ELASTOSTATIC INTERACTION OF CRACKS IN THE INFINITE PLANE

Thesis by

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ABSTRACT

The stress boundary value problem of an infinite, planar region with embedded rectilinear cracks is investigated from the viewpoint of two-dimensional, static, linear elasticity theory (plane strain or generalized stress). Any finite number of cracks may be considered. Their orientation may be arbitrary, so long as they do not intersect. Boundary loadings may take the form of quite general in-plane tractions along the crack surfaces, together with a bounded in-plane stress field at infinity.

Using Muskhelishvili's solution for colinear cracks, the problem is reduced to a set of one-dimensional Fredholm integral equations. A simple numerical technique is presented for the approximate solution of these equations. The method is established to possess an extremely high rate of convergence.

Results are presented for a number of two-crack interaction problems. As expected, the interaction of the cracks generally tends to reduce the fracture strength of a material, relative to the strength that would exist with either crack acting independently. However, for certain orientations, it is found that the interaction phemomenon can actually increase the resistance to fracture.

TABLE OF CONTENTS

TABLE OF CONTENTS

(Continued)

LIST OF FIGURES

LIST OF SYMBOLS

$\hat{\lambda}, \hat{\mu}$	Hölder indices, defined on page 17
a_n, b_n	Crack endpoints defined on page 28
A_α	Components of resultant force on a boundary
$\hat{A}_n$	Integration constants in equation (2.75)
$\hat{a}(x)$	Non-homogeneous term in Fredholm equation
$\hat{a}^i$	Vector components defined in equations (3.3)
$\underset{\sim}{\hat{a}}$	Vector defined on page 65
B	Open region occupied by body
B_R	Region defined in equation (2.6)
$\hat{B}_m$	Bernoulli numbers
$\hat{c}$	Material constant in equation (1.2)
C_R	circle of radius R
$C_{\hat{r}}, C_{\hat{r}_1}, C_{\hat{r}_2}, C_{\hat{r}_3}, C_{\hat{r}_4}$	Lines defined on page 33
$c_o, c_1, \ldots, d_o, d_1, \ldots$	Coefficients in Laurent expansions in equations (2.43)-(2.44)
c	Positive constant defined in equation (I-39)
$d^n(x_n)$	Functions defined in equation (2.83a)
$\tilde{d}^n(x_n)$	Functions defined in equation (2.95)
D	Two-dimensional region defined on page 82

LIST OF SYMBOLS

(Continued)

D_ε	Deleted region defined in equation (I-7)
E_2	Two-dimensional Euclidean space
E_c	Energy of the cracks
$\hat{E}_{2m}$	Portion of truncation error in trapezoidal rule quadrature, defined in equations (3.7)
$F^n(t_n)$	Functions defined in equation (2.62)
$f^n(t_n)$	Functions defined by equations (2.70), (2.68)
$F_c{}^n$, $f_c{}^n$	Holomorphic functions used in Appendix II
$\hat{G}^{(n)+}$, $\hat{G}^{(n)-}$	Functions defined in equations (2.23)
$G^n(t_n)$	Functions defined in equation (2.53a)
$g^n(t_n)$	Functions defined by equations (2.74)
h^n, H^n	Functions defined in equations (2.87)
$H(x,t)$	Kernel in Fredholm equation
H^{ij}	Matrix components defined in equations (3.3)
$\underset{\sim}{H}$	Matrix defined on page 65
$\hat{h}$	Mesh size for numerical quadrature formula ($=\frac{\hat{\ell}}{J}$)
$h(\varepsilon)$	Energy function defined in (I-33)
i	Unit imaginary number ($=\sqrt{-1}$)

LIST OF SYMBOLS

(Continued)

$I^n(\hat{z}_n)$	Functions defined in equation (2.81a)
$\tilde{I}^n(x_n)$	Functions defined in equation (2.84c)
$\underset{\sim}{I}$	Identity matrix
K_I, K_{II}	Stress intensity factors
K_{Icr}	Critical stress intensity factor
$k_o, k_1 \ldots k_n$	Coefficients of polynomial in equation (2.29c)
$K_{nm}, \widetilde{K}_{nm}$	Kernels of Fredholm equations
$K_I^{n,1}, K_I^{n,2}$	Symmetric stress intensity factors defined in equations (2.90)
$K_{II}^{n,1}, K_{II}^{n,2}$	Antisymmetric stress intensity factors defined in equations (2.90)
k	Positive constant defined in equation (I-28)
L_n	Line segment denoting position of crack
L	Union of all cracks
ℓ_n	Crack half-length
ℓ	Crack half-length for single crack
$\hat{\ell}$	Upper limit for integral in equation (3.7a)
$L^+_{n,\varepsilon}$, $L^-_{n,\varepsilon}$	Boundary lines defined in equations (I-8)

LIST OF SYMBOLS

(Continued)

M_u	Constant bounding the displacements, defined in equation (2.13)
N	Number of cracks
N_1, N_2	Principal stresses associated with σ_∞
N^n	Neighborhood of crack L_n, used in Appendix II
$N_\rho{}^n$	Neighborhood of crack L_n, defined in equation (II-2)
$\hat{N}^n$	Reflection of N^n through the real z_n- axis
$P_N(z)$	Polynomial defined in equation (2.29c)
$p(t)$, $q(t)$	Functions defined in equations (2.26)
$p^{n,o}(t)$, $q^{n,o}(t)$	Functions defined in equations (2.50)
$p^n(t_n)$, $q^n(t_n)$	Functions defined in equations (2.61), (2.62)
r,θ	Polar coordinates with origin at crack tip
R_c	Radius of circle defined on page 18
$\hat{r}$	Parameter defined on page 33
$R_J(x)$	Truncation error in numerical quadrature formula
R^i_j	Vector components defined in equations (3.3)

LIST OF SYMBOLS

(Continued)

$\underset{\sim}{R}_J$	Vector defined on page 65
S_n^+, S_n^-	Two faces of crack surface
S^+, S^-	Total upper and lower crack surfaces, defined in equation (2.4)
$\mathcal{S}$	Stress state defined on page 19
$\mathcal{S}^{(n)}$	Stress state expressed in coordinate system associated with z_n
$\mathcal{S}^T$	Total stress state
$\mathcal{S}^n$, $\mathcal{S}^{n,o}$	Single-crack stress states
$s_\alpha^{(n)+}$, $s_\alpha^{(n)-}$	Tractions generated on crack L_n by a stress state
$\mathcal{S}_1$, $\mathcal{S}_2$	Elastostatic states satisfying same traction boundary conditions
S_o, S_c	Two-dimensional regions defined in equations (II-15), (II-16)
$T(\theta)$	Function defined in equation (1.4b)
T_{max}	Cleavage intensity factor, defined in equation (1.5b)
T_{cr}	Critical cleavage intensity factor
$T_{x_n}^+$, $T_{y_n}^+$, $T_{x_n}^-$, $T_{y_n}^-$	External tractions acting on crack surfaces
$\mathcal{T}^{(n)+}$, $\mathcal{T}^{(n)-}$	Traction operators defined in equations (2.19)

LIST OF SYMBOLS

(Continued)

$\mathcal{T}_s^{(n)}, \mathcal{T}_a^{(n)}$	Traction operators defined in equations (2.21)
u_i, u_α	Displacement components
$\underset{\sim}{u}(x_o, y_o)$	Displacement vector field
Δu_α^n	Displacement discontinuities defined in equation (2.91)
U_D	Strain energy (per unit thickness) in D
U_ε	Strain energy (per unit thickness) in the deleted region D_ε
$\underset{\sim}{u}_r$	Displacement vector field corresponding to a unit rigid-body rotation
$\underset{\sim}{u}'$	Displacement vector field defined in equation (I-68)
$\hat{v}(t)$	Integrand in numerical quadrature formula
$\hat{v}_i$	Value of integrand at quadrature points
$\hat{V}(\hat{\theta})$	Transformed integrand
V	Three-dimensional region
∂V	Boundary of V
W	Strain energy density
$w_1(z)$, $w_2(z)$	Functions defined in equations (2.32)
X(z)	Function defined in equations (2.27)

LIST OF SYMBOLS

(Continued)

z_n	Complex coordinate $(= x_n + iy_n)$
z_{mn}	Location of origin of z_n system, given in z_m system
$\hat{z}_n$	Normalized coordinate $(= \frac{z_n}{\ell_n})$
α_∞	Angle between N_1 and the $+x_o$ axis
α_{mn}	Rotation of $+x_n$ axis with respect to $+x_m$ axis
β_{max}	Cleavage angle
$\gamma_{ij}, \gamma_{\alpha\beta}$	Strain components
γ_z	Out-of-plane strain
$\gamma(x_o, y_o)$	Strain tensor field
Γ, Γ'	Constants defined in equations (2.29)
Γ	Boundary of two-dimensional region defined on page 82
$\Gamma_\varepsilon^{n,1}, \Gamma_\varepsilon^{n,2}$	Boundaries defined in equations (I-6)
Γ_ε	Boundary curve defined in equation (I-12)
$\delta_{\alpha\beta}, \delta_{ij}$	Kronecker delta
δ_j	Weighting coefficients for a numerical quadrature formula
ε_∞	Arbitrary rotation at infinity
ε_o	Length parameter defined in (I-4)
$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$	Length parameters used in Appendix I

LIST OF SYMBOLS

(Continued)

Θ	Sum of principal stresses (= $\sigma_{11} + \sigma_{22} + \sigma_{33}$)
ϑ	Sum of principal strains (= $\gamma_{11} + \gamma_{22} + \gamma_{33}$)
$\hat{\vartheta}$	Factor in $\hat{E}_{2m}$, defined in equations (3.7)
$\hat{\theta}$	Trigonometric integration variable
λ, μ	Lamé constants for a material
$\tilde{\lambda}$	Equivalent Lamé constant defined in equation (3.12)
$\hat{\lambda}$	Parameter of Fredholm equation
$\tilde{\nu}$	Equivalent Poisson's ratio (= $\frac{\tilde{\lambda}}{2(\tilde{\lambda}+\mu)}$)
$\Pi_\varepsilon^{n,1}, \Pi_\varepsilon^{n,2}$	Regions defined in equations (I-5)
Π_e	Exterior region defined in equation (I-64)
$\partial\Pi_e$	Boundary of Π_e
ρ	Length parameter used in Appendix II
$\sigma_{ij}, \sigma_{\alpha\beta}$	Stress components in a Cartesian coordinate system
$\sigma_r, \sigma_\theta, \tau_{r\theta}$	Stress components in a polar coordinate system
σ_z	Out-of-plane stress
$\underset{\sim}{\sigma}(x_o, y_o)$	Stress tensor field

LIST OF SYMBOLS

(Continued)

σ_∞	Uniform stress field at infinity
$\underset{\sim}{\sigma}, \underset{\sim}{\sigma}'$	Stress tensors used in coordinate rotation, equations (2.55), (2.56)
$\hat{\varphi}(t)$	Arbitrary function
$\Phi(z), \Omega(z)$	Complex potentials defined in Chapter II, Section 3
$\varphi(z), \omega(z)$	Complex potentials $(\int^z \Phi(\ell)d\ell, \int^z \Omega(\ell)d\ell)$
$\Phi_o(z), \Omega_o(z)$	Complex potentials defined in equations (2.29)
$\Phi^{n,o}(z_n), \Omega^{n,o}(z_n)$	Complex potentials defined in equations (2.48), (2.49)
$\Phi^n(z_n), \Omega^n(z_n)$	Complex potentials defined in equation (2.63)
$\varphi^n(z_n)$	Complex potential $(= \int^z \Phi^n(\ell)d\ell)$
$\phi(x)$	Unknown function in Fredholm equation
ϕ^j	Vector components defined in equations (3.3)
$\underset{\sim}{\phi}$	Vector defined on page 65
$\hat{\underset{\sim}{\phi}}$	Approximate solution vector
$\hat{\Phi}^n, \hat{\varphi}^n, \hat{\Omega}^n$	Complex potentials used in Appendix II
$\varkappa$	Material constant $(= \frac{\tilde{\lambda}+3\mu}{\tilde{\lambda}+\mu} = 3-4\tilde{\nu})$

I. INTRODUCTION

1. Some Concepts from the Theory of Fracture Mechanics

The first analytic solution to the problem of a plate weakened by a crack is generally credited to Inglis [9]. Using elliptical coordinates, he derived the stress field about an elliptical hole in an infinite plate, subjected to a tension field at infinity in a direction normal to the major axis of the ellipse. If the minor axis is allowed to tend to zero, the ellipse degenerates into a straight line. This may be viewed as a crack which has no thickness, but is simply a flaw in the material along which free surfaces may exist.

Inglis' paper was first published in 1913. Since that time, the problem of a plate weakened by a single crack has been studied extensively. Solutions have been found for quite general loadings, both at infinity and along the crack surface itself.

These solutions exhibit a characteristic feature. At a typical crack endpoint, consider a local coordinate system in terms of the polar coordinates r and θ (c.f. Fig. 1.). Then the stresses exhibit the following unbounded behavior as r tends to zero*:

$$\sigma_r = \frac{1}{(2\pi r)^{\frac{1}{2}}} \cos\frac{\theta}{2} \left[K_I (1+\sin^2\frac{\theta}{2}) + \frac{3}{2} K_{II} \sin\theta - 2K_{II} \tan\frac{\theta}{2} \right] + O(1) \quad (1.1a)$$

$$\sigma_\theta = \frac{1}{(2\pi r)^{\frac{1}{2}}} \cos\frac{\theta}{2} \left[K_I \cos^2\frac{\theta}{2} - \frac{3}{2} K_{II} \sin\theta \right] + O(1) \quad (1.1b)$$

$$\tau_{r\theta} = \frac{1}{2(2\pi r)^{\frac{1}{2}}} \cos\frac{\theta}{2} \left[K_I \sin\theta + K_{II} (3\cos\theta - 1) \right] + O(1) \quad (1.1c)$$

$$\text{as } r \rightarrow 0$$

where the terms O(1) represent that portion of the stress field which is bounded at the crack tip. The parameters K_I and K_{II} are dependent upon the loading conditions, but are independent of r and θ. The factor $(2\pi r)^{-\frac{1}{2}}$ appearing in equations (1.1) explicitly exhibits the characteristic "square root singularity" in the stresses at the crack tip. Evidently, the singular term may be completely determined by specifying only the two parameters K_I and K_{II}. These numbers are known as the symmetric and antisymmetric stress intensity factors, respectively.

The singular behavior of the stresses (and thus the strains) is somewhat disconcerting. The problem is posed in the realm of linear elasticity, or the "small strain" theory. Yet, the solution yields strains in the

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*c.f. Erdogan and Sih. (5) It will be possible to verify this characteristic singular behavior from results developed in the subsequent investigation. Indeed, (1.1) will be shown to be valid even when there are multiple interacting cracks.

vicinity of the crack tips which not only fail to be "small," but are in fact unbounded. In addition, for the loading considered by Inglis, the shape of the deformed crack surface is found to be that of an ellipse (c.f. Fig. 1). This violates another tenet of linear elasticity theory, namely, that the boundary conditions may be applied in the undeformed state. In the vicinity of the crack endpoint, the sharp tip of the undeformed boundary clearly differs radically from the rounded end of the deformed boundary. This is particularly true of of the normal vectors to the two surfaces. In the undeformed state, this is always normal to the crack. In the deformed state, it may be parallel to the crack. Finally, the generalized plane stress assumption is used in deriving the solution. This presumes that the stress field is two-dimensional in nature, and that the out-of-plane stress, σ_z, is essentially zero. The latter assumption then implies that the out-of-plane strain γ_z will be given by

$$\gamma_z = \hat{c}\,(\sigma_r + \sigma_\theta) \tag{1.2}$$

where $\hat{c}$ is a constant dependent on the material properties. Hence, it follows that γ_z will also exhibit a square root singularity in the vicinity of a crack tip. This necessitates unbounded deformations in the out-of-plane direction, which clearly cannot be sustained without destroying the two-dimensional nature of the problem.

Evidently, singular solutions cannot adequately represent the mechanics of the problem in the very close vicinity of the crack tips. However, because of the relative simplicity of the linear theory in relation to more complex approaches, one might hope that such solutions could still be useful. Thus, it might be presumed that they would adequately represent the stress field away from the crack tips. Experimental evidence indicates that this presumption is valid for materials that are not too ductile. In one of the more careful studies, Dudderar and O'Regan (4) utilized laser holographic interferometry to obtain very high resolution in the vicinity of a crack tip in polymethylmethacrylate. It was found that within a certain radius of the crack tip, the stress field was dominated by three dimensional effects. This radius was generally on the order of (but somewhat less than) the plate thickness. Outside of this radius, the agreement with the theoretical singular solution was quite good.

The singular solution can thus be justified on the same basis as, for example, the point load in elasticity or the point charge in electrostatics, neither of which exists in reality. That is, it is merely a mathematical convenience which considerably simplifies the analysis, but which can yield quite valid results as long as its limitations are realized.

A very attractive feature of the singular stress distribution is that the strain energy remains bounded in any finite portion of the region, including the vicinity of the crack tips. (This can be seen by noting that near a crack tip, $\sigma_{ij} \propto r^{-\frac{1}{2}}$. Hence, the strain energy density $W \propto r^{-1}$. This remains bounded when integrated over a circle enclosing the crack tip.) It might therefore be hoped that the singular solution would provide a fairly reliable estimate of the energy in any finite region of a cracked, brittle material.

Griffith, (6,7) in his famous papers of 1920 and 1924, utilized the Inglis solution to calculate the strain energy stored within a very large ellipse encompassing what is now known as the "Griffith Crack." In considering the energy balance as the crack underwent a virtual extension, Griffith found that energy was lost from the system. Reasoning that in the elongation process new surface is formed, and that a certain amount of "surface energy" is required to form a unit area of new surface, he logically proposed that the energy lost from the system during crack growth contributes to the energy of formation of new surface. Thus, a necessary condition for actual crack growth is that the rate of energy loss from the system must equal the rate of energy gained by the formation of new surface. This "necessary" condition has since been applied as a "sufficient" condition, with quite good experimental agreement, in predicting the fracture strength of cracked, brittle materials.

Griffith's criterion is sometimes referred to as a "global" criterion, since it considers the energy balance throughout the entire plane. Irwin, [10] on the other hand, developed a fracture criterion based on a "local" energy approach. He considered the work done by forces in the vicinity of the crack tip, as the crack undergoes a virtual extension in its own line. It was found that the rate of mechanical work expenditure by the system (per unit length of crack extension) is directly proportional to K_I. At the same time, the Griffith concept of "surface energy" (which is more properly termed "fracture energy") indicates that a certain minimum energy expenditure rate is necessary to form new crack surface. Thus, the Irwin energy criterion states that crack growth may occur when K_I reaches some critical value

$$K_I = K_{Icr} \qquad (1.3)$$

where K_{Icr}, the "critical stress intensity factor," is a material property which must be determined experimentally.

It might be argued that a local energy criterion, based on a non-existent stress singularity, is unrealistic. However, Sanders [18] has derived expressions for the global energy balance across an arbitrary contour in the plane. Using his results, it is possible to show that Irwin's local criterion can be directly related to Griffith's global criterion. Thus, in spite of the fact that the stress singularity is merely a fictitious mathematical construction in the very near vicinity of the crack tip, the stress intensity criterion emerges with an importance equal to that of a global energy criterion.

The discussion thus far has been primarily applicable to so-called "brittle" materials. In ductile materials, plastic flow may occur near the crack tip. If the region of plastic flow is very large, it could significantly affect the stress distribution. In addition, the energy expended in the plastic zone upsets the energy balance. However, Irwin [(11)] and others have suggested semi-empirical modifications to the brittle fracture which would allow the extension to moderately yielding materials. This would include such common structural materials as steel and aluminum.

Until recently, energy criteria could be applied only to cracks extending in their own line. This is because analytic solutions are as yet unavailable for the stress fields about "branch cracks," that is, cracks which grow at an angle to the original crack line. Hence, there is no readily available expression for the energy of the system in the extended state. However, it is generally found that cracks will tend to branch under any loading condition other than normal tension. To account for these cases, Erdogan and Sih [5] suggested a criterion based on the maximum normal tension of the singular part of the stress field. They proposed that the crack will grow radially from the crack tip, in the direction of maximum normal (singular) tension, and at a stress level such that the coefficient of this singular tension exceeds some constant which is dependent upon the surface energy of the material. Thus, the normal tension across a radial line is given by (1.1a), which may be written as

$$\sigma_r = \frac{T(\theta)}{(2\pi r)^{\frac{1}{2}}} + O(1) \tag{1.4a}$$

where

$$T(\theta) = \cos\frac{\theta}{2}\left[K_I\left(1+\sin^2\frac{\theta}{2}\right) + \frac{3}{2}K_{II}\sin\theta - 2K_{II}\tan\frac{\theta}{2}\right] \tag{1.4b}$$

Let $T(\theta)$ attain its maximum value at

$$\theta = \beta_{max} \tag{1.5a}$$

and let this maximum be given by

$$T_{max} = T(\beta_{max}) \tag{1.5b}$$

where, in this report, T_{max} will be referred to as the "cleavage intensity."

Then the hypothesis states that crack growth will begin when T_{max} reaches some critical value,

$$T_{max} = T_{cr} \tag{1.6a}$$

and that the (initial) crack growth proceeds in a radial direction along the line

$$\theta = \beta_{max} \tag{1.6b}$$

Applying the criterion to the special case of a Griffith crack loaded by normal tension at infinity, it may be concluded from (1.3) and (1.6a) that

$$T_{cr} = K_{Icr} \tag{1.7}$$

Erdogan and Sih [5] also presented experimental evidence indicating satisfactory agreement with the theory for brittle materials.

More recently, Palaniswamy [17] devised approximate numerical techniques to analyze the branch crack problem from the viewpoint of the energy balance criterion. His calculations showed very good agreement with the cleavage intensity criterion.

From the above equations, it is evident that

T_{max} is completely determined by the stress intensity factors K_I and K_{II}. Thus, these two parameters play a critical role in fracture phenomena.

2. The Stress Boundary Value Problem

Evidently, from a knowledge of stress intensity factors and crack energies, Fracture Mechanics can say a great deal about the phenomenon of crack growth in a material. Unfortunately, these critical parameters are not always easy to obtain. In order to calculate these quantities, it is first necessary to solve a boundary value problem for the stress field in the material. Although the use of linear elasticity theory provides an enormous simplification, deriving the stress state can still be a formidable problem.

In particular, no general solution exists for the problem of an arbitrary (but finite) number of cracks embedded in the plane at arbitrary orientations. Besides Inglis' original solution, the most significant contribution in this area appears to be Muskhelishvili's (16) solution for the problem of "colinear cracks." First published in 1942, this is an integral form of solution to the problem of rectilinear cracks which are constrained to lie upon the same straight line. Otherwise, the number and lengths of the cracks are arbitrary, as are the external tractions acting on the crack surfaces, and the (bounded) stress at infinity. (Muskhelishvili credits D. I. Sherman (19) with publishing an earlier, but less simple, solution in 1940.) In addition, Muskhelishvili published the first solution to the problem of the infinite plane weakened by cuts which are arcs of circles. These cuts are constrained to lie on one and the same circle. Again, a relatively simple, integral form of solution is found.

A number of papers have been published on the problem of colinear cracks since Muskhelishvili's solution. The problems treated, however, are simply special cases of Muskhelishvili's more general solution.

In addition, quite a number of solutions have been found for the problem of the infinite plane weakened by an infinite number of cracks. These problems generally assume the cracks to be arranged in a periodic array, where one period of this array is equivalent to a single crack in an infinite strip.

