LIQUID SLOSHING IN AN ELASTIC CONTAINER

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ABSTRACT

The motion of a liquid in a flexible container is important for rocket structural dynamics. The purpose of this paper is to study the dynamic response of the liquid, the sloshing frequencies and the stability of the free surface of the liquid in an elastic container.

The variational principle for the problem of an incompressible, inviscid fluid in an elastic container is presented by considering the pressure energy of the fluid, the surface energy, and the Lagrangian of the elastic thin shell. The corresponding linearized equations are studied in terms of eigenvalues and eigenfunctions.

The effects of the gravitation, the surface tension, the rigidity of the container, the free surface contact angle and its dynamic variation, on the natural frequencies and the stability of the free surface are discussed.

It is found that the flexibility of the container always lowers the natural frequencies and also induces a mean oscillatory motion of the liquid that creates an oscillatory force on the container in the vertical direction. The equilibrium contact angle and its dynamic variation have an important effect on the limit of stability.

The motion of a liquid in a circular cylindrical container with a flat flexible bottom is worked out in detail analytically by means of eigenfunctions. Some results are presented graphically. A numerical scheme using finite elements method is developed for an arbitrary container. Methods for improving the solution systematically are indicated.

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LIST OF SYMBOLS

$\underline{a}_{\alpha}$	base vector for the free surface
$a_{\alpha\beta}$	metric tensor for the free surface
$\underset{\sim}{A}_i$	matrices [Eqs. (7.1) to (7.3)]
A_{mn}	normalization factors [Eq. (6.57)]
$\underset{\sim}{b}$	column vector of the generalized coordinates in the fluid
$b_{\alpha\beta}$	curvature tensor
$B_{\alpha\beta}$	curvature tensor of the deformed surfaces
B_M	membrane number [Eq. (6.7)]
B_σ	bond number [Eq. (6.7)]
B_p	plate number [Eq. (6.7)]
$\underset{\sim}{c}$	column vector of the generalized coordinates on the free surface
c_{mn}, d_{mn}	amplitude of the n^{th} sloshing mode
$C^{\alpha\beta\lambda\theta}$	material response function [Eq. (2.22)]
$D\ (= \frac{Eh^3}{12(1-\nu^2)})$	bending rigidity
$\mathfrak{D}(\underline{U}, H)$	functional [Eq. (4.27)]
$e_{\alpha\beta}$	infinitesimal strains [Eq. (2.11)]
$E_{\alpha\beta}$	Green's strain [Eq. (2.10)]
$\mathcal{E}$	strain energy per unit area [Eq. (2.18)]
$\underset{\sim}{\mathcal{E}}$	matrix [Eq. (7.31)]
F_z	force exerted by the fluid in z-direction

$\underline{g}(t)$	gravitational acceleration
$G(0-)$	non-dimensional gravitational acceleration for $\tau < 0$
G	non-dimensional gravitational acceleration for $\tau > 0$
$\underline{g}_\alpha$	base vector
$g_{\alpha\beta}$	metric tensor
$h_{mn}(R)$	normal modes of the free surface [Eqs. (6.61) and (6.62)]
h	thickness of the shell
H	free surface deflection [Eq. (6.6)]
$\mathcal{H}(\Phi, \underline{U})$	functional [Eq. (4.29)]
H^*	initial disturbed free surface deflection
H^{**}	initial disturbed free surface velocity
$I_i (i=1, 2, 3, 4)$	functional [Eqs. (2.2 to (2.4)]
$I_m(x)$	modified Bessel function of the first kind
J_i	functional [Eqs. (3.16) to (3.19)]
$J_m(x)$	Bessel function of the first kind
$\underset{\sim}{K}$	stiffness matrix for the shell
k_n $(=k_{on})$	n^{th} root of $J_1(x) = 0$
k_{mn}	n^{th} root of $(d/dx)J_m(x) = 0$
ℓ	depth of liquid
L	non-dimensional depth of liquid
L_m	linear operator [Eq. (6.31)]
$\underset{\sim}{M}$	mass matrix for the shell
$\tilde{m}^{\alpha\beta}$	stress moments [Eq. (2.17)], based on Kirchhoff stress

$\overline{m}^{\alpha\beta}$	prescribed stress moments [Eq. (2.21)]
$m^{\alpha\beta}$	infinitesimal stress moments [Eq. (3.3)]
$M^{\alpha\beta}$	infinitesimal stress moments [Eq. (4.7)]
$\underline{N}$, $\underline{N}_o$	unit outer normal of the mid-surface of a shell in the deformed state and initial state, respectively
$\underline{n}_i$	unit vector normal to a boundary curve
n_i	number of elements used in the numerical scheme
$\tilde{n}^{\alpha\beta}$	stress resultants [Eq. (2.17)] based on Kirchhoff stress
$\overline{n}^{\alpha\beta}$	prescribed stress resultants
$n^{\alpha\beta}$	infinitesimal stress resultants [Eq. (3.3)]
$N^{\alpha\beta}$	stress resultants [Eq. (4.7)]
N_r	mid-plane stress resultant
p, p_2	pressure
p_{mn}	n^{th} non-dimensional sloshing frequency for rigid tank [Eq. (6.44)]
P, P_2	non-dimensional pressure
$\underset{\sim}{q}$	column vector of the generalized coordinates of the shell
r_o	radius of the tank
$\underline{r}_o$	undeformed surface
(r, θ, z)	cylindrical coordinates
(R, θ, z)	non-dimensional cylindrical coordinates
S_1	free surface
S_2	flexible surface of the container

S_3	wetted portion of S_2
S_4	wetted portion of the rigid surface of the container
∂S_i	boundary curve of S_i
S^{ij}	Kirchhoff stress tensor
t	time
$\underline{u}$, $\underline{U}$	displacement vector
$\underline{U}^*$	initial disturbed deflection of the shell
$\underline{U}^{**}$	initial disturbed velocity of the shell
u^1, u^2, w	components of displacement vector
V	volume occupied by the fluid
∂V	boundary surface of V
w^*	initial disturbed deflection of the bottom
w^{**}	initial disturbed velocity of the bottom
W	deflection of the tank bottom
$w_{mn}(R)$	normal modes of the bottom deflection [Eqs. (6.63) and (6.64)]
z	vertical coordinate
Z	non-dimensional vertical coordinate

α_{mn}	see Eq. (6.70)
β_{mn}	see Eq. (6.44)
γ	hysteresis constant [Eq. (4.89)]
Γ	boundary curve of the shell surface
Γ_σ	portion of Γ where stress is prescribed

$\delta_{mn}(T)$	see Eq. (6.70)
ϵ_{mn}	see Eq. (6.70)
$\epsilon^{\alpha\beta}$	permutation tensor
$\bar{\zeta}$, $\bar{\zeta}_m$, $\bar{\zeta}_{mn}$	see Eqs. (6.10), (6.29) and (6.36) respectively
ζ, ζ_m, ζ_{mn}	see Eqs. (6.11), (6.30) and (6.38) respectively
η	free surface deflection
η^*	initial disturbed deflection of the free surface
η^{**}	initial disturbed velocity of the free surface
θ	azimuthal coordinate
θ_o	angle of contact
$\varkappa^{\alpha\beta}$	curvature tensor for the free surface
$\varkappa_{mn}$, $\bar{\varkappa}_{mn}$	see Eq. (6.44)
λ	mass ratio [Eq. (6.7)]
μ, μ_m	eigenvalue
ν_x	angle (radian) between the axis of the container and z-axis
ν_α	direction cosine
ξ	free surface deflection in z-direction
Π_i	see Eqs. (2.19) to (2.21)
ρ	density per unit volume of the shell
ρ_o	density per unit volume of the fluid
$\mathcal{S}_{mn}$	see Eqs. (6.55) and (6.56)
σ	surface tension between vapor-liquid interface
σ_{vs}, σ_{sL}	surface tension between vapor-solid and solid-liquid interface

σ_H	hysteresis constant [Eq. (4.11)]
τ	non-dimensional time [Eq. (6.6)]
ϕ	velocity potential
Φ	non-dimensional velocity potential
$\underset{\sim}{\chi}$	stiffness matrix for the free surface
ψ_o, Ψ_o	see Eqs. (6.5) and (6.6)
ω	natural frequency
Ω	characteristic frequency
Ω_M^2	frequency parameter for the membrane [Eq. (6.7)]
Ω_p^2	frequency parameter for the plate [Eq. (6.7)]
Ω_σ^2	frequency parameter for the free surface [Eq. (6.7)]
∇^2	three dimensional Laplace operator
$\bar{\nabla}^2$	two dimensional Laplace operator
Δ^2	non-dimensional two-dimensional Laplace operator
$\frac{\partial}{\partial N}$	normal derivative [Eq. (4.34)]
$\partial(\ \ )$	boundary of the quantity in the parenthesis
$(\ \ )\vert_\alpha$	covariant derivative with respect to metric tensor $g_{\alpha\beta}$ or $a_{\alpha\beta}$
$(\dot{\ \ })$	time derivative

I. INTRODUCTION

Dynamics of large liquid fuel rockets naturally involves the motion of a liquid in a flexible container. The motion of the fluid influences the pressure at the tank bottom, and hence influences successively the pressure in the pump, the combustion, the thrust, and the rocket acceleration. The important effect of fluid motion on the rocket structural dynamics is evident. In some instances the longitudinal oscillation can be so serious as to affect the safety of the vehicle. For this reason the understanding of the behavior of the liquid and the interaction between the liquid and the flexible container under forced motion is important to the design of the vehicle.

There is a voluminous literature concerning the motion of a liquid in a container. The problem was treated by Poisson (1828) and Rayleigh (1876) for a stationary rigid tank; see Lamb (Ref. 1). Subsequently many authors investigated various aspects of the motion of the free surface (sloshing). Most of them restricted their works to a rigid container, although some are concerned with flexible tanks. Bauer (Ref. 2) considered a simple beam bending mode of a cylindrical tank and treated the sloshing as a problem in forced oscillations. Miles (Ref. 3) determined the frequencies and mode shapes of the coupled system by an approximate method. The stability of the plane free surface of a liquid in a cylindrical tank in a vertical periodic motion was first investigated by Benjamin and Ursell (Ref. 4), who found that the stability was governed by a Mathieu equation. Later, Tong and Fung (Ref. 5) found that, in the case of a flexible container, the stability was governed by the coupled Hill's equations. The dynamic response of a liquid in an elastic container has seldom been treated. Tong (Ref. 6) considered the transient motion of the liquid in a circular cylindrical container with a flat flexible bottom. He found that an oscillatory displacement of the liquid relative to the tank walls in the axial direction was induced by the flexibility of the bottom and it was this relative displacement that created an axial oscillating force that acts on the system. This oscillating force vanishes in the case of a rigid tank. The oscillating force can alter the motion of the system, and, more seriously, can affect the stability of the whole vehicle. However, the analysis presented in Ref. 6 is too idealized. It is the purpose of the present study to extend the results of Ref. 6 to more realistic situations, in the hope of ultimately applying it to a practical design.

We shall consider an elastic thin shell container of an arbitrary shape. It contains an inviscid, incompressible fluid with a free surface. The liquid motion in the container is assumed to be irrotational. Surface tension of the free surface will be considered. At the ground level, the effect of the surface tension on sloshing is negligible, especially in a large container; but when the gravity is greatly reduced, the surface tension becomes important. Near the zero-gravity condition, the free surface is highly curved because of the surface tension. For example, a liquid in a vehicle orbiting the earth at approximately 100 nautical miles experiences a body force of 10^{-5} g_o, where g_o is the nominal value of the gravitational acceleration at the earth's surface. At such a reduced gravity we are usually concerned with the motion and the location of the liquid in the container.

We shall first formulate the problem by the variational method in Sections II and III. From the variational functional, the properties of eigenvalues and eigenfunctions can be easily studied. By the maximum-minimum properties (in Section IV), many general effects of the elasticity, the gravitational force, the surface tension and the contact angle between the walls and the free surface can be deduced. A particularly simple example will be given in Section VI to illustrate a method of obtaining the solution and the application of eigenfunction expansion for solving the initial value problem. In the general case of arbitrary tank, the analytic solution is hard to find. A numerical scheme using the finite elements method for obtaining approximate solution is proposed in Section VII. The convergence proof of this numerical scheme for the static problem will appear in a future report. In the case of the dynamic problems, it is subject to further investigation.

II. MATHEMATICAL FORMULATION

An inviscid incompressible fluid is in an elastic container as shown in Fig. 1. The walls of the elastic container are made of a thin shell. The fluid motion in the container is assumed to be irrotational. Thus there exists a velocity potential, which satisfies the Laplace equation, for the fluid motion. The fluid exerts a pressure, which is related to the velocity potential through Bernoulli's equation, on the elastic walls. The normal velocity of the wall is equal to that of the fluid at the liquid-shell interface. No cavitation is allowed.

Because of the geometrical complexity of the problem, we shall try solving it by a variational method. We will propose a functional and show that if we require the first variation to be identically zero, then all the differential equations and boundary conditions required to solve the problem exactly will be obtained. The functional is in the following form:

$$J = \int_{t_0}^{t_1} I \, dt \qquad (2.1)$$

where

$$I = I_1 + I_2 + I_3$$

$$I_1 = -\int p \, dv \qquad (2.2)$$

$$I_2 = \sigma \int_{S_1} ds + \sigma_{VS} \int_{S_2 - S_3} ds + \sigma_{SL} \int_{S_2} ds \qquad (2.3)$$

I_3 = Lagrangian of the elastic shell (2.4)

I_1 is the so-called pressure energy (Ref. 7) where p is the pressure in the fluid. Actually I_1 corresponds to the Lagrangian of the fluid (see Appendix A). V is the actual volume occupied by the fluid. I_2 is the surface energy. σ, σ_{VS}, σ_{SL} are the surface tensions(assumed to be constant). S_1, $S_2 - S_3$, S_3 are the actual surfaces of liquid-vapor, vapor-solid, solid-liquid interfaces respectively. I_3 is the Lagrangian for the shell. In the following, we shall give a detailed derivation of the energy expression I_3 before we show that the functional J in Eq. (2.1) really satisfies our requirements.

A. Potential Energy for Elastic Thin Shell

We shall consider geometrical nonlinearity only; the strain-stress relation obeys a linear Hooke's Law. The strain at the middle surface, the ratio of the thickness to some characteristic length of the shell (e.g., minimum radius of curvature) and the rotations are all assumed to be small compared to one. The displacement field will be restricted to the case in which Kirchhoff's assumption offers a good approximation. We will also assume that the shearing stresses are zero on the inner and outer surfaces of the shell. In addition, we assume that the products of the strains themselves, or strains and rotations, or strains and the thickness of the shell (nondimensionalized by the characteristic length); and the square of in-plane rotations, or the cubic of the out-plane rotations, are all small compared to strain itself. Under these assumptions, a rather simple strain energy expression for the shell can be obtained.

We shall use a set of intrinsic curvilinear coordinates x^α, $\alpha = 1, 2$, to describe points on the undeformed shell. The general tensor notations will be used to describe the deformation of the shell. Following Kirchhoff's assumptions, the displacement field is assumed to be of the form

$$\underline{u} = u_\alpha \underline{g}^\alpha + w \underline{N}_o + z \underline{u}_1 \tag{2.5}$$

where α ranges over 1, 2, $\underline{g}^\alpha$ are the base vectors on the middle surface of the undeformed shell, $\underline{N}_o$ is the unit outward normal, z is the distance measured along $\underline{N}_o$ from the middle surface, and $\underline{u}_1$ is a vector which depends on x^α,

$$\underline{u}_1 = \underline{N} - \underline{N}_o \tag{2.6}$$

where $\underline{N}(x^\alpha)$ is the unit outward normal after deformation. Under our assumptions, we find that

$$z\underline{u}_1 = -z\left(w|_\alpha + u^r b_{r\alpha}\right)\underline{g}^\alpha \tag{2.7}$$

where $b_{r\alpha}$ is defined by the equation

$$b_{r\alpha} = \frac{\partial \underline{g}_\alpha}{\partial x^r} \cdot \underline{N}_o = \frac{\partial \underline{g}_r}{\partial x^\alpha} \cdot \underline{N}_o \tag{2.8}$$

and shall be called the curvature tensor. A single stroke denotes the covariant differentiation with respect to the base vectors of the middle surface. Then

$$\begin{aligned} \frac{\partial \underline{u}}{\partial x^\beta} &\doteq \left(u_\alpha|_\beta - b_{\alpha\beta} w\right)\underline{g}^\alpha + \left(w|_\beta + u^r b_{r\beta}\right)\underline{N}_o \\ &\quad - z\left(w|_{\alpha\beta}\,\underline{g}^\alpha + w|_r b^r_\beta\,\underline{N}_o\right) \\ \frac{\partial \underline{u}}{\partial z} &= -\left(w|_\alpha + u_r b^r_\alpha\right)\underline{g}^\alpha . \end{aligned} \tag{2.9}$$

The Green's strain tensor, with the local base vectors approximated by the base vectors of the middle surface, is

$$E_{\alpha\beta} = \frac{1}{2}\left(\underline{g}_\alpha \cdot \frac{\partial \underline{u}}{\partial x^\beta} + \underline{g}_\beta \cdot \frac{\partial \underline{u}}{\partial x^\alpha} + \frac{\partial \underline{u}}{\partial x^\alpha} \cdot \frac{\partial \underline{u}}{\partial x^\beta}\right)$$

$$\doteq e_{\alpha\beta} + \frac{1}{2} w|_\alpha w|_\beta - z\, w|_{\alpha\beta} \qquad (2.10)$$

where

$$e_{\alpha\beta} = \frac{1}{2}\left(u_\alpha|_\beta + u_\beta|_\alpha - 2 b_{\alpha\beta} w\right) \qquad (2.11)$$

We recognize that $e_{\alpha\beta}$ are the so-called infinitesimal strains at the middle surface and $w|_\alpha$ are the out-plane rotations.

From potential energy theorem, we have the energy integral

$$\Pi = \frac{1}{2}\int_S \int_{-h/2}^{h/2} S^{\alpha\beta} E_{\alpha\beta} f\, dz\, dS - \frac{1}{2}\int_S \int_{-h/2}^{h/2} \rho \dot{\underline{u}}^2 f\, dz\, dS$$

$$- \int_S \left(S^{33} w f\right)_{-h/2}^{h/2} dS - \int_{\Gamma_\sigma} dl \int_{-h/2}^{h/2} S^{\alpha j} \nu_\alpha U_j f\, dz \qquad (2.12)$$

In this equation S^{ij} is the Kirchhoff stress, i and j range over 1, 2, 3, and Γ_σ is the portion of the boundary curve of the shell where stress is prescribed. U_j is the jth component of the displacement vector that

$$U_\beta = u_\beta - z\, w|_\beta \quad ,$$

$$U_3 = w \quad , \qquad (2.13)$$

A dot over $\underline{u}$ denotes the time derivative of $\underline{u}$. ν_α is the direction cosine of a unit vector lying on the middle surface, normal (outward) to the boundary curve Γ_σ. h is the thickness of the shell. f is the Jacobian:

$$f = 1 - z b^\alpha_\alpha + z^2 \det(b^\alpha_\beta) \doteq 1 \tag{2.14}$$

Let

$$p = (S^{33} f)^{z=h/2}_{z=-h/2} \doteq S^{33}\Big|^{h/2}_{-h/2} \tag{2.15}$$

$$q^\alpha = \int_{-h/2}^{h/2} S^{\alpha 3} f \, dz \doteq \int_{-h/2}^{h/2} S^{\alpha 3} dz \tag{2.16}$$

$$\begin{Bmatrix} \tilde{n}^{\alpha\beta} \\ \tilde{m}^{\alpha\beta} \end{Bmatrix} = \int_{-h/2}^{h/2} S^{\alpha\beta} \begin{Bmatrix} 1 \\ -z \end{Bmatrix} f \, dz$$

$$\doteq \int_{-h/2}^{h/2} S^{\alpha\beta} \begin{Bmatrix} 1 \\ -z \end{Bmatrix} dz \tag{2.17}$$

The strain energy per unit middle plane area is

$$\mathcal{E} = \frac{1}{2} \int_{-h/2}^{h/2} S^{\alpha\beta} E_{\alpha\beta} f \, dz$$

$$\doteq \frac{1}{2} \left[\tilde{n}^{\alpha\beta} \left(e_{\alpha\beta} + \frac{1}{2} w|_\alpha w|_\beta \right) + \tilde{m}^{\alpha\beta} w|_{\alpha\beta} \right] \tag{2.18}$$

Then Π becomes

$$\Pi = \Pi_1 - \Pi_2 - \int_S p w \, dS - \frac{1}{2}\int_S \rho h (\dot{u}^\alpha \dot{u}_\alpha + \dot{w}^2) \, dS, \qquad (2.19)$$

where

$$\Pi_1 = \int_S \mathcal{E} \, dS, \qquad (2.20)$$

$$\Pi_2 = \int_{\Gamma_\sigma} \bar{q}^\alpha \nu_\alpha w \, dl + \int_{\Gamma_\sigma} \bar{n}^{\alpha\beta} u_\alpha \nu_\beta \, dl + \int_{\Gamma_\sigma} \bar{m}^{\alpha\beta} w|_\alpha \nu_\beta \, dl, \qquad (2.21)$$

with a bar over a quantity indicating that quantity to be prescribed.

Let the constitutive equation be

$$S^{\alpha\beta} = C^{\alpha\beta\lambda\theta} E_{\lambda\theta}, \qquad (2.22)$$

where $C^{\alpha\beta\lambda\theta}$ are independent of $E_{\lambda\theta}$ and z. Thus, Eqs. (2.17) and (2.20) become

$$\tilde{n}^{\alpha\beta} = h C^{\alpha\beta\lambda\theta} \left(e_{\lambda\theta} + \frac{1}{2} w|_\lambda w|_\theta\right), \qquad (2.23)$$

$$\tilde{m}^{\alpha\beta} = \frac{h^3}{12} C^{\alpha\beta\lambda\theta} w|_{\lambda\theta}, \qquad (2.24)$$

$$\Pi_1 = \frac{h}{2}\int_S C^{\alpha\beta\lambda\theta} \left(e_{\alpha\beta} e_{\lambda\theta} + \frac{h^2}{12} w|_{\alpha\beta} w|_{\lambda\theta}\right) dS$$

$$+ \frac{h}{2}\int_S \frac{C^{\alpha\beta\lambda\theta}}{2} \left(e_{\alpha\beta} + \frac{1}{4} w|_\alpha w|_\beta\right) w|_\lambda w|_\theta \, dS. \qquad (2.25)$$

The first integral of Eq. (2.25) is the linearized strain energy expression under the thin shell approximation; the second integral is the contribution of the geometric nonlinearity.

By varying u_α, ω arbitrarily and using divergent theorem appropriately, for a smooth boundary curve, one gets from Eq. (2.19):

$$\begin{aligned}\int_{t_0}^{t_1}\delta\Pi\,dt = \int_{t_0}^{t_1}\Big\{\int_S\left(\rho h\ddot{u}^\beta - \tilde{n}^{\alpha\beta}|_\alpha\right)\delta u_\beta\,dS \\ + \int_S\left[\rho h\ddot{\omega} + \tilde{m}^{\alpha\beta}|_{\alpha\beta} - \tilde{n}^{\alpha\beta}b_{\alpha\beta} - (\tilde{n}^{\alpha\beta}\omega|_\beta)|_\alpha - p\right]\delta\omega\,dS \\ + \int_{\Gamma_\sigma}(\tilde{n}^{\alpha\beta} - \bar{n}^{\alpha\beta})\delta u_\beta \nu_\alpha\,dl + \int_{\Gamma_\sigma}(\tilde{m}^{\alpha\beta} - \bar{m}^{\alpha\beta})\nu_\alpha\delta\omega|_\beta\,dl \\ + \int_{\Gamma_\sigma}(\tilde{n}^{\alpha\beta}\omega|_\beta - \tilde{m}^{\alpha\beta}|_\beta - \bar{q}^\alpha)\nu_\alpha\delta\omega\,dl\Big\}\,dt\end{aligned} \tag{2.26}$$

under the condition that δu_β, $\delta\omega$ and $\delta\omega|_\beta = 0$ at t_o and t_o, and on the boundary where u_β, ω and $\delta\omega|_\beta$ are prescribed.

If the first variation of $\int_{t_o}^{t_1}\Pi\,dt$ is zero for arbitrary δu_β, $\delta\omega$ over $S + \Gamma_\sigma$ and $\delta\omega|_\alpha$ over Γ_σ, one will require the differential equations

$$\tilde{n}^{\alpha\beta}|_\beta = \rho h\ddot{u}^\alpha \tag{2.27}$$

$$-\tilde{m}^{\alpha\beta}|_{\beta\alpha} + \tilde{n}^{\alpha\beta}b_{\alpha\beta} + (\tilde{n}^{\alpha\beta}\omega|_\beta)|_\alpha = -p + \rho h\ddot{\omega} \tag{2.28}$$

to be satisfied over S and boundary conditions

$$(\tilde{n}^{\alpha\beta} - \bar{n}^{\alpha\beta})\nu_\beta = 0 \qquad \text{or} \qquad \delta u_\alpha = 0$$

$$\begin{aligned} &\frac{\partial}{\partial \ell}\left[(\tilde{m} - \bar{m}^{\alpha\beta})\nu_\alpha \frac{\partial \ell}{\partial x^\beta}\right] \\ &\quad + \left[\bar{q}^\alpha + \tilde{m}^{\alpha\beta}|_\beta - \tilde{n}^{\alpha\beta} w|_\beta\right]\nu_\alpha = 0 \qquad \text{or} \qquad \delta w = 0 \end{aligned} \tag{2.29}$$

$$(\tilde{m}^{\alpha\beta} - \bar{m}^{\alpha\beta})\nu_\alpha \nu_\beta = 0 \qquad \text{or} \qquad \nu^\alpha \delta w|_\alpha = 0$$

on ∂S with ℓ to be the curve ∂S. In the limiting case of the static deflection of a flat plate, $b_{\alpha\beta} = 0$, $\ddot{u}^\beta = 0$ and $\ddot{w} = 0$, von Karman's equation for large deformation of plate is recovered. There is a clear detailed discussion about the assumptions involved in this case in Fung's Foundations of Solid Mechanics, Chapter 16, (Ref. 8). He derived the von Karman equation directly from three-dimensional equation of equilibrium using undeformed state coordinates and linear Hookean Law relating Kirchhoff stresses and strains.

In the case of an isotropic shell with plane stress approximation and the local metric tensor approximated by that of the middle surface, one has (Ref. 9)

$$C^{\alpha\beta\lambda\theta} = G\left(g^{\alpha\lambda} g^{\beta\theta} + g^{\alpha\theta} g^{\beta\lambda} + \frac{2\nu}{1-\nu} g^{\alpha\beta} g^{\lambda\theta}\right) \tag{2.30}$$

where G is the shearing modulus, and ν is the Poisson ratio. In this case the expression for the second integral in Eq. (2.25) can be written as:

$$\Pi_{12} = \frac{h}{2}\int_S \frac{C^{\alpha\beta\lambda\theta}}{2}\left(e_{\alpha\beta} + \frac{1}{4} w|_\alpha w|_\beta\right) w|_\lambda w|_\theta \, dS$$

$$= \frac{h}{2}\int_S \left[\frac{C^{\alpha\beta\lambda\theta}}{2} u_\alpha|_\beta w|_\lambda w|_\theta - G g^{\beta\theta} b^\lambda_\beta w|_\lambda w|_\theta w \right. \tag{2.31}$$

$$\left. - \frac{G\nu}{4(1-\nu)} b^\beta_\beta w (\text{grad}\, w)^2 + \frac{G\nu}{4(1-\nu)} (\text{grad}\, w)^4 \right] dS \, .$$

Note that

$$\text{grad}\, w = w|_\alpha \underline{g}^\alpha \qquad \text{or} \qquad (\text{grad}\, w)^2 = g^{\alpha\beta} w|_\alpha w|_\beta \, .$$

A special result noted by Koiter is of interest. Consider the case of a spherical shell of radius R with $w = 0$ on the boundary. One has

$$b^\alpha_\beta = -\delta^\alpha_\beta \frac{1}{R} \, .$$

Then the second and third terms in Eq. (2.31), which involve cubic terms in w only, become simply

$$-\frac{G(1+\nu) h}{2(1-\nu)R}\int_S w (\text{grad}\, w)^2 dS$$

$$= \frac{G(1+\nu) h}{4(1-\nu)R}\int_S w^2 g^{\alpha\beta} w|_{\alpha\beta} \, dS \tag{2.32}$$

$$= \frac{G(1+\nu) h}{4(1-\nu)R}\int_S w^2 \nabla^2 w \, dS$$

If further $\tilde{\nabla}^2 w = k w$, then the above integral becomes

$$\frac{G(1+\nu)h}{4(1-\nu)R} k \int_S w^3 \, dS. \tag{2.32a}$$

Hence for such a spherical shell, we have Koiter's expression:

$$\Pi_{112} = \frac{h}{2} \int \frac{C^{\alpha\beta\lambda\theta}}{2} u_{\alpha|\beta} w|_\lambda w|_\theta + \frac{G(1+\nu)h}{4(1-\nu)R} k \int_S w^3 \, dS + O(w^4). \tag{2.33}$$

B. <u>Variational Functional</u>

Now one can proceed to prove that the functional J in Eq. (2.1) satisfies all the requirements. From Section IIA, one has

$$I_3 = \Pi_1 - \Pi_2 - \frac{1}{2} \int_{S_2} \rho h \, \dot{\underline{u}}^2 \, dS, \tag{2.34}$$

and

$$\Pi_1 = \frac{1}{2} \int_{S_2} \left[\tilde{n}^{\alpha\beta} (e_{\alpha\beta} + \frac{1}{2} w|_\alpha w|_\beta) + \tilde{m}^{\alpha\beta} w|_{\alpha\beta} \right] dS, \tag{2.35}$$

$$\Pi_2 = \int_{\Gamma_\sigma} \bar{q}^\alpha \nu_\alpha w \, dl + \int_{\Gamma_\sigma} \bar{n}^{\alpha\beta} u_\alpha \nu_\beta \, dl$$

$$+ \int_{\Gamma_\sigma} \bar{m}^{\alpha\beta} w|_\alpha \nu_\beta \, dl \tag{2.36}$$

as in Eqs. (2.20), (2.21).

The arbitrary variation of displacements over S_1, S_2 and velocity potential ϕ in V, subject to given boundary conditions

$$\begin{aligned} \delta \underline{u} = \delta w|_\alpha &= 0 \qquad \text{at} \qquad \delta S_2 - \Gamma_v , \\ \delta \underline{u} = \delta \phi &= 0 \qquad \text{at} \qquad t = t_0 , t_1 , \\ \underline{N} \cdot \delta \underline{u}_f &= \underline{N} \cdot \delta \underline{u} \qquad \text{over} \qquad S_2 , \end{aligned} \tag{2.37}$$

where $\underline{u}$ is the displacement of the shell and $\underline{u}_f$ is the displacement of the fluid or the vapor, will give:

$$\begin{aligned} \delta \int_{t_0}^{t_1} I_1 \, dt &= - \int_{t_0}^{t_1} \delta \int_V p \, dV \\ &= - \int_{t_0}^{t_1} \left(\int_V \delta p \, dV + \int_{S_1 + S_3} p \, \underline{N} \cdot \delta \underline{u}_f \, dS \right) dt \end{aligned} \tag{2.38}$$

Since

$$p = -\rho_0 \left[\frac{\partial \phi}{\partial t} - \underline{g}(t) \cdot \underline{r} + \frac{1}{2} (\nabla \phi)^2 \right] , \tag{2.39}$$

where $\underline{r}$ is the position vector, and ρ_0 is the density of the fluid, one gets

$$\delta p = -\rho_0 \left[\frac{\partial \delta \phi}{\partial t} + \nabla \phi \cdot \nabla \delta \phi \right]$$

and

$$\delta\int_{t_0}^{t_1} I_1 dt = -\int_{t_0}^{t_1}\int_{S_1+S_3} p\,\underline{N}\cdot\delta\underline{u}_f\,dS\,dt$$

$$+\int_{t_0}^{t_1}\rho_0\left(\int_V\left[\frac{\partial\delta\phi}{\partial t}-\nabla^2\phi\,\delta\phi\right]dV+\int_{S_1+S_3}\underline{N}\cdot\nabla\phi\,\delta\phi\,dS\right)dt$$

$$= -\int_{t_0}^{t_1}\int_{S_1+S_3} p\,\underline{N}\cdot\delta\underline{u}\,dS\,dt \qquad (2.40)$$

$$+\int_{t_0}^{t_1}\rho_0\int_{S_1+S_3}\left(\nabla\phi-\frac{\partial\underline{u}}{\partial t}\right)\cdot\underline{N}\,\delta\phi\,dS\,dt$$

$$-\int_{t_0}^{t_1}\rho_0\int_V \nabla^2\phi\,\delta\phi\,dV\,dt\,,$$

$$\delta\int_{t_0}^{t_1} I_2 dt = \int_{t_0}^{t_1}\delta I_2\,dt\,,$$

where

$$\delta I_2 = \sigma\delta\int_{S_1} dS + \sigma_{VS}\,\delta\int_{S_2-S_3} dS + \sigma_{LS}\,\delta\int_{S_3} dS$$

$$= -\sigma\int_{S_1} B_\alpha^\alpha\,\underline{N}\cdot\delta\underline{u}_f\,dS - \sigma_{VS}\int_{S_2-S_3} B_\alpha^\alpha\,\underline{N}\cdot\delta\underline{u}_f\,dS$$

$$-\sigma_{LS}\int_{S_3} B_\alpha^\alpha\,\underline{N}\cdot\delta\underline{u}_f\,dS + \sigma\int_{\partial S_1}\underline{n}_1\cdot\delta\underline{u}_f\,dl$$

$$-(\sigma_{LS}-\sigma_{VS})\int_{\partial S_1}\underline{n}_3\cdot\delta\underline{u}_f\,dl + \sigma_{VS}\int_{\Gamma_\sigma}\underline{n}_2\cdot\delta\underline{u}\,dl\,,$$

where $\underline{N}$ is the unit outward normal of the surfaces. $\underline{n}_1$ and $\underline{n}_3$ are the unit vectors lying in S_1 and S_3 respectively, and normal to the curve ∂S_1 which is the curve where S_1 and S_2 intersect. $\underline{n}_2$ is the unit vector lying on S_2 normal to the boundary curve Γ_σ of the shell. $B_{\alpha\beta}$ is the curvature tensor for the deformed surfaces. $\underline{n}_1$ can be decomposed into two parts, namely

$$\underline{n}_1 = \sin\theta_0 \, \underline{N} + \cos\theta_0 \, \underline{n}_3 \quad . \tag{2.41}$$

Then one obtains

$$\begin{aligned} \delta I_2 = & -\sigma \int_{S_1} B^\alpha_\alpha \underline{N} \cdot \delta \underline{u}_f \, dS - \sigma_{VS} \int_{S_2 - S_3} B^\alpha_\alpha \underline{N} \cdot \delta \underline{u}_f \, dS \\ & - \sigma_{LS} \int_{S_3} B^\alpha_\alpha \underline{N} \cdot \delta \underline{u}_f \, dS + \sigma \int_{\partial S_1} \sin\theta_0 \, \underline{N} \cdot \delta \underline{u} \, dl \\ & + \sigma_{VS} \int_{\Gamma_\sigma} \underline{n}_2 \cdot \delta \underline{u} \, dl \\ & + \int_{\partial S_1} [\sigma \cos\theta_0 - (\sigma_{VS} - \sigma_{LS})] \, \underline{n}_3 \cdot \delta \underline{u}_f \, dl \quad . \end{aligned} \tag{2.42}$$

From the result of Section IIA, we know that

$$\begin{aligned} \delta \int_{t_0}^{t_1} I_3 \, dt = \int_{t_0}^{t_1} \Big\{ & \int_{S_2^*} (\rho h \ddot{u}^\beta - \tilde{n}^{\alpha\beta}|_\alpha) \, \delta u_\beta \, dS^* \\ & + \int_{S_2^*} [\rho h \ddot{w} + \tilde{m}^{\alpha\beta}|_{\beta\alpha} - \tilde{n}^{\alpha\beta} b_{\alpha\beta} - (\tilde{n}^{\alpha\beta} w|_\beta)|_\alpha] \, \delta w \, dS^* \end{aligned}$$

$$+\int_{\Gamma_\sigma^*}(\tilde{n}^{\alpha\beta}-\bar{n}^{\alpha\beta})\delta u_\beta \nu_\alpha \, d\ell^* - \int_{\Gamma_\sigma^*}(\tilde{m}^{\alpha\beta}-\bar{m}^{\alpha\beta})\,\delta w|_\beta \, d\ell^*$$

$$+\int_{\Gamma_\sigma^*}(\tilde{n}^{\alpha\beta} w|_\beta - \tilde{m}^{\alpha\beta}|_\beta - \bar{q}^\alpha)\,\nu_\alpha \,\delta w \, d\ell^* , \tag{2.43}$$

where an asterisk * refers to undeformed state.

