ON THE USE OF FIXED ATTITUDE PROPULSIVE ARCS FOR TRANSFER BETWEEN NEIGHBORING COPLANAR KEPLERIAN ORBITS

Thesis by

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ABSTRACT

The question of whether or not a coplanar, fixed attitude, constant acceleration burn in a central force field can result in a post-burn orbit that does not intersect the pre-burn orbit was investigated for small changes in the orbit. This was done by developing a criterion for separation between coplanar elliptical orbits, solving the linearized equations of motion given in reference 1 for a coplanar, fixed attitude, constant acceleration burn, and evaluating the criterion. Regions in which an initial alignment of the acceleration vector may be chosen to result in orbital separation at some point in the burn and the alignments of the acceleration vector to do this were found numerically. However, it appears that the distances between the separated orbits and the time over which the orbits separate for a particular alignment are too small to make the phenomenon of much practical use for small orbital changes.

TABLE OF CONTENTS

LIST OF SYMBOLS

a_o	semi-major axis of the Keplerian orbit
a	semi-major axis of the perturbed orbit
e_o	eccentricity of the Keplerian orbit
e	eccentricity of the perturbed orbit
Δe	$e-e_o$
P_o	semi-latus rectum of the Keplerian orbit
P	semi-latus rectum of the perturbed orbit
ΔP	$P-P_o$
θ_o	polar angle on the Keplerian orbit, measured from apoapsis in the direction of motion
θ_i	polar angle on the Keplerian orbit at which the perturbed orbit intersects the Keplerian orbit
θ	polar angle on the perturbed orbit
$\tilde{\theta}$	polar angle of the perturbed radius vector measured from apoapsis of the Keplerian orbit
θ_S	polar angle on the Keplerian orbit at the start of the burn
$\Delta\theta$	$\tilde{\theta}-\theta_o$
Δw	shift in the line of apsides between the Keplerian and perturbed orbits
$\vec{a}$	acceleration vector
$\lvert\vec{a}\rvert$	magnitude of $\vec{a}$
$\vec{r}_o$	position vector on the Keplerian orbit
r_o	magnitude of $\vec{r}_o$
$\vec{r}$	position vector on the perturbed orbit

r	magnitude of $\vec{r}$
v_o	magnitude of the velocity vector on the Keplerian orbit
v	magnitude of the velocity vector on the perturbed orbit
μ	gravitational constant
D	discriminant in the solution for $\cos\theta_i$; defined by equation (9)
$\hat{D}$	a non-dimensional form of D
δ	$1-\cos\Delta w$
C	variable in the solution for $\cos\theta_i$; defined by equation (16); also used as the angular momentum of the perturbed orbit
Ξ	function defined by equation (35)
Δr	$r-r_o$, perturbation in radius magnitude
$\Delta r'$	$d\Delta_r/d\theta_o$
Δv	$v-v_o$
$\Delta\theta'$	$d\Delta_\theta/d\theta_o$
$\hat{\Delta e}$	non-dimensional form of Δe
$\hat{\Delta P}$	non-dimensional form of ΔP
$\hat{\Delta w}$	non-dimensional form of Δw
$\hat{\Delta\theta}$	non-dimensional form of $\Delta\theta$
$\hat{\Delta\theta}'$	non-dimensional form of $\Delta\theta'$
ΔR	non-dimensional form of Δr
$\Delta R'$	non-dimensional form of $\Delta r'$
$I_{\Delta\theta}$	integral associated with $\Delta\theta$; defined by equation (65)
F_r	component of $\vec{a}$ along the radius
F_θ	component of $\vec{a}$ perpendicular to the radius
I_r	integral defined by equation (40)

$\hat{I}_r$ non-dimensional form of I_r

α alignment of the acceleration vector to the local horizontal at the start of the burn

α_o alignment of the acceleration vector that makes D = 0

α_m alignment of the acceleration vector that gives a minimum D

A $\cos(\alpha-\theta_S)$

B $\sin(\alpha-\theta_S)$

C_o angular momentum of the Keplerian orbit

γ_S flight path angle at the start of the burn

β $\theta_o-\theta_S$, angular movement of the position vector on the Keplerian orbit during the burn

x_o r_o/P_o

x_S r_o/P_o at the start of the burn

x r/P_o

F_{BS} function defined by equation (52)

F_{BC} function defined by equation (53)

F_{AC} function defined by equation (54)

F_{AS} function defined by equation (55)

$g_1, g_2, \ldots g_6$ functions in the expressions for $\Delta\hat{P}$, $\Delta\hat{e}$, and $\Delta\hat{w}$; given by equations (106) through (111)

d_1, d_2, d_3 functions used in an expression for $\hat{D}$; given by equations (115), (116), (117)

d $d_2^2-d_1d_3$

INTRODUCTION

The equation of motion of a body of negligible mass in orbit about a central body is

$$\ddot{\vec{r}}_o + \frac{\mu}{r_o^3}\vec{r}_o = 0$$

where μ = GM is a parameter of the central (or attracting body) and $\vec{r}_o$ is the position vector from the central body to the orbiting body. If there is a force on the orbiting body other than the central force field, the equation of motion is

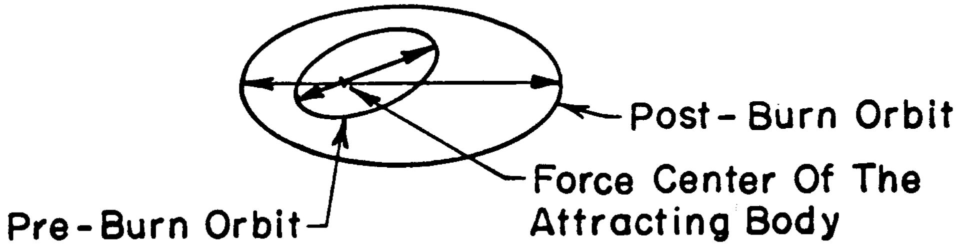
$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = \vec{a} \qquad (1)$$

where $m\vec{a}$ is the other force. In this paper, $\vec{a}$ is thought of as the thrust acceleration by a spacecraft since this study arises from my interest in the problem of spacecraft burns. However the results of the analysis are applicable to any $\vec{a}$ of the form considered here.

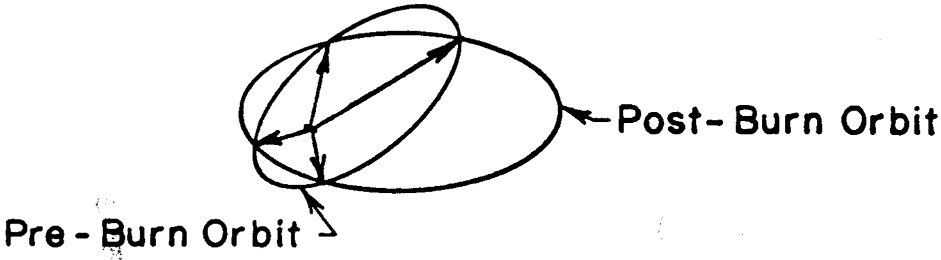
In general, equation (1) has not been solved in a closed form (although there are certain acceleration profiles for which a closed form solution may be obtained). The equation is usually handled numerically: guidance and steering equations are chosen for reasons particular to the problem being considered (e.g., to optimize a specific performance index or attain certain end conditions), and the equation numerically integrated, computing $\vec{a}$ from the instantaneous position and the guidance equations, to obtain numerical results. There may be many candidate guidance and steering equations that will work for a particular problem. Thus when considering which to use, it is natural to think of $\vec{a}$ as consisting of a magnitude,

$|\vec{a}|$, a pitch profile (motion of $\vec{a}$ in the plane of motion), and a yaw profile (motion of $|\vec{a}|$ out of the plane of motion). The simplest $\vec{a}$ has a constant inertial orientation (fixed attitude) since both the hardware and software implementation is less complex than for a variable pitch and/or yaw orientation. It would be uneconomical to use guidance and steering equations that provided a pitch and/or yaw capability when such capabilities are not needed. Thus the question about the types of orbits, with respect to the initial orbit, that result from a fixed attitude burn is of interest.

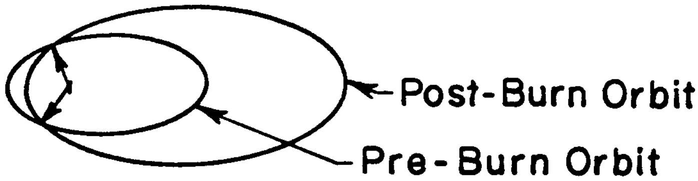
One question of this sort is whether or not the orbit resulting from the fixed attitude burn ever separates from the initial orbit, i.e., does not intersect, i.e., on no common line in space are the pre- and post-burn orbital radii equal (see fig. 1). This has some practical as well as purely academic interest. For example, small burns usually are simulated by impulsive velocity changes at a particular position, and a fixed attitude burn is found to obtain the post-impulse orbital elements (extensive experience has shown that this is easily done, for both large and small velocity changes). By this method, one spacecraft separating from another would require two burns in order to insure no recontact after one orbit. If a fixed attitude burn could be found to sufficiently separate the orbits, the second burn could be avoided. Another example is the problem of trajectory dispersions from nominal orbits. If it is desired or necessary to burn where the target (or nominal) orbit does not intersect the spacecraft orbit so that no impulsive simulation of the burn is possible, can the burn be done to the target orbit with a fixed



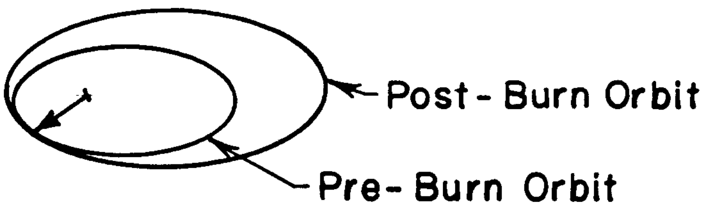
(a) Separation, Or No Intersection



(b) Intersection, Four Places



(c) Intersection, Two Places



(d) Intersection, One Place › Tangency

Figure 1

attitude?

I became interested in this question of orbital separation while working for the Manned Spacecraft Center in Houston, Texas, on the targeting for the Apollo lunar orbit insertion maneuver. To determine the capability of the guidance to flyout trajectory dispersions, extensive numerical studies of lunar orbit insertion burns (which were quite large - about 6 minutes in length, covering between 20° and 25° of central angle) were done. They indicated that fixed attitude burns for this situation not only gave intersecting pre- and post-burn orbits but also, for a small plane change, that one of these intersection points occurred at or very near the intersection of the pre- and post-burn planes, which occurred at some point in the burn arc (see figure 2). Variable pitch attitude burns gave unequal radii at this intersection of the pre- and post-burn planes (figure 2). This naturally raises the question of the generality of this phenomenon and provides the motivation for this thesis.

To start the study of this question of separation or intersection with a fixed attitude burn, a brief numerical search was made for coplanar burns. It was decided to study coplanar burns since not only is this the most restrictive condition under which separation might be found, but also one degree of freedom is eliminated from the problem. Burns were approximated by a series of impulsive velocity changes in the same inertial direction for acceleration vector alignments near the velocity vector for different places on the initial orbit. The rationale for acceleration vector alignments near the velocity vector was due to qualitative physical intuition.

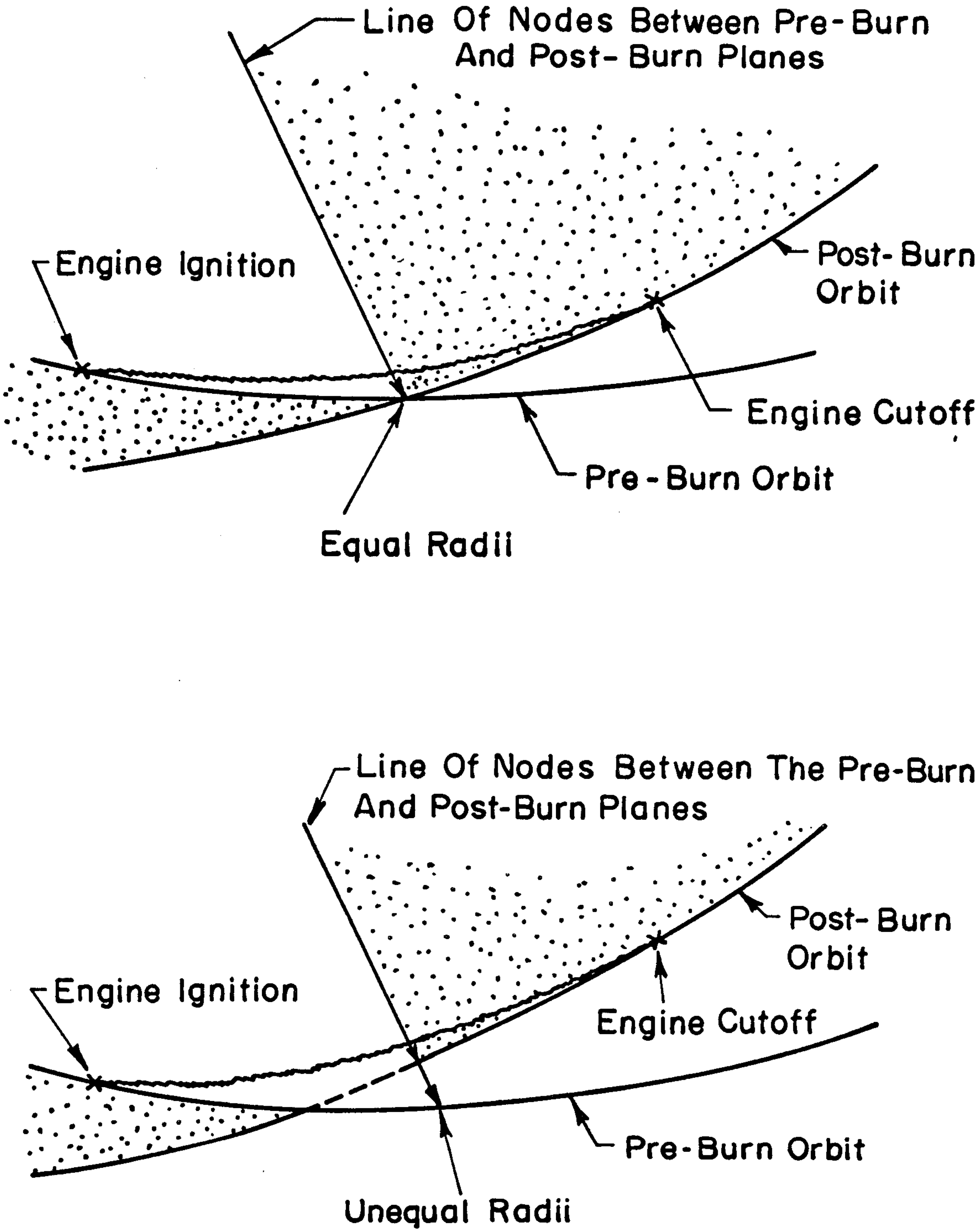
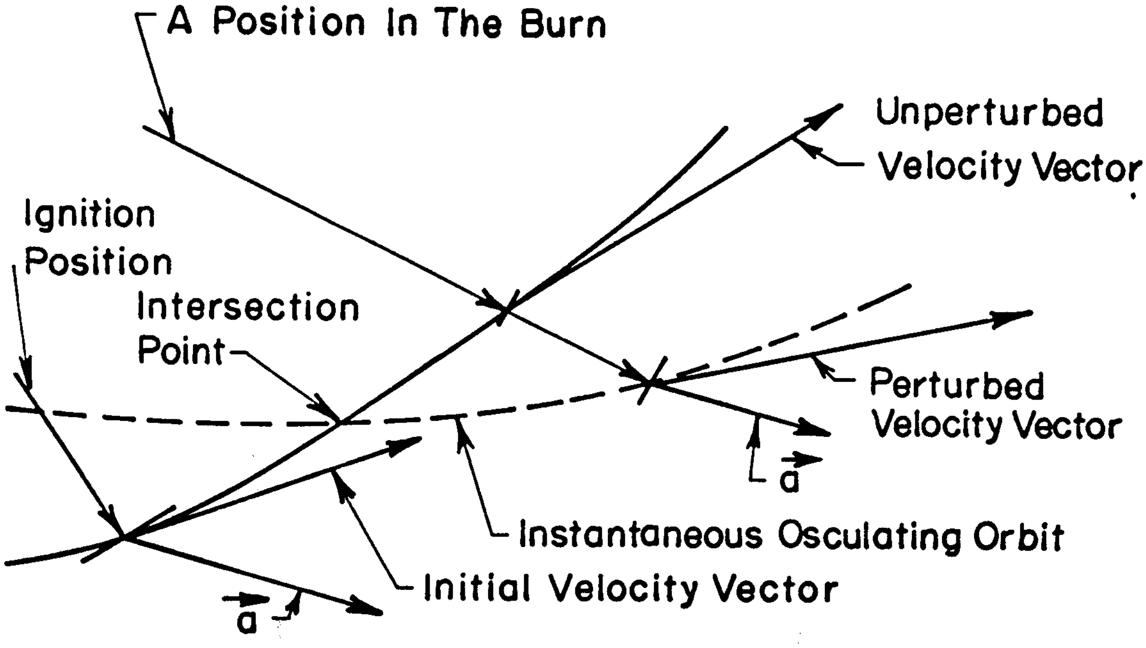

Line Of Nodes Between Pre-Burn
And Post-Burn Planes
Engine Ignition
Post-Burn
Orbit
Engine Cutoff
Pre-Burn Orbit
Equal Radii
Line Of Nodes Between The Pre-Burn
And Post-Burn Planes
Engine Ignition
Post-Burn
Orbit
Engine Cutoff
Pre-Burn Orbit
Unequal Radii

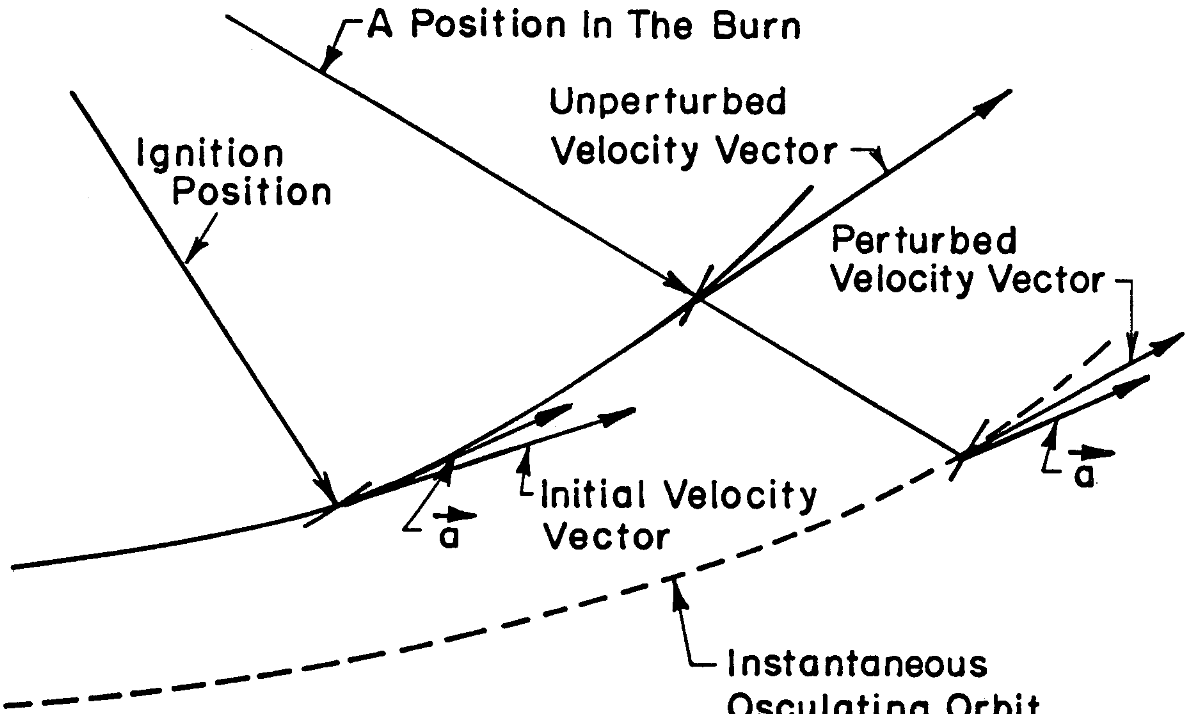
Figure 2

The thinking was that if the acceleration was aligned far from the velocity vector, the change in velocity vector orientation, i. e., flight path angle, would be large; this would cause the pre- and post-burn orbits to intersect (see fig. 3). However if the thrust vector were aligned near the velocity vector (probably below since in a local vertical coordinate system attached to the instantaneous position vector, the constant inertial acceleration vector would pitch up as the burn progressed), there would not be a drastic change in flight path angle. Thus the perturbed radius could become different in size than the initial orbit radius before a drastic change in flight path angle forced intersection (fig. 3). A criterion was developed (see the analysis portion of this paper) to determine when two orbits intersected, given their orbital elements. The pre- and post-burn orbits in this brief initial study were tested with this criterion. No separation was found. The nearest the orbits came to separating was for very small burn arcs. This suggested that calculations for small changes in the orbital elements be performed. Consequently, it was decided to do this study in the linearized limit.

The main objective was to determine if the pre- and post-burn orbits ever separated for a fixed attitude burn in the linearized limit, and if so, what the conditions for a separation were. The simplest situation was selected: the burn would be coplanar (no out of plane component of the acceleration vector), and the acceleration magnitude would be constant. The linearization and constant $|\vec{a}|$ restrictions are not quite as restrictive as it might first appear. The concept of variation of parameters can be applied to extend the



(a) Intersection



(b) No Intersection

Figure 3

theory. If there is a finite region where separation occurs (i.e., a range of some orbital parameters and/or alignments where separation occurs), then slight changes in the orbital elements should remain in that region. Thus an actual burn could result in orbital separation, even for non-linear changes in orbital elements if the separation region is large enough. Since in linear theory the changes in orbital elements should be proportional to $|\vec{a}|$, separation regions should not be a function of $|\vec{a}|$ (except that $|\vec{a}|$ must be small enough to make the linear analysis valid). Thus a variable acceleration burn could cause changes in orbital elements that remained in a separation region for a finite time.

ANALYSIS

General Criterion for Orbital Separation

The first step in determining whether or not two orbits are separated is the development of a criterion for separation. Consider two elliptical coplanar orbits as shown in figure 4 whose lines of apsides are Δw apart. The initial orbit has the subscript o. Elements of both orbits are known; the unknown factor is the intersection polar angle, the angular position on the orbit measured in the direction of motion from apoapsis.

