AN ANALYTICAL INVESTIGATION OF TURBULENT FLOW OVER A WAVY BOUNDARY

Thesis by

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ABSTRACT

The linearized, two-dimensional flow of an incompressible fully turbulent fluid over a sinusoidal boundary is solved using the method of matched asymptotic expansions in the limit of vanishing skin friction.

A phenomenological turbulence model due to Saffman (1970, 1974) is utilized to incorporate the effects of the wavy boundary on the turbulence structure.

Arbitrary lowest order wave speed is allowed in order to consider both the stationary wavy wall, and the water wave moving with arbitrary positive or negative velocity.

Good agreement is found with measured tangential velocity profiles and surface normal stress coefficients. The phase shift of the surface normal stress exhibits correct qualitative behavior with both positive and negative wave speeds, although predicted values are low.

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INTRODUCTION

The understanding of the dynamics of the wind-induced growth of a small amplitude water wave is of great practical significance. Although it has received considerable attention over the past hundred years, the problem has remained basically unsolved, principally for lack of inclusion of the effect of the wavy boundary on the turbulence structure of the wind (Miles, 1967). In this respect, it provides a valuable context in which to examine the ability of phenomenological turbulence models to predict the effects of mean streamline curvature. It is well known that large changes in turbulence structure are produced by a relatively small mean streamline curvature; in fact, significant effects on shear stress and heat transfer can occur even for radii of curvature more than a hundred times the shear layer thickness (Bradshaw, 1973, see especially Table 1).

This research concerns the two-dimensional flow of an incompressible, fully turbulent fluid over a single Fourier component of amplitude a, wave number k ($\equiv 2\pi$/wavelength), and velocity c ($\cong \pm (g/k)^{\frac{1}{2}}$ for deep water gravity waves, where $g = 9.8$ m/sec^2; the case of a rigid wall ($c \equiv 0$) is also considered). The maximum wave slope (ka) is taken to be small in order to effect a linearization of the equations of motion; observations of non-breaking oceanic gravity waves give values of ka between 0.025 and 0.30, with the maximum value predicted by Stokes being 0.45 (Kinsman, 1965). A turbulence model due to Saffman (1970) is utilized to predict the upper flow field and surface stresses. A subsequent modification of the model equations (Saffman, 1974) to include relaxational behavior of the turbulence is also investigated. The lower flow is taken to be viscous and laminar, and is analyzed according to classical techniques by linearization in ka (Lamb, 1945). The linearized equations of motion in the upper fluid, together with their appropriate boundary conditions form a singular perturbation problem in a limit of vanishing skin friction.

PART I: BRIEF HISTORY OF MATHEMATICAL MODELS OF WIND-WAVE GENERATION

The classical Kelvin-Helmholtz treatment (Lamb, 1945) assumes uniform flow in the upper and lower fluids and predicts a minimum velocity difference for wave growth of 6.46 m/s for air over water. At this difference, capillary waves of wavelength $\lambda = 1.73$ cm first appear. For the growth of a gravity wave of $\lambda = 1$ m, a velocity difference of at least 35.7 m/s (69.5 knots) is predicted as necessary. If this velocity is assumed similar to that measured on a ship's deck, the wind would qualify as a hurricane (Beaufort 12). An important characteristic of this treatment is that it yields a surface perturbation pressure in perfect anti-phase with the wave elevation (for stable waves).

Jeffries (Lamb, 1945) assumed there existed a component of surface perturbation pressure in phase with the wave slope given by

$$p_1 = \beta \, \rho_a (U-c)^2 \, \frac{\partial y_s}{\partial x}$$

where U = velocity at ∞, $y_s(x,t) = a \cos k(x-ct)$, ρ_a = density of air, and β = "sheltering" coefficient. It was hypothesized that this pressure component arose by means of a separation of the flow on the leeward side of the wave (with respect to the flow at infinity). Neglecting the rate of work by surface shear as being unimportant for wave growth, and accounting for viscous (laminar) dissipation in the water, Jeffries obtained, by energy considerations, the following criteria for growth of deep water gravity waves:

$$(U-c)^2 c > \frac{4\, \nu_w g\, \rho_w}{\beta\, \rho_a} \; .$$

No explicit means was given, however, for the calculation of β as a function of the flow characteristics.

A mechanism was introduced by Miles (1957, 1959) to account for the component of surface perturbation pressure in phase with the wave slope for a wave progressing in the direction of the wind. Miles linearized the equations of motion in ka, assuming there was no effect of the wave on the turbulent Reynolds stresses anywhere within the flow field, thereby obtaining the Orr-Sommerfeld equation as a description of the O(ka) dynamics. Viscous stresses governed the flow at the wave surface and critical layer (where the O(1) wind velocity equals the wave speed). This "inviscid laminar model," as Miles referred to it, yielded a solution for the amplitude and phase of the surface perturbation pressure for a logarithmic wind profile as a function of c/u_{*_0} and kz_0, where $u_{*_0} \equiv (\tau_{w_0}/\rho_a)^{\frac{1}{2}}$, τ_{w_0} being the O(1) surface shear and z_o being the O(1) roughness height of the surface.

Benjamin (1959) extended the work of Miles by considering arbitrary wind profiles and wave velocity.

In 1959, Miles generalized the Kelvin-Helmholtz instability (in which the entire perturbation surface pressure is in phase with the wave elevation) for parallel shear flows, concluding that it was unlikely to occur for commonly observed wind speeds over water (Miles, 1959b).

However, in 1967, Miles critized the "inviscid laminar" model:

> "It now appears fairly clear, albeit less than certain, that the inviscid laminar model under-estimates the energy transfer from wind to waves over at least a significant portion of the spectrum for an open sea."
>
> (Miles, 1967, p. 165)

> "It is conceivable that the consistent discrepancy between field observations and theoretical predictions (on the basis of the [inviscid] laminar model) of wave growth could be attributed to a consistent overestimate of z_c [the height of the critical layer]; however, this appears unlikely, and the most plausible conjecture is that the wave-induced turbulent Reynolds stresses are, in fact, not negligible over a significant portion of the gravity-wave spectrum."
>
> (Miles, 1967, p. 168)
>
> (my underline)

There have been several attempts since 1967 to account for the effect of the wavy boundary on the turbulent Reynolds stresses. Davis (1972) examined numerically two different models of the turbulence for the linearized problem, one patterned after the turbulence energy method of Bradshaw, Ferris, and Atwell (1967), and the other manifesting a viscoelastic constitutive relation for the turbulence. His results were inconclusive due to difficulties experienced in accounting for the boundary condition on the tangential velocity at the surface. Townsend (1972) treated the linearized problem using a turbulence model similar to that of Bradshaw, Ferris and Atwell (1967) and obtained rates of wave growth, based on calculated distribution of surface pressure, considerably less than experimental values. He further argued that the linearized approach would be valid only for $ka < 0.10$.

PART II: STATEMENT OF PROBLEM

A. Assumptions and Turbulence Model Equations

The coordinate system is taken attached to the wave moving at speed c with respect to the lower flow. The surface is described by

$$y_s = a \cos kx \ .$$

To facilitate handling of the boundary conditions at the surface, an orthogonal curvilinear coordinate system (ξ, η, z) introduced by Benjamin (1959) is employed where

$$\xi(x, y) = x + a\, e^{-ky} \sin kx$$

$$\eta(x, y) = y - a\, e^{-ky} \cos kx$$

As Benjamin indicated, ξ and η are the same as the velocity potential and streamfunction for irrotational motion in an inviscid fluid over a sinusoidal boundary. It is easily verified that at the surface,

$$k\eta = (ka)^2 \cos^2 kx + O(ka)^3$$

The upper flow is taken to be fully-turbulent, two-dimensional and incompressible. In addition, the following assumptions are made:

1) The flow is exactly periodic; e.g., for any dependent variable $f(\xi, \eta)$,

$$f(\xi, \eta) = f(\xi + \lambda, \eta) \quad .$$

The variables may therefore be expanded in terms of the harmonics $e^{ink\xi}$, $n = 0, 1, 2, \ldots,$ and the partial

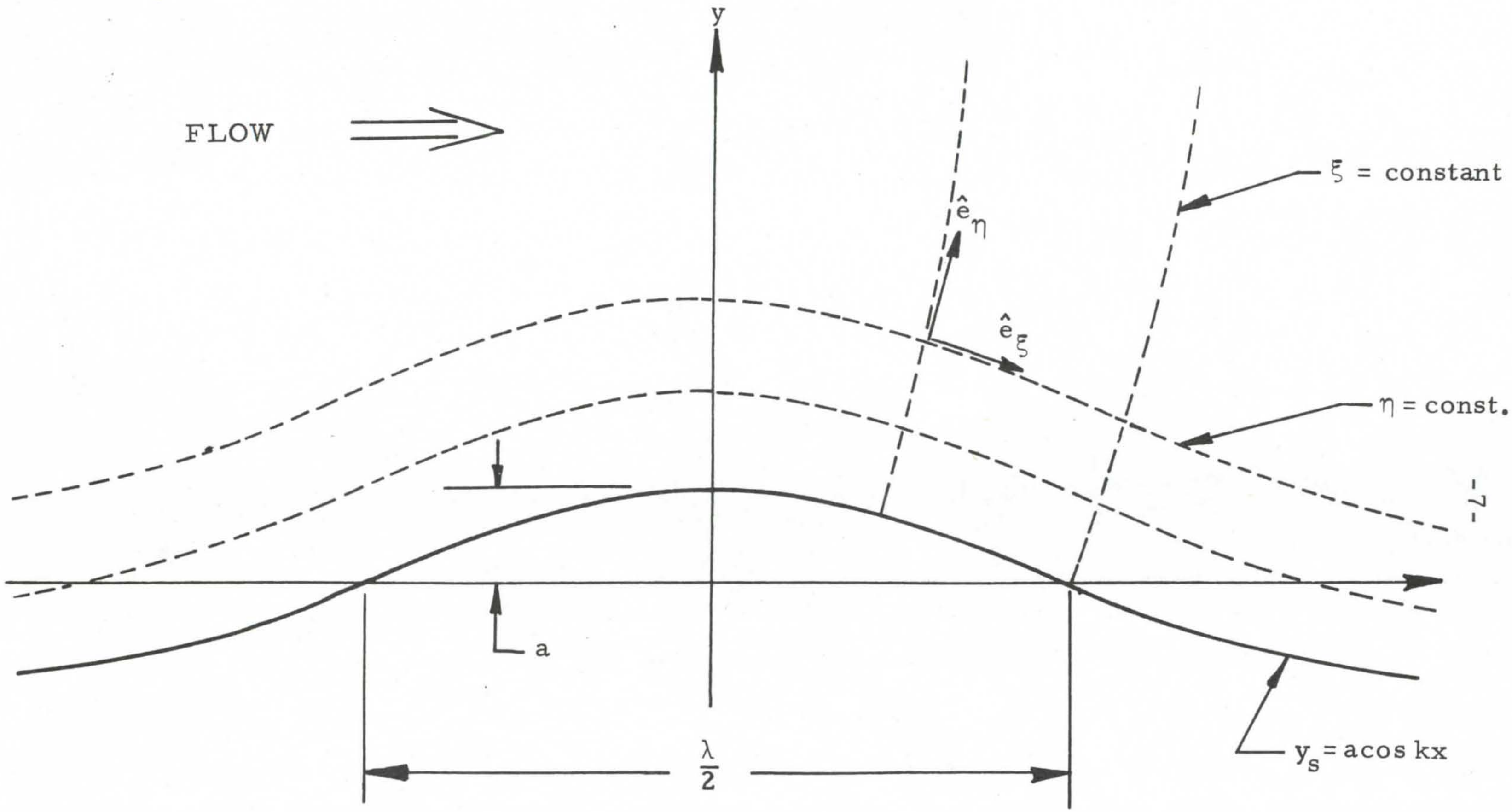

Figure 1

(Vertical Scale Exaggerated)

differential equations governing the flow are reduced to the more tractable case of an infinite hierarchy of ordinary differential equations in η. To lowest order in ka,

$$f(\xi, \eta) = f_0(\eta) + o(1) \text{ as } ka \to 0,$$

implying that the O(1) mean flow (e.g., the flow in the limit ka $\to$ 0) is independent of streamwise position, a reasonable assumption in view of the slight downstream variation of quantities in a non-separating flat-plate turbulent boundary layer. Furthermore, in the presence of the wavy boundary, the O(1) mean flow quantities, being functions of η alone, are effectively "bent" around the wave.

2) ka << 1. Only the O(ka) perturbation is obtained in this analysis; hence ka should be small for applicability of the solution. As noted previously, observed values of ka for unbroken water waves generally are less than 0.3.

3) η_{*_0} << a, where η_{*_0} is the sublayer scale of the upper flow evident in the formula for the O(1) mean velocity in the ξ-direction (see equation (3.11)):

$$U = \frac{u_{*_0}}{\kappa} \ln(\eta/\eta_{*_0}) - c$$

In the above, $u_{*_0} = (\tau_{w_0}/\rho)^{\frac{1}{2}}$, where τ_{w_0} is the O(1) surface shear, c = wave speed, and κ is von Karman's constant, taken to be equal to 0.41 (Coles, 1968).

For a smooth surface, η_{*_0} is proportional to the viscous sublayer thickness, and is given by

$$\eta_{*_0} = \frac{\nu}{u_{*_0}} e^{-B\kappa}$$

where ν = kinematic viscosity of the upper fluid, and B = 5.0 (Coles, 1968).

For a rough surface, η_{*_0} is the "roughness height."

Throughout the analysis, the term "sublayer scale" will be used to denote η_{*_0}, since both smooth and rough surfaces are considered.

The assumption that $\eta_{*_0} \ll a$ will be seen to effectively imply that the variation in the normal velocity and total shear stress (viscous plus turbulent) across the sublayer is negligible.

4) $\lambda \ll \delta$, where δ = turbulent boundary layer thickness. The mean velocity profile utilized is logarithmic, which is a reasonable approximation provided the scale on which the wave-induced perturbations decay (e.g., λ) is small compared to the boundary layer thickness.

5) For $c \neq 0$, the upper flow imposes only a slight perturbation on the speed of the water wave; e.g., Kelvin-Helmholtz instability is not considered.

In the case $c \neq 0$, the lower flow is taken to be laminar, two-dimensional, incompressible and viscous and is analyzed according to classical techniques by linearization in ka (Lamb, 1945).

A phenomenological turbulence model developed by Saffman (1970; 1974) is employed. The model specifies the following constitutive behavior for the Reynolds stress

$$-\overline{v_i' v_j'} = 2\frac{e}{\omega} S_{ij} - \tfrac{1}{3} g_{ij} \overline{q^2} - \chi_{ij} \tag{2.1}$$

where v_i = total (instantaneous) covariant velocity in i^{th} direction

$= \overline{v}_i + v_i'$, where the overbar represents an ensemble average at fixed (ξ, η, z, t)

v^i = total contravariant velocity in i^{th} direction

$= g^{ij} v_j$

g_{ij} = covariant metric tensor

$$= \begin{pmatrix} J^{-1} & 0 & 0 \\ 0 & J^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

J $= \dfrac{\partial(\xi, \eta)}{\partial(x, y)} = 1 + 2ka\, e^{-ky} \cos kx + O(ka)^2$

$-\overline{v_i' v_j'}$ = covariant Reynolds stress tensor

S_{ij} $= \frac{1}{2}(\overline{v}_{i,j} + \overline{v}_{j,i})$ = covariant rate-of-strain tensor

e = "pseudo-turbulence energy," a dependent variable which is governed by a transport equation (see below)[†]

[†] In a later development (Saffman, 1974), e has been directly identified with the turbulence energy, and differs by a constant factor from the "e" used in this analysis.

ω = "pseudo-vorticity," resembling (but not intended to be identical to) the root-mean-square fluctuating vorticity of the large eddies and also governed by a transport equation.

$\overline{q^2}$ = twice the turbulence kinetic energy $= \sum_{i=1}^{3} \overline{v_i' v^{i'}}$

χ_{ij} = covariant "relaxation" tensor, intended to account for the deviation of the principal axes of the Reynolds stress tensor from those of the mean rate-of-strain tensor noted in flows with rapid changes in the rate-of-strain.

The transport equations for the turbulence quantities introduced above are the following (Saffman, 1970, 1974) where the familiar summation notation is used unless otherwise noted:

$$\frac{\partial e}{\partial t} + e,_k \bar{v}^k = \tilde{\alpha} e \left\{ 2 S_{ij} S_{k\ell} g^{ik} g^{j\ell} \right\}^{\frac{1}{2}} - e\omega + g^{jk} \left(\tilde{\sigma} \frac{e}{\omega} e,_j \right),_k \tag{2.2}$$

$$\frac{\partial \omega}{\partial t} + \omega,_k \bar{v}^k = \alpha \omega \left\{ \bar{v}_{i,j} \bar{v}_{k,\ell} g^{ik} g^{j\ell} \right\}^{\frac{1}{2}} - \beta \omega^2 + g^{jk} \left(\sigma \frac{e}{\omega} \omega,_j \right),_k \tag{2.3}$$

$$\frac{\partial \chi_{ij}}{\partial t} + \chi_{ij,k} \bar{v}^k = 2 \frac{e}{\omega} \left(\frac{\partial S_{ij}}{\partial t} + S_{ij,k} \bar{v}^k \right) - \tilde{\lambda} \omega \chi_{ij} \tag{2.4}$$

In the rate equations for e and ω, the three terms on the right hand side represent, respectively, generation, dissipation, and diffusion. The constants α, $\tilde{\alpha}$, β, σ and $\tilde{\sigma}$ are evaluated by simple considerations (Saffman, 1970) and are not optimized for any particular set of flows.[†] Their values are restricted to

$$\begin{aligned} \tilde{\alpha} &= 0.3 \\ 0.075 < \alpha &< 0.106 \\ \frac{5}{6} < \beta &< 1 \\ \tilde{\sigma} &= 0.5 \\ \sigma &= 1.0 \end{aligned} \tag{2.5}$$

Furthermore, as indicated by Saffman (1970), matching the law of the wall boundary condition on the mean velocity requires

$$\left(\frac{\beta\tilde{\alpha}-\alpha}{\sigma}\right)^{\frac{1}{2}} = \kappa \tag{2.6}$$

where κ = Von Karman's constant. The above range of values (2.5) yields

$$0.380 < \left(\frac{\beta\tilde{\alpha}-\alpha}{\sigma}\right)^{\frac{1}{2}} < 0.475$$

The accepted value of 0.41 lies within this interval.

The transport equation for the relaxation tensor χ_{ij} manifests the characteristics that an instantaneous change in S_{ij} must leave $-\overline{v_i' v_j'}$ unaltered, and that in a steady state, the principal axes of S_{ij} and $-\overline{v_i' v_j'}$ are assumed to coincide. The constant $\tilde{\lambda}$ is assigned the value $\tilde{\lambda} = 1$ on the assumption that there is no significant approach to isotropy in decaying anisotropic homogeneous turbulence (Saffman, 1974).

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† In the original paper, (Saffman, 1970), a rate equation was written for ω^2; for convenience of this analysis, the equation was written for ω and the constants in (2.5) and (2.6) evaluated according to the procedure indicated in the original paper.

In order to evaluate the relaxation model, the problem was solved both with and without the relaxation hypothesis.

The remaining equations are:

Conservation of mass: $\bar{v}^k_{,k} = 0$ (2.7)

Conservation of momentum: (Sokolnikoff, 1967):

$$\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_{i,k}\, \bar{v}^k = -\frac{1}{\rho}\, \Phi_{,i} + g^{jk}\left[2\, \frac{e}{\omega} S_{ij} - \chi_{ij}\right]_{,k} \qquad (2.8)$$

where

$$\Phi = \bar{p} + \frac{1}{3}\, \rho\, \overline{q^2} = \text{average mean normal stress,}$$

$$\bar{p} = \text{mean static pressure .}$$

The equations are valid in fully turbulent regions only, as viscous diffusion has been neglected.

The components of the (instantaneous) vector velocity V_i are given by (Sokolnikoff, 1967, p. 122)

$$V_i = v_i/(g_{ii})^{\frac{1}{2}} \quad \text{(no summation on i)}$$

For the coordinate system used,

$$(V_1, V_2, V_3) = (J^{+\frac{1}{2}}\, v_1,\, J^{+\frac{1}{2}} v_2,\, v_3) = (J^{-\frac{1}{2}} v^1,\, J^{-\frac{1}{2}} v^2,\, v^3)$$

Since the averaged motion is two-dimensional, we may introduce a stream function satisfying (2.7), given by

$$\bar{v}_1 = J^{-\frac{1}{2}}\, \bar{V}_1 = \psi_\eta$$

$$\bar{v}_2 = J^{-\frac{1}{2}}\, \bar{V}_2 = -\psi_\xi$$

For convenience of representation, we define

$$
\begin{aligned}
u &\equiv \overline{V}_1 = +J^{+\frac{1}{2}}\,\psi_\eta \\
v &\equiv \overline{V}_2 = -J^{+\frac{1}{2}}\,\psi_\xi
\end{aligned}
\tag{2.9}
$$

and define

$$u_i' \equiv V_i' = V_i - \overline{V}_i = v_i'/(g_{ii})^{\frac{1}{2}}$$

The (physical) Reynolds stress tensor is then

$$-\overline{u_i' u_j'} = \frac{1}{\sqrt{g_{ii}\, g_{jj}}}\left(-\overline{v_i' v_j'}\right) \qquad \text{(no summation)}$$

which for the coordinate system used becomes, on account of symmetry,

$$-\overline{u_i' u_j'} = \begin{Bmatrix} J(-\overline{v_1' v_1'}) & J(-\overline{v_1' v_2'}) & 0 \\ J(-\overline{v_1' v_2'}) & J(-\overline{v_2' v_2'}) & 0 \\ 0 & 0 & -\overline{v_3' v_3'} \end{Bmatrix}$$

B. Boundary Conditions

It is easy to verify that the unit vectors in the (ξ, η) coordinate system are

$$\hat{e}_\xi \equiv \frac{(1 + kae^{-ky}\cos kx)\hat{e}_x - kae^{-ky}\sin kx\,\hat{e}_y}{(1 + 2ka\,e^{-ky}\cos kx + (ka)^2 e^{-2ky})^{\frac{1}{2}}}$$

$$\hat{e}_\eta \equiv \frac{+ka\,e^{-ky}\sin kx\,\hat{e}_x + (1 + ka\,e^{-ky}\cos kx)\hat{e}_y}{(1 + 2ka\,e^{-ky}\cos kx + (ka)^2 e^{-2ky})^{\frac{1}{2}}}$$

and that the unit tangent and normal vectors $(\hat{e}_t, \hat{e}_n)$ at the surface $y - a\cos kx = 0$ are given by

$$\hat{e}_t = \frac{\hat{e}_x - ka \sin kx\, \hat{e}_y}{(1 + (ka \sin kx)^2)^{\frac{1}{2}}}$$

$$\hat{e}_n = \frac{ka \sin kx\, \hat{e}_x + \hat{e}_y}{(1 + (ka \sin kx)^2)^{\frac{1}{2}}} \quad .$$

Therefore at $y = a \cos kx$,

$$\hat{e}_\xi = \hat{e}_t + O(ka)^2$$

$$\hat{e}_\eta = \hat{e}_n + O(ka)^2 \quad .$$

Since only the O(1) and O(ka) dynamics will be considered, it is therefore possible to identify u and v at the boundary as the tangential and normal velocity to the surface, respectively.

As the model equations are valid within the fully turbulent region only, boundary conditions must be applied asymptotically as η approaches the edge of the sublayer, e.g., as $\eta \to O^+$.† Neglecting terms of $O(ka)^2$, they are as follows:

1) $v \sim O(ka\, u_{*_0} k\eta_{*_0} e^{ik\xi}) + O(ka\, c\, k\eta_{*_0} e^{ik\xi})$ (2.10)

The derivation is given in Appendix I; the result is a consequence of the vanishing of the normal velocity at the surface. As will be seen, for the purposes of the analysis the right-hand side is negligible.

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† The model equations are valid outside the sublayer (e.g. for $\eta \gg \eta_{*_0}$). As will be shown, the sublayer scale is transcendentally small compared to the scales on which the equations are solved, and hence the notation "$\eta \to O^+$" is used to refer to η approaching the edge of the sublayer.

2) $$u \sim \frac{u_*(\xi)}{\kappa} \ln\left(\frac{\zeta}{\eta_*(\xi)}\right) - c + c\Omega\, ka e^{ik\xi} \tag{2.11}$$

where

$$u_*(\xi) = \left(\frac{\tau_w(\xi)}{\rho}\right)^{\frac{1}{2}}, \quad \tau_w(\xi) = \text{local wall shear}$$

$\eta_*(\xi)$ = local sublayer scale

Real $(c\Omega\, ka\, e^{ik\xi})$ = orbital velocity of the lower fluid at the surface to O(ka).

ζ = distance normal to the boundary $= \int_0^{\eta} J^{-\frac{1}{2}} d\eta + O(ka)^2$

κ = Von Karman's constant

The above is the familiar Law of the Wall (Coles, 1956).

3) $$e \sim \tilde{\alpha}\, u_*^2(\xi), \qquad \omega \sim \frac{\tilde{\alpha}\, u_*(\xi)}{\kappa\, \zeta} \tag{2.12}$$

As indicated by Saffman (1970), the law of the wall implies that near a solid surface with no transpiration (and outside of the sublayer), e and ω are functions only of the local wall shear and distance normal to the wall. Dimensional analysis then requires

$$e \propto u_*^2(\xi), \qquad \omega \propto \frac{u_*(\xi)}{\zeta},$$

and the proportionality factors arise from satisfying the model equations near the wall.

4) $$-\overline{u_1' u_2'} \sim u_*^2(\xi) + \left\{O\left(ka\, k\eta_{*_0}\, e^{ik\xi}\, \frac{p_1|_{SFC}}{\rho}\right) + O\left(ka\, k\eta_{*_0}\, u_{*_0}^2\, e^{ik\xi}\right) + O\left(ka\, k\eta_{*_0}\, c^2\, e^{ik\xi}\right)\right\} \tag{2.13}$$

where $-\overline{u_1' u_2'}$ = physical Reynolds shear stress

$p_1|_{SFC}$ = amplitude of the O(ka) perturbation surface pressure.

The derivation is given in Appendix II; as will be shown, for the purposes of this analysis the terms in brackets $\{ \}$ are negligible.

Furthermore, all wave-induced perturbations are required to vanish as $\eta \to \infty$.

The coupling of the upper and lower flows is provided by the condition that the normal stress and the O(ka) shear at the surface be continuous. As will be indicated later, it is not necessary to match the O(1) mean surface shear since the unsteady O(1) boundary layer which it generates within the lower fluid may be neglected over the time interval in which the O(ka) solution is valid.

C. Previous Application of Model Equations

The model equations have been applied with considerable success to a variety of cases of turbulent motion, including channel flow, Couette flow, the two-dimensional free jet and wake, the continuously separating boundary layer, and the boundary layer towards a line sink (Saffman, 1970; Govindaraju, 1970). In the case of turbulent Couette flow, a closed-form solution of equations (2.1), (2.2), (2.3), (2.4), (2.7) and (2.8) may be obtained for the mean velocity

$$U(y) = \tfrac{1}{2}\, U_w + \frac{u_*}{2\kappa}\, \ell n \left\{ \frac{1 + \sin(\frac{\pi}{2}\frac{y}{b})}{1 - \sin(\frac{\pi}{2}\frac{y}{b})} \right\} , \quad -b < y < b \qquad (2.14)$$

where U_w = velocity of upper wall (lower wall at rest)

b = half-width of channel

$u_* = (\tau_w/\rho)^{\frac{1}{2}}$ $\quad \tau_w$ = wall shear

κ = Von Kármán's constant = 0.41

The solution is valid outside the viscous sublayers at $y = \pm b$. The relation between the friction velocity u_* and wall velocity U_w, obtained by matching to the law of the wall, is

$$\frac{U_w}{2} = \frac{u_*}{\kappa}\left\{\tfrac{1}{2}\ell n(16/\pi^2) + \ell n\left(\frac{bu_*}{\nu}\right) + B\kappa\right\} \tag{2.15}$$

where ν = kinematic viscosity, B = constant in law of wall over smooth surfaces with value B = 5.0 (Coles, 1968).

The relation (2.15) is only slightly different from the corresponding one obtained using the rate equation for ω^2. In that case (Govindaraju, 1970):

$$\frac{U_w}{2} = \frac{u_*}{\kappa}\left\{0.077 + \ell n\left(\frac{bu_*}{\nu}\right) + B\kappa\right\}, \tag{2.16}$$

whereas

$$\tfrac{1}{2}\ell n(16/\pi^2) = 0.2415 .$$

As indicated in Govindaraju (1970), equation (2.16) is in good agreement with experiment.

The velocity profile (2.14) is plotted in Fig. 1, along with the data of Reichardt (1959) for a Reynolds number $U_w b/\nu = 34,000$.

PART III: SOLUTION OF THE EQUATIONS

A. Introduction

The Christoffel 3-index of the second kind for the (ξ, η, z) coordinate system is given by (no summation on any index)

$$\Gamma^i_{jk} = \frac{1}{2}\left\{ \frac{1}{(h_i)^2} \delta_{ij} \frac{\partial (h_j)^2}{\partial \xi^k} + \frac{1}{(h_i)^2} \delta_{ik} \frac{\partial (h_k)^2}{\partial \xi^j} - \frac{1}{(h_i)^2} \delta_{jk} \frac{\partial (h_j)^2}{\partial \xi^i} \right\}$$

where

$$(\xi^1, \xi^2, \xi^3) = (\xi, \eta, z)$$

$$(h_1, h_2, h_3) = (J^{-\frac{1}{2}}, J^{-\frac{1}{2}}, 1)$$

$$\delta_{ij} = \text{Kronecker delta} = \begin{cases} 0 & i \neq j \\ 0 & i = j \end{cases}$$

Note that Γ^i_{jk} is not a tensor since the coordinate transformation $(x, y, z) \longrightarrow (\xi, \eta, z)$ is not affine.

From the symmetry of the problem, it is clear that, to all orders in ka,

$$-\overline{u_1' u_3'} = -\overline{u_2' u_3'} = 0$$

and since $S_{13} = S_{23} \equiv 0$, therefore $\chi_{13} = \chi_{23} \equiv 0$. Furthermore, the definition

$$-\overline{v_i' v^{i'}} \equiv -\overline{q^2}$$

implies, from (2.1),

$$g^{ij}\chi_{ij} \equiv 0$$

which reduces to

$$J(\chi_{11} + \chi_{22}) + \chi_{33} = 0 . \qquad (3.1)$$

Clearly, symmetry of $-\overline{v_i' v_j'}$ implies $\chi_{ij} = \chi_{ji}$.

