5	20.06
101	4	31.20
201	5	76.73
401	5	149.04

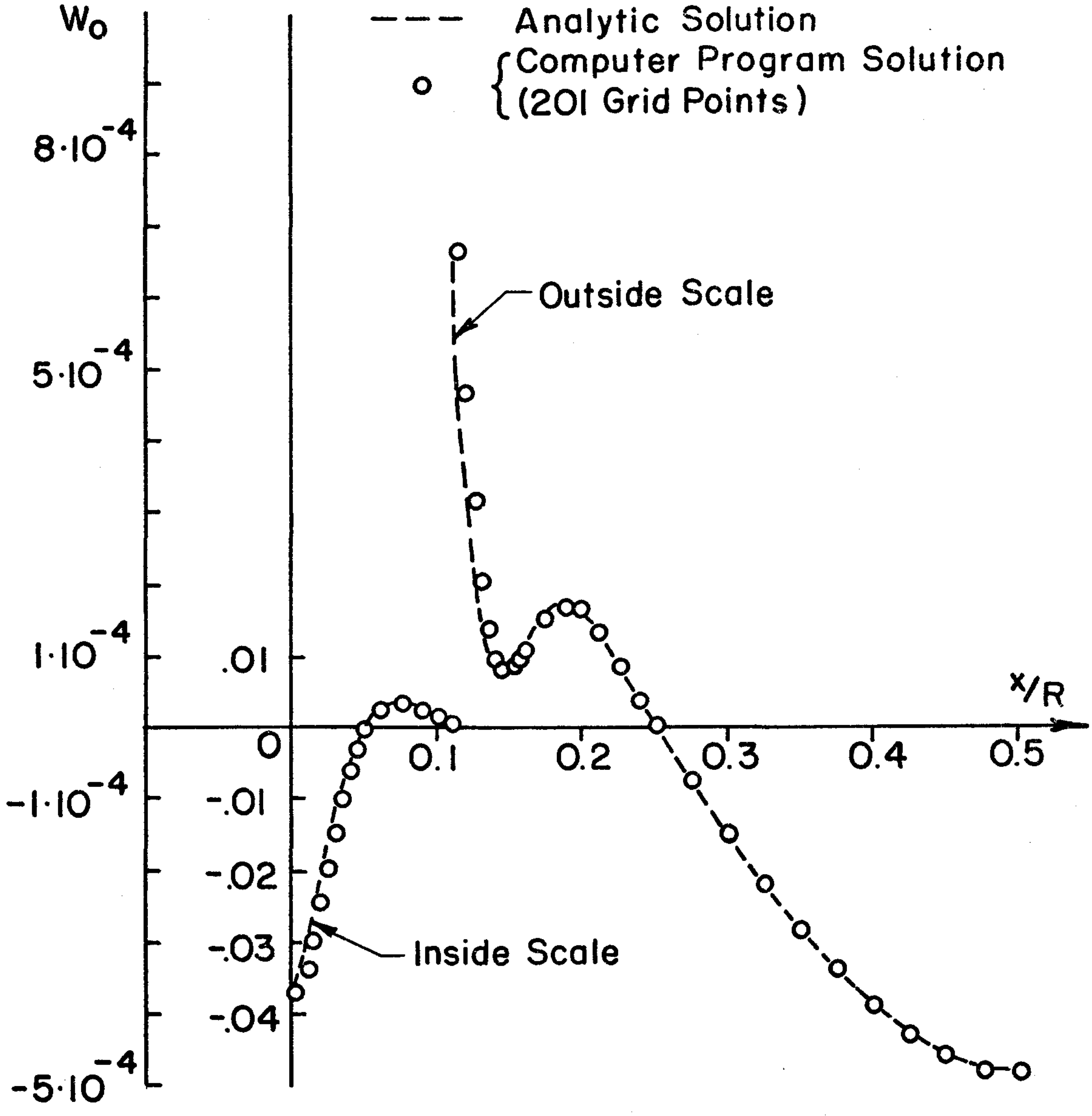

FIG. 1 AXISYMMETRIC RESPONSE W_0 – THE CASE OF AXISYMMETRIC IMPERFECTIONS ONLY.