For intersection

$$r = r_o$$

or

$$\frac{P}{1-e\cos\theta} = \frac{P_o}{1-e_o\cos\theta_i}$$

If $e_o \neq 0$, this may be solved for $\cos\theta_i$, the cosine of the polar angle in the initial orbit.

$$\cos\theta_i = \frac{1}{e_o} - \frac{P}{Pe_o} + \frac{P_o e\cos\theta}{Pe_o} \tag{2}$$

Now the polar angles on the initial and final orbits are related by

$$\theta_i = \theta + \Delta w \tag{3}$$

Note here for completeness that the sign of Δw is unimportant since the two orbits could be interchanged. Algebraically, the sign on Δw will be squared later. Using (3), $\cos\theta$ may be written in terms of θ_i and Δw. Substituting for $\cos\theta$ in (2) gives

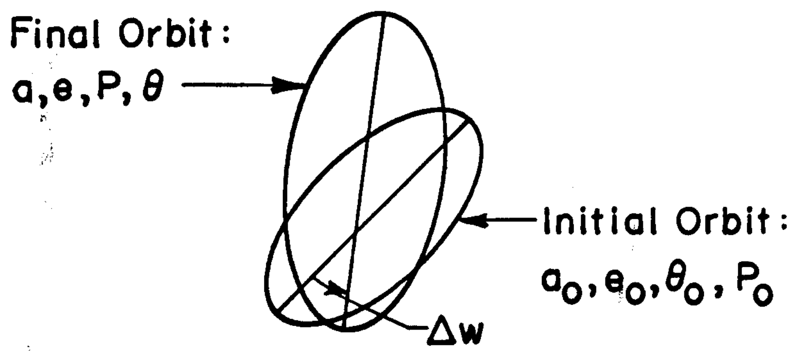


Figure 4

$$\cos\theta_i = \frac{1}{e_o} - \frac{P}{Pe_o} + \frac{P_o e}{Pe_o} (\cos\theta_i \cos\Delta w + \sin\theta_i \sin\Delta w)$$

If $e \neq 0$, this may be rearranged to give

$$\cos\theta_i \left(\frac{Pe_o}{P_o e} - \cos\Delta w\right) - \frac{P}{P_o e} + \frac{1}{e} = +\sin\Delta w \sin\theta_i \qquad (4)$$

Define

$$C_1 = \frac{P}{P_o e} - \frac{1}{e} = \frac{P-P_o}{P_o e} \qquad (5)$$

$$C_2 = \frac{Pe_o}{P_o e} - \cos\Delta w = \frac{Pe_o - P_o e\cos\Delta w}{P_o e} \qquad (6)$$

and square equation (4) to give

$$C_2^2 \cos^2\theta_i - 2C_1 C_2 \cos\theta_i + C_1^2 = \sin^2\Delta w(1-\cos^2\theta_i)$$

or

$$(C_2^2+\sin^2\Delta w)\cos^2\theta_i - 2C_1 C_2 \cos\theta_i + (C_1^2 - \sin^2\Delta w) = 0 \qquad (7)$$

which may be solved for $\cos\theta_i$ by the quadratic formula

$$\cos\theta_i = \frac{C_1 C_2 \pm \sqrt{C_1^2 C_2^2 - (C_2^2+\sin^2\Delta w)(C_1^2 - \sin^2\Delta w)}}{C_2^2 + \sin^2\Delta w} \qquad (8)$$

Note that at the start of the burn, $\cos\theta_i$ has a 0/0 form. Physically, this means that the two orbits coincide and there is an infinite number of values for θ_i. At some point into the burn, equation (8) will give four values for θ_i.

Using (5) and (6) and some algebra, the quantity under the radical may be written as

$$\frac{\sin^2\Delta w}{P_o^2\, e^2}\, D$$

where D is defined as

$$D = P^2 e_o^2 + P_o^2 e^2 - (P-P_o)^2 - 2P_o P e_o e\cos\Delta w \tag{9}$$

Thus (8) becomes

$$\cos\theta_i = \frac{C_1 C_2 \pm \frac{\sin\Delta w}{P_o e}\sqrt{D}}{C_2^2 + \sin^2\Delta w} \tag{10}$$

D can be written in terms of the initial orbit parameters and the differences between the initial and final orbits. Let

$$\begin{aligned} P &= P_o + \Delta P \\ e &= e_o + \Delta w \\ \cos\Delta w &= 1 - \delta \end{aligned} \tag{11}$$

Then using (11) in (9) and algebra, D may be written

$$D = (e_o\Delta P - P_o\Delta e)^2 - \Delta P^2 + 2P_o^2 e_o^2 \delta +$$

$$2P_o e_o\, \delta(P_o\Delta e + e_o\Delta P + \Delta P\Delta e) \tag{12}$$

which is good for any two elliptical coplanar orbits.

Now the criterion for separation is seen in equation (10): if (10) cannot be solved for θ_i, then there is separation. This can occur when the expression for $\cos\theta_i$ is complex. Thus

$$D < 0 \tag{13}$$

is sufficient for a separation. However, if the absolute value of the right-hand side of (10) is greater than 1 for both signs on the radical and for D > 0, then it again is impossible to solve for θ_i. Therefore

$$\frac{|C_1 C_2| + \left|\frac{\sin\Delta w}{P_o e}\right| \sqrt{D}}{C_2^2+\sin^2\Delta w} > 1 \tag{14}$$

also is sufficient for a separation.

Equation (14) may be considerably simplified. First write (8) as

$$\cos\theta_i = C \pm \sqrt{\tilde{D}} \tag{15}$$

where

$$C = \frac{C_1 C_2}{C_2^2+\sin^2\Delta w}$$

$$\tilde{D} = [C_1^2 C_2^2 - (C_2^2+\sin^2\Delta w)(C_1^2-\sin^2\Delta w)]/(C_2^2+\sin^2\Delta w)^2 \tag{16}$$

$\tilde{D}$ may be written as

$$\tilde{D} = \frac{C_1^2 C_2^2}{(C_2^2+\sin^2\Delta w)^2} - \frac{(C_1^2-\sin^2\Delta w)}{C_2^2+\sin^2\Delta w} \tag{17}$$

Now from (16) for $C \neq 0$ (i.e., the two orbits do not coincide)

$$C_2^2 + \sin^2\Delta w = \frac{C_1 C_2}{C}$$

and

$$\sin^2\Delta w = \frac{C_1 C_2}{C} - C_2^2$$

Using these in (17) gives

$$\tilde{D} = C^2 - \left(\frac{C_1}{C_2} + \frac{C_2}{C_1}\right) C + 1 \tag{18}$$

Define

$$f_{12} = \frac{C_1}{C_2} + \frac{C_2}{C_1} = C_{1/2} + \frac{1}{C_{1/2}}$$

There are two important facts about f_{12}. First write f_{12} as

$$f_{12} = \frac{C_1^2+C_2^2}{C_1C_2} = \frac{C_1^2+C_2^2}{(C_2^2+\sin^2\Delta w)} \frac{1}{C}$$

which shows that

$$\mathrm{Sign}(f_{12}) = \mathrm{Sign}(C) \tag{19}$$

and thus

$$-f_{12}C < 0$$

Second

$$\frac{df_{12}}{dC_{1/2}} = 1 - \frac{1}{C_{1/2}^2}$$

shows that f_{12} has extremums at $C_{1/2} = \pm 1$. Therefore for $C < 0$,

$$f_{12\ max} = -2 \tag{20}$$

and for $C > 0$

$$f_{12\ min} = +2 \tag{21}$$

Now consider curves in $\tilde{D}$ - C space. Because f_{12} does not take in all values between $\pm\infty$ (the area between ± 2 is eliminated), (18) does not allow $\tilde{D}$ to take on all values in $\tilde{D}$ - C space. The maximum value $\tilde{D}$ can assume has $f_{12}C$ minimized so that $-f_{12}C$ is as algebraically large as possible. Therefore

$$\tilde{D}_{max} = C^2 + 1 - 2|C| \tag{22}$$

Now consider $C < 0$. Here,

$$\tilde{D}_{max} = C^2 + 1 + 2C = (C+1)^2 \tag{23}$$

Now (15) never allows $\cos\theta_i > + 1$ since the maximum $\cos\theta_i$ is

$$C + \sqrt{\tilde{D}_{max}} = C + \sqrt{(C+1)^2} = 2C+1 < + 1$$

Therefore for the absolute value of the right-hand side of (15) to be greater than 1 for both signs on the radical,

$$C + \sqrt{\tilde{D}} < - 1$$

which means that

$$C < -1$$

is necessary. But the maximum $\cos\theta_i$ was

$$\cos\theta_i = 2C + 1$$

For $C < -1$, this is

$$\cos\theta_i < -2+1 = -1$$

Therefore

$$C < -1 \tag{24}$$

is both necessary and sufficient for the absolute value of the right-hand side of (15) to be greater than 1 for both signs on the radical.

Now consider $C > 0$. Here the minimum $\tilde{D}$ is

$$\tilde{D}_{min} = C^2 + 1 - 2C = (C-1)^2 \tag{25}$$

Now (15) never allows $\cos\theta_i < -1$ since the minimum $\cos\theta_i$ is

$$C - \sqrt{\tilde{D}_{min}} = C - \sqrt{(C-1)^2}$$

$$= C - \sqrt{(1-C)^2} = 2C-1 > -1$$

Therefore for the absolute value of the right-hand side of (15) to be greater than 1 for both signs on the radical

$$C - \sqrt{\tilde{D}} > 1$$

which means that

$$C > 1$$

is necessary. But the minimum $\cos\theta_i$ was

$$\cos\theta_i = 2C - 1$$

For $C > 1$,

$$\cos\theta_i > 2-1 = 1$$

Therefore

$$C > 1 \tag{26}$$

is both necessary and sufficient for the absolute value of the right-hand side of (15) to be greater than 1 for both signs on the radical.

Equations (24) and (26) may be combined as

$$|C| > 1 \tag{27}$$

Summarizing, the criterion for separation is either that

$$D < 0 \tag{13}$$

or

$$|C| > 1 \tag{27}$$

$|C| > 1$ Criterion Evaluated for Small Perturbations

Now it will be shown that $|C|$ is always ≤ 1 in the linearized limit when $D > 0$. First, write

$$C = \frac{C_1 C_2}{C_2^2+\sin^2\Delta w}$$

$$= \frac{(Pe_o-P_o e\cos\Delta w)(P-P_o)}{(Pe_o-P_o e\cos\Delta w)^2+P_o^2 e^2\sin^2\Delta w} = \frac{(P-P_o)(Pe_o-P_o e\cos\Delta w)}{P^2 e_o^2+P_o^2 e^2-2PP_o e e_o\cos\Delta w}$$

using (5) and (6) for C_1 and C_2. Now use (11) to write C in terms of the changes in orbital elements. Substituting and combining terms gives

$$C = \frac{\Delta P(e_o\Delta P - P_o\Delta e + P_o e_o \delta + P_o \Delta e \delta)}{(e_o\Delta P - P_o\Delta e)^2 + 2P_o e_o \delta(P_o e_o + P_o\Delta e + e_o\Delta P + \Delta P\Delta e)} \tag{28}$$

From its definition in (11), δ is greater than or equal to zero and second order in Δw. Thus C consists of second and third order terms divided by second, third, and fourth order terms. Keeping only the second order terms gives

$$C = \frac{\Delta P(e_o\Delta P - P_o\Delta e)}{(e_o\Delta P - P_o\Delta e)^2 + 2P_o^2 e_o^2 \delta}$$

Now let

$$\tilde{C} = \frac{\Delta P(e_o\Delta P - P_o\Delta e)}{(e_o\Delta P - P_o\Delta e)^2} = \frac{\Delta P}{e_o\Delta P - P_o\Delta e} \tag{29}$$

If $|\tilde{C}| \leq 1$, then $|C| \leq 1$. Thus

$$|\Delta P| \leq |e_o\Delta P - P_o\Delta e| \tag{30}$$

is sufficient to keep $|C| \leq 1$.

Now linearizing D (equation (12)) gives

$$D = (e_o\Delta P - P_o\Delta e)^2 - \Delta P^2 + 2P_o^2 e_o^2 \delta \tag{31}$$

Therefore

$$(e_o\Delta P - P_o\Delta e)^2 + 2P_o^2 e_o^2 \delta = D + \Delta P^2$$

Therefore

$$C = \frac{\Delta P(e_o\Delta P - P_o\Delta e)}{D + \Delta P^2}$$

Since if D is less than 0, there is a separation, consider here D greater than 0. Let

$$\hat{C} = \frac{\Delta P(e_o \Delta P - P_o \Delta e)}{\Delta P^2} = \frac{e_o \Delta P - P_o \Delta e}{\Delta P} \tag{32}$$

If $|\hat{C}| \leq 1$, then $|C| \leq 1$. Thus

$$|e_o \Delta P - P_o \Delta e| \leq |\Delta P| \tag{33}$$

is sufficient to keep $|C| \leq 1$. Thus (30) and (33) show that in the linearized limit and with $D > 0$

$$|C| \leq 1$$

and therefore

$$D < 0$$

is both necessary and sufficient for a separation.

Available Linear Relations

The equations of motion (equation (1)) will not be linearized here; this has been done in ref. 1. Results from reference 1 applicable to this problem will be used here.

Let r and v be the radius and velocity magnitudes on the final orbit and let $\tilde{\theta}$ define the position of r in space. Let

$$\begin{aligned} r &= r_o + \Delta r \\ v &= v_o + \Delta v \\ \tilde{\theta} &= \theta_o + \Delta\theta \end{aligned} \tag{34}$$

where $\tilde{\theta}$ is measured from apoapsis of the initial orbit and is <u>not</u> the polar angle of the final orbit (fig. 5). Reference 1 derives linear relations for Δr, Δv, and $d\Delta\theta/d\theta_o$. This is done by writing (1) in spherical polar coordinates, considering $\vec{a}$ as made up of radial and tangential components, and linearizing the resulting equations. The pertinent equations are numbered (6.6), (6.7), (5.10), and (6.2) in reference 1 and are reproduced here as equations (36) through (39). Equations (36) through (39) are written in terms of r_o and Δr instead of $1/r_o$ and $\Delta(1/r_o)$ as done in reference 1. Define

$$\Xi(\theta_o) = F_r \frac{r_o^4}{C_o^2} + 2\frac{r_o^3}{C_o^2} \int_{\theta_S}^{\theta_o} (+\frac{dr_o}{d\theta_o} F_r + r_o F_\theta) d\theta \tag{35}$$

where F_r and F_θ are the radial and tangential components of $\vec{a}$ and C_o is the angular momentum of the initial orbit. Then the differential equation satisfied by Δr (equation (6.6) in reference 1) is

$$\Delta r'' + \Delta r = \Xi(\theta_o) \tag{36}$$

where the primes denote differentiation with respect to θ_o. With the initial conditions

$$\Delta r(\theta_S) = 0$$

$$\Delta r'(\theta_S) = 0$$

the solution to (36) can be written

$$\Delta r(\theta_o) = \int_{\theta_S}^{\theta_o} \sin(\theta_o - \theta)\, \Xi(\theta) d\theta \tag{37}$$

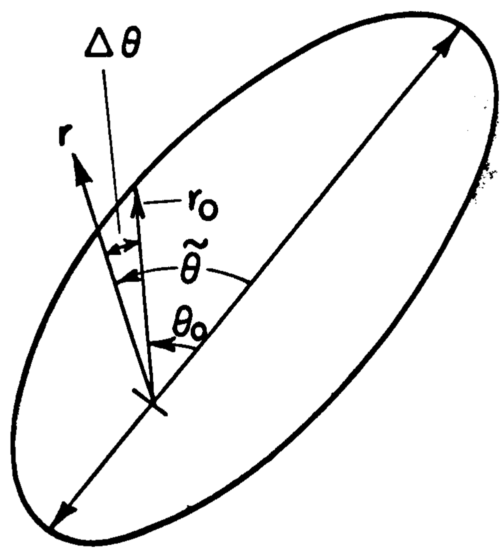


Figure 5

The per[illegible]t equation for Δv is

$$\ell \quad \frac{1}{2v_o}\dot{\theta}_o^2\{2r_o'\Delta r' \quad r_o\Delta r+2r_o^2\Delta\theta'\} \qquad (38)$$

and the [illegible]ation for Δθ' is

$$\frac{e_o\sin\theta_o}{P_o}\Delta \quad \frac{\mu r_o}{C_o^2}]$$

$$\frac{r_o^2}{C_o^2}\int_{\theta_S}^{\theta_o}(r_o'F_r \quad r \quad)d\theta \qquad (39)$$

Define f[illegible]onvenience

$$\int_{\theta_S}^{\theta_o}(r_o'F_r+r_oF \qquad (40)$$

[illegible]v (38) can b[illegible]lified by substituting (39) into (38) for Δθ' and [illegible]rrying out [illegible]$d\theta_o$. Seve[illegible]l terms cancel and using

$$= r_o^4\dot{\theta}_o^2$$

gives

$$= (-\mu\frac{\Delta r}{r_o^2} \quad r_o \qquad (41)$$

Equation ([illegible]) may be integrated with the initial condition that Δθ is zero at θ_S to obtain an expression for $\Delta\theta(\theta_o)$. Direct integration would require the integration of the expression for Δr. As will be seen later, the expression for Δr involves an arctangent. To avoid integrating this, the differential equation (equation (36)) is used.

$$\Delta\theta = \int_{\theta_S}^{\theta_o} \left(\frac{e_o \sin\theta}{P_o} \Delta r' - \frac{\Delta r}{r_o} - \frac{\mu \Delta r}{C_o^2} + \frac{r_o^2}{C_o^2} I_r\right) d\theta$$

Consider first $\int_{\theta_S}^{\theta_o} \left(\frac{e_o \sin\theta}{P_o} \Delta r' - \frac{\Delta r}{r_o}\right) d\theta$. Now $\int \frac{\Delta r}{r_o} d\theta$ can be evaluated by using (36) for Δr

$$\Delta r = \Xi - \Delta r''$$

Thus

$$-\int \frac{\Delta r}{r_o} d\theta = - \int \Xi \frac{1}{r_o} d\theta + \int \frac{\Delta r''}{r_o} d\theta$$

Integrating $\int \frac{\Delta r''}{r_o} d\theta$ by parts and using $\Delta r'(\theta_S) = 0$ gives

$$-\int \frac{\Delta r}{r_o} d\theta = - \int \Xi \frac{1}{r_o} d\theta + \frac{\Delta r'}{r_o} - \int \frac{e_o \sin\theta}{P_o} \Delta r' d\theta$$

Therefore

$$\int_{\theta_S}^{\theta_o} \left(\frac{e_o \sin\theta \Delta r'}{P_o} - \frac{\Delta r}{r_o}\right) d\theta = \frac{\Delta r'}{r_o} - \int_{\theta_S}^{\theta_o} \Xi \frac{1}{r_o} d\theta$$

Now the integral of Δr may be evaluated, again using (36).

$$\int_{\theta_S}^{\theta_o} \Delta r d\theta = \int_{\theta_S}^{\theta_o} (\Xi - \Delta r'') d\theta = - \Delta r' + \int_{\theta_S}^{\theta_o} \Xi \, d\theta$$

Thus

$$\Delta\theta = \frac{\Delta r'}{r_o} - \int_{\theta_S}^{\theta_o} \frac{\Xi}{r_o} d\theta - \frac{\mu}{C_o^2} \left[-\Delta r' + \int_{\theta_S}^{\theta_o} \Xi \, d\theta \right] + \frac{1}{C_o^2} \int_{\theta_S}^{\theta_o} r_o^2 I_r d\theta \tag{42}$$

Now equation (35) may be used to eliminate Ξ. Doing this and using

(40) gives

$$\Delta\theta(\theta_o) = \frac{\mu}{C_o^2}\Delta r' + \frac{\Delta r'}{r_o} - \frac{1}{C_o^2}\int_{\theta_S}^{\theta_o} r_o^2 I_r d\theta - \frac{2\mu}{C_o^4}\int_{\theta_S}^{\theta_o} r_o^3 I_r d\theta$$

$$- \frac{1}{C_o^2}\int_{\theta_S}^{\theta_o} r_o^3 F_r d\theta - \frac{\mu}{C_o^4}\int_{\theta_S}^{\theta_o} r_o^4 F_r d\theta \qquad (43)$$

Finally (37) may be differentiated with respect to θ_o to give an expression for $\Delta r'$.

$$\Delta r' = \int_{\theta_S}^{\theta_o} \cos(\theta_o - \theta)\, \Xi(\theta) d\theta \qquad (44)$$

It is now necessary to find F_r and F_θ for the fixed attitude, constant $|\vec{a}|$ burn. Let α be the angle between the acceleration vector and the local horizontal at ignition. This alignment angle is positive for $\vec{a}$ aligned above the local horizontal and negative for $\vec{a}$ aligned below the local horizontal. Then, referring to fig. 6

$$F_r(\theta_o) = |\vec{a}| \sin(\alpha + \theta_o - \theta_S)$$
$$F_\theta(\theta_o) = |\vec{a}| \cos(\alpha + \theta_o - \theta_S)$$

Expanding $\sin(\alpha - \theta_S + \theta_o)$ and $\cos(\alpha - \theta_S + \theta_o)$ and making the definitions

$$A = \cos(\alpha - \theta_S)$$
$$B = \sin(\alpha - \theta_S) \qquad (45)$$

gives

$$F_r = |\vec{a}| (A\sin\theta_o + B\cos\theta_o)$$
$$F_\theta = |\vec{a}| (A\cos\theta_o - B\sin\theta_o) \qquad (46)$$

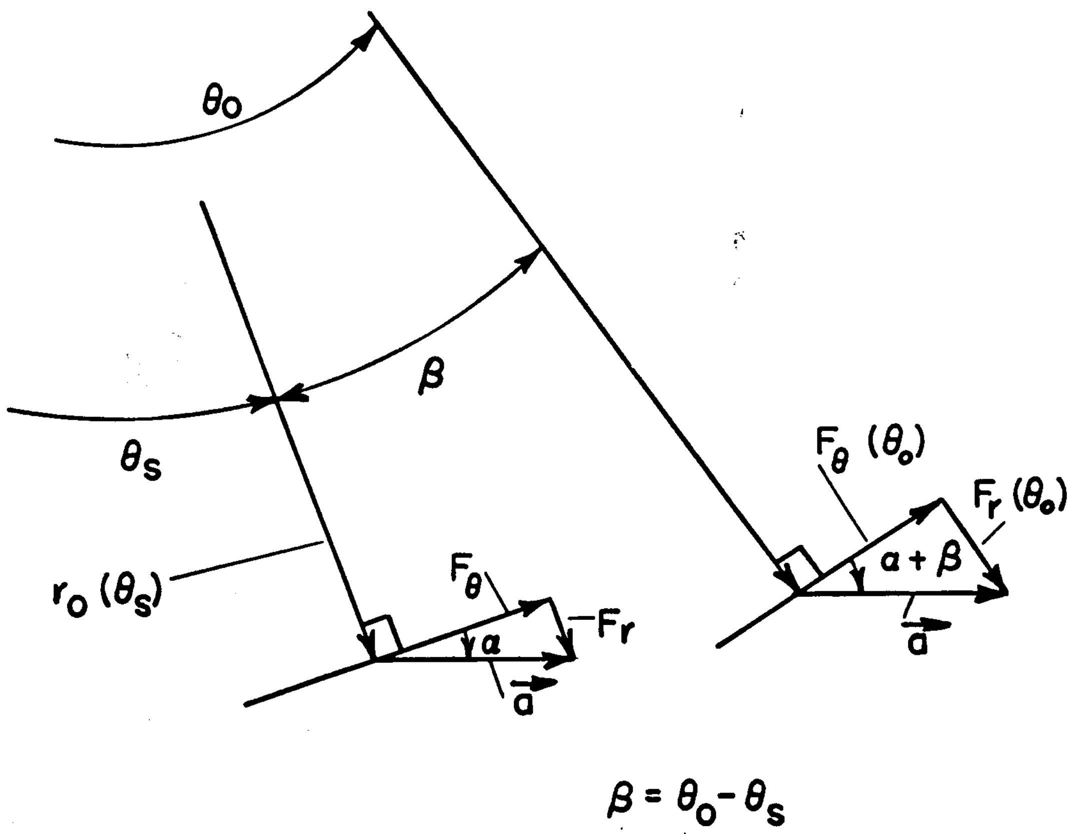

θo
β
θs
$r_o(\theta_s)$
F_θ
$-F_r$
α
$\vec{a}$
$F_\theta(\theta_o)$
$F_r(\theta_o)$
α + β
$\vec{a}$
$\beta = \theta_o - \theta_s$

Figure 6

Now note that

$$F_\theta = \frac{d}{d\theta_o}(F_r)$$

Therefore I_r can be easily evaluated

$$I_r = \int_{\theta_S}^{\theta_o} (r_o' F_r + r_o F_\theta) d\theta$$

$$= \int_{\theta_S}^{\theta_o} (r_o' F_r + r_o F_r') d\theta = \int_{\theta_S}^{\theta_o} \frac{d}{d\theta}(r_o F_r) d\theta$$

$$I_r = r_o(\theta_o) F_r(\theta_o) - r_o(\theta_S) F_r(\theta_S) \tag{47}$$

For convenience, the I_r notation will be retained.