The surface and line integrals in Eqs. (2.40), (2.42) are all integrated over the deformed state. The parts that are integrated over S_2, ∂S_1 and Γ_σ must be transformed to undeformed surface before one can compare with Eq. (2.43). Since

$$\underline{N} = \underline{N}_o - (w|_\alpha + u^\gamma b_{\gamma\alpha})\,\underline{g}^\alpha ,$$

(see Eqs. (2.6) and (2.7)) and

$$dS = (1 + E^\alpha_\alpha)\, dS^* ,$$

where $E_{\alpha\beta}$ are defined in Eq. (2.10), then one has

$$\underline{N}\cdot\delta\underline{u}\, dS = [\delta w - (w|_\alpha + u^\gamma b_{\gamma\alpha})\,\delta u^\alpha](1 + E^\beta_\beta)\, dS^* .$$

Substituting into Eqs. (2.40) and (2.42), one gets

$$\delta\int_{t_0}^{t_1} I_1\,dt = \int_{t_0}^{t_1}\Big\{ \int_{S_1+S_3} \rho_0\left(\nabla\phi - \frac{\partial \underline{u}}{\partial t}\right)\cdot\underline{N}\,\delta\phi\,dS$$

$$-\int_V \rho_0\nabla^2\phi\,\delta\phi\,dV - \int_{S_1} p\,\underline{N}\cdot\delta\underline{u}_f\,dS \qquad (2.44)$$

$$-\int_{S_3^*} p(1+E_\alpha^\alpha)\left[\delta w - (w|_\alpha + u_r b_\alpha^r)\delta u^\alpha\right]dS\Big\}\,dt,$$

$$\delta\int_{t_0}^{t_1} I_2\,dt = \int_{t_0}^{t_1}\Big\{-\sigma\int_{S_1} B_\alpha^\alpha\,\underline{N}\cdot\delta\underline{u}_f\,dS + \sigma_{VS}\int_{\Gamma_\sigma^*}\nu_\beta\,\delta u^\beta\frac{d\ell}{d\ell^*}\,d\ell^*$$

$$+\int_{\partial S_1}\left[\sigma\cos\theta_0 - (\sigma_{VS}-\sigma_{LS})\right]\underline{n}_3\cdot\delta\underline{u}_f\,d\ell$$

$$-\sigma\int_{\partial S_1^*}\sin\theta_0\left[\delta w - (w|_\alpha + u_r b_\alpha^r)\delta u^\alpha\right]\frac{d\ell}{d\ell^*}\,d\ell^* \qquad (2.45)$$

$$-\sigma_{VS}\int_{S_2^*-S_3^*} B_\alpha^\alpha(1+E_\beta^\beta)\left[\delta w - (w|_\lambda + u_r b_\lambda^r)\delta u^\lambda\right]dS^*$$

$$-\sigma_{LS}\int_{S_3^*} B_\alpha^\alpha(1+E_\beta^\beta)\left[\delta w - (w|_\lambda + u_r b_\lambda^r)\delta u^\lambda\right]dS^*\Big\}\,dt\,.$$

When

$$\delta J = \int_{t_0}^{t_1}(\delta I_1 + \delta I_2 + \delta I_3)\,dt = 0 \qquad (2.46)$$

for arbitrary variation of $\delta \underline{u}$, $\delta\phi$ and $\delta \underline{u}_f$ under condition of Eq. (2.37), one will require the following differential equations to be satisfied:

$$\nabla^2\phi = 0 \qquad \text{in} \qquad V \tag{2.47}$$

$$\left(\nabla\phi - \frac{\partial \underline{u}}{\partial t}\right)\cdot \underline{N} = 0 \qquad \text{on} \qquad S_1 \, , \, S_3 \tag{2.48}$$

$$\sigma B^{\alpha}_{\alpha} + p = 0 \qquad \text{on} \qquad S_1 \tag{2.49}$$

$$\tilde{n}^{\alpha\beta}|_{\beta} - \rho h \ddot{u}^{\alpha} = \left[\sigma_{LS} B^{\beta}_{\beta}(1+E^{\lambda}_{\lambda}) + p + \sigma \sin\theta_0 \frac{dl}{dl^*}\delta(\partial S_1^*)\right] \times (w|_{\alpha_1} + u_r b^r_{\alpha_1})\, g^{\alpha_1\alpha}$$

$$\text{on} \qquad S_3^*$$

$$= \sigma_{VS} B^{\beta}_{\beta}(1+E^{\lambda}_{\lambda})(w|_{\epsilon} + u_r b^r_{\epsilon})\, g^{\epsilon\alpha}$$

$$\text{on} \qquad S_2^* - S_3^* \tag{2.50}$$

and

$$-\tilde{m}^{\alpha\beta}|_{\beta\alpha} + \tilde{n}^{\alpha\beta} b_{\beta\alpha} + (\tilde{n}^{\alpha\beta} w|_{\beta})|_{\alpha} - \rho h \ddot{w}$$

$$= -p - \sigma_{LS} B^{\alpha}_{\alpha} + \sigma \sin\theta_0 \frac{dl}{dl^*}\delta(\partial S_1)$$

$$\text{on} \qquad S_3^*$$

$$= -\sigma_{VS} B_\alpha^\alpha \qquad \text{on} \qquad S_2^* - S_3^* \tag{2.51}$$

where $d\ell / d\ell^*$ is the ratio of the deformed length to undeformed length of the curve ∂S_1, and

$\delta(\Gamma)$ is so defined that

$\delta(\Gamma) = 0$ everywhere except on the curve Γ.

$$\int_S \delta(\Gamma) dS = \ell \qquad \text{if} \qquad \Gamma \in S. \tag{2.52}$$

where ℓ is the arc length of Γ in S.

One also requires the following boundary conditions to be satisfied:

$$\sigma_{VS} \nu^\alpha + \tilde{n}^{\alpha\beta} \nu_\beta = \bar{n}^{\alpha\beta} \nu_\beta \qquad \text{or} \qquad \delta u_\beta = 0$$

$$\begin{aligned} &\frac{\partial}{\partial \ell}\left[(\bar{m}^{\alpha\beta} - \tilde{m}^{\alpha\beta}) \nu_\alpha \frac{\partial \ell}{\partial \lambda^\beta}\right] - \bar{q}^\alpha \nu_\alpha \\ &\quad + \nu_\alpha (\tilde{n}^{\alpha\beta} w|_\beta - \tilde{m}^{\alpha\beta}|_\beta) = 0 \qquad \text{or} \qquad \delta w = 0 \end{aligned} \tag{2.53}$$

$$(\tilde{m}^{\alpha\beta} - \bar{m}^{\alpha\beta}) \nu_\alpha \nu_\beta = 0 \qquad \text{or} \qquad \nu^\alpha \delta w|_\alpha = 0$$

on the boundary ∂S_2 of the shell with ℓ representing the curve ∂S_2 and

$$\sigma \cos \theta_0 = \sigma_{LS} - \sigma_{VS} \qquad \text{or} \qquad \underline{n}_3 \cdot \delta \underline{u}_f = 0 \tag{2.54}$$

on ∂S_1. The first condition in the last equation is the constant angle of contact between the liquid and the shell. The second case is the so-called stuck condition. Thus by requiring the first variation of J to be zero, one gets the differential Eqs. (2.47) - (2.51) and the appropriate boundary conditions in Eqs. (2.53) and (2.54).

III. LINEARIZATION OF THE SYSTEM

In this section, we shall simplify the formulation of the last section by restricting ourselves to a system as shown in Fig. 2, where S_1 is the free surface in a static equilibrium configuration, S_4 and S_5 are rigid side walls, S_4 being the wetted area, S_2 is the flexible elastic shell, and ∂S_1 is the intersecting curve of the liquid-vapor-solid interface. $\underline{r}$ is the position vector of a spatial point. The motion of the liquid and the deformation of the shell from its initial state are assumed to be infinitesimal. Thus the system of the governing equation can be linearized. Finally, we assume that the surface tension between the liquid and the shell surface is so small that it can be neglected.

Let $\underline{u}_o$ be the initial deflection of the shell. If one uses a set of intrinsic curvilinear coordinates x^α ($\alpha = 1, 2$) to describe points on the initially deformed shell and writes $T^{\alpha\beta}$ and $m_o^{\alpha\beta}$ as the initial stress resultants and stress moments, one has from Eqs. (2.23) and (2.24)

$$\tilde{n}^{\alpha\beta} = T^{\alpha\beta} + h\, C^{\alpha\beta\lambda\theta}\, w_o|_\lambda w|_\theta + n^{\alpha\beta},$$

$$\tilde{m} = m_o^{\alpha\beta} + m^{\alpha\beta}.$$

Here w_o is used to denote initial normal deflection and w to denote the perturbed normal deflection of the shell. $m^{\alpha\beta}$ and $n^{\alpha\beta}$ are the perturbed stress moments and stress resultants respectively. We shall make the assumption that $w_o|_\lambda w|_\theta$ are small and can be neglected. Substituting $\tilde{n}^{\alpha\beta}$ and $\tilde{m}^{\alpha\beta}$ into Eqs. (2.48) to (2.51) and using the fact that

$$T^{\alpha\beta}|_{\beta} = 0$$

$$-m_0^{\alpha\beta}|_{\beta\alpha} + T^{\alpha\beta} b_{\alpha\beta} = p_0$$

after linearization, one gets

$$(\nabla\phi - \frac{\partial \underline{u}}{\partial t}) \cdot \underline{N}_0 = 0 \qquad \text{on} \qquad S_1 , S_2$$

$$\sigma (B_\alpha^\alpha - \kappa_\alpha^\alpha) + p' = 0 \qquad \text{on} \qquad S_1$$

$$n^{\alpha\beta}|_{\beta} - \rho h \ddot{u}^\alpha = p_0 (w|_\lambda + u_r b_\lambda^r) g^{\lambda\alpha} \qquad \text{on} \qquad S_2 \qquad (3.1)$$

$$-m^{\alpha\beta}|_{\beta\alpha} + n^{\alpha\beta} b_{\beta\alpha} + T^{\alpha\beta} w|_{\alpha\beta} - \rho h \ddot{w} = -p' \qquad \text{on} \qquad S_2$$

where $\frac{1}{2}\ \kappa_\alpha^\alpha$ is the mean curvature of the free surface at static equilibrium. p_o is the pressure on S_1 and S_2 at the equilibrium state and p' is the disturbed pressure that

$$p' = -\rho_0 \left[\frac{\partial \phi}{\partial t} - \underline{g}(t) \cdot \underline{N}_0 \eta - \underline{g}_1(t) \cdot \underline{r}_0 \right] \qquad \text{on} \qquad S_1$$

$$= -\rho_0 \left[\frac{\partial \phi}{\partial t} - \underline{g}(t) \cdot \underline{u} - \underline{g}_1(t) \cdot \underline{r}_0 \right] \qquad \text{on} \qquad S_2 \qquad (3.2)$$

where η is the perturbed free surface displacement normal to the equilibrium free surface. $\underline{r}_o$ is the vector form of the free surface, S_1 and the middle surface S_2 of the shell, $\underline{g}_1(t)$ is the disturbed gravitational acceleration and

$$n^{\alpha\beta} = h\, C^{\alpha\beta\lambda\theta} e_{\lambda\theta}$$

$$m^{\alpha\beta} = \frac{h^3}{12} C^{\alpha\beta\lambda\theta} w|_{\lambda\theta} \,. \tag{3.3}$$

If the shell is simply supported or clamped to rigid walls, then the boundary conditions for the shell are

$$u^\alpha = w = 0 \qquad \text{and either} \qquad \nu_\beta m^{\alpha\beta}\nu_\alpha = 0 \text{ or } \nu^\beta w|_\beta = 0\,. \tag{3.4}$$

For the free surface, we shall assume a general boundary condition for η on ∂S_1:

$$\frac{\partial \eta}{\partial n} + \sigma_H \eta = 0 \tag{3.5}$$

where $\partial/\partial n$ is the direction derivative along a vector $\underline{n}$. Here $\underline{n}$ is a unit vector lying on S_1 and normal to the curve ∂S_1, the positive direction of $\underline{n}$ is chosen to point toward the container. σ_H is the hysteresis coefficient. When $\sigma_H \rightarrow \infty$, we have the "stuck" condition $\eta = 0$ on ∂S_1. When σ_H is equal to a certain value (see Appendix B), depending on the local condition of the container and the equilibrium free surface on ∂S_1, we have a constant angle of contact.

To simplify the equations further, the following additional assumptions are made: (1) p_o is relatively small, the products of p_o and the components of the displacement vector or the deformation gradient can be neglected on S_2; (2) the term

$$\rho_o \underline{g}(t) \cdot \underline{u}$$

in Eq. (3.2) can be approximated as

$$\rho_o \underline{g}(t) \cdot \underline{N}_o w$$

on S_2. Expressing the terms $B^{\alpha}_{\alpha} - \kappa^{\alpha}_{\alpha}$ explicitely in terms of η (see Appendix B), and using $a_{\alpha\beta}$, $\kappa_{\alpha\beta}$ and $\underline{a}_{\alpha}$ as the metric tensor, curvature tensor and base vectors respectively for the free surface, one has the following governing equations for the linearized system:

The differential equations

$$\nabla^2 \phi = 0 \qquad \text{in} \quad V \qquad (3.6)$$

$$\sigma \left(a^{\alpha\beta} \eta |_{\beta\alpha} + \kappa^{\alpha}_{\beta} \kappa^{\beta}_{\alpha} \eta \right) + p' = 0 \qquad \text{on} \quad S_1 \qquad (3.7)$$

$$n^{\alpha\beta}|_{\beta} - \rho h \ddot{u}^{\alpha} = 0 \qquad \text{on} \quad S_2 \qquad (3.8)$$

$$-m^{\alpha\beta}|_{\beta\alpha} + n^{\alpha\beta} b_{\beta\alpha} + T^{\alpha\beta} w|_{\beta\alpha} - \rho h \ddot{w} + p' = 0 \qquad \text{on} \quad S_2 \qquad (3.9)$$

where

$$p' = -\rho_0\left[\frac{\partial\phi}{\partial t} - \underline{g}(t)\cdot\underline{N}_0\eta - \underline{g}_1(t)\cdot\underline{r}_0\right] \qquad \text{on} \quad S_1$$

$$= -\rho_0\left[\frac{\partial\phi}{\partial t} - \underline{g}(t)\cdot\underline{N}_0\varpi - \underline{g}_1(t)\cdot\underline{r}_0\right] \qquad \text{on} \quad S_2 \qquad (3.10)$$

The kinematic conditions

$$\underline{N}_0\cdot\nabla\phi = \frac{\partial\eta}{\partial t} \qquad \text{on} \quad S_1$$

$$= \frac{\partial\varpi}{\partial t} \qquad \text{on} \quad S_2 \qquad (3.11)$$

$$= 0 \qquad \text{on} \quad S_4$$

The boundary conditions

$$\underline{u}(\underline{x}, t) = 0 \qquad \text{on} \quad \partial S_2 \qquad (3.12)$$

either $\quad m^{\alpha\beta}\nu_\beta = 0 \text{ or } \varpi|_\alpha = 0 \qquad \text{on} \quad \partial S_2 \qquad (3.13)$

$$\frac{\partial\eta}{\partial n} + \gamma_H\eta = 0 \qquad \text{on} \quad \partial S_1 \qquad (3.14)$$

It can be shown that under our linearization scheme and the additional assumptions named above, the functional defined in Eq. (2.1) is reduced to the following form:

$$J = \int_{t_0}^{t_1} I \, dt ,$$

$$I = J_1 + J_2 - J_3 - J_4 , \tag{3.15}$$

where

$$J_1 = \frac{1}{2}\int_{S_1}\Big[\sigma\left(a^{\alpha\beta}\eta|_\alpha \eta|_\beta - K^\alpha_\beta K^\beta_\alpha \eta^2\right) - \rho_0 \underline{g}(t)\cdot\underline{N}_0 \eta^2$$

$$- 2\rho_0 \underline{g}_1(t)\cdot\underline{r}_0 \,\eta\Big] dS + \frac{1}{2}\int_{\partial S_1} \sigma \,\sigma_H \,\eta^2 \, d\ell \tag{3.16}$$

$$J_2 = \frac{1}{2}\int_{S_2}\Big[m^{\alpha\beta} w|_{\beta\alpha} + n^{\alpha\beta} e_{\alpha\beta} + T^{\alpha\beta} w|_\alpha w|_\beta$$

$$- \rho_0 \underline{g}(t)\cdot\underline{N}_0 w^2 - 2\rho_0 \underline{g}_1(t)\cdot\underline{r}_0 \,w\Big] \, dS \tag{3.17}$$

$$J_3 = \frac{1}{2}\int_{S_2} \rho h \,\underline{\dot{u}}\cdot\underline{\dot{u}} \, dS \tag{3.18}$$

$$J_4 = -\int_{S_1} \rho_0 \frac{\partial \phi}{\partial t}\eta \, dS - \int_{S_2} \rho_0 \frac{\partial \phi}{\partial t} w \, dS - \frac{1}{2}\int_V \rho_0 (\nabla\phi)^2 dV \tag{3.19}$$

It is straightforward to show that the requirement for the first variation of J in Eq. (3.15) to be zero, will provide us all the equations (3.6) to (3.14). The functional in the form of Eqs. (3.15) to (3.19) will be used to obtain approximate solutions which will be discussed in Section VII.

We shall call J_1 the potential energy of the free surface, and J_2 and J_3 the potential and the kinetic energy, respectively, of the shell. In order to give the physical meaning of J_4, one must write it in a slightly different form from Eq. (3.19):

$$J_4 = -\frac{d}{dt}\left(\int_{S_1} \rho_0 \phi \eta \, dS + \int_{S_2} \rho_0 \phi w \, dS\right) + \int_{S_1} \rho_0 \phi \frac{\partial \eta}{\partial t} dS + \int_{S_2} \rho_0 \phi \frac{\partial w}{\partial t} dS - \frac{1}{2}\int_V \rho_0 (\nabla\phi)^2 dV. \tag{3.20}$$

If Eqs. (3.6) and (3.11) hold, then we have

$$J_4 = -\frac{d}{dt}\left(\int_{S_1} \rho_0 \phi \eta \, dS + \int_{S_2} \rho_0 \phi w \, dS\right) + \frac{1}{2}\int \rho_0 (\nabla\phi)^2 dV.$$

Since the first term in the right hand side makes no contribution in the variation, it is clear that J_4 corresponds to the kinetic energy of the fluid.

IV. PROPERTIES OF EIGENVALUES AND EIGENFUNCTIONS

In this section, we shall assume the gravitational force $\underline{g}$ is constant, i.e. $\underline{g}_1(t) = 0$. Thus the coefficients of the system of equations are independent of time t. Then one can assume a solution proportional to a time factor

$$e^{i\omega t}$$

and eliminate the time derivatives in all equations. This enables one to study the problem in terms of the eigenfunctions.

We assume ϕ, η and $\underline{u}$ to have the form

$$\phi = i\omega\, \Phi(\underline{x}, \mu)\, e^{i\omega t},$$

$$\eta = H(\underline{x}, \mu)\, e^{i\omega t}, \tag{4.1}$$

$$\underline{u} = \underline{U}(\underline{x}, \mu)\, e^{i\omega t} = (W\, \underline{N}_o + U^\alpha \underline{g}_\alpha)\, e^{i\omega t},$$

where $\mu = \omega^2$ and Φ, H, and $\underline{U}$ are functions of spatial coordinates $\underline{x}$ only. Let us define

$$\frac{\partial \Phi}{\partial N} = \underline{N}_o \cdot \nabla \Phi \tag{4.2}$$

on the surfaces that bound the liquid. If one substitutes Eq. (4.1) into Eqs. (3.6) to (3.14) with Eqs. (3.7) to (3.9) being differentiated once with respect to time one gets

$$\nabla^2 \Phi = 0 \qquad \text{in} \qquad V \tag{4.3}$$

$$\sigma (a^{\alpha\beta} H|_{\alpha\beta} + k^{\alpha}_{\beta} k^{\beta}_{\alpha} H) + \rho_0 \mu \Phi$$

$$+ \rho_0 \underline{g}_0 \cdot \underline{N}_0 H = 0 \qquad \text{on} \qquad S_1 \tag{4.4}$$

$$N^{\alpha\beta}|_{\beta} + \mu \rho h U^{\alpha} = 0 \qquad \text{on} \qquad S_2 \tag{4.5}$$

$$-M^{\alpha\beta}|_{\beta\alpha} + N^{\alpha\beta} b_{\beta\alpha} + T^{\alpha\beta} W b_{\alpha\beta} + \rho h \mu W$$

$$= -(\rho_0 \mu \Phi + \rho_0 \underline{g}_0 \cdot \underline{N}_0 W) \qquad \text{on} \qquad S_2 \tag{4.6}$$

where $N^{\alpha\beta}$, $M^{\alpha\beta}$ are so defined that

$$n^{\alpha\beta} = N^{\alpha\beta} e^{i\omega t}$$

$$m^{\alpha\beta} = M^{\alpha\beta} e^{i\omega t} \tag{4.7}$$

The kinematic conditions become

$$\frac{\partial \Phi}{\partial N} = H \qquad \text{on} \qquad S_1$$

$$= W \qquad \text{on} \qquad S_2 \qquad (4.8)$$

$$= 0 \qquad \text{on} \qquad S_4$$

and the boundary condition in Eqs. (3.12) to (3.14) becomes

$$\underline{U} = 0 \qquad (4.9)$$

and either $\quad W|_\beta = 0 \quad$ or $\quad M^{\alpha\beta}\nu_\beta = 0 \quad$ on $\quad \partial S_2 \quad$ (4.10)

and
$$\frac{\partial H}{\partial n} + \sigma_H H = 0 \qquad (4.11)$$

on ∂S_1. The functions Φ, H, and $\underline{U}$ that satisfy Eqs. (4.3) to (4.8) and boundary conditions (4.9) to (4.11) form a set of eigenfunctions and the parameter μ is an eigenvalue of the problem.

A. Orthogonality Relation

If one takes Eq. (4.4) with

$$\begin{aligned} \Phi_m &= \Phi(\underline{x}, \mu_m) \\ H_m &= H(\underline{x}, \mu_m) \\ \underline{U}_m &= \underline{U}(\underline{x}, \mu_m) \\ \mu_m &= \mu \end{aligned} \tag{4.12}$$

and the corresponding equation with n instead of m, multiplied by H_n and H_m respectively, subtracts and integrates over S_1, with an appropriate application of the divergence theorem, we obtain

$$\rho_0 \int_{S_1} (\mu_m \Phi_m H_n - \mu_n \Phi_n H_m)\, dS = 0$$

or

$$\rho_0 \int_{S_1} \left(\mu_m \Phi_m \frac{\partial \Phi_n}{\partial N} - \mu_n \Phi_n \frac{\partial \Phi_m}{\partial N}\right) dS = 0 \tag{4.13}$$

Similarly from Eqs. (4.5) and (4.6) we have

$$\rho h(\mu_m - \mu_n) \int_{S_2} \underline{U}_m \cdot \underline{U}_n \, dS + \rho_0 \int_{S_2} \left(\mu_m \Phi_m \frac{\partial \Phi_n}{\partial N} - \mu_n \Phi_n \frac{\partial \Phi_m}{\partial N}\right) dS = 0 \tag{4.14}$$

If we add Eq. (4.14) to Eq. (4.13), we get the new equation

$$\rho h(\mu_m-\mu_n)\int_{S_2} \underline{U}_m\cdot\underline{U}_n dS+\rho_0\int_{S_1+S_2}\left(\mu_m\Phi_m\frac{\partial\Phi_n}{\partial N}-\mu_n\Phi_n\frac{\partial\Phi_m}{\partial N}\right)dS=0 \quad . \tag{4.15}$$

On noting the fact that

$$\frac{\partial\Phi_j}{\partial N}=0 \qquad (j=m,\,n) \qquad \text{on} \qquad S_4$$

$$\int_{S_1+S_2}\Phi_m\frac{\partial\Phi_n}{\partial N}dS=\int_V\nabla\Phi_m\cdot\nabla\Phi_n dV=\int_{S_1+S_2}\Phi_n\frac{\partial\Phi_m}{\partial N} \tag{4.16}$$

and $\qquad \partial V = S_1+S_2+S_4 \quad ,$

Eq. (4.15) may be written as

$$(\mu_m-\mu_n)\left(\int_{S_2}\rho h\,\underline{U}_m\cdot\underline{U}_n dS+\int_V\rho_0\nabla\Phi_m\cdot\nabla\Phi_n dV\right)=0 \tag{4.17}$$

Since $\mu_m \neq \mu_n$ if $m \neq n$, we have the simple orthogonal relation

$$\rho h\int_{S_2}\underline{U}_m\cdot\underline{U}_n dS+\rho_0\int_V\nabla\Phi_m\cdot\nabla\Phi_n dV=0 \tag{4.18}$$

or, since $\qquad \nabla^2\Phi_m=0 \quad \text{and} \quad \nabla^2\Phi_n=0$

$$\int_{S_2} \rho h \, \underline{U}_m \cdot \underline{U}_n \, dS + \int_{S_1+S_2} \rho_o \Phi_n \frac{\partial \Phi_m}{\partial N} \, dS = 0 \quad . \tag{4.19}$$

There are two simple limiting cases. First, when ρ_o tends to zero, we have

$$\int_{S_2} \rho h \, \underline{U}_m \cdot \underline{U}_n \, dS = 0 \tag{4.20}$$

which is the case of free vibration of a shell. In the second case $\underline{U}_m = 0$ on S_2, so we have

$$\int_V \rho_o \nabla \Phi_m \cdot \nabla \Phi_n \, dV = 0 \tag{4.21}$$

which is the case of a rigid tank.

If necessary, by multiplying with a constant, we can arrange that

$$\int_{S_2} \rho h (\underline{U}_m)^2 dS + \int_V \rho_o (\nabla \Phi_m)^2 dV = 1 \quad . \tag{4.22}$$

Then the functions Φ_m, $\underline{U}_m$ and H_m form a normal orthogonal set in the sense of Eqs. (4.18) and (4.22). Sometimes we can neglect the in-plane inertia, i.e. we may approximate Eq. (4.5) as

$$N^{\alpha\beta}|_\beta = 0 \quad .$$

By so doing, we shall get the orthogonality relation

$$\int_{S_2} \rho h \frac{\partial \Phi_n}{\partial N} \frac{\partial \Phi_m}{\partial N} dS + \int_V \rho_0 \nabla \Phi_m \cdot \nabla \Phi_n dV = \delta_{mn} \qquad (4.23)$$

where δ_{mn} is the Kronecker- δ . Note that

$$\frac{\partial \Phi}{\partial N} = W \qquad \text{on} \qquad S_2 \qquad .$$

If $\mu = a + iv \neq 0$ is a complex number, then its complex conjugate

$$\bar{\mu} = a + (-iv)$$

would also be an eigenvalue. Since $\Phi(\underline{x}, \bar{\mu})$, $H(\underline{x}, \bar{\mu})$ and $\underline{U}(\underline{x}, \bar{\mu})$ are the complex conjugates of $\Phi(\underline{x}, \mu)$, $H(\underline{x}, \mu)$, and $\underline{U}(\underline{x}, \mu)$ respectively, Eq. (4.18) gives

$$\int_{S_2} \rho h |\underline{U}|^2 dS + \int_V \rho_0 |\nabla \Phi|^2 dV = 0 \qquad (4.24)$$

which is impossible if $\nabla \Phi$ and $\underline{U}$ are not identically zero. Hence all eigenvalues $\mu (= \omega^2)$ are real, i.e. ω is real or purely imaginary.

For an arbitrary μ one can find the set of functions $\Phi(\underline{x}, \mu)$, $H(\underline{x}, \mu)$, and $\underline{U}(\underline{x}, \mu)$ that satisfies Eqs. (4.3) to (4.8) and all boundary conditions in Eqs. (4.9) to (4.11) except one, say

$$W(\underline{x}, \mu) = 0 \qquad \text{on} \qquad \partial S_2 \tag{4.25}$$

then Eq. (4.25) becomes the frequency equation of our problem for the determination of μ. If we have another similar set of functions $\Phi(\underline{x}, \mu')$, $H(\underline{x}, \mu')$ and $\underline{U}(\underline{x}, \mu')$ then one can easily show that

$$(\mu-\mu')\left[\int_{S_2} \rho h\, \underline{U}(\underline{x},\mu)\cdot\underline{U}(\underline{x},\mu')\,dS + \int_V \rho_0 \nabla\Phi(\underline{x},\mu)\cdot\nabla\Phi(\underline{x},\mu')\,dV\right]$$

$$= \int_{\partial S_2}\left[A^\alpha(\underline{x},\mu')W(\underline{x},\mu) - A^\alpha(\underline{x},\mu)W(\underline{x},\mu')\right]\nu_\alpha\, d\ell \tag{4.26}$$

where

$$A^\alpha(\underline{x},\mu) = N^{\alpha\beta}(\underline{x},\mu)\,W(\underline{x},\mu)|_\beta - M^{\alpha\beta}(\underline{x},\mu)\,|_\beta$$

which is not identically zero for any eigenvalue μ except the case of the trivial solution $\underline{U} = 0$. If $\mu_0 (\neq 0)$ is a double root of Eq. (4.25), then

$$W(\underline{x}, \mu_0 \pm i\nu) = O(\nu^2)$$

on ∂S_2, as $\nu \to 0$. Hence if $\mu = \mu_0 + i\nu$ and $\mu' = \mu_0 - i\nu$, the right hand side of Eq. (4.26) is $O(\nu^2)$, but, if one excludes the trivial solution Φ = constant and $\underline{U} = 0$, the left hand side is

$$\sim 2i\nu$$

Thus, one obtains a contradiction. Hence all the roots of frequency equations are simple except the possibility of existing zero roots, which may correspond to one trivial solution and one nontrivial solution.