$\bar{\xi} = 0.1$, $i = 2$, $R = 4$, $t = 0.004$, $\nu = 0.3$, $\lambda = 0.2$

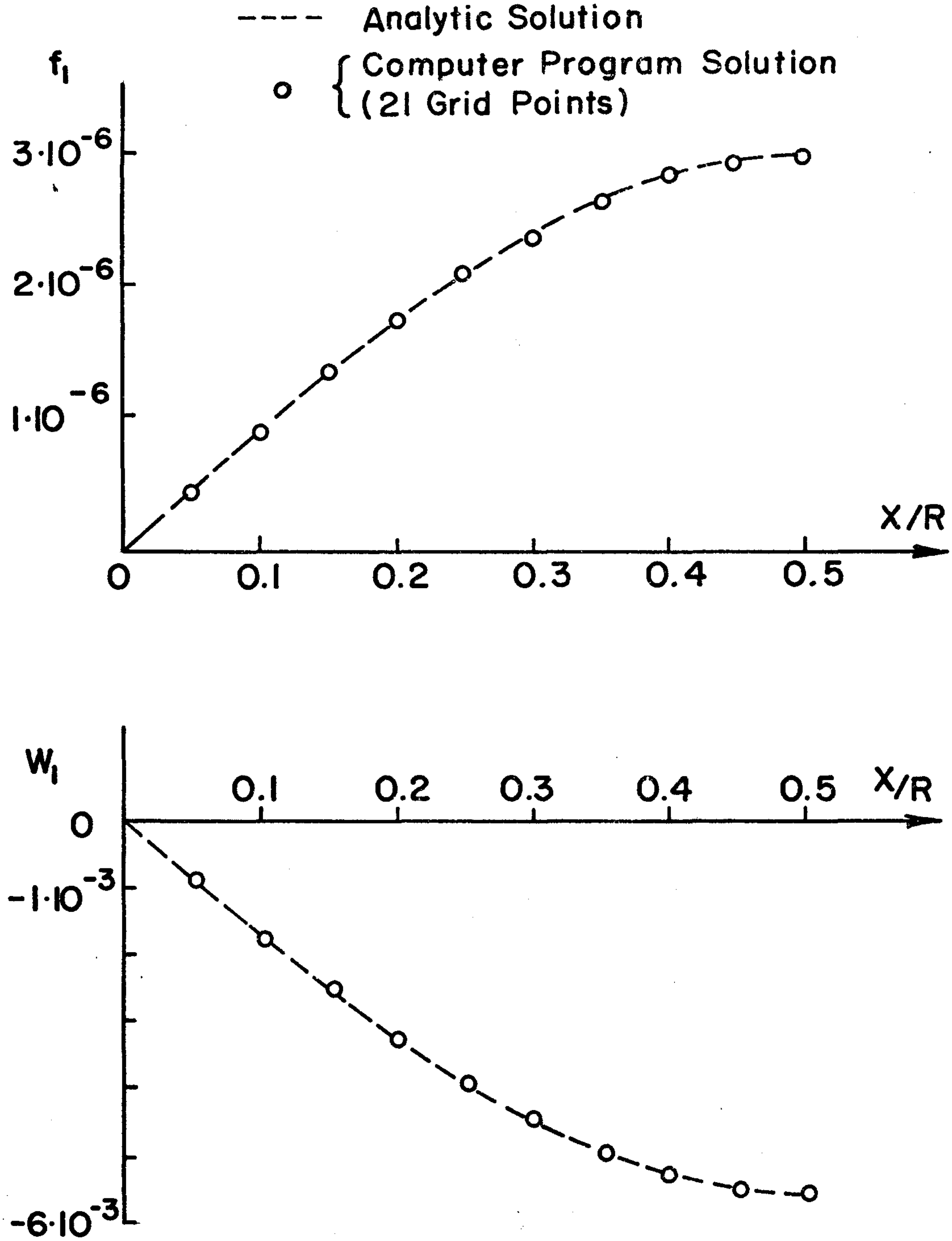

FIG. 2 RESPONSE TO ASYMMETRIC IMPERFECTION

$\overline{\xi} = -0.05,\ \eta = 13,\ \nu = 0,\ \lambda = 0.1, R = 4, L = 4, t = 0.004$