Berezhnitskii (2) has derived approximate data for two equal length, rectilinear, parallel cracks, symmetrically placed one above the other. His results were inferred from the solution for two symmetric, arc-shaped cracks.

Yokobori, Uozumi and Ichikawa (24) constructed approximate solutions for two equal length, rectilinear, parallel cracks which may be staggered. The solution was expressed in terms of an (unknown) distribution of dislocation densities. Evaluation of the boundary conditions then led to integral equations for the unknown distributions, which were then solved numerically. The approach is quite closely related to that taken here. However, the geometry and loadings are much more restrictive.

For the general problem of arbitrarily oriented rectilinear cracks, an approximate solution technique has been presented by Ishida [13]. His approach is to assume the solution in the form of a superposition of single-crack solutions. The complex potentials generating each single-crack stress field are then expressed in terms of their Laurent series outside a circle enclosing the crack. The coefficients of the series are the unknowns. They are found as solutions to infinite systems of equations which result from the application of the boundary conditions at each crack. These equations are solved by a perturbation procedure. This approach is primarily suited to problems in which the crack separation distances are not too small relative to the crack lengths. The reason for this is twofold. (1) The theoretical derivation is valid only when none of the cracks protrudes into a circle about another crack. (2) As indicated earlier, the stresses (and hence the complex potentials) will have square-root singularities near the crack tips. Such a singularity is very poorly represented by a Laurent series in the vicinity of the crack tips. Thus, when one crack tip is very close to another crack, a large number of terms will be necessary to adequately represent the stress fields. This can lead to considerable numerical difficulty, in terms of both computing time and roundoff error.

In this report, an improved numerical technique for the analysis of interacting rectilinear cracks is presented. The approach taken is to use the integral form of Muskhelishvili's colinear crack solution to reduce the problem to a set of one-dimensional integral equations. These equations, which are reasonably simple algebraically, are then solved numerically. It is shown that the method is valid for <u>any</u> orientation of the cracks, so long as they do not intersect. Further, the essential form of the solution can be retained throughout the numerical computations. Finally, an error analysis of the method shows that an extremely high rate of convergence can be established.

A computer program was written to apply the method to the problem of two interacting cracks with stress-free surfaces, subject to bounded stresses at infinity. As a check case, the program was applied to the problem of colinear cracks. The agreement is found to be excellent, and the rate of convergence is surprisingly good. Finally, a survey of some two-crack interaction problems was run, and a number of interesting conclusions drawn.

II. ANALYTICAL FORMULATION

1. Problem Description

The problem under consideration is a stress boundary value problem in the two-dimensional theory of elasticity (plane strain or generalized plane stress). The material body is assumed to be homogeneous, isotropic, and linearly elastic. Body forces are absent, and the loadings are independent of time.

The geometry and coordinate systems may be visualized with the aid of Figure 2. A two-dimensional region, infinite in extent, is acted upon by constant in-plane stresses at infinity. The far-field stresses, denoted by σ_∞ are given relative to an arbitrary global coordinate system defined by the complex variable $z_o = x_o + iy_o$. In this system, σ_∞ may be characterized by its principal stresses N_1 and N_2, together with the angle α_∞ between the directions of N_1 and the $+x_o$ axis (measured as positive in the counter-clockwise direction). A finite number (N) of cracks are embedded in the region. These cracks, denoted by the line segments $L_1, L_2, \ldots, L_N$, are assumed to be rectilinear and non-intersecting. Otherwise, the geometry is arbitrary. With each crack L_n may be associated a local coordinate system in terms of the complex variable $z_n = x_n + iy_n$. Each system is chosen such that L_n lies along the real axis $y_n = 0$, and the center of the crack lies at $z_n = 0$. The endpoints of the crack L_n then lie at $z_n = \pm \ell_n$, where ℓ_n is the half-length of the nth crack. The global coordinate z_o may be related to the local coordinate z_n by

$$z_o = z_{on} + z_n e^{i\alpha_{on}} \qquad n = 1, 2, \ldots, N \tag{2.1}$$

where z_{on} is the location of the origin of the z_n system expressed in the z_o system and α_{on} is the rotation of the $+x_n$ axis with respect to the $+x_o$ axis, measured as positive in the counter-clockwise direction. The quantities z_{on}, α_{on}, and ℓ_n completely define the geometry of the cracks, and they are presumed to be given. Equations (2.1) may be solved to relate any two coordinate systems by

$$z_m = z_{mn} + z_n e^{i\alpha_{mn}} \qquad (m, n = 0, 1, \ldots, N) \tag{2.2}$$

where $\quad z_{mn} = (z_{on} - z_{om}) e^{-i\alpha_{om}} \;, \quad \alpha_{mn} = \alpha_{on} - \alpha_{om}$

and $\quad z_{oo} = \alpha_{oo} = 0 \,.$

The local coordinate systems may be used to differentiate the two faces of each crack by defining the upper surface (S_n^+) to be that lying along $y_n = 0^+$, and the lower surface (S_n^-) to be that lying along $y_n = 0^-$. This convention may then be utilized to describe the tractions which are allowed to act along the surfaces of each crack. These tractions are assumed to be given in the local coordinate systems. Along S_n^+ there are in-plane tractions $T_{y_n}^+$ and $T_{x_n}^+$ in the $+y_n$ and $+x_n$ directions, respectively. Similarly, in-plane tractions along S_n^- are denoted by $T_{y_n}^-$ and $T_{x_n}^-$ in the $+y_n$ and $+x_n$ directions. The applied tractions may be functions of the coordinate x_n along the length of the crack, but it is assumed that the functions satisfy the Hölder condition (H condition) on L_n.*

The boundary of the body is given by

$$L = \bigcup_{n=1}^{N} L_n \tag{2.3}$$

Where it is necessary to be more specific, the boundary L may be viewed as being composed of two distinct surfaces which happen to coincide. These are given by

$$S^+ = \bigcup_{n=1}^{N} S_n^+ \quad , \quad S^- = \bigcup_{n=1}^{N} S_n^- \tag{2.4}$$

The region occupied by the material body will be denoted

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*An extensive discussion of the Hölder condition is given in Chapter I of Muskhelishvili (16). A function $\hat{\varphi}(t)$ defined on a closed interval L is said to satisfy a Hölder condition on L if positive constants $\hat{A}$ and $\hat{\mu}$ may be found such that

$$|\hat{\varphi}(t_2) - \hat{\varphi}(t_1)| \leq \hat{A}\, |t_2 - t_1|^{\hat{\mu}}$$

for all pairs of points t_1, t_2 of L.

Evidently, a <u>sufficient</u> condition for $\hat{\varphi}$ to satisfy the H condition on L is that be piecewise smooth on the closed interval L. (This result follows directly from the above definition, together with the Mean Value Theorem.)

by the open region

$$B = E_2 - L \tag{2.5}$$

where E_2 is the two-dimensional Euclidean space.

To facilitate the subsequent discussion, a one-parameter family of bounded subregions of B will be defined as follows:

$$B_R = \{(x_o, y_o) \mid (x_o, y_o) \in B \text{ and } x_o^2 + y_o^2 < R^2\} \tag{2.6}$$

That is, B_R consists of that portion of B lying within a circle of radius R centered at the origin $z_o = 0$. Let such a circle be denoted by

$$C_R = \{(x_o, y_o) \mid x_o^2 + y_o^2 = R^2\} \tag{2.7}$$

Since the cracks L_n $(n = 1,2,\ldots,N)$ are finite, both in number and in length, it follows that for some sufficiently large R all of L will lie within C_R. In the following investigation, it will be assumed that such an R has been chosen (and fixed) and denoted by R_c.

2. Preliminaries. Stress States and Traction Operators

In seeking a solution to the boundary value problem indicated above, one is generally concerned with determining the stress field $\underset{\sim}{\tau}(x_0,y_0)$ throughout the body B. However, the displacement field $\underset{\sim}{u}(x_0,y_0)$ will be seen to be of some interest. Hence, it will be convenient to speak in terms of a stress state $\mathcal{S}$, defined as the ordered array of functions

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{\tau}] \tag{2.8}$$

A stress state $\mathcal{S}$ will be called an admissible elastostatic stress state (or, simply, an elastostatic state) if the following conditions prevail. On the open region B, $\mathcal{S}$ must be continuously differentiable and, together with the strain field $\underset{\sim}{\gamma}$, $\mathcal{S}$ must satisfy the equations of two-dimensional static, linear elasticity for a homogeneous, isotropic solid in the absence of body forces:*

(i) The strain-displacement equations

$$\gamma_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \tag{2.9}$$

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*In this report, dummy indices represented by Greek letters will always assume the range (1,2,). The presence of a repeated index indicates summation over its range.

(ii) The constitutive relations

$$\sigma_{\alpha\beta} = \tilde{\lambda}\, \gamma_{\phi\phi}\, \delta_{\alpha\beta} + 2\mu\, \gamma_{\alpha\beta} \tag{2.10a}$$

or

$$\gamma_{\alpha\beta} = \frac{1}{2\mu}\, \sigma_{\alpha\beta} - \frac{\tilde{\lambda}}{4\mu(\tilde{\lambda}+\mu)}\, \sigma_{\phi\phi}\, \delta_{\alpha\beta} \tag{2.10b}$$

(iii) The equilibrium equations

$$\sigma_{\alpha\beta,\beta} = 0 \tag{2.11}$$

Here $\delta_{\alpha\beta}$ is the Kronecker delta, λ and μ are the Lamé constants for the material, and

$$\tilde{\lambda} = \begin{cases} \lambda & \text{(Plane Strain)} \\ \dfrac{2\lambda\mu}{\lambda+2\mu} & \text{(Generalized Plane Stress)} \end{cases} \tag{2.12}$$

Further, $\mathscr{S}$ must be continuous up to the boundaries S^+ and S^- of B, with the exception of the crack endpoints $z_n = \pm \ell_n (n = 1,2,\ldots,N)$. (However, $\mathscr{S}$ may be discontinuous across L.) Finally, $\underset{\sim}{u}$ and $\underset{\sim}{\sigma}$ must satisfy the following boundedness conditions:

(i) There must exist some finite constant M_u such that

$$|\underset{\sim}{u}(z_o)| \leq M_u \quad \text{for all} \quad z_o \in B_{R_c} \tag{2.13}$$

(ii) The stresses must be bounded at infinity, or

$$\underset{\sim}{\sigma}(z_o) = O(1) \quad \text{as} \quad |z_o| \to \infty \tag{2.14}$$

It may be noted that the above definition of an elastostatic state $\mathscr{S}$ permits the existence of singularities in $\mathscr{S}$ at the crack endpoints. The sole restriction on such singularities is the one implied by (2.13). That is, the displacements must be bounded up to the crack tips.

The strain energy density W is given by

$$W = \frac{1}{2} \sigma_{\alpha\beta} \gamma_{\alpha\beta} \tag{2.15}$$

Using (2.10), this may be equivalently expressed as

$$W = \frac{1}{2} \tilde{\lambda} \vartheta^2 + \mu \gamma_{\alpha\beta} \gamma_{\alpha\beta} \tag{2.16a}$$

$$W = \frac{1}{4\mu} \sigma_{\alpha\beta} \sigma_{\alpha\beta} - \frac{\tilde{\lambda}}{8\mu(\tilde{\lambda}+\mu)} \Theta^2 \tag{2.16b}$$

where $\vartheta = \gamma_{\phi\phi}$, $\Theta = \sigma_{\phi\phi}$

The above expressions directly reflect the fact that W may be expressed as a quadratic form in either the strains or the stresses. The usual assumption will be made that these are positive definite quadratic forms. It is well known that this will be true if the material constants $\tilde{\lambda}, \mu$ are such that the following relations are satisfied:

$$\mu > 0 \ , \quad -1 < \tilde{\nu} < \frac{1}{2} \tag{2.17}$$

where $\tilde{\nu} = \dfrac{\tilde{\lambda}}{2(\tilde{\lambda}+\mu)}$

From (2.8), it is evident that the definition of a stress state implicitly presumes the existence of some reference coordinate system. However, allusions will be made to the general concept of a stress state, without reference to a particular coordinate system, when no confusion will result. Conversely, a given stress state may be referred to more than one coordinate system, in which case the usual transformation properties will govern. When necessary, a reference coordinate system for $\mathcal{S}$ will be indicated by a superscript $^{(n)}$, that is,

$$\mathcal{S}^{(n)} = [\underset{\sim}{u}^{(n)}, \underset{\sim}{\sigma}^{(n)}] \qquad n = 0, 1, \ldots, N \tag{2.18}$$

where $\underset{\sim}{u}^{(n)}$, $\underset{\sim}{\sigma}^{(n)}$ are referred to the coordinate system defined by z_n.

An additional formalism which will simplify the subsequent discussion is the concept of traction operators. The traction operators $\mathcal{T}^{(n)+}$ and $\mathcal{T}^{(n)-}$ are referred to a given crack L_n and associated local coordinate z_n. They operate on a stress state $\mathcal{S}$ as follows:

$$\mathcal{T}^{(n)+}\{\mathcal{S}\}(t_n) = -s_2^{(n)+}(t_n) + i s_1^{(n)+}(t_n) = \sigma_{22}^{(n)+}(t_n) - i\,\sigma_{12}^{(n)+}(t_n) \tag{2.19a}$$

$$\mathcal{T}^{(n)-}\{\mathcal{S}\}(t_n) = s_2^{(n)-}(t_n) - i s_1^{(n)-}(t_n) = \sigma_{22}^{(n)-}(t_n) - i\,\sigma_{12}^{(n)-}(t_n) \tag{2.19b}$$

$$t_n \in (-\ell_n, \ell_n), \quad n = 1, 2, \ldots, N$$

Here $s_2^{(n)+}$ and $s_1^{(n)+}$ are the tractions in the $+y_n$ and $+x_n$ directions, respectively, which are generated by $\mathcal{S}$ on the surface S_n^+. Further, $\sigma_{22}^{(n)+}$ and $\sigma_{12}^{(n)+}$ are the indicated stresses generated by $\mathcal{S}$ on the surface S_n^+, expressed in the z_n system. Similarly, $s_2^{(n)-}$, $s_1^{(n)-}$,

$\sigma_{22}^{(n)-}$ and $\sigma_{12}^{(n)-}$ are the corresponding tractions and stresses generated by $\mathcal{S}$ on the surface S_n^-. A reference system for $\mathcal{S}$ need not be explicitly indicated.

Equations (2.19) clearly demonstrate the direct relationship between the tractions acting on S^+ and S^-, and the stress components on S^+ and S^-:

$$
\begin{aligned}
S_2^{(n)+} &= -\sigma_{22}^{(n)+} \quad , \quad S_1^{(n)+} = -\sigma_{12}^{(n)+} \\
S_2^{(n)-} &= \sigma_{22}^{(n)-} \quad , \quad S_1^{(n)-} = \sigma_{12}^{(n)-}
\end{aligned}
\tag{2.20}
$$

It is pedantically more correct to speak of tractions rather than stresses being applied to a surface. However, in the development of this problem it will be convenient to speak in terms of the applied stresses on the crack surfaces. Due to the simple relationship indicated in (2.20), no confusion should result.

Two additional operators will be defined by

$$\mathcal{T}_s^{(n)} = \frac{1}{2}\left(\mathcal{T}^{(n)+} + \mathcal{T}^{(n)-}\right) \tag{2.21a}$$

$$\mathcal{T}_a^{(n)} = \frac{1}{2}\left(\mathcal{T}^{(n)+} - \mathcal{T}^{(n)-}\right) \tag{2.21b}$$

From equations (2.19), it then follows that

$$\mathcal{T}_s^{(n)}\{\mathcal{S}\}(t_n) = \frac{1}{2}\left[\sigma_{22}^{(n)+}(t_n) + \sigma_{22}^{(n)-}(t_n)\right] - i\,\frac{1}{2}\left[\sigma_{12}^{(n)+}(t_n) + \sigma_{12}^{(n)-}(t_n)\right] \tag{2.22a}$$

$$\mathcal{T}_a^{(n)}\{\mathcal{S}\}(t_n) = \frac{1}{2}\left[\sigma_{22}^{(n)+}(t_n) - \sigma_{22}^{(n)-}(t_n)\right] - i\,\frac{1}{2}\left[\sigma_{12}^{(n)+}(t_n) - \sigma_{12}^{(n)-}(t_n)\right] \tag{2.22b}$$

$$t_n \in (-\ell_n, \ell_n) \quad , \quad n = 1, 2, \ldots, N$$

From these equations it may be seen that $\mathcal{T}_s^{(n)}$ corresponds to that portion of the applied stresses acting on L_n which is symmetric with respect to the upper and lower surfaces, S_n^+ and S_n^-. Similarly, $\mathcal{T}_a^{(n)}$ corresponds to that portion of the applied stresses which is antisymmetric with respect to S_n^+ and S_n^-.

With the aid of the above formalisms, the stress boundary value problem which was indicated in Section 1 of this chapter may be more rigorously stated as follows: Given the crack-embedded plane B, find an elastostatic state $\mathcal{S}$ defined on B and satisfying

$$\mathcal{T}^{(n)+}\{\mathcal{S}\}(t_n) = -T_{y_n}^{+}(t_n) + i\,T_{x_n}^{+}(t_n) = \hat{G}^{(n)+}(t_n) \tag{2.23a}$$

$$\mathcal{T}^{(n)-}\{\mathcal{S}\}(t_n) = T_{y_n}^{-}(t_n) - i\,T_{x_n}^{-}(t_n) = \hat{G}^{(n)-}(t_n) \tag{2.23b}$$

$$t_n \in (-\ell_n, \ell_n) \quad , \quad n = 1, 2, \ldots, N$$

$$\mathcal{T} = \mathcal{T}_\infty + o(1) \quad \text{as} \quad |z_0| \rightarrow \infty \tag{2.23c}$$

Note that the equations and continuity conditions that the solution must satisfy are implied by the requirement that $\mathcal{S}$ be an admissible elastostatic stress state. Further, since the applied tractions on L_n are assumed to satisfy the H condition on L_n, it follows that the functions $\hat{G}^{(n)+}(t)$ and $\hat{G}^{(n)-}(t)$ defined in equations (2.23) will also satisfy the H condition.

It might be noted that the strains are not explicitly exhibited in equation (2.8), the definition of a stress state. This is because the problem posed is a stress boundary value problem. Hence, the strains are not central to the problem solution, and follow merely as a result of the solution.

The displacements, which are explicitly exhibited in (2.8), are central to the problem solution in two ways: (1) uniqueness of the solution; (2) the problem of crack surface "overlap."

The difficulty with uniqueness arises because the problem being solved is a two-dimensional elasticity problem on a multiply-connected domain with stress boundary conditions. It is well known that in such a circumstance the stress equilibrium equations and boundary conditions may possess an infinity of solutions, corresponding to different multi-valued displacement fields. The correct solution may be determined only by enforcing the requirement that the displacement be single-valued. The uniqueness of this single-valued solution is proved for a wide class of admissible singular solutions in Appendix I.

The difficulty with crack surface overlap arises because the boundary conditions on the crack surfaces are assumed to be given in terms of the tractions. Hence, the displacements of the crack surfaces cannot be specified, and can only be derived as a result of the solution. For certain geometries and loadings, it may be found that the relative normal displacement of the upper and lower crack surfaces is negative over certain intervals. That is, the crack surfaces "go through" each other, or they "overlap."

A simple example is provided by the Griffith crack with traction-free surfaces. If this is subjected to a unit normal tension at infinity, then, as indicated in Chapter I, the crack surfaces open into an ellipse. Conversely, if the crack is subjected to a unit normal compression at infinity, then, by the linearity of the problem, reversing the sign of the loading should merely reverse the sign of the response. Hence the crack surfaces go through each other into an "inverted ellipse." For plane strain problems, this is clearly unacceptable from a physical viewpoint. The obvious difficulty lies in the imposition of a traction-free boundary condition on the crack surfaces. For normal compression at infinity, the physically reasonable boundary condition is a displacement condition, that is, there should be no relative displacement of the crack surfaces. The correct solution would then be merely a state of uniform compression throughout the body, as though no crack were present.

It is not difficult to conceive of situations in which the solution is not so simple. Thus, a given crack might tend to open along part of its length, and to overlap along part of its length. (An example of such a problem is considered in Chapter III.) Evidently, a physically proper formulation for such a circumstance would be in terms of a "contact problem." That is, a situation in which part of the crack surface is in contact and part is free, and the dividing point is not known beforehand. In addition, the physically reasonable boundary condition along the contact surface might be that there should be no normal displacement but that there may be sliding displacement which would depend upon, say, the shear traction and some assumed coefficient of friction.

Although it would be desirable to incorporate such considerations, the added analytical and numerical complications are beyond the scope of the present investigation. Attention will be confined to the analysis of the stress boundary value problem only. However, methods will be provided for calculating crack surface displacements, which may then be checked to verify whether the solution obtained is physically reasonable.

3. An Integral Form of Solution

Although no solution has been found for the general class of problems stated in Section 2 of this chapter, a solution has been found for one particular subclass of these problems. This is an integral form of solution for the problem of colinear cracks. (The cracks are said to be colinear when they all lie along the same line, as shown in Figure 4.) This solution, which is given in Section 120 of Muskhelishvili[16], will be presented here. It will then be applied to the special case of a single crack, resulting in a relatively simple, integral form of solution. This will be used in subsequent sections to construct solutions to the general problem, which may then be analyzed by numerical techniques.

Without loss of generality, let the global coordinate system be chosen so that the cracks all lie along the real axis (c.f. Fig. 4). To simplify notation, the global coordinate will be represented by z rather than z_o. Let the endpoint $z_n = -\ell_n$ be denoted by $z = a_n$, and the endpoint $z_n = +\ell_n$ by $z = b_n$ $(n = 1,2,\ldots,N)$. Further, assign a positive direction along L_n by the direction from a_n to b_n. Let ε_∞ be the (arbitrary) rotation at infinity. Then Muskhelishvili's solution is given in terms of a pair of sectionally holomorphic functions,* $\Phi(z)$ and $\Omega(z)$, by

$$2\mu(u_1+iu_2) = \varkappa\,\varphi(z) - \omega(\bar{z}) - (z-\bar{z})\,\overline{\Phi(z)} + \text{const.} \tag{2.24a}$$

$$\sigma_{11} + \sigma_{22} = 2\left[\Phi(z) + \overline{\Phi(z)}\right] \tag{2.24b}$$

$$\sigma_{22} - i\,\sigma_{12} = \Phi(z) + \Omega(\bar{z}) + (z-\bar{z})\,\overline{\Phi'(z)} \tag{2.24c}$$

where $\varphi(z)$ and $\omega(z)$ are defined by

$$\varphi'(z) = \Phi(z) \quad , \quad \omega'(z) = \Omega(z) \tag{2.24d}$$

and where the constant term in (2.24a) represents the (arbitrary) displacement at infinity. The factor $\varkappa$ is given by

$$\varkappa = \frac{\tilde{\lambda}+3\mu}{\tilde{\lambda}+\mu} = 3 - 4\tilde{\nu} \quad , \quad \varkappa > 0 \tag{2.25}$$

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*The usage in Muskhelishvili (16) for the terms "holomorphic" and "analytic" will be followed here. The distinction is indicated in a footnote on p. 131 of Muskhelishvili (16). A function $\hat{f} = \hat{f}_1(x,y) + i\hat{f}_2(x,y)$ of the complex variable $z = x + iy$, defined in some domain D of the complex plane, will be said to be holomorphic in D if it is <u>single valued</u> and differentiable with respect to z at every point z in D. Thus a holomorphic function is one which many authors would define as "regular" or "analytic." However, Muskhelishvili reserves the use of the term "analytic" for the broader class of multi-valued functions. A function $\hat{f}$, defined and (possibly) multi-valued on a domain D, will be said to be analytic in D if each continuously varying branch of $\hat{f}$ is holomorphic in any finite, simply connected subdomain of D.

