CRACK PROPAGATION UNDER GENERAL IN-PLANE LOADING

Thesis by

Karuppagounder Palaniswamy

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

1972

(Submitted June 7, 1971)

ACKNOWLEDGEMENTS

It is a pleasure to express my appreciation and gratitude to Prof. W.G. Knauss for suggesting this interesting problem and for his constant encouragement. Thanks are due to Mr. T. Pucik for various interesting discussions.

I am greatly indebted to Mrs. L. Palmrose for the excellent work in typing the thesis. I would also like to thank Mrs. B. Wood and Mr. P. Bhatia for their help in preparation of the figures.

Finally I would like to thank my wife and son for their understanding and patience.

Major part of the financial support of this research was provided by the National Aeronautics and Space Administration, Research Grant NsG-05-002-005.

This thesis is dedicated to my parents.

ABSTRACT

The problem of crack extension in brittle materials under general loading conditions is investigated. Methods of solution of the related two-dimensional elasto-static boundary value problem are discussed. Using Kolosov-Muskhelishvili stress functions, an approximate solution is obtained. The effect of the approximation on the results is estimated by solving two related problems exactly. Then using two postulates the critical loads and crack extension direction are determined under loading conditions unsymmetrical to the crack axis. Results are compared with those obtained using a different set of postulates.

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LIST OF SYMBOLS

a	Non-dimensionalized branch crack length $(= \frac{\ell_2}{\ell_1})$
A	Mapping parameter
b_i	Fourier coefficients
$c_i, \tilde{c}_i, C_i$	Coefficients in the strain energy expressions
C_{ij}	Coefficients in the fit for C_i
d_i	Fourier coefficients
D	Region
e_i , f_i	Fourier coefficients
$f^N, f^N_{(i)}$	Column vectors with N elements
$f_{ij}(\theta)$	Functions defining variation of stresses σ_{ij} with θ
F	A function of loading angle and geometry of the branched crack system
F^*	Limit of F
g_i	Fourier coefficients
g^N, g^N_i	Column vectors with N elements
G	Energy release rate
$\tilde{G}$	Approximate value of G
h_i	Fourier coefficients
$h(z), h_1(\xi)$	Complex stress functions
H^N	$N \times N$ complex matrix
i	Unit imaginary number $(= \sqrt{-1})$
I	Identity matrix

k	Radius of a circle $(= \frac{1}{1+\epsilon})$
K	Stress intensity factor
K_I	Stress intensity factor for symmetrical loading
K_{II}	Stress intensity factor for asymmetrical loading
K_{ij}	Kernels of integral equations
ℓ	Length of a straight crack
ℓ_1	Length of the main crack (Fig. 2)
ℓ_2	Length of the branch crack (Fig. 2)
L	Crack boundary
L_ϵ	Modified boundary
M_i	Bounds in error estimates
n	Unit normal to the crack boundary
p_i	Functions of the angle between the two crack segments
P	Total potential energy
q	Applied stress at infinity
q_{cr}	Critical value of the applied stress
$\tilde{q}_{cr}$	Approximate value of q_{cr}
(r, θ)	Polar coordinates
r_i	Fourier coefficients
r^N, r_i^N	Column vecotrs with N elements
R	Radius of a large circle
s	Surface of the solid
S	Area of crack surface
S_I, S_{II}	Sectors

T	Surface energy per unit area
$\overset{\nu}{T}_i$	Cartesian components of surface traction
T^N	$N \times N$ complex matrix
u_i	Cartesian components of displacement
(u_r, u_θ)	Polar components of displacement
$(\tilde{u}_r, \tilde{u}_\theta)$	Corrections for the displacement components (u_r, u_θ)
U	Strain energy change due to the presence of the crack
U_E	Exact value of the strain energy change
U_M	Value of the strain energy change obtained by solving the modified problem
U_R, U_R^o	Strain energy stored in the part of the solid bounded by a circle of radius R
v	Volume of the solid
V	Change in potential energy due to the presence of the crack
w	Out-of-plane displacement $(=u_3)$
w_c	Harmonic-conjugate of w
W	Strain energy density
x_i	Cartesian coordinates
z	Complex variable in the physical plane
α	Loading angle
$\alpha_1, \alpha_2, \beta_1, \beta_2$	Mapping parameters
γ	Contour of integration in the mapped plane

Γ, Γ'	Constants defining stress field at infinity
ϵ	A number very small compared to unity
κ	Material constant; function of the Poisson's ratio σ only.
λ_1	Measure of the angle $(\lambda_1 \pi)$ between the two crack segments
λ_2	$=2-\lambda_1$
λ^*	Value of λ_1 at which the energy release rate G attains the maximum
$\tilde{\lambda}^*$	Approximate value of λ^*
μ	Shear modulus
ν	Unit normal to the surface
ξ	Complex variable in the mapped plane
σ	Poisson's ratio
σ_{ij}	Stresses in the Cartesian coordinate system
$\sigma_r, \sigma_\theta, \tau_{r\theta}$	Stresses in the polar coordinate system
τ_∞	Applied out-of-plane shear stress at infinity
$\varphi, \varphi_o, \varphi_1$	Complex stress functions
Φ	Airy stress function
χ	Complex stress function
Ψ, Ψ_o, Ψ_1	Complex stress functions
ω	Mapping function
$\mathcal{S}$	Surface energy of the crack
$\mathcal{U}$	Strain energy
$\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2$	Potential energy

INTRODUCTION

In his classical papers Griffith[1, 2] proposed to find the fracture strength (i. e., load carrying capacity) of brittle solids by using the following two postulates. He first postulated the existence of flaws or cracks in materials and associated with their growth the consumption of surface energy. Secondly, he suggested that the fracture strength can be found by using an energy balance criterion which since then came to be known as Griffith's energy criterion.

A mathematical statement of this theory will be given later on. For introductory purposes it suffices to state that if a pre-existing crack is forced to extend due to the application of stress, then at the instant of incipient crack extension the rate of potential energy release should be equal to the rate of surface energy increase (due to the increase in surface area). Griffith calculated the fracture strength of the plate with a straight crack loaded normal to it by uniform tensile stress at infinity. In his calculations Griffith used Inglis'[3] solution for a plate with an elliptic perforation, and considered the crack as the limit of an ellipse when the minor axis tends to zero. Even though linear elastic theory predicts unbounded stresses at the crack tips--which are physically inadmissible--the energy contained in any finite domain is bounded; his theory of fracture may be referred to as a global one in the sense that he obtained the strain energy by integration over the entire plate.

Griffith's theory of energy balance so far has been applied only to predict failure when the loading is symmetrical to the crack. When the loading is general (i. e., both normal and shear stresses are applied at infinity) the direction of crack extension is not along the crack axis. It is of interest therefore to try the energy balance concept to this loading condition.

For the cases when the loading is symmetrical to the crack Irwin[(4)] showed that Griffith's theory of global energy balance is equivalent to a criterion based on the crack tip stress intensity factor* or the local force tending to open the crack. He developed this criterion by considering the energy released by the unloading of the stress in the vicinity of the crack tip as the crack advances. His criterion states that a condition for crack extension is that the stress intensity factor reaches a critical value, the latter being a characteristic material constant.

Barenblatt[(5)] developed a criterion which eliminates the singular behavior of the stresses at a crack tip predicted by Inglis' limit solution of the linear elastic theory. He introduced the notion of cohesive forces between the two sides of the crack which are confined to act in the region close to the crack tips and which is small compared to crack length. When the applied loads increase causing the crack faces to open, the cohesive forces also increase, the increase being such that the stresses at the crack tip are always finite. He then postulated that the cohesive forces cannot increase

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*See equation (85) page 55.

beyond a certain limit for any given material. Thus his criterion states that the condition for crack extension is the attainment of the above limit. He introduced a term called the cohesion modulus as a measure of this limit on the cohesive forces, and considered the cohesion modulus as a characteristic material constant.

Many structural materials like steel and aluminum do not behave in a brittle fashion. In order to apply the fracture theory to those materials, some attempts have been made to take into account the effect of yielding of the material in the vicinity of the crack tips. These attempts are generally in the direction of modifying the brittle fracture theory results to make them suitable for application to ductile materials. Irwin(6) suggested the use of an effective half-crack length which is defined as the sum of the half-crack length and a length factor r_y proportional to $(K/\sigma_{ys})^2$ where K is the stress intensity factor and σ_{ys} is the tensile yield stress. This length factor can be thought of as the value of the distance r from the crack tip at which the singular part of the normal stress σ_y as predicted by linear elastic theory attains the value σ_{ys}, and hence provides some measure of the plastic zone. Koskinen(7) and Irwin(8) have shown that for the longitudinal (or anti-plane) shear problem, the elasto-plastic analysis predicts a plastic zone which is a circle, with its radius proportional to the square of the ratio of the corresponding stress intensity factor to the yield stress in shear.

McClintock and Irwin[9] indicate that incorporation of this length factor r_y correlates some of the experimental results obtained for the fracture strength when the plastic zone is small compared to the crack length and the thickness of the material. Their argument of the reasons for the agreement is based on the analogy of the two modes of fracture, the in-plane opening and out-of-plane shearing. Since the analogy is not strictly valid, the results can be thought of only as semi-empirical.

Irwin[10] has given some qualitative discussion of the two types of fracture observed when specimens containing cracks are subjected to loads symmetrical to the crack axis. One can be called the "tensile" fracture when the fracture surface is normal to the plate surface and the other "shear" fracture when it is oblique. The tensile mode seems to occur when the plastic zone size is small compared to the specimen thickness and conditions close to plane strain prevail in the stress field near the crack tip. This mode is also sometimes referred to as the "brittle" or "quasi-brittle" fracture. The shear fracture seems to occur when the plastic zone is large compared to the thickness and hence is sometimes referred to as "ductile fracture". In this case conditions close to plane stress assumptions prevail in the stress field near the crack tip. Irwin also indicates that the type of fracture depends on the temperature which influences material properties notably the yield stress and hence the size of the plastic zone.

A different line of investigations began from Dugdale's[11] hypotheses. Based on experimental observations of the plastic zone in steel sheets containing slits, he calculated the size of the plastic-zone using the following postulates:

(a) Yielding occurs over some length s measured from the crack tip, and the dimension of the plastic zone in the direction normal to the crack axis is small compared to that along the crack axis.

(b) The material may be considered to deform elastically under the action of the external stress together with a uniform tensile stress σ_{ys} (yield stress in tension) distributed over the hypothetical crack segment s.

(c) The stress at the tips of the hypothetical crack is finite, when obtained through linear elasticity theory.

The results for the plastic zone-size s calculated on the basis of the above postulate agreed quite well with the experimental result reported by Dugdale.

Goodier and Field[12] suggested the use of Dugdale's hypothesis and postulated a condition for crack extension based on a critical strain in the plastic layer. Rosenfield[13] et. al., have obtained some approximate results using the postulates of Dugdale, Goodier and Field which seem to agree reaonsably well with the experimental values for the fracture stress and plastic zone, reported by the authors for steel and aluminum plates.

In concluding the remarks about the fracture strength of ductile materials, it is the writer's opinion that various features of the effect of plasticity have only been identified so far. Some approximate and often empirical calculations have been made to estimate the fracture strength. On the basis of work reported in (10) three dimensionality of the stress field seems to play an important role and it may clarify the variation of fracture toughness with sheet thickness and different types of plastic zones observed. The use of different yield conditions also need clarification. Nevertheless it is clear from the brief review that brittle fracture theory serves in many cases as a starting point for a description of fracture in ductile solids.

We have so far restricted our discussion to symmetrical loading conditions. Turning our attention to general loading conditions, very little work has been done compared to that when the loading is symmetrical, even though in practice most of the loading conditions encountered are of the former type. In each of the three seemingly different theories of brittle fracture discussed earlier a material constant appears in addition to the usual elastic constants; namely, the surface energy per unit area in Griffith's theory, the critical stress intensity factor in Irwin's theory and modulus of cohesion in Barenblatt's theory. Because these theories have been applied only to the problem of special loading, the question arises whether one will need additional material constants to adequately describe the fracture behavior under general loading conditions.

Erdogan and Sih[14] considered a case in which a plate containing a crack was subjected to general loading. They proposed to determine the failure load and the direction of crack extension using the following hypotheses: Referring to a right cylindrical-polar coordinate system with the origin at the crack tip

(a) "The crack extension starts radially from the tip"

(b) "The crack extension occurs in the plane perpendicular to the direction of greatest tension"*

(c) "The crack extension initiation under combined in-plane loading" occurs "when $(2r)^{\frac{1}{2}} \sigma_\theta$ = constant"**

Under these postulates these authors calculated the direction of crack extension and the stress intensity factors prevailing at the crack tip when the crack extended. They report experiments with plexiglass plates to corroborate their theoretical results. The experiments agreed well with the theoretical development for the direction of crack extension.+ There was large scatter in the data which determines the critical fracture load and agreement with the theory was not convincing.

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* The stresses are unbounded at the crack tips. The expression "greatest tension" is interpreted as the greatest value of the quantity $(2r)^{\frac{1}{2}} \sigma_\theta$, which is bounded. σ_θ denotes the normal stress and is the distance from the crack tip.

** σ_θ is the normal stress on the line determined by condition (b).

\+ In contrast to the critical load data, the fracture angles reported were averaged data.

In a discussion of the results obtained by these authors, McClintock(15) pointed out that if one uses a slender ellipse as the model for the flaw in the material instead of a line crack and uses the same hypotheses postulated by Erdogan and Sih, one obtains substantially different results. Cotterell(16) in a discussion of the proper model for the flaw in the material, concluded that to predict the fracture stress for a crack, it is necessary to know the stress intensity factor for a straight crack with a small branch at the tip in the direction of the principal stress trajectory. In the view of the writer, the limitation on the direction is too restrictive.

Morosov and Fridman(17) have tried to use an approximate method to predict failure loads for plates with cracks subjected to uniform tensile stress as shown in Figure 1a. They calculate the fracture load by considering the equilibrium of the plate cut from the crack tips in the direction normal to the applied stress. The normal stress distribution on the cut is assumed as shown in Figure 1b. Force equilibrium in the y-direction then determines the stress intensity factor K in terms of the applied stress q. The authors then calculate the fracture stress, using Barenblatt's criterion, which puts an upper limit on K.

In all these treatments the fracture loads are obtained from a simple knowledge of conditions prior to crack extension. It is of interest to consider all the possible modes of fracture and select the one using a suitable criterion. We will consider that the crack can extend in an arbitrary direction and try to find the probable direction of crack extension and the failure load using an energy balance concept similar to Griffith's criterion.

So it becomes necessary to calculate the change in strain energy when the crack tip extends a small distance in an arbitrary direction while the solid is subjected to general loading. The specific problem considered in this work is two-dimensional. The material contains a straight crack and is loaded in a plane by uniform tensile stress at infinity and in a direction inclined to the crack axis. The results are applicable to the plane strain conditions and plane stress, the latter with the usual limitations inherent in the generalized plane stress formulation. Both problems will be called the plane problem for brevity. The problem with two cracks originating from a common point will be called the "branched crack problem", the longer of the cracks the "main crack" and the smaller one the "branch crack"* (see Figure 2).

This thesis is divided into three parts. In Part I a formulation of Griffith's energy criterion is given, the physical problem is stated, the elasto-static problem is posed and the difficulties involved in solving it are discussed. Then the problem is modified to make it suitable for effective solution. The effects of this modification on similar problems are also discussed.

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*Andersson(18) derived a solution for the two-dimensional problem with cracks of different length originating from a common point, and subjected to uniform loads at infinity. An error in the application of boundary conditions invalidates his results.

Part II contains the solution of the modified problem. The results, discussion and conclusion are given in Part III.

PART I

1. GRIFFITH'S ENERGY CRITERION

Consider a linearly elastic solid occupying a volume v, subjected to tractions $\overset{\nu}{T}_i$ (i = 1, 2, 3) on its surface s, with ν being the unit normal to the surface at the point (x_1, x_2, x_3). Let u_i (i = 1, 2, 3) be the displacement components in the three coordinate directions. Define the potential energy $\mathcal{V}$ and strain energy $\mathcal{U}$ of the solid in the usual way[19] as

$$\mathcal{V} = \int_v W\, dv - \int_s \overset{\nu}{T}_i u_i \, ds$$
$$\mathcal{U} = \int_v W\, dv = \frac{1}{2} \int_s \overset{\nu}{T}_i u_i \, ds \qquad (1)$$

where W is the strain energy density of the solid. Let us consider two solids one without a crack and the other with a crack, which are otherwise identical. Both of them are subjected to same tractions on the common boundary. Denote by subscript '1' the energies associated with the first solid and by '2' those associated with the second.

There are several ways of presenting Griffith's energy criterion. One way is to define $V = \mathcal{V}_2 - \mathcal{V}_1$, as the change in potential energy and $U = \mathcal{U}_2 - \mathcal{U}_1$, as the change in strain energy of the solid due to the presence of the crack. The surface energy of the crack $\mathcal{S}$ is defined through the relation $\frac{d\mathcal{S}}{dS} = T$, where S is the area of the crack surface and T is the surface energy per unit area. T is assumed to be a material constant. The total potential energy of the solid containing the crack is then defined as

$$P = V + \mathcal{S} \tag{2}$$

(Although the definition $P = \mathcal{V}_2 + \mathcal{U}$, will be a more natural definition of P, the definition (2) is more appropriate when one has to deal with infinite solids--as we will later on--since in those cases $\mathcal{V}_2$ becomes infinite whereas V is finite). Then Griffith's energy criterion for a brittle material states that at the instant of incipient crack extension, the total potential energy attains a stationary value, i. e.,

$$\frac{\partial P}{\partial S} = 0 \tag{3a}$$

Using equation (2) and the definitions of V, U and $\mathcal{S}$, equation (3a) can be rewritten in the following form which is convenient for later use.

$$\frac{\partial U}{dS} = T \tag{3b}$$

Consider as an example the infinite plate containing a crack of length ℓ (Figure 3). Let it be loaded at infinity by a uniform tensile stress q in a direction perpendicular to the crack. Let the crack faces be stress free. Using Inglis'[(3)] solution for the elasto-static state the strain energy change U can be calculated by a method which will be described later on (pages 22-25) as

$$U = \frac{\pi}{32} \frac{\kappa + 1}{\mu} q^2 \ell^2 \tag{4}$$

where μ is the shear modulus of the material and κ takes the value $3-4\sigma$ for plane strain and $(3-\sigma)/(1+\sigma)$ for generalized plane stress, with σ being the Poisson's ratio of the material. Assuming the crack extends by a length $\Delta\ell$ as shown in Figure 3, we get $\Delta S = 2\Delta\ell$ and equation (3b) can be rewritten as

$$\frac{\partial U}{\partial \ell} = 2T \tag{3c}$$

Substituting the value of U from equation (4) in (3c) and rearranging one gets the following equation to determine the critical value of the applied stress q for a given crack length ℓ.

$$q_{cr} = \sqrt{\frac{32}{\pi} \frac{\mu}{\kappa+1} \frac{T}{\ell}} \tag{5}$$

In deriving equation (5) we assumed the crack extends along its own axis. This may be justified as the loading is symmetrical to the crack axis. But when loading is general, the direction of crack extension is not determined a priori. In those cases we will try to find it by an extension of the energy criterion given in the following section.

2. STATEMENT OF THE PHYSICAL PROBLEM AND THE POSTULATES:

Consider a solid with a crack of length ℓ_1 and loaded at infinity by uniform tensile stress q acting at an angle α to the crack axis as shown in Figure 4. We need to find that load q at which the crack will extend and the direction in which it will extend.

Since the problem remains invariant for a rotation of π radians about an axis normal to the plane of the plate and through the midpoint of the crack, one would expect the crack to extend from both tips as shown in Figure 5. For reasons of mathematical simplicity in solving the associated boundary value problem in elasto-statics, we will assume that the crack extension starts at one tip first--rather than starting from both tips simultaneously--which is physically probably more reasonable.

Suppose the elasto-static state for the branched crack, loaded as shown in Figure 2 is known. Then let the change in strain energy stored in the solid due to the presence of the branched crack per unit thickness be $U(\mu, \varkappa, q, \lambda_1, \ell_1, \ell_2)$*. Define the rate of change of U with respect to the branch crack length ℓ_2 as

$$\frac{\partial U}{\partial \ell_2} = \frac{\pi}{32} \frac{\varkappa + 1}{\mu} q^2 F(\alpha, \lambda_1, \ell_1, \ell_2)^{**} \tag{6a}$$

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* Arguments of the functions will be explicitly shown when the functions appear for the first time. Thereafter only the arguments which are necessary for further reasoning will be carried.