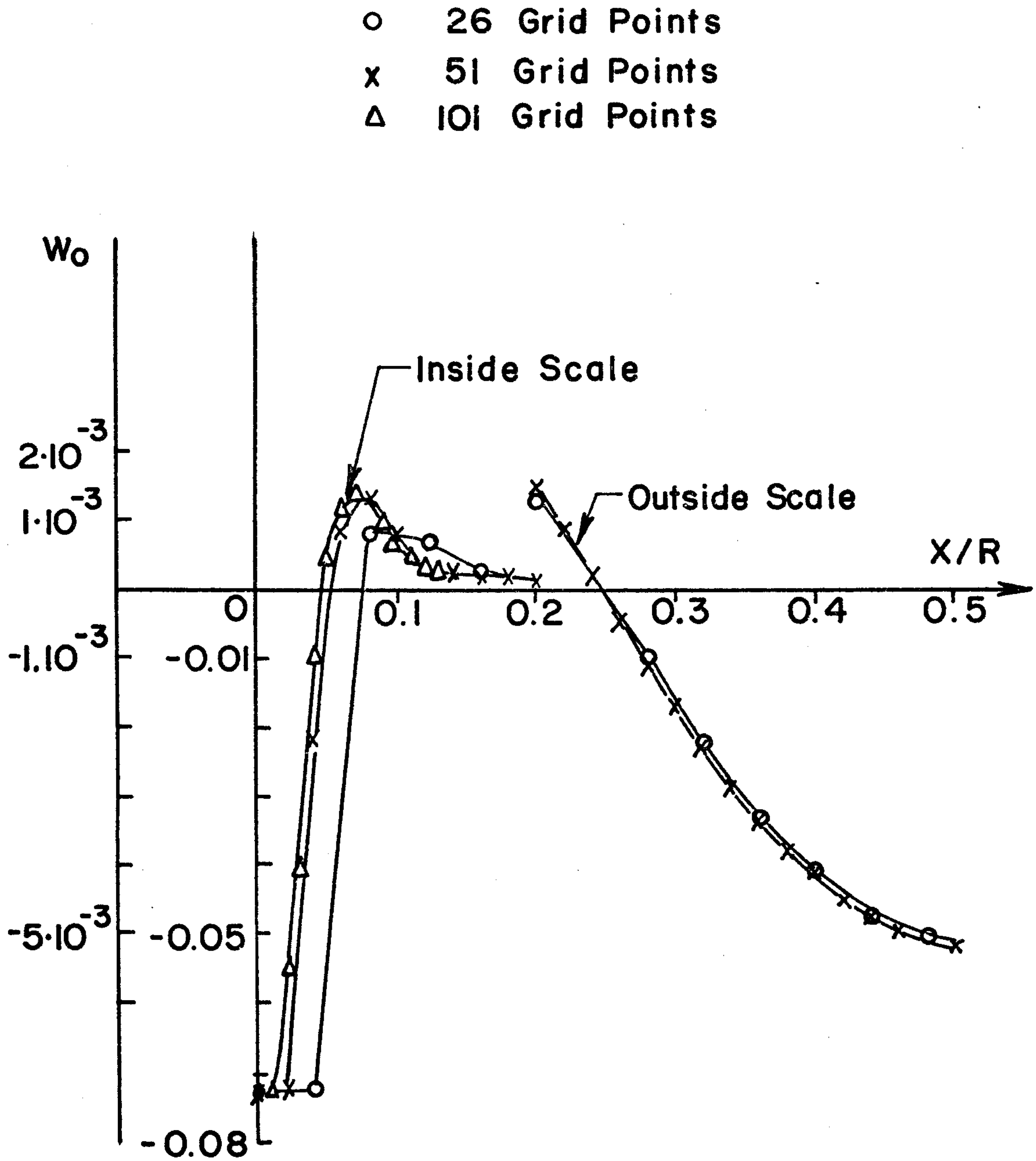

FIG. 3 DISPLACEMENT FUNCTION W_0 - THE GENERAL CASE OF AXISYMMETRIC AND ASYMMETRIC IMPERFECTIONS