The full steady equations are:

$\hat{e}_\xi$ momentum:

$$J(\psi_\eta \psi_{\eta\xi} - \psi_\xi \psi_{\eta\eta}) + \tfrac{1}{2} J_\xi(\psi_\eta^2 + \psi_\xi^2) \quad =$$

$$- \frac{1}{\rho} \Phi_\xi + \frac{e}{\omega}\frac{\partial}{\partial\eta}\left\{J(\psi_{\xi\xi} + \psi_{\eta\eta})\right\} + \left\{\frac{\partial}{\partial\xi}\left(\frac{e}{\omega}\right)\right\}\left\{2J\,\psi_{\eta\xi} + \psi_\eta J_\xi + \psi_\xi J_\eta\right\}$$

$$+ \left\{\frac{\partial}{\partial\eta}\left(\frac{e}{\omega}\right)\right\}\left\{J(\psi_{\eta\eta} - \psi_{\xi\xi}) + \psi_\eta J_\eta - \psi_\xi J_\xi\right\}$$

$$- J\left(\frac{\partial\chi_{11}}{\partial\xi} + \frac{\partial\chi_{12}}{\partial\eta}\right) - \tfrac{1}{2} J_\xi(\chi_{11} + \chi_{22}) \tag{3.2}$$

$\hat{e}_\eta$ momentum:

$$J(-\psi_\eta\psi_{\xi\xi} + \psi_\xi \psi_{\xi\eta}) + \tfrac{1}{2} J_\eta(\psi_\eta^2 + \psi_\xi^2) \ =$$

$$- \frac{1}{\rho} \Phi_\eta - \frac{e}{\omega}\frac{\partial}{\partial\xi}\left\{J(\psi_{\xi\xi} + \psi_{\eta\eta})\right\} + \left\{\frac{\partial}{\partial\xi}\left(\frac{e}{\omega}\right)\right\}\left\{J(\psi_{\eta\eta} - \psi_{\xi\xi}) + \psi_\eta J_\eta - \psi_\xi J_\xi\right\}$$

$$+ \left\{\frac{\partial}{\partial\eta}\left(\frac{e}{\omega}\right)\right\}\left\{-2J\,\psi_{\xi\eta} - \psi_\xi J_\eta - \psi_\eta J_\xi\right\}$$

$$- J\left(\frac{\partial\chi_{12}}{\partial\xi} + \frac{\partial\chi_{22}}{\partial\eta}\right) - \tfrac{1}{2} J_\eta(\chi_{11} + \chi_{22}) \tag{3.3}$$

Pseudo-Turbulence Energy:

$$J\,\psi_\eta e_\xi - J\,\psi_\xi e_\eta \ =$$

$$\tilde{\alpha}\, e \left\{ \begin{array}{l} 4\left[J\psi_{\eta\xi} + \tfrac{1}{2}(\psi_\eta J_\xi + \psi_\xi J_\eta)\right]^2 \\ +2\left[J\,\psi_{\eta\eta} + \tfrac{1}{2}(\psi_\eta J_\eta - \psi_\xi J_\xi)\right]\left[-J\,\psi_{\xi\xi} + \tfrac{1}{2}(\psi_\eta J_\eta - \psi_\xi J_\xi)\right] \\ + \left[J\psi_{\xi\xi} + \tfrac{1}{2}(\psi_\xi J_\xi - \psi_\eta J_\eta)\right]^2 + \left[J\psi_{\eta\eta} + \tfrac{1}{2}(\psi_\eta J_\eta - \psi_\xi J_\xi)\right]^2 \end{array} \right\}^{\frac{1}{2}}$$

$$- e\omega + J\frac{\partial}{\partial\xi}\left\{\tilde{\sigma}\,\frac{e}{\omega}\, e_\xi\right\} + J\frac{\partial}{\partial\eta}\left\{\tilde{\sigma}\,\frac{e}{\omega}\, e_\eta\right\} . \tag{3.4}$$

Pseudo-vorticity:

$$J\,\psi_\eta\,\omega_\xi - J\,\psi_\xi\,\omega_\eta =$$

$$\alpha\omega \left\{ \left[J\,\psi_{\xi\xi} + \tfrac{1}{2}(\psi_\xi J_\xi - \psi_\eta J_\eta)\right]^2 + \left[J\psi_{\eta\eta} + \tfrac{1}{2}(\psi_\eta J_\eta - \psi_\xi J_\xi)\right]^2 + \left[J\psi_{\eta\xi} + \tfrac{1}{2}(\psi_\eta J_\xi + \psi_\xi J_\eta)\right]^2 \right\}^{\frac{1}{2}}$$

$$- \beta\,\omega^2 + J\frac{\partial}{\partial\xi}\left\{\sigma\,\frac{e}{\omega}\,\omega_\xi\right\} + J\frac{\partial}{\partial\eta}\left\{\sigma\,\frac{e}{\omega}\,\omega_\eta\right\} \quad . \qquad (3.5)$$

Relaxation Equations:

$$J\left(\psi_\eta\frac{\partial\chi_{11}}{\partial\xi} - \psi_\xi\,\frac{\partial\chi_{11}}{\partial\eta}\right) + \chi_{11}\left(\psi_\eta J_\xi + \psi_\eta J_\eta\right) - \chi_{12}\left(\psi_\eta J_\eta + \psi_\xi J_\xi\right) =$$

$$J\,\frac{e}{\omega}\left\{\left(\psi_\eta\frac{\partial}{\partial\xi} - \psi_\xi\frac{\partial}{\partial\eta}\right)\left(2\,\psi_{\eta\xi} + \frac{\psi_\eta J_\xi}{J} + \frac{\psi_\xi J_\eta}{J}\right)\right\}$$

$$+ \frac{e}{\omega}\left\{2\,\psi_{\eta\xi} + \frac{\psi_\eta J_\xi}{J} + \frac{\psi_\xi J_\eta}{J}\right\}\left\{\psi_\eta J_\xi + \psi_\eta J_\eta\right\}$$

$$+ \frac{e}{\omega}\left\{\psi_{\eta\eta} - \psi_{\xi\xi} + \frac{\psi_\eta J_\eta}{J} - \frac{\psi_\xi J_\xi}{J}\right\}\left\{-\,\psi_\eta J_\eta - \psi_\xi J_\xi\right\} - \tilde{\lambda}\omega\chi_{11} \qquad (3.6)$$

$$J\left(\psi_\eta\frac{\partial\chi_{12}}{\partial\xi} - \psi_\xi\,\frac{\partial\chi_{12}}{\partial\eta}\right) + \chi_{12}(\psi_\eta J_\xi - \psi_\xi J_\eta) + \tfrac{1}{2}(\chi_{11} - \chi_{22})(\psi_\eta J_\eta + \psi_\xi J_\xi) =$$

$$\frac{e}{\omega}\,J\left\{\left(\psi_\eta\frac{\partial}{\partial\xi} - \psi_\xi\frac{\partial}{\partial\eta}\right)\left(\psi_{\eta\eta} - \psi_{\xi\xi} + \frac{\psi_\eta J_\eta}{J} - \frac{\psi_\xi J_\xi}{J}\right)\right\}$$

$$+ \frac{e}{\omega}\left\{\psi_{\eta\eta} - \psi_{\xi\xi} + \frac{\psi_\eta J_\eta}{J} - \frac{\psi_\xi J_\xi}{J}\right\}\left\{\psi_\eta J_\xi - \psi_\xi J_\eta\right\}$$

$$+ \frac{e}{\omega}\left\{2\,\psi_{\eta\xi} + \frac{\psi_\eta J_\xi}{J} + \frac{\psi_\xi J_\eta}{J}\right\}\left\{\psi_\eta J_\eta + \psi_\xi J_\xi\right\} - \tilde{\lambda}\omega\chi_{12} \qquad (3.7)$$

$$J\left(\psi_\eta \frac{\partial \chi_{22}}{\partial \xi} - \psi_\xi \frac{\partial \chi_{22}}{\partial \eta}\right) + \chi_{22}(\psi_\eta J_\xi - \psi_\xi J_\eta) + \chi_{12}(\psi_\eta J_\eta + \psi_\xi J_\xi) =$$

$$\frac{e}{\omega} J\left\{\left(\psi_\eta \frac{\partial}{\partial \xi} - \psi_\xi \frac{\partial}{\partial \eta}\right)\left(-2\psi_{\xi\eta} - \frac{\psi_\eta J_\xi}{J} - \frac{\psi_\xi J_\eta}{J}\right)\right\}$$

$$+ \frac{e}{\omega}\left\{-2\psi_{\xi\eta} - \frac{\psi_\eta J_\xi}{J} - \frac{\psi_\xi J_\eta}{J}\right\}\left\{\psi_\eta J_\xi - \psi_\xi J_\eta\right\}$$

$$+ \frac{e}{\omega}\left\{\psi_{\eta\eta} - \psi_{\xi\xi} + \frac{\psi_\eta J_\eta}{J} - \frac{\psi_\xi J_\xi}{J}\right\}\left\{\psi_\eta J_\eta + \psi_\xi J_\xi\right\} - \tilde{\lambda}\omega\chi_{22} \tag{3.8}$$

$$J\left(\psi_\eta \frac{\partial \chi_{33}}{\partial \xi} - \psi_\xi \frac{\partial \chi_{33}}{\partial \eta}\right) = -\tilde{\lambda}\omega\chi_{33} \tag{3.9}$$

As mentioned earlier, the problem was solved both with and without the relaxation hypothesis. The "no relaxation" case refers to the absence of χ_{ij} in the Reynolds stress constitutive relation (2.1), and therefore implies equations (3.2) and (3.3) with χ_{ij} set equal to zero, plus equations (3.4) and (3.5) for the transport of e and ω. Note that, in general, the relaxation tensor does not appear explicitly in the rate equations for e and ω.

B. The O(1) and O(ka) Equations and Boundary Conditions

The assumption of exact periodicity in ξ motivates a solution for the various dependent variables in terms of their harmonics $e^{\pm ink\xi}$, n = 0, 1, ... The effect of the wavy boundary, manifested in the derivatives of the Jacobian appearing in the equations, indicates that the expansion should proceed as follows:

$$\psi = \psi_0(\eta) + kae^{ik\xi}\,\psi_1(\eta) + O(ka)^2, \quad \text{where } \frac{d\psi_0}{d\eta} \equiv U(\eta)$$

$$\begin{aligned}
u &= U + kae^{ik\xi}\,u_1(\eta) + \cdots \\
v &= \quad kae^{ik\xi}\,v_1(\eta) + \cdots \\
\Phi &= \Phi_0(\eta) + kae^{ik\xi}\,\Phi_1(\eta) + \cdots \\
\bar{p} &= \bar{p}_0(\eta) + kae^{ik\xi}\,\bar{p}_1(\eta) + \cdots \\
e &= e_0(\eta) + kae^{ik\xi}\,N(\eta) + \cdots \\
\omega &= \omega_0(\eta) + kae^{ik\xi}\,W(\eta) + \cdots \\
\chi_{ij} &= {}_0\chi_{ij}(\eta) + kae^{ik\xi}\,{}_1\chi_{ij}(\eta) + \cdots \\
-\overline{u_i' u_j'} &= {}_0\tau_{ij}(\eta) + ka\,e^{ik\xi}\,{}_1\tau_{ij}(\eta) + \cdots \\
u_*(\xi) &= u_{*_0} + ka\,e^{ik\xi}\,u_{*_1} + \cdots \\
\eta_*(\xi) &= \eta_{*_0} + ka\,e^{ik\xi}\,\eta_{*_1} + \cdots \\
\overline{q^2} &= \overline{q^2}|_0 + ka\,e^{ik\xi}\,\overline{q^2}|_1 + \cdots
\end{aligned} \tag{3.10}$$

where the real part of the right side is implied, the quantities $\psi_1(\eta)$, $\Phi_1(\eta)$, etc. being complex, in general.

The boundary conditions near $\eta = 0$ may be expanded as follows, neglecting terms $O(ka)^2$:

1) $-ik\;ka\;\psi_1 \sim O(ka\;k\eta_{*_0}\;u_{*_0}) + O(ka\;k\eta_{*_0}\;c)$

2) $$U + ka\;e^{ik\xi}\left(\frac{d\psi_1}{d\eta} + U e^{-k\eta}\right) \sim \frac{u_{*_0}}{\kappa}\,\ell n\left(\frac{\eta}{\eta_{*_0}}\right) - c + ka\;e^{ik\xi}\left\{\frac{u_{*_1}}{\kappa}\,\ell n\left(\frac{\eta}{\eta_{*_0}}\right) - \frac{u_{*_0}}{\kappa}\left(1 + \frac{\eta_{*_1}}{\eta_{*_0}}\right) + c\Omega\right\}$$

3) $e_0 + ka\, e^{ik\xi}\, N \sim \tilde{a}\, u_{*_0}^2 + ka\, e^{ik\xi}\,(2\tilde{a}\, u_{*_1}\, u_{*_0})$

$$\omega_0 + ka\, e^{ik\xi}\, W \sim \frac{\tilde{a}\, u_{*_0}}{\kappa\eta} + ka\, e^{ik\xi}\, \frac{\tilde{a}\, u_{*_0}}{\kappa\eta}\left(\frac{u_{*_1}}{u_{*_0}} + 1\right)$$

4) Utilizing $U' \equiv dU/d\eta$

$$\frac{e_0}{\omega_0}\, U' + ka\, e^{ik\xi} \left\{ \begin{array}{l} \frac{1}{\tilde{a}}(N - \frac{e_0}{\omega_0}\, W) + \frac{e_0}{\omega_0}\left(\frac{d^2\psi_1}{d\eta^2} + k^2\psi_1 + 2U'e^{-k\eta} - 2k\, U e^{-k\eta}\right) \\ - {}_1\chi_{12} \end{array} \right\}$$

$$\sim u_{*_0}^2 + ka\, e^{ik\xi} \left\{ \begin{array}{l} 2u_{*_1}\, u_{*_0} + O(ka\; k\eta_{*_0}\; \frac{p_1|_{SFC}}{\rho}) \\ + O(ka\; k\eta_{*_0}\; u_{*_0}^2) + O(ka\; k\eta_{*_0}\; c^2) \end{array} \right\}$$

Substitution of the expansions (3.10) into the equations (3.2) through (3.9) yields the following:

$\hat{e}_\xi$ momentum:

$$0(1):\quad 0 = \frac{d}{d\eta}\left(\frac{e_0}{\omega_0}\, U' - {}_0\chi_{12}\right)$$

$$0(ka):\qquad ik\left\{\psi_1'\, U - \psi_1\, U' + e^{-k\eta}\, U^2\right\} =$$

$$- ik\, \frac{\Phi_1}{\rho} + \frac{d}{d\eta}\left\{\frac{e_0}{\omega_0}\, U'\left(\frac{N}{e_0} - \frac{W}{\omega_0}\right) + \frac{e_0}{\omega_0}\left(\psi_1'' - k^2\, \psi_1 + 2\, e^{-k\eta}\, U'\right)\right\}$$

$$+ 2(k^2\, \psi_1 - e^{-k\eta}\, k\, U)\frac{d}{d\eta}\left(\frac{e_0}{\omega_0}\right) - ik\; {}_1\chi_{11} - {}_1\chi_{12}' - 2\, e^{-k\eta}\; {}_0\chi_{12}'$$

$$- i\, k e^{-k\eta}\, ({}_0\chi_{11} + {}_0\chi_{22})$$

$\hat{e}_\eta$ momentum:

$$0(1): \quad O = -\frac{1}{\rho}\Phi_0' - {}_0\chi_{22}'$$

$$0(ka): \quad U k^2 \psi_1 - k e^{-k\eta} U^2 =$$

$$-\frac{1}{\rho}\Phi_1' + i\frac{e_0}{\omega_0}\left(k^3 \psi_1 - k\psi_1'' - 2 k e^{-k\eta} U'\right)$$

$$+ i k U' \frac{e_0}{\omega_0}\left(\frac{N}{e_0} - \frac{W}{\omega_0}\right) - i\, 2k\left(\psi_1' + Ue^{-k\eta}\right)\frac{d}{d\eta}\left(\frac{e_0}{\omega_0}\right)$$

$$- i k\, {}_1\chi_{12} - {}_1\chi_{22}' - 2e^{-k\eta}\, {}_0\chi_{22}' + ke^{-k\eta}\left({}_0\chi_{11} + {}_0\chi_{22}\right).$$

Pseudo turbulence energy:

$$0(1): \quad O = \tilde{a}\, e_0 U' - e_0\omega_0 + \frac{d}{d\eta}\left(\tilde{\sigma}\frac{e_0}{\omega_0} e_0'\right)$$

$$0(ka): \quad ik\left(NU - \psi_1 e_0'\right) =$$

$$\tilde{a}\, e_0\left(\psi_1'' + k^2\psi_1 + 2e^{-k\eta} U' - 2 ke^{-k\eta} U\right) + \tilde{a}\, U'N$$

$$-\left(N\omega_0 + e_0 W\right) + \tilde{\sigma}\left\{\frac{d}{d\eta}\left[\frac{e_0}{\omega_0} N' + \frac{e_0}{\omega_0}\left(\frac{N}{e_0} - \frac{W}{\omega_0}\right) e_0'\right]\right.$$

$$\left. - k^2\frac{e_0}{\omega_0} N + 2 e^{-k\eta}\frac{d}{d\eta}\left(\frac{e_0}{\omega_0} e_0'\right)\right\}$$

Pseudo vorticity:

$$0(1): \quad O = \tilde{a}\,\omega_0 U' - \beta\omega_0^2 + \sigma\frac{d}{d\eta}\left(\frac{e_0}{\omega_0}\omega_0'\right)$$

$$0(ka): \quad ik\left(UW - \psi_1\omega_0'\right) =$$

$$a\left\{\omega_0\left(\psi_1'' + 2 U' e^{-k\eta} - Uke^{-k\eta}\right) + WU'\right\} - 2\beta\omega_0 W$$

$$+ \sigma\left\{\frac{d}{d\eta}\left[\frac{e_0}{\omega_0} W' + \frac{e_0}{\omega_0}\left(\frac{N}{e_0} - \frac{W}{\omega_0}\right)\omega_0'\right] - k^2\frac{e_0}{\omega_0} W + 2e^{-k\eta}\frac{d}{d\eta}\left[\frac{e_0}{\omega_0}\omega_0'\right]\right\}$$

Relaxation Equations:

$O(1)$: $\quad {}_0\chi_{ij} \equiv 0 \qquad$ for $i = 1, 2, 3$; $j = 1, 2, 3$

$O(ka)$:

$$ikU\,{}_1\chi_{11} = -2U\,k^2\,\frac{e_0}{\omega_0}\left(\psi_1' - \frac{U'e^{-k\eta}}{k} + Ue^{-k\eta}\right) - \tilde{\lambda}\,\omega_0\,{}_1\chi_{11}$$

$$ikU\,{}_1\chi_{12} = ik\,\frac{e_0}{\omega_0}\left\{\begin{array}{c} U(\psi_1'' + k^2\psi_1) + 2U\,e^{-k\eta}(U' - kU) \\ - U''\,\psi_1 \end{array}\right\}$$

$$- \tilde{\lambda}\,{}_1\chi_{12}\,\omega_0$$

$$ikU\,{}_1\chi_{22} = +2Uk^2\,\frac{e_0}{\omega_0}\left(\psi_1' - \frac{U'e^{-k\eta}}{k} + Ue^{-k\eta}\right) - \tilde{\lambda}\,\omega_0\,{}_1\chi_{22}$$

$${}_1\chi_{33} \equiv 0$$

In the above $O(ka)$ relaxation equations, we have used the result that ${}_0\chi_{ij} \equiv 0$.

C. Solutions of the O(1) equations

By inspection, a solution of the O(1) equations satisfying all boundary conditions is:

$$\begin{aligned} U &= \frac{u_{*_0}}{\kappa}\,\ell n\left(\frac{\eta}{\eta_{*_0}}\right) - c \\ e_0 &= \tilde{a}\,u_{*_0}^2 \\ \omega_0 &= \tilde{a}\,u_{*_0}/\kappa\eta \\ {}_0\chi_{ij} &\equiv 0 \end{aligned} \qquad (3.11)$$

The O(1) turbulent stresses are

$$
\begin{aligned}
{}_0\tau_{11} = {}_0\tau_{22} &= {}_0\tau_{33} = -\frac{1}{3}\,\overline{q^2}\Big|_0 \qquad \dagger \\
{}_0\tau_{12} &= u_{*_0}^2
\end{aligned}
\tag{3.12}
$$

The solutions are valid within the fully turbulent region only (e.g., $\eta >> \eta_{*_0}$), since viscous effects have been neglected.

D. Solution of the O(ka) equations

The solution of the O(ka) equations is facilitated by defining (Benjamin, 1959)

$$\psi_1 \equiv S(\eta) + U e^{-k\eta}/k$$

The convective term in the mean perturbation vorticity equation, obtained by taking the curl of the momentum equation, is, in terms of ψ_1

$$ik \frac{d}{d\eta}\left\{\psi_1' U - \psi_1 U' + U^2 e^{-k\eta}\right\} - ik\, U(k^2 \psi_1 - k e^{-k\eta} U)$$

$$= ik\left\{U(\psi_1'' - k^2 \psi_1) - U'' \psi_1 + 2\, U U' e^{-k\eta}\right\}$$

In terms of $S(\eta)$, this becomes

$$ik\{U(S'' - k^2 S) - U'' S\}$$

which is similar to the convective term for the perturbation stream function in a cartesian coordinate system. Hence the $Ue^{-k\eta}\, k^{-1}$ term is a measure of the curvature effect on the perturbation

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$\dagger$ In a recent modification of the model equations (Saffman, 1974), allowance has been made for the inequality of normal turbulent stresses in a boundary layer. Further reference to this modification will be made in Part IV.

streamfunction ψ_1.

D1. The O(ka) Pseudo-Vorticity and O(ka) Mean Vorticity Equations

The O(ka) momentum equations may be combined to form the O(ka) mean vorticity equation. As e_0 is independent of η, the O(ka) pseudo turbulence energy equation provides an explicit relation between W and the derivatives of S and N. Then, too, the O(ka) relaxation equations afford a relation between ${}_1\chi_{ij}$ and the derivatives of S. Substitution of these quantities into the O(ka) mean vorticity equation and the O(ka) pseudo-vorticity equations, and use of (3.11), yields the following pair of linear, fourth-ordered, coupled homogeneous ordinary differential equations for S and N:

Pseudo-Vorticity

$$u_{*0}\left\{\begin{array}{l} b_4\eta\dfrac{d^4S}{d\eta^4}+2b_4\dfrac{d^3S}{d\eta^3}+\left[\dfrac{b_5}{\eta}-ik\tilde{a}\bar{U}\right]\dfrac{d^2S}{d\eta^2}+\left[2b_4k^2\right]\dfrac{dS}{d\eta} \\ +\left[\dfrac{\tilde{a}^2b_{11}k^2}{\eta}-b_4k^4\eta-ik^3\tilde{a}\bar{U}-\dfrac{ik\tilde{a}}{\kappa\eta^2}\right]S \end{array}\right\}$$

$$+b_6\eta^2\frac{d^4N}{d\eta^4}+5b_6\,\eta\,\frac{d^3N}{d\eta^3}+\left[b_7-2b_6k^2\eta^2-i\,b_8\,k\eta\,\bar{U}\right]\frac{d^2N}{d\eta^2} \tag{3.13}$$

$$+\left[\frac{b_9}{\eta}-5\,b_6\,k^2\eta-ikb_{10}\,\bar{U}-\frac{2i\sigma k}{\tilde{a}}\right]\frac{dN}{d\eta}$$

$$+\left[-\frac{i\sigma k}{\tilde{a}\,\eta}+\frac{\sigma}{\eta^2}+\frac{\tilde{\sigma}\,a}{\tilde{a}}\,k^2+b_6k^4\eta^2+\bar{U}\left(i\,b_8\,k^3\eta-\frac{i\,b_{11}k}{\eta}-\frac{k^2\bar{U}}{\tilde{a}}\right)\right]N=0$$

Mean Vorticity:

$$\boxed{i\,k\,H^3\left\{\bar{U}\left(\frac{d^2S}{d\eta^2} - k^2S\right) + \frac{1}{\kappa\eta^2}\,S\right\}} =$$

$$\left\{-\kappa\eta\,\bar{U}^3 + \frac{i\,2\,b_{13}\,\kappa}{k}\,\bar{U}^2 + \frac{b_{13}^2\,\kappa\,\bar{U}}{k^2\eta}\right\}\frac{d^4S}{d\eta^4} \tag{3.14}$$

$$+\left\{-2\kappa\bar{U}^3 + \frac{i\,6\,\kappa\,b_{13}\bar{U}^2}{k\eta} + \left(\frac{i\,2\,b_{13}}{k\eta} + \frac{4\kappa\,b_{13}^2}{k^2\eta^2}\right)\bar{U} + \frac{2\,b_{13}^2}{k^2\eta^2}\right\}\frac{d^3S}{d\eta^3}$$

$$+\left\{\begin{array}{l} 2\,\kappa\,k^2\eta\,\bar{U}^3 - \left(i\,4\,b_{13}\kappa\,k + \frac{1}{\eta}\right)\bar{U}^2 + \left(-\frac{2\kappa\,b_{13}^2}{\eta} + \frac{2\,b_{13}^2\,\kappa}{k^2\eta^3} + \frac{i\,b_{13}}{k\eta^2}\right)\bar{U} \\ -\frac{2\,i\,b_{13}}{\kappa\,k\eta^2} + \frac{4\,b_{13}^2}{k^2\eta^3} \end{array}\right\}\frac{d^2S}{d\eta^2}$$

$$+\left\{\begin{array}{l} 2\,\kappa\,k^2\bar{U}^3 + \left(\frac{2}{\eta^2} - \frac{i\,6\,b_{13}\,\kappa\,k}{\eta}\right)\bar{U}^2 + \left(-\frac{4\kappa\,b_{13}^2}{\eta^2} - \frac{i\,2\,b_{13}k}{\eta} + \frac{2}{\kappa\eta^2} - \frac{i2b_{13}}{k\eta^3}\right)\bar{U} \\ -\frac{2b_{13}^2}{\eta^2} - \frac{i\,2b_{13}}{\kappa\,k\eta^3} \end{array}\right\}\frac{dS}{d\eta}$$

$$+\left\{\begin{array}{l} -\kappa\,k^4\eta\,\bar{U}^3 + \left(i\,2\,b_{13}\kappa\,k^3 - \frac{k^2}{\eta} - \frac{2}{\eta^3}\right)\bar{U}^2 + \left(\frac{i\,b_{13}k}{\eta^2} + \frac{b_{13}^2\kappa\,k^2}{\eta} + \frac{2b_{13}^2\kappa}{\eta^3} - \frac{3}{\kappa\eta^3}\right)\bar{U} \\ +\frac{4\,b_{13}^2}{\eta^3} - \frac{i\,b_{13}}{\kappa\,k\eta^4} - \frac{i\,2\,b_{13}\,k}{\kappa\,\eta^2} - \frac{2}{\kappa^2\eta^3} \end{array}\right\}S$$

$$+\boxed{\left\{-4\,k^2\,\kappa\eta\,H^3\right\}\frac{d^2S}{d\eta^2} + \left\{-4\,k^2\,\kappa\,H^3\right\}\frac{dS}{d\eta}}$$

$$+\boxed{\frac{H^3}{u_{*0}}\left[\begin{array}{l} -b_1\,\eta^2\,\frac{d^4N}{d\eta^4} - 5\,b_1\eta\,\frac{d^3N}{d\eta^3} + \left\{\frac{i\,\kappa\,k\eta\,\bar{U}}{\tilde{a}^2} - b_2\right\}\frac{d^2N}{d\eta^2} \\ +\frac{dN}{d\eta}\left\{3b_1k^2\eta + \frac{i\,2\,k\,\kappa}{\tilde{a}^2}\left(\bar{U} + \frac{1}{\kappa}\right)\right\} + \left\{b_3k^2 + b_1k^4\eta^2 + \frac{i\kappa}{\tilde{a}^2}k^3\eta\left(\frac{1}{\kappa k^2\eta^2} + \bar{U}\right)\right\}N \end{array}\right]}$$

where
$$H = \bar{U} - \frac{i b_{13}}{k\eta}$$
$$\bar{U} = U/u_{*_0} = \frac{1}{\kappa} \ln\left(\frac{\eta}{\eta_{*_0}}\right) - \frac{C}{u_{*_0}}$$

$$b_1 = \tilde{\sigma}\kappa^2/\tilde{a}^2 \qquad b_8 = \frac{(\sigma+\tilde{\sigma})\kappa}{\tilde{a}}$$

$$b_2 = \frac{4\tilde{\sigma}\kappa^2}{\tilde{a}^2} - \frac{1}{\tilde{a}} \qquad b_9 = \frac{(a - 2\beta\tilde{a})\tilde{\sigma}}{\tilde{a}} - \sigma$$

$$b_3 = \frac{1}{\tilde{a}} + \frac{2\tilde{\sigma}\kappa^2}{\tilde{a}^2} \qquad b_{10} = \frac{(2\sigma\kappa + \tilde{\sigma}\kappa)}{\tilde{a}}$$

$$b_4 = \sigma\kappa\tilde{a}$$

$$b_5 = \frac{2(a - \beta\tilde{a})\tilde{a}}{\kappa} = -2b_4 \text{ since } \kappa^2 = (\beta\tilde{a} - a)/\sigma$$

$$b_6 = \frac{\sigma\tilde{\sigma}\kappa^2}{\tilde{a}} \qquad b_{11} = \frac{(a-2\beta\tilde{a})}{\kappa\tilde{a}}$$

$$b_7 = \frac{\tilde{\sigma}}{\tilde{a}}(2\sigma\kappa^2 - a) \qquad b_{12} = 2\tilde{a}\tilde{\sigma}$$

$$b_{13} = \frac{\chi\tilde{a}}{\kappa}$$

For the "no relaxation" case, only those terms in the mean vorticity equation which are enclosed within the dotted boxes are considered; the pseudo-vorticity equation is the same for either case.

D2. The Perturbation Limit $k\eta_{*_0} \to 0$

The solution of equations (3. 13) and (3. 14) is complicated by the presence of the logarithm in the coefficients of the various derivatives of S and N. Various techniques were attempted, including the coordinate transformation $\underset{\sim}{\eta} = \ln k\eta$, but no closed form solution valid for $\eta_{*_0} << \eta < \infty$ was achieved. For this reason, an asymptotic solution of (3.13) and (3.14) was obtained in the limit $k\eta_{*_0} \to 0$.

Physically, the assumption that the upper flow imposes only a slight perturbation on the wave speed c suggests that the amplitude of the O(ka) average mean normal stress at the edge of the sublayer be finite in the limit $k\eta_{*_0} \to 0$; e.g.,

$$\lim_{k\eta_{*_0}\to 0} \left| \frac{\Phi_1(0^+)}{\rho} \right| < \infty \tag{3.15}$$

In particular, it can be shown (Lamb, 1945) that an inviscid, potential flow with constant velocity U_∞ over a deep water gravity wave alters the free wave speed $c_w = (g/k)^{\frac{1}{2}}$ by a factor

$$\left(1 - \frac{P_1}{ka\ \rho_w c_w^2}\right)^{\frac{1}{2}}$$

where ρ_w = density of lower fluid, and P_1 is the amplitude of the perturbation surface pressure, given by

$$P_1 = ka\ \rho\ (U_\infty - c)^2 .$$

This suggests further that the O(1) velocity must remain finite, at fixed $k\eta$, as $k\eta_{*_0} \to 0$:

$$\lim_{\substack{k\eta_{*_0}\to 0 \\ k\eta \text{ fixed}}} U = V_0(k\eta) < \infty \tag{3.16}$$

Substituting,

$$\lim_{\substack{k\eta_{*_0}\to 0 \\ k\eta \text{ fixed}}} \left\{ \frac{u_{*_0}}{\kappa} \ln\left(\frac{1}{k\eta_{*_0}}\right) + \frac{u_{*_0}}{\kappa} \ln(k\eta) - c \right\} = V_0$$

Clearly, since $c \to c_0 < \infty$, $u_{*_0} \sim \dfrac{\kappa U_0}{\ln(1/k\eta_{*_0})} \to 0^+$

as $k\eta_{*_0} \to 0$, where U_0 is a positive (because u_{*_0} is defined positive) constant, the notation implying approach to zero through positive values, and

$$V_0 = U_0 - c_0 = \text{constant (assumed non-zero)} .$$

It is of note that the $0(1)$ velocity tends to a constant V_0 on the scale of λ as $k\eta_{*_0} \to 0$.

Defining

$$\epsilon = \frac{1}{\ln\left(\dfrac{1}{k\eta_{*_0} e^{c\kappa/u_{*0}}}\right)} \tag{3.17}$$

the velocity may be expressed as

$$U = \frac{u_{*_0}}{\kappa} \ln\left(\frac{k\eta}{k\eta_{*_0}\, e^{c\kappa/u_{*0}}}\right) = \frac{u_{*_0}}{\kappa}\left(\frac{1}{\epsilon} + \ln k\eta\right),$$

and hence (3.16) implies

$$u_{*_0} \sim \kappa V_0\, \epsilon \quad \text{as} \quad k\eta_{*_0} \to 0 \tag{3.18}$$

From (3.18), it is apparent that the limit considered is a limit in which the skin friction vanishes. For a smooth surface,

$$k\eta_{*_0} = \frac{k\,\nu}{u_{*_0}}\, e^{-B\kappa} \to 0$$

and the limit may be visualized as one in which both u_{*_0} and $\nu \to 0$, k and a being fixed with $ka \ll 1$.

Denoting

$$\epsilon' = \frac{1}{\ln(1/k\eta_{*_0})} \to 0^+ \quad \text{as} \quad k\eta_{*_0} \to 0 ,$$

then from (3.17)

$$\epsilon = \frac{1}{\ln(1/k\eta_{*_0})}\left\{\frac{1}{1 - \dfrac{c\kappa}{u_{*_0}\ln(1/k\eta_{*_0})}}\right\} = \epsilon' \frac{1}{1 - \dfrac{c_0(1+o(1))}{U_0(1+o(1))}}$$

$$\epsilon = \epsilon' \frac{U_0}{U_0 - c_0}(1 + o(1)) \quad . \qquad (3.19)$$

Clearly, $k\eta_{*_0} \to 0$ implies $|\epsilon| \to 0$, where the sign of ϵ depends on the sign of V_0. Note further that $(U_0-c_0)\epsilon \to 0^+$ and

$$k\eta_{*_0} = \exp\left\{-\left(\frac{U_0}{U_0-c_0}\right)\frac{1}{\epsilon}(1 + o(1))\right\} \quad . \qquad (3.20)$$

Hypotheses (3.15) and (3.16) are verifiable <u>a posteriori</u> (see Part IIIc).

The mathematical importance of the relation (3.18) between u_{*_0} and ϵ for solving the equations (3.13) and (3.14) in the limit $k\eta_{*_0} \to 0$ may be illuminated as follows. From the equations and boundary conditions, it is clear that a dependent variable, such as S, may be expressed as

$$\begin{aligned} S &= \lambda u_{*_0} f(k\eta, k\eta_{*_0}, c/u_{*_0}; \alpha, \widetilde{\alpha}, \beta, \cdots) \\ &= \lambda u_{*_0} F(k\eta, \epsilon, c/u_{*_0}; \alpha, \widetilde{\alpha}, \beta, \cdots) \end{aligned}$$

For $c \neq 0$, there are two small parameters: ϵ and u_{*_0}/c. In order to obtain a limit process expansion of F on the scale of λ, for example, to all orders in a single small parameter (say ϵ), the relation between u_{*_0}/c and ϵ must be known. As shall be indicated, since (3.18) is an asymptotic, not an exact, relation, the solution for $c \neq 0$ cannot be determined uniquely to all orders in ϵ. Finally, for $c \equiv 0$, it is clear that no relation is needed between u_{*_0} and ϵ, since

$$S = \lambda u_{*_0} F(k\eta, \epsilon; \alpha, \widetilde{\alpha}, \beta, \cdots) .$$

The equations (3.13) and (3.14) and their boundary conditions form a singular perturbation problem in the limit $k\eta_{*_0} \to 0$. For example, the ratio of the term in $d^4S/d\eta^4$ to that in $d^2S/d\eta^2$ on the scale of λ in the pseudo-vorticity equation is

$$\frac{O(1)}{O(\frac{1}{\epsilon})} = O(\epsilon) \to 0 \ \text{ as } k\eta_{*_0} \to 0 .$$

D3. Determination of Inner and Outer Scales

Let the variables be non-dimensionalized as follows:

Inner Region:

$$S = L_2^i u_s^i \overline{S}^i$$
$$N = (u_N^i)^2 \overline{N}^i$$
$$\eta = L_1^i \eta^*$$

Outer Region:

$$S = L_2^0 u_s^0 \overline{S}^0$$
$$N = (u_N^0)^2 \overline{N}^0$$
$$\eta = L_1^0 \overline{\eta}$$

where the velocity and length scales are to be chosen such that $\overline{S}^i$ and $\overline{N}^i$ are $O(1)$ in the inner limit, defined as

$$\begin{array}{c}\text{limit}\\ \epsilon\to 0\\ \eta^* \text{ fixed}\end{array},$$

and similarly for the outer scales in the outer limit.

From the boundary condition on e, clearly

$$u_N^i = u_{*_0} .$$

As the equations (3.13) and (3.14) are linear, only the ratio $kL_2 u_s u_{*_0} / (u_N)^2$ is significant, and we may choose

$$u_s^i = u_s^0 = u_{*_0} .$$

Asthe mean 0(1) velocity profile (3.11) introduces no boundary layer thickness, the outer length scale

$$L_1^0 = k^{-1} .$$

The inner length scale must therefore satisfy the condition

$$L_1^0 >> L_1^i >> \eta_{*_0} \quad \text{or} \quad k\eta_{*_0} << k\,L_1^i << 1 . \qquad (3.21)$$

The boundary condition (2.10) on the normal velocity

$$\lim_{\eta\to 0} (k\,L_2^i\,\overline{S}^i + \overline{U}\,e^{-k\eta}) = O\left(\frac{c}{V_0\epsilon}, e^{-\frac{U_0}{U_0-c_0}\frac{1}{\epsilon}}\right)$$

can be rewritten, noting the right side is transcendentally small in ϵ, as

$$kL_2^i\,\overline{S}^i \sim -\frac{1}{\kappa\epsilon} - \frac{1}{\kappa}\ln k\,L_1^i - \frac{1}{\kappa}\ln \eta^* + o(1) \text{ as } \eta\to 0$$

From the requirement (3.21), it can be seen that

$$\left|\frac{\ln (kL_1^i)}{\frac{1}{\epsilon}}\right| \not\longrightarrow \infty \text{ as } \epsilon \longrightarrow 0 ,$$

by the following argument.