The evaluation of the integral for Δr is straightforward, but lengthy. It is shown in some detail in Appendix A. The result is

$$\begin{aligned} \Delta r = & |\vec{a}|\, B \frac{P_o^4}{C_o^2} \{\sin\theta_o F_{BS} + \cos\theta_o F_{BC}\} \\ & + |\vec{a}|\, A \frac{P_o^4}{C_o^2} \{\cos\theta_o F_{AC} + \sin\theta_o F_{AS}\} \end{aligned} \tag{48}$$

where the following definitions were made:

$$E = 1 - e_o^2$$

$$c'_{11} = \frac{2e_o^4 + 12e_o^2 + 1}{2e_o E^3}$$

$$c'_{21} = \frac{-4e_o^4 + 3e_o^2 + 1}{2e_o E^3}$$

$$c_{31} = \frac{1}{e_o E}$$

$$c'_{41} = \frac{3(2e_o^2+3)}{2E^3}$$

$$c_{12} = \frac{2e_o^4-e_o^2-1}{2e_oE^3}$$

$$c_{22} = \frac{-1}{2e_oE}$$

$$c_{32} = -\frac{1}{e_o}$$

$$c_{42} = -\frac{3}{2E^2}$$

$$c''_{11} = \frac{2e_o^2+1}{E^2}$$

$$c''_{21} = \frac{1}{E}$$

$$c''_{41} = \frac{3e_o}{E^2}$$

$$k'_{21} = \frac{3}{2e_o^2}$$

$$k_{31} = -\frac{1}{e_o^2}$$

$$k_{22} = -\frac{1}{2e_o^2}$$

$$k_{32} = +\frac{1}{e_o^2}$$

$$k''_{21} = \frac{1}{e_o} \qquad (49)$$

$$x_o = r_o/P_o = 1/(1-e_o\cos\theta_o)$$

$$x_S = r_S/P_o = 1/(1-e_o\cos\theta_S) \qquad (50)$$

$$F_{S1} = \sin\theta_o x_o - \sin\theta_S x_S$$

$$F_{S2} = \sin\theta_o x_o^2 - \sin\theta_S x_S^2$$

$$F_{S3} = \sin\theta_o x_o^3 - \sin\theta_S x_S^3$$

$$F_2 = x_o^2 - x_S^2$$

$$F_3 = x_o^3 - x_S^3$$

$$\Delta f = \int_{\theta_S}^{\theta_o} \frac{d\theta}{(1-\theta_o\cos\theta_o)}$$

$$= \frac{2}{\sqrt{E}}\left[\tan^{-1}\left\{\sqrt{\frac{1+e_o}{1-e_o}}\tan\frac{\theta_o}{2}\right\} - \tan^{-1}\left\{\sqrt{\frac{1+e_o}{1-e_o}}\tan\frac{\theta_S}{2}\right\}\right] \qquad (51)$$

$$F_{BS} = F_{S1}(c'_{11} - x_S\frac{c''_{11}}{e_o}) + F_{S2}(c'_{21} - x_S\frac{c''_{21}}{e_o})$$

$$+ F_{S3}c_{31} + \Delta f(c'_{41} - x_S\frac{c''_{41}}{e_o}) \qquad (52)$$

$$F_{BC} = F_2(k_{22} - x_S\frac{k''_{21}}{e_o}) + F_3 k_{32} \qquad (53)$$

$$F_{AC} = F_{S1}c_{12} + F_{S2}c_{22} + F_{S3}c_{32} + \Delta f c_{42} - \sin\theta_S x_S k''_{21} F_2 \qquad (54)$$

$$F_{AS} = F_2 k'_{21} + F_3 k_{31} - \sin\theta_S x_S(c''_{11}F_{S1} + c''_{21}F_{S2} + c''_{41}\Delta f) \qquad (55)$$

An expression for $\Delta r'$ may be developed from (44) as was done for Δr or may be obtained from (48) by differentiation. An expression for $\Delta r'$ is obtained in Appendix A as

$$\Delta r' = |\vec{a}|\, B \frac{P_o^4}{C_o^2} \{\cos\theta_o F_{BS} - \sin\theta_o F_{BC}\} + |\vec{a}|\, A \frac{P_o^4}{C_o^2} \{-\sin\theta_o F_{AC} + \cos\theta_o F_{AS}\} \tag{56}$$

These expressions for Δr and $\Delta r'$ are convenient since later combinations of Δr and $\Delta r'$ multiplied by $\sin\theta_o$ and $\cos\theta_o$ are needed and the above expressions give these combinations in simple forms.

Since

$$C_o^2 = \mu P_o \tag{57}$$

the planet characteristic has factored out. Define

$$\Delta R = \frac{\Delta r}{\frac{P_o^4}{C_c^2} |\vec{a}|}$$

$$\Delta R' = \frac{\Delta r'}{\frac{P_o^4}{C_o^4} |\vec{a}|}$$

Then

$$\Delta R = B[\sin\theta_o F_{BS} + \cos\theta_o F_{BC}] + A[\cos\theta_o F_{AC} + \sin\theta_o F_{AS}] \tag{58}$$

$$\Delta R' = B[\cos\theta_o F_{BS} - \sin\theta_o F_{BC}]$$
$$+ A[-\sin\theta_o F_{AC} + \cos\theta_o F_{AS}] \quad (59)$$

Note ΔR and $\Delta R'$ are non-dimensional and functions of only α, e_o, θ_o, and θ_S. Also, note that I_r may be put in a form proportional to $|\vec{a}|$ and P_o. With the definition of F_{S1} in (51) and with

$$F_{C1} = \cos\theta_o x_o - \cos\theta_o x_S \quad (60)$$

I_r can be written

$$I_r = |\vec{a}| \, AP_o F_{S1} + |\vec{a}| \, BP_o F_{C1}$$
$$= |\vec{a}| \, P_o \, \hat{I}_r \quad (61)$$

where

$$\hat{I}_r = AF_{S1} + BF_{C1} \quad (62)$$

is a function of α, e_o, θ_o, and θ_S. Note that F_r, I_r, Δr, and $\Delta r'$ are of the form A( ) + B( ), where the alignment, α, appears only in A and B. Later, this will give the orbital elements in this form.

Now equation (43) for $\Delta\theta$ can be written

$$\Delta\theta = \frac{1}{P_o} \frac{P_o^4}{C_o^2} |\vec{a}| \, \Delta R' + \frac{1}{P_o x_o} \frac{P_o^4}{C_o^2} |\vec{a}| \, \Delta R'$$
$$- \frac{1}{C_o^2} \int_{\theta_S}^{\theta_o} P_o^2 x_o^2 \, |\vec{a}| P_o \hat{I}_r d\theta$$

$$- \frac{2}{P_o C_o^2} \int_{\theta_S}^{\theta_o} P_o^3 x_o^3 |\vec{a}| P_o \hat{I}_r d\theta - \frac{1}{C_o^2} \int_{\theta_S}^{\theta_o} P_o^3 x_o^3 |\vec{a}| \frac{F_r}{|\vec{a}|} d\theta$$

$$- \frac{1}{P_o C_o^2} \int_{\theta_S}^{\theta_o} P_o^4 x_o^4 |\vec{a}| \frac{F_r}{|\vec{a}|} d\theta$$

when $F_r/|\vec{a}| = A\sin\theta_o + B\cos\theta_o$.

Therefore

$$\Delta\theta = |\vec{a}| \frac{P_o^3}{C_o^2} \left\{\Delta R' + \frac{\Delta R'}{x_o} - I_{\Delta\theta}\right\}$$

$$= |\vec{a}| \frac{P_o^3}{C_o^2} \hat{\Delta\theta} \tag{63}$$

where

$$\hat{\Delta\theta} = \Delta R' + \frac{\Delta R'}{x_o} - I_{\Delta\theta} \tag{64}$$

is non-dimensional and a function of α, e_o, θ_o, and θ_S, and $I_{\Delta\theta}$ is the integral terms with $\frac{P_o^3}{C_o^2} |\vec{a}|$ factored out. The evaluation of $I_{\Delta\theta}$ is straightforward but lengthy. It is done in Appendix B giving

$$I_{\Delta\theta} = AI_{\Delta\theta A} + BI_{\Delta\theta B} \tag{65}$$

where

$$I_{\Delta\theta A} = -\frac{1}{e_o} F_3 - \frac{1}{e_o} F_2 - \sin\theta_S x_S (b_1' F_{S1} + b_2' F_{S2} + b_4' \Delta f)$$

$$I_{\Delta\theta B} = b_1 F_{S1} + b_2 F_{S2} + b_3 F_{S3} + b_4 \Delta f$$

$$- \cos\theta_S x_S (b_1' F_{S1} + b_2' F_{S2} + b_4' \Delta f) \tag{66}$$

and the quantities not previously defined are

$$b_1' = \frac{e_o}{E^2}(4-e_o^2)$$

$$b_2' = \frac{e_o}{E}$$

$$b_4' = \frac{3}{E^2}$$

$$b_1 = \frac{-4e_o^4+15e_o^2+4}{2E^3} \tag{67}$$

$$b_2 = \frac{-e_o^4-3e_o^2+4}{2E^3}$$

$$b_3 = 1/E$$

$$b_4 = \frac{3e_o}{2E^3}(6-e_o^2)$$

Finally, equation (39) for $\Delta\theta'$ may be written

$$\Delta\theta' = \frac{P_o^3}{C_o^2} |\vec{a}| \hat{\Delta\theta}' \tag{68}$$

where

$$\hat{\Delta\theta}' = e_o \sin\theta_o \Delta R' - \frac{\Delta R}{x_o}(1+x_o) + x_o^2 \hat{I}_r \tag{69}$$

is non-dimensional and a function of α, e_o, θ_o, and θ_S only.

In summary, and renumbering the equations here for convenience

$$\Delta r = \frac{P_o^4}{C_o^2} |\vec{a}| \Delta R \tag{70}$$

$$\Delta r' = \frac{P_o^4}{C_o^2} |\vec{a}| \Delta R' \tag{71}$$

$$\Delta\theta = \frac{P_o^3}{C_o^2} \left|\vec{a}\right| \hat{\Delta\theta} \tag{72}$$

$$\Delta\theta' = \frac{P_o^3}{C_o^2} \left|\vec{a}\right| \hat{\Delta\theta'} \tag{73}$$

where

$$\Delta R = B(\sin\theta_o F_{BS} + \cos\theta_o F_{BC})$$

$$+ A(\cos\theta_o F_{AC} + \sin\theta_o F_{AS}) \tag{74}$$

$$\Delta R' = B(\cos\theta_o F_{BS} - \sin\theta_o F_{BC})$$

$$+ A(-\sin\theta_o F_{AC} + \cos\theta_o F_{AS}) \tag{75}$$

$$\hat{\Delta\theta} = \Delta R' + \frac{\Delta R'}{x_o} - I_{\Delta\theta} \tag{76}$$

$$\hat{\Delta\theta'} = e_o \sin\theta_o \Delta R' - \frac{\Delta R}{x_o}(1 + x_o) + x_o^2 \hat{I}_r \tag{77}$$

and

$$\Delta v = \left(-\frac{\mu \Delta r}{r_o^2} + I_r\right)/v_o \tag{78}$$

Orbital Elements

The pertinent equations in reference 1 were manipulated to provide equations (70) through (78) for Δr, $\Delta r'$, $\Delta\theta$, $\Delta\theta'$, and Δv. Thus it is now necessary to express the changes in the orbital elements - ΔP, Δe, Δw - in terms of Δr, $\Delta r'$, $\Delta\theta$, $\Delta\theta'$, and Δv.

First an expression for Δa, the change in semi-major axis, can be found from the expression for velocity on a Keplerian orbit.

$$v = v_o + \Delta v = \sqrt{\mu}\left(\frac{2}{r} - \frac{1}{a}\right)^{\frac{1}{2}}$$

Using the binomial expansion on $(r_o+\Delta r)^{-1}$ and $(a_o+\Delta a)^{-1}$ and ignoring second order terms in Δr and Δa gives

$$v_o + \Delta v = \sqrt{\mu}\left(\frac{2}{r_o} - \frac{1}{a_o} - \frac{2\Delta r}{r_o^2} + \frac{\Delta a}{a_o^2}\right)^{\frac{1}{2}}$$

Using the binomial expansion again gives

$$v_o + \Delta v = \sqrt{\mu}\left[\left(\frac{2}{r_o} - \frac{1}{a_o}\right)^{\frac{1}{2}} + \tfrac{1}{2}\left(\frac{2}{r_o} - \frac{1}{a_o}\right)^{-\frac{1}{2}}\left(\frac{\Delta a}{a_o^2} - \frac{2\Delta r}{r_o^2}\right)\right]$$

But

$$v_o = \sqrt{\mu}\left(\frac{2}{r_o} - \frac{1}{a_o}\right)^{\frac{1}{2}}$$

Therefore

$$\Delta v = \frac{\mu}{2v_o}\left(\frac{\Delta a}{a_o^2} - \frac{2\Delta r}{r_o^2}\right)$$

Solving for Δa gives

$$\Delta a = \frac{2a_o^2 v_o^2}{\mu}\frac{\Delta v}{v_o} + \left(\frac{2a_o^2}{r_o}\right)\frac{\Delta r}{r_o} \qquad (79)$$

which is good for any eccentricity.

The change in angular momentum, ΔC, may be found from the definition of angular momentum

$$C = r^2 \dot{\theta}$$

with

$$r_o = r_o + \Delta r$$

$$\dot{\theta} = \dot{\theta}_o + \Delta\dot{\theta}$$

$$C = C_o + \Delta C$$

This gives, ignoring second order terms in the changes,

$$C_o + \Delta C = r_o^2 \dot{\theta}_o + r_o^2 \Delta\dot{\theta} + 2 r_o \dot{\theta}_o \Delta r$$

Since $C_o = r_o^2 \dot{\theta}_o$

$$\Delta C = r_o^2 \Delta\dot{\theta} + 2 r_o \dot{\theta}_o \Delta r$$

Now

$$\Delta\dot{\theta} = \frac{d\Delta\theta}{dt} = \frac{d\Delta\theta}{d\theta_o} \frac{d\theta_o}{dt}$$

$$= \Delta\theta' \dot{\theta}_o$$

Therefore

$$\Delta C = C_o [\Delta\theta' + 2 \frac{\Delta r}{r_o}] \qquad (80)$$

which is good for any eccentricity.

The change in semi-latus rectum, ΔP, may be found from the definition of angular momentum in terms of P

$$C = \sqrt{\mu P}$$

Using the binomial expansion on $\sqrt{P}$ gives

$$C_o + \Delta C = \sqrt{\mu}\ (P_o^{\frac{1}{2}} + \frac{1}{2}\frac{\Delta P}{P_o^{\frac{1}{2}}})$$

Therefore

$$\Delta C = \frac{1}{2}\frac{\Delta P}{\sqrt{P_o}}\sqrt{\mu}$$

Solving for ΔP and using (80) gives

$$\Delta P = 2P_o\ (\Delta\theta' + \frac{2\Delta r}{r_o}) \tag{81}$$

The change in eccentricity, Δe, may be found from the definition of P in terms of a and e

$$P = a(1-e^2)$$

with

$$P_o = a_o(1-e_o^2) = a_o E$$

$$P = P_o + \Delta P$$

$$a = a_o + \Delta a$$

$$e = e_o + \Delta e$$

and ignoring second order terms

$$\Delta P = -2ae_o\Delta e + \frac{\Delta a P_o}{a_o}$$

Solving for Δe gives

$$\Delta e = (-\Delta P + \frac{\Delta a}{a_o} P_o)/2a_o e_o$$

Substituting (81) and (79) for ΔP and Δa gives

$$\Delta e = \frac{E}{2e_o}\left[-2\Delta\theta' + \frac{2a_o v_o^2}{\mu}\frac{\Delta v}{v_o} + \frac{\Delta r}{r_o}(-4 + \frac{2a_o}{r_o})\right]$$

By using (78) for Δv and $a_o = \frac{P_o}{E}$

$$\frac{2a_o v_o^2}{\mu}\frac{\Delta v}{v_o} = -2a_o\frac{\Delta r}{r_o^2} + \frac{2P_o^2 I_r}{EC_o^2}$$

Therefore

$$\Delta e = \frac{E}{2e_o}\left[-2\Delta\theta' + \frac{2P_o^2 I_r}{EC_o^2} - 4\frac{\Delta r}{r_o}\right] \tag{82}$$

The last expression needed is for Δw, the shift in line of apsides. Referring to figure 7,

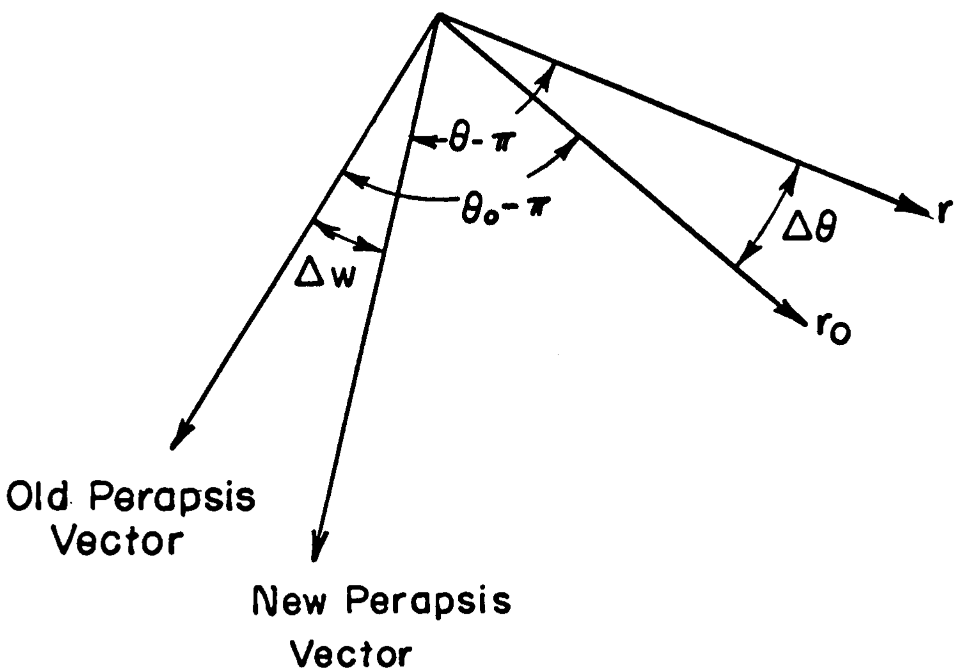
$$\Delta w + (\theta - \pi) = \theta_o - \pi + \Delta\theta$$

where θ is the polar angle on the final orbit. Thus

$$\Delta w = \theta_o - \theta + \Delta\theta$$

Thus an expression for $\theta_o - \theta$ is needed (an expression for $\Delta\theta$ has already been developed). Consider

$$\sin(\theta_o - \theta) = \sin\theta_o\cos\theta - \cos\theta_o\sin\theta \tag{83}$$



$$\Delta w + (\theta - \pi) = (\theta_o - \pi) + \Delta\theta$$

Figure 7

Expressions for cosθ and sinθ may be obtained by equating r to r on the osculating orbit and the rate of change of r with angle with the rate of change of r with angle on the osculating orbit.

$$r_o + \Delta r = P/(1 - e\cos\theta)$$

Solving for ecosθ gives

$$e\cos\theta = 1 - \frac{P}{r_o + \Delta r}$$

Using the binomial expansion for $(r_o + \Delta r)^{-1}$ gives

$$e\cos\theta = 1 - \frac{P}{r_o}\left(1 - \frac{\Delta r}{r_o}\right) \tag{84}$$

Now

$$\frac{dr}{d\theta} = \frac{d(r_o + \Delta r)}{d(\theta_o + \Delta\theta)}$$

$$d(\theta_o + \Delta\theta) = d\theta_o(1 + \Delta\theta')$$

Therefore by using the binomial expansion and ignoring second order terms

$$\frac{dr}{d\theta} = \frac{d(r_o + \Delta r)}{d\theta_o(1 + \Delta\theta')} = \frac{d(r_o + \Delta r)}{d\theta_o}(1 - \Delta\theta')$$

$$\frac{dr}{d\theta} = \frac{dr_o}{d\theta_o} + \Delta r' - \Delta\theta' \frac{dr_o}{d\theta_o}$$

Now

$$\frac{dr}{d\theta} = \frac{-e\sin\theta\, r^2}{P}$$

$$\frac{dr_o}{d\theta_o} = \frac{-e_o \sin\theta_o\, r_o^2}{P_o}$$

Thus

$$\frac{-e\sin\theta r^2}{P} = \frac{-e_o\sin\theta_o r_o^2}{P_o} + \Delta r' + \frac{e_o\sin\theta_o r_o^2}{P_o}\Delta\theta'$$

Solving for esinθ gives

$$e\sin\theta = \frac{P}{P_o}\,\frac{1}{r^2}\,[-e_o\sin\theta_o r_o^2 - \Delta r' P_o + e_o\sin\theta_o r_o^2\Delta\theta']$$

with $r = r_o + \Delta r$, the binomial expansion, and ignoring second order terms

$$e\sin\theta = \frac{P}{P_o}\,[e_o\sin\theta_o - \frac{\Delta r' P_o}{r_o^2} - e_o\sin\theta_o\Delta\theta' - 2e_o\sin\theta_o\frac{\Delta r}{r_o}] \quad (85)$$

Equation (83) gives an expression for $\theta_o - \theta$ for small $\theta_o - \theta$

$$\theta_o - \theta = \sin\theta_o\cos\theta - \cos\theta_o\sin\theta$$

Using (84) and (85) gives

$$e(\theta_o - \theta) = -\cos\theta_o[e_o\sin\theta_o - 2e_o\sin\theta_o\frac{\Delta r}{r_o} - \frac{\Delta r' P_o}{r_o^2}$$

$$-e_o\sin\theta_o\Delta\theta' + \frac{\Delta P}{P_o}e_o\sin\theta_o]$$

$$+\sin\theta_o\,[1 - \frac{P_o}{r_o}(1 - \Delta r) - \frac{\Delta P}{r_o}]$$

The first order terms cancel leaving

$$e(\theta_o - \theta) = \frac{\Delta r}{r_o}[2e_o\sin\theta_o\cos\theta_o + \frac{P_o}{r_o}\sin\theta_o]$$

$$-\frac{\Delta P}{P_o}[e_o\sin\theta_o\cos\theta_o + \frac{P_o}{r_o}\sin\theta_o]$$

$$+\frac{\Delta r'}{r_o^2}P_o\cos\theta_o + e_o\sin\theta_o\cos\theta_o\Delta\theta'$$

The first two brackets simplify leaving

$$e(\theta_o - \theta) = \sin\theta_o \frac{\Delta r}{r_o} + e_o \sin\theta_o \cos\theta_o \frac{\Delta r}{r_o} - \sin\theta_o \frac{\Delta P}{P_o}$$

$$+ \frac{\Delta r' P_o}{r_o^2} \cos\theta_o + e_o \sin\theta_o \cos\theta_o \Delta\theta'$$

Since $\frac{1}{e} = \frac{1}{e_o} - \frac{\Delta e}{e_o^2}$, to first order

$$\theta_o - \theta = \frac{1}{e_o} [\sin\theta_o \frac{\Delta r}{r_o} + e_o \sin\theta_o \cos\theta_o \frac{\Delta r}{r_o} - \sin\theta_o \frac{\Delta P}{P_o}$$

$$+ \frac{\Delta r' P_o}{r_o^2} \cos\theta_o + e_o \sin\theta_o \cos\theta_o \Delta\theta'] \qquad (86)$$

Thus

$$\Delta w = \theta_o - \theta + \Delta\theta$$

when $\Delta\theta$ is found from (72) and (76) and $(\theta_o - \theta)$ from (86).