The meaning of the term "sectionally holomorphic" is explained in Section 106 of Muskhelishvili (16).

The functions $\Phi(z)$ and $\Omega(z)$ may be found in terms of the applied loads as follows: Let L be as given in (2.3), and assign a positive direction to L in terms of the positive directions of the line segments composing L. Let t denote a point on L, and let

$$p(t) = \frac{1}{2}\left[\hat{G}^{(n)+}(t) + \hat{G}^{(n)-}(t)\right] \tag{2.26a}$$

$$q(t) = \frac{1}{2}\left[\hat{G}^{(n)+}(t) - \hat{G}^{(n)-}(t)\right] \tag{2.26b}$$

$$t \in L_n \quad , \quad n = 1, 2, \ldots, N$$

where the functions on the right are defined in (2.23). Also, let

$$X(z) = \prod_{n=1}^{N} (z - a_n)^{\frac{1}{2}} (z - b_n)^{\frac{1}{2}} \tag{2.27a}$$

where the branch cuts of $X(z)$ are chosen to lie on the line segments L_n $(n = 1, 2, \ldots, N)$ and the branch of $X(z)$ is chosen such that

$$X(z) = + z^N + c_{N-1} z^{N-1} + \cdots \quad \text{as } |z| \to \infty \tag{2.27b}$$

Then $\Phi(z)$ and $\Omega(z)$ are given by

$$\Phi(z) = \Phi_0(z) + \frac{P_N(z)}{X(z)} + \frac{1}{2}(\Gamma - \bar{\Gamma} - \bar{\Gamma}') \tag{2.28a}$$

$$\Omega(z) = \Omega_0(z) + \frac{P_N(z)}{X(z)} - \frac{1}{2}(\Gamma - \bar{\Gamma} - \bar{\Gamma}') \tag{2.28b}$$

where

$$\Phi_o(z) = \frac{1}{2\pi i X(z)} \int_L \frac{X(t)\,p(t)}{t-z}\,dt + \frac{1}{2\pi i}\int_L \frac{q(t)}{t-z}\,dt \tag{2.29a}$$

$$\Omega_o(z) = \frac{1}{2\pi i X(z)} \int_L \frac{X(t)\,p(t)}{t-z}\,dt - \frac{1}{2\pi i}\int_L \frac{q(t)}{t-z}\,dt \, . \tag{2.29b}$$

$$P_N(z) = k_o z^N + k_1 z^{N-1} + \cdots + k_N \tag{2.29c}$$

$$\Gamma = \frac{1}{4}(N_1 + N_2) + i\,\frac{2\mu\varepsilon_\infty}{1+\varkappa} \tag{2.29d}$$

$$\Gamma' = -\frac{1}{2}(N_1 - N_2)\, e^{-i2\alpha_\infty} \tag{2.29e}$$

$$k_o = \frac{1}{2}(\Gamma + \overline{\Gamma} + \overline{\Gamma'}) \tag{2.29f}$$

The quantity X(t) appearing under the integral in (2.29a) and (2.29b) is understood to be $X^+(t)$, the value taken by X(z) on S^+. The constants $k_1, k_2, \ldots, k_N$ are determined by the system of equations

$$2(\varkappa+1)\int_{L_n} \frac{P_N(t_n)}{X(t_n)}\,dt_n + \varkappa\int_{L_n}[\Phi_o^+(t_n) - \Phi_o^-(t_n)]\,dt_n + \int_{L_n}[\Omega_o^+(t_n) - \Omega_o^-(t_n)]\,dt_n = 0 \tag{2.29g}$$

$$n = 1, 2, \ldots, N$$

Equations (2.29g) are a result of the requirement that the displacements be single valued about each crack. They form a system of N linear, algebraic equations for the N coefficients $k_1, k_2, \ldots, k_N$.

The assertion that $\Phi(z)$ and $\Omega(z)$ are sectionally holomorphic follows from equations (2.26), together with the fact that under the hypothesis of the problem, the functions $\hat{G}^{(n)+}$ and $\hat{G}^{(n)-}$ satisfy the H condition on L_n $(n = 1, 2, \ldots, N)$ (c.f. Muskhelishvili [16]).

If the above solution is applied to the special case of a single crack, the expressions for the generating functions may be considerably simplified. For a crack of length 2ℓ, lying on the real axis and centered at the origin, with vanishing stresses and rotation at infinity, equations (2.26) - (2.29) yield

$$\Phi(z) = \Phi_0(z) + \frac{k_1}{\sqrt{z^2-\ell^2}} \quad , \quad \Omega(z) = \Omega_0(z) + \frac{k_1}{\sqrt{z^2-\ell^2}} \tag{2.30}$$

where

$$\Phi_0(z) = w_1(z) + w_2(z) \quad , \quad \Omega_0(z) = w_1(z) - w_2(z) \tag{2.31}$$

and

$$w_1(z) = \frac{1}{2\pi i \sqrt{z^2-\ell^2}} \int_{-\ell}^{\ell} \frac{p(t)\sqrt{t^2-\ell^2}}{t-z} \, dt \tag{2.32a}$$

$$w_2(z) = \frac{1}{2\pi i} \int_{-\ell}^{\ell} \frac{q(t)}{t-z} \, dt \tag{2.32b}$$

The functions p(t) and q(t) are given in equations (2.26), where the dummy coordinate t may be taken to be the global coordinate along the real axis. As indicated above, p(t) and q(t) satisfy the H condition on L. Consequently, it may be shown that the functions $w_1(z)$ and $w_2(z)$ defined in equations (2.32) are sectionally holomorphic.

The constant k_1 is determined by (2.29g) which, together with (2.32) yields

$$-2(\varkappa+1)k_1 \int_{-\ell}^{\ell} \frac{dt}{\sqrt{t^2-\ell^2}} = (\varkappa+1)\int_{-\ell}^{\ell} (w_1^+ - w_1^-)\,dt + (\varkappa-1)\int_{-\ell}^{\ell} (w_2^+ - w_2^-)\,dt \tag{2.33}$$

The integral on the left is given by

$$\int_{-\ell}^{\ell} \frac{dt}{\sqrt{t^2-\ell^2}} = -i\pi \tag{2.34}$$

The first integral on the right of (2.33) is identically zero for <u>any</u> w_1 generated by any $p(t)$ which satisfies the H condition on L. This is most easily seen by contour integration. Referring to Figure 5, consider the following one-parameter families of contours:

(1) C_R, depending on the parameter R, defined by
C_R: $|z| = R$, $R > \ell$

(2) $C_{\hat{r}}$, depending on the parameter $\hat{r}$, defined by
$C_{\hat{r}}$: $C_{\hat{r}_1} + C_{\hat{r}_2} + C_{\hat{r}_3} + C_{\hat{r}_4}$

where

$$\left.\begin{aligned} C_{\hat{r}_1}: &\quad z = x + i0^+ \\ C_{\hat{r}_2}: &\quad z = x + i0^- \end{aligned}\right\} -\ell+\hat{r}<x<\ell-\hat{r}$$

$$C_{\hat{r}_3}: \quad |z-\ell| = \hat{r}$$

$$C_{\hat{r}_4}: \quad |z+\ell| = \hat{r}$$

$$0<\hat{r}<\ell$$

Since $w_1(z)$ is sectionally holomorphic, it is continuous up to the boundary L, both from above and below, at every point of L with the exception of the endpoints $z = \pm\ell$. Further, the following may be shown without undue difficulty: The fact that $w_1(z)$ is continuous up to the smooth boundary L implies that the boundary values, denoted by $w_1^+(t)$ and $w_1^-(t)$, are themselves continuous functions of the coordinate t along L, for $-\ell < t < \ell$. Thus the function $w_1(z)$ can be seen to be continuous on the contours $C_{\hat{r}}$ and C_R, for $0<\hat{r}<\ell$ and $R>\ell$. Moreover, $w_1(z)$ is holomorphic at every point in the domain bounded by $C_{\hat{r}}$ and C_R. Hence, the strong form of the Cauchy-Goursat Theorem* may be utilized to justify the "deformation of contours," that is,

$$\oint_{C_{\hat{r}}} w_1(z)\,dz = \oint_{C_R} w_1(z)\,dz \qquad (0<\hat{r}<\ell,\ R>\ell) \tag{2.35}$$

Thus

$$\int_{-\ell+\hat{r}}^{\ell-\hat{r}} [w_1^+(t)-w_1^-(t)]\,dt = \int_{C_{\hat{r}_3}+C_{\hat{r}_4}} w_1(z)\,dz - \oint_{C_R} w_1(z)\,dz \tag{2.36}$$

where the integrals on the right are taken in their positive (counter-clockwise) direction. But in the vicinity of $z = \pm\ell$, $w_1(z) = O(\hat{r}^{-\frac{1}{2}})$ while the length of the contours $C_{\hat{r}_3}$ and $C_{\hat{r}_4}$ are proportional to $\hat{r}$. Thus, taking the limit of equation (2.36) as $\hat{r}\to 0$, it follows that the integral around $C_{\hat{r}_3}$ and $C_{\hat{r}_4}$ will vanish leaving

$$\int_{-\ell}^{\ell} [w_1^+(t)-w_1^-(t)]\,dt = -\oint_{C_R} w_1(z)\,dz \tag{2.37}$$

From (2.32a) it is clear that $w_1(z) = O(R^{-2})$ as $R\to\infty$,

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*c.f. Copson (3), Sections 4.22 and 4.3.

while the length of the contour C_R is proportional to R. Therefore the right hand side of (2.37) vanishes as $R \to \infty$, resulting in

$$\int_{-\ell}^{\ell} [w_1^+(t) - w_1^-(t)]dt = 0 \tag{2.38}$$

From (2.32b), $w_2(z)$ is defined by a Cauchy Integral of a function q(t) which is presumed to satisfy the H condition of L. Thus, from the Plemelj Formulae* one has

$$w_2^+(t) - w_2^-(t) = q(t) \qquad t \in (-\ell, \ell) \tag{2.39}$$

Combining (2.33), (2.34), (2.38) and (2.39) yields

$$k_1 = \left(\frac{\varkappa - 1}{\varkappa + 1}\right) \cdot \frac{1}{2\pi i} \int_{-\ell}^{\ell} q(t)\,dt \tag{2.40}$$

From (2.26b) and equations (2.23), it might be noted that q(t) is the symmetric component of the tractions acting on the crack surface. Thus the integral on the right of equation (2.40) has a simple physical interpretation.

$$\int_{-\ell}^{\ell} q(t)\,dt = -A_2 + i A_1 \tag{2.41}$$

where A_1, A_2 are the net resultant forces (per unit

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*c.f. Muskhelishvili (15,16)

thickness) acting on the crack surface in the +x and +y directions, respectively. Evidently k_1 will be zero if there is no net load acting on the crack.

Combining equations (2.30), (2,31), (2.32) and (2.40) yields the following simplified expressions for the generating functions of the unique single-crack stress state with single-valued displacements:

$$\Phi(z)=\frac{1}{2\pi i\sqrt{z^2-\ell^2}}\int_{-\ell}^{\ell}\frac{p(t)\sqrt{t^2-\ell^2}}{t-z}dt+\frac{1}{2\pi i}\int_{-\ell}^{\ell}\left[\frac{1}{t-z}+\frac{\left(\frac{\varkappa-1}{\varkappa+1}\right)}{\sqrt{z^2-\ell^2}}\right]q(t)dt \quad (2.42a)$$

$$\Omega(z)=\frac{1}{2\pi i\sqrt{z^2-\ell^2}}\int_{-\ell}^{\ell}\frac{p(t)\sqrt{t^2-\ell^2}}{t-z}dt-\frac{1}{2\pi i}\int_{-\ell}^{\ell}\left[\frac{1}{t-z}-\frac{\left(\frac{\varkappa-1}{\varkappa+1}\right)}{\sqrt{z^2-\ell^2}}\right]q(t)dt \quad (2.42b)$$

Since $\Phi(z)$ and $\Omega(z)$ are sectionally holomorphic in the entire plane, and vanish at infinity, they may be expanded in Laurent series outside of a circle enclosing L.

$$\Phi(z)=\sum_{n=1}^{\infty}c_n z^{-n} \quad , \quad \Omega(z)=\sum_{n=1}^{\infty}d_n z^{-n} \quad (|z|>\ell) \quad (2.43)$$

Thus, from (2.24), $\varphi(z)$ and $\omega(z)$ may be expressed as

$$\varphi(z)=c_1\log(z)+c_0-\sum_{n=1}^{\infty}\frac{c_{n+1}}{n}z^{-n} \quad (2.44a)$$

$$\omega(z)=d_1\log(z)+d_0-\sum_{n=1}^{\infty}\frac{d_{n+1}}{n}z^{-n} \quad (2.44b)$$

where c_o and d_o are arbitrary integration constants.

From these expressions, it is evident that if c_1 and d_1 are non-zero, then $\varphi(z)$ and $\omega(z)$ will be multi-valued (and thus merely analytic). By utilizing equations (2.42), it is easily shown that

$$c_1 = \left(-1 + \frac{\varkappa-1}{\varkappa+1}\right) \cdot \frac{1}{2\pi i} \int_{-\ell}^{\ell} q(t)\,dt \tag{2.45a}$$

$$d_1 = \left(1 + \frac{\varkappa-1}{\varkappa+1}\right) \cdot \frac{1}{2\pi i} \int_{-\ell}^{\ell} q(t)\,dt \tag{2.45b}$$

By comparing (2.45) to (2.41), it is clear that c_1 and d_1 will be zero if the net resultant force (per unit thickness) acting on the crack surface vanishes. In this event, it is evident from (2.44) that $\varphi(z)$ and $\omega(z)$ will be holomorphic. Further, by considering the behavior of φ and ω near L, it is easily shown that they will be sectionally holomorphic.

4. Reduction of the Problem to Integral Equations

The single-crack solution given in the preceding section is an integral form of solution. This immediately suggests an approach which is quite commonly used when an integral form of solution is known for a special case of a more general, linear problem. One merely assumes that the solution to the general problem of N arbitrarily oriented cracks may be represented as the superposition of N single crack solutions, where the nth single crack solution is referred to L_n. Each single crack stress field is assumed to be generated by some (as yet) unknown loading along the crack surface. The boundary conditions are then enforced along each crack L_m $(m = 1,2,\ldots,N)$. This results in a system of one-dimensional integral equations for the unknown single-crack loadings. That is, the problem is " reduced to integral equations."

This reduction has a number of advantages. (1) The number of dimensions involved in the problem is reduced from two to one. Instead of seeking a stress field $\sigma(x,y)$ satisfying field equations in the two-dimensional plane, one need only seek loadings $p^n(t)$, $q^n(t)$ satisfying one-dimensional integral equations along the crack lines. (2) Many of the powerful theorems of integral equations may be invoked to reach analytical conclusions. (3) Integral equations are often admirably suited to approximate, numerical solution. (4) The essential singular character of the solution may be "built in" a priori.

With regard to (2) above, the theory of integral equations will make it possible to justify the assumed decomposition of the solution into a superposition of single-crack stress states. It is not immediately obvious that such a decomposition is valid. Thus, it will be necessary to proceed at first in a formal manner. Once the problem is reduced to integral equations, the theory of integral equations can then be used to justify the assumed decomposition.

With regard to (4) above, it may be seen that the characteristic "square root singularity" of the stresses at the crack tips is explicitly displayed in equations (2.42). The explicit representation will be carried throughout the subsequent analytical and numerical analyses. This might be contrasted with the approach described in Chapter I, in which the singularity is "buried" in a Laurent expression for the complex potentials generating the solution.

In the subsequent analysis, stress states may be denoted by non-parenthetical (and possibly multiple) superscripts. Thus

$$\mathcal{S}^n = [\underset{\sim}{u}^n, \underset{\sim}{\sigma}^n] \tag{2.46}$$

The presence of the superscripts will serve merely to identify the stress state. This may be compared with the notation concerning parenthetical superscripts indicated in equation (2.18).

Before proceeding with the reduction to integral equations, an initial reduction will be made which will prove quite helpful in the subsequent analytical and numerical work. The total stress state $\mathcal{S}^T$ will be expressed as

$$\mathcal{S}^T = \sum_{n=0}^{N} \mathcal{S}^{n,0} + \mathcal{S} \tag{2.47}$$

Here $\mathcal{S}^{0,0}$ is the uniform stress state corresponding to the applied $\underset{\sim}{\sigma}_\infty$ at infinity. $\mathcal{S}^{n,0}$ is the single-crack stress state corresponding to the applied loadings $\hat{G}^{(n)+}$ and $\hat{G}^{(n)-}$ acting on L_n, with vanishing stresses and rotation at infinity ($n = 1,2,\ldots,N$). $\mathcal{S}$ is the residual stress state that remains after the above superposition is effected.

The generating functions for $\mathcal{S}^{0,0}$ cannot be found as a special case of the colinear crack solution since this solution presumes the existence of at least one crack. Rather, it may be easily verified that the generating functions are given by

$$\Phi^{0,0}(z_o) = \Gamma \quad , \quad \Omega^{0,0}(z_o) = \overline{\Gamma} + \overline{\Gamma'} \tag{2.48}$$

where Γ and Γ' are defined in equations (2.29). The generating functions for the states $\mathcal{S}^{n,o}$ $(n = 1,2,\ldots,N)$ may be found by applying the single-crack solution to each L_n. Thus, from (2.42) and (2.26),

$$\Phi^{n,o}(z_n) = \frac{1}{2\pi i\sqrt{z_n^2-\ell_n^2}} \int_{-\ell_n}^{\ell_n} \frac{p^{n,o}(t_n)\sqrt{t_n^2-\ell_n^2}}{t_n - z_n} dt_n + \frac{1}{2\pi i}\int_{-\ell_n}^{\ell_n} \left[\frac{1}{t_n - z_n} + \frac{\left(\frac{\chi-1}{\chi+1}\right)}{\sqrt{z_n^2-\ell_n^2}}\right] q^{n,o}(t_n)dt_n \tag{2.49a}$$

$$\Omega^{n,o}(z_n) = \frac{1}{2\pi i\sqrt{z_n^2-\ell_n^2}} \int_{-\ell_n}^{\ell_n} \frac{p^{n,o}(t_n)\sqrt{t_n^2-\ell_n^2}}{t_n - z_n} dt_n - \frac{1}{2\pi i}\int_{-\ell_n}^{\ell_n} \left[\frac{1}{t_n - z_n} - \frac{\left(\frac{\chi-1}{\chi+1}\right)}{\sqrt{z_n^2-\ell_n^2}}\right] q^{n,o}(t_n)dt_n \tag{2.49b}$$

$$p^{n,o}(t_n) = \tfrac{1}{2}\left[\hat{G}^{(n)+}(t_n) + \hat{G}^{(n)-}(t_n)\right] \tag{2.50a}$$

$$-\ell_n \leq t_n \leq \ell_n$$

$$q^{n,o}(t_n) = \tfrac{1}{2}\left[\hat{G}^{(n)+}(t_n) - \hat{G}^{(n)-}(t_n)\right] \tag{2.50b}$$

$$n = 1, 2, \ldots, N$$

The generating functions $\Phi^{n,o}$ and $\Omega^{n,o}$ are, of course, referred to the z_n system, as is the state $\mathcal{S}^{n,o}$ which they generate $(n = 0,1,\ldots,N)$.

From equations (2.21) and (2.23), it is easily seen that the boundary conditions on L_n can be written in the completely equivalent form

$$\mathcal{T}_s^{(n)}\{\mathcal{S}^T\}(t_n) = \tfrac{1}{2}\left[\hat{G}^{(n)+}(t_n) + \hat{G}^{(n)-}(t_n)\right] = p^{n,o}(t_n) \tag{2.51a}$$

$$\mathcal{T}_a^{(n)}\{\mathcal{S}^T\}(t_n) = \frac{1}{2}\left[\hat{G}^{(n)+}(t_n) - \hat{G}^{(n)-}(t_n)\right] = q^{n,o}(t_n) \qquad (2.51b)$$

$$t_n \in (-\ell_n, \ell_n) \quad , \quad n = 1, 2, \ldots, N$$

where the last equalities follow from (2.50). Thus the function $p^{n,o}$ corresponds to the symmetric portion of the applied stress, while $q^{n,o}$ corresponds to the antisymmetric portion of the applied stress.

But by the construction of the single-crack stress state, $\mathcal{S}^{n,o}$ must also satisfy

$$\mathcal{T}_s^{(n)}\{\mathcal{S}^{n,o}\}(t_n) = \frac{1}{2}\left[\hat{G}^{(n)+}(t_n) + \hat{G}^{(n)-}(t_n)\right] = p^{n,o}(t_n) \qquad (2.52a)$$

$$\mathcal{T}_a^{(n)}\{\mathcal{S}^{n,o}\}(t_n) = \frac{1}{2}\left[\hat{G}^{(n)+}(t_n) - \hat{G}^{(n)-}(t_n)\right] = q^{n,o}(t_n) \qquad (2.52b)$$

$$t_n \in (-\ell_n, \ell_n) \,, \quad n = 1, 2, \ldots, N$$

Thus (2.47), (2.51) and (2.52) imply the following boundary conditions on the residual state $\mathcal{S}$:

$$\mathcal{T}_s^{(n)}\{\mathcal{S}\}(t_n) = -\sum_{\substack{m=0\\m\neq n}}^{N} \mathcal{T}_s^{(n)}\{\mathcal{S}^{m,o}\}(t_n) = G^n(t_n) \qquad (2.53a)$$

$$\mathcal{T}_a^{(n)}\{\mathcal{S}\}(t_n) = -\sum_{\substack{m=0\\m\neq n}}^{N} \mathcal{T}_a^{(n)}\{\mathcal{S}^{m,o}\}(t_n) = 0 \qquad (2.53b)$$

$$t_n \in (-\ell_n, \ell_n) \quad , \quad N = 1, 2, \ldots, N$$

$$\mathcal{T} = o(1) \quad \text{as} \quad |z_o| \to \infty \qquad (2.53c)$$

Note that the second equality in equation (2.53b) follows because the stress states $\mathcal{S}^{m,o}$ ($m = 0,1,\ldots,N; m \neq n$) are all analytic across L_n. Thus they can produce only symmetric stresses on the surfaces of L_n, since

$$\mathcal{T}^{(n)+}\{\mathcal{S}^{m,o}\}(t_n) = \mathcal{T}^{(n)-}\{\mathcal{S}^{m,o}\}(t_n) \qquad (m \neq n) \tag{2.54}$$

$$t_n \in (-\ell_n, \ell_n)\ , \quad m = 0,1,\ldots,N\ ;\ n = 1,2,\ldots,N$$

Thus the antisymmetric component, represented by $\frac{1}{2}(\mathcal{T}^{(n)+} - \mathcal{T}^{(n)-})$ must be identically zero.

By use of superposition and the single-crack solution, the problem can always be reduced to one in which the applied stresses are symmetric across the crack surfaces, and the stresses vanish at infinity. The symmetry of the applied stresses represent a considerable simplification, both analytically and numerically. Further, the fact that the applied stresses on L_n are generated by stress states which are analytic in a neighborhood of L_n implies certain analytic properties of the solution which are quite beneficial in terms of the convergence rate of numerical procedures that will be used.

