OPTIMIZATION OF ARCH AND SHELL STRUCTURES

Thesis by

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To Séverine, my daughter.

May she always be a "best design."

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ABSTRACT

Structural optimization of structures with respect to their shape and thickness distribution is studied using a variational approach. The behavioral constraint is either the state of stress or the stiffness.

The boundary value problems, derived using Optimal Control theory, are solved with the parallel shooting technique. For statically determinant arches subjected to a uniform pressure, the contribution of the shear force is included in the behavioral constraints to prevent the problems from being singular. For the case of membrane shells of revolution supporting a combined pressure and end traction loading case, solutions were obtained up to a critical value of the load coefficient. A physical interpretation of the singularity is obtained by including the possibility of discrete rings in the formulation. A set of optimality conditions for shells of revolution described by the bending theory, satisfying a stiffness constraint, is derived. The problem is found to be ill posed when the shear contribution is not included in the structural operator.

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I. INTRODUCTION

The purpose of structural optimization is to provide systematic ways of obtaining "better" structures. In economically oriented societies, one would like to minimize the cost likely to be associated with the structure during its lifetime. Considering the price of the material and the indirect cost associated with the weight of a structure, as for example the fuel needed for the transportation of aerospace structures, the minimization of the material volume can often be chosen as the primary objective. Furthermore the traditional design procedure of trial and error relies heavily on previous experience in order to maintain the number of design and analysis cycles as small as possible. In areas for which previous knowledge is not readily available the structural optimization techniques provide the engineer with guidelines for the conception of the structures and with a reference to evaluate the merits of the final design.

Since the early 60's the field of structural optimization underwent a rapid development. A review article published in 1963 by Wasiuntynski [1] describes the history of the field, starting with Galileo. Barnett [2] in 1966 presented a survey of the important design techniques and principles together with a number of significant results. The review by Sheu and Prager [3] covers the progress made between 1963 and 1968. An extensive reference list of the recent areas of research is contained in the article by Niordson and Pedersen [4].

Two reasons can be found for the extensive work for the past 15 years in the field of structural optimization. The space programs provided the designers with new challenges in which the structural weight was much more critical than the construction cost. Also the development of large computers made it feasible to automate the design process. It was felt that the optimization procedures would provide means of finding the best possible structures in those domains for which previous experience was limited. But the day where "black boxes" will design structural systems is still to come.

A structural optimization problem consists of finding the "best" possible element in a given class of admissible designs. One or several behavorial constraints characterize, in part, the feasibility of a given design. Prager and Taylor [5] presented a uniform method of treating problems of optimal design of sandwich structures subjected to constraint on the stiffness, on the fundamental frequency or on the buckling load. Huang [6] treated the problem of a solid circular plate supporting a uniform pressure with a stiffness constraint. Niordson [7] analyzed the optimal elastic design of solid beams with a prescribed natural frequency. Cases where the buckling load is imposed have been treated by Budiansky and Frauenthal [8] for arches, by Frauenthal [9] for circular plates, and by Taylor [10] for sandwich columns. A general formulation of the mass minimization of structures subjected to stress constraints has been given by Giraudbit [11]. A review and an assessment of the state of the art in optimal aeroelastic design has been presented by Stroud [12].

Having discussed some of the behavioral constraints entering in the formulation of a structural optimization problem, let us review the methods of solution. An analytical formulation, using the Calculus of Variations to obtain the optimality conditions can be made when unknown design functions are chosen. Optimization of simple structural elements only can be treated in this fashion due to the highly non-linear character of the resulting differential equations. For large structural systems, an a priori discretization is performed. The unknown design variables are reduced into a set of parameters. They can be determined directly using the mathematical programming techniques of Ref. [13], or indirectly by means of optimality criteria, cf. Ref. [14]. An increase of the efficiency of the mathematical programming techniques can be achieved by using approximative concepts, cf. Ref. [15], or by utilizing the geometric programming methods, cf. Ref. [16]. To size complex structures a combination of the mathematical programming and the optimality criteria approach is used in Ref. [17]. An important difference exists between the analytical and the "discretized" formulations. The variational approach is concerned with finding the optimal solution as the mathematical programming techniques improve feasible designs. When the optimal solution does not exist, a solution technique based on the second approach, may yield a "better" design even when there is no "best" design. Investigations using the analytical method are therefore necessary to determine which classes of problems are, or are not, well posed. The aim of this thesis is to investigate optimization problems in which two design functions are unknown. One of our objectives is to determine formulations of the investigated cases, leading to well-posed problems. The behavioral constraint is either the state of stress or the stiffness of the structure. A definition of the problems is given in Chapter II. The Optimal Control Theory and the Shooting Techniques, described in Chapter III, are used respectively to derive the optimality conditions and to solve the resulting two points boundary value problems. Statically determinant arches of unknown middle line shape, unknown thickness distribution and unknown slope of the support are investigated in Chapter IV. The study of membrane shells of unknown geometry, under a combined edge and pressure loading case, is contained in Chapter V. In Chapter VI the problem of determining the axially symmetric shells of revolution, including the bending effects, is formulated. In evaluating the following work, one has to remember the assessment made on the analytical approach by Niordson and Pedersen [4]:

"Due to the fact that the differential equations are often highly non-linear and have no regular solution (singularities very often appear at the boundaries) such problems may be rather cumbersome and tricky to solve and the whole field seems rather unclarified."

II. DEFINITION OF THE PROBLEMS

The problem of determining structures of minimum material volume can be defined as:

> "In a given class of admissible structures, find the one of minimum material volume."

It is to be understood that only the definition of the class of admissible structures will determine the optimal design. The minimization techniques are only means of finding the solution.

The definition of the class of admissible structures should specify:

a) the type of structures and the approximative theory used to derive their governing equations

b) the geometrical constraints

c) the loading case

d) the imposed boundary conditions

e) the design requirements which constrain the design variables.

The design variables constitute a set of parameters and of functions with respect to which the minimization procedure is to be performed.

In the present analysis, two types of design requirements are taken into consideration. When the state of stress in the structure is constrained to be admissible, the problem will be referred to as "stress case." The work done by the prescribed external forces on the structure is constrained for the "stiffness case."

A general discussion of those two problems is included in this chapter, for the purpose of defining the minimization tools needed to find their solutions.

2-1. Definitions

Let us consider only structures which can be described using one coordinate x only. Let $\underset{\sim}{s}(x)$ be the set of design parameters. Let $\underset{\sim}{u}(x)$ and $\underset{\sim}{\sigma}(x)$ represent respectively the field of displacement and the field of stress in the structure. Let $\underset{\sim}{F}(x)$ and $\underset{\sim}{p}(x)$ be respectively the set of concentrated forces and the pressure applied on the structure.

The governing equations of the structure provide the following two operators:

a) the equilibrium equations

$$L_1(\underset{\sim}{s}, \underset{\sim}{u}, \underset{\sim}{F}, \underset{\sim}{p}) = 0 \qquad (2.1)$$

b) the stress displacement relations

$$L_2(\underset{\sim}{s}, \underset{\sim}{u}, \underset{\sim}{\sigma}) = 0 \qquad (2.2)$$

The boundary conditions are included in the operator L_1. Let us remark that, using approximations of the linear theory of elasticity, the operators L_1 and L_2 are linear with respect to $\underset{\sim}{u}$, $\underset{\sim}{\sigma}$, $\underset{\sim}{F}$ and $\underset{\sim}{p}$, but in general they are non-linear with respect to $\underset{\sim}{s}$.

The material volume of the structure can be written as:

$$V = \int v(\underset{\sim}{s})\, dx \qquad (2.3)$$

where the integral is to be evaluated on the entire structure.

2-2. Stress Case

Let us assume the existence of a function $f(\underset{\sim}{\sigma})$ which describes the state of stress at each point of the structure. An admissible state of stress is defined as being such that:

$$f(\underset{\sim}{\sigma}) \leq 0 \tag{2.4}$$

The problem of minimizing the material volume of a structure can be formulated as:

$$\min_{\underset{\sim}{s}} \; V = \int v(\underset{\sim}{s}) \, ds \tag{2.5}$$

subjected to:

$$L_1(\underset{\sim}{s}, \underset{\sim}{u}, \underset{\sim}{F}, \underset{\sim}{p}) = 0 \tag{2.6}$$

$$L_2(\underset{\sim}{s}, \underset{\sim}{u}, \underset{\sim}{\sigma}) = 0 \tag{2.7}$$

$$f(\underset{\sim}{\sigma}) \leq 0 \tag{2.8}$$

To solve the present problem, optimality conditions which include inequality constraints are needed.

2-3. Stiffness Case

Rather than requiring the state of stress in the structure to be admissible, a limit on the displacements occurring during the deformation of the structure can be imposed. This can be achieved by requiring the work done by the external forces, during the deformation process, to be equal to a given amount. In the previous formulation, the average displacement of the structure, using the applied loads as weighting functions, is prescribed. Let $U(\underset{\sim}{s}, \underset{\sim}{F}, \underset{\sim}{p})$

be the work done by the external forces. Its value is given by

$$U(\underset{\sim}{s}, \underset{\sim}{F}, \underset{\sim}{p}) = \frac{1}{2} \{\Sigma \underset{\sim}{F} \cdot \underset{\sim}{u} + \int \underset{\sim}{p} \cdot \underset{\sim}{u} \, dx\} \tag{2.9}$$

where the summation is to be made for all the concentrated external forces, and the integral is to be evaluated over the entire structure. The field of displacement $\underset{\sim}{u}$ is obtained from the equilibrium operator L_1.

As it has been shown by Wasiutynski [18], the problem of minimizing the material volume of the structure subjected to a stiffness constraint is equivalent to the problem of minimizing the work done by the external forces in which the material volume is imposed as a constraint. For both problems the equilibrium equations have to be enforced.

The problem of minimizing the material volume of a structure can therefore be formulated in an alternate way as:

$$\underset{\underset{\sim}{s}}{\text{Min}} \, U = \frac{1}{2} \{\Sigma \underset{\sim}{F} \cdot \underset{\sim}{u} + \int \underset{\sim}{p} \cdot \underset{\sim}{u} \, dx\} \tag{2.10}$$

subjected to:

$$L_1(\underset{\sim}{s}, \underset{\sim}{u}, \underset{\sim}{F}, \underset{\sim}{p}) = 0 \tag{2.11}$$

$$\int v(\underset{\sim}{s}) \, dx = V_0 \tag{2.12}$$

To solve the present problem, optimality conditions which include integral constraints are needed.

It is to be noted that, in the context of linear elasticity theory, or of its approximations, the work done by the external forces is equal to the strain energy of the structure in its equilibrium configuration. This provides an alternate formulation found to be convenient for statically determinant structures. In this particular case the computation of the strain energy density which requires the evaluation of the stress field $\underset{\sim}{\sigma}$ can be made without considering the field of displacement $\underset{\sim}{u}$. The equilibrium equations for statically determinant structures can be expressed as:

$$L_3(\underset{\sim}{s}, \underset{\sim}{\sigma}, \underset{\sim}{F}, \underset{\sim}{p}) = 0 \tag{2.13}$$

The problem of minimizing the material volume of a statically determinant structure can therefore be formulated in an alternate way as:

$$\text{Min } J = \int W(\underset{\sim}{\sigma})\, dx \tag{2.14}$$

subjected to:

$$L_3(\underset{\sim}{s}, \underset{\sim}{\sigma}, \underset{\sim}{F}, \underset{\sim}{p}) = 0 \tag{2.15}$$

$$\int v(\underset{\sim}{s})\, dx = V_0 \tag{2.16}$$

2-4. Summary

We have shown that the problem of minimizing the material volume of a structure will involve inequality constraints for the stress case and integral constraints for the stiffness case. The method of deriving the optimality conditions and the numerical techniques used to find their solutions are described in the following chapter.

III. OPTIMALITY CONDITIONS AND METHOD OF SOLUTION

When the design variables consist of unknown functions, rather than a set of parameters, the optimality conditions are obtained by means of the Calculus of Variations. Due to the highly non-linear character of the resulting system of differential equations, the solution is generally obtained by numerical integration.

The Optimal Control Theory is used to derive the necessary first order optimality conditions. This formalism was preferred to the classical Calculus of Variations since it provides directly a system of first order differential equations, well suited for numerical integration. The resulting Two Point Boundary Value Problems are solved using the parallel shooting technique. A general purpose computer code which can solve a one parameter family of n-point boundary value problems was developed.

In order to define the terminology used in the sequel, the first order optimality conditions are recalled. A detailed derivation of them can be found in Ref. [19]. The parallel shooting technique and the computer code capabilities are described in the second part of this chapter. A general analysis of the numerical methods available for solving two points boundary value problems is given in Ref. [20].

3-1. Optimality Conditions

The unconstrained minimization problem is first defined and the corresponding set of first order conditions is derived. Optimality conditions for different variations of the simplest problem are then obtained.

3-1-1. Unconstrained Minimization

Let the governing equations of a physical system be:

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}(x), \underset{\sim}{u}(x)) \tag{3. 1}$$

where:

$\underset{\sim}{y}$ is the state variables vector with m components

$\underset{\sim}{u}$ is the control variables vector with n components.

The equations (3. 1) are called the state equations of the system.

Let us consider the following problem:

$$\underset{\underset{\sim}{u}(x)}{\text{Min}} \quad J = \varphi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) + \int_{x_0}^{x_1} L(x, \underset{\sim}{y}(x), \underset{\sim}{u}(x))\, dx \tag{3. 2}$$

subjected to

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}(x), \underset{\sim}{u}(x)) \tag{3. 3}$$

$\underset{\sim}{y}(x_0)$ prescribed (a)

(3. 4)

$\underset{\sim}{y}(x_1)$ not prescribed (b)

The objective function is J and L is the Lagrangian of the problem.

The first order necessary conditions for J to be a minimum are obtained by requiring J to be stationary with respect to arbitrary variations of the control variables $\underset{\sim}{u}(x)$. Let us define

$$\tilde{J} = J + \int_{x_0}^{x_1} \underset{\sim}{\lambda}^T \cdot [\underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{u}) - \frac{d\underset{\sim}{y}}{dx}]\ dx \tag{3.5}$$

$$H = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} \tag{3.6}$$

where: a) $\underset{\sim}{\lambda}^T$ is a vector of m Lagrange multipliers which will be chosen such that the coefficient of $\delta \underset{\sim}{y}(x)$ in $\delta\tilde{J}$ vanishes.

b) H is the Hamiltonian of the system.

The variations $\delta\tilde{J}$ of $\tilde{J}$ are:

$$\delta\tilde{J} = \left[\frac{\partial\varphi}{\partial \underset{\sim}{y}(x_0)} + \underset{\sim}{\lambda}^T\right]_{x=x_0} \delta\underset{\sim}{y}(x_0) + \left[\frac{\partial\varphi}{\partial \underset{\sim}{y}(x_1)} - \underset{\sim}{\lambda}^T\right]_{x=x_1} \delta\underset{\sim}{y}(x_1)$$

$$+ \int_{x_0}^{x_1} \left[\left\{L_{\underset{\sim}{u}} + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f}_{\underset{\sim}{u}}\right\} \delta\underset{\sim}{u}(x) + \left\{L_{\underset{\sim}{y}} + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f}_{\underset{\sim}{y}} + \frac{d\underset{\sim}{\lambda}^T}{dx}\right\} \delta y(x)\ dx\right] \tag{3.7}$$

where $L_{\underset{\sim}{u}} \equiv \dfrac{\partial L(x, \underset{\sim}{y}, \underset{\sim}{u})}{\partial \underset{\sim}{u}}$

Choosing the Lagrange's multipliers $\underset{\sim}{\lambda}^T$ such that

$$\frac{d\underset{\sim}{\lambda}^T}{dx} = -[L_{\underset{\sim}{y}} + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f}_{\underset{\sim}{y}}] \tag{3.8}$$

the variation $\delta\tilde{J}$ will vanish if:

$$L_{\underset{\sim}{u}} + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f}_{\underset{\sim}{u}} = 0 \tag{3.9}$$

$$\underset{\sim}{\lambda}^T(x_1) = \frac{\partial\varphi}{\partial\underset{\sim}{y}(x_1)} \tag{3.10}$$

The Lagrange's multipliers are not specified at $x = x_0$ since $\underset{\sim}{y}(x_0)$ is specified. It is to be noted that the previous conditions are only sufficient conditions for $\tilde{J}$ to be stationary because we assumed that the defined Lagrange's multipliers do exist.

Let us summarize the previous results:

a) Problem

$$\underset{\underset{\sim}{u}(x)}{\text{Min}}\ J = \varphi(y(x_0), y(x_1)) + \int_{x_0}^{x_1} L(x, \underset{\sim}{y}(x), \underset{\sim}{u}(x))dx \qquad (3.11)$$

subjected to

$$\frac{d\underset{\sim}{y}}{dx} = f(x, \underset{\sim}{y}, \underset{\sim}{u}) \qquad (3.12)$$

$\underset{\sim}{y}(x_0)$ specified (3.13)

b) First order optimality conditions are

$$H \equiv L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} \qquad (3.14)$$

$$H_{\underset{\sim}{u}} = 0 \qquad (3.15)$$

$$\frac{d\underset{\sim}{\lambda}^T}{dx} = - H_{\underset{\sim}{y}} \qquad (3.16)$$

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{u}) \qquad (3.17)$$

$\underset{\sim}{y}(x_0)$ prescribed (3.18a)

$$\underset{\sim}{\lambda}^T(x_1) = \frac{\partial\varphi}{\partial \underset{\sim}{y}(x_1)} \qquad (3.18b)$$

The equations (3.15) define a set of n algebraic equations to compute the control variables $\underset{\sim}{u}$. The equations (3.16), (3.17) along with the boundary conditions (3.18) define a two point boundary value problem.

Due to their highly non-linear character, it is often not possible to solve directly the equations (3.15). However they can be used to generate a system of differential equations for the control variables $\underset{\sim}{u}$. Since (3. 15) is satisfied for all x in $[x_0, x_1]$ then

$$\frac{dH_{\underset{\sim}{u}}}{dx} = 0 \qquad (3.19)$$

$$H_{\underset{\sim}{u}}\Big|_{x=x^*} = 0, \quad x^* \in [x_0, x_1] \qquad (3.20)$$

The equations (3. 19) reduce to:

$$H_{\underset{\sim}{u}\underset{\sim}{u}} \frac{d\underset{\sim}{u}}{dx} = - [H_{\underset{\sim}{u}x} + H_{\underset{\sim}{u}\underset{\sim}{y}}\underset{\sim}{f} - H_{\underset{\sim}{u}\underset{\sim}{\lambda}} H_{\underset{\sim}{y}}] \qquad (3.21)$$

The equations (3. 21) determine $d\underset{\sim}{u}/dx$ uniquely when:

$H_{\underset{\sim}{u}\underset{\sim}{u}}$ is not singular, i. e. :

$$\det [H_{\underset{\sim}{u}\underset{\sim}{u}}] \neq 0 \qquad (3.22)$$

The set II of optimality conditions for the problem under considera-tion, provided (3. 22) holds, can be written as:

$$\frac{d\underset{\sim}{u}}{dx} = - H^{-1}_{\underset{\sim}{u}\underset{\sim}{u}} [H_{\underset{\sim}{u}x} + H_{\underset{\sim}{u}\underset{\sim}{y}}\underset{\sim}{f} - H_{\underset{\sim}{u}\underset{\sim}{\lambda}} H_{\underset{\sim}{y}}] \qquad \text{(a)}$$

$$\frac{d\underset{\sim}{\lambda}^T}{dx} = - H_{\underset{\sim}{y}} \qquad \text{(b)} \qquad (3.23)$$

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{u}) \qquad \text{(c)}$$

$$H_{\underset{\sim}{u}}\Big|_{x=x^*} = 0 \qquad \text{(a)}$$

$$\underset{\sim}{y}(x_0) \text{ specified} \qquad \text{(b)} \qquad (3.24)$$

$$\underset{\sim}{\lambda}^T(x_1) = \frac{\partial \varphi}{\partial \underset{\sim}{y}(x_1)} \qquad \text{(c)}$$

The equations (3.23) define a system of differential equations for which the boundary conditions (3.24) are specified at different values of x.

3-1-2. Problem Depending on an Unknown Coefficient

Problem:

$$\min_{\underset{\sim}{u}(x),\, a} \quad J = \int_{x_2}^{x_1} L(x, \underset{\sim}{y}(x), \underset{\sim}{u}(x), a)\, dx \qquad (3.25)$$

$$\text{subjected to:} \quad \frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{u}, a) \qquad (3.26)$$

The previous results can be applied by considering the extended state vector $\bar{y}$:

$$\bar{\underset{\sim}{y}} = \begin{bmatrix} \underset{\sim}{y} \\ \\ a \end{bmatrix} \qquad (3.27)$$

The state equation associated with y_{m+1} is:

$$\frac{dy_{m+1}}{dx} = 0 \qquad (3.28)$$

Since the value of $y_{m+1}(x_0)$ is not specified, the admissible $\delta y_{m+1}(x_0)$ are not zero. The optimality conditions are given by (3.23) and (3.24) where the Hamiltonian H is given by:

$$H = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} + \lambda_{m+1} \times (0) \qquad (3.29)$$

The boundary conditions on λ_{m+1} are:

$$\lambda_{m+1}(x_0) = \lambda_{m+1}(x_1) = 0 \qquad (3.30)$$

3-1-3. Some State Variables Prescribed at $x = x_1$

Problem:

$$\underset{u(x)}{\text{Min}} \quad J = \varphi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) + \int_{x_2}^{x_1} L(x, \underset{\sim}{y}, \underset{\sim}{u})\, dx \tag{3.31}$$

subjected to:

$$\frac{d\underset{\sim}{y}}{dx} = f(x, \underset{\sim}{y}, \underset{\sim}{u})$$

$y_i(x_0)$ prescribed , $i \in I$

$y_j(x_1)$ prescribed , $j \in J$

The optimality conditions (3.15) to (3.17) hold, provided the system is controllable. The boundary conditions are in this case:

$$y_i\,(x_0) \text{ prescribed} \qquad i \in I \tag{3.32}$$

$$\lambda_k(x_0) = -\frac{\partial \varphi}{\partial y_k(x_0)} \qquad k \notin I \tag{3.33}$$

$$y_j(x_1) \text{ prescribed} \qquad j \in J \tag{3.34}$$

$$\lambda_\ell(x_1) = \frac{\partial \varphi}{\partial y_\ell(x_1)} \qquad \ell \notin J \tag{3.35}$$

Note: If the state variables must also satisfy a constraint $\psi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) = 0$, the boundary conditions can be obtained by adjoining ψ to φ with a Lagrange's multiplier ν

$$\tilde{\varphi} = \varphi + \nu\psi \tag{3.36}$$

The boundary conditions become

$$y_i(x_0) \text{ prescribed} \qquad i \in I \tag{3.37}$$

$$\lambda_k(x_0) = -\left[\frac{\partial \varphi}{\partial y_k(x_0)} + \nu \frac{\partial \psi}{\partial y_k(x_0)}\right] \quad k \notin I \tag{3.38}$$

$$y_j(x_1) \text{ prescribed} \qquad j \in J \tag{3.39}$$

$$\lambda_\ell(x_1) = \left[\frac{\partial \varphi}{\partial y_\ell(x_1)} + \nu \frac{\partial \psi}{\partial y_\ell(x_1)}\right] \quad \ell \notin J \tag{3.40}$$

$$\psi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) = 0 \tag{3.41}$$

3-1-4. Problem with an Integral Constraint

Problem:

$$\underset{\underset{\sim}{u}(x)}{\text{Min}} \qquad J = \varphi(\underset{\sim}{y}(x_0), \underset{\sim}{y}(x_1)) + \int_{x_0}^{x_1} L(x, \underset{\sim}{y}, \underset{\sim}{u})\, dx \tag{3.42}$$

subjected to:

$$\frac{d\underset{\sim}{y}}{dx} = f(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.43}$$

$$\int_{x_0}^{x_1} g(x, \underset{\sim}{y}, \underset{\sim}{u})\, dx = c \tag{3.44}$$

The previous result can be applied by considering the extended state vector

$$\bar{\underset{\sim}{y}} = \begin{bmatrix} \underset{\sim}{y} \\ \\ y_{m+1} \end{bmatrix} \tag{3.45}$$

The state equation associated with y_{m+1} is:

$$\frac{dy_{m+1}}{dx} = g(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.46}$$

The boundary conditions on y_{m+1} are:

$$y_{m+1}(x_0) = 0, \quad y_{m+1}(x_1) = c \tag{3.47}$$

The Hamiltonian of the problem becomes

$$H = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} + \lambda_{m+1}\, g(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.48}$$

Since the state variable y_{m+1} does not appear in the Lagrangian L and the state equations (3.43) the governing equation for λ_{m+1} is:

$$\frac{d\lambda_{m+1}}{dx} = 0 \tag{3.49}$$

The boundary conditions on λ_{m+1} are unspecified since the values of y_{m+1} are given. The Lagrange's multiplier λ_{m+1} appears as an unknown constant. Its value is to be determined from the satisfaction of the integral constraint (3.44).

3-1-5. Problem with Inequality Constraint

Problem

$$\underset{\underset{\sim}{u}(x)}{\text{Min}} \quad J = \int_{x_0}^{x_1} L(x, \underset{\sim}{y}, \underset{\sim}{u})\, dx \tag{3.50}$$

subjected to:

$$\frac{d\underset{\sim}{y}}{dx} = f(x, \underset{\sim}{y}, \underset{\sim}{u}) \tag{3.51}$$

$$C(x, \underset{\sim}{y}, \underset{\sim}{u}) \leqslant 0 \tag{3.52}$$

Let H^* be defined as:

$$H^* = L + \lambda^T \cdot f \tag{3.53}$$

The variations of $\tilde{J}$ given by (3.7) are:

$$\delta \tilde{J} = \int_{x_0}^{x_1} H^*_u \cdot \delta u(x)\, dx \tag{3.54}$$

The constraint (3.52) is said to be "effective" on an arc of the solution if:

$$C(x, y, u) = 0 \tag{3.55}$$

When the strict inequality is satisfied the constraint is said to be "not effective."

In order for $\tilde{J}$ to be a minimum, we must have $\delta \tilde{J} \geq 0$ for all admissible variations of the control variable.

When the constraint is effective, the admissible δu are such that:

$$C_u \cdot \delta u \leq 0 , \quad \int_{x_0}^{x_1} H^*_u \cdot \delta u \, dx \geq 0 \tag{3.56}$$

A sufficient condition of optimality is obtained by introducing a non-negative Lagrange multiplier η such that

$$H^*_u + \eta C_u = 0 \tag{3.57}$$

When the constraint is not effective the class of admissible δu is not restricted by the constraint. The optimality conditions (3.15) hold.

A set of optimality conditions analogous to the equations (3. 15) and (3. 16) can be obtained by a transformation of the inequality constraint (3. 52) into an equality constraint using an additional control variable μ such that:

$$C(x, \underset{\sim}{y}, \underset{\sim}{u}) + \mu^2 = 0 \tag{3.58}$$

The Hamiltonian H is obtained by adjoining to (3. 53) the equality constraint (3. 57) using a non-negative Lagrange's multiplier η

$$H = L + \underset{\sim}{\lambda}^T \cdot \underset{\sim}{f} + \eta \left[C(x, \underset{\sim}{y}, \underset{\sim}{u}) + \mu^2\right] \tag{3.59}$$

3-1-6. Summary

The first order optimality conditions recalled in this section are only sufficient conditions. However, the first order conditions are nothing but necessary conditions for a solution to be optimal. To determine whether or not a solution corresponds to a local minimum a second order sufficiency condition, the convexity condition, will be used. In addition a restricted perturbation of the solutions will be performed in some cases.

3-2. Shooting Technique

The shooting technique is a general method to solve systems of first order ordinary differential equations for which the boundary conditions are specified at several points. The technique is first illustrated for the simplest case of a two points boundary value problem. The capabilities of the general shooting program are then described.

3-2-1. Method

Problem 1:

Find the solution of

$$\left.\begin{aligned} &\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}(x)) && \text{(n equations)} && \text{(a)} \\ &\text{with the boundary conditions} \\ &y_i(x_0) = a_i && i \in I \quad \text{(p conditions)} && \text{(b)} \\ &y_j(x_1) = b_j && j \in J \quad \text{(n-p conditions)} && \text{(c)} \end{aligned}\right\} \quad (3.60)$$

The present problem is not an initial value problem since some of the initial conditions are not specified, i.e.,

$$y_k(x_0) \text{ unspecified} \quad k \notin I \quad \text{(n-p values)} \qquad (3.61)$$

However the solution of problem 1 can be obtained from the following initial value problem.

Problem 2

Find the solution of

$$\left.\begin{aligned} &\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}(x)) && \text{(a)} \\ &\text{with the initial values} \\ &y_i(x_0) = a_i \quad i \in I && \text{(b)} \\ &y_k(x_0) = s_k \quad k \notin I && \text{(c)} \end{aligned}\right\} \quad (3.62)$$

where the initial values s_k are such that

$$y_j(x_1) = b_j \qquad j \in J \tag{3.63}$$

Let $\underset{\sim}{u}(x, s_k)$ be the solution of problem 2. It will be a solution of problem 1 iff

$$\Phi_j(s_k) \equiv u_j(x_1, s_k) - b_j = 0 \qquad j \in J, \quad k \notin I \tag{3.64}$$

The function Φ of the unknown initial values s_k is called the "mismatch" function. Problem 1 is reduced into an initial value problem, where the initial values s_k must be determined from the solution of the n-p algebraic equations (3.64).

The algebraic equations are solved with the Newton-Ralphson's method, which requires the solution of a sequence of initial value problems 2 where:

$$y_k(x_0) = t_k^{\nu} \qquad k \notin I \tag{3.65}$$

The approximation $t_k^{\nu+1}$ of the solution s_k is obtained from the previous iterate t_k^{ν} using:

$$\frac{\partial \Phi_j}{\partial s_k} (t_k^{\nu+1} - t_k^{\nu}) = -\Phi(t_k^{\nu}) \qquad j \in J, \ k \notin I \tag{3.66}$$

where:

$$\frac{\partial \Phi_j}{\partial s_k} = \frac{\partial y_j(x_1)}{\partial y_k(x_0)} \qquad j \in J, \quad k \notin I \tag{3.67}$$

The gradient of the final values $y_j(x_1)$ with respect to the unknown initial values $y_k(x_0)$ is obtained by integration of the variational system of (3.62a):

$$\left.\begin{aligned} &\frac{d}{dx}\ \frac{\partial y_j}{\partial y_k(x_0)} = \frac{\partial f_j}{\partial y_\ell}\ \frac{\partial y_\ell}{\partial y_k(x_0)} && \text{(a)} \\ &\text{with the initial values:} && \\ &\frac{\partial y_\ell}{\partial y_k(x_0)} = \begin{cases} 0 & \text{if} \quad k \neq \ell \\ 1 & \text{if} \quad k = \ell \end{cases} && \text{(b)} \end{aligned}\right\} \quad (3.68)$$

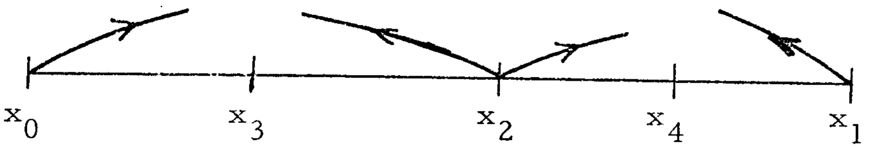
Therefore the evaluation of $\frac{\partial \Phi_j}{\partial s_k}$ requires the integration of $(n-p) \times n$ linear differential equations.

The solution of the two points boundary value problem 1 is obtained by solving a sequence of initial value problems 2. The total number of differential equations to be solved for each iteration is $n + n\,(n-p)$. The iterative process requires an initial guess t_k^o of the unknown initial values.

Remarks:

a) The choice of the initial guess of the unknown initial values should be made carefully, especially when the solution of problem 1 is not unique.

b) Due to the exponential character of the solutions of the variational system (3.68), and to reduce the numerical errors, the integration interval $[x_0, x_1]$ is divided into subintervals called shooting intervals. Initial values are guessed at each starting point of the shooting intervals. The solutions are matched at each final point of the shooting intervals. This variation of the previous method is called parallel shooting technique. The following diagram illustrates the method.



where: x_0, x_2, x_1 are the shooting points

x_3, x_4 are the matching points

$[x_0, x_3]$, $[x_2, x_3]$, $[x_2, x_4]$ and $[x_4, x_1]$ are the shooting intervals.

3-2-2. General Shooting Program

When using the shooting method, most of the programming effort is spent to construct the variational matrix (3.64) for each particular problem. To alleviate this practical problem, a general computer program was realized. It utilizes the parallel shooting technique to solve the following problem:

$$\frac{d\underset{\sim}{y}}{dx} = \underset{\sim}{f}(x, \underset{\sim}{y}, \underset{\sim}{a}, \gamma) \qquad \text{n equations} \qquad \text{(a)}$$

$$\frac{d\underset{\sim}{u}}{dx} = \underset{\sim}{g}(x, \underset{\sim}{y}, \underset{\sim}{u}, \underset{\sim}{a}, \gamma) \qquad \text{m equations} \qquad \text{(b)}$$

$$\int_{x_0}^{x_1} \underset{\sim}{h}(x, \underset{\sim}{y}, \underset{\sim}{a}, \gamma) = \underset{\sim}{a} \qquad \text{q integrals} \qquad \text{(c)}$$

with the boundary conditions:

$$y_{m_k}(x_k) \text{ prescribed} \qquad \text{(d)}$$

$$\left.\begin{array}{ll} y_{j_k}(x_k) \text{ not prescribed} & \text{(e)} \\ y_{n_\ell}(z_\ell) \text{ prescribed} & \text{(f)} \\ y_{i_\ell}(z_\ell) \text{ not prescribed} & \text{(g)} \\ N(y_{j_k}(x_k),\ y_{i_\ell}(z_\ell)) = 0 & \text{(h)} \end{array}\right\} \quad (3.69)$$

where:

x_k, $k = 1, \ldots K$ are the shooting points

z_ℓ, $\ell = 1, \ldots L$ are the matching points

$\underset{\sim}{a}$ is a vector of unknown coefficients

γ is a parameter with respect to which a one parameter family of problems is defined

$\underset{\sim}{u}$ a vector to be evaluated on the solution

$N(y_{j_k}(x_k),\ y_{i_\ell}(z_\ell)) = 0$ is a set of non-linear equations to be satisfied by the solutions.

The set of differential equations (3.69a), (3.69b), the integrals (3.69c), as well as their variational system, are specified by a user's written subroutine. The non-linear conditions (3.69c) are also defined by a subroutine. The boundary conditions and the characteristics of the different shooting and matching points are specified by data cards. The program was found to be very convenient since the number of shooting and matching points could be modified as needed with a minimum amount of work.

3-3. Summary

The optimization problems of the following sections will consist of these three steps:

a) definition of the problem

b) derivation of a set of differential equations with boundary values prescribed at several points

c) resolution of the resulting n-points boundary value problems using, in general, the shooting technique. It is to be noted again that only the first step, i. e., the definition of the problem, will determine the solutions.

IV. ARCH STRUCTURES

The problem of determining the optimal thickness distribution only of beams or arches of known geometry has been studied extensively in the past. Huang [21] treated the case of circular sandwich beams for the stiffness case. For sandwich sections, the bending and the extensional rigidities of the cross section are linear functions of the face sheet thickness. Giraudbit [11] investigated the case of a clamped circular arch with a solid cross section subjected to a stress constraint.

In this chapter the problem of determining the thickness distribution, the shape of the middle line and the slope of the support of statically determinant arches of minimum material volume, satisfying either a stress or a stiffness constraint, is treated. The applied load is a uniform pressure normal to the middle line of the structures. One of the objectives is to determine whether or not the best geometry is such that the bending moment vanishes everywhere for the optimal structure.

A formal problem definition is first given. The optimality problem for a stiffness constraint is then investigated. The shear force contribution is included in the strain energy definition. The same problem is then treated for the stress constraint case. The shear force contribution is also taken into account in the failure criterion.

4-1. Problem Definition

Let S be the set of arches such that:

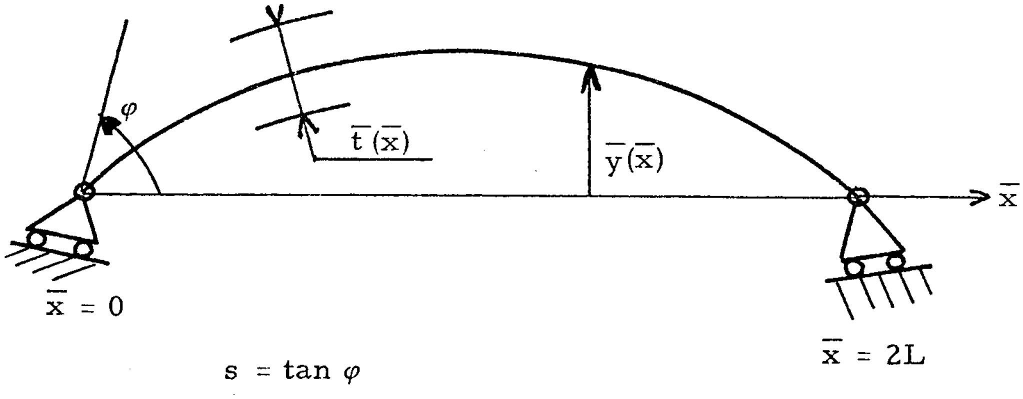
a) their governing equations include the effects of the shear stress.

b) they are subjected to a uniform pressure normal to their middle line and have simple support type of boundary conditions.

c) they satisfy either a given stress or a given stiffness constraint.