The changes ΔP, Δe, and Δw can be written as $\frac{P_o^3}{C_o^2} |\vec{a}|$ times a function of $(\alpha, P_o, \theta_o, \theta_S)$ since the terms in their expressions may be written in this way. First, equation (81) for ΔP with use of (70) for Δr and (73) for $\Delta\theta'$ gives

$$\Delta P = 2 \frac{P_o^4}{C_o^2} |\vec{a}| (\hat{\Delta\theta}' + 2 \frac{\Delta R}{x_o})$$

Define

$$\hat{\Delta P} = 2(\hat{\Delta\theta}' + 2 \frac{\Delta R}{x_o}) \qquad (87)$$

Then

$$\Delta P = \frac{P_o^4}{C_o^2} |\vec{a}| \hat{\Delta P} \qquad (88)$$

Next, using (82) for Δe and (70) for Δr, (73) for $\Delta\theta'$, and (61) for I_r gives

$$\Delta e = \frac{P_o^3}{C_o^2} |\vec{a}| \frac{E}{2e_o} (-\frac{4\Delta R}{x_o} - 2\hat{\Delta\theta}' + \frac{2\hat{I}_r}{E})$$

Define

$$\hat{\Delta e} = \frac{E}{2e_o} (-\frac{4\Delta R}{x_o} - 2\hat{\Delta\theta}' + \frac{2\hat{I}_r}{E}) \tag{89}$$

Then

$$\Delta e = \frac{P_o^3}{C_o^2} |\vec{a}| \hat{\Delta e} \tag{90}$$

Finally using (86) for $\theta_o - \theta$, (70) for Δr, (88) for ΔP, (71) for $\Delta r'$, and (73) for $\Delta\theta'$ gives

$$\theta_o - \theta = -\frac{P_o^3}{C_o^2} |\vec{a}| \frac{1}{e_o} (\sin\theta_o \hat{\Delta P} - e_o \sin\theta_o \cos\theta_o \frac{\Delta R}{x_o} - e_o \sin\theta_o \cos\theta_o \hat{\Delta\theta}' - \frac{\Delta R'}{x_o^2} \cos\theta_o - \sin\theta_o \frac{\Delta R}{x_o})$$

Now using (72) for $\Delta\theta$ and (87) for $\hat{\Delta P}$ and remembering that $\Delta w = \theta_o - \theta + \Delta\theta$

$$\Delta w = \frac{P_o^3}{C_o^2} |\vec{a}| [-\frac{1}{e_o} (\sin\theta_o \hat{\Delta P} - e_o \sin\theta_o \cos\theta_o \frac{\Delta R}{x_o} - \frac{\Delta R'}{x_o^2} \cos\theta_o - \sin\theta_o \frac{\Delta R}{x_o}) + \hat{\Delta\theta}]$$

or

$$\Delta w = \frac{P_o^3}{C_o^2} |\vec{a}| \hat{\Delta w} \tag{91}$$

where

$$\hat{\Delta w} = -\frac{2}{e_o}\sin\hat{\Delta\theta}' - \frac{3}{e_o}\frac{\Delta R}{x_o}\sin\theta_o + e_o\sin\theta_o\cos\theta_o\frac{\Delta R}{x_o} + \frac{\Delta R'}{e_o x_o^2}\cos\theta_o + \hat{\Delta\theta} \tag{92}$$

Finally, $\hat{\Delta P}$, $\hat{\Delta e}$, and $\hat{\Delta w}$ can be written as

$$A\ fn(\theta_o, \theta_S, e_o) + B\ fn(\theta_o, \theta_S, e_o) \tag{93}$$

since the terms in their expression can be written in this way. First using (87) for $\hat{\Delta P}$, (62) for $\hat{I}_r$, (74) for ΔR and (77)) for $\hat{\Delta\theta}'$ gives

$$\hat{\Delta P} = 2[e_o(\sin\theta_o\Delta R' - \cos\theta_o\Delta R) + x_o^2(AF_{S1} + BF_{C1})]$$

Here the forms used for ΔR and $\Delta R'$ (equations (74) and (75) eliminate algebra since

$$\sin\theta_o\ \Delta R' - \cos\theta_o\Delta R = -BF_{BC} - AF_{AC} \tag{94}$$

Therefore

$$\hat{\Delta P} = g_1 A + g_2 B \tag{95}$$

where

$$\begin{aligned} g_1 &= -2e_o F_{AC} + 2x_o^2 F_{S1} \\ g_2 &= -2e_o F_{BC} + 2x_o^2 F_{C1} \end{aligned} \tag{96}$$

Next, using (89) for $\hat{\Delta e}$, (74) for ΔR, (77) for $\hat{\Delta\theta}'$ and (62) for $\hat{I}_r$ gives

$$\hat{\Delta e} = \frac{E}{2e_o}\left[2e_o(\cos\theta_o\Delta R - \sin\theta_o\Delta R') + \left(\frac{2}{E} - 2x_o^2\right)(AF_{S1} + BF_{C1})\right]$$

and using (94)

$$\hat{\Delta e} = g_3 A + g_4 B \tag{97}$$

where

$$\begin{aligned} g_3 &= E\,F_{AC} + \frac{F_{S1}}{e_o} - \frac{Ex_o^2}{e_o} F_{S1} \\ g_4 &= E\,F_{BC} + \frac{F_{C1}}{e_o} - \frac{Ex_o^2}{e_o} F_{C1} \end{aligned} \tag{98}$$

Finally, $\hat{\Delta w}$ may be written as

$$\hat{\Delta w} = g_5 A + g_6\, B \tag{99}$$

Using (92) for $\hat{\Delta w}$, (77) for $\hat{\Delta\theta}'$, (74) for ΔR, (75) for $\Delta R'$, (76) for $\hat{\Delta\theta}$, and (62) for $\hat{I}_r$ gives

$$\begin{aligned} \hat{\Delta w} = &-\frac{2}{e_o}\sin\theta_o\left[e_o\sin\theta_o\Delta R' - \frac{\Delta R'}{x_o} - \Delta R + x_o^2\,\hat{I}_r\right] \\ &- \frac{3}{e_o}\,\frac{\Delta R}{x_o}\,\sin\theta_o + e_o\sin\theta_o\cos\theta_o\frac{\Delta R}{x_o} + \frac{\Delta R'}{\theta_o x_o^2}\cos\theta_o \\ &+ \Delta R' + \frac{\Delta R'}{x_o} - I_{\Delta\theta} \end{aligned}$$

Using (74) for ΔR, (75) for $\Delta R'$, (65) for $I_{\Delta\theta}$ and arranging the equation in the form of (99) gives

$$g_5 = F_{AS}\left[+\frac{1}{e_o x_o} + \frac{(1-2\cos^2\theta_o)\cos\theta_o}{x_o} + (2-\cos\theta_o)\cos\theta_o\right]$$

$$+ F_{AC}\left[+2\sin\theta_o + \frac{2\sin\theta_o\cos^2\theta_o}{x_o} + (e_o\cos\theta_o - 2)\sin\theta_o\right]$$

$$-2x_o^2 F_{S1}\frac{\sin\theta_o}{e_o} - I_{\Delta\theta A}$$

$$g_6 = F_{BS}\left[+\frac{1}{e_o x_o} + \frac{(1-2\cos^2\theta_o)\cos\theta_o}{x_o} + (2-\cos\theta_o)\cos\theta_o\right]$$

$$+ F_{BC}\left[2\sin\theta_o + \frac{2\sin\theta_o\cos^2\theta_o}{x_o} + (e_o\cos\theta_o - 2)\sin\theta_o\right]$$

$$-2x_o^2\frac{\sin\theta_o}{e_o}F_{C1} - I_{\Delta\theta B}$$

Note that the coefficients of F_{AS} and F_{BS} are the same, those of F_{AC} and F_{BC} are the same, and those of F_{S1} and F_{C1} are the same also. The coefficients of F_{AS} and F_{AC} can be simplified. Define

$$P_{51} = +\frac{1}{e_o} + 2\cos\theta_o - 2e_o\cos^2\theta_o - 2\cos^3\theta_o + 2e_o\cos^4\theta_o$$

$$P_{52} = (+e_o\cos\theta_o + 2\cos^2\theta_o - 2e_o\cos^3\theta_o)\sin\theta_o \tag{100}$$

Then

$$g_5 = P_{51}F_{AS} + P_{52}F_{AC} - \frac{2x_o^2\sin\theta_o}{e_o}F_{S1} - I_{\Delta\theta A} \tag{101}$$

$$g_6 = P_{51}F_{BS} + P_{52}F_{BC} - \frac{2x_o^2\sin\theta_o}{e_o}F_{C1} - I_{\Delta\theta B} \tag{102}$$

Summarizing and renumbering the equations

$$\hat{\Delta P} = g_1 A + g_2 B \tag{103}$$

$$\hat{\Delta e} = g_3 A + g_4 B \tag{104}$$

$$\hat{\Delta w} = g_5 A + g_6 B \tag{105}$$

$$g_1 = -2e_o F_{AC} + 2x_o^2 F_{S1} \tag{106}$$

$$g_2 = -2e_o F_{BC} + 2x_o^2 F_{C1} \tag{107}$$

$$g_3 = (1-e_o^2)F_{AC} + \frac{F_{S1}}{e_o} - \frac{(1-e_o^2)x_o^2}{e_o} F_{S1} \tag{108}$$

$$g_4 = (1-e_o^2)F_{BC} + \frac{F_{C1}}{e_o} - \frac{(1-e_o^2)x_o^2}{e_o} F_{C1} \tag{109}$$

$$g_5 = P_{51}F_{AS} + P_{52}F_{AC} - \frac{2x_o^2 \sin\theta_o}{e_o} F_{S1} - I_{\Delta\theta A} \tag{110}$$

$$g_6 = P_{51}F_{BS}P_{52}F_{BC} - \frac{2x_o^2 \sin\theta_o}{e_o} F_{C1} - I_{\Delta\theta B} \tag{111}$$

Evaluation of D

In previous sections, it was shown that in the linearized limit $D < 0$ was both necessary and sufficient for a separation. D was linearized in equation (31) as

$$D = (e_o \Delta P - P_o \Delta e)^2 - \Delta P^2 + 2P_o^2 e_o^2 \delta \tag{31}$$

Equation (11) defined δ. In the linearized limit

$$\delta = \frac{\Delta w^2}{2}$$

Therefore

$$D = (e_o \Delta P - P_o \Delta e)^2 - \Delta P^2 + P_o^2 e_o^2 \Delta w^2$$

Now equations (88) for ΔP, (90) for Δe, and (91) for Δw may be used to factor out $\frac{P_o^8}{C_o^4} |\vec{a}|^2$. Defining

$$\hat{D} = (\hat{\Delta P} - \frac{\hat{\Delta e}}{e_o})^2 - \frac{\hat{\Delta P}^2}{e_o^2} + \hat{\Delta w}^2 \tag{112}$$

gives

$$D = \frac{P_o^8}{C_o^4} |\vec{a}|^2 \; e_o^2 \hat{D} \tag{113}$$

For completeness, note that equation (10) for $\cos\theta_i$ can be written as

$$\cos\theta_i = fn(\alpha, e_o, \theta_o, \theta_S)$$

as P_o, $|\vec{a}|$, and C_o all divide out.

The quantities $\hat{\Delta P}$, $\hat{\Delta e}$, and $\hat{\Delta w}$ in equation (112) have been written previously in the form

$$A \; fn(e_o, \theta_o, \theta_S) + B \; fn(e_o, \theta_o, \theta_S)$$

Thus $\hat{D}$ can be written in the form

$$\hat{D} = d_1 A^2 + 2d_2 AB + d_3 B^2 \tag{114}$$

where each d_i is made up of combinations of the g_i which are given by equations (106) through (111). Using (103) through (105) for $\hat{\Delta P}$,

$\hat{\Delta e}$, and $\hat{\Delta w}$ gives

$$d_1 = \frac{-E}{e_o^2} g_1 + \frac{g_3^2}{e_o^2} - \frac{2g_1g_2}{e_o} + g_5^2 \tag{115}$$

$$d_2 = \frac{-E}{e_o^2} g_1g_2 + \frac{g_3g_4}{e_o^2} - \frac{g_2g_3}{e_o} - \frac{g_1g_4}{e_o} + g_5g_6 \tag{116}$$

$$d_3 = \frac{-E}{e_o^2} g_2^2 + \frac{g_4^2}{e_o^2} - \frac{2g_2g_4}{e_o} + g_6^2 \tag{117}$$

The sign of D is determined by the sign of $\hat{D}$. Thus the sign on $\hat{D}$ must be investigated. To do this, note that $\hat{D}$ has been written as a general quadratic form. Thus it can be factored into the form

$$\hat{D} = d_1[A + B\,(\frac{d_2+\sqrt{d}}{d_1})]\ [A+B\,(\frac{d_2-\sqrt{d}}{d_1})] \tag{118}$$

where

$$d = d_2^2 - d_1d_3 \tag{119}$$

First, consider d greater than zero so that $\sqrt{d}$ is real. Remember that

$$\begin{aligned} A &= \cos(\alpha-\theta_S) \\ B &= \sin(\alpha-\theta_S) \end{aligned} \tag{45}$$

and that the d_i's are not dependent on α. Thus equating a linear factor of $\hat{D}$ to zero gives a relation for the alignment, α_o, to make $\hat{D}$ equal zero. This can be written

$$A + B \left(\frac{d_2 \pm \sqrt{d}}{d_1}\right) = 0 \tag{120}$$

so that

$$\frac{B}{A} = \tan(\alpha_o - \theta_S) = -\frac{d_1}{d_2 \pm \sqrt{d}} \tag{121}$$

This gives two alignments, $\alpha_o^{(1)}$ and $\alpha_o^{(2)}$, one for each sign on the radical. Further, since the tangent has a period of π, (121) really gives four values of α that make $\hat{D} = 0$.

$$\begin{aligned} \alpha_o^{(1)} &= \theta_S + \tan^{-1}\left(-\frac{d_1}{d_2 + \sqrt{d}}\right) \pm \pi \\ \alpha_o^{(2)} &= \theta_S + \tan^{-1}\left(-\frac{d_1}{d_2 - \sqrt{d}}\right) \pm \pi \end{aligned} \tag{122}$$

Only $\alpha_o^{(1)}$ and $\alpha_o^{(2)}$ will be mentioned from here on, with the understanding that if they are selected to be between $+90^o$ and -90^o, there is a corresponding pair 180^o away. Finally, since $d \neq 0$, $\alpha_o^{(1)}$ and $\alpha_o^{(2)}$ are different (there is no double root for $\hat{D} = 0$), and thus $\hat{D}$ is positive on one side of an α_o and negative on the other side.

Second, consider d less than zero. Here $\sqrt{d}$ is imaginary, and it is not possible to find a physically existing alignment to give $\hat{D} = 0$. Thus $\hat{D}$ will not change sign. Since $\alpha = \theta_S$ gives $\hat{D} = d_1 A^2$, the sign of $\hat{D}$ will be the sign of d_1. ($\hat{D}$ could also have been written in the form of (118) by factoring out d_3 instead of d_1; the above argument would be equally valid for d_3.)

Thus the criterion for separation of $D < 0$ has become a criterion on d and d_1, specifically D will be less than zero if

$$d < 0 \quad \text{and} \quad d_1 < 0$$

or an alignment can be chosen to make D less than zero if

$$d > 0$$

The expressions for d_1, d_2, d_3, and d can be written in terms of F_{BC}, F_{AC}, etc. by using equations (106) through (111) for the g_i's. The resulting expressions can be rewritten by using the definitions of F_{AC}, F_{BC}, etc. This was tried, but simple expressions for d and d_1 were not found. Thus the signs of d and d_1 could not be determined analytically. However, since the d_i's are functions of only e_o, θ_S, and θ_o, it is not unreasonable to find the signs of d and d_1 numerically and thus find regions where separation is possible, if such regions exist.

NUMERICAL RESULTS

Linear Theory

To search numerically for regions where separation is possible, a program was written for the IBM 370/155 to calculate d_1 and d (along with other pertinent quantities) with e_o, θ_S and θ_o as inputs. For a given θ_S, θ_o was scanned from $\theta_o = \theta_S$ to $\theta_o = \theta_S + \pi$ to compute d_1 and d; θ_S was changed and θ_o scanned again, etc. This was done for e_o = .1, .5, and .9. Tables 1 through 3 show the various θ_S and θ_o combinations used for each e_o. These tables are presented since not all values of θ_S and θ_o could be done, and there is always the question of whether or not any regions of separation were missed. Tables 1 through 3 indicate the size, in terms of θ_o and θ_S, of any regions that possibly were missed.

The computations never resulted in d_1 being less than zero when d was less than zero. Thus there is no point (i.e., combination of θ_o and θ_S) where the orbits always separate as a result of a fixed attitude burn, regardless of the alignment. This is to be expected; it is known from experience that a fixed attitude burn can be fit to an impulse and, since θ_o is equivalent in a sense to time, all θ_o points can be generated by increasing the length of the impulsive ΔV vector. Consequently the rest of the results concerning separation regions are based on the $d > 0$ criterion; $d_1 < 0$ and $d < 0$ will no longer be considered.

Separation regions were found based on the $d > 0$ criterion. The initial computer run for each e_o was made for $\theta_S = 0$ to $\theta_S = 330^o$, inclusive, in steps of 30^o. When separation regions were noted, more cases were run to determine the boundaries of the regions, i.e., for a given θ_S, the value of θ_o for d = 0. No attempt was made to find this value more accurately than $\pm .5^o$ (except for a few cases where the separation regions were small). The separation regions are tabulated in tables 4 through 6 in terms of $\beta = \theta_o - \theta_S$, which is the angle on the initial orbit associated with the burn time. These separation regions also are plotted in figures 8 through 10. These figures are to be interpreted as follows. For example for $e_o = .1$ and $\theta_S = 120^o$, it is possible to select an alignment at ignition that will result in an orbital separation at $\beta = 10^o$. It also is possible to select an alignment at ignition that will result in an orbital separation at $\beta = 5^o$; but this alignment may not be the one which gave a separation at $\beta = 10^o$ and, in fact, may result in an intersection at $\beta = 10^o$.

Figures 8 through 10 are shown for θ_S between 0 and 360^o, with slight extensions beyond 360^o and before 0^o. It is unnecessary to investigate any other θ_S values since everything is periodic in θ_S with a period of 360^o (see equation (48) and the following equations).

Note the small region around $\theta_S = 0$ on the $e_o = .5$ plot. It is conceivable that regions like this could have been missed in the scan (and hence tables 1 through 3 are presented). When two orbits are separated, there is some point of closest approach between the two, a minimum separation distance. When an alignment at ignition produces separation for a small time (i.e., small $\Delta\beta$), this distance cannot get large with respect to the minimum separation distance resulting from an alignment that produces separation over a much larger period of time. It will be seen later that these small regions have associated alignments that produce separation only over very small $\Delta\beta$'s. Thus any missed regions are probably negligible for practical purposes, since the minimum separation distance is probably very small.

It must be remembered that this analysis does not hold for $\beta = 0$, i.e., at the start of the burn (although d = 0 there). This is because the expression for $\cos\theta_i$ (from which the criterion for separation was developed) had a 0/0 form at the start of the burn. Since the two orbits coincide at the start of the burn, there must be intersection at $\beta = 0$. In order to see if this intersection continues for β near zero or becomes a separation region,* θ_S values from 0° to 330° in steps of 30° and $\beta = .1^\circ, .2^\circ, \ldots, .9^\circ, 1^\circ$ were computed. The results of these computations were consistent with the rest of the $\beta \neq 0$ computations. Thus the separation regions are shown going down to the $\beta = 0$ axis with the understanding that $\beta = 0$ is excluded or is the boundary of the separation region.

It is possible to formulate the equations by using

$$\theta_o = \theta_S + \Delta\beta$$

where $\Delta\beta$ is small so that the variation of parameters technique could be used to calculate numerically results for non-linear changes

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*For example, on the $e_o = .1$ plot, does the separation region extend down to the $\beta = 0$ axis or is there a small strip between the $\beta = 0$ axis and the separation region?

in the orbital elements. This has not been done, but the cases computed near $\beta = 0$ imply that there would be no trouble encountered in doing this.

Also calculated in the program were the alignments $\alpha_o^{(1)}$ and $\alpha_o^{(2)}$ to give d = 0 at a particular θ_S and θ_o. Equation (122) was used for this, selecting the roots to give a posigrade alignment. (As pointed out in the previous section, if $\alpha_o^{(1)}$ results in d = 0 at some point, then $\alpha_o^{(1)} \pm \pi$ also results in d = 0 at that point.) Posigrade alignments only will be discussed from here on, but the ideas hold for retrograde alignments.