**That this form of representation for $\frac{\partial U}{\partial \ell_2}$ is possible will be shown on pages 22-26.

where F is some function of the geometry of the branched crack system and the loading angle α and let

$$F^*(\alpha, \lambda_1, \ell_1) = \lim_{\ell_2 \to 0} F(\alpha, \lambda_1, \ell_1, \ell_2). \tag{6b}$$

Define the energy release rate G at the instant of incipient crack extension as

$$G(\kappa, \mu, q, \alpha, \mu) = \frac{\pi}{32} \frac{\kappa + 1}{\mu} q^2 F^*(\alpha, \lambda_1, \ell_1) \tag{7}$$

We now propose to determine the crack extension angle λ^* and the critical value of the applied stress q_{cr} at which the crack will extend using the following two postulates:

1. The direction of crack extension is that for which the energy release rate G attains a maximum.

2. The critical value of the applied stress is the one for which the above maximum energy release rate reaches the value $2T$ as the applied stress q is increased slowly from zero.

In mathematical form, the direction of crack extension λ^* is determined by

$$\frac{\partial G}{\partial \lambda_1} = 0 \text{ (Maximum of } G \text{ with respect to } \lambda_1) \tag{8}$$

and the critical value of the applied stress q is determined by

$$G\big|_{\lambda_1 = \lambda^*} = 2T \tag{9}$$

Substituting the expression for G in equation (9) and rearranging, we obtain the following relation to determine the critical value of the applied stress q.

$$q_{cr} = \sqrt{\frac{32}{\pi} \frac{\mu}{(\kappa+1)} \frac{2T}{F^*(\alpha, \lambda^*, \ell_1)}} \tag{10}$$

Hence, once the change in strain energy of the solid due to the presence of the branched crack is known, we can calculate the direction of crack extension and the value of the loading stress which will cause the crack to extend.

In order to calculate the change in strain energy, it becomes necessary to solve a boundary value problem in elastostatics. The boundary value problem is posed in the next section and methods of solution are discussed.

3. THE BOUNDARY VALUE PROBLEM

The boundary value problem can be formulated using the Airy stress function Φ. For convenience the crack boundary will be denoted by L and the region outside the crack boundary by D. The stresses defined by (Figure 6)

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2} \tag{11}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right)$$

satisfy the equilibrium equations. The compatibility equation then reduces to

$$\nabla^4 \Phi(r, \theta) = 0 \text{ in } D \tag{12}$$

where ∇^4 denotes the two-dimensional bi-harmonic operator, while the boundary conditions are

$$\left.\begin{aligned} \sigma_\theta &= 0 \\ \tau_{r\theta} &= 0 \end{aligned}\right\} \quad \text{on } L \quad \left\{\begin{aligned} &\theta = 0, \ 0<r<\ell_1 \\ &\theta = \lambda_1\pi, \ 0<r<\ell_2 \end{aligned}\right\} \tag{13}$$

At infinity we have the asymptotic behavior

$$\sigma_r = \frac{q}{2}[1 + \cos 2(\alpha - \theta)] + O(\frac{1}{r^2})$$

$$\sigma_\theta = \frac{q}{2}[1 - \cos 2(\alpha - \theta)] + O(\frac{1}{r^2}) \quad \text{as } r \to \infty \tag{14}$$

$$\tau_{r\theta} = \frac{q}{2}\sin 2(\alpha - \theta) + O(\frac{1}{r^2})$$

(We say $F(\xi) = O(\xi)$ as $\xi \to 0$ (or infinity) if and only if, $F(\xi) \leq c|\xi|$ for sufficiently small (or large) values of $|\xi|$, where c is a finite constant).

One method of solution of the boundary value problem as posed in equations (11) to (14) is as follows.

Consider the region D as a union of two sectors $S_I(0 < \theta < \lambda_1 \pi)$ and $S_{II}(\lambda_1 \pi < \theta < 2\pi)$. Then using Mellin transforms[20] the boundary value problem can be reduced to a problem of determining four functions $f_i(t)$ satisfying four coupled integral equations of the following form. (Details are given in Appendix I).

$$\left.\begin{aligned} f_i(t) &= g_i(t) + \sum_{j=1}^{4}\left(\int_0^{r_j} K_{ij}(t, r)\, f_j(r)\, dr\right), \quad 0 < t < r_i \\ &(i = 1, 2, 3, 4;\ r_1 = r_2 = \ell_1;\ r_3 = r_4 = \ell_2) \end{aligned}\right\} \tag{15}$$

The four functions $f_i(t)$ are related to the four displacement discontinuities (two across each crack segment) in a way similar to the one given below

$$u_r(r,0) - u_r(r,2\pi) = \int_r^{\ell_1} \frac{f_1(t)\,dt}{\sqrt{t^2 - r^2}} \,, \quad 0 < r < \ell_1 \tag{16}$$

The integral equations (15) are obtained after several formal interchanges of integrations and differentiations. The kernels $K_{ii}(t, r)$ become singular like $\frac{1}{r}$ as one approaches the origin along the line $t = r$ in the (t, r) plane. The author was unable to confirm whether this singular behavior is due to the formal way in which the integral equations were derived or due to the singular nature of the boundary value problem itself. For this reason it was decided to use an alternate approach to solve the boundary value problem. This method is discussed below.

The biharmonic function $\Phi(r, \theta)$ can be represented in terms of two functions φ and χ, which are functions of the complex variable $z = re^{i\theta}$, by[21]

$$\Phi(r, \theta) = \text{Re}\left[\bar{z}\,\varphi(z) + \chi(z)\right] \tag{17}$$

where 'Re' denotes the 'real part of' and a bar denotes the complex conjugate. In this representation for Φ, the stresses and displacements are defined in terms of the two functions $\varphi(z)$ and $\psi(z)$, the later being the derivative of $\chi(z)$, as

$$\sigma_r + \sigma_\theta = 4 \text{ Re } [\varphi'(z)]$$

$$\sigma_\theta + i\tau_{r\theta} = 2 \text{ Re } [\varphi'(z)] + \frac{z}{\bar{z}}[\bar{z}\,\varphi''(z) + \Psi'(z)] \tag{18}$$

$$u_r + i u_\theta = \frac{1}{2\mu} e^{-i\theta} [\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\Psi(z)}]$$

Hence the determination of the biharmonic function Φ reduces to finding two functions $\varphi(z)$ and $\Psi(z)$ which are holomorphic (single valued analytic functions) in the region D outside the crack boundary. The stress free boundary conditions (13) become

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\Psi(z)} = 0 \text{ on } L \tag{19}$$

and the conditions at infinity (14) reduce to

$$\left.\begin{aligned} \varphi(z) &= \Gamma z + \varphi_o(z) \\ \Psi(z) &= \Gamma' z + \Psi_o(z) \end{aligned}\right\} \text{ as } |z| \to \infty \tag{20}$$

where

$$\left.\begin{aligned} \Gamma &= \frac{q}{4} \text{ and} \\ \Gamma' &= -\frac{q}{2} e^{-2i\alpha} \end{aligned}\right\} \tag{20a}$$

and $\varphi_o(z)$, $\Psi_o(z)$ are functions holomorphic in the region D including the point at infinity.

Before proceeding to solve for the functions $\varphi(z)$ and $\Psi(z)$ we will derive the expression for the strain energy change in the solid due to the presence of the branched crack in terms of these functions.

4. THE EXPRESSION FOR THE STRAIN ENERGY CHANGE

Let us calculate the strain energy U_R stored in that part of the solid bounded by the circle of radius R (Figure 7). Using Clapeyron's theorem[22], one obtains

$$\begin{aligned} U_R &= \frac{1}{2}\int_0^{2\pi}\left[\sigma_r\, u_r + \tau_{r\theta}\, u_\theta\right]_{r=R} \cdot R\,d\theta \\ &= \frac{1}{2}\,\mathrm{Re}\int_0^{2\pi}\left[(\sigma_r - i\tau_{r\theta})\,(u_r + iu_\theta)\right]_{r=R} R\,d\theta \end{aligned} \qquad (21)$$

In order to calculate U_R let us expand $\varphi(z)$ and $\psi(z)$ in power series valid for $|z| > \ell_1$ in the following way.

$$\begin{aligned} \varphi(z) &= \Gamma z + d_o + \frac{d_1}{z} + \frac{d_2}{z^2} + \ldots \\ \psi(z) &= \Gamma' z + e_o + \frac{e_1}{z} + \frac{e_2}{z^2} + \ldots \end{aligned} \qquad (22)$$

Substituting these series in equations (18), using the relation $z = re^{i\theta}$ and rearranging we get for $r = R$,

$$\left.\begin{aligned} \sigma_r - i\tau_{r\theta} &= \left[2\Gamma - \Gamma' e^{2i\theta}\right] + \left(e_1 - 3d_1 e^{-2i\theta} - \bar{d}_1 e^{2i\theta}\right)\frac{1}{R^2} \\ &\quad + O\left(\frac{1}{R^3}\right) \\ 2\mu(u_r + iu_\theta) &= \left[(\kappa - 1)\Gamma - \bar{\Gamma}' e^{-2i\theta}\right] R + \{\kappa d_o - \bar{e}_o\} e^{-i\theta} \\ &\quad + \left(\kappa d_1 e^{-2i\theta} + \bar{d}_1 e^{2i\theta} - \bar{e}_1\right)\frac{1}{R} + O\left(\frac{1}{R^2}\right) \end{aligned}\right\} \qquad (23)$$

The coefficients d_n, e_n $(n = 0, 1, 2, \ldots)$ are, in general, complex, except e_1 which is real due to the fact that the moment of the applied boundary forces is zero and hence

$$\int_0^{2\pi} [(\tau_{r\theta} \cdot r) \cdot r] \Big|_{r=R} d\theta = 0$$

This real behavior of e_1 will be used in deriving equations (25) and (26) on page 24 in this section.

The terms in square brackets are the only ones which will prevail if there is no crack in the solid. The terms in curly brackets represent rigid body displacements. The rest of the terms are generated by the presence of the crack. In order to meaningfully compare the strain energy of the two solids, one with the crack and one without, and for which stress boundary conditions are prescribed at the outer boundary, one needs to satisfy the stress boundary condition on the large circle of radius R to order $\frac{1}{R^2}$. That is, the stress field must behave like

$$\sigma_r - i\tau_{r\theta} = (2\Gamma - \Gamma' e^{2i\theta}) + O\left(\frac{1}{R^3}\right) \qquad (24)$$

The necessity of this behavior of stresses when the strain energy change is calculated using (21) was first shown by Spencer[23]. The method used here follows that of Sih and Liebowitz[24]. The necessary modification of the stress field given in equation (23) to make it conform with equation (24) can be derived from the following stress functions $\tilde{\varphi}$ and $\tilde{\psi}$

$$\left.\begin{aligned} \tilde{\varphi}(z) &= \left(-\frac{e_1}{2R^2}\right) z + \left(\frac{\bar{d}_1}{R^4}\right) z^3 \\ \tilde{\psi}(z) &= \left(-\frac{4\bar{d}_1}{R^2}\right) z \end{aligned}\right\} \tag{25}$$

Adding these stress functions to $\varphi(z)$ and $\psi(z)$ eliminates the terms of order $\frac{1}{R^2}$ for the stresses in the expression (23) on the boundary of the circle. The following additional displacements are caused due to this modification of the stress field

$$2\mu(\tilde{u}_r + i\tilde{u}_\theta) = \left[-(\kappa - 1)\frac{e_1}{2} + \kappa\bar{d}_1\, e^{2i\theta} + d_1\, e^{-2i\theta}\right]\frac{1}{R} \tag{26}$$

Adding these terms to the displacement expressions (23), there results

$$\begin{aligned} 2\mu(u_r + iu_\theta) &= \left\{(\kappa-1)\Gamma + \overline{\Gamma' e^{2i\theta}}\right\} R + \left\{\kappa d_o - \bar{e}_o\right\} e^{-i\theta} \\ &\quad + (\kappa+1)\left\{-\frac{e_1}{2} + \bar{d}_1 e^{2i\theta} + d_1 e^{-2i\theta}\right\}\frac{1}{R} \\ &\quad + O\left(\frac{1}{R^2}\right) \end{aligned} \tag{27}$$

Substituting the expressions (24) and (27) in (21) we get after integration

$$U_R = \frac{\pi}{2\mu}\left\{[2(\varkappa-1)\,\Gamma^2 + \Gamma'\overline{\Gamma'}\,]\,R^2 + (\varkappa+1)\,\mathrm{Re}\,(-\Gamma_1 e_1 - \Gamma' d_1)\right\} + O\left(\frac{1}{R}\right) \tag{28}$$

Upon letting $\Gamma = \frac{q}{4}$ and $\Gamma' = -\frac{q}{2}\,e^{-2i\alpha}$ (see definition (20a)) and rearranging one obtains

$$U_R = \frac{\pi(\varkappa+1)q^2}{8\,\mu}\,R^2 + \frac{\pi(\varkappa+1)\,q}{16\,\mu} \times \mathrm{Re}\left\{-2e_1 + 4\,d_1 e^{-2i\alpha}\right\} + O\left(\frac{1}{R}\right) \tag{29}$$

The term proportional to R^2 represents the energy stored when there is no crack present. Let us denote this part by U_R^o. The rest of the terms are due to the presence of the crack. Hence taking the limit of $\{U_R - U_R^o\}$ as $R \to \infty$, we get the required expression for the strain energy change in an infinite solid due to the crack as

$$U = \frac{\pi(\varkappa+1)\,q}{16\,\mu}\,\mathrm{Re}\left[-2\,e_1 + 4d_1 e^{-2i\alpha}\right] \tag{30}$$

It will be shown later that e_1 and d_1 are representable in the following form*

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*This representation follows from the linearity of the functions $\varphi(z)$ and $\psi(z)$ with respect to $\Gamma = q/4$ and $\Gamma' = -\frac{q}{2}\,e^{-2i\alpha}$. However, it will be explicitly derived later on pages 46-50.

$$d_1 = q\,[d_{11} + d_{12}\cos 2\alpha + d_{13}\sin 2\alpha]$$

$$e_1 = q\,[e_{11} + e_{12}\cos 2\alpha + e_{13}\sin 2\alpha]$$

where the terms with two subscripts are functions of the geometry only. That is for example

$$e_{11} = e_{11}(\lambda_1, \ell_1, \ell_2)$$

where ℓ_1 and ℓ_2 are the lengths of the two connected crack segments and λ_1 represents the angle between them (Figure 2). Substituting the expressions for e_1 and d_1 in equation (30) and collecting like terms one obtains

$$U = \frac{\pi(\varkappa+1)q^2}{32\mu}\,[c_1 + c_2\cos 2\alpha + c_3\sin 2\alpha + c_4\cos 4\alpha + c_5\sin 4\alpha] \tag{31}$$

where c_m are functions of λ_1, ℓ_1 and ℓ_2 only, and they are determined completely in terms of the coefficients of $\frac{1}{z}$ terms in the expansions (22) of $\varphi(z)$ and $\Psi(z)$.

The coefficients $c_m (m = 1, 2, \ldots, 5)$ are related to the terms d_{1j}, e_{1j} $(j = 1, 2, 3)$ in the following way.

$$\left.\begin{aligned}
c_1 &= 4\,\mathrm{Re}\,[d_{12} - i d_{13} - e_{11}] \\
c_2 &= 4\,\mathrm{Re}\,[2 d_{11} - e_{12}] \\
c_3 &= 4\,\mathrm{Re}\,[-2 i d_{11} - e_{13}] \\
c_4 &= 4\,\mathrm{Re}\,[d_{12} + i d_{13}] \\
c_5 &= 4\,\mathrm{Re}\,[d_{13} - i d_{12}]
\end{aligned}\right\} \tag{31a}$$

Before proceeding to solve for $\varphi(z)$ and $\psi(z)$, in order to calculate the coefficients d_1 and e_1, we will normalize the length scales. The main crack length will be taken as unity and the non-dimensionalized branch crack length will be denoted by 'a' where

$$a = \frac{\ell_2}{\ell_1} = \ell_2 \text{ for } \ell_1 = 1$$

The boundary value problem for the functions $\varphi(z)$ and $\psi(z)$ is considered next.

5. THE PROBLEM OF SOLVING FOR THE STRESS FUNCTIONS $\varphi(z)$ AND $\psi(z)$

It is convenient to consider the problem of determining the functions $\varphi(z)$ and $\psi(z)$ satisfying conditions (19) and (20) by conformally mapping the region outside the crack boundary L in the physical plane (z-plane) into the region exterior to the unit circle in the ξ plane. This will be achieved by the use of the following mapping function as given by Darwin[25].

$$z = \omega(\xi) = \frac{A}{\xi}(\xi - e^{i\alpha_1})^{\lambda_1} (\xi - e^{i\alpha_2})^{\lambda_2} \tag{32}$$

For purposes of later discussion, the derivative of the mapping function is given below

$$\begin{aligned} \omega'(\xi) &= \frac{A}{\xi^2}(\xi - e^{i\alpha_1})^{\lambda_1 - 1} (\xi - e^{i\alpha_2})^{\lambda_2 - 1} (\xi - e^{i\beta_1})(\xi - e^{i\beta_2}) \\ &= \omega(\xi)\,\frac{(\xi - e^{i\beta_1})(1 - e^{i\beta_2})}{\xi(\xi - e^{i\alpha_1})(\xi - e^{i\alpha_2})} \end{aligned} \tag{33}$$

The parameters $A, \alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1$ and λ_2 are real and $\lambda_1 + \lambda_2 = 2$.* The branches of the functions in (32) are so chosen that

$$z = A\xi + O(1) \text{ as } |\xi| \to \infty \tag{34}$$

*See Ref. (18) and Appendix II for details.

As the point ξ traverses the unit circle the point z moves along the crack boundary (Figure 8). The points $B'(\xi = e^{i\beta_1})$, $D'(\xi = e^{i\alpha_1})$, $C'(\xi = e^{i\beta_2})$ and $O'(\xi = e^{i\alpha_2})$ on the unit circle $|\xi| = 1$ go over the points B, D, C and O, respectively in the physical plane on the crack boundary.

Note that the derivative of the mapping function has simple zeros at points $\xi = e^{i\beta_1}$ and $\xi = e^{i\beta_2}$ which are images of the two crack tips and has branch points at $\xi = e^{i\alpha_1}$ and $\xi = e^{i\alpha_2}$. The branch points correspond to the angular corners of the branched crack system. Note also that for the range of values of λ_1 in which we are interested, $0 < \lambda_1 < 1$, the mapping function is not rational. (The functions φ, Ψ are known[21] for the cases $\lambda_1 = 1$ and $\lambda_1 = 0$, i. e., for the cases when the branched crack system becomes a straight crack). Details of the dependence of the parameters, $A, \alpha_1, \alpha_2, \beta_1$ and β_2 on the non-dimensional branch crack length 'a' and the angle between them are given in Appendix II.