B. Extremum Properties of Eigenvalues

a. Classical extremum properties

For further study of the properties of the eigenvalues, we introduce the quadratic functional expressions $\mathcal{D}(\underline{U}, H)$ and $\mathcal{H}(\Phi, \underline{U})$ defined as

$$\mathcal{D}(\underline{U}, H) = D(\underline{U}, H) + \int_{\partial S_1} \sigma \sigma_H H^2 d\ell \tag{4.27}$$

with $D(\underline{U}, H) = D_1(H) + D_2(\underline{U})$

$$D_1(H) = \int_{S_1} \sigma \left[a^{\alpha\beta} H|_\alpha H|_\beta - k^\alpha_\beta k^\beta_\alpha H^2 - \frac{\rho_0 \underline{g}_0 \cdot \underline{N}_0 H^2}{\sigma} \right] dS \tag{4.28}$$

$$D_2(\underline{U}) = \int_{S_2} \left[h C^{\alpha\beta\lambda\theta} e_{\alpha\beta} e_{\lambda\theta} + T^{\alpha\beta} W|_\alpha W|_\beta + \frac{h^3}{12} C^{\alpha\beta\lambda\theta} W|_{\alpha\beta} W|_{\lambda\theta} - \rho_0 \underline{g}_0 \cdot \underline{N}_0 W^2 \right] dS$$

and

$$\mathcal{H}(\Phi, \underline{U}) = \int_{S_2} \rho h (\underline{U})^2 dS + \int_V \rho_0 (\nabla \Phi)^2 dV \quad . \tag{4.29}$$

The associated forms

$$\mathfrak{D}(\underline{U},H;\underline{U}',H') = D(\underline{U},H;\underline{U}',H') + \int_{\partial S_1} \sigma\sigma_H H H' d\ell \tag{4.30}$$

$$\begin{aligned} D(\underline{U},H;\underline{U}',H') = &\int_{S_1} [\sigma(a^{\alpha\beta} H|_\beta H'|_\alpha - \varkappa^\alpha_\beta \varkappa^\beta_\alpha H H') - \rho_0 \underline{g}_0 \cdot \underline{N}_0 H H'] dS \\ &+ \int_{S_2} [h C^{\alpha\beta\lambda\theta} e_{\alpha\beta} e'_{\lambda\theta} + T^{\alpha\beta} W|_\alpha W'|_\beta + \frac{h^3}{12} C^{\alpha\beta\lambda\theta} W|_{\alpha\beta} W'|_{\lambda\theta} \\ &\qquad - \rho_0 \underline{g}_0 \cdot \underline{N}_0 W W'] \, dS \end{aligned} \tag{4.31}$$

$$\mathcal{H}(\Phi,\underline{U};\Phi',\underline{U}') = \int_{S_2} \rho h \, \underline{U}\cdot\underline{U}' dS + \int_V \rho_0 \nabla\Phi \cdot \nabla\Phi' dV \tag{4.32}$$

satisfy the relations

$$\begin{aligned} &\mathfrak{D}(\underline{U}+\underline{U}', H+H') = \mathfrak{D}(\underline{U},H) + 2\mathfrak{D}(\underline{U},H;\underline{U}',H') + \mathfrak{D}(\underline{U}',H') \\ &\mathcal{H}(\Phi+\Phi',\underline{U}+\underline{U}') = \mathcal{H}(\Phi,\underline{U}) + 2\mathcal{H}(\Phi,\underline{U};\Phi',\underline{U}') + \mathcal{H}(\Phi',\underline{U}') \end{aligned} \tag{4.33}$$

where $\underline{U} = U^\alpha \underline{g}^\alpha + W \underline{N}_0$

$$2e_{\alpha\beta} = U_\alpha|_\beta + U_\beta|_\alpha - 2b_{\alpha\beta} W$$

$$\begin{aligned} \frac{\partial\Phi}{\partial N} &= H \quad \text{on } S_1 \\ &= W \quad \text{on } S_2 \\ &= 0 \quad \text{on } S_4 \end{aligned} \tag{4.34}$$

and Φ', U'^{α}, W', and H' satisfy the similar relation. We require the argument functions to be continuous in their corresponding closed domains, where the functions are defined. We also require $\nabla\Phi$ to be piecewise continuous in V. $\underline{U}$ and $\partial\Phi/\partial N$ must have piecewise continuous first derivativesin S_2 and S_1 respectively and $\partial\Phi/\partial N$ must have a piecewise continuous second derivative in S_2.

One obtains the eigenvalue μ_m and the associated eigenfunction sets Φ_m, $\underline{U}_m$, H_m of the differential equations (4.3) to (4.6) from the minimum properties, similar to that of Ref. 10, p. 399, under the conditions

$$
\begin{aligned}
&\mathcal{H}(\Phi, \underline{U}) = 1 && \\
&\nabla^2 \Phi = 0 && \text{in} \quad V \\
&\frac{\partial\Phi}{\partial N} = H && \text{in} \quad S_1 \\
&\qquad = W && \text{in} \quad S_2 \\
&\qquad = 0 && \text{in} \quad S_4 \quad . \qquad (4.35)
\end{aligned}
$$

The admissi ble functions which minimize the expression $\mathfrak{D}(\underline{U}, H)$comprise a set of eigenfunctions Φ_1, $\underline{U}_1$,H_1 for the differential Eqs. (4.3) to (4.6) and satisfy the boundary conditions in Eqs. (4.8) to (4.11). The minimum value of $\mathfrak{D}$ is the corresponding eigenvalue. If the harmonic function Φ satisfies not only the conditions in Eq. (4.35) but also the orthogonality condition

$$\mathcal{H}(\Phi, \underline{U}; \Phi_1, \underline{U}_1) = 0 \qquad (4.36)$$

then the solution is again an eigenfunction set Φ_2, $\underline{U}_2$, and H_2 of Eqs. (4.3) to (4.6) satisfying the same boundary conditions. The minimum value $\mathfrak{D}(\underline{u}_2, H_2) = \mu_2$ is the associated eigenvalue. The successive minimum problems, $\mathfrak{D}(\underline{U}, H)$ = minimum subject to the conditions in Eqs. (4.8) to (4.11), (4.35) and to the auxiliary conditions

$$\mathfrak{H}(\Phi, \underline{U}; \Phi_n, \underline{U}_n) = 0 \qquad (n = 1, 2, \cdots, m-1) \tag{4.37}$$

define the eigenfunction sets Φ_m, $\underline{U}_m$, and H_m of Eqs. (4.3) to (4.6) with the boundary conditions in Eqs. (4.8) to (4.11). The associated eigenvalue μ_m equals the minimum value $\mathfrak{D}(\underline{U}_m, H_m)$. Thus the eigenfunction sets of the variational problem satisfy the orthogonality relations

$$\mathfrak{H}(\Phi_m, \underline{U}_m; \Phi_n, \underline{U}_n) = \delta_{mn} \tag{4.38}$$

$$\begin{aligned} \mathfrak{D}(\underline{U}_m, H_m; \underline{U}_n, H_n) &= 0 \quad \text{for} \quad m \neq n \\ &= \mu_m \quad \text{for} \quad m = n \quad . \end{aligned} \tag{4.39}$$

If one assumes the minimum problem of the above type to possess solutions with sufficient continuity requirements, it will be easy to show that the solutions of the variational problem are also eigenfunctions for our differential equations (see Ref. 10, Chapter VI). Eq. (4.38) simply follows from the definition of Φ_m and $\underline{U}_m$. Eq. (4.39) also follows immediately from Eqs. (4.3) to (4.6).

However, we still have to show that they furnish all the eigenfunctions, except the trivial one (Φ = constant and $\underline{U} = 0$, $H = 0$ with its associated eigenvalues to be zero), by showing that the system of functions Φ_1, Φ_2 .... obtained from the variational problem is semi-complete, i.e. any harmonic function in V can be approximated by the linear combination of Φ_m up to an additive constant. This assertion will be proved later.

b. The maximum-minimum property of eigenvalues and its direct consequences

One can define the mth eigenvalue and its associate eigenfunction without reference to the preceding eigenvalues and eigenfunctions. Instead of stipulating $\mathcal{H}(\Phi, \underline{U};\ \Phi_n, \underline{U}_n) = 0$ $(n = 1, 2, \ldots, m - 1)$ one imposes the m - 1 modified conditions,

$$\mathcal{H}(\Phi, \underline{U};\ \Psi_n, \underline{V}_n) = 0 \qquad (n = 1, 2, \cdots, m-1) \tag{4.40}$$

where $\underline{V}_1$, $\underline{V}_2$, ... $\underline{V}_{n-1}$ are arbitrary piecewise continuous functions on S_2 and Ψ_1, Ψ_2, ... Ψ_{m-1} are m - 1 arbitrary chosen functions which are harmonic in V. Under the conditions imposed, the functional $\mathcal{H}(\underline{U}, H)$ with $\underline{U}$ and H satisfying Eqs. (4.9) and (4.34) has a greatest lower bound, which depends on Ψ_n and $\underline{V}_n$ $(n = 1, 2, \ldots, m - 1)$ and will be denoted by $d_m(\Psi_n, \underline{V}_n)$. Then one has the following theorem.

μ_m is equal to the largest values assumed by $d_m(\Psi_n, \underline{V}_n)$ if Ψ_n and $\underline{V}_n$ (n = 1, 2, ... m - 1) range over all the sets of the admissible functions. The maximum-minimum is attained when $\Psi_n = \Phi_n$ and $\underline{V}_n = \underline{U}_n$ (n = 1, 2, ... , m - 1) where Φ_n and $\underline{U}_n$ are the first m - 1 eigenfunctions.

To prove the theorem, we first notice by definition

$$d_m(\Phi_m, \underline{U}_m) = \mu_m \quad .$$

All we have to show is that μ_m is the maximum. Let us choose a set of functions Φ, $\underline{U}$, H to be the linear combination of the first m eigenfunctions such that

$$\Phi = \sum_{n=1}^{m} a_n \Phi_n$$

$$\underline{U} = \sum_{n=1}^{m} a_n \underline{U}_n$$

$$H = \sum_{n=1}^{m} a_n H_n$$

where a_n are constants. For arbitrary m - 1 function set Ψ_n and $\underline{V}_n$ the m - 1 orthogonal relations in Eq. (4.40) will lead to m - 1 homogeneous conditions for m quantities $a_1, a_2, \ldots, a_m$ and thus those conditions can always be satisfied. From the equation (4.38) one has

$$\mathcal{H}(\Phi, \underline{U}) = \sum_{n=1}^{m} a_n^2 = 1 \quad .$$

Since

$$\mathfrak{D}(\underline{U}_m, H_m ; \underline{U}_j, H_j) = 0$$

for $m \neq j$, and

$$\mathfrak{D}(\underline{U}_n, H_n) = \mu_n \quad ,$$

one gets

$$\mathfrak{D}(\underline{U}, H) = \sum_{n=1}^{m} a_n^2 \mu_n \leq \mu_m \quad .$$

Thus the minimum $d_m(\Psi_n, \underline{V}_n)$ is certainly no greater than μ_m. The proof is complete.

Two simple principles, exactly the same as that of Ref. 10, p. 407, can be concluded from the maximum-minimum property. (1) By strengthening the conditions in a minimum problem we do not diminish the value of the minimum; conversely, by weakening the conditions the minimum does not increase. (2) Given two minimum problems with the same class of admissible functions, for every set of admissible functions, the functional to be minimized in the first problem is greater than or equal to that in the second problem, so the minimum for the first problem is greater than or equal to the minimum of the second problem.

From the maximum-minimum property and the last two principles some important conclusions can be drawn for our physical problem. (1) Flexibility of the container lowers the eigenvalues. (2) If two problems have the same static equilibrium free surface, the problem with the larger σ_H everywhere on ∂S_1 will have larger eigenvalues. Therefore the stuck condition for the free surface gives the largest eigenvalues. The sloshing frequencies are the square roots of the corresponding eigenvalues, one can immediately see the dependence of sloshing frequencies on the flexibility of the container and boundary condition of the free surface.

c. Number of eigenvalues and infinite growth of the eigenvalues

The eigenvalues and their associated eigenfunctions are infinite in number. If there are only m eigenvalues, from the m sets of the associated eigenfunctions Φ_n, $\underline{U}_n$ and H_n, we can always choose m points $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_m$ such that the m x m determinant

$$\det \left| \Phi_i(\underline{x}_j) \right| \neq 0 \quad . \tag{4.41}$$

Let Φ, $\underline{U}$ and H be a set of functions satisfying Eq. (4.35) and only at the m points $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_m$

$$\begin{aligned} \Phi(\underline{x}_n) &= \Phi_m(\underline{x}_n) \\ \underline{U}(\underline{x}_n) &= \underline{U}_m(\underline{x}_n) \\ H(\underline{x}_n) &= H_m(\underline{x}_n) \qquad (n = 1, 2, \cdots, m) \end{aligned} \tag{4.42}$$

but these functions do not equal to Φ_m, $\underline{U}_m$ and H_m respectively elsewhere. Let

$$\Phi' = \Phi - \sum_{n=1}^{m} a_n \Phi_n$$

$$\underline{U}' = \underline{U}' - \sum_{n=1}^{m} a_n \underline{U}_n \qquad (4.43)$$

where

$$a_n = \mathcal{H}(\Phi, \underline{U}; \Phi_n, \underline{U}_n) \qquad (n=1,2,\cdots m). \qquad (4.44)$$

Obviously Φ' and $\underline{U}'$ are orthogonal to all the eigenfunction sets, i.e.

$$\mathcal{H}(\Phi', \underline{U}'; \Phi_n, \underline{U}_n) = 0 \qquad (n=1,2,\cdots m).$$

We claim that Φ' and $\underline{U}'$ are not necessarily constants, for all $\Phi, \underline{U}$ satisfying Eqs. (4.35) and (4.42). If Φ' and $\underline{U}'$ are constants, by evaluating Φ', at $\underline{x}_1$, $\underline{x}_2$, ..., $\underline{x}_m$ and by Eqs. (4.41) to (4.43) one can determine a_n uniquely. When so determined, a_n is independent of the function itself. But by Eq. (4.44), the values of a_n should depend on Φ and $\underline{U}$. Thus, we have a contradiction and the constancy assumption is untenable. We have now shown that there exists at least one harmonic function, Φ' with $\mathcal{H}(\Phi', \underline{U}') \neq 0$ and orthogonal to all the eigenfunctions. Hence, we can construct another new eigenfunction. This is contradictory to the original assumption. Thus the number of eigenvalues cannot be finite.

In order to prove the infinite growth of the eigenvalues of our problem, we shall consider another problem first. Let us consider the extremum of the functional

$$\mathfrak{I}''(W) = \int_{S_2} g^{\alpha\beta}\, W|_\alpha W|_\beta \, ds \tag{4.45}$$

for all W that have appropriate continuity property and satisfy the auxiliary condition in Eq. (4.35) with $U^\alpha = 0$ and satisfy an optional boundary condition

$$W = 0 \quad \text{on} \quad \partial S_2 \quad . \tag{4.46}$$

The minimums of the functional $\mathfrak{I}''(W)$ will be the eigenvalues of the equations

$$\begin{aligned}
g^{\alpha\beta} W|_{\alpha\beta} + \mu''(\rho_0 \Phi + \rho h W) = 0 \quad &\text{on} \quad S_2 \\
\Phi = 0 \quad &\text{on} \quad S_1 \\
\nabla^2 \Phi = 0 \quad &\text{in} \quad V \\
\frac{\partial \Phi}{\partial N} = 0 \quad &\text{on} \quad S_4 \\
= W \quad &\text{on} \quad S_2 \\
W = 0 \quad \text{or} \quad W|_\alpha \nu^\alpha = 0 \quad &\text{on} \quad \partial S_2 \quad .
\end{aligned} \tag{4.47}$$

We have shown that the number of eigenvalues is infinite; consequently so is the number of eigenfunctions. Let Φ_n, W_n be the associated eigenfunction set of μ_n''. Because $\mathcal{H}(\Phi_n, \underline{U}_n) = 1$,

$$\rho h \int_{S_2} W_n^2 \, dS \le 1 \qquad (n = 1, 2, \cdots)$$

is uniformly bounded. If the eigenvalues are also bounded from above, i.e., if $\mathcal{T}''(W)$ is uniformly bounded for all the eigenfunctions, then we can select a subsequence W_n' from the set of eigenfunctions W_n for which (Ref. 11)

$$\lim_{n,m \to \infty} \int_{S_4} (W_n' - W_m')^2 \, dS = 0 \qquad .$$

From the harmonicity property of Φ and the boundary conditions

$$\Phi = 0 \quad \text{on} \quad S_1$$

$$\begin{aligned} \frac{\partial \Phi}{\partial N} &= 0 \quad \text{on} \quad S_4 \\ &= W \quad \text{on} \quad S_2 \, , \end{aligned}$$

one has

$$\begin{aligned} \int_V [\nabla(\Phi_n' - \Phi_m')]^2 dV &= \int_{S_2} (\Phi_n' - \Phi_m') \frac{\partial(\Phi_n' - \Phi_m')}{\partial N} dS \\ &= \int_{S_2} (\Phi_n' - \Phi_m')(W_n' - W_m') \, dS \end{aligned}$$

$$\leq \left[\int_{S_2} (\Phi_n' - \Phi_m')^2 dS \int_{S_2} (W_n' - W_m')^2 dS \right]^{1/2}$$

$$\longrightarrow 0 \qquad \text{as} \qquad m, n \longrightarrow \infty ,$$

then, with the assumption $U_\alpha = 0$, as m, $n \to \infty$,

$$\mathcal{H}(\Phi_n' - \Phi_m', \underline{U}_n' - \underline{U}_m') = \rho h \int_{S_2} (W_n' - W_m')^2 dS + \rho_0 \int_V [\nabla(\Phi_n' - \Phi_m')]^2 dV$$

$$= 0 \qquad . \tag{4.48}$$

But on the other hand from the orthogonal relation in Eq. (4.38)

$$\mathcal{H}(\Phi_n' - \Phi_m', \underline{U}_n' - \underline{U}_m') = 2 \qquad .$$

This contradiction proves that the eigenvalue μ_n'' cannot be bounded from above.

By the case just proved, we can prove the infinite growth of the eigenvalues of another problem with the functional

$$J'(W) = \int_{S_2} [A(g^{\alpha\beta} W|_{\alpha\beta})^2 + B g^{\alpha\beta} W|_\alpha W|_\beta] dS \tag{4.49}$$

to be minimized under conditions in Eq. (4.35) with $U_\alpha = 0$ and boundary conditions

$$W = 0$$

$$W|_\alpha = 0 \qquad \text{on} \qquad \partial S_2$$

(the last condition is optional). Here $A(>0)$ and B are constants. First we shall show that $\mathcal{J}'(W)$ is bounded from below. Let W_n, Φ_n be the eigenfunction sets associated with the functional $\mathcal{J}''(W)$ in Eq. (4.45), then any continuous function W on S_2 with its second derivative to be square integrable, satisfying the rigid boundary condition on ∂S_2, can be expressed as

$$W = \sum_{n=1}^{\infty} a_n W_n \qquad , \qquad (4.50)$$

where a_n are constants, the function Φ associated with this particular W will be

$$\Phi = \sum_{n=1}^{\infty} a_n \Phi_n \qquad . \qquad (4.51)$$

Then by Eq. (4.47), one has

$$g^{\alpha\beta} W|_{\alpha\beta} = \sum_{n=1}^{\infty} a_n g^{\alpha\beta}(W_n)|_{\alpha\beta}$$

$$= -\rho h W' - \rho_0 \Phi' \qquad .$$

where

$$W' = \sum_{n=1}^{\infty} \mu_n'' a_n W_n$$

$$\Phi' = \sum_{n=1}^{\infty} \mu_n'' a_n \Phi_n \quad .$$

Substituting into Eq. (4.49), by the orthogonality relation, $\mathcal{J}'(W)$ becomes

$$\sum_{n=1}^{\infty} a_n^2 \mu_n'' (\mu_n'' A + B) + A \int_{S_2} (\rho h W' + \rho_0 \Phi') \rho_0 \Phi' dS$$

$$= \sum_{n=1}^{\infty} a_n^2 \mu_n'' (\mu_n'' A + B) + A \left[\int_{S_2} \rho_0^2 \Phi'^2 dS + \int_V \rho_0 \rho h (\nabla \Phi') dV \right]$$

Because we have assumed that the square of the second derivative of W is integrable, then

$$\sum_{n=1}^{\infty} a_n^2 (\mu_n'')^2$$

will be a convergent series. So the integration just carried out is valid.

Since μ_n'' tents to positive infinite as $n \to \infty$, one may choose an integer m such that for $n > m$, μ_n'' and $\mu_n'' A + B$ are both positive. Then, one has

$$\mathcal{J}'(W) \geq \sum_{n=1}^{m} a_n^2 \mu_n'' (\mu_n'' A + B) \quad . \tag{4.52}$$

Therefore for all given $A(>0)$ and $B(>-\infty)$, $\mathcal{J}'(W)$ is bounded from below.

For the proof of infinite growth of its eigenvalues, we take

$$a_n = 0 \qquad \text{for} \qquad n \leq m-1$$

$$\neq 0 \qquad \text{for} \qquad n = m$$

in Eqs. (4.50) and (4.51). Then the so defined functions Φ, and W will be orthogonal to the m - 1 function sets Φ_n and W_n. From Eq. (4.52), one has

$$\mathcal{J}'(W) \geq a_m^2 \mu_m'' (\mu_m'' A + B) = A_m \quad .$$

Then A_m will be less than or equal to the greatest lower bound $d_m(\Phi_n, \underline{U}_n)$, (n = 1, 2, ..., m - 1) (defined in IV-B-b). By the maximum-minimum property, the mth eigenvalue, say μ_m, associated with $\mathcal{J}'(W)$ will not be less than A_m. But on the other hand, A_m tends to positive infinity as m tends to infinity, so μ_m will also tend to positive infinity.

For the proof of our problem we shall restrict ourselves to the following case.

$$\int_{S_1} \{ \sigma (a^{\alpha\beta} H|_\alpha H|_\beta - k^\alpha_\beta k^\beta_\alpha H^2) - \rho_0 \underline{g}_0 \cdot \underline{N}_0 H^2 \} dS \qquad (4.53)$$

is bounded from below by a constant for all $H(=\partial\Phi/\partial N$ on $S_1)$ under the conditions in Eq. (4.11), and

$$\left(C^{\alpha\beta\lambda\theta} - A g^{\alpha\beta} g^{\lambda\theta}\right) x_{\lambda\theta}\, x_{\alpha\beta} > 0 \tag{4.54}$$

for all $x_{\alpha\beta} \not\equiv 0$ and where A is a positive constant. Eq. (4.54) is always true if $C^{\alpha\beta\lambda\theta}$ is in the form of Eq. (2.30), i.e. $C^{\alpha\beta\lambda\theta}$ are the linear isotropic elastic constants.

For the time being we shall assume

$$\sigma_H \geq 0$$

everywhere on ∂S_1, later we shall remove this restriction. Let B be such a constant that

$$\left(T^{\alpha\beta} - B g^{\alpha\beta}\right) \alpha_\alpha \alpha_\beta \geq 0 \qquad \forall\ x_\alpha \not\equiv 0 . \tag{4.55}$$

Then obviously

$$\mathfrak{D}(\underline{u}, H) + C \geq \mathfrak{J}'(w) \tag{4.56}$$

where C is some finite constant. By the second maximum-minimum principle of eigenvalues in IV-B-b, $\mathfrak{D}$ ($\underline{U}$, H) must be bounded from below and the sequence of the associated eigenvalues will be unbounded from above.

The assumption that

$$\sigma_H \geq 0$$

can be removed, since by the result in Ref. 10, p. 418,

$$\left| \int_{\partial S_1} \sigma_H H^2 d\ell \right| \leq C_1 (D_1(H))^{1/2} + C_2$$

where c_1 and c_2 are finite constant. Therefore the term $\int_{\partial S_1} \sigma_H H^2 d\ell$ will not affect the unboundness of $\mathfrak{D}(\underline{U}, H)$.

C. Completeness of Eigenfunctions

Let Φ_n, $\underline{U}_n$, H_n be the eigenfunction sets and μ_n be the eigenvalues associated with the quotient $\mathfrak{D}(\underline{U}, H)/\mathfrak{H}(\Phi, \underline{U})$ subject to auxiliary conditions in Eq. (4.35). We shall prove that the system of eigenfunctions is semi-complete in the following sense: For any given functions Φ and $\underline{U}$ where Φ is harmonic in V and $\underline{U}$ is defined on S_2 with

$$\frac{\partial \Phi}{\partial N} = \underline{U} \cdot \underline{N}_0 \qquad \text{on} \qquad S_2$$

and any given ϵ (> 0), arbitrary small, we can find a finite linear combination

$$\sum_{n=1}^{N} a_n \Phi_n = \Psi_N$$
$$\sum_{n=1}^{N} a_n \underline{U}_n = \underline{V}_N \qquad (4.57)$$

such that

$$\mathcal{H}(\Phi - \Psi_N, \underline{U} - \underline{V}_N) \leq \epsilon \quad . \qquad (4.58)$$

The best approximation, i.e. the smallest $\mathcal{H}(\Phi - \Psi_N, \underline{U} - \underline{V}_N)$, is obtained with

$$a_n = c_n = \mathcal{H}(\Phi, \underline{U}; \Phi_n, \underline{U}_n) \quad . \qquad (4.59)$$

These will satisfy the semi-completeness relation

$$\mathcal{H}(\Phi, \underline{U}) = \sum_{n=1}^{\infty} c_n^2 \quad . \qquad (4.60)$$

Any harmonic function Φ and vector function $\underline{U}$ with

$$\mathcal{H}(\Phi, \underline{U}) = 0$$

will imply that Φ is constant almost everywhere in V and $\underline{U}$ is zero on S_2.

The last assertion is obvious, since

$$\mathcal{H}(\Phi, \underline{U}) = \int_{S_2} \rho h\, \underline{U} \cdot \underline{u}\, dS + \int_V \rho_0 (\nabla \Phi)^2 dV$$

will imply $\nabla \Phi = 0$ almost everywhere in V and $\underline{U} = 0$ almost everywhere on S_2. Therefore Φ must be a constant in V.

The proof of the best mean approximation of Φ and $\underline{U}$ with respect to $\mathcal{H}$ is standard:

$$\mathcal{H}(\Phi - \sum_{n=1}^{N} a_n \Phi_n, \underline{U} - \sum_{n=1}^{N} a_n \underline{U}_n)$$

$$= \mathcal{H}(\Phi - \sum_{n=1}^{N} c_n \Phi_n + \sum_{n=1}^{N} (c_n - a_n) \Phi_n, \underline{U} - \sum_{n=1}^{N} (c_n + a_n - c_n) \underline{U}_n)$$

$$= \mathcal{H}(\Phi, \underline{U}) - \sum_{n=1}^{N} c_n^2 + \sum_{n=1}^{N} (c_n - a_n)^2$$

$$\geq \mathcal{H}(\Phi, \underline{U}) - \sum_{n=1}^{N} c_n^2 = \mathcal{H}(\Phi - \sum_{n=1}^{N} c_n \Phi_n, \underline{U} - \sum_{n=1}^{N} c_n \underline{U}_n)$$

$$\geq 0 \tag{4.61}$$

From here one can also conclude the convergence of $\sum_{n=1}^{\infty} c_n^2$ and

$$\mathcal{H}(\Phi, \underline{U}) \geq \sum_{n=1}^{\infty} c_n^2 . \tag{4.62}$$

We shall show not only the above inequality, but also the semi-completeness relation (4.60). Let us assume the set of functions Φ, $\underline{U}$, H satisfy the conditions in Eqs. (4.9) to (4.11) and (4.35). Then the function

$$\varphi_N = \Phi - \sum_{n=1}^{N} c_n \Phi_n \tag{4.63}$$

and its associated functions

$$\underline{v}_N = \underline{U} - \sum_{n=1}^{N} c_n \underline{U}_n$$

$$\eta_N = H - \sum_{n=1}^{N} c_n H_n \tag{4.64}$$

satisfy the orthogonality relations

$$\mathcal{D}(\underline{v}_N, \eta_N; \underline{U}_n, H_n) = \mu_n \mathcal{H}(\varphi_N, \underline{v}_N; \Phi_n, \underline{U}_n) = 0$$

and

$$\mathcal{H}(\varphi_N, \underline{v}_N; \Phi_n, \underline{U}_n) = 0 \tag{4.65}$$

for $n = 1, 2, \ldots, N$.

By the minimum property of μ_{N+1}, we have

$$\mathfrak{D}(\underline{V}_N, \eta_N) \geqslant \mu_{N+1} \mathcal{H}(\varphi_N, \underline{V}_N) \quad . \tag{4.66}$$

On the other hand $\mathfrak{D}(\underline{V}_N, \eta_N)$ is bounded. It can be shown that

$$\mathfrak{D}(\underline{U}, H) = \sum_{n=1}^{N} \mu_n c_n^2 + \mathfrak{D}(\underline{V}_N, \eta_N) \quad .$$

Let m be so large that

$$\mu_N > 0 \qquad \text{for all} \qquad N > m$$

then one has

$$\sum_{n=1}^{N} \mu_n c_n^2 \geqslant \sum_{n=1}^{m} \mu_n c_n^2$$

and

$$\mathfrak{D}(\underline{V}_N, \eta_N) \leq \mathfrak{D}(\underline{U}, H) - \sum_{n=1}^{m} \mu_n c_n^2 \tag{4.67}$$

for all $N > m$. That is, $\mathfrak{D}(\underline{V}_N, \eta_N)$ is bounded from above. By Eq. (4.66), as N tends to infinity, we have

$$\mathcal{H}(\varphi_N, \underline{V}_N) \longrightarrow 0 \quad .$$

Since

$$\mathcal{H}(\Phi, \underline{U}) = \sum_{n=1}^{N} c_n^2 + \mathcal{H}(\varphi_N, \underline{V}_N)$$

therefore we have

$$\mathcal{H}(\Phi, \underline{U}) = \sum_{n=1}^{\infty} c_n^2 \quad .$$

Any set of functions Φ, $\underline{U}$, H, satisfying Eq. (4.35) but failing to satisfy the conditions in Eqs. (4.9) to (4.11) can be approximated by another set of functions Φ', $\underline{U}'$, H' that satisfies Eqs. (4.9) to (4.11) and (4.35) such that

$$\mathcal{H}(\Phi - \Phi', \underline{U} - \underline{U}') \leq \frac{\epsilon}{6}$$

for the given arbitrary small positive ϵ. Then one can approximate Φ', $\underline{U}'$, H' by the linear combination

$$\varphi_N = \sum_{n=1}^{N} c_n \Phi_n$$

$$\underline{V}_N = \sum_{n=1}^{N} c_n \underline{U}_n$$

$$\eta_N = \sum_{n=1}^{N} c_n H_n$$

with the property that

$$\mathcal{H}(\underline{\Phi}' - \varphi_N, \underline{U}' - \underline{V}_N) \leq \frac{\epsilon}{6}$$

then

$$\mathcal{H}(\underline{\Phi} - \varphi_N, \underline{U} - \underline{V}_N) = \mathcal{H}(\underline{\Phi} - \underline{\Phi}', \underline{U} - \underline{U}')$$

$$+ 2\,\mathcal{H}(\underline{\Phi} - \underline{\Phi}', \underline{U} - \underline{U}'; \underline{\Phi}' - \varphi_N, \underline{U}' - \underline{V}_N) + \mathcal{H}(\underline{\Phi}' - \varphi_N, \underline{U}' - \underline{V}_N). \qquad (4.68)$$

By Schwartz's inequality,

$$\mathcal{H}(\underline{\Phi} - \underline{\Phi}', \underline{U} - \underline{U}'; \underline{\Phi}' - \varphi_N, \underline{U}' - \underline{V}_N) \leq \left[\int_{S_2} \rho h (\underline{U} - \underline{U}')^2 dS \int_{S_2} \rho h (\underline{U}' - \underline{V}_N)^2 dS \right]^{1/2}$$

$$+ \left[\int_V \rho_0 \{\nabla(\underline{\Phi} - \underline{\Phi}')\}^2 dV \int_V \rho_0 \{\nabla(\underline{\Phi}' - \varphi_N)\}^2 dV \right]^{1/2}$$

$$\leq \frac{\epsilon}{6} + \frac{\epsilon}{6} \qquad . \qquad (4.69)$$

Therefore one has

$$\mathcal{H}(\underline{\Phi} - \varphi_N, \underline{U} - \underline{V}_N) \leq \epsilon \qquad . \quad \text{q.e.d.}$$

This proves the semi-completeness theorem for any harmonic function Φ with $\partial\Phi/\partial N = 0$ on S_4, i.e. Φ can be approximated by the eigenfunctions up to an arbitrary additive constant. This is because, from the variational method, we cannot obtain the trivial eigenfunction set, Φ = constant, $\underline{U} = 0$ and $H = 0$ and its associated eigenvalue (zero) of the differential equations. If this additional eigenfunction is used, we can approximate the harmonic function Φ in the sense of Eq. (4.58).

This theorem just proved is also the completeness theorem for any continuous function H on S_1 and $\underline{U}$ on S_2 with the property

$$\int_{S_1} H\, dS + \int_{S_2} \underline{U}\cdot\underline{N}_0\, dS = 0 \qquad . \qquad (4.70)$$

Since for such given H on S_1 and $\underline{U}$ on S_2 we can determine uniquely, up to an additive constant, the harmonic function Φ such that Eq. (4.35) is satisfied. The series, with $c_n = \mathcal{Y}(\Phi, \underline{U}; \Phi_n, \underline{U}_n)$,

$$\sum_{n=1}^{\infty} c_n \Phi_n \qquad (4.71)$$

is equal to Φ almost everywhere up to an additive constant. Then the series

$$\sum_{n=1}^{\infty} c_n H_n \qquad (4.72)$$

and

$$\sum_{n=1}^{\infty} c_n \underline{U}_n \tag{4.73}$$

will be equal to H and $\underline{U}$ almost everywhere on S_1 and S_2 respectively.