Finally, the alignment for a minimum $\hat{D}$, α_m, was calculated. The equation for this was developed from the equation for $\hat{D}$ by differentiating with respect to α.

$$\frac{d\hat{D}}{d\alpha} = 2A\frac{dA}{d\alpha} d_1 + 2d_2 \left(A\frac{dB}{d\alpha} + B\frac{dA}{d\alpha}\right) + 2\,B\frac{dB}{d\alpha}\,d_3$$

Using the definitions of A and B from equation (45) gives

$$\frac{dA}{d\alpha} = -B \qquad \frac{dB}{d\alpha} = +A \tag{123}$$

Thus an extremum of $\hat{D}$ occurs at

$$\tan 2(\alpha-\theta_S) = \frac{2d_2}{d_1-d_3} \tag{124}$$

This gives four extremum values of α between $\pm\,\pi$. The two posigrade alignments were selected, $\hat{D}$ calculated for each, and α_m selected as the α associated with the minimum $\hat{D}$.

The alignments $a_o^{(1)}$, a_m, and $a_o^{(2)}$ are tabulated in tables 7 through 9 for various values of θ_S and β; the alignments $a_o^{(1)}$ and $a_o^{(2)}$ are plotted in figures 11 through 14, the a_o envelopes. The particular θ_S values were chosen to be in different types of separation regions. For example, for $e_o = .9$, $\theta_S = 0$ goes through a no separation region, then a separation region; $\theta_S = 60$ stays in a separation region; $\theta_S = 150$ goes through a separation region, then a no separation region, etc.

A curve for a particular θ_S on the a_o envelope plots is shown coming together at the end of a separation region (for $\beta \neq 0$). This is because $a_o^{(1)} = a_o^{(2)} = a_m$ at the value of β where $d = 0$. To see this, note first that for $d = 0$, equation (122) can be written

$$\tan(a_o-\theta_S) = -\frac{d_1}{d_2 \pm \sqrt{d}} = -\frac{d_1}{d_2} \qquad (125)$$

which means that $a_o^{(1)}$ and $a_o^{(2)}$ are identical. Now equation (124) may be expanded by the double angle formula for the tangent to give

$$\frac{2\tan(a-\theta_S)}{1-\tan^2(a-\theta_S)} = \frac{2d_2}{d_1-d_3}$$

which can be written

$$\tan^2(a-\theta_S) + \frac{d_1-d_3}{d_2}\tan(a-\theta_S) - 1 = 0$$

This can be solved by the quadratic formula. Noting that $d = d_2^2 - d_1 d_3 = 0$ gives

$$\tan(a-\theta_S) = \frac{d_3-d_1}{2d_2} \pm \frac{1}{2d_2}(d_1+d_3)$$

Using the - sign gives

$$\tan(\alpha-\theta_S) = -\frac{d_1}{d_2} = \tan(\alpha_o-\theta_S) \tag{126}$$

A feeling for the size in some sense of this phenomenon can be obtained from the α_o envelope plots. For example, consider e_o = .1 and $\theta_S = 150^o$. If an alignment of -5^o were chosen, then separation would occur between about $\beta = 4.8^o$ and $\beta = 5.6^o$, a $\Delta\beta$ of $.8^o$. Conversely, if a separation were desired at $\beta = 5^o$, alignments between -4.7^o and -5.1^o must be used, a $\Delta\alpha$ of $.4^o$. Thus the β range where separation occurs for a given alignment is small relative to the β range over which separation is possible (for this example, roughly $.8^o$ vs. 12^o), and the α range for separation at a particular point is small relative to the total α range which will result in separation at some point (for this example, roughly $.4^o$ vs. 5.5^o). This holds for e_o = .1 and .5 and for some of the e_o = .9 curves. In particular, note the α_o envelope for the small separation region around $\theta_S = 0$, e_o = .5 (figure 12). Here a particular separation alignment gives separation over a very small $\Delta\beta$ relative to other separation regions. Thus as mentioned earlier in this section, the separation distances between orbits in these small regions are probably negligible for practical purposes. The smallness of the $\Delta\beta$ over which separation occurs for some initial alignment is probably why the numerical search described in the introduction for orbital separations failed.

Recall that in the Introduction, a qualitative idea of what possibly was happening physically during a separation was advanced.

The angle between the velocity vector at the start of the burn and the acceleration vector was important to this idea; thus the flight path angles at the start of the burn are indicated on the α_o envelope plots. For the $e_o = .1$ and $.5$ and some of the $e_o = .9$ curves, this qualitative idea seems to be valid. The α_o envelopes slope down to the right in the plots meaning that for separations at larger and larger β's, alignments farther and farther below the initial velocity vector must be used. This idea is not meant to explain separation for if it always held, then separation would be possible everywhere. There is something else happening that prevents separation most of the time.

The $e_o = .9$ α_o envelope needs to be mentioned specifically since it is so different from the α_o envelopes for $e_o = .1$ and $e_o = .5$. The $\theta_S = 150$ and the first part of the $\theta_S = 200$ curves look like the $e_o = .1$ and $.5$ curves. Note that the separation regions (figure 10) are more the size of the $e_o = .1$ and $.5$ regions. The $\theta_S = 300$ curve is for a separation region about 2 1/2 times as large as the $\theta_S = 150$ and $\theta_S = 120$ regions so the expanded size of the α_o envelope does not appear unreasonable. However, the $\theta_S = 350$, 0, and 60 α_o envelopes show drastic differences from the other curves in this plot and from the $e_o = .1$ and $.5$ α_o envelopes. The alignment regions for separation and the $\Delta\beta$ over which separation occurs for a particular alignment is large (except for the first part of the $\theta_S = 350$ curve). The top of the envelope for $\theta_S = 0$ and 60 have a different shape than is seen elsewhere. Here the qualitative explanation advanced earlier does not appear to hold. Note that the separation region (figure 10) is considerably different from that seen elsewhere and appears to extend for a very long distance (no cases were run to find the end of the region). In the next section, numerical integration will be used to check this linear theory; because of the difference of the $\theta_S = 0$, $e_o = .9\ a_o$ envelope from other such curves, this was a case numerically integrated.

TABLE 1

$e_o = .1$ cases run, inclusive

θ_S	θ_o
0	.1-(.1)-.9; 1-(1)-4; 5-(5)-180
30	30.1-(.1)-30.9; 31-(1)-34; 35-(5)-210
60	60.1-(.1)-60.9; 61-(1)-64; 65-(5)-240
70	71-(1)-80
80	81-(1)-90
85	86-(1)-95
87	88-(1)-97
89	90-(1)-99
90	90.1-(.1)-90.9; 91-(1)-94; 95; 100-(1)-104; 105-(5)-270
100	101-(1)-114
110	111-(1)-124
120	120.1-(.1)-120.9; 121-(1)-124; 125; 130-(1)-134; 135-(5)-300
135	145-(1)-149
150	150.1-(.1)-150.9;151-(1)-154;155;160-(1)-164;165-(5)-330
170	171-(1)-180
180	180.1-(.1)-180.9;181-(1)-189;190-(5)-360
185	186-(1)-195
190	191-(1)-204
210	210.1-(.1)-210.9;211-(1)-214;215;220-(1)-224;225-(5)-390
225	235-(1)-239
240	240.1-(.1)-240.9;241-(1)-244;245;250-(1)-254;255-(5)-420

θ_S	θ_o
255	265-(1)-269
270	270.1-(.1)-270.9;271-(1)-279;280-(5)-420
271	272-(1)-281
272	273-(1)-282
273	274-(1)-283
274	275-(1)-284
275	276-(1)-285
280	281-(1)-290
290	291-(1)-300
300	300.1-(.1)-300.9;301-(1) -304;305-(5)-480
330	330.1-(.1)-330.9;331-(1)-334;335-(5)-510

The numbers in ( ) indicate the increment in θ_o used to step from the value of θ_o on the left to the value of θ_o on the right.

Example: $\theta_S = 0$; θ_o = .1,.2,.3,......9,1.0,2.,3.,4.,5.,10., 15.,20.,.....,180.

TABLE 2

e_o = .5 cases run, inclusive

θ_S	
0	.1-(.1)-.9;1-(.5)-10;15-(5)-180
5	5.5-(.5)-15
10	10.5-(.5)-20;25-(5)-180
12.5	13-(.5)-17.5
15	15.5-(.5)-25
20	20.5-(.5)-30;35-(5)-200
25	30-(5)-75
27.5	47.5-(1)-66.5
28	48-(1)-72
29	49-(1)-73
30	30.1-(.1)-30.9;31-(1)-34;35-(5)-50;51-(1)-74;75-(5)-210
45	55-(1)-62;65-(5)-85;89-(1)-100;105-(5)-115
60	60.1-(.1)-60.9;61-(1)-65;70-(1)-74;75-(5)-100;101-(1)-104; 105-(5)-240
70	75;80-(1)-84;85-(5)-100;100-(1)-104;105-(5)-250
75	80;85-(1)-90;95-(1)-100;105-(5)-125
77.5	87.5-(1)-106.5
80	85-(5)-260
90	90.1-(.1)-90.9;91-(1)-94;95-(5)-270
120	120.1-(.1)-120.9;121-(1)-124;125-(5)-300
150	150.1-(.1)-150.9;151-(1)-154;155-(5)-330
180	180.1-(.1)-180.9;181-(1)-184;185-(5)-360

θ_S	
210	210.1-(.1)-210.9;211-(1)-214;215-(5)-390
240	240.1-(.1)-240.9;241-(1)-244;245-(5)-420
250	255-(5)-430
255	260-(5)-305
257.5	282.5-(1)-301.5
260	265-(5)-285;286-(1)-289;290-(5)-300;305-(1)-310;315-(5)-460
270	270.1-(.1)-270.9;271-(1)-274;275;280-(1)-284; 285-(5)-320;321-(1)-324;325-(5)-450
285	294-(1)-301;305-(5)-325;327-(1)-334
300	300.1-(.1)-300.9;301-(1)-305;310-(1)-315;320-(5)-330; 331-(1)-335;340-(5)-480
310	315;320-(1)-325;330-(1)-337;340-(5)-510
312.5	322.5-(1)-341.5
315	320-(5)-365
320	325-(5)-520
330	330.1-(.1)-330.9;331-(1)-334;335-(5)-510
340	340.5-(.5)-350
342.5	343-(.5)-347.5
345	345.5-(.5)-355
346	347-(1)-354
347	348-(1)-355
348	349-(1)-352

θ_S	
350	350.5-(.5)-360
355	355.5-(.5)-360

The numbers in ( ) indicate the increment in θ_o used to step from the value of θ_o on the left to the value of θ_o on the right.

Example: θ_S = 90; θ_o = 90.1, 90.2, . . . , 90.9, 91., 92., 93., 94., 95., 100., 105., ... , 270.

TABLE 3

$e_o = .9$, cases run

θ_S	
0	.1-(.1)-.9;1-(1)-30;35-(5)-180
5	6-(1)-20
10	11-(1)-25
15	16-(1)-30
20	21-(1)-40
25	26-(1)-45;50-(5)-75
27.5	28-(.5)-32.5
30	30.1-(.1)-30.9;31-(1)-34;35-(5)-210
40	45-(5)-200;201-(1)-205;210-(5)-220
50	55-(5)-190;191-(1)-194;195-(5)-210
60	60.1-(.1)-60.9;61-(1)-64;65-(5)-180;180-(1)-184;185-(5)-240
75	80-(5)-125
90	90.1-(.1)-90.9;91-(1)-94;95-(5)-155;156-(1)-159;160-(5)-270
120	120.1-(.1)-120.9;121-(1)-124;125-(5)-155;156-(1)-159; 160-(5)-300
150	150.1-(.1)-150.9;151-(1)-154;155-(5)-175;176-(1)-179; 180-(5)-325;326-(1)-330
180	180.1-(.1)-180.9;181-(1)-184;185-(5)-205;206-(1)-209; 210-(5)-295;295-(1)-299;300-(5)-360
185	190-(5)-210;211-(1)-214;215-(5)-365
190	195-(5)-215;216-(1)-219;220-(5)-280;281-(1)-284;285-(5)-370

θ_S	
195	200-(5)-375
200	205-(5)-230;231-(1)-234;235-(5)-270;271-(1)-274; 275-(5)-370;371-(1)-375;380
205	210-(5)-240;241-(1)-244;245-(5)-265;266-(1)-269; 270-(5)-385
207	237-(1)-266
209	210-(1)-268
210	210.1(.1)-210.9;211-(1)-214;215-(5)-370;371-(1)-374; 375-(5)-390
240	240.1-(.1)-240.9;241-(1)-244;245-(5)-370;370-(1)-374; 375-(5)-420
255	260-(5)-305
270	270.1-(.1)-270.9;271-(1)-274;275-(5)-370; 371(1)-379;380-(5)-450
300	300.1-(.1)-300.9;301-(1)-304;305-(5)-375;376-(1)-379; 380-(5)-480
315	320-(5)-360
330	330.1-(.1)-330.9;331(-1)-334;335-(5)-370;371-(1)-379; 380-(5)-510
331	332-(1)-336
332	333-(1)-337
332.5	333.5-(1)-337.5
335	335.5-(.5)-340
340	340.5-(.5)-345;350-(5)-365;366-(1)-369;370-(5)-520

θ_S	
345	346-(1)-375;380-(5)-405;406-(1)-409;410-(5)-465; 466-(1)-469;470-(5)-520
350	351-(1)-395;400-(5)-530
352.5	353-(.5)-357.5
355	356-(1)-395
357.5	358.5-(1)-377.5

The numbers in ( ) indicate the increment in θ_o used to step from the value of θ_o on the left to the value of θ_o on the right.

Example: θ_S = 330; θ_o = 330.1, 330.2, . . . , 330.9, 331., 332., 333., 334., 335., 340., . . . , 370., 371., 372., ... , 379., 380.,385., . . . 510.

TABLE 4

$e_o = .1$

Separation Regions

θ_S	β
85	2/3 - 7/8
87	0/1 - 8/9
89	0/1 - 9/10
90	0/.1 - 10/11
100	0/1 - 12/13
110	0/1 - 12/13
120	0/.1 - 13/14
135	- 12/13
150	0/.1 - 11/12
170	0/1 - 9/10
180	0/.1 - 9/10
185	0/1 - 9/10
190	0/1 - 10/11
210	0/.1 - 12/13
225	- 12/13
240	0/.1 - 12/13
255	- 11/12
270	0/.1 - 5/6
271	0/1 - 4/5
272	0/1 - 3/4
273	0/1 - 1/2

a/b - c/d means that separation starts between $\beta = a$ and $\beta = b$ and terminates between $\beta = c$ and $\beta = d$.

-c/d means that cases were not run to determine a/b

TABLE 5

$e_o = .5$

Separation Regions

θ_S	β
0	0/.1 - 4/4.5
5	0/.5 - 4/4.5
10	0/.5 - 3/3.5
12.5	0/.5 - 2/2.5
28	30/31 - 35/36
29	23/24 - 41/42
30	20/21 - 44/45
45	10/11 - 51/52
60	10/11 - 42/43
70	10/11 - 31/32
75	11/12 - 24/25
77.5	12/13 - 20/21
260	26/27 - 49/50
270	12/13 - 52/53
285	10/11 - 44/45
300	10/11 - 32/33
310	11/12 - 20/21
312.5	12/13 - 15/16
346	0/1 - 3/4
347	0/1 - 3/4
348	0/1 - 3/4
350	0/.5 - 4/4.5
355	0/.5 - 4/4.5

a/b - c/d means that separation starts between β = a and β = b and terminates between β = c and β = d.

TABLE 6

$e_o = .9$

Separation Regions

θ_S	β
0	7/8 - > 180
5	5/6 -
10	5/6 -
15	4/5 -
20	4/5 -
25	3/4 -
27.5	2.5/3 -
30	0/.1 - > 180
40	0/5 - 163/164
50	0/5 - 142/143
60	0/.1 - 123/124
90	0/.1 - 68/69
120	0/.1 - 35/36
150	0/.1 - 26/27; 175/176 - > 180
180	0/.1 - 25/26; 116/117 - > 180
185	0/5 - 26/27; 105/110 - > 180
190	0/5 - 27/28; 94/95 - > 180
195	0/5 - 25/30; 80/85 - 175/180
200	0/5 - 32/33; 72/73 - 172/173
205	0/5 - 36/37; 60/61 - 165/170
207	- 39/40; 53/54
210	0/.1 - 163/164
240	0/.1 - 134/135
270	0/.1 - 105/106
300	0/.1 - 75/76
330	0/.1 - 42/43
331	3/4 -
332	4/5 -
332.5	4/5 -

θ_S	β
335	4.5/5 -
340	4.5/5 - 28/29
345	5/6 - 21/22; 60/61 - 121/122
350	5/6 -12/13; 32/33 - > 180
355	18/19 -
357.5	12/13 -

a/b - c/d; e/f - g/h means that separation starts between β = a and β = b and terminates between β = c and β = d, then starts again between β = e and β = f and terminates between β = g and β = h.
a/b - means that cases were not run to determine c/d.

TABLE 7

$e_o = .1$

Separation Alignments

θ_S	β	$a_o^{(1)}$	a_m	$a_o^{(2)}$	γ_S
85	3	-7.19	-7.23	-7.27	-5.74
	4	-7.67	-7.72	-7.79	
	5	-8.15	-8.21	-8.28	
	6	-8.64	-8.70	-8.76	
120	.1	-4.46	-4.76	-5.07	-4.72
	1	-4.88	-5.19	-5.49	
	3	-5.82	-6.13	-6.43	
	5	-6.76	-7.06	-7.37	
	10	-9.13	-9.38	-9.63	
	11	-9.63	-9.84	-10.06	
	12	-10.14	-10.30	-10.46	
	13	-10.72	-10.76	-10.79	
150	.1	-2.43	-2.68	-2.93	-2.63
	1	-2.84	-3.09	-3.34	
	3	-3.76	-4.01	-4.25	
	5	-4.68	-4.92	-5.16	
	10	-7.05	-7.20	-7.34	
	11	-7.57	-7.65	-7.73	
180	.1	.12	-.05	-.21	.00
	1	-.29	-.45	-.62	
	3	-1.20	-1.36	-1.53	
	5	-2.43	-2.27	-3.12	
	7	-3.04	-3.18	-3.32	
	9	-4.01	-4.09	-4.18	

θ_S	β	$a_o^{(1)}$	a_m	$a_o^{(2)}$	γ_S
210	.1	2.84	2.59	2.34	2.63
	1	2.43	2.17	1.92	
	3	1.51	1.25	1.00	
	5	.58	.33	.07	
	10	-1.80	-2.00	-2.20	
	11	-2.31	-2.47	-2.62	
	12	-2.86	-2.94	-3.01	
240	.1	4.97	4.67	4.36	4.72
	1	4.55	4.24	3.94	
	3	3.59	3.29	2.99	
	5	2.63	2.33	2.03	
	10	.15	-.08	-.32	
	11	-.38	-.57	-.77	
	12	-.93	-1.06	-1.19	
270	.1	5.79	5.66	5.54	5.71
	1	5.34	5.22	5.10	
	3	4.32	4.22	4.12	
	4	3.80	3.72	3.63	
	5	3.28	3.21	3.15	

TABLE 8

$e_o = .5$

Separation Alignments

θ_S	β	$a_o^{(1)}$	a_m	$a_o^{(2)}$	γ_S
346	1	12.32	12.30	12.27	13.22
	2	11.39	11.36	11.33	
	3	10.44	10.41	10.39	
0	.1	-.02	-.10	-.18	0
	1	-.92	-1.00	-1.08	
	2	-1.92	-2.00	-2.08	
	3	-2.93	-3.00	-3.07	
	4	-3.94	-3.99	-4.04	
12.5	.5	-12.37	-12.40	-12.43	-11.94
	1.5	-13.31	-13.33	-13.35	
	2	-13.78	-13.79	-13.80	
28	31	-40.01	-40.04	-40.07	-22.80
	32	-40.35	-40.42	-40.50	
	34	-41.06	-41.16	-41.25	
	35	-41.44	-41.51	-41.59	
45	11	-34.45	-34.50	-34.56	-28.68
	15	-36.09	-36.30	-36.51	
	20	-37.93	-38.32	-38.71	
	25	-39.55	-40.15	-40.72	
	30	-41.01	-41.77	-42.53	
	35	-42.34	-43.25	-44.16	
	40	-43.62	-44.60	-45.58	
	45	-44.66	-45.61	-46.56	
	50	-46.43	-47.02	-47.61	
	51	-46.83	-47.25	-47.67	

θ_S	β	$a_o^{(1)}$	a_m	$a_o^{(2)}$	γ_S
60	11	-34.86	-34.93	-34.99	-30.00
	15	-36.27	-36.46	-36.65	
	20	-37.89	-38.21	-38.54	
	25	-39.35	-39.81	-40.27	
	30	-40.71	-41.26	-41.82	
	35	-42.03	-42.60	-43.18	
	40	-43.43	-43.85	-44.27	
	41	-43.76	-44.09	-44.42	
	42	-44.15	-44.33	-44.51	
77.5	13	-33.67	-33.71	-33.74	-28.70
	15	-34.32	-34.38	-34.45	
	17	-34.97	-35.04	-35.11	
	19	-35.61	-35.67	-35.74	
	20	-35.94	-35.98	-36.02	
260	27	11.87	11.79	11.72	24.37
	30	10.25	10.07	9.90	
	35	7.37	7.06	6.75	
	40	4.27	3.85	3.43	
	45	.89	.45	.02	
	47	-.58	-.96	-1.33	
	49	-2.19	-2.40	-2.60	
285	11	23.73	23.68	23.62	29.02
	15	21.65	21.46	21.27	
	20	18.84	18.48	18.11	
	25	15.81	15.25	14.68	
	30	12.53	11.77	11.01	
	35	8.96	8.06	7.15	
	40	5.01	4.12	3.24	
	42	3.26	2.49	1.72	
	44	1.32	.83	.34	

θ_S	β	$a_o^{(1)}$	a_m	$a_o^{(2)}$	γ_S
300	11	23.93	23.86	23.79	30.00
	15	21.51	21.30	21.08	
	20	18.21	17.83	17.46	
	25	14.58	14.09	13.61	
	30	10.50	10.09	9.68	
	31	9.58	9.26	8.94	
	32	8.52	8.42	8.32	
312.5	13	20.58	20.56	20.54	29.10
	14	19.86	19.81	19.77	
	15	19.09	19.06	19.02	

TABLE 9

$e_o = .9$

Separation Alignments

θ_S	β	$a_o^{(1)}$	a_m	$a_o^{(2)}$	γ_S
350	6	42.04	41.69	41.35	53.97
	8	36.76	35.94	35.12	
	10	30.40	29.28	28.16	
	12	22.64	21.81	20.99	
350	33	-37.99	-39.08	-40.17	
	35	-36.59	-41.21	-45.83	
	40	-36.73	-45.15	-53.58	
	50	-37.54	-49.71	-61.89	
	60	-38.40	-52.29	-66.18	
	70	-39.37	-54.00	-68.64	
	80	-40.42	-55.27	-70.13	
	90	-41.51	-56.28	-71.05	
0	8	-33.44	-33.81	-34.17	0
	10	-38.16	-39.28	-40.41	
	15	-45.45	-48.60	-51.75	
	20	-48.48	-53.76	-59.09	
	25	-49.17	-56.60	-64.02	
	30	-48.86	-58.16	-67.47	
	40	-47.41	-59.62	-71.84	
	50	-46.22	-60.30	-74.39	
	60	-45.57	-60.78	-76.00	
	70	-45.36	-61.21	-77.06	
	80	-45.43	-61.61	-77.79	
	90	-45.70	-61.99	-78.29	

