NUMERICAL SOLUTION OF STEADY, SYMMETRIC AND LAMINAR FLOW AROUND A CIRCULAR CYLINDER

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ABSTRACT

A numerical integration of the Navier-Stokes equations is given for the steady, symmetric flow around a circular cylinder. The problem is formulated in terms of a streamfunction and the vorticity. The method used is the semi-analytical one of series truncation, in which the streamfunction and the vorticity are expanded in a finite Fourier sine series with argument β, the polar angle. Substitution of the truncated series into the Navier-Stokes equations yields a system of non linear, coupled, ordinary differential equations for the Fourier coefficients which is subjected to boundary conditions on the cylinder and at infinity. In order to be able to do a numerical calculation the free stream conditions at infinity are replaced by the asymptotic Oseen conditions at a finite distance from the cylinder. The resulting two point boundary value problem for the system of differential equations is solved numerically by a finite difference method. This method gives rise to a non linear system of algebraic difference equations. Four different iteration methods are discussed to solve this algebraic system. The most efficient iteration method seems to be Newton's method, which needs only about three iterations to converge to a solution of the difference equations. The approximation of the solution of the finite difference equations to the exact solution of the differential equations is improved by performing a Richard-

son extrapolation.

It can be concluded that a very efficient scheme has been obtained to solve the system of ordinary differential equations which follow from the application of the method of series truncation. It has been found however that the number of terms in the Fourier series needed to describe the flow adequately and correspondingly the computation time increase considerably with the Reynolds number. Nevertheless, it is believed that the method developed here is much more efficient than previous ones. Calculations have been done for $R = 0.5$, 2.0, 3.5 and 5.0 where $R = Ua/\nu$ (a is the radius of the cylinder). The results compare reasonably well with previous numerical calculations of Keller-Takami and Dennis-Chang.

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INTRODUCTION

In this report we discuss the numerical calculation of the symmetric, steady, laminar flow around a circular cylinder for low Reynolds numbers.

The reason for studying this problem is to understand better the flow behind bluff bodies. In particular the limit for the flow as $R \to \infty$ is a matter of great interest. Because an exact solution of the Navier-Stokes equations cannot be expected for this case and because the experiments are restricted to very low Reynolds numbers because of instability and turbulence, this problem is usually approached by numerical methods.

Several other investigators, such as Keller and Takami, Dennis and Chang, Underwood, have treated this problem numerically. The solutions by these authors agree well for parameters such as the drag coefficient in the case of Reynolds numbers of about $R = 1 - 30$, where $R = Ua/\nu$, and a is the radius of the cylinder. Detailed information about the rest of the flow field is not found in their work.

Because the amount of computer time for doing these calculations is rather large for the higher Reynolds numbers, there is considerable incentive for developing an efficient calculation scheme. Though there has been much work on the numerical calculation of the flow around the cylinder, the numerical problem is far from being solved. It is very difficult to find a numerical scheme in which the iterations converge rapidly and do not employ too much computer time for the higher Reynolds numbers.

In this report several numerical schemes and their efficiencies are discussed. A method has been developed, which reduces the number of iterations to find a numerical solution to about 5 - 7. All of these schemes have the following fact in common. The partial differential equations are transformed by means of a Fourier series expansion to a system of ordinary differential equations.

The aim of the calculations in this report is to obtain not only the global constants of the flow field such as the drag coefficient but also detailed information about the flow far from the cylinder.

I. EQUATIONS OF MOTION.

There are two basically different methods for the numerical solution of the flow around a cylinder.

The first method uses the time dependent Navier-Stokes equations. The problem is treated as a marching problem in time and the calculation proceeds as follows. An inital flow is chosen at time t=0. From the equations of motion the derivatives with respect to time are calculated. The flow at time $t+\Delta t$ then can be calculated. This process continues until a steady flow has been obtained.

In the second method the equations of motion for a steady state are solved. In this report the numerical solution of the flow around a cylinder will be attempted using this second method.

Consider the steady flow field around a circular cylinder in a uniform, incompressible flow. This flow field will satisfy the Navier-Stokes equations in two dimensions

$$\begin{aligned} &\operatorname{div} \underset{\sim}{q}^{*} = 0 \\ &(\operatorname{grad} \underset{\sim}{q}^{*})\, \underset{\sim}{q}^{*} = -1/\rho \operatorname{grad} p^{*} + \nu \Delta \underset{\sim}{q}^{*} \qquad 1.1 \\ &\qquad \text{where } \underset{\sim}{q}^{*} = (u^{*}, v^{*}) \end{aligned}$$

These equations can be made dimensionless with the undisturbed flow at infinity $\underset{\sim}{q}^{*}_{\infty} = (U, 0)$ and a characteristic body dimension. For this case the characteristic length is chosen to be the radius of the cylinder, a. The dimensionless equations are

$$\begin{aligned} &\operatorname{div} \underset{\sim}{q} = 0 \\ &(\operatorname{grad} \underset{\sim}{q})\, \underset{\sim}{q} = -\operatorname{grad} p + 1/R\, \Delta \underset{\sim}{q} \\ &\underset{\sim}{q} = \underset{\sim}{q}^*/U \\ &p = p^*/\rho U^2 \\ &R = Ua/\nu \end{aligned} \qquad 1.2$$

The flow is assumed to be symmetric so that the numerical calculation only has to be done for a half plane. The boundary conditions which must then be satisfied are

$$\begin{aligned} &\text{on the cylinder} && \underset{\sim}{q} = 0 \\ &\text{at infinity} && \underset{\sim}{q} \rightarrow (1,\ 0) \\ &\text{on the symmetry line} && v = 0,\ \partial u/\partial y = 0 \end{aligned} \qquad 1.3$$

Introduce a streamfunction ψ and a vorticity ζ, which are defined by

$$\begin{aligned} &u = \partial\psi/\partial y\,, \qquad v = -\,\partial\psi/\partial x \\ &\zeta = -\,\partial u/\partial y + \partial v/\partial x \end{aligned} \qquad 1.4$$

Elimination of the pressure from equation 1.2 and substituting the streamfunction in the definition of the vorticity yields the two following scalar equations for ψ and ξ

$$\begin{aligned} \Delta\psi &= -\xi \\ \Delta\xi &= R\,\partial(\xi,\psi)/\partial(x,y) \end{aligned} \qquad 1.5$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$

The boundary conditions in terms of the vorticity and the streamfunction become

on the cylinder $\psi = \partial\psi/\partial n = 0$

at infinity $\psi \rightarrow y,\ \xi \rightarrow 0$ 1.6

on the symmetry line $\psi = \xi = 0$

The half plane $y > 0$ outside the circular cylinder is transformed into a region in the (α, β)-plane by the following conformal transformation

$$\alpha + i\beta = \ln(x + iy) \qquad 1.7$$

This transformation transforms the region $y > 0$ outside the cylinder in the (x, y)-plane into a rectangular region $0 \leqslant \alpha < \infty$, $0 \leqslant \beta \leqslant \pi$ in the (α, β)-plane. (See figure 1). The cylinder wall corresponds with $\alpha = 0$.

The transformation of the equations (1.5) into the new coordinate system yields

$$\Delta\psi = -H^{-2}\zeta$$
$$\Delta\zeta = R\,\partial(\zeta,\psi)/\partial(\alpha,\beta) \qquad 1.8$$
$$\text{where } H = e^{-\alpha},\quad \Delta = \partial^2/\partial\alpha^2 + \partial^2/\partial\beta^2$$

The boundary conditions become

$$\psi = \partial\psi/\partial\alpha = 0 \qquad \alpha = 0$$
$$\psi = \zeta = 0 \qquad \beta = 0,\pi \qquad 1.9$$
$$\psi \to e^{\alpha}\sin\beta,\ \zeta \to 0 \qquad \alpha \to \infty$$

The velocity components in the (α, β)-plane are given by

$$u_\alpha = H\,\partial\psi/\partial\beta$$
$$u_\beta = -H\,\partial\psi/\partial\alpha \qquad 1.10$$

These equations are derived in appendix III.

The equations (1.8) form a system of two second order, coupled, non linear differential equations, which must be solved to find the flow field around the cylinder.

It must be noted that no boundary condition for the vorticity is given on the cylinder. An iteration procedure, which solves the vorticity equation and the streamfunction equation separately, needs such a boundary condition. On the other hand the solution of the streamfunction equation is overdetermined by too many boundary conditions on the cylinder. Therefore an approximation to the boundary condition for the vorticity in relation to the boundary conditions for the streamfunction must be found or else the iterations must employ both equations simultaneously.

Because of the fact that the numerical solution cannot proceed to $\alpha \rightarrow \infty$, there must be imposed some artificial boundary at a large but finite α_m. The boundary conditions for the streamfunction and the vorticity, which have to be applied on this boundary, are discussed in chapter III.

Different methods have been applied for the solution of this problem. The most important results will be discussed below.

Keller and Takami (1) use a finite difference approximation in both partial differential equations. This finite difference problem is then solved by an iteration procedure. Dennis and Chang (2) use a finite difference approximation in the non linear partial differential equation for the vorticity. The linear equation for the streamfunction is solved by the expansion of the streamfunction in a Fourier sine series. The substitution of this series, which is truncated after a certain number of terms, into the equation yields ordinary differential equations for the Fourier coefficients. These ordinary differential equations then are solved subject to boundary conditions on the cylinder and at infinity. Underwood (3) expands both the vorticity and the streamfunction in a Fourier sine series. Truncation and substitution in the differential equations yield a non linear system of ordinary differential equations for the Fourier coefficients of both series.

The method used in this report to calculate the flow around a cylinder is in principle the same as the method used by Underwood. However there is a great difference between the two approaches in the scheme used to solve the system of ordinary differential equations.

The advantage of expanding both the streamfunction and the vorticity in a Fourier sine series is the fact that only ordinary differential equations have to be solved. In general numerical schemes for the solution of ordinary differential equations are more efficient than schemes for partial differential equations. The method we employ treats all the ordinary differential equations in the same way and is thus equivalent to a simultaneous solution of the equations for the vorticity and the streamfunction. Therefore the difficulty in finding a boundary condition for the vorticity on the cylinder, which was mentioned above, does not arise here.

This method of solving the equations of motion for the flow around a circular cylinder will be discussed in more detail in the next chapter.

II. METHODS

The streamfunction and the vorticity are both expanded in a Fourier sine series. The boundary conditions for $\beta = 0$ and π are then automatically satisfied

$$\psi = \sum_{1}^{\infty} f_n(\alpha) \sin n\beta \qquad\qquad 2.1$$
$$\zeta = \sum_{1}^{\infty} g_n(\alpha) \sin n\beta$$

Substitution of these series into equation (1.8) and using the orthogonality of the sine functions yield the following ordinary differential equations for f_n and g_n

$$f_n'' - n^2 f_n = -H^2 g_n \qquad \text{for } n = 1,2,\ldots\ldots \qquad 2.2$$
$$g_n'' - n^2 g_n = R\, r_n$$
$$r_n = 2/\pi \int_0^{\pi} \Big(\sum_{k=1}^{\infty} g_k' \sin k\beta \sum_{p=1}^{\infty} f_p\, p \cos p\beta - \sum_{k=1}^{\infty} g_k\, k \cos k\beta \sum_{p=1}^{\infty} f_p' \sin p\beta \Big) \sin n\beta \, d\beta$$

The r_n can be evaluated in the following form

$$r_n = 1/2 \Big\{ \sum_{k=1}^{\infty} k(g_{n-k}' f_k - f_{n-k}' g_k) + \sum_{k=1}^{\infty} k(g_{n+k}' f_k - f_{n+k}' g_k) \Big\} \qquad 2.3$$

using the definitions $\quad g_{-i}' = -g_i' \quad g_0' = 0$

$\qquad f_{-i}' = -f_i' \quad f_0' = 0$

The boundary conditions for the equations 2.2 are

$$f_n(0) = f'_n(0) = 0$$

$$f_1 \to e^{\alpha}, \quad \underset{n \neq 1}{f_n} \to 0 \qquad \alpha \to \infty \qquad 2.3$$

$$g_n \to 0 \qquad \alpha \to \infty$$

The boundary conditions at infinity will be replaced by boundary conditions at a finite distance $\alpha = \alpha_m$ from the cylinder. The form of these boundary conditions is discussed in chapter III.

The infinite system of non linear equations (2.2) cannot be solved in closed form, to our knowledge. Thus approximate solutions are sought and the first approximation made is to truncate all infinite sums by retaining only harmonics of some finite order, N. The vorticity and the streamfunction are then represented by a finite Fourier series of N terms.

The resulting finite system of ordinary differential equations is strongly coupled and non linear, so that exact solutions are still not available. Approximate solutions are usually sought by means of iterations and finite difference approximations. There are two general ways to proceed: (A) Devise iterative schemes for solving the differential equations and then approximate the iterates by numerical (difference) methods; (B)Devise numerical methods for approximating the differential equations and then solve the resulting algebraic system by iterations. The latter procedure is followed in all of our calculations and the numerical method used is Euler centered finite differences using two net points (see Appendix I). Four different iteration schemes will be discussed for solving the difference equations. However it is somewhat easier to describe the ideas in the iteration schemes by adopting the procedure (A). We do this here for ease of exposition.

The equations (2.2) for each harmonic can be written in the form of a first order system

$$\underset{\sim}{F}_n' = A_n \underset{\sim}{F}_n + \underset{\sim}{R}_n \qquad n = 1, \ldots\ldots N$$

$$\underset{\sim}{F}_n = \begin{bmatrix} u_n = f_n' \\ v_n = g_n' \\ f_n \\ g_n \end{bmatrix} \tag{2.4}$$

$$A_n = \begin{bmatrix} 0 & 0 & n^2 & -H^2 \\ 0 & 0 & 0 & n^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad R_n = \begin{bmatrix} 0 \\ R\, r_n \\ 0 \\ 0 \end{bmatrix}$$

The boundary conditions (2.3) are written in the following way

$$B^{\circ} F_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } \alpha = 0$$

$$B' F_n = \begin{bmatrix} Rf \\ Rg \end{bmatrix} \quad \text{for } \alpha = \alpha_m \tag{2.5}$$

$$B^{o} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As has been discussed above all iteration procedures will be explained using the differential equation (2.4). As will be shown on the following pages each iteration procedure will give rise to a differential equation of the form (2.4). However in each case the matrix A_n and the $\underset{\sim}{R}_n$ will be defined differently. In Appendix I the details are given of the solution of an equation of the form (2.4) by finite differences subjected to the boundary conditions (2.5) for a general form of the matrix A_n and the vector $\underset{\sim}{R}_n$. This solution then can be applied to all the iteration schemes discussed in this chapter.

The first iteration procedure solves at each iteration step the linear part of equation (2.2) with the non linear terms calculated from the previous iteration as a forcing term. This is the iteration scheme used by R. Underwood (3). There however the iterates are each evaluated by a Runge-Kutta numerical integration. The procedure can be described in the form (2.4) in the following way: if $\underset{\sim}{F}_n^r$ is known, define $\underset{\sim}{F}_n^*$ as the solution of:

$$\underset{\sim}{F}'^{*}_{n} = A_n \underset{\sim}{F}^{*}_{n} + \underset{\sim}{R}_n \qquad n = 1,\ldots\ldots N \qquad 2.6$$

where A_n and $\underset{\sim}{R}_n$ are defined in the same way as in equation (2.4)

The r_n is evaluated completely from the r^{th} iterate $\underset{\sim}{F}^{r}_{n}$. The boundary conditions satisfied by $\underset{\sim}{F}^{*}_{n}$ are of the form (2.5).

Then $\underset{\sim}{F}^{r+1}_{n}$ is defined as

$$\underset{\sim}{F}^{r+1}_{n} = \underset{\sim}{F}^{r}_{n} + D\,(\,\underset{\sim}{F}^{*}_{n} - \underset{\sim}{F}^{r}_{n}\,)$$

where 2.7

$$D = \begin{bmatrix} Fk & & & \\ & Vk & & 0 \\ 0 & & Fk & \\ & & & Vk \end{bmatrix}$$

The Fk and Vk are defined as acceleration parameters. These parameters are chosen in such a way to improve the convergence of the iteration scheme.

Physically, this iteration procedure balances at each iteration step the viscous forces with the inertial forces calculated from the previous iterate.

The convergence to a solution for this scheme has been found to be reasonably fast only for small Reynolds numbers ($R \leqslant 0.5$). For example, it took 25 iterations for $R = 0.5$ --with $Fk = 0.5$, $Vk = 0.5$ and starting from a calculated solution for R = 0.2-- to reduce the difference between two consecutive iterations to the following value

$$\|\underline{F}^{\nu+1} - \underline{F}^{\nu}\|_{\infty} \leqslant 10^{-4}$$

For larger Reynolds numbers the number of iterations needed to find a convergent solution increases very rapidly. For R = 1 the number of iterations needed was of the order of 100. In order to find a convergent behaviour at all the parameters, Fk and Vk had to be chosen rather small: Fk, Vk ~ 0.1-0.01. Therefore the conclusion must be that this iteration procedure is not efficient for Reynolds numbers R > 0.5. The reason for this must be that at each iteration step the viscous terms are calculated from the equation, whereas the inertial terms are obtained from the previous iteration. However for the higher Reynolds numbers the inertial forces are important in most of the flow field compared to the viscous forces. So the iteration procedure can only be supposed to work efficiently in the small boundary layer regions where the viscous forces are important. In other words this iteration procedure is basically wrong for the higher Reynolds numbers, because it iterates on a value which is almost everywhere negligible.

In the second iteration procedure we attempt to solve the system by incorporating some of the non linear convective terms in the calculation. The iteration scheme written in the form of equation (2.4) and satisfying boundary conditions of the form (2.5) then becomes

$$\underset{\sim}{F}_n'^{*} = \bar{A}_n \underset{\sim}{F}_n^{*} + \bar{\underset{\sim}{R}}_n \qquad n = 1,\ldots.N \qquad 2.8$$

$$\bar{A}_n = \begin{bmatrix} 0 & 0 & n^2 & -H^2 \\ n\,R\,g_{2n} & -n\,R\,f_{2n} & \tfrac{1}{2}n\,R\,f_{2n}' & -\tfrac{1}{2}n\,R\,g_{2n} + n^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\bar{\underset{\sim}{R}}_n = \begin{bmatrix} 0 \\ R\,\bar{r}_n \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{r}_n = 1/2\left\{\sum_{\substack{k=1\\k\neq 2n}}^{N} k\,(\,g_{n-k}' f_k - f_{n-k}' g_k\,) + \sum_{\substack{k=1\\k\neq n}}^{N-n} k\,(\,g_{n+k}' f_k - f_{n+k}' g_k\,)\right\}$$

the newest available iterate of $\underset{\sim}{F}_n$ is used to calculate $\bar{r}_n$.

So the calculation of the new iterate for $\underset{\sim}{F}_n$ uses the non linear terms of r_n with harmonic index n by bringing these terms inside the matrix A_n. The $\nu + 1$ iterate of $\underset{\sim}{F}_n$ is calculated in the same way as is done in equation (2.7).

As expected this iteration procedure seems to work better for the higher Reynolds numbers. For example for the calculation of the flow for R = 1.0, starting from a calculated flow for R = 0.5, about 23 iterations were needed to reduce the difference between the drag coefficients calculated from two consecutive iterates to the following value

$$|Cd^{r+1} - Cd^{r}| \leq 0.2\ 10^{-5} \qquad 2.9$$

For these Reynolds numbers an optimal convergence behaviour was found when the acceleration parameters Fk and Vk had the following values

$$Fk - 0.5, \qquad Vk - 1/(2R) \qquad 2.10$$

The number of iterations needed to find a convergent solution increased when the Reynolds number was increased. For the calculation of the flow at Reynolds number R = 2.5 at least about 80 iterations were needed to find a convergent solution. But compared to the several hundred iterations that would have been needed to find a convergent solution using the first iteration scheme, the new iteration procedure is a considerable improvement. More precise results of the convergence behaviour of this method at different Reynolds numbers will be given in chapter IV.

The success of the new iteration procedure indicates that the convergence behaviour of the method is closely related to the treatment of the non linear convective terms. Therefore an iteration procedure, which takes all of the non linear terms into account, may give better results. We proceed to find a third iteration scheme which has these properties.

Define the calculation of the new iterate in the following way

$$\underset{\sim}{F}_n^{\nu+1} = \underset{\sim}{F}_n^{\nu} + \delta \underset{\sim}{F}_n^{\nu}$$

where

$$\delta \underset{\sim}{F}_n = \begin{bmatrix} \delta u_n = \delta f_n' \\ \delta v_n = \delta g_n' \\ \delta f_n \\ \delta g_n \end{bmatrix} \qquad 2.11$$

Substitute this expression in equation (2.4). Using the fact that $\delta \underset{\sim}{F}_n^{\nu}$ will be small we can find the following linear equation for $\delta \underset{\sim}{F}_n$

$$\delta \underset{\sim}{F}_n' = (A_n + R_{n,n})\, \delta \underset{\sim}{F}_n + \sum_{\substack{m=1 \\ m \neq n}}^{N} R_{n,m}\, \delta \underset{\sim}{F}_m + (-\underset{\sim}{F}_n^{\nu\prime} + A_n \underset{\sim}{F}_n^{\nu} + \underset{\sim}{R}_n^{\nu}) \qquad 2.12$$

where the Jacobian $R_{n,m} = \partial \underset{\sim}{R}_n / \partial \underset{\sim}{F}_m$ is defined as

$$\sum_{m=1}^{N} R_{n,m}\, \delta \underset{\sim}{F}_m = \begin{bmatrix} 0 \\ R \quad \delta r_m \\ 0 \\ 0 \end{bmatrix}$$

$$\delta r_m = 1/2 \Big\{ \sum_{k=1}^{N} k\big[(\delta g'_{m-k} f_k - \delta f'_{m-k} g_k) + (g'_{m-k} \delta f_k - f'_{m-k} \delta g_k)\big] + \sum_{k=1}^{N-m} k\big[(\delta g'_{m+k} f_k - \delta f'_{m+k} g_k) + (g'_{m+k} \delta f_k - f'_{m+k} \delta g_k)\big] \Big\}$$

Essentially we linearised all non linear terms in the equation so that this method can be compared with Newton's method. Recalling the fact that in the calculation procedures all iteration schemes are done on the algebraic system of difference equations this method then exactly becomes Newton's method for non linear algebraic systems.

For Newton's method for algebraic systems it can be proved that when the first iterate is sufficiently close to the root of the system this scheme has quadratic convergence. One of the main properties of quadratic convergence is the fact that, when sufficiently close to a root, at each iteration step the number of significant digits doubles. This implies that the convergence of this method is very fast and it can be observed in the results to insure that convergence is actually occurring.

In order to find the convergence behaviour connected with Newton's method the equation (2.12) must be solved simultaneously for all harmonics because these equations are strongly coupled. The order of the matrices involved is rather large. Therefore a direct solution of the Newton equation will not be attempted at first, but some kind of inner iteration procedure will be used to find the Newton corrections $\delta \underset{\sim}{F}_n$.

The most obvious inner iteration procedure is to solve the equation for each harmonic separately, because all the previous iteration schemes are set up for systems of the form (2.4). This method essentially can be described as a block Gauss-Seidel iteration of equation (2.12), where the blocks are formed by the equation for each harmonic.