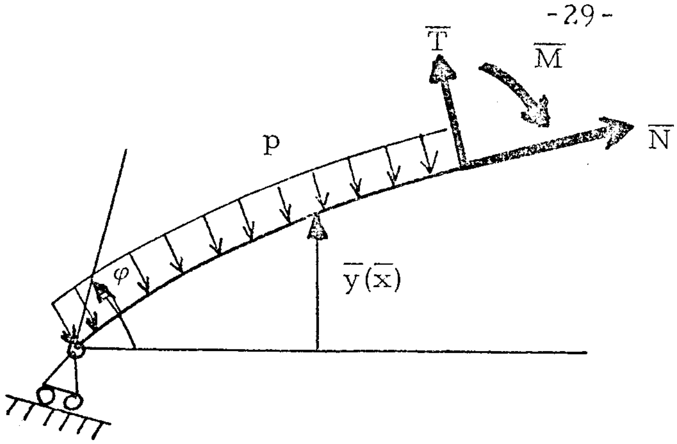
d) their length is 2L, their shape is such that $\bar{y}(o) = \bar{y}(2L) = 0$ and they have a constant unit width.

We seek the middle line shape $\bar{y}(\bar{x})$, the thickness distribution $\bar{t}(\bar{x})$ and the slope of the support s of the element in S of minimum material volume.



4-2. Equilibrium Equations

Since the arches are simply supported, the normal force $\bar{N}$, the bending moment $\bar{M}$ and the shear force $\bar{T}$ can be computed directly from the three equilibrium equations of a plane structure.



$s = \tan\varphi$

For symmetric arches with respect to $\bar{x} = L$:

$$\left.\begin{aligned}
\bar{N} &= \frac{pL}{\sqrt{1+\bar{y},_{\bar{x}}^{2}}}\left[\bar{y},_{\bar{x}}\left(\frac{\bar{x}}{L}-1\right)-\left(\frac{1}{s}+\frac{\bar{y}}{L}\right)\right] && \text{(a)}\\
\bar{T} &= \frac{pL}{\sqrt{1+\bar{y},_{\bar{x}}^{2}}}\left[\left(\frac{\bar{x}}{L}-1\right)+\bar{y},_{\bar{x}}\left(\frac{1}{s}+\frac{\bar{y}}{L}\right)\right] && \text{(b)}\\
\bar{M} &= pL^{2}\left[\tfrac{1}{2}\left(\frac{\bar{y}}{L}\right)^{2}+\frac{\bar{y}}{Ls}+\tfrac{1}{2}\left(\frac{\bar{x}}{L}\right)^{2}-\frac{\bar{x}}{L}\right] && \text{(c)}
\end{aligned}\right\} \qquad (4.1)$$

Note that the solution (4. 1) satisfies the simply supported boundary conditions since

$$\bar{M}\Big|_{\bar{x}=0} = \bar{M}\Big|_{\bar{x}=2L} = 0 \qquad (4.2)$$

Upon introduction of the following dimensionless variables:

$$\left.\begin{aligned}
&x = \frac{\bar{x}}{L},\quad z_1 = \frac{\bar{y}}{L},\quad z_2 = \bar{y},_{\bar{x}},\quad z_3 = s\\
&N = \frac{\bar{N}}{pL},\quad T = \frac{\bar{T}}{pL},\quad M = \frac{\bar{M}}{pL^{2}}
\end{aligned}\right\} \qquad (4.3)$$

the equilibrium equations (4. 1) reduce to:

$$\left.\begin{aligned} N &= \frac{1}{\sqrt{1+z_2^2}}\left[z_2(x-1) - \left(\frac{1}{z_3} + z_1\right)\right] && \text{(a)} \\ T &= \frac{1}{\sqrt{1+z_2^2}}\left[(x-1) + z_2\left(\frac{1}{z_3} + z_1\right)\right] && \text{(b)} \\ M &= \tfrac{1}{2} z_1^2 + \frac{z_1}{z_3} + \tfrac{1}{2} x^2 - x && \text{(c)} \end{aligned}\right\} \qquad (4.\ 4)$$

A shape on which $M = 0$ everywhere, hereafter referred to as "membrane design," is a circular arch defined by:

$$(z_1)_M = -\frac{1}{z_3} + \sqrt{\frac{1}{z_3^2} + 2\left(x - \frac{x^2}{2}\right)} \qquad (4.\ 5a)$$

when $\frac{1}{z_3} > 0$.

On such a design, the shear force T vanishes and the normal traction N is constant

$$N = -\sqrt{1 + \left(\frac{1}{z_3}\right)^2} \qquad (4.\ 5b)$$

4-3. Optimal Arches with a Stiffness Constraint

Since the structure is statically determinant the strain energy of the structure, in its equilibrium configuration, is used as the objective function $\bar{J}$.

$$\bar{J} = \tfrac{1}{2}\int_0^{2L}\left(\frac{\bar{N}^2+\beta^2\bar{T}^2}{EA} + \frac{\bar{M}^2}{EI}\right)\sqrt{1 + \bar{y}_{,\bar{x}}^2}\; d\bar{x} \qquad (4.\ 6)$$

where:

E ..... material Young's modulus

A = t..... cross section area

$I = \frac{\bar{t}^3}{12}$ ..... cross section moment of inertia

$\beta^2 = \frac{E}{G} k$ ... where G is the material shear modulus and k is a coefficient to take into account the real distribution of the shear stress in the cross section.

Note that the width of the structure does not enter in the previous expressions since it is assumed to be constant and unity.

The material volume V_0 is used as a constraint

$$V_0 = \int_0^{2L} A\sqrt{1 + \bar{y},_{\bar{x}}^2} \; d\bar{x} \tag{4.7}$$

Upon introduction of the previously defined dimensionless variables (4.3) and of:

$$t = \bar{t} \left(\frac{2L}{V_0}\right) \tag{4.8}$$

the dimensionless objective function J and the volume constraint reduce respectively to:

$$J = \frac{1}{2} \int_0^2 \left(\frac{N^2+\beta^2 T^2}{t} + \frac{12\alpha^2 M^2}{t^3}\right) \sqrt{1 + z_2^2} \; dx \tag{4.9}$$

$$\int_0^2 t \sqrt{1+z_2^2} \; dx = 2 \tag{4.10}$$

where: $\alpha = \frac{2L^2}{V_0}$ characterizes the length to the thickness ratio of a straight beam of uniform thickness satisfying the volume constraint:

$$\bar{J} = \frac{p^2 L^2}{E} \alpha J \tag{4.11}$$

4-3-1. Formulation

Because of the symmetry in the imposed boundary conditions the problem can be formulated as:

$$\text{Min } J = \int_0^1 \left(\frac{N^2+\beta^2 T^2}{t} + \frac{12M^2\alpha^2}{t^3} \right) \sqrt{1+z_2^2}\ dx \qquad (4.12)$$

subjected to:

i) the material volume constraint

$$\int_0^1 t\sqrt{1+z_2^2}\ dx = 1 \qquad (4.13)$$

ii) the state equations

$$\left.\begin{aligned} \frac{dz_1}{dx} &= z_2 \qquad &\text{(a)} \\ \frac{dz_3}{dx} &= 0 \qquad &\text{(b)} \end{aligned}\right\} \qquad (4.14)$$

iii) the equilibrium equations (4.4)

iv) the boundary conditions:

$$\left.\begin{aligned} &z_1(0) = 0 \qquad && z_1(1) \text{ unknown} \\ &z_3(0) \text{ unknown} \qquad && z_3(1) \text{ unknown} \end{aligned}\right\} \qquad (4.15)$$

Note: a) the unknown parameter z_3 is treated as a state variable bounded to be constant by (4.14b). b) the symmetry condition $z_2(1)=0$ cannot be inforced directly since z_2 is not a state variable. It will be a consequence of the boundary conditions since the symmetry information is contained in the equilibrium equations (4.4).

The Hamiltonian H of the system is obtained by adjoining to its Lagrangian the material volume constraint (4.13), the state equations (4. 14) by means of the Lagrange's multipliers λ_4, λ_1, and λ_3. The state variables are z_1 and z_3 and the control variables are z_2 and t.

$$H = \left(\frac{N^2+\beta^2T^2}{t} + \frac{12M^2\alpha^2}{t^3} + \lambda_4 t\right)\sqrt{1+z_2^2} + \lambda_1 z_2 + \lambda_3 \times(0) \qquad (4.16)$$

4-3-2. Optimality Conditions

The first order optimality conditions are:

$$H_t = \left(\lambda_4 - \frac{N^2+\beta^2T^2}{t^2} - \frac{36M^2\alpha^2}{t^4}\right)\sqrt{1+z_2^2} = 0 \qquad \text{(a)}$$

$$H_{z_2} = \left(\frac{12M^2\alpha^2}{t^3} + \frac{N^2+\beta^2T^2}{t} + \lambda_4 t\right)\frac{z_2}{\sqrt{1+z_2^2}} + \lambda_1 + \frac{2}{t}\sqrt{1+z_2^2}\left(N\frac{\partial N}{\partial z_2} + \beta^2 T\frac{\partial T}{\partial z_2}\right) = 0 \qquad \text{(b)}$$

$$(4.17)$$

$$\frac{d\lambda_1}{dx} = -H_{z_1} = -\frac{24M\alpha^2}{t^3}(z_1 + \frac{1}{z_3})\sqrt{1+z_2^2} + 2\frac{N}{t} - 2\beta^2 z_2\frac{T}{t} \qquad \text{(a)}$$

$$\frac{d\lambda_3}{dx} = -H_{z_3} = \frac{1}{z_3^2}\left[24\frac{M^2\alpha^2}{t^3} z_1\sqrt{1+z_2^2} - \frac{2N}{t} + 2\beta^2 z_2\frac{T}{t}\right] \qquad \text{(b)}$$

$$(4.18)$$

The material volume constraint (4. 13) and the state equation (4. 14) are also part of the optimality conditions. The previous conditions define the set I of optimality conditions.

The boundary conditions are:

at $x = 0$

$$\left.\begin{array}{lll} z_1(0) = 0 & \lambda_1(0) \text{ unknown} & \text{(a)} \\ z_3(0) \text{ unknown} & \lambda_3(0) = 0 & \text{(b)} \\ \text{at } x = 1 & & \\ z_1(1) \text{ unknown} & \lambda_1(1) = 0 & \text{(c)} \\ z_3(1) \text{ unknown} & \lambda_3(1) = 0 & \text{(d)} \end{array}\right\} \quad (4.19)$$

Note that the condition (4. 19c) is implied by (4. 17b) when $z_2(1) = 0$.

To avoid the difficulty of solving (4. 17a) and (4. 17b) for the control variables z_2 and t the following set of equations is used to generate a differential equation for z_2.

$$\left.\begin{array}{ll} \dfrac{dH_t}{dx} = 0 \quad , & H_t = 0 \quad x^*,\ x^* \epsilon\ [0, 1] \\ \dfrac{dH_{z_2}}{dx} = 0 \quad , & H_{z_2} = 0 \quad \text{at } x = 1 \end{array}\right\} \quad (4.20)$$

Solving (4. 20) for $\dfrac{dz_2}{dx}$ and t, keeping only the positive root for the thickness, one obtains:

$$\left.\begin{array}{ll} t = \sqrt{\dfrac{B_1 + \sqrt{B_1^2 + 144 M^2 \alpha^2 \lambda_4}}{2\lambda_4}} & \text{(a)} \\ \dfrac{dz_2}{dx} = -(1+z_2^2)^{3/2}\, N\, \dfrac{T^2 B_2 + B_3}{C} & \text{(b)} \end{array}\right\} \quad (4.21)$$

where
$$\left.\begin{aligned} B_1 &= N^2+\beta^2T^2 && \text{(a)}\\ B_2 &= \frac{12M\alpha^2}{t^2}(4\beta^2-3)+\beta^2 && \text{(b)}\\ B_3 &= \left(\frac{12M\alpha^2}{t^2}+2-\beta^2\right)\left(\frac{72M^2\alpha^2}{t^2}+N^2\right) && \text{(c)} \end{aligned}\right\} \qquad (4.22)$$

$$\left.\begin{aligned} C &= M^2C_1+\beta^2C_2 && \text{(a)}\\ C_1 &= \frac{24\alpha^2}{t^2}\left[\frac{72M^2\alpha^2}{t^2}+N^2(1+3\beta^2)+T^2(3+\beta^2)\right] && \text{(b)}\\ C_2 &= (N^2+T^2)^2 && \text{(c)} \end{aligned}\right\} \qquad (4.23)$$

The equivalence between the systems of equations (4. 17) and (4. 22) hold only when the Jacobian of (4. 17), which is proportional to C, does not vanish, i. e.,

$$C \sim H_{tt}H_{z_2z_2} - (H_{tz_2})^2 \neq 0 \qquad (4.24)$$

<u>Theorem 4-1</u>

If the effects of the shear forces are not taken into account in the strain energy density, i. e., $\beta^2 = 0$, then $C = 0$ whenever $M = 0$.

<u>Proof</u>

(4. 23) and $\beta^2 = 0 \Longrightarrow C = \frac{24\alpha^2}{t^2}M^2\left[\frac{72M^2\alpha^2}{t^2}+N^2+3T^2\right]$ q. e. d.

This case is undesirable since the Hamiltonian H becomes locally linear in the control variable z_2. Under these conditions it can be shown that in order for $\frac{dz_2}{dx}$ to remain finite (i. e., the radius of curvature of the middle line to be non-zero) the traction N has to vanish. A physical interpretation could be found from the fact that when $\beta = 0$ a penalty on the system is set for only N and M but not for the shear force T. When $M = 0$, the condition $N = 0$ implies that all the force is transmitted to the structure as a shear force. This difficulty was alleviated here by including the contribution of the shear force into the strain energy density.

Since λ_1 does not appear in (4.22) and because the boundary condition (4.19c) is satisfied setting $z_2(1) = 0$, the optimality conditions reduce to the equations (4.21), (4.18b) along with the material volume constraint (4.13). In the boundary conditions (4.19), (4.19c) is replaced by $z_2(1) = 0$. This defines the set II of optimality conditions.

4-3-3. Straight Design

Theorem 4-2

The straight design defined by:

$$z_1(x) = 0 , \qquad \frac{1}{z_3} = 0$$

satisfies all the optimality conditions.

Proof

Since $z_1(x) = 0 \quad \forall\, x \in [0, 1], \quad z_2(x) = 0$

By virtue of (4.4), $N = 0$.

Furthermore (4.18b), (4.19b) imply that $\lambda_3 = 0$ which satisfies (4.19c). The thickness distribution is obtained from (4.22a) and λ_4 is computed from the material volume constraint (4.13) which reduces to:

$$\int_0^1 \sqrt{D_1 + \sqrt{D_1^2 + \lambda_4 \alpha^2 D_2}}\; dx = \sqrt{2\lambda_4} \tag{4.25}$$

where $D_1 = \beta^2 (x-1)^2$, $\qquad D_2 = 144(\frac{x^2}{2} - x)^2$

The integral equation (4.25) can be solved easily by taking an arbitrary value of $\lambda_4 a^2$ and then deducing, from the evaluation of the integral, the values of λ_4 and a^2.

4-3-4. Membrane Design

Theorem 4-3

The membrane design defined by (4.5) does not satisfy the optimality conditions.

Proof

It is most convenient to use, for the present proof, the original set of optimality conditions. If z_1 as defined by (4.5) were a solution then

$$(4.17) \Rightarrow \quad \lambda_1 + 2\lambda_4 t \frac{z_2}{\sqrt{1+z_2^2}} = 0 \tag{4.26}$$

$$(4.18a) \Rightarrow \frac{d\lambda_1}{dx} = \frac{2N}{t} \tag{4.27}$$

Since $z_2(x) \geq 0 \quad \forall x, \ x \in [0,1]$, (4.26) implies that $\lambda_1(x) \leq 0$ $\forall x, \ x \in [0,1]$. But (4.5b) implies that $N < 0 \ \forall x, \ x \in [0,1]$ therefore according to (4.27) $\frac{d\lambda_1}{dx} < 0$. This is a contradiction since $\lambda_1(1) = 0$.

q.e.d.

It is to be noted that the derivatives of λ_1 given respectively by (4.27) and the derivative of (4.26) differ only by their sign.

The membrane designs are a one parameter family of designs with respect to the slope of their support. Let us seek in that subclass of admissible designs the "best" element according to our stiffness criterion. We consider t and z_3 to be respectively an unknown function and an unknown parameter. The shape z_1 and its derivative z_2 are two known functions of z_3 defined by (4.5). The optimality condition (4.17a) remains valid, and a direct minimization of the strain energy with respect to z_3 is performed.

$$(4.17a) \implies \frac{N^2}{t^2} = \lambda_4 \tag{4.28}$$

The volume constraint and the objective function become respectively

$$\int_0^1 |N| \sqrt{1 + z_2^2}\, dx = \sqrt{\lambda_4} \tag{4.29}$$

$$J = \int_0^1 \frac{N^2}{t} \sqrt{1+z_2^2}\, dx = \lambda_4 \tag{4.30}$$

Therefore λ_4 has to be minimized with respect to z_3. The evaluation of the integral (4.29), after substitution of the value of N defined in (4.5b) yields:

$$\sqrt{\lambda_4} = a \sin^{-1} \left(\frac{1}{\sqrt{a}}\right) \tag{4.31}$$

where $a = 1 + \left(\frac{1}{z_3}\right)^2$

Noting that: $$\sin^{-1}\left(\frac{1}{\sqrt{a}}\right) = \tan^{-1} (z_3) \tag{4.32}$$

one obtains:

$$\frac{d\sqrt{\lambda_4}}{dz_3} = \frac{1}{z_3^2} \left[- \frac{2 \tan^{-1} z_3}{z_3} + 1\right] \tag{4.33}$$

A stationary value, here a minimum, of $\sqrt{\lambda_4}$ is obtained for the positive root of:

$$z_3 = 2 \tan^{-1} z_3, \quad z_3 \neq 0 \tag{4.34}$$

The minimum was found to be:

$$\sqrt{\lambda_4} = 1.3800\,, \qquad \lambda_4 = 1.9044$$
$$\text{at } \frac{1}{z_3} = 0.4290 \tag{4.35}$$

4-3-5. Results

Since no closed form solutions beside the straight design were found, a numerical solution of the two point boundary value problem defined by the set II of optimality conditions was performed. The integration was performed up to $\alpha^2 = 8$, the value at which numerical difficulties occurred due to the sensitivity of the solution with respect to the unknown initial slope $z_2(0)$. The results for the middle line shape, the thickness distribution, and the slope of the support as well as the initial slope of the structures are plotted respectively in figure 1 through 3 for different values of α and $\beta^2 = 2.5$. The value of the objective function corresponding to the straight designs, the optimal designs and the "best" membrane design are plotted on figure 4.

4-3-6. Conclusion

For the shape and the thickness optimization of simply supported arches, subjected to a uniform pressure, satisfying a stiffness constraint, it has been found necessary to include the effects of the shear force contribution in the strain energy definition. Two families of local optimal solutions have been found. The straight design which satisfies all the necessary optimality conditions was not found to be a global minimum for the achieved numerical solutions. A one parameter family of optimal designs, with respect to an average thickness to length ratio $(\frac{1}{\alpha})$ was generated. Although the "best" membrane design does not satisfy the optimality conditions, it represents for practical values of $\alpha (\alpha > 5)$ a very good approximation to the "best" design.

4-4. Optimal Arches with a Stress Constraint

The approximative two-dimensional state of stress used for the present analysis is defined as follows:

a) On an element normal to the middle line of the arch the stress vector is composed of a normal stress $\bar{\sigma}_t$, due to the normal force $\bar{N}$ and the bending moment $\bar{M}$, and of a shear stress $\bar{\tau}$ due to the shear force $\bar{T}$.

b) On an element parallel to the middle line, the normal stress is zero and the shear stress is $-\bar{\tau}$.

The normal stress $\bar{\sigma}_t$ is a linear function of the distance $\bar{v}$ from the middle line

$$\bar{\sigma}_t = \frac{\bar{M}\,\bar{v}}{I} + \frac{\bar{N}}{A}, \quad -\frac{t}{2} \leqslant \bar{v} \leqslant \frac{t}{2} \tag{4.36}$$

where $I = \frac{\bar{t}^3}{12}$: cross section moment of inertia

$A = \bar{t}$: cross section area

Note: the width of the section does not appear in those formulas since it was assumed to be a unit constant.

Although theoretically this is not true, the shear stress $\bar{\tau}$ is assumed to be a constant on the cross section.

$$\bar{\tau} = \tfrac{1}{2}\beta\,\frac{\bar{T}}{A}$$

where $\frac{1}{2}\beta$ is a coefficient which can be used to study the influence of the shear stress on the solution.

The maximum shear failure criterion defines the admissible state of stress. At each point of the cross section, there exists a direction for which the shear stress is maximum. The state of stress is said to be admissible if the magnitude of the maximum shear is less than or equal to a given limit value $\bar{\tau}_{adm}$. Given our approximative state of stress:

$$T_{max}(\bar{v}) = \frac{\sqrt{\sigma_t^2(\bar{v}) + 4\tau^2}}{2} \tag{4.38}$$

Since the magnitude of the tensile stress, in a given cross section, is maximum at the top or bottom fiber, the stress criterion can be formulated as:

$$\left.\begin{array}{ll} \frac{1}{\bar{t}^2}\left(\frac{6\bar{M}}{\bar{t}} + \bar{N}\right)^2 + \beta^2 \frac{\bar{T}^2}{t^2} \leqslant 4\,\bar{\tau}^2_{adm} & \text{(a)} \\ \frac{1}{t^2}\left(\frac{6\bar{M}}{\bar{t}} - \bar{N}\right)^2 + \beta^2 \frac{T^2}{t^2} \leqslant 4\,\bar{\tau}^2_{adm} & \text{(b)} \end{array}\right\} \tag{4.39}$$

Our assumption on the shear stress distribution $\bar{\tau}$ in the cross section was made in order to avoid the search of the location of the fiber in a cross section for which the shear has its largest value.

The material volume is the objective function $\bar{J}$.

$$\bar{J} = \int_0^{2L} \bar{t}\ \sqrt{1 + \bar{y},_{\bar{x}}^2}\ d\bar{x} \tag{4.40}$$

Upon introduction of the dimensionless variables defined by (4.3) and of

$$t = \frac{\bar{t}}{L} \quad \frac{2\,\bar{\tau}_{adm}}{p} \tag{4.41}$$

the objective function (4.40) and the stress constraint (4.39) reduce respectively to

$$J = \frac{1}{2} \int_0^2 t \sqrt{1+z_2^2}\ dx \tag{4.42}$$

$$\begin{aligned} &\frac{1}{t^2}\left(\frac{6M\gamma}{t} + N\right)^2 + \frac{\beta^2 T^2}{t^2} - 1 \leq 0 \\ &\frac{1}{t^2}\left(\frac{6M\gamma}{t} - N\right)^2 + \frac{\beta^2 T^2}{t^2} - 1 \leq 0 \end{aligned} \tag{4.43}$$

where

$$\left.\begin{aligned} \bar{J} &= \frac{pL^2}{\tau_{adm}}\, J \qquad &(a) \\ \gamma &= \frac{2\tau_{adm}}{p} \qquad &(b) \end{aligned}\right\} \tag{4.44}$$

For large values of γ, referred to as the load coefficient, the applied pressure is small with respect to the admissible maximum shear.

Since the result cannot depend on the sign of the pressure, it will be taken as positive. Therefore we will limit our investigations to the case $\gamma > 0$, and $\frac{1}{z_3} \geq 0$.

4-4-1. Formulation

Because of the symmetry in the imposed boundary conditions, the problem can be formulated as:

$$\text{Min } J = \int_0^1 t \sqrt{1+z_2^2}\ dx \tag{4.45}$$

subjected to:

i) the state equations

$$\left.\begin{aligned} \frac{dz_1}{dx} &= z_2 && \text{(a)} \\ \frac{dz_3}{dx} &= 0 && \text{(b)} \end{aligned}\right\} \quad (4.46)$$

ii) the stress constraints:

$$\left.\begin{aligned} \frac{1}{t^2}\left(\frac{6M\gamma}{t} + N\right)^2 + \beta^2 \frac{T^2}{t^2} - 1 + \mu_1^2 &= 0 && \text{(a)} \\ \frac{1}{t^2}\left(\frac{6M\gamma}{t} - N\right)^2 + \beta^2 \frac{T^2}{t^2} - 1 + \mu_2^2 &= 0 && \text{(b)} \end{aligned}\right\} \quad (4.47)$$

iii) the equilibrium equations (4. 4)

iv) the boundary conditions

$$\left.\begin{aligned} &z_1(0) = 0 && z_1(1) \text{ unknown} \\ &z_3(0) \text{ unknown} && z_3(1) \text{ unknown} \end{aligned}\right\} \quad (4.48)$$

Note. a) the unknown slope of the support is treated as a state variable bounded to be a constant by (4. 46b).

b) the inequality constraints (4. 43) have been transformed into the equality constraints (4. 47) by introducing two additional control variables μ_1 and μ_2. A stress constraint (4. 46) will be said to be "effective" when its corresponding μ_i $(i = 1, 2)$ is zero.

The Hamiltonian H of the system is obtained by adjoining to its Lagrangian the state equations (4. 46) and the stress constraint (4. 47) by means of the Lagrange's multipliers λ_1, λ_3, η_1 and η_3. The state variables are z_1 and z_2. The control variables are t,

z_2, μ_1 and μ_2.

$$H = t\sqrt{1+z_2^2} + \lambda_1 z_2 + \lambda_3 x(0)$$
$$+ \eta_1 \left[\frac{1}{t^2}\left(\frac{6M\gamma}{t} + N\right)^2 + \beta^2 \frac{T^2}{t^2} - 1 + \mu_1^2\right]$$
$$+ \eta_2 \left[\frac{1}{t^2}\left(\frac{6M\gamma}{t} - N\right)^2 + \beta^2 \frac{T^2}{t^2} - 1 + \mu_2^2\right] \qquad (4.49)$$

4-4-2. Optimality Conditions

The first order optimality conditions are:

$$H_{\mu_i} = 2\mu_i\eta_i = 0 \qquad i = 1, 2 \qquad \text{(a)}$$

$$H_t = \sqrt{1+z_2^2} - \frac{2\eta_1}{t}\left[1 + \frac{6M\gamma}{t^3}\left(\frac{6M\gamma}{t} + N\right) - \mu_1^2\right]$$
$$+ \frac{2\eta_2}{t}\left[1 + \frac{6M\gamma}{t^3}\left(\frac{6M\gamma}{t} - N\right) - \mu_2^2\right] \qquad \text{(b)} \quad (4.50)$$

$$H_{z_2} = \frac{t z_2}{\sqrt{1+z_2^2}} + \frac{2\eta_1}{t^2}\left[\left(\frac{6M\gamma}{t} + N\right)\frac{\partial N}{\partial z_2} + \beta^2 T \frac{\partial T}{\partial z_2}\right]$$
$$+ \frac{2\eta_2}{t^2}\left[-\left(\frac{6M\gamma}{t} - N\right)\frac{\partial N}{\partial z_2} + \beta^2 T \frac{\partial T}{\partial z_2}\right] + \lambda_1 = 0 \qquad \text{(c)}$$

$$\frac{d\lambda_i}{dx} = - H_{z_i} = - \frac{2\eta_1}{t^2}\left[\left(\frac{6M\gamma}{t} + N\right)\left(\frac{6\gamma}{t}\frac{\partial M}{\partial z_i} + \frac{\partial N}{\partial z_i}\right)\right.$$
$$\left. + \beta^2 T \frac{\partial T}{\partial z_i}\right] - \frac{2\eta_2}{t^2}\left[\left(\frac{6M\gamma}{t} - N\right)\left(\frac{6\gamma}{t}\frac{\partial M}{\partial z_i} - \frac{\partial N}{\partial z_i}\right) + \beta^2 T \frac{\partial T}{\partial z_i}\right]$$
$$i = 1 \text{ or } 3 \qquad (4.51)$$

The stress constraints (4.47) are also part of the optimality conditions. Since they result from the transformation of the inequality constraints (4. 43), their associated Lagrange's multipliers must be non-negative on an optimal solution. The boundary conditions are:

at $x = 0$

$$\left.\begin{array}{lll} z_1(0) = 0 & \lambda_1(0) \text{ unknown} & (a) \\ z_3(0) \text{ unknown} & \lambda_3(0) = 0 & (b) \\ \text{at } x = 1 & & \\ z_1(1) \text{ unknown} & \lambda_1(1) = 0 & (c) \\ z_3(0) \text{ unknown} & \lambda_3(1) = 0 & (d) \end{array}\right\} \quad (4.52)$$

As for the stiffness case, the condition (4. 52c) is implied by the symmetry condition:

$$z_2(1) = 0 \qquad (4.53)$$

Two constraints enter into the problem, therefore three cases should be considered:

a) case where no constraint is effective

b) case where one constraint is effective

c) case where two constraints are effective

Theorem 4-4

At every point of the optimal structure, at least one stress constraint has to be effective.

Proof

Let us suppose that no stress constraint is effective, i. e., that $\mu_1 \neq 0$ and $\mu_2 \neq 0$. Then (4. 50a) implies that $\eta_1 = \eta_2$ which contradicts (4.50b).

The previous theorem rules out the first possibility, as expected from physical considerations. Since the structure is statically determinant, a change of thickness at a point does not affect the values of M, N and T at the other locations. Therefore if the value of the thickness at a point was higher than the one required to satisfy the stress constraints, it could be reduced until one constraint becomes effective.

Theorem 4-5

A feasible state of stress for which the stress constraint (4.47a) is effective, is such that

$$NM \geq 0$$

Proof

Let us suppose $NM < 0$. Then the constraint (4.47b) is violated. q.e.d.

Theorem 4-6

When two constraints are effective, then either $N = 0$ or $M = 0$.

Proof

When the two constraints are effective, then $\mu_1 = \mu_2 = 0$. The two constraints (4.49) imply that $NM = 0$. A segment of arch on which $N = 0$ everywhere is, from (4.4a), a straight line. Using (4.4b), M is found to be non-zero on such an arc, except at $x = 0$.

q.e.d.

When two constraints are effective, the shape and the thickness distribution are fully determined by the stress conditions. However the optimality conditions will determine whether or not such structures are optimal.

4-4-3. Membrane Design

Theorem 4-7

The "best" membrane design, i.e., the membrane design with $\frac{1}{z_3} = 0.429$ satisfies all the optimality conditions provided $\gamma \geq 1/3$.

Proof

We will first show that any membrane design satisfies all the optimality conditions, except the boundary condition on λ_3. For a membrane design $M = 0 \quad \forall x, x \in [0,1]$. Therefore the two stress constraints are effective and $T = 0$.

Let us compute the two Lagrange's multipliers η_1 and η_2 from the optimality conditions.

$$(4.47) \Rightarrow \quad \frac{N^2}{t^2} = 1 \tag{4.54}$$

$$(4.5b) \Rightarrow \quad t = \sqrt{1 + \left(\frac{1}{z_3}\right)^2} \tag{4.55}$$

$$(4.50b) \Rightarrow \quad \frac{2\eta_1}{t} + \frac{2\eta_2}{t} = \sqrt{1+z_2^2} \tag{4.56}$$

$$(4.50c) \text{ and } (4.5a) \Rightarrow \lambda_1 = \frac{t\, z_2}{\sqrt{1+z_2^2}} = (x-1) \tag{4.57}$$

$$(4.57) \Rightarrow \frac{d\lambda_1}{dx} = 1 \tag{4.58}$$

Using (4.51) with i = 1, (4.56) and (4.58) one obtains:

$$\frac{2\eta_1}{t} - \frac{2\eta_2}{t} = \frac{1}{3\gamma} \sqrt{1+z_2^2} \qquad (4.59)$$

(4.59) and (4.56) define a system of two linear equations for η_1 and η_2 which has the following solution:

$$\left.\begin{aligned} \eta_1 &= \frac{t\sqrt{1+z_2^2}}{4}\left(1 + \frac{1}{3\gamma}\right) && \text{(a)} \\ \eta_2 &= \frac{t\sqrt{1+z_2^2}}{4}\left(1 - \frac{1}{3\gamma}\right) && \text{(b)} \end{aligned}\right\} \qquad (4.60)$$

when $0 < \gamma < \frac{1}{3}$ then $\eta_2 < 0$ which indicates that the membrane design is not optimal. When $\gamma > \frac{1}{3}$ then $\eta_1 > 0$ and $\eta_2 > 0$. In this case the membrane design of minimum material volume, i.e., satisfying also the boundary conditions on λ_3, is a local optimum. This solution corresponds to the "best" membrane defined in section 4-3-4. The slope of the support is given by $\frac{1}{z_3} = 0.429$ and the material volume is 1.3800.

4-4-4. Straight Design

Theorem 4-8

The straight design $z_1(x) = 0$, $\frac{1}{z_3} = 0$ satisfies all the optimality conditions for all values of γ.

Proof

For the straight design N = 0, the two constraints (4.47) are effective.

$$(4.47) \Longrightarrow \quad t = \sqrt{\frac{\beta^2 T^2 + \sqrt{\beta^4 T^4 + 144\, M^2 \gamma^2}}{2}} \qquad (4.61)$$

where T and M are given by the equilibrium equation (4. 4):

$$T = x-1 , \quad M = \frac{x^2}{2} - x \tag{4.62}$$

$$(4.50b) \Longrightarrow (\frac{2\eta_1}{t} + \frac{2\eta_2}{t}) (1 + \frac{36M^2\gamma^2}{t^4}) = 1 \tag{4.63}$$

Using (4. 51) with i = 1 and the boundary condition $\lambda_2(1) = 0$ one obtains:

$$\frac{d\lambda_1}{dx} = 0 \quad \Longrightarrow \quad \lambda_1(x) = 0 \tag{4.64}$$

$$(4.50c) \Longrightarrow 2 \{\eta_1 - \eta_2\}[\frac{6M\gamma}{t^3} T] = 0 \tag{4.65}$$

Since M and T are not identically zero for all x in the interval $[0, 1]$:

$$(4.63) \text{ and } (4.65) \Longrightarrow \eta_1 = \eta_2 = \frac{t}{4(1 + \frac{36M^2\gamma^2}{t^4})} \tag{4.66}$$

A direct computation of λ_3 from (4. 51) with i = 3 shows that $\lambda_3(x) = 0$. Therefore the straight design satisfies all the optimality conditions.

The material volume of the straight design V_s can be computed by quadratures using (4. 61) and (4. 62). A lower bound of the material volume is obtained when $\beta = 0$.

$$[V_s]_{\beta=0} = \sqrt{3\gamma} \; \frac{\pi}{4} \tag{4.67}$$

The straight design corresponds to a local optimum, but it cannot be a global optimum for all values of γ, since the material volume associated with the "best" membrane design is independent of γ.

4-4-5. Designs with One Effective Constraint

The membrane design ceased to be a local optimum for $\gamma < \frac{1}{3}$ because η_2, the Lagrange's multiplier associated with the stress constraint (4. 47b) became negative. This gives an indication that better designs, on which the constraint (4. 47b) is not effective, might be found. Let us give a physical interpretation of the previous argument. On a membrane design:

$$z_1(x) = [z_1(x)]_M, \quad M = 0, \; N < 0$$

A direct substitution in (4. 4c) for the case $\frac{1}{z_3} > 0$ shows that:

a) if $z_1(x) > [z_1(x)]_M$ then $M > 0$

b) if $z_1(x) < [z_1(x)]_M$ then $M < 0$

Therefore the length of the middle line of an arch on which $M > 0$ everywhere, cannot be shorter than the one of the membrane design having the same initial slope. A study of the one parameter family of designs:

$$z_1(x) = [z_1(x)]_M + \varepsilon\, x^2(-2x+3)$$

showed that the "best" membrane design was improved for $\varepsilon < 0$ and $\gamma \leq \gamma_\ell < \frac{1}{3}$. The decrease of the material volume was obtained even though there was an increase of the average thickness of the arch, due to the bending moment. However the decrease in the middle line length more than compensated for the increase of the average thickness. For the investigated family of designs, $M \times N$

is positive when $\varepsilon < 0$. These were the determining factors for investigating the case where only the constraint (4.47a) is effective.

When the constraint (4.47a) only is effective, the optimality conditions are obtained from the ones defined in the section 4-4-2 setting

$$\mu_1 = \eta_2 = 0 \qquad (4.68)$$

$$(4.50b) \Longrightarrow \frac{2\eta_1}{t} = \frac{\sqrt{1+z_2^2}}{1+\frac{6M\gamma}{t^3}\left(\frac{6M\gamma}{t}+N\right)} \qquad (4.69)$$

When the state of stress constrained by (4.47a) only is feasible, theorem 4-5 and (4.69) imply that $\frac{2\eta_1}{t} > 0$.