θ_S	β	$a_o^{(1)}$	a_m	$a_o^{(2)}$	γ_S
60	.1	-54.28	-54.82	-55.35	-54.79
	1	-54.54	-55.08	-55.63	
	5	-55.59	-56.16	-56.73	
	10	-56.61	-57.27	-57.92	
	20	-57.72	-58.88	-60.04	
	30	-57.91	-59.92	-61.92	
	40	-57.64	-60.59	-63.55	
	50	-57.20	-61.08	-64.95	
	60	-56.79	-61.47	-66.14	
	70	-56.56	-61.85	-67.15	
	80	-56.59	-62.28	-67.96	
	90	-56.98	-62.77	-68.56	
150	.1	-13.42	-14.20	-14.99	-14.14
	1	-13.66	-14.44	-15.22	
	5	-14.72	-15.48	-16.24	
	10	-16.01	-16.75	-17.48	
	20	-18.65	-19.23	-19.80	
	25	-20.18	-20.46	-20.74	
	26	-20.63	-20.71	-20.80	
300	.1	55.29	54.76	54.22	54.79
	1	55.01	54.48	53.95	
	5	53.63	53.12	52.61	
	10	51.69	51.11	50.53	
	20	47.42	45.89	44.36	
	30	42.29	38.84	35.39	
	40	35.85	29.61	23.37	
	50	27.47	17.28	7.08	
	60	15.99	.78	-14.43	
	70	-1.36	-16.00	-30.63	
	75	-16.53	-22.76	-28.99	

θ_S	β	$a_o^{(1)}$	a_m	$a_o^{(2)}$	γ_S
200	5	8.84	8.12	7.40	9.47
	10	7.48	6.75	6.01	
	20	4.53	3.83	3.13	
	25	2.86	2.26	1.66	
	30	.99	.61	.23	
	31	.56	.27	-.02	
	32	.07	-.08	-.22	
200	73	-18.34	-18.75	-19.16	
	75	-18.89	-19.93	-20.96	
	80	-20.92	-22.97	-25.02	
	85	-23.23	-26.18	-29.12	
	90	-25.71	-29.53	-33.35	

$e_o = .1$

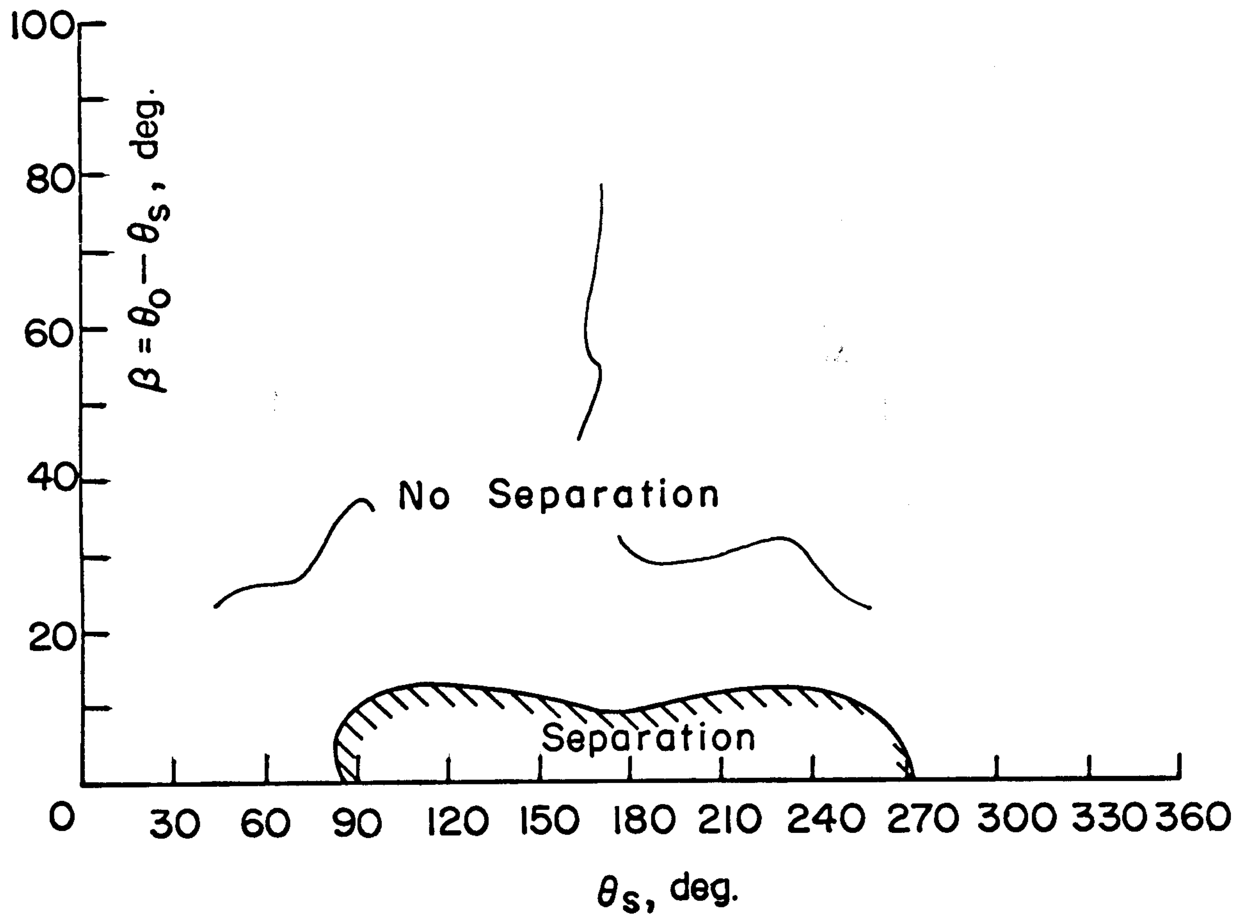

100
80
60
40
20
0
β = θo − θs , deg.
No Separation
Separation
30
60
90
120
150
180
210
240
270
300
330
360
θs, deg.

Figure 8

$e_o = .5$

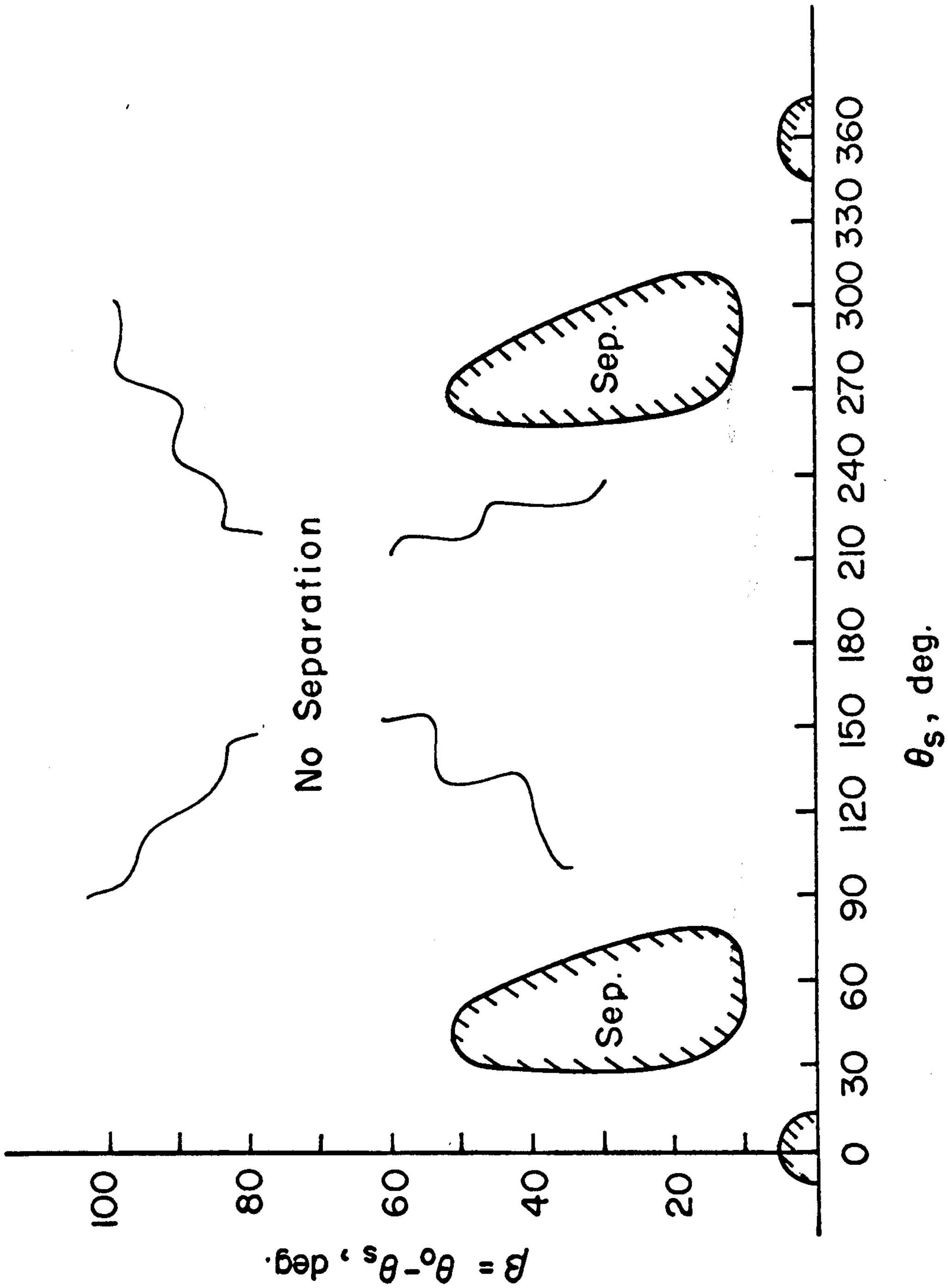

No Separation
Sep.
Sep.
0 30 60 90 120 150 180 210 240 270 300 330 360
θ_s, deg.
20 40 60 80 100
$\beta = \theta_o - \theta_s$, deg.

Figure 9

$e_o = .9$

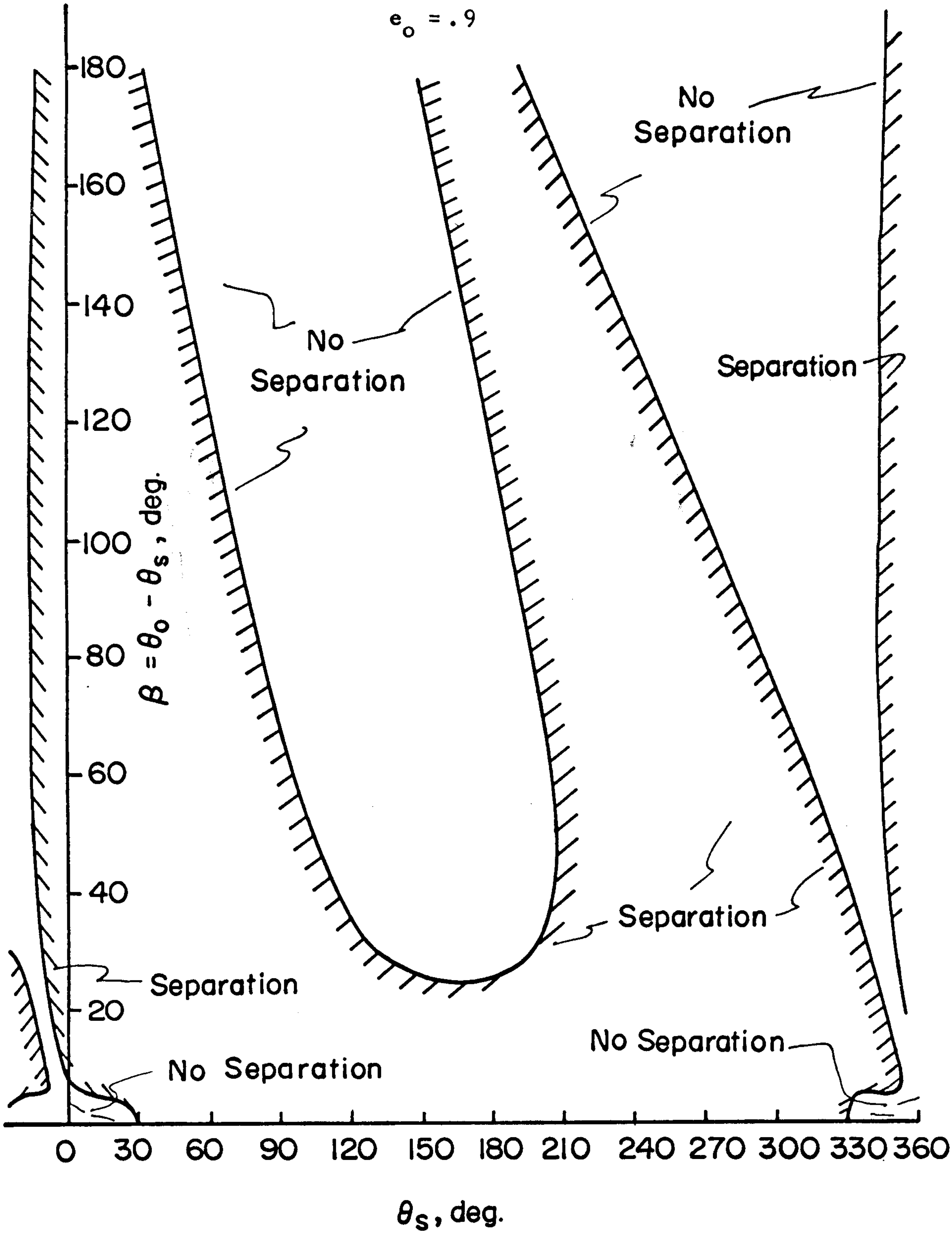
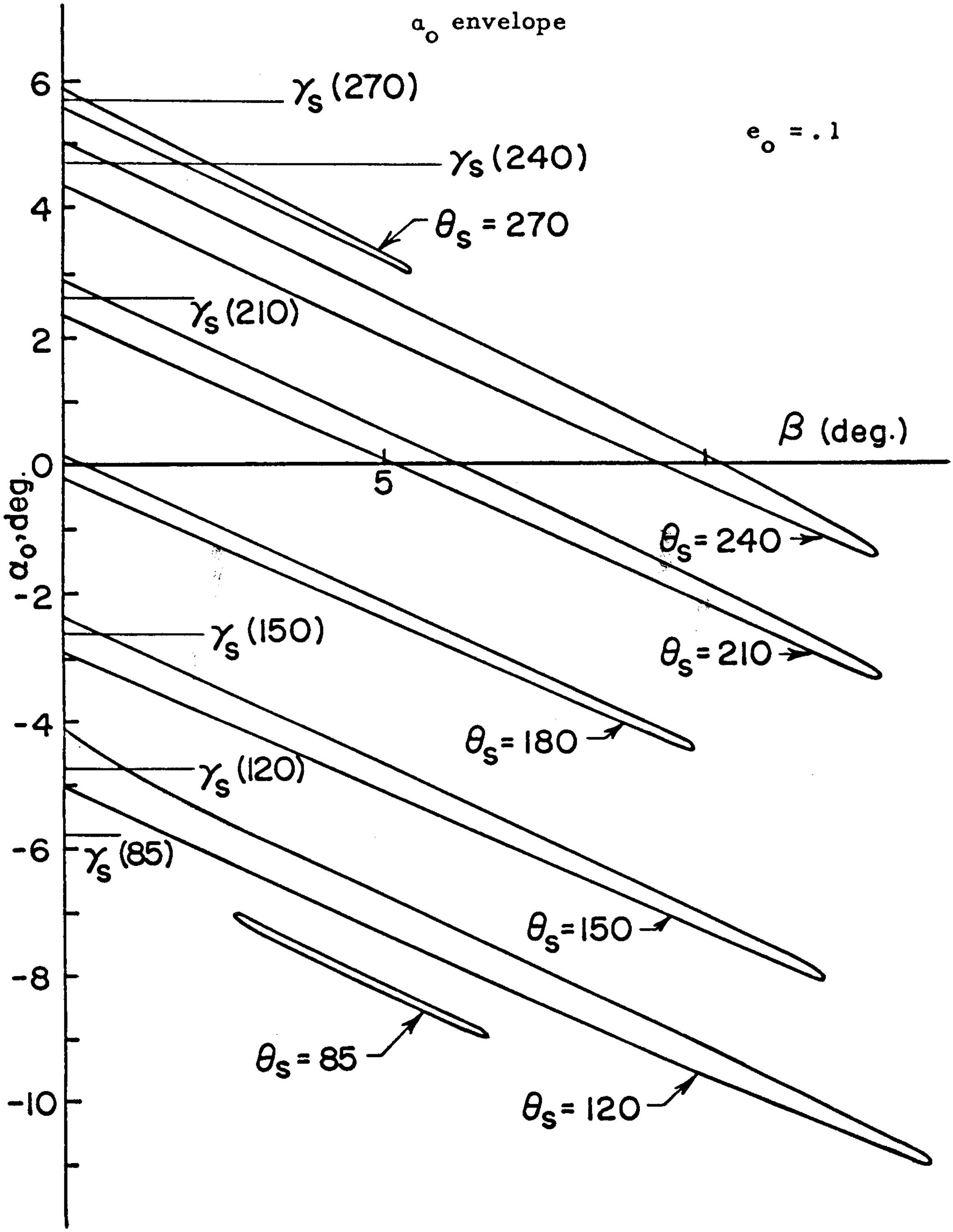
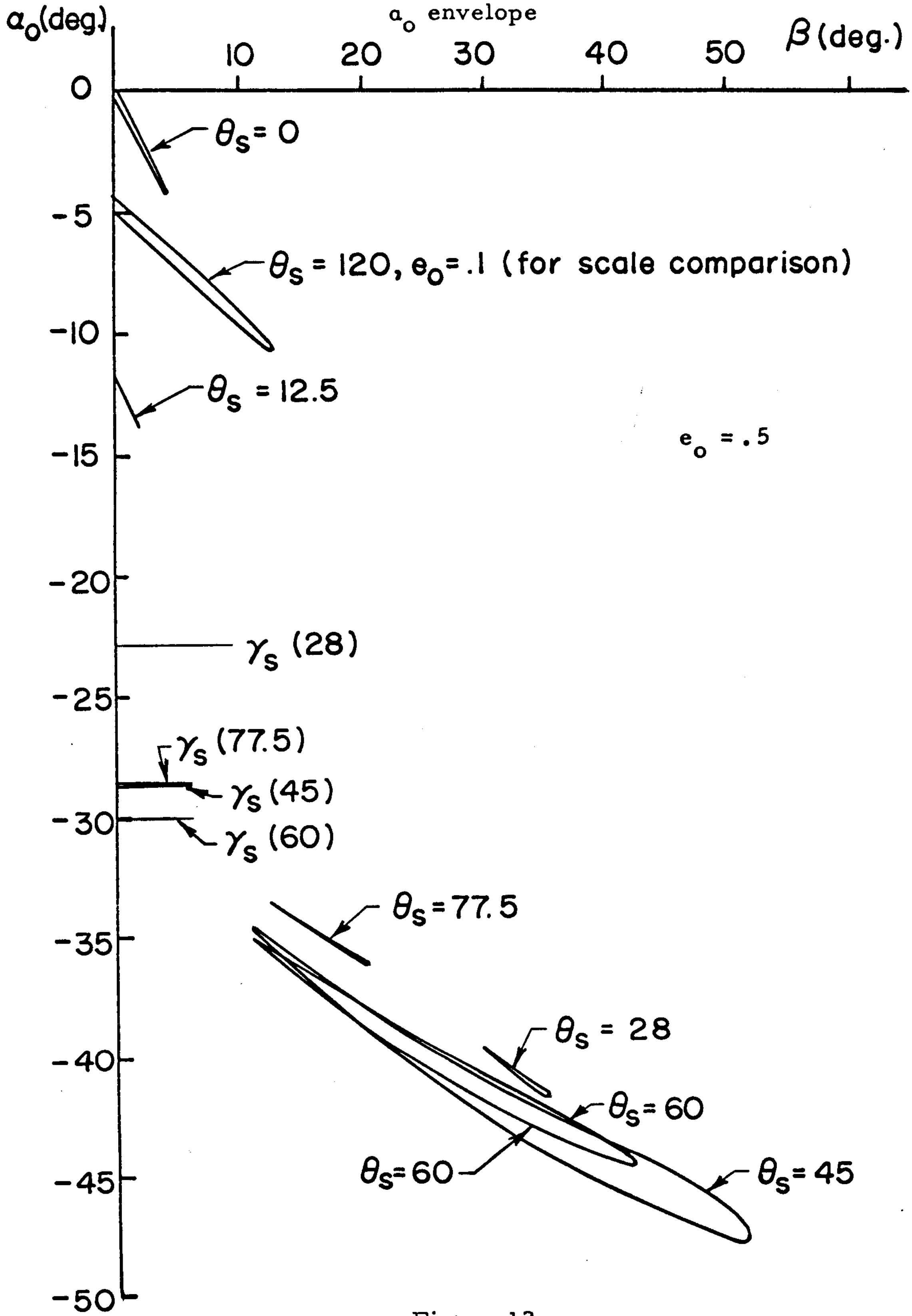


Figure 10



a_o envelope
$e_o = .1$
γ_s (270)
γ_s (240)
$\theta_s = 270$
γ_s (210)
β (deg.)
$\theta_s = 240$
$\theta_s = 210$
γ_s (150)
$\theta_s = 180$
γ_s (120)
γ_s (85)
$\theta_s = 150$
$\theta_s = 85$
$\theta_s = 120$
a_o, deg.
6
4
2
0
-2
-4
-6
-8
-10
5

Figure 11



a_o envelope
a_o(deg)
β (deg.)
10
20
30
40
50
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
θ_s = 0
θ_s = 120, e_o = .1 (for scale comparison)
θ_s = 12.5
e_o = .5
γ_s (28)
γ_s (77.5)
γ_s (45)
γ_s (60)
θ_s = 77.5
θ_s = 28
θ_s = 60
θ_s = 60
θ_s = 45