This inner iteration procedure can be written in the form of equation (2.4) in the following way

$$\delta \underset{\sim}{F}'^{*}_n = \tilde{A}_n \, \delta \underset{\sim}{F}^{*}_n + \tilde{\underset{\sim}{R}}_n \qquad n = 1,\ldots\ldots N \qquad 2.13$$

$$\tilde{A}_n = A_n + R_{n,n}$$

$$\tilde{\underset{\sim}{R}}_n = \sum_{m=1}^{n-1} R_{n,m} \, \delta \underset{\sim}{F}^{\mu+1}_m + \sum_{m=n+1}^{N} R_{n,m} \, \delta \underset{\sim}{F}^{\mu}_m + \left(-\underset{\sim}{F}'^{\nu}_n + A_n \underset{\sim}{F}^{\nu}_n + \underset{\sim}{R}^{\nu}_n\right)$$

where the most recent calculated iterate of $\underset{\sim}{F}_n$ is used to evaluate $\underset{\sim}{R}^{\nu}_n$.

Then a new iterate for the Newton corrections is calculated by

$$\delta \underset{\sim}{F}^{\mu+1}_n = \delta \underset{\sim}{F}^{\mu}_n + D \left(\delta \underset{\sim}{F}^{*}_n - \delta \underset{\sim}{F}^{\mu}_n \right) \qquad 2.14$$

where D is defined as in equation (2.7)

When a sufficiently good approximation to the Newton corrections has been obtained, one step in the outer iteration (Newton's iteration) can be completed to find a new approximation to the solution

$$\underline{F}_n^{\nu+1} = \underline{F}_n^{\nu} + \delta \underline{F}_n \qquad n = 1,\ldots\ldots N \qquad 2.15$$

The quadratic convergence for this scheme has been observed for R = 1.0, where about 10 iterations were needed with Fk = 0.5 and Vk = 0.5 to obtain convergence for the inner iteration. For R = 3 quadratic convergence could not be found because of the poor convergence behaviour of the inner iteration. So the convergence problem seems to have been transferred to the inner iteration procedure. The fact however that quadratic convergence has been found for R = 1.0 seems to indicate that Newton's method works. Therefore the only alternative is to solve the total Newton equations (2.12) directly, without the use of any inner iteration scheme.

This means that we must solve the equation (2.12) simultaneously for all harmonics n = 1,.....N. In order to do the numerical calculation it must be noted that the equation (2.12) for all harmonics can be represented as one linear differential equation of the following form

$$\delta \underline{F}' = A\, \delta \underline{F} + \underline{R} \qquad 2.16$$

$$\delta \underline{F} = \begin{bmatrix} \delta \underline{F}_1 \\ \vdots \\ \delta \underline{F}_N \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 + R_{1,1} & R_{1,2} & \cdots & R_{1,N} \\ R_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_{N-1,N} \\ R_{N,1} & \cdots & R_{N,N-1} & A_N + R_{N,N} \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} -\underline{F}_1'^{\nu} + A_1 \underline{F}_1^{\nu} + \underline{R}_1^{\nu} \\ \vdots \\ -\underline{F}_N'^{\nu} + A_N \underline{F}_N^{\nu} + \underline{R}_N^{\nu} \end{bmatrix}$$

Essentially this equation is of the form (2.4) and therefore can be solved by approximately the same methods which are discussed in Appendix I. At the end of Appendix I the solution of equation (2.16) is discussed in more detail.

This method, which has eliminated the inner iteration procedure in the Newton iteration scheme, works very well and shows the desirable quadratic convergence. Detailed results will be given in chapter IV.

It must be noted that the boundary conditions at the artificial boundary, as will be shown in chapter III, will depend on the drag coefficient. Therefore these boundary conditions are a functional of the solution.

This problem can be solved by incorporating these boundary conditions in the iteration scheme. First a drag coefficient is specified in order to be able to calculate the outer boundary conditions. Then the numerical solution yields a new drag coefficient, which is used in the boundary condition for the second calculation. This procedure is continued until convergence is obtained. In general this will not take more than 2-3 iterations. This follows from the fact that a good approximation to the drag coefficient can be used as the first iterate. Further, since the equations of motion tend to become of the parabolic type far from the body, a slight difference in the boundary conditions far downstream will not have much influence on the solution. Details about this iteration scheme are given in chapter IV.

As will be discussed in Appendix I, the truncation error in replacing the differential equation by difference equations is $O(h^2)$, where h is the stepsize in the radial direction. An additional improvement of the accuracy can be obtained by doing a Richardson extrapolation, also discussed in Appendix I. By this method the error can be decreased to $O(h^4)$. This method requires the solution of the system of differential equations on a net of half the size of the original net. This solution must be found using the same iteration procedure described in this chapter. To start the iteration the solution on the original net can be used, so this procedure introduces still another iteration. Results of using Richardson extrapolation are discussed in chapter IV.

III. ASYMPTOTIC BOUNDARY CONDITIONS.

As has been discussed in chapter I, it is not possible to extend the numerical calculations to an infinite distance from the cylinder. Therefore it is necessary to limit the calculation to a finite region around the cylinder by imposing some artificial boundary. In this chapter the conditions which must be applied on this boundary will be discussed.

One can argue that far from the cylinder the flow is almost equal to the undisturbed parallel flow. However the application of free stream conditions at the boundary of the finite region is not a very good one, because of the following reason. The cylinder will cause a loss of momentum of the flow connected with the drag of the cylinder. Integration along a contour surrounding the cylinder must yield the same momentum loss for all contours. If at a certain distance from the cylinder the flow conditions have been set equal to the free stream values, the momentum loss found by integration along a contour in this region will be zero. This discontinuity in the drag coefficient clearly indicates that the free stream flow is not the right boundary condition to use.

A consistent set of boundary conditions can be found by doing an asymptotic expansion on the equations of motion far from the cylinder. This asymptotic solution is known by the name of Oseen's solution and is given by

Imai (4). Another derivation, which makes use of the ideas of Dennis and Chang (2) is given in this chapter.

The equations of motion and the boundary conditions in the (α,β)-plane are given by (1.8) and (1.9)

$$\begin{aligned} &\Delta\psi = -H^{-2}\zeta \\ &\Delta\zeta = R\,\partial(\zeta,\psi)/\partial(\alpha,\beta) \quad \text{where } H = e^{-\alpha} \\ &\psi = \partial\psi/\partial\alpha = 0 \qquad \alpha = 0 \\ &\psi = \zeta = 0 \qquad \beta = 0,\pi \\ &\psi \to e^{\alpha}\sin\beta \quad \zeta\to 0 \qquad \alpha\to\infty \end{aligned} \tag{3.1}$$

We expand the streamfunction and the vorticity in an asymptotic series about the free stream solution. The first term in the series for the streamfunction will clearly be the free stream. Substitution of both series in the equation for the vorticity and retaining only first order terms yield the following linear equation for the first term in the asymptotic series for the vorticity

$$1/R\ \ \Delta\zeta = e^{\alpha}\cos\beta\,\partial\zeta/\partial\alpha - e^{\alpha}\sin\beta\,\partial\zeta/\partial\beta \tag{3.2}$$

We look for a solution in the following form

$$\zeta = e^{F(\alpha,\beta)}\phi(\alpha,\beta) \tag{3.3}$$

Then substitute this expression in equation (3.2)

$$\partial^2\phi/\partial\alpha^2 + \partial^2\phi/\partial\beta^2 - 1/4\ R^2\ e^{2\alpha}\phi = 0 \qquad 3.4$$

$$\text{if } F = 1/2\ R\ e^{\alpha}\cos\beta$$

Transformation to a new variable χ yields

$$\chi^2\ \partial^2\phi/\partial\chi^2 + \chi\ \partial\phi/\partial\chi - \chi^2\phi + \partial^2\phi/\partial\beta^2 = 0$$

$$\text{where } \chi = 1/2\ R\ e^{\alpha} \qquad 3.5$$

A solution of this equation can be found by separation of variables. The separated equations and the boundary conditions for $\beta = 0, \pi$ imply that the proper eigenfunction expansion of ϕ in the β-direction will be a Fourier sine series. Therefore we substitute the following series into equation (3.5)

$$\phi = \sum_{n=1}^{\infty} F_n(\chi)\ \sin n\beta \qquad 3.6$$

The equation for the Fourier coefficients is then found to be

$$\chi^2 F_n'' + \chi F_n' - (n^2 + \chi^2)\ F_n = 0 \qquad 3.7$$

This equation is known as the modified Bessel equation. Two linearly independent solutions are $I_n(\chi)$ and $K_n(\chi)$. As $\chi \to \infty$ I_n grows exponentially whereas K_n decays exponentially. Therefore by the boundary condition for the vorticity at infinity only the K_n can be used for the solution of (3.7). So the solution for the vorticity becomes

$$\zeta = e^{\chi \cos\beta} \sum_{n=1}^{\infty} c_n \, K_n(\chi) \sin n\beta \qquad 3.8$$

The unknown coefficients c_n must follow from boundary conditions on the cylinder.

Substitution of the asymptotic form of $K_n(\chi)$ for $\chi \to \infty$ into equation (3.8) yields the asymptotic solution for the vorticity far from the cylinder

$$\zeta = G(\beta)\; 1/\chi^{1/2} \;\; e^{-\chi(1-\cos\beta)} \qquad 3.9$$

where $G(\beta) = (\pi/2)^{1/2} \sum_{n=1}^{\infty} c_n \sin n\beta$

$K_n(\chi) \simeq (\pi/2\chi)^{1/2} \;\; e^{-\chi}$ for $\chi \to \infty$

From this equation it follows that the vorticity becomes exponentially small for $\chi \to \infty$ except in a region near $\beta = 0$. This region of vorticity can be described by the following equation

$$\chi(1-\cos\beta) = \beta_0^{\,2} \qquad 3.10$$

where $\beta_0^{\,2}$ is some arbitrary small number whose magnitude is chosen such that for $\chi(1-\cos\beta) < \beta_0^{\,2}$ $\exp(-\chi(1-\cos\beta)) = O(1)$. As it is clear that β will be small in this region, $\cos\beta$ can be replaced by the first two terms in its series expansion. Then equation (3.10) describing the region of vorticity becomes

$$\beta = \beta_0 (2/\chi)^{1/2} \qquad \text{when} \quad \chi \to \infty \qquad 3.11$$

In terms of the (x, y) coordinates equation (3.11) becomes

$$y = cx^{1/2} \qquad 3.12$$

So the region of vorticity in the (x, y)-plane is a parabolic region behind the cylinder.

Because inside the region of vorticity β is small, $\sin n\beta = n\beta$ is substituted into equation (3.9). This can be done only when the c_n in (3.9) go sufficiently strongly to zero so that the terms for large n do not contribute. Then the asymptotic form of the vorticity becomes

$$\zeta = (\pi/2)^{1/2} \beta/\chi^{1/2} \, K \, e^{-\chi\beta^2/2} \qquad 3.13$$

$$\text{where } K = \sum_{n=1}^{\infty} n \, c_n$$

With this expression for the vorticity it is possible to solve the equation (3.1) for the streamfunction. A solution of this equation can be found by separation of variables. By the same reasons used for equations (3.5) the proper eigenfunction expansion in the β-direction is a Fourier sine series. Therefore substitute the following series in the equation for the streamfunction (3.1)

$$\psi = \sum_{n=1}^{\infty} f_n \sin n\beta \qquad 3.14$$

The equation for the Fourier coefficient then becomes

$$f_n'' - n^2 f_n = r_n \qquad 3.15$$

$$\text{where } r_n = -2/\pi \int_0^{\pi} e^{2\alpha} \zeta \sin n\beta \, d\beta$$

Substituting expression (3.8) for the vorticity in equation (3.15) the integral for r_n can be evaluated

$$r_n = -\sum_{p=1}^{\infty} c \, K_p(\chi) \left\{ I_{p-n}(\chi) - I_{p+n}(\chi) \right\} \qquad 3.16$$

$$\text{where } I_n = 1/\pi \int_0^{\pi} e^{\chi \cos\beta} \cos n\beta \, d\beta$$

Substitution of the asymptotic form for $K_n(\chi)$ (3.9) and the asymptotic form for I_n yields

$$r_n = -4n/R^2 \sum_{1}^{\infty} n \, c_n = -4n/R^2 \, K = -n \, C \qquad 3.17$$

$$\text{where } I_n(\chi) \rightarrow 1/\sqrt{2\pi\chi} \, (1 - (4n^2-1)/8\chi) e^{\chi}$$

$$\text{as } \chi \rightarrow \infty$$

The solution of equation (3.15) using the boundary conditions (3.1) and the asymptotic form for the r_n (3.17) then becomes

$$f_n = C/n + \delta/n \, e^{n\alpha} + O(e^{-n\alpha}) \quad \text{for } \alpha \to \infty \qquad 3.18$$

$$\text{where } \delta = 1 \qquad n = 1$$
$$\delta = 0 \qquad n \neq 1$$

So the solution for the streamfunction is

$$\psi = e^{\alpha} \sin \beta + \sum_{1}^{\infty} C/n \sin n\beta \qquad 3.19$$

This Fourier series can be summed to give the following expression for the streamfunction

$$\psi = e^{\alpha} \sin \beta + \pi/2 \; C \; (1 - \beta/\pi) \qquad 3.20$$

The first term in this expression clearly represents the undisturbed parallel flow.

It must be noted that the function (3.20) is harmonic everywhere. Therefore the streamfunction (3.20) must represent an irrotational flow. It has been found however that there is a region of concentrated vorticity near $\beta = 0$ for which the streamfunction of equation (3.20) will not give the right solution. This problem can be solved by doing a boundary layer type argument for the region near $\beta = 0$. The streamfunction (3.20) then constitutes an outer solution, which is a valid solution everywhere except in the boundary layer region. We proceed to find the solution inside the boundary layer or wake region

Because the wake region is parabolic far in the wake $y \ll x$. So the scale in the y-direction is very small compared to the scale in the x-direction. Therefore the boundary layer equations can be used

$$\partial u/\partial x + \partial v/\partial y = 0 \qquad 3.21$$
$$u\, \partial u/\partial x + v\, \partial u/\partial y = U_e\, dU_e/dx + 1/R\, \partial^2 u/\partial y^2$$

The pressure term in equation (3.21) can be calculated from the solution (3.20). It is then found that this pressure term is essentially of second order and therefore can be neglected.

Far down stream the flow will be almost equal to the free stream. Therefore we can linearize equation (3.21) in the following form

$$\partial u^*/\partial x = 1/R\, \partial^2 u^*/\partial y^2 \qquad 3.22$$

where $u^* = 1 - u$

The boundary conditions become

$$u^* = \partial u^*/\partial y = 0 \qquad \text{for } y \rightarrow \pm\infty \qquad 3.23$$

Conservation of momentum implies

$$D = \rho U^2 l \int_{-\infty}^{\infty} u \, (1 - u) \, dy \qquad 3.24$$

Linearization of this condition yields

$$Cd = \int_{-\infty}^{\infty} u^* \, dy \qquad 3.25$$

We can solve equation (3.22) by transformation techniques. Define the Fourier transform and its corresponding inverse transform

$$\tilde{f}\,(\lambda) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} e^{i\lambda y} \, f\,(y)\, dy$$
$$f\,(y) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda y} \, \tilde{f}\,(\lambda)\, d\lambda \qquad 3.26$$

The application of this transformation to equation (3.22) and condition (3.25) yields

$$d\tilde{u}/dx = - \lambda^2/R \, \tilde{u}$$
$$\tilde{u}(0) = 1/\sqrt{2\pi} \, Cd \qquad 3.27$$

The solution of this equation for the Fourier transform then becomes

$$\tilde{u} = Cd/\sqrt{2\pi} \; e^{-\lambda^2/R \, x} \qquad 3.28$$

By applying the inverse transformation we find

$$u^* = Cd/2\sqrt{\pi}\ \sqrt{R/x}\ e^{-(y^2R)/(4x)} \qquad 3.29$$

The streamfunction for the inner solution then is given by

$$\psi^* = y - \psi = \int_0^y u^*\, dy =$$

$$Cd/2\ \mathrm{erf}\ (\ y/2\ \sqrt{R/x}\ ) \qquad 3.30$$

$$\text{where } \mathrm{erf}\ x = 2/\sqrt{\pi} \int_0^x e^{-s^2}\ ds$$

The unknown constant C in the outer streamfunction (3.20) can be evaluated by applying the following matching condition

$$\lim_{y/\sqrt{x} \to \infty} \psi^*_{inner} = \lim_{y/x \to 0} \psi^*_{outer} \qquad 3.31$$

Application of this condition yields

$$C = -\ Cd/\pi \qquad 3.32$$

A composite solution for the streamfunction, which is valid everywhere, can be found by adding equation (3.30) and (3.20) and substracting the common part

$$\psi = e^{\alpha} \sin\beta - Cd/2\ (\ 1 - \beta/\pi\ ) +$$

$$Cd/2 \; \mathrm{erfc} \; (y/2\sqrt{R/x}) \qquad 3.33$$

$$\text{where } \mathrm{erfc}\,(x) = 1 - \mathrm{erf}\,(x)$$

This solution for the streamfunction (3.23) and for the vorticity (3.13) is identical to the solution found by Imai (4) only for small values of β. However this can be expected because the derivation in this chapter has been restricted to small values of β. Imai's solution is

$$\psi = e^{\alpha}\sin\beta - Cd/2 \; (1 - \beta/\pi) + Cd/2 \; \mathrm{erfc}\; Q$$

$$\zeta = - (Cd\; R)/(2\sqrt{\pi}) \; Q/e^{\alpha} \; e^{-Q^2} \qquad 3.34$$

$$\text{where } Q = \sqrt{R\; e^{\alpha}}\sin\beta/2$$

It must be noted that in the calculation procedures, Imai's solution has been used to evaluate the asymptotic boundary conditions.

With the asymptotic solution known there must be made an estimate of the distance from the cylinder at which this solution can be applied.

In order to make some rough estimates we have to know the most important characteristics of a flow behind a bluff body such as the cylinder. Therefore we suppose that the flow behind the cylinder satisfies the theoretical, asymptotic model of Sychyev (5) for the laminar, symmetric and incompressible flow behind a bluff body in the limit $R \to \infty$.

The Sychyev model has the following characteristics. The wake bubble behind the bluff body has the form of an ellipse with major axis $O(R)$ and minor axis $O(R^{1/2})$. Inside and outside this wake bubble the flow is non viscous. These two flow regions are separated along the boundary of the ellipse by a free shear layer. At the end of the wake bubble the two free shear layers join to form a wake which must diffuse in the free stream. The distance after the wake bubble, in which the velocity has sufficiently approached the free stream velocity such that asymptotic solutions can be applied, is taken to be the boundary where the asymptotic condition can be applied.

First consider the free shear layer near the end of the wake bubble region. According to Sychyev (5) the following estimates can be made for the coordinates and the velocities in this region

$$x = O(R) \qquad y = O(1)$$
$$u = O(1) \qquad v = O(R^{-1}) \qquad 3.35$$
$$p = O(R^{-1/2})$$

where x and y are the dimensionless coordinates along and perpendicular to the free shear layer.

Define the corresponding coordinates of O(1) inside this region.

$$\tilde{x} = x\,R^{-1} \qquad \tilde{y} = y$$
$$\tilde{u} = u \qquad \tilde{v} = R\,v \qquad 3.36$$
$$\tilde{p} = R^{1/2}\,p$$

Substitution into the equations of motion and retension of only first order terms yield the following equation and boundary conditions

$$\partial \tilde{u}/\partial \tilde{x} + \partial \tilde{v}/\partial \tilde{y} = 0$$
$$\tilde{u}\,\partial \tilde{u}/\partial \tilde{x} + \tilde{v}\,\partial \tilde{v}/\partial \tilde{y} = \partial^2 \tilde{u}/\partial \tilde{y}^2 \qquad 3.37$$
$$\tilde{u} = 1 \qquad \tilde{y} \to \infty$$
$$\tilde{u} = 0 \qquad \tilde{y} \to 0$$

By the introduction of a streamfunction the continuity equation is satisfied. The streamfunction can be found by assuming the solution to be of the following form

$$\psi = (\tilde{x})^{1/2}\, f(\eta) \qquad \eta = (\tilde{x})^{-1/2}\,\tilde{y} \qquad 3.38$$

Substitution into equation and boundary condition (3.27) yields

$$f''' + 1/2\, f\, f'' = 0$$

$$f' = 1 \qquad \eta \to \infty \tag{3.39}$$
$$f' \to 0 \qquad \eta \to -\infty$$
$$f(0) = 0 \qquad \eta = 0$$

The last boundary condition implies that the zero streamline is fixed at y = 0. This problem can be solved numerically and the results can be found in R.C. Lock (6).

According to the model of Sychyev the velocity distribution at the end of the wake bubble consists of the two outside parts of the free shear layers joined together.

If we consider the outside part of the free shear layer,the solution of (3.39) for $\eta>0$, then it follows that the velocity on the centerline at the end of the wake bubble is given by

$$u = 0.5873 \tag{3.40}$$

The momentum thickness, the displacement thickness and the form factor are found by a simple numerical integration

$$\delta^* = \int_0^\infty (1-\tilde{u})\, d\tilde{y} = \tilde{x}^{1/2} \int_0^\infty (1-f')\, d\eta = 0.56\ \tilde{x}^{1/2}$$
$$\theta = \int_0^\infty \tilde{u}(1-\tilde{u})\, d\tilde{y} = \tilde{x}^{1/2} \int_0^\infty f'(1-f')\, d\eta = 0.44\ \tilde{x}^{1/2}$$
$$H = \delta^*/\theta = 1.25 \tag{3.41}$$

Application of the principle of conservation of momentum yields

$$Cd = 2\,\theta \qquad 3.42$$

Using the results of Dennis and Chang (2) for the wake bubble length, $L_{\bullet} = 0.36\ R$, the drag coefficient corresponding to Sychyev's model can be found to be

$$Cd = 0.53 \qquad 3.43$$

To give an estimate of the length after the wake bubble, at which asymptotic conditions can be applied, we use the results of Charwat and Schneider (7). From their results can be found that the distance to reach asymptotic conditions starting with a velocity profile with a centerline velocity and a form factor given by (3.40) and (3.41) is given by

$$\Delta x = (Cd^2\ R)/0.8 \qquad 3.44$$

From this equation and the knowledge of the wake bubble length, the distance behind the cylinder at which asymptotic conditions are valid can be estimated. Obviously this approximation will be very crude, because an asymptotic solution for $R \rightarrow \infty$ has been used, whereas the calculations are done for a comparatively small Reynolds number.

In figure (2) the estimates for this distance have been plotted. There are plotted two curves calculated with equation (3.44).

The first curve follows from substituting the actual drag coefficient found by Dennis and Chang (2) in (3.44). The calculation of the second curve uses the drag coefficient consistent with Sychyev's model (3.43).

From the figure can be seen that there is a big difference between the two curves. Therefore it must be concluded that the position of the outer boundary will be quite arbitrarily within these specified bounds.

IV. NUMERICAL ERROR EVALUATION.

As has been described in chapter II, three different iteration schemes are involved in the there-discussed numerical solution of a flow around a cylinder: the iterative solution of the algebraic equations, which follow from the finite difference approximation to the ordinary differential equations; the determination of the boundary condition at the artificial boundary far from the cylinder; Richardson's extrapolation. The error introduced by each of these iteration procedures will be discussed in this chapter.

First a comparison will be made between the different iteration schemes of chapter II for solving the non linear algebraic equations. The number of iterations which are needed to find a convergent solution of the algebraic system, will be considered as a function of the Reynolds numbers. This number therefore does not include the iterations which must be done in order to find the outer boundary conditions and to do Richardson's extrapolation. A comparison with the iteration schemes used by other authors is not possible because none of them except Keller-Takami (8) give their number of iterations. The iteration schemes in table 1 are numbered in the same order as they are discussed in chapter II.