Let us use the following notation

$$\left.\begin{aligned} \varphi_1(\xi) &= \varphi[\omega(\xi)] \\ \Psi_1(\xi) &= \Psi[\omega(\xi)] \end{aligned}\right\} \qquad (35)$$

Noting that

$$\frac{d\varphi(z)}{dz} = \frac{\frac{d\varphi_1(\xi)}{d\xi}}{\frac{d\omega}{d\xi}} = \frac{\varphi_1'(\xi)}{\omega'(\xi)}$$

the stress free boundary condition (19) is transformed into

$$\varphi_1(\xi) + \frac{\omega(\xi)}{\overline{\omega'(\xi)}} \overline{\varphi_1'(\xi)} + \overline{\psi_1(\xi)} = 0 \text{ on } |\xi| = 1 \tag{36}$$

The conditions (20) at infinity become

$$\left.\begin{aligned} \varphi_1(\xi) &= \Gamma\,\omega(\xi) + \varphi_o(\xi) \\ \psi_1(\xi) &= \Gamma' A\xi + \psi_o(\xi) \end{aligned}\right\} \text{ as } |\xi| \to \infty \tag{37}$$

where $\varphi_o(\xi)$ and $\psi_o(\xi)$ are functions holomorphic outside the unit circle $|\xi| = 1$ including the point at infinity. In the right hand side of the second of the equations (37) $A\xi$ is used instead of $\omega(\xi)$ for convenience in later use. The regular part of $\omega(\xi)$--that part which is bounded as $|\xi| \to \infty$ -- can be absorbed into $\psi_o(\xi)$. Substituting the expressions for φ_1 and ψ_1 from (37) into (36) and rearranging one obtains

$$\varphi_o(\xi) + \frac{\omega(\xi)}{\overline{\omega'(\xi)}} \overline{\varphi_o'(\xi)} + \overline{\psi_o(\xi)} = -2\Gamma\,\omega(\xi) - \overline{\Gamma' A\xi} \text{ on } |\xi| = 1 \tag{38}$$

Hence the boundary value problem originally posed in equations (11) to (14) reduces to finding two functions $\varphi_o(\xi)$ and $\psi_o(\xi)$ which are holomorphic (single valued analytic functions) in the region $|\xi| > 1$ including the point at infinity and satisfying the boundary condition (38) on the unit circle. For the mapping function ω, using equation (33) we get

$$\frac{\omega}{\overline{\omega'}} = \frac{\omega}{\omega'} \overline{\left[\frac{\xi(\xi - e^{i\alpha_1})(\xi - e^{i\alpha_2})}{(\xi - e^{i\beta_1})(\xi - e^{i\beta_2})}\right]} \qquad (39)$$

For rational mapping functions closed form solutions for $\varphi_o(\xi)$ and $\psi_o(\xi)$ could be obtained[21] since $\frac{\omega}{\omega'}$ can be expressed as a ratio of two polynomials in ξ on $|\xi| = 1$*. Fourier expansion of the boundary conditions (38) cannot be made as $\omega'(\xi)$ vanishes at points $\xi = e^{i\beta_1}$ and $\xi = e^{i\beta_2}$ which are images of the crack tips. Further simple poles for $\psi_o(\xi)$ at those points are admissible. Unique local expansions for the functions $\varphi(\xi)$ and $\psi(\xi)$ around the singular points $\xi = e^{i\alpha_1}$, $\xi = e^{i\alpha_2}$, $\xi = e^{i\beta_1}$ and $\xi = e^{i\beta_2}$ could not be made as the homogeneous problem has non-trivial local expansions around these points as was shown by Williams[26]

Due to these difficulties the boundary value problem will be modified by considering a different "crack boundary". The new crack boundary corresponds to the circle of radius $|\xi| = 1 + \epsilon$ in the mapped plane, (Figure 9), where $0 < \epsilon << 1$.

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* Andersson[18] due to an error in selecting the branches of ω and ω' evaluated the expression $\frac{\omega}{\omega'}$ incorrectly. Hence his results are invalid.

Let us denote U_E the strain energy change due to the presence of the crack when the boundary conditions are applied on the crack itself and by U_M that when they are applied on the modified boundary. Even though the magnitude of $(U_E - U_M)$ cannot be rigorously ascertained, it will be shown that for two singular elastostatic problems related to our problem, it is only of the order of ϵ. These two examples are given in the following section.

6. EFFECT OF MODIFYING THE BOUNDARY ON THE STRAIN ENERGY

In this section we will argue that the modification of the crack boundary in the manner discussed in section 5 affects the strain energy change only to order ϵ.

Example 1. Straight Crack

Consider the special case of our general problem, namely $\lambda_1 = 1$. We have a straight crack of length $(1 + a)$. The mapping function in this case reduces to

$$z = \omega(\xi) = A\left(\xi + \frac{1}{\xi}\right)^{*} \tag{40}$$

where $A = \dfrac{1 + a}{4}$.

Since the mapping function is rational, the solution for $\varphi_1(\xi)$ and $\Psi_1(\xi)$ satisfying equations (36) and (37) can be found in closed form[21].

$$\begin{aligned} \varphi_1(\xi) &= A\left[\Gamma\xi - (\Gamma + \overline{\Gamma'}\, m^2)\,\frac{1}{\xi}\right] \\ \Psi_1(\xi) &= A\left[\Gamma'\xi + \overline{\Gamma'}\, m^4\,\frac{1}{\xi} + \left(\frac{2\Gamma}{m^2} + \overline{\Gamma'}\right)\left(1 + m^4\right)\frac{\xi}{1 - \xi^2}\right] \end{aligned} \tag{41}$$

where $m = 1$ corresponds to the case when the boundary conditions are applied on the crack and with $m = 1 + \epsilon$ we get the solution when they are applied on the modified boundary corresponding to the

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*We have neglected a constant term on the R.H.S., which corresponds to a translation of origin, which does not affect the strain energy change.

circle $|\xi| = 1 + \epsilon$ in the mapped plane. From (40) we obtain

$$\left.\begin{aligned} \xi &= \frac{z}{A} - \frac{A}{z} + O\left(\frac{1}{z^3}\right) \\ \frac{1}{\xi} &= \frac{A}{z} + O\left(\frac{1}{z^3}\right) \end{aligned}\right\} \quad \text{as } |z| \to \infty \tag{42}$$

Using (41), (42), (35), and (30) one obtains

$$U_E = \frac{\pi(\kappa+1)q^2}{32\mu}\left\{\left(\frac{1}{2} + a + \frac{a^2}{2}\right) - \left(\frac{1}{2} + a + \frac{a^2}{2}\right)\cos 2\alpha\right\} \tag{43}$$

$$U_M = \frac{\pi(\kappa+1)q^2}{32\mu}\left\{\left[\left(\frac{1}{2} + a + \frac{a^2}{2}\right)\cdot(1+\epsilon) + \epsilon(1+a)^2\right]\right.$$

$$\left. - \left[\frac{1}{2} + a + \frac{a^2}{2}\right]\cos 2\alpha\right\} + O(\epsilon^2) \tag{43a}$$

where U_E denotes the exact strain energy change obtained for the original problem and U_M that calculated by solving the modified problem.

Comparing with equation (31) we see that the coefficients $c_i(\lambda_1, a)$ are affected only to order ϵ and hence the total change in strain energy is also affected to order ϵ, only. Since we will be comparing quantities of order 'a' as seen from equation (43) we shall take $\epsilon = a^4$ in our later calculation when solving the modified boundary value problem.

It might be pointed out here that when $\alpha = 0$, $U_E = 0$ while $U_M = O(\epsilon)$. This fact is of no serious consequence to us as we are interested only in the values of the strain energy in the neighborhood of its maximum with respect to λ_1.

Now we proceed to the second example.

Example 2. Anti-Plane Strain Problem

Consider an infinitely long cylinder with its cross section D independent of the axial coordinate x_3. Let L denote the boundary of the cross-section D and n the normal to the boundary. Then by anti-plane shear problem it is meant that the cylinder is loaded by shear tractions σ^*_{n3} only on the boundary which acts in the direction parallel to the axis of the cylinder. Further it will be assumed that the applied tractions are independent of the axial coordinate x_3. In such a case the in-plane displacements u_1 and u_2 vanish and the three normal stresses σ_{11}, σ_{22} and σ_{33} and the shear stress σ_{12} also vanish. The out-of-plane (or warping) displacement $w = u_3$, is independent of the axial coordinate x_3.

The stress-strain relations combined with the strain-displacement relations reduce to

$$\left.\begin{aligned} \sigma_{13} &= \mu \frac{\partial w}{\partial x_1} \\ \sigma_{23} &= \mu \frac{\partial w}{\partial x_2} \end{aligned}\right\} \tag{44}$$

and the equilibrium equations become

$$\nabla^2 w(x_1, x_2) = 0 \quad \text{in} \quad D \tag{45}$$

while the boundary conditions reduce to

$$\sigma_{n3} = \mu \frac{\partial w}{\partial n} = \sigma^*_{n3} \quad \text{on } L \tag{46}$$

Let us consider the case when the region D is the one exterior to the crack boundary L and the loading is at infinity by uniform shear stresses τ_∞ as shown in Figure 10. Then the boundary value problem reduces to

$$\nabla^2 w = 0 \quad \text{in } D \tag{47}$$

$$\sigma_{n3} = \mu \frac{\partial w}{\partial n} = 0 \quad \text{on } L \tag{48}$$

$$\left.\begin{aligned} \underset{r\to\infty}{\text{Limit}}\ \sigma_{13} &= \tau_\infty \cos\alpha \\ \underset{r\to\infty}{\text{Limit}}\ \sigma_{23} &= \tau_\infty \sin\alpha \end{aligned}\right\} \tag{49}$$

Sih[27] derived a solution only for the stress intensity factors at the crack tips (points B and C in Figure 8) but not for those at the angular corners (points O and D), by making use of the complex representation of w and mapping the region D into a half-plane. He, furthermore, did not calculate the energy release rate. The method of solution adopted in this work, using the mapping of D onto the region exterior to the unit circle, facilitates the complete solution of the original and modified problems analogous to those of the in-plane problems. This method could also be used to obtain an exact solution when the boundary consists of an arbitrary but finite number of cracks of unequal lengths originating from a common point.

We shall re-state the boundary value problem as posed in equations (47) to (49) in order to make use of the complex representation of w.

Let w_c be the harmonic conjugate of w and let

$$h(z) = w + i \cdot w_c \tag{50}$$

where $z = x_1 + i x_2$

Condition (48) will be rewritten in terms of w_c as

$$\frac{\partial w_c}{\partial s} = 0 \quad \text{on} \quad L$$

where s denotes the coordinate along L. Without loss of generality this can be written as

$$w_c = 0 \quad \text{on} \quad L \tag{51}$$

From (50) we get

$$\frac{dh}{dz} = \frac{\partial w}{\partial x_1} + i \frac{\partial w_c}{\partial x_2}$$

and using the Cauchy-Riemann conditions there results

$$\begin{aligned} \frac{dh}{dz} &= \frac{\partial w}{\partial x_1} - i \frac{\partial w}{\partial x_2} \\ &= \frac{1}{\mu} (\sigma_{13} - i \sigma_{23}) \end{aligned} \tag{52}$$

Hence the boundary value problem as posed in equations (47) to (49) reduces to finding a function h(z), holomorphic-single valued and analytic-in the region D excluding the point at infinity. The boundary condition (48), using (51) becomes

$$h(z) - \overline{h(z)} = 0 \quad \text{on} \quad L \tag{53}$$

while the conditions at infinity (49) reduce to

$$\operatorname*{Limit}_{|z|\to\infty} \frac{dh}{dz} = \frac{\tau_\infty}{\mu} e^{-i\alpha} \tag{54}$$

By use of the mapping function $\omega(z)$ stated in equation (32), the region D can be mapped into the region outside the unit circle in the ξ-plane. Let $h_1(\xi) = h(\omega(\xi))$. Then conditions (53) and (54) lead to

$$\left.\begin{aligned} h_1(\xi) - \overline{h_1(\xi)} &= 0 \quad \text{on} \quad |\xi| = 1 \\ \operatorname*{Limit}_{|\xi|\to\infty} \left(\frac{h_1(\xi)}{\omega'(\xi)}\right) &= \frac{\tau_\infty}{\mu} e^{-i\alpha} \end{aligned}\right\} \tag{55}$$

Noting that $\operatorname*{Limit}_{|\xi|\to\infty} \omega'(\xi) = A$, the solution to (55) is obtained as

$$h_1(\xi) = \frac{A\tau_\infty}{\mu}\left[e^{-i\alpha}\xi + \frac{e^{i\alpha}}{\xi}\right] \tag{56}$$

Having found the solution, the strain energy can be calculated using a method[28] similar to the one described for in-plane problems (see pages 22-25).

$$U_E = \frac{2\pi A^2 \tau_\infty^2}{\mu} \operatorname{Re}\left[1 - \omega_2 e^{-2i\alpha}\right] \tag{57}$$

where $\omega_2 = \frac{1}{2}\left[\lambda_1(\lambda_1 - 1)e^{2i\alpha_1} + 2\lambda_1\lambda_2 e^{i(\alpha_1+\alpha_2)} + \lambda_2(\lambda_2 - 1)e^{2i\alpha_2}\right]$

The solution to the modified problem, namely the one in which the stress free boundary condition is applied on a smooth curve corresponding to $|\xi| = 1 + \epsilon$ instead of on the crack itself, can be obtained in a similar way by replacing the first of the conditions in equation (55) by

$$h_1(\xi) - \overline{h_1(\xi)} = 0 \quad \text{on} \quad |\xi| = 1 + \epsilon \tag{58}$$

One obtains in this case

$$h_1(\xi) = \frac{A\tau_\infty}{\mu}\left[e^{-i\alpha}\,\xi + (1+\epsilon)^2\,\frac{e^{i\alpha}}{\xi}\right] \tag{59}$$

and the strain energy change as

$$U_M = \frac{2\pi A^2\tau_\infty^2}{\mu}\,\mathrm{Re}\,[(1+\epsilon)^2 - \omega_2 e^{-2i\alpha}] \tag{60}$$

From equations (57) and (60) it follows that

$$U_E - U_M = \frac{2\pi A^2\tau_\infty^2}{\mu}\,[2\epsilon + \epsilon^2] = O(\epsilon) \text{ as } \epsilon \to o \tag{61}$$

For purposes of later discussion the expansion of U_E in terms of the branch crack length 'a', obtained by substituting the expansions for A and ω_2 (given in Appendix II, equation (2.5) and (2.10)) in equation (57), is given below.

$$U_E = \frac{2\pi\tau_\infty^2}{\mu}\,[\tilde{c}_1 + \tilde{c}_2 a + \tilde{c}_3 a^{3/2} + O(a^2)] \tag{62}$$

where

$$
\left.
\begin{aligned}
\tilde{c}_1 &= \frac{\sin^2\alpha}{8} = \text{term corresponding to a single crack of length unity} \\
\tilde{c}_2 &= \frac{1}{4}\,\lambda_1^{1-\lambda_1}\,\lambda_2^{1-\lambda_2}\,\sin^2\alpha \\
\tilde{c}_3 &= +\,\frac{p_1^3\lambda_1\lambda_2}{12}\left[4\lambda_2^2(\lambda_1-1)-(\lambda_1-\lambda_2)^3-4\lambda_1^2(\lambda_2-1)\right]\sin 2\alpha \\
&\text{with } p_1^2 = \lambda_1^{-\lambda_1}\,\lambda_2^{-\lambda_2}
\end{aligned}
\right\} \quad (63)
$$

In both the examples considered the modification of the boundary affects the strain energy change only to the order ϵ. These examples will be taken as supportive evidence of the validity of the approximation made for the general case. Solution of the modified boundary value problem is discussed in the following part.

PART II

1. SOLUTION OF THE MODIFIED BOUNDARY VALUE PROBLEM

We now normalize the radius of the circle in the ξ-plane, corresponding to the modified smooth boundary in the physical plane, to unity by changing the mapping function as follows.

$$z = \omega(\xi) = \frac{A}{k\xi}(\xi - ke^{i\alpha_1})^{\lambda_1}(\xi - ke^{i\alpha_2})^{\lambda_2}, \quad (k = \frac{1}{1+\epsilon}) \tag{64}$$

In this representation, the branched crack will correspond to the circle $|\xi| = k$ (Figure 11). For all our further discussion it will always be taken that $0 < \epsilon << 1$. We need to determine two functions $\varphi_o(\xi)$ and $\psi_o(\xi)$, holomorphic in the region $|\xi| > 1$ including the point at infinity and satisfying the following boundary condition

$$\left.\begin{aligned} \overline{\varphi_o(\xi)} + \frac{\overline{\omega(\xi)}}{\overline{\omega'(\xi)}}\varphi_o'(\xi) + \psi_o(\xi) = -2\Gamma\,\omega(\xi) - \frac{\Gamma' A}{k}\xi \\ \text{on } |\xi| = 1 \end{aligned}\right\} \tag{65}$$

There are essentially two methods of solution for the above type of problem as given by Muskhelishvili[21].

Method I

In this method, which uses the properties of Cauchy integrals, the problem can be reduced to an integral equation for the boundary values of the function $\varphi_o'(\xi)$, of the following form

$$\varphi_o'(\eta) = f(\eta) + \frac{1}{2\pi i} \oint_\gamma K(\eta, \xi)\, \overline{\varphi_o'(\xi)}\, d\xi \tag{66}$$

where γ is the unit circle in the ξ-plane and $|\eta| = 1$. This equation can be reduced to a Fredholm integral equation of the second kind. Once the boundary values of φ_o' are determined from equation (66), $\varphi_o(\xi)$ and $\Psi_o(\xi)$ can be determined with the help of Cauchy integrals and using equation (65).

Method II

Let

$$\left.\begin{aligned} \varphi_o(\xi) &= \sum_{m=o}^{\infty} f_m \xi^{-m} & |\xi| \geq 1 \\ \Psi_o(\xi) &= \sum_{m=o}^{\infty} b_m \xi^{-m} & |\xi| \geq 1 \end{aligned}\right\} \tag{67}$$

$$\left.\begin{aligned} \frac{\overline{\omega(\xi)}}{\omega'(\xi)} &= \sum_{n=-\infty}^{\infty} h_n \xi^n & |\xi| = 1 \\ -2\Gamma\overline{\omega(\xi)} - \frac{\Gamma' A}{k}\xi &= \sum_{n=-\infty}^{\infty} g_n \xi^n & |\xi| = 1 \end{aligned}\right\} \tag{68}$$

Substituting for φ_o, Ψ_o from equations (67) and (68) in (65) and remembering that on $|\xi| = 1$, $\bar{\xi} = \frac{1}{\xi}$ we get

$$\left.\begin{aligned} &\sum_{m=o}^{\infty} \bar{f}_m \xi^m - \left(\sum_{n=-\infty}^{\infty} h_n \xi^n\right)\left(\sum_{m=1}^{\infty} m f_m \xi^{-m-1}\right) \\ &+ \sum_{m=o}^{\infty} b_m \xi^{-m} = \sum_{n=-\infty}^{\infty} g_n \xi^n \end{aligned}\right\} \tag{69}$$

Equating the coefficients of equal positive powers of ξ on both sides of the equation (69) one obtains

$$\bar{f}_n - \sum_{m=1}^{\infty} m\, h_{n+m+1}\, f_m = g_n \qquad (n = 1, 2, \ldots) \tag{70}$$

We shall write down only the equation corresponding to the negative power ξ^{-1} as the terms b_n for $n > 1$ do not contribute to the energy integral, since we need only the coefficient of the $\frac{1}{z}$ term in $\Psi(z)$ as was shown in pages 22-25 and also $\frac{1}{\xi} = \frac{A}{kz} + O(\frac{1}{z^2})$ as $|z| \to \infty$.

$$b_1 - \sum_{m=1}^{\infty} m\, h_m\, f_m = g_{-1} \tag{71}$$

The multiplication of the two series in equation (69) is justified because both series are absolutely convergent. Once the coefficients f_n are determined from equations (70), b_1 can be calculated from (71).

The writer first attempted to solve the integral equations (66) but difficulties were encountered. The kernel $K(\eta, \xi)$ is complicated and the iteration methods seemed to lead nowhere as even the first iterate could not be obtained analytically. As for approximating the kernel $K(\eta, \xi)$ by a P-G kernel[29] of the form $\tilde{K}(\eta,\xi) = \sum_{n=0}^{N} F_n(\eta)\, G_n(\xi)$, which is one way to solve the integral equations approximately, appropriate functions F_n and G_n could not be found other than the polynomials which lead to equations of the form (70).

The numerical solution of the integral equation (66) was attempted. In the method which the author tried, the integration on the right hand side was replaced by summation using quadrature rules (four point and six point Simpson's rules were tried) using discreet values of the kernel K and the function $\omega_o'(\xi)$. The intervals at which these discreet values were used were non-uniform, smaller in the region where the variation of the kernel was great and larger where the variation was smooth. Using this method the values of $\omega_o'(z)$ were calculated at discreet points on the unit circle. Before using this method for the general case, the author used the method for the special case for which the exact solution is known; namely, the straight crack with its modified boundary being an ellipse. The error in the energy expression due to this approximate method was about 10%. Then the author felt this method was not suitable for numerical calculations, mainly because the kernel is not well-behaved.

Due to the above difficulties encountered in solving the integral equation (66), the author decided to use the second method.