For the application of the completeness theorem, we shall point out that in order to obtain the expansion for H and $\underline{U}$, one does not have to determine the harmonic function Φ first. Instead one can use the expansions of H and $\underline{U}$ to determine Φ in V, since

$$c_n = \mathcal{H}(\Phi, \underline{U}\, ;\, \Phi_n, \underline{U}_n)$$

$$= \int_{S_2} \rho h\, \underline{U}\cdot\underline{U}_n dS + \int_V \rho_0 \nabla\Phi\cdot\nabla\Phi_n dV \tag{4.74}$$

$$= \int_{S_2} \rho h\, \underline{U}\cdot\underline{U}_n dS + \int_{S_2} \rho_0\, \underline{U}\cdot\underline{N}_0 \Phi_n dS + \int_{S_1} \rho_0 H \Phi_n dS \quad .$$

We have another kind of expansion for any continuous function, say f defined on S_1 and S_2 and U_α on S_2. In principle, one can determine the unique harmonic Φ in V such that

$$\rho_0 \Phi = f \qquad \text{on} \qquad S_1$$

$$\rho_0 \Phi + \rho h \frac{\partial \Phi}{\partial N} = f \qquad \text{on} \qquad S_2 \tag{4.75}$$

$$\frac{\partial \Phi}{\partial N} = 0 \qquad \text{on} \qquad S_4 \quad .$$

Let us define a vector $\underline{U}$ on S_2 such that

$$\underline{U} = U_\alpha \underline{g}^\alpha + \frac{\partial \Phi}{\partial N} \underline{N}_0 \quad .$$

If one expands Φ in terms of the eigenfunctions, then the series

$$\sum_{n=1}^{\infty} c_n \Phi_n$$

will satisfy the boundary condition

$$\rho_0 \Phi = \sum_{n=1}^{\infty} \rho_0 c_n \Phi_n + \text{CONSTANT} = f \qquad \text{on} \qquad S_1$$

$$\rho_0 \Phi + \rho h \frac{\partial \Phi}{\partial N} = \sum_{n=1}^{\infty} c_n \left(\rho_0 \Phi_n + \frac{\partial \Phi_n}{\partial N}\right) + \text{constant} = f \qquad \text{on} \qquad S_2 \quad . \tag{4.76}$$

One also has

$$U^\alpha = \sum_{n=1}^{\infty} c_n U_n^\alpha \qquad \text{on} \qquad S_2 \quad . \tag{4.77}$$

Actually in order to determine c_n, one does not have to know Φ to begin with, since

$$c_n = \mathcal{H}(\Phi, \underline{U}; \Phi_n, \underline{U}_n) = \int_{S_2} \rho h\, \underline{U}\cdot\underline{U}_n dS + \int_V \rho_0 \nabla\Phi\cdot\nabla\Phi_n dV$$

$$= \int_{S_2} \rho h\, U_\alpha U_n^\alpha dS + \int_{S_2} (\rho h \frac{\partial \Phi}{\partial N} + \rho_0 \Phi) \frac{\partial \Phi_n}{\partial N} dS + \int_{S_1} \rho_0 \Phi \frac{\partial \Phi_n}{\partial N} dS \tag{4.78}$$

$$= \int_{S_2} \rho h\, U_\alpha U_n^\alpha dS + \int_{S_2} f \frac{\partial \Phi_n}{\partial N} dS + \int_{S_1} f \frac{\partial \Phi_n}{\partial N} dS \quad .$$

Obviously the series only approximates f up to an additive constant.

D. Uniqueness of Solution

In the case that all eigenvalues obtained by the variational functional are positive, we can show a simple uniqueness theorem. If ϕ, $\underline{u}$, η is a solution set of our problem, then by multiplying Eq. (3.7) by $\partial\eta/\partial t$, Eq. (3.8) by $\partial u_\alpha/\partial t$ and Eq. (3.9) by $\partial w/\partial t$ and integrating over its corresponding surface and adding, after some appropriate application of the divergence theorem, one gets

$$\frac{1}{2}\frac{d}{dt}\left[\mathfrak{D}(\underline{u}, \eta) + \int_{S_2} \rho h (\dot{\underline{u}})^2 dS + \int_V \rho_0 (\nabla\phi)^2 dV\right] = 0 .$$

That simply means

$$\mathfrak{D}(\underline{u}, \eta) + \int_{S_2} \rho h (\dot{\underline{u}})^2 dS + \int_V \rho_0 (\nabla\phi)^2 dV = c \tag{4.79}$$

where c is a constant for all time. If ϕ, $\underline{u}$, η satisfy the zero initial condition, i.e.

$$\nabla\phi = \underline{u} = \eta = 0$$

at time t = 0, c must be zero for all time. By the minimum property of eigenvalues, one has

$$\mathcal{D}(\underline{u}, \eta) / \mathcal{H}(\phi, \underline{u}) \geq \mu_1 > 0$$

when $\phi \neq$ constant and $\underline{u}$, $\eta \neq 0$. $\mathcal{H}(\phi, \underline{u})$ is always greater than zero, in this case, one has

$$\mathcal{D}(\underline{u}, \eta) > 0$$

unless $\eta = \underline{u} = 0$, then by Eq. (4.79), we have

$$\mathcal{D}(\underline{u}, \eta) = 0$$

$$\dot{\underline{u}} = 0$$

$$\nabla\phi = 0$$

or

$$\underline{u} = \nabla\phi = \eta = 0$$

for all time. Thus there can be only one solution that satisfies the same initial conditions and boundary conditions.

E. Stability of Liquid Gas Interface and the Shell

We shall call a system unstable if there exists a solution that satisfies Eqs. (4.3) to (4.6) and boundary conditions (4.8) to (4.11) and tends to infinity as time increases indefinitely. From the form of solution in Eq. (4.1) it is evident that whenever there exists an ω with a negative imaginary part, the corresponding solution will have an exponential growth in time. Since $\omega = \pm\sqrt{\mu}$, if all the eigenvalues μ are positive, the solution will be always bounded. Since it has been shown that the eigenvalues μ are real, the sign of the lowest eigenvalue, say μ_1, corresponding to a non-trivial solution, can be used as the stability criteria. From Eqs. (4.27) and (4.28), it can be shown that there exist parameters which make μ_1 negative, and because the eigenvalue μ_1 is a continuous function of the parameters,

$$\mu_1 = 0$$

can be used as the condition of neutral stability. By Eq. (4.27), it can be shown that $\mu_1 = 0$ is a hyper-surface which separates the parameters corresponding to different sign of μ_1. The condition $\mu_1 = 0$ is equivalent to the condition that there exists a non-trivial solution of the following equations:

$$\sigma(a^{\alpha\beta}H|_{\beta\alpha} + \varkappa^{\beta}_{\alpha}\varkappa^{\alpha}_{\beta}H) + \rho_0 \underline{g}_0 \cdot \underline{N}_0 H = 0 \quad \text{on} \quad S_1$$

$$N^{\alpha\beta}|_{\beta} = 0 \quad \text{on} \quad S_2 \tag{4.80}$$

$$-M^{\alpha\beta}|_{\beta\alpha} + N^{\alpha\beta}b_{\alpha\beta} + T^{\alpha\beta}W|_{\beta\alpha} + \rho_0 \underline{g}_0 \cdot \underline{N}_0 W = 0 \quad \text{on} \quad S_2$$

where $\underline{U}$ and H satisfy Eqs. (4.9) to (4.11) and

$$\int_{S_1} H\, dS + \int_{S_2} \underline{U} \cdot \underline{N}_0\, dS = 0 \tag{4.81}$$

and no solution exists for Eqs. (4.3) to (4.11) for negative μ. Let us call $\underline{U}$ and H admissible if they satisfy appropriate smoothness conditions and satisfy Eqs. (4.9) to (4.11) and (4.81). Then the neutral stability condition is also equivalent to the condition that $\mathfrak{D}(\underline{U}, H)$ is positive semidefinite, i.e.

$$\mathfrak{D}(\underline{U}, H) \geq 0 \tag{4.82}$$

for all admissible $\underline{U}$ and H and the equality sign is actually attained by at least a nontrivial admissible set of $\underline{U}$ and H. Evidently, from the dependence of Eq. (4.80), or the functional in Eq. (4.82), and the admissible conditions for functions $\underline{U}$ and H, the existence of such a set of functions $\underline{U}$ and H is determined by the surfaces S_1, S_2 and the value of the parameters on these surfaces. We can conclude that the neutral stability condition is independent of the size and shape of the tank as long as the surfaces S_1 and S_2, and the conditions on these surfaces are the same. Also from the form of these equations, the neutral stability conditions will be symmetric about the contact angle $\theta_0 = 90^\circ$ on ∂S_1.

$\underline{g}_0$ is an important parameter that affects the stability of the system. Let us write

$$\begin{aligned} \underline{g}_0 \cdot \underline{N}_0 &= -g_0 \cos\varphi \qquad \text{on} \quad S_1 \\ &= g_0 \cos\varphi \qquad \text{on} \quad S_2 \end{aligned} \tag{4.83}$$

In the case that $\cos\varphi$ is positive on S_1 and S_2, by Eqs. (4.27) and (4.82) we can see that for a given problem there exists an upper bound B_u and a lower bound B_L for g_o between which the system is stable. By the principles described in Section IV-B-b, we can show that B_u is less than or equal to stability boundary B_u of g_o in the same system but with the free surface S_1 replaced by a rigid lid and B_L is greater than or equal to the stability boundary B_L of g_o for the corresponding rigid container. Because if we strengthen the admissible condition by requiring

$$\int_{S_1} H\, dS = 0 \qquad \text{and} \qquad \int_{S_2} \underline{U} \cdot \underline{N}_0\, dS = 0 \tag{4.84}$$

instead of Eq. (4.82), obviously B_u is the least upper bound and B_L is greatest lower bound of g_o, such that there exists at least a function H or $\underline{U} \neq 0$ satisfying the strengthened admissible condition and

$$\mathfrak{D}(\underline{u}, H) = 0 \quad .$$

In the particular case if S_1 is axially symmetric, we shall show that

$$B_L = \bar{B}_L \tag{4.85}$$

i.e. the flexibility of the container has no effect on the lower boundary of g_0. In order to show this, let us use the cylindrical coordinates (r, θ, z) with the axis of symmetry being coincidental with the z-axis, and use ξ, the deflection in the z-direction, instead of H. Then, from Eq. (4.27), one has (see Appendix B)

$$D_1^{**} = D_1(H) + \int_{\partial S_1} \sigma \sigma_H H^2 d\ell = D_1^*(\xi) - \sigma \int_{\partial S_1} \gamma \xi^2 d\ell \tag{4.86}$$

where

$$D_1^*(\xi) = \int_{S_0} \sigma \left[\frac{\xi_r^2}{(1+F_r^2)^{3/2}} + \frac{\frac{1}{r^2}\xi_\theta^2}{(1+F_r^2)^{1/2}} + \frac{\rho_0 g_0}{\sigma} \xi^2 \right] r\,dr\,d\theta \quad . \tag{4.87}$$

S_0 is the project area of S_1 on the r-θ-plane, and F is the initial deflection of the free surface. The corresponding Euler Equation is

$$\frac{1}{r} \frac{\partial}{\partial r} \frac{r \xi_r}{(1+F_r^2)^{3/2}} + \frac{1}{r^2} \frac{\xi_{\theta\theta}}{(1+F_r^2)^{1/2}} - \frac{\rho_0 g_0}{\sigma} \xi = 0 \tag{4.88}$$

with the boundary condition

$$\frac{\partial \xi}{\partial r} = \gamma \xi \quad . \tag{4.89}$$

The stability boundary of the free surface is determined by the condition that D_1^{**} of Eq. (4.86) is positive semidefinite or the existence of the solution of Eq. (4.88) satisfying Eqs. (4.84) and (4.89). Since there is no axially symmetric solution of (4.88) satisfying (4.84) and (4.89) if $\gamma \neq 0$, therefore, if $\gamma \neq 0$, the unstable solution must be of the form

$$\xi \sim e^{in\theta} \quad .$$

But in this case, Eqs. (4.81) and (4.84) are equivalent. Therefore B_L and $\bar{B}_L$ must be the same.

From the form of Eq. (4.87), one more important conclusion can be drawn immediately. If one lets

$$\bar{D}_1(\xi) = \int_{S_0} \sigma \left[\frac{\xi_r^2}{(1+\bar{F}_r^2)^{3/2}} + \frac{\frac{1}{r^2}\xi_\theta^2}{(1+\bar{F}_r^2)^{1/2}} + \frac{\rho_0 g_0}{\sigma} \xi^2 \right] r\,dr\,d\theta$$

then for all admissible ξ, one has

$$D_1^*(\xi) \geq \bar{D}_1(\xi)$$

if

$$\bar{F}_r^2 \geq F_r^2$$

everywhere on S_0. Evidently the neutral stability boundary of g_o will increase as F_r^2 increases, i.e. for the highly curved initial surface, the system becomes unstable at higher values of g_o. This result is checked numerically in Ref. 12 in the case of a rigid circular cylindrical container.

V. APPLICATION OF EIGENFUNCTION EXPANSION TO AN INITIAL VALUE PROBLEM

A given initial value problem is well defined when the deflection and velocity of the free surface and the flexible surface of the container are prescribed at time zero,

$$\left.\begin{aligned} \eta\big|_{t=0} &= \eta^{*}(\underline{x}) \\ \frac{\partial \eta}{\partial t}\bigg|_{t=0} &= \eta^{**}(\underline{x}) \end{aligned}\right\} \quad \text{on } S_1 \tag{5.1}$$

$$\left.\begin{aligned} \underline{u}\big|_{t=0} &= \underline{u}^{*}(\underline{x}) \\ \frac{\partial \underline{u}}{\partial t}\bigg|_{t=0} &= \underline{u}^{**}(\underline{x}) \end{aligned}\right\} \quad \text{on } S_2 \tag{5.2}$$

with

$$\int_{S_1} \eta^{**} dS + \int_{S_2} \underline{N}_0 \cdot \underline{u}^{**} dS = 0 \quad . \tag{5.3}$$

We want to determine the motion of the liquid in the container, the force and the moment induced by the motion of the liquid.

We shall assume a solution of the following form

$$\phi = \phi^* t + \phi_1(\underline{x},t) + \phi_2(\underline{x},t)$$

$$\eta = \eta^*(\underline{x}) + \eta_1(\underline{x},t) + \eta_2(\underline{x},t) \qquad (5.4)$$

$$\underline{u} = \underline{u}^*(\underline{x}) + \underline{u}_1(\underline{x},t) + \underline{u}_2(\underline{x},t)$$

where ϕ^* is a constant and

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \sum_{n=1}^{\infty} \Phi_n(\underline{x})\, \omega_n \begin{Bmatrix} a_n \sin \omega_n t \\ b_n \cos \omega_n t \end{Bmatrix} \qquad (5.5)$$

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \sum_{n=1}^{\infty} H_n(\underline{x}) \begin{Bmatrix} a_n (1 - \cos \omega_n t) \\ b_n \sin \omega_n t \end{Bmatrix} \qquad (5.6)$$

$$\begin{Bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{Bmatrix} = \sum_{n=1}^{\infty} \underline{U}_n(\underline{x}) \begin{Bmatrix} a_n (1 - \cos \omega_n t) \\ b_n \sin \omega_n t \end{Bmatrix} \qquad (5.7)$$

a_n and b_n are constants, Φ_n, H_n and $\underline{U}_n$ are the eigenfunctions of Eqs. (4.3) to (4.11).

These functions ϕ, η and $\underline{u}$ satisfy Eqs. (4.1), (4.3), the kinematic conditions (4.5), the boundary conditions (4.6) and (4.7), and the initial deflection condition. However, a_n and b_n must be chosen appropriately so that the given initial velocity condition and the Eqs. (4.2) to (4.4) are satisfied.

By Eqs. (5.4) to (5.7), one has

$$\frac{\partial \eta}{\partial t} = \sum_{n=1}^{\infty} b_n \omega_n H_n(\underline{x}) \qquad \text{on} \qquad S_1$$

$$\frac{\partial \underline{u}}{\partial t} = \sum_{n=1}^{\infty} b_n \omega_n \underline{U}_n(\underline{x}) \qquad \text{on} \qquad S_2 \tag{5.8}$$

at t = 0. By Eq. (4.74), if

$$b_n \omega_n = \int_{S_1} \rho_o \eta^{**}(\underline{x}) \Phi_n \, dS + \int_{S_2} \rho_o \underline{N}_o \cdot \underline{u}^{**} \Phi_n \, dS + \int_{S_2} \rho h \, \underline{u}^{**} \cdot \underline{U}_n \, dS \tag{5.9}$$

the equations

$$\left.\frac{\partial \eta}{\partial t}\right|_{t=0} = \eta^{**}(\underline{x}) \qquad \text{on} \qquad S_1$$

$$\left.\frac{\partial \underline{u}}{\partial t}\right|_{t=0} = \underline{u}^{**}(\underline{x}) \qquad \text{on} \qquad S_2$$

will be satisfied.

Substituting Eq. (5.4) into Eqs. (4.2) to (4.4), with $\underline{g}(t)$ = constant and $\underline{g}_1(t) = 0$, we have

$$\zeta(\chi) - \sum_{n=1}^{\infty} a_n \mu_n \rho_o \Phi_n - \rho_o \phi^* = 0 \qquad \text{on} \quad S_1 \tag{5.10}$$

$$N^{\alpha\beta}|_\beta - \sum_{n=1}^{\infty} a_n \mu_n U_n^\alpha = 0 \qquad \text{on} \quad S_2 \tag{5.11}$$

$$\xi(\chi) - \sum_{n=1}^{\infty} a_n \mu_n (\rho h W_n + \rho_o \Phi_n) - \rho_o \phi^* = 0 \qquad \text{on} \quad S_2 \tag{5.12}$$

where

$$\zeta(\chi) = \sigma \left\{ a^{\alpha\beta} \eta^*|_{\alpha\beta} + k^\alpha_\beta x^\beta_\alpha \eta^* \right\} + \rho_o \underline{g}_o \cdot \underline{N}_o \eta^*$$

$$\xi(\chi) = -M^{\alpha\beta}|_{\beta\alpha} + N^{\alpha\beta} b_{\alpha\beta} + T^{\alpha\beta} W^*|_{\alpha\beta} + \rho_o \underline{g}_o \cdot \underline{N}_o W^*$$

where $N^{\alpha\beta}$, $M^{\alpha\beta}$ are the stress resultants and stress moments [see Eq. (3.3)] associated with the displacement field $\underline{u}^*(\underline{\chi})$.
If

$$a_n \mu_n = \rho_o \int_{S_1} \zeta(\chi) H_n dS + \rho_o \int_{S_2} \xi(\chi) \underline{U}_n \cdot \underline{N}_o dS + \int_{S_2} \rho h \underline{U}^* \cdot \underline{U}_n dS \tag{5.13}$$

by Eqs. (4.78), (5.10) and (5.11), we have satisfied Eqs. (4.2) to (4.4) up to an additive constant. This constant can be fixed by proper choice of ϕ^* e.g. by integrating Eq. (5.10) over S_1, one gets

$$\rho_0 \phi^* = \int_{S_1} \left[\zeta(\underline{x}) - \sum_{n=1}^{\infty} a_n \mu_n \Phi_n(\underline{x}) \right] dS \Big/ \int_{S_1} dS \quad . \qquad (5.14)$$

Thus the initial value problem is completely solved provided that we have predetermined the eigenfunctions. The force and the moment acting on the container can be obtained by appropriate integration of the pressure of the liquid over the wetted surfaces. A particular example of a circular cylindrical container with rigid side walls and a flat flexible bottom will be discussed in detail in the next section.

VI. EXAMPLE: LIQUID MOTION IN A CIRCULAR CYLINDRICAL CONTAINER WITH A FLEXIBLE BOTTOM

We shall consider a particular case of a circular cylindrical container of radius r_o with rigid side walls and a flat flexible bottom containing a liquid with a free surface. The tank walls are subjected to an axial acceleration along the z-axis which is approximately coincident with the axis of the container. A constant pressure exists above the liquid surface and underneath the tank bottom. The situation is pictured in Fig. 3. The problem is to determine the motion of the liquid caused by a sudden change in the acceleration of the container. As a further simplification we assume the deviation of the initially curved free surface of the liquid from the mean plane free surface is small, so that the boundary values may be evaluated on the mean plane surface, which is perpendicular to the z-axis. A similar assumption of infinitesimal deflection is made for the rigid side walls and the flexible bottom.

As shown in Fig. 3, a cylindrical polar coordinate system, which moves with the container translationally, is chosen so that the positive z-direction is directed upward away from the liquid. The zero of this axis being fixed on the mean free surface where time is zero. In this coordinate system, Eqs. (3.6) to (3.14) become

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right]\phi = 0 \quad \text{in} \quad V \tag{6.1}$$

$$\frac{\sigma}{\rho_0}\bar{\nabla}^2\eta = \left.\frac{\partial\phi}{\partial t}\right|_{z=0} + g(t)\eta \qquad \text{on} \quad z=0 \qquad (6.2)$$

$$D\bar{\nabla}^2\bar{\nabla}^2 w - N_r\bar{\nabla}^2 w + \rho h\frac{\partial^2 w}{\partial t^2}$$

$$= \rho_0\left[\left.\frac{\partial\phi}{\partial t}\right|_{z=-\ell} + g(t)w + \left(\frac{\rho h}{\rho_0}+\ell\right)g(t)\right] + p_2 \qquad \text{on} \quad z=-\ell \qquad (6.3)$$

and

$$\frac{\partial\phi}{\partial r} = 0 \qquad \text{on} \quad r=r_0$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} \qquad \text{on} \quad z=0$$

$$\frac{\partial\eta}{\partial r} = \cot\theta_0 + \nu_x \sin\theta \qquad \text{(zero capillary-hysteresis)} \quad \text{on} \quad z=0$$

$$(6.4)$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial w}{\partial t} \qquad \text{on} \quad z=-\ell$$

$$w = w^*(\theta) \qquad \text{at} \quad r=r_0$$

$$\frac{\partial w}{\partial r} = 0 \qquad \text{(clamped edge)} \quad \text{at} \quad r=r_0$$

where η and w are the deflections of the free surface and of the flexible bottom, respectively, measured from its corresponding mean plane surface. D is the bending rigidity, N_r is the initial stress resultant, ℓ is the depth of the liquid, p_2 is the pressure acting underneath the bottom and ν_x is the angle between the z-axis and the axis of the container.

A. Dimensionless Equations

For convenience, we shall try to find the solution in the following form:

$$\phi = \tilde{\phi} + \psi_o t$$

$$\eta = \tilde{\eta}(r,\theta,t) + \eta^*(r,\theta) \tag{6.5}$$

$$w = \tilde{w}(r,\theta,t) + w^*(r,\theta)$$

where η^* and w^* are the given initial deflection of the free surface and the elastic bottom respectively, and ψ_o is a constant.

Taking the radius of the cylinder r_o as the characteristic length, a_o as the characteristic acceleration, and Ω as the characteristic frequency, we define the dimensionless variables

$$R = \frac{r}{r_o}, \qquad Z = \frac{z}{r_o}, \qquad L = \frac{\ell}{r_o},$$

$$\tau = \Omega t, \qquad \Psi_o = \frac{\psi_o}{\Omega r_o^2}, \qquad \Phi = \frac{\tilde{\phi}}{\Omega r_o^2},$$

$$H = \frac{\tilde{\eta}}{r_0}, \qquad W = \frac{\tilde{w}}{r_0}, \qquad G = \frac{g(t)}{a_0}$$

$$G_0 = \frac{g(0_-)}{a_0}, \qquad P = \frac{p}{\rho_0 \Omega^2 r_0} \qquad P_2 = \frac{p_2}{\rho_0 \Omega^2 r_0} \tag{6.6}$$

the dimensionless parameters

Bond number = $B_\sigma = \dfrac{\rho_0 a_0 r_0^2}{\sigma}$

Membrane number = $B_M = \dfrac{\rho_0 a_0 r_0^2}{N_r}$

Plate number = $B_p = \dfrac{\rho_0 a_0 r_0^4}{D}$ with $D = \dfrac{E h^3}{12(1-\nu^2)}$

Frequency parameter for surface tension = $\Omega_\sigma^2 = \dfrac{\rho_0 r_0^3 \Omega^2}{\sigma}$

Frequency parameter for the membrane = $\Omega_M^2 = \dfrac{\rho_0 r_0^3 \Omega^2}{N_r}$

Frequency parameter for the plate = $\Omega_p^2 = \dfrac{\rho_0 r_0^5 \Omega^2}{D}$

Mass ratio = $\lambda = \dfrac{\rho h}{\rho_0 r_0}$ (6.7)

and the operator

$$\Delta^2 = \frac{1}{R}\frac{\partial}{\partial R} R \frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} = r_0^2 \nabla^2$$

Then the equations become

$$(\Delta^2 + \frac{\partial^2}{\partial Z^2}) \Phi = 0 \tag{6.8}$$

$$\frac{1}{\Omega_\sigma^2} \Delta^2 H - \left(\frac{\partial \Phi}{\partial Z}\right)_{Z=0} - \frac{B_\sigma G(\tau)}{\Omega_\sigma^2} H = \bar{\zeta}(R,\theta,\tau) + \Psi_0 \tag{6.9}$$

$$\frac{1}{\Omega_p^2}\Delta^2\Delta^2 W - \frac{1}{\Omega_M^2}\Delta^2 W + \lambda\frac{\partial^2 W}{\partial\tau^2} - \frac{\partial\Phi}{\partial\tau}\bigg|_{Z=-L} - \frac{B_M}{\Omega_M^2}G(\tau)W \qquad (6.10)$$

$$= \zeta(R,\theta) + \Psi_0$$

where

$$\Omega_\sigma^2\bar{\zeta} = -\Delta^2\frac{\eta^*}{r_0} + B_\sigma G(\tau)\left[\frac{\eta^*}{r_0} + \nu_x R\sin\theta\right]$$

$$\Omega_M^2\zeta = -\Delta^2\Delta^2\frac{w^*}{r_0}\frac{\Omega_M^2}{\Omega_p^2} + \Delta^2\frac{w^*}{r_0} + B_M G\left[\frac{w^*}{r_0} + \nu_x R\sin\theta - (\lambda + D)\right] + P_z\Omega_M^2 \qquad (6.11)$$

with the boundary conditions

$$\frac{\partial\Phi}{\partial R} = 0 \qquad \text{on} \qquad R = 1 \qquad (6.12)$$

$$\frac{\partial\Phi}{\partial Z} = \frac{\partial W}{\partial\tau} \qquad \text{on} \qquad Z = -L \qquad (6.13)$$

$$\frac{\partial\Phi}{\partial Z} = \frac{\partial H}{\partial\tau} \qquad \text{on} \qquad Z = 0 \qquad (6.14)$$

$$W = 0 \qquad \text{on} \qquad R = 1 \qquad (6.15)$$

$$\frac{\partial W}{\partial R} = 0 \qquad \text{on} \qquad R = 1 \qquad (6.16)$$

$$\frac{\partial H}{\partial R} = 0 \qquad \text{on} \qquad R = 1 \qquad (6.17)$$

initial conditions

$$W = H = 0\,, \qquad \frac{\partial W}{\partial \tau} = W^{**}(R,\theta)\,, \quad \frac{\partial H}{\partial \tau} = H^{**}(R,\theta) \quad \text{at} \quad \tau = 0\,. \quad (6.18)$$

Eqs. (6. 8) to (6.18) show that the problem depends on the parameters

$$\Omega_\sigma^2\,,\ B_\sigma\,,\ \Omega_M^2\,,\ \Omega_p^2\,,\ B_M\,,\ B_p\,,\ \lambda\,,\ L\,,\ G\,,\ G_0\,,\ \nu_x\,.$$

These dimensionless parameters are not all independent; for

$$\frac{B_M}{\Omega_M^2} = \frac{B_\sigma}{\Omega_\sigma^2} = \frac{B_p}{\Omega_p^2}\,.$$

However, we retain the symbols

$$\Omega_\sigma^2\,,\ B_\sigma\,,\ \Omega_M^2\,,\ B_M\,,\ \Omega_p^2\,,\ B_p$$

because these three pairs of parameters are not likely to be all important.

$\Omega_p^2\,, B_p \rightarrow 0$, or $\Omega_M^2, B_M \rightarrow 0$, if the tank bottom is rigid.

$\Omega_\sigma^2\,, B_\sigma \rightarrow \infty$ if the surface tension has no effect.

$\Omega_p^2\,, B_p \rightarrow \infty$ if the bottom behaves like a membrane.

Finally, the pressure P in V, is

$$P(R,\theta,Z;\tau) = -\left[\frac{\partial\Phi}{\partial\tau} + \frac{B_M}{\Omega_M^2}G(\tau)Z + \Psi_0\right] \quad . \qquad (6.19)$$

B. Series Solution

In this subsection we shall discuss how to construct the solution. To construct the solution in the form of an infinite series we will encounter some basic difficulties as were pointed out in Ref. 5, Sec. 5. A solution of Eq. (6.1) may be posed as

$$\Phi = \sum_{m=0}^{\infty} \varphi_m(R,Z,\tau)\, e^{im\theta} \qquad (6.20)$$

where

$$\varphi_0 = \dot{d}_0(\tau)Z + \dot{c}_0(\tau) + \sum_{n=1}^{\infty} \frac{J_0(k_nR)}{J_0(k_n)}\left(\dot{c}_{on}\frac{\cosh k_nZ}{\sinh k_nL} + \dot{d}_{on}\frac{\sinh k_nZ}{\cosh k_nL}\right) \qquad (6.21)$$

$$\varphi_m = \sum_{n=1}^{\infty} \frac{J_m(k_{mn}R)}{J_m(k_{mn})}\left(\dot{c}_{mn}\frac{\cosh k_{mn}Z}{\sinh k_{mn}L} + \dot{d}_{mn}\frac{\sinh k_{mn}Z}{\cosh k_{mn}L}\right) \quad . \qquad (6.22)$$

With

$$J_1(k_n) = 0 \ , \qquad J_m'(k_{mn}) = 0 \ , \quad (n = 1, 2, \cdots)$$

both Eqs. (6.12) and (6.17) are satisfied. Then Eqs. (6.13) and (6.14) give

$$H = \sum_{m=0} h_m(R,\tau)\, e^{im\theta} \tag{6.23}$$

$$W = \sum_{m=0} w_m(R,\tau)\, e^{im\theta} \tag{6.24}$$

where

$$h_0 = d_0(\tau) + \sum_{n=1}^{\infty} \frac{k_n J_0(k_n R)}{J_0(k_n)} \frac{d_{on}(\tau)}{\cosh k_n L} \tag{6.25}$$

$$h_m = \sum_{n=1}^{\infty} \frac{k_{mn} J_m(k_{mn} R)}{J_m(k_{mn})} \frac{d_{mn}(\tau)}{\cosh k_{mn} L} \tag{6.26}$$

$$w_0 = d_0(\tau) + \sum_{n=1}^{\infty} \frac{J_0(k_n R)}{J_0(k_n)} k_n (d_{on} - c_{on}) \tag{6.27}$$

$$w_m = \sum_{n=1}^{\infty} \frac{J_m(k_{mn} R)}{J_m(k_{mn})} k_{mn} (d_{mn} - c_{mn}) \tag{6.28}$$

and Eqs. (6.9) and (6.10) become

$$L_m h_m - \Omega_\sigma^2 \left(\frac{\partial \delta_m}{\partial \tau}\right)_{z=0} - B_\sigma G\, h_m = \bar{\zeta}_m(R,\tau) \tag{6.29}$$

$$L_m\left[\frac{\Omega_M^2}{\Omega_p^2} L_m - 1\right] w_m + \lambda \Omega_M^2 \frac{\partial^2 w_m}{\partial \tau^2} - \Omega_M^2 \left.\frac{\partial \delta_m}{\partial \tau}\right|_{z=-L} - B_M G(\tau)\, w_m = \zeta_m(R,\tau) \tag{6.30}$$

where

$$L_m = \frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R} - \frac{m^2}{R^2} \qquad (m = 0, 1, 2 \cdots) \tag{6.31}$$

$\bar{\zeta}_m(R, \tau)$, $\zeta_m(R, \tau)$ are the mth coefficientsof the Fourier series of the functions $\bar{\zeta}(R, \theta, \tau)$, $\zeta(R, \theta, \tau)$, respectively.

Since the assumed series satisfies condition (6.16) we do not have to worry about the convergence of the series ω_m after one term-by-term differentiation, but it may become divergent after four times term-by-term differentiations. To avoid this difficulty, we multiply Eq. (6.30) by R^{m+1}, integrate from zero to R, divide by R^{2m+1}, integrate with respect to R again, and then multiply by R^m to get

$$\frac{\Omega_M^2}{\Omega_p^2} L_m \omega_m - \omega_m + R^m \int \frac{dR}{R^{2m+1}} \left\{ \int_0^R \left[\lambda \Omega_M^2 \frac{\partial^2 \omega_m}{\partial \tau^2} - \Omega_M^2 \left(\frac{\partial \zeta_m}{\partial \tau}\right)_{Z=-L} - B_M G(\tau) \omega_m \right] R^{m+1} dR \right\}$$

$$= R^m \int \frac{dR}{R^{2m+1}} \left\{ \int_0^R \zeta_m(R, \tau) R^m dR \right\} + A_m(\tau) R^m \tag{6.32}$$

where $A_m(\tau)$ is an arbitrary function of time from indefinite integration, and it will be determined by boundary condition (6.15).

Instead of using Eq. (6.30) we substitute Eqs. (6.21), (6.22), and (6.25) to (6.28) into (6.29) and (6.32), using the property of Fourier Bessel series, one finds the governing equations for c_{mn}, d_{mn} as follows:

$$\Omega_\sigma^2\ \ddot{c}_o(\tau) + B_\sigma G(\tau)\, d_o(\tau) = 0 \tag{6.33}$$

$$\Omega_\sigma^2\ \ddot{c}_{on}(\tau) + k_n(\tanh k_n L)(k_n^2 + B_\sigma G(\tau))\, d_{on} = \bar{\zeta}_{on}(\tau) \tag{6.34}$$

$$\Omega_M^2\left[\ddot{c}_{on}(\tau)\,(\coth k_n L + \lambda k_n) - \ddot{d}_{on}(\tanh k_n L + \lambda k_n)\right]$$

$$+ k_n\left[\frac{\Omega_M^2}{\Omega_p^2}\, k_n^4 + k_n^2 - B_M G(\tau)\right](c_{on} - d_{on})$$

$$= -(L+\lambda)\,\Omega_M^2\, \ddot{d}_o + \zeta_{on}(\tau) \tag{6.35}$$

$$\Omega_\sigma^2\,\ddot{c}_{mn} + \left[k_{mn}(k_{mn}^2 + B_\sigma G(\tau)) \tanh k_{mn} L\right] d_{mn} = \bar{\zeta}_{mn}(\tau) \tag{6.36}$$

$$\mathcal{L}_{mn} = \mathcal{L}_{ms} \qquad (m, n, s = 1, 2, \cdots) \tag{6.37}$$

where

$$\mathcal{L}_{mn} = \frac{k_{mn}^2 - m^2}{k_{mn}^2}\left[\left\{\ddot{c}_{mn}\,(\coth k_{mn} L + \lambda k_{mn}) - \ddot{d}_{mn}(\tanh k_{mn} L + \lambda k_{mn})\right\}\Omega_M^2\right.$$

$$\left. + \left\{k_{mn}\left(\frac{k_{mn}^4}{\Omega_p^2} + \frac{k_{mn}^2}{\Omega_M^2} - \frac{B_M}{\Omega_M^2} G(\tau)\right)(c_{mn} - d_{mn})\right\}\Omega_M^2 - \zeta_{mn}\right] \tag{6.38}$$

$\bar{\zeta}_{mn}(\tau)$, $\zeta_{mn}(\tau)$ are the coefficients of the $J_m(k_{mn}R)$-Fourier-Bessel series of the functions $\bar{\zeta}_m(R,\tau)$ and

$$R^m \int \frac{dR}{R^{2m+1}} \left[\int_0^R \zeta_m(R_1,\tau) R_1^m dR_1 \right]$$

respectively.