From this table it is clear that though method I seems to be an improvement on the results of Keller-Takami, a rapid increase in the number of iterations can be expected for the larger Reynolds numbers. For instance, the calculation using method I at Reynolds number $R = 3.5$ was terminated after about 120 iterations without having reached any indication that the iterations did converge. In order to check for this case in what way a solution of the flow field had been obtained, the drag coefficient was calculated along circular contours of different radii enclosing the cylinder (see Appendix II). This drag coefficient, which theoretically must have a constant value for all contours, fluctuated very violently especially far from the cylinder. This shows that no solution of the Navier-Stokes equations had been obtained. Also in method I the relaxation parameters Fk and Vk had to be chosen very small in order not to obtain a rapidly divergent solution. Because of these properties method I proves not to be very efficient for the higher Reynolds numbers.

Method II, which includes some of the non linear terms in the numerical solution, is a considerable improvement on method I. However the number of iterations increases with the Reynolds number; especially for Reynolds numbers $R > 3$ the number of iterations increased very rapidly, whereas the optimal value of the parameter Vk decreased very rapidly.

Method III works very well as far as Newton's part of the iteration scheme is concerned. The problem now is the convergence of the inner iteration scheme. For the higher Reynolds numbers the inner iteration did converge, but far too slowly. Because of the fact that the inner iteration was terminated after 20 iterations, the Newton equations were not completely solved and therefore the fast quadratic convergence was not always observed.

Therefore the only way to get Newton's method to work was to solve the Newton equation directly and this was done in method IV. This method works extremely well at all Reynolds numbers and therefore is a great improvement on all the other methods, because it reduces the number of iterations. A disadvantage however is the fact that the order of the blocks in the matrix, which follows from the finite difference approximation, increases linearly with the number of terms in the Fourier series. As is shown in Appendix I, the number of operations for this method is proportional to the cubic power of the number of Fourier terms. Therefore the computer time increases very rapidly when more terms in the Fourier series are taken into account. Nevertheless this method is superior to all the other methods presented in this report, because of the presence of quadratic convergence. The calculation of the data for the four Reynolds numbers discussed in this report is done using this method.

Next the errors in the numerical scheme will be discussed. There are five different error sources present:

the error in solving the nonlinear algebraic difference equations iteratively; the truncation error in approximating the differential equation by finite difference equations; the error introduced by the fact that the boundary conditions on the artificial boundary far from the cylinder are asymptotic conditions and depend on the drag coefficient; the error due to the finite number of terms in the Fourier series; the error following from round off in the calculations.

The first error due to the iterative solution of the algebraic non linear equations can be controlled very well, because of the quadratic convergence properties of Newton's method for solving these equations. A solution of the difference equations is said to have converged when the following conditions are satisfied. The difference between two drag coefficients calculated on the cylinder for two consecutive iterations must be smaller than 10^{-5}. At the same time the condition must be satisfied that when the solution is resubstituted into the difference equations the maximum absolute value of the residual at all net points and for all harmonics is less than 10^{-3}. These conditions are most of the time satisfied after 3-4 iterations for all Reynolds numbers.

The truncation error in approximating the differential equations by finite difference equations can be controlled by doing a Richardson extrapolation, described in Appendix I. This procedure reduces the truncation error to about $O(h^4)$, where h is the radial stepsize. Taking into account that in these calculations $h \sim 0.1$, the truncation error will be $\sim 10^{-4}$. The truncation error and the iteration error therefore will be of about the same order of magnitude. It is assumed that both these errors are so small that the other sources contribute the largest error, which cannot be estimated so easily.

The drag coefficient, which is used in the outer boundary conditions, is a functional of the solution and therefore the outer boundary conditions can only be calculated iteratively. First an estimate for the drag coefficient is used in evaluating the boundary conditions at the outer boundary. Then two Newton iterations are done to find a new estimate for this drag coefficient. This procedure is repeated until the new calculated drag coefficient differs only slightly in second decimal place from the former drag coefficient. The application of Newton's method is then continued until a convergent solution satisfying the conditions specified above is found. A more elaborate outer iteration is not done because of the negligible effect that a small change in the outer boundary has on the solution near the cylinder. At the same time, because the calculated drag coefficient as a function of the radial distance oscillates slightly, especially far from the cylinder, it is not clear which drag coefficient must be used in evaluating the outer boundary conditions. In most cases this outer iteration procedure does not take longer than 2-3 steps consisting of two Newton iterates.

The error involved in imposing asymptotic boundary conditions at a finite boundary is very difficult to determine. One of the main problems is that it is not known how the solution tends to its asymptotic form. In chapter III it has been tried to give some estimates of the distances at which asymptotic conditions can be applied. These estimates however are very crude, because of the many assumptions that had to be made in order to find them. Apart from Oseen's solution and the free stream conditions which must be considered inferior as follows from the discussion in chapter III, no other alternatives for the outer boundary conditions are given by the other investigators, who have treated the numerical solution of the flow around a cylinder. In most of the work the error, that is introduced by applying the asymptotic boundary conditions at a finite distance, is only considered by performing several calculations with the outer boundary conditions imposed on different distances from the cylinder. In this report the calculations are performed using asymptotic boundary conditions at only one fixed distance, which is assumed to be large enough for asymptotic conditions to be valid. It is believed, however, that apart from possible inconsistencies in the boundary conditions themselves, a consistent solution has been obtained, because of the fact that the calculated drag coefficient is reasonably constant throughout the whole flow field and matches the drag coefficient used in evaluating Oseen's boundary conditions.

The error made by the truncation of the Fourier series after a fixed number of terms is very difficult to treat, because there is no expression for the remainder term in the Fourier series.

Therefore an estimate of this error can only be obtained by performing the solution for different numbers of terms in the Fourier series. It has been found that five terms in the Fourier series are sufficient only for very small Reynolds numbers. For the higher Reynolds numbers substantial differences are found between the solution with five terms and the solution with 10 terms. This is expressed in this report by the results presented for R = 2.0, R = 3.5, and R = 5.0. This contradicts the conclusions made by Underwood (3). The results indicate that with increasing Reynolds numbers the number of terms in the Fourier series and therefore also the computer time must increase. This is largely due to the fact that the representation of the vorticity in a finite Fourier series is very poor, because the vorticity is everywhere almost zero except in a small region near the centerline behind the cylinder. It must be concluded therefore that the main error source in these calculations probably will be the truncation of the Fourier series.

The errors due to round off are related to the computer that is used for the calculations. The results in this report are calculated in single precision on the IBM 370/150 of the California Institute of Technology. This computer has a mantissa of 7 figures in single precision, so that at most 7 significant digits can be expected. However the round off error usually accumulates so fast that only about 4 digits or less can be expected to be significant depending on the number of operations carried out. In the case of the Richardson extrapolation for R = 5, quadratic convergence could not be found, most probably because of round off errors. However the solution did converge linearly and took about 8 iterations. A way to decrease the round off errors is to apply the method of iterative improvement on the solution of the linear system, which follows from the application of Newton's method. This method of iterative improvement is described by G. Forsythe and C.B. Moler (9). Recent results using this method seem to indicate that the effects of round off errors can be considerably reduced this way.

V. RESULTS AND ASSESSMENT.

In this report the flows for the following Reynolds numbers have been calculated: R = 0.5, 2.0, 3.5 and 5.0. In each of these calculations the outer boundary was set at $\alpha_m = \pi$ and the stepsize h in the radial direction was chosen to be $h = \pi/30$. In table 2 the number of iterations are given for each Reynolds number to find a convergent solution. These numbers include the iterations that must be done to determine the outer boundary condition.

The amount of computer time used in these calculations is of the order of 10 seconds for one iteration for the solution using a Fourier series of five terms. Because the number of operations is proportional to the cubic power of the number of Fourier terms, the computing time for one iteration using a Fourier series of ten terms took about 80 seconds. So it is clear that computer time increases very rapidly with the number of Fourier terms. It is difficult to compare these computer times with the results of other authors, because we could not find any statement in their work about the amount of computer time they used. However it must be taken into account that the computer time depends very much on the type of computer used. In our case a comparatively slow computer was used.

Nevertheless we believe that the calculations presented here are more efficient than those carried out by the other investigators for the same Reynolds numbers. The main reason for this is the observation of quadratic convergence in the solution of the difference equations. This has considerably reduced the number of iterations needed and also insures that a convergent solution was actually occurring. From the physical point of view it is clear that the drag coefficient must be a constant throughout the whole flow field. Therefore a good check to insure whether the numerical solution is actually a solution of the Navier-Stokes equations consists of calculating the drag coefficient along different contours enclosing the cylinder (for details see Appendix II).

In the tables 3-6 the drag coefficient is tabulated as a function of the radial distance from the cylinder. Also a comparison is made with results of Keller-Takami and Dennis-Chang. From these tables it follows that the drag coefficient is very satisfactorily constant throughout the whole flow field, especially for the solution calculated with Richardson extrapolation. It is interesting to note that a smaller number of terms in the Fourier series decreases the drag coefficient. The solution, however, retains the property that the drag coefficient is kept constant throughout the flow field. The meaning of this effect is not quite clear.

In figure 3 the vorticity distribution on the cylinder is plotted for various Reynolds numbers and compared to the results of Keller-Takami and Dennis-Chang. The comparison is reasonably good. The influence of the number of Fourier terms is seen to be very strong, especially for R = 5.0.

In figure 4 the pressure coefficient along the cylinder surface is plotted for different Reynolds numbers together with some results of Keller-Takami and Dennis-Chang. The comparison with their results is not as good for this case. However it must be noted that there exist also some differences between the results of both authors. It also follows from this figure that the pressure coefficient depends very much on the number of terms in the Fourier series. From these results can be concluded that it is very difficult to find accurate values for the pressure coefficient.

In figure 5 the velocity on the centerline behind the cylinder is shown together with some results of Keller-Takami. The agreement is not very good, especially at a larger distance from the cylinder.

In the figures 6 and 7 the pressure coefficient on the centerline is plotted. These figures also show clearly the large influence of the number of Fourier terms.

In the figures 8-23 contour plots are given of the streamfunction, the vorticity, the pressure and the velocity. In order to make a pressure plot the pressure coefficient has to be calculated throughout the whole flow field. Details of this calculation are given in Appendix III.

The pressure coefficient is calculated along a contour connecting infinity with the point where the pressure coefficient is to be calculated. The use of different contours then must lead to the same result. This enables us to do another check on the accuracy of the numerical solution. The pressure coefficient at the base of the cylinder is calculated twice using different integration contours. The first contour starts at infinity and proceeds along the centerline before the cylinder and the cylinder surface to the base of the cylinder. The second contour starts at infinity and proceeds along the centerline behind the cylinder to the base of the cylinder. For the latter case the result must be corrected, because the Oseen boundary conditions are applied at a finite distance from the cylinder (see Appendix III).

For the Reynolds numbers $R = 0.5$ and 2.0 both pressure coefficients on the base of the cylinder, which were calculated using the different contours, were equal within a few percent. For the higher Reynolds numbers some difference between the two pressure coefficients was found. This may be due to the finite Fourier series representation of the solution and to the application of asymptotic conditions at a finite distance from the cylinder.

From the results presented in this report can be concluded that a very accurate numerical solution of the flow around a cylinder has been obtained for the Reynolds numbers R = 0.5 and R = 2.0. The solutions for the higher Reynolds numbers, especially for R = 5.0 cannot be considered to be of the same order of accuracy. The main reason for this seems to be the influence of the finite number of Fourier terms used to represent the solution. A solution involving more Fourier terms, however, has not been attempted because of the large amount of computer time involved. A solution for higher Reynolds numbers using only ten terms in the Fourier series has not been considered because of the probably very doubtful results that can be expected. This would be in contrast with the goal of this report to find an efficient and accurate calculation scheme.

As far as the results presented in the literature are concerned, in my opinion accurate solutions exist only for $R \leq 10$. The solution presented for the higher Reynolds numbers is not of the same order of accuracy as can be obtained for the solution of the small Reynolds numbers. The calculation schemes for this problem must be much more thoroughly developed before accurate results for the higher Reynolds numbers can be expected. In my opinion the misunderstanding concerning the accuracy arises largely from the many authors, a few excepted, who do not discuss all error sources properly and call a numerical solution too easily "exact".

Where the method presented in this report is concerned, all effort must be directed towards improving the representation of the solution by the finite Fourier series. Not until then can an accurate solution for higher Reynolds numbers be expected.

APPENDIX I.

Consider the solution of the following first order system of differential equations

$$\underset{\sim}{F}' = A\,\underset{\sim}{F} + \underset{\sim}{S} \qquad \text{I.1}$$

$$\underset{\sim}{F} = \begin{bmatrix} u \\ v \\ f \\ g \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & n^2 & -H \\ t_1 & t_2 & t_3 & t_4 + n^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \underset{\sim}{S} = \begin{bmatrix} 0 \\ r \\ 0 \\ 0 \end{bmatrix}$$

It must be noted that the matrix A and the vector S are known.

This system must be solved on the interval $0<\alpha<\alpha_m$ subjected to the following boundary conditions

$$B^\circ \underset{\sim}{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \alpha = 0$$

$$B' \underset{\sim}{F} = \begin{bmatrix} Rf \\ Rg \end{bmatrix} \qquad \alpha = \alpha_m \qquad \text{I.2}$$

$$B^\circ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Divide the interval $[0,\alpha_m]$ in J steps of magnitude h. Then a finite difference approximation to the equation (I.1) on this grid is given by

$$(\underline{F}_j - \underline{F}_{j-1})/ h = A_{j-1/2} (\underline{F}_j + \underline{F}_{j-1})/2 + \underline{S}_{j-1/2} \qquad \text{I.3}$$

$$j=1,\ldots..J$$

The subscript j-½ means that the matrix A and the vector S are evaluated at the argument between the grid points j and j-1.

Rewriting equation (I.3) we find

$$-L_j \underline{F}_{j-1} + R_j \underline{F}_j = h \underline{S}_{j-1/2} \qquad \text{I.4}$$

$$R_j = I - h/2 \; A_{j-1/2}$$

$$L_j = I + h/2 \; A_{j-1/2}$$

Together with the boundary conditions (I.2) the equation (I.4) can be written in block matrix form of the order 4J*4J

$$\begin{bmatrix} B^o & & & \\ -L_1 & R_1 & & \\ & \ddots & \ddots & \\ & & -L_J & R_J \\ & & & B' \end{bmatrix} \begin{bmatrix} \underline{F}_o \\ \underline{F}_1 \\ \vdots \\ \\ \underline{F}_J \end{bmatrix} = \begin{bmatrix} 0 \\ h\underline{S}_{1/2} \\ \vdots \\ h\underline{S}_{J-1/2} \\ R_f \\ R_J \end{bmatrix} \qquad T\,\bar{\underline{F}} = \bar{\underline{S}} \qquad \text{I.5}$$

The matrix in equation (I.5) can be written in block tridiagonal form

$$T = \begin{bmatrix} A_0 & C_0 & & \\ B_1 & \ddots & \ddots & \\ & \ddots & \ddots & C_{J-1} \\ & & B_J & A_J \end{bmatrix} \qquad \text{I.6}$$

where the blocks B_i ,A_i ,C_i are of the order 4*4

This special form makes a very efficient LU-decomposition of the matrix T possible

$$\begin{bmatrix} A_0 & C_0 & & \\ B_1 & \ddots & \ddots & \\ & \ddots & \ddots & C_{J-1} \\ & & B_J & A_J \end{bmatrix} = \begin{bmatrix} X_0 & & & \\ B_1 & \ddots & & \\ & \ddots & \ddots & \\ & & B_J & X_J \end{bmatrix} \begin{bmatrix} I & Y_0 & & \\ & \ddots & \ddots & \\ & & \ddots & Y_{J-1} \\ & & & I \end{bmatrix}$$

$$T = L\ U \qquad \text{I.7}$$

$$X_0 = A_0 , \quad X_0 Y_0 = C_0$$

$$B_i Y_{i-1} + X_i = A_i , \ X_i Y_i = C_i \qquad 1 \leqslant i \leqslant J-1$$

$$B_J Y_{J-1} + X_J = A_J$$

A necessary condition for this factorization to exist and to be unique is the fact that X_i must be non singular.

The solution of equation (I.5) can now be given in two steps

$$L \bar{Z} = \bar{S} \qquad \text{forward substitution} \qquad \text{I.8}$$

$U \bar{F} = \bar{Z}$ backward substitution

So the problem has been reduced to finding a LU-decomposition for the matrix T. Because all the blocks in T are sparse, some simplification can be expected in calculating the LU-decomposition. As a matter of fact it must be clear that the first two rows of each C_i and the last two rows of each B_i must contain only zero elements. Then it is not difficult to show that X_i must be of the following form

$$X_j = \begin{bmatrix} x_j & y_j & z_j & w_j \\ k_j & l_j & m_j & n_j \\ -1 & 0 & n^2 h/2 & h/2\ H\ (j+\frac{1}{2}) \\ -\frac{h}{2}t_1(j+\frac{1}{2}) & -1-\frac{h}{2}t_2(j+\frac{1}{2}) & -\frac{h}{2}t_3(j+\frac{1}{2}) & -\frac{n^2h}{2}-\frac{h}{2}t_4(j+\frac{1}{2}) \end{bmatrix}$$

$$0 < j < J-1$$

$$X_J = \begin{bmatrix} x_J & y_J & z_J & w_J \\ k_J & l_J & m_J & n_J \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{I.9}$$

So at each step of the LU-decomposition only the eight quantities in the first two rows of X_j must be calculated, instead of sixteen quantities for an ordinary 4*4 matrix.

Assume that the inverse of X_j has the following form

$$X_j^{-1} = \begin{bmatrix} Q_{11}(j) & Q_{12}(j) & Q_{13}(j) & Q_{14}(j) \\ Q_{21}(j) & Q_{22}(j) & Q_{23}(j) & Q_{24}(j) \\ Q_{31}(j) & Q_{32}(j) & Q_{33}(j) & Q_{34}(j) \\ Q_{41}(j) & Q_{42}(j) & Q_{43}(j) & Q_{44}(j) \end{bmatrix} \qquad \text{I.10}$$

Then the recursion relations for the x_j, y_j, z_j, w_j, k_j, l_j, m_j, n_j are

$$x_0 = 1 \quad y_0 = 0 \quad z_0 = 0 \quad w_0 = 0$$

$$k_0 = 0 \quad l_0 = 0 \quad m_0 = 1 \quad n_0 = 0$$

$$\begin{aligned}
x_j &= -a + \{Q_{33}(j-1) + a\, Q_{13}(j-1)\} + \\
&\quad \{Q_{34}(j-1) + a\, Q_{14}(j-1)\}\, t_1(j-\tfrac{1}{2})\, h/2 \\
y_j &= \{Q_{34}(j-1) + a\, Q_{14}(j-1)\}\{1 - h/2\; t_2(j-\tfrac{1}{2})\} \\
z_j &= 1 - \frac{n^2 h}{2}\{Q_{33}(j-1) + a\, Q_{13}(j-1)\} - \qquad \text{I.11} \\
&\quad h/2\; t_3(j-\tfrac{1}{2})\{Q_{34}(j-1) + a\, Q_{14}(j-1)\} \\
w_j &= h/2\; H(j-\tfrac{1}{2})\{Q_{33}(j-1) + a\, Q_{13}(j-1)\} - \\
&\quad \{\frac{n^2 h}{2} + h/2\; t_4(j-\tfrac{1}{2})\}\{Q_{34}(j-1) + a\, Q_{14}(j-1)\} \\
k_j &= \{Q_{43}(j-1) + a\, Q_{23}(j-1)\} - \\
&\quad h/2\; t_1(j-\tfrac{1}{2})\{Q_{44}(j-1) + a\, Q_{24}(j-1)\} \\
l_j &= -a + \{Q_{44}(j-1) + a\, Q_{24}(j-1)\}\{1 - h/2\; t_2(j-\tfrac{1}{2})\} \\
m_j &= -\frac{n^2 h}{2}\{Q_{43}(j-1) + a\, Q_{23}(j-1)\} - \\
&\quad h/2\; t_3(j-\tfrac{1}{2})\{Q_{44}(j-1) + a\, Q_{24}(j-1)\} \\
n_j &= 1 + h/2\; H(j-\tfrac{1}{2})\{Q_{43}(j-1) + a\, Q_{23}(j-1)\} -
\end{aligned}$$

$$\left\{\frac{n^2 h}{2} + h/2\ t_4(j-\tfrac{1}{2})\right\}\left\{ Q_{44}(j-1) + a\ Q_{24}(j-1)\right\}$$

where $a = h/2 \quad 1 \leqslant j \leqslant J$

It must be noted that a very efficient calculation is possible if the LU-decomposition and the forward substitution are performed simultaneously.

The recursion formulas for the forward substitution are

$$\xi_j = Q_{11}(j)\ (\ a\xi_{j-1} + \zeta_{j-1}) + Q_{12}(j)\ (\ a\eta_{j-1} + \theta_{j-1}) + Q_{14}(j)\ h\ r_{j+1/2}$$

$$\eta_j = Q_{21}(j)\ (\ a\xi_{j-1} + \zeta_{j-1}) + Q_{22}(j)\ (\ a\eta_{j-1} + \theta_{j-1}) + Q_{24}(j)\ h\ r_{j+1/2}$$

$$\zeta_j = Q_{31}(j)\ (\ a\xi_{j-1} + \zeta_{j-1}) + Q_{32}(j)\ (\ a\eta_{j-1} + \theta_{j-1}) + Q_{34}(j)\ h\ r_{j+1/2}$$

$$\theta_j = Q_{41}(j)\ (\ a\xi_{j-1} + \zeta_{j-1}) + Q_{42}(j)\ (\ a\eta_{j-1} + \theta_{j-1}) + Q_{44}(j)\ h\ r_{j+1/2}$$

where $a = h/2 \quad 0 \leqslant j \leqslant J-1$ I.12

A reduction in storage can be obtained because for the backward substitution only the Y_i must be known. These blocks can be calculated from the last two columns of X_i^{-1}. So only these columns have to be stored.

The recursion relation for the backward substitution then becomes.

$$
\begin{aligned}
u_j = \xi_j - \Big[& \{Q_{13}(j) - h/2\ t_1(j+\tfrac{1}{2})\ Q_{14}(j)\}\ u_{j+1} - \\
& \{1 - h/2\ t_2(j+\tfrac{1}{2})\}\ Q_{14}(j)\ v_{j+1} + \\
& \{\tfrac{n^2 h}{2}\ Q_{13}(j) + h/2\ t_3(j+\tfrac{1}{2})\ Q_{14}(j)\}\ f_{j+1} + \\
& \{-h/2\ H(j+\tfrac{1}{2})\ Q_{13}(j) + [\tfrac{n^2 h}{2} + h/2\ t_4(j+\tfrac{1}{2})]\ Q_{14}(j)\}\ g_{j+1} \Big] \\
v_j = \eta_j - \Big[& \{Q_{23}(j) - h/2\ t_1(j+\tfrac{1}{2})\ Q_{24}(j)\}\ u_{j+1} - \\
& \{1 - h/2\ t_2(j+\tfrac{1}{2})\}\ Q_{24}(j)\ v_{j+1} + \\
& \{\tfrac{n^2 h}{2}\ Q_{23}(j) + h/2\ t_3(j+\tfrac{1}{2})\ Q_{24}(j)\}\ f_{j+1} + \\
& \{-h/2\ H(j+\tfrac{1}{2})\ Q_{23}(j) + [\tfrac{n^2 h}{2} + h/2\ t_4(j+\tfrac{1}{2})]\ Q_{24}(j)\}\ g_{j+1} \Big] \\
f_j = \zeta_j - \Big[& \{Q_{33}(j) - h/2\ t_1(j+\tfrac{1}{2})\ Q_{34}(j)\}\ u_{j+1} - \\
& \{1 - h/2\ t_2(j+\tfrac{1}{2})\}\ Q_{34}(j)\ v_{j+1} + \qquad\qquad \text{I.13} \\
& \{\tfrac{n^2 h}{2}\ Q_{33}(j) + h/2\ t_3(j+\tfrac{1}{2})\ Q_{34}(j)\}\ f_{j+1} + \\
& \{-h/2\ H(j+\tfrac{1}{2})\ Q_{33}(j) + [\tfrac{n^2 h}{2} + h/2\ t_4(j+\tfrac{1}{2})]\ Q_{34}(j)\}\ g_{j+1} \Big] \\
g_j = \theta_j - \Big[& \{Q_{43}(j) - h/2\ t_1(j+\tfrac{1}{2})\ Q_{44}(j)\}\ u_{j+1} - \\
& \{1 - h/2\ t_2(j+\tfrac{1}{2})\}\ Q_{44}(j)\ v_{j+1} + \\
& \{\tfrac{n^2 h}{2}\ Q_{43}(j) + h/2\ t_3(j+\tfrac{1}{2})\ Q_{44}(j)\}\ f_{j+1} + \\
& \{-h/2\ H(j+\tfrac{1}{2})\ Q_{43}(j) + [\tfrac{n^2 h}{2} + h/2\ t_4(j+\tfrac{1}{2})]\ Q_{44}(j)\}\ g_{j+1} \Big]
\end{aligned}
$$

$$0 \leqslant j \leqslant J-1$$

Next we consider the accuracy of the solution of the finite difference scheme. The existence, uniqueness and stability of the solution of the finite difference equations, which are an approximation to the differential equation (I.1), can be proved when the matrix T is non singular and the inverse of T is bounded. (See Keller (10)). For this case we suppose that the matrix T is sufficiently well behaved inside the region of integration.