In this method the equations (70) for the coefficients are approximately solved by assuming $f_n = 0$ for $n>N$. Then the relations (70) reduce to the following finite set of equations in f_n.

$$\overline{f}_n - \sum_{m=1}^{N} m\, h_{n+m+1} f_m = g_n \quad (n = 1, 2, \ldots, N) \tag{70a}$$

Let us denote the solution of this system of equations for f_n as f_n^N to distinguish it from the solution of the system (70). The corresponding relation for b_1 reduces to the following by assuming $f_m = 0$ for $m > N$ in equation (71).

$$b_1^N = g_{-1} + \sum_{m=1}^{N} m\, h_m f_m^N \tag{71a}$$

Using the following vector notation,

$$\left.\begin{aligned} f^N &= (f_1^N, f_2^N, \ldots, f_N^N)^T \\ g^N &= (g_1, g_2, \ldots, g_N)^T \end{aligned}\right\} \tag{72}$$

and defining the square matrix H^N by

$$H^N = [H_{ij}] = [j\, h_{i+j+1}] \quad (i, j = 1, 2, \ldots, N) \tag{73}$$

equation (70a) can be written in the matrix form as given below

$$\overline{f^N} - H^N f^N = g^N \tag{70b}$$

Taking the conjugate we obtain,

$$f^N - \overline{H^N}\,\overline{f^N} = \overline{g^N} \tag{70c}$$

Substituting for $\overline{f^N}$ from (70b) in (70c) and using the notation

$$\left.\begin{aligned} T^N &= I^N - \overline{H^N} H^N \\ r^N &= \overline{g^N} + \overline{H^N}\, g^N \end{aligned}\right\} \tag{74}$$

where I^N is the $N \times N$ identity matrix there results,

$$T^N f^N = r^N \tag{70d}$$

which needs to be solved for f^N for given r^N. Once the solution for

f^N is known b_1^N can be calculated using the relation (71a).

From the left hand side of the second of the equations (68) we see that the terms g_m can be represented as

$$g_m = \Gamma\, g_{m1} + \Gamma'\, g_{m2} \qquad (m = 0, \pm 1, \pm 2 \ldots\ldots) \tag{75}$$

where

$$\begin{aligned} g_{m1} &= \frac{1}{2\pi i} \oint_{\gamma} -2\, \omega(\xi)\, \xi^{-(m+1)}\, d\xi \quad (\gamma: |\xi| = 1) \\ g_{m2} &= \begin{cases} -\frac{A}{k} & \text{if } m = 1 \\ 0 & \text{if } m \neq 1 \end{cases} \end{aligned} \tag{76}$$

Using the notation

$$g_i^N = (g_{1i}, g_{2i}, g_{3i}, \ldots, g_{Ni})^T \qquad (i = 1, 2) \tag{77}$$

and making use of the representation (75) for g_i, one obtains

$$g^N = \Gamma\, g_1^N + \Gamma'\, g_2^N \tag{78}$$

Substituting for g^N from (78) in the second of equations (74) we get

$$r^N = \Gamma\, r_1^N + \overline{\Gamma'}\, r_2^N + \Gamma'\, r_3^N \tag{79a}$$

where

$$\overline{r_1^N} = \overline{g_1^N} + \overline{H^N} \; g_1^N$$

$$r_2^N = \overline{g_2^N} \qquad (79b)$$

$$r_3^N = \overline{H_N} \; g_2^N$$

Let us denote by $f_{(i)}^N$ (i, =1, 2, 3) the solution of (70d) when r_i^N is used on the right hand side. Then there results

$$f^N = \Gamma \; f_{(1)}^N + \overline{\Gamma'} \; f_{(2)}^N + \Gamma' \; f_{(3)}^N \qquad (80)$$

Using the same notation for $f_{(i)}^N$ as that of g^N in equation (77), and remembering the notation for f^N as given in equation (72), one obtains

$$f_1^N = \Gamma \; f_{11} + \overline{\Gamma'} \; f_{12} + \Gamma' \; f_{13} \qquad (80a)$$

From equations (71a) and (80) we get

$$b_1^N = g_{-1} + \sum_{m=1}^{N} m \, h_m \, [\Gamma \, f_{m1}^N + \overline{\Gamma'} \; f_{m2}^N + \Gamma' \, f_{m3}^N] \qquad (71b)$$

Also from (75) and (76) there results,

$$g_{-1} = \Gamma \; g_{-11} \qquad (75a)$$

Combining (75a) and (71b) and using the notation

$$b_{11} = g_{-11} + \sum_{m=1}^{N} m\, h_m\, f_{m1}^N$$

$$b_{12} = \sum_{m=1}^{N} m\, h_m\, f_{m2}^N$$

$$b_{13} = \sum_{m=1}^{N} m\, h_m\, f_{m3}^N$$

we obtain

$$b_1^N = \Gamma\, b_{11} + \overline{\Gamma'}\, b_{12} + \Gamma'\, b_{13} \tag{80b}$$

Once the values of f_1^N and b_1^N which are the approximate values of f_1 and b_1, the coefficients of $\frac{1}{\xi}$ terms of $\varphi_o(\xi)$, $\Psi_o(\xi)$ in the expansions (67) are known then d_1 and e_1 which are the coefficients of $\frac{1}{z}$ terms of $\varphi(z)$ and $\Psi(z)$ in the expansions (22) can be calculated as follows.

From the definitions (35) and (37), we get

$$\varphi(z) = \varphi_1(\xi) = \Gamma\omega(\xi) + \varphi_o(\xi)$$

$$\Psi(z) = \Psi_1(\xi) = \frac{\Gamma' A}{k}\xi + \Psi_o(\xi)$$

Using the expansions for $\varphi_o(\xi)$ and $\Psi_o(\xi)$ from (67) one obtains

$$\left.\begin{aligned} \varphi(z) &= \Gamma z + f_o + \frac{f_1}{\xi} + O\left(\frac{1}{\xi^2}\right) \\ \Psi(z) &= \frac{\Gamma' A}{k}\xi + b_o + \frac{b_1}{\xi} + O\left(\frac{1}{\xi^2}\right) \end{aligned}\right\} \text{ as } |\xi| \to \infty$$

Substituing the expressions for ξ and $\frac{1}{\xi}$ from equations (2.12) and (2.13) in Appendix II, page 82, there results

$$\left.\begin{aligned} \varphi(z) &= \Gamma z + f_o + f_1 \frac{A}{k} \cdot \frac{1}{z} + O\left(\frac{1}{z^2}\right) \\ \Psi(z) &= \frac{\Gamma' A}{k}\left(\frac{k}{A} z - k\omega_1 - A k \frac{\omega_2}{z}\right) + b_o + b_1 \cdot \frac{A}{k} \cdot \frac{1}{z} + O\left(\frac{1}{z^2}\right) \end{aligned}\right\} \text{ as } |\xi| \to \infty$$

Comparing these expressions with the expansions (22), one gets the following expressions for d_1 and e_1 in terms of f_1 and b_1.

$$\left.\begin{aligned} d_1 &= \frac{A}{k} \cdot f_1 \\ e_1 &= -\Gamma' A^2 \omega_2 + \frac{A}{k} \cdot b_1 \end{aligned}\right\} \qquad (81a)$$

Substitution of the approximate values of f_1 and b_1 from equations (80a) and (80b) in the above expressions results in

$$\left.\begin{aligned} d_1 &= \frac{A}{k} [\Gamma f_{11} + \overline{\Gamma'} f_{12} + \Gamma' f_{13}] \\ e_1 &= \frac{A}{k} [\Gamma b_{11} + \overline{\Gamma'} b_{12} + \Gamma' (b_{13} - A k \omega_2)] \end{aligned}\right\} \qquad (81b)$$

Use of the definitions (20a) for Γ and Γ' yields

$$\left.\begin{aligned} d_1 &= q [d_{11} + d_{12} \cos 2\alpha + d_{13} \sin 2\alpha] \\ e_1 &= q [e_{11} + e_{12} \cos 2\alpha + d_{13} \sin 2\alpha] \end{aligned}\right\} \qquad (81c)$$

where

$$\left.\begin{aligned}
d_{11} &= \frac{A}{4k} f_{11} \\
d_{12} &= -\frac{A}{2k} (f_{12} + f_{13}) \\
d_{13} &= -\frac{A}{2k} i (f_{12} - f_{13}) \\
e_{11} &= \frac{A}{4k} b_{11} \\
e_{12} &= -\frac{A}{2k} (b_{12} + b_{13} - Ak\omega_2) \\
e_{13} &= -\frac{A}{2k} i (b_{12} - b_{13} + Ak\omega_2)
\end{aligned}\right\} \qquad (81d)$$

Recalling the expression (31) for the strain energy change due to the branched crack system and subtracting the energy change due to the main crack (which corresponds to $c_1 = 0.5$, $c_2 = 0.5$ with the rest of the coefficients vanishing in equation (31)) we get the required expression for the strain energy change when the branch crack extends from a length of zero to a length of 'a', as

$$\begin{aligned}
U(\lambda_1, a) = \frac{\pi(\kappa + 1)q^2}{32\mu} [&C_1(\lambda_1, a) + C_2(\lambda_1, a) \cos 2\alpha + C_3(\lambda_1, a) \sin 2\alpha \\
&+ C_4(\lambda_1, a) \cos 4\alpha + C_5(\lambda_1, a) \sin 4\alpha]
\end{aligned} \qquad (82)$$

where

$$\begin{aligned}
C_1 &= c_1 - 0.5 \\
C_2 &= c_2 + 0.5 \\
C_m &= c_m \quad (m = 3, 4, 5)
\end{aligned} \qquad (82a)$$

with $c_m (m=1, 2, \ldots, 5)$ related to the d_{ij} and e_{ij} $(j = 1, 2, 3)$ by the expressions (31 a) derived earlier in page 27.

The method of obtaining the coefficients C_i, and then determining the failure load and branching angle from them are discussed in the following section.

2. METHOD OF OBTAINING THE CRACK BRANCHING ANGLE AND THE CRITICAL LOAD:

For discreet values of the crack branching angle λ_1 and the branch crack length 'a', the matrix equations (70d) were solved using Gaussian elimination with maximal pivoting[30]. To check for possible numerical errors the solutions for f_n^N $(n = 1, 2, \ldots, N)$ were substituted back in the left hand side in equations (70a) and the results compared with the right hand side. They differed only in the fourth significant place and this was considered sufficiently accurate. The coefficient b_1^N, is then calculated using equation (71a). Having calculated f_1^N and b_1^N, the coefficients d_1 and e_1 are obtained using equation (81a). The terms C_i appearing in the expression (82) for the strain energy change are calculated using equations (80a), (80b), (81d), (82a) and (31a). For each value of 'a' the value of N was doubled until the values of the coefficients $C_i(\lambda_1, a)$ from two successive calculations differed by less than 2 per cent. The range of values of 'a' for which the numerical results are obtained is from 0.1 to 0.005. The corresponding range of N is from 20 to 100. The values of λ_1 are increased from 0.5 to 1.0 in steps of 0.05 (nine degrees). Once having calculated the coefficients $C_i(\lambda_1, a)$ for discreet values of 'λ_1' and 'a', the direction of crack extension and the critical value of the applied stress for given direction of loading are obtained according to the following procedure.

For each given value of the loading angle we need to calculate the energy release rate $G(\alpha, \lambda_1) = \lim_{a \to 0} \frac{\partial U}{\partial a}$ and find the maximum of

G with respect to λ_1. Let $\lambda^*(\alpha)$ be the value of λ_1 at which G attains the maximum. Then $\lambda^*(\alpha)$ defines the crack branching angle and the energy release rate $G(\alpha, \lambda^*)$ determines the critical value of the applied stress $q_{cr}(\alpha)$ through equation (9), page 15.

Since 'a' is small compared to unity and by definition $U\Big|_{a=0} = 0$ the author replaced $\frac{\partial U}{\partial a}$ by $\frac{U}{a}$ in the initial calculations. Then for each value of 'a' and 'α' the maximum of $\frac{U}{a}$ with respect to λ_1 was found using a parabolic fit for the three values of $\frac{U}{a}$. near the maximum. Let us denote by $\tilde{G}(\alpha, a)$ the maximum value of $\frac{U}{a}$ and by $\tilde{\lambda}^*(\alpha, a)$ the value of λ_1 at which the maximum is attained. Using $\tilde{G}(\alpha, a)$ as the energy release rate G in equation (9) the critical loading stress $\tilde{q}_{cr}(\alpha, a)$ was calculated. Then in the limit $a \to 0$, $\tilde{q}_{cr}(\alpha, a)$ and $\tilde{\lambda}^*(\alpha, a)$ should tend to the values $q_{cr}(\alpha)$ and $\lambda^*(\alpha)$ respectively. When the value of 'a' was decreased from 0.1, the values of $\tilde{q}_{cr}$ and $\tilde{\lambda}^*$ converged in the range for which $\tilde{\lambda}^* > 0.67$. In the range $\tilde{\lambda}^* < 0.67$, the author was not able to obtain the limit values this way, as the successive values were still differing by about 3% for $\tilde{q}_{cr}$ and about 2° for the crack branching angle, when $a = 0.005$. The main reason for the slow convergence in this range $(\lambda_1 < 0.67)$ may be due to the presence of the terms of order $a^{3/2}$ in the expansion for U for small values of 'a', similar to those in equation (62) for the anti-plane shear problem solved in pages 35-40. In the expression (62) for U, the coefficient of the $a^{3/2}$ term vanishes when $\lambda_1 = 1$ and is of the same order as that of the 'a' term when $\lambda_1 = 0.5$. Because of this term, replacing $\frac{\partial U}{\partial a}$ by $\frac{U}{a}$ could cause an error of the order of 6¢ when a = 0.005. Since the author could not reduce the value of 'a' below 0.005 due to space limitations on the computer, he decided to obtain the limit values as follows.

In order to calculate $\lim_{a \to 0} \frac{\partial U(\lambda_1, a)}{\partial a}$, we need to know $\lim_{a \to 0} \frac{\partial C_i(\lambda_1, a)}{\partial a}$ $(i = 1, 2, \ldots, 5)$. These limits are obtained by making the following fit for the values of $C_i(\lambda, a)$, calculated for discreet but small values of 'a'.

$$C_i(\lambda_1, a) = C_{i1}(\lambda_1)\, a + C_{i2}(\lambda_1)\, a^{3/2} + C_{i3}(\lambda_1)\, a^2 \tag{83}$$

The above fit is based on the knowledge of the behavior of the corresponding coefficients (equation (62)) for the anti-plane shear problem. The fit was made for the values of $C_i(\lambda_1, a)$ calculated for every three consecutive values of 'a', and the results so obtained for $C_{i1}(\lambda_1)$, which govern the limits, differed less than 2¢. Recalling the expression (82) for U, and since $G(\alpha, \lambda_1) = \lim_{a \to 0} \frac{\partial U}{\partial a}$, we get the following expression for the energy release rate at the instant of incipient crack extension

$$G(\alpha, \lambda_1) = \frac{\pi}{32} \frac{(\kappa + 1) q^2}{\mu} [C_{11}(\lambda_1) + C_{21}(\lambda_1) \cos 2\alpha + C_{31}(\lambda_1) \sin 2\alpha + C_{41}(\lambda_1) \cos 4\alpha + C_{51}(\lambda_1) \sin 4\alpha] \tag{84}$$

Having calculated the energy release rate, the crack branching angle $\lambda^*(\alpha)$ was determined by finding the value of λ_1 for which G attains the maximum for given α, using a three point parabolic fit for the three values of G near the maximum. A four point least square fit was also used and the results differed less than one per cent. Using the maximum value of G in the equation (9), the critical value of the applied stress $q_{cr}(\alpha)$ was calculated.

At this point it becomes necessary to define the stress intensity factors in order to compare the results with those obtained by Erdogan and Sih(14) using a different set of postulates. Referring to Figure 12, the stress field in the vicinity of the crack tip can be represented in the following form when the loading is symmetrical to the crack

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f^I_{ij}(\theta) + O(1) \quad \text{as } r \to 0 \tag{85}$$

and when it is asymmetrical

$$\sigma_{ij} = \frac{K_{II}}{\sqrt{2\pi r}} f^{II}_{ij}(\theta) + O(1) \quad \text{as } r \to 0 \tag{86}$$

$(i = 1, 2;\ j = 1, 2)$

K_I is called the stress intensity factor for the opening mode and K_{II} that for the sliding mode. $f^I_{ij}(\theta)$ and $f^{II}_{ij}(\theta)$ are functions independent of loading and geometry of the solid and are given in equations (78) and (80), page 216 of reference (28). K_I and K_{II} are functions of loading and geometry only. Once the values of q_{cr} are known, the values of the stress intensity factors K_I and K_{II} at the instant of incipient crack extension are calculated using the following relations connecting q to the stress intensity factors derived by Sih, Paris and Erdogan(31).

$$\left.\begin{aligned} K_I &= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} q\, \ell^{\frac{1}{2}} \sin^2\alpha \\ K_{II} &= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} q\, \ell^{\frac{1}{2}} \sin\alpha \cos\alpha \end{aligned}\right\} \qquad (87)$$

3. ANALYTICAL ERROR ESTIMATES

Some attempts made to estimate the error due to the truncation of the system (70) are described below. Let us rewrite the system of equations (70) in the following form

$$\bar{f}_n - \sum_{m=1}^{N} m\, h_{n+m+1}\, f_m = g_n + \left(\sum_{m=N+1}^{\infty} m\, h_{n+m+1}\, f_m \right) = g_n + (\delta g_n) \quad (n=1,\ldots,N) \tag{88}$$

where δg_n represents the summed terms on the right hand side. Suppose the values of f_m for $m > N$ are known. Then the exact values of f_n for $n = 1,\ldots,N$ can be calculated by solving the finite system (88). The effect of truncation in system (70) is equivalent to neglecting δg_n in the system (88). The error made due to this approximation can be estimated as follows.

Let us use the following notation,

$$x = (\bar{f}_1, \sqrt{2}\, \bar{f}_2, \sqrt{3}\, \bar{f}_3, \ldots\ldots, \sqrt{N}\, \bar{f}_N, f_1, \sqrt{2}\, f_2, \ldots\ldots, \sqrt{N}\, f_N)^T$$

$$y = (g_1, \sqrt{2}\, g_2, \ldots\ldots, \sqrt{N}\, g_n, \bar{g}_1, , \sqrt{2}\, \bar{g}_2, \ldots\ldots, \sqrt{N}\, \bar{g}_N)^T$$

i.e., x and y are column vectors with $2N$ elements as shown. Define the $2N \times 2N$ matrix C as follows

$$C = \begin{bmatrix} O & A \\ A & O \end{bmatrix}$$

where A is an N x N symmetric matrix with elements

$$A_{ij} = \sqrt{ij} \quad h_{i+j+1}$$

Also let $B = I - C$ where I is the 2N x 2N identity matrix and let $z = y + \delta y$.

In the above representation $\sqrt{n}\ f_n$ and $\sqrt{n}\ \bar{f}_n$ are chosen as independent variables in order to make the matrix A symmetric and hence the matrices C and B Hermitian. This choice simplified the error analysis as properties of Hermitian matrices are better known[31]. In this representation equations (88) and their conjugates can be written in the following form

$$B x = z \tag{89}$$

The coefficients h_n can be calculated only by numerical integration approximately. In the numerical calculations they are replaced by $k^{n+1}\ \tilde{h}_n$ where $\tilde{h}_n$ can be calculated exactly. An estimate of the magnitude of $(h_n - k^{n+1}\ \tilde{h}_n)$ is given in equation (3. 4), Appendix III. Let us denote the error in the matrix B, due to this approximation by δB. Then the exact values of x satisfy (89) while the calculated values $(x + \delta x)$ satisfy

$$(B + \delta B)(x + \delta x) = z + \delta z \tag{90}$$

where $\delta z = -\delta y$.