Assume that
$$\left|\frac{\ln(kL_1^i)}{\frac{1}{\epsilon}}\right| \longrightarrow \infty \text{ as } k\eta_{*_0} \to 0 .$$

Then
$$\frac{\ln\left(\frac{kL_1^i}{k\eta_{*_0}}\right)}{\ln(kL_1^i)} = 1 + \frac{\ln\left(\frac{1}{k\eta_{*_0}}\right)}{\ln(kL_1^i)} = 1 + \frac{\left(\frac{U_0}{V_0}\right)\frac{1}{\epsilon}(1+o(1))}{\ln(kL_1^i)} \to 1$$

However, $kL_1^i \to 0^+$, hence $\ln kL_1^i \to -\infty$; $kL_1^i/k\eta_{*_0} \to +\infty$, hence $\ln(kL_1^i/k\eta_{*_0}) \to +\infty$, and the ratio above must be negative, which is a contradiction. Thus, provided

$$-\frac{1}{\epsilon} - \ln kL_1^i \neq 0$$

we may take

$$kL_2^i = {}^1/_\epsilon .$$

The case

$$\ln kL_1^i = {}^1/_\epsilon$$

implies $L_1^i = \eta_{*_0} e^{c\kappa/u_{*_0}} \equiv \eta_c$.

A careful analysis of this situation was made. For $c_0 \leq 0$, this length scale is not allowed, as it violates the condition $k\eta_{*_0} \ll kL_1^i$. For $c_0 > 0$ and $V_0 < 0$, it is also unallowed, as it violates the condition $kL_1^i \ll 1$. For $c_0 > 0$ and $V_0 > 0$, it was found that, using such an inner length scale, it was impossible to match the inner and outer solutions, succeeding terms in the inner expansions of S and N becoming larger and larger in order of ϵ in any intermediate (matching) limit. This length η_c is the so-called critical height, since $U(k\eta_c) = 0$.

Noting $\overline{U} = O(1/\epsilon)$ in the inner limit, the pseudo-vorticity equation becomes

$$b_4 \eta^* \frac{d^4\overline{S}}{d\eta^{*4}} + 2\, b_4 \frac{d^3\overline{S}}{d\eta^{*3}} + \left[\frac{b_5}{\eta^*} - i\,\tilde{\alpha}(k L_1^i)\overline{U}\right] \frac{d^2\overline{S}}{d\eta^{*2}}$$

$$+ o(1) \quad - i\,\tilde{\alpha}(k L_1^i)^3\, \overline{U}\, \overline{S} \quad - \quad \frac{\overline{U}^2}{\tilde{\alpha}} (k L_1^i)^3\, \epsilon\, \overline{N} = 0$$

where, by choice of length and velocity scales, $\overline{S}$ and $\overline{N}$ are $O(1)$ in the inner limit. Clearly, therefore

$$k L_1^i = |\epsilon| \ .$$

A similar investigation of the mean vorticity equation in the inner limit yields the same result, both with and without relaxation.

The matching of the inner and outer expansions of S and N implies that, with the velocity scale, u_{*_0}, the proper outer length scale for S is, again, L_2^i, and that the proper outer velocity scale for N is u_{*_0}.

D4. The Perturbation Expansions and O(ka) Boundary Conditions

The presence of log $|\epsilon|$ in the mean velocity on the inner scale implies the following expansions:

Inner:

$$\overline{S}^i = S_0^i + \epsilon \ln|\epsilon|\, S_1^i + \epsilon\, S_2^i + \epsilon^2 \ln^2|\epsilon|\, S_3^i$$

$$+ \epsilon^2 \ln|\epsilon|\, S_4^i + \epsilon^2\, S_5^i + \cdots \tag{3.22}$$

$$\overline{N}^i = N_0^i + \epsilon \ln|\epsilon|\, N_1^i + \cdots$$

where S_ℓ^i, N_ℓ^i are functions of η^*.

Outer:

$$\overline{S}^0 = S_0^0 + \epsilon \, \ell n|\epsilon| \, S_1^0 + \epsilon \, S_2^0 + \cdots$$
$$\overline{N}^0 = N_0^0 + \epsilon \, \ell n|\epsilon| \, N_1^0 + \epsilon \, N_2^0 + \cdots \qquad (3.23)$$

where S_ℓ^0, N_ℓ^0 are functions of $\tilde{\eta}$.

The scalar parameters of the flow are also functions of ϵ, and are expanded as follows:

$$\frac{u_{*_1}}{u_{*_0}} = \delta_0 + \epsilon \, \ell n|\epsilon| \, \delta_1 + \epsilon \, \delta_2 + \cdots$$
$$c = c_0(1 + \epsilon \, \ell n|\epsilon| \, \Lambda_1 + \epsilon \, \Lambda_2 + \cdots)$$
$$c\Omega = c_0(\Omega_0 + \epsilon \, \ell n|\epsilon| \, \Omega_1 + \epsilon \, \Omega_2 + \cdots) \qquad (3.24)$$
$$\frac{\eta_{*_1}}{\eta_{*_0}} = \mathcal{Z}_0 + \epsilon \, \ell n|\epsilon| \, \mathcal{Z}_1 + \epsilon \, \mathcal{Z}_2 + \cdots$$
$$\frac{u_{*_0}}{\kappa V_0} = \epsilon \, (1 + \epsilon \, \ell n|\epsilon| \, A_1 + \cdots)$$

Note that, in general, the δ_i, Ω_i, $\mathcal{Z}_i$ are complex, whereas the Λ_i and A_i are, by definition, real. Furthermore, the $\mathcal{Z}_i$ are determined by the nature of the surface. If it is aerodynamically smooth, then

$$\eta_*(\xi) = \frac{\nu}{u_*(\xi)} \, e^{-B\kappa}$$

where ν is the kinematic viscosity of the upper fluid, and B = 5.0 (Coles, 1968), and therefore

$$\mathcal{Z}_n = -\delta_n \qquad n = 0, 1, 2, \cdots .$$

For an oceanic water wave, the surface is nearly aerodynamically rough at typically observed wind speeds (R. W. Stewart, 1961), and the determination of the variation of roughness along the surface requires further hypotheses than presented herein.

As indicated previously, the A_i are not specified by this analysis, and serve only to indicate explicitly, for $c \neq 0$, the order at which certain expansions become indeterminate.

The boundary conditions on the O(ka) quantities as $\eta \to 0$ may be rewritten as follows:

1) No normal velocity at the wall

$$\overline{S}^i \sim -\frac{1}{\kappa} - \frac{\epsilon \ln|\epsilon|}{\kappa} - \frac{\epsilon}{\kappa} \ln \eta^* + o(1) \tag{3.25}$$

2) Law of the wall behavior for the tangential velocity

$$\begin{aligned}\frac{d\overline{S}^i}{d\eta^*} &\sim \epsilon \left\{-\frac{1}{\kappa\eta^*} + \frac{\mathrm{sgn}(\epsilon)}{\kappa}\left(\delta_0 + \frac{c_0\Omega_0}{V_0} + \frac{c_0\delta_0}{V_0}\right)\right\} \\ &+ \epsilon|\epsilon|\ln|\epsilon|\left\{\frac{\delta_0+\delta_1}{\kappa} + \frac{c_0}{\kappa V_0}(\Omega_1 - \Omega_0 A_1 + \delta_1 + \Lambda_1\delta_0 - A_1\delta_0)\right\} \\ &+ \epsilon|\epsilon|\left\{-\frac{\mathcal{Z}_0}{\kappa} + \frac{\delta_0 \ln \eta^*}{\kappa} + \frac{\delta_2}{\kappa} + \frac{c_0}{\kappa V_0}(\Omega_2 - \Omega_0 A_2 + \delta_2 + \Lambda_2\delta_0 - A_2\delta_0)\right\}\end{aligned}$$

$+ o(1)$ plus terms of higher order in ϵ. (3.26)

where $$\mathrm{sgn}(\epsilon) = \begin{cases} +1 & \epsilon > 0 \\ -1 & \epsilon < 0 \end{cases}$$

3) Law of the wall behavior for ϵ and ω

$$\overline{N}^i \sim 2\tilde{\alpha}(\delta_0 + \epsilon \ln|\epsilon|\, \delta_1 + \epsilon\, \delta_2 + \cdots) + o(1) \tag{3.27}$$

and utilizing the relation between W and the derivatives of N and S, the boundary condition on W becomes

$$\frac{d^2\overline{S}^i}{d\eta^{*2}} + \epsilon^2\,\overline{S}^i + \frac{\tilde{\sigma}\,\kappa}{\tilde{\alpha}^2}\,\epsilon|\epsilon|\left(\frac{d\overline{N}}{d\eta^*} + \eta^*\frac{d^2\overline{N}}{d\eta^{*2}}\right) \sim$$

$$+ \frac{\epsilon}{\kappa\eta^{*2}} + \frac{\epsilon|\epsilon|}{\kappa\eta^*}\left(\delta_0 + \epsilon\,\ln|\epsilon|\,\delta_1 + \cdots\right)$$

$$+ o\left(\frac{1}{\eta^*}\right) \qquad (3.28)$$

4) Shear stress

As indicated, the terms in brackets { } in equation (2.13), according to (3.15) and (3.20), are transcendentally small in ϵ. Hence, using

No relaxation:

$$\kappa\eta^*\,\frac{d^2\overline{S}^i}{d\eta^{*2}} \sim + \frac{\epsilon}{\eta^*} + \epsilon|\epsilon|\left(\delta_0 + \epsilon\,\ln|\epsilon|\,\delta_1 + \cdots\right)$$

$$+ o(1)$$

(3.29)

With relaxation:

$$\kappa\eta^*\left\{\frac{d^2\overline{S}^i}{d\eta^{*2}} - \frac{\overline{U}\,\frac{d^2\overline{S}^i}{d\eta^{*2}} + \frac{1}{\kappa\eta^{*2}}\,\overline{S}}{\left(\overline{U} - \frac{i\,b_{13}}{|\epsilon|\eta^*}\right)}\right\} \sim + \frac{\epsilon}{\eta^*} + \epsilon|\epsilon|\left(\delta_0 + \cdots\right)$$

$$+ o(1)$$

The non-dimensional pseudo-vorticity equation (3.13) is, omitting superscripts on $\overline{S}$ and $\overline{N}$:

(see next page)

Inner scale:

$$
\begin{aligned}
0 = {} & b_4 \eta^* \frac{d^4\overline{S}}{d\eta^{*4}} + 2b_4 \frac{d^3\overline{S}}{d\eta^{*3}} + \left\{ \frac{b_5}{\eta^*} - i\,\widetilde{\alpha}\,|\epsilon|\,\overline{U} \right\} \frac{d^2\overline{S}}{d\eta^{*2}} \\
& + 2b_4\,\epsilon^2 \frac{d\overline{S}}{d\eta^*} + \left\{ -\frac{i\,\widetilde{\alpha}|\epsilon|}{\kappa\,\eta^{*2}} + \frac{\widetilde{\alpha}^2 b_{11}}{\eta^*}\,\epsilon^2 - i\alpha|\epsilon|\epsilon^2\,\overline{U} - b_4\,\epsilon^4\eta^* \right\} \overline{S} \\
& + \epsilon|\epsilon| \left[\begin{aligned}
& + b_6 \eta^{*2} \frac{d^4\overline{N}}{d\eta^{*4}} + 5b_6\eta^* \frac{d^3\overline{N}}{d\eta^{*3}} + \left\{ b_7 - i\,b_8|\epsilon|\eta^*\overline{U} - 2b_6\epsilon^2\eta^{*2} \right\} \frac{d^2\overline{N}}{d\eta^{*2}} \\
& + \left\{ \frac{b_9}{\eta^*} - i\,b_{10}|\epsilon|\overline{U} - \frac{i\,2\,\sigma}{\widetilde{\alpha}}|\epsilon| - 5b_6\,\epsilon^2\eta^* \right\} \frac{d\overline{N}}{d\eta^*} \\
& + \left\{ \begin{aligned} & \frac{\sigma}{\eta^{*2}} - \frac{\epsilon^2}{\widetilde{\alpha}}\,\overline{U}^2 - \frac{i\,b_{11}}{\eta^*}\,|\epsilon|\,\overline{U} - \frac{i\sigma}{\widetilde{\alpha}}\,\frac{|\epsilon|}{\eta^*} \\ & + \frac{\widetilde{\sigma}\alpha}{\widetilde{\alpha}}\,\epsilon^2 + i\,b_8\eta^*|\epsilon|^3\,\overline{U} + b_6\,\eta^{*2}\,\epsilon^4 \end{aligned} \right\} \overline{N}
\end{aligned} \right]
\end{aligned}
\tag{3.30}
$$

Outer scale:

$$
\begin{aligned}
0 = {} & b_4\,\widetilde{\eta}\,\frac{d^4\overline{S}}{d\widetilde{\eta}^4} + 2b_4 \frac{d^3\overline{S}}{d\widetilde{\eta}^3} + \left\{ -i\,\widetilde{\alpha}\,\overline{U} + \frac{b_5}{\widetilde{\eta}} \right\} \frac{d^2\overline{S}}{d\widetilde{\eta}^2} + 2\,b_4 \frac{d\overline{S}}{d\widetilde{\eta}} \\
& + \left\{ -i\,\widetilde{\alpha}\,\overline{U} + \frac{\widetilde{\alpha}^2 b_{11}}{\widetilde{\eta}} - b_4\widetilde{\eta} - \frac{i\widetilde{\alpha}}{\kappa\widetilde{\eta}^2} \right\} \overline{S} + \epsilon\, b_6\,\widetilde{\eta}^2 \frac{d^4\overline{N}}{d\widetilde{\eta}^4} + 5b_6\,\epsilon\,\widetilde{\eta}\,\frac{d^3\overline{N}}{d\widetilde{\eta}^3} \\
& + \left\{ -i\,b_8\widetilde{\eta}\,\epsilon\,\overline{U} + \epsilon\,b_7 - 2b_6\epsilon\widetilde{\eta}^2 \right\} \frac{d^2\overline{N}}{d\widetilde{\eta}^2} + \left\{ -i\,b_{10}\epsilon\overline{U} + \frac{\epsilon\,b_9}{\widetilde{\eta}} - 5b_6\epsilon\widetilde{\eta} - \frac{i2\sigma\epsilon}{\widetilde{\alpha}} \right\} \frac{d\overline{N}}{d\widetilde{\eta}} \\
& + \left\{ -\frac{\epsilon}{\widetilde{\alpha}}\,\overline{U}^2 - \frac{i\,\sigma\,\epsilon}{\widetilde{\alpha}\widetilde{\eta}} + \frac{\sigma\,\epsilon}{\widetilde{\eta}^2} + \frac{\widetilde{\sigma}\alpha}{\widetilde{\alpha}}\,\epsilon + b_6\epsilon\,\widetilde{\eta}^2 + \epsilon\,\overline{U}\left[i\,b_8\widetilde{\eta} - \frac{i\,b_{11}}{\widetilde{\eta}} \right] \right\} \overline{N}
\end{aligned}
\tag{3.31}
$$

The mean vorticity equation (3.14) is:

Inner Scale:

$$\boxed{+ i H^3\left[\overline{U}\left(\frac{d^2\overline{S}}{d\eta^{*2}} - \epsilon^2\overline{S}\right) + \frac{1}{\kappa\eta^{*2}}\ \overline{S}\ \right]} =$$

$$\left\{- \frac{\kappa\eta^*\overline{U}^3}{|\epsilon|} + \frac{i\,2b_{13}\kappa\overline{U}^2}{\epsilon^2} + \frac{b_{13}^2\kappa\overline{U}}{|\epsilon|^3\eta^*}\right\}\frac{d^4\overline{S}}{d\eta^{*4}}$$

$$+ \left\{- \frac{2\kappa\overline{U}^3}{|\epsilon|} + \frac{i\,6\kappa b_{13}}{\epsilon^2\eta^*}\overline{U}^2 + \left(\frac{i\,2b_{13}}{\eta^*\epsilon^2} + \frac{4\kappa b_{13}^2}{|\epsilon|^3\eta^{*2}}\right)\overline{U} + \frac{2b_{13}^2}{|\epsilon|^3\eta^{*2}}\right\}\frac{d^3\overline{S}}{d\eta^{*3}}$$

$$\left\{\begin{array}{l} + 2\kappa\eta^*|\epsilon|\overline{U}^3 + \left(i4b_{13}\kappa - \frac{1}{\eta^*|\epsilon|}\right)\overline{U}^2 + \left(-\frac{2\kappa b_{13}^2}{\eta^*|\epsilon|} + \frac{2b_{13}^2\kappa}{\eta^{*3}|\epsilon|^3} + \frac{i\,b_{13}}{\eta^{*2}\epsilon^2}\right)\overline{U} \\ - \frac{i\,2b_{13}}{\kappa\epsilon^2\eta^{*2}} + \frac{4b_{13}^2}{\eta^{*3}|\epsilon|^3} \end{array}\right\}\frac{d^2\overline{S}}{d\eta^{*2}}$$

$$+ \frac{d\overline{S}}{d\eta^*}\left\{\begin{array}{l} 2\kappa|\epsilon|\overline{U}^3 + \left(-\frac{i\,6b_{13}\kappa}{\eta^*} + \frac{2}{|\epsilon|\eta^{*2}}\right)\overline{U}^2 + \left(-\frac{4\kappa b_{13}^2}{|\epsilon|\eta^{*2}} - \frac{i2b_{13}}{\eta^*} + \frac{2}{|\epsilon|\kappa\eta^{*2}} - \frac{i\,2b_{13}}{\eta^{*3}\epsilon^2}\right)\overline{U} \\ - \frac{i\,2b_{13}}{\kappa\epsilon^2\eta^{*3}} - \frac{2b_{13}^2}{|\epsilon|\eta^{*2}} \end{array}\right\}$$

$$+ \left\{\begin{array}{l} -\kappa\eta^*|\epsilon|^3\overline{U}^3 + \left(i\,2b_{13}\kappa\epsilon^2 - \frac{|\epsilon|}{\eta^*} - \frac{2}{|\epsilon|\eta^{*3}}\right)\overline{U}^2 + \frac{4b_{13}^2}{|\epsilon|\eta^{*3}} - \frac{i\,b_{13}}{\kappa\eta^{*4}\epsilon^2} \\ + \left(\frac{i\,b_{13}}{\eta^{*2}} + \frac{b_{13}^2\kappa|\epsilon|}{\eta^*} + \frac{2b_{13}^2\kappa}{|\epsilon|\eta^{*3}} - \frac{3}{\kappa\eta^{*3}|\epsilon|}\right)\overline{U} - \frac{i\,2b_{13}}{\kappa\eta^{*2}} - \frac{2}{|\epsilon|\kappa^2\eta^{*3}} \end{array}\right\}\overline{S}$$

$$\boxed{- 4\kappa\eta^*|\epsilon|H^3\frac{d^2\overline{S}}{d\eta^{*2}} - 4\kappa|\epsilon|H^3\frac{d\overline{S}}{d\eta^*}} \qquad (3.32)$$

(continued on next page)

$$
\boxed{
\begin{aligned}
+ H^3 \left\{
\begin{aligned}
&- \epsilon\, b_1 \eta^{*2} \frac{d^4\overline{N}}{d\eta^{*4}} - 5\, b_1\, \epsilon\, \eta^* \frac{d^3\overline{N}}{d\eta^{*3}} + \left\{ \frac{i\kappa}{\tilde{\alpha}^2}\, \eta^* \epsilon |\epsilon| \overline{U} - b_2 \epsilon \right\} \frac{d^2\overline{N}}{d\eta^{*2}} \\
&+ \left\{ 3 b_1 \eta^* \epsilon^3 + \frac{i\, 2\, \kappa}{\tilde{\alpha}^2}\, \epsilon |\epsilon| \left(\overline{U} + \frac{1}{\kappa} \right) \right\} \frac{d\overline{N}}{d\eta^*} \\
&+ \left\{ b_3\, \epsilon^3 + b_1\, \epsilon^5\, \eta^{*2} + \frac{i\,\kappa}{\tilde{\alpha}^2}\, \eta^* |\epsilon| \epsilon \left(\frac{1}{\kappa \eta^{*2}} + \epsilon^2 \overline{U} \right) \right\} \overline{N}
\end{aligned}
\right\}
\end{aligned}
}
$$

Outer region:

$$
\boxed{ + i\, H^3 \left[\overline{U} \left(\frac{d^2\overline{S}}{d\tilde{\eta}^2} - \overline{S} \right) + \frac{1}{\kappa \tilde{\eta}^2}\, \overline{S} \right] } \; =
$$

$$
\begin{aligned}
& \left\{ - \kappa \tilde{\eta}\, \overline{U}^3 + i\, 2\, b_{13} \kappa\, \overline{U}^2 + \frac{b_{13}^2 \kappa}{\tilde{\eta}}\, \overline{U} \right\} \frac{d^4\overline{S}}{d\tilde{\eta}^4} \\
+ & \left\{ -2\kappa \overline{U}^3 + \frac{i\, 6\, \kappa\, b_{13}}{\tilde{\eta}}\, \overline{U}^2 + \left(\frac{i\, 2\, b_{13}}{\tilde{\eta}} + \frac{4\kappa\, b_{13}^2}{\tilde{\eta}^2} \right) \overline{U} + \frac{2\, b_{13}^2}{\tilde{\eta}^2} \right\} \frac{d^3\overline{S}}{d\tilde{\eta}^3} \\
+ & \left\{
\begin{aligned}
& 2\kappa \tilde{\eta}\, \overline{U}^3 - \left(i 4 b_{13} \kappa + \frac{1}{\tilde{\eta}} \right) \overline{U}^2 + \left(- \frac{2\kappa\, b_{13}^2}{\tilde{\eta}} + \frac{2 b_{13}^2 \kappa}{\tilde{\eta}^3} + \frac{i\, b_{13}}{\tilde{\eta}^2} \right) \overline{U} \\
& - \frac{i\, 2\, b_{13}}{\kappa\, \tilde{\eta}^2} + \frac{4\, b_{13}^2}{\tilde{\eta}^3}
\end{aligned}
\right\} \frac{d^2\overline{S}}{d\tilde{\eta}^2} \\
+ & \left\{
\begin{aligned}
& 2\kappa \overline{U}^3 + \left(- \frac{i\, 6 b_{13} \kappa}{\tilde{\eta}} + \frac{2}{\tilde{\eta}^2} \right) \overline{U}^2 + \left(- \frac{4\kappa b_{13}^2}{\tilde{\eta}^2} - \frac{i\, 2 b_{13}}{\tilde{\eta}} + \frac{2}{\kappa \tilde{\eta}^2} - \frac{i 2 b_{13}}{\tilde{\eta}^3} \right) \overline{U} \\
& - \frac{2\, b_{13}^2}{\tilde{\eta}^2} - \frac{i\, 2 b_{13}}{\kappa \tilde{\eta}^3}
\end{aligned}
\right\} \frac{d\overline{S}}{d\tilde{\eta}} \\
+ & \left\{
\begin{aligned}
& - \kappa \tilde{\eta}\, \overline{U}^3 + \overline{U}^2 \left(+ i 2 b_{13} \kappa - \frac{1}{\tilde{\eta}} - \frac{2}{\tilde{\eta}^3} \right) \\
& + \left(\frac{i\, b_{13}}{\tilde{\eta}^2} + \frac{\kappa\, b_{13}^2}{\tilde{\eta}} + \frac{2 b_{13}^2 \kappa}{\tilde{\eta}^3} - \frac{3}{\kappa \tilde{\eta}^3} \right) \overline{U} \\
& + \frac{4 b_{13}^2}{\tilde{\eta}^3} - \frac{i\, b_{13}}{\kappa \tilde{\eta}^4} - \frac{i 2 b_{13}}{\kappa\, \tilde{\eta}^2} - \frac{2}{\kappa^2 \tilde{\eta}^3}
\end{aligned}
\right\} \overline{S}
\end{aligned}
\qquad (3.33)
$$

(Continued)

(Equation (3. 33) continued)

$$
\begin{aligned}
& - 4\kappa\tilde{\eta}\, H^3 \frac{d^2\overline{S}}{d\tilde{\eta}^2} - 4\kappa H^3 \frac{d\overline{S}}{d\tilde{\eta}} \\
& + H^3 \Bigg[- b_1 \tilde{\eta}^2 \epsilon \frac{d^4\overline{N}}{d\tilde{\eta}^4} - 5 b_1 \tilde{\eta}\epsilon \frac{d^3\overline{N}}{d\tilde{\eta}^3} + \left\{ \frac{i\kappa\tilde{\eta}\epsilon}{\tilde{\alpha}^2}\overline{U} - b_2\epsilon \right\} \frac{d^2\overline{N}}{d\tilde{\eta}^2} \\
& \quad + \left\{ 3b_1 \epsilon\tilde{\eta} + \frac{i2\kappa\epsilon}{\tilde{\alpha}^2}\left(\overline{U} + \frac{1}{\kappa}\right) \right\} \frac{d\overline{N}}{d\tilde{\eta}} \\
& \quad + \left\{ \epsilon b_3 + b_1 \epsilon \tilde{\eta}^2 + \frac{i\kappa\tilde{\eta}}{\tilde{\alpha}^2}\left(\frac{\epsilon}{\kappa\tilde{\eta}^2} + \epsilon\overline{U} \right) \right\} \overline{N} \Bigg] .
\end{aligned}
$$

where

$$H = \overline{U} - \frac{i b_{13}}{\tilde{\eta}}$$

$$\overline{U} = \begin{cases} \frac{1}{\epsilon\kappa}\left(1 + \epsilon \ln|\epsilon| + \epsilon \ln \eta^*\right) & \text{Inner scale} \\ \frac{1}{\epsilon\kappa}\left(1 + \epsilon \ln \tilde{\eta}\right) & \text{Outer scale} \end{cases}$$

D5. Analysis of the Equations on the Inner and Outer Scales

The substitution of the inner expansions into the equations yields the following:

Inner Mean Vorticity Equation	Inner Pseudo-Vorticity Transport
O(1): NR: $\frac{d^2 S_0}{d\eta^{*2}} = 0$	$\mathcal{L}_2 S_0 = 0$
R: $\mathcal{L}_1 S_0 = 0$	

where "NR," "R" imply the "no relaxation" and "relaxation"models, respectively, and

$$\mathscr{L}_1 = \left(-\frac{\eta^*}{\kappa^2}\ \mathrm{sgn}(\epsilon) + \frac{i\,2\,b_{13}}{\kappa} + \frac{b_{13}^2\,\mathrm{sgn}(\epsilon)}{\eta^*}\right)\frac{d^4}{d\eta^{*4}}$$

$$+\left(-\frac{2\ \mathrm{sgn}(\epsilon)}{\kappa^2} + \frac{i\,6\,b_{13}}{\kappa\eta^*} + \frac{4b_{13}^2\,\mathrm{sgn}(\epsilon)}{\eta^{*2}}\right)\frac{d^3}{d\eta^{*3}} + \left(\frac{2b_{13}^2}{\eta^{*3}}\ \mathrm{sgn}(\epsilon) \quad + \right.$$

$$\left. -\frac{i}{\kappa}\left[\frac{1}{\kappa} - \frac{i\,b_{13}\mathrm{sgn}(\epsilon)}{\eta^*}\right]^3\right)\frac{d^2}{d\eta^{*2}}$$

$$\mathscr{L}_2 = b_4\eta^*\frac{d^4}{d\eta^{*4}} + 2b_4\frac{d^3}{d\eta^{*3}} + \left\{\frac{b_5}{\eta^*} - i\mathrm{sgn}(\epsilon)\frac{\tilde{\alpha}}{\kappa}\right\}\frac{d^2}{d\eta^{*2}}$$

The only solution satisfying both the inner ω-equation and either inner mean vorticity equation is

$$d^2S_0/d\eta^{*2} \equiv 0$$

which, with (3.25) and (3.26) yields

$$S_0^i = -\frac{1}{\kappa} \tag{3.34}$$

$O(\epsilon\,\ell n|\epsilon|)$: NR: $\dfrac{d^2S_1}{d\eta^{*2}} = 0 \qquad \mathscr{L}_2 S_1 = 0$

R: $\mathscr{L}_1 S_1 = 0$

which becomes

$$S_1^i = -\frac{1}{\kappa}\quad . \tag{3.35}$$

$O(\epsilon)$ NR: $\dfrac{d^2S_2}{d\eta^{*2}} = +\dfrac{1}{\kappa\eta^{*2}}\ ;\ \mathscr{L}_2 S_2 = -\dfrac{i\tilde{\alpha}\,\mathrm{sgn}(\epsilon)}{\kappa^2\eta^{*2}}$

R: $\mathscr{L}_1 S_2 = -\dfrac{i}{\kappa^2\eta^{*2}}\left(\dfrac{1}{\kappa} - \dfrac{i\,b_{13}\mathrm{sgn}(\epsilon)}{\eta^*}\right)^3 - \dfrac{2\mathrm{sgn}(\epsilon)}{\kappa^3\eta^{*3}}$

where the solution for S_0 has been used. In either case, the solution is

$$\frac{d^2S_2}{d\eta^{*2}} = \frac{1}{\kappa\eta^{*2}}$$

which yields, with (3.25), (3.26), (3.28) and (3.29),

$$S_2^i = -\frac{\ell n\,\eta^*}{\kappa} + \frac{1}{\kappa}\left(\delta_0 + \frac{c_0\Omega_0}{V_0} + \frac{c_0\delta_0}{V_0}\right)\eta^*\mathrm{sgn}(\varepsilon) \tag{3.36}$$

The outer equations to $O(\varepsilon)$ are as follows, omitting the superscript "0":

	Outer Mean Vorticity	Outer Pseudo-Vorticity Transport
O(1)	NR: $\dfrac{d^2S_0}{d\tilde{\eta}^2} - S_0 = 0$	$0 = +\dfrac{i\tilde{\alpha}}{\kappa}\left(\dfrac{d^2S_0}{d\tilde{\eta}^2} + S_0\right) + \dfrac{N_0}{\tilde{\alpha}\kappa^2}$
	R: $\dfrac{d^2S_0}{d\tilde{\eta}^2} - S_0 = 0$	

Thus, since the perturbations must vanish as $\tilde{\eta} \to \infty$,

$$\begin{aligned} S_0^0 &= C_0\, e^{-\tilde{\eta}} \\ N_0^0 &= -\,i\,2\tilde{\alpha}^2\kappa\, C_0\, e^{-\tilde{\eta}} \end{aligned} \tag{3.37}$$

where C_0 is a complex constant.