An estimate of the truncation error of the finite difference scheme then can be made as follows.

Define the difference between the exact solution of equation (I.1): $\underset{\sim}{F}^*$ and the solution of the difference equations (I.3): $\underset{\sim}{F}_j$ as

$$\underset{\sim}{E}_j = \underset{\sim}{F}_j - \underset{\sim}{F}^*_j \qquad 0 \leqslant j \leqslant J \qquad \text{I.14}$$

Application of the finite difference equations (I.3) on the $\underset{\sim}{E}_j$ yields

$$\begin{aligned} &(\underset{\sim}{E}_j - \underset{\sim}{E}_{j-1})/h - A_{j-1/2}(\underset{\sim}{E}_j + \underset{\sim}{E}_{j-1})/2 = \\ &(\underset{\sim}{F}^*_j - \underset{\sim}{F}^*_{j-1})/h - A_{j-1/2}(\underset{\sim}{F}^*_j + \underset{\sim}{F}^*_{j+1})/2 \; + \\ &\underset{\sim}{F}'^*_{j-1/2} - A \;\; \underset{\sim}{F}^*_{j-1/2} \qquad 1 \leqslant j \leqslant J \end{aligned} \qquad \text{I.15}$$

A Taylor expansion of F* around the point $\alpha_{j-1/2}$ yields

$$\begin{aligned} &\underset{\sim}{F}'^*_{j-1/2} - (\underset{\sim}{F}^*_j - \underset{\sim}{F}^*_{j-1})/h = -1/24 \; h^2 \; \underset{\sim}{F}'''^*_{j-1/2} + O(h^4) \\ &\underset{\sim}{F}^*_{j-1/2} - (\underset{\sim}{F}^*_j + \underset{\sim}{F}^*_{j-1})/2 = +1/8 \;\; h^2 \; \underset{\sim}{F}''^*_{j-1/2} + O(h^4) \end{aligned} \qquad \text{I.16}$$

Substitution into (I.15) yields

$$T \, \underset{\sim}{\tilde{E}} = -1/24 \; h^2 \; \underset{\sim}{\tilde{F}}'''{*} + \;\; 1/8 \; h^2 \; \underset{\sim}{\tilde{F}}''{*} + O(h^4)$$

$$\bar{E} = \begin{bmatrix} E_0 \\ | \\ | \\ E_J \end{bmatrix} \quad \bar{F}^{m} = \begin{bmatrix} 0 \\ 0 \\ h\ F^{m}_{1/2} \\ | \\ h\ F^{m}_{J-1/2} \\ 0 \\ 0 \end{bmatrix} \quad \bar{F}^{I} = \begin{bmatrix} 0 \\ 0 \\ h\ F^{I}_{1/2} \\ | \\ h\ F^{I}_{J-1/2} \\ 0 \\ 0 \end{bmatrix} \qquad \text{I.17}$$

So the truncation error $\|E\|$ is $O(h^2)$ whenever the matrix T is not too ill conditioned.

The accuracy as far as the truncation error is concerned can be improved considerably by doing a Richardson extrapolation. The following assumption then must be made. There exists a function $e(\alpha)$ independent of the grid size h such that the error (I.14) can be written as (see Keller (10))

$$E_j = h^2\, e(\alpha_j) + O(h^4) \qquad \text{I.18}$$

Then in practice Richardson extrapolation is done the following way. First solve the finite difference equations to obtain the solution on the original grid: F_j. Next do the same calculation on a smaller grid (for instance half the original grid size) to obtain: F_{2j}. Then equation (I.18) implies

$$\begin{aligned} F^*_j - F_j &= h^2\, e(\alpha_j) + O(h^4) \\ F^*_j - F_{2j} &= h^2/4\, e(\alpha_j) + O(h^4) \end{aligned} \qquad \text{I.19}$$

Elimination yields

$$\underset{\sim}{F}^*_j - \{4\ \underset{\sim}{F}_{2j}(\alpha_j) - \underset{\sim}{F}_j(\alpha_j)\}/3 = O(h^4) \qquad \text{I.20}$$

So in two steps a very good approximation to the solution of the differential equation has been obtained.

It must be noted that the error terms are also dependent on the higher derivatives of $\underset{\sim}{F}^*$. For this case it follows that these derivatives are strongly dependent on the harmonic variable n. Therefore for a large n (a series with many terms) an accurate solution can only be expected when the product nh is small.

Finally some remarks will be made on the solution of the total Newton iteration given by the equation (2.16). It is clear that this equation is written in the form of equation (I.1). The order of the matrix A however, will be 4N*4N, where N is the number of harmonics in the Fourier series.

The solution of this equation can be found by the same methods described for equation (I.1) in the beginning of this appendix. The differential equation must be written in finite difference form. The resulting matrix equation (I.5) with blocks of the order 4N*4N then can be solved by making the matrix block tridiagonal.

The solution of this block tridiagonal system is almost identical to the solution described before. The only differnce lies in the fact that the order of the blocks X_i will be considerably larger then 4. So it is no longer possible to calculate the inverse of X_i manually. Therefore the equations involving the X_i must be solved by Gaussian elimination. Taking this fact into account the solution proceeds further completely identically.

Because the order of the blocks may become very large an estimate will be made of the number of operations involved.

At each radial net point a LU-decomposition of the matrix X_i must be made. Using the operational count for Gaussian elimination, the number of operations for this part of the program becomes: $\sim 1/3\ (4N)^3 J$. Further in order to find the matrix Y_i on the J-1 net points a matrix multiplication must be done. This accounts for $(4N)^3(J-1)$ operations. So the total number of operations will be approximately proportional to

$$\text{ops.} \sim 256/3\ N^3 J \qquad \text{I.21}$$

It is clear that for increasing N the number of operations will increase very rapidly.

APPENDIX II

The drag coefficient is given by the generalized Blasius formula (Imai (4))

$$Cd = -\text{Real}\Big\{ 2i \oint (\partial\psi/\partial z)^2 \, dz + i\oint \zeta \bar{z} \, d\psi + 2/R \oint \partial\zeta/\partial\bar{z} \; \bar{z} \, d\bar{z} \Big\} \qquad \text{II.1}$$

The streamfunction and the vorticity in this formula are real functions of the two complex variables z and $\bar{z}$. The integrals are line integrals along a closed contour enclosing the cylinder.

Transform equation (I.1) to polar coordinates using the following formulas

$$z = r\,e^{i\theta} \qquad \bar{z} = r\,e^{-i\theta}$$

$$\partial f/\partial z = \partial f/\partial r \;\partial r/\partial z + \partial f/\partial\theta \;\partial\theta/\partial z = 1/2\,(\partial f/\partial r - i\,1/r\;\partial f/\partial\theta)\,e^{-i\theta}$$

$$\partial f/\partial\bar{z} = \partial f/\partial r \;\partial r/\partial\bar{z} + \partial f/\partial\theta \;\partial\theta/\partial\bar{z} = 1/2\,(\partial f/\partial r + i\,1/r\;\partial f/\partial\theta)\,e^{i\theta} \qquad \text{II.2}$$

Then the drag coefficient evaluated by integrating along a circular contour of radius r becomes using the fact that ψ and ζ are even functions of β

$$Cd = 1/r\int_0^{\pi} \Big[\big\{(\partial\psi/\partial\alpha)^2 - (\partial\psi/\partial\beta)^2\big\}\cos\beta \;\; 2\,\partial\psi/\partial\alpha\;\partial\psi/\partial\beta\;\sin\beta\Big]\,d\beta \; -$$

$$2\ r \int_0^{\pi} \zeta\, \partial\psi/\partial\beta\ \sin\beta\ d\beta\ + \qquad \text{II.3}$$
$$2/R\ r \int_0^{\pi} (\partial\zeta/\partial\alpha\ -\ \zeta\ )\ \sin\beta\ d\beta$$
$$\text{where } r = e^{\alpha}\ , \quad \beta = \theta$$

The streamfunction, the vorticity and their derivatives with respect to α are expanded in a Fourier sine series

$$\psi = \sum_1^{\infty} f_n \sin n\beta \quad \partial\psi/\partial\alpha = \sum_1^{\infty} u_n \sin n\beta \qquad \text{II.4}$$
$$\zeta = \sum_1^{\infty} g_n \sin n\beta \quad \partial\zeta/\partial\alpha = \sum_1^{\infty} v_n \sin n\beta$$

Substitution of these series into equation (II.3) yields the following sum for the drag coefficient

$$Cd = \pi/2 \Big\{ 1/r \sum_{p=1}^{\infty} \big[u_{p+1} u_p - (p+1)p\ f_p f_{p+1} - \qquad \text{II.5}$$
$$u_{p+1}\ p\ f_p + u_p\ (p+1)\ f_{p+1}) \big] -$$
$$r \sum_{p=1}^{\infty} \big[g_{p+1}\ p\ f_p - g_p\ (p+1)\ f_{p+1} \big] + 2/R\ r\ \big[v_1 - g_1 \big] \Big\}$$

The evaluation of this sum at each α-coordinate then yields the drag coefficient calculated along a circular contour through this coordinate point and enclosing the cylinder.

APPENDIX III

The Navier-Stokes equations in dimensionless form are given by

$$\text{div}\ \underline{q} = 0$$
$$(\ \text{grad}\ \underline{q}\ )\ \underline{q} = -\ \text{grad}\ p + 1/R\ \text{div def}\ \underline{q} \qquad \text{III.1}$$

After the use of some vector identities these equations become

$$\text{div}\ \underline{q} = 0 \qquad \text{III.2}$$
$$\text{grad}\ \underline{q}.\underline{q}/2 + (\ \text{curl}\ \underline{q}\ ) * \underline{q} = -\text{grad}\ p + 1/R\ \text{curl curl}\ \underline{q}$$

Because only a two dimensional flow is considered, the vorticity will only have a component in the z-direction. The continuity equation can be integrated by introducing a streamfunction. Therefore we define

$$\zeta\ \underset{\sim}{k} = \text{curl}\ \underset{\sim}{q}$$
$$\underset{\sim}{q} = \text{grad}\ \psi * \underset{\sim}{k} \qquad \text{III.3}$$

where $\underset{\sim}{k}$ is the unit vector in the z-direction.

The streamfunction will be uniquely determined if the condition is imposed that grad ψ must be perpendicular to $\underset{\sim}{k}$. Using these definitions the equation (III.2) can be written as

$$1/2 \text{ grad } (\text{grad}\,\psi \cdot \text{grad}\,\psi) + \xi \text{ grad}\,\psi = - \text{grad } p - 1/R \text{ curl } (\xi \underset{\sim}{k}) \qquad \text{III.4}$$

From this equation the pressure can be calculated as a function of the streamfunction and vorticity.

To be able to perform the integration we must write equation (III.4) in its component form in the (α, β) coordinates. Therefore we consider the coordinate transformation given by (1.7)

$$\begin{aligned} x &= e^{\alpha} \cos \beta \\ y &= e^{\alpha} \sin \beta \\ z &= z \end{aligned} \qquad \text{III.5}$$

The scale factors of this transformation are given by

$$\begin{aligned} h_1 &= |\partial \underset{\sim}{r} / \partial \alpha| = e^{\alpha} \\ h_2 &= |\partial \underset{\sim}{r} / \partial \beta| = e^{\alpha} \\ h_3 &= |\partial \underset{\sim}{r} / \partial z| = 1 \end{aligned} \qquad \text{III.6}$$

where $\underset{\sim}{r} = x\,\underset{\sim}{i} + y\,\underset{\sim}{j} + z\,\underset{\sim}{k}$

Then the components of the grad and curl in this coordinate system are

$$\text{grad}\,\psi = (1/h_1\, \partial\psi/\partial\alpha,\ 1/h_2\, \partial\psi/\partial\beta,\ 1/h_3\, \partial\psi/\partial z) = (e^{-\alpha} \partial\psi/\partial\alpha,\ e^{-\alpha} \partial\psi/\partial\beta,\ 0)$$

$$\text{curl}\ \zeta\ \underset{\sim}{k} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \underset{\sim}{e}_\alpha & h_2 \underset{\sim}{e}_\beta & h_3 \underset{\sim}{e}_z \\ \partial/\partial\alpha & \partial/\partial\beta & \partial/\partial z \\ 0 & 0 & \zeta \end{vmatrix} = \qquad \text{III.7}$$

$$(e^{-\alpha}\partial\zeta/\partial\beta, -e^{-\alpha}\partial\zeta/\partial\alpha , 0)$$

Substituting these results into equations (III.4) equations for the partial derivatives of the pressure can be found

$$\partial p/\partial\alpha = -1/R\ \partial\zeta/\partial\beta - \zeta\partial\psi/\partial\alpha - 1/2\ \partial/\partial\alpha \left\{ (\partial\psi/\partial\alpha)^2 + (\partial\psi/\partial\beta)^2 \right\} e^{-2\alpha}$$

$$\partial p/\partial\beta = 1/R\ \partial\zeta/\partial\alpha - \zeta\partial\psi/\partial\beta - 1/2\ \partial/\partial\beta \left\{ (\partial\psi/\partial\alpha)^2 + (\partial\psi/\partial\beta)^2 \right\} e^{-2\alpha} \qquad \text{III.8}$$

Integration of the first equation along a line β= const. and integration of the second equation along a line α = const. yields

$$Cp\ (\alpha) = 1 - e^{-2\alpha} \left\{ (\partial\psi/\partial\alpha)^2 + (\partial\psi/\partial\beta)^2 \right\}\Big|_{\alpha} + 2\int_{\alpha}^{\infty} (1/R\ \partial\zeta/\partial\beta + \zeta\partial\psi/\partial\alpha)\, d\alpha \qquad \text{III.9}$$

$$Cp\ (\beta) = Cp\ (\beta_0) + e^{-2\alpha} \left\{ (\partial\psi/\partial\alpha)^2 + (\partial\psi/\partial\beta)^2 \right\}\Big|_{\beta}^{\beta_0} + 2\int_{\beta_0}^{\beta} (1/R\ \partial\zeta/\partial\alpha - \zeta\partial\psi/\partial\beta)\, d\beta$$

where $Cp = (p^* - p^*(\infty))/\ 1/2\, \rho U^2$

To calculate the pressure coefficient at each coordinate (α, β) we use the following procedure. Use the first formula of (III.9) along the line $\beta = \pi$ to find Cp (α, $\beta = \pi$). Then use the second formula along the line α is constant. The combination of both formulas then yields

$$Cp(\alpha,\beta) = 1 - e^{-2\alpha}\left\{(\partial\psi/\partial\alpha)^2 + (\partial\psi/\partial\beta)^2\right\}\Big|_{\alpha,\beta} + 2\int_{\alpha}^{\infty} 1/R\ \partial\zeta/\partial\beta\ d\alpha + 2\int_{\pi}^{\beta}\left(1/R\ \partial\zeta/\partial\alpha - \zeta\partial\psi/\partial\beta\right) d\beta \qquad \text{III.10}$$

Because of the fact that the numerical calculations are terminated at a finite distance α_m from the cylinder the integration in the α- direction along the line $\beta = \pi$ cannot be extended to infinity and must also be terminated at α_m. Therefore the following correction must be added to the pressure coefficient calculated from equation (III.10)

$$2/R \int_{\alpha_m}^{\infty} \partial\zeta/\partial\beta\ d\alpha \qquad \text{along } \beta = \pi \qquad \text{III.11}$$

In order to calculate this correction we assume that for $\alpha \geqslant \alpha_m$ the asymptotic solution determined in chapter III is valid. Substituting equation (3.34) for the vorticity into the integral (III.11) it is found that the correction term is equal to zero, which is no coincidence because the Oseen solution is constructed this way. This is true only if we do the integration along the line $\beta = \pi$.

The calculation of the correction term along the line $\beta = 0$ for instance yields the following result

$$2/R \int_{\alpha_m}^{\infty} \partial\xi/\partial\beta \, d\alpha = - Cd \sqrt{R/\pi} \; e^{-\alpha_m/2} \quad \text{for} \quad \beta = 0 \qquad \text{III.12}$$

Therefore it is clear that in order to find the pressure coefficient in the whole flow the integration in the α-direction is done along the line $\beta = \pi$, because for that case the pressure coefficient will not depend on the asymptotic conditions or on the distance at which they are applied.

The streamfunction and the vorticity and their derivatives with respect to α are expanded in a Fourier sine series (see (II.4)).

Substitution of these series into equation (III.10) yields the following expression for the pressure coefficient

$$\begin{aligned} Cp = {} & 1 - e^{-2\alpha}\left\{ \left(\sum_1^{\infty} u_n \sin n\beta \right)^2 + \left(\sum_1^{\infty} f_n n \cos n\beta \right)^2 \right\} \\ & + 2/R \sum_1^{\infty} (-1)^n \, n \int_{\alpha}^{\infty} g_n(\alpha) \, d\alpha \\ & + 2/R \sum_1^{\infty} v_n \left\{ (-1)^n - \cos n\beta \right\}/n \\ & - 2\sum_1^{\infty}\sum_1^{\infty} \; n \, g_k f_n \, 1/2 \left[\left\{ (-1)^{k+n} - \cos (k+n)\beta \right\}/(k+n) \right. \\ & \left. + \left\{ (-1)^{n-k} - \cos (n-k)\beta \right\} /(k-n) \right] \end{aligned} \qquad \text{III.13}$$

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Table 1: Number of iterations for the different iteration schemes of chapter II

R	Keller - Takami	I	II	III inner	III outer	IV
0.5		25				
1.0	1000	100	25	4	10	3
1.5			40			
2.0	700		70			3
2.5			80			
3.0			150	6	20	
3.5				6	20	3
5.0	500					4

Table 2: Number of iterations.

R	0.5	2.0	3.5	5.0
N= 5, -	6	4	7	8
N= 5, Rich.	3	3	3	3
N=10, -		5	4	8
N=10, Rich.		3	3	8

Rich. : Richardson's extrapolation
N : Number of Fourier terms

Table 3: Dragcoefficient Cd for R = 0.5

Keller-Takami Cd = 10.227

Dennis-Chang Cd = -

r	Cd N=5, -	Cd N=5, Rich.
1.000	10.262	10.313
1.110	10.261	10.313
1.233	10.260	10.313
1.369	10.259	10.313
1.520	10.258	10.313
1.688	10.257	10.313
1.874	10.255	10.313
2.081	10.254	10.313
2.311	10.253	10.313
2.566	10.252	10.313
2.850	10.251	10.313
3.164	10.250	10.313
3.514	10.248	10.313
3.901	10.247	10.313
4.332	10.246	10.313
4.810	10.245	10.313
5.342	10.244	10.313
5.931	10.243	10.313
6.586	10.243	10.313
7.313	10.242	10.313
8.121	10.242	10.313
9.017	10.242	10.313
10.012	10.242	10.313
11.117	10.242	10.313
12.345	10.242	10.313
13.708	10.242	10.313
15.221	10.242	10.313
16.901	10.241	10.313
18.768	10.242	10.313
20.839	10.244	10.313
23.140	10.255	10.313

Rich.: Richardson's extrapolation

N : Number of Fourier terms

Table 4: Dragcoefficient Cd for R = 2.0

Keller-Takami Cd = 4.438

Dennis-Chang Cd = -

r	Cd N=5, -	Cd N=5, Rich.	Cd N=10, -	Cd N=10, Rich
1.000	4.303	4.327	4.554	4.581
1.110	4.302	4.327	4.552	4.581
1.233	4.301	4.327	4.551	4.581
1.369	4.299	4.327	4.550	4.581
1.520	4.298	4.327	4.548	4.581
1.688	4.297	4.327	4.547	4.581
1.874	4.296	4.327	4.546	4.581
2.081	4.295	4.327	4.545	4.581
2.311	4.294	4.327	4.544	4.581
2.566	4.294	4.327	4.544	4.581
2.850	4.294	4.327	4.544	4.581
3.164	4.295	4.327	4.544	4.581
3.514	4.295	4.327	4.545	4.581
3.901	4.296	4.327	4.546	4.581
4.332	4.297	4.327	4.546	4.581
4.810	4.298	4.327	4.547	4.581
5.342	4.299	4.327	4.547	4.581
5.931	4.298	4.327	4.546	4.581
6.586	4.297	4.327	4.545	4.581
7.313	4.295	4.327	4.543	4.581
8.121	4.292	4.327	4.542	4.581
9.017	4.288	4.327	4.539	4.581
10.012	4.285	4.327	4.536	4.581
11.117	4.281	4.327	4.533	4.581
12.345	4.277	4.327	4.529	4.581
13.708	4.273	4.327	4.525	4.581
15.221	4.271	4.327	4.522	4.581
16.901	4.268	4.328	4.516	4.582
18.768	4.287	4.323	4.525	4.578
20.839	4.257	4.335	4.475	4.593
23.140	4.311	4.326	4.616	4.580

Rich.:Richardson's extrapolation

N : Number of Fourier terms

Table 5: Dragcoefficient Cd for R = 3.5

Keller-Takami Cd = 3.30

Dennis-Chang Cd = 3.42

r	Cd N=5, -	Cd N=5, Rich.	Cd N=10, -	Cd N=10, Rich.
1.000	2.903	2.919	3.394	3.413
1.110	2.901	2.919	3.391	3.413
1.233	2.899	2.919	3.387	3.413
1.369	2.897	2.919	3.385	3.413
1.520	2.896	2.919	3.383	3.413
1.688	2.894	2.919	3.381	3.413
1.874	2.894	2.919	3.380	3.413
2.081	2.894	2.919	3.380	3.413
2.311	2.895	2.919	3.381	3.413
2.566	2.896	2.919	3.382	3.413
2.850	2.897	2.919	3.383	3.413
3.164	2.899	2.919	3.384	3.413
3.514	2.900	2.919	3.385	3.413
3.901	2.901	2.919	3.385	3.413
4.332	2.901	2.919	3.384	3.413
4.810	2.900	2.919	3.383	3.413
5.342	2.898	2.919	3.381	3.413
5.931	2.895	2.919	3.379	3.413
6.586	2.892	2.919	3.376	3.413
7.313	2.889	2.919	3.374	3.413
8.121	2.886	2.919	3.370	3.413
9.017	2.882	2.919	3.367	3.413
10.012	2.879	2.919	3.363	3.413
11.117	2.876	2.919	3.358	3. 413
12.345	2.875	2.919	3.355	3.413
13.708	2.867	2.920	3.346	3.414
15.221	2.881	2.915	3.355	3.410
16.901	2.847	2.927	3.324	3.419
18.768	2.914	2.906	3.389	3.399
20.839	2.864	2.932	3.245	3.447
23.140	2.832	2.917	3.502	3.408