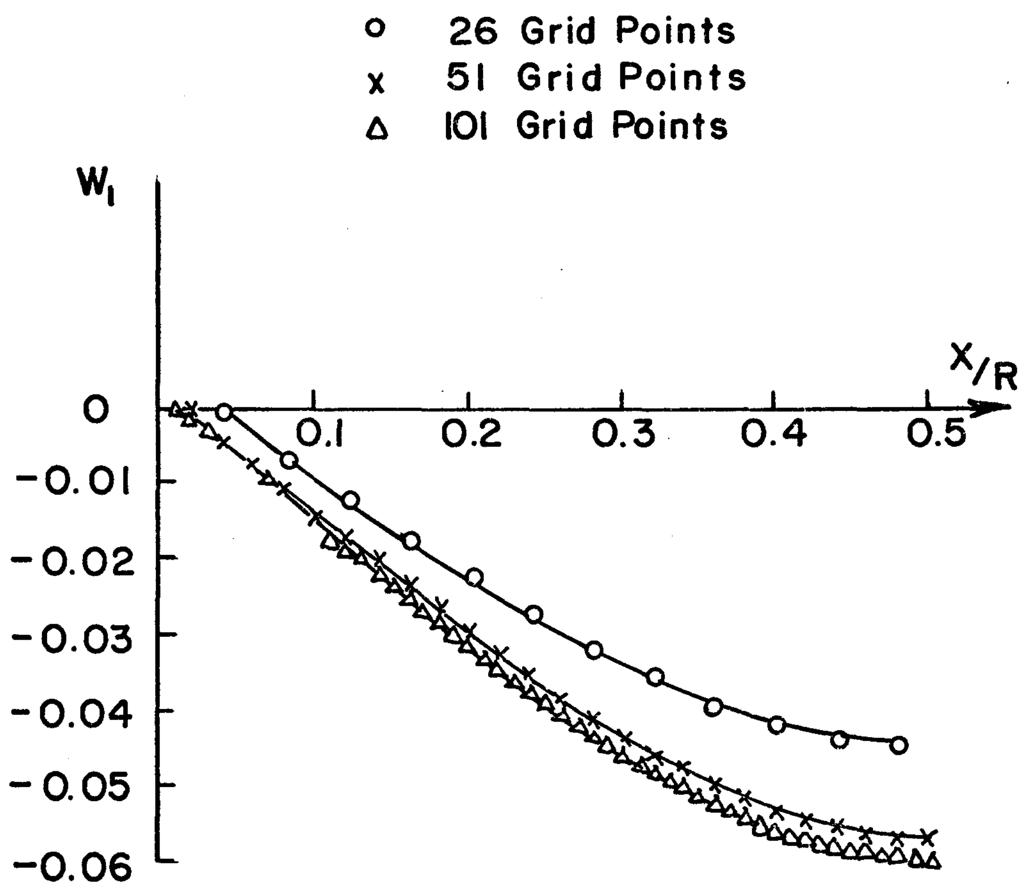

FIG. 4 DISPLACEMENT FUNCTION W_1 — THE GENERAL CASE OF AXISYMMETRIC AND ASYMMETRIC IMPERFECTIONS