Figure 12

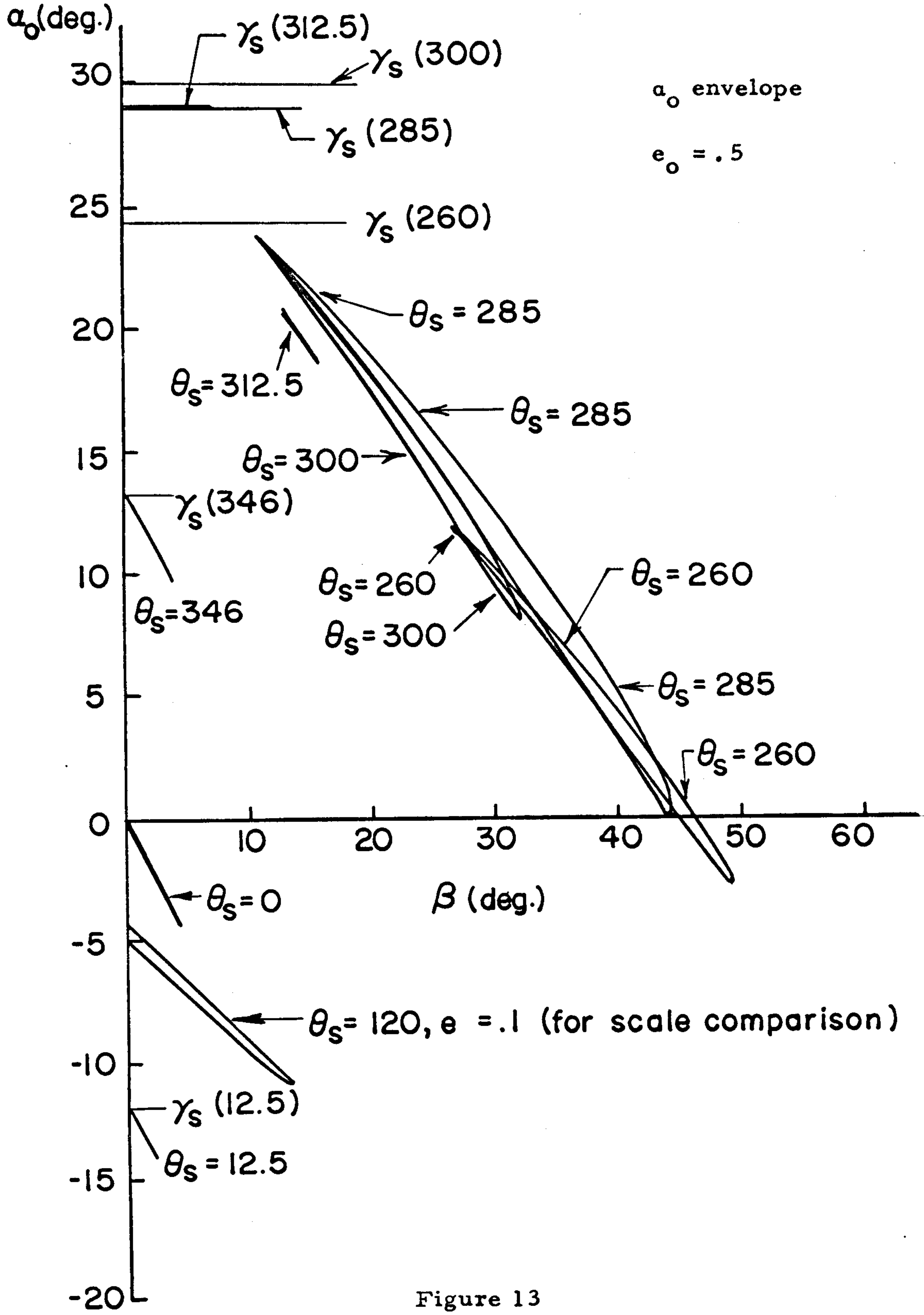

a_o(deg.)
γ_s (312.5)
γ_s (300)
γ_s (285)
γ_s (260)
a_o envelope
e_o = .5
θ_s = 285
θ_s= 312.5
θ_s= 285
θ_s= 300
γ_s(346)
θ_s= 260
θ_s= 260
θ_s=346
θ_s= 300
θ_s= 285
θ_s= 260
θ_s= 0
β (deg.)
θ_s= 120, e =.1 (for scale comparison)
γ_s (12.5)
θ_s = 12.5
30
25
20
15
10
5
0
-5
-10
-15
-20
10
20
30
40
50
60

Figure 13

a_o envelope

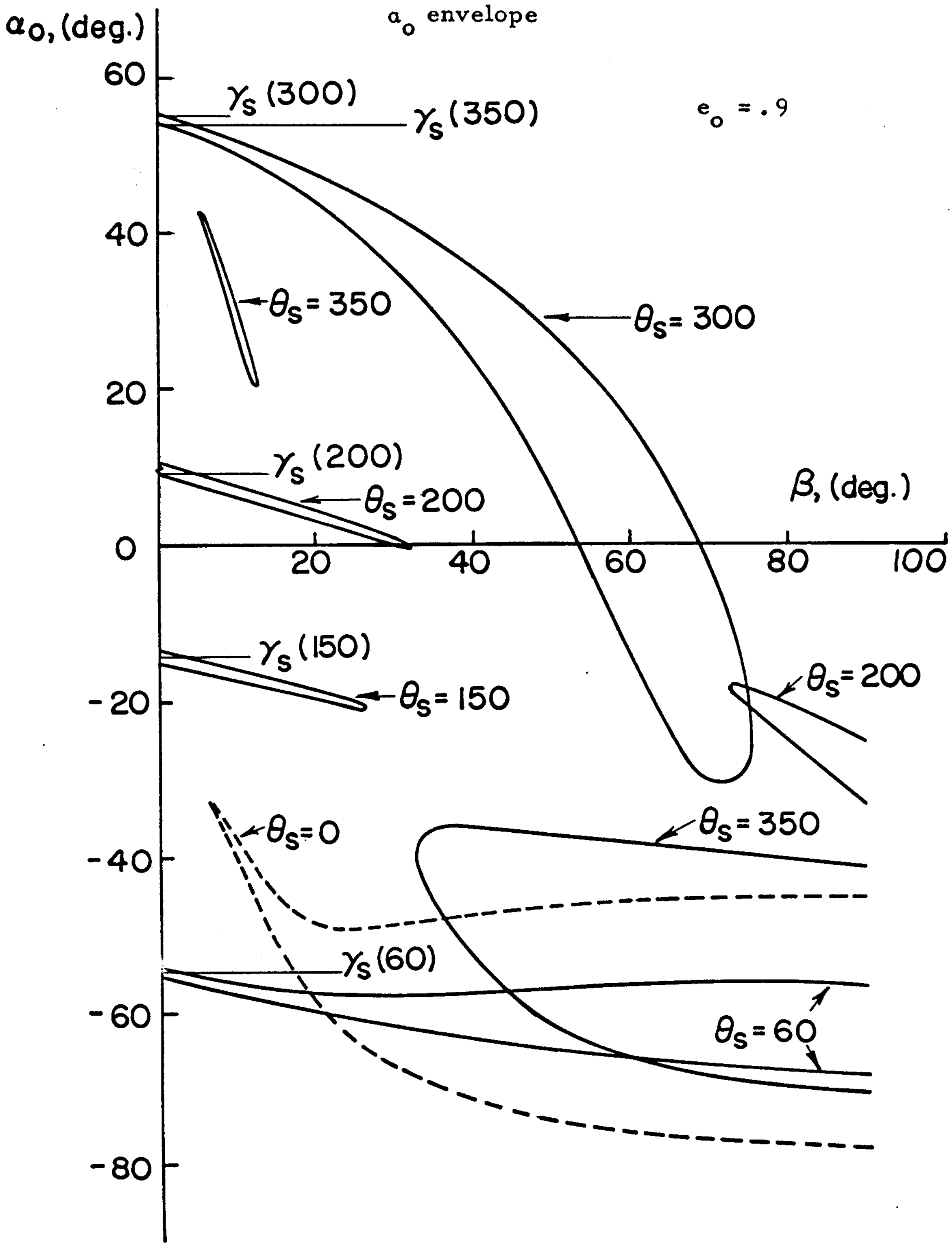


Figure 14

Numerical Integration Verification

In order to check this theory, equation (1) with a coplanar acceleration was integrated numerically to obtain the actual ΔP, Δe, and Δw for two e_o-θ_S-α combinations that, according to the theory, should result in an orbital separation. Then the $D < 0$ criterion was applied to verify that separation occurred when and where it should. The integration procedure and equations are discussed in Appendix C.

Tables 10 and 11 compare the non-dimensionalized orbital elements obtained from the numerical integration state vector with those obtained from the linear theory for two different eccentricities and acceleration magnitudes. (The linear theory orbital elements were calculated in two special runs, not tabulated in tables 1 and 3.) Table 10 is a low acceleration, long burn time case while table 11 is a higher acceleration, shorter burn time case.

Tables 12 and 13 show D (equation (12)) as a function of β. Also shown are the various terms that make up the linearized version of D (equation (31)) as well as the first neglected term in the linearization. Using equation (31) for the linearized version of D (and calling it D_L here to distinguish it from D of equation (12)) and the series expression for δ, i.e.,

$$\delta = \frac{\Delta w^2}{2} + O(\Delta w^4)$$

D may be written

$$D = D_L + \Delta w^2 P_o e_o (P_o \Delta e + e_o \Delta P) + O(\Delta w^4) + O(\Delta w^2 \Delta P \Delta e) \qquad (127)$$

It may be seen from tables 12 and 13 that the terms of $O(\Delta w^4)$ and $O(\Delta w^2 \Delta P \Delta e)$ do not affect the sign of D. However the first neglected term can be important in the sign of D.

Thus tables 12 and 13 show that the acceleration magnitude must be chosen such that the higher order terms in D are smaller than might normally be used in a linear theory. Consider table 12, $\beta = 17.376$, for example. Note that D_L is four orders of magnitude less than its largest component term, ΔP^2. Thus if the neglected term was only three orders of magnitude less than ΔP^2, the neglected term would decide the sign of D, i.e., decide if there was separation or not. For this particular case the neglected term, $\Delta w^2(P_o^2 e_o \Delta e + P_o e_o^2 \Delta P)$, is one order of magnitude less than D_L, and thus the linear theory gives the correct answer. This numerical cancellation of the terms of D_L was to be expected since for $D_L = 0$, the terms in D_L must cancel completely. This numerical cancellation requires some care in the integration; the significant figures in the state vector that determine the sign of D are near the end of single precision accuracy.

It has been mentioned previously that the minimum distance between separated orbits was expected to be small since separation did not happen over a large $\Delta\beta$. If linear orbital elements were used to calculate that distance, and these linear orbital elements were only an order of magnitude or so larger than their squares, then this distance would be significant (some calculations were done this way). However, due to the previously mentioned numerical cancellation problem, linear orbital elements much smaller than these must be used. When this is done, there is trouble in calculating the minimum separation distance due to uncertainty about how to round the last single precision figure. But the distance is quite small relative to using the larger linear orbital elements, e. g., O(100 ft) for $e_o = .1$.

Tables 12 and 13, numerical cancellation notwithstanding, demonstrate orbital separation and are considered verification for the linear theory. The alignment used in table 12, $\alpha = -48.6^o$, should first give no separation, then separation, then no separation again, and finally separation (see figure 14 and/or table 9). And this is what occurs. Where the linear theory predicts a separation, one occurs and vice versa.

Table 13 was generated to see if the results of the linear theory could be extended to substantial accelerations. An acceleration of .2 earth radii/hr^2 is like that produced by the Apollo reaction control system engine, which is used for small orbital maneuvers as well as for attitude control. The D_L of the last entry is negative while the actual D is positive; this is because this point is very close to the d = 0 boundary and numerical cancellation in D_L is much more pronounced. (This same explanation applies to table 12, $\beta = 20.281$ also.)

TABLE 10

β =	12.186	14.575	17.376	20.821	25.377	32.190
$\hat{\Delta P}_{int} \times 10^{-3}$ =	.259645	.295040	.328750	.360173	.388538	.412778
$\hat{\Delta P}_{linear} \times 10^{-3}$ =	.259582	.294958	.328649	.360052	.388399	.412624
$\hat{\Delta e}_{int} \times 10^{-2}$ =	-.245539	-.276263	-.304454	-.329393	-.350150	-.365436
$\hat{\Delta e}_{linear} \times 10^{-2}$ =	-.245494	-.276208	-.304391	-.329325	-.350080	-.365372
$\hat{\Delta w}_{int} \times 10^{-2}$ =	.300830	.368677	.442505	.521811	.605646	.692111
$\hat{\Delta w}_{linear} \times 10^{-2}$ =	.300737	.368549	.442337	.521599	.605387	.691804

$e_o = .9$, $a_o = 1$ er, $|\vec{a}| = .002$ er/hr^2, $\alpha = -48.6^\circ$, $\theta_S = 0^\circ$, μ of the earth in er^3/hr^2

er = earth radius

TABLE 11

$\beta =$	.573	1.290	2.009
$\hat{\Delta P}_{int} \times 10 =$	.171838	.386656	.601467
$\hat{\Delta P}_{linear} \times 10 =$	.171801	.386596	.601349
$\hat{\Delta e}_{int} \times 10 =$	.104368	.236187	.369503
$\hat{\Delta e}_{linear} \times 10 =$	.104253	.235637	.368153
$\hat{\Delta w}_{int} =$	.147991	.331948	.514725
$\hat{\Delta w}_{linear} =$	.148101	.332649	.516475

$e_o = .1$, $a_o = 1$ er, $|\vec{a}| = .2$ er/hr^2, $\alpha = -5.3^o$, $\theta_S = 120^o$,

μ of the earth

er = earth radius

TABLE 12

β	$(e_o \Delta P - P_o \Delta e)^2$	$-\Delta P^2$	$(P_o e_o \Delta w)^2$	$\Delta w^2 (P_o^2 e_o \Delta e + e_o^2 P_o \Delta P)$	D_{linear}	D
12.186	$.316587 \times 10^{-7}$	$-.320055 \times 10^{-7}$	$.348011 \times 10^{-9}$	$.293251 \times 10^{-12}$	$+.12 \times 10^{-11}$	$.15 \times 10^{-11}$
14.575	$.408020 \times 10^{-7}$	$-.413263 \times 10^{-7}$	$.522686 \times 10^{-9}$	$.501060 \times 10^{-12}$	$-.17 \times 10^{-11}$	$-.11 \times 10^{-11}$
17.376	$.505538 \times 10^{-7}$	$-.513093 \times 10^{-7}$	$.752984 \times 10^{-9}$	$.805325 \times 10^{-12}$	$-.26 \times 10^{-11}$	$-.18 \times 10^{-11}$
20.821	$.605386 \times 10^{-7}$	$-.615867 \times 10^{-7}$	$.104707 \times 10^{-8}$	$.122865 \times 10^{-11}$	$-.11 \times 10^{-11}$	$+.16 \times 10^{-12}$
25.377	$.702599 \times 10^{-7}$	$-.716691 \times 10^{-7}$	$.141055 \times 10^{-8}$	$.178846 \times 10^{-11}$	$-.13 \times 10^{-10}$	$+.31 \times 10^{-11}$
32.190	$.790458 \times 10^{-7}$	$-.808907 \times 10^{-7}$	$.184205 \times 10^{-8}$	$.248615 \times 10^{-11}$	$-.28 \times 10^{-11}$	$-.32 \times 10^{-12}$
32.366	$.792061 \times 10^{-7}$	$-.810604 \times 10^{-7}$	$.185129 \times 10^{-8}$	$.250135 \times 10^{-11}$	$-.31 \times 10^{-11}$	$-.56 \times 10^{-12}$

$e_o = .9$, $a_o = 1er$, $|\vec{a}| = .002\ er/hr^2$, $\alpha = -48.6^\circ$, $\theta_S = 0^\circ$, μ of the earth

er = earth radius

TABLE 13

β	$(e_o \Delta P - P_o \Delta e)^2$	$-\Delta P^2$	$(P_o e_o \Delta w)^2$	$\Delta w^2 (P_o^2 e_o \Delta e + e_o^2 P_o \Delta P)$	D_{linear}	D
.573	$.722158 \times 10^{-8}$	$-.280539 \times 10^{-7}$	$.208076 \times 10^{-7}$	$.249016 \times 10^{-10}$	$-.25 \times 10^{-10}$	$+.17 \times 10^{-12}$
1.290	$.370665 \times 10^{-7}$	$-.142038 \times 10^{-6}$	$.104688 \times 10^{-6}$	$.283294 \times 10^{-9}$	$-.28 \times 10^{-9}$	$-.48 \times 10^{-12}$
2.009	$.909225 \times 10^{-7}$	$-.343699 \times 10^{-6}$	$.251713 \times 10^{-6}$	$.106478 \times 10^{-8}$	$-.11 \times 10^{-8}$	$+.15 \times 10^{-11}$

$e_o = .1$, $a_o = 1$ er, $|\vec{a}| = .2$ er/hr^2, $\alpha = -5.13^\circ$, $\theta_S = 120^\circ$, μ of the earth

er = earth radius

CONCLUSIONS

As a result of the preceding investiation, the following conclusions may be drawn:

1. It is possible to obtain neighboring, separated, Keplerian orbits by using a fixed attitude thrust.

2. The acceleration level that will do this is not insignificant; no conclusion can be drawn from this theory about how large this acceleration level may be.

3. This orbital separation phenomenon does not appear too useful for orbital trim maneuvers to remove dispersions since the alignment ranges to give a separation are small and the region over which the orbits separate for a given alignment is small (except possibly for some ignition polar angles for large eccentricities). The phenomenon may be useful for avoiding recontact using only one burn, but its usefulness depends on the necessary margin for error. However, the separation regions and alignments have been identified and can be investigated if desired.

REFERENCES

1. Wu, Jain-Ming, A Satellite Theory and its Applications, Ph. D. Thesis, California Institute of Technology, 1965.

APPENDIX A

Evaluation of Δr and $\Delta r'$

Using equations (35), (40, and (47), equation (37) for Δr may be written

$$\Delta r = \int_{\theta_S}^{\theta_o} \sin(\theta_o - \theta)\left[F_r \frac{r_o^4}{C_o^2} + \frac{2r_o^3}{C_o^2}(r_o F_r - r_o(\theta_S)F_r(\theta_S))\right] d\theta \qquad \text{(A. 1)}$$

where

$$F_r(\theta_o) = |\vec{a}|\,(A\sin\theta_o + B\cos\theta_o) \qquad \text{(A. 2)}$$

The procedure for finding Δr is straightforward. Equation (A. 2) is substituted into (A. 1), $\sin(\theta_o-\theta)$ expanded, r_o expressed as $P_o/(1-e_o\cos\theta)$ and the resulting expression integrated term by term.

Let

$$r_o = P_o x_o \qquad \text{(A. 3)}$$

$$r_o(\theta_S) = P_o x_S \qquad \text{(A. 4)}$$

$$C_I = F_r(\theta_S)x_s/|\vec{a}| \qquad \text{(A. 5)}$$

Then the indefinite integral for Δr is

$$\Delta r = \frac{|\vec{a}|P_o^4}{C_o^2}\int [\sin\theta_o\cos\theta - \cos\theta_o\sin\theta]$$
$$[A\sin\theta x_o^4 + B\cos\theta x_o^4 + 2x_o^3(A\sin\theta x_o + B\cos\theta x_o - C_I)]\; d\theta$$

Expanding and combining terms gives

$$\Delta r = \frac{|\vec{a}|P_o^4}{C_o^2}\left\{\sin\theta_o\left[3A\int \sin\theta\cos\theta x_o^4 d\theta + 3B\int \cos^2\theta x_o^4 d\theta - 2C_I\int \cos\theta x_o^3 d\theta\right]\right. \tag{A.6}$$

$$\left. -\cos\theta_o\left[3A\int x_o^4 d\theta - 3A\int \cos^2\theta x_o^4 d\theta + 3B\int \sin\theta\cos\theta x_o^4 d\theta - 2C_I\int \sin\theta x_o^3 d\theta\right]\right\}$$

Now $\int \sin\theta x_o^3 d\theta$ may be evaluated by the substitution $\rho = 1 - e_o\cos\theta$ since $\sin\theta d\theta = d\rho/e_o$. Also, by a partial fraction decomposition,

$$\int \cos^2\theta x_o^4 d\theta = \int \left(\frac{x_o^2}{e_o} - \frac{2x_o^3}{e_o^2} + \frac{x_o^4}{e_o^2}\right)d\theta$$

A general formula containing $\int x_o^k d\theta$ and $\int \cos\theta x_o^k d\theta$ is

$$\int \frac{M+N\cos\theta}{(m+n\cos\theta)^k}\,d\theta = \frac{1}{(k-1)(m^2-n^2)}\left\{\frac{mN-Mn}{(m+n\cos\theta)^{k-1}}\right.$$

$$\left. + \int \frac{(k-1)(Mm-Nn)+(k-2)(mN-nM)\cos\theta}{(m+n\cos\theta)^{k-1}}\,d\theta\right\}$$

for $m \neq n$ and $k \neq 1$. And finally

$$\int x_o d\theta = \int \frac{d\theta}{(1-e_o\cos\theta)} = \frac{2}{\sqrt{1-e_o^2}}\tan^{-1}\left[\sqrt{\frac{1+e_o}{1-e_o}}\tan\frac{\theta}{2}\right]$$

The above allows the individual integrals of (A.6) to be evaluated. The results are

$$\int \sin\theta x_o^3 d\theta = -\frac{x^2}{2e_o}$$

$$\int \sin\theta\cos\theta x_o^4 d\theta = \frac{x^2}{2e_o^2} - \frac{x^3}{3e_o^2}$$

$$\int \cos^2\theta x_o^4 d\theta = \frac{\sin\theta}{6e_o(1-e_o^2)^3}[(6e_o^4+10e_o^2-1)x+(-6e_o^4+7e_o^2-1)x^2$$

$$+(2e_o^4-4e_o^2+2)x^3] + \frac{12e_o^2+3}{6(1-e_o^2)^3}\int x_o d\theta$$

$$\int \cos\theta x_o^3 d\theta = \frac{\sin\theta}{2(1-e_o^2)^2}[(2e_o^2+1)x+(1-e_o^2)x^2] + \frac{3e_o}{2(1-e_o^2)^3}\int x_o d\theta$$

$$\int x_o^4 d\theta = \frac{\sin\theta}{6(1-e_o^2)^3}[(11e_o+4e_o^3)x+5e_o(1-e_o^2)x^2+2e_o(1-e_o^2)^2x^3]$$

$$+ \frac{6+9e_o^2}{6(1-e_o^2)^2}\int x_o d\theta$$

where the $\int x_o d\theta$ is not evaluated for notational convenience.

Note the similarity of the first two integrals and the similarity of the other integrals. The first two are of the form

$$(\ )x^2 + (\ )x^3$$

while the rest are of the form

$$\sin\theta[(\ )x + (\ )x^2 + (\ )x^3] + (\ )\int x_o d\theta$$

where ( ) denotes a ratio of polynomials in e_o. Thus the indefinite integral for Δr can be put in the form

$$\frac{|\vec{a}|P_o^4}{C_o^2}\left\{\sin\theta[(\ )x+(\ )x^2+(\ )x^3]+(\ )\int x_o d\theta+(\ )x^2+(\ )x^3]\right\}$$

where now A, B, C_I, $\sin\theta_o$ and $\cos\theta_o$ are in the ( ). The expression for Δr may be rearranged as

$$\Delta r = \frac{|\vec{a}|P_o^4}{C_o^2}\Big\{B\sin\theta_o\Big[\sin\theta x\Big(\frac{6e_o^4+10e_o^2-1}{2e_oE^3} - x_S\cos\theta_S\frac{2e_o^2+1}{E^2}\Big)$$

$$+\sin\theta x^2\Big(\frac{-6e_o^4+7e_o^2-1}{2e_oE^3} - \frac{x_S\cos\theta_S}{E}\Big)+\sin\theta x^3\Big(\frac{1}{e_oE}\Big)+\Big(\frac{12e_o^2+3}{2E^3}$$

$$-x_S\cos\theta_S\frac{3e_o}{E^3}\Big)\int x_o d\theta\Big] + B\cos\theta_o\Big[\Big(-\frac{3}{2e_o^2}\frac{-x_S\cos\theta_S}{e_o}\Big)x^2 + \frac{x^3}{e_o^2}\Big]$$

$$+A[\sin\theta_o(k'_{21}x^2+k_{31}x^3-\sin\theta_S x_S(c''_{11}\sin\theta x+c''_{21}\sin\theta x^2+c''_{41}\int x_o d\theta))]$$

$$+A[\cos\theta_o(c_{12}\sin\theta x+c_{22}\sin\theta x^2+c_{32}\sin\theta x^3+c_{42}\int x_o d\theta)$$

$$-\cos\theta_o(\sin\theta_S x_S)(k''_{21}x^2)]\Big\} \qquad (A.7)$$

where the k's and c's are given by equation (49) in the text.