Rich. : Richardson's extrapolation

N : Number of Fourier terms

Table 6: Dragcoefficient Cd for R = 5.0

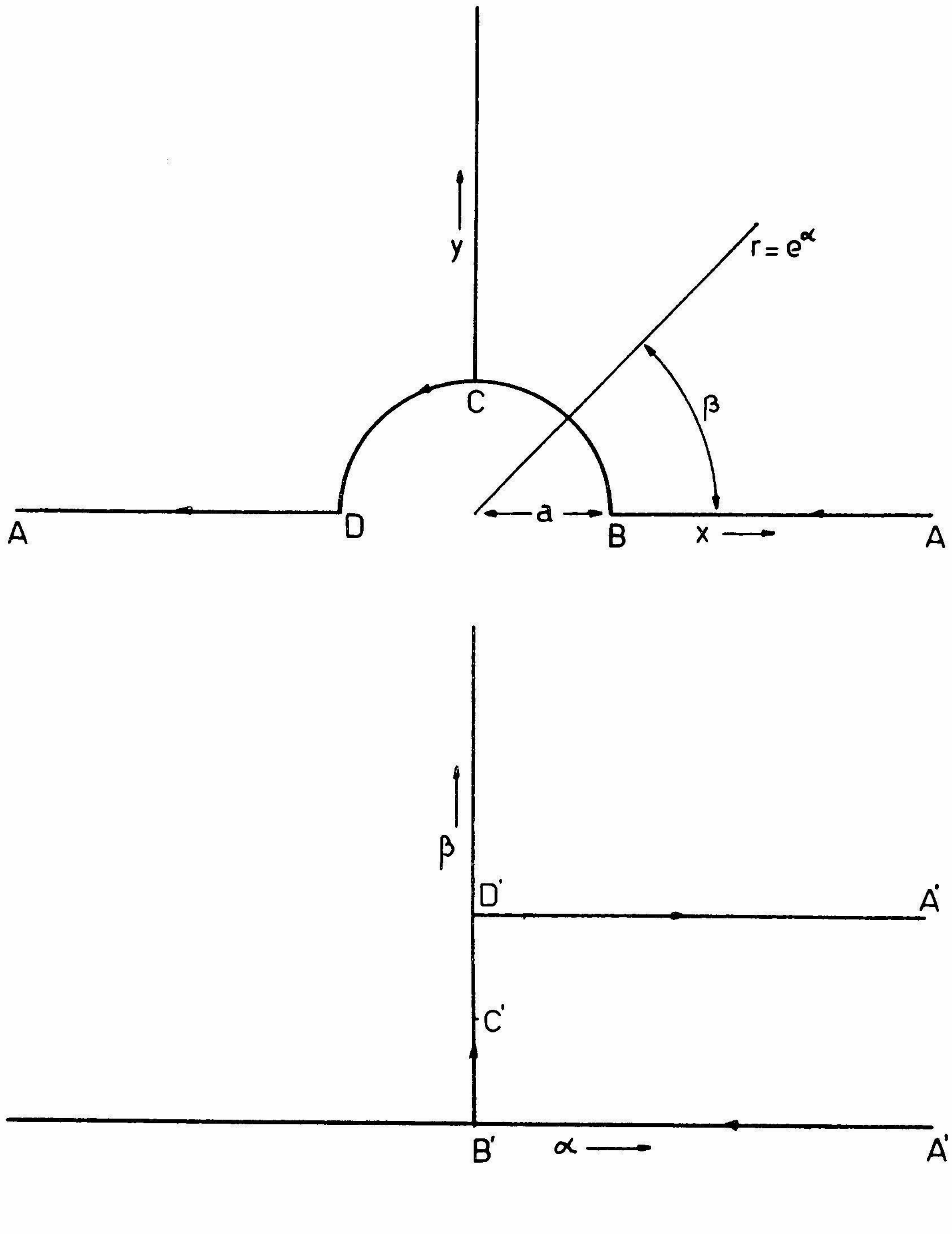
Keller-Takami Cd = 2.77
Dennis-Chang Cd = 2.85

r	Cd N=5, -	Cd N=5, Rich.	Cd N=10, -	Cd N=10, Rich.
1.000	2.221	2.231	2.882	2.895
1.110	2.217	2.231	2.877	2.895
1.233	2.215	2.231	2.872	2.895
1.369	2.213	2.231	2.868	2.895
1.520	2.211	2.231	2.865	2.895
1.688	2.211	2.231	2.864	2.895
1.874	2.211	2.231	2.864	2.895
2.081	2.212	2.231	2.865	2.895
2.311	2.213	2.231	2.867	2.895
2.566	2.215	2.231	2.869	2.895
2.850	2.217	2.231	2.870	2.895
3.164	2.219	2.231	2.871	2.895
3.514	2.220	2.231	2.870	2.895
3.901	2.218	2.231	2.869	2.895
4.332	2.217	2.231	2.867	2.895
4.810	2.214	2.231	2.864	2.895
5.342	2.212	2.231	2.862	2.895
5.931	2.209	2.231	2.859	2.895
6.586	2.206	2.231	2.856	2.895
7.313	2.203	2.231	2.853	2.895
8.121	2.201	2.231	2.848	2.895
9.017	2.198	2.231	2.844	2.895
10.012	2.197	2.231	2.840	2.895
11.117	2.189	2.232	2.831	2.897
12.345	2.200	2.228	2.834	2.894
13.708	2.171	2.237	2.811	2.900
15.221	2.223	2.220	2.850	2.887
16.901	2.140	2.248	2.776	2.912
18.768	2.247	2.216	2.908	2.870
20.839	2.220	2.237	2.671	2.955
23.140	2.042	2.232	3.023	2.885

Rich. : Richardson's extrapolation
N : Number of Fourier terms

Figure 1

Domain of integration



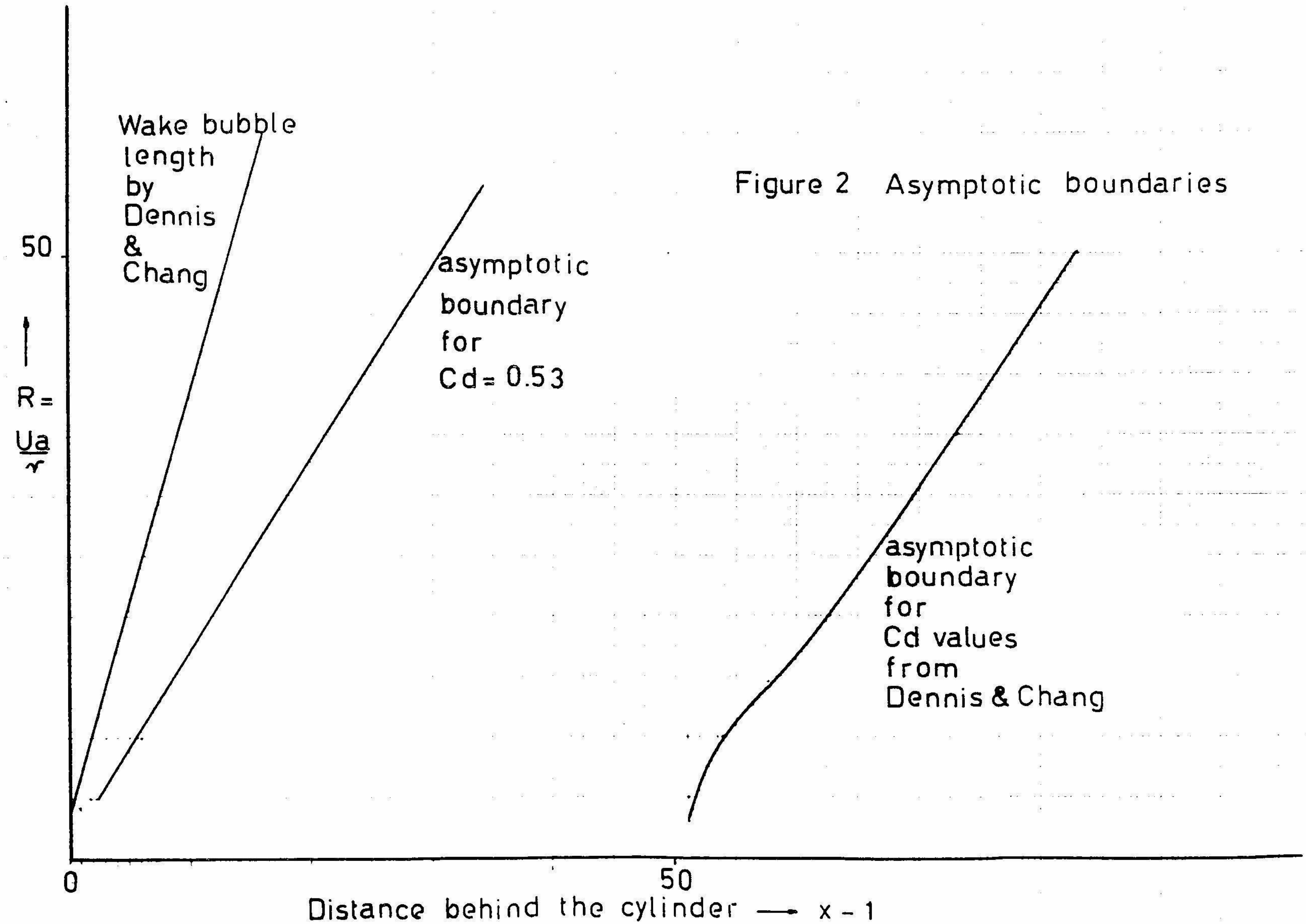
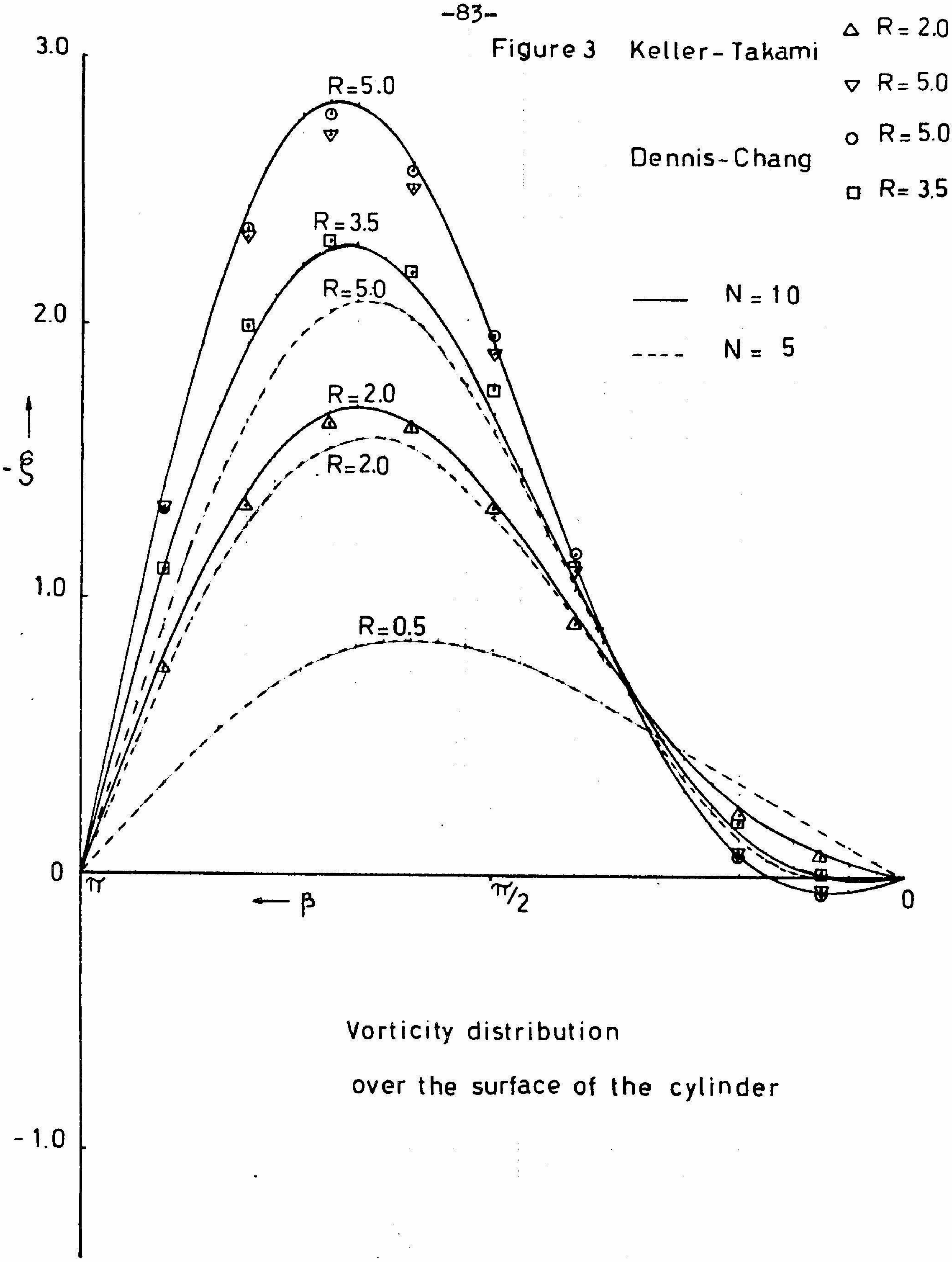


Figure 2 Asymptotic boundaries

Figure 3



Vorticity distribution over the surface of the cylinder

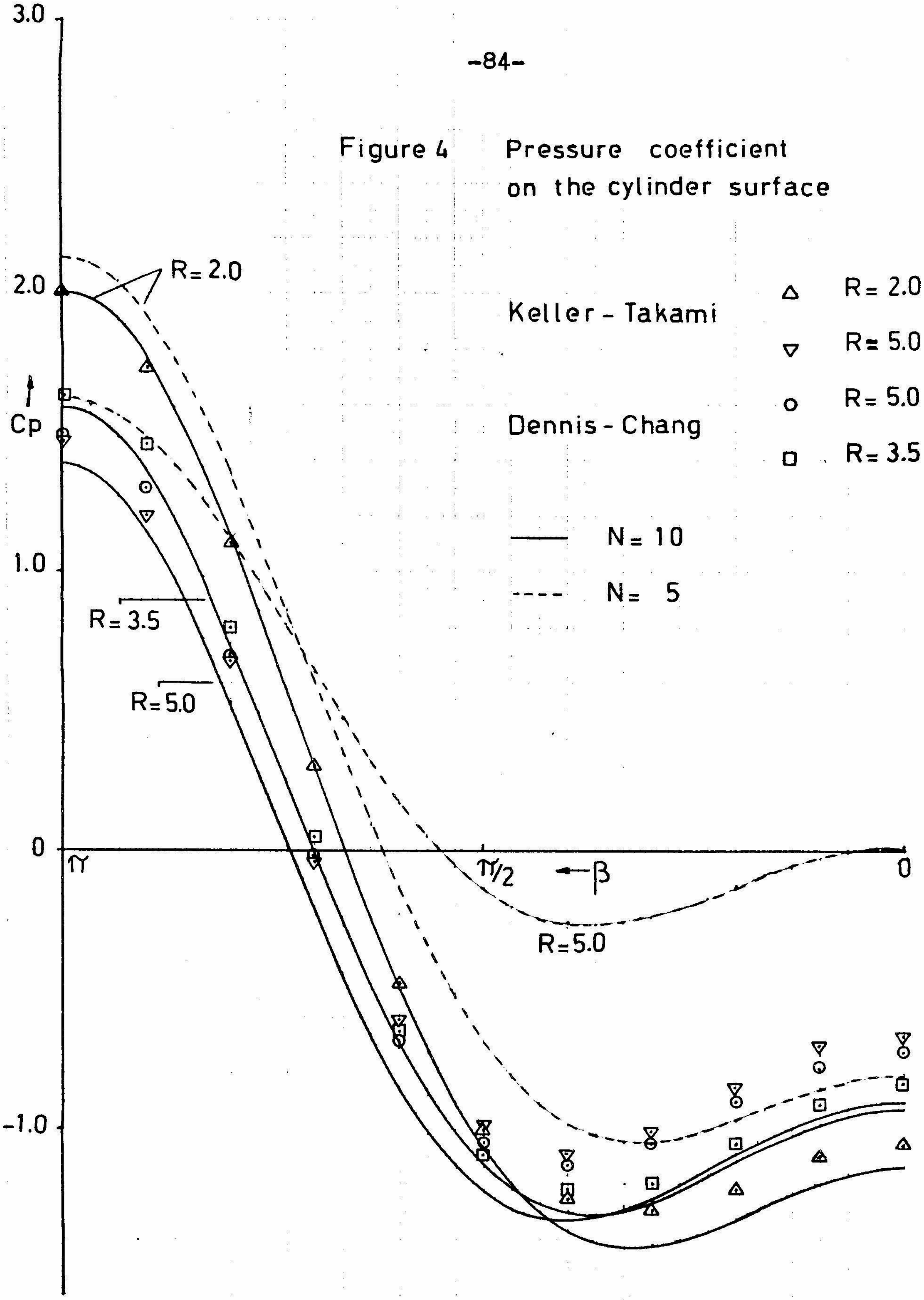


Figure 4 Pressure coefficient on the cylinder surface
3.0
2.0
1.0
0
-1.0
Cp
R= 2.0
R= 3.5
R= 5.0
R=5.0
π
π/2
β
0
Keller - Takami
Dennis - Chang
R= 2.0
R= 5.0
R= 5.0
R= 3.5
N= 10
N= 5

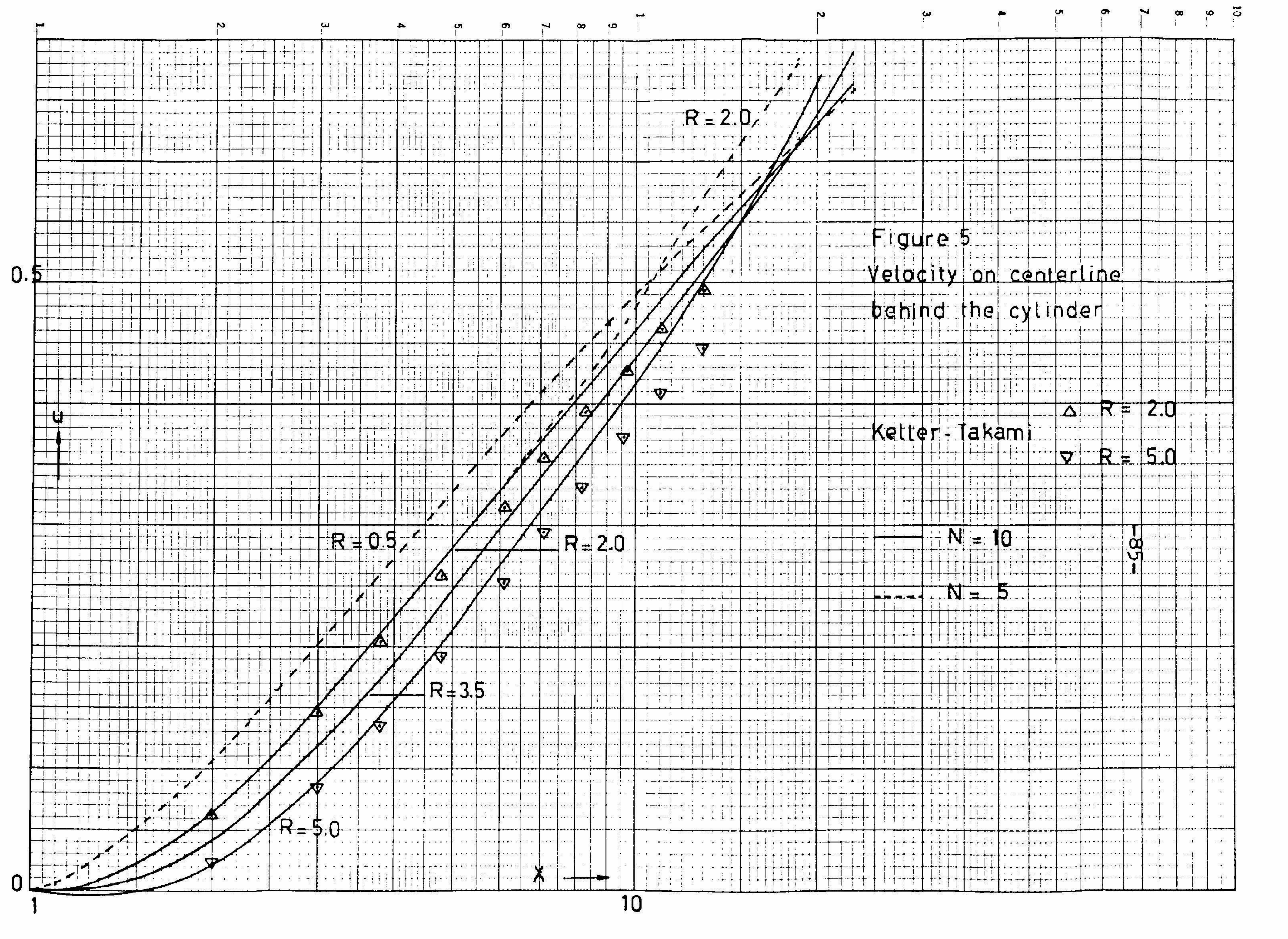


Figure 5
Velocity on centerline
behind the cylinder
Ketter - Takami
△ R = 2.0
▽ R = 5.0
N = 10
N = 5
R = 2.0
R = 0.5
R = 2.0
R = 3.5
R = 5.0
0.5
0
u
x
1
10
-85-