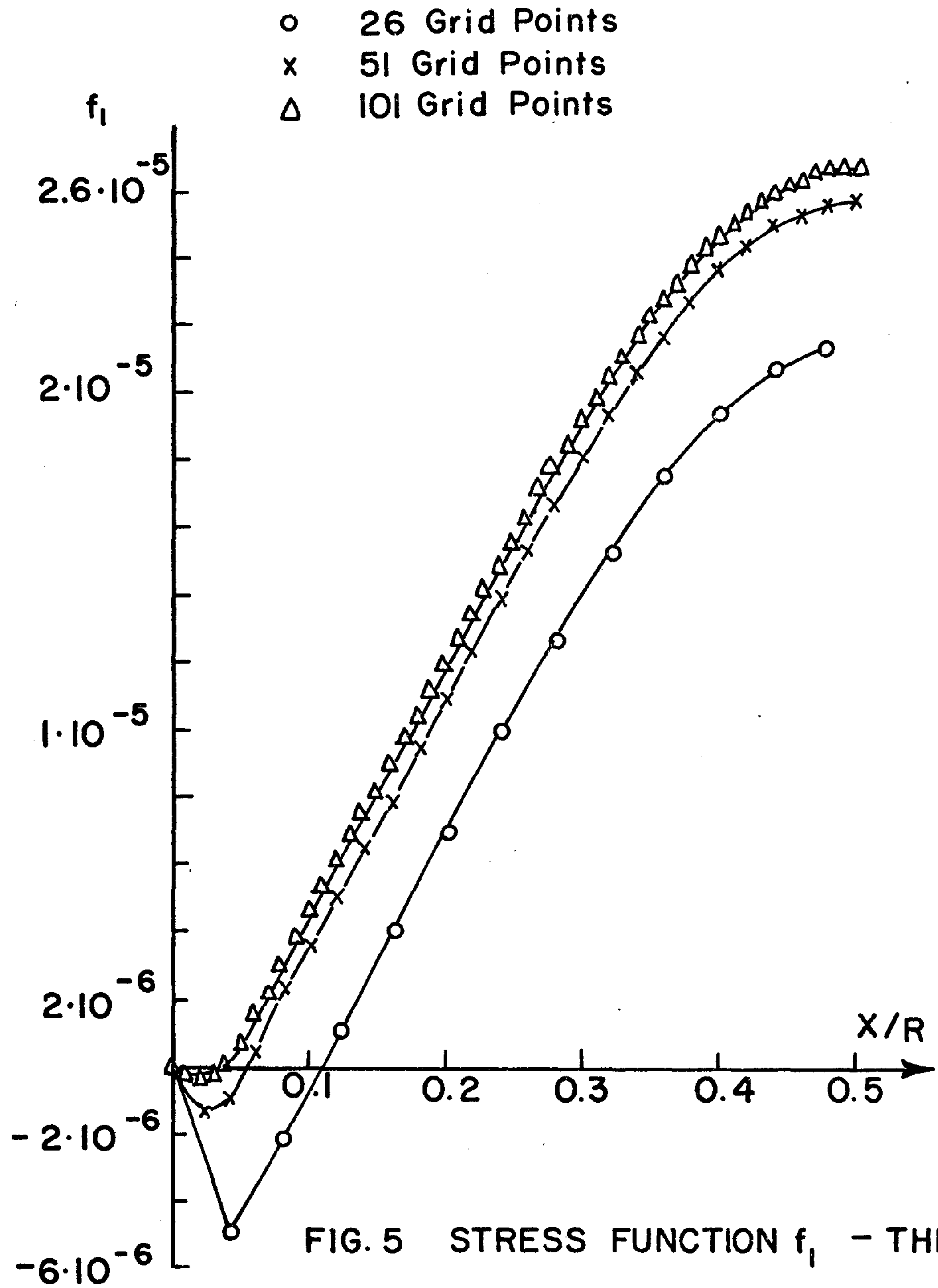

FIG. 5 STRESS FUNCTION f_1 – THE GENERAL CASE OF AXISYMMETRIC AND ASYMMETRIC IMPERFECTIONS