Assume B to be non-singular and

$$\|\delta B\| \, \|B^{-1}\| < 1 \tag{91}$$

where $\|\cdot\|$ denotes the Euclidean norm[30]. Then we get[30]

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|B^{-1}\| \, \|B\|}{1 - \|B^{-1}\| \, \|\delta B\|} \left\{ \frac{\|\delta z\|}{\|z\|} + \frac{\|\delta B\|}{\|B\|} \right\} \tag{92}$$

On the basis of the behavior of the displacements near the corners and crack tips as shown by Williams[26], it will be assumed that for $n > N$,

$$|f_n| \leq \frac{d}{n^{1.5}} \tag{93}$$

where d is a constant independent of ϵ. It is shown in Appendix III that

$$|h_n| \leq \left\{ M_1(\lambda_1)\sqrt{a} \;\; \epsilon \log\left(\frac{1}{\epsilon}\right) + M_2(\lambda_1) \;\; \frac{1}{(n-2)^2} \right\} k^{n+1} \tag{94}$$

where

$$M_1(\lambda_1) = \frac{8}{\sqrt{2}\,\pi} \; p_2(1-\lambda_1)$$

$$M_2(\lambda_1) = \frac{2 \sin \lambda_1 \pi}{\pi}$$

and $\qquad k = \dfrac{1}{1+\epsilon}$

Using (93) and (94), and the definition of δg_n (equation (88)) we get

$$|\delta g_n| \leq d \sum_{m=N+1}^{\infty} \left\{ \frac{M_1 \sqrt{a}\ \epsilon \log\left(\frac{1}{\epsilon}\right)}{m^{0.5}} + \frac{M_2}{m^{0.5}(m+n-1)} \right\} k^{m+n+1}$$

$$\leq d \left\{ M_1 (\pi a \epsilon)^{0.5} \log\left(\frac{1}{\epsilon}\right) + \frac{M_2}{N^{0.5}(N+n-1)} \right\} k^{N+n}$$

Noting that $\delta z_n = \sqrt{n}\ \delta g_n$, it follows that

$$\|\delta z\| \leq d \left[M_1 \left(\frac{\pi}{2} a \epsilon\right)^{0.5} \log\left(\frac{1}{\epsilon}\right)(N+1) + \frac{M_2}{N^{0.5}} \right] (1-\epsilon)^N$$

Since the problem is linear (see equation (88)) the coefficients g_n can be normalized such that, $\|z\| = 1$. Hence one obtains

$$\frac{\|\delta z\|}{\|z\|} \leq d \left[M_1 \left(\frac{\pi}{2} a \epsilon\right)^{0.5} \log\left(\frac{1}{\epsilon}\right)(N+1) + \frac{M_2}{N^{0.5}} \right] (1-\epsilon)^N \quad (95)$$

From the magnitude estimates of h_n given in Appendix III, and Greshgorin's theorem[30], it follows that

$$\|C\| \leq 2M_2 + M_3 \quad (96)$$

$$\|\delta B\| \leq M_3 \quad (97)$$

and

$$\|B\| \leq 1 + 2M_2 + M_3 \quad (98)$$

where $M_3 = \frac{M_1}{1.5} \quad N^2 \sqrt{a} \quad \epsilon \log\left(\frac{1}{\epsilon}\right)$.

Also from the following theorem[32],

"If $\quad \|C\| < 1, \qquad (99)$

then

$$\|(I-C)^{-1}\| \leq \frac{1}{1 - \|C\|} \text{"}$$

it follows that

$$\|B^{-1}\| \leq \frac{1}{1-[2M_2 + M_3]} \tag{100}$$

Using the expressions (97), (98), and (100) in (92) one obtains that

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{(1+2M_2+M_3)\frac{\|\delta z\|}{\|z\|} + M_3}{1-2(M_2+M_3)} \tag{101}$$

Noting that $|\sin\lambda_1\pi| \leq \pi(1-\lambda_1)$, using the inequality (95) for $\frac{\|\delta z\|}{\|z\|}$ and substituting the expressions for M_1, M_2 and M_3 in (101), there results,

$$\frac{\|\delta x\|}{\|x\|} \leq \left\{ \frac{\left[d\,\{1 + (1-\lambda_1)[4 + 4\cdot 2\, N^2\sqrt{a}\;\epsilon\;\log(\frac{1}{\epsilon})]\}\cdot \{7.9\,(a\epsilon)^{0.5}\log(\frac{1}{\epsilon})\,(N+1) + \frac{2}{N^{0.5}}\}(1-\epsilon)^N + 4.2\,N^2\sqrt{a}\;\epsilon\log(\frac{1}{\epsilon}) \right](1-\lambda_1)}{1-(1-\lambda_1)\{4 + 8.2\,N^2\sqrt{a}\;\epsilon\log(\frac{1}{\epsilon})\}} \right\} \tag{102}$$

In order to satisfy condition (99) the range of λ_1 has to be restricted to $\lambda_1 > 0.75$. For this range it is possible to select N and ϵ for given 'λ_1' and 'a' to give the desired accuracy in the value of the coefficients f_n. However, since the error bounds are in general not the least upper-bounds, the value of N required will be in general large. In the numerical calculations made, the value of N was doubled until two successive values of the coefficients C_i in the energy expression (82) differed by less than 2%.

4. "MAXIMUM NORMAL STRESS" THEORY AND THE "ENERGY" THEORY

For purposes of comparison the results derived by Erdogan and Sih[14] are given below. Referring to Figure 13a, the stress σ_θ in the vicinity of the crack tip can be expanded as follows.[14]

$$\left.\begin{aligned} \sigma_\theta &= \frac{f(\theta)}{\sqrt{2\pi r}} + O(1) \quad \text{as } r \to 0 \\ \text{where } f(\theta) &= \sqrt{\frac{\pi}{2}}\,\sin\frac{\theta}{2}\left[\sin^2\alpha\,\sin^2\frac{\theta}{2} + \frac{3}{2}\sin\alpha\cos\alpha\sin\theta\right] \end{aligned}\right\} \tag{103}$$

Erdogan and Sih postulate to find the direction of crack extension θ_e using a "maximum normal stress" criterion; namely,

$$\frac{\partial f(\theta)}{\partial \theta} = 0, \text{ at } \theta = \theta_e \quad \text{(maximum of f)} \tag{104}$$

and the critical value of the stress q using the postulate that

$$f(\theta_e) = \overline{K}_I \tag{105}$$

where $\overline{K}_I$ is the critical value of the stress intensity factor for the opening mode when the applied loads are symmetrical to the crack. This theory will here be referred to as the "maximum normal stress" theory. In this formulation one needs to know only the solution of the elasto-static boundary value problem with a straight crack.

The change in strain energy can be rewritten in the following form. Referring to Figure (13a) denote by superscript "I" the stresses and displacements of problem "I" where we have a straight crack and by "II" that of problem "II" (Figure 13b) in which there is also a branch crack. Then an application of Betti's reciprocity theorem[19] gives the expression for the strain energy change as

$$U(\Phi, a) = \frac{1}{2} \int_0^a [\sigma_\theta^I (u_\theta^{II}) + \tau_{r\theta}^I (u_r^{II})] \bigg|_{\theta = \Phi} dr \tag{106}$$

where (u_θ^{II}) and (u_r^{II}) denote the difference in the displacement values across the crack, and for the energy release rate $G(\theta)$ we get

$$G(\theta) = \underset{a \to 0}{\text{Limit}} \frac{\partial U}{\partial a} (\theta, a) \tag{107}$$

Then according to the energy criterion given in pages (14-16) of this work, the direction of crack extension is determined by the maximum of G and the critical value of the applied load by this maximum energy release rate attaining a critical value.

In the following part the results obtained in this work are presented and the conclusions are given.

PART III

RESULTS AND CONCLUSIONS

Before presenting the results let us summarize the assumptions made.

1. Linearly elastic two-dimensional theory assumed.
2. Existence and uniqueness of the branched crack problem is assumed.
3. Strain energy change in the solid with branched crack is assumed to be nearly equal to that obtained using a modified crack boundary.

Due to assumption (1), the application of the results are to be restricted to brittle materials only. For materials with plastic flow at the tips or for materials with other forms of energy dissipation, all the relevant energies should be included in the calculations of the total potential energy in equation (2). As regards the application of the results to the plane stress case namely a plate with a crack which is a more easily realized model in experiments than plane strain conditions, the effect of three-dimensionality of the stress field when the branch crack is small compared to the thickness of the plate may affect the results. Quantitative estimates of error due to the three-dimensionality of the stress field are not yet published even for the straight crack.

Proof of existence and uniqueness for singular problems are very difficult. Only for relatively very few cases such proofs are known. For instance Magnaradze[33] gives a proof of uniqueness and existence for the function $\omega(z)$ for regions with boundaries having a finite number of points at which the tangent turns less than π radians as one moves with the boundary. In our case at the crack tips the tangents do turn by π radians and hence we are unable to use those proofs.

Regarding the third assumption even though it cannot be rigorously proved that the change in strain energy due to the modification is small, we have shown that the energies differ to the order of ε for two elasto-static singular problems for which the exact solutions can be obtained. Since ϵ is taken to equal to a^4 and we are comparing quantities of order 'a' to determine the crack branching angle and the critical load, the author feels that this approximation will not affect the end results seriously.

RESULTS

The results obtained in this work are presented in the form of three figures. The crack branching angle for various values of the loading angle are plotted in Figure 14. For comparison purposes those obtained by Erdogan and Sih[14] using the "maximum normal stress theory" are also plotted. That portion of the results using the extrapolation fit as discussed earlier are shown in dashed lines. From the results it is clear that both the theories predict very close results (maximum difference was about 4^o).

The values of stress intensity factors prevailing at the crack tips are plotted in Figure 15. They are normalized so that $K_I = 1$ when $\alpha = \pi/2$. In this figure also those results obtained in reference (14) using the maximum normal stress theory are plotted for comparison purposes. As before the results predicted by the two theories differ only about two per cent. It might be pointed out here that the experimental results reported in reference (14) using plexiglass plates were generally in the area above the curves and only at both ends (i.e., near the points $K_I = 0$ and $K_{II} = 0$) of the curves some experimental points were near the theoretical results.

In Figure (16) we have plotted the critical value of the loading stress q_{cr} versus α, the angle of loading. The values of q_{cr} are normalized to unity for $\alpha = \pi/2$.

The following explanation may give an indication as to the reasons for the closeness of the results predicted by the two theories.

The stresses in the vicinity of the crack tips can be expanded as follows. Referring to Figure (13a) for notation

$$\sigma_\theta^{(1)}(r,\theta) = \frac{f_1(\theta,\alpha)}{\sqrt{2\pi r}} + O(1) \quad \text{as} \quad r \to 0$$

$$\tau_{r\theta}^{(1)}(r,\theta) = \frac{f_2(\theta,\alpha)}{\sqrt{2\pi r}} + O(1) \quad \text{as} \quad r \to 0 \tag{108}$$

In the maximum normal stress theory the branching angle θ_e is determined by the maximum of $f_1(\theta,\alpha)$ for given value of α. At this value of θ, $f_2(\theta_e,\alpha) = 0$. Also f_1 and f_2 can be expanded in Taylor's series as follows in the vicinity of $\theta = \theta_e$

$$f_1(\theta,\alpha) = f_{10}(\theta_e,\alpha) + f_{11}(\theta_e,\alpha)(\theta-\theta_e) + O(\theta-\theta_e)^2$$
$$f_2(\theta,\alpha) = \quad 0 \quad + f_{21}(\theta_e,\alpha)(\theta-\theta_e) + O(\theta-\theta_e)^2 \tag{109}$$

If one now assumes (u_θ^{II}) and (u_r^{II}) have the same order of magnitude in the neighborhood of $\theta = \theta_e$, then we see from the equation (106) that the dominant term for the strain energy change comes from the f_{10} term and hence the strain energy rate may also attain the maximum in the neighborhood of θ_e. This is probably the reason why the two theories predict results which differ by only a few per cent.

CONCLUSION

The usefulness of any theory is based on its ability to predict results which can be confirmed by experiments. Unfortunately the author was not yet able to get experimental results. We were able to realize only the difficulties involved in performing the experiments at this stage. Since both the "normal" stress" theory and the results obtained in this thesis using the energy postulates predict close results, the author feels that it is necessary to obtain experimental results before proceeding to modify the theories.

If experimental confirmation of the theoretical results has been established, the stress intensity factor curve can be used as a failure envelope similar to yield surfaces. If one can determine the stress intensity factors at the crack tips when the loading is general, either analytically or experimentally, for example as indicated by

Irwin[4], and if the point falls below the curve in Figure 15, then one can expect that the crack will not extend. With use of suitable safety factors, the curve can be used to predict safe loads for given crack sizes or for given loading conditions to estimate maximum allowable crack lengths.

As indicated in the introduction, practically no work has been done to predict fracture strength when the loading is general. The work contained in this thesis provides an estimate in such a case. The solution method in this work can also be extended for the cases when there is an arbitrary stress distribution prescribed on the crack faces and for any given uniform stress distribution at infinity.

Further work along the lines of investigation by McClintock and Irwin[9] for the symmetrical loading will be necessary to incorporate the effect of yielding on the value of the fracture load, when the loading is general. A three-dimensional solution of the straight crack problem, which is not yet available in the literature, will be helpful in understanding the effect of the three-dimensionality of the stress field on the fracture behavior. It may also help to explain the appearance of different fracture surfaces, which seem to depend on the thickness of the specimen and the length of the crack as mentioned in the introduction.

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APPENDIX I

AN ATTEMPT AT USING THE MELLIN TRANSFORM METHOD FOR SOLVING THE BOUNDARY VALUE PROBLEM

Consider the boundary value problem of a plate with a branched crack subjected to uniform tension at infinity. The solution to this problem can be obtained by superposition of the solutions of the following two problems.

1. A plate without any crack subjected to the uniform tension at infinity.

2. A plate with the branched crack with given constant normal and shear stresses on the crack faces and vanishing stresses at infinity.

The solution to the first problem can be written down easily. A solution method for the second problem will now be outlined. Using the formulation in terms of Airy stress function and polar coordinates, (see Figure 6),

Field Equations:

$$\nabla^4 \Phi(r, \theta) = 0 \quad \text{in the region } D \tag{1.1}$$

Boundary Conditions:

$$\left.\begin{aligned}
&\left.\begin{aligned}
\sigma_\theta &= \frac{\partial^2\Phi}{\partial r^2} = k_1 \\
\tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) = k_2
\end{aligned}\right\} \begin{aligned}&\text{on } \theta = 0 \\ &0 < r < \ell_1\end{aligned} \\
&\left.\begin{aligned}
\sigma_\theta &= \frac{\partial^2\Phi}{\partial r^2} = k_3 \\
\tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) = k_4
\end{aligned}\right\} \begin{aligned}&\text{on } \theta = \lambda_1\pi, \\ &0 < r < \ell_2\end{aligned}
\end{aligned}\right\} \qquad (1.2)$$

where k_i $(i = 1, 2, 3, 4)$ are constants.

For field conditions:

$$\sigma_\theta,\ \sigma_r,\ \tau_{r\theta} = O\left(\frac{1}{r^2}\right) \quad \text{as } r \to \infty \qquad (1.3)$$

The problem can be re-formulated by considering the region R as the union of two sectors S_I and S_{II} where

$$S_I = \{0 < r < \infty, \quad 0 < \theta < \lambda_1\pi\}$$

$$S_{II} = \{0 < r < \infty, \quad \lambda_1\pi < \theta < 2\pi\}$$

Let

$$\sigma_\theta\big|_{\theta=0} = g_1(r)$$

$$\tau_{r\theta}\big|_{\theta=0} = g_2(r)$$

$$\sigma_\theta\big|_{\theta=\lambda_1\pi} = g_3(r)$$

$$\tau_{r\theta}\big|_{\theta=\lambda_1\pi} = g_4(r)$$

$$\overline{g_i}(s) = \int_0^\infty g_i(r)\, r^{s-1}\, dr = \text{Mellin Transform of } g_i(r)$$

The variable s is complex and its real part is so chosen that the transforms exist. Then the stresses and displacements in the sectors S_I and S_{II} can be found in terms of contour integrals of the functions $\overline{g_i}(s)$ (see Ref. 20). We have the following conditions for the determination of the four functions $\overline{g_i}(s)$.

$$\left.\begin{array}{l} \left.\begin{array}{l} \sigma_{\theta_I}\Big|_{\theta=0} = \sigma_{\theta_{II}}\Big|_{\theta=2\pi} = k_1 \\ \tau_{r\theta_I}\Big|_{\theta=0} = \tau_{r\theta_{II}}\Big|_{\theta=2\pi} = k_2 \end{array}\right\} \quad 0<r<\ell_1 \\ \left.\begin{array}{l} \sigma_{\theta_I} = \sigma_{\theta_{II}} = k_3 \\ \tau_{r\theta_I} = \tau_{r\theta_{II}} = k_4 \end{array}\right\} \quad \theta = \lambda_1\pi,\ 0<r<\ell_2 \end{array}\right\} \tag{1.4}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} u_{r_I}\Big|_{\theta=0} = u_{r_{II}}\Big|_{\theta=2\pi} \\ u_{\theta_I}\Big|_{\theta=0} = u_{\theta_{II}}\Big|_{\theta=2\pi} \end{array}\right\} \quad \ell_1<r<\infty \\ \left.\begin{array}{l} u_{r_I} = u_{r_{II}} \\ u_{\theta_I} = u_{\theta_{II}} \end{array}\right\} \quad \theta = \lambda_1\pi, \ell_2<r<\infty \end{array}\right\} \tag{1.5}$$

These conditions lead to four sets of coupled dual integral equations of the following form

$$\left.\begin{aligned}
&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{g}_t(s)\, r^{-s}\, ds = k_t \qquad 0<r<r_t \\
&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\sum_{j=1}^{4} F_{tj}(s,\lambda)\, \bar{g}_j(s)\right) r^{-s}\, ds = 0 \qquad r>r_t \\
&\qquad (r_1 = r_2 = \ell_1;\ r_3 = r_4 = \ell_2)
\end{aligned}\right\} \qquad (1.6)$$

The equation (1.6) can be reduced to four coupled integral equations of the following form, for four unknown functions f_i

$$f_i(t) = g_i(t) + \sum_{j=1}^{4} \int_0^{r_j} \left\{ K_{ij}(t,\tau)\, f_j(\tau) \right\} d\tau \qquad 0<t<r_i \qquad (1.7)$$

where $f_i(t)$ are related to the displacement discontinuity along the crack faces in the following way (for example the radial disp. u)

$$u_{r_{II}}(r, 2\pi) - u_{r_I}(r, 0) = \int_r^{\ell_1} \frac{f_1(t)\, dt}{\sqrt{t^2 - r^2}} \qquad (0<r<\ell_1)$$

The method of reduction is similar to the one given by Srivastav[34]. The reduction to the integral equations (1.7) from (1.6) is very formal (during this reduction several interchanges of integrations and integration and differentiation are involved) and also the kernels K_{ii} become singular like $\frac{1}{t}$ when one approaches the origin in the (t, τ) plane along the diagonal $t = \tau$.