To avoid the difficulty of solving the constraint (4.47a) and the equation (4.50c) for z_2 and t, total derivatives of those equations with respect to x are used to generate a system of two linear equations for $\frac{dt}{dx}$ and $\frac{dz_2}{dx}$. The total derivative of the stress constraint (4.47a) with respect to x can be written as:

$$B_1 \frac{dt}{dx} + B_2 \frac{dz_2}{dx} = B_3 \qquad (4.70)$$

where

$$\left.\begin{aligned} B_1 &= 4t^3 - 2t(\beta^2T^2+N^2) - 12\,MN\gamma && \text{(a)}\\ B_2 &= -\frac{2t^2}{1+z_2^2}\,T\left[N(1-\beta^2) + \frac{6M\gamma}{t}\right] && \text{(b)}\\ B_3 &= 2T\sqrt{1+z_2^2}\,\left[\beta^2t^2 + 6\gamma\,tN + 36\gamma^2M\right] && \text{(c)} \end{aligned}\right\} \quad (4.71)$$

The conditions (4.50) can be expressed as:

$$
\left.
\begin{aligned}
&H_{z_2} = \frac{tz_2}{\sqrt{1+z_2^2}} + DC + \lambda_1 = 0 && \text{(a)}\\
&\text{where } D \equiv \frac{2\eta_1}{t} && \text{(b)}\\
&C \equiv \frac{T}{t(1+z_2^2)} \left[\frac{6M\gamma}{t} + N(1-\beta^2)\right] && \text{(c)}
\end{aligned}
\right\} \quad (4.72)
$$

The total derivatives with respect to x of D and C are respectively:

$$
\left.
\begin{aligned}
&\frac{dD}{dx} = D_1 \frac{dt}{dx} + D_2 \frac{dz_2}{dx} + D_3 && \text{(a)}\\
&\text{where: } D_1 = \frac{D^2}{\sqrt{1+z_2^2}} \frac{18M\gamma}{t^4} \left(\frac{8M\gamma}{t} + N\right) && \text{(b)}\\
&D_2 = \frac{D}{\sqrt{1+z_2^2}} \left(\frac{z_2}{\sqrt{1+z_2^2}} - \frac{6M\gamma}{t^3} D \frac{\partial N}{\partial z_2}\right) && \text{(c)}\\
&D_3 = -D^2 T \frac{6\gamma}{t^3} \left(\frac{12M\gamma}{t} + N\right) && \text{(d)}
\end{aligned}
\right\} \quad (4.73)
$$

$$
\left.
\begin{aligned}
&\frac{dC}{dx} = C_1 \frac{dt}{dx} + C_2 \frac{dz_2}{dx} + C_3 && \text{(a)}\\
&\text{where } C_1 = -\frac{T}{t^2(1+z_2^2)} \left[\frac{12M\gamma}{t} + N(1-\beta^2)\right] && \text{(b)}\\
&C_2 = \frac{1}{t^2(1+z_2^2)} \left[\left(-\frac{6M\gamma}{t}(N+2Tz_2)\right) + \right.\\
&\qquad \left.(1-\beta^2)(T^2-N^2-2TNz_2)\right] && \text{(c)}\\
&C_3 = \frac{1}{t\sqrt{1+z_2^2}} \left[\frac{6}{t}(T^2+M) + N(1-\beta^2)\right] && \text{(d)}
\end{aligned}
\right\} \quad (4.74)
$$

The total derivative with respect to x of the condition (4.50c) can be written as

$$\frac{dH_{z_2}}{dx} = F_1 \frac{dt}{dx} + F_2 \frac{dz_2}{dx} - F_3 = 0 \qquad (4.75)$$

where

$$\left.\begin{aligned} F_1 &= \frac{z_2}{\sqrt{1+z_2^2}} + DC_1 + D_1C && \text{(a)} \\ F_2 &= \frac{t}{(1+z_2^2)^{3/2}} + DC_2 + D_2C && \text{(b)} \\ F_3 &= -\frac{d\lambda_1}{dx} - DC_3 - D_3C && \text{(c)} \end{aligned}\right\} \qquad (4.76)$$

$$\frac{d\lambda_1}{dx} = -\frac{D}{t}\left[\left(\frac{6M\gamma}{t} + N\right)\left(\frac{6\gamma}{t}\left\{z_1 + \frac{1}{z_3}\right\} - \frac{1}{\sqrt{1+z_2^2}}\right) + \beta^2 T \frac{z_2}{\sqrt{1+z_2^2}}\right] \qquad (4.77)$$

The relation (4.77) is derived from the condition (4.51) with i = 1. The conditions (4.70) and (4.75) define a linear system of equations for $\frac{dz_2}{dx}$ and $\frac{dt}{dx}$ which has a unique solution when

$$G \equiv B_1F_2 - B_2F_1 \neq 0 \qquad (4.78)$$

The solution is given by:

$$\left.\begin{aligned} \frac{dz_2}{dx} &= \frac{B_1F_3 - B_3F_1}{G} && \text{(a)} \\ \frac{dt}{dx} &= \frac{B_3F_2 - B_2F_3}{G} && \text{(b)} \end{aligned}\right\} \qquad (4.79)$$

Theorem 4-9

If $\beta = 0$ then $G = 0$ whenever $M = 0$.

Proof

The thickness can be computed directly from (4.47a)

$$t^2 = N^2 \tag{4.80}$$

A direct evaluation of the variables defined by (4.71) through (4.77) gives:

$$B_1 = 2N^2 t\ , \qquad B_2 = -\frac{2N^2}{1+z_2^2} NT$$

$$D = \sqrt{1+z_2^2}\ , \qquad D_1 = 0\ , \quad D_2 = \frac{z_2}{\sqrt{1+z_2^2}}$$

$$C = \frac{NT}{t(1+z_2^2)}\ , \qquad C_1 = -\frac{NT}{t^2(1+z_2^2)}$$

$$C_2 = \frac{1}{t(1+z_2^2)^2} \left\{T^2 - 2TNz_2 - N^2\right\}$$

$$F_1 = \frac{z_2}{\sqrt{1+z_2^2}} - \frac{NT}{t^2\sqrt{1+z_2^2}}$$

$$F_2 = \frac{t}{1+z_2^2)^{3/2}} \left[1 + \frac{1}{t^2}(T^2 - TNz_2 - N^2)\right]$$

Using (4.78) to evaluate G, one finds:

$$G = \frac{2N^2}{(1+z_2^2)^{3/2}} \left[t^2 - N^2 + T^2\left(1 - \frac{N^2}{t^2}\right)\right]$$

Therefore by virtue of (4.80) : $G = 0$ q.e.d.

The shear force contribution was introduced into the stress criterion to insure that $\frac{dt}{dx}$ and $\frac{dz_2}{dx}$ can be uniquely determined from the equations (4.70) and (4.75).

The system of equations (4.79), along with equation (4.51) with i = 3 define the set II of optimality field equations. The boundary conditions are given by (4.52), where (4.52c) is replaced by (4.53). The initial value of the thickness is given by the stress constraint (4.47a) evaluated at x = 0. This defines the set II of optimality conditions.

Numerical solution of the set II of optimality conditions was first performed for $\gamma = .3$ using a direct shooting technique as described in Chapter III. The unknown initial values were $z_2(0)$ and $\frac{1}{z_3}$. The matching conditions were $z_2(1) = \lambda_3(1) = 0$. Due to the peculiar changes in the unknown initial conditions from one iteration to the next one, this search method was abandoned. A direct computation of $z_2(1)$ and $\lambda_3(1)$ as a function of $z_2(0)$ and $\frac{1}{z_3}$ was performed for different values of γ and $\beta = 2$. Figure 6 shows the loci of the points in the $(z_2(0), \frac{1}{z_3})$ plane for which either $z_2(1) = 0$ or $\lambda_3(1) = 0$. A solution of our problem, which must satisfy both conditions, is an intersection point of the two loci. For $\gamma = .25$ the figure 6a shows that no intersection point exists. When $\gamma = .3$ there exist two intersection points A and B as shown on figure 6b. The point A corresponds to a local minimum of the material volume, but the point B is neither a minimum nor a maximum. As γ was increased up to the value 1/3, the point A moved toward point C which represents the best membrane design. For $\gamma = 0.35$ only one intersection point B of the two loci was found as shown on figure 6c. The problems encountered during the iterative search can be explained by the fact that the solutions correspond to the intersection points of two curves which are almost tangent when $\gamma = .3$.

When one constraint only is effective, the number and the character of the solutions to the optimality conditions depend on γ in the following manner:

a) $\gamma < \gamma^*$: no solution. It was found that $\gamma^* = 0.27$ for $\beta = 2$.

b) $\gamma^* < \gamma < \frac{1}{3}$: two solutions. Only one of which corresponds to a local minimum of the material volume.

c) $\gamma > \frac{1}{3}$: one solution which is not a local minimum.

To verify these surprising results, the problem of determining the "best" parabolic arch satisfying the stress constraints (4.47) was investigated. The thickness t(x) is considered as an unknown function. The initial slope of the structure $z_2(0)$ and the slope of the support are two unknown parameters. The dependence of the solutions on the parameter γ, when one stress constraint only is effective, was found to be similar to the one of the general problem.

Remark: In the previous analysis we considered the cases where the same set of constraints was effective on the entire structure. Theoretically solutions on which several sets of constraints are effective on different arcs of the solution should be investigated. Two types of switching points could exist:

a) switching point between an arc on which the two constraints are effective and an arc on which only one constraint is effective,

b) switching point between an arc on which one of the two stress constraints is effective and an arc on which the other stress constraint is effective.

A detailed analysis of those cases was not performed, since they can be ruled out on physical considerations. When one constraint is effective, its associated Lagrange's multiplier is positive. It indicates that no branching to another type of arc will be locally improving as long as the other constraint is of course not violated. This was found to be verified during our numerical computations.

4-4-6. Results

Figure 5 shows the value of the material volume as a function of γ for the different types of solutions. The numerical computations were made for the case $\beta = 2.0$.

The straight design is a local optimum for any value of γ, and corresponds to the global minimum for $\gamma < \gamma^+$. The "best" membrane design is a local optimum for $\gamma > \frac{1}{3}$ and corresponds to the global minimum for $\gamma > \gamma^+$. When one stress constraint only is effective, solutions to the optimality conditions exist when $\gamma > \gamma^*$. The local minimum found for $\gamma^* < \gamma < \frac{1}{3}$ did not appear to be a global minimum. The values of γ^+ and γ^* obtained in our computations are respectively $\gamma^* = 0.27$ and $\gamma^+ = 0.44$. These values are functions of the coefficient β.

4-4-7. Conclusion

The influence of the shear force was introduced in the definition of the admissible state of stress. The maximum shear was used as the failure criterion, introducing two stress constraints.

For the considered cases the global optimum was obtained when the two stress constraints were effective. It was either the "best" membrane design, when the load coefficient γ is larger than γ^+, or the straight design when $\gamma < \gamma^+$. Furthermore the "best" membrane design ceased to be a local optimum for $\gamma < \frac{1}{3}$.

When only one stress constraint was effective, two solutions of the optimality conditions were shown to exist for $\gamma^* < \gamma < \frac{1}{3}$. One of these only was a local minimum. When $\gamma > \gamma^*$ one solution of this type did exist, but it was neither a maximum nor a minimum.

Although for practical applications the "best" membrane design corresponds to the minimal material volume design, it has been shown that other solutions to the optimality conditions do exist.

V. SHELL STRUCTURES: "MEMBRANE THEORY"

The problem of determining the shape and the thickness distribution of shells of revolution, subjected to axisymmetric loads, described by the membrane theory, received some consideration in the past. Chan [22] investigated the case of shells of revolution under edge loading for both a stress or a stiffness constraint, and found an analytical solution for the case of an edge traction which did not satisfy the "natural" boundary conditions. Stroud [23] treated the case where the shell is subjected to end tractions and a normal pressure with a stress constraint. Using a Ritz method to determine the unknown coefficients of the assumed trigonometric series for the shape, non-convergent cases were found. The same problem is reformulated here, but with an emphasis on the influence of the boundary condition definition on the optimality conditions.

A formal problem definition is first given. The optimality problem for a stress constraint is investigated for the case of a combined edge and normal pressure loading case. The influence of including the possibility of discrete rings in the formulation is then investigated. The same problems are then treated for a stiffness constraint.

5-1. Problem Definitions

Let S be the set of shells of revolutions such that:

a) the governing equations are derived from the membrane theory of shells

b) they support axisymmetric end tractions and a pressure normal to their middle surface (different classes of problems depending on the definition of the end tractions will be investigated)

c) they satisfy either a given stress or stiffness constraint

d) their length is 2L and their end radii are r_0.

We seek the shape $\bar{r}(\bar{x})$ and the thickness distribution $\bar{t}(\bar{x})$ of the element in S of minimum mass.

5-2. Equilibrium Equations

When a shell is described using membrane theory, the principal stress resultant $\bar{n}_1$ and $\bar{n}_2$ can be computed directly from the two equilibrium equations:

$$\frac{d\bar{n}_1}{d\bar{x}} = -\frac{\bar{r},_{\bar{x}}}{\bar{r}}[\bar{n}_1-\bar{n}_2] \tag{5. 1}$$

$$\frac{\bar{r},_{\bar{x}\bar{x}}\,\bar{n}_1}{(1+\bar{r},_{\bar{x}}^2)^{3/2}} - \frac{\bar{n}_2}{\bar{r}\sqrt{1+\bar{r},_{\bar{x}}^2}} + p = 0 \tag{5. 2}$$

When the applied pressure, p, does not vanish the following dimensionless variables can be introduced:

$$r = \frac{\bar{r}}{L}, \quad x = \frac{\bar{x}}{L}$$
$$n_1 = \frac{\bar{n}_1}{\frac{pr_0}{2}}, \quad n_2 = \frac{\bar{n}_2}{\frac{pr_0}{2}} \tag{5. 3}$$

The equilibrium equations (5. 1) and (5. 2), in terms of the dimensionless variables, become:

$$\frac{dn_1}{dx} = -\frac{r,_x}{r}[n_1-n_2] \tag{5. 4}$$

$$\frac{r,_{xx}\, n_1}{(1+r,_x^2)^{3/2}} - \frac{n_2}{r\sqrt{1+r,_x^2}} + \gamma = 0 \tag{5.5}$$

where $\gamma = \frac{2L}{r_0}$ (5.6)

5-3. Optimum Membrane Shell with Stress Constraint

The Von Mises yield criterion is used to define the admissible state of stress. The objective function and the optimality conditions are first derived for a combined edge and pressure loading case. Several classes of problems in which the normal pressure vanishes are investigated. Numerical solutions for the combined loading case are presented which show that the problem becomes singular for some range of the parameters.

The Von Mises yield criterion is given by

$$\bar{n}_1^2 - \bar{n}_1\,\bar{n}_2 + \bar{n}_2^2 \leq \sigma_a^2\,\bar{t}^2 \tag{5.7}$$

where σ_a is some experimentally determined critical stress such as the yield stress. Upon introduction of the previously defined dimensionless variables and:

$$t = \frac{2\sigma_a}{pr_0}\,\bar{t} \tag{5.8}$$

the Von Mises conditions reduces to

$$n_1^2 - n_1 n_2 + n_2^2 \leq t^2 \tag{5.9}$$

The material volume is taken as an objective function $\bar{J}$ for the present problem:

$$\overline{J} = \int_{x_0}^{x_0+2L} 2\pi \overline{r}\,\overline{t}\,\sqrt{1+\overline{r},_{\overline{x}}^2}\; d\overline{x} \tag{5.10}$$

It reduces in dimensionless form to:

$$J = \int_0^2 r\,t\,\sqrt{1+r,_x^2}\; dx \tag{5.11}$$

where

$$\overline{J} = 2\pi L^2 \frac{pr_0}{\sigma_a} J \tag{5.12}$$

5-3-1. Formulation:

The mass minimization of membrane shells of revolution for a stress constraint can be formulated, in dimensionless variables as:

$$\text{Min} \int_0^2 r\,t\,\sqrt{1+r,_x^2}\; dx \tag{5.13}$$

subjected to:

i) the state equations

$$\left.\begin{aligned} &\frac{dr}{dx} = r,_x && \text{(a)} \\ &\frac{dr,_x}{dx} = r,_{xx} && \text{(b)} \\ &\frac{dn_1}{dx} = -\frac{r,_x}{r}\,[n_1 - n_2] && \text{(c)} \end{aligned}\right\} \tag{5.14}$$

ii) the constraints

$$\left.\begin{aligned} &\frac{r,_{xx}\, n_1}{(1+r,_x^2)^{3/2}} - \frac{n_2}{r\sqrt{1+r,_x^2}} + \gamma = 0 && \text{(a)} \\ &n_1^2 - n_1 n_2 + n_2^2 + \mu^2 = t^2 && \text{(b)} \end{aligned}\right\} \tag{5.15}$$

where μ is an additional control variable introduced to transform the inequality constraint (5. 9) into an equality constraint according to (3. 58).

iii) the boundary conditions (different classes will be investigated later since they do not influence the optimality field equations)

The Hamiltonian H of the system is obtained by adjoining to its Lagrangian the state equations (5. 14) and the constraints (5. 15). The state variables are r, $r,_x$, n_1 and the control variables are $r,_{xx}$, t, n_2 and μ.

$$H = rt\sqrt{1+r,_x^2} + \lambda_1 r,_x + \lambda_2 r,_{xx} - \lambda_3 \frac{r,_x}{r}[n_1-n_2]$$

$$+ \eta_1\left[\frac{r,_{xx}\, n_1}{(1+r,_x^2)^{3/2}} - \frac{n_2}{r\sqrt{1+r,_x^2}} + \gamma\right] + \eta_2[n_1^2-n_1n_2+n_2^2-t^2+\mu^2] \tag{5. 16}$$

5-3-2. Optimality Conditions

The first order optimality conditions are:

$$\left.\begin{aligned}
H_\mu &= 2\eta_2\mu = 0 && \text{(a)}\\
H_t &= r\sqrt{1+r,_x^2} - 2\eta_2 t = 0 && \text{(b)}\\
H_{r,_{xx}} &= \lambda_2 + \frac{\eta_1 n_1}{(1+r,_x^2)^{3/2}} = 0 && \text{(c)}\\
H_{n_2} &= \lambda_3\frac{r,_x}{r} + \eta_2[2n_2-n_1] - \eta_1\frac{n_2}{r^2\sqrt{1+r,_x^2}} = 0 && \text{(d)}
\end{aligned}\right\} \tag{5. 17}$$

$$\frac{d\lambda_1}{dx} = -H_r = -t\sqrt{1+r,_x^2} - \lambda_3 \frac{r,_x}{r^2}[n_1 - n_2] - \eta_1 \frac{n_2}{r^2\sqrt{1+r,_x^2}} \qquad \text{(a)}$$

$$\frac{d\lambda_2}{dx} = -H_{r,_x} = \frac{t\,r\,r,_x}{\sqrt{1+r,_x^2}} - \lambda_1 + \lambda_3 \frac{n_1 - n_2}{r}$$

$$+ \eta_1 \left[\frac{3r,_x r,_{xx} n_1}{(1+r,_x^2)^{5/2}} - \frac{r,_x}{r(1+r,_x^2)^{3/2}} n_2\right] \qquad \text{(b)}$$

$$\frac{d\lambda_3}{dx} = -H_{n_1} = \frac{\lambda_3 r,_x}{r} - \eta_2 [2n_1 - n_2] - \eta_1 \frac{r,_{xx}}{(1+r,_x^2)^{3/2}} \qquad \text{(c)}$$

(5.18)

The state equations (5.14) and the constraints (5.15) are also part of the optimality conditions. Note: Since the equality constraint (5.15) results from the transformation of the inequality (5.9), η_2 should be non-negative.

Let us eliminate the unknown Lagrange's multipliers η_1, η_2 and compute the control variables from the optimality conditions.

i) (5.17.b) $\Longrightarrow$ a) $\eta_2 = 0$ only if $r = 0$ (a)

b) $t = 0$ only if $r = 0$ (b) (5.19)

c) when $r \neq 0$ $\eta_2 = \dfrac{r\sqrt{1+r,_x^2}}{2t}$ (c)

ii) (5.17.a) and (5.19.c) $\Longrightarrow$ $\mu = 0$ (5.20)

iii) (5.17.c) and $n_1 \neq 0$ $\Longrightarrow$ $\eta_1 = \dfrac{\lambda_2(1+r,_x^2)^{3/2}}{n_1}$ (5.21)

iv) (5.17.d) $\Longrightarrow n_2 = \frac{1}{2} n_1 - \frac{\lambda_2 \sqrt{1+r,_x^2}}{r^2} \frac{t}{n_1} - \frac{\lambda_3 t\, r,_x}{r^2 \sqrt{1+r,_x^2}}$ (a)

(5.15.b) and (5.20) $\Longrightarrow t = [n_1^2 - n_1 n_2 + n_2^2]^{\frac{1}{2}}$ (b)

(5.15.a) and $n_1 \neq 0 \Longrightarrow r,_{xx} = \frac{n_2}{n_1} \frac{1+r,_x^2}{r} - \frac{\gamma}{n_1}(1+r,_x^2)^{3/2}$ (c)

(5.22)

v) $$\frac{d\lambda_1}{dx} = \lambda_2 \frac{n_2}{n_1} \frac{1+r,_x^2}{r^2} - \lambda_3 \frac{r,_x}{r} [n_1 - n_2] - t\sqrt{1+r,_x^2} \quad \text{(a)}$$

$$\frac{d\lambda_2}{dx} = -\lambda_1 - \lambda_2 \left[\frac{3r,_x\, r,_{xx}}{(1+r,_x^2)} - \frac{r,_x}{r} \frac{n_2}{n_1} \right] + \lambda_3 \frac{n_1 - n_2}{r} - \frac{r\,t\,r,_x}{\sqrt{1+r,_x^2}} \quad \text{(b)}$$

$$\frac{d\lambda_3}{dx} = \lambda_2 \frac{r,_{xx}}{n_1} + \lambda_3 \frac{r,_x}{r} - \frac{r\sqrt{1+r,_x^2}}{2t} [2n_1 - n_2] \quad \text{(c)}$$

(5.23)

vi) $$\frac{dr}{dx} = r,_x \quad \text{(a)}$$

$$\frac{dr,_x}{dx} = r,_{xx} \quad \text{(b)}$$

$$\frac{dn_1}{dx} = -\frac{r,_x}{r}[n_1 - n_2] \quad \text{(c)}$$

(5.24)

5-3-3. Case where $p = 0$

Despite the fact that the previous non-dimensionalization was performed for the case $p \neq 0$, the optimality conditions for the case p = 0 are obtained from (5. 19) through (5. 24) setting $\gamma = 0$ in (5. 22c) only and using the following as a definition for the dimensionless variables:

$$n_1 = \frac{\overline{n_1}}{\sigma_a r_0}, \quad n_2 = \frac{\overline{n_2}}{\sigma_a r_0}, \quad x = \frac{\overline{x}}{L}, \quad r = \frac{\overline{r}}{L}$$
$$t = \frac{\overline{t}}{r_0}, \quad J = 2\pi L^2 r_0 \overline{J}, \quad \gamma = \frac{2L}{r_0} \qquad (5.25)$$

Several types of problems can be considered depending on the definition of the boundary conditions for the values of $n_1(0)$ and $n_1(2)$. They cannot be specified independently of each other since they have to satisfy the overall equilibrium equation:

$$\left.\frac{r n_1}{\sqrt{1+r,_x^2}}\right|_{x=0} - \left.\frac{r n_1}{\sqrt{1+r,_x^2}}\right|_{x=2} = 0 \qquad (5.26)$$

a) <u>Problem 1</u>

Let the magnitude of the end traction n_1 be specified at $x = 0$. Its direction will be aligned with the slope of the structure at $x = 0$, i. e.:

$$r(0) = r(2) = \frac{2}{\gamma} \qquad n_1(0) = F \qquad (5.27a)$$

$r,_x(0)$, $r,_x(2)$, $n_1(2)$ are unknown but satisfy (5. 26). The boundary conditions on the Lagrange multipliers derived from

are:

$$\left.\begin{aligned}
&\lambda_1(0),\ \lambda_1(2),\ \lambda_3(0) \text{ are unknown.}\\
&\lambda_2(0) = \frac{\nu n_1\, r,_x}{(1+r,_x^2)^{3/2}}\Bigg|_{x=0}\\
&\lambda_2(2) = \frac{-\nu n_1\, r,_x}{(1+r,_x^2)^{3/2}}\Bigg|_{x=2}\\
&\lambda_3(2) = \frac{-\nu r}{\sqrt{1+r,_x^2}}\Bigg|_{x=2}
\end{aligned}\right\} \qquad (5.27b)$$

where ν is the Lagrange's multiplier associated with (5.26).

b) Problem 2

Let the magnitude of the axial projection of the end traction n_1 be specified at $x = 0$, i.e.,

$$n_1(0) = F\sqrt{1+r,_x^2(0)}\,, \qquad r(0) = r(2) = \frac{2}{\gamma}$$

$r,_x(0)$, $r,_x(2)$ and $n_1(2)$ are unknown but satisfy (5.26) which reduces here to: (5.28a)

$$n_1(2) = F\sqrt{1+r,_x^2(2)} \qquad \text{since } r(0) = r(2)$$

The boundary conditions on the Lagrange's multipliers derived from are:

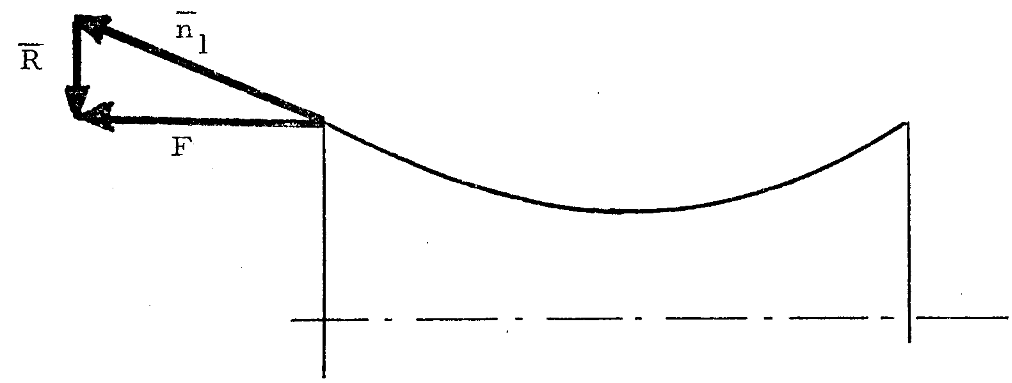
$$\left.\begin{aligned}
&\lambda_1(0),\ \lambda_1(2) \text{ are unknown}\\
&\lambda_2(0) = \nu_1 \frac{F\, r,_x(0)}{\sqrt{1+r,_x^2(0)}}\\
&\lambda_3(0) = -\nu_1
\end{aligned}\right\} \qquad (5.28b)$$

$$\left.\begin{aligned} \lambda_2(2) &= -\nu_2 \frac{F\, r,_x(2)}{\sqrt{1 + r,_x^2(2)}} \\ \lambda_3(2) &= \nu_2 \end{aligned}\right\} \tag{5.28b}$$

where ν_1 and ν_2 are respectively the Lagrange's multipliers associated with the boundary conditions on $n_1(0)$ and $n_1(2)$.

c) Problem 3

Let the magnitude of an axial force be defined on the boundary. Whenever the slope at the ends of the structure is not zero a ring, satisfying the same stress constraint as the remaining part of the structure, will be added to support the load $\overline{R}$ not taken by the shell.



To our previous integral objective function, discrete terms corresponding to the possible end-rings must be added. The cross section $\overline{A}$ of a ring supporting a radial force $\overline{R}$ per unit length for our stress constraint is:

$$\overline{A} = \frac{|\overline{R}|\, r_0}{\sigma_a} \tag{5.29a}$$

The volume $\overline{V}_R$ of the ring is:

$$\overline{V}_R = \frac{2\pi r_0^2 \, |\overline{R}|}{\sigma_a} \tag{5.29b}$$

Upon non-dimensionalization according to (5.25) our dimensionless objective function becomes:

$$J = r^2(0)\, F\, |r,_x(0)| + r^2(2)\, F\, |r,_x(2)| + \int_0^2 rt\sqrt{1+r,_x^2}\, dx \tag{5.30}$$

the boundary conditions are:

$$r(0) = r(2) = \frac{2}{\gamma}$$

$$r,_x(0),\ r,_x(2) \text{ are unknown} \tag{5.31}$$

$$n_1(0) = F\sqrt{1+r,_x^2(0)} \quad \text{and} \quad n_1(2) = F\sqrt{1+r,_x^2(2)}$$

The variation of δJ due to admissible variations of the value of the state variables on the boundary is given by:

$$\begin{aligned} \delta J = {} & r^2(0)\, F \left\{ |r,_x(0) + \delta r,_x(0)| - |r,_x(0)| \right\} \\ & + r^2(2)\, F \left\{ |r,_x(2) + \delta r,_x(2)| - |r,_x(2)| \right\} \\ & + \lambda_2(0)\, \delta r,_x(0) - \lambda_2(0)\, \delta r,_x(2) \\ & + \lambda_3(0) \frac{\partial n_1(0)}{\partial r,_x(0)}\, \delta r,_x(0) - \lambda_3(2) \frac{\partial n_1(2)}{\partial r,_x(2)}\, \delta r,_x(2) \end{aligned} \tag{5.32}$$

Let us suppose $r,_x(0) < 0$ and $r,_x(2) > 0$

$$\begin{aligned} \text{then } \delta J = {} & \left\{ -r^2(0)F + \lambda_2(0) - \lambda_3(0) \times \frac{Fr,_x(0)}{\sqrt{1+r,_x^2(0)}} \right\} \delta r,_x(0) \\ & - \left\{ -r^2(2)\, F + \lambda_2(2) - \lambda_3(2) \times \frac{Fr,_x(2)}{\sqrt{1+r,_x^2(2)}} \right\} \delta r,_x(2) \end{aligned} \tag{5.33}$$

The boundary conditions on λ_2 and λ_3 must be such that the coefficients in (5.33) of the arbitrary $\delta r,_x(0)$ and $\delta r,_x(2)$ vanish. Similar conditions would be obtained for $r,_x(0) < 0$ and $r,_x(2) > 0$. Let us now suppose that $r,_x(0) = r,_x(2) = 0$. Because of the absolute values, two cases must be considered

i) $\delta r,_x(0) > 0$, $\delta r,_x(2) > 0$

$$\delta J = \{r^2(0)\ F + \lambda_2(0)\}\ \delta r,_x(0) + \{r^2(2)\ F - \lambda_2(2)\}\ \delta r,_x(2) \quad (5.34a)$$

ii) $\delta r,_x(0) < 0$, $\delta r,_x(2) < 0$

$$\delta J = \{-r^2(0)\ F - \lambda_2(0)\}\ (-\delta r,_x(0)) + \{r^2(2) F + \lambda_2(2)\}\ (-\delta r,_x(2))$$

(5.34b)

In order to have $\delta J \geqslant 0$ for all admissible perturbations of the boundary conditions we must have:

$$|\lambda_2(0)| \leqslant r^2(0)\ F\ , \quad |\lambda_2(2)| \leqslant r^2(2)\ F \quad (5.35)$$

It is to be noted that here the solution is not stationary with respect to the end slopes when the strict inequalities are satisfied.

d) <u>Problem 4</u>

The class of admissible structures is restricted to the subset of S having a given initial and final slope. The end traction is also given. For a symmetrical case the boundary conditions on the state variables are

$$\left.\begin{aligned} r(0) &= r(2) = \frac{2}{\gamma} \\ r,_x(0) &= -r,_x(2) = \alpha \end{aligned}\right\} \quad (5.36a)$$

$$\left.\begin{array}{l} n_1(0) = F \\ n_1(2), \text{ determined from (5.26), is considered given.} \end{array}\right\} \quad (5.36a)$$

The boundary conditions on the Lagrange's multipliers derived from (3.32) are:

$$\lambda_1(0),\ \lambda_1(2),\ \lambda_2(0),\ \lambda_2(2),\ \lambda_3(0),\ \lambda_3(2) \text{ are unknown.} \quad (5.36b)$$

Theorem 5-1

The function $r(x) = r(0)$ generates a solution of problem 3 but not of problem 1 or problem 2. It is also a solution of problem 4 provided the imposed initial and final slopes are zero.

Let us first show that the field optimality equations are satisfied.

(5.22c) $\Rightarrow$ $\quad n_2 = 0$

(5.24a) $\Rightarrow$ $\quad \dfrac{dn_1}{dx} = 0$

(5.22b) $\Rightarrow$ $\quad t = |n_1|$

(5.23a) $\Rightarrow$ $\quad \dfrac{d\lambda_1}{dx} = -t$

$$\lambda_1 = -tx + K_1$$

(5.23c) $\Rightarrow$ $\quad \dfrac{d\lambda_3}{dx} = -\dfrac{n_1 r}{t}$

$$\lambda_3 = -\frac{n_1 r}{t} x + K_3$$

(5.23b) $\Rightarrow$ $\quad \dfrac{d\lambda_2}{dx} = -\lambda_1 + \lambda_3 \dfrac{n_1}{r} \Rightarrow \lambda_2 = (\dfrac{K_3 n_1}{r} - K_1)\, x + K_2$

Let us now substitute the solutions for λ_1, λ_2, λ_3 in (5.22a) which is the last equation to be satisfied.

$$(5.22a) \Longrightarrow \qquad n_2 = \tfrac{1}{2} n_1 - \left\{\left(\frac{K_3 n_1}{r} - K_1\right) x + K_2\right\} \frac{t}{n_1 r^2} = 0$$

This will be satisfied iff:

$$\left.\begin{aligned} &K_3 \frac{n_1}{r} = K_1 && \text{(a)} \\ &\tfrac{1}{2} n_1 = \frac{K_2 t}{n_1 r^2} && \text{(b)} \\ &\text{i.e., iff } \lambda_2 = K_2 && \text{(c)} \end{aligned}\right\} \qquad (5.37)$$

Let us now verify that the boundary conditions are not satisfied for problems 1 and 2.

i) <u>Problem 1</u>

$$(5.27b) \Longrightarrow \qquad \lambda_2(0) = \lambda_2(2) = 0$$

$$(5.37c) \Longrightarrow \qquad K_2 = 0$$

but this contradicts (5.37b). q.e.d.

The proof for problem 2 is identical.

ii) <u>Problem 3</u>

Let us now show that the boundary conditions are satisfied for problem 3. Since the cylinder satisfies $r,_x(0) = r,_x(2) = 0$ and $n_1(0) = n_1(2) = F$, then from (5.37) $\lambda_2 = K_2 = \frac{1}{2} F r^2$, which implies that the conditions (5.35) are satisfied. q.e.d.

iii) <u>Problem 4</u>

Since the boundary conditions on the Lagrange's multiplier are arbitrary in this problem, the curve $r(x) = r(0)$ is a solution provided that the imposed slopes are zero.

Theorem 5-2

The curve $r(x) = K \cosh \frac{x-1}{K}$ generates a solution of problem 4, provided that the boundary conditions are compatible with such a solution. It is not a solution of problems 1 through 3.

Proof:

Let us first remark that the shape under consideration corresponds to the surface of revolution of minimum area of length 2 and of end radii

$$r(0) = r(2) = \frac{2}{\gamma} = K \cosh \frac{1}{K}$$

A detailed discussion of this problem is given in Ref. [19].

Using the shape under consideration in the equilibrium equations (5. 4) and (5. 5) one obtains:

$$\left.\begin{aligned} &n_1(x) = n_2(x) \\ &\frac{dn_1}{dx} = 0 \end{aligned}\right\} \qquad (5.38)$$

Upon substitution in the optimality conditions the following solution for the Lagrange's multipliers is obtained:

$$\left.\begin{aligned} \lambda_1 &= -\tfrac{1}{2} Kt \left[\frac{x+c}{K \cosh \frac{x-1}{K}} + 2 \sinh \frac{x-1}{K} \right] \\ \lambda_2 &= \tfrac{1}{2} K^2 t \left[\frac{x+c}{K} \frac{\sinh \frac{x-1}{K}}{\cosh^2 \frac{x-1}{K}} - \frac{1}{\cosh \frac{x-1}{K}} \right] \\ \lambda_3 &= -\tfrac{1}{2} \frac{K^2 t}{n_1} \left[\sinh \frac{x-1}{K} \cosh \frac{x-1}{K} + \frac{x+c}{K} \right] \end{aligned}\right\} \qquad (5.39)$$

where c is a constant of integration to be determined from the boundary conditions on the Lagrange's multipliers. Because of the symmetry of the shape under consideration the following symmetry relations must be satisfied:

$$\left.\begin{aligned} \lambda_1(x) &= -\lambda_1(2-x) \\ \lambda_2(x) &= \lambda_2(2-x) \\ \lambda_3(x) &= -\lambda_3(2-x) \end{aligned}\right\} \qquad (5.40)$$

Those relations imply that $c = -1$.

i) Problem 1

$$(5.27b) \Rightarrow \quad \lambda_2(0) = -\lambda_2(2)$$

$$(5.40) \Rightarrow \quad \lambda_2(0) = \lambda_2(2) = 0$$

which is not satisfied in (5.39). q.e.d.

ii) Problem 2

$$(5.40) \text{ and } (5.28b) \Rightarrow \quad \nu_1 = -\nu_2$$

The remaining part of the proof is identical to the one for problem 1.

iii) Problem 3

Since $r,_x(0) \neq 0$, the boundary conditions implied by (5.33) are applicable here. A direct substitution shows that they are not satisfied.

iv) Problem 4

Since the boundary conditions on the Lagrange's multipliers are arbitrary in this problem, the curve $r(x) = K \cosh \frac{x-1}{K}$ is a solution provided that it exists and that the imposed slopes are compatible with such a solution.

5-3-4. Case where $p \neq 0$

When the applied pressure on the shell is non-zero, closed form solutions of the optimality conditions were not found. Numerical solutions are sought only for the case of shells with a prescribed zero initial slope in the interval $x \in [0, 1]$.