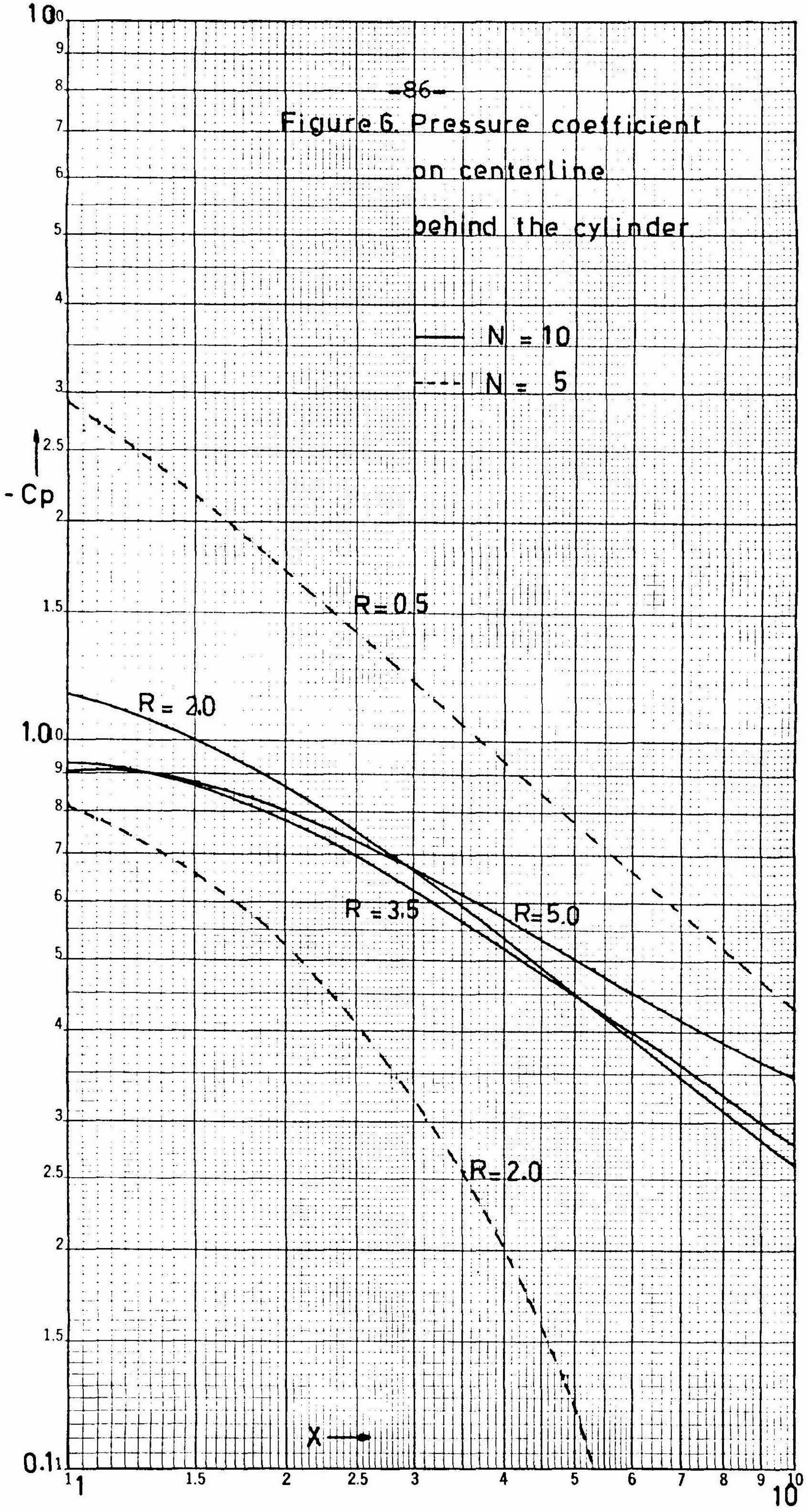


Figure 6. Pressure coefficient on centerline behind the cylinder

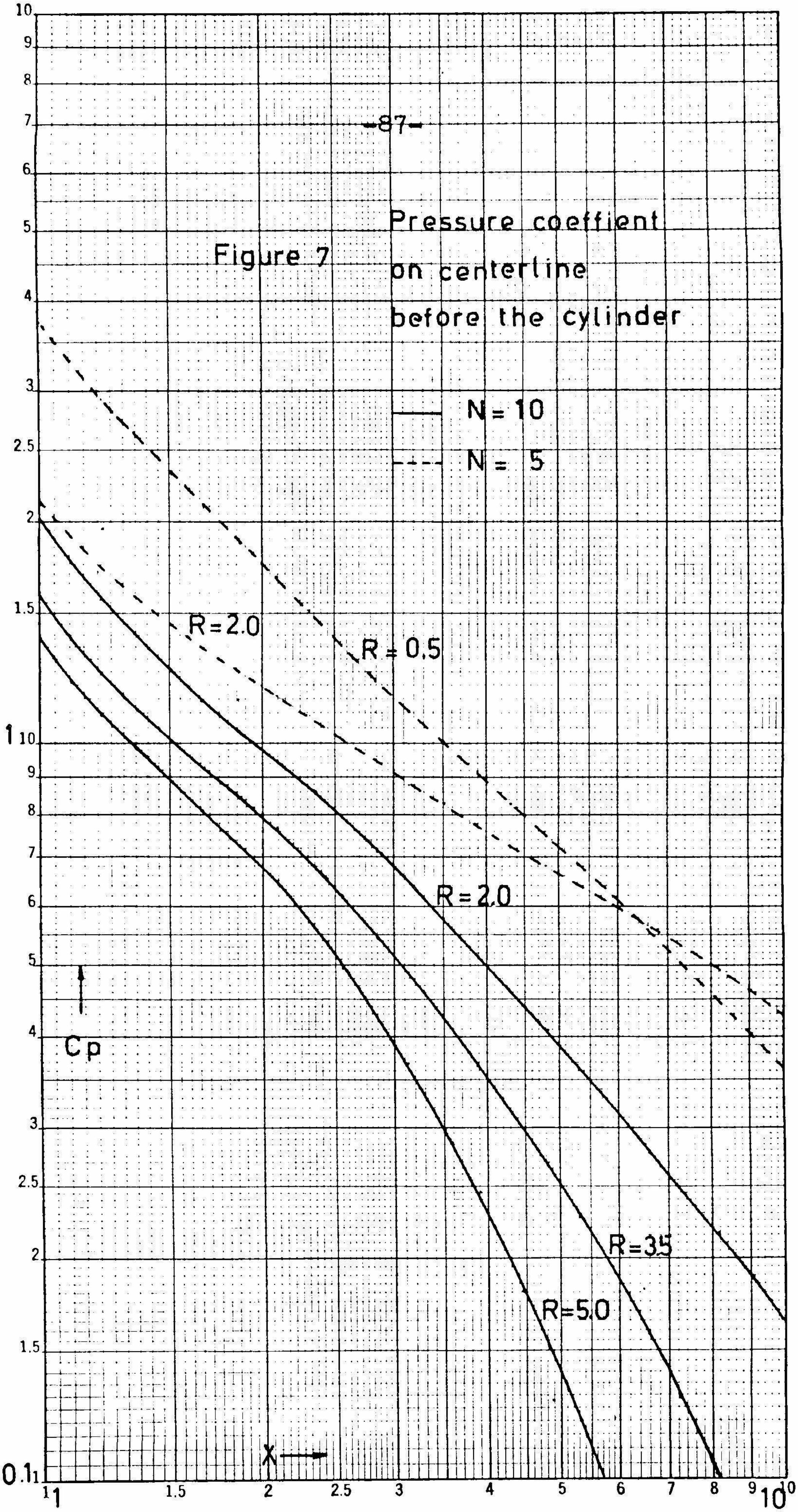
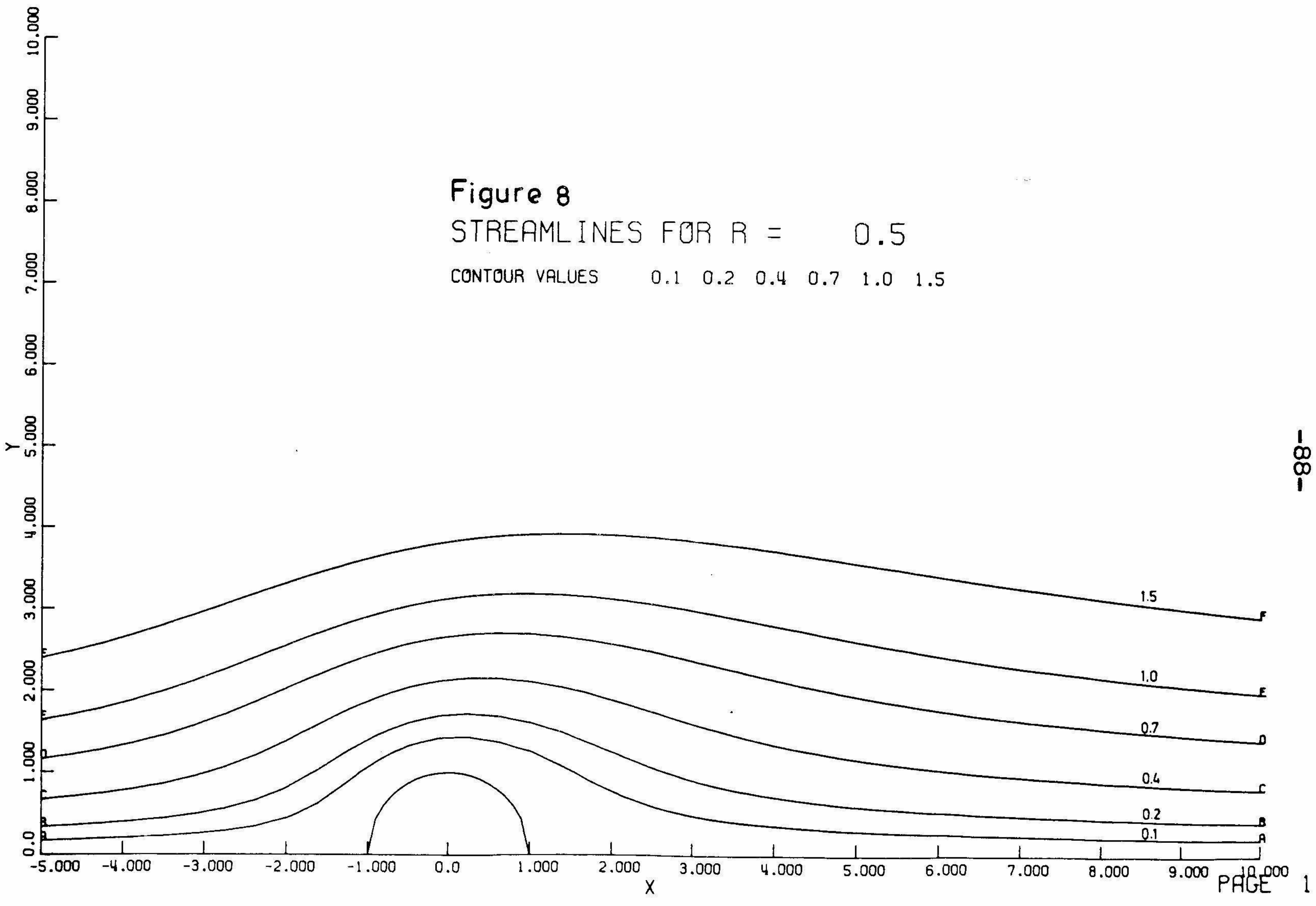


Figure 7
Pressure coeffient
on centerline
before the cylinder
N = 10
N = 5
R = 2.0
R = 0.5
R = 2.0
R = 3.5
R = 5.0
Cp
X
10
9
8
7
6
5
4
3
2.5
2
1.5
1
0.1
1.5
2
2.5
3
4
5
6
7
8
9
10

Figure 8

STREAMLINES FØR R = 0.5

CONTOUR VALUES 0.1 0.2 0.4 0.7 1.0 1.5



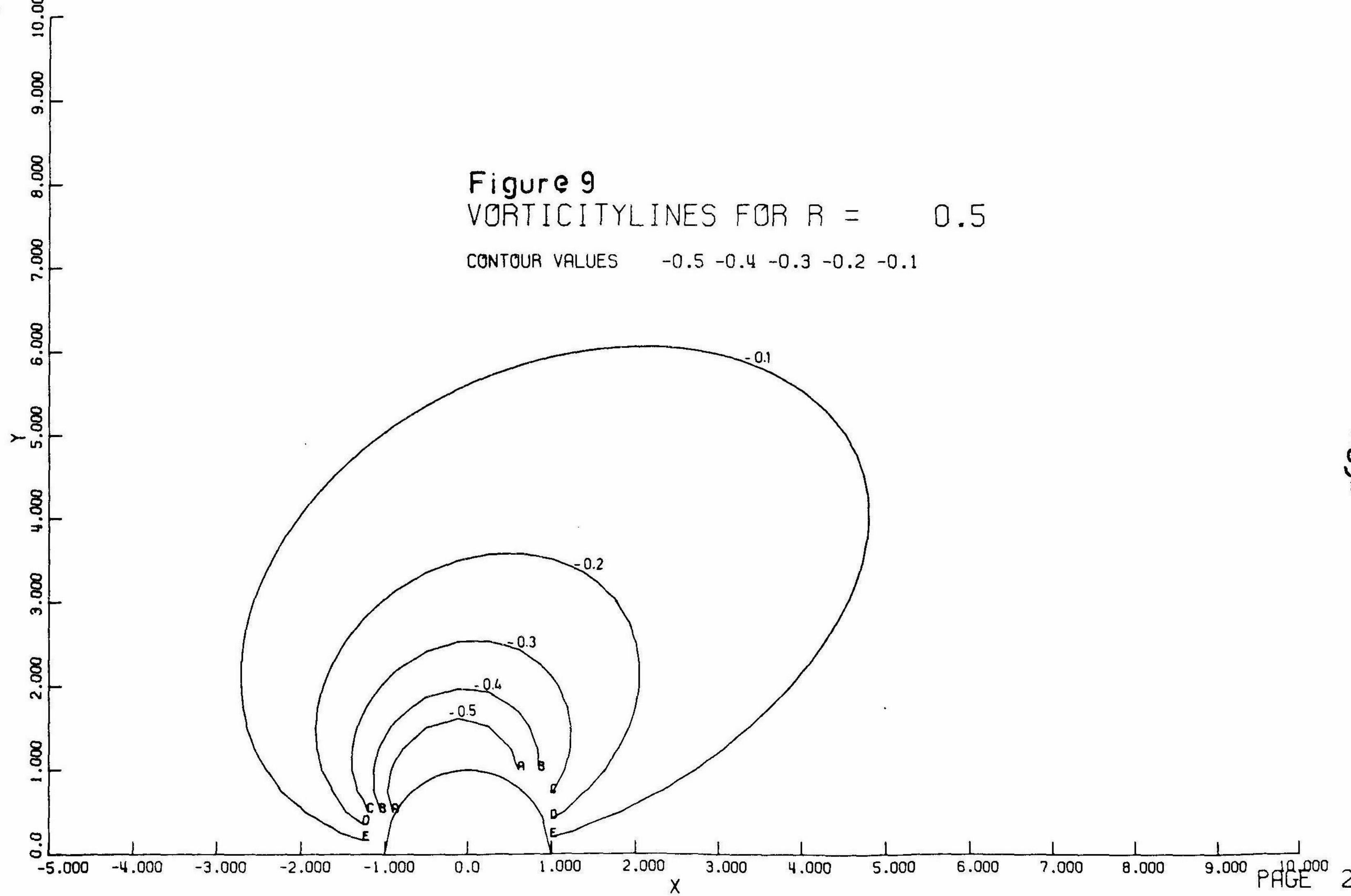


Figure 9
VORTICITYLINES FOR R = 0.5

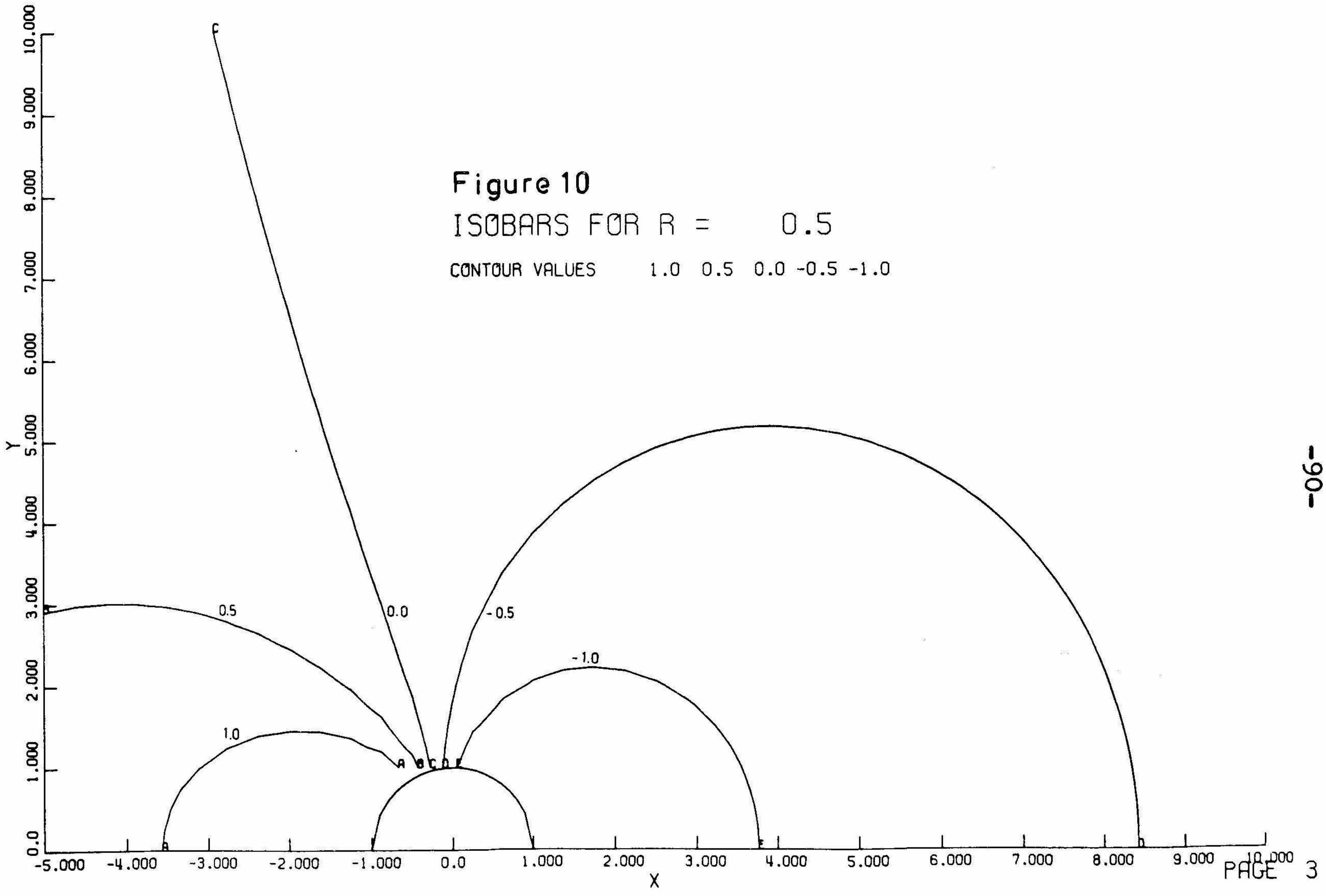


Figure 10
ISOBARS FOR R = 0.5
CONTOUR VALUES 1.0 0.5 0.0 -0.5 -1.0
0.5
1.0
0.0
-0.5
-1.0
Y
10.000
9.000
8.000
7.000
6.000
5.000
4.000
3.000
2.000
1.000
0.0
-5.000
-4.000
-3.000
-2.000
-1.000
0.0
1.000
2.000
3.000
4.000
5.000
6.000
7.000
8.000
9.000
10.000
X
PAGE 3