Matching with (3.34)

$$C_0 = -\frac{1}{\kappa}$$

	Outer Mean Vorticity	Outer Pseudo-Vorticity Transport
$O(\varepsilon\,\ell n|\varepsilon|)$:	NR: $\dfrac{d^2S_1}{d\tilde{\eta}^2} - S_1 = 0$	$0 = +\dfrac{i\tilde{\alpha}}{\kappa}\left(\dfrac{d^2S_1}{d\tilde{\eta}^2} + S_1\right) + \dfrac{N_1}{\tilde{\alpha}\kappa^2}$
	R: $\dfrac{d^2S_1}{d\tilde{\eta}^2} - S_1 = 0$	

$$S_1^0 = C_1\, e^{-\tilde{\eta}}, \quad N_1^0 = -\,i\,2\tilde{\alpha}^2\, C_1\,\kappa\, e^{-\tilde{\eta}} \tag{3.38}$$

The O(ε) outer mean vorticity equation with no relaxation is

$$+\frac{i}{\kappa}\left(\frac{d^2S_2}{d\tilde{\eta}^2}-S_2\right)+\frac{i\ln\tilde{\eta}}{\kappa}\left(\frac{d^2S_0}{d\tilde{\eta}}-S_0\right)+\frac{i}{\kappa\tilde{\eta}^2}S_0 =$$

$$-4\kappa\tilde{\eta}\frac{d^2S_0}{d\tilde{\eta}^2}-4\kappa\frac{dS_0}{d\tilde{\eta}}+\frac{i}{\tilde{\alpha}^2}\left(\tilde{\eta}\frac{d^2N_0}{d\tilde{\eta}^2}+\frac{2dN_0}{d\tilde{\eta}}+\tilde{\eta}N_0\right)$$

yielding

$$O(\epsilon)\quad \text{NR:}\quad S_2^0 = C_2e^{-\tilde{\eta}}+\frac{e^{+\tilde{\eta}}}{\kappa}E_1(2\tilde{\eta}), \tag{3.39}$$

$$\text{where}\quad E_1(2\tilde{\eta}) \equiv \int_{2\tilde{\eta}}^{\infty}\frac{e^{-x}}{x}\,dx\,.$$

The O(ε) outer mean vorticity equation with relaxation is:

$$+\frac{i}{\kappa}\left(\frac{d^2S_2}{d\tilde{\eta}^2}-S_2\right)+\frac{i\ln\tilde{\eta}}{\kappa}\left(\frac{d^2S_0}{d\tilde{\eta}^2}-S_0\right)+\frac{iS_0}{\kappa\tilde{\eta}^2} =$$

$$-\kappa\tilde{\eta}\frac{d^4S_0}{d\tilde{\eta}^4}-2\kappa\frac{d^3S_0}{d\tilde{\eta}^3}-2\tilde{\eta}\kappa\frac{d^2S_0}{d\tilde{\eta}^2}-2\kappa\frac{dS_0}{d\tilde{\eta}}-\kappa\tilde{\eta}S_0 +$$

$$+\frac{i}{\tilde{\alpha}^2}\left(\tilde{\eta}\frac{d^2N_0}{d\tilde{\eta}^2}+2\frac{dN_0}{d\tilde{\eta}}+N_0\tilde{\eta}\right)$$

whose solution is, again,

$$O(\epsilon)\quad \text{R:}\quad S_2^0 = C_2e^{-\tilde{\eta}}+\frac{e^{+\tilde{\eta}}}{\kappa}E_1(2\tilde{\eta}) \tag{3.40}$$

The O(ε) outer pseudo-vorticity equation is

$$O = b_4\tilde{\eta}\frac{d^4S_0}{d\tilde{\eta}^4}+2b_4\frac{d^3S_0}{d\tilde{\eta}^3}-\frac{i\tilde{\alpha}}{\kappa}\frac{d^2S_2}{d\tilde{\eta}^2}+\left(\frac{b_5}{\tilde{\eta}}-\frac{i\tilde{\alpha}}{\kappa}\ln\tilde{\eta}\right)\frac{d^2S_0}{d\tilde{\eta}^2}$$

$$+2b_4\frac{dS_0}{d\tilde{\eta}}-\frac{i\tilde{\alpha}}{\kappa}S_2+\left(\frac{\tilde{\alpha}^2b_{11}}{\tilde{\eta}}-b_4\tilde{\eta}-\frac{i\tilde{\alpha}}{\kappa}\ln\tilde{\eta}-\frac{i\tilde{\alpha}}{\kappa\tilde{\eta}^2}\right)S_0$$

$$-\frac{ib_8\tilde{\eta}}{\kappa}\frac{d^2N_0}{d\tilde{\eta}^2}-\frac{ib_{10}}{\kappa}\frac{dN_0}{d\tilde{\eta}}-\frac{N_2}{\tilde{\alpha}\kappa^2}+N_0\left(-\frac{2\ln\tilde{\eta}}{\tilde{\alpha}\kappa^2}+\frac{ib_8\tilde{\eta}}{\kappa}-\frac{ib_{11}}{\kappa\tilde{\eta}}\right)$$

yielding

$$N_2^0 = -\left(2\,\tilde{\sigma}\tilde{\alpha}^2\kappa^2 + \frac{\alpha\tilde{\alpha}^2}{\tilde{\eta}}\right)e^{-\tilde{\eta}} \qquad (3.41)$$
$$- i\,2\,\tilde{\alpha}^2\left\{e^{+\tilde{\eta}}\,E_1(2\tilde{\eta}) + C_2\kappa\, e^{-\tilde{\eta}} + \ln\tilde{\eta}\; e^{-\tilde{\eta}}\right\}$$

The expansions for S must be matched at $O(\epsilon)$, not at $O(\epsilon \ln|\epsilon|)$, since $\epsilon\, S_2^i$ is $O(\epsilon\ln|\epsilon|)$ in the "matching" or "intermediate" limit. At $O(\epsilon)$, the matching variable $\zeta\eta_\zeta = k\eta$ (Cole, 1968) is found to satisfy

$$|\epsilon| \ll \zeta \ll |\epsilon|^{\frac{1}{2}}$$

and the following results are obtained†

$$\delta_0 + \frac{c_0\Omega_0}{V_0} + \frac{c_0\delta_0}{V_0} = 1 \qquad (3.42)$$

$$C_1 = 0$$

$$C_2 = \frac{1}{\kappa}(\ln 2 + \gamma), \quad \gamma = 0.577\cdots = \text{Euler's constant}$$

The inner equations to higher order are:

	Inner Mean Vorticity	Inner Pseudo-vorticity transport
$O(\epsilon^2\ln^2\|\epsilon\|)$:	NR: $\dfrac{d^2S_3}{d\eta^{*2}} + \dfrac{d^2S_1}{d\eta^{*2}} = 0$	$\mathcal{L}_2 S_3 - \dfrac{i\tilde{\alpha}}{\kappa}\,\mathrm{sgn}(\epsilon)\dfrac{d^2S_1}{d\eta^{*2}} = 0$
	R: $\mathcal{L}_1 S_3 = 0$	

Thus, using (3.25) and (3.26),

$$S_3^i \equiv 0 \qquad (3.43)$$

† $E_1(x) \sim -\ln x - \gamma + x - \dfrac{x^2}{4} - \dfrac{2}{3}x^3\ln x + O(x^3)$ as $x \to 0$
$\gamma = 0.577\cdots$

$O(\epsilon^2 \ell n|\epsilon|)$:

$$\text{NR:} \qquad \frac{d^2 S_4}{d\eta^{*2}} + \frac{d^2 S_2}{d\eta^{*2}} + \frac{d^2 S_1}{d\eta^{*2}} + \frac{S_1}{\eta^{*2}} = 0 \;;$$

$$\mathcal{L}_2 S_4 - \frac{i\tilde{\alpha}}{\kappa}\,\mathrm{sgn}(\epsilon)\left\{\ell n\eta^* \frac{d^2 S_1}{d\eta^{*2}} + \frac{S_1}{\eta^{*2}} + \frac{d^2 S_2}{d\eta^{*2}}\right\} = 0$$

and the $O(\epsilon^2 \ell n|\epsilon|)$ inner mean vorticity equation with relaxation is

$O(\epsilon^2 \ell n|\epsilon|)$: R:

$$\mathcal{L}_1 S_4 - \frac{i}{\kappa}\left(\frac{1}{\kappa} - \frac{i b_{13}\mathrm{sgn}(\epsilon)}{\eta^*}\right)^3\left[\frac{S_1}{\eta^{*2}} + \frac{d^2 S_2}{d\eta^{*2}}\right] - \frac{i\,3}{\kappa^2}\left(\frac{1}{\kappa} - \frac{i b_{13}\mathrm{sgn}(\epsilon)}{\eta^*}\right)^2 \cdot$$

$$\cdot\left(\frac{d^2 S_2}{d\eta^{*2}} + \frac{S_0}{\eta^{*2}}\right) + \left(-\frac{3\eta^*\,\mathrm{sgn}(\epsilon)}{\kappa^2} + \frac{i\,4\,b_{13}}{\kappa} + \frac{b_{13}^2\,\mathrm{sgn}(\epsilon)}{\eta^*}\right)\frac{d^4 S_2}{d\eta^{*4}}$$

$$+\left(-\frac{6\,\mathrm{sgn}(\epsilon)}{\kappa^2} + \frac{i\,12\,b_{13}}{\kappa\,\eta^*} + \frac{4\,b_{13}^2\mathrm{sgn}(\epsilon)}{\eta^{*2}}\right)\frac{d^3 S_2}{d\eta^{*3}} + \frac{2\,b_{13}^2\mathrm{sgn}(\epsilon)}{\eta^{*3}}\,\frac{d^2 S_2}{d\eta^{*2}}$$

$$-\frac{2\,\mathrm{sgn}(\epsilon)}{\kappa^2\,\eta^{*3}}\,(S_1 + 2S_0) = 0$$

Again, the solution is identical either with or without relaxation:

$$\frac{d^2 S_4}{d\eta^{*2}} = 0$$

thus

$$S_4^i = \frac{\mathrm{sgn}(\epsilon)}{\kappa}\left\{\delta_0 + \delta_1 + \frac{c_0}{V_0^2}(\Omega_1 - \Omega_0 A_1 + \delta_1 + \Lambda_1\delta_0 - A_1\delta_0)\right\}\eta^* \qquad (3.44)$$

The perturbation pseudo-turbulence energy first appears at $O(\epsilon^2)$ in the inner expansion. Specifically, the $O(\epsilon^2)$ inner pseudo vorticity transport is

$$\mathcal{L}_2 S_5 - \frac{i\tilde{\alpha}}{\kappa}\ \mathrm{sgn}(\epsilon)\left(\ell n\,\eta^* \frac{d^2S_2}{d\eta^{*2}} + \frac{S_2}{\eta^{*2}}\right) \tag{3.45}$$

$$+\left(2b_4 + \frac{\tilde{\alpha}^2 b_{11}}{\eta^*} - \frac{i\tilde{\alpha}}{\kappa}\ \mathrm{sgn}(\epsilon)\right)S_0 + \mathrm{sgn}(\epsilon)\left\{b_6\,\eta^{*2}\frac{d^4N_0}{d\eta^{*4}} + 5b_6\frac{d^3N_0}{d\eta^{*3}} + \right.$$

$$\left.\left(b_7 - \frac{i\,b_8\,\eta^*\mathrm{sgn}(\epsilon)}{\kappa}\right)\frac{d^2N_0}{d\eta^{*2}} + \left(\frac{b_9}{\eta^*} - \frac{i\,b_{10}\mathrm{sgn}(\epsilon)}{\kappa}\right)\frac{dN_0}{d\eta^*} + \frac{\sigma}{\eta^{*2}}N_0\right\} = 0$$

The $O(\epsilon^2)$ inner mean vorticity equation without relaxation is

$$+\frac{i}{\kappa}\left(\frac{d^2S_5}{d\eta^{*2}} + \ell n\,\eta^*\frac{d^2S_2}{d\eta^{*2}} - S_0 + \frac{S_2}{\kappa\eta^{*2}}\right) = -b_1\,\eta^{*2}\frac{d^4N_0}{d\eta^{*4}} - 5b_1\eta^*\frac{d^3N_0}{d\eta^{*3}} +$$

$$+\left(\frac{i\,\kappa\,\eta^*\mathrm{sgn}(\epsilon)}{\tilde{\alpha}^2} - b_2\right)\frac{d^2N_0}{d\eta^{*2}} +$$

$$+\ \frac{i\,2\ \mathrm{sgn}(\epsilon)}{\tilde{\alpha}^2}\ \frac{dN_0}{d\eta^*} \tag{3.46}$$

where knowledge of lower order solutions has been used.

The $O(\epsilon^2)$ inner mean vorticity equation with relaxation becomes

$$\mathcal{L}_1 S_5 + \frac{i}{\kappa}\left(\frac{1}{\kappa} - \frac{i\,b_{13}\mathrm{sgn}(\epsilon)}{\eta^*}\right)^3\left\{S_0 - \frac{S_2}{\eta^*} - \ell n\,\eta^*\frac{d^2S_2}{d\eta^{*2}}\right\} + \left\{-\frac{3\eta^*\mathrm{sgn}(\epsilon)}{\kappa^2} + \frac{i\,4b_{13}}{\kappa} + \right.$$

$$\left.+\frac{b_{13}^2\,\mathrm{sgn}(\epsilon)}{\eta^*}\right\}\ell n\eta^*\frac{d^4S_2}{d\eta^{*4}} + \left\{\left(-\frac{6\,\mathrm{sgn}(\epsilon)}{\kappa^2} + \frac{i\ 12\,b_{13}}{\kappa\eta^*} + \frac{4\,b_{13}^2\mathrm{sgn}(\epsilon)}{\eta^{*2}}\right)\ell n\eta^* \right. +$$

$$\left.+\frac{i\,2\,b_{13}}{\kappa\eta^*} + \frac{2\,b_{13}^2\mathrm{sgn}(\epsilon)}{\eta^{*2}}\right\}\frac{d^3S_2}{d\eta^{*3}} + \left\{-\frac{\mathrm{sgn}(\epsilon)}{\kappa^2\eta^*} + \frac{2\,b_{13}^2\mathrm{sgn}(\epsilon)\ell n\eta^*}{\eta^{*3}} + \frac{i\,b_{13}}{\kappa\,\eta^{*2}}\right. +$$

$$\left.+\ \frac{4\,b_{13}^2\,\mathrm{sgn}(\epsilon)}{\eta^{*3}}\right\}\frac{d^2S_2}{d\eta^{*2}} + \left\{\frac{2\,\mathrm{sgn}(\epsilon)}{\kappa^2\eta^{*2}} - \frac{i\,2\,b_{13}}{\kappa\,\eta^{*3}}\right\}\frac{dS_2}{d\eta^*} - \frac{2\,\mathrm{sgn}(\epsilon)}{\kappa^2\,\eta^{*3}}S_2 +$$

$$+\left\{-\frac{4\mathrm{sgn}(\epsilon)\ell n\eta^*}{\kappa^2\eta^{*3}} - \frac{i\,b_{13}}{\kappa\eta^{*4}} + \frac{2\,b_{13}^2}{\eta^{*3}}\ \mathrm{sgn}(\epsilon) - \frac{3\,\mathrm{sgn}(\epsilon)}{\kappa^2\eta^{*3}}\right\}S_0 +$$

$$-\ \frac{i\,3}{\kappa^2}\ell n\,\eta^*\left(\frac{1}{\kappa} - \frac{i\,b_{13}\mathrm{sgn}(\epsilon)}{\eta^*}\right)^2\left(\frac{d^2S_2}{d\eta^{*2}} + \frac{S_0}{\eta^{*2}}\right) + \tag{3.47}$$

(Continued on next page)

(Equation (3.47) continued)

$$+\left(\frac{1}{\kappa}-\frac{i\,b_{13}\mathrm{sgn}(\epsilon)}{\eta^*}\right)^3\left\{\begin{aligned}&-b_1\eta^{*2}\frac{d^4N_0}{d\eta^{*4}}-5b_1\eta^*\frac{d^3N_0}{d\eta^{*3}}+\left(\frac{i\eta^*\mathrm{sgn}(\epsilon)}{\tilde{\alpha}^2}-b_2\right)\frac{d^2N_0}{d\eta^{*2}}\\&+\frac{i\,2\,\mathrm{sgn}(\epsilon)}{\tilde{\alpha}^2}\frac{dN_0}{d\eta^*}\end{aligned}\right\}=0$$

Even in the simpler case of no relaxation no closed-form solution was obtainable for N_0^i and S_5^i in terms of known functions.

The higher order outer equations are:

Outer Mean Vorticity	Outer Pseudo Vorticity Transport

$O(\epsilon^2 \ln^2|\epsilon|)$: NR: $\frac{d^2S_3}{d\tilde{\eta}^2}-S_3=0$ $\qquad +\frac{i\tilde{\alpha}}{\kappa}\left(\frac{d^2S_3}{d\tilde{\eta}^2}+S_3\right)+\frac{1}{\tilde{\alpha}\kappa^2}N_3=0$

R: $\frac{d^2S_3}{d\tilde{\eta}^2}-S_3=0$

Thus

$$S_3^0=C_3e^{-\tilde{\eta}}\,,\qquad N_3=-i2\tilde{\alpha}^2\kappa C_3e^{-\tilde{\eta}} \tag{3.48}$$

$O(\epsilon^2\ln|\epsilon|)$: NR: $\frac{d^2S_4}{d\tilde{\eta}^2}-S_4=0$ $\qquad +\frac{i\tilde{\alpha}}{\kappa}\left(\frac{d^2S_4}{d\tilde{\eta}^2}+S_4\right)+\frac{1}{\tilde{\alpha}\kappa^2}N_4=0$

R: $\frac{d^2S_4}{d\tilde{\eta}}-S_4=0$

where knowledge of lower order solutions has been used.

$$\therefore\ S_4^i=C_4e^{-\tilde{\eta}},\quad N_4=-i2\tilde{\alpha}^2\kappa C_4e^{-\tilde{\eta}} \tag{3.49}$$

At $O(\epsilon^2)$, there is, as on the inner scale, a marked difference between the "no relaxation" and "relaxation" models. The $O(\epsilon^2)$ outer mean vorticity equation without relaxation is

$$\text{NR:}\quad \frac{d^2S_5}{d\tilde{\eta}^2} - S_5 = \frac{-1}{\kappa\tilde{\eta}^2}\left\{e^{+\tilde{\eta}}\, E_1(2\tilde{\eta}) + e^{-\tilde{\eta}}\,\ell n\,\tilde{\eta} + (\gamma+\ell n\, 2)e^{-\tilde{\eta}}\right\}$$

$$+\, e^{-\tilde{\eta}}\left\{+\frac{i\,2\,\kappa}{\tilde{\eta}} + 2\kappa(2\tilde{\alpha}-\alpha)\right\}$$

yielding, for no relaxation,

$$\text{NR:}\quad S_5^0 = e^{-\tilde{\eta}}\left\{C_5 - i\,\kappa\,\ell n\tilde{\eta} - \kappa(2\tilde{\alpha}-\alpha)\tilde{\eta}\right\}$$

$$-\,\frac{E_1(2\tilde{\eta})e^{+\tilde{\eta}}}{\kappa}\left\{+\,\ell n\tilde{\eta} + \gamma + \ell n\, 2 + i\,\kappa^2\right\} - \frac{e^{+\tilde{\eta}}}{\kappa}\int_{2\tilde{\eta}}^{\infty}\frac{E_1(x)}{x}\,dx$$

$$-\,\frac{e^{-\tilde{\eta}}}{\kappa}\int_{2\tilde{\eta}}^{\infty}\frac{e^{+x}\,E_1(x)}{x}\,dx \quad . \tag{3.50}$$

However, the $O(\epsilon^2)$ outer mean vorticity equation with relaxation reduces to

$$\frac{d^2S_5}{d\tilde{\eta}^2} - S_5 = -\frac{1}{\kappa\tilde{\eta}^2}\left\{e^{+\tilde{\eta}}\, E_1(2\tilde{\eta}) + e^{-\tilde{\eta}}\ell n\tilde{\eta} + (\gamma+\ell n\, 2)e^{-\tilde{\eta}}\right\}$$

$$+\, e^{-\tilde{\eta}}\, 2\kappa(2\tilde{\alpha}-\alpha)$$

obtaining

$$\text{R:}\quad S_5 = e^{-\tilde{\eta}}\left\{\tilde{C}_5 - \kappa(2\tilde{\alpha}-\alpha)\tilde{\eta}\right\} - \frac{E_1(2\tilde{\eta})e^{+\tilde{\eta}}}{\kappa}(\ell n\tilde{\eta} + \gamma + \ell n\, 2)$$

$$-\,\frac{e^{+\tilde{\eta}}}{\kappa}\int_{2\tilde{\eta}}^{\infty}\frac{E_1(x)}{x}\,dx - \frac{e^{-\tilde{\eta}}}{\kappa}\int_{2\tilde{\eta}}^{\infty}\frac{e^{+x}E_1(x)}{x}\,dx \tag{3.51}$$

Note that the term

$$-\,i\,\kappa\,\ell n\,\tilde{\eta}\; e^{-\tilde{\eta}} - i\,\kappa\, E_1(2\tilde{\eta})e^{+\tilde{\eta}}$$

is not present in (3.51). In general, C_5 and $\tilde{C}_5$ will be different.

The experience of matching at $O(\epsilon)$ would certainly caution against matching at $O(\epsilon^2\,\ell n^2|\epsilon|)$ or $O(\epsilon^2\ell n|\epsilon|)$ in S. Due to the difficulty of the $O(\epsilon^2)$ inner equations, S_5^i is not known; thus, complete matching at $O(\epsilon^2)$ is not possible.

It is possible, however, to determine C_3 and C_4 (for both the "no relaxation" and "relaxation" case), as follows. Consider matching S at $O(\epsilon^2)$. Clearly, the contribution from S_3^0 and S_4^0 can only be cancelled by terms in S^i. Inspection of (3.34), (3.35), (3.36), (3.43) and (3.44) indicates that $\left\{S_k^i\right\}_{k=0}^{4}$ do not affect S_3^0 or S_4^0. From the nature of the solutions S_3^0, S_4^0, it is clear that, to their respective lowest orders (e.g., $\epsilon^2 \ln^2|\epsilon|\, C_3$ and $\epsilon^2 \ln|\epsilon| C_4$), they cannot be affected by $\left\{S_k^i\right\}_{k=6}^{\infty}$. Thus, they can only be cancelled by S_5^i, and clearly, $C_3\epsilon^2\ln^2|\epsilon|$ can only be matched by a term $\epsilon^2\ln^2\eta^*$ in S_5^i, and $C_4\epsilon^2\ln|\epsilon|$ by a term $\sim \epsilon^2\ln\eta^*$.

However,
$$\epsilon^2\ln\eta^* = \epsilon^2\ln\zeta\,\eta_\zeta - \epsilon^2\ln|\epsilon|$$
$$\epsilon^2\ln^2\eta^* = \epsilon^2\ln^2\zeta\,\eta_\zeta - 2\epsilon^2\ln\zeta\,\eta_\zeta\ln|\epsilon| + \epsilon^2\ln^2|\epsilon|$$

and thus S_5^0 must match the $\epsilon^2\ln^2\zeta\eta_\zeta$, $\epsilon^2(\ln\zeta\eta_\zeta)\ln|\epsilon|$, and $\epsilon^2\ln\zeta\eta_\zeta$ behavior.

However, from (3.50) and (3.51),

$$S_5^0 \sim O(1) \quad \text{as} \quad \tilde{\eta} \to 0$$

for both the R and NR **cases**. Assuming the matching proceeds satisfactorily at $O(\epsilon^2)$, NR & R: $C_3 = C_4 = 0$.

D6. The Inner and Outer Solutions for S and N

We therefore have the following:

Inner Expansions:

Both cases:

$$\overline{S}^i = -\frac{1}{\kappa} - \frac{\epsilon \ln|\epsilon|}{\kappa} + \epsilon\left\{-\frac{\ln\eta^*}{\kappa} + \frac{1}{\kappa}\eta^* \operatorname{sgn}(\epsilon)\right\}$$
$$+ \epsilon^2 \ln|\epsilon| \frac{\operatorname{sgn}(\epsilon)}{\kappa}\left\{\delta_0+\delta_1+\frac{c_0}{V_0}(\Omega_1-\Omega_0 A_1+\delta_1+\Lambda_1\delta_0-A_1\delta_0)\right\}\eta^*$$
$$+ O(\epsilon^2) \tag{3.52}$$

S_5^i and $\overline{N}^i$ were not obtainable in closed form.

Outer Expansions:

No relaxation:

$$\overline{S}^0 = -\frac{e^{-\widetilde{\eta}}}{\kappa} + \epsilon\left\{\frac{(\gamma+\ln 2)}{\kappa}e^{-\widetilde{\eta}} + \frac{e^{+\widetilde{\eta}}}{\kappa}E_1(2\widetilde{\eta})\right\}$$
$$+\epsilon^2\left\{\begin{array}{l} e^{-\widetilde{\eta}}(C_5 - i\kappa\ln\widetilde{\eta} - \kappa(2\widetilde{\alpha}-\alpha)\widetilde{\eta}) - \dfrac{E_1(2\widetilde{\eta})e^{+\widetilde{\eta}}}{\kappa}(\ln\widetilde{\eta}+\gamma+\ln 2+i\kappa^2) \\ -\dfrac{e^{+\widetilde{\eta}}}{\kappa}\displaystyle\int_{2\widetilde{\eta}}^{\infty}\frac{E_1(x)}{x}\,dx - \frac{e^{-\eta}}{\kappa}\int_{2\widetilde{\eta}}^{\infty}\frac{e^{+x}E_1(x)}{x}\,dx \end{array}\right\}$$
$$+ O(\epsilon^3\ln^3|\epsilon|) \tag{3.53}$$

Relaxation:

$$\overline{S}^0 = -\frac{e^{-\widetilde{\eta}}}{\kappa} + \epsilon\left\{\frac{(\gamma+\ln 2)}{\kappa}e^{-\widetilde{\eta}} + \frac{e^{+\widetilde{\eta}}}{\kappa}E_1(2\widetilde{\eta})\right\}$$
$$+\epsilon^2\left\{\begin{array}{l} e^{-\widetilde{\eta}}(\widetilde{C}_5 - \kappa(2\widetilde{\alpha}-\alpha)\widetilde{\eta}) - \dfrac{E_1(2\widetilde{\eta})e^{+\widetilde{\eta}}}{\kappa}(\ln\widetilde{\eta}+\gamma+\ln 2) \\ -\dfrac{e^{+\widetilde{\eta}}}{\kappa}\displaystyle\int_{2\widetilde{\eta}}^{\infty}\frac{E_1(x)}{x}\,dx - \frac{e^{-\widetilde{\eta}}}{\kappa}\int_{2\widetilde{\eta}}^{\infty}\frac{e^{+x}E_1(x)}{x}\,dx \end{array}\right\} + O(\epsilon^3\ln^3|\epsilon| \tag{3.54}$$

$$\overline{N}^0 = +i\,2\widetilde{\alpha}^2 e^{-\widetilde{\eta}} + \epsilon\left\{-(2\widetilde{\sigma}\,\widetilde{\alpha}^2\kappa^2 + \frac{\alpha\widetilde{\alpha}^2}{\widetilde{\eta}})e^{-\widetilde{\eta}}\right. \tag{3.55}$$
$$\left. - i\,2\widetilde{\alpha}^2[e^{+\widetilde{\eta}}E_1(2\widetilde{\eta}) + (\gamma+\ln 2)e^{-\widetilde{\eta}} + \ln\widetilde{\eta}\,e^{-\widetilde{\eta}}]\right\} + O(\epsilon^2)$$

An approximation to $S(\eta,\epsilon)$, uniformly valid to $O(\epsilon)$ for $\eta \in (0^+, \infty)$, can be constructed in the usual fashion by adding the inner and outer expansions and subtracting the common part (Cole, 1968). It is of interest that the outer expansion contains the inner expansion to $O(\epsilon)$, and that therefore the composite expansion to $O(\epsilon)$ is simply the outer expansion to that order. Judging from the complexity of the $O(\epsilon^2)$ inner equations, it is doubtful that such a situation will persist to $O(\epsilon^2)$. The inner solution for $\overline{S}$ is genuine, however, in the sense that the outer solution for $\overline{S}$ is incapable of satisfying all the boundary conditions on $\overline{S}$ and its derivatives to lowest order at $\eta^* = 0^+$.

E. Surface Stresses and Flow Variables

An important characteristic of the flow is the magnitude and phase of the $O(ka)$ surface normal stress. It is easily derived that the jump in the total mean normal stress across the sublayer is (see Appendix II)

$$\Delta\left\{\frac{\overline{p}_1 - \rho_1\tau_{22}}{\rho}\, kae^{ik\xi} - \nu\frac{\partial v}{\partial\eta}\right\}\Bigg|_{\substack{\text{across}\\ \text{sublayer}}} = O(u_{*_0}^2\, k\eta_{*_0}\, kae^{ik\xi}) + O(c^2 k\eta_{*_0}\, kae^{ik\xi}) \qquad (3.56)$$

By definition (see (2.8)),

$$\Phi_1(\eta) = \overline{p}_1(\eta) + \frac{1}{3}\rho\overline{q^2}\Big|_1(\eta)$$

However, from (2.1) and (3.10),

$$_1\tau_{11} = i\,2k\frac{e_0}{\omega_0}\left(\frac{dS}{d\eta} + \frac{U'e^{-k\eta}}{k}\right) - {}_1\chi_{11} - \frac{1}{3}\overline{q^2}\Big|_1$$

$$_1\tau_{22} = -i\,2k\frac{e_0}{\omega_0}\left(\frac{dS}{d\eta} + \frac{U'e^{-k\eta}}{k}\right) - {}_1\chi_{22} - \frac{1}{3}\overline{q^2}\Big|_1$$

$$_1\tau_{33} = -\frac{1}{3}\overline{q^2}\Big|_1$$

which, with the boundary condition on the tangential velocity yields, as $\eta \to 0^+$ (e.g., the edge of the sublayer)

$$ {}_1\tau_{11}(0^+) = {}_1\tau_{22}(0^+) = {}_1\tau_{33}(0^+) = -\frac{1}{3}\overline{q^2}\Big|_1\Big|_{\eta=0^+} \tag{3.57} $$

Hence

$$ \Phi_1(0^+) = \overline{p}_1(0^+) - \rho\, {}_1\tau_{22}(0^+) $$

which, neglecting molecular viscosity in the fully turbulent region, is the complex amplitude of the O(ka) mean normal stress at the edge of the sublayer. Noting that the right side in (3.56) is transcendentally small compared to $\Phi_1(0^+, \epsilon)$ by (3.20), we obtain the fact that the O(ka) mean normal stress at the wave is

$$ ka e^{ik\xi}\, \Phi_1(0^+;\epsilon) \,. $$

One method of evaluating $\Phi_1(0^+;\epsilon)$ is to simply substitute the inner expansions for S and N into the O(ka) $\hat{e}_\xi$-momentum equation, and take the limit as $\eta^* \longrightarrow 0$. This furnishes

$$ \frac{\Phi_1(0^+;\epsilon)}{\rho} = \left(\frac{u_{*0}}{\epsilon}\right)^2 \left\{ -\frac{1}{\kappa^2} - \frac{\epsilon \ln|\epsilon|}{\kappa^2}\left[1+\delta_0+\delta_1+\frac{c_0}{V_0}(\Omega_1-\Omega_0 A_1+\delta_1+\Lambda_1\delta_0-A_1\delta_0)\right] + O(\epsilon)\right\} \tag{3.58} $$

where, unfortunately, the $\left(\frac{u_{*0}}{\epsilon}\right)^2 O(\epsilon)$ term is not capable of evaluation since it requires knowledge of various derivatives of N_0^i and S_5^i. Note the supposition that $\frac{\Phi_1(0^+, \epsilon)}{\rho}$ be finite as $\epsilon \to 0$ is consistent with the above relation using (3.18).

Another method to determine $\Phi_1(0^+;\epsilon)$ is through integration of the O(ka) $\hat{e}_\eta$-momentum equation, which can be written in the form

$$-\frac{1}{\rho}\frac{d\Phi_1}{d\eta} = Uk^2S - ik_1\,{}_1\tau_{12} - \frac{d}{d\eta}\underbrace{\left\{-i\,2k\frac{e_0}{\omega_0}\left(\frac{dS}{d\eta} + \frac{U'e^{-k\eta}}{k}\right) - {}_1\chi_{22}\right\}}_{{}_1\tau_{22} + \frac{1}{3}\overline{q^2}\big|_1} + i2k\frac{e_0}{\omega_0}U'e^{-k\eta} \tag{3.59}$$

The final term on the right side arises from the fact that ${}_1\tau_{12}$ is the complex amplitude of the O(ka) physical (not covariant) turbulent Reynolds stress. In particular

$$ {}_1\tau_{12} = \frac{N}{\tilde{\alpha}} - \frac{\tilde{\sigma}\kappa^2\eta}{\tilde{\alpha}^2}\left(\frac{dN}{d\eta} + \eta\frac{d^2N}{d\eta^2} - k^2\eta N\right) + \frac{i\kappa k\eta \bar{U} N}{\tilde{\alpha}^2} - \left[\kappa u_{*_0}\eta \;\; \frac{\left(U\frac{d^2S}{d\eta^2} + Uk^2S - U''S\right)}{\left(U - \frac{i b_{13} u_{*_0}}{k\eta}\right)}\right] \tag{3.60}$$

where the last term is $-{}_1\chi_{12}$.

For the no relaxation case, the ${}_1\chi_{12}$ and ${}_1\chi_{22}$ terms are neglected, and the appropriate expansions used.

For the relaxation case use of the boundary condition on the tangential velcocity as $\eta \to 0$ and the condition that all perturbations and their derivatives must vanish as $\eta \to \infty$ gives

$$\frac{\Phi_1(0^+, \epsilon)}{\rho} = \int_0^\infty U k^2 S d\eta - ik\int_0^\infty {}_1\tau_{12}\, d\eta + i\,2u_{*_0}^2 \tag{3.61}$$

In the no relaxation case, the same boundary condition as $\eta \to 0$ and the assumption that $dS/d\eta = o(\frac{1}{\tilde{\eta}})$ as $\tilde{\eta} \to \infty$ (which is quite reasonable, since S may be expected to decay exponentially to all orders in ϵ--see (3.53)) yield (3.61).

The integrals in equation (3.61) may be evaluated by dividing the range of integration into the intervals $(0, \zeta\eta_\zeta)$ and $(\zeta\eta_\zeta, \infty)$, the ranges over which, respectively, the inner and outer expansions are assumed to be uniformly valid approximations (Lagerstrom and Casten, 1972). Note that from (3.42), $|\epsilon| \ll \zeta \ll |\epsilon|^{\frac{1}{2}}$.

Consider the "no-relaxation" case first. Using (3.60), we obtain

$$\text{NR:} \qquad -ik\int_0^{\zeta\eta_\zeta} {}_1\tau_{12}\, d\eta = \left(\frac{u_{*_0}}{\epsilon}\right)^2 o(\epsilon^2)$$

This result utilizes the boundary condition on N, and the fact that N_0^i is bounded, independent of ϵ, for $\eta^* \epsilon [0,\infty)$, since $N_0^i(0) = 2\widetilde{\alpha}\,\delta_0$, $N_0^i(\infty) = N_0^0(0) = +i\,2\widetilde{\alpha}^2$, and the differential equation for N_0^i has no singular points except $\eta^* = 0$ and $\eta^* = \infty$. Thus, for example,

$$\int_0^{\zeta\eta_\zeta} N\, d\widetilde{\eta} = u_{*_0}^2 \int_0^{\zeta\eta_\zeta} (N_0^i + \epsilon \ln|\epsilon| N_1^i + \cdots)\, d\widetilde{\eta} \tag{3.62}$$

and $\quad \left|u_{*_0}^2 \int_0^{\zeta\eta_\zeta} N_0^i\, d\widetilde{\eta}\right| \le \max\limits_{0\le\eta^*<\infty} |N_0^i|\, u_{*_0}^2\, \zeta\eta_\zeta = o(u_{*_0}^2) = \left(\frac{u_{*_0}}{\epsilon}\right)^2 o(\epsilon^2)$

Succeeding integrals involving N_1^i, N_2^i, etc. are certainly expected to be $o(u_{*_0}^2)$.

Furthermore, since

$$\text{NR:} \quad {}_1\tau_{12}\Big|_{\substack{\text{outer}\\ \text{scale}}} = u_{*_0}^2 \left\{-\frac{2\widetilde{\eta}e^{-\widetilde{\eta}}}{\epsilon} + i(2\widetilde{\alpha}-\alpha)e^{-\widetilde{\eta}} + 2\widetilde{\eta}\left[e^{+\widetilde{\eta}}E_1(2\widetilde{\eta}) + (\gamma + \ln 2)e^{-\widetilde{\eta}}\right] + O(\epsilon)\right\}, \tag{3.63}$$

thus

$$\text{NR:} \; +i2u_{*_0}^2 - ik\int_\zeta^\infty {}_1\tau_{12}\, d\eta = \left(\frac{u_{*_0}}{\epsilon}\right)^2 \left\{+i\,2\,\epsilon + \epsilon^2\left(2\widetilde{\alpha} - \frac{\alpha}{\kappa} - i\,6\,\gamma\right) + o(\epsilon^2)\right\} \tag{3.64}$$

Employing (3.52) and (3.53), we obtain the result

$$\text{NR:}\quad \int_0^\infty Uk^2 S\,d\eta = \left(\frac{u_{*0}}{\epsilon}\right)^2 \left\{-\frac{1}{\kappa^2} + \epsilon\left[\frac{2(\gamma+\ln 2)}{\kappa^2}\right] + O(\epsilon^2)\right\} \tag{3.65}$$

where the $O(\epsilon^2)$ terms involve C_5, which could not be evaluated. Thus for no relaxation,

$$\text{NR:}\quad \frac{\Phi_1(0^+;\epsilon)}{\rho} = \left(\frac{u_{*0}}{\epsilon}\right)^2 \left\{-\frac{1}{\kappa^2} + \epsilon\left[\frac{2(\gamma+\ln 2)}{\kappa^2} + i\,2\right] + O(\epsilon^2)\right\} \tag{3.66}$$

Utilizing the expansion for u_{*_0} in (3.24), it is clear that

$$\lim_{\epsilon\to 0} \frac{\Phi_1(0^+;\epsilon)}{\rho} = -V_0^2$$

which is identical to the result for a potential flow with velocity V_0 over the wavy wall. This lowest order behavior is expected, as in the limit $\epsilon \to 0$ the O(1) velocity on both inner and outer scales approaches V_0.