From Eq. (6.15)

$$d_o(\tau) = \sum_{n=1}^{\infty} k_n \left[c_{on}(\tau) - d_{on}(\tau) \right] \tag{6.39}$$

$$0 = \sum_{n=1}^{\infty} k_{mn} \left[c_{mn}(\tau) - d_{mn}(\tau) \right] \quad . \tag{6.40}$$

If the edge on ∂S_2 is not clamped, but simply supported, the assumed series for W will be divergent after only one term-by-term differentiation. Then we have to repeat the process that operates on Eq. (6.30), to Eq. (6.32) again, thus introducing another arbitrary function of time. If one substitutes H, Φ, and W into these new equations, one obtains the relations between c_{mn}, d_{mn} and computes $L_m \omega_m$, $\partial \omega_m / \partial R$ etc., from these new equations (not from term-by-term differentiation of the series), one can then solve the problem. The actual algebra is complicated, and will not be presented here.

In the case of a rigid tank, Ω_M^2, B_M and Ω_p^2, B_p and ζ_{mn} are zero. We must have, from Eqs. (6.35) to (6.40),

$$c_{mn} = d_{mn} \qquad (m = 0, 1, 2, \cdots; \; n = 1, 2, 3, \cdots\cdots)$$

$$d_0 = 0 \qquad . \tag{6.41}$$

Then Eqs. (6.34) and (6.36) are exactly the same equations obtained by Benjamin and Ursell (Ref. 7). In this case all the modes are uncoupled. On the other hand, in a flexible container, all modes are coupled, and the coupling of different modes could cause instability of the system. If $G(\tau)$ is a periodic function and if the system of equations has been truncated to a finite set of equations, coupling instability has been shown explicitly in Refs. 5 and 13.

If $G(\tau)$ is a constant for $\tau > 0$, Eqs. (6.33) to (6.37) can be solved analytically with the aid of the eigenfunctions discussed in the previous sections. We shall consider this solution in the next subsection.

C. Eigenvalues and Eigenfunctions

a. Eigenvalues

Let us consider the homogeneous solution of Eqs. (6.33) to (6.37), with G a constant. Let

$$c_{mn} = C_{mn} e^{i\omega_m \tau}$$

$$d_{mn} = D_{mn} e^{i\omega_m \tau} \tag{6.42}$$

then

$$C_{oo} = \frac{B_\sigma G}{\Omega_\sigma^2 \mu_o} D_{oo}$$

$$C_{mn} = \frac{p_{mn}^2}{\mu_m} D_{mn}$$

(6.43)

$$\beta_{on}(\mu_o) D_{on} = (\lambda + L)\, \mu_o D_{oo}$$

$$\frac{k_{mn}^2 - m^2}{k_{mn}^2} \beta_{mn}(\mu_m) D_{mn} = \frac{k_{ms}^2 - s^2}{k_{ms}^2} \beta_{ms}(\mu_m) D_{ms}$$

where

$$p_{mn}^2 = k_{mn}(k_{mn}^2 + B_\sigma G) \tanh k_{mn} L / \Omega_\sigma^2$$

$$\beta_{mn}(\mu) = -(p_{mn}^2 \varkappa_{mn} - \mu \bar{\varkappa}_{mn})$$

$$+ k_{mn}\left(\frac{k_{mn}^4}{\Omega_p^2} + \frac{k_{mn}^2}{\Omega_M^2} - \frac{B_M}{\Omega_M^2} G\right)\left(\frac{p_{mn}^2}{\mu} - 1\right) \qquad (6.44)$$

$$\varkappa_{mn} = \coth k_{mn} L + \lambda k_{mn}$$

$$\bar{\varkappa}_{mn} = \tanh k_{mn} L + \lambda k_{mn}$$

$$\mu_m = \omega_m^2 \quad .$$

Substituting into Eqs. (6.39) and (6.40), one gets the frequency equations

$$f_0(\mu) = \frac{1}{(\lambda+L)\mu} + \sum_{n=1}^{\infty} \frac{(\mu - p_{on}^2)\, k_{on}}{\mu\, \beta_{on}(\mu)} = 0 \qquad (6.45)$$

$$f_m(\mu) = \sum_{n=1}^{\infty} \frac{k_{mn}^3\, (\mu - p_{mn}^2)}{(k_{mn}^2 - m^2)\, \beta_{mn}(\mu)\, \mu} = 0 \qquad (6.46)$$

We have shown in Section IV that all the eigenvalues μ must be real and simple, and that there exists an upper bound and a lower bound of B_σ (or B_M) between which all the eigenvalues are positive. These upper and lower bounds will be determined explicitly below.

From Eq. (4.45), if

$$\frac{k_{01}^4}{\Omega_p^2} + \frac{k_{01}^2}{\Omega_M^2} > \frac{B_M G}{\Omega_M^2} > - \frac{k_{01}^2}{\Omega_\sigma^2} \qquad (6.47)$$

where $k_{01} = 3.83$, there can be no negative root of $f_o(\mu) = 0$, because in this range of $(B_M G/\Omega_M^2)$, $p_{on} > 0$ and $\beta_{on}(\mu) > 0$ for $\mu < 0$. A similar situation is true for $f_m(\mu)$, $m \geq 1$, if

$$\frac{k_{11}^4}{\Omega_p^2} + \frac{k_{11}^2}{\Omega_M^2} > \frac{B_M G}{\Omega_M^2} > - \frac{k_{11}^2}{\Omega_\sigma^2} \qquad (6.48)$$

where $k_{11} = 1.84$.

On the other hand, if

$$\frac{k_{01}^4}{\Omega_p^2} + \frac{k_{01}^2}{\Omega_M^2} > \frac{B_M G}{\Omega_M^2} \geq \frac{k_{11}^4}{\Omega_p^2} + \frac{k_{11}^2}{\Omega_M^2} \qquad (6.49)$$

one cannot see directly from (6.46) whether $f_m(\mu)$ has negative roots or not. If one differentiates Eq. (6.46) with respect to μ, one gets

$$\frac{df_m}{d\mu} = -\sum_{n=1}^{\infty} \frac{p_{mn}^4 k_{mn}^3}{k_{mn}^2 - m^2} \frac{\left(1 - \frac{\mu}{p_{mn}^2}\right)^2 \bar{\varkappa}_{mn} + (\coth k_{mn}L - \tanh k_{mn}L)}{[\beta_{mn}(\mu)\,\mu]^2} \,. \qquad (6.50)$$

One sees that $df_m/d\mu$ are always less than zero. Therefore $f_m(\mu)$ is a monotonically decreasing function except at some isolated points where the function is discontinuous and has an infinite jump from $-\infty$ to ∞. This implies that the function f_m can have one and only one zero between each pair of consecutive singularities of $f_m(\mu)$. Since $f_m(\mu)$ tends to $0-$ (i.e. f_m is negative) as μ tends to $-\infty$, there is no root of Eq. (6.46) for

$$-\infty < \mu < \text{first zero of the denominators of } f_m(\mu).$$

In the case of (6.49) the first zero of the denominators of $f_1(\mu)$ is negative, call it $-R_1$; but the second one is positive, call it R_2. Therefore the first and only the first root of $f_1(\mu) = 0$ must lie between $-R_1$ and R_2. Let $\mu = R_1 + \delta$. As $\delta > 0$ and tends to 0, we have

$$f_1(\mu) \longrightarrow \infty$$

whereas at $\mu = 0$, one has

$$f_1(0) = -\sum_{n=1}^{\infty} \frac{k_{1n}^3}{(k_{1n}^2 - 1)\left(\frac{k_{1n}^4}{\Omega_p^2} + \frac{k_{1n}^2}{\Omega_M^2} - \frac{B_M G}{\Omega_M^2}\right)}$$

$$= -\frac{\Omega_p^2}{2\left[\left(\frac{\Omega_p^2}{\Omega_M^2}\right)^2 + \frac{4B_M G\,\Omega_p^2}{\Omega_M^2}\right]^{1/2}} \left[\frac{J_1(a)}{J_1'(a)} - \frac{I_1\left(\sqrt{\left(\frac{\Omega_p^2}{\Omega_M^2}\right)^2 + a^2}\right)}{I_1'\left(\sqrt{\left(\frac{\Omega_p^2}{\Omega_M^2}\right)^2 + a^2}\right)}\right] \quad (6.51)$$

where a^2 is a positive constant so defined that

$$\frac{B_M G}{\Omega_M^2} = \frac{a^4}{\Omega_p^2} + \frac{a^2}{\Omega_M^2} \quad . \quad (6.52)$$

Evidently, from the properties of the modified Bessel functions,

$$-I_1\left(\left[\left(\frac{\Omega_p^2}{\Omega_M^2}\right)^2 + a^2\right]^{1/2}\right) \Big/ I_1'\left(\left[\left(\frac{\Omega_p^2}{\Omega_M^2}\right)^2 + a^2\right]^{1/2}\right)$$

is always less than zero. If k_{11} (= 1.84) $<$ a $<$ k_{01}(= 3.83), one has

$$J_1(a) / J_1'(a) < 0$$

so

$$f_1(0) > 0 \quad .$$

Therefore, the first root of $f_1(\mu) = 0$ must be greater than zero. From Eqs. (6.47) to (6.49)

$$\frac{k_{01}^4}{\Omega_p^2} + \frac{k_{01}^2}{\Omega_M^2} > \frac{B_M G}{\Omega_M^2} > -\frac{k_{11}^2}{\Omega_\sigma^2} \tag{6.53}$$

then the roots of Eqs. (6.45) and (6.46) are always real, and the system is stable.

From Eqs. (6.45) and (6.46) one sees that the eigenvalues μ are complicated functions of the parameters. In a special case that the flexible bottom is a membrane, i.e. Ω_p^2 and B_p are infinite, and $B_M G$ and $B_\sigma G$ are both small compared to one, it can be shown that the eigenvalues μ depend on $B_\sigma G/B_M G$, λ and L only.

b. <u>Eigenfunctions</u>

From Eqs. (6.21) and (6.22), the eigenfunctions are

$$\Phi = \zeta_{ms}(R, z)\, e^{im\theta} \begin{cases} \omega_{ms} \cos \omega_{ms}\tau \\ \omega_{ms} \sin \omega_{ms}\tau \end{cases} \tag{6.54}$$

where ω_{ms}^2 is the sth root of the function $f_m(\omega_m^2) = 0$, $(m = 0, 1, 2 \ldots)$.

The functions ζ_{ms} are

$$\zeta_{os} = A_{os}\Big[z + \frac{B\sigma G}{\Omega_\sigma^2 \omega_{os}^2} + \sum_{n=1}^{\infty} \frac{J_o(k_n R)\,(\lambda + L)\,\omega_{os}^2}{J_o(k_n)\ \ \beta_{on}(\omega_{os}^2)} \times\Big(\frac{p_{on}^2}{\omega_{os}^2}\frac{\cosh k_n z}{\sinh k_n L} + \frac{\sinh k_n z}{\cosh k_n L}\Big)\Big] \qquad (6.55)$$

$$\zeta_{ms} = A_{ms} \sum_{n=1}^{\infty} \frac{J_m(k_{mn}R)}{J_m(k_{mn})}\frac{k_{mn}^3\,\omega_{ms}^2}{(k_{mn}^2 - m^2)\,\beta_{mn}(\omega_{ms}^2)} \times\Big(\frac{p_{mn}^2}{\omega_{ms}^2}\frac{\cosh k_{mn} z}{\sinh k_{mn} L} + \frac{\sinh k_{mn} z}{\cosh k_{mn} L}\Big) \qquad (6.56)$$

(s, m = 1, 2 ...) where

$$A_{os} = \pi^{-1/2}\Big\{ L + \lambda + \sum_{n=1}^{\infty} k_n \Big[\frac{\omega_{os}^2 (L+\lambda)}{\beta_{on}(\omega_{os}^2)}\Big]^2 \times\Big[\Big(\frac{p_{on}^2}{\omega_{os}^2}\Big)^2 (\coth k_n L - \tanh k_n L) + \Big(\frac{p_{on}^2}{\omega_{os}^2} - 1\Big)^2 \bar{\varkappa}_{on}\Big]\Big\}^{-1/2} \qquad (6.57)$$

$$A_{ms} = \pi^{-1/2}\Big\{ \sum_{n=1}^{\infty} \frac{k_{mn}^3}{k_{mn}^2 - m^2}\Big[\frac{\omega_{mn}^2}{\beta_{mn}(\omega_{mn}^2)}\Big]^2 \times\Big[\Big(\frac{p_{mn}^2}{\omega_{ms}^2}\Big)^2 (\coth k_{mn} L - \tanh k_{mn} L) + \Big(\frac{p_{mn}^2}{\omega_{ms}^2} - 1\Big)^2 \bar{\varkappa}_{ms}\Big]\Big\}^{-1/2}$$

A's are so chosen that φ_{ms} is normalized in the sense of Eq. (4.22) i.e.

$$\int_{S_2} \lambda \frac{\partial \varphi_{ms}}{\partial N} \frac{\partial \varphi_{nr}}{\partial N} dS + \int_{S_1+S_2} \varphi_{ms} \frac{\partial \varphi_{nr}}{\partial N} dS = \delta_{mn} \delta_{sr} \tag{6.58}$$

where δ_{mn} is the Kronecker-δ.

By Eqs. (6.23) and (6.24), one has

$$H = h_{ms}(R) e^{im\theta} \begin{cases} \sin \omega_{ms} \tau \\ -\cos \omega_{ms} \tau \end{cases} \tag{6.59}$$

$$W = W_{ms}(R) e^{im\theta} \begin{cases} \sin \omega_{ms} \tau \\ -\cos \omega_{ms} \tau \end{cases} \tag{6.60}$$

where

$$h_{os}(R) = A_{os} \left[1 + \sum_{n=1}^{\infty} \frac{J_o(k_n R)}{J_o(k_n)} \frac{k_n}{\cosh k_n L} \frac{(\lambda + L) \omega_{os}^2}{\beta_{on}(\omega_{os}^2)} \right] \tag{6.61}$$

$$W_{os}(R) = A_{os} \left[1 + \sum_{n=1}^{\infty} \frac{J_o(k_n R)}{J_o(k_n)} \left(1 - \frac{p_{on}^2}{\omega_{os}^2}\right) \frac{k_n (\lambda + L) \omega_{os}^2}{\beta_{on}(\omega_{os}^2)} \right] \tag{6.62}$$

$$h_{ms}(R) = A_{ms} \sum_{n=1}^{\infty} \frac{J_m(k_{mn} R)}{J_m(k_{mn})} \frac{k_{mn}^3}{k_{mn}^2 - m^2} \frac{\omega_{ms}^2}{\beta_{mn}(\omega_{ms}^2) \cosh k_{mn} L} \tag{6.63}$$

$$W_{ms}(R) = A_{ms}\sum_{n=1}^{\infty} \frac{J_m(k_{mn}R)}{J_m(k_{mn})} \frac{k_{mn}^3}{k_{mn}^2 - m^2} \left(1 - \frac{p_{mn}^2}{\omega_{ms}^2}\right) \frac{\omega_{ms}^2}{\beta_{mn}(\omega_{ms}^2)} \quad . \tag{6.64}$$

Similar to the eigenvalues, when $\Omega_p^{\ 2}$ and B_p are infinite and when $B_M G$ and $B_\sigma G$ are small compared to one, all the eigenfunctions depend on λ, L and $B_\sigma G/B_M G$ only.

D. Initial Value Problem

In this section, a linear combination of the eigenfunctions given above will be determined to satisfy the inhomogeneous Eqs. (6.9) and (6.10) with the initial conditions,

$$W = H = 0 \qquad \text{at} \qquad \tau = 0 \tag{6.65}$$

$$\frac{\partial \Phi}{\partial N} = \frac{\partial H}{\partial \tau} = H^{**}(R,\theta) \qquad \text{on} \qquad S_1 \ , \quad \tau = 0 \tag{6.66}$$

$$= -\frac{\partial W}{\partial \tau} = -W^{**}(R,\theta) \qquad \text{on} \qquad S_2 \ , \quad \tau = 0 \ .$$

Let

$$\Phi = \Phi_1 + \Phi_2 + \int_0^{\tau} \Phi_3(R,\theta,Z,\tau - T; T)\,dT \tag{6.67}$$

$$H = H_1 + H_2 + \int_0^\tau H_3(R, \theta, \tau - T; T)\, dT \tag{6.68}$$

$$W = W_1 + W_2 + \int_0^\tau W_3(R, \theta, \tau - T; T)\, dT \tag{6.69}$$

where

$$\begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} = \sum_{m,s} \mathcal{S}_{ms}(R,Z) e^{im\theta} \begin{Bmatrix} \alpha_{ms}\,\omega_{ms} \cos \omega_{ms}\tau \\ \epsilon_{ms}\,\omega_{ms} \sin \omega_{ms}\tau \\ \delta_{ms}(T)\,\omega_{ms} \sin \omega_{ms}(\tau - T) \end{Bmatrix} \tag{6.70}$$

$$\begin{Bmatrix} H_1 \\ H_2 \\ H_3 \end{Bmatrix} = \sum_{m,s} h_{ms}(R) e^{im\theta} \begin{Bmatrix} \alpha_{ms} \sin \omega_{ms}\tau \\ \epsilon_{ms}(1 - \cos \omega_{ms}\tau) \\ \delta_{ms}(T)\,(1 - \cos \omega_{ms}(\tau - T)) \end{Bmatrix} \tag{6.71}$$

$$\begin{Bmatrix} W_1 \\ W_2 \\ W_3 \end{Bmatrix} = \sum_{m,s} w_{ms}(R) e^{im\theta} \begin{Bmatrix} \alpha_{ms} \sin \omega_{ms}\tau \\ \epsilon_{ms}(1 - \cos \omega_{ms}\tau) \\ \delta_{ms}(T)\,(1 - \cos \omega_{ms}(\tau - T)) \end{Bmatrix} \tag{6.72}$$

where ς_{ms}, h_{ms}, w_{ms} are eigenfunctions given in Eqs. (6.61) to (6.63) and where α_{ms}, ϵ_{ms}, $\delta_{ms}(T)$ are to be determined. Evidently H and W satisfy zero initial conditions. If

$$\alpha_{ms} = \int_{S_1} \frac{1}{\omega_{ms}} H^{**}(R,\theta)\, \varsigma_{ms}(R,0)\, e^{-im\theta}\, dS$$

$$+ \int_{S_2} -\frac{1}{\omega_{ms}} W^{**}(R,\theta)\left[\varsigma_{ms}(R,-L) - \lambda \frac{\partial \varsigma_{ms}}{\partial Z}\Big|_{Z=-L}\right] e^{-im\theta}\, dS, \qquad (6.73)$$

by the results proved in Section V, the conditions

$$\frac{\partial \Phi}{\partial N} = H^{**}(R,\theta) \left(= \frac{\partial H}{\partial \tau} = \frac{\partial H_1}{\partial \tau} \right) \qquad \text{on } S_1 \qquad (6.74)$$

$$= -W^{**}(R,\theta) \left(= -\frac{\partial W}{\partial \tau} = \frac{\partial W_1}{\partial \tau} \right) \qquad \text{on } S_2 \qquad (6.75)$$

are satisfied at $\tau = 0$. Substituting Φ, H and W into Eqs. (6.9) and (6.10), one gets

$$\frac{1}{\Omega_\sigma^2} \Delta^2 H_2 - \left(\frac{\partial \Phi_2}{\partial \tau}\right)_{Z=0} - \frac{B_\sigma G}{\Omega_\sigma^2} H_2 + \int_0^\tau \left[\frac{1}{\Omega_\sigma^2} \Delta^2 H_3 - \left(\frac{\partial \Phi_3}{\partial \tau}\right)_{Z=0} - \frac{B_\sigma G}{\Omega_\sigma^2} H_3\right](R,\theta,\tau-T;T)\, dT$$

$$= -\sum_{m,s} \epsilon_{ms}\, \omega_{ms}^2\, \varsigma_{ms}(R,0)\, e^{im\theta} - \int_0^\tau \sum_{m,s} \delta_{ms}(T)\, \omega_{ms}^2\, \varsigma_{ms}(R,0)\, e^{im\theta}\, dT \qquad (6.76)$$

$$= \bar{\zeta}(R,\theta;\tau) + \Psi_0 = \bar{\zeta}(R,\theta;0) + \int_0^\tau \frac{\partial}{\partial T} \bar{\zeta}(R,\theta;T)\, dT + \Psi_0$$

$$\frac{1}{\Omega_p^2}\Delta^2\Delta^2 W_2 - \frac{1}{\Omega_M^2}\Delta^2 W_2 - \frac{B_M G}{\Omega_M^2} W_2 - \frac{\partial \Phi_2}{\partial \tau}\Big|_{z=-L}$$

$$+\int_0^\tau \left[\frac{1}{\Omega_p^2}\Delta^2\Delta^2 W_3 - \frac{1}{\Omega_M^2}\Delta^2 W_3 - \frac{B_M G}{\Omega_M^2} W_3 - \frac{\partial \Phi_3}{\partial \tau}\Big|_{z=-L}\right](R,\theta,\tau-T;T)\,dT$$

$$= -\sum_{m,s} \epsilon_{ms}\,\omega_{ms}^2 \left(\varphi_{ms} + \lambda \frac{\partial \varphi_{ms}}{\partial N}\right)_{z=-L} e^{im\theta} \tag{6.77}$$

$$- \int_0^\tau \sum_{m,s} \delta_{ms}(T)\,\omega_{ms}^2 \left(\varphi_{ms} + \lambda \frac{\partial \varphi_{ms}}{\partial N}\right)_{z=-L} e^{im\theta}\,dT$$

$$= \zeta(R,\theta,\tau) + \Psi_0 = \zeta(R,\theta,0) + \int_0^\tau \frac{\partial}{\partial T}\zeta(R,\theta;T)\,dT + \Psi_0\ .$$

If one lets

$$\epsilon_{ms} = -\frac{1}{\omega_{ms}^2}\left[\int_{S_1} \bar{\zeta}(R,\theta,0)\frac{\partial \varphi_{ms}}{\partial z}\Big|_{z=0} e^{-im\theta}\,dS - \int_{S_2} \zeta(R,\theta,0)\frac{\partial \varphi_{ms}}{\partial z}\Big|_{z=0} e^{-im\theta}\,dS\right] \tag{6.78}$$

$$\delta_{ms}(T) = -\frac{1}{\omega_{ms}^2}\left[\int_{S_1} \frac{\partial \bar{\zeta}}{\partial T}(R,\theta,T)\frac{\partial \varphi_{ms}}{\partial z}\Big|_{z=-L} e^{-im\theta}\,dS - \int_{S_2} \frac{\partial \zeta}{\partial T}(R,\theta,T)\frac{\partial \varphi_{ms}}{\partial z}\Big|_{z=-L} e^{-im\theta}\,dS\right] \tag{6.79}$$

then by the results proved in Section V, Eqs. (6.76) and (6.77) are satisfied up to a constant. This constant can always be fixed by a proper choice of Ψ_0. Thus by integrating Eq. (6.76) over S_1, one gets (note that $\int_{S_1} (\partial \bar{\zeta}/\partial \tau)\, dS_1 = 0$)

$$\Psi_0 = -\sum_{s=1}^{\infty} \epsilon_{0s} A_{0s} \frac{B_\sigma G}{\Omega_\sigma^2} + 2\frac{\cot\theta_0}{\Omega_\sigma^2} \tag{6.80}$$

which corresponds to a constant pressure field. The problem is thus solved.

Let us consider a specific example. Let the liquid in the tank be initially in equilibrium with a constant axial acceleration G(0-). At time zero, the axial acceleration is changed to G. To make the example as simple as possible, we shall consider only the case where the bottom behaves like a membrane, i.e. $\Omega_p^2 \to \infty$. Thus we have an initial value problem with

$$W = H = \frac{\partial W}{\partial \tau} = \frac{\partial H}{\partial \tau} = 0 \quad \text{at} \quad \tau = 0$$

and

$$G(\tau) = G(0_-) \quad \text{for} \quad \tau < 0$$

$$= G \quad \text{for} \quad \tau > 0 \quad .$$

If $G(0-) \neq 0$, we shall choose, for convenience, the characteristic acceleration a_o to be $G(0-)$, thus the initial deflection of the free surface and bottom are

$$\frac{\eta^*}{r_0} = \frac{\cot\theta_0}{\sqrt{B_\sigma}\, I_1(\sqrt{B_\sigma})} \left[I_0(\sqrt{B_\sigma} R) - \frac{2 I(\sqrt{B_\sigma})}{\sqrt{B_\sigma}} \right] + \nu_x \sin\theta \, \frac{I_1(\sqrt{B_\sigma} R)}{\sqrt{B_\sigma}\, I_1'(\sqrt{B_\sigma})}$$

(6.81)

$$\frac{w^*}{r_0} = \left(\lambda + L - P_2 \frac{\Omega_M^2}{B_M} - \frac{2\cot\theta_0}{B_\sigma} \right) \left[1 - \frac{\sqrt{B_M}\, J_0(\sqrt{B_M} R)}{J_1(\sqrt{B_M})} \right] + \frac{J_1(\sqrt{B_M} R)}{J_1(\sqrt{B_M})} \nu_x \sin\theta \quad .$$

Consequently, $\bar{\zeta}$ and ζ defined in Eqs. (6.9) and (6.10) become

$$\bar{\zeta} = \frac{B_\sigma}{\Omega_\sigma^2}(G-1)\frac{\eta^*}{r_o} - \frac{2\cot\theta_o}{\Omega_\sigma^2}$$

$$\zeta = \frac{B_M}{\Omega_M^2}(G-1)\left[\frac{w^*}{r_o} - (\lambda + L)\right] - \frac{2\cot\theta_o}{\Omega_\sigma^2} \qquad (6.82)$$

From Eqs. (6.78) and (6.82), trigonometric orthogonality implies

$$\epsilon_{ms} = 0 \qquad \text{for} \qquad |m| \geq 2 \qquad \text{and all} \quad s \; .$$

Because of zero initial velocity, one has

$$\alpha_{ms} = 0 \qquad \text{for all} \quad m, s \; .$$

Because there is no lateral or rotational oscillation, so that $\bar{\zeta}$ and ζ are independent of time, one has

$$\delta_{ms}(T) = 0 \qquad \text{for all} \quad m, s \; ,$$

and

$$\epsilon_{os} = -\int_0^{2\pi}\left[\int_0^1 \bar{\zeta}\,\frac{\partial \varphi_{os}}{\partial z}\bigg|_{z=0} R\,dR - \int_0^1 \zeta\,\frac{\partial \varphi_{os}}{\partial z}\bigg|_{z=-L} R\,dR\right]\frac{1}{\omega_{os}^2}\,d\theta$$

$$= -2\pi A_{os}\frac{B_\sigma}{\Omega_\sigma^2}\,\frac{G-1}{\omega_{os}^2}(\lambda+L)\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\omega_{os}^2 k_n}{\beta_{os}(\omega_{os}^2)}\left[\frac{\cot\theta_0}{(k_n^2+B_\sigma^2)\cosh k_n L}\right.\right.$$

$$\left.\left. - \left(1-\frac{p_{on}^2}{\omega_{os}^2}\right)\frac{(\lambda+L-P_3\Omega_M^2/B_M - 2\cot\theta_0/B_\sigma)B_M}{2(k_n^2-B_M)}\right]\right\}. \quad (6.83)$$

If one defines $\bar{\epsilon}_{1s} = i\,(\epsilon_{1,s} - \epsilon_{-1,s})$, where $i = \sqrt{-1}$, one gets for the antisymmetric case,

$$\bar{\epsilon}_{1s} = -2\int_0^1\left[\int_0^{2\pi}\left(\bar{\zeta}\,\frac{\partial \varphi_{1s}}{\partial z}\bigg|_{z=0} - \zeta\,\frac{\partial \varphi_{1s}}{\partial z}\bigg|_{z=-L}\right)\sin\theta\,d\theta\right] R\,dR$$

$$= -2\pi\,\frac{B_\sigma}{\Omega_\sigma^2}(G-1)\,\nu_x\,A_{1s}\sum_{n=1}^{\infty}\frac{k_{1n}^3}{(k_{1n}^2-1)\,\beta_{1n}(\omega_{1s}^2)}\left[\frac{1}{(k_{1n}^2+B_\sigma)\cosh k_{1n}L}\right.$$

$$\left. - \frac{\sqrt{B_M}\,J_1'(\sqrt{B_M})}{J_1(\sqrt{B_M})}\left(1-\frac{p_{1n}^2}{\omega_{1s}^2}\right)\frac{1}{k_{1n}^2-B_M}\right] \quad (6.84)$$

for $s = 1, 2 \ldots$. Thus one obtains the final solution for the free surface and the bottom deflection:

$$H=\sum_{s=1}^{\infty} \epsilon_{os} h_{os}(R)(1-\cos\omega_{os}\tau)$$

$$+\sin\theta \sum_{s=1}^{\infty} \bar{\epsilon}_{1s} h_{1s}(R)(1-\cos\omega_{1s}\tau) \tag{6.85}$$

$$W=\sum_{s=1}^{\infty} \epsilon_{os} W_{os}(R)(1-\cos\omega_{os}\tau)$$

$$+\sin\theta \sum_{s=1}^{\infty} \bar{\epsilon}_{1s} W_{1s}(R)(1-\cos\omega_{1s}\tau) \tag{6.86}$$

for $\tau>0$.

The frequency equations and the transient deflection of surfaces are quite complex. Some numerical results are presented in Figs. 4 to 10, and will be discussed in Section VI-F.

E. Force and Moment

The force and moment exerted on the tank by the fluid, for $\tau \geq 0$, can be easily obtained by appropriate integration of the pressure over the wetted surfaces.

$$F_z=\int_{S_2} P\Big|_{z=-L+W+\frac{w^*}{r_o}} dS + \pi\frac{\cot\theta_o}{\Omega_\sigma^2}$$

$$=-\int_0^1\int_0^{2\pi}\left[\frac{\partial\Phi}{\partial\tau}\Big|_{z=-L}+G\frac{B_M}{\Omega_M^2}\left(W+\frac{w^*}{r_o}-L\right)+\Psi_o\right]d\theta\, R\,dR \tag{6.87}$$

$$+\pi\frac{\cot\theta_o}{\Omega_\sigma^2} = \pi G\frac{B_M}{\Omega_M^2}L - \pi F_z'$$

where,

$$F_z' = \sum_{m=1}^{\infty} \epsilon_{om} A_{om} \omega_{om}^2 \cos \omega_{om} \tau \quad . \tag{6.88}$$

F_z' is an oscillatory force which is positive if pointed along the negative z-axis. F_z' tends to zero as B_M tends to zero, because $\epsilon_{om} \Omega_{om}^2$ remains finite, and it can be shown that A_{om} as defined in Eq. (6.57) tends to zero as $B_M \to 0$. Hence in the case of a rigid tank, the oscillatory force F_z' in the z-direction is identically zero. The fluid in the tank acts as a rigid body, as far as the inertial force in the z-direction is concerned. When B_M is not zero, the flexibility of the bottom acting like a spring, allows the fluid to have an oscillating displacement relative to the tank, and an oscillatory force is created that acts on the tank.

The force in the y-direction and the moment about the x-axis can be obtained by

$$F_y = -\int_0^{2\pi} \int_{-L}^{0} P|_{R=1} \sin\theta \, dZ \, d\theta \tag{6.89}$$

$$M_x = -\int_{S_2} P|_{Z=-L+W+\frac{W^*}{\bar{r}_0}} R \sin\theta \, dS - \int_{-L}^{0} \int_0^{2\pi} P|_{R=1} Z \sin\theta \, d\theta \, dZ \quad . \tag{6.90}$$

F. Numerical Example

We have studied numerically the case of a circular cylindrical container with rigid side walls and a flat flexible bottom containing an inviscid incompressible fluid (Fig. 3). The numerical results on sloshing frequencies, oscillatory force and transient deflection of the free surface for various parameters will be discussed in this section.

All of the results obtained in Sections VI-D and VI-E are in the form of infinite series; in actual computation we have to truncate the series. Since the series converges rapidly, we only take twelve terms of the series in our computation.

The sloshing frequencies are computed from Eqs. (6.45) and (6.46), for $\lambda = 0.04$, $L = 0.4$, $B_\sigma G = 0.01$ and various $B_M G$. One recalls that from Eqs. (6.6) and (6.7), $\lambda\ (= \rho h/\rho_o r_o)$ is the mass ratio, $L\ (= \ell / r_o)$ is the ratio of the depth of the liquid to the radius of the tank, $B_\sigma\ (= \rho_o a_o r_o^{\,2}/\sigma)$ is called the Bond number, $B_M (= \rho_o a_o r_o^{\,2}/N_r)$ is called the membrane number, and G is the ratio of the gravitational acceleration to the chosen characteristic acceleration a_o. As far as the sloshing frequencies are concerned, the problem is independent of the choice of a_o. In Fig. 4, the first symmetric and the first asymmetric ($m = 1$, see Eq. (6.46)) sloshing frequency are plotted versus $B_M G$. In this plot the frequencies are normalized by the first symmetric and the first asymmetric sloshing frequency, respectively, in the corresponding rigid container. In the range of $B_M G$ considered, the flexibility of the bottom has a larger effect on the symmetric frequencies than on the asymmetric frequencies. One also sees from Fig. 4 that the frequencies decrease as $B_M G$ increases. Physically it says that the sloshing frequencies decrease as the bottom of the tank becomes more flexible. At first the rate of decreasing is very small until $B_M G$ reaches a certain value; then the symmetric frequency drops sharply. This kind of transition phenomenon also occurs for the frequencies of higher modes, as can be seen from Fig. 5. But the transition points occur at smaller values of $B_M G$ for higher modes. In Fig. 5, the frequency spectra of the symmetric modes for the same parameters as Fig. 4 are plotted. For a given $B_M G$, there are some frequencies whose values are not close to any of the rigid tank sloshing frequencies. Modes corresponding to these frequencies have large mean deflections on the free surface. On the other hand, for those modes whose frequencies are close to a sloshing frequency of a rigid tank, the mean free surface deflections are approximately zero. A typical example is plotted in Fig. 6 for $\lambda = 0.04$, $L = 0.4$, $B_\sigma G = 0.01$ and $B_M G = 0.0003$. Those modes whose frequencies are close to the rigid tank modes also have approximately the same free surface deflection. As we have pointed out in Sections VI - C, when $B_\sigma G$ and $B_M G$ are both small compared to one, the eigenvalues and the eigenfunctions depend approximately only on $B_M G/B_\sigma G$, λ and L. Therefore the curves in Figs. 4 to 6 should be a good approximation for the same $B_M G/B_\sigma G$, as long as $B_\sigma G$ is small compared to one.