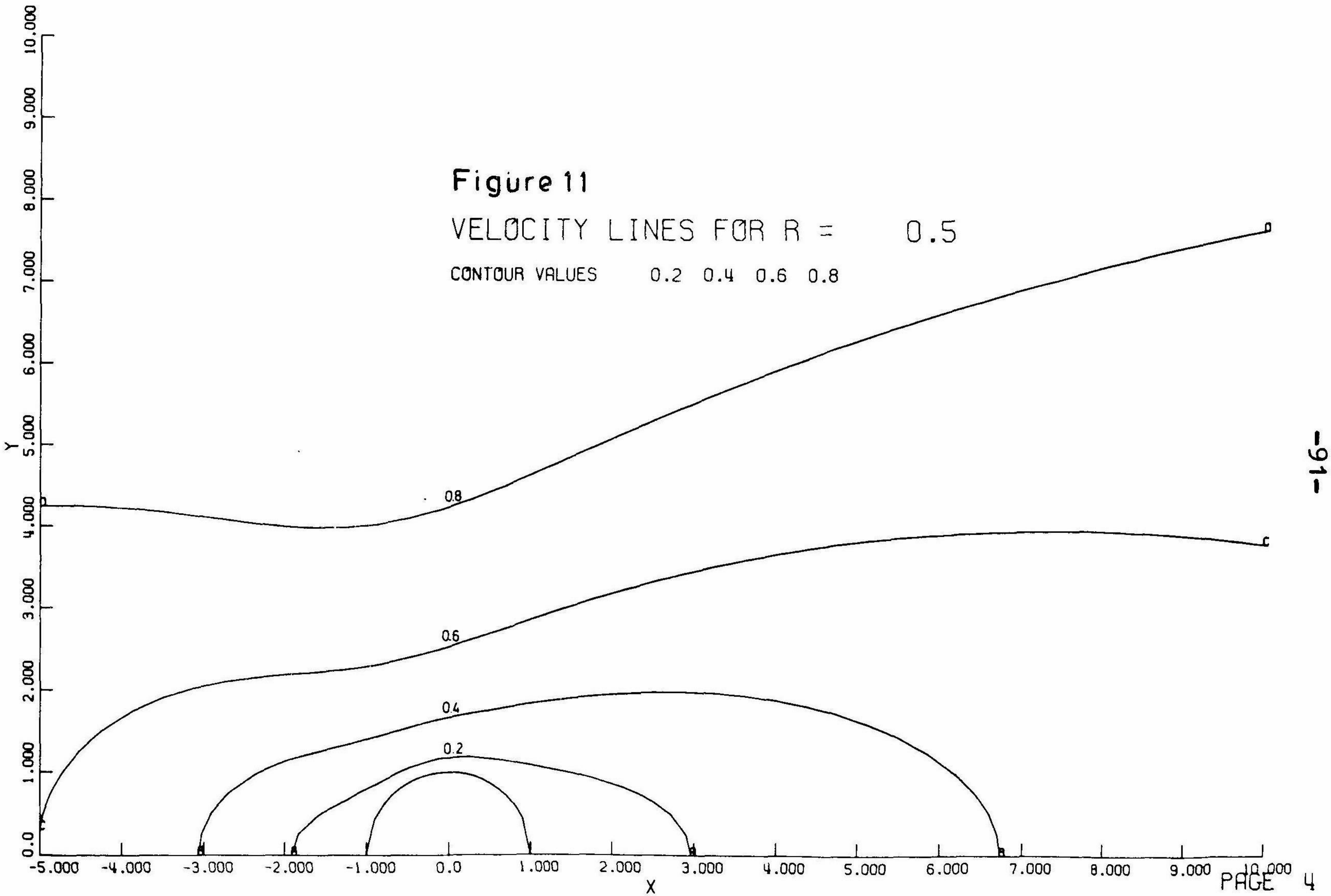


Figure 11

VELOCITY LINES FØR R = 0.5

CONTOUR VALUES 0.2 0.4 0.6 0.8

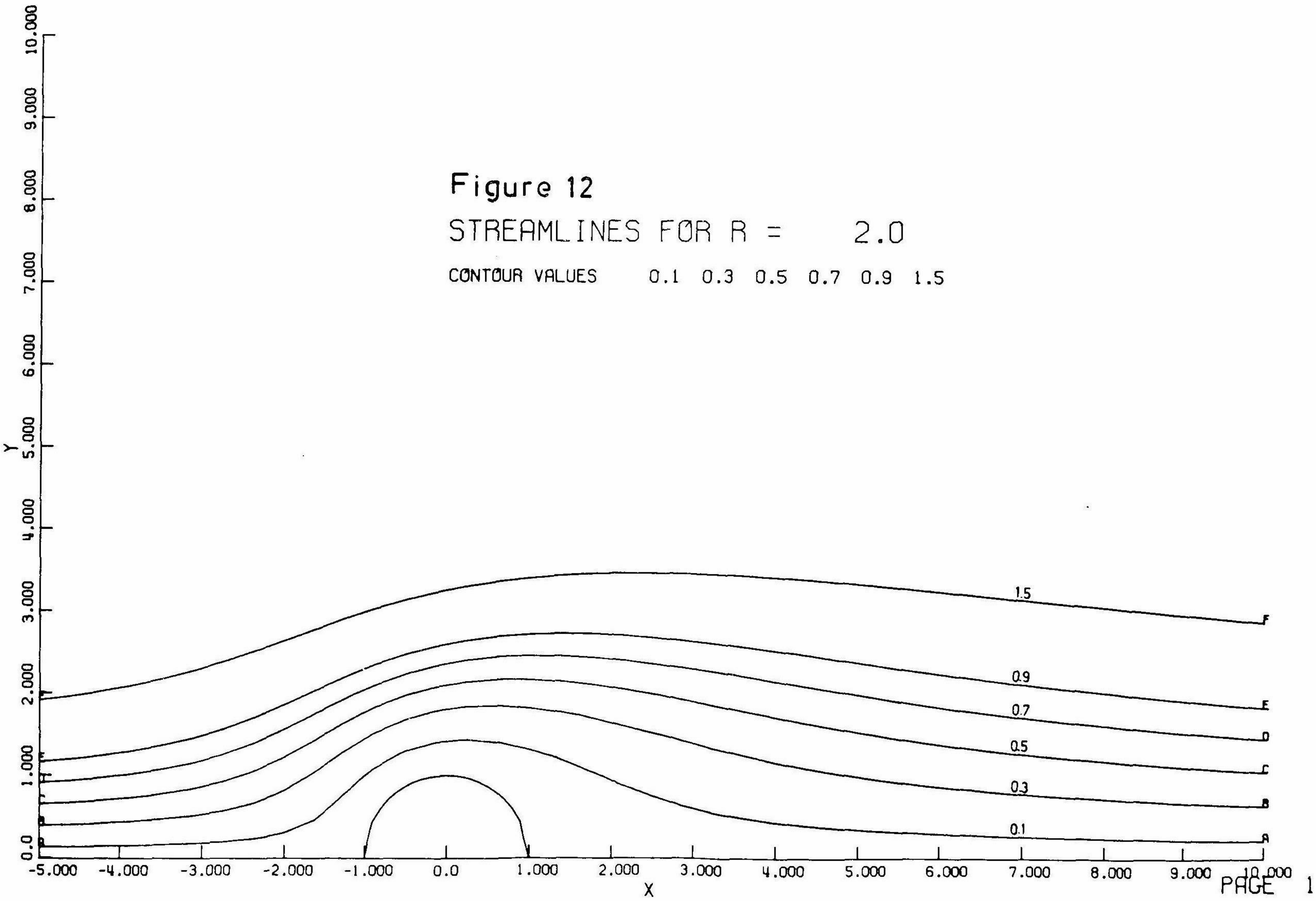
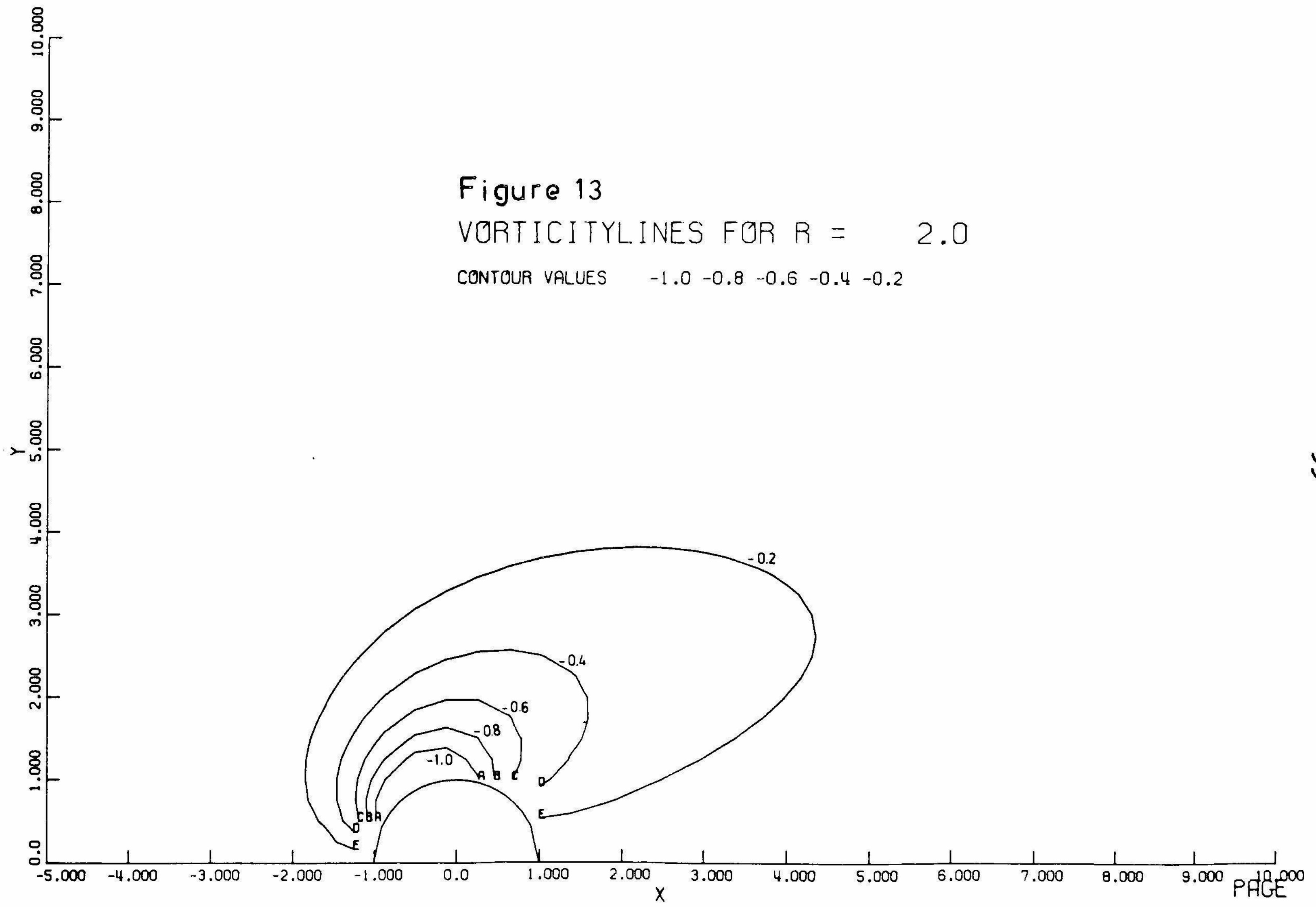

STREAMLINES FØR R = 2.0
CONTOUR VALUES 0.1 0.3 0.5 0.7 0.9 1.5
1.5
0.9
0.7
0.5
0.3
0.1
A
B
C
D
E
F
X
Y
-5.000 -4.000 -3.000 -2.000 -1.000 0.0 1.000 2.000 3.000 4.000 5.000 6.000 7.000 8.000 9.000 10.000
0.0 1.000 2.000 3.000 4.000 5.000 6.000 7.000 8.000 9.000 10.000
PAGE 1

Figure 12

Figure 13

VØRTICITYLINES FØR R = 2.0

CØNTOUR VALUES -1.0 -0.8 -0.6 -0.4 -0.2



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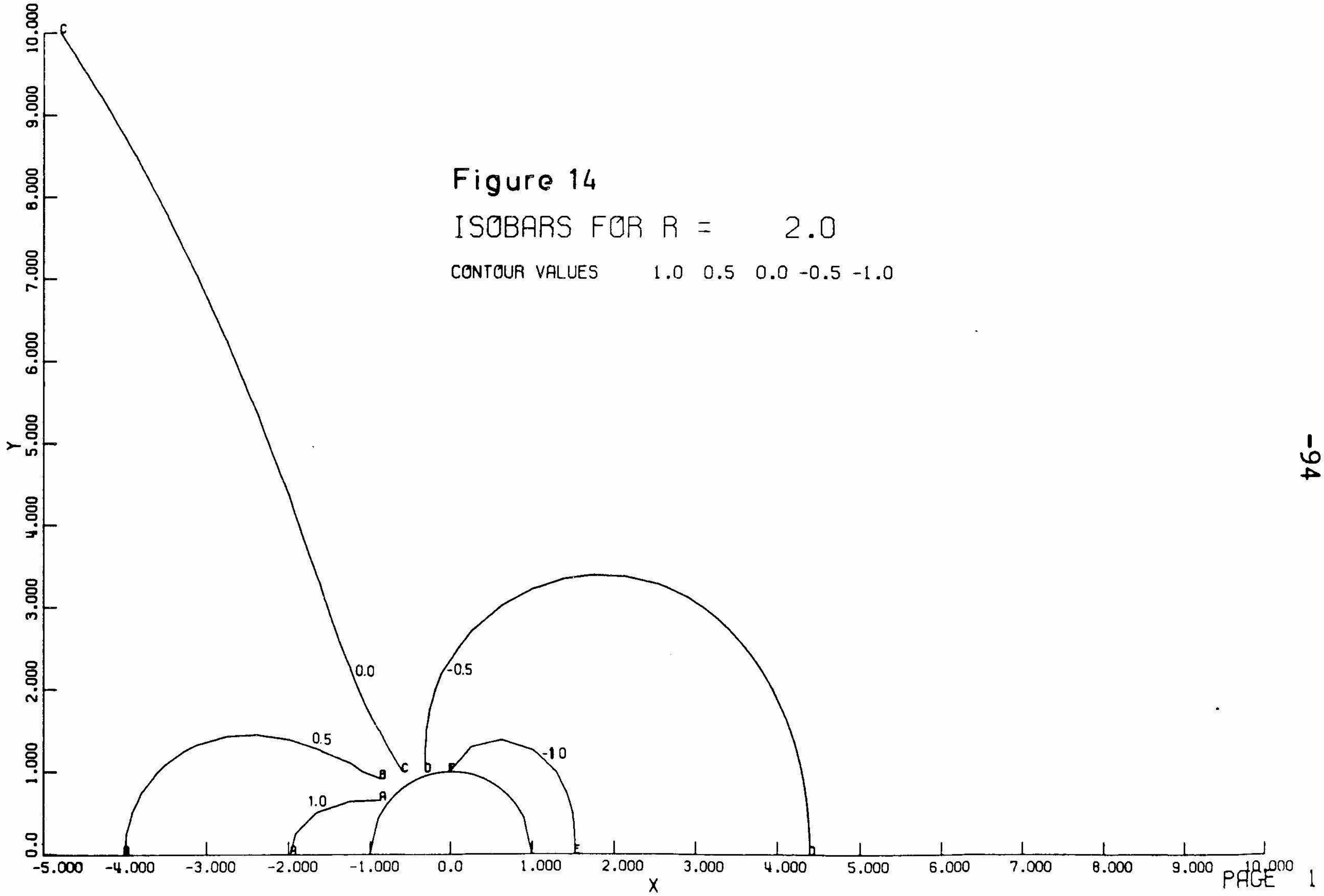

ISØBARS FØR R = 2.0
CØNTØUR VALUES 1.0 0.5 0.0 -0.5 -1.0
0.0
-0.5
0.5
-10
1.0
Y
10.000
9.000
8.000
7.000
6.000
5.000
4.000
3.000
2.000
1.000
0.0
-5.000
-4.000
-3.000
-2.000
-1.000
0.0
1.000
2.000
3.000
4.000
5.000
6.000
7.000
8.000
9.000
10.000
X
PAGE 1

Figure 14

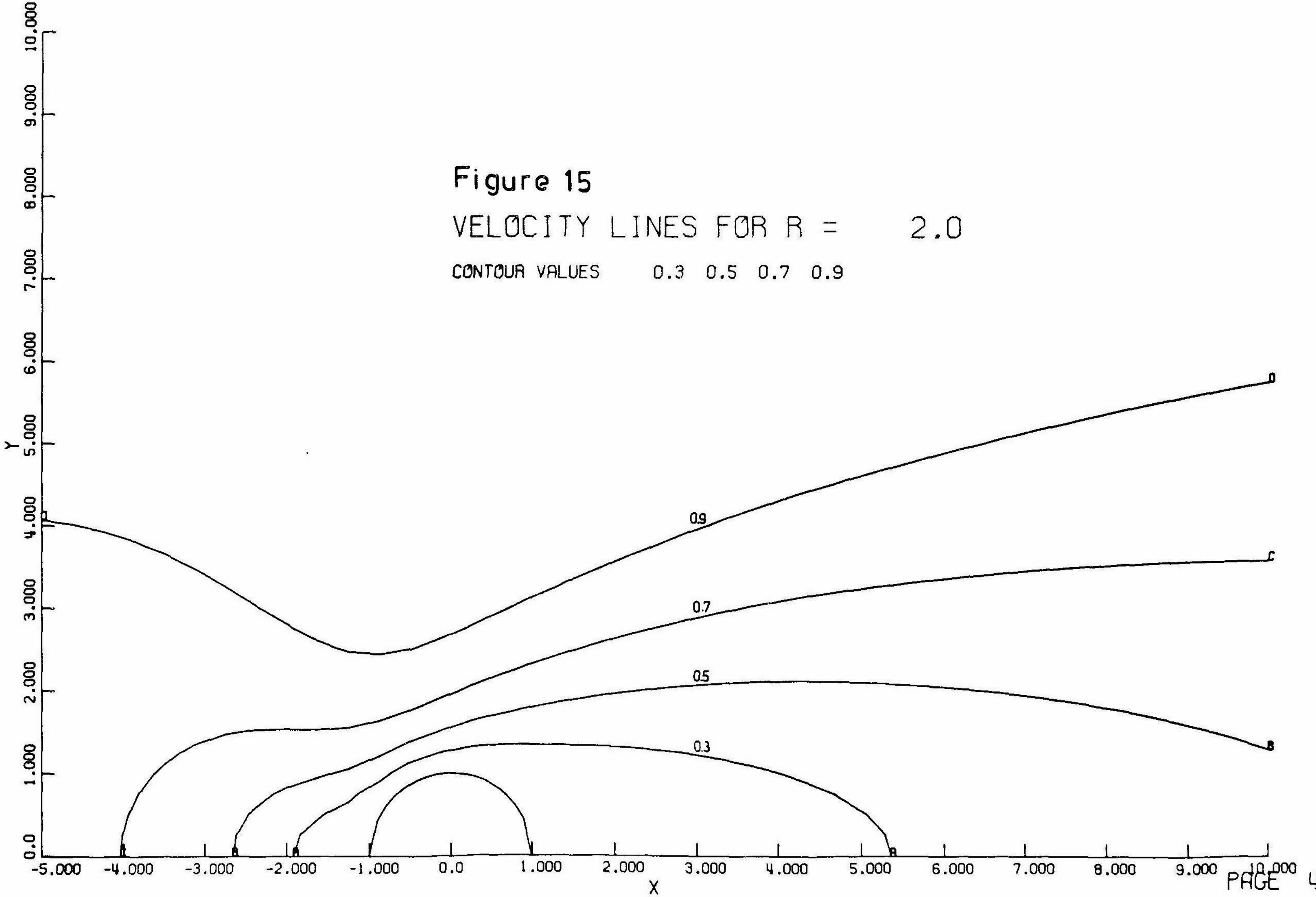


Figure 15

VELOCITY LINES FØR R = 2.0

CØNTOUR VALUES 0.3 0.5 0.7 0.9

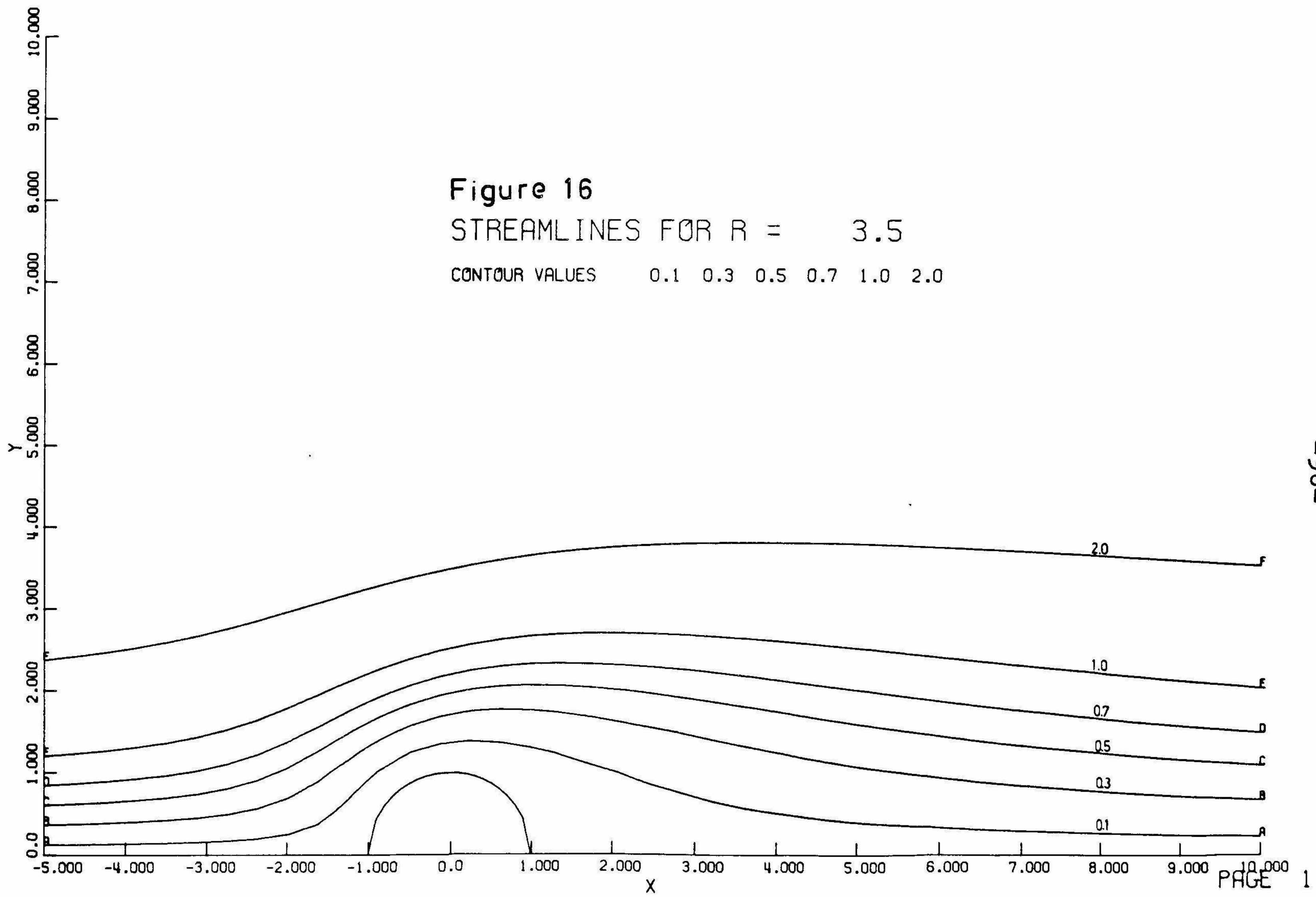


Figure 16
STREAMLINES FØR R = 3.5
CØNTØUR VALUES 0.1 0.3 0.5 0.7 1.0 2.0

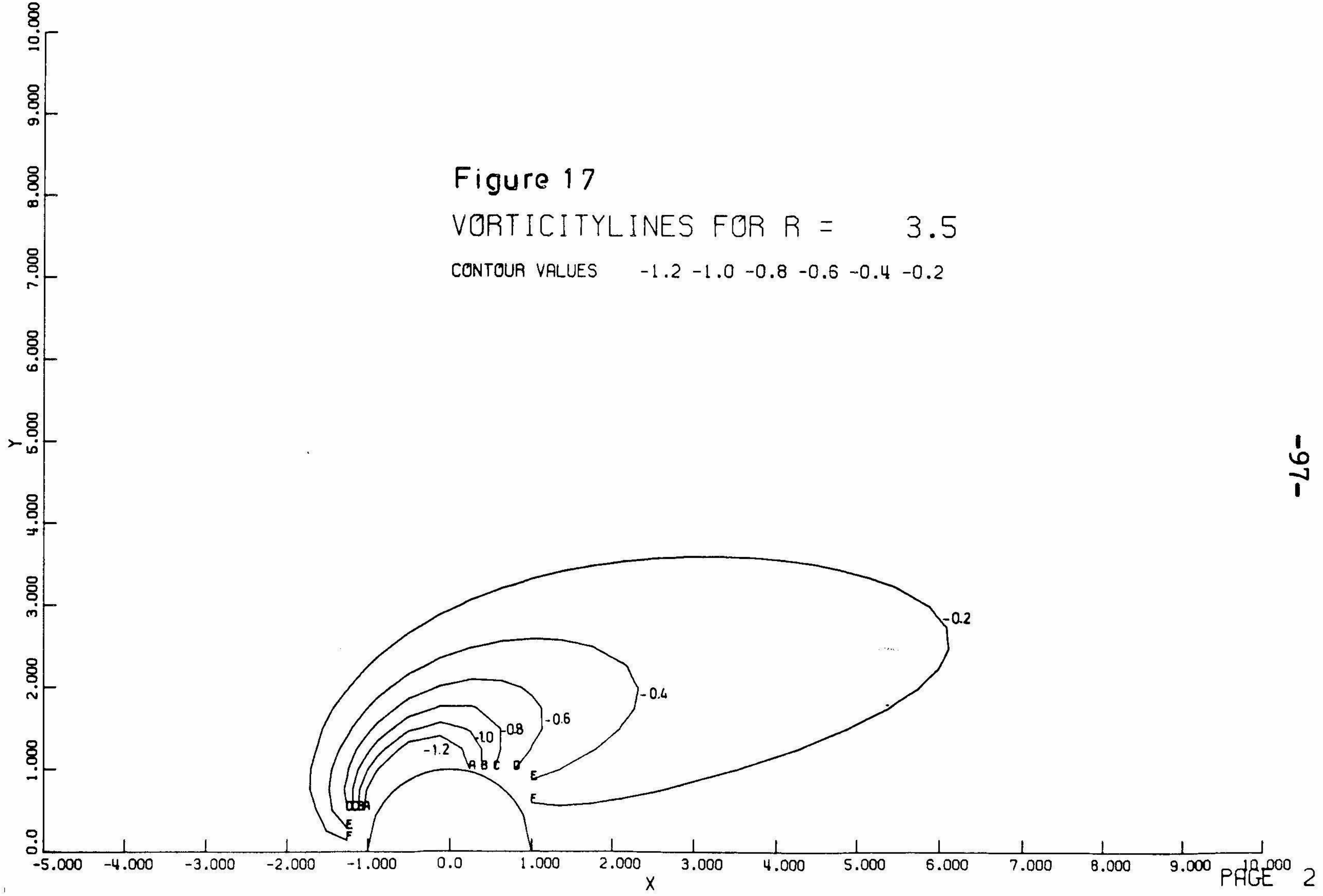

VØRTICITYLINES FØR R = 3.5
CONTOUR VALUES -1.2 -1.0 -0.8 -0.6 -0.4 -0.2
-0.2
-0.4
-0.6
-0.8
-1.0
-1.2
A B C D E
E
DCBA
E
F
Y
10.000
9.000
8.000
7.000
6.000
5.000
4.000
3.000
2.000
1.000
0.0
-5.000
-4.000
-3.000
-2.000
-1.000
0.0
1.000
2.000
3.000
4.000
5.000
6.000
7.000
8.000
9.000
10.000
X
PAGE 2