It is notable that, to $O(\epsilon)$, the phase shift of the O(ka) surface normal stress is due to the turbulent Reynolds shear.

Comparing (3.58) and (3.66), we obtain

$$1 + \delta_0 + \delta_1 + \frac{c_0}{V_0}(\Omega_1 - \Omega_0 A_1 + \delta_1 + \Lambda_1\delta_0 - A_1\delta_0) \equiv 0\,,$$

a relation that will also be shown to hold in the relaxation case.

Consider the relaxation case. Again, from (3.52) and (3.53).

$$\int_0^\infty Uk^2 S\,d\eta = \left(\frac{u_{*0}}{\epsilon}\right)^2 \left\{-\frac{1}{\kappa^2} + \epsilon\left[\frac{2(\gamma+\ln 2)}{\kappa^2}\right] + O(\epsilon^2)\right\}$$

where the $O(\epsilon^2)$ terms involve $\widetilde{C}_5$, which could not be evaluated. It can be shown that in the relaxation case (see Appendix III),

$$\text{R:}\quad -i\int_0^{\zeta\eta_\zeta} {}_1\tau_{12}\,d\widetilde{\eta} = \left(\frac{u_{*0}}{\epsilon}\right)^2 o(\epsilon^2)$$

Furthermore, since

$$\text{R:} \qquad {}_1\tau_{12}\Big|_{\substack{\text{outer}\\ \text{scale}}} = u_{*_0}^2 \frac{\{+i(2b_{13}\kappa + 2\tilde{\alpha} - \alpha)e^{-\tilde{\eta}} + O(\epsilon)\}}{\left\{1 + \epsilon\left(\ell n\tilde{\eta} - \dfrac{ib_{13}\kappa}{\tilde{\eta}}\right)\right\}}, \tag{3.67}$$

therefore (see Appendix IV)

$$\text{R:} \qquad -ik\int_{\zeta\eta_\zeta}^{\infty} {}_1\tau_{12}\, d\eta = \left(\frac{u_{*0}}{\epsilon}\right)^2 \Big((2b_{13}\kappa + 2\tilde{\alpha} - \alpha)\epsilon^2 + o(\epsilon^2)\Big) \tag{3.68}$$

Hence, for the relaxation case,

$$\text{R:} \qquad \frac{\Phi_1(0^+;\epsilon)}{\rho} = \left(\frac{u_{*0}}{\epsilon}\right)^2 \left\{-\frac{1}{\kappa^2} + \frac{2\epsilon(\gamma + \ell n 2)}{\kappa^2} + O(\epsilon^2)\right\} \tag{3.69}$$

Note that the phase shift at $O(\epsilon)$ has been lost with the inclusion of the relaxation effect.

The normal Reynolds stresses within the flow may be readily determined. In general,

$$ {}_1\tau_{11} + \frac{1}{3}\overline{q^2}\Big|_1 = -{}_1\tau_{22} - \frac{1}{3}\overline{q^2}\Big|_1 = +i2k\kappa u_{*_0}\eta\left(\frac{dS}{d\eta} + \frac{U'e^{-k\eta}}{k}\right) - {}_1\chi_{11},$$

$$ {}_1\tau_{33} = -\frac{1}{3}\overline{q^2}\Big|_1 .$$

On the outer scale ($\tilde{\eta} >> \epsilon$):

$$\text{NR:} \quad {}_1\tau_{11} + \frac{1}{3}\overline{q^2}\Big|_1 = +i2\kappa\frac{u_{*0}^2}{\epsilon}\left\{\tilde{\eta}\frac{e^{-\tilde{\eta}}}{\kappa} + \epsilon\left[-\tilde{\eta}\frac{(\gamma + \ell n 2)}{\kappa}e^{-\tilde{\eta}} + \tilde{\eta}\frac{e^{+\tilde{\eta}}E_1(2\tilde{\eta})}{\kappa}\right] + O(\epsilon^2)\right\} \tag{3.70}$$

$$\text{R:}\quad {}_1\tau_{11}+\frac{1}{3}\overline{q^2}\Big|_1 = \frac{2\kappa^2 b_{13}u_{*0}^2}{\left\{1+\epsilon\left(\ell n\tilde{\eta}-\frac{i b_{13}\kappa}{\tilde{\eta}}\right)\right\}}\left\{\frac{e^{-\tilde{\eta}}}{\kappa}+\frac{i}{\kappa^2 b_{13}}\right.$$

$$\left.+\epsilon\left[-\frac{(\gamma+\ell n2)}{\kappa}e^{-\tilde{\eta}}+\frac{e^{+\tilde{\eta}}E_1(2\tilde{\eta})}{\kappa}-\frac{e^{-\tilde{\eta}}}{\kappa\tilde{\eta}}+\frac{1}{\kappa\tilde{\eta}}+\frac{i\ell n\tilde{\eta}}{\kappa^2 b_{13}}\right]+O(\epsilon^2)\right\}$$

(3.71)

and on the inner scale ($\eta = O(\epsilon)$):

$$\text{NR:}\quad {}_1\tau_{11}+\frac{1}{3}\overline{q^2}\Big|_1 = +i2\kappa\frac{u_{*0}^2\eta^*}{\epsilon}\left\{\frac{\epsilon\,\mathrm{sgn}(\epsilon)}{\kappa}-\frac{\epsilon^2\ell n|\epsilon|\mathrm{sgn}(\epsilon)}{\kappa}+O(\epsilon^2)\right\} \quad (3.72)$$

$$\text{R:}\quad {}_1\tau_{11}+\frac{1}{3}\overline{q^2}\Big|_1 = \left\{\frac{2\kappa^2 b_{13}u_{*0}^2}{\left\{1-\frac{i b_{13}\mathrm{sgn}(\epsilon)}{\eta^*}\kappa+\epsilon\ell n|\epsilon|+\epsilon\ell n\eta^*\right\}}\right\}\left\{\frac{1}{\kappa}+\frac{i}{\kappa^2 b_{13}}+\frac{\epsilon\ell n|\epsilon|}{\kappa}\left(\frac{i}{\kappa b_{13}}-1\right)+O(\epsilon)\right\}$$

(3.73)

It is of interest to note that, to the extent the inner expansions are known, they are contained within the respective outer expansions. Furthermore, the NR expansion on the outer scale is an order of magnitude (in ϵ) greater that the R expansion.

The expression for $\Phi_1(\tilde{\eta};\epsilon)$ on the outer scale, obtained either through the $\hat{e}_\xi$ momentum equation or by integrating the $\hat{e}_\eta$ momentum equation, is

$$\text{NR:}\quad \frac{\Phi_1(\tilde{\eta};\epsilon)}{\rho} = \left(\frac{u_{*0}}{\epsilon}\right)^2\left\{-\frac{e^{-\tilde{\eta}}}{\kappa^2}+\frac{\epsilon}{\kappa^2}\left[(\gamma+\ell n2)e^{-\tilde{\eta}}-e^{+\tilde{\eta}}E_1(2\tilde{\eta})-\ell n\tilde{\eta}e^{-\tilde{\eta}}+i2\kappa^2e^{-\tilde{\eta}}\right]+O(\epsilon^2)\right\}$$

(3.74)

Another quantity of interest is the so-called "wave-induced" Reynolds stress. In this instance, it is conveniently defined as (see (2.9)),

$$-\langle\tilde{u}\,\tilde{v}\rangle \equiv -\frac{1}{2\pi}\int_0^{2\pi}(u(\xi,\eta)-U(\eta))\,v(\xi,\eta)dk\xi\,.$$

The two models exhibit dissimilar behavior; specifically, on the outer scale,

$$\text{NR:}\quad -\langle\tilde{u}\tilde{v}\rangle = (u_{*_0}\,ka)^2\left\{-\frac{e^{-2\tilde{\eta}}}{2\kappa}\,\mathrm{Imag}(C_5)+\tfrac{1}{2}\ln\tilde{\eta}\,e^{-2\tilde{\eta}}+\frac{E_1(2\tilde{\eta})}{2}+o(1)\right\}$$

$$+\,O(ka)^3$$

$$\text{R:}\quad -\langle\tilde{u}\tilde{v}\rangle = (u_{*_0}\,ka)^2\left\{-\frac{e^{-2\tilde{\eta}}}{2\kappa}\,\mathrm{Imag}(\tilde{C}_5)+o(1)\right\}+O(ka)^3$$

The complex amplitudes of the O(ka) perturbation velocities are

$$\text{Outer:}\quad u_1 = \frac{u_{*_0}}{\epsilon}\left\{+\frac{e^{-\tilde{\eta}}}{\kappa}+\epsilon\left(-\frac{(\gamma+\ln 2)}{\kappa}e^{-\tilde{\eta}}+\frac{e^{+\tilde{\eta}}}{\kappa}E_1(2\tilde{\eta})\right)+O(\epsilon^2)\right\} \qquad (3.75)$$

$$v_1 = -\frac{i\,u_{*_0}}{\epsilon}\left\{\epsilon\left[\frac{(\gamma+\ln 2)}{\kappa}e^{-\tilde{\eta}}+\frac{e^{+\tilde{\eta}}E_1(2\tilde{\eta})}{\kappa}+\frac{e^{-\tilde{\eta}}\ln\tilde{\eta}}{\kappa}\right]+O(\epsilon^2)\right\} \qquad (3.76)$$

where the $O(\epsilon^2)$ terms involve S_5^0.

$$\text{Inner:}\qquad u_1 = \frac{u_{*_0}}{\epsilon}\left\{\frac{1}{\kappa}-\frac{\epsilon\ln|\epsilon|}{\kappa}+O(\epsilon)\right\}$$

$$v_1 = \frac{-i\,u_{*_0}}{\epsilon}\left\{-2\frac{\epsilon^2\ln|\epsilon|}{\kappa}\,\mathrm{sgn}(\epsilon)\eta^*+O(\epsilon^2)\right\}$$

where the unwritten terms involve S_5^i.

Again, to the order in the inner solutions indicated, the inner solutions are contained within the outer solutions.

F. Solutions of the Lower Flow and Calculation of Scalar Constants

In order to calculate the various scalar constants appearing in (3.24) for $c \neq 0$, the lower flow field must be treated. In the following, the subscript "w" refers to the lower fluid; in keeping with the previous convention, similar quantities corresponding to the upper fluid are not subscripted.

The following assumptions are made:

1) The lower flow is two-dimensional, viscous and laminar.

Although generally quite valid for laboratory studies, the assumption of laminar flow is questionable for oceanic application.

2) $ka \ll 1$

As in the upper flow, the O(ka) dynamics are solved. The analysis provides the wave amplitude with a temporal behavior of the form

$$a(t) = A\, e^{nt} \tag{3.77}$$

where n and A are both real. Clearly, the solution will be valid for a time interval of order n^{-1}. As with the other scalar quantities in (3.24), n has an expansion of the form

$$n + 2\,\nu_w\, k^2 = \epsilon\, n_2 + o(\epsilon) \tag{3.78}$$

The $-2\,\nu_w k^2$ term is the viscous decay rate for a free wave. In the limit of $\epsilon \to 0$ the phase shift of the O(ka) surface normal stress vanishes and therefore, neglecting Kelvin-Helmholtz instability, there can be no wave growth. Thus, the leading term on the right side of (3.77) is $O(\epsilon)$.

3) $kh >> 1$ and $\nu_w k/c_w << 1$, where h = depth of lower fluid and $c_w = (g/k)^{\frac{1}{2}}$; e.g., only deep water gravity waves are considered. For typical laboratory studies of wind-generated water waves (R. H. Stewart, 1970; Shemdin and Hsu, 1967), $c_w/\nu_w k > 10^{+4}$, and for deep water gravity waves, $c_w/\nu_w k > 10^{+5}$ for $\lambda > 1$ meter. For simplification of the analysis, h is taken as infinite, and $\nu_w k/c_w$ of order ϵ (typical values of ϵ for data considered were between 0.1 and 0.3).

4. At $t = 0$, the lower O(1) horizontal velocity is spatially uniform and equal to $-c$. As will be indicated later, this enables the O(1) mean surface shear to be neglected.

5) The upper flow imposes only a slight perturbation on the water wave. As will be shown, this implies

$$\frac{\rho}{\rho_w}\left(\frac{V_0}{c_w}\right)^2 << 1$$

and rules out Kelvin-Helmholtz instability <u>a priori.</u> For wind-generated water waves, $\frac{\rho}{\rho_w} = 1.22\ 10^{-3}$. From (3.17), it is apparent that V_0 is essentially the velocity at $k\eta = 1$, which is generally of the same magnitude as c_w.

The equations of motion in the upper flow ((3.2) to (3.9)) neglected time variations. These can be taken into consideration through use of (3.77) and (3.78). The effect, however, is insignificant. The expression for $\overline{N}_2^0$ has an added term

$$B\,\frac{\rho}{\rho_w}\,e^{-\tilde{\eta}}$$

where B is a numerical constant of order 1, and $\rho/\rho_w = 1.22\ 10^{-3}$ (for air-water). Similarly, terms with factors of ρ/ρ_w are added to S_5^0 and N_5^0, and to the equations for S_5^i and N_0^i. The expressions (3.66) and (3.69) for $\Phi_1(0^+;\varepsilon)$ are unchanged to the order given.

The analysis of the lower fluid is essentially that given in Lamb, § 349, and will therefore not be repeated here. The only significant differences are 1) the O(ka) shear stress along the surface is matched with that of the upper flow = $2\rho\ u_{*_1} u_{*_0} ka\, e^{ik\xi}$; 2) the normal sfc. stress is matched with $-\Phi_0 - ka\, e^{ik\xi}\,\Phi_1(0^+;\varepsilon)$; 3) surface tension effects are neglected.

From the velocity tangent to the surface we obtain the complex amplitude of the O(ka) orbital velocity as

$$\frac{c\Omega}{c_w} = \frac{c}{c_w} + \frac{i(n+2\nu_w k^2)}{k c_w} - i\,\frac{2\nu_w k}{c_w}\left(1 + \frac{n}{\nu_w k^2} - \frac{ic}{\nu_w k}\right)^{\frac{1}{2}}$$
$$- i\,2\left(\frac{\rho}{\rho_w}\right)\frac{u_{*_1} u_{*_0}}{c_w^2}\left(\frac{c}{c_w} + \frac{in}{c_w k}\right)^{-1}\left(1 - \left(1 + \frac{n}{\nu_w k^2} - \frac{ic}{\nu_w k}\right)^{\frac{1}{2}}\right) \qquad (3.79)$$

where, by a convention of the analysis, the real part of $\left(1 + \frac{n}{\nu_w k^2} - \frac{ic}{\nu_w k}\right)^{\frac{1}{2}}$ is positive, in order that the perturbations to the flow field vanish as $y \to -\infty$.

Matching of the normal surface stress, together with the other boundary conditions, yields the following relation between $\Phi_1(0^+;\varepsilon)$, c, and n:

$$0 = -\frac{\Phi_1(0^+,\varepsilon)}{\rho_w c_w^2} - \left\{1 + \left(\frac{n+2\nu_w k^2}{k c_w} - \frac{ic}{c_w}\right)\right\} + 4\left(\frac{\nu_w k}{c_w}\right)^{\frac{3}{2}}\left(-\frac{ic}{c_w} + \frac{n+\nu_w k^2}{k c_w}\right)^{\frac{1}{2}}$$
$$+ 2\left(\frac{\rho}{\rho_w}\right)\frac{u_{*_1} u_{*_0}}{c_w^2}\left(\frac{c}{c_w} + \frac{in}{kc_w}\right)^{-1}\left\{1 - 2\left(\frac{\nu_w k}{c_w}\right)^{\frac{1}{2}}\left(-\frac{ic}{c_w} + \frac{n+k^2\nu_w}{k c_w}\right)^{\frac{1}{2}}\right\} \qquad (3.80)$$

Utilizing (3.78), (3.10) and the NR solution (3.66) for $\Phi_1(0^+;\epsilon)$ we obtain

$$\text{NR:} \qquad n = -2\nu_w k^2 + \epsilon \frac{\rho}{\rho_w} \frac{V_0^2}{c_0} k\kappa + o(\epsilon) \tag{3.81}$$

where the second term is proportional to the imaginary term in (3.66),

$$c = c_0(1 + \epsilon \ln|\epsilon| \Lambda_1 + \epsilon \Lambda_2 + \cdots)$$

where

$$c_0 = c_w\left(1 - \frac{\rho}{\rho_w}\left(\frac{V_0}{c_w}\right)^2\right)^{\frac{1}{2}}, \quad c_w = \pm(g/k)^{\frac{1}{2}}$$

$$\Lambda_1 = -\frac{\rho}{\rho_w} A_1 \left(\frac{V_0}{c_0}\right)^2$$

$$\Lambda_2 = \frac{\rho}{\rho_w}\left(\frac{V_0}{c_0}\right)^2 (\ln 2 + \gamma - A_2)$$

$$c\Omega = c_0 + o(1)$$

and thus, from (3.40),

$$\delta_0 = \frac{1 - \frac{c_0}{V_0}}{1 + \frac{c_0}{V_0}}$$

The square root in c_0 is taken positive, the sign of c being indicated by the sign of c_w.

Several characteristics are apparent in the above. First, the solution is not unique to all orders, since too few equations exist to specify all the constants. Secondly, the term $\epsilon \frac{\rho}{\rho_w} \frac{V_0^2}{c_0} k\kappa$ in (3.81) implies a generation effect for $V_0 c_w > 0$ (e.g., waves moving in the direction of the wind with $U_0 > c_0$) and a damping effect for $V_0 c_w < 0$ (e.g., waves moving faster than U_0, or waves moving in opposite direction to the wind).

In the previous analysis, the O(1) shear stress at the surface was not matched, it in fact being assumed zero. As the O(1) surface shear acts to set up a non-uniform current in the lower fluid, such an assumption is warranted provided that the current generated during the time interval over which the analysis is valid is small compared to -c. Assuming the O(1) horizontal velocity is -c throughout the lower flow at t = 0, a non-uniform current is generated in a time interval of $O(h^2/\nu_w)$ (Lamb, 1945, § 334). The ratio of this current development time scale to that of viscous decay is

$$\frac{h^2}{\nu_w}(\nu_w k^2) = (kh)^2 >> 1$$

Since the induced surface current cannot exceed $O(V_0) \simeq U(k\eta = 1)$ without the surface shear changing appreciably, it is therefore clear that the O(1) surface shear may be neglected.

G. Stationary Wall

In this case, from (3.42), (3.66) and (3.58),

$$\delta_0 = +1$$

$$\delta_1 = -2$$

and, as mentioned previously, the expansion (3.24) for u_{*_0} is not utilized.

PART IV: COMPARISON WITH EXPERIMENT

The theoretical results have been compared with the experimental data of Kendall (1970), Shemdin and Hsu (1967), and Sigal (1971). The principal characteristics of these studies are listed in Table 1.

A. $u(\xi, \eta)$

In Figures 2 through 4, the total velocity $\mathbf{u}(\xi, \eta)$ in the ξ-direction (3.75) at various constant values of η is compared with the data of Sigal, which has been transformed to the (ξ, η) coordinate system. The values $k\eta_{*_0} = 6.91 \times 10^{-5}$, $c_f = 2(u_{*_0}/U_\infty)^2 = 2.98\ 10^{-3}$ are used, corresponding to the mean velocity profile obtained at the same Reynolds number and position in the wind tunnel with a flat lower surface.

As can be seen, the theory predicts both the phase and amplitude of the u-fluctuations with accuracy.

It is evident from (3.75) that the u-fluctuations are dominated by $\overline{S}_0^0(\tilde{\eta})$. The outer mean vorticity equation (3.33) can be written as

$$\frac{\overline{U}}{\epsilon}\left(\frac{d^2\overline{S}}{d\tilde{\eta}^2} - \overline{S}\right) + \frac{S}{\epsilon\kappa\tilde{\eta}^2} = -i\left\{\frac{d^2}{d\tilde{\eta}^2}\left({}_1\tau_{12}/u_{*_0}^2\right) + \frac{{}_1\tau_{12}}{u_{*_0}^2}\right\} \qquad (3.82)$$

$$+ 2\frac{d}{d\tilde{\eta}}\left(\frac{{}_1\tau_{11} + \frac{1}{3}\overline{q^2}\big|_1}{u_{*_0}^2}\right) + i\,4\,e^{-\tilde{\eta}}$$

The left-side (convective terms) are $O(\epsilon^{-2})$, whereas in either case the mean vorticity generating terms on the right are $o(\epsilon^{-2})$. Hence,

TABLE 1

	Experimental Apparatus	ka	δ/λ	$U_\infty \lambda/\nu$
Kendall (1970)	Deformable neoprene rubber sheet	0.196	1.0	$(1.9 \rightarrow 7.2) \times 10^4$
Shemdin and Hsu (1967)	Water waves	0.100	$<< 1$	$(1.61 \rightarrow 4.03) 10^{+6}$ †
Sigal (1971) ‡	Stationary, rigid wall in wind tunnel ($c \equiv 0$)	0.1755	0.28	$3.06\ 10^{+5}$

† Reynolds number based on "V_{max}" in Table 1 of article.

‡ Model "WWI"

$\overline{S}_0^0$ is an "inviscid" term, as we expect since, to lowest order, the O(ka) normal surface stress is that of a potential flow.

B. $v(\xi, \eta)$

In order to facilitate comparison with the data of Kendall, the velocity in the y-direction at constant values of y is plotted in Figures 5 and 6, using equation (3.76). The values of $k\eta_{*_0}$ and u_{*_0}/U_∞ are from Table 3 of Kendall (1970). Note that the wave speed is non-zero.

It is apparent that the amplitude is predicted within 20%, although the phase is in error by approximately 35° at both elevations.

The velocity in the y-direction, $\tilde{v}(x, y)$, is related to $u(\xi, \eta)$ by

$$\frac{\tilde{v}(x, y)}{u_{*_0}} = \frac{-\,ka\,e^{-ky}\sin kx\; u(\xi, \eta)/u_{*_0} + (1 + ka\,e^{-ky}\cos kx)\, v(\xi,\eta)/u_{*_0}}{\sqrt{1 + 2\,kae^{-ky}\cos kx + (ka)^2\, e^{-2ky}}}$$

using

$$u = U(\eta) + kae^{ik\xi}\, u_1(\eta) + O(ka)^2$$

$$v = kae^{ik\xi}\, v_1(\eta) + O(ka)^2 = -J^{+\frac{1}{2}}\, \psi_\xi = -ikae^{ik\xi}(kS + Ue^{-k\eta})$$

and

$$e^{ik\xi} = e^{ikx} + O(ka)$$

$$e^{-ky} = e^{-k\eta} + O(ka)$$

thus

$$\frac{\tilde{v}(x, y)}{u_{*_0}} = +\,i\,kae^{ik\xi}\, e^{-k\eta}\, \overline{U}(\eta) - i\,kae^{ik\xi}\left(\frac{kS}{u_{*_0}} + \overline{U}\, e^{-k\eta}\right) + O(ka)^2$$

and therefore, to O(ka)

$$\frac{\tilde{v}(x,y)}{u_{*_0}} = -i\,ka e^{ikx} \frac{kS}{u_{*_0}} = -i\,ka e^{ikx} \frac{\bar{S}}{\epsilon}$$

where

$$\bar{S}^0 = -\frac{e^{-\tilde{\eta}}}{\kappa} + \epsilon\left\{\frac{(\gamma+\ell n 2)}{\kappa} e^{-\tilde{\eta}} + \frac{e^{+\tilde{\eta}} E_1(2\tilde{\eta})}{\kappa}\right\} + O(\epsilon^2)\,.$$

It is evident that the measured phase shift of $\tilde{v}(x, y)$ implies an imaginary term in $\bar{S}^0$ at $O(\epsilon)$, e.g., in $\bar{S}^0_2$. If there was a term in $\bar{S}^0_2$ of the form

$$+i(\text{positive function of } \tilde{\eta}) \tag{3.83}$$

the theoretical curve would be shifted towards the experimental data (e.g., the maximum in $\tilde{v}(x, y)$ would move from $kx = -\pi/2$ towards $kx \doteq -55^{\circ}$). Considering again the outer mean vorticity equation (3.82), the diffusion terms on the right side, using (3.63), (3.70), (3.67) and (3.71) are

$$\text{NR: Diffusion terms} = +2(2\tilde{\alpha}-\alpha)e^{-\tilde{\eta}} + \frac{i\,2e^{-\tilde{\eta}}}{\tilde{\eta}} + o(1)$$

$$\text{R: Diffusion terms} = O(1)\,.$$

This therefore implies that, contrary to our expectations, $\bar{S}^0_2$ is also an "inviscid" term. Note that in the no relaxation case, ${}_1\tau_{12}/u^2_{*_0}$ and $({}_1\tau_{11} + \frac{1}{3}\overline{q^2}\big|_1)/u^2_{*_0}$ are each $O(\epsilon^{-1})$, but their $O(\epsilon^{-1})$ contributions to the diffusion terms exactly cancel. It is reasonable to infer from this that the phase of $\tilde{v}(x, y)$ is sensitive to the predictions of the Reynolds stresses, a result that will be considered later in detail.

In Figs. 7 through 9, comparison is made with the data of Sigal ($c \equiv 0$). The amplitude is predicted within 10% at all three elevations, although the phase is in error by approximately 4^{o}. Note that the measured profile is again shifted downstream with respect to the theoretical predictions.

C. Reynolds Stresses

Figures 10 through 12 indicate the data of Sigal (1971) for the Reynolds shear stress in the (ξ, η) coordinate system and the theoretical calculations using the relaxation and no relaxation models. The theory implies that the O(1) Reynolds shear (3.12) is equal to $\rho\, u_{*_0}^2$ throughout the flow; however, the turbulent boundary layer thickness in Sigal's experiment was a fraction of the wavelength, and the [O(1) Reynolds shear]$/(\frac{1}{2}\rho U_\infty^2)$ at $k\eta = 0.262$ and 0.524 was $2.25\ 10^{-3}$ and $2.0\ 10^{-3}$. For this reason, the values $2(u_{*_0}/U_\infty)^2 = 2.25\ 10^{-3}$ and $2.0\ 10^{-3}$ were used in the calculations of the Reynolds stress at $k\eta = 0.262$ and 0.524, respectively, using (3.10), (3.12), (3.63) and (3.67).

Several results are evident. First, the data clearly indicate both significant non-linear (higher harmonic) effects, and a region of separation (zero shear stress) within the flow near the wall. Secondly, the no relaxation model predicts the magnitude of the shear stress with reasonable accuracy, but not the phase, whereas the relaxation model predicts the phase accurately at larger values of $\tilde{\eta}$ while not accounting for the amplitude. It is of note also that the O(ka) shear is decaying on the scale of δ, while the theoretical values decay on a scale of $\lambda (\cong 3.6\delta)$ due to the assumption of an infinite log profile.

The importance of relaxation within the flow can be illuminated by the following. The time scale of the turbulence is, by (2.4), $(\tilde{\lambda}\omega)^{-1} = \omega^{-1}$, while the time required for a fluid parcel to experience a complete cycle of the fluctuating rate of strain is $\lambda/U(\eta)$. The ratio of the reaction time of the turbulence to this convection time is, for c = 0,

$$\text{Ratio} = \frac{\omega^{-1}}{\lambda U^{-1}} = \frac{\tilde{\eta}}{2\pi\tilde{\alpha}} \ln(\tilde{\eta}/\eta_{*_0})$$

For the values of $\tilde{\eta}$ in Figs. 10 through 12, we have

$\tilde{\eta}$	Ratio
0.157	0.64
0.262	1.14
0.524	2.5

The O(ka) rate of strain $kae^{ik\xi}\ {}_1S_{12}$ is, on the outer scale,

$$kae^{ik\xi}\ {}_1S_{12} = ka \cos k\xi \frac{u_{*_0}}{\kappa}\left\{-\frac{2e^{-\tilde{\eta}}}{\epsilon} + 2\left[(\gamma + \ln 2)e^{-\tilde{\eta}} + e^{+\tilde{\eta}}E_1(2\tilde{\eta})\right] + o(1)\right\}$$

which is in anti-phase with the wave. The above argument would imply that for $\tilde{\eta}$ small, the O(ka) Reynolds stress should be roughly in phase with the corresponding Reynolds O(ka) rate of strain, and as $\tilde{\eta}$ increases, a significant phase shift should occur. The data of Sigal (Figs. 9 through 11) indicate the maximum of the Reynolds shear moving from $\approx 225^{\circ}$ at $\tilde{\eta} = 0.157$ to 270° at $\tilde{\eta} = 0.524$.

The defect of the relaxation model is its failure to predict the correct amplitude of the Reynolds shear at large $\tilde{\eta}$, and its inability to account for the close phase relation between the rate of strain and the Reynolds stress at low $\tilde{\eta}$. As a result, the relaxation model generates no contribution to the surface normal stress at $O(\epsilon)$ or $O(\epsilon^2)$ (see (3.67), (3.68)).

In a recent modification of the model equations, Saffman (1974) has included additional terms representing a non-linear interaction between the mean vorticity and the rate of strain in both the constitutive equation for the Reynolds stress (2.1) and the relaxation equation (2.4). As a result, the model is capable of accounting for the inequalities in the normal Reynolds stresses in a constant shear-stress boundary layer (compare (3.12)). The equations become (in cartesian coordinates)

$$-\overline{u_i' u_j'} = 2A \frac{e}{\omega} S_{ij} - \frac{2}{3} e\, \delta_{ij} + \frac{\tilde{\tilde{\lambda}} e}{\omega^2} (\epsilon_{ikm} S_{jk} + \epsilon_{jkm} S_{ik}) \Omega_m - \chi_{ij} \quad (3.84)$$

$$\frac{D\chi_{ij}}{Dt} = 2A \frac{e}{\omega} \frac{D}{Dt} S_{ij} + \frac{\tilde{\tilde{\lambda}} e}{\omega^2} \Omega_m \frac{D}{Dt} (\epsilon_{ikm} S_{jk} + \epsilon_{jkm} S_{ik}) - \omega \chi_{ij} \quad (3.85)$$

where ϵ_{ijk} is the cyclic tensor, $\tilde{\tilde{\lambda}} = 1.0$, and Ω_m is the mean vorticity. The quantity e is identified directly with the turbulence kinetic energy, and A is a constant with value 0.09. In order to roughly estimate the effect of the additional terms, the solutions obtained previously were substituted into the right-side of (3.84). The outer shear stress was changed to the lowest order, indicating one might expect significant changes.

As Kendall (1970) did not measure the normal Reynolds stress in the y-direction, the turbulent shear stress must be compared in the (x, y) coordinate frame. The transformation is

$$\tau_{xy} = \left\{{}_0\tau_{12} + ka\ \text{Real}({}_1\tau_{12}\, e^{ik\xi})\right\} [1 - 2\sin^2\varphi]$$

$$+ \left\{{}_0\tau_{11} - {}_0\tau_{22} + ka\ \text{Real}\,([{}_1\tau_{11} - {}_1\tau_{22}] e^{ik\xi}\right\} \sin\varphi \cos\varphi$$

where φ is the angle between $\hat{e}_\xi$ and $\hat{e}_x$

$(\hat{e}_x \wedge \hat{e}_\xi = +\hat{e}_z \sin\varphi)$ and

$$\sin\varphi = -ka\,e^{-ky} \sin kx \left\{1 + 2\,ka e^{-ky} \cos kx + (ka e^{-ky})^2\right\}^{-\frac{1}{2}} .$$

The model equations utilized predict ${}_0\tau_{11} - {}_0\tau_{22} \equiv 0$ (see (3.12)), whereas in a flat plate zero pressure gradient turbulent boundary layer the ratio ${}_0\tau_{22}/{}_0\tau_{11}$ varies from approximately 0.25 to 0.50 for $y/\delta \in (0.1, 0.7)$ (Hinze, 1959).[†] As $\overline{-u_1 u_2}/\overline{q^2}$ is roughly 0.075 in a flat plate boundary layer for $y/\delta \in (0.1, 0.8)$, the comparison with the data of Kendall is therefore somewhat qualitative.

Fig. 13 indicates both the relaxation and no relation cases, using (3.12), and the same calculations assuming ${}_0\tau_{22} = \frac{1}{2}\, {}_0\tau_{11} =$ $= -\frac{1}{2}(0.01)U_\infty^2$ (Kendall, 1970, p. 277).[‡] The contribution from the O(ka) normal stresses ${}_1\tau_{11}$ and ${}_1\tau_{22}$ was less than 5% of that due to $\tau_{12}(\xi, \eta, ka)$. It is apparent that the no-relaxation case reasonably predicts the amplitude of the O(ka) component of the shear in anti-phase with the wave, which is one source of the phase shift in $\Phi_1(O^+;\epsilon)$ (see equation (3.61)).

[†] See above, p. 77.

[‡] It is possible that the ordinate on Fig. 13 of Kendall has been incorrectly labeled as "$-2 < u_t' v_t' > / U_\infty^2$." As the edge of the sublayer is approached from above, the Reynolds shear stress should fluctuate about a mean which is approximately u_{*0}^2. Since $C_f = 2(u_{*0}/U_\infty)^2 = 4.5 \times 10^{-3}$ for $U_\infty = 5.5$ m/s (Table 2 of Kendall), it is possible that the ordinate should read "$-< u_t' v_t' > / U_\infty^2$." In Fig. 13, both cases are shown.

D. Phase Shift of the Surface Normal Stress

In the limit of $\epsilon \to 0$, the O(ka) normal stress at the surface is exactly in anti-phase with the wave. It is reasonable, then, to define the phase shift relative to $kx = -\pi$, a positive shift ϕ_{p_1} implying a shift downstream with respect to the wave trough.