The values of the frequencies, which deviate considerably from all the sloshing frequencies of the corresponding rigid tank, depend on the parameters of the problem. In the case that $B_\sigma G \ll 1$, we find that these frequencies are approximately linearly proportional to the square root of the ratio $B_\sigma G/B_M G$ (Fig. 7). If $B_\sigma G$ is fixed, it means these frequencies are approximately linearly proportional to the inverse of the square root of $B_M G$. In Fig. 7, the first six symmetric frequencies of Fig. 5 are replotted versus $(B_M G)^{-1/2}$.

We use solid lines to denote the frequencies corresponding to modes with large mean free surface deflections. One sees that the solid lines are practically straight lines. The first line has value approximately equal to 0.324 of the first symmetric vibration frequency of the membrane. The second one is approximately 0.403 of the second symmetric frequency of the membrane.

The magnitude of the force acting on the tank in the z-direction by the mth symmetric mode can be shown to be

$$A_{om}\,\omega_{om}^2\,\pi$$

where A_{om} is defined in Eq. (6.57) and is actually the mean (spatial) free surface displacement. Therefore, for a given motion of the liquid in the tank, one may expect those modes with large mean amplitude to make a large contribution to the total oscillatory force on the tank. In the case that the motion of liquid is caused by a sudden change of gravitational force, one sees that this expectation is true. For this kind of induced liquid motion, the force contributed by the mth symmetric mode, from Eq. (6.88), is

$$\epsilon_{om}\,A_{om}\,\omega_{om}^2\,\pi \quad .$$

In Tables I - IX, we have listed the first ten $\epsilon_{om}A_{om}\omega_{om}^{2}$ and ω_{om}/p_{ol} for various parameters for the case that the gravitational force is suddenly changed to 10^{-4} of its original value at time $\tau = 0$.

We have also computed the transient free surface shape after the sudden change of the gravitational force from Eq. (6.85). In Figs. 8 to 10, the transient deflections of the free surface versus $B_M G$ are plotted for $B_\sigma G = 0.01$ and 0.05, $L = 0.4$, $\lambda = 0.04$. The contact angle θ_o between the free surface and the rigid wall is assumed to be 75^o. The angle ν_x between the axis of the tank and the z-axis is 7.5^o. The pressure above the free surface and beneath the tank bottom is assumed to be the same. The gravitational acceleration ratio G is changed to 10^{-4} of its original value at $\tau = 0$. The original value of the gravitational acceleration is assumed to be non zero and is chosen as the characteristic acceleration. The first symmetric sloshing frequency of the corresponding rigid tank is chosen to be the characteristic frequency. In all the cases, the free surface shapes are more or less the same. A central dome appears at the center of the tank. Its height depends on the value of B_σ and B_M. The larger B_σ is, the higher is the dome. If the direction of the body force does not coincide with the axis of the tank, as the case we have considered, the dome will gradually shift to one side and the free surface deflection at the other side will grow rapidly. The elastic effect contributes mainly to a mean oscillatory displacement of the free surface. The amplitude and frequency of this mean oscillatory motion depends on the stiffness of the bottom. As we have pointed out before, it is this mean oscillatory displacement of the liquid that creates a force acting on the tank. This force is zero in the case of a rigid tank.

VII. NUMERICAL SCHEME FOR OBTAINING AN APPROXIMATE SOLUTION

In the general case an analytic solution cannot be easily obtained. In order to obtain quantitative results for application to the design of a vehicle, we have to resort to numerical computation. Here we shall develop a numerical scheme that will enable us to obtain an approximate solution which can be improved systematically.

For simplicity the scheme is described for an axially symmetric container. At static equilibrium, the free surface is also axially symmetric and the axis of symmetry coincides with that of the container. The matrix displacement method is to be used. The basic idea is to represent the conditions in the region $V + \partial V$ by a finite number of generalized displacements. To be more explicit, let $c_1, c_2 \ldots c_{n_1}$ be the generalized coordinates for the free surface, $q_1, q_2 \ldots q_{n_2}$ those for the shell surface and $b_o, b_1, b_2 \ldots b_{n_3}$ those for fluid. We seek to represent the functional in Eqs. (3.16) to (3.20) in the following form:

$$J_1 = \frac{1}{2} \underset{\sim}{c}^T \underset{\sim}{\chi} \underset{\sim}{c} - \underset{\sim}{A}_6 \underset{\sim}{c} \tag{7.1}$$

$$J_2 = \frac{1}{2} \underset{\sim}{q}^T \underset{\sim}{K} \underset{\sim}{q} - \underset{\sim}{A}_3 \underset{\sim}{q} \tag{7.2}$$

$$J_3 = \frac{1}{2} \dot{\underset{\sim}{q}}^T \underset{\sim}{M} \dot{\underset{\sim}{q}} \tag{7.3}$$

$$J_4 = -\dot{\underset{\sim}{b}}^T \underset{\sim}{A}_1 \underset{\sim}{c} - \dot{\underset{\sim}{b}}^T \underset{\sim}{A}_2 \underset{\sim}{q} - \frac{1}{2} \underset{\sim}{b}^T \underset{\sim}{B} \underset{\sim}{b} - b_o (\underset{\sim}{A}_5 \underset{\sim}{c} + \underset{\sim}{A}_4 \underset{\sim}{q}) \tag{7.4}$$

where $\underset{\sim}{\chi}$, $\underset{\sim}{K}$, $\underset{\sim}{M}$ and $\underset{\sim}{B}$ are square symmetric matrices. The matrix $\underset{\sim}{A}_1$ is $n_3 \times n_1$, $\underset{\sim}{A}_2$ is $n_3 \times n_2$, $\underset{\sim}{A}_3$ and $\underset{\sim}{A}_4$ are $1 \times n_2$, and $\underset{\sim}{A}_6$, $\underset{\sim}{A}_5$ are $1 \times n_1$. $\underset{\sim}{\chi}$ and $\underset{\sim}{K}$ are called the stiffness matrices for the free surface and shell respectively. $\underset{\sim}{M}$ and $\underset{\sim}{B}$ are the mass matrices for the shell and the fluid respectively. $\underset{\sim}{c}$, $\underset{\sim}{q}$ and $\underset{\sim}{b}$ are column vectors for the generalized coordinates c_i, q_i and b_i respectively. $(\quad)^T$ denotes the transpose of the matrix inside the parenthesis and $(\dot{\quad})$ denotes the time derivative of the quantity inside the parenthesis. We shall discuss later how to obtain these matrices.

The first variation of J of Eq. (3.15) is

$$\delta J = \int_{t_0}^{t_1} (\delta J_1 + \delta J_2 - \delta J_3 - \delta J_4)\, dt$$

$$= \int_{t_0}^{t_1} [\, \underset{\sim}{c}^T \underset{\sim}{\chi}\, \delta \underset{\sim}{c} + \underset{\sim}{q}^T \underset{\sim}{K}\, \delta \underset{\sim}{q} - \underset{\sim}{A}_3 \delta \underset{\sim}{q} - \dot{\underset{\sim}{q}}^T \underset{\sim}{M}\, \delta \dot{\underset{\sim}{q}} + \dot{\underset{\sim}{b}}^T \underset{\sim}{A}_1 \delta \underset{\sim}{c} - \underset{\sim}{A}_6 \delta \underset{\sim}{c}$$

$$+ \dot{\underset{\sim}{b}}^T \underset{\sim}{A}_2 \delta \underset{\sim}{q} + \delta \dot{\underset{\sim}{b}}^T (\underset{\sim}{A}_1 \underset{\sim}{c} + \underset{\sim}{A}_2 \underset{\sim}{q}) + \dot{\underset{\sim}{b}}^T \underset{\sim}{B}\, \delta \underset{\sim}{b}$$

$$+ (\underset{\sim}{A}_5 \underset{\sim}{c} + \underset{\sim}{A}_4 \underset{\sim}{q})\, \delta b_0 + b_0 \underset{\sim}{A}_5 \delta \underset{\sim}{c} + b_0 \underset{\sim}{A}_4 \delta \underset{\sim}{q}\,]\, dt$$

$$= \int_{t_0}^{t_1} \Big\{ [\, \underset{\sim}{c}^T \underset{\sim}{\chi} + \dot{\underset{\sim}{b}}^T \underset{\sim}{A}_1 + b_0 \underset{\sim}{A}_5 \underset{\sim}{I} - \underset{\sim}{A}_6 \underset{\sim}{I}\,]\, \delta \underset{\sim}{c} + [\, \underset{\sim}{q}^T \underset{\sim}{K}$$

$$+ \dot{\underset{\sim}{b}}^T \underset{\sim}{A}_2 + \ddot{\underset{\sim}{q}}^T \underset{\sim}{M} - \underset{\sim}{A}_3 \underset{\sim}{I} + b_0 \underset{\sim}{A}_4 \underset{\sim}{I}\,]\, \delta \underset{\sim}{q}$$

$$+ \delta \underset{\sim}{b}^T [\, \underset{\sim}{B} \underset{\sim}{b} + \underset{\sim}{A}_1 \dot{\underset{\sim}{c}} + \underset{\sim}{A}_2 \dot{\underset{\sim}{q}}\,] + (\underset{\sim}{A}_5 \underset{\sim}{c} + \underset{\sim}{A}_4 \underset{\sim}{q})\, \delta b_0 \Big\}\, dt \qquad (7.5)$$

for $\delta \underset{\sim}{b}^T = 0$, $\delta \underset{\sim}{q}$ at $t = t_o$ and t_1. $\underset{\sim}{I}$ is the identity matrix. If

$$\delta J = 0$$

is to be satisfied for arbitrary δb_o, $\delta \underset{\sim}{c}$, $\delta \underset{\sim}{q}$ and $\delta \underset{\sim}{b}^T$ for $t_o < t < t_1$, we have

$$\underset{\sim}{c}^T \underset{\sim}{\chi} + \dot{\underset{\sim}{b}}^T \underset{\sim}{A}_1 + \underset{\sim}{A}_5 \underset{\sim}{I}\, \dot{b}_o = \underset{\sim}{A}_6 \underset{\sim}{I}$$

$$\ddot{\underset{\sim}{q}}^T \underset{\sim}{M} + \underset{\sim}{q}^T \underset{\sim}{K} + \dot{\underset{\sim}{b}}^T \underset{\sim}{A}_2 + \underset{\sim}{A}_4 \underset{\sim}{I}\, \dot{b}_o = \underset{\sim}{A}_3 \underset{\sim}{I}$$

$$\underset{\sim}{A}_5 \underset{\sim}{c} + \underset{\sim}{A}_4 \underset{\sim}{q} = 0$$

$$\underset{\sim}{B}\, \underset{\sim}{b} + \underset{\sim}{A}_1 \dot{\underset{\sim}{c}} + \underset{\sim}{A}_2 \dot{\underset{\sim}{q}} = 0 \quad .$$

Or using the fact that $\underset{\sim}{\chi}$, $\underset{\sim}{M}$ and $\underset{\sim}{K}$ are symmetric, we have

$$\underset{\sim}{\chi}\, \underset{\sim}{c} + \underset{\sim}{A}_1^T \dot{\underset{\sim}{b}} + \underset{\sim}{A}_5^T \underset{\sim}{I}\, \dot{b}_o = \underset{\sim}{A}_6^T \underset{\sim}{I} \tag{7.6}$$

$$\underset{\sim}{M}\, \ddot{\underset{\sim}{q}} + \underset{\sim}{K}\, \underset{\sim}{q} + \underset{\sim}{A}_2^T \dot{\underset{\sim}{b}} + \underset{\sim}{A}_4^T \underset{\sim}{I}\, \dot{b}_o = \underset{\sim}{A}_3^T \underset{\sim}{I} \tag{7.7}$$

$$\underset{\sim}{b} = -\underset{\sim}{B}^{-1} (\underset{\sim}{A}_1 \dot{\underset{\sim}{c}} + \underset{\sim}{A}_2 \dot{\underset{\sim}{q}}) \quad . \tag{7.8}$$

Eliminating $\underset{\sim}{b}$ from the equations above, we get

$$\underset{\sim}{A_5}\,\underset{\sim}{c} + \underset{\sim}{A_4}\,\underset{\sim}{q} = 0 \qquad (7.9)$$

$$\underset{\sim}{\chi}\,\underset{\sim}{c} - \underset{\sim}{A_1}^T\,\underset{\sim}{B}^{-1}\left(\underset{\sim}{A_1}\,\ddot{\underset{\sim}{c}} + \underset{\sim}{A_2}\,\ddot{\underset{\sim}{q}}\right) + \underset{\sim}{A_5}^T\,\underset{\sim}{I}\,\dot{b}_o = \underset{\sim}{A_6}^T\,\underset{\sim}{I} \qquad (7.10)$$

$$\left(\underset{\sim}{M} + \underset{\sim}{A_2}^T\,\underset{\sim}{B}^{-1}\,\underset{\sim}{A_2}\right)\ddot{\underset{\sim}{q}} + \underset{\sim}{K}\,\underset{\sim}{q} - \underset{\sim}{A_2}^T\,\underset{\sim}{B}^{-1}\,\ddot{\underset{\sim}{c}} + \underset{\sim}{A_4}^T\,\underset{\sim}{I}\,\dot{b}_o = \underset{\sim}{A_3}^T\,\underset{\sim}{I} \quad . \qquad (7.11)$$

The natural frequency or dynamic response of the system can be obtained by solving Eqs. (7.9) to (7.11).

The center of interest lies in the formation of those matrices just mentioned. In the following the finite element method will be used. The region of interest is idealized into a series of finite elements, and the motion in each of these elements is approximated. Axial symmetry is assumed, and only the nth harmonic needs to be considered. Let us idealize the free surface and the shell into a series of curved frusta joined at nodal circles, which lie in the original surface. The region occupied by the fluid will be idealized into a series of rings whose cross-sections are acute or right triangles. Each of these frusta or rings will be called an element. The behavior of these elements is characterized by the displacement variables or the generalized coordinates. The cylindrical coordinates (r, θ, z) are used in the analysis. The z-axis coincides with the axis of symmetry of the container. The idealization of the system is illustrated in Fig. 11. Let ξ be the axial displacement of a point on the free surface, u, v and ω be the displacement of the shell in the meridional, tangential and normal directions, (H) be the rotations, in a plane containing the axis of the shell, of the meridian in the shell passing through the corresponding points. Specializing for the nth harmonic, we write at node p on the free surface.

$$\xi_p(\theta) = c_p \begin{array}{c}\cos n\theta \\ \text{or} \\ \sin n\theta\end{array} \tag{7.12}$$

whereas on the shell,

$$\begin{Bmatrix} u_p(\theta) \\ w_p(\theta) \\ \Theta_p(\theta) \end{Bmatrix} = \begin{Bmatrix} q_{4p-3} \\ q_{4p-1} \\ q_{4p} \end{Bmatrix} \begin{array}{c}\cos n\theta \\ \text{or} \\ \sin n\theta\end{array} \tag{7.13a}$$

$$v_p(\theta) = q_{4p-2} \begin{array}{c}\sin n\theta \\ \text{or} \\ \cos n\theta\end{array} \tag{7.13b}$$

and in the region V,

$$\phi_p(\theta) = \delta(n) b_o + b_p \begin{array}{c}\cos n\theta \\ \text{or} \\ \sin n\theta\end{array} \tag{7.14}$$

where

$$\delta(n) = 1 \quad \text{for} \quad n = 0$$
$$= 0 \quad \text{for} \quad n \neq 0 \quad . \qquad (7.15)$$

In what follows only sin nθ will be taken in Eq. (7.13b) and cos nθ in other equations, because this suffices for the computation of the desired matrices. We shall determine first the matrices for each element. The matrices for the complete system are obtained by superposition of the matrices of the individual elements.

A. Determination of Stiffness Matrix for the Free Surface

The existing form of J_1 in Eq. (3.16) is good for analytic investigation, but not convenient for numerical computation. If we express η in terms of ξ, the axial displacement, we get

$$J_1 = \frac{\sigma}{2}\int_{S_1}\left[\frac{\xi_r^2}{(1+F_r^2)^2} + \frac{\xi_\theta^2}{r^2(1+F_r^2)} + \frac{\rho_0 g}{\sigma}\frac{\xi^2}{(1+F_r^2)^{1/2}}\right]dS$$

$$+ \frac{\sigma}{2}\int_{\partial S_1}\frac{\xi^2}{(1+F_r^2)}\left(\sigma_H + \frac{d}{dr}\frac{1}{(1+F_r^2)^{1/2}}\right) r\,d\theta - \int_{S_1}\frac{\rho_0 \underline{r}_0 \cdot \underline{g}_1 \xi}{(1+F_r^2)^{1/2}}dS \qquad (7.16)$$

where z = F is the equation for the static equilibrium free surface and is assumed to be known (Ref. 14). The second integral in Eq. (7.16) contributes only to boundary conditions of ξ on ∂S_1 and can be dropped in our present consideration.

Thus the potential energy for each element is

$$\mathcal{V}_1 = \frac{\sigma}{2}\int_S \left[\frac{\xi_r^2}{(1+F_r^2)^2} + \frac{\xi_\theta^2}{r^2(1+F_r^2)} + \frac{\rho_0 g}{\sigma}\,\frac{\xi^2}{(1+F_r^2)^{1/2}}\right] dS \qquad (7.17)$$

where S is the area of the element being considered. The displacement field of the $p+1^{\text{th}}$ element, the one between node p and node $p+1$, is

$$\xi_p = \left[c_{p-1} f_1(s) + c_p f_2(s)\right] \cos n\theta \qquad (7.18)$$

where s is the arc length measured along the meridian from the node p. The functions $f_i(s)$ satisfy the following conditions:

$$f_1(0) = f_2(s_0) = 1$$
$$f_1(s_0) = f_2(0) = 0 \qquad (7.19)$$

where s_0 is the arc length along the meridian from node p to node $p+1$. The first approximation to the stiffness matrix is to assume that f_i are polynomials, such as

$$f_1 = 1 - \frac{s}{s_0}$$
$$f_2 = \frac{s}{s_0} \qquad (7.20)$$

A substitution of Eq. (7.18) into (7.17) yields

$$\mathcal{J}_1 = \frac{1}{2} \underset{\sim}{c}_p^T \underset{\sim}{\chi}_p \underset{\sim}{c}_p \tag{7.21}$$

where

$$\underset{\sim}{c}_p = \begin{Bmatrix} c_{p-1} \\ \epsilon_p \end{Bmatrix}$$

$$\underset{\sim}{\chi} = (\chi_{ij})$$

$$= \left(\sigma \int_S \left[\frac{\frac{df_i}{d\ell} \frac{df_j}{d\ell}}{(1+F_r^2)^2} \cos^2 n\theta + \frac{n^2 f_i f_j}{(1+F_r^2)} \sin^2 n\theta + \frac{\rho_0 g}{\sigma} \frac{f_i f_j}{(1+F_r^2)^{1/2}} \cos^2 n\theta \right] dS \right) \tag{7.22}$$

$(i, j = 1, 2)$. Clearly $\chi_{ij} = \chi_{ji}$, and

$$\frac{d}{d\ell} = \frac{1}{(1+F_r^2)^{1/2}} \frac{d}{dr} \qquad \text{and} \qquad dS = r\,dr\,d\theta\,(1+F_r^2)^{1/2}. \tag{7.23}$$

The element of the free surface, which touches the boundary ∂S_1 must be considered special. If ∂S_1 is on a flexible surface of the container as shown in Fig. 1, the boundary condition will involve coupling with the motion of the shell on ∂S_1. If ∂S_1 is on the rigid portion of the container as shown in Fig. 2, the boundary condition for ξ (see Eq. (3.14) and Appendix B) is

$$\frac{\partial \xi}{\partial r} = \gamma \xi \qquad \text{on} \qquad \partial S_1 \quad . \tag{7.24}$$

Thus Eq. (7.21) must be written as

$$\begin{aligned} \mathcal{V}_1 &= \frac{1}{2} \underset{\sim}{c}_{n_1}^T \underset{\sim}{\chi}^*_{n_1} \underset{\sim}{c}_{n_1} - \frac{1}{2} \gamma (1 + \delta_{0n}) \pi c_{n_1}^2 \\ &= \frac{1}{2} \underset{\sim}{c}_{n_1}^T \underset{\sim}{\chi}_{n_1} \underset{\sim}{c}_{n_1} \quad . \end{aligned} \tag{7.25}$$

The element for the matrix $\underset{\sim}{\chi}_{n_1}$ is the same as Eq. (7.22) except that a term

$$-\gamma (1 + \delta_{0n}) \pi$$

is added to χ_{22}.

If the stuck condition prevails so that $|\gamma| = \infty$, $c_{n_1} = 0$, then the matrix $\underset{\sim}{\chi}_{n_1}$ should have only one element χ_{11}.

For the element that intersects the axis of the container, we must have the deflection at z-axis to be zero if $n \neq 0$. Thus we have

$$\xi_1(\theta) = c_1 f_2(a) \cos n\theta \tag{7.26}$$

and the matrix $\underset{\sim}{\chi}_1$ has only one element, i.e. χ_{22}.

The integration required in Eq. (7.22) can be carried out numerically. But if the element is sufficiently small, F_r^2 can be approximated by a constant, then the integration for each element can be carried out analytically. Even if the element is not very small, one can separate the integration into several parts and approximate F_r^2 by an appropriate constant in each part.

We renumber the indices of the elements of the matrix $\underset{\sim}{\chi}_p$ as follows:

$$\underset{\sim}{\chi}_p = \begin{Bmatrix} a_{p-1,p-1} & x^*_{p-1,p} \\ x^*_{p,p-1} & x^*_{p,p} \end{Bmatrix} . \qquad (7.27)$$

Then the column vector $\underset{\sim}{c}$ and stiffness matrix $\underset{\sim}{\chi}$ for the axial symmetric deformation

$$\underset{\sim}{c} = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n_1} \end{Bmatrix} ; \quad \underset{\sim}{\chi} = \begin{Bmatrix} a_{00} & x^*_{01} & 0 & \cdots & \cdots & \cdots \\ x^*_{10} & x^*_{11}+a_{11} & x^*_{12} & 0 & \cdots & \cdots \\ 0 & x^*_{21} & x^*_{22}+a_{22} & x^*_{23} & 0 & \cdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \\ \vdots & \vdots & & & x^*_{n_1,n_1-1} & x^*_{n_1,n_1} \end{Bmatrix}$$

(7.28)

where $\chi^*_{ij} = \chi^*_{ji}$. For the asymmetric case, the form of $\underset{\sim}{\chi}$ is the same as that in Eq. (7.28) except that we remove the first row and first column from that matrix.

We recall n_1 is the number of nodal circles on the free surface. From (7.28) and (7.1), we can see that there are $n_1 + 1$ degrees of freedom for the free surface except for the case of the stuck condition. The matrix $\underset{\sim}{A}_6$ can be simply obtained from the assumed displacement functions and the third integral of Eq. (7.16).

B. <u>Determination of Matrices for the Shell</u>

The procedure of obtaining the matrices required in Eq. (7.2) is similar to that of free surface and has been discussed in Refs. 15 and 16. Therefore we shall only sketch the basic steps here. For simplicity, we shall consider only the case that the initial stress is axially symmetric and each element of the shell is isotropic and linearly elastic.

Let us first use physical components of strain and stress resultant and write Eq. (3.17) in matrix form

$$\mathcal{J}_2 = \frac{1}{2}\int_S [\underset{\sim}{e}^T \underset{\sim}{\sigma} + \underset{\sim}{w}'^T \underset{\sim}{T}\, \underset{\sim}{w}' - \rho_0 \underline{g}\cdot\underline{N}_0 w^2 - 2\rho_0 \underline{g}_1(t)\cdot\underline{r}_0 w]\, dS \qquad (7.29)$$

where S is the area of the pth element, and

$$\underset{\sim}{\sigma} = \begin{Bmatrix} n_{ss} \\ n_{\theta\theta} \\ 2n_{s\theta} \\ m_{ss} \\ m_{\theta\theta} \\ 2m_{s\theta} \end{Bmatrix} ; \quad \underset{\sim}{e} = \begin{Bmatrix} e_{ss} \\ e_{\theta\theta} \\ e_{s\theta} \\ \kappa_{ss} \\ \kappa_{\theta\theta} \\ \kappa_{s\theta} \end{Bmatrix} \quad \underset{\sim}{w}' = \begin{Bmatrix} \frac{\partial w}{\partial s} \\ \frac{\partial w}{r\partial\theta} \end{Bmatrix} \quad \underset{\sim}{T} = \begin{Bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{Bmatrix} \tag{7.30}$$

where T_{ij} is the initial stress resultant (physical component). $\underset{\sim}{\sigma}$ and $\underset{\sim}{e}$ are called the stress resultant and strain vectors respectively and are related by the equation (see Appendix C or Ref. 15)

$$\underset{\sim}{\sigma} = \underset{\sim}{\mathcal{E}}\,\underset{\sim}{e} \quad . \tag{7.31}$$

Then defining an appropriate approximate displacement field for each element related to the generalized coordinates (see Appendix C) we obtain

$$\underset{\sim}{e} = \underset{\sim}{W}_p\,\underset{\sim}{q}_p \qquad \underset{\sim}{w}' = \underset{\sim}{U}_p\,\underset{\sim}{q}_p \qquad w = \underset{\sim}{E}_p\,\underset{\sim}{q}_p \tag{7.32}$$

$\underset{\sim}{q}_p$ is the column vector of the generalized coordinates in pth element.

Substituting Eqs. (7.30) to (7.32) into (7.29), one gets

$$\gamma_2 = \frac{1}{2} \underset{\sim}{q}_p^T \underset{\sim}{K}_p \underset{\sim}{q}_p - \underset{\sim}{a}_p \underset{\sim}{q}_p \tag{7.33}$$

where

$$\underset{\sim}{K}_p = \int_S \left[\underset{\sim}{W}_p^T \underset{\sim}{\mathcal{E}} \underset{\sim}{W}_p + \underset{\sim}{U}_p^T \underset{\sim}{I} \underset{\sim}{U}_p - \rho_o \underline{g} \cdot \underline{N}_o \underset{\sim}{E}_p^T \underset{\sim}{E}_p \right] dS$$

$$\underset{\sim}{a}_p = \int_S \rho_o \, \underline{g}_1(t) \cdot \underline{r}_o \underset{\sim}{E}_p \, dS \tag{7.34}$$

$\underset{\sim}{k}_p$ is the stiffness matrix for the pth element.

In order to obtain the mass matrix, we write the kinetic energy of the shell as

$$\gamma_3 = \frac{1}{2} \int_S \rho h \left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right) dS \tag{7.35}$$

for one element. A substitution of the assumed displacements into Eq. (7.35) for the corresponding element yields (See Appendix C)

$$\gamma_3 = \frac{1}{2} \dot{\underset{\sim}{q}}_p^T \underset{\sim}{m}_p \underset{\sim}{q}_p \tag{7.36}$$

where $\underset{\sim}{m}_p$ is the mass matrix for the pth element.

By appropriate superposition of all individual matrices $\underset{\sim}{m}_p$, $\underset{\sim}{k}_p$ and $\underset{\sim}{a}_p$ one obtains the mass matrix $\underset{\sim}{M}$, the stiffness matrix $\underset{\sim}{K}$ and matrix $\underset{\sim}{A}_2$ for the whole surface.

C. Determination of Matrices for the Fluid

As before, we shall consider each individual element. Now all the elements are acute or right triangles on the r-z plane. Let us consider the element m with vertices (r_i, z_i), $r_j, z_j)$ and (r_k, z_k) and with area Δ_m (Fig. 12). The generalized coordinates associated with these vertices are b_i, b_j and b_k, which are the values of ϕ at the corresponding vertices. Then, inside the triangle, one has

$$\phi = \delta(n)\, b_0 + [\, g_1(r,z)\, b_i + g_2(r,z)\, b_j + g_3(r,z)\, b_k \,] \cos n\theta \; . \tag{7.37}$$

The simplest interpolation functions $g_e(r, z)$ are

$$g_1 = \frac{1}{\Delta_m}[(r_j - r)(z_k - z) - (r_k - r)(z_j - z)]$$

$$g_2 = \frac{1}{\Delta_m}[(r_k - r)(z_i - z) - (r_i - r)(z_k - z)] \tag{7.38}$$

$$g_3 = \frac{1}{\Delta_m}[(r_i - r)(z_j - z) - (r_j - r)(z_i - z)] \; .$$

Then the kinetic energy for this particular element m is

$$\mathcal{J}_4 = \int_0^{2\pi} \int_{\Delta_m} \frac{1}{2}(\nabla\phi)^2 r\,dr\,dz\,d\theta$$

$$= \frac{\pi}{2}(1+\delta(n)) \int_{\Delta_m} \left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 + \frac{n^2}{r^2}\right] r\,dr\,dz$$

$$= \frac{1}{2}\, \underset{\sim}{b}_m^T \underset{\sim}{\beta}_m \underset{\sim}{b}_m \tag{7.39}$$

where

$$\underset{\sim}{\beta}_m = (\beta_{pq})$$

$$= \left(\frac{\pi}{2}(1+\delta(n)) \int_{\Delta_m} \left[\frac{\partial g_p}{\partial r}\frac{\partial g_q}{\partial r} + \frac{\partial g_p}{\partial z}\frac{\partial g_q}{\partial z} + \frac{n^2 g_p g_q}{r^2}\right] r\,dr\,dz\right). \tag{7.40}$$

By appropriate superposition of each of the individual mass matrices $\underset{\sim}{\beta}_m$, we get the mass matrix for the fluid.

From the assumed displacement field in Sections VII - A and VII - B, and ϕ in Eq. (7.37), we can easily write down

$$\frac{\partial\phi}{\partial t}\,\frac{\xi}{(1+F_r^2)^{1/2}} = \underset{\sim}{b}^T \underset{\sim}{\alpha}_1 \underset{\sim}{c} + b_0\, \underset{\sim}{\alpha}_5 \underset{\sim}{c}\, \delta(n) \tag{7.41}$$

for the free surface, and

$$\frac{\partial\phi}{\partial t}\frac{w}{(1+F_r^2)^{1/2}} = \underset{\sim}{b}^T \underset{\sim}{\alpha}_2 \underset{\sim}{q} + b_0 \underset{\sim}{\alpha}_4 \underset{\sim}{q}\,\delta(n) \tag{7.42}$$

for the shell. Then $\underset{\sim}{A}_i$ and $\underset{\sim}{A}_j$ of Eq. (7.4) is given by

$$\begin{aligned} A_i &= \int_{S_1} \alpha_i(s)\, dS \\ A_j &= \int_{S_2} \alpha_j(s)\, dS \end{aligned} \tag{7.43}$$

where $i = 1, 5,$ and $j = 2, 4$.

In practice, we only use those interpolation functions of ϕ that involve the vertices on the boundary ∂V to compute $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2 \cdots$ of Eqs. (7.41) and (7.42). We also make those vertices coincide with the nodes of the free surface or the shell; e.g., the vertices (r_i, z_i) and (r_j, z_j) lying on the shell surface and coinciding with nodes p and p + 1 of the shell. Then

$$\frac{\partial\phi}{\partial t} w = \left[\delta(n)\, \dot{b}_0 + (\dot{b}_i\, g_1(r,z) + \dot{b}_j\, g_2(r,z) + b_k\, g_3(r,z))\cos n\theta\right]$$

$$\times \left[q_{4p-1} f_3(s) + q_{4p} f_3(s) + q_{4p+3} f_7(s) + q_{4p+4} f_8(s)\right] \cos n\theta$$

for the portion of the surface between node p and node p + 1. In this case, the integration required in Eq. (7.43) is also relatively simple.

VIII. DISCUSSION AND CONCLUSIONS

We have formulated the problem of the liquid motion in an elastic container by the variational method. The orthogonality relation, the completeness of eigenfunctions and the uniqueness of the solution in the case of constant axial acceleration have been proved. From the variational functional and by the maximum-minimum properties of eigenvalues, we conclude that (1) the eigenvalues are simple and real valued. (2) Flexibility of the container lowers the eigenvalues. (3) The stuck condition of the free surface raises the eigenvalues. (4) The neutral stability condition is an even function of the angle $\psi = \theta_o - 90^o$, where θ_o is the equilibrium contact angle of the free surface. (5) In the case as shown in Fig. 2, the neutral stability conditions are independent of the size and shape of the rigid portion of the container as long as the flexible surfaces remain the same. (6) For the gravitational force there exists an upper bound and a lower bound, which are functions of other parameters, between which the system is stable. The upper bound is less than or equal to that of the same system when the free surface is replaced by a rigid lid. The lower bound is greater than or equal to that of a similar container with a rigid wall. In the case that the static-equilibrium free surface is axially symmetric and the hysteresis constant γ in Eq. (4.89) is not zero, the lower bound is equal to the stability boundary of the liquid in the rigid container. (7) For the highly curved static equilibrium free surface, the lower boundary of the neutral stability condition for g_o is higher.