APPENDIX II

THE MAPPING FUNCTION

Consider the mapping function

$$z = \omega(\xi) = A(\xi - e^{i\alpha_1})^{\lambda_1} (\xi - e^{i\alpha_2})^{\lambda_2} \tag{2.1}$$

where A, α_1, α_2, λ_1 and λ_2 are real constants. This function maps the region outside the unit circle in the ξ-plane into the region outside the two radial lines OB and OC in the z-plane (Figure 8). Let OB be along the x-axis and the OC make an angle $\lambda_1\pi$ with the x-axis. Let the line OB be of length ℓ_1 and OC of ℓ_2 and let the points $\xi_1 = e^{i\beta_1}$ and $\xi_2 = e^{i\beta_2}$ correspond to the points B and C, respectively. Then the parameters $A, \alpha_1, \alpha_2, \beta_1, \beta_2$, and λ_2 are determined in terms of ℓ_1, ℓ_2, and λ_1 by the following relations (see Reference 18)

$$\lambda_1 + \lambda_2 = 2 \tag{2.2}$$

$$\left.\begin{aligned}
&\lambda_1\alpha_1 + \lambda_2\alpha_2 = 2\pi \\
&\lambda_1 \cot\frac{\beta_1 - \alpha_1}{2} + \lambda_2 \cot\frac{\beta_1 - \alpha_2}{2} = 0 \\
&\lambda_1 \cot\frac{\beta_2 - \alpha_1}{2} + \lambda_2 \cot\frac{\beta_2 - \alpha_2}{2} = 0 \\
&4A \left|\sin\frac{\alpha_1 - \beta_1}{2}\right|^{\lambda_1} \left|\sin\frac{\alpha_2 - \beta_1}{2}\right|^{\lambda_2} = \ell_1 \\
&4A \left|\sin\frac{\alpha_1 - \beta_2}{2}\right|^{\lambda_1} \left|\sin\frac{\alpha_2 - \beta_2}{2}\right|^{\lambda_2} = \ell_2
\end{aligned}\right\} \tag{2.3}$$

Also

$$\omega'(\xi) = \frac{d\omega}{d\xi} = \frac{\omega(\xi)\,(\xi - e^{i\beta_1})\,(\xi - e^{i\beta_2})}{\xi(\xi - e^{i\alpha_1})\,(\xi - e^{i\alpha_2})} \tag{2.4}$$

When $\ell_2 << \ell_1$, equations (2.3) can be solved in series form for A, α_1, α_2, β_1 and β_2 in terms of the small parameters $a = \ell_2/\ell_1$. These expressions are given as

$$\left.\begin{aligned}
A &= \frac{\ell_1}{4}\,[1 + p_1^2 \lambda_1 \lambda_2\, a + O(a^2)] \\
\alpha_1 &= \pi - \lambda_2 p_2 a^{\frac{1}{2}} - \lambda_2 p_3 a^{3/2} + O(a^2) \\
\alpha_2 &= \pi + \lambda_1 p_2 a^{\frac{1}{2}} + \lambda_2 p_3 a^{3/2} + O(a^2) \\
\beta_1 &= 2\pi + (\lambda_1 - \lambda_2) p_4 a^{3/2} + O(a^2) \\
\beta_2 &= \pi + (\lambda_1 - \lambda_2)\, p_2 a^{\frac{1}{2}} + (\lambda_1 - \lambda_2)(p_3 - p_4)\, a^{3/2} \\
&\qquad + O(a^2)
\end{aligned}\right\} \tag{2.5}$$

where

$$\left.\begin{aligned}
p_1^2 &= \frac{1}{\lambda_1^{\lambda_1} \cdot \lambda_2^{\lambda_2}} \\
p_2 &= 2p_1 \\
p_3 &= \frac{p_1^3}{6}(\lambda_1^2 + \lambda_2^2 - 6\lambda_1\lambda_2) \\
p_4 &= \frac{2p_1^3}{3}\lambda_1\lambda_2
\end{aligned}\right\} \tag{2.6}$$

Far field expansions:

From (2.1) it can be shown that

$$z = A\xi\left(1 + \frac{\omega_1}{\xi} + \frac{\omega_2}{\xi}\right) + O\left(\frac{1}{\xi^3}\right) \text{ as } |\xi| \to \infty \qquad (2.7)$$

where

$$\omega_1 = -\left(\lambda_1 e^{i\alpha_1} + \lambda_2^{\,i\alpha_2}\right)$$

$$\omega_2 = \frac{1}{2}\left[\lambda_1(\lambda_1 - 1)\, e^{2i\alpha_1} + 2\lambda_1\lambda_2\, e^{i(\alpha_1 + \alpha_2)} + \lambda_2(\lambda_2 - 1)\, e^{2i\alpha_2}\right] \qquad (2.8)$$

Also from equation (2.7), we get

$$\xi = \frac{1}{A} z - \omega_1 - A\,\omega_2 \frac{1}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as} \quad |z| \to \infty \qquad (2.9)$$

Furthermore, it can be shown using (2.5) and (2.8), that

$$\omega_2 = 1 + i c\, a^{3/2} + O(a^2) \quad \text{as} \quad a \to 0$$

where

$$c = \frac{p_2^3 \lambda_1 \lambda_2}{6}\left[4\lambda_2^2(\lambda_1 - 1) - (\lambda_1 - \lambda_2)^3 - 4\lambda_1^2(\lambda_2 - 1)\right] \qquad (2.10)$$

For the modified mapping function

$$\omega(\xi) = \frac{A}{k\xi}\left(\xi - k e^{i\alpha_1}\right)^{\lambda_1}\left(\xi - k e^{i\alpha_2}\right)^{\lambda_2} \qquad (2.11)$$

one obtains

$$\xi = \frac{k z}{A} - k \omega_1 - A k \frac{\omega_2}{z} + O(\frac{1}{z^2}) \text{ as } |z| \to \infty \qquad (2.12)$$

and

$$\frac{1}{\xi} = \frac{A}{kz} + O(\frac{1}{z^2}) \text{ as } |z| \to \infty \qquad (2.13)$$

APPENDIX III

ESTIMATE OF THE MAGNITUDE OF THE COEFFICIENTS h_n

From equation (68) we have,

$$h_n = \frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\xi)}}{\omega(\xi)} \frac{\xi(\xi-ke^{i\alpha_1})(\xi-ke^{i\alpha_2})}{(\xi-ke^{i\beta_1})(\xi-ke^{i\beta_2})}\, \xi^{-(n+1)}\, d\xi$$

$$(\gamma:\ |\xi| = 1)$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{\frac{A}{k}\xi\left(\frac{1}{\xi}-ke^{-i\alpha_1}\right)^{\lambda_1}\left(\frac{1}{\xi}-ke^{-i\alpha_2}\right)^{\lambda_2}\ \xi\ \ \xi^{-(n+1)}\, d\xi}{\frac{A}{k\xi}(\xi-ke^{i\alpha_1})^{\lambda_1-1}(\xi-ke^{i\alpha_2})^{\lambda_2-1}(\xi-ke^{i\beta_1})(\xi-ke^{i\beta_2})} \tag{3.1}$$

The integrand is analytic in region $k < |\xi| < \frac{1}{k}$. Letting $\xi = \frac{\eta}{k}$ and deforming the contour of the integration to $|\eta| = 1$, one obtains

$$(h_n k^{-(n+1)} - \tilde{h}_n) = \frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega_o}}{\omega_o} \cdot \frac{\eta(\eta-e^{i\alpha_1})(\eta-e^{i\alpha_2})}{(\eta-k^2e^{i\beta_1})(\eta-k^2e^{i\beta_2})}\ \times$$

$$\times\left\{\frac{(\eta-e^{i\alpha_1})^{\lambda_1-1}(\eta-e^{i\alpha_2})^{\lambda_2-1}}{(\eta-k^2e^{i\alpha_1})^{\lambda_1-1}(\eta-k^2e^{i\alpha_2})^{\lambda_2-1}} - 1\right\}\eta^{-(n+1)}\, d\eta \tag{3.2}$$

where $$\tilde{h}_n = \frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega_o}}{\omega_o} \frac{\eta(\eta-e^{i\alpha_1})(\eta-e^{i\alpha_2})}{(\eta-k^2e^{i\beta_1})(\eta-k^2e^{i\beta_2})}\, \eta^{-(n+1)}\, d\eta \tag{3.3}$$

and $\omega_o = \frac{A}{\eta}(\eta-e^{i\alpha_1})^{\lambda_1}(\eta-e^{i\alpha_2})^{\lambda_2}$.

The coefficients $\tilde{h}_n$ can be calculated exactly whereas h_n could be obtained only approximately by numerical integration. It can be shown from equation (3.2) that

$$|h_n k^{-(n+1)} - \tilde{h}_n| \leq \frac{8}{\sqrt{2}\pi} p_2 (1-\lambda_1) \{\sqrt{a}\ \epsilon \log (\frac{1}{\epsilon})\} + O(\epsilon\sqrt{a}) \qquad (3.4)$$

where p_2 is a function of λ_1 given in equation (2.6), Appendix II. From equation (3.3) we get

$$\frac{\overline{\omega}_o}{\omega_o} \cdot \frac{(\eta-e^{i\alpha_1})(\eta-e^{i\alpha_2})}{\eta} \cdot (1-\frac{k^2 e^{i\beta_1}}{\eta})^{-1} (1-\frac{k^2 e^{i\beta_2}}{\eta})^{-1} = \sum_{-\infty}^{\infty} \tilde{h}_n \eta^n$$

By Fourier expansion of the function $\frac{\overline{\omega}_o'}{\omega_o} \frac{(\eta-e^{i\alpha_1})(\eta-e^{i\alpha_2})}{\eta}$ on $|\eta| = 1$ and expanding the rest of the terms in series it can be shown that

$$\tilde{h}_n = F_1 S_1 + F_2 S_2 \qquad (n>2) \qquad (3.5)$$

where

$$\left.\begin{aligned} F_1 &= \left\{\frac{1-e^{-2i\lambda_1\pi}}{2\pi i}\right\} \left(e^{i\alpha_2} - e^{i\alpha_1}\right) \frac{e^{-i\frac{\beta_1+\beta_2}{2}}}{2\sin\frac{\beta_1-\beta_2}{2}}\, i \\ F_2 &= \left\{\frac{1-e^{-2i\lambda_1\pi}}{2\pi i}\right\} \left(e^{i\alpha_2} + e^{i\alpha_1}\right) \frac{e^{-i\frac{\beta_1+\beta_2}{2}}}{2\sin\frac{\beta_1-\beta_2}{2}}\, i \end{aligned}\right\} \qquad (3.6)$$

$$S_1 = \sum_{m=n}^{\infty} k^{(m-n)} \frac{e^{-mi\alpha_2} + e^{-mi\alpha_1}}{(m^2-1)} \left\{e^{i(m-n+1)\beta_2} - e^{i(m-n+1)\beta_1}\right\}$$

and

$$S_2 = \sum_{m=n}^{\infty} k^{2(m-n)} \frac{e^{-mi\alpha_2} - e^{-mi\alpha_1}}{m(m^2-1)} \left\{ e^{i(m-n+1)\beta_2} - e^{i(m-n+1)\beta_1} \right\}$$

Consider the series

$$S_3 = \sum_{m=n}^{\infty} \left(e^{-mi\alpha_2} + e^{-mi\alpha_1} \right) \left(e^{i(m-n+1)\beta_2} - e^{i(m-n+1)\beta_1} \right)$$

Its partial sums

$$\sigma_N = \sum_{m=n}^{N} \left(e^{-mi\alpha_2} + e^{-mi\alpha_1} \right) \left(e^{i(m-n+1)\beta_2} - e^{i(m-n+1)} \right)$$

are bounded by

$$f = \frac{1}{\left|\sin\frac{\beta_2 - \alpha_2}{2}\right|} + \frac{1}{\left|\sin\frac{\beta_2 - \alpha_1}{2}\right|} + \frac{1}{\left|\sin\frac{\beta_1 - \alpha_2}{2}\right|} + \frac{1}{\left|\sin\frac{\beta_1 - \alpha_1}{2}\right|} \tag{3. 7}$$

Hence using Abel's lemma[35], we get

$$|S_1| \leq \frac{f}{n^2-1} \tag{3. 8}$$

Also one obtains from (3. 6) that

$$|S_2| \leq \sum_{m=n}^{\infty} \frac{4}{m(m^2-1)}$$

$$\leq \frac{2}{(n-2)^2} \tag{3.9}$$

From (3. 6) it can be shown that

$$\left.\begin{aligned} |F_1| &\leq \frac{\sin \lambda_1\pi}{2\pi} \quad p_2\sqrt{a} + O(a) \\ |F_2| &\leq \frac{\sin \lambda_1\pi}{2\pi} \end{aligned}\right\} \qquad (3.10)$$

Using the expression for $\alpha_1, \alpha_2, \beta_1$, and β_2 from Appendix II, in equation (3. 7) one obtains

$$|f| \leq \frac{2}{p_2\sqrt{a}} \left[\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right] + O(1) \quad \text{as } a \to 0 \qquad (3.11)$$

From equation (3. 5) and equations (3.6) to (3. 11) it follows that

$$|\tilde{h}_n| \leq \frac{2 \sin \lambda_1\pi}{\pi} \quad \frac{1}{(n-2)^2} \quad (n>2) \qquad (3.12)$$

From (3. 4) and (3. 12) we get that

$$|h_n| \leq \left[\frac{8}{\sqrt{2}\ \pi} p_2(1-\lambda_1)\sqrt{a}\ \ \epsilon \log\left(\frac{1}{\epsilon}\right) + \frac{2 \sin \lambda_1\pi}{\pi} \frac{1}{(n-2)^2} \right] k^{n+1} \qquad (3.13)$$

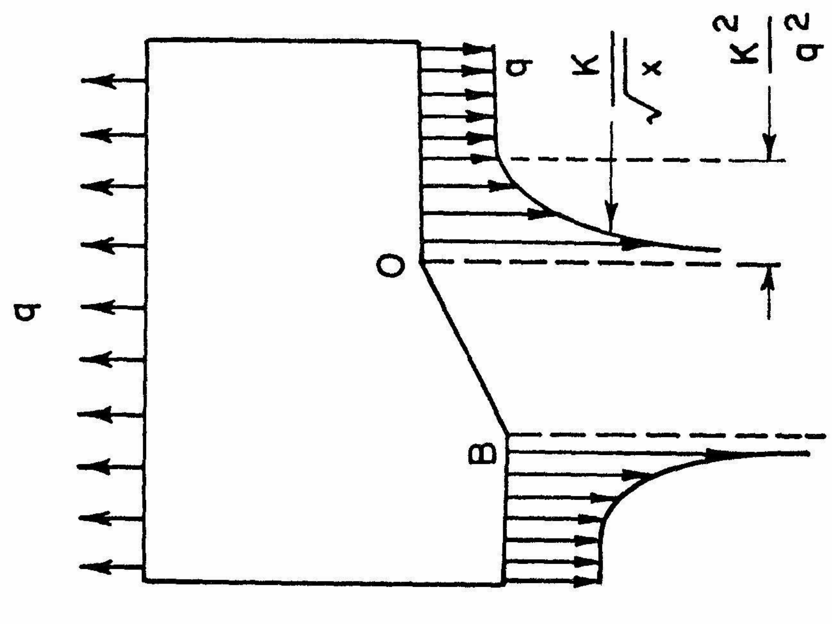


FIG. (1b)

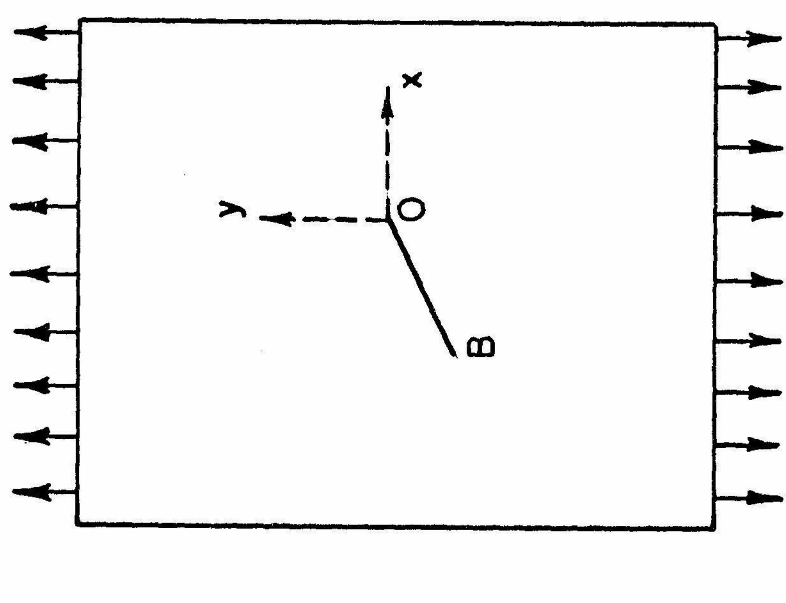
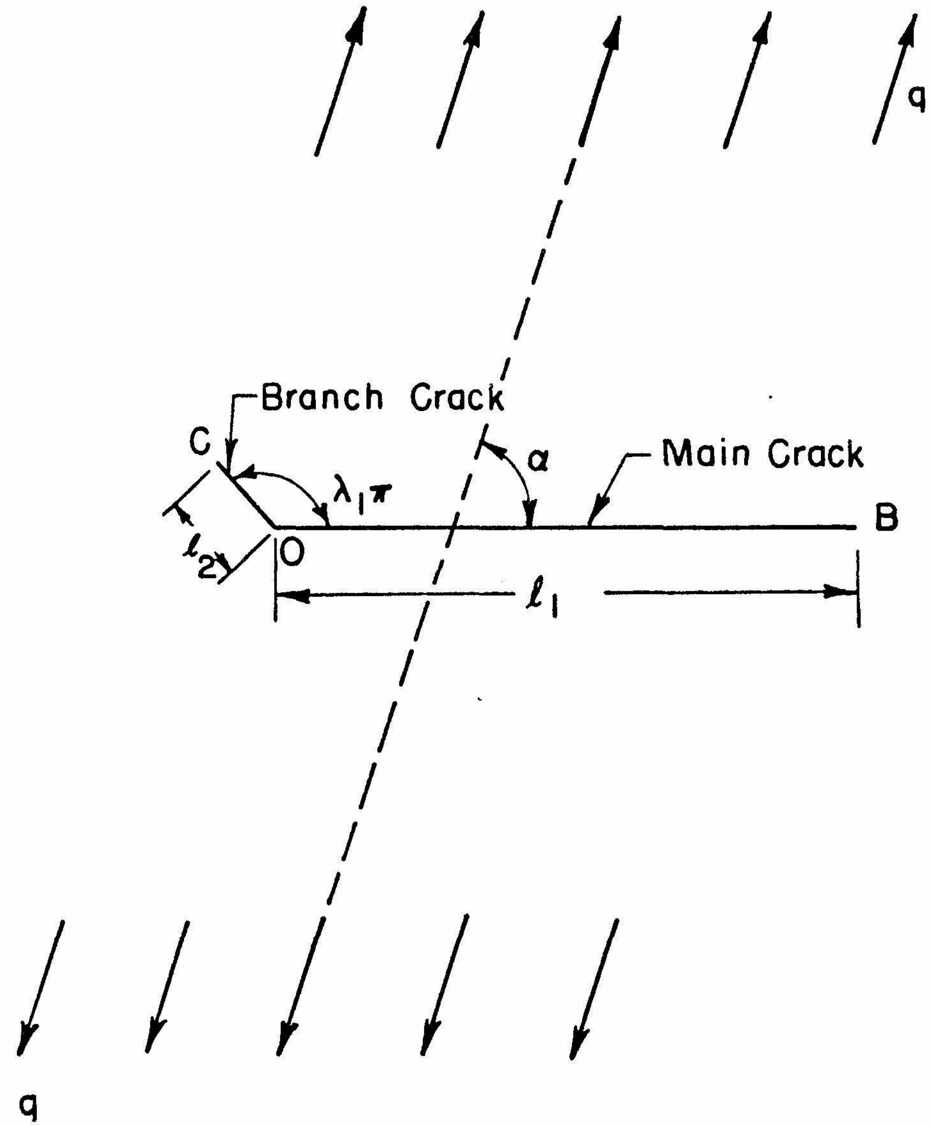


FIG. (1a)