For the no relaxation case, from (3.66)

$$\phi_{p_1} = \tan^{-1}\left\{ \frac{+2\epsilon\kappa^2 + O(\epsilon^2) + O(ka)^2}{1 - 2(\gamma + \ln 2)\epsilon + O(\epsilon^2) + O(ka)^2} \right\}$$

It is clear from (3.81) that the phase shift ϕ_{p_1} is directly responsible for the generation term in n, where $a = Ae^{nt}$. The net drag on the wavy surface in the x-direction due to the normal surface stress (including all higher harmonics) is

$$C_{D_p} = \frac{ka}{2} c_{p_1} \sin\phi_{p_1} \tag{3.86}$$

where c_{p_1} is the amplitude coefficient of the fundamental harmonic of the surface normal stress:

$$c_{p_1} = ka\left(\sqrt{\Phi_1(0^+)\Phi_1^*(0^+)} + O(ka)^2\right) / \tfrac{1}{2}\rho U_\infty^2$$

Note that ϕ_{p_1} depends non-linearly on ka through terms $\propto (ka)^3 e^{ikx}$ in the expansion for $\Phi(\xi, \eta)$.

The data of Sigal indicate a value of ϕ_{p_1} between $+3^\circ$ and $+4.5^\circ$, based on his calculated value of C_{D_p} and the maximum and minimum measured values of c_{p_1}. Using $k\eta_{*_0} = 6.91\ 10^{-5}$ ($\epsilon = 0.104$; see part IVA) the no relaxation model predicts $+2.6^\circ$.

The phase shift ϕ_{p_1} vs. $c\kappa/u_{*_0}$ for the data of Shemdin & Hsu

(1967) is presented in Fig. 14. The theoretical predictions of Miles (1959a) are also shown. Although the no relaxation model predicts the correct qualitative behavior with c, it underestimates ϕ_{p_1} by 10°-20° for $1.5 \leq c\kappa/u_{*_0} \leq 4.0$. Part of this discrepancy can perhaps be attributed to higher order terms in ϵ--indeed, ϵ increases from 0.216 to 0.347 as $c\kappa/u_{*_0}$ grows from 1.5 to 4.0.

The data of Kendall (1970) are plotted in Fig. 15, along with the theoretical predictions of Miles (1959a), and the calculations of the no relaxation model, using the values of $k\eta_{*_0}$ and u_{*_0}/U_{∞} from Table 2 of Kendall (1970). The theory manifests the correct qualitative behavior with $c\kappa/u_{*_0}$ over a wide range of both positive and negative wave speeds. As Miles' theory relies on the existence of a critical layer within the flow field, it is inapplicable for $c \leq 0$. The theory also indicates qualitatively the observed downshift of the ϕ_{p_1} vs. $c\kappa/u_{*_0}$ curve with increasing Reynolds number. However, it clearly underestimates the phase shift. As in the data of Shemdin and Hsu, ϵ increases with increasing wave speed, and at large positive wave velocities ($c\kappa/u_{*_0} \gtrsim 2.5$), higher order terms in ϵ will be significant. Furthermore, non-linear effects may be important in Kendall's data (ka = 0.196); Townsend (1972) has argued that the linear approximation fails for ka $\gtrsim$ 0.10. Nevertheless, the data near c = 0 appear to indicate that the no relaxation model is lacking a positive term of $O(\epsilon)$ in $\tan \phi_{p_1}$.

From (3.61),

$$\frac{\Phi_1(o^+;\epsilon)}{\rho} = \left\{ \frac{u_{*_0}^2}{\epsilon^2} \right\} \left\{ \frac{1}{\kappa} \int_0^{\infty} (1+\epsilon \ln \tilde{\eta}) \bar{S} \, d\tilde{\eta} - i\epsilon^2 \int_0^{\infty} \frac{{}_1\tau_{12}}{u_{*_0}^2} d\tilde{\eta} + i\,2\,\epsilon^2 \right\}$$

For the no relaxation case, we obtained ((3.65), (3.64))

$$\frac{1}{\kappa}\int (1 + \epsilon \ln\tilde{\eta})\overline{S}\, d\tilde{\eta} = -\frac{1}{\kappa^2} + \frac{2\epsilon(\gamma + \ln 2)}{\kappa^2} + O(\epsilon^2)$$

$$- i\,\epsilon^2 \int_0^\infty \frac{{}_1\tau_{12}}{u_{*_0}^2}\, d\tilde{\eta} = + i\, 2\epsilon + O(\epsilon^2)$$

As indicated in (3.83), the discrepancy between the predicted and measured values of the phase of $\tilde{v}(x, y)$ could be accounted for by a term in $\overline{S}_2^0(\tilde{\eta})$ of form

$$+i(\text{positive function of } \tilde{\eta}) .$$

Clearly, such a term would also add a positive term of $O(\epsilon)$ to $\tan \phi_{p_1}$. As was shown previously, the phase of $\overline{S}_2^0(\tilde{\eta})$ is sensitive to the functional form of the O(ka) Reynolds stresses, and it is consistent to conclude that the underestimation of ϕ_{p_1} is due, in part, to misprediction of the O(ka) Reynolds stresses.

In addition, both the relaxation and no relaxation models predict (see (3.57))

$${}_1\tau_{22} + \frac{1}{3}\, \overline{q^2}\Big|_1 \to 0 \quad \text{as} \quad \tilde{\eta} \to 0^+ ,$$

thus the contribution from $\frac{d}{d\eta}\left({}_1\tau_{22} + \frac{1}{3}\, \overline{q^2}\Big|_1\right)$ to $\Phi_1(0^+;\epsilon)$ is zero (see (3.59)). As in a constant-stress flat plate boundary layer, one does not expect equality of the O(ka) normal Reynolds stresses at the edge of the sublayer; however, the contribution to $\Phi_1(0^+;\epsilon)$ would be $O(u_{*_0}^2) = (u_{*_0}/\epsilon)^2\, O(\epsilon^2)$.

E. c_{p_1}

The variation of c_{p_1} with wave speed for the data of Kendall is indicated in Fig. 16. The no relaxation case predicts the pressure coefficient within 15% for $-0.4 \leq c/U_\infty \leq 0.4$. For comparison, c_{p_1} for a potential flow of velocity $U_\infty - c$ is also presented.

PART V: CONCLUSIONS

The following conclusions can be drawn:

1. Both the relaxation and no-relaxation models accurately predict the O(ka) horizontal velocity.

2. Neither the relaxation or no-relaxation models adequately represent the O(ka) turbulent Reynolds stresses. A more suitable relaxation hypothesis (such as (3.84) and (3.85)) is needed.

3. The no-relaxation model predicts the correct qualitative behavior for the phase shift of the surface normal stress for a wide range of positive and negative wave speeds; quantitative predictions are deficient due to inadequate representation of the turbulent Reynolds stresses and higher order effects in ϵ and ka.

4. Both the relaxation and no-relaxation models accurately predict the coefficient of the fundamental harmonic of the surface normal stress over a broad range of positive and negative wave speeds.

Appendix I: The charge in the normal velocity across the sublayer

Consider the wavy surface to be smooth. From continuity (utilizing the notation of (2.9)),

$$v(\xi,\eta) = -J^{+\frac{1}{2}} \int_{k\eta_s}^{k\eta} \frac{\partial}{\partial k\xi} \left(J^{-\frac{1}{2}} u\right) dk\eta$$

where $\quad k\eta_s = k\eta \Big|_{y=a\cos kx} = (ka)^2 \cos^2 k\xi + O(ka)^3$

and $\quad J(\xi,\eta) = 1 + 2\,kae^{-k\eta} \cos k\xi + O(ka)^2.$

Assume that, within the sublayer $\eta_s \le \eta \le \eta_{sL}$, where $\eta_{sL} \equiv \eta \Big|_{\text{Edge of sublayer}} << a$,

$$u = u_*(\xi)\, f\left(\frac{\zeta u_*}{\nu}\right) - c + kae^{ik\xi}\, c\Omega + O(ka)^2 \qquad \text{(I. 1)}$$

where $\quad \zeta = \eta - \eta_s + a\, e^{ik\xi}\, e^{-k\eta_s}(e^{-k(\eta-\eta_s)} - 1) + a\, O(ka)$ is the distance normal to the surface; $f(\zeta^+)$ is a universal function, independent of $u_{*_0}/k\nu$, with $f(o) = 0$; and $c\Omega kae^{ik\xi}$ is the orbital velocity at the surface.

Expanding the integrand using (3.10) yields, to O(ka),

$$v = -i\, kae^{ik\xi} \int_0^{k\eta} \left\{ u_{*_1} \frac{d}{d\eta^+}(f(\eta^+)\eta^+) + u_{*_0} \frac{df}{d\eta^+} \frac{u_{*_0}}{k\nu}(e^{-k\eta} - 1) - e^{-k\eta} u_{*_0} f(\eta^+) + c\Omega + e^{-k\eta} c \right\} dk\eta$$

where $\quad \eta^+ = \dfrac{\eta u_{*_0}}{\nu}$.

Now

$$\int_0^{k\eta} u_{*_1} \frac{d}{d\eta^+}(\eta^+ f(\eta^+))\, dk\eta = u_{*_1} \frac{k\nu}{u_{*_0}} f(\eta^+_{sL})\, \eta^+_{sL}$$

where η^+_{sL} is a constant, and $k\eta_{sL} \propto k\nu/u_{*0}$.

Since we expect $\left|\frac{df(\eta^+)}{d\eta^+}\right|$ to be bounded in $0 < \eta^+ < \eta^+_{sL}$,

$$\left|\int_0^{k\eta} u_{*_0} \frac{df}{d\eta^+} \frac{u_{*_0}}{k\nu} (e^{-k\eta} - 1) d\,k\eta\right| \le u_{*_0} \frac{u_{*_0}}{k\nu} \max_{\text{in } sL}\left|\frac{df}{d\eta^+}\right| \int_0^{k\eta} (1 - e^{-k\eta})\,dk\eta$$

$$= u_{*_0} \frac{u_{*_0}}{k\nu} \max\left|\frac{df}{d\eta^+}\right| \left(\frac{(k\eta)^2}{2} + O(k\eta)^3\right)$$

Similarly,

$$\left|\int_0^{k\eta} e^{-k\eta} u_{*_0} f(\eta^+) d\,k\eta\right| \le u_{*_0} \max|f(\eta^+)| (k\eta + O(k\eta)^2)$$

and
$$\int_0^{k\eta} (c\Omega + c\,e^{-k\eta}) d\,k\eta = c\Omega\,k\eta + c(1 - e^{-k\eta})$$

Thus, writing $k\eta_{*_0} = k\eta_{sL} \propto \frac{k\nu}{u_{*_0}}$, and using the fact that $u_{*_1} = O(u_{*_0})$, and $c\Omega = O(c)$,

$$v(\xi, \eta_{sL}) = ka e^{ik\xi} \left\{O(u_{*_0} k\eta_{*_0}) + O(c\,k\eta_{*_0})\right\} . \qquad \text{(I. 2)}$$

For a rough wavy wall, the functional form of the tangential velocity within the sublayer is not known, since it depends on the shape of the roughness elements, and (I. 2) is taken as an assumption.

Appendix II: The change in the stresses across the sublayer

Consider the curvilinear coordinate system in Fig. II wherein (dX, dY) are incremental lengths parallel and normal to the wall, respectively, and u and v are the corresponding average velocities.

The equations of motion within the thin roughness layer, making the boundary layer approximation, are the following (Goldstein (1938)):

$$\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} + \frac{v}{R} = 0$$

$$u\frac{\partial u}{\partial X} + v\frac{\partial u}{\partial Y} + \frac{uv}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \nu\frac{\partial^2 u}{\partial Y^2} + \frac{1}{\rho}\frac{\partial \tau}{\partial Y} + \frac{2\tau}{R\rho}$$

$$+\frac{u^2}{R} = -\frac{1}{\rho}\left(P - \mu\frac{\partial v}{\partial Y}\right)$$

where $\tau = -\rho\,\overline{u'v'}$ is the turbulent shear stress

$P = \bar{p} + \rho\overline{v'^2}$, $\bar{p}$ is mean (static) pressure

R = radius of curvature of surface

$= (-ka\;k\sin kx + \cdots)^{-1}$

Writing

$$u = u_0(Y) + kae^{ikX}\,u_1(Y) + O(ka)^2$$

$$v = kae^{ikX}\,v_1(Y) + \cdots$$

$$\tau = \tau_0(y) + kae^{ikX}\,\tau_1(y) + \cdots$$

$$P = P_0(y) + kae^{ikX}\,P_1(y) + \cdots$$

the boundary conditions are

$$u(X, 0) = -c + kae^{ikX}\,c\Omega + O(ka)^2$$

$$v(X, 0) = 0$$

$$\tau(X, 0) = 0$$

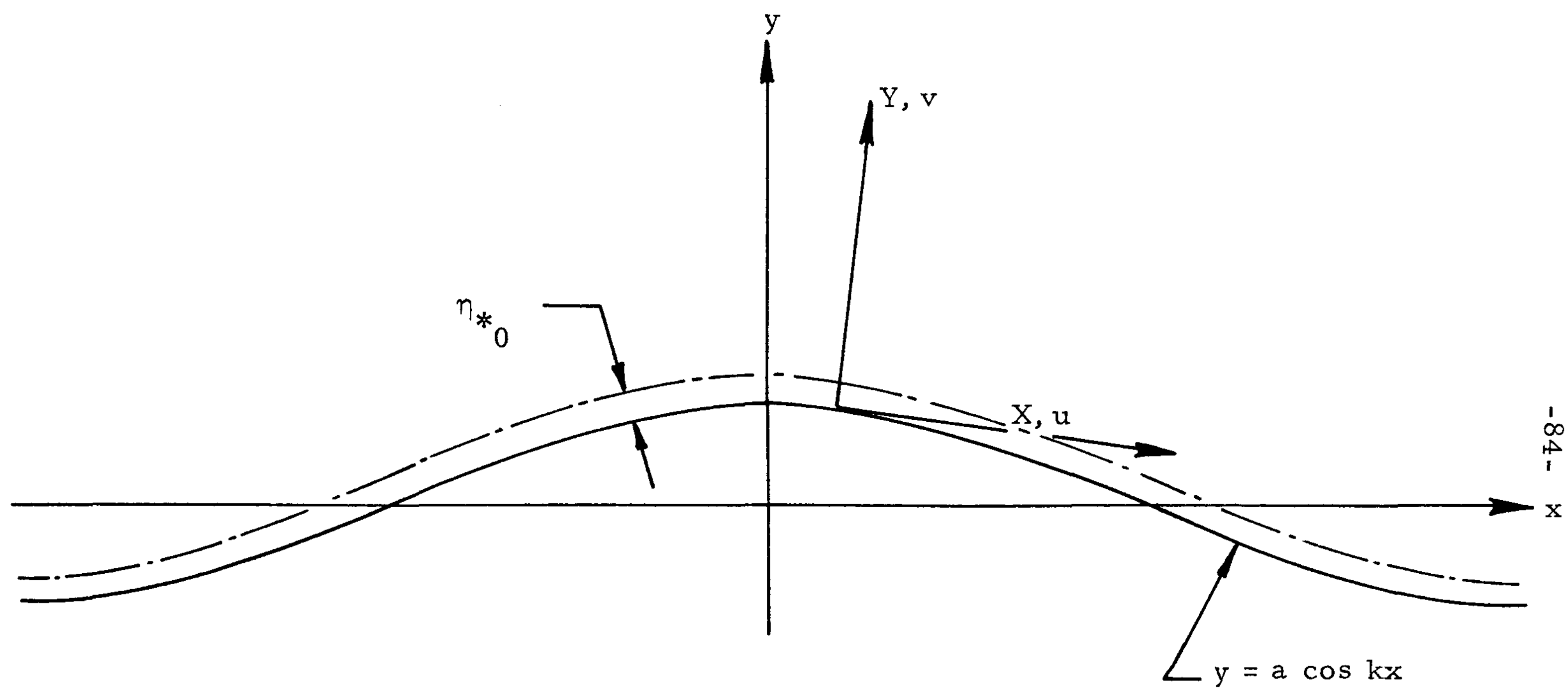


Figure II

(Vertical scale exaggerated)

Substituting,

$$0 = \frac{\partial}{\partial Y}\left(\nu \frac{du_0}{dY} + \frac{\tau_0(y)}{\rho}\right); \quad \frac{dp_0}{dy} = 0$$

$$iku_1 + \frac{dv_1}{dy} = 0\,; \quad iku_0^2 = -\frac{1}{\rho}\frac{d}{dY}\left(P_1 - \mu\frac{dv_1}{dy}\right)$$

$$iku_0u_1 + v_1\frac{du_0}{dy} = -\frac{ik}{\rho}P_1 + \frac{d}{dy}\left(\nu\frac{du_1}{dy} + \frac{\tau_1}{\rho}\right) + \frac{i2k\tau_0(y)}{\rho}$$

Thus

$$\boxed{\nu\frac{du_0}{dy} + \frac{\tau_0(y)}{\rho} = u_{*_0}^2} \qquad \text{(II. 1)}$$

Writing for the O(1) velocity (assuming a smooth surface),

$$u_0 = u_{*_0} f\left(\frac{Yu_{*0}}{\nu}\right) - c$$

we have

$$\Delta\left.\frac{(P_1 - \mu\frac{dv_1}{dy})}{\rho}\right|_{\substack{\text{Across}\\\text{sublayer}}} = -iu_{*_0}^2 k\eta_{*_0}\int_0^1 f^2(\zeta)d\zeta + i2cu_{*_0}k\eta_{*_0}\int_0^1 f(\zeta)d\zeta - ic^2k\eta_{*_0}$$

The integrals are independent of $k\eta_{*_0}$; thus

$$\Delta\left.\frac{(P_1 - \mu\frac{dv_1}{dy})}{\rho}\right|_{\substack{\text{Across}\\\text{sublayer}}} = -iu_{*_0}^2 k\eta_{*_0}\vartheta_1 + i2u_{*_0}k\eta_{*_0}\vartheta_2 c - ic^2k\eta_{*_0}$$

where ϑ_1, ϑ_2 are real constants.

Using the notation of (3. 10), (II. 2)

$$\boxed{\Delta\left\{\left(\frac{\overline{p}_1}{\rho} - {}_1\tau_{22}\right)ka e^{ik\xi} - \nu\frac{\partial v}{\partial\eta}\right\}\Bigg|_{\substack{\text{Across}\\\text{sublayer}}} = -ika e^{ik\xi}k\eta_{*_0}\left\{u_{*_0}^2\vartheta_1 - 2u_{*_0}c\vartheta_2 + c^2\right\}}$$

Since $$k\eta_{*_0} \sim \exp\left(-\frac{U_0}{(U_0-c_0)\epsilon}\right)$$

where $(U_0-c_0)\epsilon \to 0^+$, the change in the total normal stress across the sublayer is transcendentally small compared to any of the terms in $kae^{ik\xi}\,\Phi_1(o^+;\epsilon)$.

Similarly,

$$\frac{d}{dY}\left(\nu\frac{du_1}{dY} + \frac{\tau_1}{\rho}\right) = v_1\frac{du_0}{dY} - u_0\frac{dv_1}{dY} + \frac{ik}{\rho}P_1(y) - i\,\frac{2k}{\rho}\tau_0(Y)$$

where $\nu\frac{du_1}{dY} + \frac{\tau_1}{\rho}$ is the complex amplitude of the total O(ka) kinematic shear stress.

Then

$$\Delta\left(\nu\frac{du_1}{dy} + \frac{\tau_1}{\rho}\right)\Bigg|_{\substack{\text{Across}\\ \text{sublayer}}} = +\frac{i}{\rho}\int_0^{k\eta_{*_0}} P_1(kY)dkY - i\,\frac{2}{\rho}\int_0^{k\eta_{*_0}}\tau_0(kY)dkY + \int_0^{k\eta_{*_0}}\left(v_1\frac{du_0}{dkY} - u_0\frac{dv_1}{dkY}\right)dkY$$

Now, using (II. 2),

$$\frac{i}{\rho}\int_0^{k\eta_{*_0}} P_1(kY)dkY = O\left(k\eta_{*_0}\,\frac{P_1}{\rho}\Big|_{SFC}\right)$$

and since $0 \le \frac{\tau_0(y)}{\rho} \le u_{*_0}^2$ in sublayer (as $d^2u_0/dY^2 \le 0$ in sublayer of a flat plate zero-pressure-gradient boundary layer),

$$\int_0^{k\eta_{*_0}}\frac{\tau_0}{\rho}\,dkY \le k\eta_{*_0}\,u_{*_0}^2$$

Also

$$\int_0^{k\eta_{*_0}} dkY\left(v_1\frac{du_0}{dkY} - u_0\frac{dv_1}{dkY}\right) = u_0v_1\Big|_0^{k\eta_{*_0}} + 2i\int_0^{k\eta_{*_0}} u_1u_0\,dkY\ .$$

$$u_0 v_1 \Big|_0^{k\eta_{*_0}} = O(u_{*_0}^2 \, k\eta_{*_0}) + O(c^2 \, k\eta_{*_0}) \quad \text{using (I. 1)} .$$

Furthermore,

$$2\, u_1\, u_0 \le u_0^2 + u_1^2$$

and assuming in sublayer (for smooth wall),

$$u = u_*(X)\, f\left(\frac{Y\, u_*(X)}{\nu}\right) - c + c\Omega ka e^{ikX} + O(ka)^2 ,$$

$$2i \int_0^{k\eta_{*_0}} u_1 u_0 \, dkY = O(u_{*_0}^2 \, k\eta_{*_0}) + O(c^2 k\eta_{*_0}) .$$

Thus

$$\Delta\left(\nu \frac{du_1}{dy} + \frac{\tau_1}{\rho}\right)\Bigg|_{\substack{\text{Across}\\ \text{sublayer}}} = O\left(k\eta_{*_0} \frac{P_1\big|_{SFC}}{\rho}\right) + O(u_{*_0}^2 \, k\eta_{*_0}) + O(c^2 k\eta_{*_0}) \qquad \text{(II.3)}$$

As indicated, the change in the complex amplitude of the total O(ka) kinematic shear stress across the sublayer is transcendentally small compared to any of the terms in ${}_1\tau_{12}(\eta)$.

For a rough wavy wall, the functional form of the tangential velocity within the sublayer is dependent on the shape of the roughness elements, and (II. 2) and (II. 3) are taken as assumptions.

Appendix III: Contribution of the Inner Shear Term to Surface Normal Stress--relaxation included

From (3.60),

$$\frac{{}_1\tau_{12}}{u^2_{*_0}} = \frac{\bar{N}}{\tilde{\alpha}}\left\{1 + \frac{\tilde{\sigma}\kappa^2 k^2\eta^2}{\tilde{\alpha}} + i\kappa k\eta\frac{\bar{U}}{\tilde{\alpha}} - \frac{\tilde{\sigma}\kappa^2}{\tilde{\alpha}}\right\} - \frac{\tilde{\sigma}\kappa^2}{\tilde{\alpha}^2}\frac{d}{d\tilde{\eta}}\left\{\tilde{\eta}^2\frac{d\bar{N}}{d\tilde{\eta}} - \tilde{\eta}\bar{N}\right\}$$

$$- \frac{\kappa\tilde{\eta}}{\epsilon}\;\frac{\left\{\bar{U}\, d^2\bar{S}/d\tilde{\eta}^2 + \bar{U}\,\bar{S} + \frac{1}{\kappa\tilde{\eta}^2}\bar{S}\right\}}{\left(\bar{U} - \frac{ib_{13}}{\tilde{\eta}}\right)}$$

As before (3.62),

$$\int_0^\zeta \frac{\bar{N}}{\tilde{\alpha}}\left\{1 + \frac{\tilde{\sigma}\kappa^2\tilde{\eta}^2}{\tilde{\alpha}} - \frac{\tilde{\sigma}\kappa^2}{\tilde{\alpha}}\right\}d\tilde{\eta} = o(1)\,.$$

Similarly

$$\left|\int_0^\zeta \bar{N}\tilde{\eta}\,\bar{U}\,d\tilde{\eta}\right| = \left|\int_0^\zeta \bar{N}\tilde{\eta}\left(\frac{1}{\epsilon\kappa}\right)(1 + \epsilon\,\ell n\tilde{\eta})d\tilde{\eta}\right|$$

$$\le \frac{1}{|\epsilon|\kappa}\max_{0\le\tilde{\eta}\le\zeta}|\bar{N}|\left(\frac{\zeta^2}{2} - |\epsilon|\left(\frac{\zeta^2}{2}\right)(\ell n\zeta - \tfrac{1}{2})\right) = o(1)$$

Since $|\epsilon| \ll \zeta \ll |\epsilon|^{\frac{1}{2}}$ (see (3.42)).

Also

$$\int_0^\zeta \frac{d}{d\tilde{\eta}}\left(\tilde{\eta}^2\frac{d\bar{N}}{d\tilde{\eta}} - \tilde{\eta}\,\bar{N}\right)d\tilde{\eta} = \tilde{\eta}^2\frac{d\bar{N}}{d\tilde{\eta}}\Bigg|_0^\zeta - \tilde{\eta}\,\bar{N}\Bigg|_0^\zeta = o(1)$$

Now in the inner region,

$$\bar{U}\left(\frac{d^2\bar{S}}{d\tilde{\eta}^2} + \bar{S}\right) + \frac{1}{\kappa\tilde{\eta}^2}\bar{S} = \frac{\bar{U}}{\epsilon^2}\left(\frac{d^2\bar{S}}{d\eta^{*2}} + \epsilon^2\bar{S}\right) + \frac{\bar{S}}{\kappa\epsilon^2\eta^{*2}}$$

$$= \frac{1}{\epsilon\kappa}\left\{\frac{d^2 S_5^i}{d\eta^{*2}} - \frac{1}{\kappa} + \frac{\mathrm{sgn}(\epsilon)}{\kappa\eta^*} + o(1)\right\}$$

Thus in inner region,

$$-\frac{\kappa\tilde{\eta}}{\epsilon}\;\frac{\left\{\overline{U}\,(d^2\overline{S}/d\tilde{\eta}^2-\overline{S})+\frac{1}{\kappa\tilde{\eta}^2}\,\overline{S}\right\}}{(\overline{U}-\frac{i\,b_{13}}{\tilde{\eta}})} =$$

$$-\,i\kappa\,\mathrm{sgn}(\epsilon)\;\frac{\left\{\eta^{*2}\left(\frac{d^2S_5^i}{d\eta^{*2}}-\frac{1}{\kappa}\right)+\frac{\eta^*\mathrm{sgn}(\epsilon)}{\kappa}+o(1)\right\}}{b_{13}\kappa\,\mathrm{sgn}(\epsilon)+i\eta^*(1+\epsilon\ell n|\epsilon|+\epsilon\ell n\eta^*)}$$

Now, $d^2S_5^i/d\eta^{*2}$ is not known. However, from the boundary condition on $-\overline{u_1'u_2'}$ (equation (3.29)), it can be shown that $\eta^{*2}d^2S_5^i/d\eta^{*2}\to 0$ as $\eta^*\to 0$. Furthermore, $\eta^{*2}\,d^2\,S_5^i/d\eta^{*2}$ should tend towards $\tilde{\eta}^2d^2\,S_5^0/d\tilde{\eta}^2$ as $\eta^*\to\infty$, and from (3.51),

$$\tilde{\eta}^2\,\frac{d^2S_5^0}{d\tilde{\eta}^2}\to 0 \quad\text{as}\quad \tilde{\eta}\to 0\,.$$

Inspection of the equations for S_5^i and N_0^i ((3.45) and (3.47)) indicates that the only singular points are $\eta^*=0,\infty$. Hence we replace $\eta^{*2}\,d^2\,S_5^i\,/d\eta^{*2}$ by its maximum absolute value for $0\le\eta^*\le\zeta/|\epsilon|$ (which is independent of ϵ), and write

$$\left|\int_0^{\zeta}\frac{-i\kappa\,\mathrm{sgn}(\epsilon)\,\eta^{*2}\,d^2S_5^i/d\eta^{*2}\,d\tilde{\eta}}{b_{13}\kappa\,\mathrm{sgn}(\epsilon)+i\eta^*(1+\epsilon\ell n|\epsilon|+\epsilon\ell n\eta^*)}\right| \le \frac{1}{b_{13}}\max\left|\eta^{*2}d^2S_5^i/d\eta^{*2}\right|\zeta$$

$$= o(1)\,.$$

Similarly, since $\zeta^2=o(\epsilon)$

$$\left|\int_0^{\zeta}\frac{-\,i\,\eta^*\,d\tilde{\eta}}{b_{13}\kappa\,\mathrm{sgn}(\epsilon)+i\eta^*(1+\epsilon\ell n|\epsilon|+\epsilon\ell n\eta^*)}\right| \le \frac{|\epsilon|}{b_{13}\kappa}\,\frac{\zeta^2}{\epsilon^2} = o(1)$$

Finally, it is easy to show that

$$\left| \int_0^{\zeta/|\epsilon|} \frac{i|\epsilon|\, \mathrm{sgn}(\epsilon)\,\eta^{*2}\, d\eta^*}{\{b_{13}\,\kappa\, \mathrm{sgn}(\epsilon) + i\eta^*(1+\epsilon\,\ell n|\epsilon| + \epsilon\,\ell n\,\eta^*)\}} \right| = o(1)$$

by dividing the integration into $0 \le \eta^* \le \frac{e^{-1/2\epsilon}}{\epsilon}$ and $\frac{e^{-1/2\epsilon}}{\epsilon} \le \eta^* \le \zeta$ when $\epsilon > 0$, and by simply using $1 + \epsilon\,\ell n\tilde{\eta} \ge 1$ for $0 \le \tilde{\eta} \le \zeta$ when $\epsilon < 0$.

Thus,

$$\int_0^{\zeta} {}_1\tau_{12}\, d\tilde{\eta} = \left(\frac{u_{*0}}{\epsilon}\right)^2 o(\epsilon^2)\,.$$

Appendix IV: Contribution of Outer Shear Term to Surface Normal Stress - relaxation included

From (3.61) and (3.67),

$$-i\int_{\zeta\eta_\zeta}^{\infty} {}_1\tau_{12}\, d\widetilde{\eta} = u_{*_0}^2 (2b_{13}\kappa + 2\widetilde{\alpha}-\alpha)\int_{\zeta\eta_\zeta}^{\infty} \frac{\{e^{-x} + O(\epsilon)\}dx}{\left\{1 + \epsilon\left(\ln x - \frac{i b_{13}\kappa}{x}\right)\right\}}$$

Now

$$\int_\zeta^\infty \frac{e^{-x}\, dx}{1+\epsilon\left(\ln x - \frac{i b_{13}\kappa}{x}\right)} = 1 + O(\zeta) + \epsilon\int_\zeta^\infty \frac{i b_{13}\kappa\, e^{-x}\, dx}{x\left(1+\epsilon\left(\ln x - \frac{i b_{13}\kappa}{x}\right)\right)}$$

$$-\epsilon\int_\zeta^\infty \frac{\ln x\, e^{-x}\, dx}{1+\epsilon\left(\ln x - \frac{i b_{13}\kappa}{x}\right)}$$

The cases $\epsilon > 0$ and $\epsilon < 0$ must be considered separately. Note that $|\epsilon| \ll \zeta \ll |\epsilon|^{\frac{1}{2}} \ll 1$.

Case I: $\epsilon > 0$

First, we show that $\epsilon \ln\zeta = o(1)$, as follows:

a) $\epsilon \ln\zeta \not\to \infty$

Assume $|\epsilon \ln\zeta| \to \infty$ as $\epsilon \to 0$.

$$\text{Then } \frac{\ln(\zeta e^{1/\epsilon})}{\ln\zeta} = 1 + \frac{1}{\epsilon \ln\zeta} \longrightarrow 1 \qquad \text{(IV.1)}$$

But $\zeta \gg \epsilon \gg e^{-1/\epsilon}$, $\Rightarrow \zeta e^{1/\epsilon} \to +\infty$, and therefore $\ln(\zeta e^{1/\epsilon}) \to +\infty$. However, since $\zeta \to 0$, $\ln\zeta \to -\infty$. This contradicts the result (IV.1), and $\therefore$ $\epsilon \ln \not\to \infty$.

b) $\epsilon \ln \zeta \neq O(1)$

Assume $\epsilon \ln \zeta = A + o(1)$, A is real, $A \neq 0$.

Then $\zeta = \exp\left(\frac{A}{\epsilon}(1+o(1))\right)$ and $\zeta << 1 \Rightarrow A < 0$.

However $\zeta >> \epsilon \Rightarrow A > 0 \Rightarrow$ contradiction and $A \equiv 0$.

Hence $\epsilon \ln \zeta = o(1)$.