The form (A. 7) leads to a convenient form for Δr when the indefinite integral is evaluated. Making the definitions for F_{S1}, F_{S2}, F_{S3}, and Δf of equation (51) and for F_{BS}, F_{BC}, F_{AC}, and F_{AS} of equations (52) through (55) and evaluating the indefinite integral with $x_S\cos\theta_S = \frac{x_S-1}{e_o}$ gives

$$\Delta r = \frac{P_o^4}{C_o^2}|\vec{a}|\Big\{B[\sin\theta_o F_{BS}+\cos\theta_o F_{BC}] +$$

$$A[\cos\theta_o F_{AC}+\sin\theta_o F_{AS}]\Big\} \qquad (A.8)$$

where $\Delta r(\theta_S) = 0$ is used.

To derive an expression for $\frac{d\Delta r}{d\theta_o} = \Delta r'$, two methods may be used. First, differentiation of (A. 1) gives

$$\Delta r' = \int_{\theta_S}^{\theta_o} \cos(\theta_o - \theta)\left[F_r \frac{r_o^4}{C_o^2} + 2\frac{r_o^3}{C_o^2}(r_o F_r - r_o(\theta_S)F_r(\theta_S))\right] d\theta$$

This may be evaluated by the same procedure as (A. 1). The second method is to differentiate (A. 8) to give

$$\begin{aligned} \Delta r' = {} & \frac{P_o^4}{C_o^2}\, |\vec{a}|\, B\{\cos\theta_o F_{BS} - \sin\theta_o F_{BC}\} \\ & + \frac{P_o^4}{C_o^2}\, |\vec{a}|\, A\{-\sin\theta_o F_{AC} + \cos\theta_o F_{AS}\} \\ & + \frac{P_o^4}{C_o^2}\, |\vec{a}|\, \{B[\sin\theta_o F'_{BS} + \cos\theta_o F'_{BC}] + \\ & A[\cos\theta_o F'_{AC} + \sin\theta_o F'_{AS}]\} \end{aligned} \tag{A. 9}$$

It may be shown that

$$\begin{aligned} \sin\theta_o F'_{BS} + \cos\theta_o F'_{AC} &= 0 \\ \cos\theta_o F'_{AC} + \sin\theta_o F'_{AS} &= 0 \end{aligned} \tag{A. 10}$$

To do this, compute F'_{BS}, F'_{BC}, F'_{AC}, and F'_{AS} by differentiation. Rearrange the expressions in powers of $\sin\theta_o$ and $\cos\theta_o$ by factoring out the highest power of x_o (e. g. , $x_o^2 = x_o^4\,(1-e_o\cos\theta_o)^2$). Most of the coefficients multiplying the powers of $\cos\theta_o$ will cancel, and the other coefficients will simplify. The results are

$$F'_{BS} = -\frac{2x_S}{e_o} x_o^3\cos\theta_o + \frac{2x_o^4}{e_o}\cos\theta_o + x_o^4\cos^2\theta_o \tag{A. 11}$$

$$F'_{BC} = \frac{2x_S}{e_o} x_o^3 \sin\theta_o - \frac{2x_o^4}{e_o} \sin\theta_o - \sin\theta_o \cos\theta_o x_o^4 \qquad (A.12)$$

$$F'_{AC} = 2\sin\theta_S x_S x_o^3 \sin\theta_o - 3x_o^4 \sin^2\theta_o \qquad (A.13)$$

$$F'_{AS} = -2\sin\theta_S x_S x_o^3 \cos\theta_o + 3x_o^4 \sin\theta_o \cos\theta_o \qquad (A.14)$$

Thus using (A.11) and (A.12)

$$\sin\theta_o F'_{BS} + \cos\theta_o F'_{BC} = 0$$

and using (A.13) and (A.14)

$$\cos\theta_o F'_{AC} + \sin\theta_o F'_{AS} = 0$$

Therefore

$$\Delta r' = \frac{P_o^4}{C_o^2} |\vec{a}| \Big\{ B[\cos\theta_o F_{AS} - \sin\theta_o F_{BC}]$$

$$+ A[-\sin\theta_o F_{AC} + \cos\theta_o F_{AS}] \Big\} \qquad (A.15)$$

For completeness, it can be shown that Δr and $\Delta r'$ satisfy the differential equation for Δr by differentiating (A.15) to give $\Delta r''$ and adding to (A.8) for Δr.

APPENDIX B

Evaluation of $I_{\Delta\theta}$

The integral $I_{\Delta\theta}$ was introduced in equation (63) of the text. It is written as

$$I_{\Delta\theta} = \int_{\theta_S}^{\theta_o} [x_o^2 \hat{I}_r + 2x_o^3 \hat{I}_r + x_o^3 \frac{F_r}{|\vec{a}|} + x_o^4 \frac{F_r}{|\vec{a}|}]\, d\theta \qquad (B.1)$$

Now $\hat{I}_r$ can be written as

$$\hat{I}_r = x_o \frac{F_r}{|\vec{a}|} - x_S \frac{F_r(\theta_S)}{|\vec{a}|} \qquad (B.2)$$

Thus

$$I_{\Delta\theta} = \int_{\theta_S}^{\theta_o} [2x_o^2 \frac{F_r}{|\vec{a}|} + 3x_o^3 \frac{F_r}{|\vec{a}|}]\, d\theta - x_S \frac{F_r(\theta_S)}{|\vec{a}|} \int_{\theta_S}^{\theta_o} (x_o^2 + x_o^3) d\theta \qquad (B.3)$$

Using

$$\frac{F_r(\theta_o)}{|\vec{a}|} = A\sin\theta_o + B\cos\theta_o$$

the first integral of (B.3) becomes

$$2A \int_{\theta_S}^{\theta_o} \sin\theta x_o^3 d\theta + 3A \int_{\theta_S}^{\theta_o} \sin\theta x_o^4 d\theta + 2B \int_{\theta_S}^{\theta_o} \cos\theta x_o^3 d\theta$$
$$+ 3B \int_{\theta_S}^{\theta_o} \cos\theta_o x_o^4 d\theta$$

In Appendix A, $\int \sin\theta x_o^3 d\theta$ and $\int \cos\theta x_o^3 d\theta$ were evaluated. The substitution $\rho = 1 - e_o\cos\theta$ allows $\int \sin\theta x_o^3 d\theta$ to be evaluated. The use of the general formulas for $\int \frac{M+N\cos\theta}{(m+n\cos\theta)^k} d\theta$ in Appendix A will give $\int \cos\theta x_o^4 d\theta$. This general formula also gives $\int x_o^2 d\theta$ and $\int x_o^3 d\theta$.

The results are

$$\int \sin\theta x_o^3 d\theta = -\frac{x^2}{2e_o}$$

$$\int \sin\theta x_o^4 d\theta = -\frac{x^3}{3e_o}$$

$$\int \cos\theta x_o^3 d\theta = \frac{\sin\theta}{2(1-e_o^2)2}[(2e_o^2+1)x+(1-e_o^2)^2x^2]+\frac{3e_o}{2(1-e_o^2)^2}\int x_o d\theta$$

$$\int \cos\theta x_o^4 d\theta = \frac{\sin\theta}{6(1-e_o^2)^3}[(13e_o^2+2)x+(-3e_o^4+e_o^2+2)x^2$$

$$+2(\theta_o-2e_o^4+1)x^3] + \frac{3e_o^3+12e_o}{6(1-e_o^2)^3}\int x_o d\theta$$

$$\int x_o^2 d\theta = \frac{e_o \sin\theta}{1-e_o^2}x + \frac{1}{(1-e_o^2)}\int x_o d\theta$$

$$\int x_o^3 d\theta = \frac{e_o\sin\theta}{2(1-e_o^2)^2}[3x+(1-e_o^2)x^2] + \frac{2+e_o^2}{2(1-e_o^2)}\int x_o d\theta$$

Thus the first integral of (B. 3) becomes

$$-A[\frac{x^2}{e_o}+\frac{x^3}{e_o}] + B[\sin\theta(b_1x+b_2x^2+b_3x^3)+b_4\int x_o d\theta]$$

where the b's are defined by equation (67) in the text. The second integral of (B. 3) becomes

$$\sin\theta[b_1'x+b_2'x^2]+b_4'\int x_o d\theta$$

where the b's are defined in equation (67) also. Now using

$$\frac{F_r(\theta_S)}{|\vec{a}|} = A\sin\theta_S x_S + B\cos\theta_S x_S$$

evaluating the indefinite integral, and using the definitions of (51) through (55) in the text

$$I_{\Delta\theta} = A\left\{-\frac{F_2}{e_o} - \frac{F_3}{e_o} - \sin\theta_S x_S[b_1' F_{S1} + b_2' F_{S2} + b_4' \Delta f]\right\}$$

$$+ B\left\{b_1 F_{S1} + b_2 F_{S2} + b_3 F_{S3} + b_4 \Delta f - \cos\theta_S x_S [b_1' F_{S1} + b_2' F_{S2} + b_4' \Delta f]\right\} \tag{B.4}$$

APPENDIX C

Integration of the Equations of Motion

Because the perturbations to the orbit are small and a double precision integration routine is not available, it is desirable to integrate the differences between the Keplerian and actual orbits rather than integrate the equations of motion directly. This is known as the Encke method. The perturbations to the orbit (ΔP, Δe, and Δw) can be written in terms of these differences. To do the integration, an Adams-Moulton predictor-corrector with a Runge-Kutta-Gill starter was used (which was available in the CIT Computing Center).

The coordinate system lies in the Keplerian plane of motion. The variables ξ and η are associated with the Keplerian motion and x and y are associated with the perturbed motion. The x and ξ axes point toward apoapsis and the y and η axes are 90° from the x and ξ axes in the direction of motion.

The accelerations are constant and given by

$$a_x = |\vec{a}| \cos(\theta_S + \frac{\pi}{2} - \alpha)$$
$$a_y = |\vec{a}| \sin(\theta_S + \frac{\pi}{2} - \alpha) \qquad (C.1)$$

At the start of the burn

$$x = \xi = r_S \cos\theta_S$$
$$y = \eta = r_S \sin\theta_S$$
$$\dot{x} = \dot{\xi} = v_S \cos(\theta_S + \frac{\pi}{2} - \gamma_S) \qquad (C.2)$$
$$\dot{y} = \dot{\eta} = v_S \sin(\theta_S + \frac{\pi}{2} - \gamma_S)$$

where

$$r_S = \frac{P_o}{1-e_o\cos\theta_S}$$

$$v_S = \left(\mu\left(\frac{2}{r_S} - \frac{1}{a_o}\right)\right)^{\frac{1}{2}}$$

$$\tan\gamma_S = \frac{e_o\sin\theta_S}{1-e_o\sin\theta_S} \tag{C.3}$$

$$-\frac{\pi}{2} \leq \gamma_S \leq \frac{\pi}{2}$$

For the Keplerian motion

$$\ddot{\xi} = -\frac{\mu}{\zeta^3}\,\xi$$

$$\ddot{\eta} = -\frac{\mu}{\zeta^3}\,\eta \tag{C.4}$$

$$\zeta^2 = \xi^2 + \eta^2$$

and for the perturbed motion

$$\ddot{x} = -\frac{\mu}{r^3}x + a_x$$

$$\ddot{y} = -\frac{\mu}{r^3}y + a_y \tag{C.5}$$

$$r^2 = x^2 + y^2$$

To obtain equations for the perturbations from the Keplerian orbit, define

$$\Delta x = x - \xi$$

$$\Delta y = y - \eta \tag{C.6}$$

using (C. 5) for x and y and subtract $\ddot{\xi}$ from $\ddot{x}$ and $\ddot{\eta}$ from $\ddot{y}$ to give

$$\Delta\ddot{x} = \frac{-\mu\Delta x}{r^3} - \mu\left(\frac{1}{r^3} - \frac{1}{\zeta^3}\right)\xi + a_x$$
$$\Delta\ddot{y} = \frac{-\mu\Delta y}{r^3} - \mu\left(\frac{1}{r^3} - \frac{1}{\zeta^3}\right)\eta + a_y \qquad (C.7)$$

where

$$\frac{1}{r^3} = \frac{1}{[(\xi+\Delta x)^2+(\eta+\Delta y)^2]^{3/2}}$$
$$\frac{1}{\zeta^3} = \frac{1}{(\xi^2+\eta^2)^{3/2}} \qquad (C.8)$$

The expression for $\frac{1}{r^3}$ can be expanded by the binomial expansion to give

$$\frac{1}{r^3} = \frac{1}{\zeta^3} + \Phi \qquad (C.9)$$

where Φ is made up terms containing Δx and Δy

$$\Phi = -\frac{3}{\zeta^5}\xi\Delta x - \frac{3}{\zeta^5}\eta\Delta y + \left(-\frac{3}{2\zeta^5} + \frac{15\xi^2}{2\zeta^7}\right)\Delta x^2 + \left(-\frac{3}{2\zeta^5} + \frac{15\eta^2}{2\zeta^7}\right)\Delta y^2 + \frac{15\xi\eta}{\zeta^7}\Delta x\Delta y + O(\Delta x^3, \Delta y^3, \Delta x^2\Delta y, \Delta y^2\Delta x) \qquad (C.10)$$

Now the equations (C. 7) can be written as a first order system. Define

$$z = \begin{Bmatrix} \xi \\ \eta \\ \dot{\xi} \\ \dot{\eta} \\ \Delta x \\ \Delta y \\ \Delta\dot{x} \\ \Delta\dot{y} \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{Bmatrix} \qquad \text{(C.11)}$$

Then

$$\dot{z} = \begin{Bmatrix} z_3 \\ z_4 \\ \ddot{\xi} \\ \ddot{\eta} \\ z_7 \\ z_8 \\ \Delta\ddot{x} \\ \Delta\ddot{y} \end{Bmatrix}$$

or

$$\begin{Bmatrix} z_3 \\ z_4 \\ -\frac{\mu}{\zeta^3} z_1 \\ -\frac{\mu}{\zeta^3} z_2 \\ z_7 \\ z_8 \\ -\mu(\frac{1}{\zeta^3} + \Phi)z_5 - \mu\Phi z_1 + a_x \\ -\mu(\frac{1}{\zeta^3} + \Phi)z_6 - \mu\Phi z_2 + a_y \end{Bmatrix} \qquad \text{(C.12)}$$

where

$$\frac{1}{\zeta^3} = \frac{1}{(z_1^2+z_2^2)^{3/2}}$$

$$\Phi = -\frac{3}{\zeta^5} z_1 z_5 - \frac{3}{\zeta^5} z_2 z_6 + \left(-\frac{3}{2\zeta^5} + \frac{15 z_1^2}{2\zeta^7}\right) z_5^2 \tag{C.13}$$

$$+ \left(-\frac{3}{2\zeta^5} + \frac{15 z_2^2}{2\zeta^7}\right) z_6^2 + \frac{15 z_1 z_2}{\zeta^7} z_5 z_6$$

Equation (C. 12) is the form needed for the integration routine.

An equation for $\Delta P = P - P_o$ may be developed from the definition of P as

$$P = \frac{|\vec{r} \times \vec{v}|^2}{\mu}$$

$$= \frac{[x\dot{y} - y\dot{x}]^2}{\mu} \tag{C.14}$$

Using (C. 6) to substitute for x and y in terms of ξ, η, Δx, and Δy gives

$$P = [\xi\dot{\eta} - \eta\dot{\xi} + \xi\Delta\dot{y} - \dot{\xi}\Delta y + \dot{\eta}\Delta x - \eta\Delta\dot{x} + \Delta x\Delta\dot{y} - \Delta y\Delta\dot{x}]^2 / \mu$$

since

$$P_o = \frac{[\xi\dot{\eta} - \eta\dot{\xi}]^2}{\mu}$$

thus

$$\Delta P = \frac{2[\xi\dot{\eta} - \eta\dot{\xi}]}{\mu} [\xi\Delta\dot{y} - \dot{\xi}\Delta y + \dot{\eta}\Delta x - \eta\Delta\dot{x} + \Delta x\Delta\dot{y} - \Delta y\Delta\dot{x}]$$

$$+ [\xi\Delta\dot{y} - \dot{\xi}\Delta y + \dot{\eta}\Delta x - \eta\Delta\dot{x} + \Delta x\Delta\dot{y} - \Delta y\Delta\dot{x}] / \mu \tag{C.15}$$

An expression for Δe may be obtained from

$$\Delta e = e - e_o = \sqrt{1 - P/a} - e_o \qquad (C.16)$$

where a is the semi-major axis of the perturbed orbit provided the calculations are in double precision and a is accurately known. An expression for a can be obtained from an exact integral of the equations of motion. This integral is obtained by the same procedure that gives the vis-viva integral. First note that

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{x}{r^3}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{r^3}$$

and thus (C. 5) can be written as

$$\ddot{x} = \frac{\partial}{\partial x}\left(\frac{\mu}{r} + a_x x\right)$$

$$\ddot{y} = \frac{\partial}{\partial y}\left(\frac{\mu}{r} + a_y y\right) \qquad (C.17)$$

Multiplying the first equation of (C. 17) by $\dot{x}$ and the second by $\dot{y}$ and adding gives

$$\frac{d}{dt}\left(\frac{\dot{x}^2+\dot{y}^2}{2}\right) = \dot{x}\,\frac{\partial}{\partial x}\left(\frac{\mu}{r} + a_x x\right) + \dot{y}\,\frac{\partial}{\partial y}\left(\frac{\mu}{r} + a_y y\right)$$

$$= \frac{d}{dt}\left(\frac{\mu}{r}\right) + \frac{d}{dt}(a_x x) + \frac{d}{dt}(a_y y)$$

Therefore

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) = \frac{d}{dt}\left(\frac{\mu}{r} + a_x x + a_y y\right)$$

or

$$\frac{v^2}{2} = \frac{\mu}{r} + a_x x + a_y y + \text{constant} \qquad (C.18)$$

To evaluate the constant, note that at the start of the burn

$$\frac{v_S^2}{2} = \frac{\mu}{r_S} - \frac{\mu}{2a_o}$$

from the vis-viva integral. Therefore

$$\frac{\mu}{r_S} + a_x x_S + a_y y_S + \text{constant} = \frac{\mu}{r_S} - \frac{\mu}{2a_o}$$

gives the constant. Thus

$$\frac{v^2}{2} = \frac{\mu}{r} - \frac{\mu}{2a_o} + a_x(x-x_S) + a_y(y-y_S) \qquad \text{(C. 19)}$$

Finally, noting that

$$\frac{v^2}{2} = \frac{\mu}{r} - \frac{\mu}{2a}$$

and equating to the right-hand side of (C. 19) gives an equation involving a. This may be solved for a to give

$$a = a_o \left[\frac{1}{1 - \frac{2a_o}{\mu}[a_x(x-x_S)+a_y(y-y_S)]} \right] \qquad \text{(C. 20)}$$

Expansion of (C. 20) by the binomial expansion gives a to use in (C. 16) to obtain Δe.

Finally, an expression for Δw may be obtained from

$$\cos\theta = \frac{1}{e}\left(1 - \frac{P}{r}\right) \qquad \text{(C. 21)}$$

$$\Delta w = \theta - \theta_o - \Delta\theta \qquad \text{(C. 22)}$$

where $\theta - \theta_o = \delta\theta$ may be found from

$$\cos\theta = \cos(\theta_o + \delta\theta) = \cos\theta_o + \Delta C \tag{C.23}$$

and ΔC found from (C.21).

Using

$$e = e_o + \Delta e$$

$$r = (x^2+y^2)^{\frac{1}{2}} = (\xi^2+\eta^2+2\xi\Delta x+2\eta\Delta y+\Delta x^2+\Delta y^2)^{\frac{1}{2}}$$

$$P = P_o + \Delta P$$

and the binomial expansion, (C.21) can be written

$$\cos\theta = \cos\theta_o + \Delta C$$

where to second order

$$\Delta C = -\frac{\Delta P}{e_o r_o} - \frac{\Delta e}{e_o^2} + \frac{P_o}{r_o e_o}\left[\frac{\Delta e}{e_o} + \frac{\xi \Delta x}{r_o^2} + \frac{\eta \Delta y}{r_o^2}\right]$$

$$+ \frac{\Delta e^2}{e_o^3} - \frac{P_o}{e_o r_o}\left[\frac{\Delta e^2}{e_o^2} + \frac{\Delta e}{e_o}\left(\frac{\xi \Delta x}{r_o^2} + \frac{\eta \Delta y}{r_o^2}\right) + \Delta x^2\left(\frac{3\xi^2}{2r_o^4} - \frac{1}{2r_o^2}\right)\right.$$

$$\left. + \Delta y^2\left(\frac{3\eta^2}{2r_o^4} - \frac{1}{2r_o^2}\right) + \Delta x \Delta y\left(\frac{3\xi\eta}{r_o^4}\right)\right]$$

$$+ \frac{\Delta P}{e_o r_o}\left[\frac{\Delta e}{e_o} + \frac{\xi \Delta x}{r_o^2} + \frac{\eta \Delta y}{r_o^2}\right] \tag{C.24}$$

Now (C.23) may be solved for $\delta\theta$ by expanding the cosine and keeping terms of second order in $\delta\theta$.

$$\cos\theta_o(1 - \frac{\delta\theta^2}{2}) - \sin\theta_o\delta\theta = \cos\theta_o + \Delta C$$

Solving for $\delta\theta$ to third order in ΔC

$$\delta\theta = -\tan\theta_o \pm \left\{\tan\theta_o - \frac{\Delta C}{\sin\theta_o} - \frac{\Delta C^2\cos\theta_o}{2\sin^3\theta_o} - \frac{\Delta C^3\cos^2\theta_o}{2\sin5\theta_o}\right.$$

or

$$\delta\theta = -\frac{\Delta C}{\sin\theta_o} - \frac{\Delta C^2\cos\theta_o}{2\sin^3\theta_o} - \frac{\Delta C^3\cos^2\theta_o}{2\sin^5\theta_o} \qquad (C.25)$$

(Note the singularity at $\sin\theta_o = 0$. This was avoided in the computer program by avoiding θ_o's near 0 or π.) Finally $\Delta\theta$ may be found from

$$\Delta\theta = \sin^{-1}\left(\frac{|\bar{r} \times \bar{r}_o|}{r\,r_o}\right) \qquad (C.26)$$

which allows Δw to be calculated.