The functions $G^n(t_n)$ may be expressed in terms of the generating functions $\Phi^{m,o}(z_m)$, $\Omega^{m,o}(z_m)$ by utilizing some simplified transformation formulae for the plane theory of elasticity. Consider two coordinate systems, a primed system and an unprimed system (c.f. Fig. 6). The primed system is rotated by an angle α in the positive (counter-clockwise) direction. Then the relation between the stresses in the two systems is given by

$$\sigma'_{x'x'} + \sigma'_{y'y'} = \sigma_{xx} + \sigma_{yy} \tag{2.55a}$$

$$\sigma'_{y'y'} - \sigma'_{x'x'} + 2i\,\sigma'_{x'y'} = (\sigma_{yy} - \sigma_{xx} + 2i\,\sigma_{xy})e^{i2\alpha} \tag{2.55b}$$

(c.f. Muskhelishvili [16] p.25). Adding equations (2.55) and taking the complex conjugate of the resulting equation yields the second of the following equations.

$$\sigma'_{x'x'} + \sigma'_{y'y'} = \sigma_{xx} + \sigma_{yy} \tag{2.56a}$$

$$\sigma'_{y'y'} - i\,\sigma'_{x'y'} = \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy})(1 - e^{-i2\alpha}) + (\sigma_{yy} - i\,\sigma_{xy})e^{-i2\alpha} \tag{2.56b}$$

Equations (2.56) give the transformation formulae in a form suitable for use with equations (2.24).

From (2.1), (2.24), (2.53), (2.56), $G^n(t_n)$ is given by

$$G^n(t_n) = \sum_{\substack{m=0\\ m\neq n}}^{N} \Big\{ \big[\Phi^{m,0}(z_m) + \overline{\Phi^{m,0}(z_m)}\big](1 - e^{-i2\alpha_{mn}}) + \big[\Phi^{m,0}(z_m) + \Omega^{m,0}(\bar{z}_m) + (z_m - \bar{z}_m)\overline{\Phi^{m,0\,\prime}(z_m)}\big]e^{-i2\alpha_{mn}} \Big\} \tag{2.57a}$$

$$n = 1, 2, \ldots, N$$

where $$z_m = z_m(t_n) = z_{mn} + t_n e^{i\alpha_{mn}}, \quad t_n \in (-\ell_n, \ell_n) \tag{2.57b}$$

Proceeding on to the reduction to integral equations, it will be assumed that the residual state $\mathcal{S}$ may be represented as a superposition of single-crack stress states (this will later be justified). Thus

$$\mathcal{S} = \sum_{n=1}^{N} \mathcal{S}^n \tag{2.58}$$

where $\mathcal{S}^n$ is a single-crack stress state referred to L_n, and is generated by potentials of the form (2.42). These potentials, in turn, are generated by some (as yet unknown) symmetric and antisymmetric loadings, $p^n(t_n)$ and $q^n(t_n)$, respectively. Since potentials of this form generate vanishing stresses at infinity, it is clear that the assumed superposition in (2.58) automatically satisfies the boundary condition on $\mathcal{S}$ at infinity, equation (2.53c). Further, substituting (2.58) into (2.53b) yields

$$\sum_{m=1}^{N} \mathcal{T}_a^{(n)}\{\mathcal{S}^m\}(t_n) = 0 \tag{2.59}$$

$$t_n \in (-\ell_n, \ell_n), \quad n = 1, 2, \ldots, N$$

But $\mathcal{S}^m$ is analytic **across** L_n ($m = 1,2,\ldots,N$, $m \neq n$) and hence can generate no antisymmetric stresses across L_n ($n = 1,2,\ldots,N$). Thus

$$\mathcal{T}_a^{(n)}\{\mathcal{S}^m\}(t_n) = 0 \qquad (m \neq n) \tag{2.60}$$

$$t_n \in (-\ell_n, \ell_n) \quad , \quad m, n = 1, 2, \ldots, N$$

Substituting (2.60) into (2.59) gives

$$\mathcal{T}_a^{(n)}\{\mathcal{S}^n\}(t_n) = q^n(t_n) = 0 \tag{2.61}$$

$$n = 1, 2, \ldots, N$$

That is, the antisymmetric portion of the (unknown) crack loading generating $\mathcal{S}^n$ is identically zero. Hence, one of the "unknown" generating loadings is completely determined beforehand. It is necessary to consider only one unknown loading, $p^n(t_n)$, rather than two. This represents a considerable simplification, particularly when it comes to numerical analysis of the resulting equations. It might be noted that the disappearance of $q^n(t_n)$ is a direct result of the initial reduction, equation (2.47).

In the subsequent analysis, the (unknown) symmetric loading will be denoted by

$$p^n(t_n) = F^n(t_n) = F_1^n(t_n) + i F_2^n(t_n) \qquad t_n \in (-\ell_n, \ell_n) \tag{2.62}$$

Then from (2.42), the generating functions of $\mathcal{S}^n$ are given by

$$\Phi^n(z_n) = \Omega^n(z_n) = \frac{1}{2\pi i \sqrt{z_n^2 - \ell_n^2}} \int_{-\ell_n}^{\ell_n} \frac{F^n(t_n)\sqrt{t_n^2 - \ell_n^2}}{t_n - z_n}\, dt_n \tag{2.63}$$

$$n = 1, 2, \ldots, N$$

Substituting (2.63) in (2.24) and utilizing (2.56), the final boundary condition, equation (2.53a), may be evaluated to give

$$F^n(x_n)+\sum_{\substack{m=1\\m\neq n}}^{N} e^{-i\alpha_{mn}}\Big\{e^{i\alpha_{mn}}\Phi^m(z_m)+e^{-i\alpha_{mn}}\Phi^m(\bar{z}_m) \tag{2.64a}$$

$$+(e^{i\alpha_{mn}}-e^{-i\alpha_{mn}})\overline{\Phi^m(z_m)}+e^{-i\alpha_{mn}}(z_m-\bar{z}_m)\overline{\Phi^{m\prime}(z_m)}\Big\}=G^n(x_n)$$

$$z_m = z_m(x_n) = z_{mn}+x_n e^{i\alpha_{mn}} \quad , \quad x_n \in (-\ell_n,\ell_n) \tag{2.64b}$$

From equation (2.63), $\Phi^m(z_m)$ is given in terms of an integral of F^m. Thus equations (2.64) are a set of integral equations of the form

$$F^n(x_n)+\sum_{\substack{m=1\\m\neq n}}^{N}\Big\{\int_{-\ell_m}^{\ell_m}K_{nm}(x_n,t_m)F^m(t_m)\,dt_m+\int_{-\ell_m}^{\ell_m}\widetilde{K}_{nm}(x_n,t_m)\overline{F^m(t_m)}\,dt_m\Big\}=G^n(x_n) \tag{2.65a}$$

$$x_n \in (-\ell_n,\ell_n) \quad , \quad n=1,2,\ldots,N$$

where

$$K_{nm}(x_n,t_m)=\frac{\sqrt{t_m^2-\ell_m^2}\,e^{-i\alpha_{mn}}}{2\pi i}\,Re\left\{\frac{e^{i\alpha_{mn}}}{\sqrt{z_m^2-\ell_m^2}\,(t_m-z_m)}\right\} \tag{2.65b}$$

$$\widetilde{K}_{nm}(x_n,t_m)=\frac{\sqrt{t_m^2-\ell_m^2}\,e^{-i\alpha_{mn}}}{2\pi i\sqrt{\bar{z}_m^2-\ell_m^2}\,(t_m-\bar{z}_m)}\Big[(e^{i\alpha_{mn}}-e^{-i\alpha_{mn}})$$

$$+e^{-i\alpha_{mn}}(z_m-\bar{z}_m)\Big(\frac{1}{t_m-\bar{z}_m}-\frac{1}{\bar{z}_m^2-\ell_m^2}\Big)\Big] \tag{2.65c}$$

$$z_m=z_m(x_n)=z_{mn}+x_n e^{i\alpha_{mn}}$$

Equations (2.65) constitute a well-behaved set of Fredholm integral equations of the second kind.

By separating the real and imaginary parts of equations (2.65), one might view them as a system of 2N Fredholm equations for the 2N unknown functions

$F_1^1, F_2^1, F_1^2, F_2^2, \ldots, F_1^N, F_2^N$. Alternately, it can be shown that any system of Fredholm equations can always be reduced to a single equation of the standard type (c.f. Tricomi [23], Section 3.17). Thus without carrying out the details, equations (2.65) can always be expressed as a single equation of the form

$$F(x) + \hat{\lambda} \int_0^1 K(x,t) F(t)\, dt = G(x) \qquad 0 \leq x \leq 1$$

For analytical arguments, it will frequently be convenient to consider the equations in the **above form**.

The homogeneous Fredholm equations (2.65) cannot have a non-trivial, quadratically integrable solution. This is easily shown by contradiction. Suppose a non-trivial solution were to exist. From (2.65b) and (2.65c), it is easily seen that $K_{nm}(x_n, t_m)$ and $\widetilde{K}_{nm}(x_n, t_m)$ are infinitely continuously differentiable with respect to x_n in the rectangle $-\ell_n \leq x_n \leq \ell_n$, $-\ell_m \leq t_m \leq \ell_m$. Using (2.65a) to express $F^n(x_n)$ in terms of the integrals on the right, it then follows that $F^n(x_n)$ is infinitely continuously differentiable on the interval $-\ell_n \leq x_n \leq \ell_n$ $(n = 1, 2, \ldots, N)$. Thus, the F^n certainly satisfy the H condition on their intervals of definition. From (2.63) it follows that Φ^n and Ω^n are sectionally holomorphic, non-trivial potentials. Working backwards, it is easily seen that these potentials generate a non-trivial solution to the homogeneous stress boundary problem. But this is impossible by the uniqueness theorem (c.f. App.I). Thus, by contradiction, there can be no non-trivial, quadratically integrable solution to the homogeneous Fredholm equations.

The alternative Theorem of Fredholm equations may now be applied (c.f. Tricomi, p. 64). Since the homogeneous equations have only a trivial solution, it follows that the corresponding non-homogeneous equations must have one and only one solution. Working backwards as before, this solution forms a solution to the original (non-homogeneous) stress boundary value problem. Again applying the uniqueness theorem, it follows that this must be <u>the</u> solution.

The above argument justifies the assumed superposition in (2.58). Further, it indicates that the approach is valid for <u>any</u> arbitrary orientation of the cracks, so long as they do not intersect.

5. Expressions for Quantities of Interest

In this section, expressions are presented for various quantities of interest. As seen in Chapter I, Section 1, this will include the stress intensity factors at the crack tips, as well as the "energy of the cracks," that is, the energy difference due to the presence of the cracks. As indicated in Section 2 of this chapter, some attention must also be given to the relative displacements of the crack surfaces, to check for the overlap problem.

The various quantities will be derived only for the residual state $\mathscr{S}$. The analysis of the additional states in (2.47) can be done in a nearly analogous manner.

Since displacements will be of interest, it can be seen from (2.24a) that it might be more convenient to work in terms of φ^n rather than Φ^n. From (2.61), the antisymmetric component of the stresses generating Φ^n vanishes. From (2.45a), it follows that the logarithmic term in (2.44a) disappears. Thus φ^n is sectionally holomorphic. Further, from (2.24d), φ^n may be written

$$\varphi^n(z_n) = \int_{\bar{z}_n}^{z_n} \Phi^n(\zeta_n)\, d\zeta_n \;+\; const \qquad (2.66)$$

$$n = 1, 2, \ldots, N$$

where $\bar{z}_n$ is an arbitrary point in the open region B. Equation (2.63) indicates that Φ^n has only a square-root singularity at the crack endpoints $z_n = \pm \ell_n$. Since this is integrable, it follows from (2.66) that φ^n is bounded at the crack tips. Further, from (2.44a), φ^n is holomorphic at infinity. It can be shown that under these conditions, $\varphi^n(z_n)$ may be represented as

$$\varphi^n(z_n) = \frac{\sqrt{z_n^2 - \ell_n^2}}{2\pi i} \int_{-\ell_n}^{\ell_n} \frac{f^n(t_n)}{t_n - z_n} \cdot \frac{dt_n}{\sqrt{t_n^2 - \ell_n^2}} \tag{2.67a}$$

where

$$f^n(t_n) = \varphi^{n+}(t_n) + \varphi^{n-}(t_n) \qquad -\ell_n \leq t_n \leq \ell_n \tag{2.67b}$$

$$n = 1, 2, \ldots, N$$

(c.f. Muskhelishvili [16] pp. 454-455). The arbitrary integration constant in (2.44a) can be chosen so that φ^n vanishes at infinity. From (2.67), this implies

$$\int_{-\ell_n}^{\ell_n} f^n(t_n) \frac{dt_n}{\sqrt{t_n^2 - \ell_n^2}} = 0 \tag{2.68}$$

$$n = 1, 2, \ldots, N$$

Applying the Plemelj Formulae to (2.63) gives

$$F^n(t_n) = \Phi^{n+}(t_n) + \Phi^{n-}(t_n) \qquad -\ell_n < t_n < \ell_n \tag{2.69}$$

$$n = 1, 2, \ldots, N$$

Equations (2.66), (2.67b) and (2.69) then imply

$$\frac{df^n(t_n)}{dt_n} = F^n(t_n) \qquad n = 1, 2, \ldots, N \tag{2.70}$$

Since F^n is uniquely determined, f^n is also uniquely determined by (2.68) and (2.70).

By differentiating (2.64b), one gets

$$\frac{dZ_m}{dx_n} = e^{i\alpha_{mn}} \quad , \quad \frac{d\bar{Z}_m}{dx_n} = e^{-i\alpha_{mn}} \tag{2.71}$$

Also, since φ^m is analytic across L_n for $m \neq n$, it follows that

$$\frac{d\varphi^m(Z_m)}{dx_n} = e^{i\alpha_{mn}} \Phi^m(Z_m) \tag{2.72a}$$

$$\frac{d\varphi^m(\bar{Z}_m)}{dx_n} = e^{-i\alpha_{mn}} \Phi^m(\bar{Z}_m) \tag{2.72b}$$

$$\frac{d\overline{\varphi^m(Z_m)}}{dx_n} = e^{-i\alpha_{mn}} \overline{\Phi^m(Z_m)} \tag{2.72c}$$

$$\frac{d\overline{\Phi^m(Z_m)}}{dx_n} = e^{-i\alpha_{mn}} \overline{\Phi^{m\prime}(Z_m)} \tag{2.72d}$$

Using (2.70)-(2.72) in (2.64a) results in the following equivalent expression:

$$\frac{d}{dx_n}\Big\{f^n(x_n)+\sum_{\substack{m=1\\m\neq n}}^{N} e^{-i\alpha_{mn}}\big[\varphi^m(z_m)+\varphi^m(\bar z_m)+(z_m-\bar z_m)\overline{\Phi^m(z_m)}\big]\Big\}=G^n(x_n) \quad (2.73)$$

$$z_m=z_m(x_n)=z_{mn}+x_n e^{+i\alpha_{mn}} \quad , \quad x_n\in(-\ell_n,\ell_n)\ , \ n=1,2,\dots,N$$

Let $g^n(x_n)$ be defined by

$$\frac{dg^n(x_n)}{dx_n}=G^n(x_n) \qquad -\ell_n<x_n<\ell_n \quad (2.74a)$$

$$\int_{-\ell_n}^{\ell_n} g^n(t_n)\frac{dt_n}{\sqrt{t_n^2-\ell_n^2}}=0 \quad (2.74b)$$

$$n=1,2,\dots,N$$

where the arbitrary integration constant in g^n is fixed by (2.74b). Note that from (2.53a), G^n is well-behaved on L_n. Hence, g^n will also be well-behaved, as will the integral in (2.74b).

Substituting (2.74a) into (2.73) and integrating gives

$$f^n(x_n)+\sum_{\substack{m=1\\m\neq n}}^{N} e^{-i\alpha_{mn}}\big[\varphi^m(z_m)+\varphi^m(\bar z_m)+(z_m-\bar z_m)\overline{\Phi^m(z_m)}\big]=g^n(x_n)+\hat A_n \quad (2.75)$$

$$z_m=z_m(x_n)=z_{mn}+x_n e^{+i\alpha_{mn}} \quad , \quad x_n\in(-\ell_n,\ell_n)\ , \ n=1,2,\dots,N$$

where $\hat A_n$ are integration constants which may be determined by (2.68).

Equations (2.75) together with (2.68) result in a system of Fredholm equations of the second kind for the unknown functions $f_1^1, f_2^1, f_1^2, f_2^2, \ldots, f_1^N, f_2^N$. As before, this can be made equivalent to a single Fredholm equation of the form

$$f(x) + \hat{\lambda}\int_0^1 k(x,t) f(t)\, dt = g(t) \qquad 0 \le x \le 1 \tag{2.76}$$

Since (2.64a) has one and only one solution, it follows from (2.70) and (2.68) that there is one and only one solution for the f^n.

In the following analysis, it will be assumed that the equations are solved for the f^n. The quantities of interest will then be calculated in terms of this solution.

Expression (2.67a) is not well suited for later numerical work, since for large $|z_n|$, φ^n appears to approach a constant. Although this constant is zero by (2.68), it would contribute greatly to roundoff error if the integrals are evaluated numerically. Hence it is advantageous to eliminate this constant analytically by using (2.68) in (2.67a) to get

$$\varphi^n(z_n) = \frac{\sqrt{z_n^2 - \ell_n^2}}{2\pi i}\int_{-\ell_n}^{\ell_n}\left[\frac{1}{t_n - z_n} - \frac{1}{z_n}\right] f^n(t_n)\frac{dt_n}{\sqrt{t_n^2 - \ell_n^2}}$$

or

$$\varphi^n(z_n) = \frac{\sqrt{z_n^2 - \ell_n^2}}{2\pi i\, z_n}\int_{-\ell_n}^{\ell_n}\frac{t_n f^n(t_n)}{t_n - z_n}\cdot\frac{dt_n}{\sqrt{t_n^2 - \ell_n^2}} \tag{2.77}$$

This form explicitly exhibits the z_n^{-1} behavior of φ^n for large $|z_n|$.

In (2.77) let

$$\hat{z}_n = \frac{z_n}{\ell_n} \tag{2.78a}$$

$$t_n = -\ell_n \cos\theta_n \qquad 0 \leq \theta_n \leq \pi \tag{2.78b}$$

Then

$$\varphi^n(z_n) = -\frac{\sqrt{\hat{z}_n^2 - 1}}{2\pi \hat{z}_n} \int_0^\pi \frac{\cos\theta_n \; f^n(-\ell_n \cos\theta_n)}{\hat{z}_n + \cos\theta_n}\, d\theta_n \tag{2.79}$$

Differentiating gives

$$\Phi^n(z_n) = \frac{1}{\ell_n \sqrt{\hat{z}_n^2 - 1}}\; I^n(\hat{z}_n) \tag{2.80a}$$

$$\Phi^{n\prime}(z_n) = \frac{-1}{\ell_n^2 \sqrt{\hat{z}_n^2 - 1}} \left[\frac{\hat{z}_n}{\hat{z}_n^2 - 1} I^n(\hat{z}_n) + I^{n\prime}(\hat{z}_n) \right] \tag{2.80b}$$

$$\Phi^{n\prime\prime}(z_n) = \frac{1}{\ell_n^3 \sqrt{\hat{z}_n^2 - 1}} \left[\frac{2\hat{z}_n^2 + 1}{(\hat{z}_n^2 - 1)^2} I^n(\hat{z}_n) + \frac{2\hat{z}_n}{\hat{z}_n^2 - 1} I^{n\prime}(\hat{z}_n) + 2 I^{n\prime\prime}(\hat{z}_n) \right] \tag{2.80c}$$

where

$$I^n(\hat{z}_n) = \frac{1}{2\pi} \int_0^\pi \left[\frac{\cos\theta_n}{\hat{z}_n + \cos\theta_n} + \frac{\sin^2\theta_n}{(\hat{z}_n + \cos\theta_n)^2} \right] f^n(-\ell_n \cos\theta_n)\, d\theta_n \tag{2.81a}$$

$$I^{n\prime}(\hat{z}_n) = \frac{1}{2\pi} \int_0^\pi \left[\frac{\cos\theta_n}{(\hat{z}_n + \cos\theta_n)^2} + \frac{2\sin^2\theta_n}{(\hat{z}_n + \cos\theta_n)^3} \right] f^n(-\ell_n \cos\theta_n)\, d\theta_n \tag{2.81b}$$

$$I^{n\prime\prime}(\hat{z}_n) = \frac{1}{2\pi} \int_0^\pi \left[\frac{\cos\theta_n}{(\hat{z}_n + \cos\theta_n)^2} + \frac{3\sin^2\theta_n}{(\hat{z}_n + \cos\theta_n)^4} \right] f^n(-\ell_n \cos\theta_n)\, d\theta_n \tag{2.81c}$$

Further, (2.64a) and its derivative give

$$F^{n}(x_n) = G^{n}(x_n) - \sum_{\substack{m=1 \\ m\neq n}}^{N} e^{-i\alpha_{mn}}\Big[e^{i\alpha_{mn}} \Phi^{m}(z_m) + e^{-i\alpha_{mn}} \Phi^{m}(\bar{z}_m) + \left(e^{i\alpha_{mn}} - e^{-i\alpha_{mn}}\right) \overline{\Phi^{m}(z_m)} + e^{-i\alpha_{mn}}(z_m - \bar{z}_m)\, \overline{\Phi^{m\prime}(z_m)}\Big] \tag{2.82a}$$

$$F^{n\prime}(x_n) = G^{n\prime}(x_n) - \sum_{\substack{m=1 \\ m\neq n}}^{N} e^{-i\alpha_{mn}}\Big[e^{i2\alpha_{mn}} \Phi^{m\prime}(z_m) + e^{-i2\alpha_{mn}} \Phi^{m\prime}(\bar{z}_m) + 2e^{-i\alpha_{mn}}\left(e^{i\alpha_{mn}} - e^{-i\alpha_{mn}}\right) \overline{\Phi^{m\prime}(z_m)} + e^{-i2\alpha_{mn}}(z_m - \bar{z}_m)\, \overline{\Phi^{m\prime\prime}(z_m)}\Big] \tag{2.82b}$$

Thus, if the f^n are known, then F^n and $F^{n\prime}$ can be calculated through (2.80)-(2.82).

Note that the existence of $F^{n\prime}$ can be shown from the construction of F^n. Indeed, it is shown in Appendix II that F^n has some quite remarkable analytic properties.

Let

$$d^{n}(x_n) = \varphi^{n+}(x_n) - \varphi^{n-}(x_n) \tag{2.83a}$$

$$d^{n\prime}(x_n) = \Phi^{n+}(x_n) - \Phi^{n-}(x_n) \tag{2.83b}$$

where the second equation follows from the first by (2.66). From (2.63), (2.67a) and the Plemelj Formulae, these quantities may be expressed as

$$d^n(x_n) = \frac{\sqrt{x_n^2 - \ell_n^2}}{\pi i} \fint_{-\ell_n}^{\ell_n} \frac{f^n(t_n)}{t_n - x_n} \cdot \frac{dt_n}{\sqrt{t_n^2 - \ell_n^2}} \tag{2.84a}$$

$$d^{n\prime}(x_n) = \frac{1}{\sqrt{x_n^2 - \ell_n^2}} \, \tilde{I}^n(x_n) \tag{2.84b}$$

$$\tilde{I}^n(x_n) = \frac{1}{\pi i} \fint_{-\ell_n}^{\ell_n} \frac{F^n(t_n)\sqrt{t_n^2 - \ell_n^2}}{t_n - x_n} \, dt_n \tag{2.84c}$$

where the presence of a bar on an integral indicates that the Cauchy principal value is to be taken. Also, as before, all square roots evaluated on L_n are understood to be taken in terms of their values on S_n^+.