Now for $\epsilon > 0$, $\zeta \leq x < \infty$,

$$1 + \epsilon \ln x \geq 1 + \epsilon \ln \zeta$$

and therefore, since $\zeta >> \epsilon$,

$$\left| \epsilon \int_\zeta^\infty \frac{e^{-x}\,dx}{x\left(1 + \epsilon\left(\ln x - \frac{i b_{13}\kappa}{x}\right)\right)} \right| \leq \frac{\epsilon}{\zeta} \int_\zeta^\infty \frac{e^{-x}\,dx}{1 + \epsilon \ln \zeta} = \frac{\epsilon}{\zeta}\ \frac{(1 + O(\zeta))}{(1 + \epsilon \ln \zeta)} = o(1)$$

Similarly,

$$\left| \epsilon \int_\zeta^\infty \frac{\ln x\, e^{-x}\,dx}{1 + \epsilon\left(\ln x - \frac{i b_{13}\kappa}{x}\right)} \right| \leq \frac{\epsilon}{1+\epsilon \ln\zeta} \left\{ - \int_\zeta^1 \ln x\, e^{-x} dx + \int_1^\infty \ln x\, e^{-x}\, dx \right\}$$

$$= \frac{\epsilon}{1+\epsilon \ln \zeta} \left\{ \gamma + 2\,E_1(1) + o(1) \right\} = O(\epsilon)\ .$$

Thus for $\epsilon > 0$,

$$\int_\zeta^\infty \frac{e^{-x}}{1 + \epsilon\left(\ln x - \frac{i b_{13}\kappa}{x}\right)} = 1 + o(1)\quad .$$

Case II: $\epsilon < 0$

Divide the regions of integration into two: $\zeta \leq x \leq e^{-1/2\epsilon}$ and $e^{-1/2\epsilon} \leq x < \infty$. For $\epsilon < 0$ and $\zeta \leq x \leq e^{-1/2\epsilon}$,

$$1 + \epsilon \ln x \geq 1 + \epsilon \ln e^{-1/2\epsilon} = \tfrac{1}{2}\ .$$

Now

$$\left| \epsilon \int_{\zeta}^{e^{-1/2\epsilon}} \frac{e^{-x}\, dx}{x\left(1+\epsilon(\ell n x - \frac{i b_{13}{}^{\kappa}}{x})\right)} \right| \leq \frac{|\epsilon|}{\zeta} \frac{1}{(\frac{1}{2})} (1 + o(1)) = o(1)$$

and

$$\left| \epsilon \int_{e^{-1/2\epsilon}}^{\infty} \frac{e^{-x}}{x\left(1+\epsilon(\ell n x - \frac{i b_{13}{}^{\kappa}}{x})\right)} \right| \leq \int_{e^{-1/2\epsilon}}^{\infty} \frac{e^{-x}}{b_{13}{}^{\kappa}}\, dx = o(1)$$

Also

$$\left| \epsilon \int_{\zeta}^{e^{-1/2\epsilon}} \frac{\ell n x\, e^{-x}\, dx}{\left(1+\epsilon(\ell n x - \frac{i b_{13}{}^{\kappa}}{x})\right)} \right| \leq 2|\epsilon| \left\{ -\int_{\zeta}^{1} \ell n x\, e^{-x} dx + \int_{1}^{e^{-1/2\epsilon}} \ell n x\, e^{-x}\, dx \right\}$$

$$\leq 2|\epsilon| \left\{ \gamma + 2\, E_1(1) + o(1) \right\} = O(\epsilon)$$

and

$$\left| \epsilon \int_{e^{-1/2\epsilon}}^{\infty} \frac{\ell n x\, e^{-x}\, dx}{\left(1+\epsilon(\ell n x - \frac{i b_{13}{}^{\kappa}}{x})\right)} \right| \leq \int_{e^{-1/2\epsilon}}^{\infty} \frac{x \ell n x\, e^{-x}\, dx}{b_{13}{}^{\kappa}} = o(1)$$

Thus for $\epsilon < 0$,

$$\int_{\zeta}^{\infty} \frac{e^{-x}\, dx}{1+\epsilon\left(\ell n x - \frac{i b_{13}{}^{\kappa}}{x}\right)} = 1 + o(1)$$

and therefore, for $|\epsilon| \to 0$,

$$-i \int_{\zeta \eta_{\zeta}}^{\infty} {}_{1}\tau_{12}\, d\tilde{\eta} = \left(\frac{u_{*0}}{\epsilon}\right)^2 \left\{ (2b_{13}{}^{\kappa} + 2\tilde{\alpha} - \alpha)\epsilon^2 + o(\epsilon^2) \right\} .$$

Appendix V: A Note on the Critical Layer

It is important to distinguish between high Reynolds number laminar flow and fully turbulent flow over a sinusoidal boundary with small slope. For the former case, the O(ka) mean vorticity equation can be written as (Benjamin, 1959)

$$U\left(\frac{d^2S}{d\eta^2} - k^2S\right) - \frac{d^2U}{d\eta^2}\,S =$$

$$-\frac{i\nu}{\kappa}\left\{\frac{d^4S}{d\eta^4} - 2k^2\frac{d^2S}{d\eta^2} + k^4S + \left(\frac{1}{\kappa}\frac{d^4U}{d\eta^4} - 2\frac{d^3U}{d\eta^3}\right)e^{-k\eta}\right\}$$

which is recognized as essentially the Orr-Sommerfeld equation, written in (ξ, η) coordinates. Solutions for the high Reynolds number laminar case have been given by Miles (1957, 1959) and Benjamin (1959). The dynamics are governed by the inviscid terms (Rayleigh equation) except at the critical layer, where the Rayleigh equation is singular and viscous stresses are important, and at the boundary, where viscous stresses must be included to satisfy the no-slip condition.

However, for the fully turbulent case, the molecular stresses are unimportant outside the viscous sublayer, and the O(ka) mean vorticity equation is, using Saffman's turbulence model,

$$U\left(\frac{d^2S}{d\eta^2} - k^2S\right) - \frac{d^2U}{d\eta^2}S = -\frac{i}{k}\left(\frac{d^2\,{}_1\tau_{12}}{d\eta^2} + k^2\,{}_1\tau_{12}\right)$$

$$+ 2\frac{d}{d\eta}\,{}_1\tau_{11} + i\,4u_{*_0}^2\,e^{-k\eta}$$

where $U = \frac{u_{*_0}}{\kappa}\ln\left(\frac{\eta}{\eta_{*_0}}\right) - c$, and ${}_1\tau_{12}$, ${}_1\tau_{11}$ are the complex amplitudes of the O(ka) turbulent Reynolds stresses (see (3.10)).

It is not reasonable to assume a priori that the turbulent Reynolds stresses are unimportant except at the critical layer vis-a-vis the laminar case. The critical layer is a pertinent mathematical characteristic of the Orr-Sommerfeld equation; for our analysis involving the limit of vanishing skin friction, it held no particular significance. This result is not unique to our investigation. For example, Townsend (1972) indicated in his numerical study of fully turbulent flow over a sinusoidal boundary that the critical layer held no special importance, either. In fact, for many of his calculations, the critical layer lay outside his range of numerical integration, lying between the wavy surface and his inner boundary conditions which were taken at a position outside the viscous sublayer.

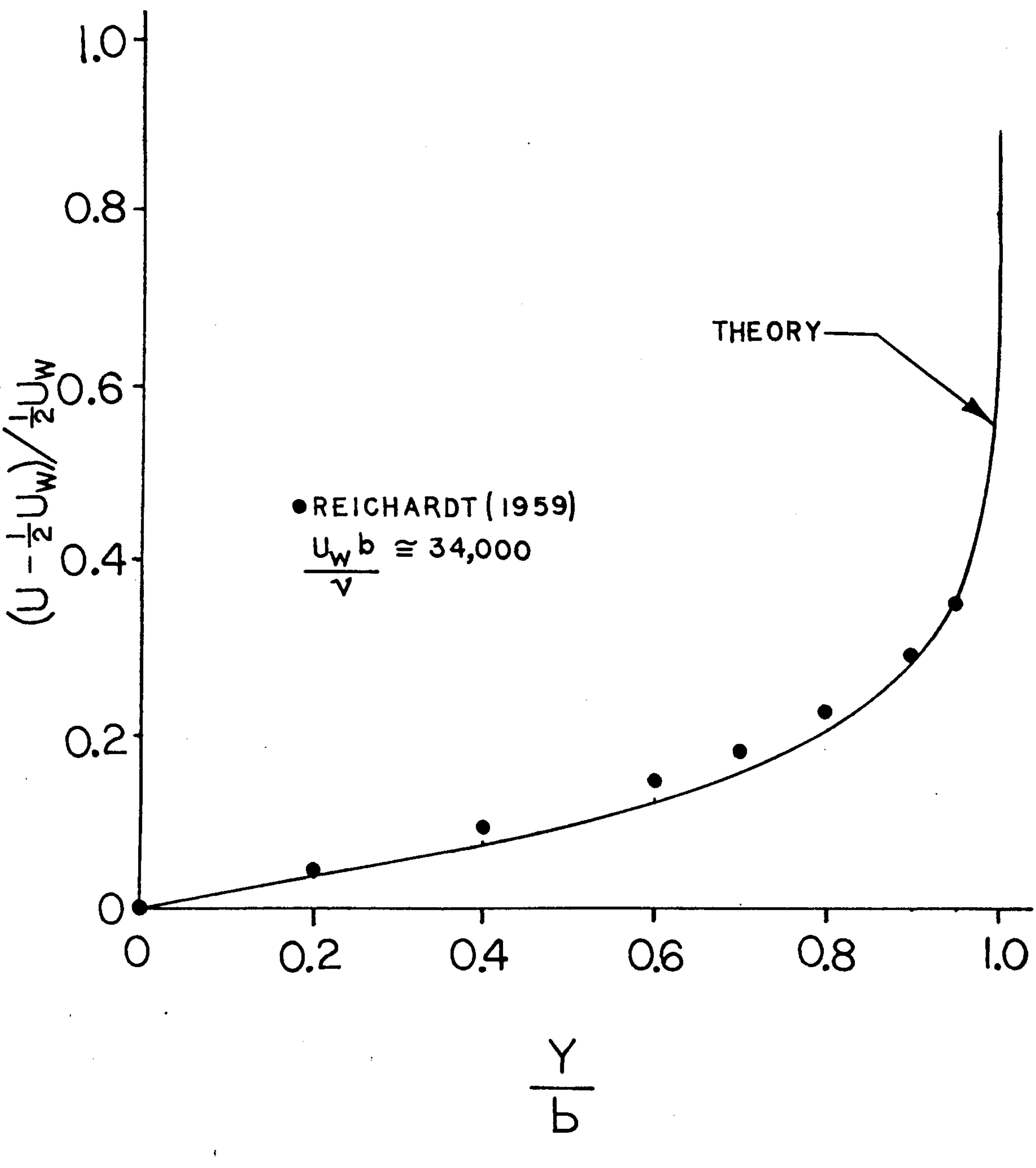
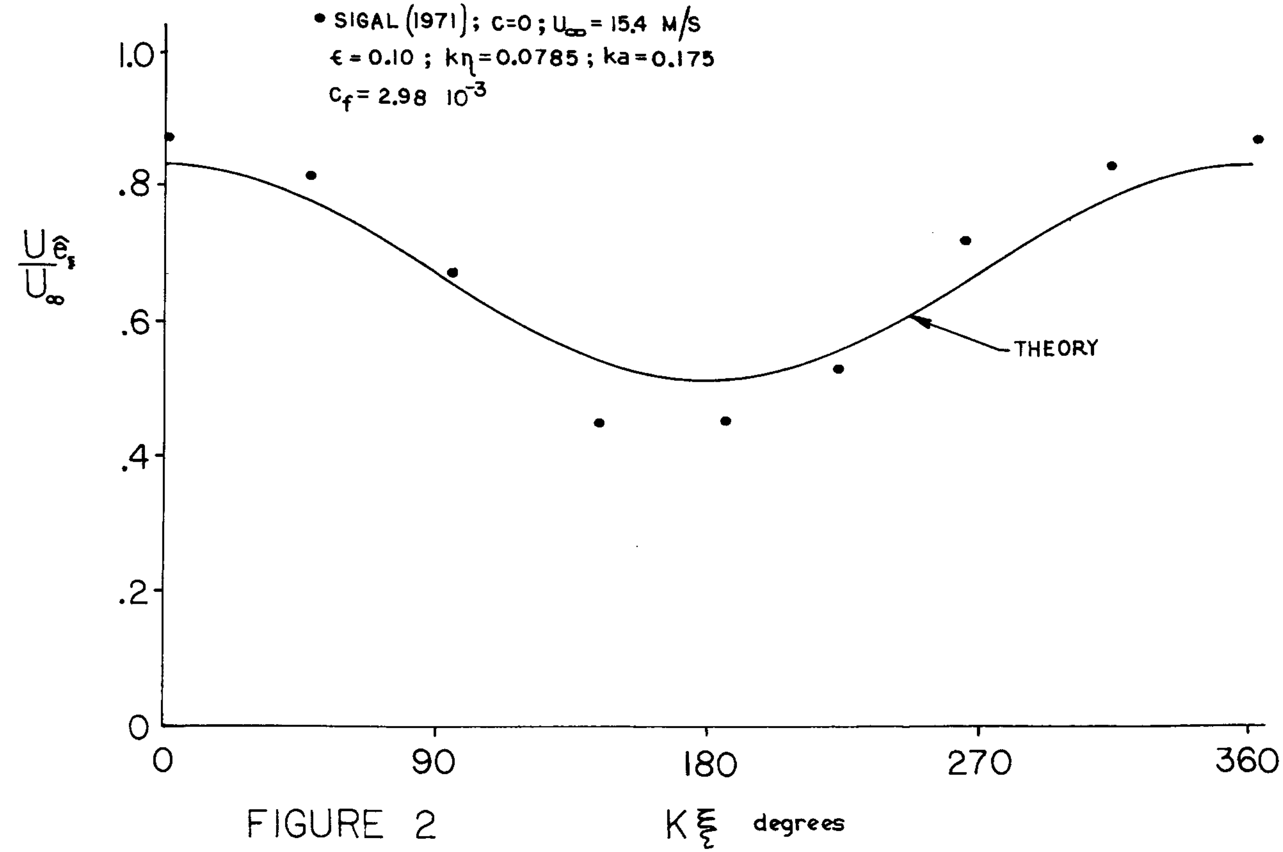
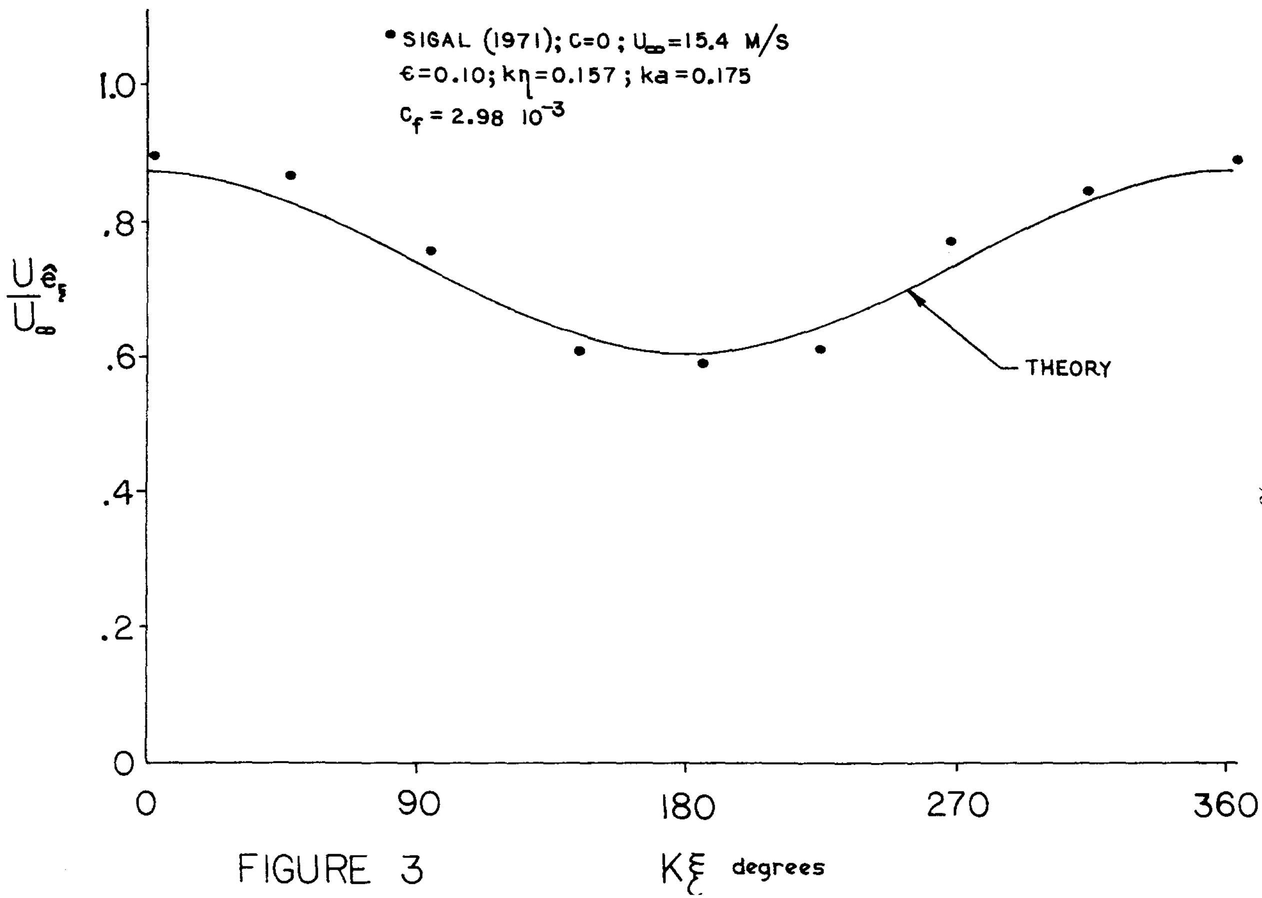

1.0
0.8
0.6
0.4
0.2
0
$(U - \frac{1}{2}U_W)/\frac{1}{2}U_W$
THEORY
● REICHARDT (1959)
$\frac{U_W b}{\nu} \cong 34{,}000$
0
0.2
0.4
0.6
0.8
1.0
$\frac{Y}{b}$

FIGURE 1



• SIGAL (1971); C=0; U_∞ = 15.4 M/S
ϵ = 0.10 ; $k\eta$ = 0.0785 ; ka = 0.175
c_f = 2.98 10^{-3}
THEORY
$\frac{U\hat{e}_\xi}{U_\infty}$
1.0
.8
.6
.4
.2
0
90
180
270
360
$K\xi$ degrees

FIGURE 2



• SIGAL (1971); C=0 ; U_∞=15.4 M/S
ϵ=0.10; $k\eta$=0.157 ; ka =0.175
c_f = 2.98 10^{-3}
THEORY
$\frac{U\hat{e}_\xi}{U_\infty}$
1.0
.8
.6
.4
.2
0
0
90
180
270
360
$K\xi$ degrees

FIGURE 3

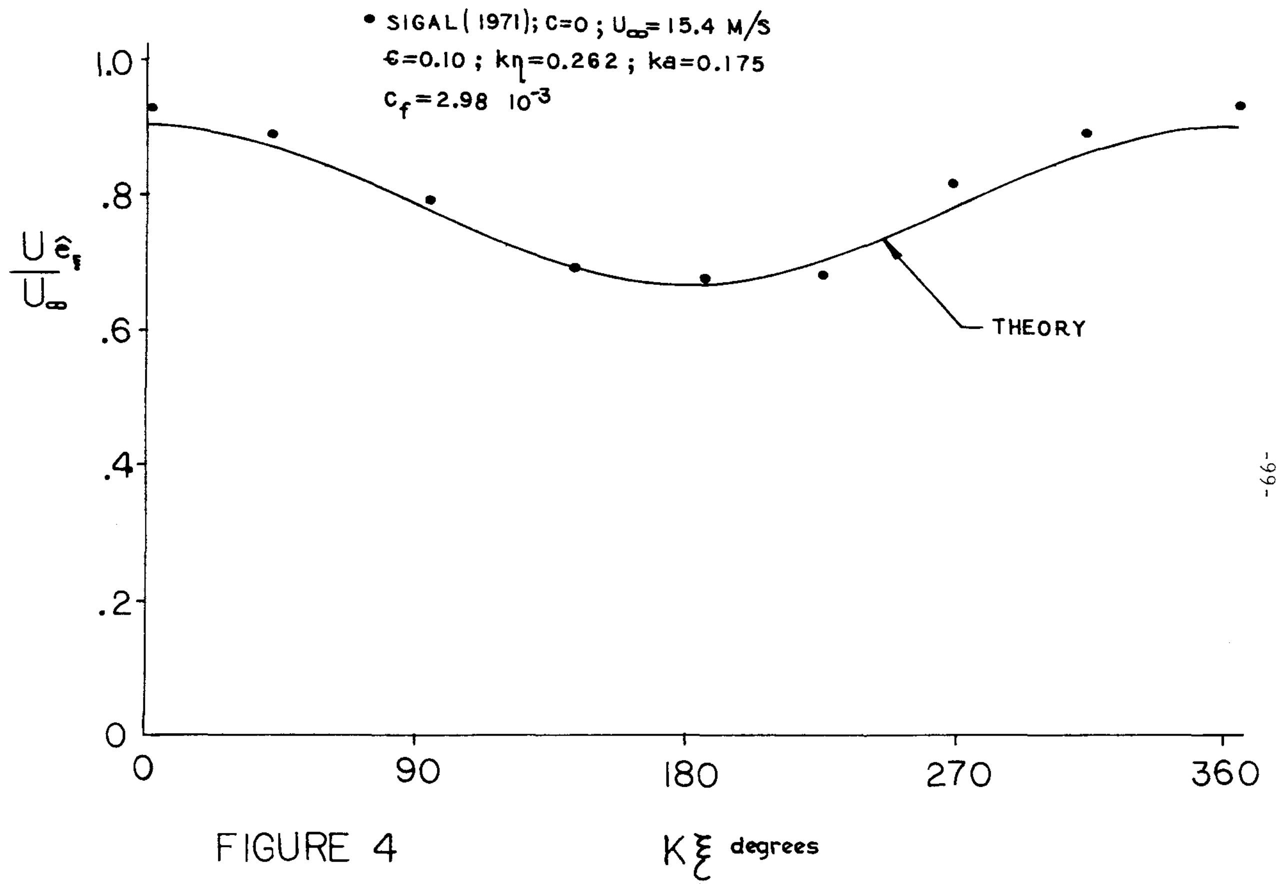

• SIGAL (1971); C=0 ; U_∞= 15.4 M/S
ϵ=0.10 ; $k\eta_l$=0.262 ; ka=0.175
c_f=2.98 10^{-3}
THEORY
$\frac{U\hat{e}_\xi}{U_\infty}$
1.0
.8
.6
.4
.2
0
0
90
180
270
360
$K\xi$ degrees

FIGURE 4

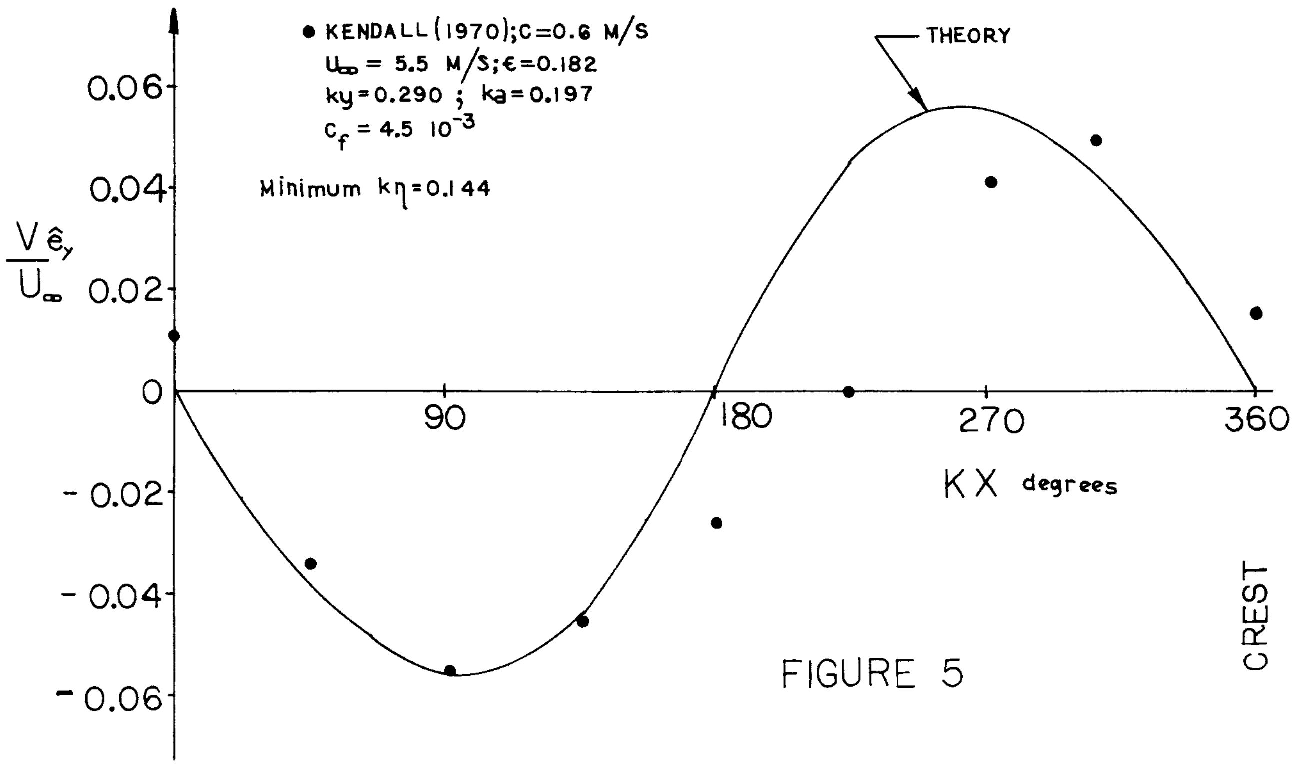

● KENDALL (1970); C=0.6 M/S
U_∞ = 5.5 M/S; ϵ=0.182
ky=0.290 ; ka=0.197
c_f = 4.5 10^{-3}
Minimum kη=0.144
THEORY
$\frac{V\hat{e}_y}{U_\infty}$
0.06
0.04
0.02
0
- 0.02
- 0.04
- 0.06
90
180
270
360
KX degrees
CREST

FIGURE 5

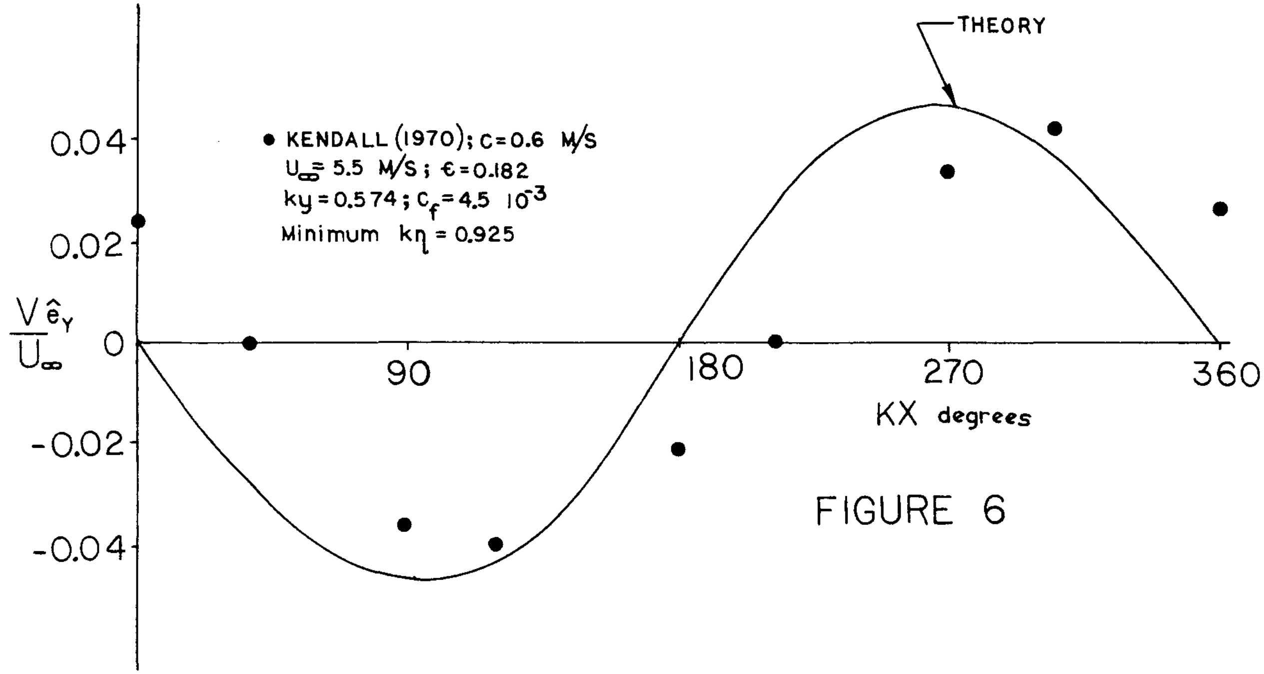
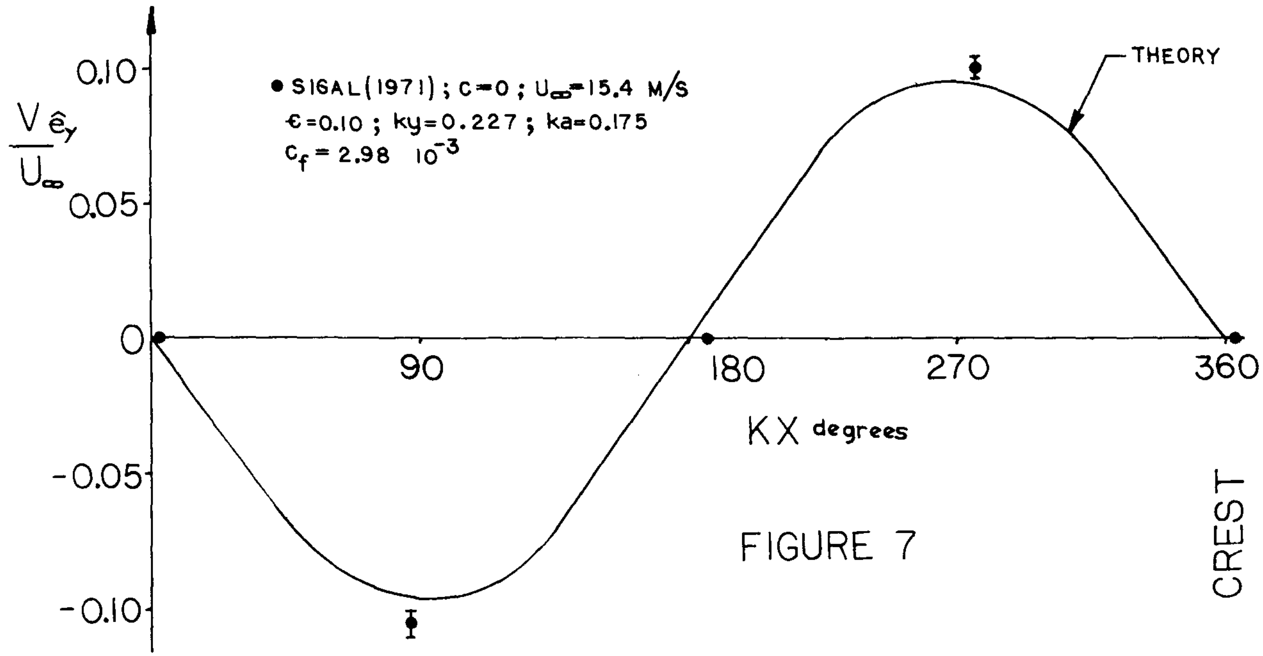

THEORY
• KENDALL (1970); c=0.6 M/S
U∞= 5.5 M/S ; ϵ=0.182
ky=0.574 ; cf=4.5 10-3
Minimum kη = 0.925
0.04
0.02
0
-0.02
-0.04
V ê_Y / U∞
90
180
270
360
KX degrees

FIGURE 6



● SIGAL(1971); C=0; U∞=15.4 M/S
ε=0.10; ky=0.227; ka=0.175
cf = 2.98 10^-3
THEORY
V ê_y / U∞
0.10
0.05
0
-0.05
-0.10
90
180
270
360
KX degrees
CREST

FIGURE 7

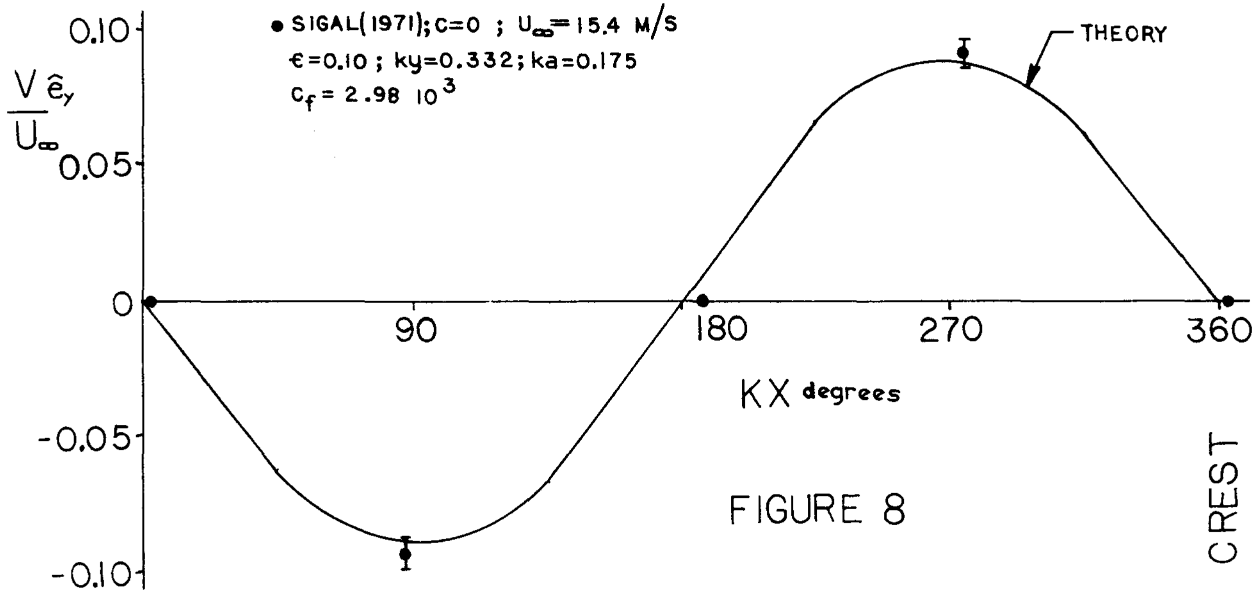
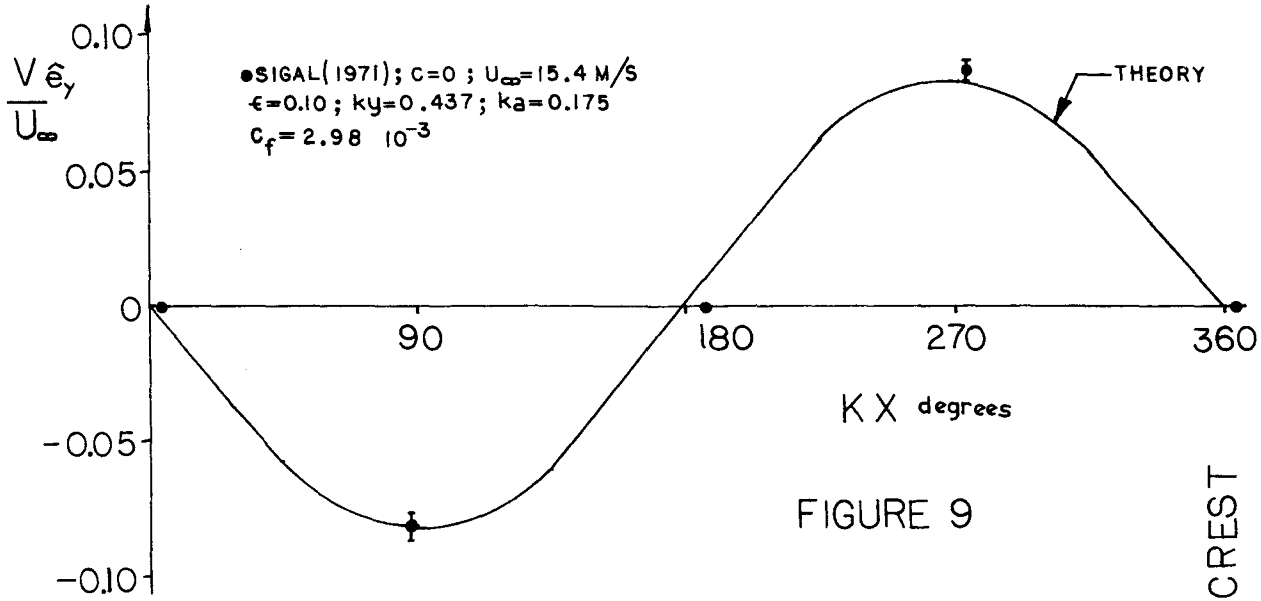