The end traction per unit length on the shell is supposed to be formed of two parts:

i) the traction a pressure vessel head would transmit to the shell when subjected to the pressure p : $\frac{1}{2} pr_0$

ii) an additional, externally applied load per unit length of circumference $\overline{F}_0$, not associated with the pressure. A parameter ω which describes the loading is defined as:

$$\omega = \frac{2\overline{F}_0}{pr_0} \tag{5.41}$$

Let us solve the equations (5.22) for the control variables t, n_2 and $r,_{xx}$, defining:

$$g \equiv \frac{1}{r\sqrt{1+r,_x^2}} \left(\frac{\lambda_2}{n_1} \frac{1+r,_x^2}{r} + \lambda_3 \frac{r,_x}{r} \right) \tag{5.42}$$

When $g^2 < 1$ then:

$$\left.\begin{aligned} t &= \tfrac{1}{2}\sqrt{\frac{3n_1^2}{1-g^2}} && \text{(a)} \\ n_2 &= \tfrac{1}{2} n_1 - gt && \text{(b)} \end{aligned}\right\} \tag{5.43}$$

$$r,_{xx} = \frac{n_2}{n_1}\frac{1+r,_x^2}{r} - \frac{\gamma(1+r,_x^2)^{3/2}}{n_1} \qquad (5.43)$$

Equations (5.43) along with equations (5.23) and (5.24) define a system of differential equations for which the boundary conditions are:

at $x = 0$

$r = \frac{2}{\gamma}$	λ_1 unknown	(a)	
$r,_x = 0$	λ_2 unknown	(b)	
$n_1 = 1 + \omega$	λ_3 unknown	(c)	

at $x = 1$ (5.44)

r unknown	$\lambda_1 = 0$	(d)	
$r,_x = 0$	λ_2 unknown	(e)	
n_1 unknown	$\lambda_3 = 0$	(f)	

The numerical results for $r(x)$, $t(x)$, $n_1(x)$ and $n_2(x)$ are presented in figures 7 through 18 for $\gamma = 1.0$, $\gamma = 1.5$ and $\gamma = 2.0$ respectively with different values of the load coefficient ω. The material volume is plotted in figure 20.

Discussion of the results for $\gamma = 1.5$

For large values of ω, i.e., when the additional end traction is large with respect to the applied pressure, the optimal structure tends, as expected, to a cylindrical shell, which is the solution of problem 4 in the previous section. As ω decreases a neck down appears in the shape of the structure and both the values of the thickness t and of the hoop stress n_2 increase near the middle of the shell. Finally, numerical solutions could not be obtained for $\omega < 0.25$. Figure 19 gives a plot of the value of $|g|$ at $x = 1$ as a function of ω.

A physical interpretation of the condition $|g| = 1$ can be found by including the possibility of a ring if $r,_x \neq 0$ at $x = 1$, in a similar manner as for problem 3 in the previous section. The boundary conditions on the Lagrange's multipliers λ_2 and λ_3 at $x = 1$ become:

if $\quad r,_x = 0 \quad$ then:

$$\left.\begin{cases} |\lambda_2| < r^2 n_1 & \text{(a)} \\ \lambda_3 = 0 & \text{(b)} \end{cases}\right\} \tag{5.45}$$

if $\quad r,_x < 0 \quad$ then:

$$\frac{1}{r\sqrt{1+r,_x^2}}\left(\frac{\lambda_2}{n_1}\,\frac{1+r,_x^2}{r} + \lambda_3\,\frac{r,_x}{r}\right) + 1 = 0 \tag{5.46}$$

The left hand side of (5.46) corresponds to g as defined in (5.42), i.e., $\quad g+1 = 0$

if $\quad r,_x > 0 \quad$ then: $\; g-1 = 0$

Therefore, the optimal structure would present a discrete ring at $x = 1$ when $|g| = 1$.

5-3-5. Conclusion

We have shown the influence of the boundary conditions, on the shape function r(x), on the optimality character of the cylindrical shell when the applied pressure vanishes.

For the combined edge traction and applied pressure loading case, numerical solutions of the optimality conditions were found up to a critical value of the load coefficient ω where a discrete ring would appear at the middle of the shell. However it was not possible to find those solutions on which the slope at the middle of the structure would be non-zero.

The problem of a membrane shell was also treated by Stroud [23]. In his treatment the shape was expanded in a truncated trigonometric series symmetric about the center ($x = 1$). Stroud then found solutions for the series coefficients for a range of values of ω and γ. The results showed an oscillatory behavior of the thickness for the case where $\omega = 0.2$, $\gamma = 1.5$. An explanation of the oscillatory behavior of his results for this case can be found from the fact that his truncated expansion is attempting to represent a discontinuity in slope at the center as well as what would appear to be a ring. In Stroud analysis this would take the form of an infinite thickness (finite area ring) at $x = 1$. From the formulation above, it is clear that for $\omega < .25$ ($\gamma = 1.5$) the formulation of the problem without a ring leads to an ill-posed problem. One would suspect that the inclusion of a ring at the center, as in the previous section, would alleviate this problem. Attempts to do so were unsuccessful.

When the membrane theory is used as a model of the shape to perform both a shape and a thickness optimization, with a stress constraint, the possibility of discrete rings has to be included in the formulation. However a critical analysis of the results would be necessary since the compatibility of the displacements between the ring and the remaining part of the structure would not be necessarily satisfied.

5-4. Optimum Membrane Shell with a Stiffness Constraint

Since the stress resultants n_1 and n_2 can be computed directly from (5. 1) and (5. 2), the strain energy, in the equilibrium configuration, is used as an objective function and the material volume is imposed as a constraint. A detailed discussion of the case $p = 0$ is not repeated, but the same results hold as for the stress case. The optimality conditions are derived for a combined edge and pressure loading. Numerical solutions, for the case of a zero imposed initial slope, are presented which show that again the problem becomes singular for some range of the parameters.

The strain energy of the structure is given by:

$$\bar{J} = \frac{1}{2} \int_{x_0}^{x_0+2L} 2\pi \, \frac{\bar{n}_1^2 - 2\nu\, \bar{n}_1 \bar{n}_2 + \bar{n}_2}{E\,\bar{t}} \, \bar{r} \sqrt{1+\bar{r},_{\bar{x}}^2} \, d\bar{x} \tag{5. 47}$$

where E is the material Young's modulus.

The material volume is:

$$\bar{V} = \int_{x_0}^{x_0+2L} 2\pi \, \bar{r}\, \bar{t} \sqrt{1 + \bar{r},_{\bar{x}}^2} \, d\bar{x} \tag{5. 48}$$

Upon introduction of dimensionless variables defined by (5. 3) and of:

$$\bar{t} = \frac{4\pi L^2 t}{V} \tag{5.49}$$

the objective function and the material constraint reduce respectively to

$$J = \frac{1}{2}\int_0^2 \frac{n_1^2 - 2\nu n_1 n_2 + n_2^2}{t} \; r\sqrt{1+r,_x^2}\; dx \tag{5.50}$$

$$\int_0^2 r\, t\sqrt{1+r,_x^2}\; dx = 2 \tag{5.51}$$

where

$$\bar{J} = 2\,\frac{p^2}{E}\left(\frac{r_0}{2L}\right)^2 \frac{(2\pi L^3)^2}{V}\; J \tag{5.52}$$

5-4-1. Formulation

Because of the symmetry in the imposed boundary conditions the problem can be formulated as:

$$\text{Min } J = \int_0^1 \frac{n_1^2 - 2\nu n_1 n_2 + n_2^2}{t} \; r\sqrt{1+r,_x^2}\; dx \tag{5.53}$$

subjected to:

i) the material volume constraint

$$\int_0^1 r\, t\sqrt{1+r,_x^2}\; dx = 1 \tag{5.54}$$

ii) the state equations

$$\left.\begin{aligned} \frac{dr}{dx} &= r,_x && \text{(a)} \\ \frac{dr,_x}{dx} &= r,_{xx} && \text{(b)} \\ \frac{dn_1}{dx} &= -\frac{r,_x}{r}\left[n_1 - n_2\right] && \text{(c)} \end{aligned}\right\} \tag{5.55}$$

iii) the constraint

$$\frac{r,_{xx} n_1}{(1+r,_x^2)^{3/2}} - \frac{n_2}{r\sqrt{1+r,_x^2}} + \gamma = 0 \qquad (5.56)$$

iv) the boundary conditions:

$r(0) = \frac{2}{\gamma}$ $\qquad$ $r(1)$ unknown

$r,_x(0) = 0$ $\qquad$ $r,_x(1) = 0$

$n_1(0) = 1 + \omega$ $\qquad$ $n_1(1)$ unknown

The Hamiltonian H of the system is obtained by adjoining to its Lagrangian the state equations (5.55), the material volume constraint (5.54) and the constraint (5.56). The state variables are r, $r,_x$, n_1 and the control variables are $r,_{xx}$, t and n_2.

$$H = \frac{n_1^2 - 2\nu n_1 n_2 + n_2^2}{t}\, r\sqrt{1+r,_x^2} + \lambda_4\, r t \sqrt{1+r,_x^2}$$

$$+ \lambda_1 r,_x + \lambda_2 r,_{xx} - \lambda_3 \frac{r,_x}{r}(n_1 - n_2)$$

$$+ \eta_1 \left[\frac{n_1 r,_{xx}}{(1+r,_x^2)^{3/2}} - \frac{n_2}{r\sqrt{1+r,_x^2}} + \gamma\right] \qquad (5.57)$$

where λ_4 is the Lagrange's multiplier associated with the material volume (5.54).

5-4-2. Optimality Conditions

The first order optimality conditions are:

$$H_t = -\frac{n_1^2 - 2\nu n_1 n_2 + n_2^2}{t^2} + \lambda_4 = 0 \qquad (a)$$

$$H_{r,_{xx}} = \lambda_2 + \eta_1 \frac{n_1}{(1+r,_x^2)^{3/2}} = 0 \qquad (b)$$

$$H_{n_2} = \lambda_3 \frac{r,_x}{r} + \frac{2(n_2 - \nu n_1)}{t} r\sqrt{1+r,_x^2} - \eta_1 \frac{1}{r\sqrt{1+r,_x^2}} = 0 \qquad (c)$$

(5.58)

Using (5.58a) the Lagrange's multipliers λ_1, λ_2, λ_3 are given by:

$$\frac{d\lambda_1}{dx} = -H_r = -\frac{\lambda_3 r,_x}{r^2} - \eta_1 \frac{n_2}{r^2\sqrt{1+r,_x^2}} - 2\lambda_4 t\sqrt{1+r,_x^2} \qquad (a)$$

$$\frac{d\lambda_2}{dx} = -H_{r,_x} = -\lambda_1 + \frac{\lambda_3}{r}(n_1 - n_2) +$$

$$\eta_1 \left(\frac{3n_1 r,_x r,_{xx}}{(1+r,_x^2)^{5/2}} - \frac{n_2\, r,_x}{r(1+r,_x^2)^{3/2}} \right) - 2\lambda_4 rt \frac{r,_x}{\sqrt{1+r,_x^2}} \qquad (b)$$

$$\frac{d\lambda_3}{dx} = -H_{n_1} = -\frac{2(n_1 - \nu n_2)}{t} r\sqrt{1+r,_x^2} + \lambda_3 \frac{r,_x}{r}$$

$$-\eta_1 \frac{r,_{xx}}{(1+r,_x^2)^{3/2}} \qquad (c)$$

(5.59)

The volume material constraint (5.54), the state equations (5.55) and the constraint (5.56) are also part of the optimality conditions.

Eliminating the unknown Lagrange's multiplier η_1, and solving (5.58) for the control variables one obtains if $n_1 \neq 0$ and $g^2 < \lambda_4$:

$$g \equiv \frac{1}{2r\sqrt{1+r,_x^2}} \quad \lambda_3 \frac{r,_x}{r} + \frac{\lambda_2}{n_1} \frac{1+r,_x^2}{r} \tag{5.60}$$

$$\left.\begin{aligned} t &= \sqrt{\frac{n_1^2(1-\nu^2)}{\lambda_4-g^2}} && \text{(a)} \\ n_2 &= \nu n_1 - gt && \text{(b)} \end{aligned}\right\} \tag{5.61}$$

$$\left.\begin{aligned} \frac{dr,_x}{dx} &= \frac{n_2}{n_1} \frac{1+r,_x^2}{r} - \frac{\gamma(1+r,_x^2)^{3/2}}{n_1} && \text{(a)} \\ \frac{dn_1}{dx} &= -\frac{r,_x}{r}(n_1-n_2) && \text{(b)} \end{aligned}\right\} \tag{5.62}$$

$$\frac{d\lambda_1}{dx} = \lambda_2 \frac{n_2}{n_1} \frac{1+r,_x^2}{r^2} - \lambda_3 \frac{r,_x}{r^2}(n_1-n_2) - 2\lambda_4 t\sqrt{1+r,_x^2} \qquad \text{(a)}$$

$$\frac{d\lambda_2}{dx} = -\lambda_1 - \lambda_2\left(\frac{2n_2}{n_1}\frac{r,_x}{r} - \frac{3\gamma r,_x\sqrt{1+r,_x^2}}{n_1}\right) + \frac{\lambda_3}{r}(n_1-n_2) - 2\lambda_4 rt\frac{r,_x}{\sqrt{1+r,_x^2}} \qquad \text{(b)}$$

$$\frac{d\lambda_3}{dx} = \frac{\lambda_2}{n_1}\left(\frac{n_2}{n_1}\frac{1+r,_x^2}{r} - \frac{\gamma(1+r,_x^2)^{3/2}}{n_1}\right) + \lambda_3\frac{r,_x}{r} - \frac{2(n_1-\nu n_2)}{t} r\sqrt{1+r,_x^2} \qquad \text{(c)}$$

(5.63)

$$\int_0^1 rt\sqrt{1+r,_x^2}\, dx = 1 \tag{5.64}$$

The boundary conditions are:

$$
\left.\begin{array}{llllr}
\text{at } x = 0 & r(0) = \frac{2}{\gamma} & \lambda_i(0) \text{ unknown} & (a) \\
 & r,_x = 0 & \lambda_2(0) \text{ unknown} & (b) \\
 & n_1 = 1+\omega & \lambda_3(0) \text{ unknown} & (c) \\
\text{at } x = 1 & r(1) \text{ unknown} & \lambda_1(1) = 0 & (d) \\
 & r,_x(1) = 0 & \lambda_2(1) \text{ unknown} & (e) \\
 & n_1(1) \text{ uknown} & \lambda_3(1) = 0 & (f)
\end{array}\right\} (5.65)
$$

The value of λ_4 is determined by the satisfaction of the material volume constraint (5.64).

5-4-3. Results

The numerical results for $r(x)$, $t(x)$, $n_1(x)$ and $n_2(x)$ are presented in the figures 21 through 28 for $\gamma = 1.0$ and $1 = 1.5$ respectively with different values of ω and $\nu = 0.25$. The strain energy value is plotted in figure 30.

The dependence of the optimal structure on the load coefficient ω follows the same trends as for the stress constraint case. A plot of $\frac{g^2}{\lambda_4}$ is given in figure 29 for the case $\gamma = 1.5$.

A physical interpretation of the condition $g^2 = \lambda_4$ can again be found by including the possibility of a ring if $r,_x \neq 0$ at $x = 1$, in a similar manner as of problem 3 in the previous section.

Let A be the dimensionless cross section of the ring at $x = 1$. The modified objective function augmented of the modified material constraint by means of its Lagrange's multiplier λ_4 is:

$$J = \left[r \left(\frac{n_1 r r,_x}{\sqrt{1+r,_x^2}} \right)^2 \frac{1}{A} \right]_{x=1} + \int_0^1 \frac{n_1^2 - 2\nu n_1 n_2 + n_2^2}{t} \, r\sqrt{1+r,_x^2}\, dx$$

$$+ \lambda_4 (rA - \int_0^1 rt\sqrt{1+r,_x^2}\, dx - 1 \qquad (5.66)$$

where $A > 0$ iff $r,_x \neq 0$

Let us compute the variation δJ of our objective function for admissible variations $\delta r(1)$, $\delta r,_x(1)$ and $\delta n_1(1)$ satisfying the overall equilibrium equation of the shell:

$$\left. \frac{n_1}{\sqrt{1+r,_x^2}} \right|_{x=1} = \frac{r(0)}{r(1)} \omega + \frac{r(1)}{r(0)} \qquad (5.67)$$

$$\delta J = \left\{ -\frac{1}{A^2} \left(\frac{n_1 r r,_x}{\sqrt{1+r,_x^2}} \right)^2 + \lambda_4 \right\} r \Big|_{x=1} \delta A$$

$$+ \left\{ \frac{3}{A} \left(\frac{n_1 r r,_x}{\sqrt{1+r,_x^2}} \right)^2 - \lambda_1 \right\}_{x=1} \delta r(1)$$

$$+ \left\{ \frac{1}{A} r^3 n_1^2 \frac{2r,_x}{(1+r,_x^2)^2} - \lambda_2 \right\}_{x=1} \delta r,_x(1)$$

$$+ \left\{ \frac{2}{A} n_1 r \left(\frac{r,_x r}{\sqrt{1+r,_x^2}} \right)^2 - \lambda_3 \right\}_{x=1} \delta n_1(1) \qquad (5.68)$$

Since a relation between $\lambda_2(1)$, $\lambda_3(1)$ and λ_4 is sought it is sufficient to consider the variations of J, called $\delta\tilde{J}$, with respect to A and $r,_x(1)$ only, where $\delta n_1(1)$ is expressed as a function of $\delta r,_x(1)$ using the constraint (5.67).

$$\delta\tilde{J} = \left[\left(-\frac{1}{A^2}\left(\frac{n_1 r r,_x}{1+r,_x^2}\right)^2 + \lambda_4\right) r\right]_{x=1} \delta A$$

$$+ \left[\frac{2}{A} r^3 n_1^2 \frac{r,_x}{1+r,_x^2} - \lambda_2 - \lambda_3 \frac{r,_x n_1}{1+r,_x^2}\right]_{x=1} \delta r,_x(1) \qquad (5.69)$$

In order to have $\delta\tilde{J} = 0$ for arbitrary $\delta r,_x(1)$ and δA, which are admissible when $A > 0$, the following relations giving respectively the cross section area of the ring and the boundary condition on λ_2 and λ_3 must hold:

$$\frac{1}{A^2}\left(\frac{n_1 r r,_x}{\sqrt{1+r,_x^2}}\right)^2 \Bigg|_{x=1} = \lambda_4 \qquad (5.70)$$

$$\left\{\left(\frac{\lambda_2}{n_1}\frac{1+r,_x^2}{r} + \lambda_3 \frac{r,_x}{r}\right) \times \frac{1}{2r\sqrt{1+r,_x^2}}\right\}^2 \Bigg|_{x=1} = \lambda_4 \qquad (5.71)$$

The left hand side of (5.71) corresponds to g^2 as defined by (5.60). Therefore the optimal structure would present a discrete ring at $x = 1$ if $g^2 = \lambda_4$.

5-4-4. Conclusion

As for the stress case, numerical solutions of the optimality equations were found up to a critical value of the load coefficient ω, where a discrete ring would appear at the middle of the shell. However it was not possible to find these solutions.

VI. SHELL STRUCTURES WITH BENDING EFFECTS

In Chapter V we treated the optimum design of axisymmetric membrane shells for either a stress or a stiffness constraint. In the present chapter we study the optimization problem of axisymmetric shells, including the bending effects in their description, for a stiffness constraint.

The governing equations of shells of revolution subjected to axisymmetric loads are first obtained using Koiter's best linear theory of shells, modified according to [24], to include the effects of shear deformation. The objective function and the optimality conditions for the mass minimization of shells subjected to a stiffness constraint are derived. It is then shown that the problem can be ill-posed.

Problem Definition

Let S be the set of shells of revolution such that

a) their governing equations include the bending and shear effects

b) their length is 2L and their end radii are r_0

c) they support axisymmetric end tractions and a normal pressure to their middle surface

d) they have prescribed boundary conditions

e) they satisfy a given stiffness constraint in their equilibrium configuration.

We seek the shape $\overline{r}(\overline{x})$ and the thickness distribution $\overline{t}(\overline{x})$ of the element in S of minimum mass.

6-1. Governing Equations

The detailed derivation of the governing equations of shells of revolution subjected to a normal pressure p and axisymmetric end loads is contained in Appendix A.

The three equilibrium equations, the strain measures-stress resultant relations and the strain measures-displacement relations are combined to obtain a system of 6 first order linear ordinary differential equations:

$$\frac{d\bar{z}}{d\bar{x}} = \bar{B}\,\bar{z} + \bar{f} \qquad (6.1)$$

where:

i) $\bar{B}$ is a 6 x 6 matrix, function only of the meridianal curve $\bar{r}(\bar{x})$, its derivative $\bar{r}_{,\bar{x}}$ and the thickness $\bar{t}$. The components of $\bar{B}$ are given in Table 6-I.

ii)

$$z \equiv \begin{bmatrix} \bar{u} & \text{.... axial displacement} \\ \bar{w} & \text{.... radial displacement} \\ \bar{T} & \text{.... axial force per unit length} \\ \bar{\varphi} & \text{.... rotation of the normal} \\ \bar{m} & \text{.... bending moment per unit length} \\ \bar{R} & \text{.... radial force per unit length} \end{bmatrix} \qquad (6.2)$$

The sign conventions used for the variables are best illustrated in Figure 31.

$$\text{iii)} \quad \bar{\underset{\sim}{f}} \equiv \begin{bmatrix} 0 \\ 0 \\ p\bar{r},_{\bar{x}} \\ 0 \\ 0 \\ -p \end{bmatrix} \tag{6.3}$$

Let $\overline{N}$, $\overline{Q}$, $\bar{u}_p$ and $\bar{w}_n$ be respectively the in-plane and normal components of the stress resultant and displacement vectors.

$$\overline{N} = \frac{1}{\sqrt{1+\bar{r},_{\bar{x}}^2}} \left[\overline{T} + \bar{r},_{\bar{x}} \overline{R} \right] \tag{6.4a}$$

$$\overline{Q} = \frac{1}{\sqrt{1+\bar{r},_{\bar{x}}^2}} \left[-\bar{r},_{\bar{x}} \overline{T} + \overline{R} \right] \tag{6.4b}$$

$$\bar{u}_p = \frac{1}{\sqrt{1+\bar{r},_{\bar{x}}^2}} \left[\bar{u} + \bar{r},_{\bar{x}} \bar{w} \right] \tag{6.4c}$$

$$\bar{w}_n = \frac{1}{\sqrt{1+\bar{r},_{\bar{x}}^2}} \left[-\bar{r},_{\bar{x}} \bar{u} + \bar{w} \right] \tag{6.4d}$$

6-2. Objective Function Definition

Since it is not possible to evaluate the stress resultants without computing the displacement field when the bending effects are included in the description of the shell behavior, the work done by the external forces during the deformation is used as the objective function $\bar{J}$.

$$\bar{J} = W_1 + W_2 \tag{6.5}$$

where:

i) W_1: the work done by the applied pressure on the shell middle surface is given by:

$$W_1 = \frac{1}{2} \int_{x_0}^{x_0+2L} 2\pi \bar{r}\, p\, \bar{w}_n \sqrt{1+\bar{r}_{,\bar{x}}^{2}}\, d\bar{x} \tag{6.6a}$$

ii) W_2: the work done by the prescribed edge loads is given by

$$W_2 = \frac{1}{2}\left[2\pi\bar{r}\left\{S_T\,\bar{u}\,\bar{T} + S_R\,\bar{w}\,\bar{R} + S_m\,\bar{m}\,\bar{\varphi}\right\}\right]_{x_0+2L} - \frac{1}{2}\left[2\pi\bar{r}\left\{S_T\,\bar{u}\,\bar{T} + S_R\,\bar{w}\,\bar{R} + S_m\,\bar{m}\,\bar{\varphi}\right\}\right]_{x_0} \tag{6.6b}$$

$S_F = 0$ if $\bar{F}$ is not a prescribed edge load

$S_F = 1$ if $\bar{F}$ is a prescribed edge load

The constraints to be imposed on the system are as previously, the prescribed volume of the material in the structure and the state equations which include the shell governing equations.

6-3. Non-dimensionalization

Upon introduction of the following dimensionless variables,

$$u = \frac{\bar{u}}{\frac{pr_0}{EV_0}\pi L^3}, \quad w = \frac{\bar{w}}{\frac{pr_0}{EV_0}\pi L^3}, \quad \varphi = \frac{\bar{\varphi}}{\frac{pr_0}{EV_0}\pi L^2}$$

$$T = \frac{\bar{T}}{\frac{pr_0}{2}}, \quad R = \frac{\bar{R}}{\frac{pr_0}{2}}, \quad m = \frac{m_0}{\frac{pr_0}{2}L}$$

$$t = \frac{\bar{t}}{\left(\frac{V_0}{2\pi L^2}\right)}, \quad x = \frac{\bar{x}}{L}, \quad r = \frac{\bar{r}}{L} \tag{6.7}$$

the dimensionless governing equations, the objective function and the volume constraint depend only on the two following parameters:

$$\text{i)} \quad \alpha^2 = 12 \, {\frac{2\pi L^3}{V_0}}^{2} \tag{6.8}$$

which characterizes the ratio between the membrane and bending stiffnesses.

$$\text{ii)} \quad \gamma = \frac{2L}{r_0} \tag{6.9}$$

which characterizes the shell "aspect ratio."

The dimensionless objective function J is given by:

$$J = \frac{12\gamma}{\alpha^2} \frac{E}{p^2 V_0} \overline{J} \tag{6.10}$$

The dimensionless governing equations are

$$\frac{d\underset{\sim}{z}}{dx} = B \underset{\sim}{z} + \underset{\sim}{f} \tag{6.11}$$

where the components of $\underset{\sim}{z}$ correspond to the dimensionless components of $\overline{\underset{\sim}{z}}$. The components of B and $\underset{\sim}{f}$ are described in Tables 6.2 and 6.3.

6-4. Optimality Conditions

The Hamiltonian, H, of the problem is constructed by adjoining to the Lagrangian the state equations and the volume constraint by means of Lagrange multipliers.

$$H = a + \underset{\sim}{A}^T \underset{\sim}{z} + \underset{\sim}{\lambda}^T [B \underset{\sim}{z} + \underset{\sim}{f}] + \eta \, r,_x \tag{6.12}$$

where:

i) $a = \lambda_v rt \sqrt{1 + r,_x^2}$ (6.13)

ii) $A^T = [-r,_x r \,|\, r \,|\, 0 \,|\, 0 \,|\, 0 \,|\, 0]$ (6.14)

iii) λ^T : Lagrange multipliers associated with the governing state equations (6.11).

iv) η : Lagrange multiplier associated with the shape state equation.

v) λ_v : Lagrange multiplier associated with the material volume constraint.

Since the only two control variables are $r,_x$ and t the first order optimality conditions are given by:

$$H_{r,_x} = a_{r,_x} + \underset{\sim}{A}^T_{r,_x} \underset{\sim}{z} + \underset{\sim}{\lambda}^T [B_{r,_x} \underset{\sim}{z} + \underset{\sim}{f}_{r,_x}] + \eta = 0 \qquad (6.15a)$$

$$H_t = a_r + \underset{\sim}{\lambda}^T B_t \underset{\sim}{z} = 0 \qquad (6.15b)$$

$$\frac{d\underset{\sim}{\lambda}}{dx} = -B^T \underset{\sim}{\lambda} - \underset{\sim}{A} \qquad (6.15c)$$

$$\frac{d\eta}{dx} = -a_r - A_r^T \underset{\sim}{z} - \underset{\sim}{\lambda}^T [B_r \underset{\sim}{z} + \underset{\sim}{f}_r] \qquad (6.15d)$$

$$\frac{d\underset{\sim}{z}}{dx} = B \underset{\sim}{z} + \underset{\sim}{f} \qquad (6.15e)$$

$$\frac{dr}{dx} = r,_x \qquad (6.15f)$$

$$\int_o^1 tr \sqrt{1 + r,_x^2}\, dx = 1 \qquad (6.15g)$$

where (6. 15f) and (6. 15g) are respectively the dimensionless shape state equation and the volume constraint.

Instead of solving the two nonlinear equations (6.15a and b) for $r_{,x}$ and t, the following equivalent set of equations (6.16) is used to generate the two linear equations (6.17) in $\frac{dr_{,x}}{dx}$ and $\frac{dt}{dx}$.

$$\frac{dH_{r,x}}{dx} = 0 \ , \qquad H_{r,x} = 0 \quad \text{for some } x \epsilon [0,1] \tag{6.16a}$$

$$\frac{dH_t}{dx} = 0 \ , \qquad H_t = 0 \quad \text{for some } x \epsilon [0,1] \tag{6.16b}$$

$$F \begin{bmatrix} \frac{dr_{,x}}{dx} \\ \frac{dt}{dx} \end{bmatrix} = \underset{\sim}{g} \tag{6.17}$$

where

$$\text{i)} \quad F = \begin{bmatrix} H_{r,_x r,_x} & H_{tr,_x} \\ H_{tr,_x} & H_{tt} \end{bmatrix} \tag{6.18a}$$

is the matrix of the second variation of the Hamiltonian with respect to the control variables

$$H_{r,_x r,_x} = \frac{\lambda_v t r}{(1+r_{,x}^2)^{3/2}} + \underset{\sim}{\lambda}^T B_{r,_x r,_x} \underset{\sim}{z} \tag{6.18b}$$

$$H_{tr,_x} = \frac{\lambda_v r r_{,x}}{\sqrt{1+r_{,x}^2}} + \underset{\sim}{\lambda}^T B_{r,_x t} \underset{\sim}{z} \tag{6.18c}$$

$$H_{tt} = \underset{\sim}{\lambda}^T B_{tt} \underset{\sim}{z} \tag{6.18d}$$

$$\text{ii)} \quad \underset{\sim}{g} = \begin{bmatrix} \underset{\sim}{\lambda}^T D \underset{\sim}{z} + \gamma \underset{\sim}{\lambda}^T \underset{\sim}{f}_1 + \underset{\sim}{f}_2 \underset{\sim}{z} + \lambda_v \dfrac{t}{\sqrt{1 + r,_x^2}} \\ \underset{\sim}{\lambda}^T E \underset{\sim}{z} - \lambda_v r,_x \sqrt{1 + r,_x^2} \end{bmatrix} \tag{6.19a}$$

$$D = [BB_{r,_x} - B_{r,_x} B + B_r - B_{rr,_x} r,_x] \tag{6.19b}$$

$$\underset{\sim}{f}_1 = B \underset{\sim}{f}_{r,_x} - B_{r,_x} \underset{\sim}{f} \tag{6.19c}$$

$$\underset{\sim}{f}_2 = \underset{\sim}{A}_r - \underset{\sim}{A}_{r,_x r} r,_x + B^T_{r,_x} \underset{\sim}{A} - B^T \underset{\sim}{A}_{r,_x} \tag{6.19d}$$

$$E = [B B_t - B_t B - B_{tr} r,_x] \tag{6.19e}$$

The content of the different matrices is described in Tables 6.4 through 6.10.

Let us remark that the system (6.17) can be solved only if

$$\det [F] = H_{tt} H_{r,_x r,_x} - (H_{tr,_x})^2 \neq 0 \tag{6.20}$$

which is the condition for the original system (6.15a and b) to have an isolated solution.

6-5. Boundary Conditions

After the derivation of the optimality equations of the previous section we now have to derive the boundary conditions on the state variables and the corresponding Lagrange's multipliers.

The boundary conditions at x = 1, i.e., at the middle of the shell will be classified as a "symmetry condition," but the boundary conditions at x = 0 will depend on the type of support.

i) Symmetry Conditions

at $x = 1$:

$$\left.\begin{array}{llll} u = 0 & & \lambda_1 & \text{unknown} \\ w & \text{unknown} & \lambda_2 = 0 & \\ T & \text{unknown} & \lambda_3 = 0 & \\ \varphi = 0 & & \lambda_4 & \text{unknown} \\ m & \text{unknown} & \lambda_5 = 0 & \\ R = 0 & & \lambda_6 & \text{unknown} \\ r & \text{unknown} & \eta = 0 & \end{array}\right\} \qquad (6.21)$$

$(H_{r,x})_{x=1} = 0$ is satisfied if $r,_x(1) = 0$

t is obtained from $(H_t)_{x=1} = 0$

ii) Free end condition:

at $x = 0$:

The prescribed forces are

$$T = 1 + \omega, \qquad m = 0, \qquad R = 0 \qquad (6.22a)$$

where ω is a load coefficient identical as the one defined in (5.41).

The work done by the edge loads is:

$$- \left[\frac{r}{\gamma} \{u\ T + wR + m\varphi\} \right]_{x=0} \qquad (6.22b)$$

The corresponding boundary conditions are:

$$\left.\begin{array}{llll} u & \text{unknown} & \lambda_1 = \frac{r}{\gamma}\,T = \frac{r}{\gamma}\,(1+\omega) \\ w & \text{unknown} & \lambda_2 = \frac{r}{\gamma}\,R = 0 \\ T = 1+\omega & & \lambda_3 \text{ unknown} \\ \varphi & \text{unknown} & \lambda_4 = \frac{r}{\gamma}\,m = 0 \\ m = 0 & & \lambda_5 \text{ unknown} \\ R = 0 & & \lambda_6 \text{ unknown} \\ r = \frac{2}{\gamma} & & \eta \text{ unknown} \end{array}\right\} \quad (6.23)$$

$r_{,x}$ and t are unknown.

iii) Simply supported edge boundary conditions

at x = 0:

a) The prescribed edge forces are

$$T = 1 + \omega \;, \qquad m = 0 \qquad (6.24a)$$

b) The work done by the prescribed edge loads is

$$-\left[\frac{r}{\gamma}\left\{uT + m\varphi\right\}\right]_{x=0} \qquad (6.24b)$$

c) The boundary conditions are:

$$\left.\begin{array}{llll} u & \text{unknown} & \lambda_1 = \frac{r}{\gamma}\,T = \frac{r}{\gamma}\,(1+\omega) \\ w = 0 & & \lambda_2 \text{ unknown} \\ T = 1+\omega & & \lambda_3 \text{ unknown} \\ \varphi & \text{unknown} & \lambda_4 = \frac{r}{\gamma}\,m = 0 \\ m = 0 & & \lambda_5 \text{ unknown} \\ R & \text{unknown} & \lambda_6 = 0 \\ r = \frac{2}{\gamma} & & \eta \text{ unknown} \end{array}\right\} \quad (6.25)$$

iv) Clamped edge boundary conditions

at $x = 0$:

a) the prescribed edge load is

$$T = 1 + \omega \tag{6.26a}$$

b) the work done by the prescribed edge load is

$$-\left[\frac{r}{\gamma}\{uT\}\right]_{x=0} \tag{6.26b}$$

c) the boundary conditions are:

$$\left.\begin{array}{ll} u \quad \text{unknown} & \lambda_1 = \frac{r}{\gamma}T = \frac{r}{\gamma}(1+\omega) \\ w = 0 & \lambda_2 \text{ unknown} \\ T = 1+\omega & \lambda_3 \text{ unknown} \\ \varphi = 0 & \lambda_4 \text{ unknown} \\ m \quad \text{unknown} & \lambda_5 = 0 \\ R \quad \text{unknown} & \lambda_6 = 0 \end{array}\right\} \tag{6.27}$$

6-6. Solution for the Lagrange's Multipliers $\underset{\sim}{\lambda}$.

<u>Theorem 6-1:</u>

When forces or homogeneous displacement boundary conditions are imposed, the following set of relations hold:

$$\begin{array}{lll} \lambda_1 = \frac{r}{\gamma}T, & \lambda_2 = \frac{r}{\gamma}R, & \lambda_3 = -\frac{r}{\gamma}u \\ \lambda_4 = \frac{r}{\gamma}m, & \lambda_5 = -\frac{r}{\gamma}\varphi, & \lambda_6 = -\frac{r}{\gamma}w \end{array} \tag{6.28}$$

Proof: It can be seen by inspection that the Lagrange multipliers $\underset{\sim}{\lambda}$ and the corresponding combination of state variables satisfy the same set of ordinary differential equations.

If the imposed displacement boundary conditions are homogeneous, then the boundary conditions on the Lagrange's multipliers and the corresponding combination of state variables are identical.

The equality (6.28) follows from the uniqueness of the solution of the two points boundary value problem for the $\underset{\sim}{z}$ vector. q.e.d.

6-7. Characteristics of F

Theorem 6-2 : Let the hypothesis of theorem 6-1 hold. Then the matrix F is not positive definite.

Proof: Since H_t and $H_{r,x}$ vanish on the solution we can add any of their combinations to the components of F without changing the value of its determinant.

By substitution one obtains:

$$H_t = -\frac{\sqrt{1+r,_x^2}}{t^2}\left[N^2(1-\nu^2)+\beta^2 Q^2\right]$$

$$-\frac{3a^2}{t^4}(1-\nu^2)\sqrt{1+r,_x^2}\; m^2 - 3\left(\frac{r,_x}{r}\right)^2 \frac{t^2\varphi^2}{a^2\sqrt{1+r,_x^2}}$$

$$-\frac{\sqrt{1+r,_x^2}}{r^2}\, w^2 + \lambda_v r\sqrt{1+r,_x^2} = 0 \qquad (6.29)$$

$$\frac{\gamma}{r}\left\{H_{r,_x r,_x} - \frac{t H_t}{(1+r,_x^2)^{5/2}}\right\} \doteq \frac{2}{t(1+r,_x^2)^{3/2}}\left[Q^2(1-\nu^2)+\beta^2N^2\right]$$

$$+ \frac{4a^2}{t^3}\frac{(1-\nu^2)m^2}{(1+r,_x^2)^{3/2}} m^2 - \frac{t^3}{a^2r^2}\frac{(2-4r,_x^2)}{(1+r,_x^2)^{5/2}}\varphi^2 \qquad (6.30)$$

$$\frac{\gamma}{r} H_{tt} = \frac{2\sqrt{1+r,_x^2}}{t^3}\left[N^2(1-\nu^2) + \beta^2Q^2\right]$$

$$+ \frac{12a^2}{t^5}(1-\nu^2)\sqrt{1+r,_x^2}\, m^2 - 6\left(\frac{r,_x}{r}\right)^2 \frac{t\varphi^2}{a^2\sqrt{1+r,_x^2}} \qquad (6.31)$$

$$\frac{r}{\gamma}\left(H_{tr,_x} - \frac{r,_x}{1+r,_x^2}\right) = \frac{2}{t^2\sqrt{1+r,_x^2}}\left[\beta^2-(1-\nu^2)\right] NQ$$

$$- \frac{6t^2}{a^2r^2}\frac{\varphi^2 r,_x}{(1+r,_x^2)^{3/2}} \qquad (6.32)$$

Upon evaluation one obtains

$$\left(\frac{r}{\gamma}\right)^2 \times \det F = \frac{8a^2}{t^6} m^2 \frac{(1-\nu^2)}{1+r,_x^2}\left\{N^2\left[(1-\nu^2) + 3\beta^2\right]\right.$$

$$\left. + Q^2\left[3(1-\nu^2)+\beta^2\right] + \frac{6a^2}{t^2} m^2(1-\nu^2)\right\}$$

$$+ \frac{2\beta^2}{t^4(1+r,_x^2)}(1-\nu^2)(N^2+Q^2)^2$$

$$+ \frac{2\varphi^2}{a^2r^2(1+r,_x^2)^2}\left\{\beta^2\left[(-4+8r,_x^2)\, Q^2-3r,_x^2 N^2+6r,_x NQ\right]\right.$$

$$\left. + (1-\nu^2)\left[N^2(-4+8r,_x^2) - 3r,_x^2N^2-6r,_x NQ\right]\right\}$$

$$+ \frac{48(1-\nu^2)}{r^2 t^2 (1+r,_x^2)} m^2 \varphi^2$$

$$- \frac{12t^4}{a^4 r^4} \frac{\varphi^4 r,_x^2}{1+r,_x^2} \qquad (6.33)$$

For a free end such that

$$N(0) = Q(0) = m(0) = 0 \qquad (6.34)$$

$$\det F \leq 0 \quad \text{at } x = 0$$

Furthermore the symmetry conditions imply that det $F > 0$ at $x = 1$. Therefore $\exists\, x^*, x^* \in [0, 1] \ni \det F \big|_{x^*} = 0$. At x^* equation (6.19) cannot be solved. Since det F is not a positive definite form in the variables m, N, Q and φ, F is not a positive definite matrix. q.e.d.