It can be shown that

$$\fint_{-\ell_n}^{\ell_n} \frac{1}{t_n - x_n} \cdot \frac{dt_n}{\sqrt{t_n^2 - \ell_n^2}} = 0 \tag{2.85a}$$

$$\frac{1}{\pi i} \fint_{-\ell_n}^{\ell_n} \frac{\sqrt{t_n^2 - \ell_n^2}}{t_n - x_n} = -x_n \tag{2.85b}$$

These results are most easily shown using the theory of sectionally holomorphic functions, but they may be shown by contour integration or any other means.

Using (2.85) in (2.84) gives

$$d^n(x_n) = \frac{\sqrt{x_n^2 - \ell_n^2}}{\pi i} \int_{-\ell_n}^{\ell_n} h^n(x_n, t_n) \frac{dt_n}{\sqrt{t_n^2 - \ell_n^2}} \tag{2.86a}$$

$$\tilde{I}^n(x_n) = \frac{1}{\pi i}\int_{-\ell_n}^{\ell_n} H^n(x_n, t_n)\sqrt{t_n^2 - \ell_n^2}\, dt_n - x_n F^n(x_n) \tag{2.86b}$$

where

$$h^n(x_n, t_n) = \begin{cases} \dfrac{f^n(t_n) - f^n(x_n)}{t_n - x_n} & (t_n \neq x_n) \\ F^n(x_n) & (t_n = x_n) \end{cases} \tag{2.87a}$$

$$H^n(x_n, t_n) = \begin{cases} \dfrac{F^n(t_n) - F^n(x_n)}{t_n - x_n} & (t_n \neq x_n) \\ F^{n\prime}(x_n) & (t_n = x_n) \end{cases} \tag{2.87b}$$

$(x_n, t_n) \in L_n \times L_n \quad , \quad n = 1, 2, \ldots, N$

It might be noted that the bars have been removed from the integrals in (2.86). This is because the integrands are no longer singular at $x_n = t_n$, and a Cauchy principal value is no longer required. In fact, it is shown in Appendix II that the functions defined in (2.87) are not only continuous in their range of definition, but are infinitely continuously differentiable there.

The quantities of interest are all basically related to the d^n and $\tilde{I}^n$ given in (2.86). Evidently these can be calculated from the f^n through equations (2.80)-(2.82) and (2.86)-(2.87).

Consider the singular behavior of the stress field near the crack endpoints $z_n = \pm \ell_n$. The states $\mathcal{S}^m$ ($m = 1,2,\ldots,N$; $m \neq n$) are all analytic across L_n. Hence, the only state that can contribute singular stresses is $\mathcal{S}^n$ itself. As in (1.1), consider a local polar coordinate system in the neighborhood of $z_n = +\ell_n$ (c.f. Fig. 1). Then

$$z_n = \ell_n + r e^{i\theta} \tag{2.88}$$

From (2.63) it follows that

$$\Phi^n = \Omega^n = \frac{e^{-i\frac{\theta}{2}}}{2(2\pi r)^{\frac{1}{2}}} \cdot \sqrt{\frac{\pi}{\ell_n}} \cdot \tilde{I}^n(\ell_n) + O(1) \tag{2.89}$$

Substituting (2.89) into (2.24) and utilizing the transformation equations (2.56) to express the result in polar coordinates, it is not difficult to show that the singular portion of the stress field exhibits the asymptotic behavior claimed in (1.1). Further, by considering the special case $\theta = 0$, it is easily seen that

$$K_I^{n,1} + i K_{II}^{n,1} = -\sqrt{\frac{\pi}{\ell_n}}\ \overline{\tilde{I}^n(-\ell_n)} \tag{2.90a}$$

$$K_I^{n,2} + i K_{II}^{n,2} = \sqrt{\frac{\pi}{\ell_n}}\ \overline{\tilde{I}^n(\ell_n)} \tag{2.90b}$$

$$n = 1, 2, \ldots, N$$

where (2.90a) gives the values of the stress intensity factors at $z_n = -\ell_n$, while (2.90b) gives the values at $z_n = +\ell_n$. These values are, of course, referred to the z_n system.

The relative displacement of the crack surfaces may be found directly in terms of the d^n. Note that from the point of view of the overlap phenomenon, the absolute displacements are immaterial and only the relative displacements are of importance. Across a typical crack L_n, the stress states $\mathcal{S}^m$ ($m \neq n$) are analytic. Thus they may contribute to absolute displacements, but they cannot contribute to relative displacements. The only state that may contribute to relative displacements is $\mathcal{S}^n$ itself. Let these relative displacements be given by

$$\Delta u_\alpha^n(t_n) = u_\alpha^{n+}(t_n) - u_\alpha^{n-}(t_n) \tag{2.91}$$

$$t_n \in (-\ell_n, \ell_n) \quad , \quad n = 1, 2, \ldots, N$$

where the displacement components u_α^n are referred to the z_n system.

Note that the arbitrary integration constant in ω^n may be chosen such that ω^n vanishes at infinity. It then follows that

$$\omega^n(z_n) = \varphi^n(z_n) \tag{2.92}$$

Equations (2.24a), (2.67a), (2.84a), (2.92), and the Plemelj Formulae then give

$$\Delta u_1^n(x_n) + i\,\Delta u_2^n(x_n) = \frac{\varkappa+1}{2\mu}\, d^n(x_n) \tag{2.93}$$

Of course, in checking for overlap, it is only Δu_2^n that is of interest. Overlap is indicated if and only if Δu_2^n is negative somewhere along the crack line.

Suppose there is no net load acting on any of the crack surfaces. Then from (2.41) and (2.49), it follows that $\Phi^{n,o}$ and $\Omega^{n,o}$ are independent of the material properties. From (2.57), it follows that the G^n are independent of the material properties, as are the F^n by (2.64). It then follows that the stress field will be independent of the material properties. (This is simply a special case of a general result. c.f. Muskhelishvili [15], Section 44.) Further, it can be seen from (2.67) and (2.84a) that the d^n will be independent of the material properties. One may therefore conclude from (2.93) that the material properties enter only through a multiplicative factor in the relative displacements. Since only the <u>sign</u> of Δu_2^n is important in checking overlap, it follows that the validity of the solution with regard to the overlap problem will also be independent of the material properties.

Finally, the "energy of the cracks" will be calculated. Intuitively, the "crack energy" is a simple idea. In practice, it has resulted in considerable confusion (c.f. Spencer [22] and Sih and Liebowitz [20]). The expressions given here will be for the case of traction-free internal cracks with some arbitrary load at infinity. (Note that in this circumstance, the G^n will be constants.) Consider the region B_R given in (2.6), for $R>R_c$. Suppose tractions are applied to B_R which would, in the absence of cracks, produce the uniform stress state $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}_{\infty}$. In addition, suppose that tractions are applied to S^+ and S^- in such a way that the uniform stress state is produced. Now, suppose that the tractions on S^+ and S^- are allowed to relax until they vanish. In the process of relaxing, the tractions remove a certain amount of energy from the system. It is this energy which will be called the "energy of the cracks." The case of the infinite region under consideration will be treated as the limit of this model as $R\rightarrow\infty$. By utilizing the Reciprocal Theorem of Betti and Rayleigh it can be shown that the energy of the cracks, denoted by E_c, is given by

$$\frac{2\mu}{\varkappa+1} E_c = -\sum_{n=1}^{N} I_m \left\{ \frac{1}{2} \overline{G^n} \int_{-\ell_n}^{\ell_n} d^n(t_n)\, dt_n \right\} \tag{2.94}$$

Here Im{ } denotes the imaginary part of the term in the curly brackets.

Since the cracks are traction-free, it follows from earlier discussions that the term on the right of (2.94) is independent of the material properties.

Some interesting analytic properties of $d^n(t_n)$ are demonstrated in Appendix II. It can be shown that the function

$$\tilde{d}^n(x_n) = \frac{d^n}{\sqrt{x_n^2-\ell_n^2}} = \frac{1}{\pi i}\int_{-\ell_n}^{\ell_n} \frac{f^n(t_n)}{t_n - x_n} \cdot \frac{dt_n}{\sqrt{t_n^2-\ell_n^2}} \tag{2.95}$$

is infinitely continuously differentiable with respect to x_n in the closed interval $-\ell_n \leq x_n \leq \ell_n$. This will be quite useful in later numerical analysis.

III. NUMERICAL ANALYSIS

1. An Approximate Solution Technique

Analytical solutions are not presently available for the general stress boundary value problem discussed in Chapter II. Therefore, a numerical approach to the solution of the integral equations (2.57) or (2.76) is indicated. When such solutions are considered, convergence rate of the numerical technique can be an important factor. A rapidly convergent scheme has obvious advantages in terms of cumputing efficiency, truncation error, and roundoff error. Further, it will be seen that the convergence rate sets the limit on the range of geometries that may be practicably considered.

Consider a Fredholm equation of the second kind in standard form,

$$\phi(x) + \hat{\lambda} \int_0^{\hat{\ell}} H(x,t)\, \phi(t)\, dt = \hat{a}(x) \qquad 0 \leq x \leq \hat{\ell} \tag{3.1}$$

A quite common approach for the numerical solution of such an equation is to replace the integral equation by a system of algebraic equations. The integral in (3.1) is approximated by some mechanical quadrature formula with associated weights δ_j. Thus

$$\phi(x) + \hat{\lambda}\, \hat{h} \sum_{j=0}^{J} \delta_j\, H(x, t^j)\, \phi(t^j) + R_J(x) = \hat{a}(x) \quad 0 \leq x \leq \hat{\ell} \tag{3.2}$$

where t^j are the interpolation points associated with the quadrature formula, $\hat{h} = \frac{\ell}{J}$ is the step size, and $R_J(x)$ is the truncation error. Let

$$\phi^j = \phi(t^j) \quad , \quad H^{ij} = H(x^i, t^j) \tag{3.3}$$

$$R_J^i = R_J(x^i) \quad , \quad \hat{a}^i = \hat{a}(x^i)$$

Evaluating (3.2) at the points $x = x^i$ $(i = 0, 1, \ldots, J)$ leads to the system of linear, algebraic equations

$$\phi^i + \hat{\lambda} \sum_{j=0}^{J} \hat{h}\, \delta_j H^{ij} \phi^j + R_J^i = \hat{a}^i \qquad i = 0, 1, \ldots, J \tag{3.4a}$$

or

$$(\underset{\sim}{I} + \hat{\lambda} \underset{\sim}{H}) \underset{\sim}{\phi} + \underset{\sim}{R}_J = \underset{\sim}{\hat{a}} \tag{3.4b}$$

where

$$\underset{\sim}{I} = (\delta_{ij}) \quad , \quad \underset{\sim}{H} = (\hat{h}\, \delta_j H^{ij})$$

$$\underset{\sim}{\phi} = (\phi^j) \quad , \quad \underset{\sim}{R}_J = (R_J^j) \quad , \quad \underset{\sim}{\hat{a}} = (\hat{a}^j)$$

It can be shown that so long as $\hat{\lambda}$ is not an eigenvalue of (3.2), the matrix $(\underset{\sim}{I} + \hat{\lambda} \underset{\sim}{H})$ is non-singular for sufficiently large J (c.f. Hoheisel [8]). Consequently, (3.4b) may be solved to give

$$\underset{\sim}{\phi} = (\underset{\sim}{I} + \hat{\lambda} \underset{\sim}{H})^{-1} \underset{\sim}{\hat{a}} - (\underset{\sim}{I} + \hat{\lambda} \underset{\sim}{H})^{-1} \underset{\sim}{R}_J \tag{3.5}$$

Assuming the second term on the right to be "small," an approximate solution is then given by

$$\hat{\underset{\sim}{\phi}} = (\underset{\sim}{I} + \hat{\lambda}\underset{\sim}{H})^{-1}\hat{\underset{\sim}{a}} \tag{3.6}$$

Evidently, the second term on the right of (3.5) represents the truncation error in $\hat{\underset{\sim}{\phi}}$. It can be shown that so long as $\hat{\lambda}$ is not an eigenvalue of (3.1), the norm of $(\underset{\sim}{I} + \hat{\lambda}\underset{\sim}{H})^{-1}$ remains bounded for sufficiently large J (c.f. Hoheisel [8]). Consequently the truncation error in $\hat{\underset{\sim}{\phi}}$ is directly related to the truncation error R_J in the approximate quadratic formula. Clearly the error in $\hat{\underset{\sim}{\phi}}$ depends on just how "small" R_J is. This in turn depends upon the inherent accuracy of the quadrature formula and the number of points taken, as well as the behavior of the integrand.

For most composite quadrature formulas, one is quaranteed that $R_J \rightarrow 0$ as $J \rightarrow \infty$. However, in solving the system of algebraic equations leading to (3.6), the computer storage capacity required is proportional to J^2, and the computing time is proportional to J^3. Further, in solving a large system of equations, one might generate a great deal of roundoff error (in addition to the truncation error $(\underset{\sim}{I} + \hat{\lambda}\underset{\sim}{H})^{-1}\underset{\sim}{R}_J$). Thus, much care must be taken to select a quadrature formula which is inherently well suited to the problem, so that a given accuracy may be achieved with a smaller number of points.

It happens that the trapezoidal rule is particularly well suited to the present problem. Ordinary expressions for the truncation error in trapezoidal rule integration indicate that $R_J \propto \hat{h}^2$. This does not seem to be a terribly rapid rate of convergence. However, for certain classes of integrands the convergence rate can be far superior. This can be seen by considering the Euler-Maclaurin Summation Formula (c.f. Abramowitz and Stegun [1]). This is given by

$$\int_0^{\hat{\ell}} \hat{v}(x)\,dx = \hat{h}\left[\tfrac{1}{2}\hat{v}_0 + \hat{v}_1 + \cdots + \hat{v}_{J-1} + \tfrac{1}{2}\hat{v}_J\right] - \sum_{m=1}^{M} \frac{\hat{B}_{2m}\,\hat{h}^{2m}}{(2m)!}\left[\hat{v}^{(2m-1)}(\hat{\ell}) - \hat{v}^{(2m-1)}(0)\right] + \hat{E}_{2M} \tag{3.7a}$$

where

$$\hat{h} = \frac{\hat{\ell}}{J} \tag{3.7b}$$

$$\hat{E}_{2m} = \frac{\hat{\vartheta}\,\hat{\ell}\,\hat{B}_{2M+2}\,\hat{h}^{2M+2}}{(2M+2)!} \cdot \max_{0 \leq x \leq \hat{\ell}} \left|\hat{v}^{(2M+2)}(x)\right| \tag{3.7c}$$

$$-1 \leq \hat{\vartheta} \leq 1 \tag{3.7d}$$

and $\hat{B}_m$ are the Bernoulli numbers. Here it is assumed that $\hat{v}$ possesses 2M+2 continuous derivatives in the <u>closed</u> interval $0 \leq x \leq \hat{\ell}$.

For an arbitrary integrand $\hat{v}$, one would not expect that $\hat{v}^{(1)}(\hat{\ell}) = \hat{v}^{(1)}(0)$. Thus the first term in the series in (3.7a) will in general be non-zero. Hence R_J, which is composed of the series plus $\hat{E}_{2M}$, will generally exhibit the $\hat{h}^2$ behavior usually attributed to trapezoidal rule integration.

However, if it happens that $\hat{v}^{(1)}(\hat{\ell}) = \hat{v}^{(1)}(0)$, then R_J will behave at worst as $\hat{h}^4$. If, in addition, $\hat{v}^{(3)}(\hat{\ell}) = \hat{v}^{(3)}(0)$, then R_J will behave at worst as $\hat{h}^6$, and so on. (This presumes, of course, that $\hat{v}$ possesses sufficient smoothness). In particular, for periodic functions integrated over a period, it is easily seen that $\hat{v}^{(2m-1)}(\hat{\ell}) = \hat{v}^{(2m-1)}(0)$ for all orders of the indicated derivatives (if they exist). As pointed out in Isaacson and Keller [12], this is the basis for the "remarkable" accuracy which experience has shown to hold for the trapezoidal rule when integrating periodic functions over a period.

Consider integrals of the form

$$I = \int_{-\hat{\ell}}^{\hat{\ell}} \hat{v}(t) \frac{dt}{\sqrt{\hat{\ell}^2 - t^2}} \tag{3.8}$$

where $\hat{v}(t)$ is infinitely continuously differentiable on the <u>closed</u> interval $-\hat{\ell} \leq t \leq \hat{\ell}$. In (3.8), let

$$t = -\hat{\ell} \cos \hat{\theta} \qquad 0 \leq \hat{\theta} \leq \pi \tag{3.9}$$

Then

$$I = \int_{0}^{\pi} \hat{V}(\hat{\theta})\, d\hat{\theta} \tag{3.10a}$$

where

$$\hat{V}(\hat{\theta}) = \hat{v}(-\hat{\ell}\cos\hat{\theta}) \tag{3.10b}$$

By the smoothness of $\hat{v}(t)$, it follows that $\hat{V}(\hat{\theta})$ will be infinitely continuously differentiable on the closed interval $0 \leq \hat{\theta} \leq \pi$. Further, from (3.10b) it is easily verified that

$$\hat{V}^{(2m-1)}(\pi) = \hat{V}^{(2m-1)}(0) = 0 \qquad m = 1, 2, \ldots \tag{3.11}$$

If the trapezoidal rule is applied to (3.10), it then follows from (3.4) that the result converges faster than $\hat{h}^{2M+2}$. But M may be taken as large as desired, since $\hat{V}$ is infinitely continuously differentiable. Consequently, it may be concluded that the truncation error goes to zero faster than any power of the mesh size $\hat{h}$.

The integrals defining $\varphi^n(z_n)$, which are given in (2.67) and (2.68), are of the form indicated in (3.8). Although $f^n(x_n)$ is unknown, it is shown in Appendix II that it must be infinitely continuously differentiable in the closed interval $-\ell_n \leq x_n \leq \ell_n$. Consequently, if the integrals defining $\varphi^n(z_n)$ are transformed as in (3.9) and then approximated by the trapezoidal rule, the result converges faster than any power of the mesh size. If this approximation is used to reduce the integral equations (2.75) to an approximate set of algebraic equations, it follows that the solution will converge faster than any power of the mesh size. Using the approximate solution the stress intensity factors, the relative displacements of the crack surfaces, and the energy of the cracks may be calculated from the formulas given in Chapter II, Section 5. It can be verified that all of these quantities are expressed in terms of integrals of the form indicated in (3.8). Further, it is shown in Appendix II that the equivalent $\hat{v}(t)$ for each of these integrals is infinitely continuously differentiable in the closed interval $-\ell \leq t \leq \ell$. Consequently, it may be concluded that the errors in the various quantities of interest will go to zero faster than any power of the mesh size.

2. Results

A computer program was written to apply the foregoing approach to the problem of two traction-free cracks of arbitrary orientation, subjected to a constant stress field at infinity. The convergence of the method was found to be quite good. Some surveys of two-crack interaction problems were performed, yielding interesting results.

The trapezoidal rule was applied to a known integral to test the rate of convergence. The integral considered is given by

$$f(z) = \frac{1}{\pi i} \int_{-\ell}^{\ell} \frac{1}{t - z} \cdot \frac{dt}{\sqrt{t^2 - \ell^2}} \tag{3.12}$$

From the arguments of the previous section, it is easily seen that the integral in (3.12) is of the type which, upon transformation to the trigonometric variable, should converge faster than any power of the mesh size under trapezoidal rule integration.

Thus letting

$$t = -\ell \cos \theta \tag{3.13}$$

in (3.12) yields

$$f(z) = \frac{1}{\pi} \int_{0}^{\pi} \frac{d\theta}{\ell \cos\theta + z} \tag{3.14}$$

Let the trapezoidal rule approximation be represented by

$$f_h(z) = h \sum_{n=0}^{N} \frac{\delta_n}{\ell\cos(nh) + z} \tag{3.15}$$

$$h = \frac{\pi}{N}\ , \qquad \delta_n = \begin{cases} \frac{1}{2} & (n=0 \text{ or } n=N) \\ 1 & (n=2,3,\ldots,N-1) \end{cases}$$

By contour integration or other means, it can be shown from (3.12) that f(z) has the known value

$$f(z) = \frac{1}{\sqrt{z^2-\ell^2}} \tag{3.16}$$

Let the <u>relative</u> error of f_h be given by

$$\delta(h;z) = \frac{|f(z) - f_h(z)|}{|f(z)|} \tag{3.17}$$

The behavior of $\delta(h;z)$ with respect to h is shown in Figure 7. Note that the scales are logarithmic. The normalization on h is chosen so that succeeding integral values along the abscissa represent a halving of the mesh size. Also, note that (negative) integral values on the ordinate represent (approximately) the number of figures of accuracy in the integration.

The integrand in (3.12) varies on a scale determined by the characteristic length S. Since the mesh points are uniformly distributed along the θ axis, they are nonuniformly distributed along the t axis. However, one might define an "average" mesh length by

$$t_a = \frac{2\ell}{N} \tag{3.18}$$

It might be noted that $\delta(h;z)$ exhibits a broad "knee," or turning point, when t_a is of the order of (but somewhat less than) the characteritic length S. As h continues to decrease past this knee, the convergence becomes extremely rapid. Locally,

$$\mathrm{Log}_{10}\,\delta = C_1 - C_2\,\mathrm{Log}_{10}\,h \tag{3.19}$$

or

$$\delta = 10^{C_1} \cdot h^{-C_2} \tag{3.20}$$

where C_2 is the slope of the curve. Since the curve gets steeper and steeper, it follows from (3.20) that eventually the method does indeed converge faster than any power of the mesh size.

The convergence rate of the computer program for interacting cracks was tested and found to exhibit the same characteristic behavior. The error in the approximate program was determined by comparison to the special case of colinear cracks, for which the analytic solution is known.

It might be noted that in the case of interacting cracks, there are basically two length scales. These are the length of the shortest crack, and the minimum separation distance between cracks. The relative importance of the two scales depends upon the specific geometry.

A survey was performed for the problem of two equal-length, parallel cracks, subjected to a tension field at infinity which is normal to the crack lines. Some of the results are shown in Figures 8-10.

The crack energy might be used as a simple scalar measure of the degree of interaction of the cracks. For large separation, the energy for two cracks should be twice that of a single crack. If it deviates from this value, then interaction is taking place. From Figure 8, it may be seen that there is virtually no interaction if the cracks are separated by eight or more crack lengths. Further, a <u>strong</u> interaction does not occur until the separation is less than two to three crack lengths. This conclusion is strengthened by the behavior of the cleavage angle and cleavage intensity shown in Figures 9 and 10.

When the cracks are in an overlapping configuration and the vertical separation becomes quite small, the crack energy seemed to approach that of a single, equivalent, elongated crack. This observation was used to extend the energy curves along the dotted lines shown in Figure 8. Further, the cleavage intensity at the outer crack tips seemed to approach that of the equivalent elongated crack. Hence, the curves in Figure 10 were extended in a similar manner.

An interesting observation may be made from the data presented in Figure 10. The fracture strength of a material may be increased due to the interaction of the cracks. This is evidenced by the fact that in the intermediate separation range, the cleavage intensity actually dips below that of a single crack of length 2ℓ in an infinite sheet. This seems to be connected with the variation of the cleavage angle, as indicated in Figure 9.

Finally, an interaction problem leading to crack surface overlap was considered. The geometry and some of the results are shown in Figure 11. The curve is dotted where crack surface overlap was indicated. It might be noted that as s decreases, the onset of overlap coincides with the point where the cleavage intensity at A becomes negative. The singularity at A is then a "compression singularity" rather than a "tension singularity." The inset shows the shape assumed by the surfaces of the vertical crack for a separation

$$\frac{s}{2\ell} = 0.1 \tag{3.21}$$

The separation shown in the inset is also drawn to this scale.