● SIGAL(1971); C=0 ; U_∞=15.4 M/S
ϵ=0.10 ; ky=0.332 ; ka=0.175
c_f = 2.98 10^3
THEORY
$\frac{V\hat{e}_y}{U_\infty}$
0.10
0.05
0
-0.05
-0.10
90
180
270
360
KX degrees
CREST

FIGURE 8



$\frac{V \hat{e}_y}{U_\infty}$
0.10
0.05
0
-0.05
-0.10
●SIGAL(1971); C=0 ; U_∞=15.4 M/S
ϵ=0.10 ; ky=0.437 ; ka=0.175
c_f=2.98 10^{-3}
THEORY
90
180
270
360
KX degrees
CREST

FIGURE 9

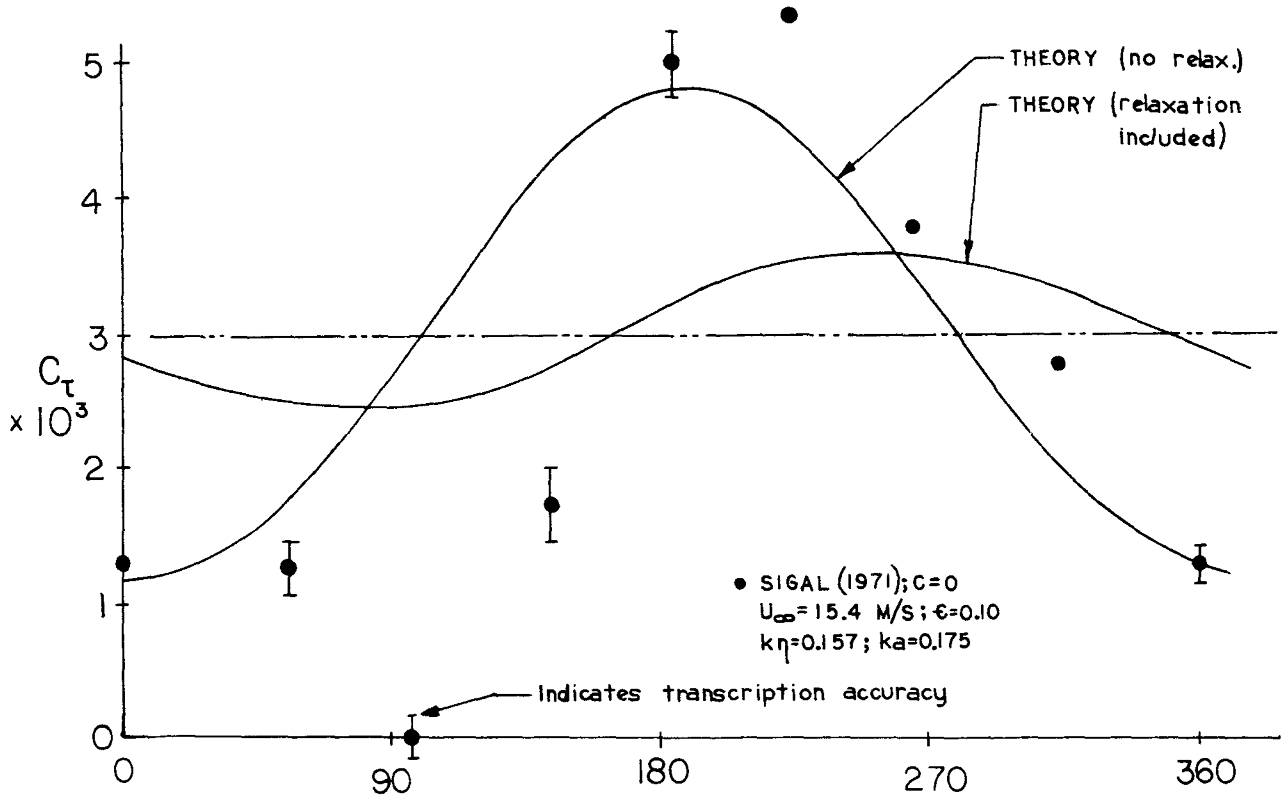

THEORY (no relax.)
THEORY (relaxation included)
$C_\tau \times 10^3$
5
4
3
2
1
0
0
90
180
270
360
• SIGAL (1971); C=0
U_∞= 15.4 M/S ; ε=0.10
kη=0.157 ; ka=0.175
Indicates transcription accuracy
Kξ degrees

FIGURE 10

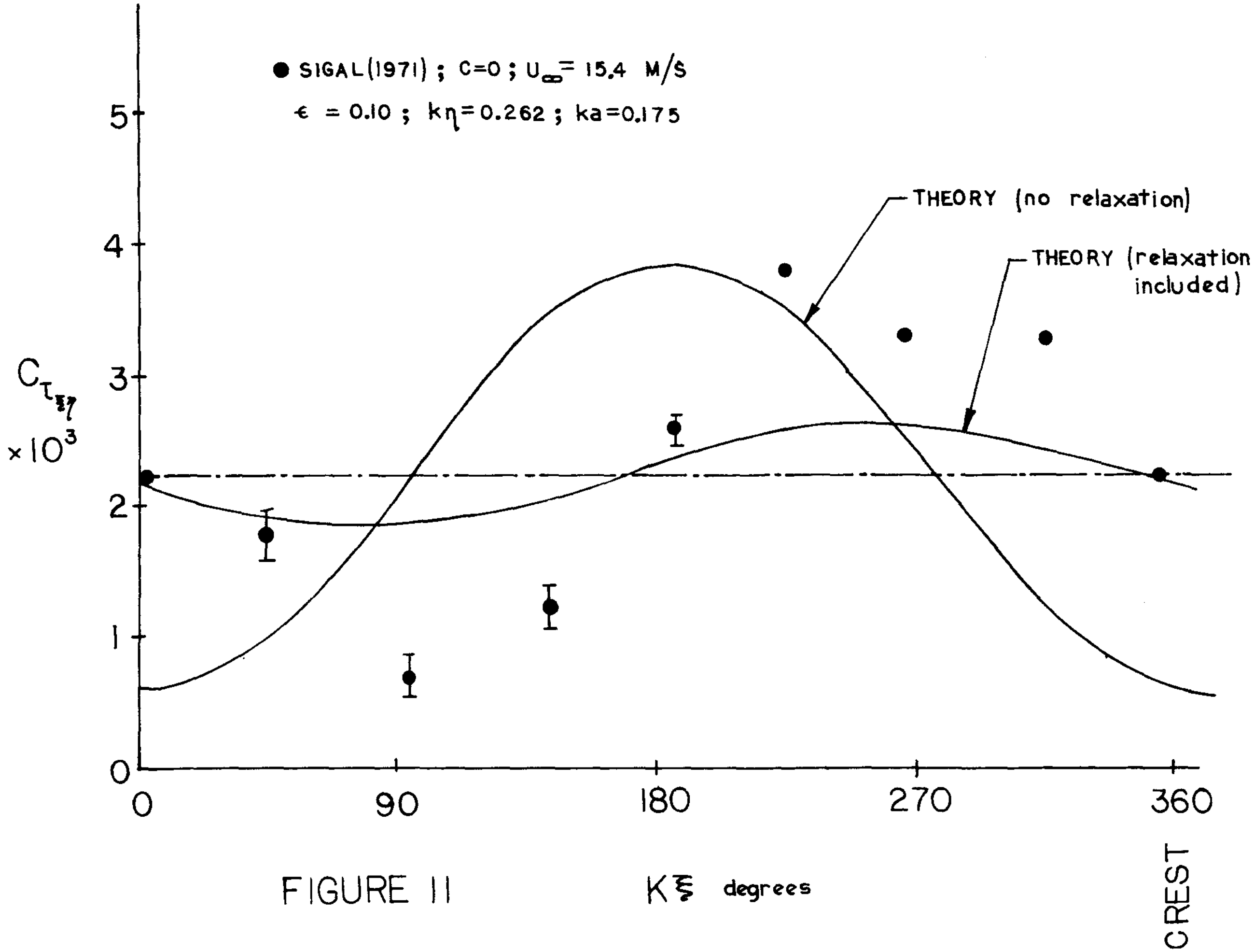

● SIGAL(1971) ; C=0 ; U_∞= 15.4 M/S
ϵ = 0.10 ; $k\eta$=0.262 ; ka=0.175
THEORY (no relaxation)
THEORY (relaxation included)
$C_{\tau_{\xi\eta}} \times 10^3$
0
1
2
3
4
5
0
90
180
270
360
CREST
$K\xi$ degrees

FIGURE 11

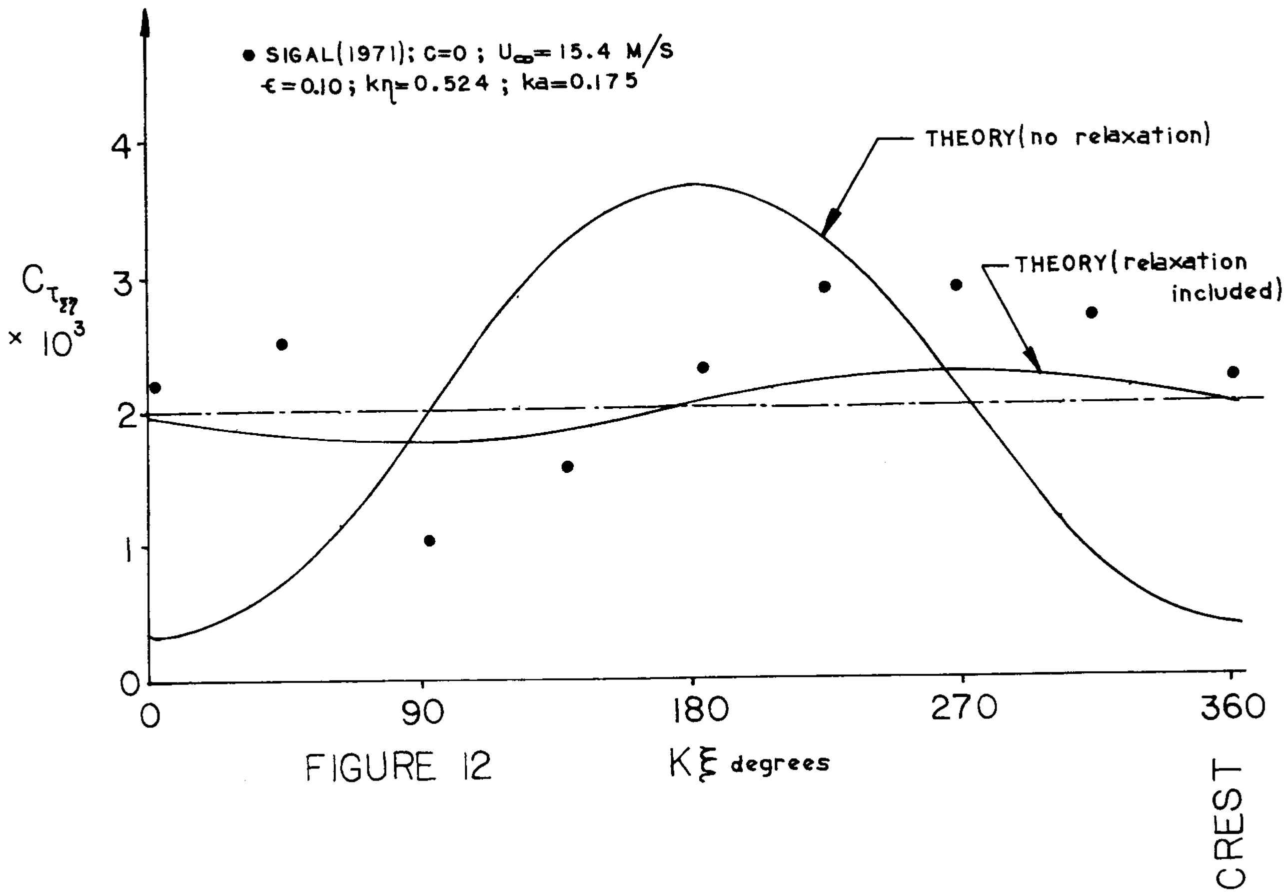
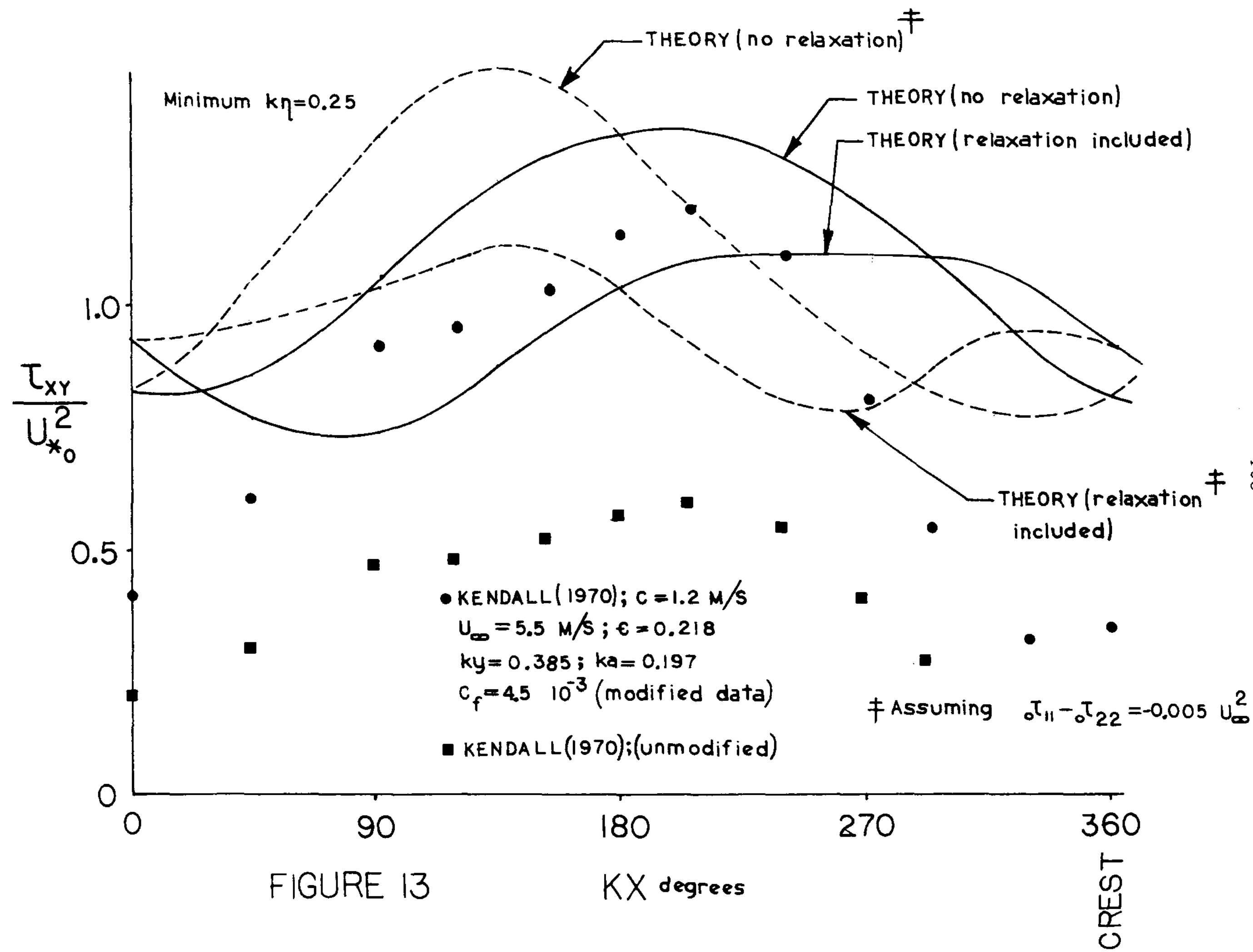

• SIGAL(1971); C=0 ; U_∞= 15.4 M/S
ϵ=0.10 ; $k\eta$=0.524 ; ka=0.175
THEORY(no relaxation)
THEORY(relaxation included)
$C_{\tau_{\xi\eta}} \times 10^3$
4
3
2
1
0
0
90
180
270
360
$K\xi$ degrees
CREST

FIGURE 12



THEORY (no relaxation) ‡
Minimum kη=0.25
THEORY (no relaxation)
THEORY (relaxation included)
1.0
0.5
0
$\frac{\tau_{XY}}{U_{*o}^2}$
THEORY (relaxation included) ‡
● KENDALL (1970); C = 1.2 M/S
U∞ = 5.5 M/S ; ε = 0.218
ky = 0.385 ; ka = 0.197
Cf = 4.5 10⁻³ (modified data)
■ KENDALL (1970); (unmodified)
‡ Assuming $_o\tau_{11} - {_o\tau_{22}} = -0.005\ U_\infty^2$
0
90
180
270
360
CREST
KX degrees

FIGURE 13

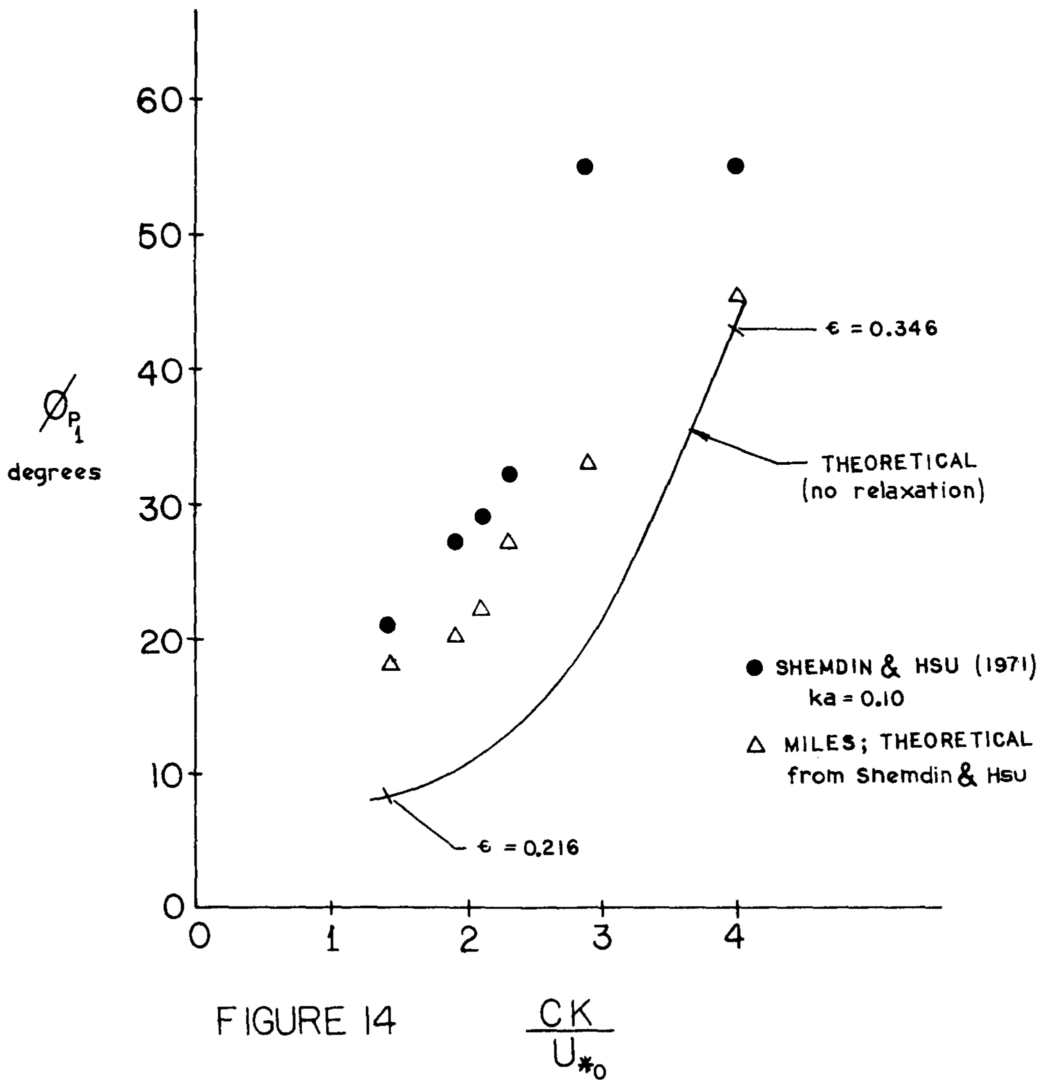

60
50
40
30
20
10
0
0
1
2
3
4
ϕ_{P_1}
degrees
ϵ = 0.346
THEORETICAL
(no relaxation)
ϵ = 0.216
SHEMDIN & HSU (1971)
ka = 0.10
MILES; THEORETICAL
from Shemdin & Hsu
$\frac{CK}{U_{*0}}$

FIGURE 14

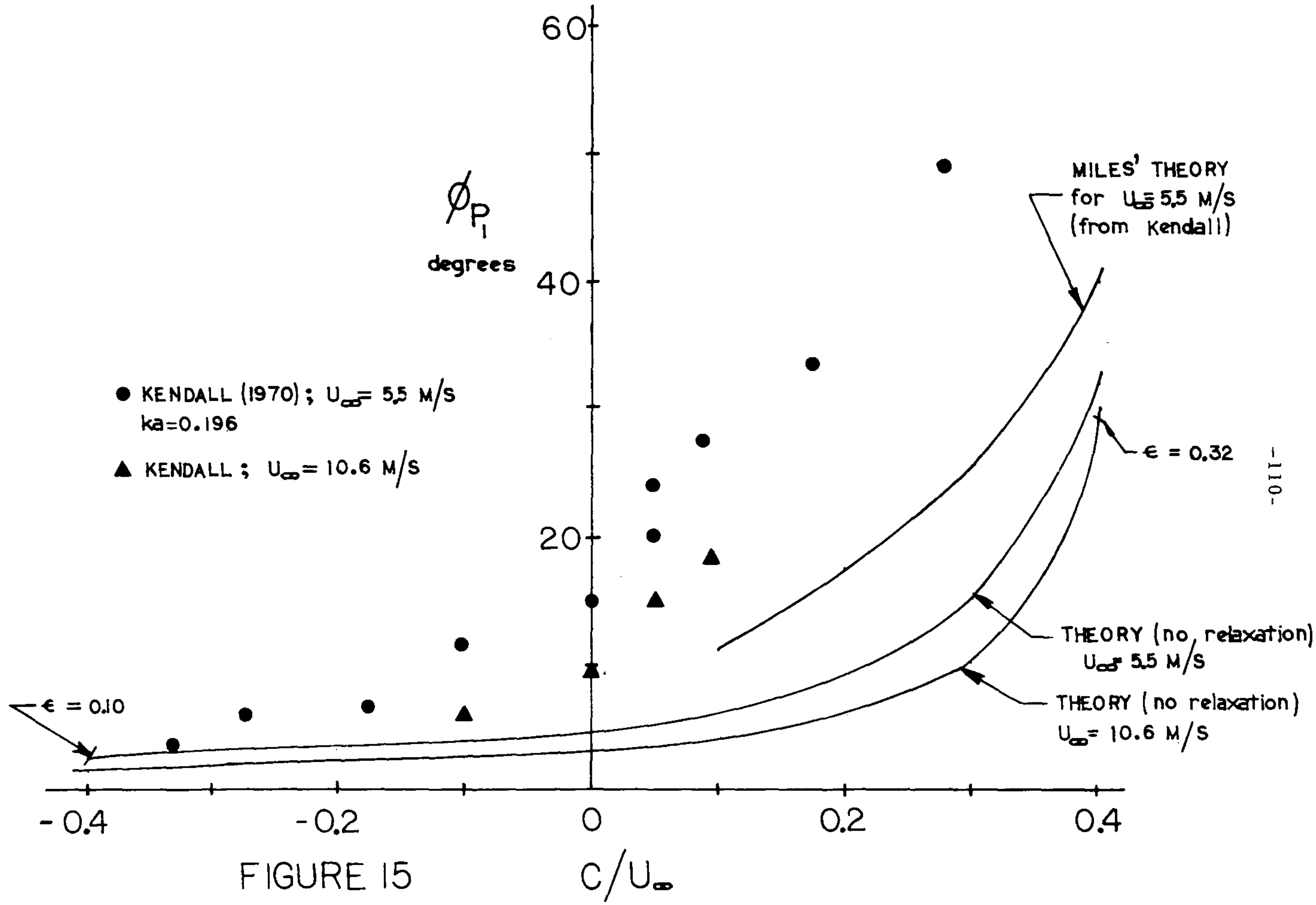
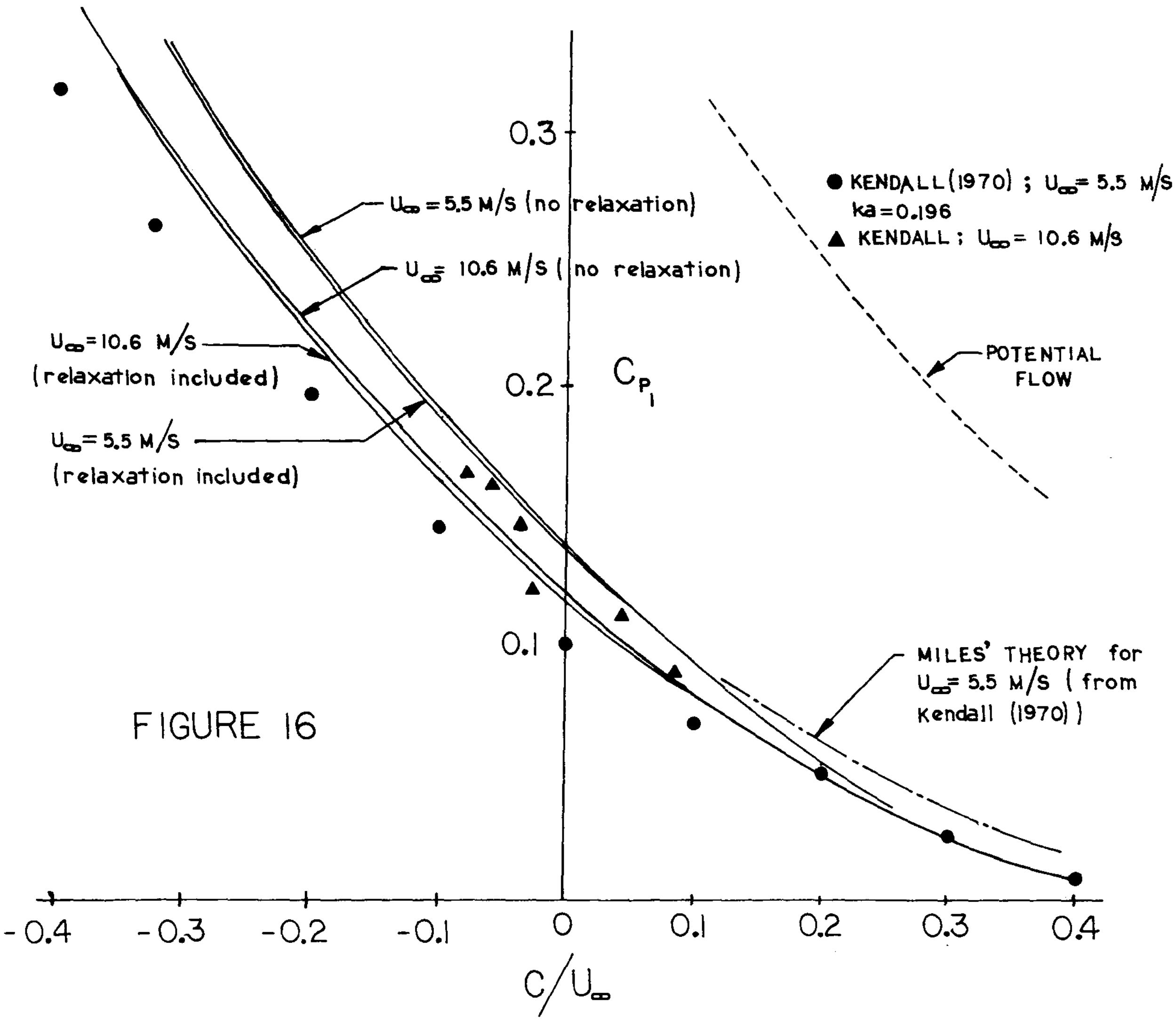

60
40
20
ϕ_{P_1}
degrees
● KENDALL (1970) ; U_∞= 5.5 M/S
ka=0.196
▲ KENDALL ; U_∞= 10.6 M/S
MILES' THEORY
for U_∞= 5.5 M/S
(from Kendall)
ϵ = 0.32
THEORY (no relaxation)
U_∞= 5.5 M/S
THEORY (no relaxation)
U_∞= 10.6 M/S
ϵ = 0.10
-0.4
-0.2
0
0.2
0.4
C/U_∞

FIGURE 15



KENDALL (1970) ; U_∞ = 5.5 M/S
ka = 0.196
KENDALL ; U_∞ = 10.6 M/S
POTENTIAL FLOW
U_∞ = 5.5 M/S (no relaxation)
U_∞ = 10.6 M/S (no relaxation)
U_∞ = 10.6 M/S (relaxation included)
U_∞ = 5.5 M/S (relaxation included)
MILES' THEORY for U_∞ = 5.5 M/S (from Kendall (1970))
C_{P_1}
0.3
0.2
0.1
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
C/U_∞

FIGURE 16

Bibliography

Benjamin, T. Brooke (1959). "Shearing Flow Over a Wavy Boundary," Journal of Fluid Mechanics, 6, 2.

Bradshaw, P., Ferris, D. H., and Atwell, N. P. (1967). "Calculation of Boundary-Layer Development Using the Turbulent Energy Equation," Journal of Fluid Mechanics, 28, 3.

Bradshaw, P. (1973). Effects of Streamline Curvature on Turbulent Flow. Agardograph No. 169. Advisory Group for Aerospace Research and Development, North Atlantic Treaty Organization.

Cole, Julian D. (1968). Perturbation Methods in Applied Mathematics. Blaisdell (Ginn and Company).

Coles, Donald E. (1956). "The Law of the Wake in the Turbulent Boundary Layer," Journal of Fluid Mechanics, 1, 2.

Coles, Donald E., and Hirst, E. A. (1968). Proceedings of the 1968 AFOSR-IFP-Stanford Conference on Computation of Turbulent Boundary Layers, Volume 2, Stanford University.

Davis, Russ E. (1972). "On Prediction of the Turbulent Flow Over a Wavy Boundary," Journal of Fluid Mechanics, 52, 2.

Goldstein, Sydney (1938). Modern Developments in Fluid Dynamics. The Clarendon Press, Oxford.

Govindaraju, S. P. (1970). "A Study of Some Turbulent Flows Using a Model for Inhomogeneous Turbulence," Ph.D. Thesis, Part II, California Institute of Technology, Pasadena, California.

Hinze, J. O. (1959). Turbulence. McGraw-Hill, p. 488.

Bibliography (Cont'd)

Kendall, James M. (1970). "The Turbulent Boundary Layer Over a Wall with Progressive Surface Waves," Journal of Fluid Mechanics, 41, 2

Kinsman, Blair (1965). Wind Waves. Prentice-Hall, Inc., page 11.

Lagerstrom, P. A., and Casten, R. G. (1972). "Basic Concepts Underlying Singular Perturbation Techniques," SIAM Review, 14, 1

Lamb, H. (1945). Hydrodynamics. 6th edition, Dover, § 268, 348, 349.

Miles, J. W. (1957). "On the Generation of Surface Waves by Shear Flows," Journal of Fluid Mechanics, 3, 2.

____________ (1959a). "On the Generation of Surface Waves by Shear Flows, Part 2," Journal of Fluid Mechanics, 6, 4.

____________ (1959b). "On the Generation of Surface Waves by Shear Flows, Part 3: Kelvin-Helmholtz Instability," Journal of Fluid Mechanics, 6, 4.

____________ (1967). "On the Generation of Surface Waves by Shear Flows, Part 5," Journal of Fluid Mechanics, 30, 1.

Reichardt, H. (1959). "Gesetzmäßigkeiten der Geradlinigen Turbulenten Couette Strömung," Mitteilungen aus dem Max-Planck-Institut für Stromung s forschung, Nr. 22, Göttingen.

Saffman, P. G. (1970). "A Model for Inhomogeneous Turbulent Flow," Proc. Roy. Soc. London A, 317.

____________ (1974). "Model Equations for Turbulent Shear Flow," Studies in Applied Mathematics, 53, 1.

Bibliography (Cont'd)

Shemdin, Omar H., and Hsu, En Yun (1967). "Direct Measurement of Aerodynamic Pressure Above a Simple Progressive Gravity Wave," Journal of Fluid Mechanics, 30, 2.

Sigal, A. (1971). "An Experimental Investigation of the Turbulent Boundary Layer Over a Wavy Wall," Ph. D. Thesis, California Institute of Technology, Pasadena, California.

Sokolnikoff, I. S. (1964). Tensor Analysis. J. Wiley and Sons, Inc.

Stewart, R. H. (1970). "Laboratory Studies of the Velocity Field over Deep Water Waves," Journal of Fluid Mechanics, 42, 4.

Stewart, R. W. (1961). "The Drag of Wind Over Water," Journal of Fluid Mechanics, 10, 2.

Townsend, A. A. (1972). "Flow in a Deep Turbulent Boundary Layer Over a Surface Distorted by Water Waves," Journal of Fluid Mechanics, 55, 4.