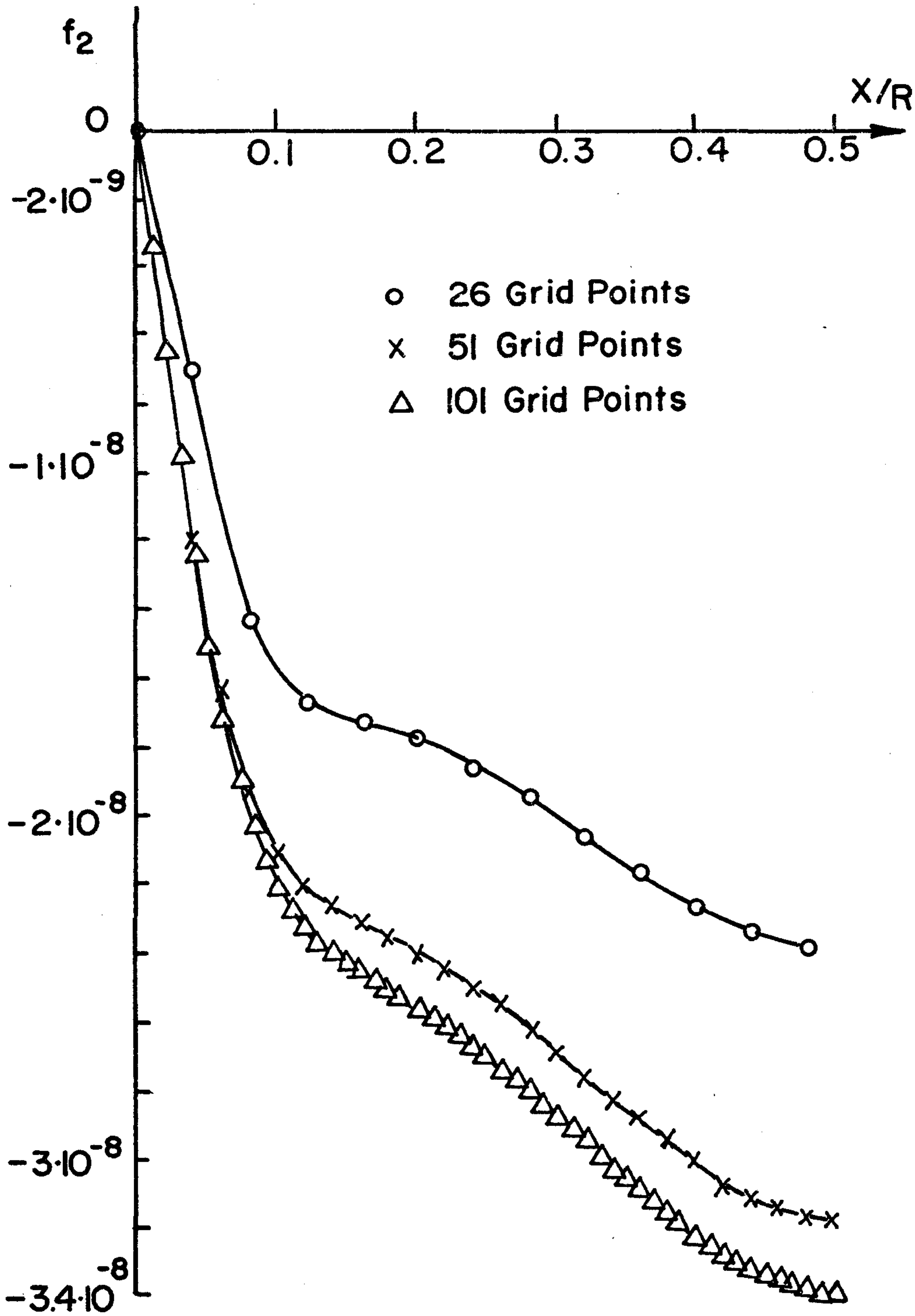

FIG. 6 STRESS FUNCTION f_2 – THE GENERAL CASE OF AXISYMMETRIC AND ASYMMETRIC IMPERFECTIONS