Overlap problems such as that shown in Figure 11 might be solved by an approximate, iterative procedure. When overlap is detected, the problem could be restated with some assumed compression tractions along the overlap surface. These tractions might be taken in proportion to the degree of overlap. The problem could then be solved again. In ideal circumstances, the overlap in the second iteration would be less severe. The process could be repeated until the overlap was reduced to an arbitrarily small amount. Since the general method put forth in this investigation is applicable to stress boundary value problems with loads on the crack surfaces, it would be useful in such an iteration procedure.

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APPENDIX I: ON THE UNIQUENESS OF THE SOLUTION

The concept of "uniqueness" should always be understood to mean "uniqueness within a particular class of admissible solutions." The ordinary uniqueness theorem of elastostatics (Kirchoff's uniqueness theorem) is confined to the class of regular solutions. Thus, whenever singular solutions are considered in elastostatics, the question of uniqueness must be re-examined.

A typical proof of the ordinary uniqueness theorem for a bounded, three dimensional region may be found in Sokolnikoff[21]. The result follows directly from the work-energy identity (Clapeyron's Theorem). In the absence of body forces, this is given by**

$$\int_V W \, dv = \frac{1}{2} \int_{\partial V} s_i u_i \, dS \tag{I-1}$$

where V is a three-dimensional, bounded, regular region, and ∂V is the surface of V. In equation (I-1), the expression on the right may be deduced from the expression on the left by a straightforward application of the

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*The results presented in this section were developed in consultation with Professor James K. Knowles of the California Institute of Technology, Pasadena, California.

**In this report, dummy indices represented by Latin letters will assume the range (1,2,3,) unless otherwise indicated. The presence of a repeated index indicates summation over its range.

divergence theorem (Gauss' theorem). A rigorous statement of the divergence theorem reveals that the vector field $\underset{\sim}{v}$ with components $v_i = \sigma_{ij}u_j$ must be presumed to be continuous up to the boundary ∂R (c.f. Kellogg[14]). Thus, the ordinary uniqueness proof carries the implicit assumption that the admissible solutions are confined to the class of regular solutions.

In the crack problem under consideration here, it is necessary to expand the class of admissible solutions to include solutions having singular stresses at the crack tips. If singularities of arbitrary order are admitted at the crack tips, then the solution may cease to be unique. For example, in the case of the Griffith crack it is not difficult to construct an infinite number of singular solutions, with each one having its own characteristic singularity at the crack endpoints.

Thus, the objective must be to expand the class of admissible solutions to include the "correct" singular solution, but not to expand the class so far that multiple solutions may occur. Evidently, some sort of restriction must be placed on the singularities to limit their order. Of course, a less restrictive limitation will lead to a more powerful uniqueness theorem, since it will apply to a broader class of admissible singular solutions.

It will be shown below that the only limitation that is necessary is the one implied by (2.13). That is, the displacements should be bounded (but not necessarily even continuous) at the crack tips. Physically, this is a quite reasonable restriction. Mathematically, it admits a wide class of singular solutions.

To facilitate the uniqueness proof, the following useful result will be proved. Consider an open, bounded, two-dimensional region D, whose outer boundary consists of a simple, piecewise smooth contour Γ, and whose inner boundaries consist of line segments $L_1, L_2, \ldots, L_N$. The geometry and coordinate systems will be as in Section 1 of Chapter II, with the additional restriction that the L_n do not intersect Γ. Let admissible elastostatic stress state $\mathcal{S}$ be defined on D as in Section 2 of Chapter II, except that in place of the condition at infinity, equation (2.14), it will be required that $\mathcal{S}$ be continuous up to Γ. Suppose a state $\mathcal{S}$ is given such that

$$s_\alpha = 0 \qquad \text{on} \quad S^+, S^- \tag{I-2}$$

Then the strain energy U_D (per unit thickness in the out-of-plane direction) is bounded, and is given by

$$U_D = \int_D W\, dA = \frac{1}{2} \int_\Gamma s_\alpha u_\alpha \, d\ell \tag{I-3}$$

where the first integral in (I-3) is understood to be taken as an improper integral in the vicinity of the crack endpoints.

Note that this result is to be expected on physical grounds, and the proof is quite simple in the absence of singularities. However, the presence of singularities restricted only by the weak limitations implied by (2.13) causes the proof to be somewhat more difficult.

Let

$$\varepsilon_0 = \min_{n=1,2,\ldots,N} [L_n] \tag{I-4}$$

Let P represent a point in the plane, and let $z_n(P)$ be its coordinate in the z_n system. Let

$$\Pi_\varepsilon^{n,1} = \{ P \mid |z_n(P) + \ell_n| < \varepsilon \} \tag{I-5a}$$

$$0 < \varepsilon < \varepsilon_0$$

$$\Pi_\varepsilon^{n,2} = \{ P \mid |z_n(P) - \ell_n| < \varepsilon \} \tag{I-5b}$$

$$n = 1, 2, \ldots, N$$

That is, $\Pi_\varepsilon^{n,1}$ consists of all points lying within a circle of radius ε centered at $z_n = -\ell_n$, while $\Pi_\varepsilon^{n,2}$ consists of all points lying within a circle of radius ε centered at $z_n = +\ell_n$ (c.f. Fig. 3). The boundaries of these regions are then given by the circles

$$\Gamma_\varepsilon^{n,1} = \{ P \mid |z_n(P) + \ell_n| = \varepsilon \} \tag{I-6a}$$

$$\Gamma_\varepsilon^{n,2} = \{ P \mid |z_n(P) - \ell_n| = \varepsilon \} \tag{I-6b}$$

Let

$$D_\varepsilon = D - \bigcup_{n=1}^{N} (\overline{\Pi}_\varepsilon^{n,1} \cup \overline{\Pi}_\varepsilon^{n,2}) \qquad 0 < \varepsilon < \varepsilon_0 \tag{I-7}$$

where the presence of a bar over a region represents the closure of the region, that is, the region plus its boundary points. Thus D_ε consists of that portion of D lying outside circles of radius ε which are centered at the crack tips. The region D_ε then has a boundary ∂D_ε comprised of an outer boundary Γ and inner boundaries consisting of the circles $\Gamma_\varepsilon^{n,1}$ and $\Gamma_\varepsilon^{n,2}$ (n = 1,2,...N) together with the line segments

$$L^+_{n,\varepsilon} = \{P \mid z_n(P) = x_n + i0^+,\ -\ell_n+\varepsilon \leq x_n \leq \ell_n - \varepsilon\} \qquad \text{(I-8a)}$$

$$L^-_{n,\varepsilon} = \{P \mid z_n(P) = x_n + i0^-,\ -\ell_n+\varepsilon \leq x_n \leq \ell_n - \varepsilon\} \qquad \text{(I-8b)}$$

$$n = 1, 2, \ldots, N$$

Consider the three-dimensional region V_ε defined by the two-dimensional region D_ε together with two planes a unit distance apart in the out-of-plane direction. Then the three-dimensional elastostatic state generated by the plane strain state $\mathscr{S}$ will be regular in V_ε. Hence the work energy identity, equation (I-1), may be applied to to the three-dimensional state on V_ε. From the plane strain assumption, the displacement u_3 is identically zero, and the tractions s_1 and s_2 vanish on the planar surfaces. Consequently the integrand on the right hand side of (I-1) vanishes identically on the planar surfaces. Also, the term s_3u_3 vanishes on the lateral surfaces. Further, the in-plane displacements, strains, stresses and tractions are independent of the out-of-plane coordinate. By equations (2.15) and (2.16), the strain energy density W must also be independent of the out-of-plane coordinate. Consequently, the application of (I-1) to the three-dimensional elastostatic state defined on V_ε yields

$$U_\varepsilon = \int_{D_\varepsilon} W \, dA = \frac{1}{2} \int_{\partial D_\varepsilon} s_\alpha u_\alpha \, d\ell \tag{I-9}$$

where U_ε is the strain energy contained in the region D_ε , per unit thickness in the out-of-plane direction.

From (I-2) it follows that

$$s_\alpha = 0 \quad \text{on } L^+_{n,\varepsilon} \, , \, L^-_{n,\varepsilon} \qquad 0 < \varepsilon < \varepsilon_0 \tag{I-10}$$

$$n = 1, 2, \ldots, N$$

Thus, combining (I-10) with (I-9),

$$U_\varepsilon = \frac{1}{2} \int_{\Gamma_\varepsilon} s_\alpha u_\alpha \, d\ell + \frac{1}{2} \int_{\Gamma} s_\alpha u_\alpha \, d\ell \tag{I-11}$$

where

$$\Gamma_\varepsilon = \bigcup_{n=1}^{N} \left(\Gamma_\varepsilon^{n,1} \cup \Gamma_\varepsilon^{n,2} \right) \tag{I-12}$$

Note that

$$\left| \int_{\Gamma_\varepsilon} s_\alpha u_\alpha \, d\ell \right| \leq \int_{\Gamma_\varepsilon} | s_\alpha u_\alpha | \, d\ell \tag{I-13}$$

Applying the Cauchy-Schwartz inequality to the integrand on the right yields

$$\left|\int_{\Gamma_\varepsilon} s_\alpha u_\alpha \, d\ell\right| \leq \int_{\Gamma_\varepsilon} \sqrt{s_\alpha s_\alpha}\,\sqrt{u_\beta u_\beta}\, d\ell \tag{I-14}$$

Or, by squaring each side of (I-14)

$$\left[\int_{\Gamma_\varepsilon} s_\alpha u_\alpha \, d\ell\right]^2 \leq \left[\int_{\Gamma_\varepsilon} \sqrt{s_\alpha s_\alpha}\,\sqrt{u_\beta u_\beta}\, d\ell\right]^2 \tag{I-15}$$

Applying the Cauchy-Schwartz integral inequality to the integral on the right then gives

$$\left[\int_{\Gamma_\varepsilon} s_\alpha u_\alpha \, d\ell\right]^2 \leq \left[\int_{\Gamma_\varepsilon} s_\alpha s_\alpha \, d\ell\right] \cdot \left[\int_{\Gamma_\varepsilon} u_\beta u_\beta \, d\ell\right] \tag{I-16}$$

Dropping the summation convention momentarily, recall that

$$s_\alpha = \sum_{\beta=1}^{2} \sigma_{\alpha\beta} n_\beta \tag{I-17}$$

or

$$s_\alpha^2 = \left[\sum_{\beta=1}^{2} \sigma_{\alpha\beta} n_\beta\right]^2 \tag{I-18}$$

Hence, by the Cauchy-Schwartz inequality

$$s_\alpha^2 \leq \left[\sum_{\beta=1}^{2} \sigma_{\alpha\beta}^2\right] \cdot \left[\sum_{\beta=1}^{2} n_\beta^2\right] \tag{I-19}$$

But

$$\sum_{\beta=1}^{2} n_\beta^2 = 1 \tag{I-20}$$

Thus, using (I-20) and summing over α,

$$\sum_{\alpha=1}^{2} S_\alpha^2 \leq \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \sigma_{\alpha\beta}^2 \tag{I-21}$$

Or, reintroducing the summation convention, this reads

$$S_\alpha S_\alpha \leq \sigma_{\alpha\beta} \sigma_{\alpha\beta} \tag{I-22}$$

Now consider equation (2.16b), which may be written as

$$\frac{1}{4\mu} \sigma_{\alpha\beta} \sigma_{\alpha\beta} - W = \frac{\tilde{\lambda}}{8\mu(\tilde{\lambda}+\mu)} \Theta^2 \tag{I-23}$$

Again dropping the summation convention for the moment, Θ^2 may be expressed as

$$\Theta^2 = \left[\sum_{\alpha=1}^{2} 1_\alpha \, \sigma_{\alpha\alpha} \right]^2 \tag{I-24}$$

where $1_\alpha = 1 \quad (\alpha = 1, 2)$

Applying the Cauchy-Schwartz inequality thus implies

$$\Theta^2 \leq 2 \sum_{\alpha=1}^{2} \sigma_{\alpha\alpha}^2 \tag{I-25}$$

It then clearly follows that

$$\Theta^2 \leq 2 \, \sigma_{\alpha\beta} \sigma_{\alpha\beta} \tag{I-26}$$

where the summation convention is again resumed.

It can be shown from (2.17) that the factor multiplying Θ^2 in (I-23) is always positive. This, together with (I-23) and (I-26) then gives

$$k^2 \sigma_{\alpha\beta} \sigma_{\alpha\beta} \leqslant W \qquad (I-27)$$

where

$$k^2 = \frac{1}{4(\lambda+\mu)} = \frac{(1+\tilde{\nu})(1-2\tilde{\nu})}{2E} \quad , \quad k^2 > 0 \qquad (I-28)$$

The fact that k^2 is always a positive (non-zero) constant follows from (2.17). It might be noted that k^2 is dependent only on the material properties.

Using (I-22) and (I-27), it then follows that

$$s_\alpha s_\alpha \leqslant \frac{1}{k^2} W \qquad (I-29)$$

Or, integrating along Γ_ε

$$\int_{\Gamma_\varepsilon} s_\alpha s_\alpha \, d\ell \leqslant \frac{1}{k^2} \int_{\Gamma_\varepsilon} W \, d\ell \qquad (I-30)$$

Further, (2.13) and (I-12) imply

$$\int_{\Gamma_\varepsilon} u_\beta u_\beta \, d\ell \leqslant 4N\pi M_u^2 \varepsilon \qquad (I-31)$$

Thus, (I-16), (I-30) and (I-31) yield

$$\left[\int_{\Gamma_\varepsilon} s_\alpha u_\alpha \, d\ell \right]^2 \leqslant \frac{4N\pi M_u^2}{k^2} \varepsilon \int_{\Gamma_\varepsilon} W \, d\ell \qquad (I-32)$$

Define

$$h(\varepsilon) = \int_{D_\varepsilon} W \, dA - \frac{1}{2} \int_{\Gamma} s_\alpha u_\alpha \, d\ell \qquad 0 < \varepsilon < \varepsilon_0 \qquad (I-33)$$

Note that if it can be shown that h(ε) tends to zero as ε tends to zero, then the desired conclusion given in equation (I-3) will have been achieved. Now h(ε) may be expressed as

$$h(\varepsilon) = \sum_{n=1}^{N} \left[\int_{\varepsilon}^{\varepsilon_0} \int_{-\pi}^{\pi} W r_{n,1}\, d\theta_{n,1}\, dr_{n,1} + \int_{\varepsilon}^{\varepsilon_0} \int_{-\pi}^{\pi} W r_{n,2}\, d\theta_{n,2}\, dr_{n,2} \right] \qquad \text{(I-34)}$$

$$+ \int_{D_{\varepsilon_0}} W\, dA - \frac{1}{2} \int_{\Gamma} s_\alpha u_\alpha\, d\ell$$

where $r_{n,1}$, $\theta_{n,1}$, $r_{n,2}$, $\theta_{n,2}$ are local polar coordinates based at the indicated crack tips. It might be noted that the last two integrals in (I-34) are mere constants. Further, the quantity W in the remaining integrals is considered to be a function of the local polar coordinates, and is independent of ε . Thus the only appearance of ε is in the lower limits of the outer integrals in the sum. Consequently, differentiating (I-34) with respect to ε yields

$$h'(\varepsilon) = - \sum_{n=1}^{N} \left[\int_{-\pi}^{\pi} W \varepsilon\, d\theta_{n,1} + \int_{-\pi}^{\pi} W \varepsilon\, d\theta_{n,2} \right] \qquad \text{(I-35)}$$

Or, from (I-6) and (I-12),

$$h'(\varepsilon) = - \int_{\Gamma_\varepsilon} W\, d\ell \qquad \text{(I-36)}$$

Now, (I-9), (I-11) and (I-33) give

$$h(\varepsilon) = \frac{1}{2} \int_{\Gamma_\varepsilon} s_\alpha u_\alpha\, d\ell \qquad \text{(I-37)}$$

Hence, combining (I-32), (I-36) and (I-37) results in

$$h^2(\varepsilon) \leq -c^2 \varepsilon h'(\varepsilon) \tag{I-38}$$

where

$$c^2 = \frac{\sqrt{N\pi}\, M_y}{k} \quad , \qquad c^2 > 0 \tag{I-39}$$

From (I-33), h is a well-defined, smooth function of ε in the open interval $(0, \varepsilon_0)$. By simple logic, it follows that either one or the other of the following two possibilities must be true (but not both):

Case I:

$$h(\varepsilon_1) > 0 \quad \text{for some} \quad \varepsilon_1 \in (0, \varepsilon_0) \tag{I-40}$$

Case II:

$$h(\varepsilon) \leq 0 \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_0) \tag{I-41}$$

Since W is positive definite, it may be seen from either (I-33) or (I-36) that

$$h'(\varepsilon) \leq 0 \qquad 0 < \varepsilon < \varepsilon_0 \tag{I-42}$$

That is, $h(\varepsilon)$ increases monotonically with <u>decreasing</u> ε. Consequently, if Case I is true, then it follows that h will be greater then zero for all ε in the open interval $(0, \varepsilon_1)$. Further, if Case II is true, and if

$$h(\varepsilon_2) = 0 \quad \text{for some} \quad \varepsilon_2 \in (0, \varepsilon_0)$$

then

$$h(\varepsilon) = 0 \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_2)$$

Thus Case I may be restated, and Case II may be decomposed into two further categories as follows:

Case I:

$$h(\varepsilon) > 0 \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_1) \tag{I-43}$$

Case II(a):

$$h(\varepsilon) = 0 \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_2) \tag{I-44}$$

Case II(b):

$$h(\varepsilon) < 0 \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_0) \tag{I-45}$$

Now if Case II(a) is true, then there is nothing left to prove, since (I-33) and (I-44) immediately imply the desired conclusion given in equation (I-3). Thus, assume that either Case I or Case II(b) is true. Let

$$\varepsilon_5 = \begin{cases} \varepsilon_1 & \text{for Case I} \\ \varepsilon_0 & \text{for Case II(b)} \end{cases} \tag{I-46}$$

Then for either Case I or Case II(b), it follows that $h^2(\varepsilon)>0$ in the open interval $(0, \varepsilon_5)$. Consequently, (I-38) may be rearranged to give

$$\frac{1}{h^2}\frac{dh}{d\varepsilon} + \frac{1}{c^2\varepsilon} \leq 0 \qquad 0 < \varepsilon < \varepsilon_5 \tag{I-47}$$

Let ε_3, ε_4 be chosen such that $0 < \varepsilon_3 < \varepsilon_4 < \varepsilon_5$. Integrating

(I-47) from ε_3 to ε_4 then gives

$$-\frac{1}{h(\varepsilon_4)} + \frac{1}{h(\varepsilon_3)} + \frac{1}{c^2}\log\frac{\varepsilon_4}{\varepsilon_3} \leq 0 \tag{I-48}$$

Case I: Inequality (I-48) may be rearranged to give

$$\frac{1}{h(\varepsilon_4)} \geq \frac{1}{c^2}\log\frac{\varepsilon_4}{\varepsilon_3} + \frac{1}{h(\varepsilon_3)} \geq \frac{1}{c^2}\log\frac{\varepsilon_4}{\varepsilon_3} \tag{I-49}$$

where the last inequality follows by (I-43). Keeping in mind that $h(\varepsilon_4)$ is positive while rearranging the outer inequalities in (I-49) yields

$$h(\varepsilon_4) \leq \frac{c^2}{\log\frac{\varepsilon_4}{\varepsilon_3}} \tag{I-50}$$

Letting ε_3 tend to zero in (I-50) yields

$$h(\varepsilon_4) \leq 0 \tag{I-51}$$

But by (I-43), $h(\varepsilon_4)$ was assumed to be strictly greater than zero. Hence a contradiction is achieved, and it may be concluded that Case I can never occur.

Case II(b): Inequality (I-48) may be rearranged to give

$$\frac{1}{h(\varepsilon_3)} + \frac{1}{c^2}\log\frac{\varepsilon_4}{\varepsilon_3} \leq \frac{1}{h(\varepsilon_4)} \leq 0 \tag{I-52}$$

where the last inequality follows by (I-45). Hence

$$\frac{1}{h(\varepsilon_3)} \leq -c^2\log\frac{\varepsilon_4}{\varepsilon_3} \tag{I-53}$$

Since h is presumed to be negative, multiplying inequality (I-53) by h(ε_3) reverses the inequality. Thus

$$1 \geq -h(\varepsilon_3) \cdot \frac{1}{C^2} \log \frac{\varepsilon_4}{\varepsilon_3} \tag{I-54}$$

or

$$0 \leq -h(\varepsilon_3) \leq \frac{C^2}{\log \frac{\varepsilon_4}{\varepsilon_3}} \tag{I-55}$$

where the first inequality holds by (I-45). Letting ε_3 tend to zero in (I-55), it follows that

$$\lim_{\varepsilon \to 0} h(\varepsilon) = 0 \tag{I-56}$$

Therefore, in this final case, (I-3) follows from (I-56) together with (I-33).

Returning now to the question of uniqueness, it might be noted that for a bounded region, the uniqueness of the solution to the stress boundary value problem follows directly from (I-3). Thus, suppose two states, $\mathcal{S}_1$ and $\mathcal{S}_2$, satisfy the same traction boundary conditions on S^+, S^- and Γ. Then defining

$$\mathcal{S} = \mathcal{S}_1 - \mathcal{S}_2 \quad \text{on } D \tag{I-57}$$

it follows that

$$s_\alpha = 0 \quad \text{on } S^+, S^-, \Gamma \tag{I-58}$$

Consequently, (I-3) may be applied to yield

$$\int_D W\, dA = 0 \qquad \text{(I-59)}$$

From (2.16a) and (2.17), W is a positive definite quadratic form in the strains. Thus (I-59) requires

$$\gamma_{\alpha\beta} = 0 \quad \text{on } D \qquad \text{(I-60)}$$

In the usual manner, it may then be argued that $\mathcal{S}_1$ and $\mathcal{S}_2$ differ at most by an infinitesimal rigid body motion, and the uniqueness proof is complete (for bounded regions).

For the infinite region, the proof is complicated somewhat by the fact that the only restriction on the behavior of $\mathcal{S}$ at infinity is the weak limitation given in (2.14). Consider the stress boundary value problem for the infinite region, as stated in Section 2 of **Chapter II.** Suppose two elastostatic states, $\mathcal{S}_1$ and $\mathcal{S}_2$, satisfy the same boundary value problem. Define

$$\mathcal{S} = \mathcal{S}_1 - \mathcal{S}_2 \quad \text{on } B \qquad \text{(I-61)}$$

Evidently $\mathcal{S}$ satifies

$$s_\alpha = 0 \quad \text{on } \mathcal{S}^+, \mathcal{S}^- \qquad \text{(I-62a)}$$

$$\underset{\sim}{\tau} = o(1) \quad \text{as } |z_0| \to \infty \qquad \text{(I-62b)}$$

Consider B_R as given in (2.6), for $R \geqslant R_c$. Then on the bounded region B_R, $\mathscr{S}$ is an elastostatic state satisfying (I-2). Consequently, (I-3) may be applied to $\mathscr{S}$ on B_R to yield

$$\int_{B_R} W \, dA = \frac{1}{2} \int_{C_R} s_\alpha u_\alpha \, d\ell \tag{I-63}$$

where C_R is given by (2.7).

If it could be shown that the integral on the right in (I-63) vanishes as R tends to infinity, then the proof of uniqueness would follow in the usual manner. It might not seem that the weak restriction given in (I-62b) is sufficient to guarantee the vanishing of this integral. However, a more detailed analysis indicates otherwise.