Remarks:

a) Using the transformation

$$r(x) = r(0) + y(x)$$

and taking the limit of the equations as $r(0) \to \infty$ similar equations to the arch optimality equations are recovered.

b) An attempt to solve the two points boundary value problem for the case $\beta = 0$, i.e., when the shear contribution is not taken into account, was unsuccessful.

6-8. Conclusion

In chapter IV we treated the optimality problem of arch structures for a stiffness constraint. It was shown that the problem

A microscope glass slide is cut in half along its length. The cut side is sanded smooth and one end is tapered (Fig. 7). Platinum films are baked into the front face of the slide, indium contacts are soldered from the front face to the back, and then copper wires are soldered to the indium contacts on the back face. The flat end of the slide is epoxied perpendicular to the test section bottom flange (Fig. 5).

The films are several hundred Angstroms thick, ~ .03-inch high, and ~ .5-inch wide. Room temperature resistances range from 300 to 600 ohms and the resistances are roughly 30% lower at 4.2^{o}K. Film height and the distance between films are measured on a Kodak optical comparator. Repeatability of the film separation distances (4 - 5 cm) is within ± .1% for six measurements taken over three months. The slide is perpendicular to the flange to within ± .5 of a degree. The cross sectional area of the slide represents 3.2% of the tube cross section.

B.3.g. <u>Additional Instrumentation</u>

Thermocouple and thermistor data are taken using a multi-function meter* capable of reading five significant figures. The voltage output of the films is amplified a thousand times and displayed on an oscilloscope†. A time-mark** is made on each

---

* Hewlett-Packard; Model 3450A.

† Tektronix; Model 555 dual beam using type "L" plug-ins.

** Using a Tektronix Time Mark Generator; Model 180A.

VII. CONCLUSION

The material volume minimization with respect to the shape and the thickness has been investigated for three types of structures subjected to either a stress or a stiffness constraint. The optimality conditions have been derived using the Optimal Control Theory, and a general purpose computer program solving n-points boundary value problems with the parallel shooting techniques has been developed.

The inclusion of the contribution of the shear force in the strain energy density or in the failure criterion was found sufficient to obtain well-posed problems when dealing with statically determinant arches. For the stiffness problem, a membrane design, i.e., a structure on which the bending moment is identically zero, does not correspond to the optimal structure. However it was found to be a very good approximation of the "best" design for practical cases. When the maximum shear failure criterion is imposed, the "best" membrane design corresponds to the true optimum for practical values of the load coefficient. However the number of solutions of the first order optimality conditions was found to depend on the value of the load coefficient.

The optimization problems of membrane shell were investigated for the cases of edge traction only and combined edge loads and pressure. For the detailed discussion of the influence of the boundary conditions on the optimality character of the cylindrical shell, when edge tractions only are applied, the possibility of discrete rings was included in the formulation. Solutions for the combined loading case were obtained up to given values of the load coefficient. The singularity which seems to appear then could also be physically interpreted by including the possibility of a discrete ring at the middle of the shell in the formulation.

When the shear force contribution is not taken into account in the governing equations of shells, derived using the bending theory, the material volume minimization problem with a stiffness constraint was found to be ill-posed. However the inclusion of the shear force contribution does not seem sufficient to always obtain a well-posed problem.

Evidence has been shown that a modification of the governing equations of the structures to include some of the effects judged insignificant in structural analysis may transform an ill-posed optimization problem into a well-posed problem. Furthermore the inclusion of discrete elements in the formulation of the variational problems has been found helpful to interpret the singularities which appeared. A systematic investigation of these two techniques will contribute toward the clarification of the analytical approach in structural optimization.

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APPENDIX A

GOVERNING EQUATIONS FOR AXISYMMETRIC SHELLS OF REVOLUTION INCLUDING BENDING EFFECTS

In this appendix the governing equations for axisymmetric shells of revolutions are derived first using Koiter's "best" linear theory. The equations are then modified to take into account the shear deformation.

1. General Linear Shell Equations

The middle surface of the shell, in a three-dimensional Euclidean space is described using the vector field $\underset{\sim}{r}(x^{\alpha})$ depending on two variables only: the surface coordinates.

The base vectors, along the coordinate lines are given by:

$$\underset{\sim}{a}_{\alpha} = \frac{\partial \underset{\sim}{r}}{\partial x^{\alpha}} = r,_{\alpha} \qquad \text{(A. 1)}$$

The normal unit vector to the tangent plane is:

$$\underset{\sim}{n} = \frac{\underset{\sim}{a}_1 \times \underset{\sim}{a}_2}{|\underset{\sim}{a}_1 \times \underset{\sim}{a}_2|} \qquad \text{(A. 2)}$$

The reciprocal base vectors $\underset{\sim}{a}^{\alpha}$ are such that

$$\underset{\sim}{a}^{\alpha} \cdot \underset{\sim}{a}_{\beta} = \delta^{\alpha}_{\beta} \qquad \text{(A. 3)}$$

$$\underset{\sim}{a}^{\alpha} \cdot \underset{\sim}{n} = 0 \qquad \text{(A. 4)}$$

Let $a_{\alpha\beta}$ and $b_{\alpha\beta}$ be respectively the covariant components of the metric and the curvature tensor of the middle surface

$$a_{\alpha\beta} = \underset{\sim}{a}_{\alpha} \cdot \underset{\sim}{a}_{\beta} \qquad \text{(A. 5)}$$

$$b_{\alpha\beta} = \underset{\sim}{n} \cdot r,_{\alpha\beta} \qquad \text{(A. 6)}$$

The covariant derivative of the covariant components of a surface vector is:

$$u_{\alpha|\beta} = u_{\alpha,\beta} - A^{\kappa}_{\alpha\beta}\, u_{\kappa} \qquad \text{(A. 7)}$$

where $A^{\kappa}_{\alpha\beta}$ are the Christoffel symbols of the second kind

$$A^{\kappa}_{\alpha\beta} = \frac{1}{2} a^{\kappa\lambda} \left[a_{\lambda\alpha,\beta} + a_{\lambda\beta,\alpha} - a_{ab,\lambda}\right] \qquad \text{(A. 8)}$$

1-1. Strain Measure-Displacement Relations

Let u_{α} and w be respectively the covariant components and the normal component of the displacement vector of a point of the middle surface. The covariant components of the linearized strain tensor are given by:

$$\gamma_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} w \qquad \text{(A. 9)}$$

The covariant components of the change in curvature tensor $\rho_{\alpha\beta}$ are

$$\rho_{\alpha\beta} = \frac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} - b^{\kappa}_{\alpha}\omega_{\kappa\beta} - b^{\kappa}_{\beta}\omega_{\kappa\alpha}) \qquad \text{(A. 10)}$$

where

i) φ_{α} are the covariant components of the rotation of the normal

$$\varphi_{\alpha} = w,_{\alpha} + b^{\kappa}_{\alpha}\, u_{\kappa} \qquad \text{(A. 11)}$$

ii) $\omega_{\kappa\alpha}$ are the covariant components of the linearized rotation tensor in the middle surface

$$\omega_{\kappa\alpha} = \frac{1}{2}(u_{\alpha}|_{\beta} - u_{\beta}|_{\alpha}) \qquad (A.12)$$

1-2. Equilibrium Equations

Let $n_{\alpha\beta}$ and $m_{\alpha\beta}$ be respectively the components of the in-plane stress resultant and of the stress couple tensors.

The in-plane and normal equilibrium equations are respectively:

$$\left[n^{\alpha\beta} + \frac{1}{2}(b^{\alpha}_{\kappa}\, m^{\kappa\beta} - b^{\beta}_{\kappa}\, m^{\alpha\kappa})\right]\Big|_{\beta} + b^{\alpha}_{\kappa}\, m^{\beta\kappa}\Big|_{\beta} = 0 \qquad (A.13)$$

$$m^{\alpha\beta}\Big|_{\alpha\beta} - b_{\alpha\beta}\, n^{\alpha\beta} - p = 0 \qquad (A.14)$$

where p is the pressure normal to the shell middle surface.

1-3. Stress-strain Relations

The following equations relate respectively the stress resultant tensor to the strain tensor and the stress couple tensor to the change in curvature tensor:

$$\gamma_{\alpha\beta} = \overline{A}\left[(1+\nu)\, n_{\alpha\beta} - \nu a_{\alpha\beta}\, n^{\kappa}_{\kappa}\right] \qquad (A.15)$$

$$m_{\alpha\beta} = \overline{D}\left[(1-\nu)\,\rho^{\alpha\beta} + \nu a^{\alpha\beta}\,\rho^{\kappa}_{\kappa}\right] \qquad (A.16)$$

where

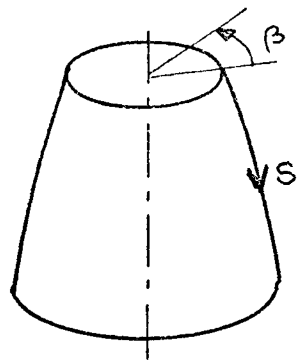
$$\overline{A} = \frac{1}{E\overline{t}}$$

$$\overline{D} = \frac{E\overline{t}^{3}}{12(1-\nu^{2})}$$

$\overline{t}$: shell thickness

2. Axially Symmetric Case without Torsion

2-1. Coordinate System



$x^1 = s$

$x^2 = \beta$: circumferential angle

The components of the metric tensor are:

$$a_{11} = 1, \qquad a_{22} = r^2, \qquad a_{12} = a_{21} = 0$$

$$a^{11} = 1, \qquad a^{22} = \frac{1}{r^2}, \qquad a^{12} = a^{21} = 0 \qquad \text{(A. 17)}$$

The components of the curvature tensor are:

$$b_1^1 = -\frac{r,_{11}}{\sqrt{1 - r,_1^2}}, \qquad b_2^2 = -\frac{\sqrt{1 - r,_1^2}}{r} \qquad \text{(A. 18)}$$

The only non-vanishing Christoffel symbols are:

$$A_{12}^2 = A_{21}^2 = \frac{r,_1}{r}$$

$$A_{22}^1 = -r\, r,_1 \qquad \text{(A. 19)}$$

2-2. Choice of the Unknowns

When there is no torsion applied to the shell then: n_2^1, m_2^1,

ρ^1_2, γ^1_2 and φ_2 are zero. Since we are seeking a first order system of differential equations, the following variables are taken as unknowns:

$$
\begin{aligned}
\bar{u}_p &\equiv u_1 \\
\bar{w}_n &\equiv w \\
\bar{N} &\equiv n^1_1 \\
\bar{\varphi} &\equiv \varphi_1 \\
\bar{M} &\equiv m^1_1 \\
\bar{Q} &\equiv -m^a_1\big|_a
\end{aligned}
\tag{A. 20}
$$

2-3. Equation for $u_{1,1}$

The strain displacement relations (A. 9) and the strain strain relations reduce to:

$$\gamma^1_1 = u_{1,1} - b^1_1 w = A(n^1_1 - \nu n^2_2)$$

$$\gamma^2_2 = \frac{r,_1}{r} u_1 - b^2_2 w = A(n^2_2 - \nu n^1_1)$$

By elimination, one obtains

$$u_{1,1} = \bar{A}(1-\nu^2)\, n^1_1 - \nu \frac{r,_1}{r} u_1 + \left[-\frac{\nu}{r} \frac{\sqrt{1-r,_1^2}}{r} + \frac{r,_{11}}{\sqrt{1-r,_1^2}} \right] w \tag{A. 21}$$

$$n^2_2 = \frac{1}{\bar{A}} \left(\frac{r,_1}{r} u_1 + \frac{\sqrt{1-r,_1^2}}{r} w \right) + \nu n^1_1 \tag{A. 22}$$

2-4. Equation for $w,_1$

The differential equation for w is obtained from the definition of the rotation of the normal φ_1

$$w,_1 = \varphi_1 - \frac{r,_{11}}{\sqrt{1-r,_1^2}} u_1 \tag{A. 23}$$

2-5. Equation for $\varphi_{1,1}$

The differential equation for $\varphi_{1,1}$ is obtained from the change of curvature-rotations relations (A. 10) and the change of curvature-stress couples relations (A. 16) which reduce to:

$$m_1^1 = \overline{D} [\rho_1^1 + \nu \rho_2^2] \tag{A. 24}$$

$$m_2^2 = \overline{D} [\rho_2^2 + \nu \rho_1^1] \tag{A. 25}$$

$$\rho_1^1 = \varphi_{1,1} \tag{A. 26}$$

$$\rho_2^2 = \frac{r,_1}{r} \varphi_1 \tag{A. 27}$$

By elimination one obtains:

$$\varphi_{1,1} = \frac{1}{\overline{D}} m_1^1 - \nu \frac{r,_1}{r} \varphi_1 \tag{A. 28}$$

$$m_2^2 = \overline{D} \frac{r,_1}{r} (1-\nu^2) \varphi_1 + \nu m_1^1 \tag{A. 29}$$

2-6. Equation for $n_{1,1}^1$

The two in-plane equilibrium equations (A. 13) reduce to one equation giving

$$n^1_{1,1} = \frac{1}{A}\left(\frac{r,_1}{r}\right)^2 u_1 + \frac{1}{A}\frac{r,_1}{r}\frac{1-r,_1^2}{r} w - \frac{r,_1}{r}(1-\nu)\, n^1_1$$

$$+ \frac{r,_{11}}{\sqrt{1-r,_1^2}}\,\overline{Q} = 0 \qquad (A.\ 30)$$

2-7. Equation for $m^1_{1,1}$

The equation for $m^1_{1,1}$ is obtained from the definition $\overline{Q}$ which is in fact the moment equilibrium equation.

$$\overline{Q} = -\, m^a_1\big|_a = -m^1_{1,1} + \frac{r,_1}{r}\left[m^1_1 - m^2_2\right] \qquad (A.\ 31)$$

$$m^1_{1,1} = -\,\overline{Q} + \left(\frac{r,_1}{r}\right)^2 D(1-\nu^2)\,\varphi - \frac{r,_1}{r}(1-\nu)m^1_1 \qquad (A.\ 32)$$

2-8. Equation for $\overline{Q},_1$

The equation for $\overline{Q},_1$ is obtained from the normal equilibrium equation (A. 14)

$$\overline{Q},_1 = \frac{1}{A}\frac{r,_1}{r}\frac{\sqrt{1-r,_1^2}}{r} u_1 + \frac{1}{A} w - \frac{r,_{11}}{\sqrt{1-r,_1^2}} - \frac{\nu\sqrt{1-r,_1^2}}{r}\, n^1_1$$

$$-\frac{r,_1}{r}\,\overline{Q}_1 - p = 0 \qquad (A.\ 33)$$

3. Inclusion of the Shear Deformation

The deformation due to the shear force can be taken into account by removing the assumption that a normal to the middle surface in the undeformed configuration will remain normal in the deformed state of the shell. The definition of the rotation of the normal (A. 23) is changed to:

$$w,_1 = \varphi + \beta^2 \frac{\overline{Q}}{\overline{E}} - \frac{r,_{11}}{\sqrt{1-r,_1^2}} u_1 \qquad (A.34)$$

where $\beta^2 = k \frac{E}{G}$, G is the material shear modulus and k is a coefficient to take into account the real distribution of the shear stress in the cross section.

4. Transformations of Equations

Two transformations are still required to obtain the desired equations:

a) substitute the axial and radial components of the stress resultants and displacements

b) take the derivatives with respect to the axial coordinate $\overline{x}$ instead of s.

Let us define:

$$\overline{r}(\overline{x}(s)) = r(s) \qquad (A.35)$$

$$\overline{u} = \frac{1}{\sqrt{1+\overline{r},_{\overline{x}}^2}} [\overline{u}_p - r,_x \overline{w}_n]$$

$$\overline{w} = \frac{1}{\sqrt{1+\overline{r},_{\overline{x}}^2}} [+ \overline{r},_{\overline{x}} \overline{u}_p + \overline{w}_n] \qquad (A.36)$$

$$\overline{T} = \frac{1}{\sqrt{1+\overline{r},_{\overline{x}}^2}} [\overline{N} - \overline{r},_{\overline{x}} \overline{Q}]$$

$$\overline{R} = \frac{1}{\sqrt{1+\overline{r},_{\overline{x}}^2}} [\overline{r},_{\overline{x}} \overline{N} + \overline{Q}]$$

$$\underset{\sim}{\bar{z}} \equiv \begin{cases} \bar{u} \;\ldots\; \text{axial displacement} \\ \bar{w} \;\ldots\; \text{radial displacement} \\ \bar{T} \;\ldots\; \text{axial force per unit length} \\ \bar{\varphi} \;\ldots\; \text{rotation of the normal} \\ \bar{m} \;\ldots\; \text{bending moment per unit length} \\ \bar{R} \;\ldots\; \text{radial force per unit length} \end{cases} \qquad \text{(A. 37)}$$

Then the previous equations reduce to:

$$\frac{d\underset{\sim}{\bar{z}}}{d\bar{x}} = \bar{B}\,\underset{\sim}{\bar{z}} + \underset{\sim}{\bar{f}} \qquad \text{(A. 38)}$$

where the components of $\bar{B}$ are given in Table 1.

$$\underset{\sim}{\bar{f}} = \begin{bmatrix} 0 \\ 0 \\ p\bar{r},_{\bar{x}} \\ 0 \\ 0 \\ -p \end{bmatrix} \qquad \text{(A. 39)}$$

It is to be noted that $\bar{B}$ is a function of $\bar{t}$, $\bar{r}$, $\bar{r},_{\bar{x}}$ but not of $\bar{r},_{\bar{x}\bar{x}}$.

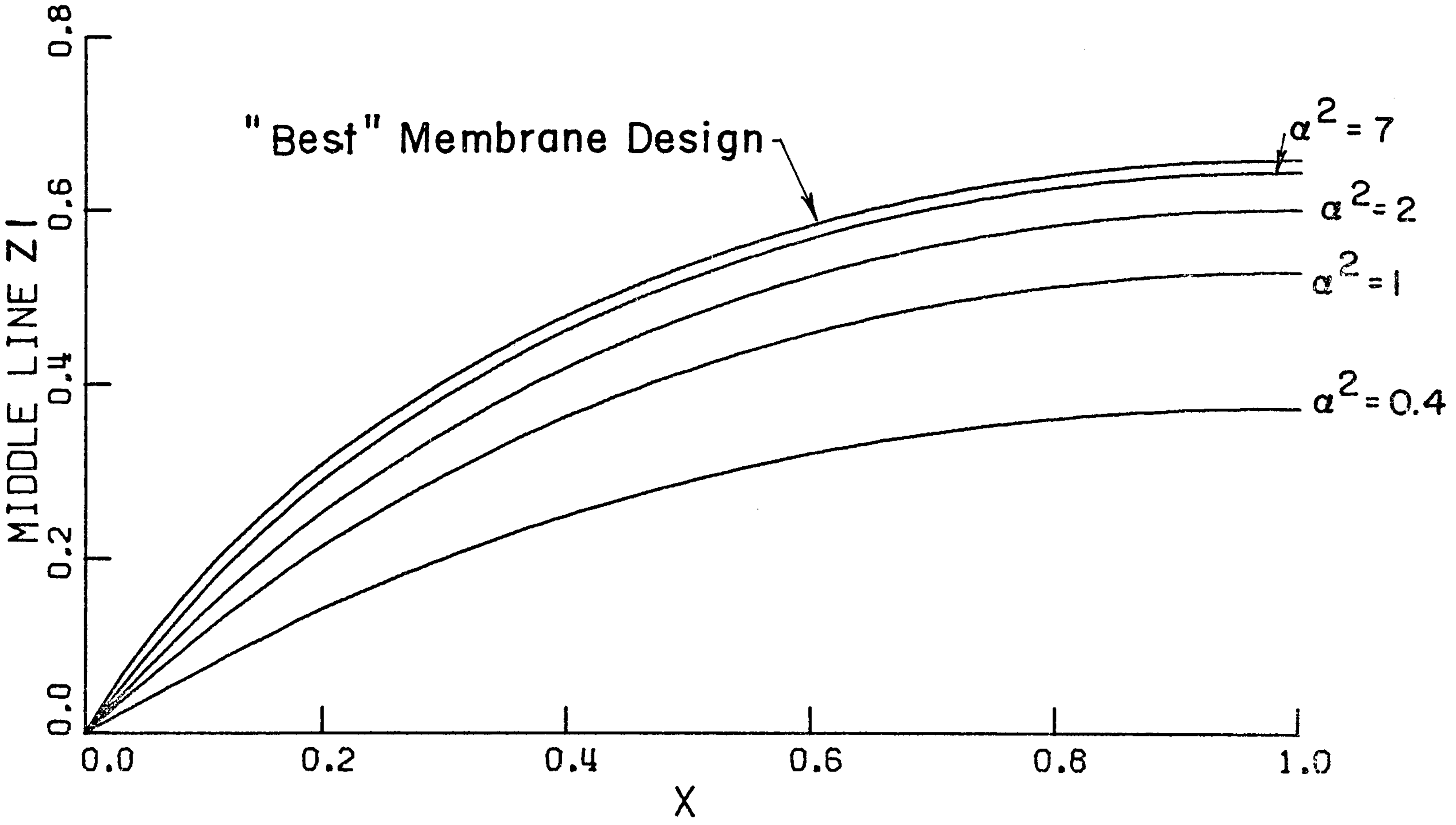


FIG. I ØPTIMAL ARCH
STIFFNESS CØNSTRAINT
MIDDLE LINE SHAPE Z1 $\beta^2 = 2.5$

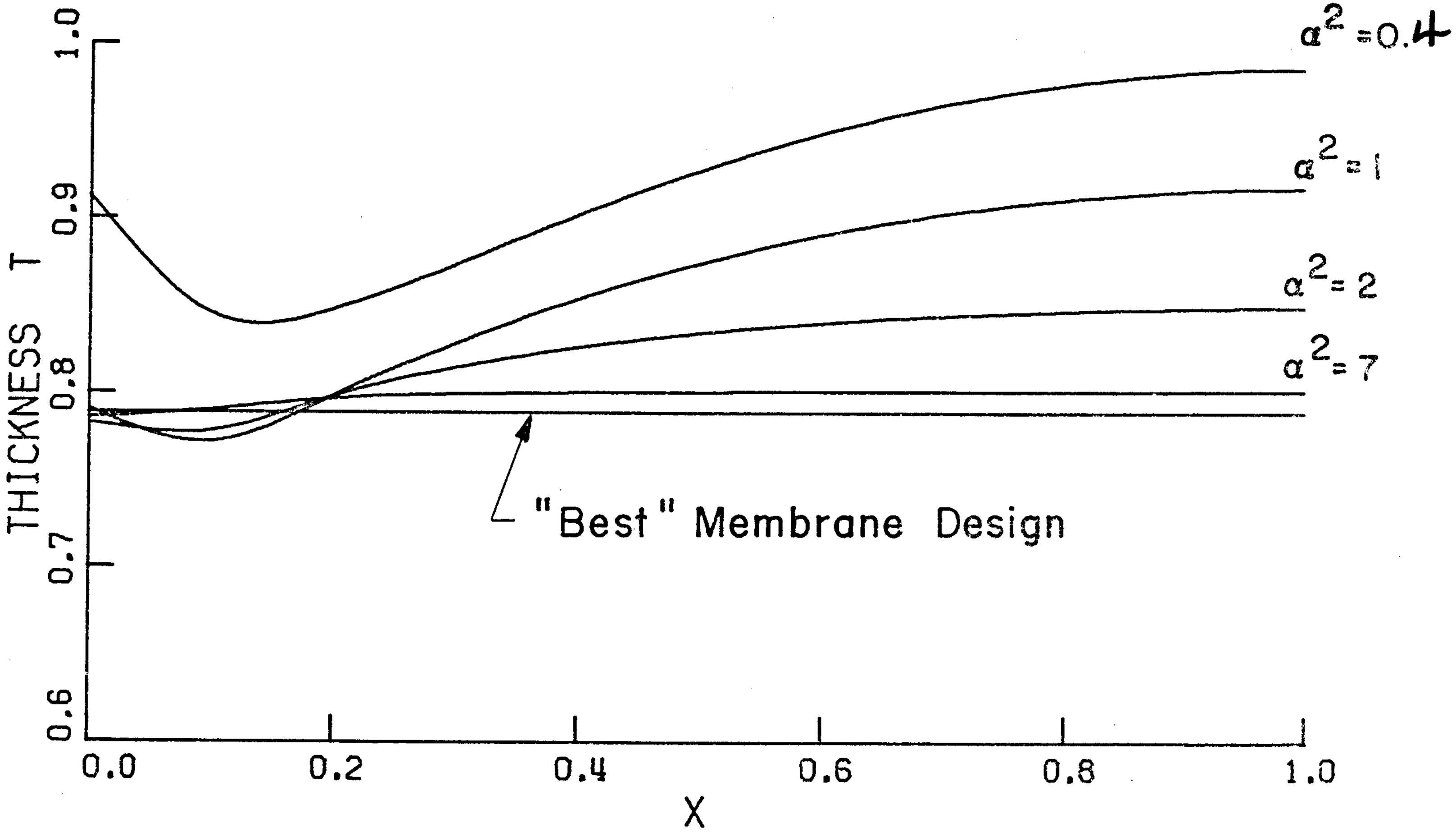
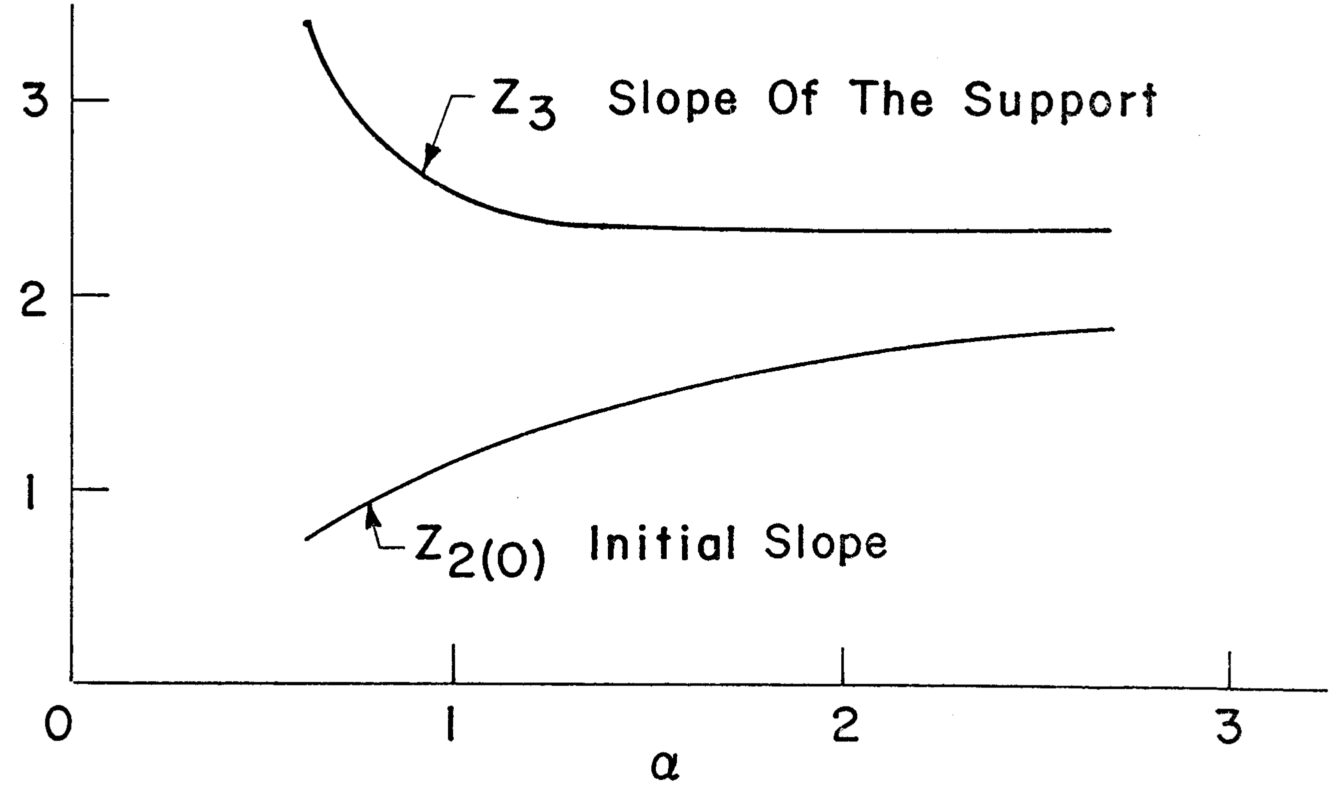
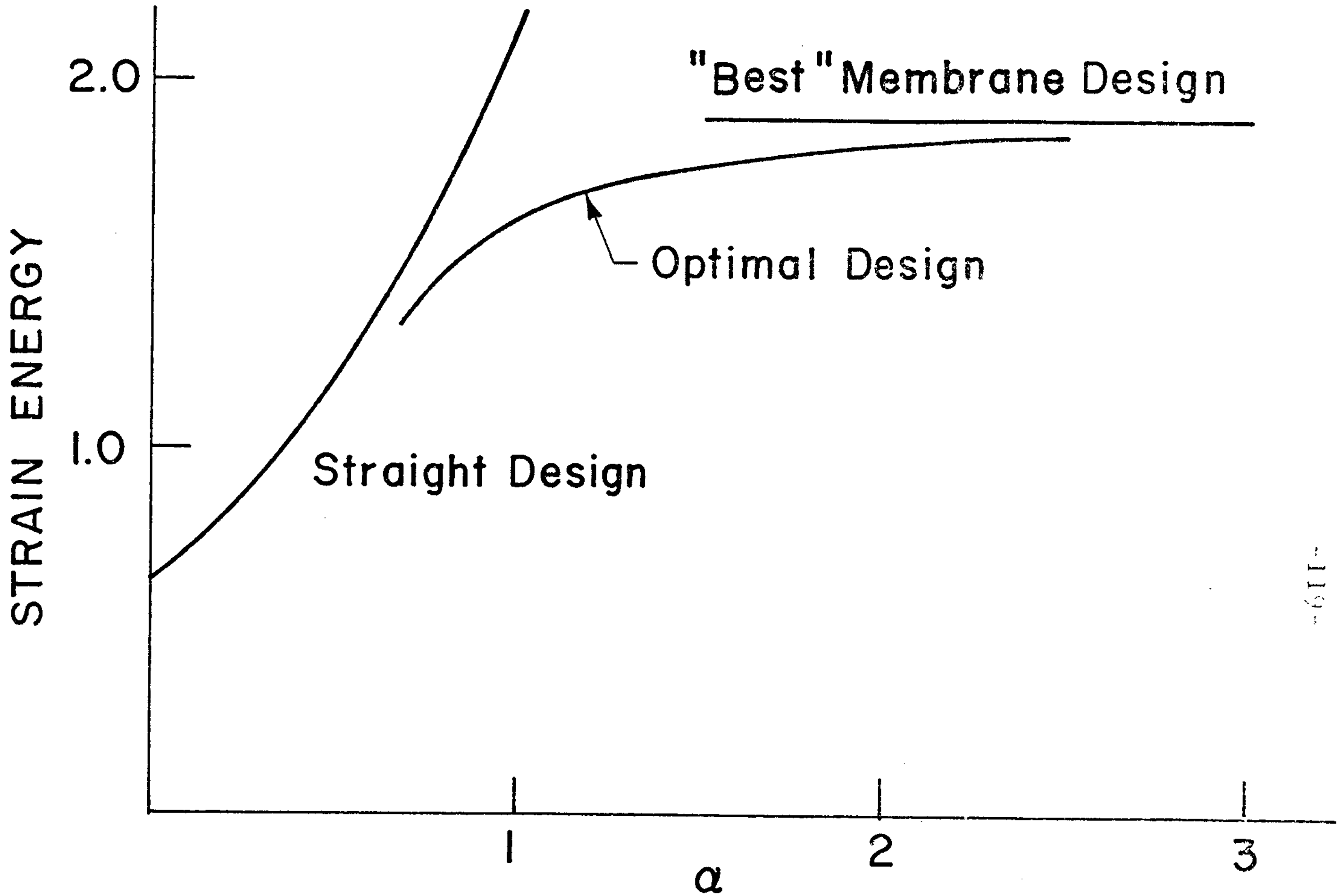