The flexibility of the container will induce a mean (spatial) oscillatory motion of the liquid in the axial direction of the tank, which in turn creates an oscillatory force acting on the system in that direction. This has been shown explicitly in an example in Section VI. This oscillatory force is dominated by a few particular frequencies. The lowest of these frequencies is usually much higher than the first sloshing frequency and much lower than the first free vibration frequency of the bottom membrane, and is linearly proportional to the free vibration frequency of the membrane if the product of the membrane number B_M and the product of Bond number B with the gravitational acceleration G is small, i.e. $B_\sigma G << 1$, and $B_M G << 1$. The magnitude of the oscillatory force depends on the rigidity of the bottom and the initial disturbance. The numerical example in Section VI also indicates that when the acceleration is suddenly decreased, a dome appears in the center of the free surface. If the acceleration does not coincide with the axis of the container, the central dome will gradually shift to one side and the deflection of the free surface at the other side will also grow rapidly. The rate of growth of the central dome depends on the Bond number and the membrane number.

In the case of a forced periodic motion of the container, the stability of the system is governed by the coupled Hill's equations. The instability may be caused by the coupling of different modes, which does not exist in the case of a rigid container.

The example given in Section VI has such a particularly simple geometry that an analytical solution can be obtained. In the general cases, recourse to some approximate method is necessary in order to obtain a reasonably satisfactory solution. The Rayleigh-Ritz method can be used, but for a complex system it is hard to construct approximate functions. Therefore, a new numerical scheme is developed in Section VII, in which the finite elements method is used.

The accuracy can be improved by refining the elements, or by introducing a larger number of generalized coordinates, or by using a higher degree of polynomials for the interpolation functions. The convergence proof of such a numerical scheme for a static problem will appear in a future report. In the case of a dynamic problem, it is subject to further investigation.

APPENDIX A

THE RELATION BETWEEN THE PRESSURE ENERGY AND THE LAGRANGIAN OF THE FLUID

We shall show that the pressure energy is actually the Lagrangian of the fluid. The pressure energy is

$$I_1 = -\int_V \rho_o \left[\frac{\partial\phi}{\partial t} - \underline{g} \cdot \underline{r} + \frac{1}{2}(\nabla\phi)^2\right] dv \quad ,$$

Let

$$T = \frac{1}{2} \int_V \rho_o (\nabla\phi)^2 \, dv$$

$$V = -\int_V \rho_o \; \underline{g} \cdot \underline{r} \; dv$$

where T and V are the kinetic energy and potential energy respectively of the fluid. Let

$$U = \int_V \rho_o \; \frac{\partial\phi}{\partial t} \; dV \quad .$$

Then

$$U = \frac{d}{dt} \int_V \rho_o \phi dv - \int_{\partial V} \rho_o \phi \, \underline{N} \cdot \nabla\phi \, dS \; ,$$

$$= \frac{d}{dt} \int_V \rho_o \phi \, dv - \int_V (\nabla\phi)^2 \, dv \quad ,$$

since $\nabla\phi$ is the velocity of the fluid and

$$\nabla^2 \phi = 0 \qquad \text{in } V \; .$$

Now we have

$$I_1 = V - T + \frac{d}{dt} \int_V \rho_o \phi \, dv \quad .$$

We notice that the last term in the equation above has no contribution to the variational equation.

APPENDIX B

THE EQUATION AND BOUNDARY CONDITIONS FOR THE FREE SURFACE

We shall derive Eq. (3.7) and work out the explicit condition for constant angle of contact of the free surface when the wall is stationary. Let us use $a_{\alpha\beta}$, $\kappa_{\alpha\beta}$ and $\underline{a}_\alpha$ as the metric tensor, curvature tensor and base vectors respectively for the free surface and $a'_{\alpha\beta}$, $\kappa'_{\alpha\beta}$ and $\underline{a}'_\alpha$ for the stationary wall surface. If the equilibrium free surface is $\underline{r}_o$, then the disturbed free surface can be represented by

$$\underline{r} = \underline{r}_o + \eta \underline{N}_o \tag{B-1}$$

where η is an infinitesimal quantity of the first order. Then

$$\begin{aligned} \underline{r}_{,\alpha} &= \underline{a}_\alpha - \eta \kappa_{\alpha\beta} \underline{a}^\beta + \eta_{,\alpha} \underline{N}_o \\ \underline{r}_{,\alpha\beta} &= \underline{a}_\alpha - \left[(\kappa^\lambda_\alpha \omega)_{,\beta} + \Gamma^\lambda_{\vartheta\beta} \kappa^\vartheta_\alpha \eta \right] \underline{a}_\lambda \\ \underline{N} &= \underline{N}_o - \eta_{,\alpha} \underline{a}^\alpha \end{aligned} \tag{B-2}$$

and the curvature tensor for the disturbed free surface is

$$B_{\alpha\beta} = \underline{r}_{,\alpha\beta} \cdot \underline{N} = \kappa_{\alpha\beta} + \eta_{\alpha\beta} - \kappa_{\lambda\alpha} \kappa^\lambda_\beta \eta \quad . \tag{B-3}$$

Let

$$A_{\alpha\beta} = \underline{r}_{,\alpha} \cdot \underline{r}_{,\beta} \tag{B-4}$$

then we have the mean curvature

$$\begin{aligned} B^{\alpha}_{\alpha} &= A^{\alpha\beta} \; B_{\alpha\beta} \\ &= \frac{\det (a_{\alpha\beta})}{\det (A_{\alpha\beta})} \, \epsilon^{\alpha\lambda} \epsilon^{\beta\theta} \, A_{\lambda\theta} \; B_{\alpha\beta} \\ &\doteq (1 + 2\,\kappa^{\theta}_{\theta} \; \eta)(a^{\alpha\beta} - 2\,\epsilon^{\alpha\lambda}\epsilon^{\beta\theta} \, \kappa_{\lambda\theta} \; \eta) B_{\alpha\beta} \\ &\doteq \kappa^{\alpha}_{\alpha} + a^{\alpha\beta} \, \eta|_{\alpha\beta} + \kappa^{\alpha}_{\beta} \, \kappa^{\beta}_{\alpha} \, \eta \end{aligned} \tag{B-5}$$

where $\epsilon^{\alpha\beta}$ is the permutation tensor. Substituting into Eq. (3.1), one gets Eq. (3.7).

If one uses x^{α} ($\alpha = 1, 2$) as the real variables for the stationary wall and

$$\underline{x}_o(x^{\alpha})$$

as the surface of the wall, and y^1, y^2 as the real variables for the free surface, then

$$\underline{x}_o(x_o^\alpha) = \underline{r}_o(y_o^\alpha) \tag{B-6}$$

represents the intersecting curve ∂S_1 of the free surface S_1 and the stationary wall at static equilibrium state. Then

$$\underline{x}_o(x_o^\alpha + \Delta x^\alpha) = \underline{r}_o(y_o^\alpha + \Delta y^\alpha) + \underline{N}_o \eta \tag{B-7}$$

will represent the intersecting curve of the disturbed free surface and the wall.

In the case of the stuck condition for the free surface, one has

$$\eta = 0 \qquad \text{at} \qquad \partial S_1 \tag{B-8}$$

and $\Delta x^\alpha = \Delta y^\alpha = 0$. In the case of constant contact angle between the free surface and the stationary wall, one has, from Eqs. (B-6) and (B-7), by using Taylor's expansion,

$$\underline{a}'_\alpha \ \Delta x^\alpha = \underline{N}_o \eta + \underline{a}_\alpha \ \Delta y^\alpha + \text{higher order} \tag{B-9}$$

(note that $(\partial \underline{x}_o / \partial x^\alpha) = \underline{a}'_\alpha$ and $\partial \underline{r}_o / \partial y^\alpha = \underline{a}_\alpha$).
Let $\underline{T}$ be the unit tangent of the curve ∂S_1. One can always have

$$(\underline{T} \cdot \underline{a}_\alpha) \ \Delta y^\alpha = 0 \quad . \tag{B-10}$$

Let $\underline{N}_w$ be the unit outward normal of the wall, then

$$\underline{N}_w (x_o^\alpha) \cdot \underline{N}_o (y_o^\alpha) = - \cos \theta_o \tag{B-11}$$

where θ_o is the angle of contact at ∂S_1.

The normal of the disturbed free surface at $y^\alpha = y_o^\alpha + \Delta y^\alpha$ is

$$\underline{N} (y_o^\alpha + \Delta y^\alpha) = \underline{N}_o (y_o^\alpha + \Delta y^\alpha) - a^{\alpha\beta} \eta_{,\beta} \; \underline{a}_\alpha$$

$$= \underline{N}_o (y_o^\alpha) - [a^{\alpha\beta} \eta_{,\beta} + \kappa_\beta^\alpha \Delta y^\beta] \underline{a}_\alpha \quad . \tag{B-12}$$

Because of the angle of contact, at the new intersecting curve between the disturbed free surface and the stationary wall,

$$\underline{N}_w (x_o^\alpha + \Delta x^\alpha) \cdot \underline{N} (y_o^\alpha + \Delta y^\alpha) = - \cos \theta_o \tag{B-13}$$

must hold, or

$$(a^{\alpha\beta} \eta_{,\beta} + \kappa_\beta^\alpha \Delta y^\beta) \underline{a}_\alpha \cdot \underline{N}_w + \kappa'^\alpha_\beta \Delta x^\beta \underline{a}'_\alpha \cdot \underline{N}_o = 0 \quad . \tag{B-14}$$

By Eqs. (B-9) and (B-10), one obtains

$$
\begin{aligned}
(\underline{T} \cdot \underline{a}_\alpha)\ \Delta y^\alpha &= 0 \\
(\underline{N}_\omega \cdot \underline{a}_\alpha)\ \Delta y^\alpha &= \cos\theta_o\ \eta \\
(\underline{T} \cdot \underline{a}'_\alpha)\ \Delta x^\alpha &= 0 \\
(\underline{N}_o \cdot \underline{a}'_\alpha)\ \Delta x^\alpha &= \eta
\end{aligned}
\qquad \text{(B-15)}
$$

Since the coefficients determinant of Δx^α and Δy^α in Eq. (B-15) never vanish, Δx^α and Δy^α can be solved uniquely in terms of η, say

$$
\begin{aligned}
\Delta x^\alpha &= c^\alpha \eta \\
\Delta y^\alpha &= d^\alpha \eta
\end{aligned}
$$

Substituting into Eq. (B-14), one gets

$$
a^{\alpha\beta}\ \eta|_\beta\ (\underline{a}_\alpha \cdot \underline{N}_\omega) + \sigma'\ \eta = 0 \qquad \text{(B-16)}
$$

where σ' is a function of metric tensor and curvature tensor of the free surface and the stationary wall at ∂S_1 and the contact angle.

The functional form of σ' is, in general, quite complex, therefore we shall not write out its general form. However, for a special choice of orthogonal coordinates

$$-\underline{a}_1 = \underline{T} = \underline{a}_1' \qquad \text{on} \qquad \partial S_1$$

and

$$\underline{N}_o = \frac{\underline{a}_1 \times \underline{a}_2}{|\underline{a}_1 \times \underline{a}_2|}$$

$$\underline{N}_w = \frac{\underline{a}_1' \times \underline{a}_2'}{|\underline{a}_1' \times \underline{a}_2'|}$$

where $\underline{N}_o$ and $\underline{N}_w$ are the unit normals pointing away from the fluid of the free surface and the stationary wall respectively. One has, from Eq. (B-15)

$$\Delta x^1 = \Delta y^1 = 0$$

$$\underline{N}_w \cdot \underline{a}_2 \quad \Delta y^2 = \cos\theta_o \, \eta \qquad \text{(B-17)}$$

$$\underline{N}_o \cdot \underline{a}_2' \quad \Delta x^2 = \eta \quad .$$

Substituting into Eq. (B-14), one obtains

$$a^{22} \eta_{,2} \;\; \underline{a}_2 \cdot \underline{N}_w + (\kappa_2^2 \cos\theta_o + \kappa_2'^2)\eta = 0 \text{ on } \partial S_1 \quad . \qquad \text{(B-18)}$$

Let us define another vector as

$$\underline{n}_1 = \underline{T} \times \underline{N}_o \qquad \text{at} \qquad \partial S_1 \quad . \qquad \text{(B-19)}$$

Since $\underline{N}_o$, $\underline{n}_1$ and $\underline{N}_{\varpi}$ at ∂S_1 lie on the same plane, one has

$$\underline{N}_{\varpi} = - \underline{N}_o \cos\theta_o + \underline{n}_1 \sin\theta_o \quad . \qquad \text{(B-20)}$$

Substituting into Eq. (B-16), one gets

$$a^{\alpha\beta}\, \eta|_{\beta}\, \underline{a}_{\alpha} \cdot \underline{n}_1 + \sigma''\eta = 0 \qquad \text{on } \partial S_1 \qquad \text{(B-21)}$$

where $\sigma'' = \sigma' / \sin\theta_o$ and σ' is defined in Eq. (B-16). If one uses the special coordinate just mentioned, the Eq. (B-21) becomes

$$\frac{1}{(a_{22})^{1/2}} \frac{\partial\eta}{\partial y^2} + \sigma''\eta = 0 \qquad \text{at} \qquad \partial S_1 \qquad \text{(B-22)}$$

where

$$\sigma'' = (\kappa'^2_2 + \kappa^2_2 \cos\theta_o) / \sin\theta_o \quad . \qquad \text{(B-23)}$$

In Section VII, for practical convenience, ξ, the disturbed deflection of the free surface in z-direction, is used instead of the normal deflection H . Therefore one has to transform

$$D_1(H) + \int_{\partial S_1} \sigma \ \sigma_H H^2 \, d\ell \qquad \text{(B-24)}$$

of Eqs. (4.27) and (4.28), in terms of ξ. We shall consider only the case that the static equilibrium free surface $z = F(r)$ is axially symmetric. Cylindrical polar coordinates with the z-axis coinciding with the axis of symmetry are used. Let

$$Fr = \frac{dF}{dr} \quad ,$$

then

$$\xi = f H$$

$$\kappa_2^2 = \frac{Frr}{f^3} = \frac{d}{dr}\left(\frac{Fr}{f}\right)$$

$$\kappa_2^1 = \kappa_1^2 = 0 \qquad \text{(B-25)}$$

$$\kappa_1^1 = \frac{Fr}{rf}$$

$$a^{22} = \frac{1}{f^2}$$

$$a^{11} = \frac{1}{r^2}$$

$$H_2 = \frac{\partial H}{\partial r}$$

$$H_1 = \frac{\partial H}{\partial \theta}$$

where $f = (1 + Fr^2)^{1/2}$. Substituting into (B-24) one gets

$$D_1(H) + \int_{\partial S_1} \sigma\, \sigma_H H^2 d$$

$$= \sigma \int_{S_1} \left[a^{11}(H_1)^2 + a^{22}(H_2)^2 - ((\kappa_1^1)^2 + (\kappa_2^2)^2) H^2 + \frac{\rho_o}{\sigma} \underline{g}_o \cdot \underline{N}_o H^2 \right] dS$$

$$+ \int_{\partial S_1} \sigma\, \sigma_H H^2 \, d\ell$$

$$= \sigma \int_{S_1} \left[\frac{1}{f^2} \left(\frac{\xi_r}{f} + \xi \frac{d}{dr} \frac{1}{f} \right)^2 + \frac{\xi_\theta^{\,2}}{r^2 f^2} + \frac{\rho_o g_o}{\sigma} \frac{\xi^2}{f^3} - \frac{\xi^2}{f^2} \right.$$

$$\left. \times \left\{ \left(\frac{d}{dr} \left(\frac{Fr}{f} \right) \right)^2 + \left(\frac{Fr}{rf} \right)^2 \right\} \right] dS + \int_{\partial S_1} \sigma\, \sigma_H \frac{\xi^2}{f^2} \, d\ell \, . \qquad \text{(B-26)}$$

After some algebraic manipulation and using the fact that

$$\frac{d}{dr} \left(\frac{Fr}{f} \right) + \frac{Fr}{rf} - \frac{\rho_o g_o}{\sigma} F = 0 \qquad \text{(B-27)}$$

(the static equilibrium equation), Eq. (B-24) becomes

$$D_1(H) + \int_{\partial S_1} \sigma\, \sigma_H H^2 \, d\ell$$

$$= \sigma \int_{S_1} \left(\frac{\xi_r^{\,2}}{f^3} + \frac{\xi_\theta^{\,2}}{r^2 f} + \frac{\rho_o g_o}{\sigma} \xi^2 \right) r dr d\theta + \sigma \int_{\partial S_1} \left[\frac{\sigma_H}{f^2} + \frac{1}{f^2} \frac{d}{dr} \left(\frac{1}{f} \right) \right] \xi^2 d\ell \, . \text{(B-28)}$$

From the form of (B-28), one must have

$$\frac{\partial \xi}{\partial r} = \gamma \xi \qquad \text{on} \qquad \partial S_1 \tag{B-29}$$

as the boundary condition of ξ at ∂S_1, where

$$\gamma = -\sigma_H - \frac{d}{dr}\left(\frac{1}{f}\right) \quad . \tag{B-30}$$

In the case of constant angle of contact and circular cylindrical container, we have

$$\kappa'^2_2 = 0$$

and

$$\sigma_H = \sigma'' = \cot \theta_o \frac{d}{dr}\left(\frac{Fr}{f}\right) = \frac{FrFrr}{f^3}$$

then

$$\gamma = -\left[\frac{FrFrr}{f^3} - \frac{d}{dr}\left(\frac{1}{f}\right)\right] = -\left[\frac{FrFrr}{f^3} - \frac{FrFrr}{f^3}\right] = 0 \quad .$$

APPENDIX C

MATRICES FOR AN ELEMENT OF THE SHELL

We shall use (θ, s) as the coordinates for each element of the shell where s is the arc length measured along the meridian. Let us assume first the displacement for each element. There are three kinds of elements:

(1) The element that intersects the axis of the container. There are two cases:

Axial symmetric deflections

$$\begin{aligned} u_1 &= f_1(s)\, q_1 \\ v_1 &= f_2(s)\, q_2 \\ w_1 &= f_o(s)\, q_o + f_3(s)\, q_3 + f_4(s)\, q_4 \quad . \end{aligned} \tag{C-1}$$

Asymmetric deflections

$$\begin{aligned} u_1 &= \left\{ \left[f_1(s) + a\bar{f}(s) \right] q_1 + b\bar{\bar{f}}(s)\, q_2 \right\} \cos n\theta \\ v_1 &= \left\{ \left[f_2(s) - b\bar{\bar{f}}(s) \right] q_2 - a\bar{f}(s)\, q_1 \right\} \sin n\theta \\ w_1 &= \left\{ f_o(s)\, q_o + f_3(s)\, q_3 + f_4(s)\, q_4 \right\} \cos n\theta \end{aligned} \tag{C-2}$$

where $f_i(s)$ satisfy the conditions

$$f_i(0) = 0 \qquad f_i(s_o) = 1 \qquad (i = 1,\ 2)$$

$$f_o(0) = 1 \qquad \frac{df_o}{ds} = 0 \qquad \text{if } n = 0$$

$$= 0 \qquad\qquad = 1 \qquad \text{if } n \neq 0$$

$$f_o(s_o) = \frac{df_o}{ds}(s_o) = 0$$

$$f_3(s_o) = \frac{df_4}{ds}(s_o) = 1$$

$$f_3(s_o) = f_4(0) = \frac{df_3}{ds}(0) = \frac{df_4}{ds}(0) = \frac{df_3}{ds_o}(s_0) = f_4(s_0) = 0$$

$$\bar{f}(0) = \bar{\bar{f}}(0) = 1$$

$$\bar{f}(s_o) = \bar{\bar{f}}(s_o) = 0 \qquad \text{(C-3)}$$

The terms $a\bar{f}(s)$, $b\bar{\bar{f}}(s)$ in Eq. (C-2) are introduced to allow rigid body motion. In the case of polynomial approximation,

$$\bar{f}(s) = \bar{\bar{f}}(s) = 1 - \frac{s}{s_o}$$

$$f_i(s) = \frac{s}{s_o} \qquad (i = 1,\ 2)$$

$$f_o(s) = 1 - \frac{3}{s_o^2}\, s^2 + \frac{2}{s_o^3}\, s^3 \qquad \text{if } n = 0$$

$$= s - \frac{2s^2}{s_o} + \frac{s^3}{s_o^2} \qquad \text{if } n = 1$$

$$f_3(s) = \frac{3s^2}{s_o^2} - \frac{2s^3}{s_o^3}$$

$$f_4(s) = -\frac{s^2}{s_o} + \frac{s^3}{s_o^2} \quad . \qquad \text{(C-4)}$$

The value of a and b in Eq. (C-2) is to be optimized by the principle of minimum potential energy, i.e.

$$\frac{\partial \mathcal{V}_2}{\partial a} = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{V}_2}{\partial b} = 0 \quad .$$

$\mathcal{V}_2$ is defined in Eq. (7.31)

(2) The elements bounded by node p and node p + 1 ($p \geq 2$). The displacement field is

$$u_p = \left[q_{4p-3}\, f_1(s) + q_{4p+1} f_5(s)\right] \cos n\theta$$

$$v_p = \left[q_{4p-2}\, f_2(s) + q_{4p+2}\, f_6(s)\right] \sin n\theta \qquad \text{(C-5)}$$

$$w_p = \left[q_{4p-1}\, f_3(s) + q_{4p}\, f_4(s) + q_{4p+3}\, f_7(s) + q_{4p+4}\, f_8(s)\right]$$

$$\times \cos n\theta$$

where f_i satisfies the conditions

$$\left.\begin{aligned} f_i(0) &= f_{i+4}(s_o) = 1 \\ f_i(s_o) &= f_{i+4}(0) = f_4(0) = f_4(s_o) = f_8(0) = f_8(s_o) = 0 \end{aligned}\right\} \quad (i = 1, 2, 3)$$

$$\frac{df_i}{ds}(0) = \frac{df_i}{ds_o}(s_o) = 0 \qquad (i = 3.5)$$

$$\frac{df_7}{ds}(0) = \frac{df_8}{ds}(s_o) = 1$$

$$\frac{df_7}{ds}(s_o) = \frac{df_8}{ds}(0) = 0 \qquad \text{(C-6)}$$

For the polynomial approximation,

$$f_i = 1 - \frac{s}{s_o} \qquad (i = 1, 2)$$

$$= \frac{s}{s_o} \qquad (i = 5, 6)$$

$$f_3(s) = 1 - \frac{3}{s_o^2} s^2 + \frac{2}{s_o^3} s^3$$

$$f_4(s) = s - \frac{2s^2}{s_o} + \frac{s^3}{s_o^2}$$

$$f_7(s) = f_3(s_o - s)$$

$$f_8(s) = -f_4(s_o - s) \qquad \text{(C-7)}$$

(3) The element which touches the boundary. In our particular case, it is the last element of the shell. By Eqs. (3.12) and (3.13), one has

$$u_{n_2-1} = q_{4n_2-7}\, f_1(s) \cos n\,\theta$$

$$v_{n_2-1} = q_{4n_2-6}\, f_2(s) \sin n\,\theta \qquad \text{(C-8)}$$

$$w_{n_2-1} = \left[q_{4n_2-5}\, f_3(s) + q_{4n_2-4}\, f_4(s) + q_{4n_2}\, f_8(s)\right] \cos n\,\theta$$

where $f_i(s)$ are the same as Eq. (C-7). In the clamped condition, one must set $q_{4n_2} = 0$.

If one writes

$$w_p = \underset{\sim}{E}_p\, \underset{\sim}{q}_p \qquad \text{(C-9)}$$

and

$$\underset{\sim}{w}_p{}' = \begin{Bmatrix} \dfrac{\partial w_p}{\partial s} \\[2ex] \dfrac{\partial w_p}{r\partial \theta} \end{Bmatrix} = \underset{\sim}{U}_p\, \underset{\sim}{q}_p \qquad \text{(C-10)}$$

from the displacement field defined in Eqs. (C-1), (C-2), (C-5) and (C-8), one can easily obtain $\underset{\sim}{E}_p$ and $\underset{\sim}{U}_p$ for each element.

The strain-displacement relations for the thin shell are

$$
\begin{aligned}
e_{ss} &= \frac{\partial u}{\partial s} + \frac{w}{R_2} \\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\frac{dr}{ds} + \frac{w}{R_1} \\
e_{\theta s} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial rv}{\partial s} - \frac{2v}{r}\frac{dr}{ds}\right) \\
k_{ss} &= \frac{\partial^2 w}{\partial s^2} \\
k_{\theta\theta} &= \frac{1}{r^2}\left(\frac{\partial^2 w}{\partial \theta^2} + r\frac{\partial w}{\partial s}\frac{dr}{ds}\right) \\
k_{\theta s} &= \frac{1}{2}\frac{1}{r}\left(\frac{\partial^2 w}{\partial s \partial \theta} - \frac{1}{r}\frac{dr}{ds}\frac{\partial w}{\partial \theta}\right)
\end{aligned}
\qquad \text{(C-11)}
$$

where R_1, R_2 are the principle radii of curvature with R_2 in the plane containing the meridian. Then

$$\underset{\sim}{e} = \begin{Bmatrix} e_{ss} \\ e_{\theta\theta} \\ \vdots \\ k_{\theta s} \end{Bmatrix} = \underset{\sim}{W}_p \, \underset{\sim}{q}_p \quad . \qquad \text{(C-12)}$$

By Eq. (C-11) and the assumed displacement field, $\underset{\sim}{W}_p$ can easily be obtained.

The strain-stress resultant relations of Eq. (7.33) are

$$\underset{\sim}{\sigma} = \begin{Bmatrix} \frac{Eh}{1-\nu^2} & \frac{Eh}{1-\nu^2} & 0 & 0 & 0 & 0 \\ \frac{hE}{1-\nu^2} & \frac{hE}{1-\nu^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{Eh}{1+\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{Eh^3}{12(1-\nu^2)} & \frac{Eh^3}{12(1-\nu^2)} & 0 \\ 0 & 0 & 0 & \frac{Eh^3}{12(1-\nu^2)} & \frac{Eh^3}{12(1-\nu^2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{Eh^3}{12(1+\nu)} \end{Bmatrix} \underset{\sim}{e}$$

(C-13)

The matrix mass $\underset{\sim}{m}_p$ defined in Eq. (7.36) is

$$\underset{\sim}{m}_p = (m_{ij}) = \left(\pi(1 + \delta(n)) \int_o^{s_o} f_i f_j \, r \frac{dr}{ds} \, ds \right) \quad . \qquad \text{(C-14)}$$

If a and b of Eq. (C-2) are not zero, the expression of m_{ij} should be modified accordingly. We recall that

$$\delta(n) = 1 \quad \text{if} \quad n = 0$$
$$= 0 \quad \text{if} \quad n \neq 0 \quad .$$

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COEFFICIENTS OF THE FIRST TEN TERMS OF THE OSCILLATORY FORCE EQ. (6.88)

WITH $B_\sigma G = 0.01$, $L = 0.4$, $= 0.04$, $P_2 = 0.0$, $\theta_o = 75^o$, $G = 10^{-4}$

(ANY QUANTITY LESS THAN 10^{-4} LISTED AS ZERO)

Freq. $= \omega_{om}/p_{o1}$, Ampl. $= \omega^2_{om} A_{om} \epsilon_{om}$

TABLE I $B_M G = 0.33 \times 10^{-4}$

Freq.	.9997	2.5857	4.5303	6.7916	9.3348	9.5671	12.132	15.165	18.411	21.862
Ampl.	0	0	0	0	0	1.5403	0	0	0	0

TABLE II $B_M G = 0.65 \times 10^{-4}$

Freq.	.9994	2.5856	4.5303	6.7916	6.7956	9.3348	12.132	15.163	18.411	18.214
Ampl.	0.0001	0	0	0.0010	1.5482	0	0	0	0	0.2312

TABLE III $B_M G = 0.75 \times 10^{-4}$

Freq.	.9993	2.5856	4.5303	6.3478	6.7916	9.3348	12.132	15.163	17.968	18.411
Ampl.	0.001	0	0	1.5733	0	0	0	0	0.2278	0

TABLE IV $B_M G = 1.46 \times 10^{-4}$

Freq.	.9987	2.5854	4.5296	4.5374	6.7916	9.3348	12.132	12.832	15.163	18.411
Ampl.	0.0003	0.0002	0.1433	1.4924	0	0	0.0002	0.1971	0	0

TABLE V $B_M G = 1.463 \times 10^{-4}$

Freq.	.9987	2.5854	4.5289	4.5335	6.7916	9.3348	12.132	12.810	15.163	18.411
Ampl.	0.0003	0.0002	.5122	1.1164	0	0	0.0002	0.1970	0	0

TABLE VI $B_M G = 1.465 \times 10^{-4}$

Freq.	.9987	2.5854	4.5274	4.5319	6.7918	9.3348	12.182	12.801	15.163	18.411
Ampl.	0.0003	0.0002	1.0664	0.5629	0	0	0	0.1969	0	0

TABLE VII $B_M G = 1.475 \times 10^{-4}$

Freq.	.9986	2.5854	4.5144	4.5306	6.7916	9.3348	12.132	12.758	15.163	18.411
Ampl.	.0003	0.0002	1.6168	0.0232	0	0	0	0.1964	0	0

TABLE VIII $B_M G = 3.1 \times 10^{-4}$

Freq.	.9971	2.5850	3.1173	4.5303	6.7916	8.8043	9.3348	12.132	15.163	16.128
Ampl.	0.0012	0.0013	1.8024	0	0	0.1101	0	0	0	0.0218

TABLE IX $B_M G = 4.0 \times 10^{-4}$

Freq.	.9962	2.5842	2.7468	4.5303	6.7916	7.7527	9.3348	12.132	14.639	15.163
Ampl.	0.0021	0.0111	1.9027	0	0	0.0487	0	0	0.0024	0