Let Π_e be the exterior region given by

$$\Pi_e = \{(x_o, y_o) \mid x_o^2 + y_o^2 > R_c^2\} \tag{I-64}$$

The boundary of Π_e is then given by the simple, smooth contour

$$\partial \Pi_e = C_{R_c} \tag{I-65}$$

Suppose the resolvent force on C_{R_c} vanishes, that is

$$A_\alpha = \int_{C_{R_c}} s_\alpha \, d\ell = 0 \tag{I-66}$$

It can be shown that (I-66) and (I-62b), together with the fact that Π_e is an exterior region bounded by a simple, smooth contour, imply the following far-field behavior for $\mathcal{S}$:*

$$\underset{\sim}{u} = \varepsilon_\infty \underset{\sim}{u}_r + O(1) \;, \quad \underset{\sim}{\mathcal{T}} = O(|z_0|^{-2}) \text{ as } |z_0| \to \infty \qquad \text{(I-67)}$$

Here $\underset{\sim}{u}_r$ is the displacement field due to a unit rigid body rotation, and ε_∞ is the magnitude of the rotation. A rigid body rotation generates no stresses. Thus, by (2.16b), the integral on the left of (I-63) must be independent of ε_∞. Further, the only appearance of ε_∞ in the expression for the displacements and strains is in the term indicated in (I-67). Thus, $\underset{\sim}{u}$ may be written as

$$\underset{\sim}{u} = \varepsilon_\infty \underset{\sim}{u}_r + \underset{\sim}{u}' \qquad \text{(I-68)}$$

where $\underset{\sim}{u}'$ is independent of ε_∞ and, from (I-67)

$$\underset{\sim}{u}' = O(1) \quad \text{as } |z_0| \to \infty \qquad \text{(I-69)}$$

Then (I-63) may be written as

$$\int_{B_R} W\,dA = \frac{\varepsilon_\infty}{2} \int_{C_R} s_\alpha u_\alpha \,d\ell + \frac{1}{2} \int_{C_R} s_\alpha u'_\alpha \,d\ell \qquad \text{(I-70)}$$

where all indicated integrals are independent of ε_∞. Consequently, the integral on the left can only be

---

*This follows from the expressions given in Section 36 of Muskhelishvili (16) for the form of the complex potentials which generate $\mathcal{S}$ on Π_e.

independent of ε_∞ if the first integral on the right vanishes. Further, using (I-67) and (I-69) to estimate the second integral on the right yields

$$\int_{B_R} W\, dA = O(R^{-1}) \quad \text{as} \quad R \to \infty \tag{I-71}$$

or

$$\int_B W\, dA = 0 \tag{I-72}$$

Thus, as in the uniqueness argument for the case of a bounded region given above, it then follows that $\mathcal{S}_1$ and $\mathcal{S}_2$ may differ at most by a rigid body motion. Hence, the uniqueness proof for infinite regions will be complete if the supposition in (I-66) can be proved.

Let B_{R_c} be as in (2.6), and let $D = B_{R_c}$ in the definitions in equations (I-4) through (I-8). Note that $\mathcal{T}$ is regular on D_ε. Consequently, if (2.11) are integrated over D_ε, then Gauss' theorem may be applied to yield

$$0 = \int_{D_\varepsilon} \sigma_{\alpha\beta,\beta}\, dA = \int_{\partial D_\varepsilon} \sigma_{\alpha\beta} n_\beta\, d\ell \tag{I-73}$$

Or using (I-12) and (I-62a) in (I-73)

$$|A_\alpha| = \left| \int_{\Gamma_\varepsilon} S_\alpha\, d\ell \right| \tag{I-74}$$

Using the Cauchy-Schwartz integral inequality gives

$$A_\alpha A_\alpha \leqslant 4N\pi\varepsilon \int_{\Gamma_\varepsilon} S_\alpha S_\alpha \, d\ell \tag{I-75}$$

Therefore, from (I-30), (I-36) and (I-75)

$$\frac{A_\alpha A_\alpha}{\varepsilon} \leqslant -\frac{4N\pi}{k^2} h'(\varepsilon) \tag{I-76}$$

Pick ε_1, ε_2 such that $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$, and integrate (I-76) from ε_1 to ε_2 to get

$$0 \leqslant A_\alpha A_\alpha \leqslant \frac{4N\pi\,[h(\varepsilon_1) - h(\varepsilon_2)]}{\log \frac{\varepsilon_2}{\varepsilon_1}} \tag{I-77}$$

By (I-42), the quantity in brackets is positive, and by earlier analysis $h(\varepsilon)$ tends to zero as ε tends to zero. Thus letting ε_1 tend to zero in (I-77), the numerator on the right remains positive and bounded, and one concludes

$$A_\alpha A_\alpha = 0 \tag{I-78}$$

Hence, the proof is complete.

It might be noted that in the above proofs, no particular use is made of the fact that the cracks are rectilinear. Consequently, the proofs apply equally well to cracks in the shape of non-intersecting, piecewise smooth curves, so long as stress singularities are permitted only at the crack endpoints.

APPENDIX II: SOME ANALYTIC PROPERTIES OF THE SOLUTION TO THE INTEGRAL EQUATIONS

In this section, some analytic properties are established for the function $F^n(x_n)$ on $f^n(x_n)$, and also for some of the integrands appearing in Chapter II, Section 5. These analytic properties are necessary to verify the convergence rate of the numerical scheme developed in Chapter III, Section 1.

Let $F^1, F^2, \ldots, F^n$ be a set of functions satisfying the integral equations (2.64) or (2.65), with each F^n defined on L_n $(n = 1,2,\ldots,N)$. Then for each n, there exists a neighborhood N^n of L_n, and a function $F^n_c(z_n)$ defined on N^n, such that:

1.) N^n is a simply-connected domain and $L_n \subset N^n$;

2.) F^n_c is a holomorphic function of z_n for all $z_n \in N^n$; and

$$F_c^n(x_n) = F^n(x_n) \quad \text{for all } x_n \in L_n \tag{II-1}$$

For any arbitrary integer $n \in (1,N)$, consider the family of domains depending on the parameter $\rho > 0$, and defined by

$$N_\rho^n = \left\{ (x_n, y_n) \mid (x_n - \hat{x}_n)^2 + y_n^2 < \rho^2 \ \text{for some } \hat{x}_n \in L_n \right\} \tag{II-2}$$

Since the cracks L_n do not intersect, it follows that for some sufficiently small ρ, N^n_ρ will contain no points of the cracks L_m ($m = 1,2,\ldots,N$; $m \neq n$). Let such a ρ be picked and denoted by ρ_0, and let

$$N^n = N^n_{\rho_0} \tag{II-3}$$

Note that N^n is easily shown to satisfy (1). Further, it may be noted that N^n is symmetric with respect to the real exis $y_n = 0$. That is, let

$$\hat{N}^n = \{ z_n \mid \bar{z}_n \in N^n \} \tag{II-4}$$

Then, $\hat{N}^n$ is the "reflection" of N^n through the real axis. It is easily seen that

$$\hat{N}^n = N^n \tag{II-5}$$

From (2.53a), (2.58), and (2.62),

$$F^n(x_n) = \mathcal{T}_s^{(n)} \{ \hat{\mathcal{S}}^n \} \qquad -\ell_n \leq x_n \leq \ell_n \tag{II-6}$$

where

$$\hat{\mathcal{S}}^n = -\mathcal{S}^{0,0} - \sum_{\substack{m=1 \\ m \neq n}}^{N} (\mathcal{S}^m + \mathcal{S}^{m,0}) \tag{II-7}$$

From (II-7) and (II-3) it is easily verified that $\hat{\mathcal{S}}^n$ is analytic throughout N^n. Consequently, the stress field $\hat{\mathcal{T}}^n$ may be expressed as

$$\hat{\sigma}_{11}^n + \hat{\sigma}_{22}^n = 2 [\hat{\Phi}^n(z_n) + \overline{\hat{\Phi}^n(z_n)}] \tag{II-8a}$$

$$\hat{\sigma}_{22}^{n} - \hat{\sigma}_{11}^{n} + 2i\,\hat{\sigma}_{12}^{n} = 2[\bar{z}_n\,\hat{\Phi}^{n\prime}(z_n) + \hat{\Psi}(z_n)] \qquad \text{(II-8b)}$$

where $\hat{\Phi}^n$ and $\hat{\Psi}^n$ are holomorphic functions of z_n for all $z_n \in N^n$. Define

$$\hat{\Omega}^n(z_n) = \overline{\hat{\Phi}^n(\bar{z}_n)} + z_n\,\overline{\hat{\Phi}^{n\prime}(\bar{z}_n)} + \overline{\hat{\Psi}^n(\bar{z}_n)} \qquad \text{(II-9)}$$

It can be shown that $\hat{\Omega}^n$ is a holomorphic function of z_n for all $z_n \in \hat{N}^n$ (c.f. Muskhelishvili [16], Section 76). Thus, by (II-5), $\hat{\Omega}^n$ is a holomorphic function of z_n for all $z_n \in N^n$. Combining (II-8) with (II-9) gives

$$\hat{\sigma}_{22}^{n} - i\,\hat{\sigma}_{12}^{n} = \hat{\Phi}^n(z_n) + \hat{\Omega}^n(\bar{z}_n) + (z_n - \bar{z}_n)\,\overline{\hat{\Phi}^{n\prime}(z_n)} \qquad \text{(II-10)}$$

Evaluating (II-10) for $z_n \in L_n$ and using (II-6) together with (2.22a) gives

$$F^n(x_n) = \hat{\Phi}^n(x_n) + \hat{\Omega}^n(x_n) \qquad -l_n \le x_n \le l_n \qquad \text{(II-11)}$$

Thus, letting

$$F_c^n(z_n) = \hat{\Phi}^n(z_n) + \hat{\Omega}^n(z_n) \quad \text{for all } z_n \in N^n \qquad \text{(II-12)}$$

it follows that F_c^n has the properties claimed in (2).

Statements (2) above can also be shown to be true for f^n. Thus let

$$f_c^n(z_n) = \int_0^{z_n} F_c^n(\xi)\,d\xi + f^n(0) \quad \text{for all } z_n \in N^n \qquad \text{(II-13)}$$

where the integral is taken along any path lying wholly in N^n. Then $f_c^n(z_n)$ is easily seen to be holomorphic throughout N^n, and satisfies

$$f_c^n(x_n) = f^n(x_n) \quad \text{for all} \quad x_n \in L_n \tag{II-14}$$

Since L_n is interior to N_n, it then follows that f^n (and F^n) are infinitely continuously differentiable on the closed interval $-\ell_n \le x_n \le \ell_n$. Further, it follows trivially that the functions $h^n(x_n, t_n)$ and $H^n(x_n, t_n)$ defined in equations (2.87) are infinitely continuously differentiable on the open square

$$S_0 = \{(x_n, t_n) \,|\, -\ell_n - \rho_0 < x_n < \ell_n + \rho_0, \; -\ell_n - \rho_0 < t_n < \ell_n + \rho_0\} \tag{II-15}$$

Hence they are certainly infinitely continuously differentiable on the closed square

$$S_c = \{(x_n, t_n) \,|\, -\ell_n \le x_n \le \ell_n, \; -\ell_n \le t_n \le \ell_n\} \tag{II-16}$$

Finally, from (2.95) and (2.86a), $\tilde{d}^n$ may be written as

$$\tilde{d}^n(x_n) = \frac{1}{\pi i} \int_{-\ell_n}^{\ell_n} h^n(x_n, t_n) \frac{dt_n}{\sqrt{t_n^2 - \ell_n^2}} \tag{II-17}$$

In (II-17), let

$$t_n = -\ell_n \cos \tau_n \qquad 0 \le \tau_n \le \pi \tag{II-18}$$

Then

$$\tilde{d}^n(x_n) = -\frac{1}{\pi}\int_0^\pi h^n(x_n, -\ell_n \cos \tau_n)\, d\tau_n \qquad \text{(II-19)}$$

By the analyticity of h^n on S_0, it follows from (II-19) that $\tilde{d}^n$ is infinitely continuously differentiable on the closed interval $-\ell_n \le x_n \le \ell_n$.

$$\sigma_r = \frac{1}{(2\pi r)^{1/2}} \cos\frac{\theta}{2}\left[K_I\left(1+\sin^2\frac{\theta}{2}\right)+\frac{3}{2}K_{II}\sin\theta - 2K_{II}\tan\frac{\theta}{2}\right]+O(1)$$

$$\sigma_\theta = \frac{1}{(2\pi r)^{1/2}} \cos\frac{\theta}{2}\left[K_I\cos^2\frac{\theta}{2}-\frac{3}{2}K_{II}\sin\theta\right]+O(1)$$

$$\tau_{r\theta} = \frac{1}{2(2\pi r)^{1/2}} \cos\frac{\theta}{2}\left[K_I\sin\theta + K_{II}(3\cos\theta - 1)\right]+O(1)$$