FIG. 2 OPTIMAL ARCH
STIFFNESS CONSTRAINT
THICKNESS T

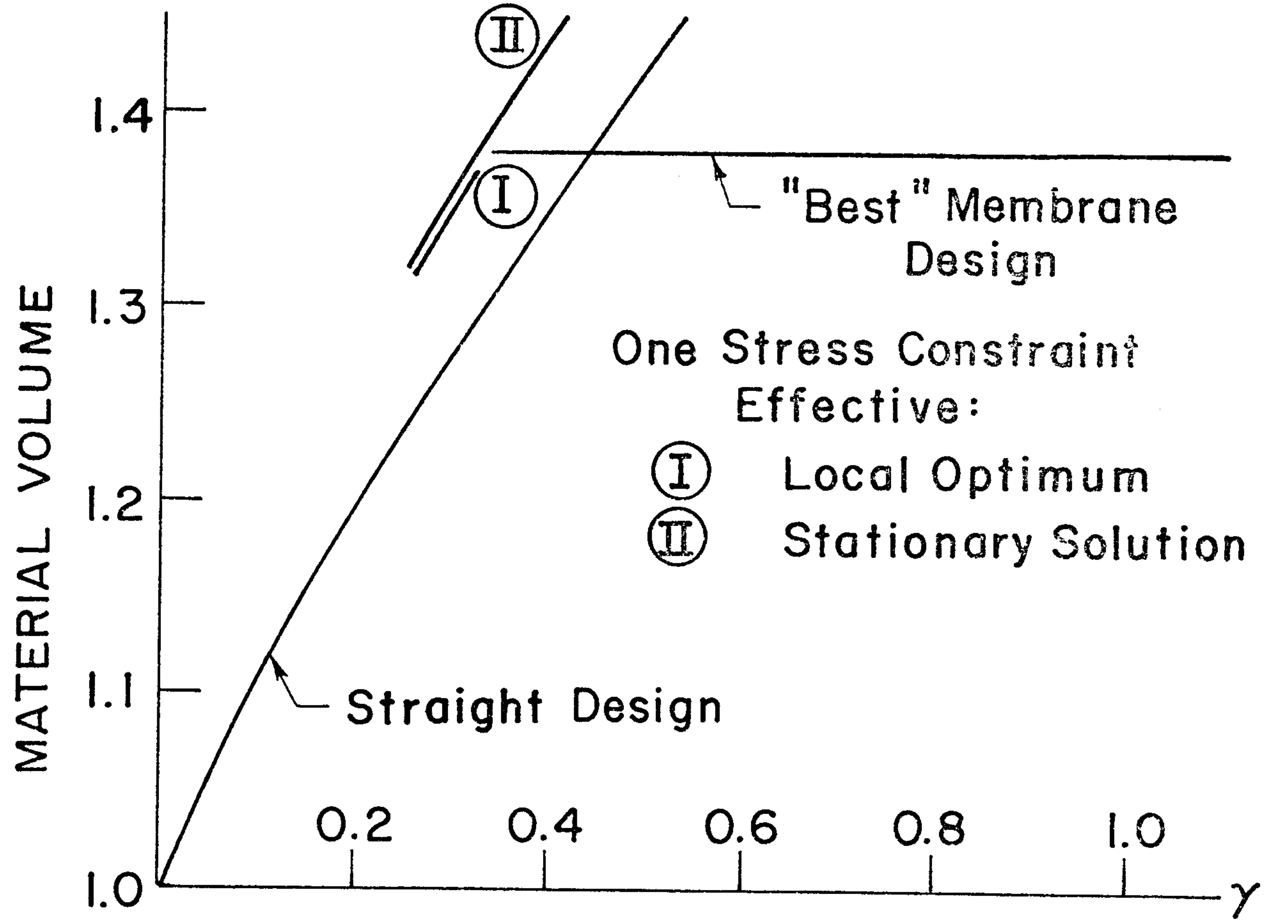
$\beta^2 = 2.5$



OPTIMAL ARCH
STIFFNESS CONSTRAINT $\beta^2 = 2.5$
FIG. 3 SLOPE OF THE SUPPORT AND INITIAL SLOPE



OPTIMAL ARCH
STIFFNESS CONSTRAINT $\beta^2 = 2.5$
FIG. 4 STRAIN ENERGY



OPTIMAL ARCH STRESS CONSTRAINT
FIG. 5 MATERIAL VOLUME

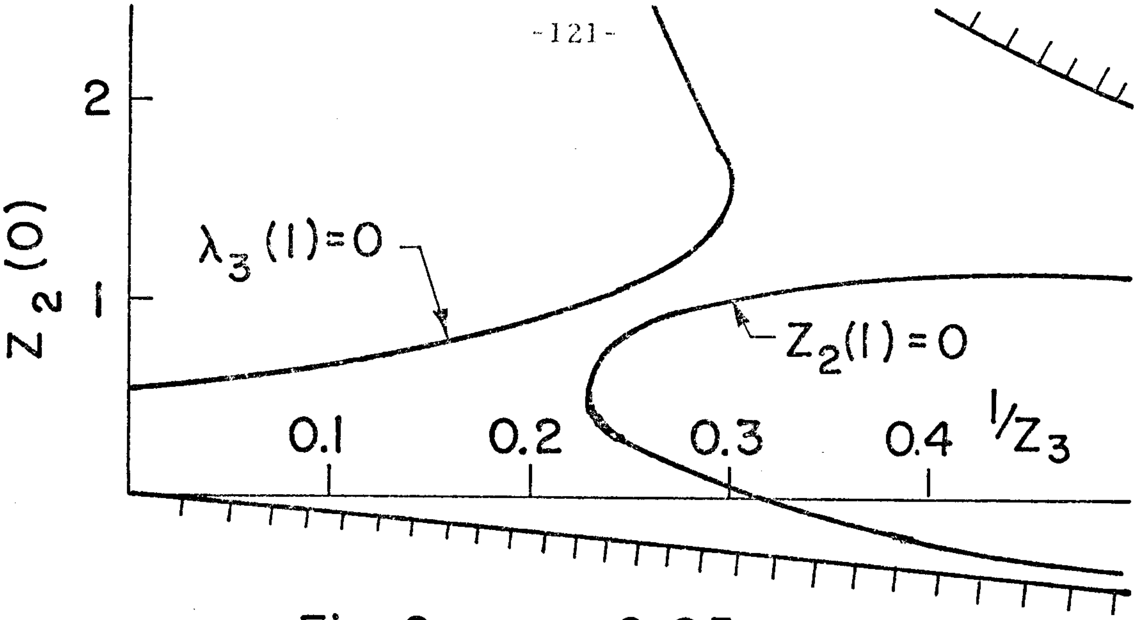


Fig. 6a $\gamma = 0.25$

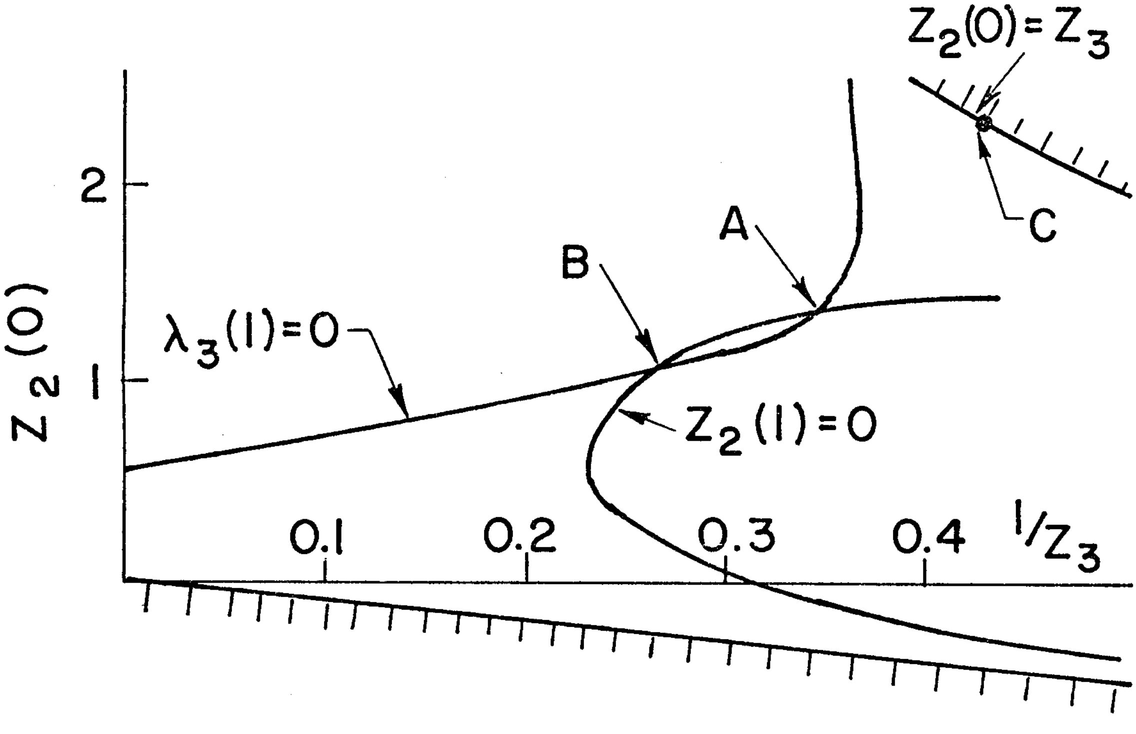


Fig. 6b $\gamma = 0.3$

OPTIMAL ARCH STRESS CONSTRAINT

FIG. 6 LOCI OF $Z_2(1)=0$, $\lambda_3(1)=0$

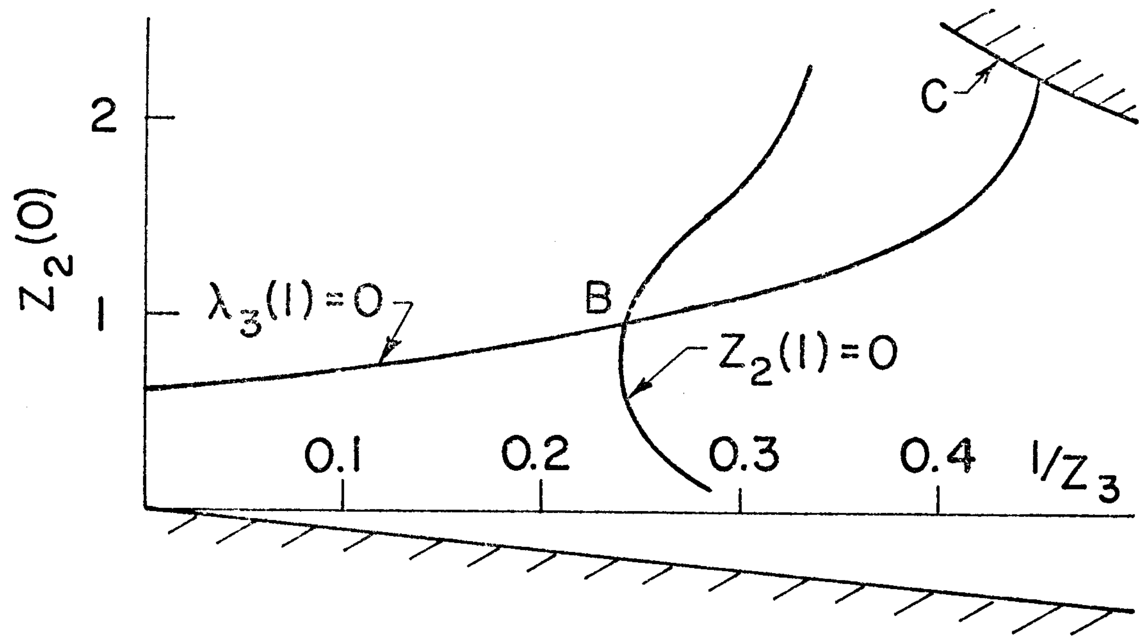


Fig. 6c $\gamma = 0.35$

OPTIMAL ARCH STRESS CONSTRAINT

FIG. 6 LOCI OF $Z_2(1)=0, \lambda_3(1)=0$

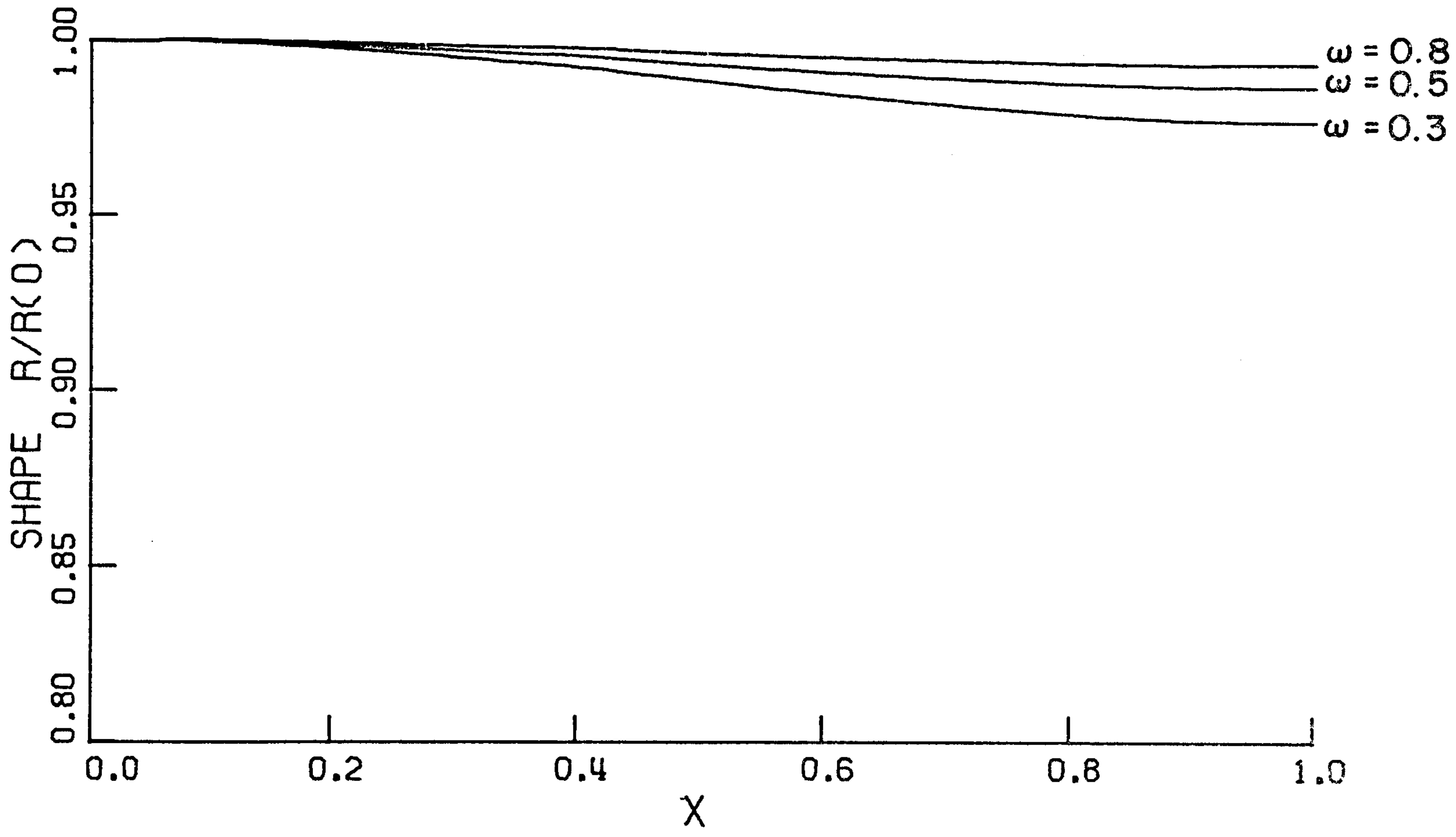


FIG. 7 OPTIMAL MEMBRANE SHELL
STRESS CONSTRAINT
SHAPE R/R(0) $\gamma = 1.0$

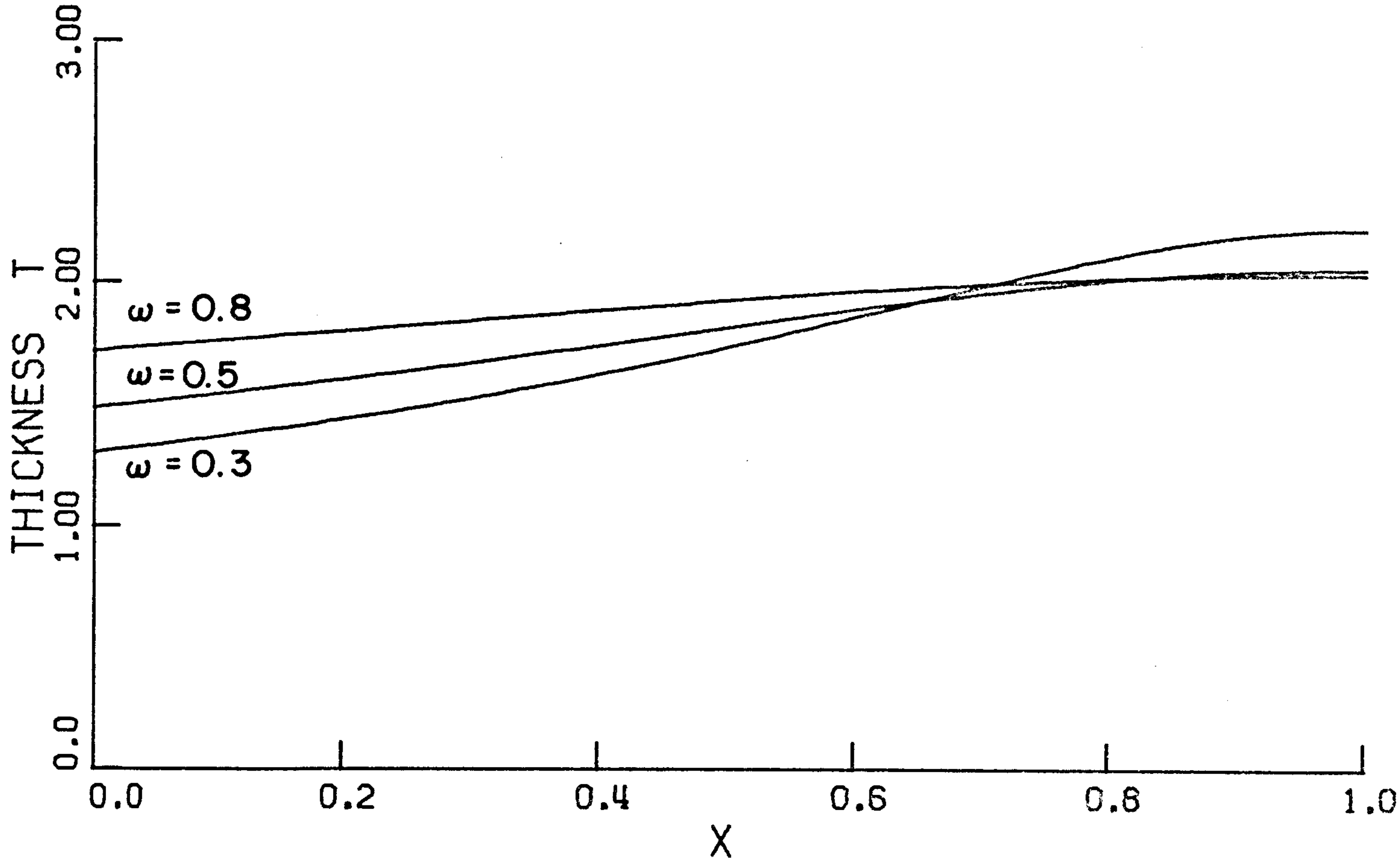


FIG. 8 ØPTIMAL MEMBRANE SHELL
STRESS CØNSTRAINT
THICKNESS T $\gamma = 1.0$

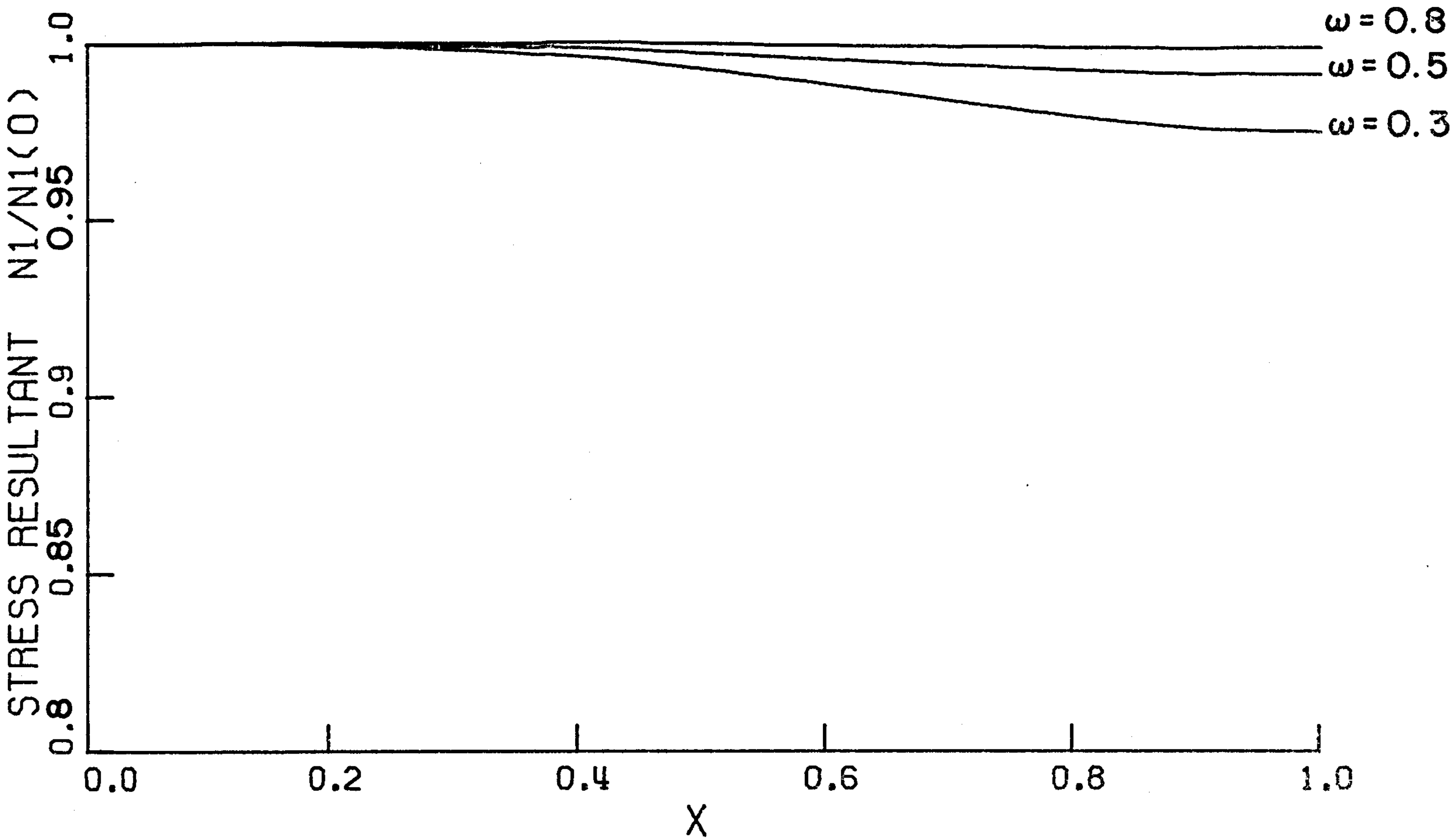


FIG.9 OPTIMAL MEMBRANE SHELL
STRESS CONSTRAINT
STRESS RESULTANT N1/N1(0) $\gamma = 1.0$

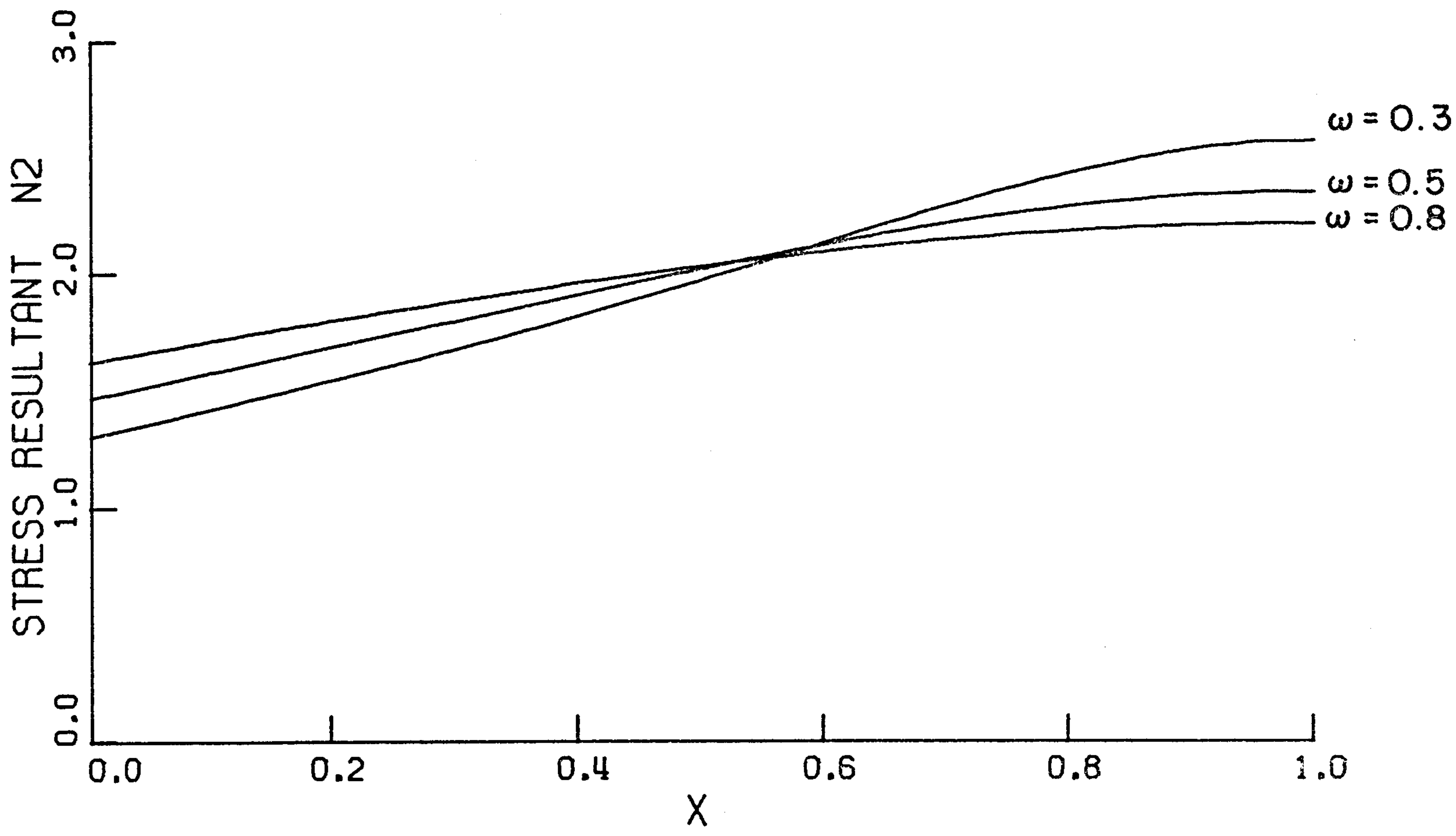


FIG.10 ØPTIMAL MEMBRANE SHELL
STRESS CØNSTRAINT
STRESS RESULTANT N2 $\gamma = 1.0$

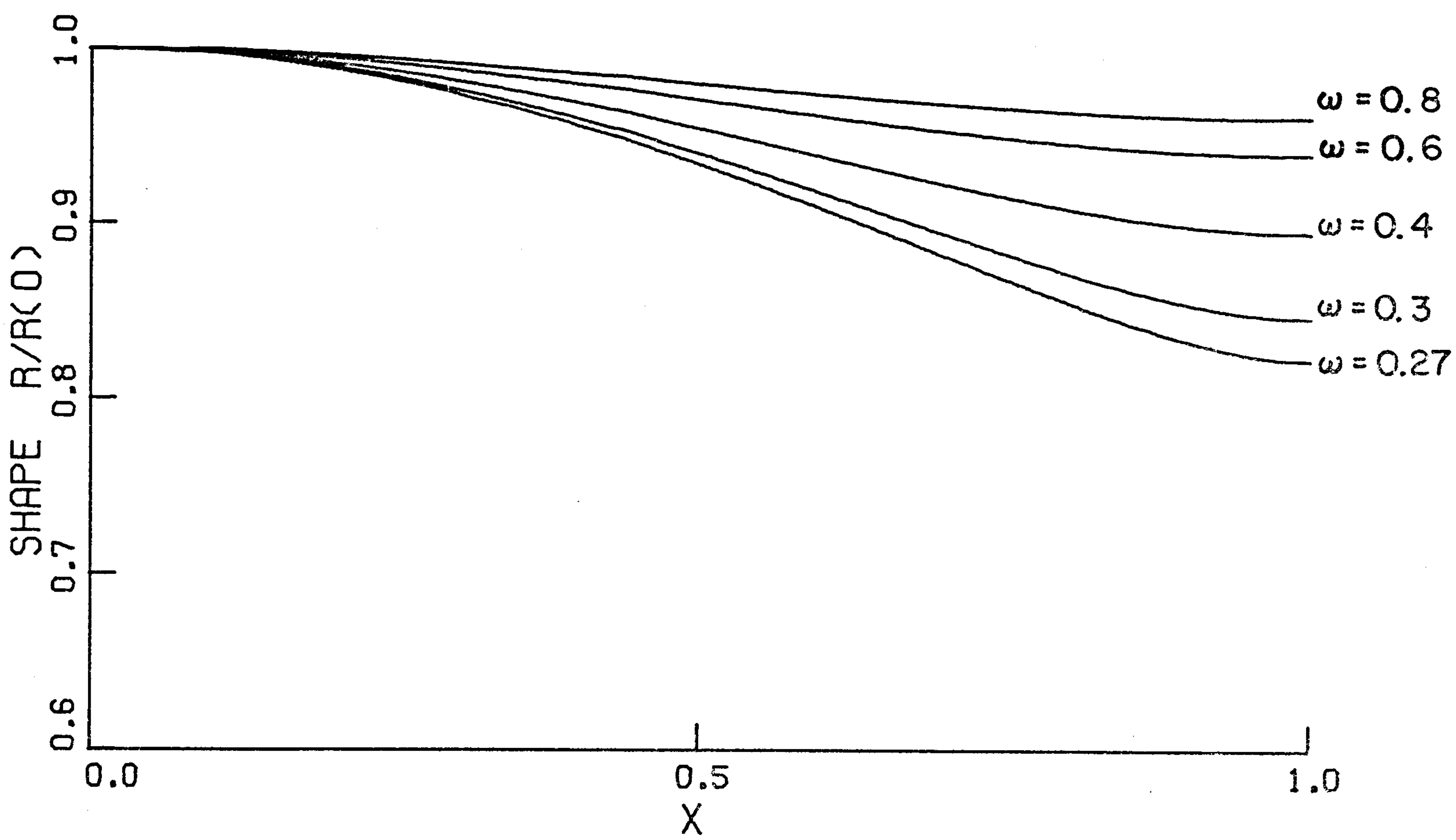


FIG.11 OPTIMAL MEMBRANE SHELL
STRESS CONSTRAINT
SHAPE R/R(0) $\gamma = 1.5$

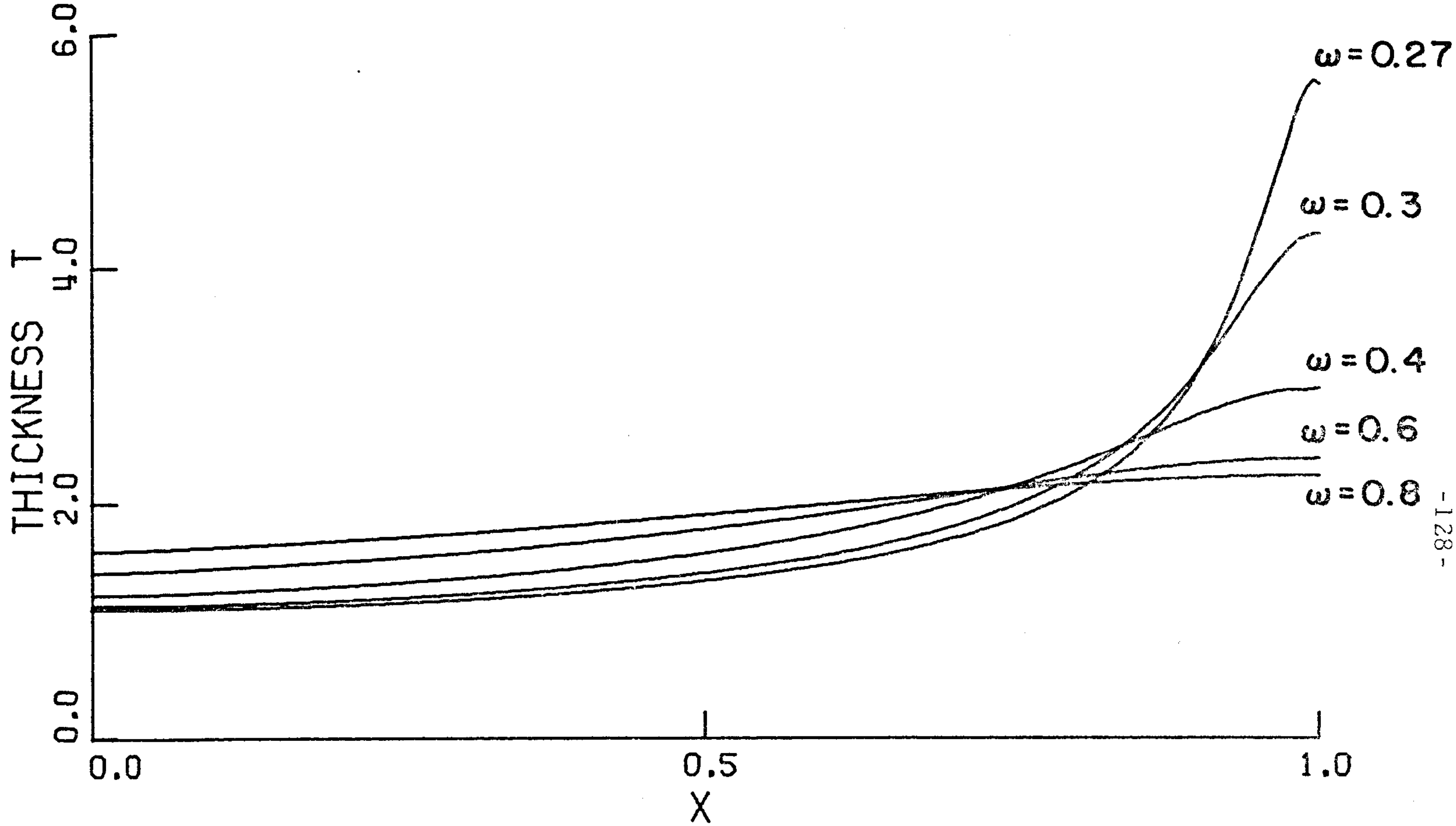