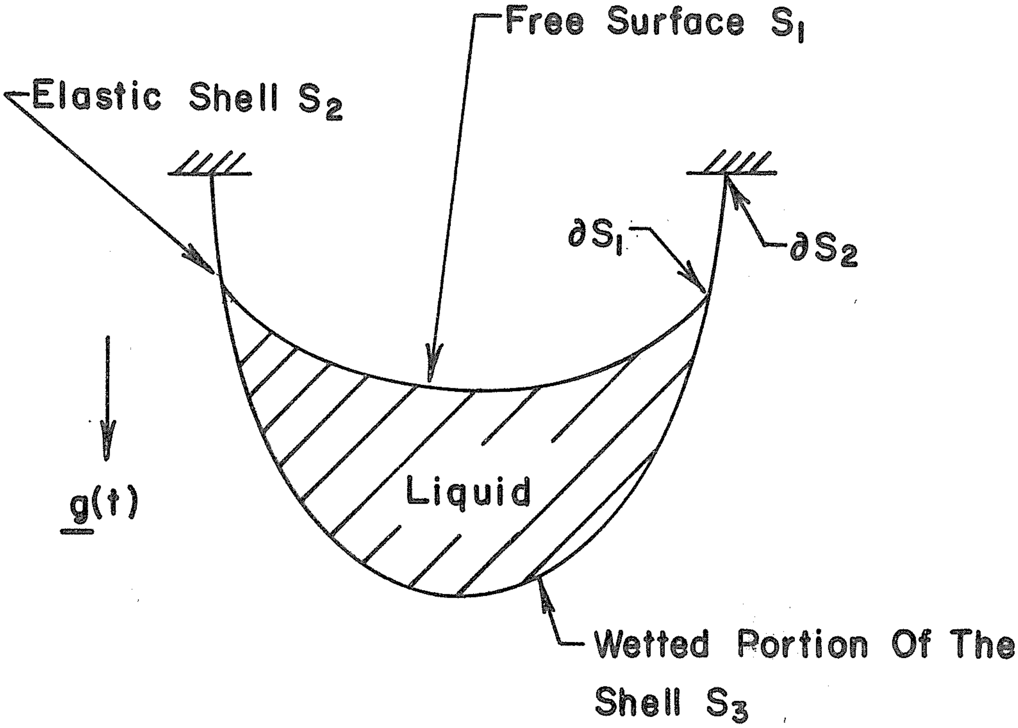

FIG. 1 GEOMETRIC NOTATIONS

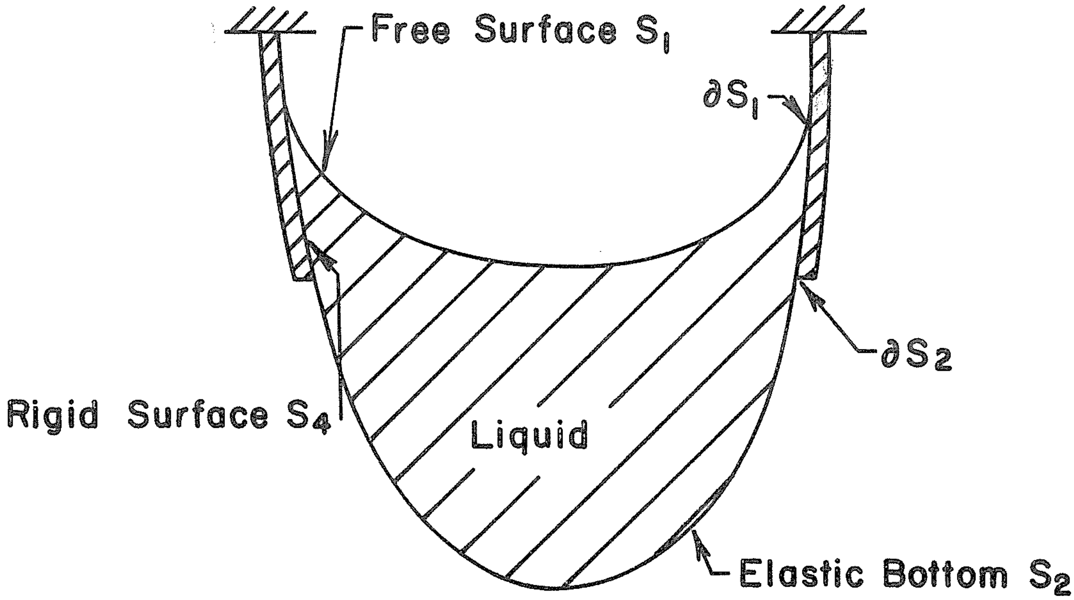

FIG. 2 NOTATIONS USED IN SECTION 3

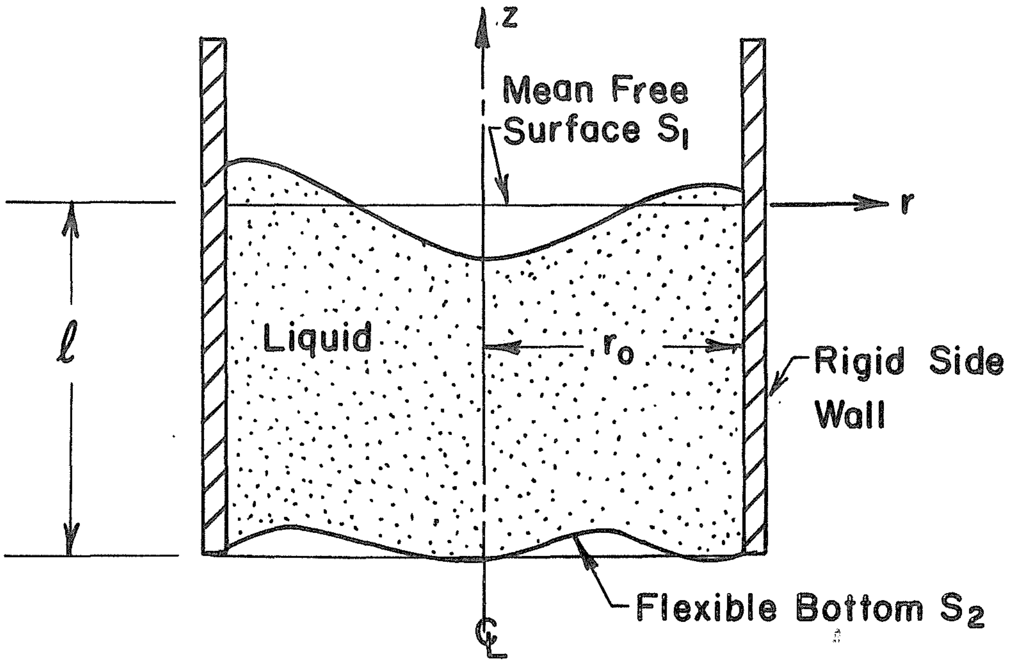

FIG. 3 EXAMPLE: A CYLINDRICAL TANK WITH A FLEXIBLE BOTTOM

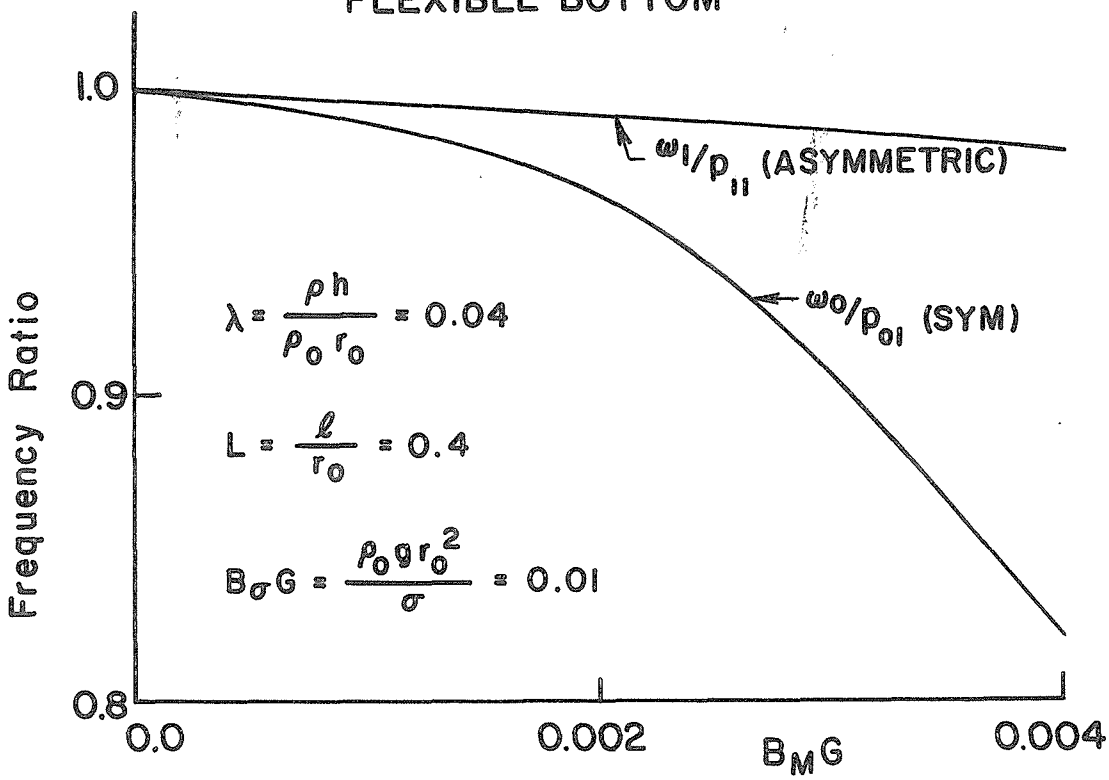

FIG. 4 FREQUENCY RATIOS OF THE FIRST SYMMETRIC AND THE FIRST ASYMMETRIC MODES FROM EQS. (6.45) AND (6.46) NOTE: $p_{11} << p_{01}$, $B_M G \; \frac{\rho_0 g r_0^2}{N_r}$

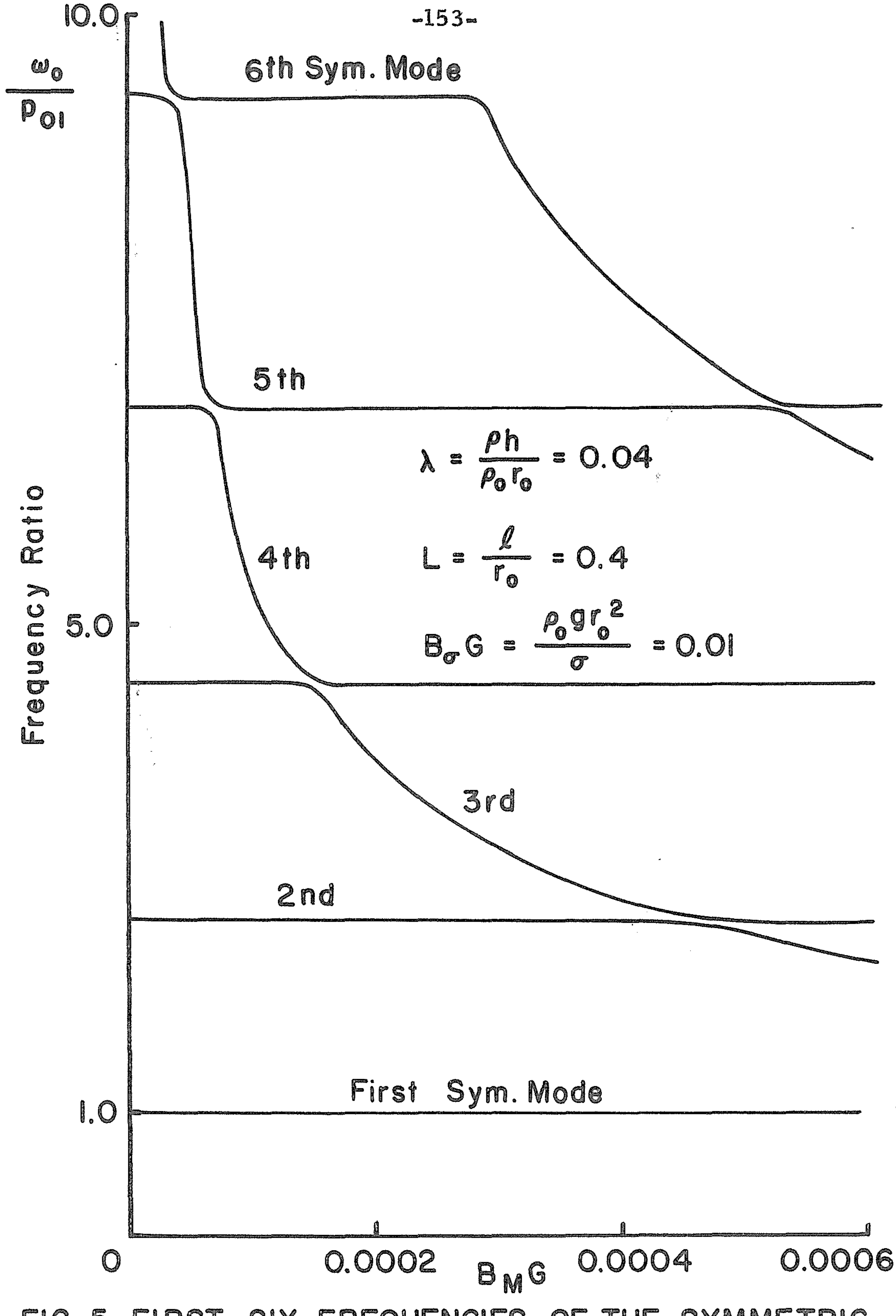

FIG. 5 FIRST SIX FREQUENCIES OF THE SYMMETRIC MODES FROM EQ.(6.45) $B_M G = \frac{\rho_0 g r_0^2}{N r}$

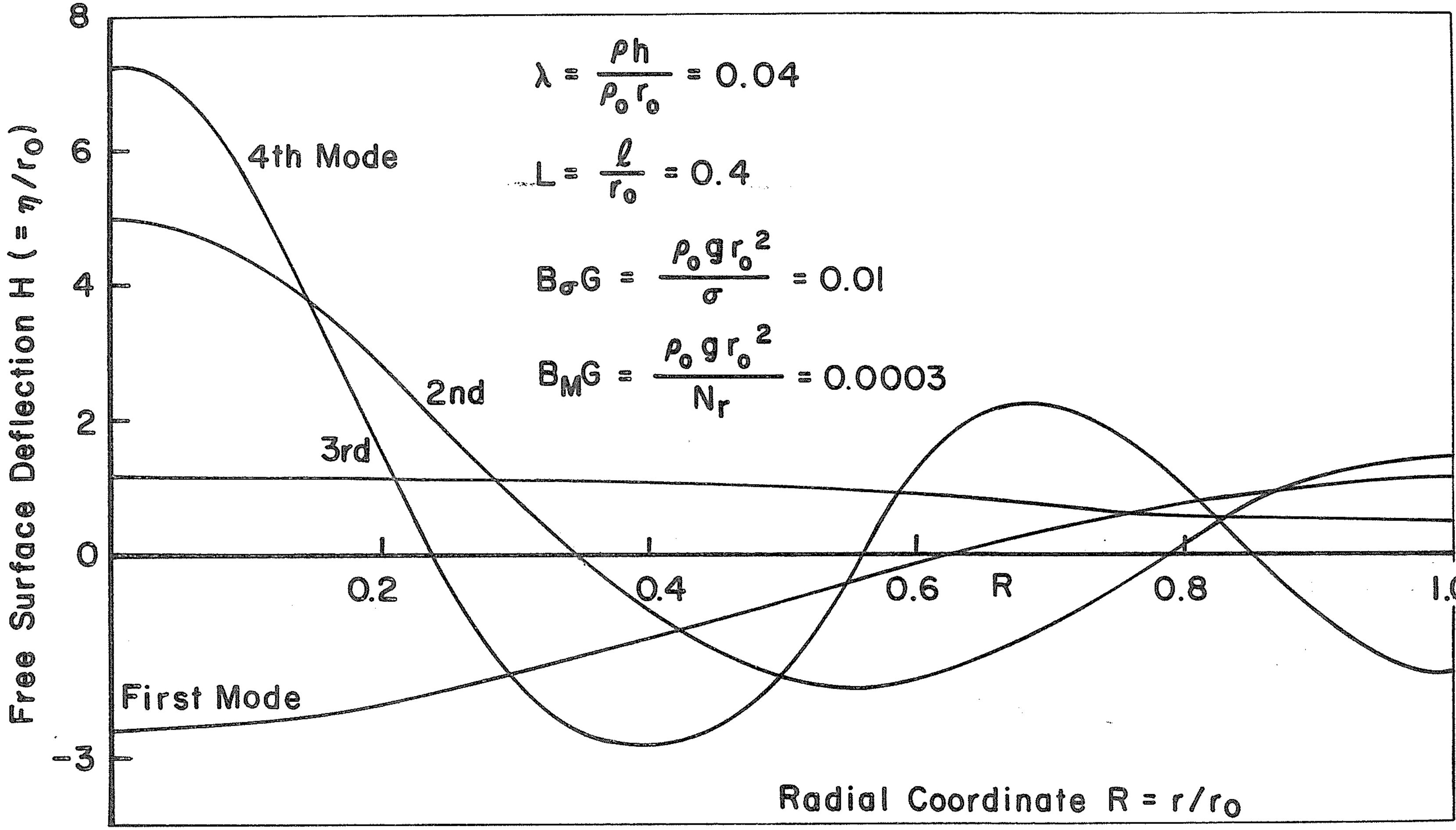

FIG. 6 FREE SURFACE DEFLECTION OF THE FIRST FOUR SYMMETRIC MODES FROM EQ (6.61)

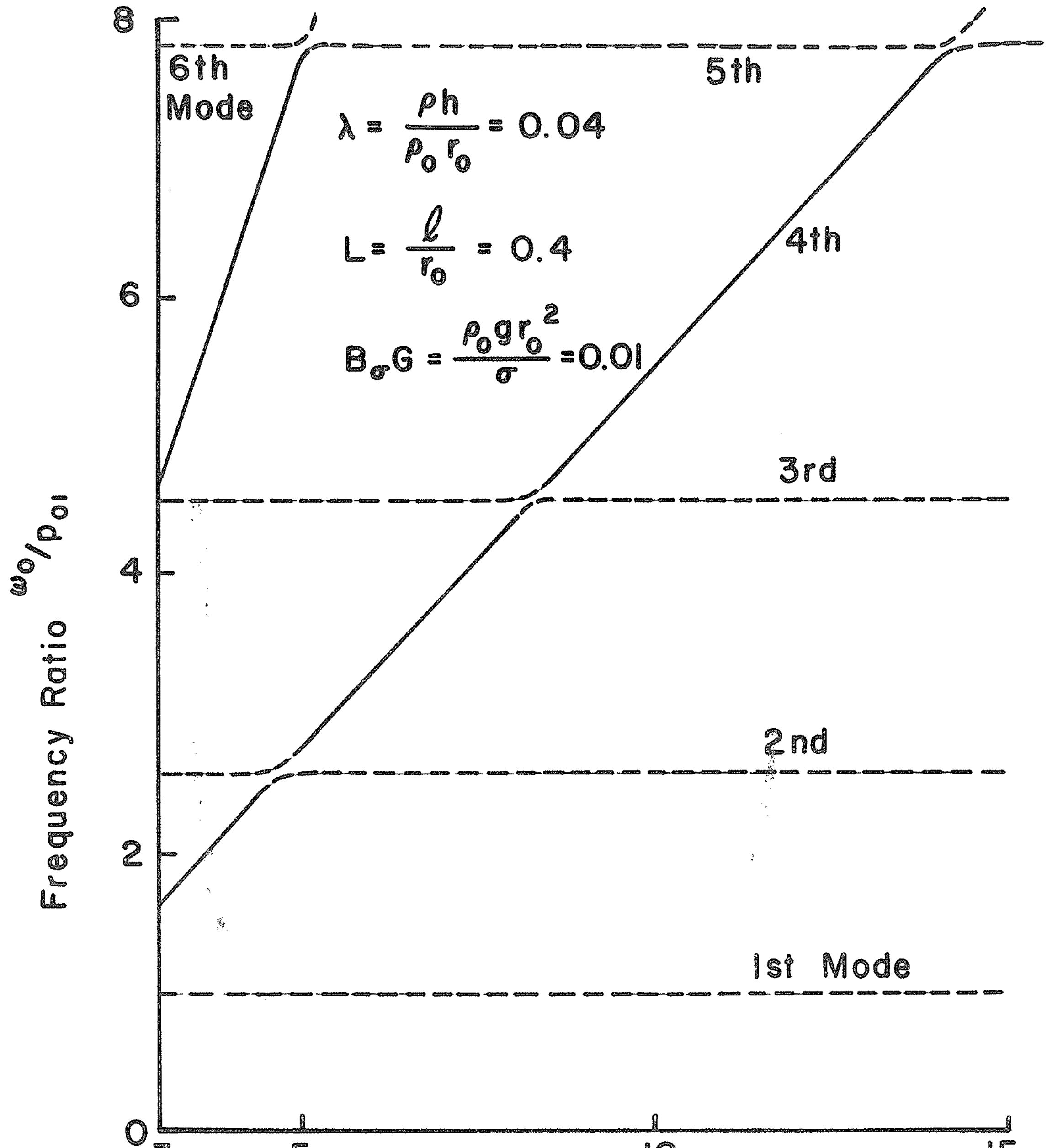

FIG. 7 FREQUENCIES FOR THE FIRST SIX SYMMETRIC MODES FROM EQ (6.45). PORTION ON SOLID CURVES CORRESPOND TO THOSE MODES WITH LARGE MEAN DEFLECTION ON THE FREE SURFACE

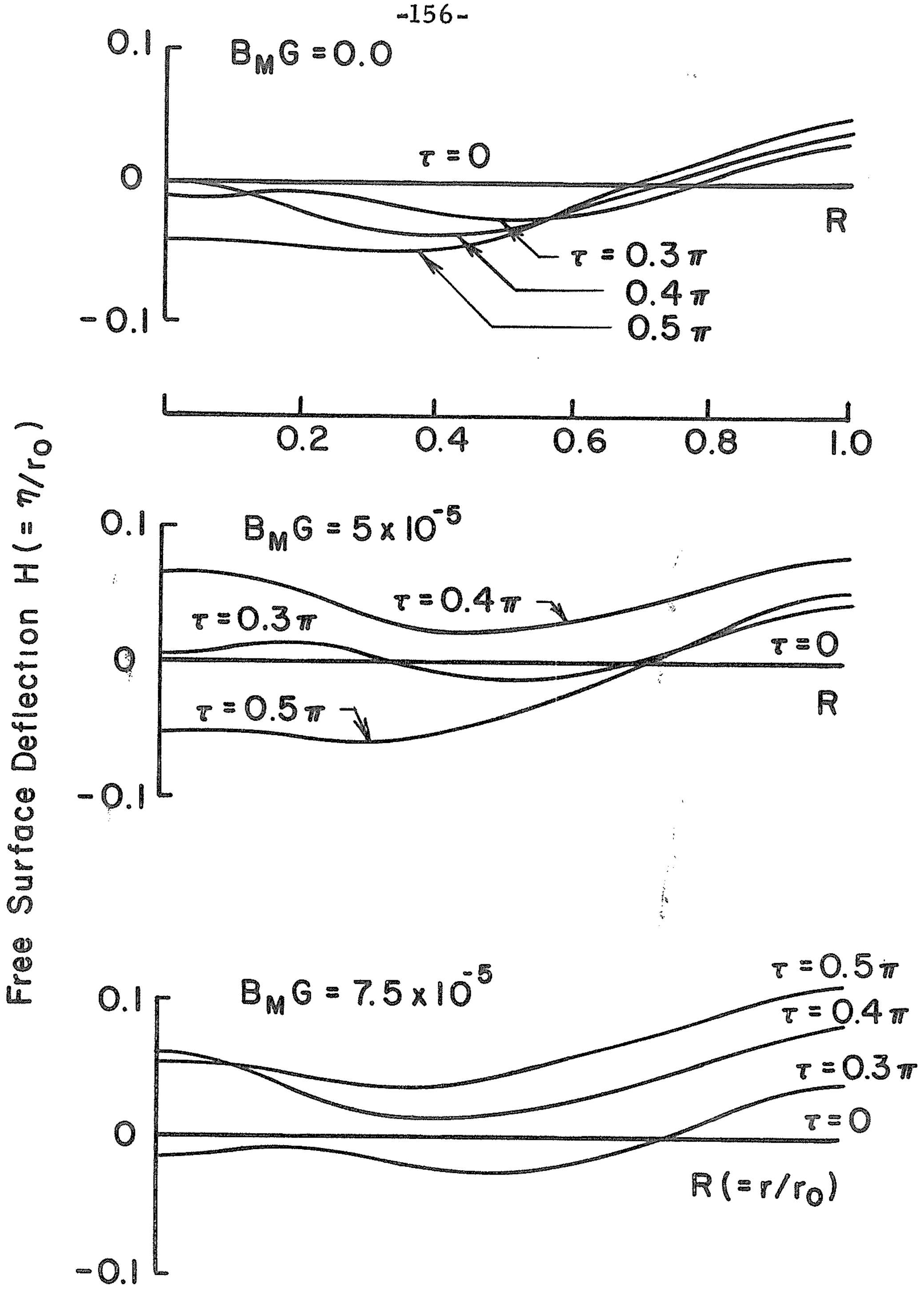

FIG.8 SYMMETRIC DEFLECTION OF THE FREE SURFACE FROM EQ(6.85). $\lambda = 0.04$, $L = 0.4$, $B_\sigma G = 0.01$, $G = 10^{-4}$, $\theta_0 = 75°$, $\nu_X = 0$ AND $P_2 = 0$

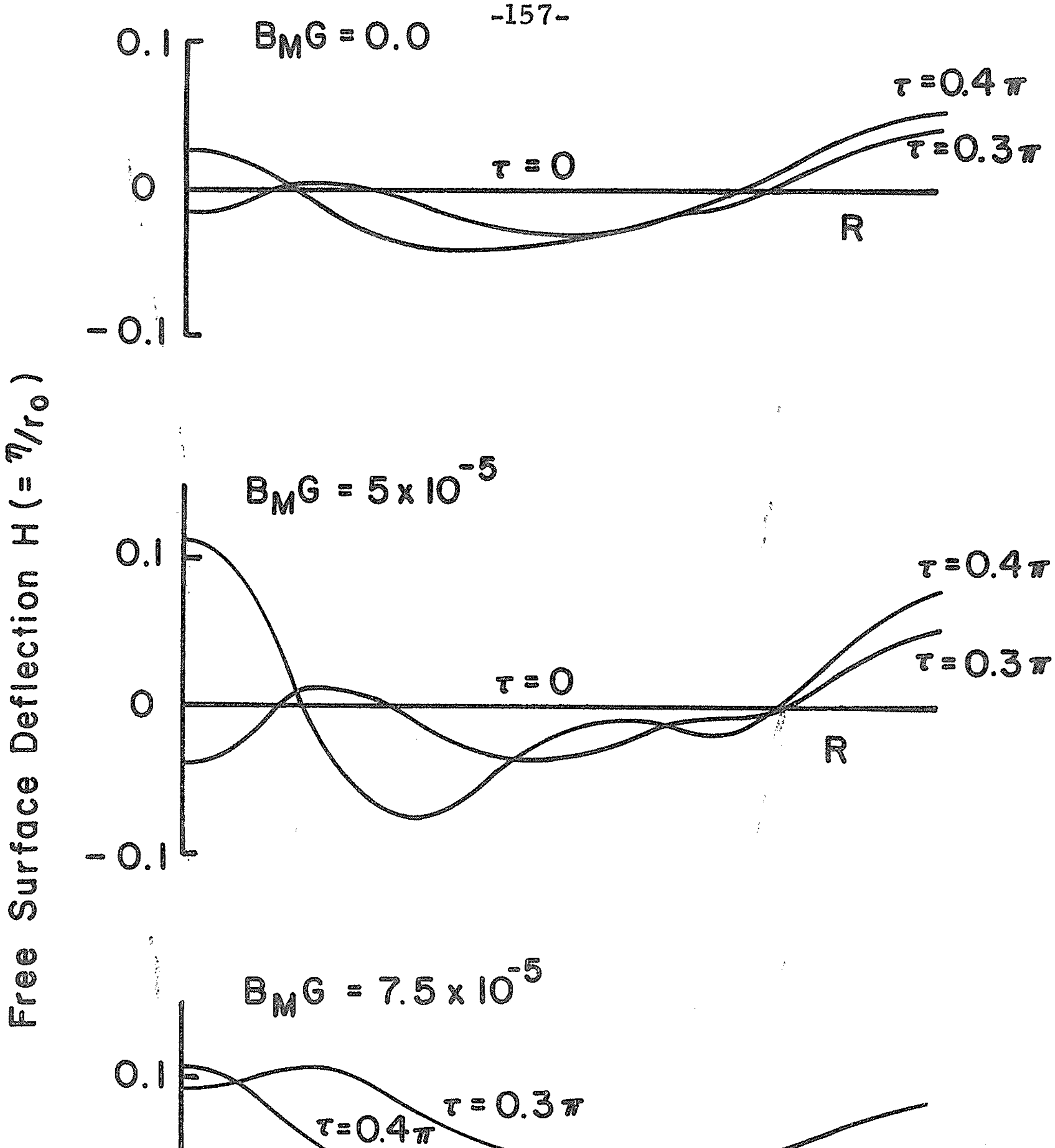

FIG. 9 SYMMETRIC DEFLECTION OF THE FREE SURFACE FROM EQ (6.85). $\lambda = 0.04$, $L = 0.4$, $B_\sigma G = 0.05$, $G = 10^{-4}$, $\theta_0 = 75°$, $\nu_x = 7.5°$ AND $P_2 = 0$

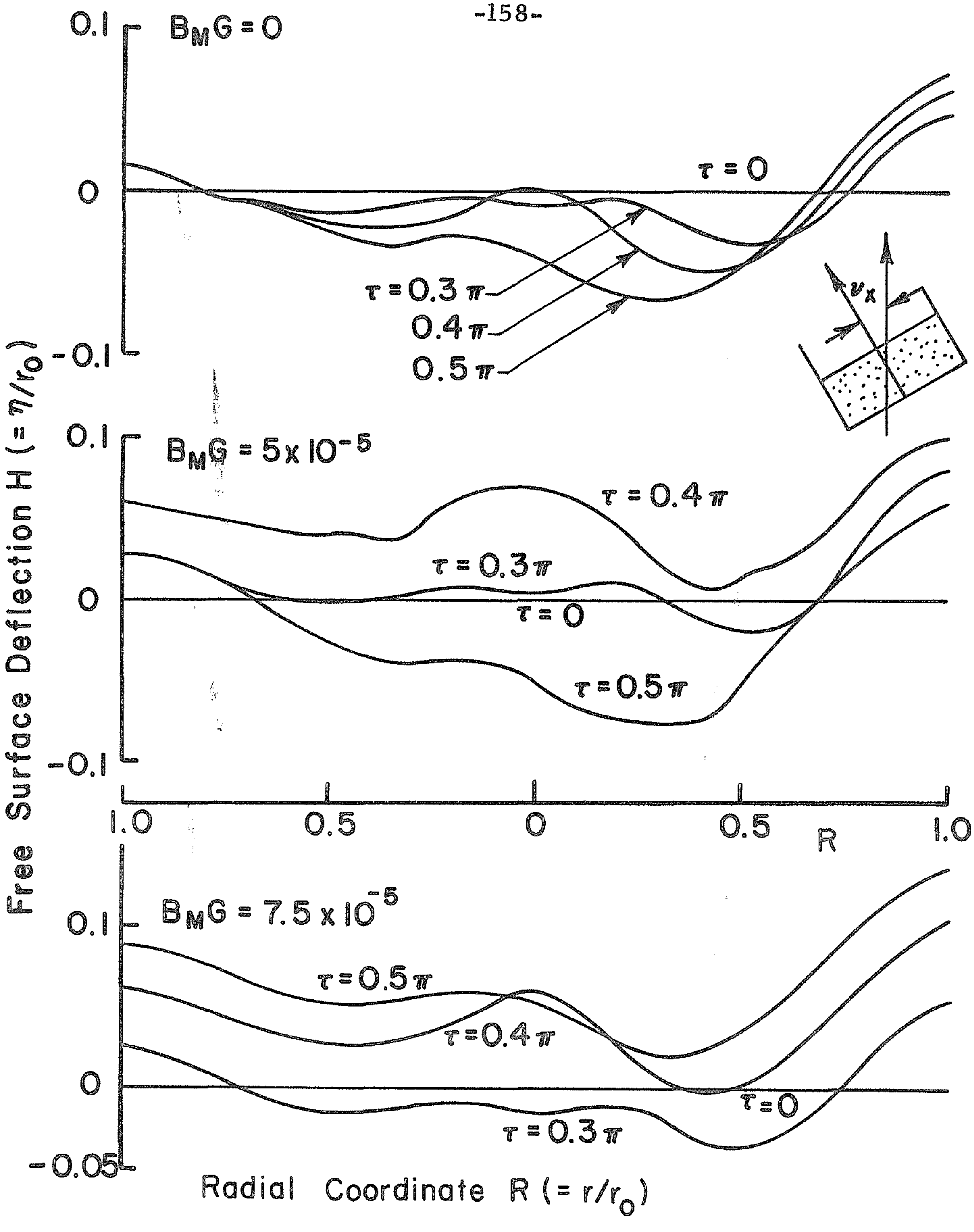

FIG. 10 TRANSIENT FREE SURFACE DEFLECTION AT $\theta = \pm 90°$ FROM EQ (6.85). $\lambda = 0.04$, L=0.4, $B_\sigma G = 0.01$, $G = 10^{-4}$, $\theta_0 = 75°$, $\nu_x = 7.5°$ AND $P_2 = 0$

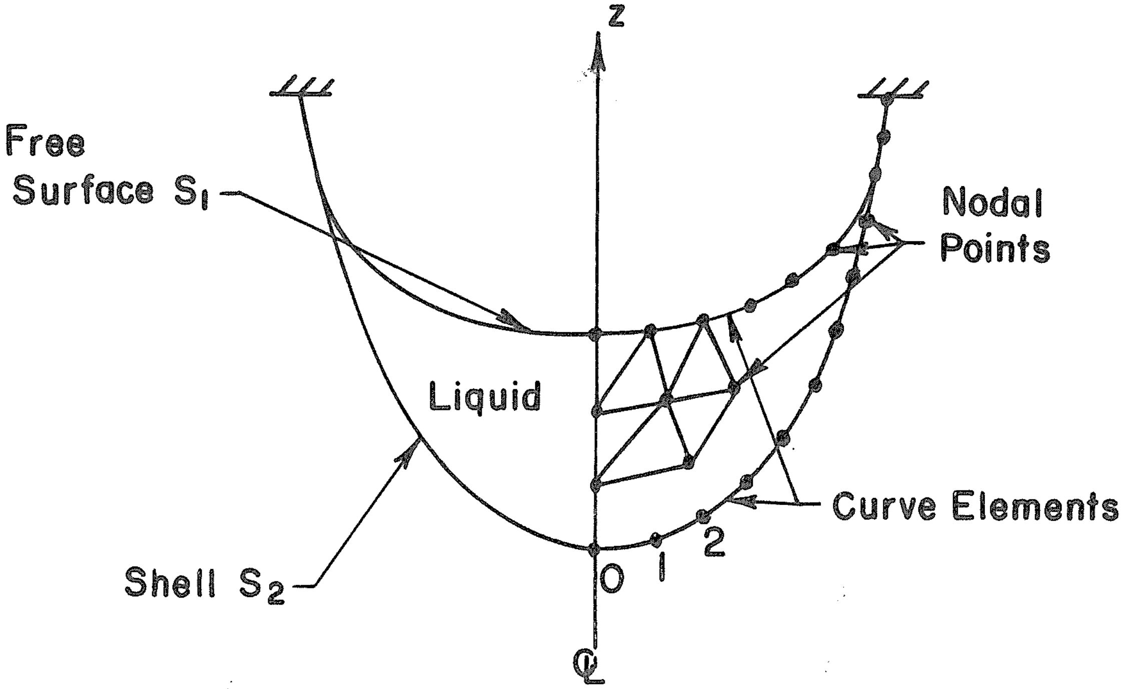

FIG.11 DIAGRAM TO ILLUSTRATE THE SUBDIVISION OF THE SYSTEM FOR NUMERICAL ANALYSIS

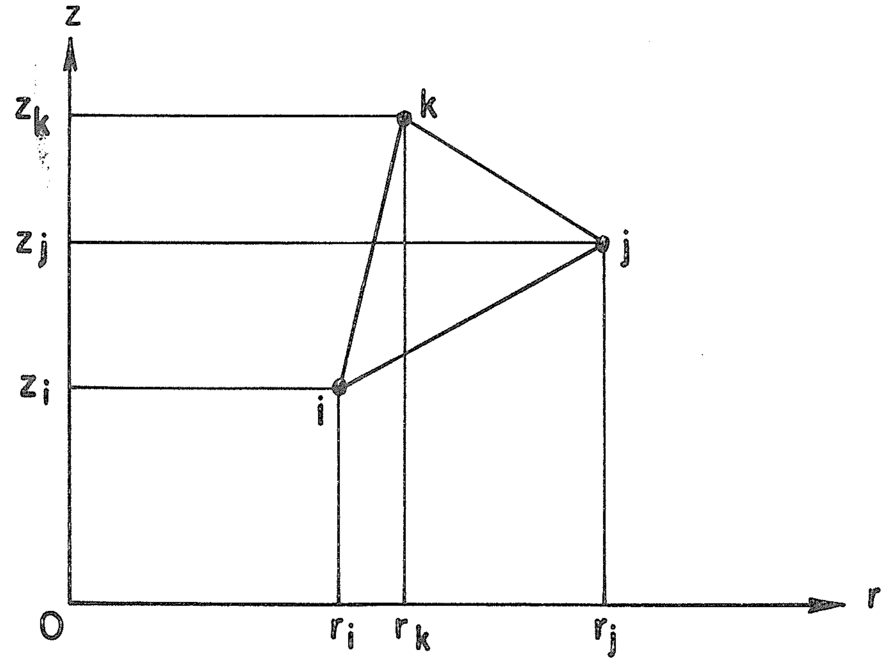

FIG.12 A TYPICAL ELEMENT FOR THE REGION OCCUPIED BY THE FLUID