BRANCHED CRACK SYSTEM

FIG. 2

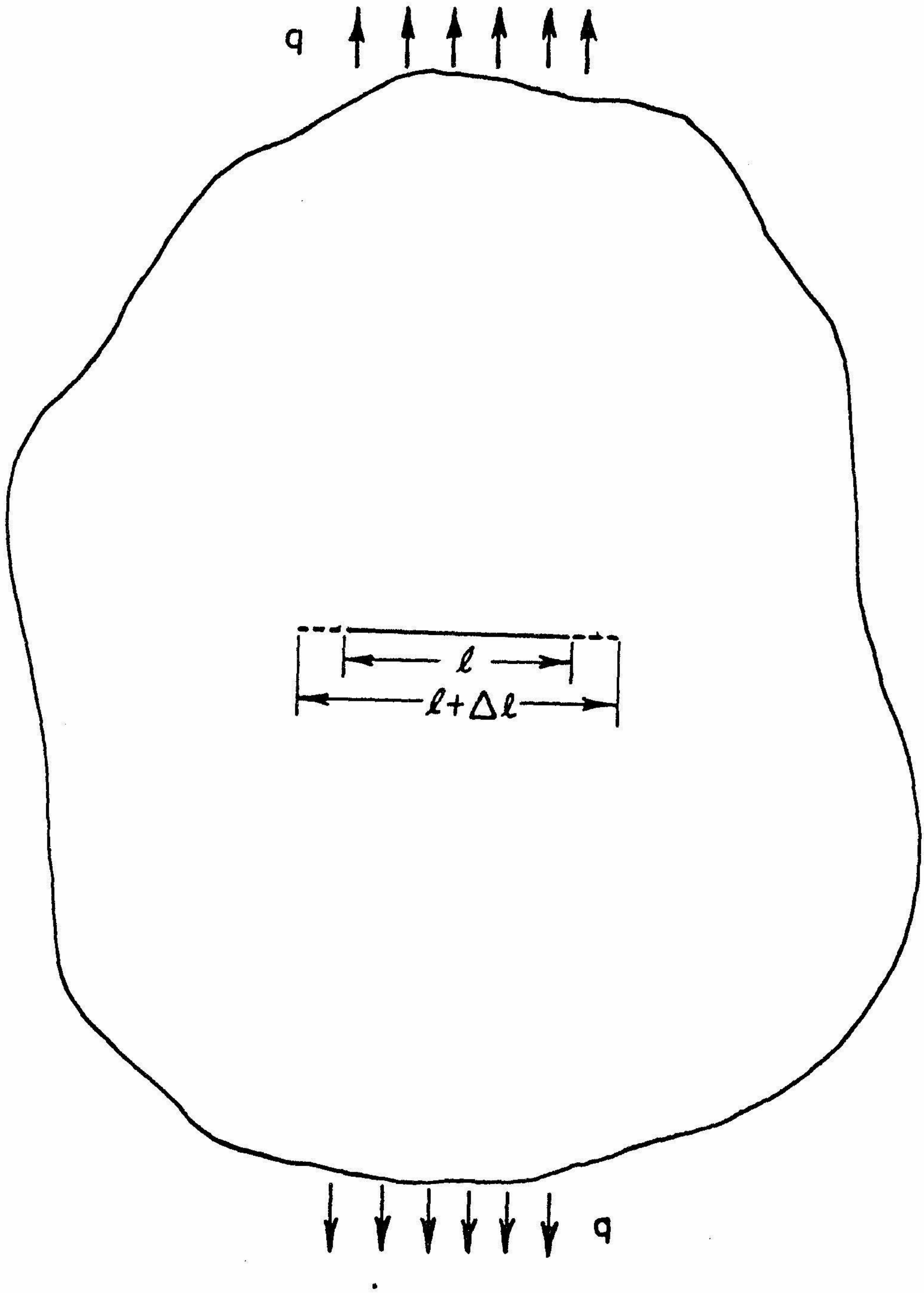


FIG. 3

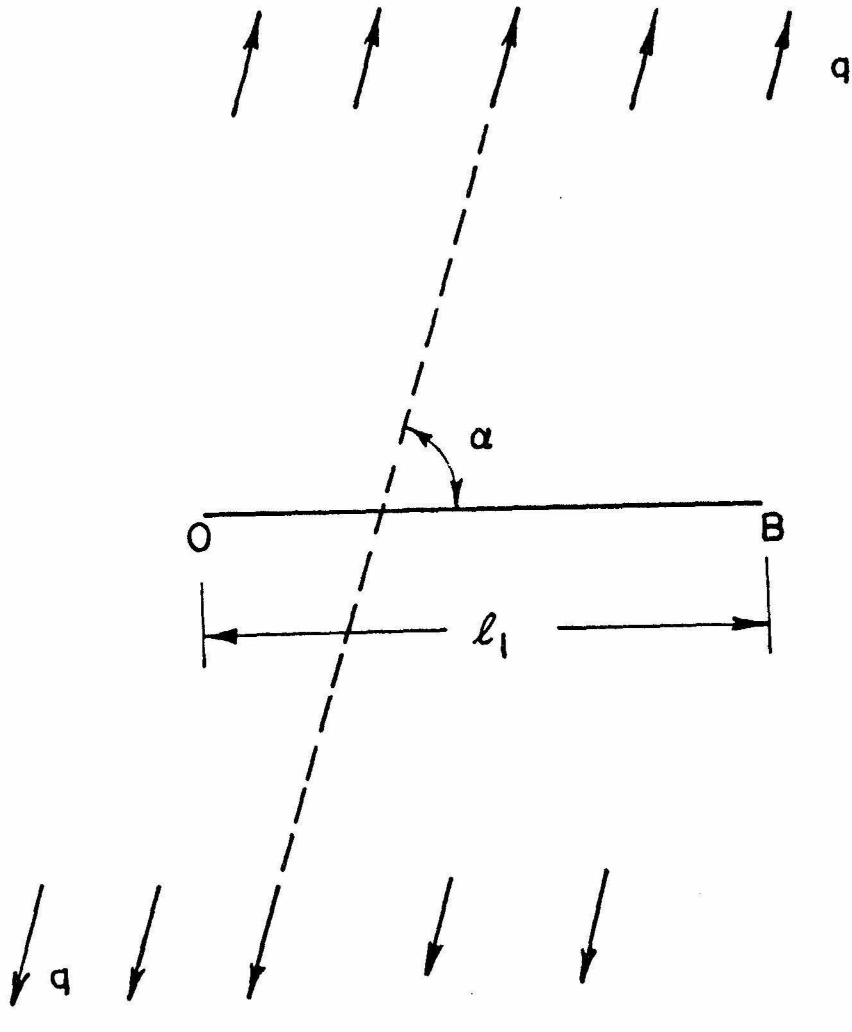


FIG. 4

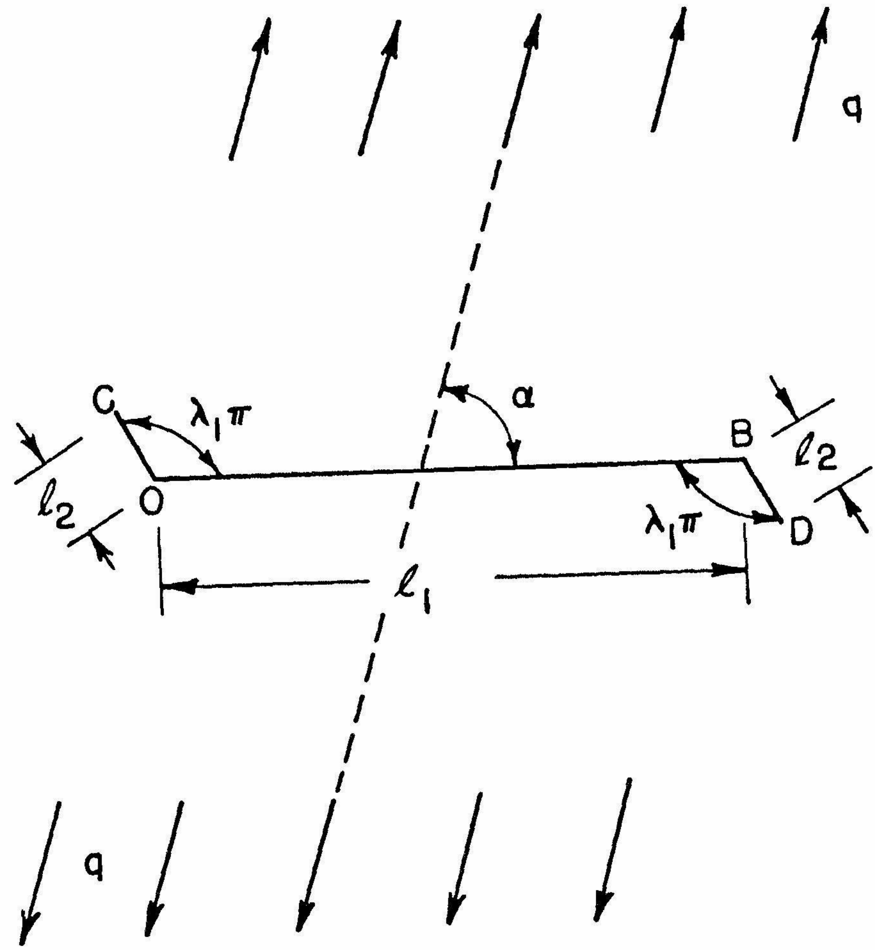


FIG. 5

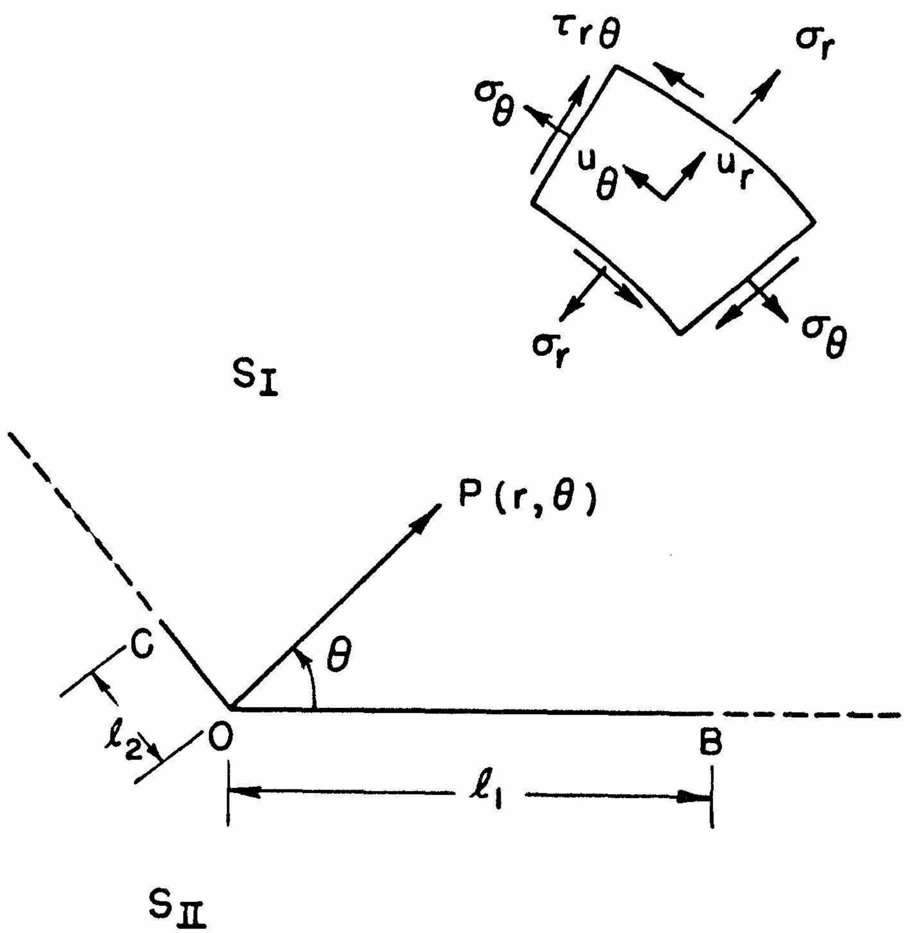


FIG. 6

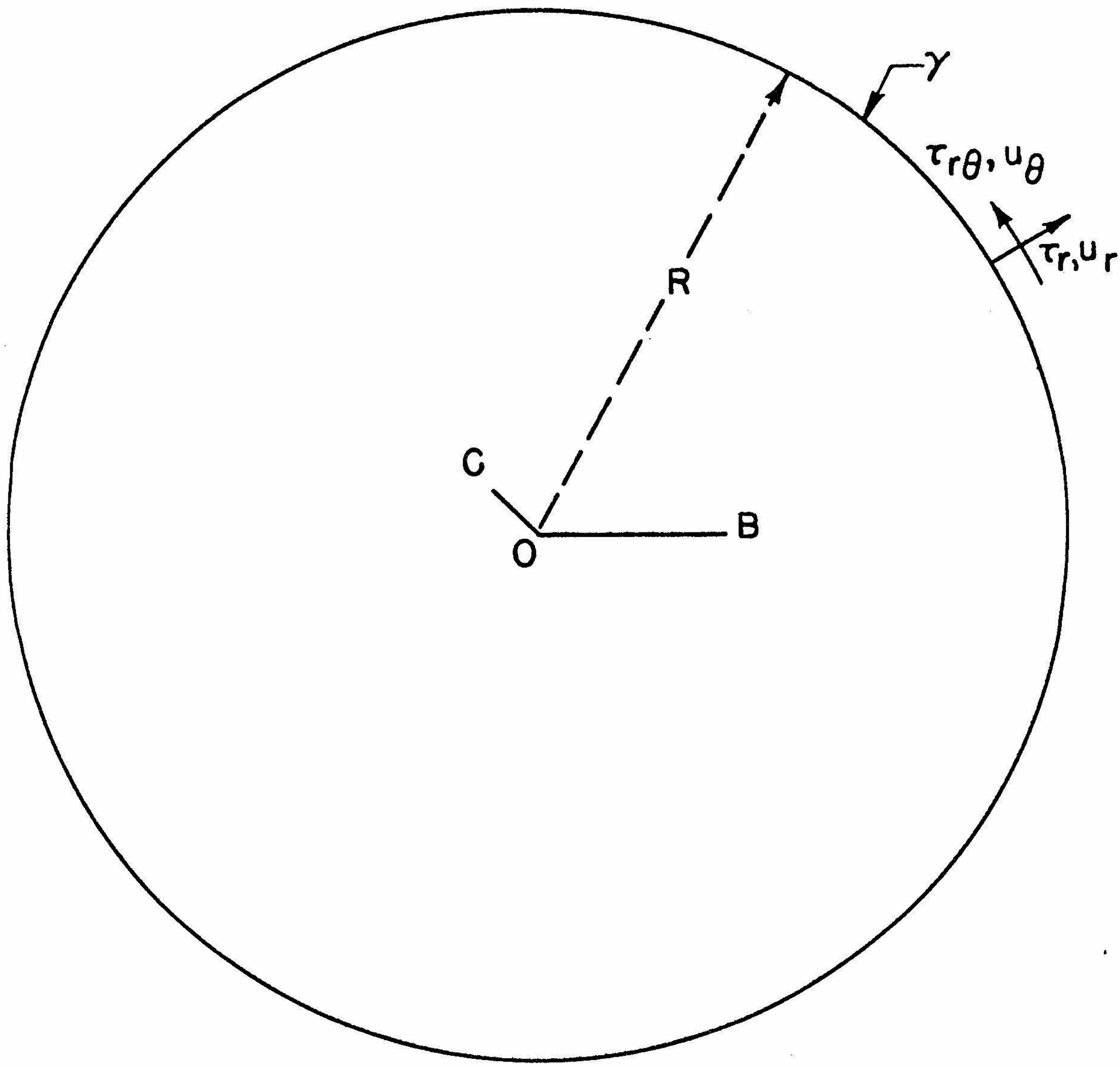


FIG. 7

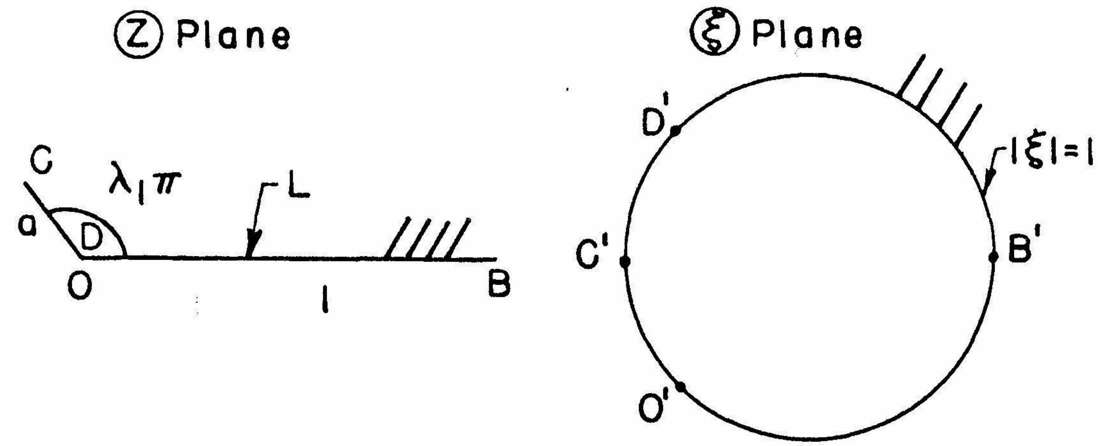


FIG. 8

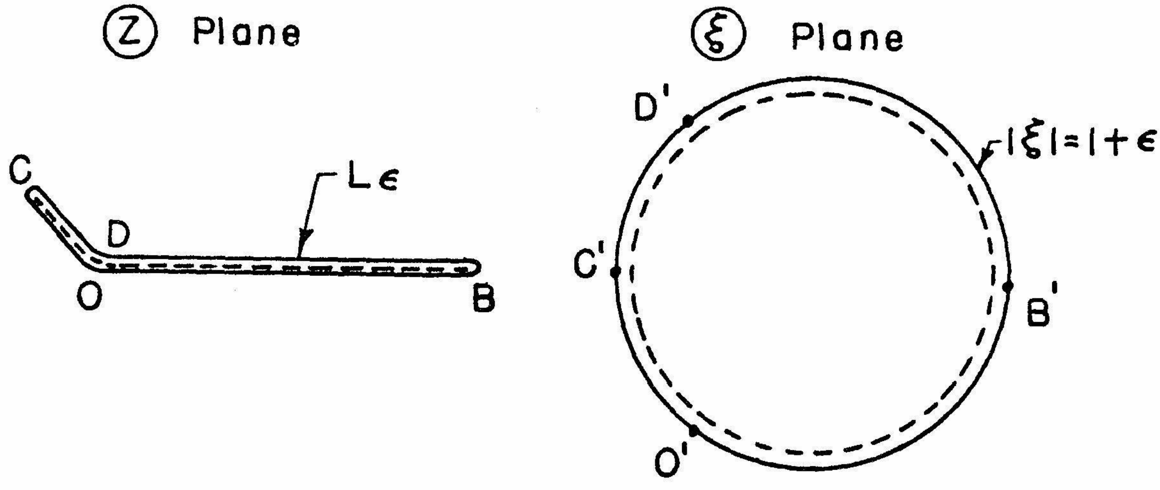
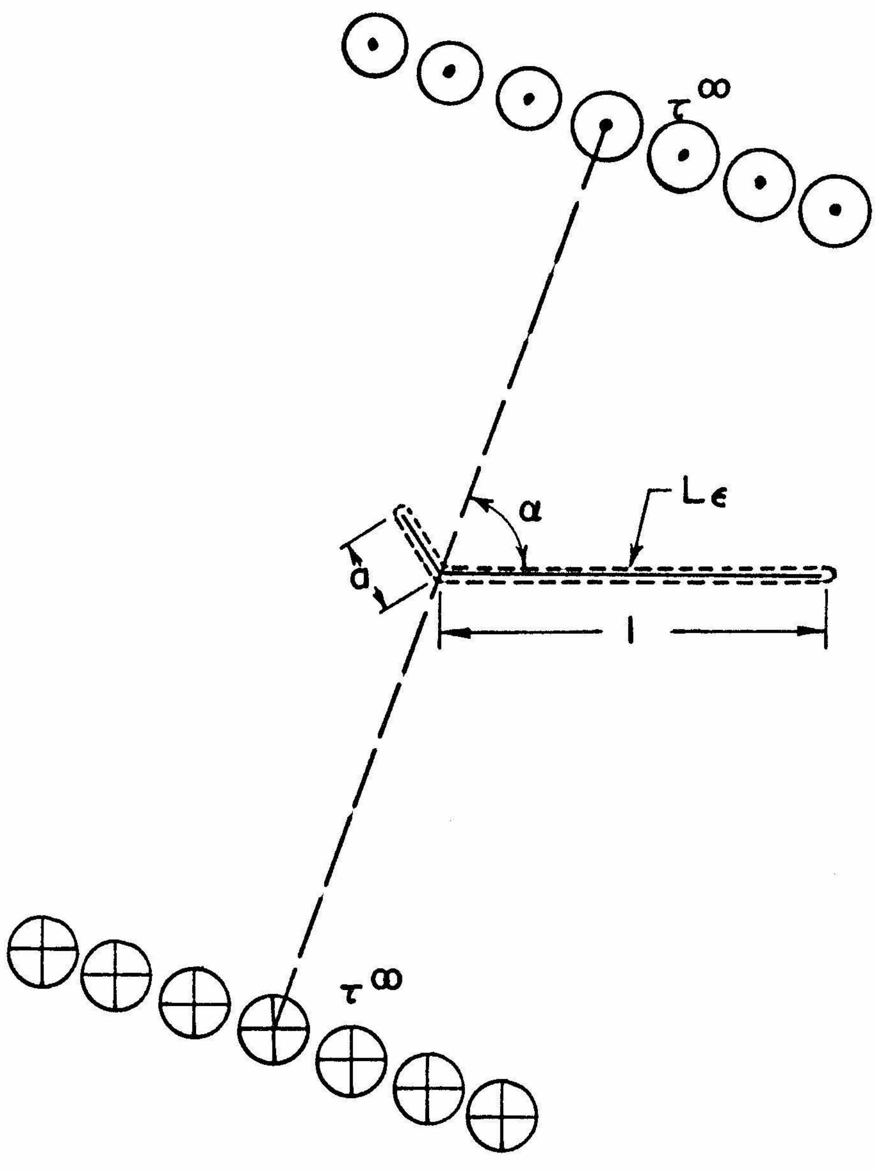