FIG. 12 ØPTIMAL MEMBRANE SHELL
STRESS CØNSTRAINT
THICKNESS T $\gamma = 1.5$

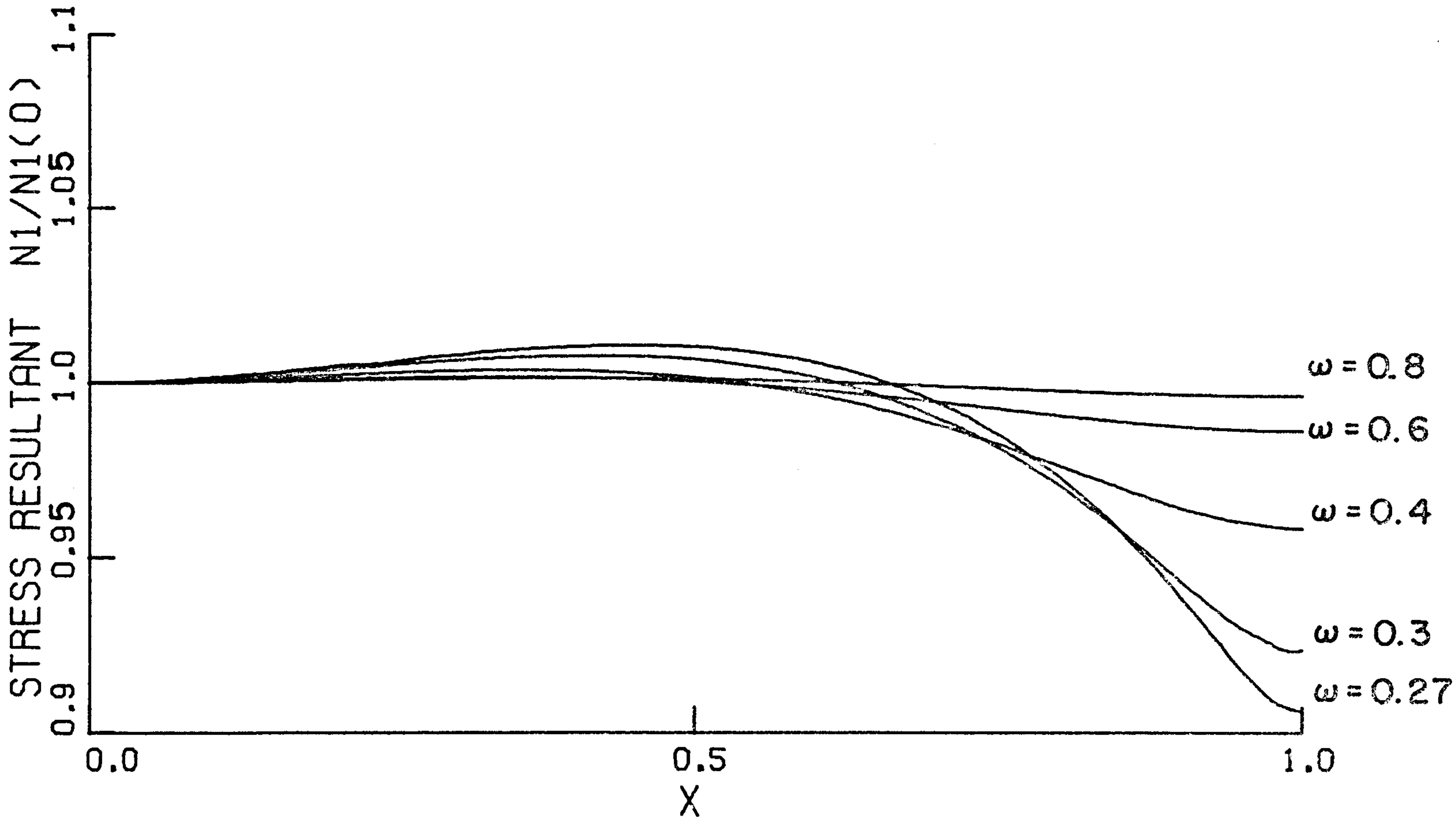


FIG.13 ØPTIMAL MEMBRANE SHELL
STRESS CØNSTRAINT
STRESS RESULTANT N1/N1(O) $\gamma = 1.5$

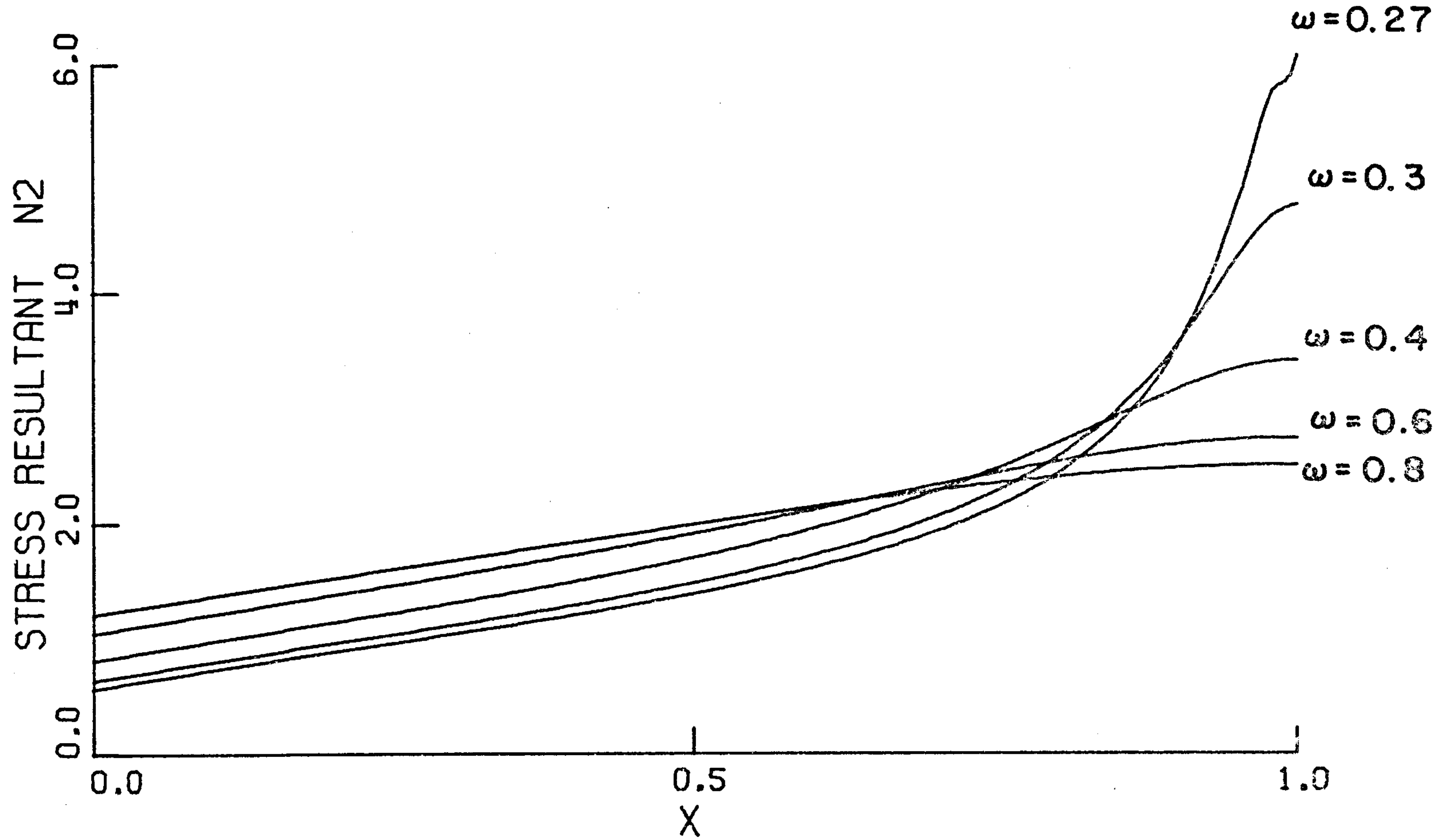


FIG. 14 ØPTIMAL MEMBRANE SHELL
STRESS CØNSTRAINT
STRESS RESULTANT N2 $\gamma = 1.5$

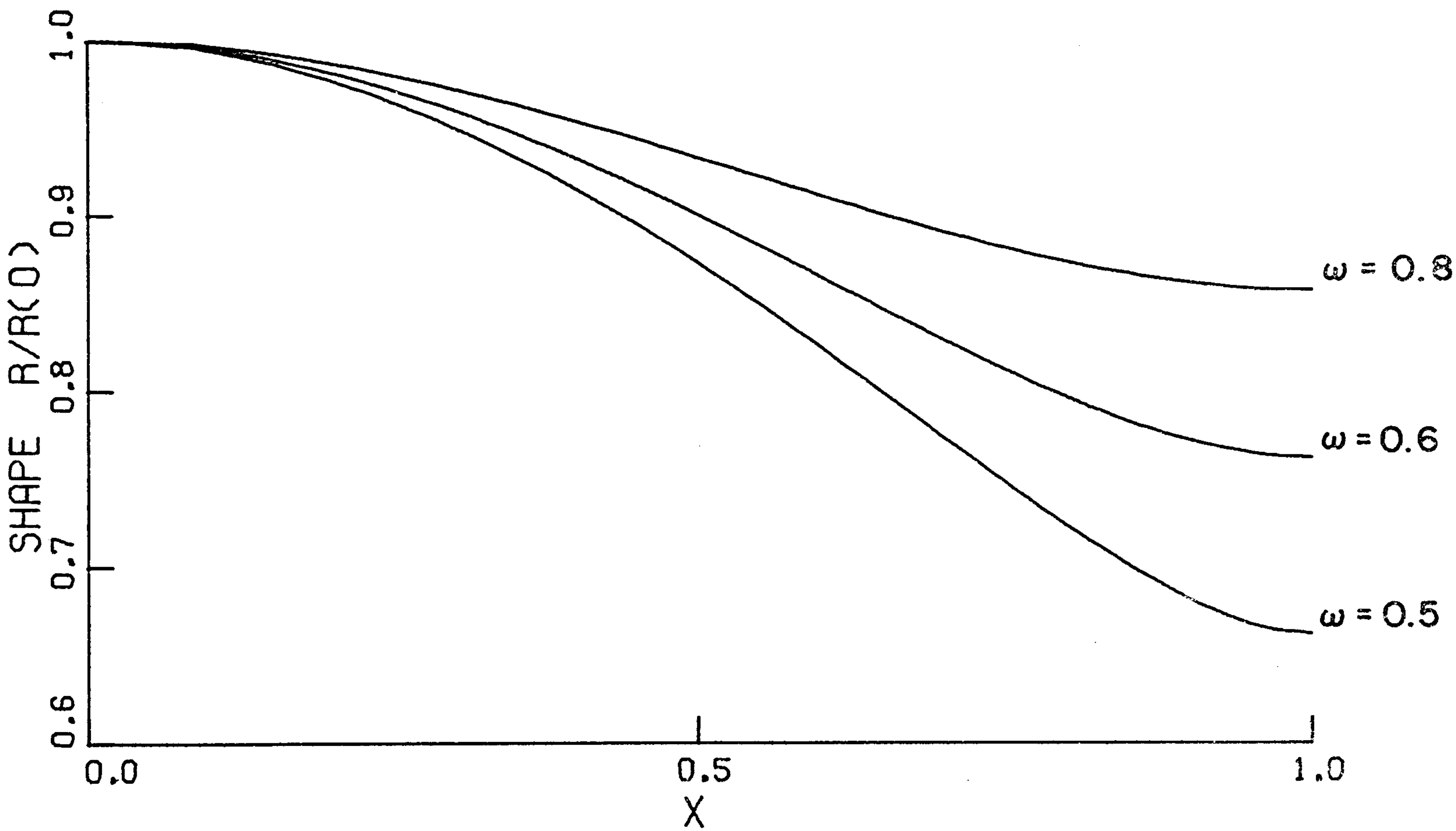


FIG. 15 ØPTIMAL MEMBRANE SHELL
STRESS CØNSTRAINT
SHAPE R/R(0) $\gamma = 2.0$

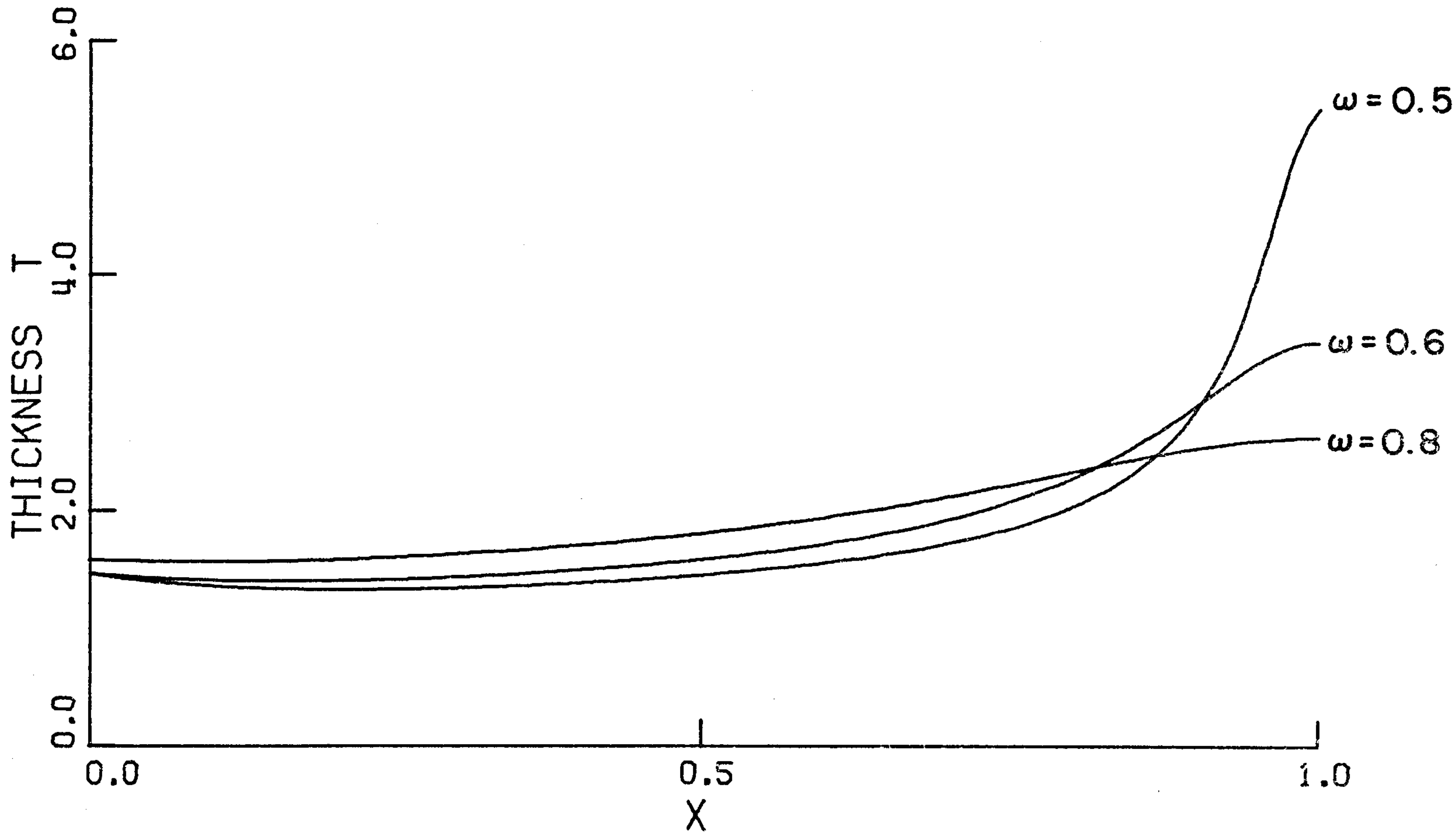


FIG.16 OPTIMAL MEMBRANE SHELL
STRESS CONSTRAINT
THICKNESS T

$\gamma = 2.0$

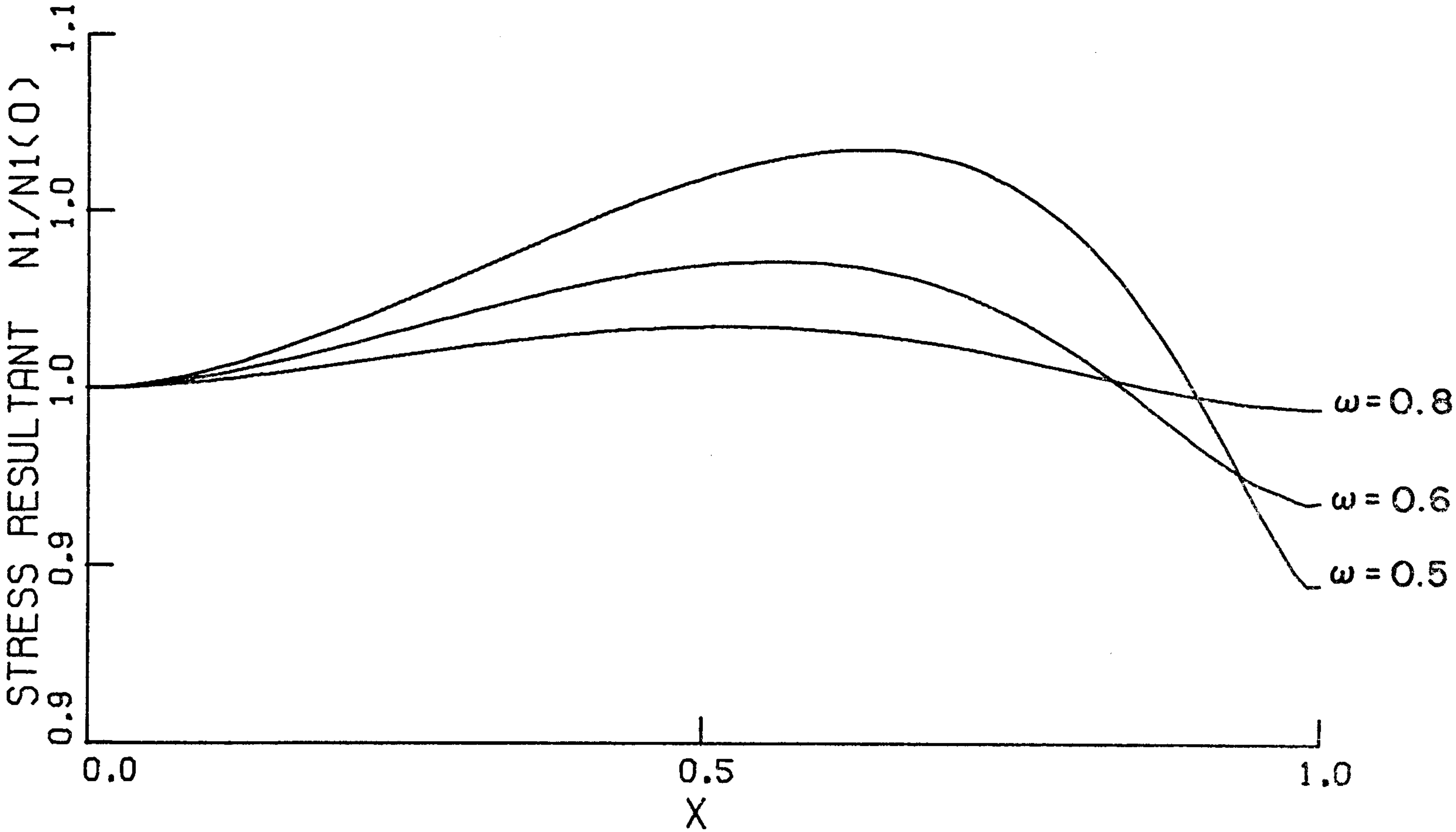


FIG.17 ØPTIMAL MEMBRANE SHELL
STRESS CØNSTRAINT
STRESS RESULTANT N1/N1(O) $\gamma = 2.0$

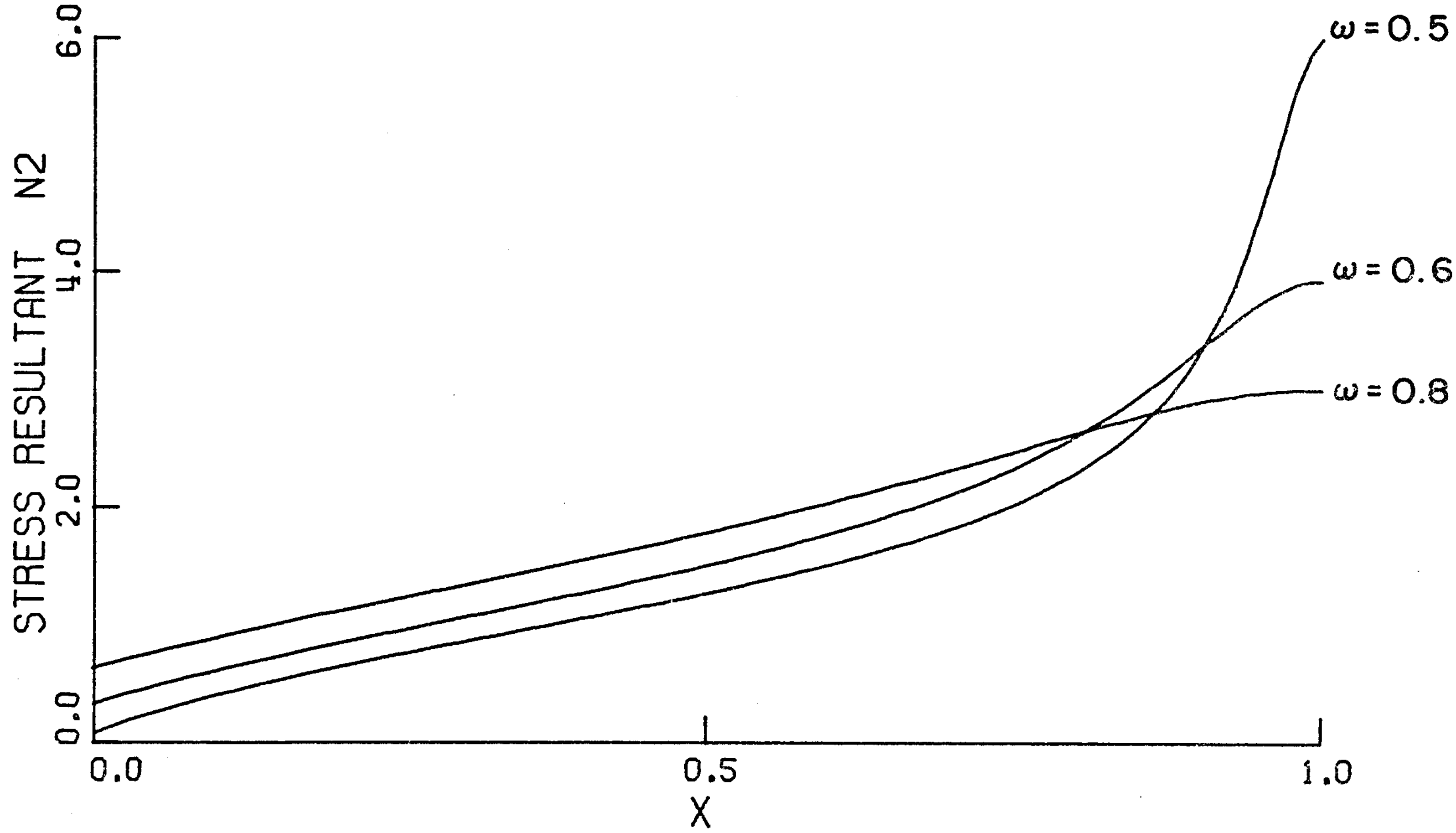
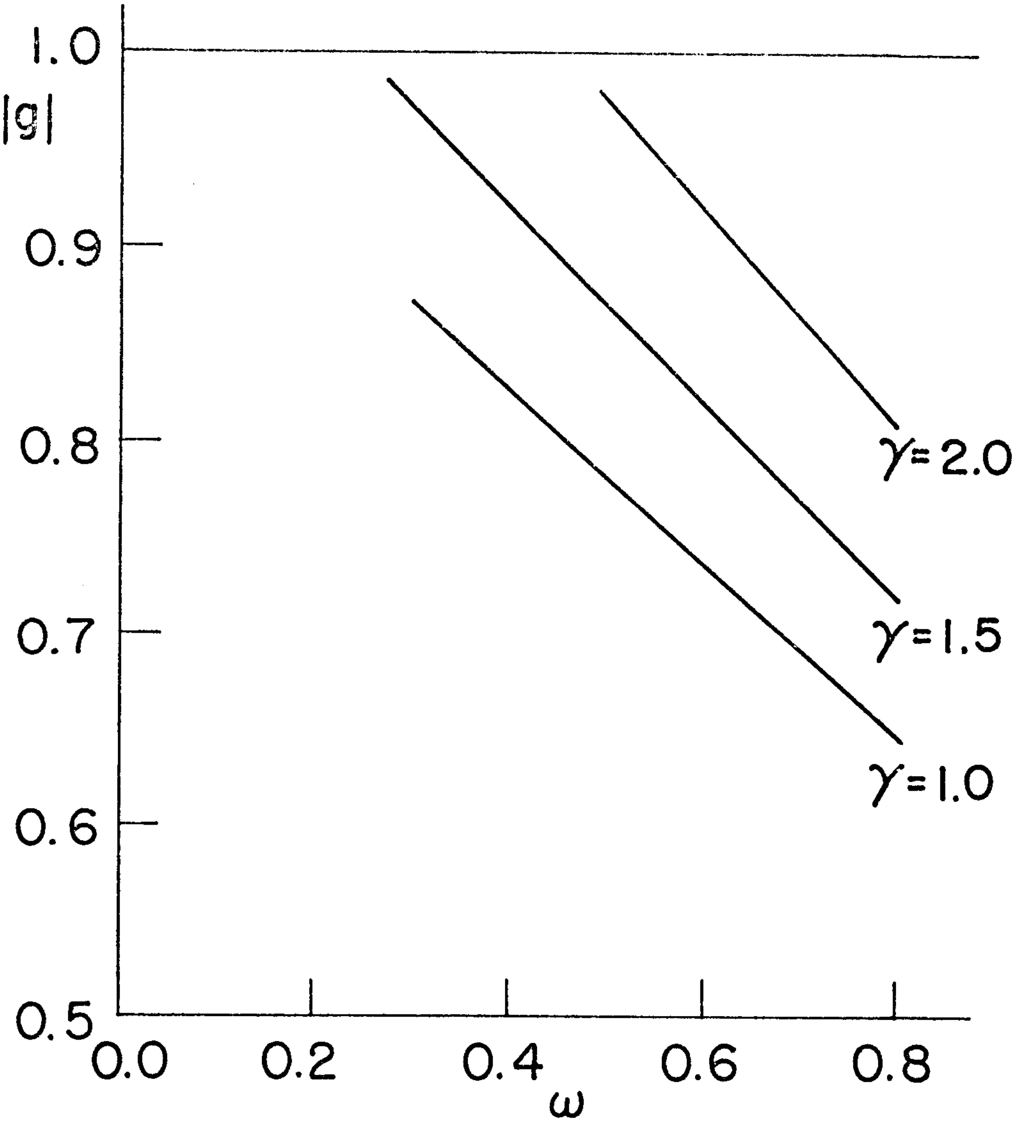


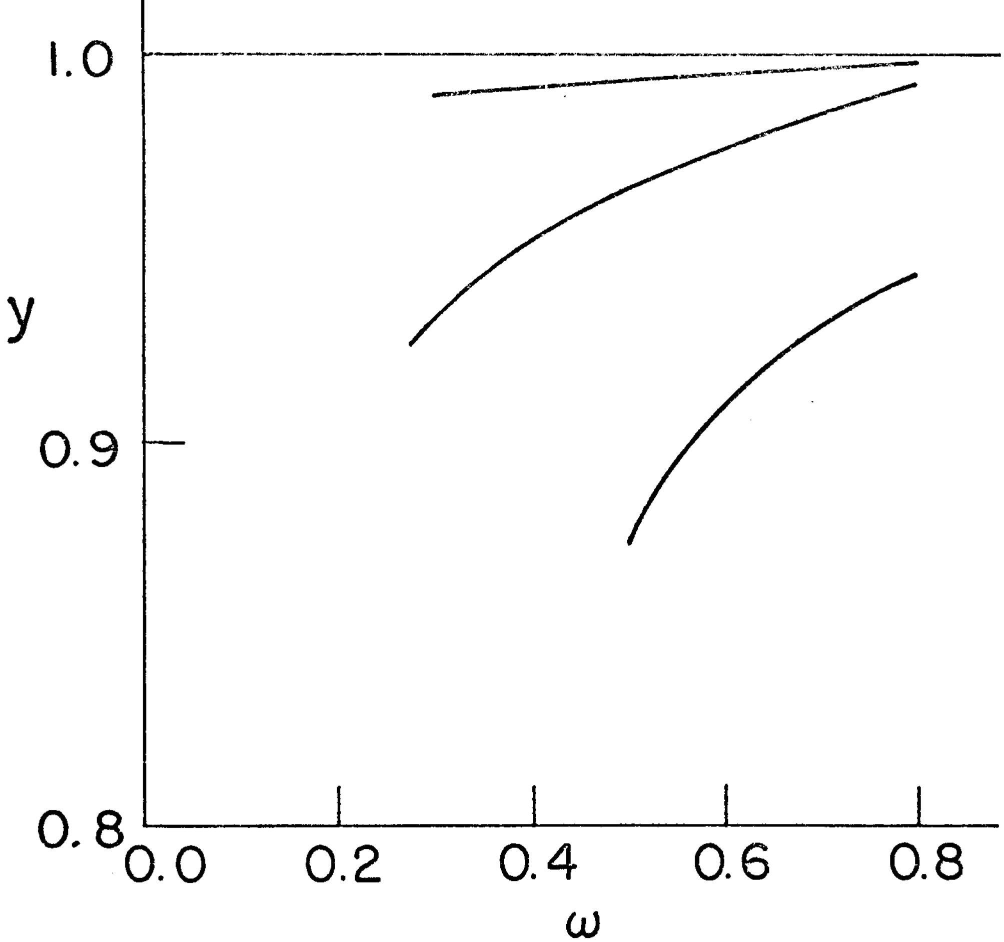
FIG.18 OPTIMAL MEMBRANE SHELL
STRESS CONSTRAINT
STRESS RESULTANT N2

$\gamma = 2.0$



$$t = \frac{1}{2}\sqrt{\frac{3n_1^2}{1-g^2}} \qquad (5.43\text{-a})$$

OPTIMAL MEMBRANE SHELL
STRESS CONSTRAINT
FIG. 19 |g| AS A FUNCTION OF ω



$$y \equiv \frac{\text{Material Volume Optimal Shell}}{\text{Material Volume Cylindrical Shell}}$$

OPTIMAL MEMBRANE SHELL
STRESS CONSTRAINT
FIG. 20 MATERIAL VOLUME

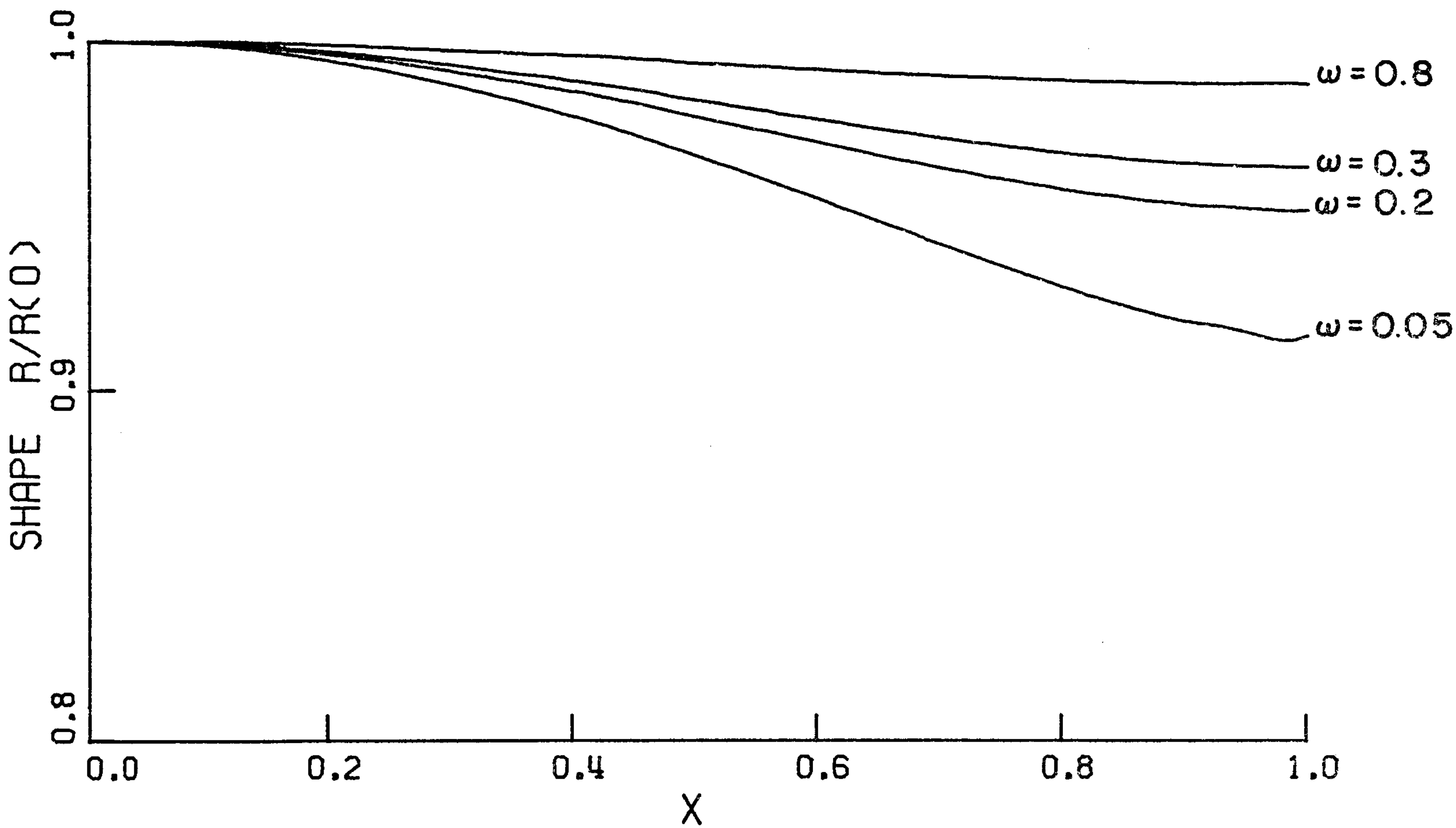


FIG.21 OPTIMAL MEMBRANE SHELL STIFFNESS CONSTRAINT SHAPE R/R(0)