Figure 17

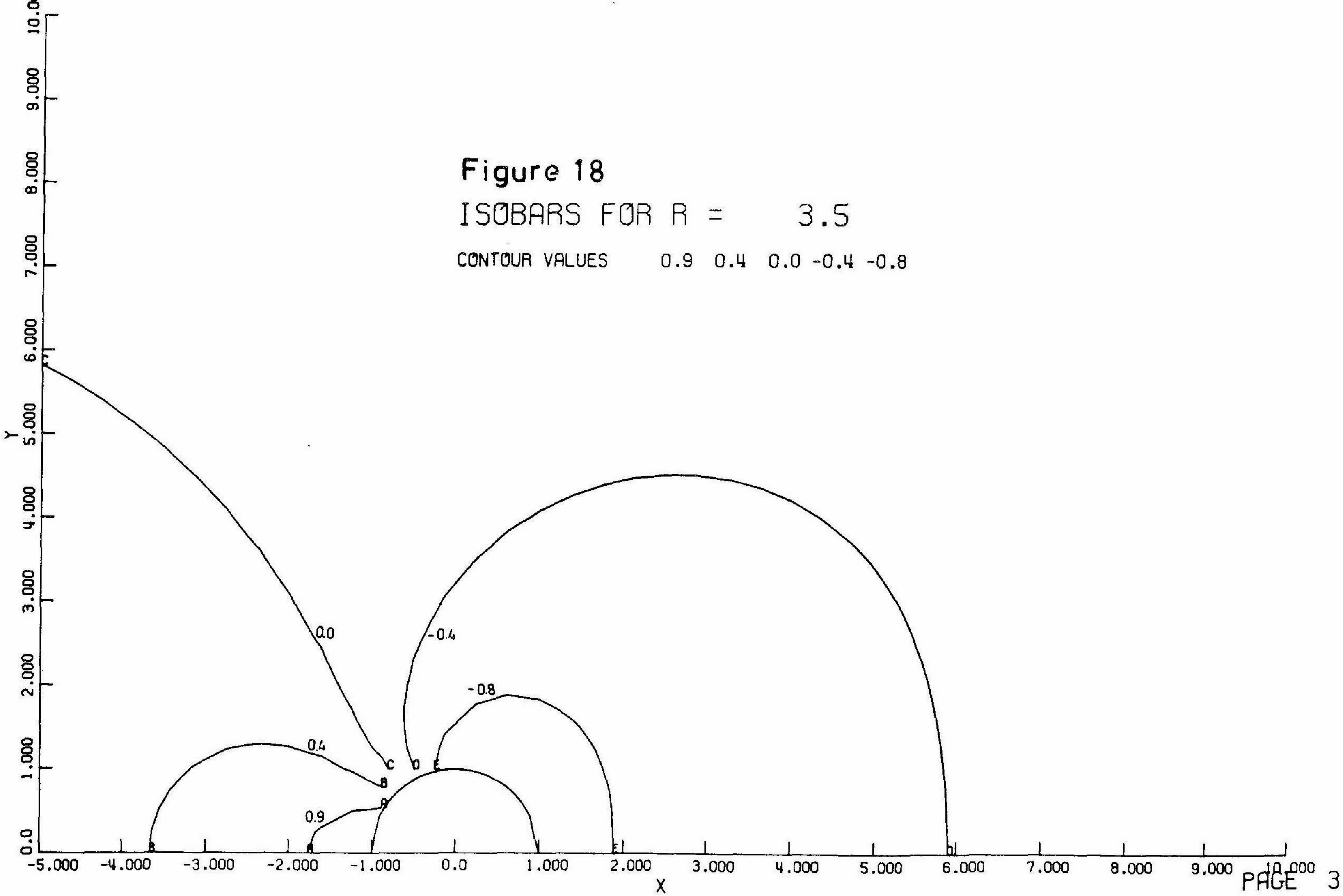

ISØBARS FØR R = 3.5
CONTOUR VALUES 0.9 0.4 0.0 -0.4 -0.8
0.0
-0.4
-0.8
0.4
0.9
Y
X
-5.000 -4.000 -3.000 -2.000 -1.000 0.0 1.000 2.000 3.000 4.000 5.000 6.000 7.000 8.000 9.000 10.000
0.0 1.000 2.000 3.000 4.000 5.000 6.000 7.000 8.000 9.000 10.000
PAGE 3

Figure 18

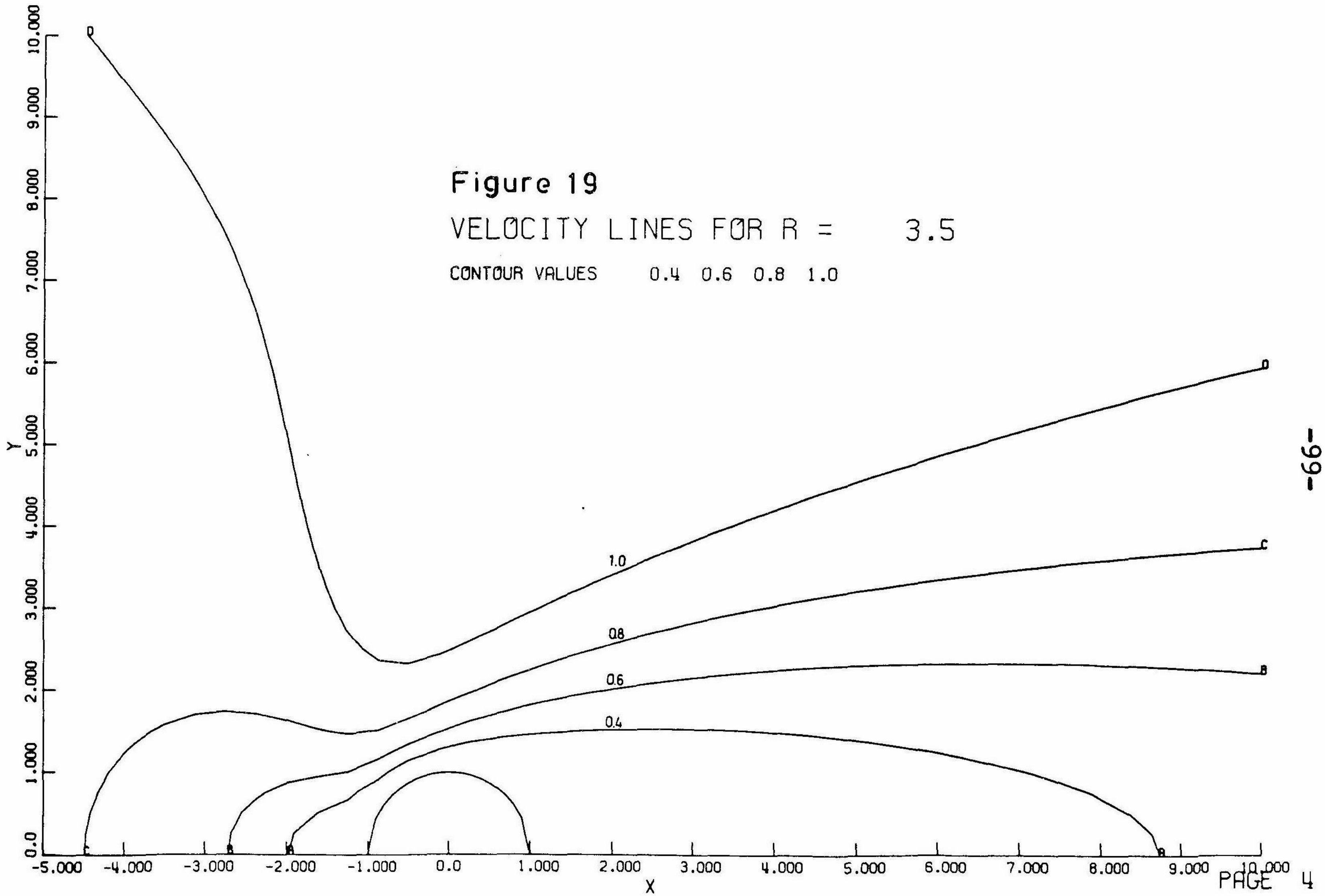
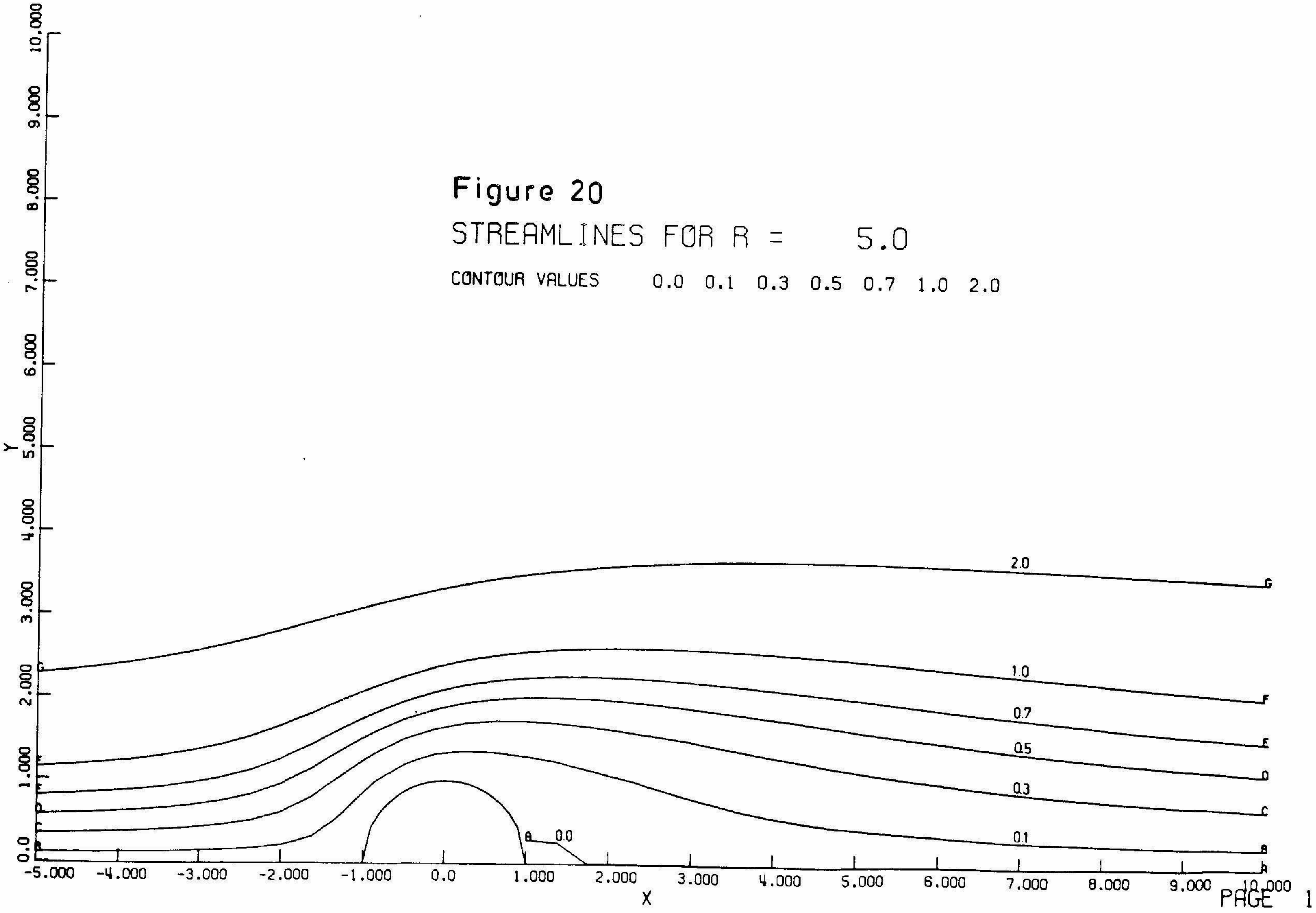


Figure 19
VELOCITY LINES FOR R = 3.5
CONTOUR VALUES 0.4 0.6 0.8 1.0
1.0
0.8
0.6
0.4
Y
0.0
1.000
2.000
3.000
4.000
5.000
6.000
7.000
8.000
9.000
10.000
X
-5.000
-4.000
-3.000
-2.000
-1.000
0.0
1.000
2.000
3.000
4.000
5.000
6.000
7.000
8.000
9.000
10.000
PAGE 4

Figure 20

STREAMLINES FØR R = 5.0

CONTOUR VALUES 0.0 0.1 0.3 0.5 0.7 1.0 2.0



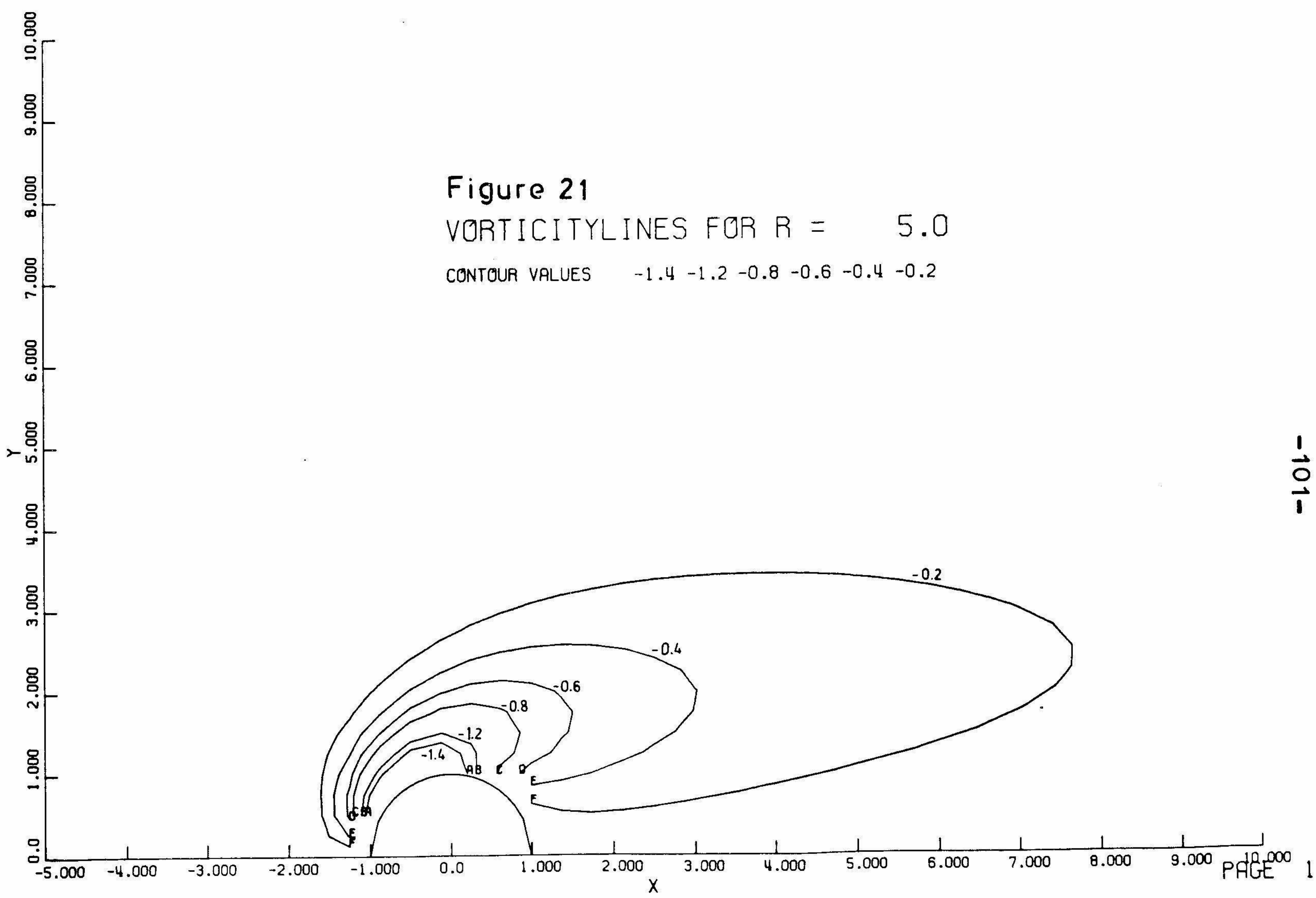


Figure 21
VØRTICITYLINES FØR R = 5.0
CONTOUR VALUES -1.4 -1.2 -0.8 -0.6 -0.4 -0.2
-0.2
-0.4
-0.6
-0.8
-1.2
-1.4
Y
X
-5.000 -4.000 -3.000 -2.000 -1.000 0.0 1.000 2.000 3.000 4.000 5.000 6.000 7.000 8.000 9.000 10.000
0.0 1.000 2.000 3.000 4.000 5.000 6.000 7.000 8.000 9.000 10.000
PAGE 1

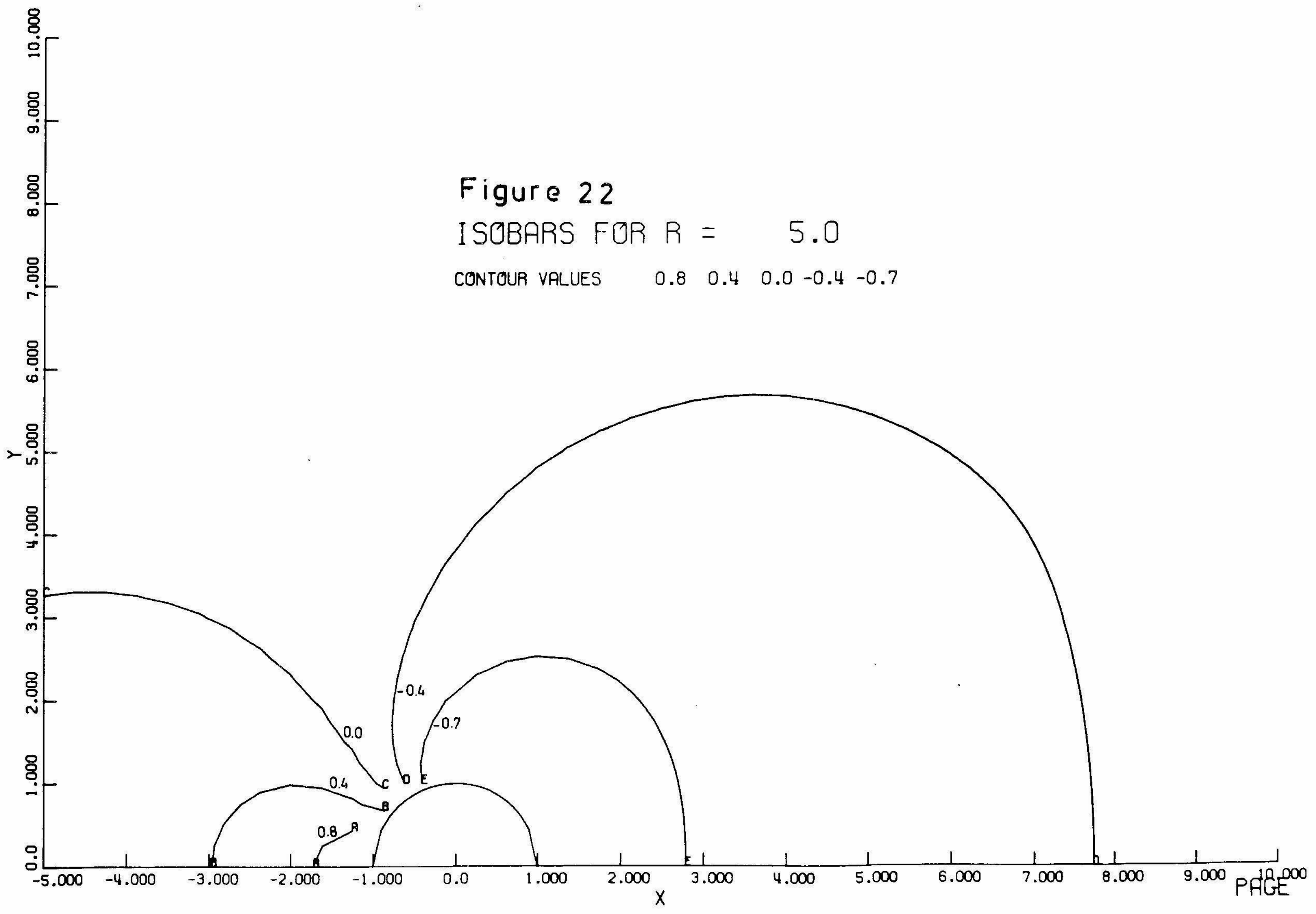


Figure 22

ISØBARS FØR R = 5.0

CØNTØUR VALUES 0.8 0.4 0.0 -0.4 -0.7

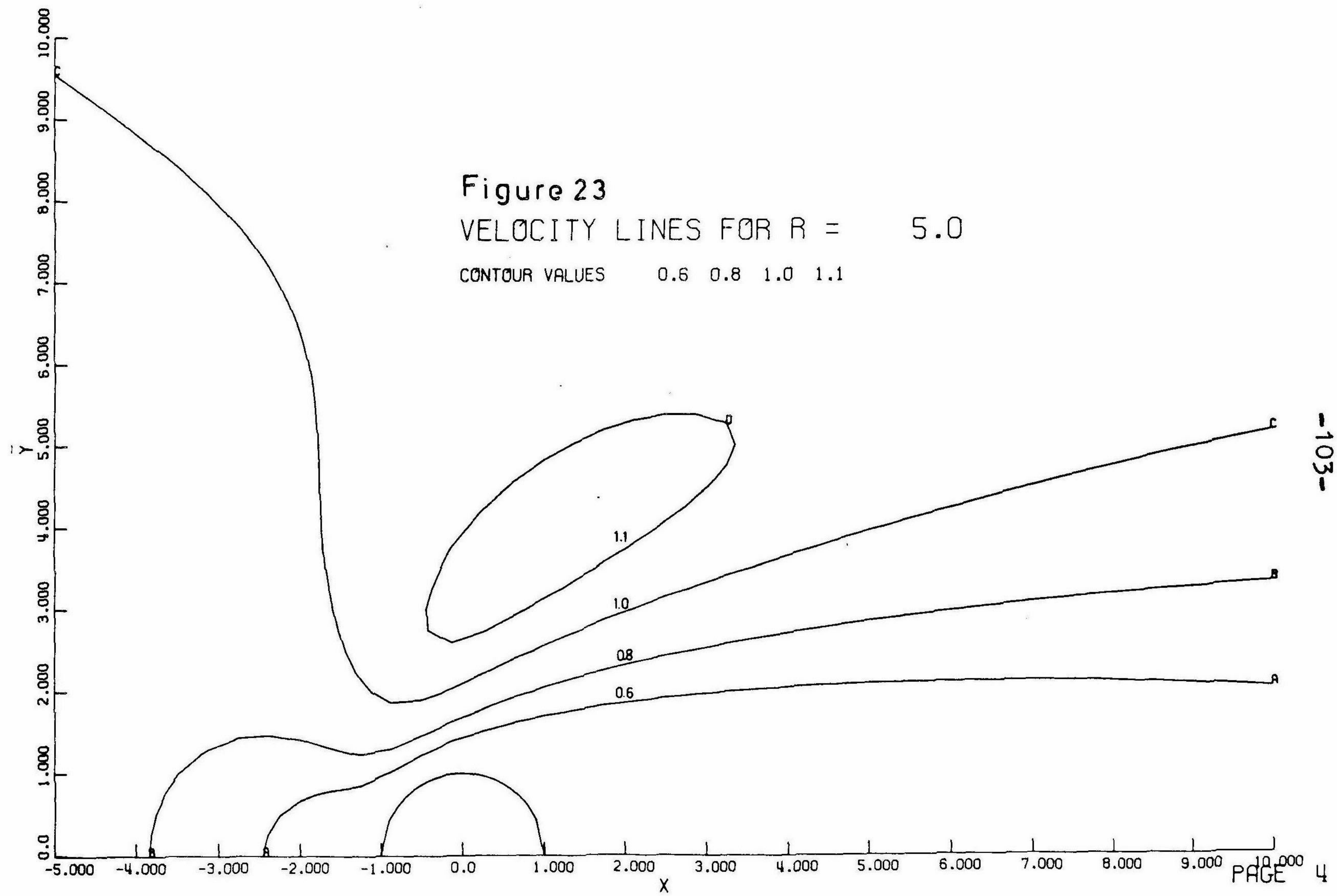


Figure 23

VELØCITY LINES FØR R = 5.0

CØNTØUR VALUES 0.6 0.8 1.0 1.1