K_I = Symmetric Component of Stress Intensity Factor

K_{II} = Antisymmetric Component of Stress Intensity Factor

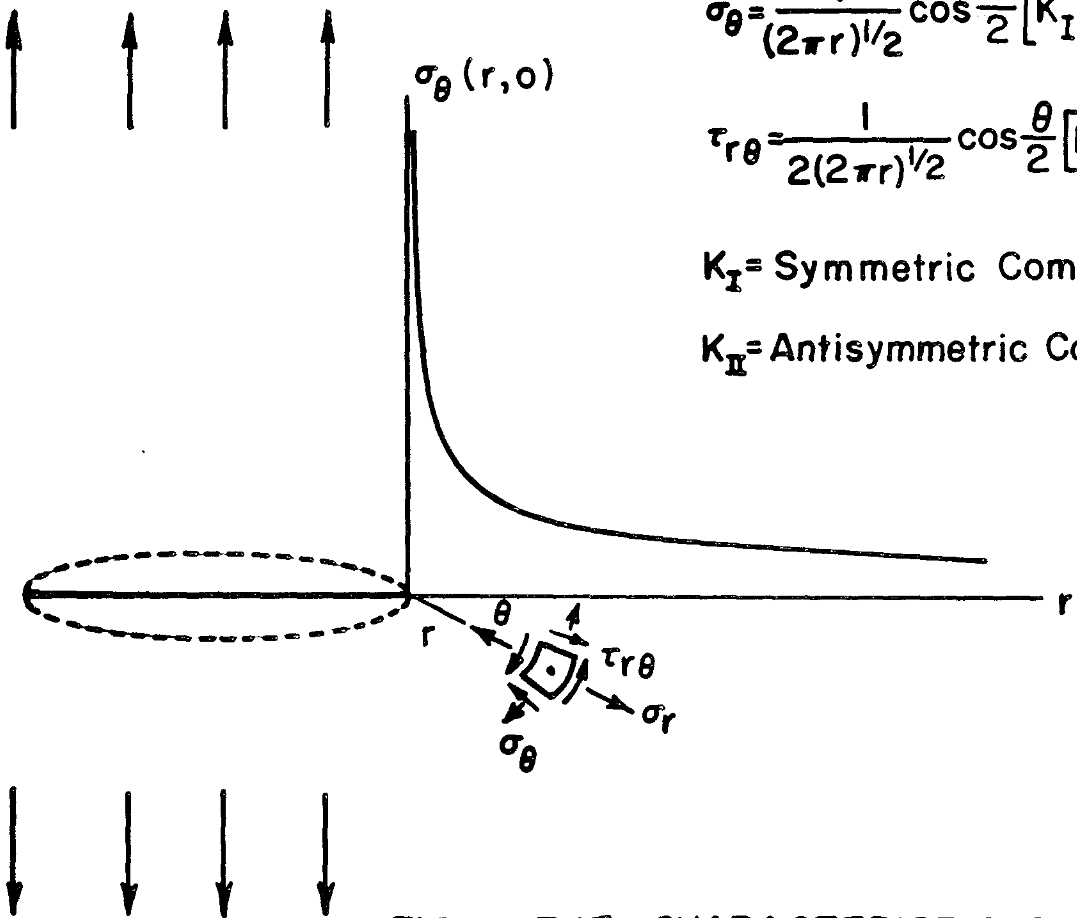


FIG. I THE CHARACTERISTIC SINGULARITY

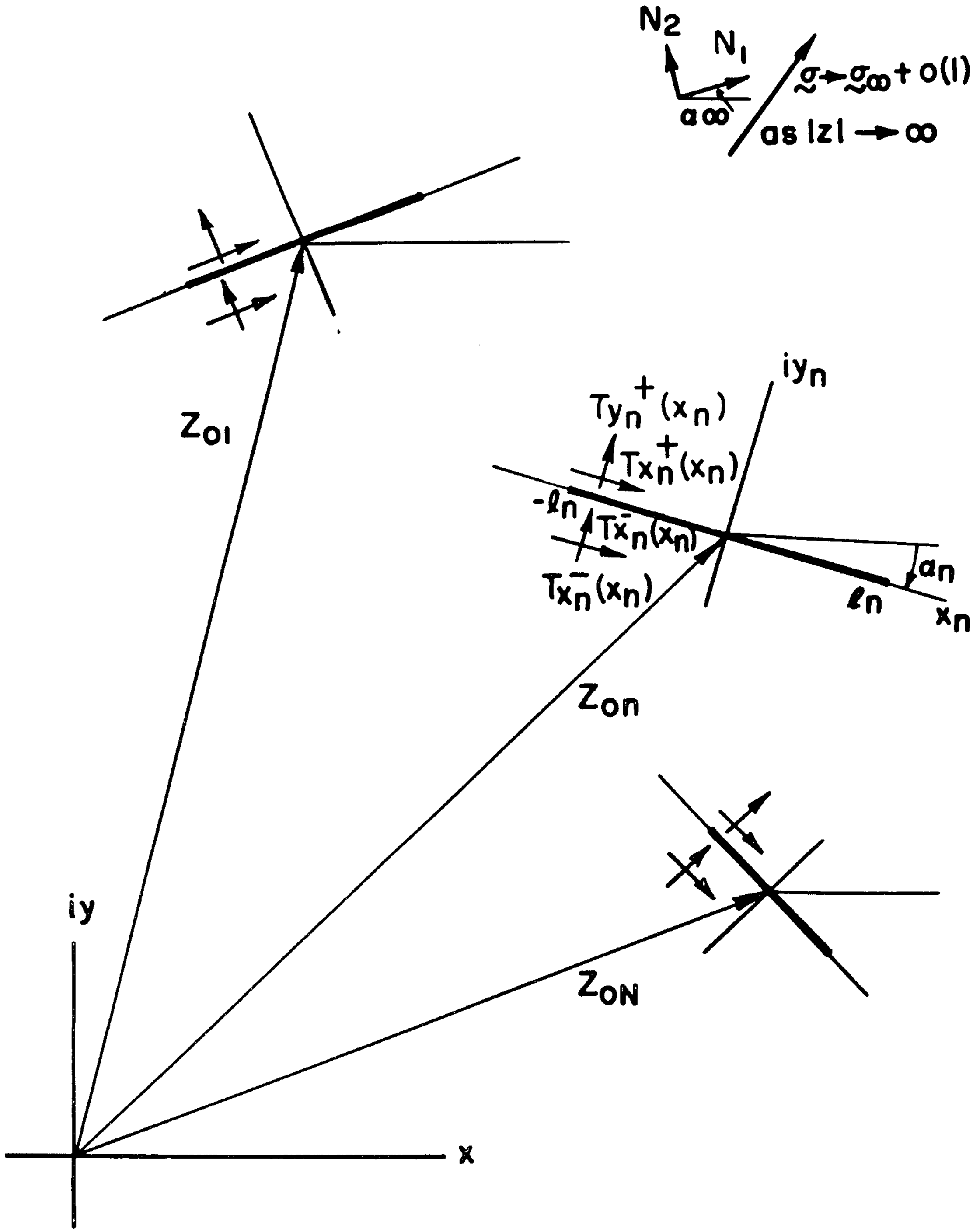


FIG. 2 GEOMETRY, COORDINATE SYSTEMS, AND LOADINGS

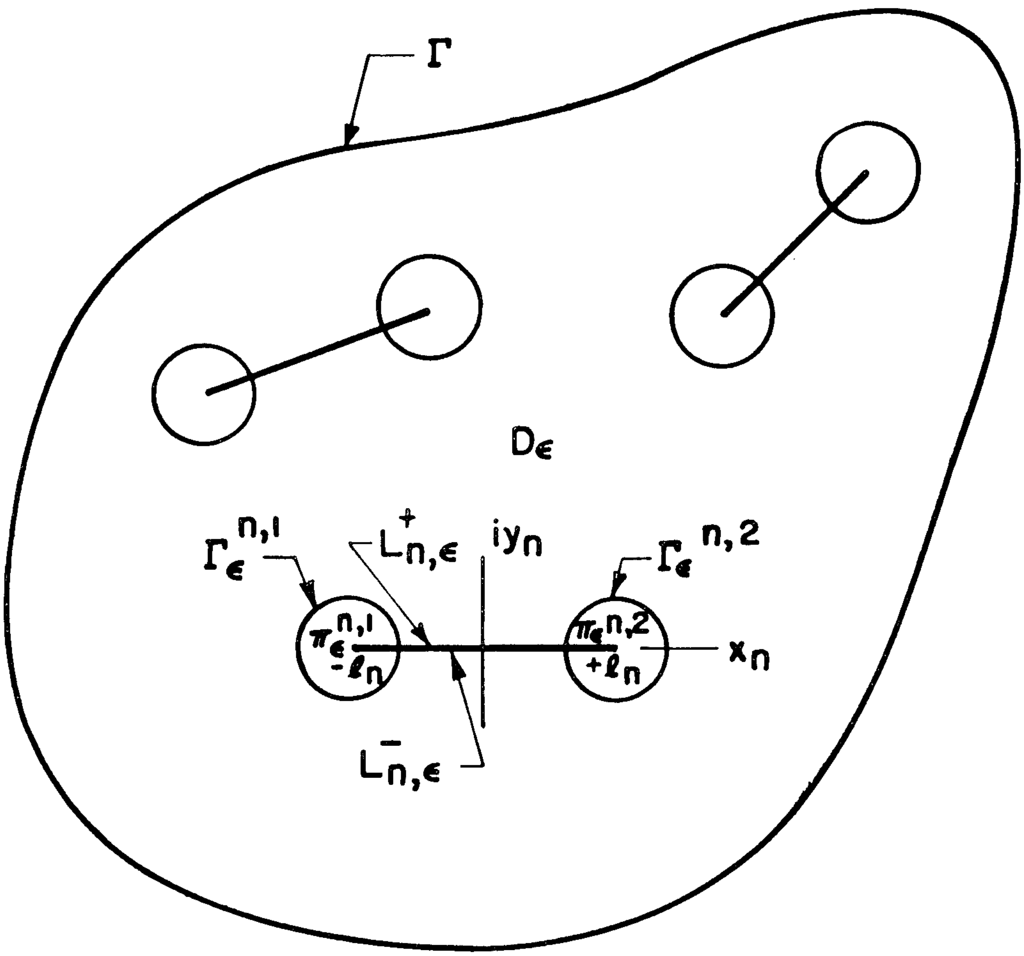


FIG. 3 THE DELETED REGION

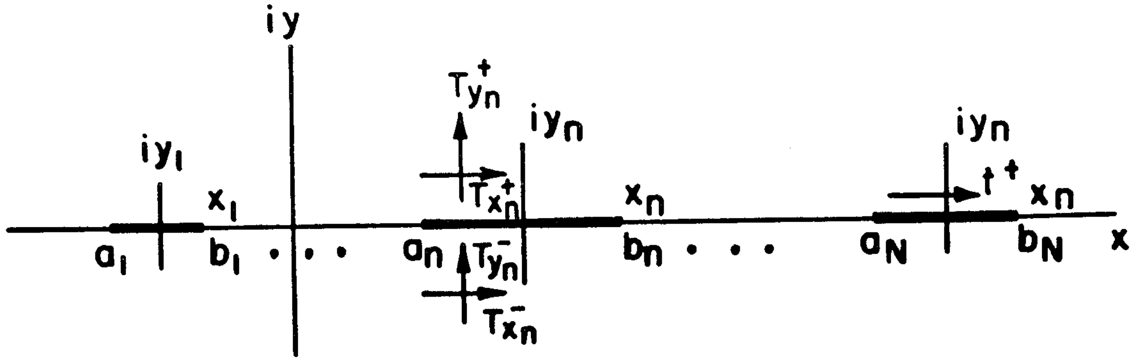


FIG. 4 COLINEAR CRACKS

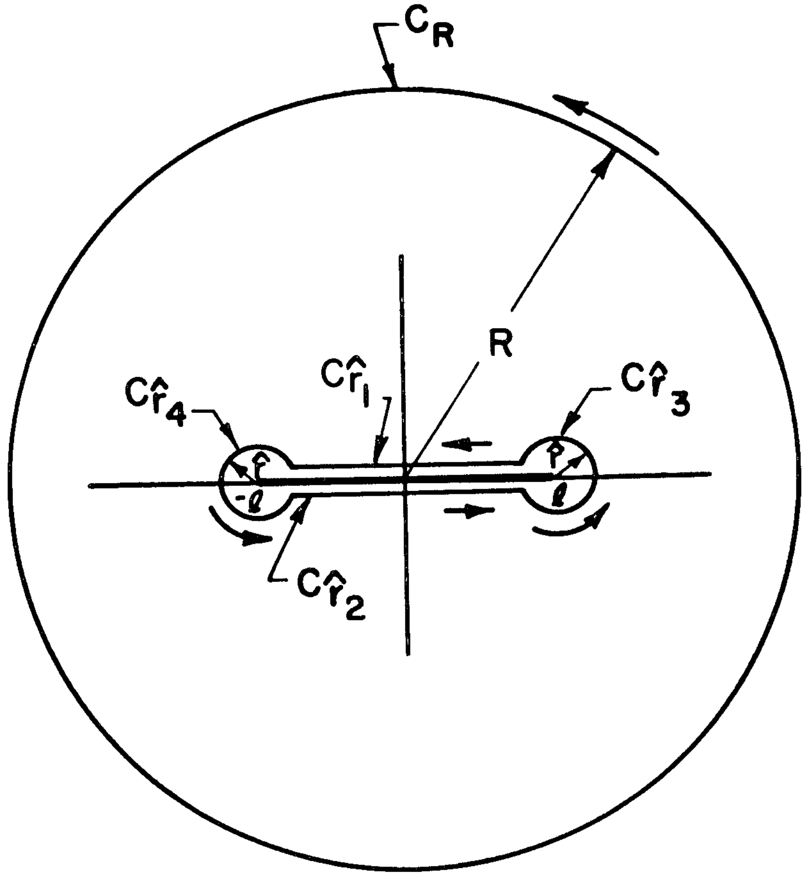


FIG. 5 INTEGRATION CONTOURS

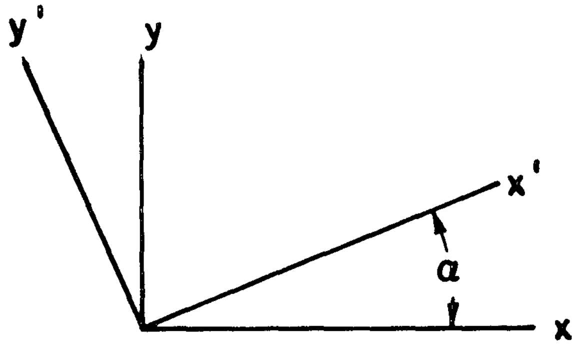
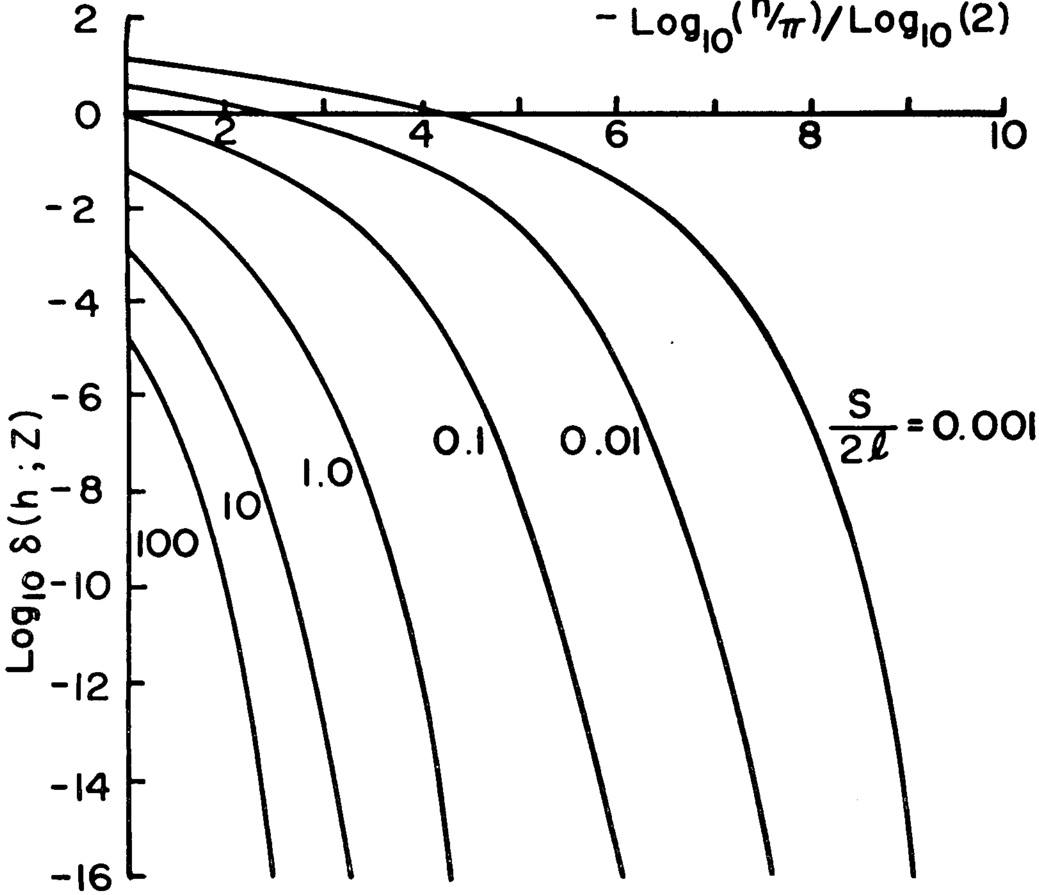
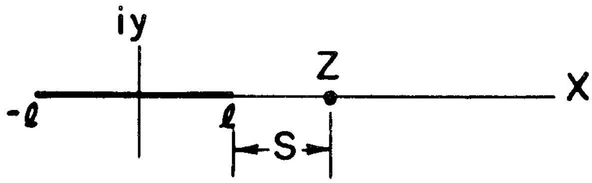


FIG. 6 COORDINATE ROTATION





$$f(Z)=\frac{1}{\sqrt{Z^2-\ell^2}}=\frac{1}{\pi i}\int_{-\ell}^{\ell}\frac{1}{t-Z}\cdot\frac{dt}{\sqrt{t^2-\ell^2}}=\frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{\ell\cos\theta+Z}\,(t=-\ell\cos\theta)$$

$$f_h(Z)=h\sum_{n=0}^{N}\frac{\delta_n}{\ell\cos(nh)+Z}\qquad\left(h=\frac{\pi}{N}\right)$$

$$\delta(h;Z)=\frac{|f(Z)-f_h(Z)|}{|f(Z)|}$$

Locally: $\operatorname{Log}\delta = C_1 - C_2 \operatorname{Log} h$

$$\therefore\ \delta = 10^{C_1}\cdot h^{-C_2}$$

FIG. 7 CONVERGENCE STUDY

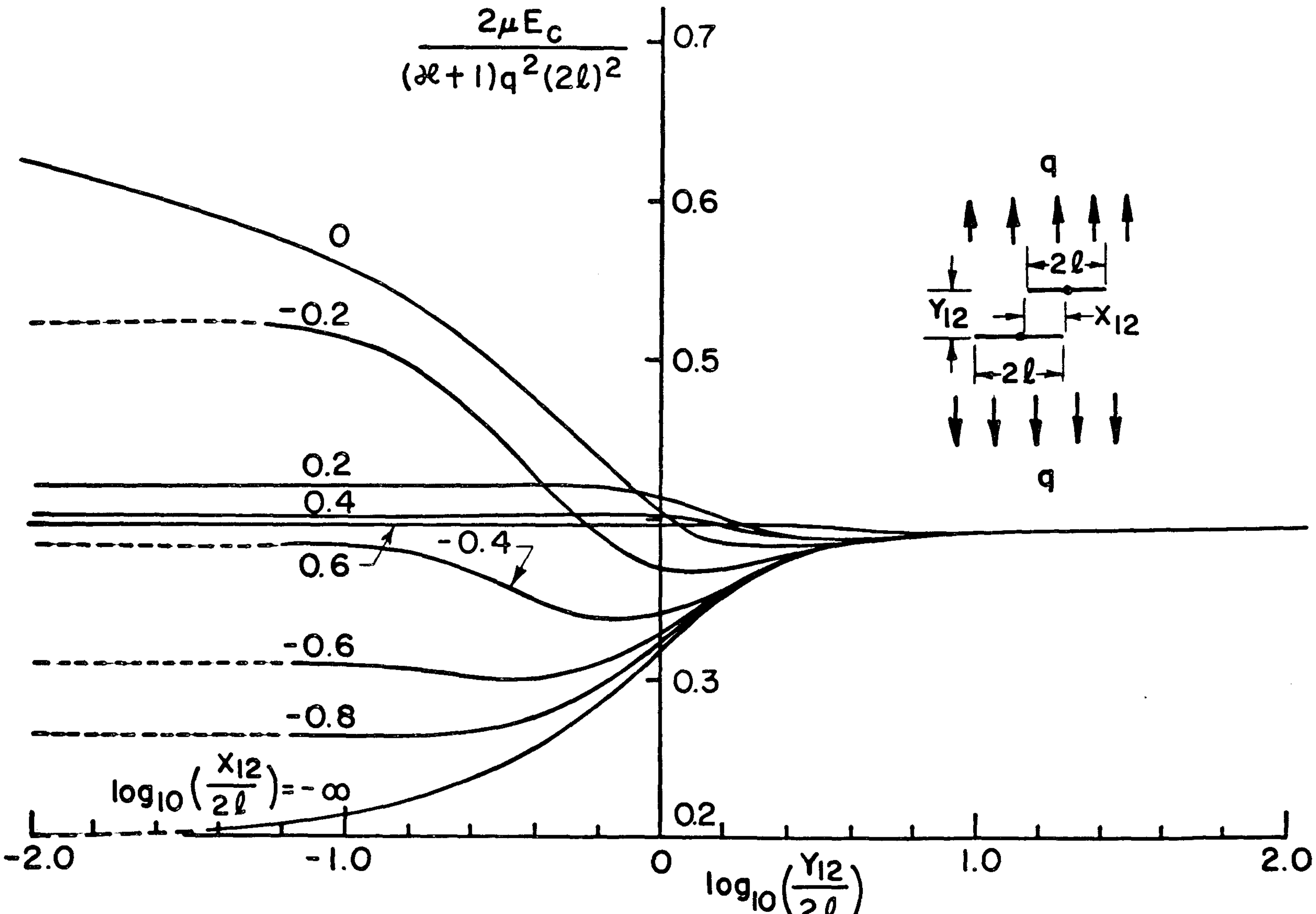


FIG. 8 CRACK ENERGY VS SEPARATION

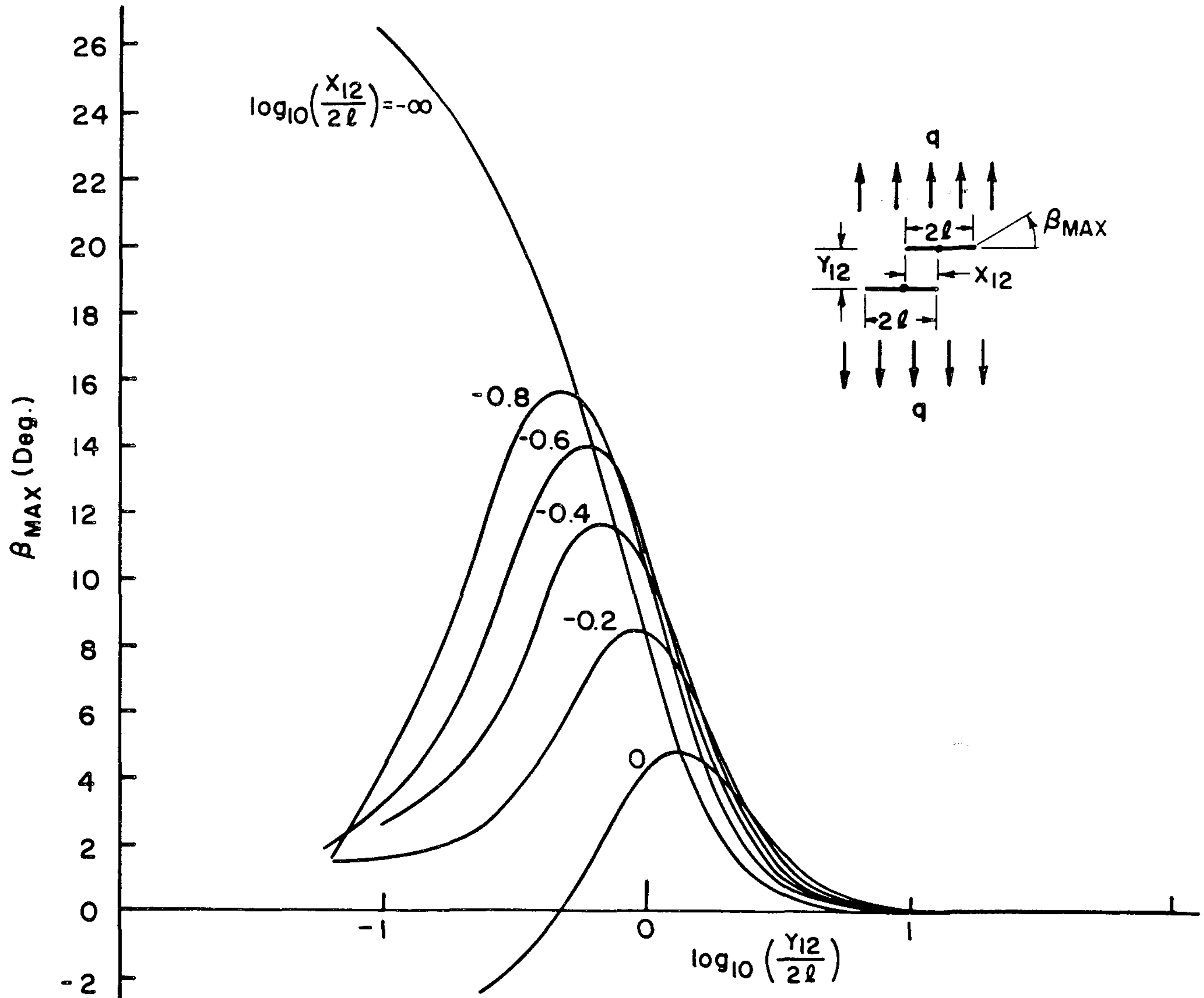


FIG.9 CLEAVAGE ANGLE, β_{MAX} VS. SEPARATION

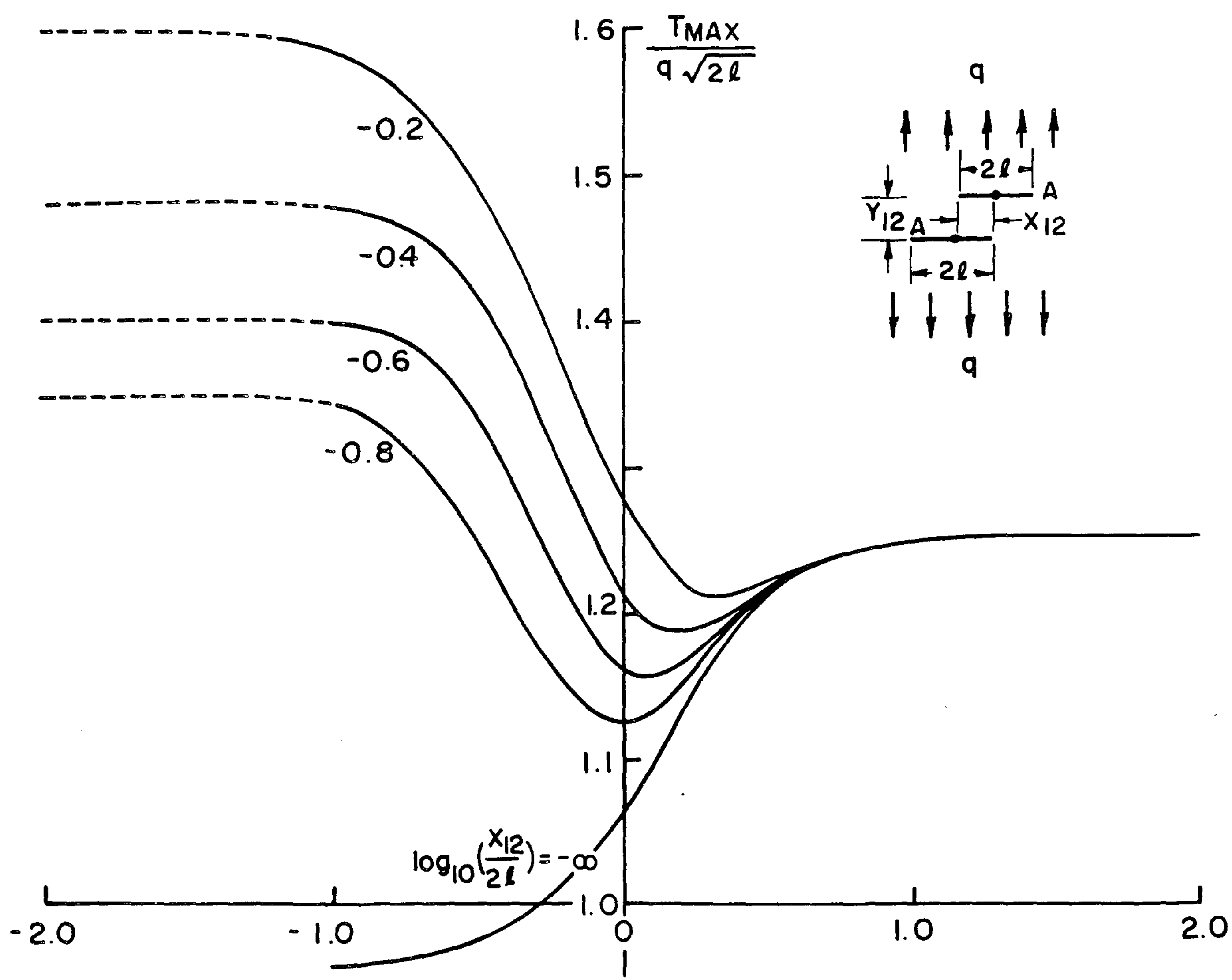


FIG.10 CLEAVAGE INTENSITY FACTOR (T_{MAX}) AT A VS. SEPARATION

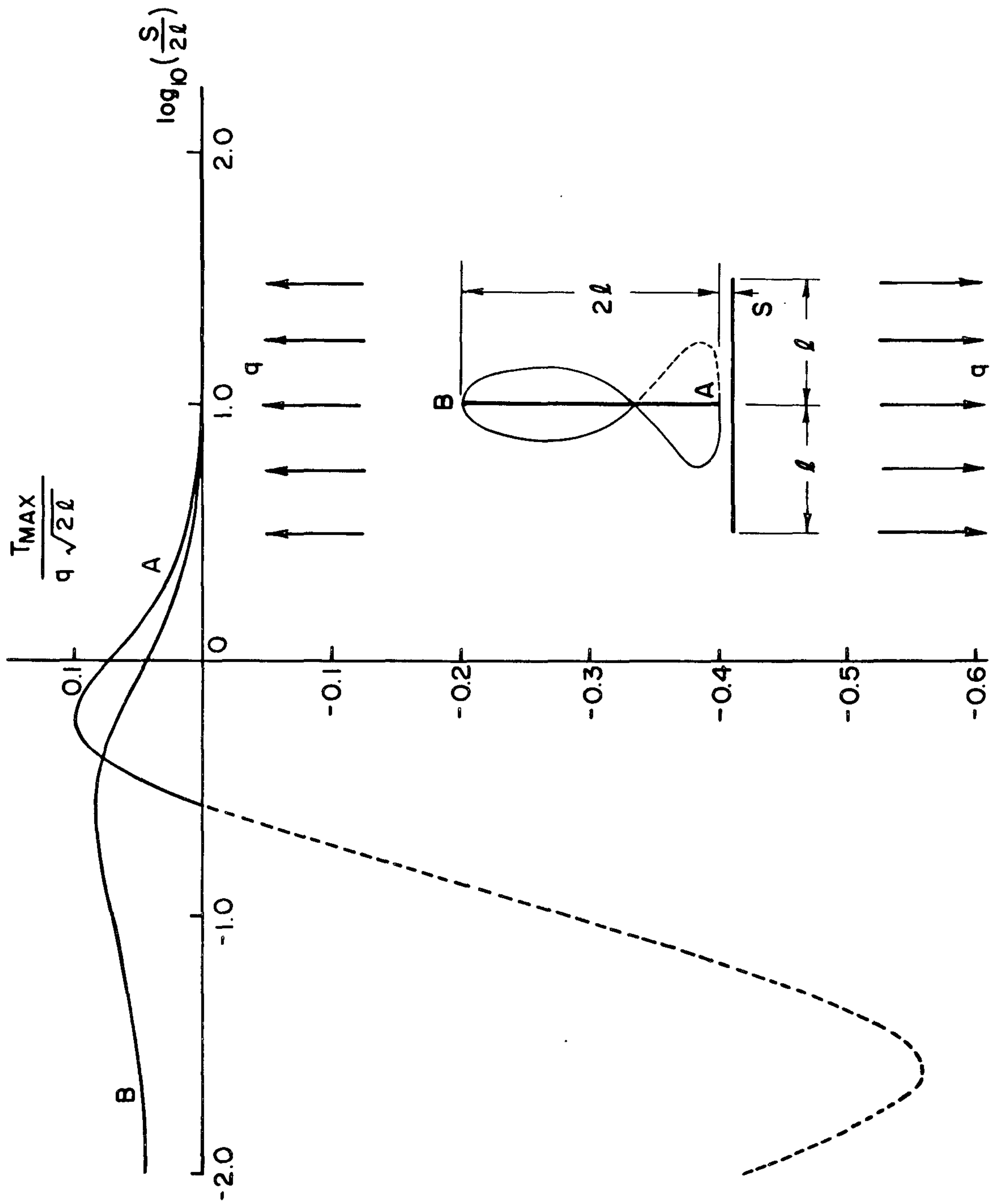


FIG. 11 CLEAVAGE INTENSITY FACTOR (T) VS. SEPARATION (S)