FIG. 9



ANTIPLANE SHEAR PROBLEM

FIG. 10

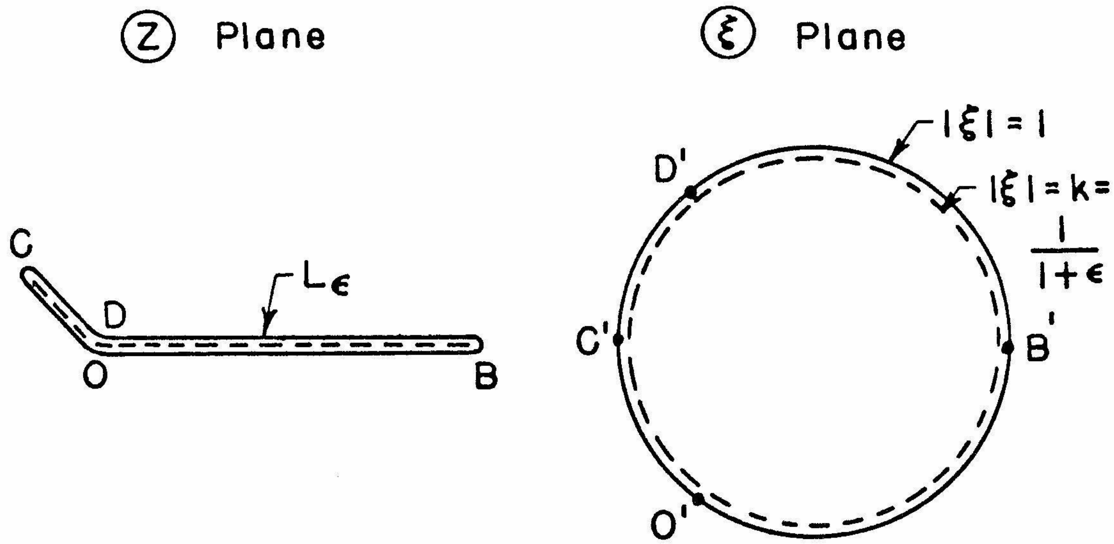
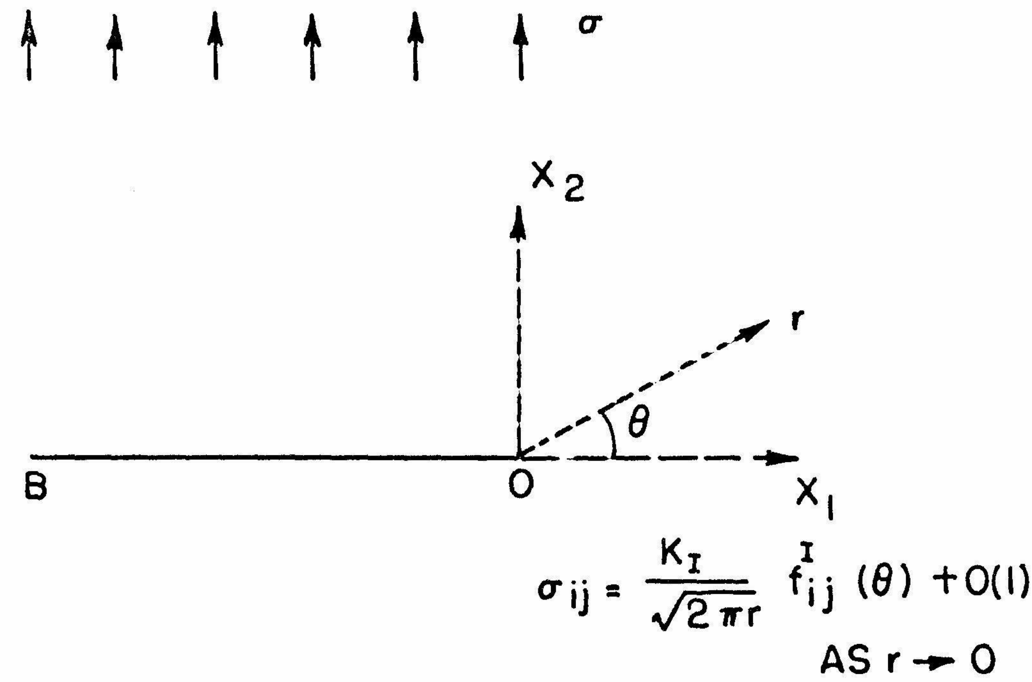


FIG. 11



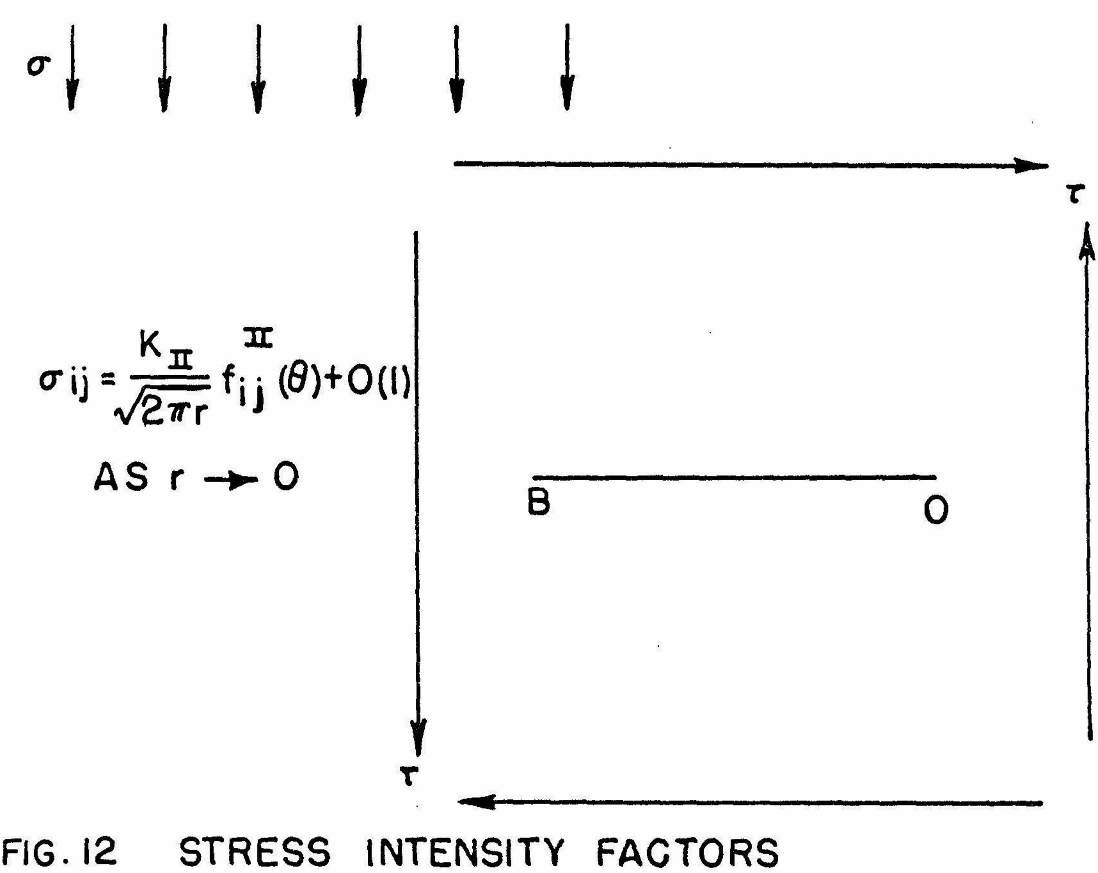
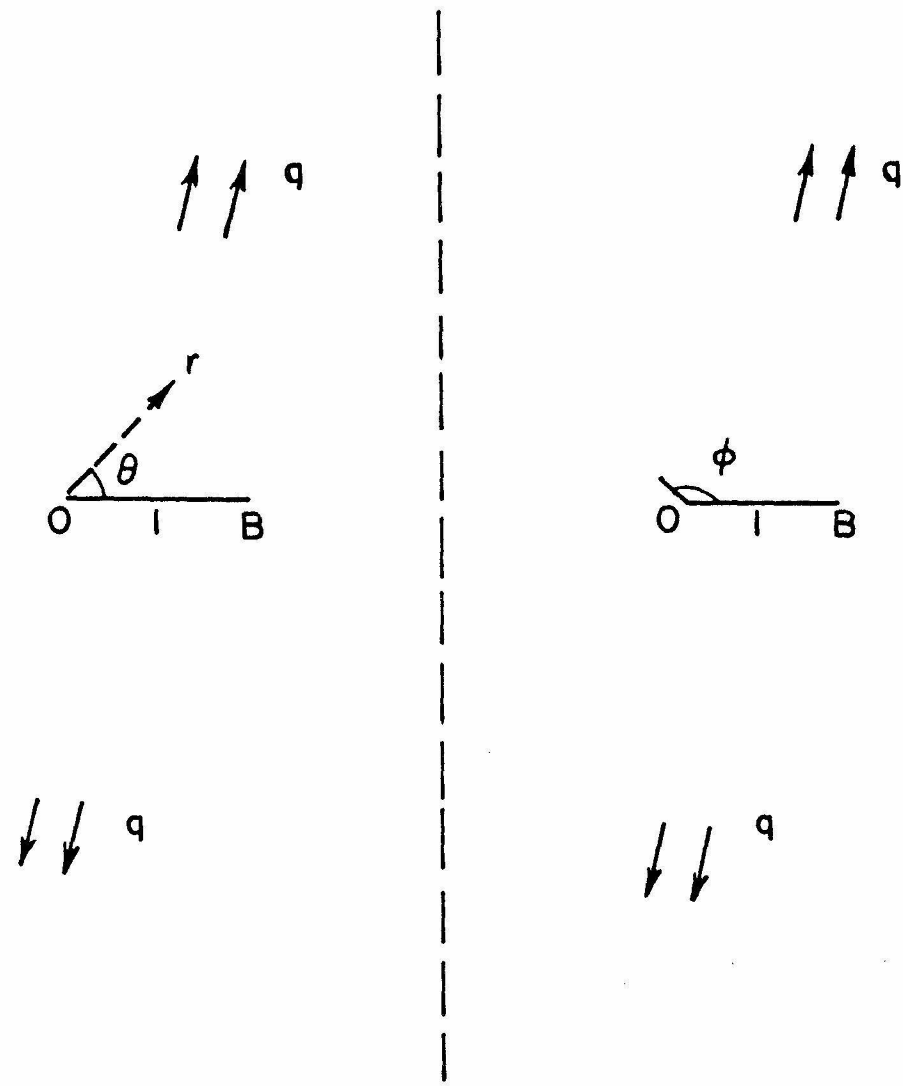


FIG. 12 STRESS INTENSITY FACTORS



(a) Problem I

(b) Problem II

FIG. 13

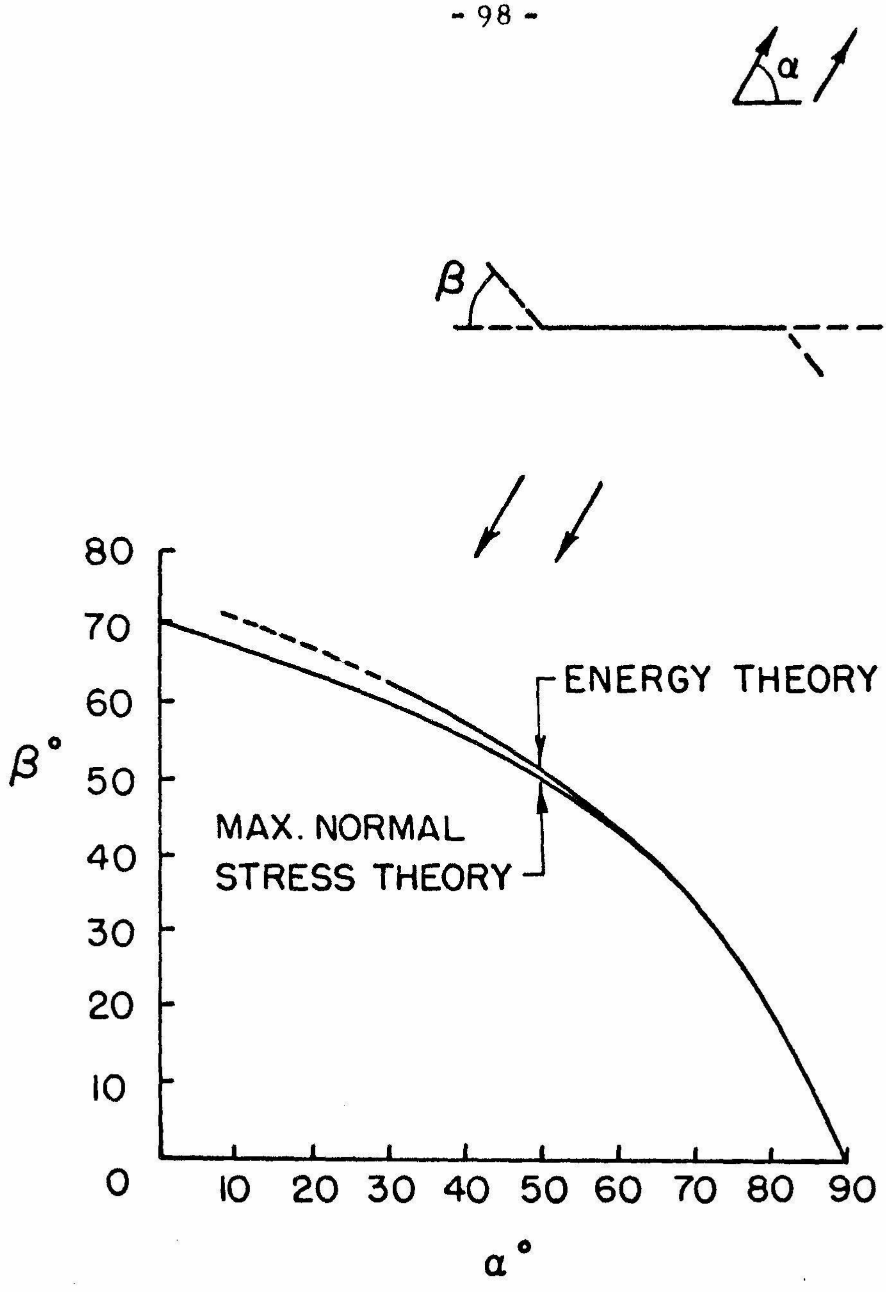

α
β
80
70
60
β°
50
40
30
20
10
0
10 20 30 40 50 60 70 80 90
α°
ENERGY THEORY
MAX. NORMAL
STRESS THEORY

FIG. 14

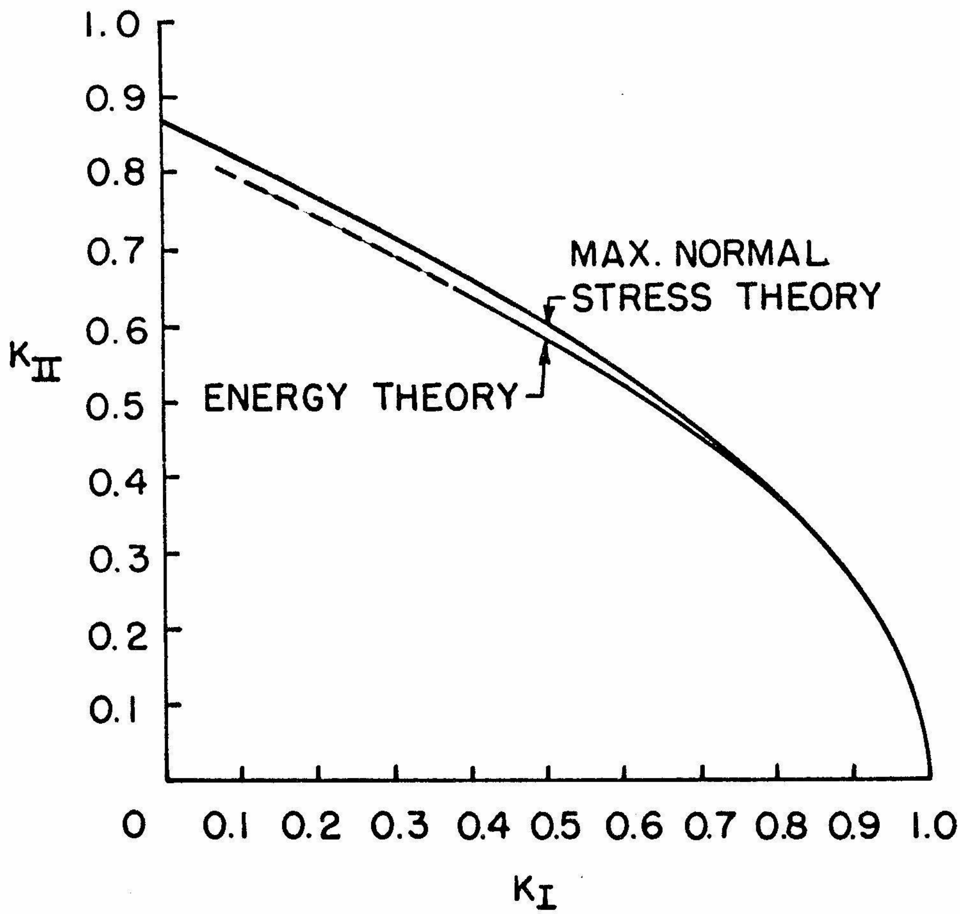

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
K_{II}
MAX. NORMAL STRESS THEORY
ENERGY THEORY
0
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
K_I

FIG.15

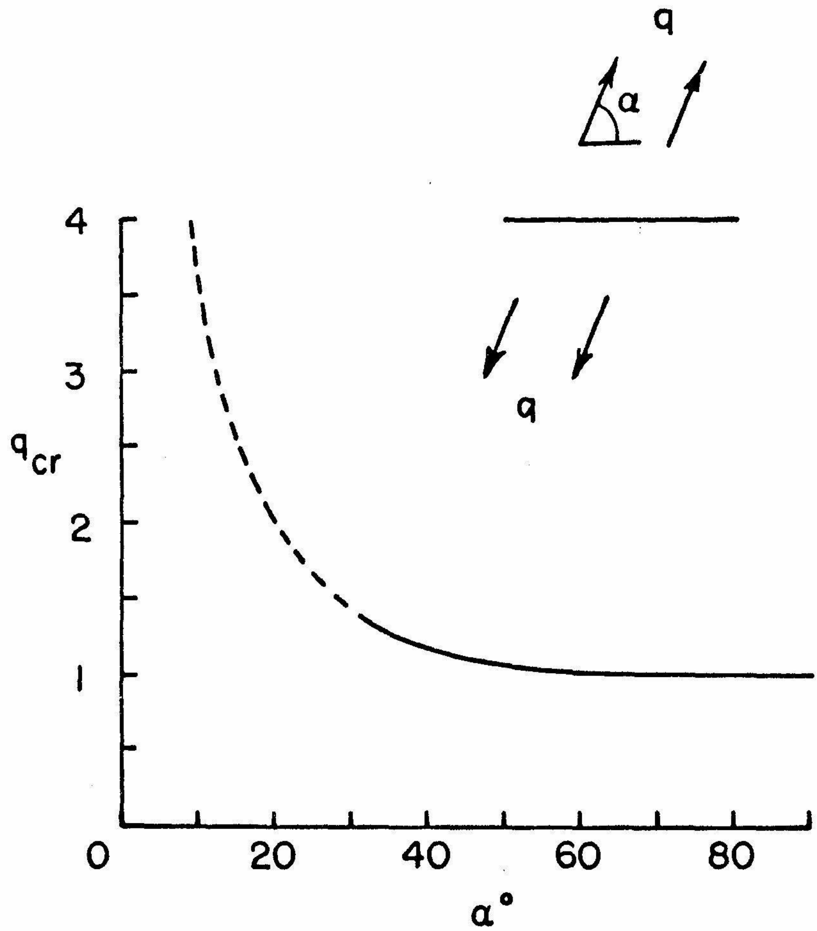


FIG. 16