$\gamma = 1.0$

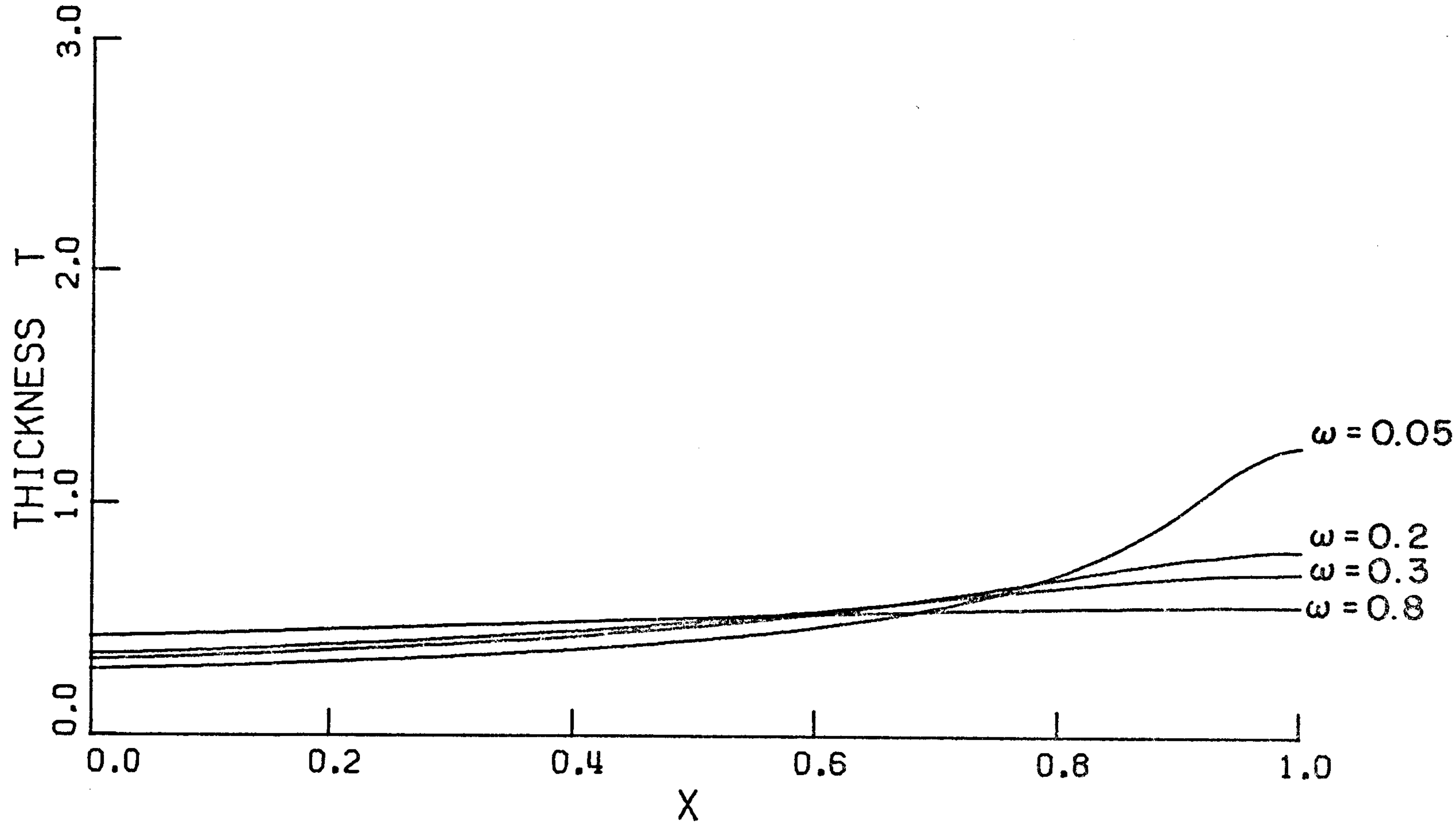


FIG.22 ØPTIMAL MEMBRANE SHELL
STIFFNESS CØNSTRAINT
THICKNESS T

$\gamma = 1.0$

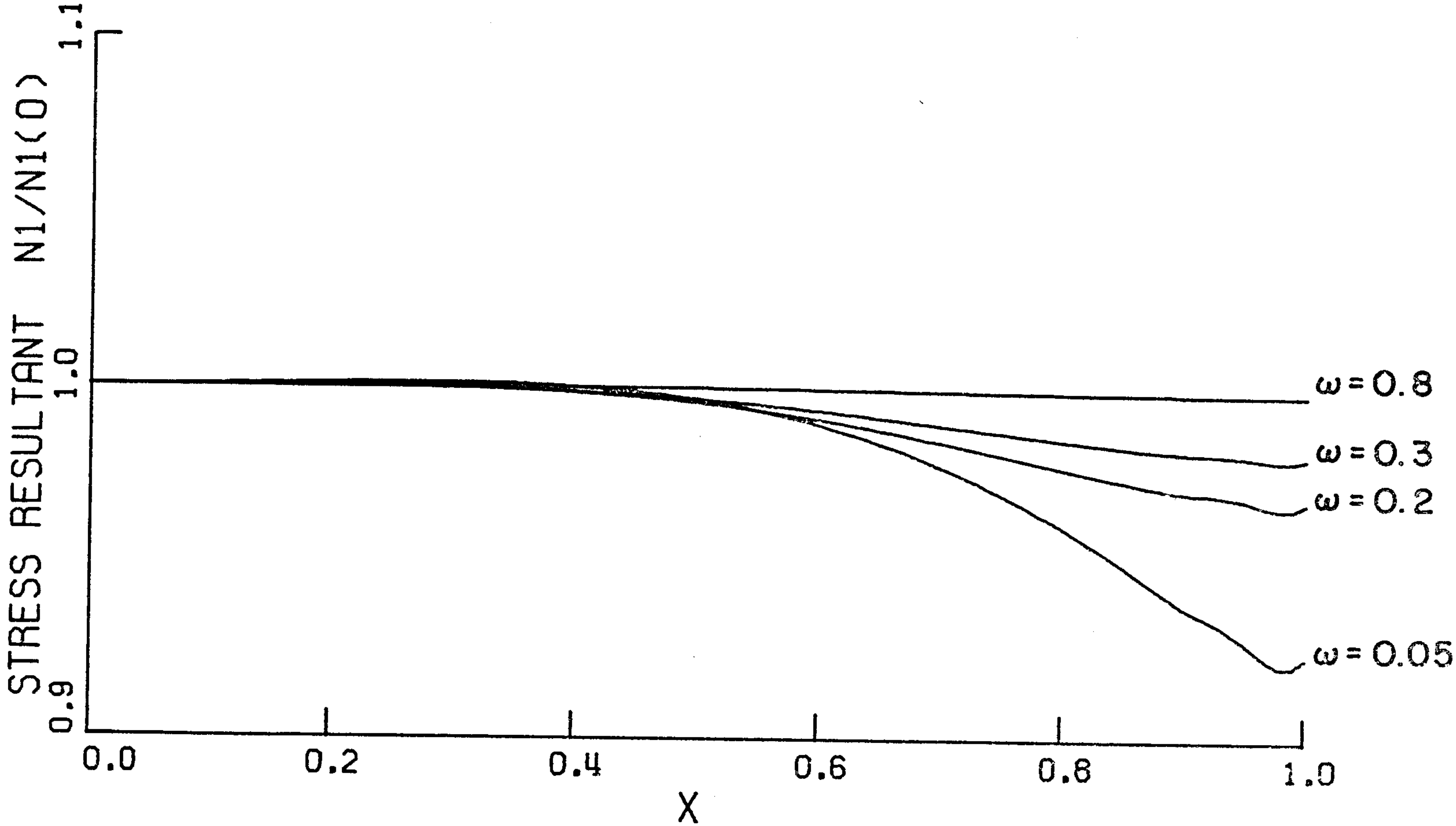


FIG.23 ØPTIMAL MEMBRANE SHELL
STIFFNESS CØNSTRAINT
STRESS RESULTANT N1/N1(O) $\gamma = 1.0$

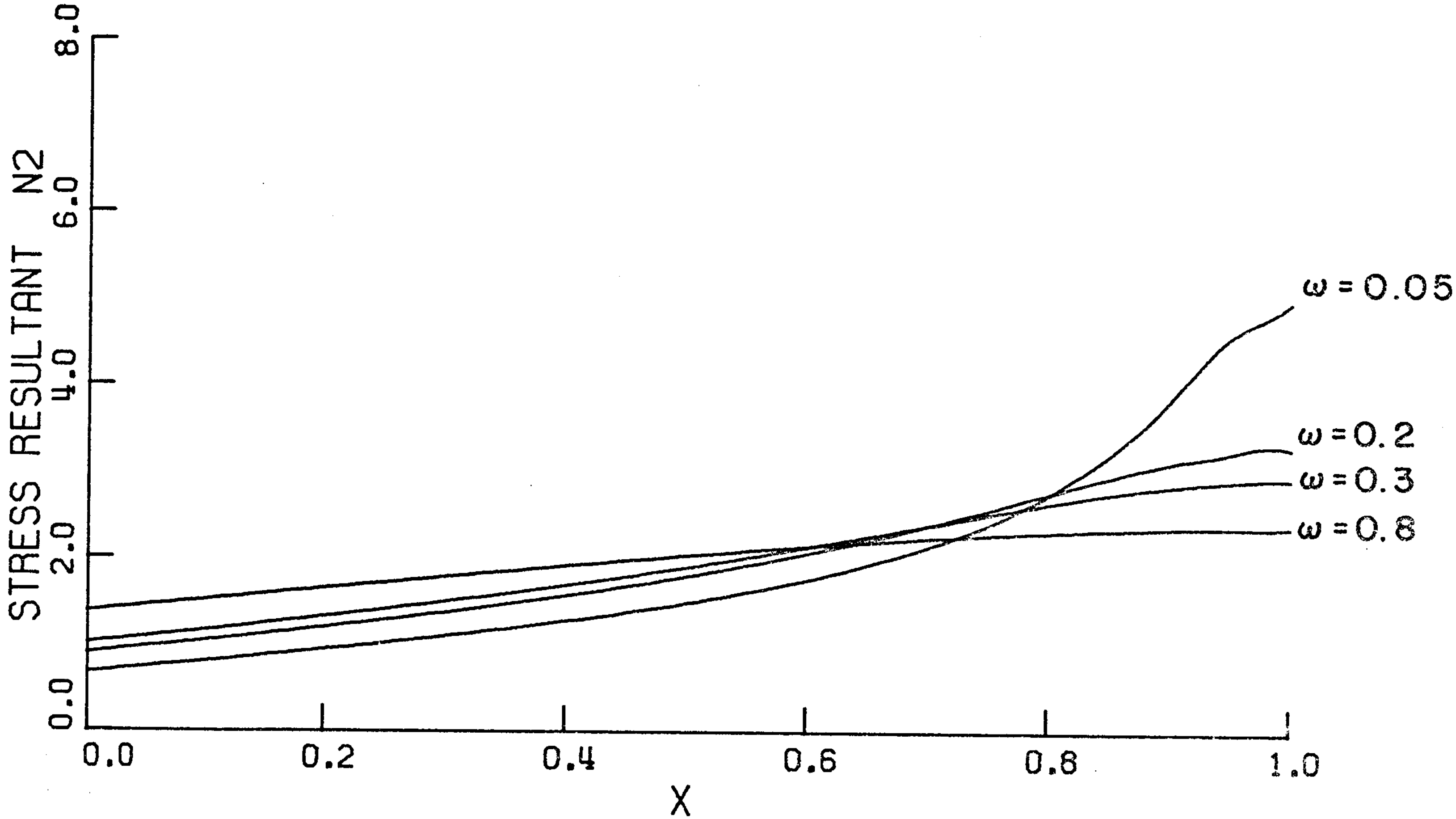


FIG.24 ØPTIMAL MEMBRANE SHELL
STIFFNESS CØNSTRAINT
STRESS RESULTANT N2 $\gamma = 1.0$

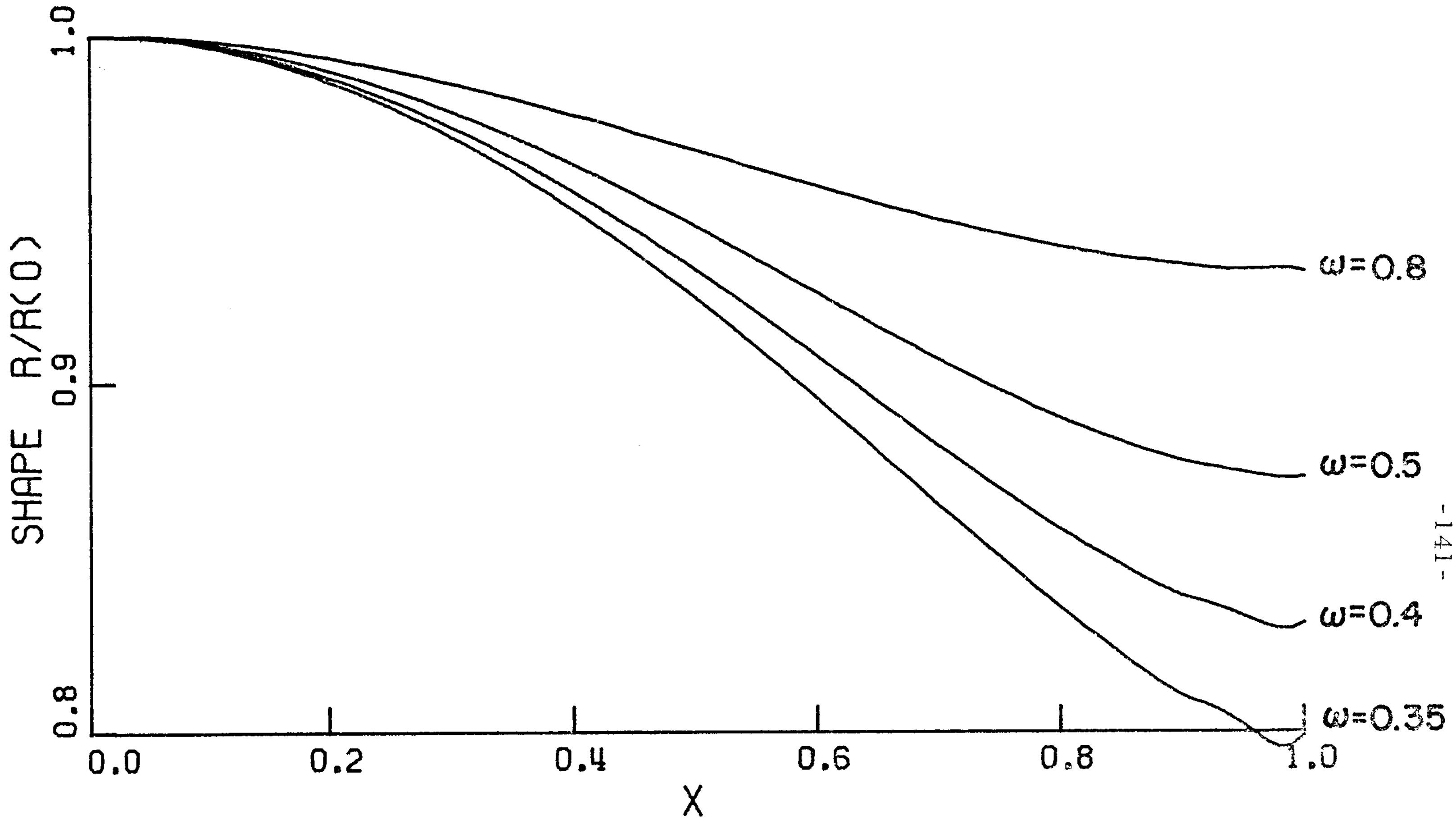


FIG.25 ØPTIMAL MEMBRANE SHELL
STIFFNESS CØNSTRAINT
SHAPE R/R(0) $\gamma = 1.5$

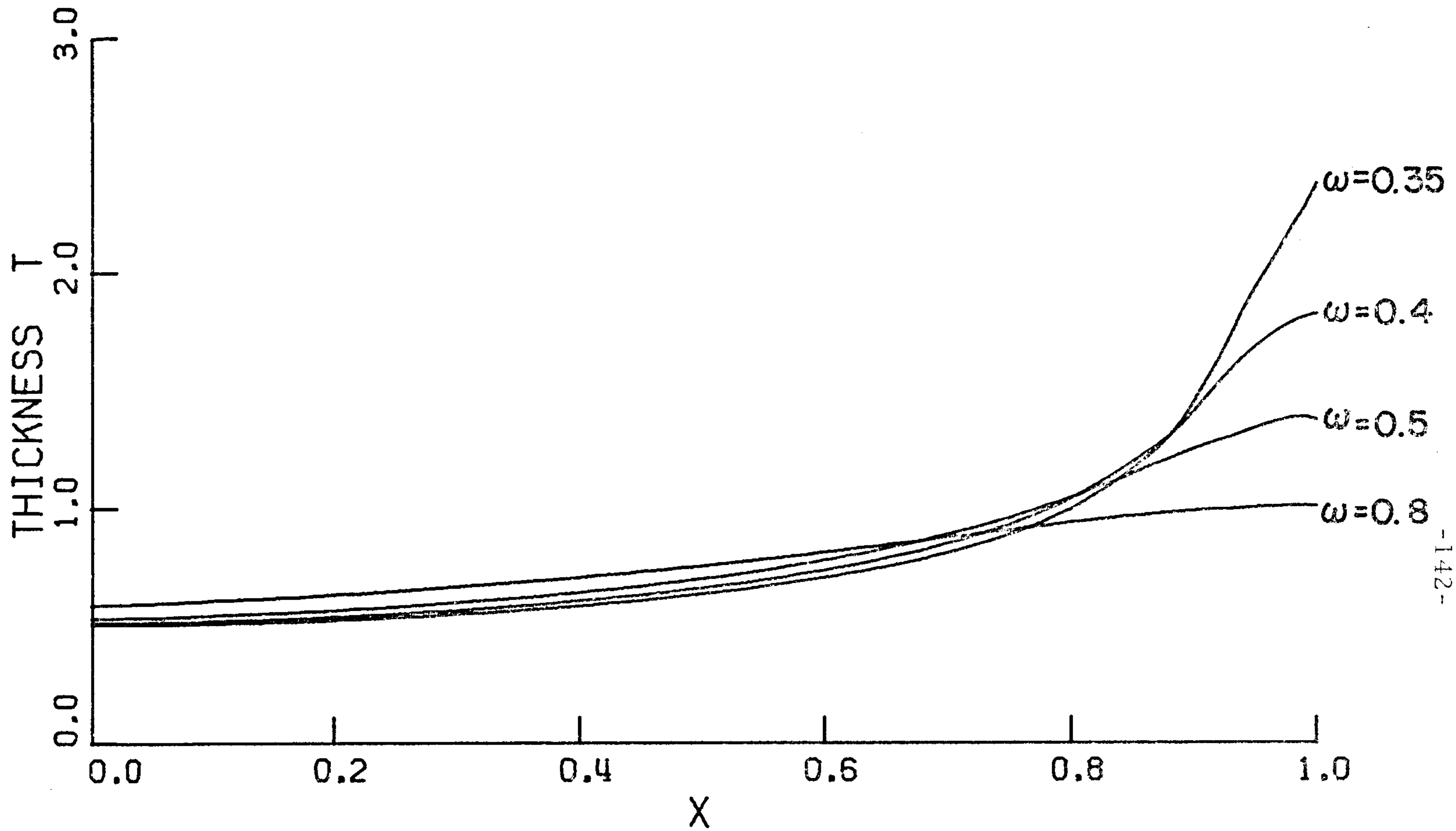


FIG. 26 ØPTIMAL MEMBRANE SHELL STIFFNESS CØNSTRAINT THICKNESS T $\gamma = 1.5$

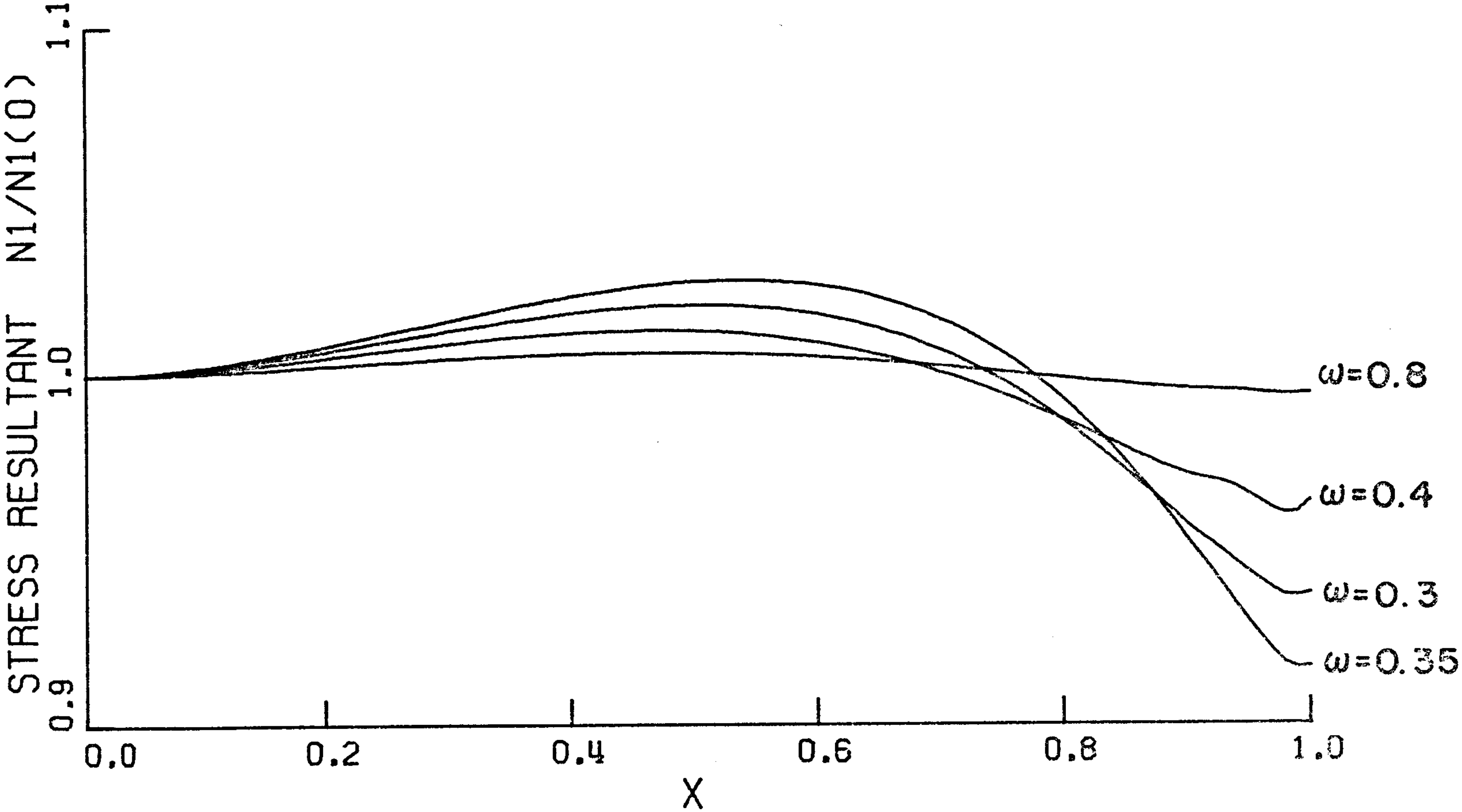


FIG.27 ØPTIMAL MEMBRANE SHELL
STIFFNESS CØNSTRAINT
STRESS RESULTANT N1/N1(O) $\gamma=1.5$

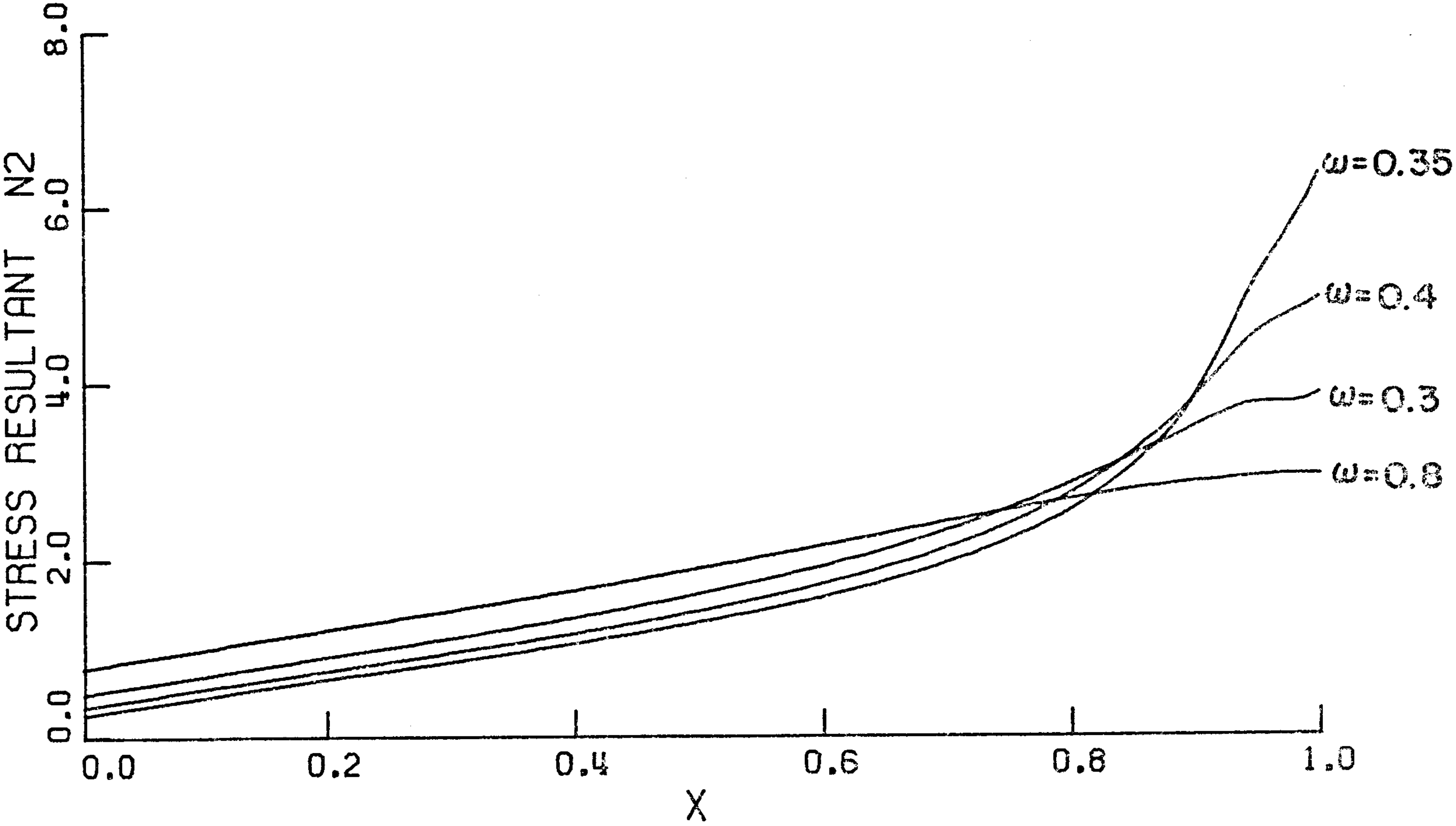
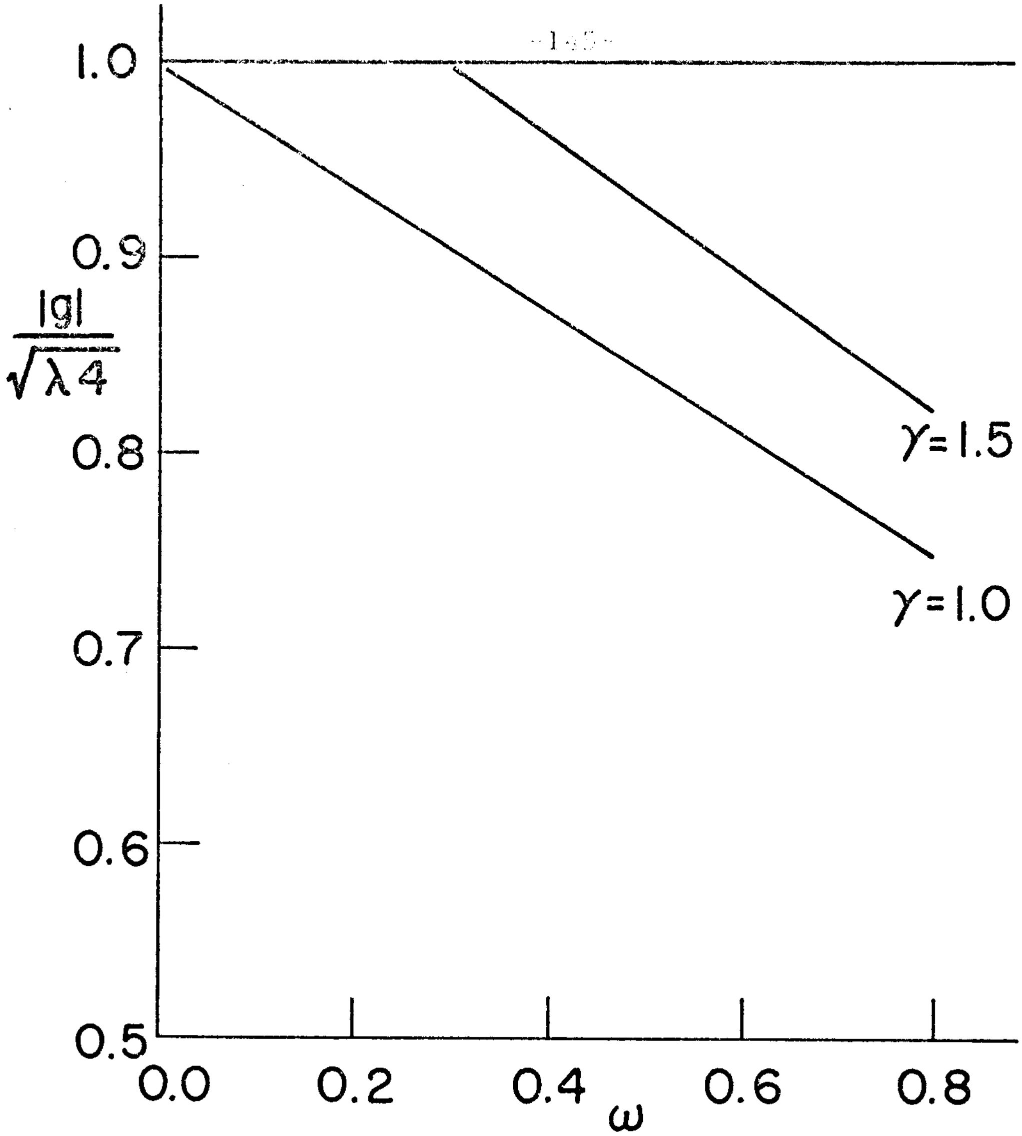


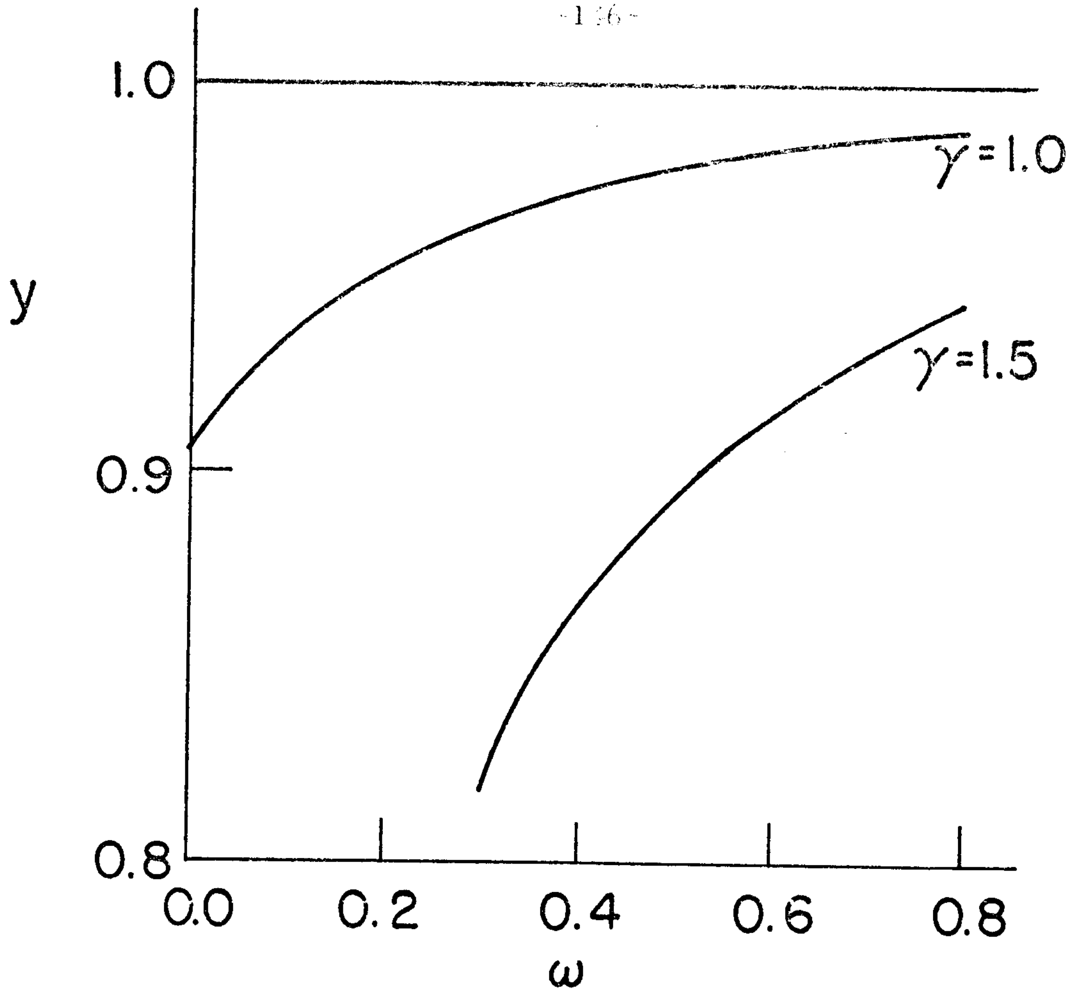
FIG.28 OPTIMAL MEMBRANE SHELL
STIFFNESS CONSTRAINT
STRESS RESULTANT N2 $\gamma = 1.5$



$$t = \sqrt{\frac{n_1(1-\nu^2)}{\lambda 4 - g^2}}$$

OPTIMAL MEMBRANE SHELL STIFFNESS CONSTRAINT

FIG. 29 $\frac{|g|}{\sqrt{\lambda 4}}$ AS A FUNCTION OF ω



$$y \equiv \frac{\text{Strain Energy Optimal Shell}}{\text{Strain Energy Cylindrical Shell}}$$

OPTIMAL MEMBRANE SHELL
STIFFNESS CONSTRAINT
FIG.30 STRAIN ENERGY

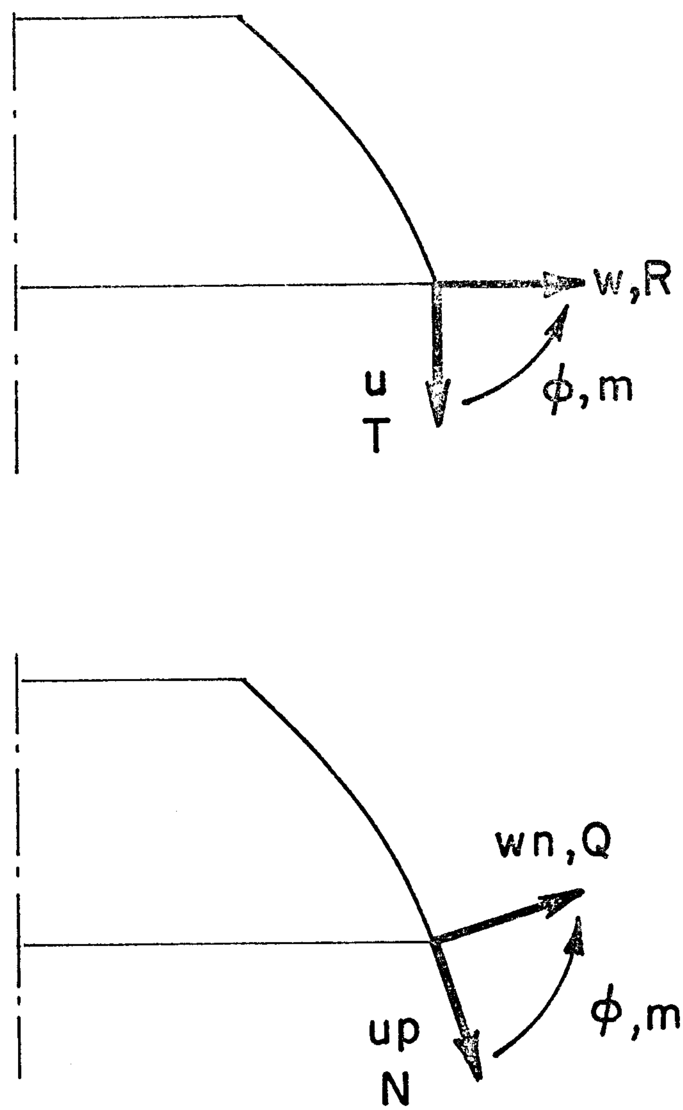


FIG. 31 SIGN CONVENTIONS

	$-\frac{\nu}{\bar{r}}$	$\frac{A}{\sqrt{1+\bar{r}_{,\bar{x}}^2}} \times \{(1-\nu^2)+\bar{r}_{,\bar{x}}^2 \beta^2\}$	$-\bar{r}_{,\bar{x}}$		$\frac{A\,\bar{r}_{,\bar{x}}}{\sqrt{1+\bar{r}_{,\bar{x}}^2}} \{(1-\nu^2)-\beta^2\}$
	$-\frac{\nu\,\bar{r}_{,\bar{x}}}{\bar{r}}$	$\frac{A\,\bar{r}_{,\bar{x}}}{\sqrt{1+\bar{r}_{,\bar{x}}^2}} \times \{(1-\nu^2)-\beta^2\}$	1		$\frac{A}{\sqrt{1+\bar{r}_{,\bar{x}}^2}} \{\bar{r}_{,\bar{x}}^2(1-\nu^2)+\beta^2\}$
		$-\frac{\bar{r}_{,\bar{x}}}{\bar{r}}$			
			$-\nu\frac{\bar{r}_{,\bar{x}}}{\bar{r}}$	$\frac{1}{D}\sqrt{1+\bar{r}_{,\bar{x}}^2}$	
		$\bar{r}_{,\bar{x}}$	$\frac{\bar{r}_{,\bar{x}}}{r^2} \times \frac{D(1-\nu^2)}{\sqrt{1+\bar{r}_{,\bar{x}}^2}}$	$-\frac{\bar{r}_{,\bar{x}}}{\bar{r}}(1-\nu)$	-1
	$\frac{\sqrt{1+\bar{r}_{,\bar{x}}^2}}{A\,\bar{r}^2}$	$\frac{\nu}{\bar{r}}$			$-\frac{\bar{r}_{,\bar{x}}}{\bar{r}}(1-\nu)$

$A = \frac{1}{E\bar{t}}$ $\qquad$ $D = \frac{E\bar{t}^3}{12(1-\nu^2)}$

Table 6-1 Matrix $\bar{E}$

	$\frac{-\nu}{r}$	$\frac{1}{t\sqrt{1+r_{,x}^2}} \times \{(1-\nu^2)+r_{,x}^2\beta^2\}$	$-r_{,x}$		$\frac{r_{,x}}{t\sqrt{1+r_{,x}^2}}\{(1-\nu^2)-\beta^2\}$
	$\nu\frac{r_{,x}}{r}$	$\frac{r_{,x}}{t\sqrt{1+r_{,x}^2}} \times \{(1-\nu^2)-\beta^2\}$	1		$\frac{1}{t\sqrt{1+r_{,x}^2}}\{r_{,x}^2(1-\nu^2)+\beta^2\}$
		$-\frac{r_{,x}}{r}$			
			$-\nu\frac{r_{,x}}{r}$	$\frac{\alpha^2}{t^3}(1-\nu^2)\sqrt{1+r_{,x}^2}$	
		$r_{,x}$	$\left(\frac{r_{,x}}{r}\right)^2\frac{t^3}{\alpha^2\sqrt{1+r_{,x}^2}}$	$-\frac{r_{,x}}{r}(1-\nu)$	-1
	$\frac{t\sqrt{1+r_{,x}^2}}{r^2}$	$\frac{\nu}{r}$			$-\frac{r_{,x}}{r}(1-\nu)$

Table 6-2 Matrix B

$$F \equiv \gamma \begin{bmatrix} 0 \\ 0 \\ r_{,x} \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Table 6-3 Vector f

		$\frac{r_{,x}}{t(1+r_{,x}^2)^{3/2}} \times \{-(1-\nu^2) + \beta^2(2+r_{,x}^2)\}$	-1		$\frac{1}{t(1+r_{,x}^2)^{3/2}} \times \{(1-\nu^2) - \beta^2\}$
	$-\frac{\nu}{r}$	$\frac{1}{t(1+r_{,x}^2)^{3/2}} \times \{(1-\nu^2) - \beta^2\}$			$\frac{r_{,x}}{t(1+r_{,x}^2)^{3/2}} \times \{(1-\nu^2)(2+r_{,x}^2) - \beta^2\}$
		$-\frac{1}{r}$			
			$-\frac{\nu}{r}$	$\frac{\alpha^2}{t^3}(1-\nu^2)\frac{r_{,x}}{\sqrt{1+r_{,x}^2}}$	
		1	$\frac{t^3}{\alpha^2 r^2}\frac{r_{,x}(2+r_{,x}^2)}{(1+r_{,x}^2)^{3/2}}$	$-\left(\frac{1-\nu}{r}\right)$	
	$\frac{t\, r_{,x}}{r^2\sqrt{1+r_{,x}^2}}$				$-\frac{(1-\nu)}{r}$

Table 6-4 Matrix $B_{r,x}$

		$\frac{-1}{t^2\sqrt{1+r_{,x}^2}} \times \{(1-\nu^2)+r_{,x}^2\beta^2\}$			$\frac{-r_{,x}}{t^2\sqrt{1+r_{,x}^2}} \times \{(1-\nu^2)-\beta^2\}$
		$\frac{-r_{,x}}{t^2\sqrt{1+r_{,x}^2}} \times \{(1-\nu^2)-\beta^2\}$			$-\frac{1}{t^2\sqrt{1+r_{,x}^2}} \times \{r_{,x}^2(1-\nu^2)+\beta^2\}$
				$-\frac{3\alpha^2}{t^4}(1-\nu^2)\sqrt{1+r_{,x}^2}$	
			$3\left(\frac{r_{,x}}{r}\right)^2\frac{t^2}{\alpha^2\sqrt{1+r_{,x}^2}}$		
	$\frac{\sqrt{1+r_{,x}^2}}{r^2}$				

Table 6-5 Matrix B_t

		$\frac{1}{t(1+r_{,x}^2)^{5/2}} \times \{(1-\nu^2)(2r_{,x}^2-1) + \beta^2(2-r_{,x}^2)\}$			$\frac{-3r_{,x}}{t(1+r_{,x}^2)^{5/2}} \times \{(1-\nu^2)-\beta^2\}$
		$\frac{-3r_{,x}}{t(1+r_{,x}^2)^{5/2}} \{(1-\nu^2)-\beta^2\}$			$\frac{1}{t(1+r_{,x}^2)^{5/2}} \{(1-\nu^2)(2-r_{,x}^2) + \beta^2(2r_{,x}^2-1)\}$
				$\frac{\alpha^2}{t^3}\frac{(1-\nu^2)}{(1+r_{,x}^2)^{3/2}}$	
			$\frac{t^3}{\alpha^2 r^2}\frac{2-r_{,x}^2}{(1+r_{,x}^2)^{5/2}}$		
	$\frac{t}{r^2(1+r_{,x}^2)^{3/2}}$				

Table 6-6 Matrix $E_{r,_x r,_x}$

		$\frac{r_{,x}}{t^2(1+r_{,x}^2)^{3/2}} \times \{(1-\nu^2) - \beta^2(2+r_{,x}^2)\}$			$\frac{-1}{t^2(1+r_{,x}^2)^{3/2}} \times \{(1-\nu^2) - \beta^2\}$
		$\frac{-1}{t^2(1+r_{,x}^2)^{3/2}} \times \{(1-\nu^2) - \beta^2\}$			$\frac{-r_{,x}}{t^2(1+r_{,x}^2)^{3/2}} \times \{(2+r_{,x}^2)(1-\nu^2) - \beta^2\}$
				$-\frac{3\alpha^2}{t^4}(1-\nu^2)\frac{r_{,x}}{\sqrt{1+r_{,x}^2}}$	
			$\frac{3t^2}{\alpha^2 r^2}\frac{r_{,x}(2+r_{,x}^2)}{(1+r_{,x}^2)^{3/2}}$		
	$\frac{r_{,x}}{r^2\sqrt{1+r_{,x}^2}}$				

Table 6-7 Matrix $B_{tr,x}$

		$\frac{2}{t^3\sqrt{1+r_{,x}^2}}\{(1-\nu^2)+r_{,x}^2\beta^2\}$			$\frac{2\, r_{,x}}{t^3\sqrt{1+r_{,x}^2}}\{(1-\nu^2)-\beta^2\}$
		$\frac{2\, r_{,x}}{t^3\sqrt{1+r_{,x}^2}}\{(1-\nu^2)-\beta^2\}$			$\frac{2}{t^3\sqrt{1+r_{,x}^2}}\{r_{,x}^2(1-\nu^2)+\beta^2\}$
				$\frac{12\alpha^2}{t^5}(1-\nu^2)\sqrt{1+r_{,x}^2}$	
			$\frac{6t}{r^2\alpha^2}\frac{r_{,x}^2}{\sqrt{1+r_{,x}^2}}$		

Table 6-8 Matrix B_{tt}

			$-\dfrac{6\, r_{,x}^2\, t^2}{r^3 \alpha^2 \sqrt{1+r_{,x}^2}}$		
	$-\dfrac{2\sqrt{1+r_{,x}^2}}{r^3}$				

Table 6-9 Matrix E_{tr}

	$\frac{\nu}{r^2}$				
		$\frac{1}{r^2}$			
			$\frac{\nu}{r^2}$		
			$\frac{-2t^3}{\alpha^2 r^3} \frac{r_{,x}(2+r_{,x}^2)}{(1+r_{,x}^2)^{3/2}}$	$\frac{(1-\nu)}{r^2}$	
	$\frac{-2t\, r_{,x}}{r^3\sqrt{1+r_{,x}^2}}$				$\frac{(1-\nu)}{r^2}$

Table 6-10 Matrix $B_{rr,x}$