Geography and Botany of Irreducible Symplectic 4-Manifolds with Abelian Fundamental Group

Thesis by

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Abstract

In this thesis the geography and botany of irreducible symplectic 4-manifolds with abelian fundamental group of small rank are studied. It resembles an anthology of the contribution obtained by the author during his infatuation with 4-dimensional topology by studying its recent developments. As such, each chapter is independent from each other and the reader is welcomed to start reading whichever one seems more appealing. We now give an outline for the sake of convenience.

The first chapter of the thesis deals with the existence and (lack of) uniqueness of smooth irreducible symplectic non-spin 4-manifolds with cyclic fundamental group (both finite and infinite). Chapter 2 does the same for 4-manifolds with abelian, yet non-cyclic π_1; the use of the homeomorphism criteria on these manifolds due to I. Hambleton and M. Kreck is of interest. In Chapter 3, the Spin geography for abelian fundamental groups of small rank is studied. A couple of subtle relations between simply connected and non-simply connected exotic 4-manifolds are explored through out the fourth chapter.

Chapter 5 gives closure to a question raised in Chapter 4, and describes current research projects pursued by the author. These projects came naturally through the results presented in previous chapters. The thesis ends by describing two research progress that are being pursued. Chapter 6 contains the current situation regarding the geography and botany of spin manifolds with zero signature.

The current state of the joint work of the author with Jonathan Yazinski (at McMaster University at the time of writing) is described in the seventh and final chapter.

Contents

Chapter 1

Nonspin symplectic 4-manifolds with cyclic π_1

The geography and botany of smooth/symplectic 4-manifolds with cyclic fundamental group are addressed. For all the possible lattice points that correspond to non-spin manifolds of negative signature and a given homeomorphism type, an irreducible symplectic manifold and an infinite family of pairwise non-diffeomorphic non-symplectic irreducible manifolds are constructed. In the same fashion, a region of the plane for manifolds with non-negative signature is filled in.

1.1 Introduction

Our understanding of simply connected smooth 4-manifolds has witnessed a drastic improvement in recent years. A quick description of the blueprint to the chain of fresh successes of 4-dimensional topologists can be achieved through (great) oversimplification, by attributing them to two factors: an increase in the repertoire of techniques that manufacture small symplectic 4-manifolds and a new perspective on the usage of already existing mechanisms. The idea of using symplectic sums (see [28]) of non-simply connected building blocks along genus 2 surfaces to kill fundamental groups in an efficient way was introduced in [1]. Its immediate outcome was the construction of an exotic symplectic $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ and, later on, the existence of an exotic symplectic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ (cf. [4]) was put on display. Shortly after, Luttinger surgery ([45], [8]) was invited to the game in [11] and in [57]. The combinations of these techniques produced another exotic symplectic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ in [11]. Several of these constructions trace their origins to [20], where symplectic sums of products of 2-manifolds and surgery along nullhomologous tori were employed to construct symplectic and non-symplectic exotic 4-manifolds.

Concerning the (lack of) uniqueness of smooth structures on irreducible 4-manifolds, the article [22] introduces a technique to produce infinite families of distinct smooth structures on many smooth 4-manifolds. The influx of these rather elegant geometric-topological manufacturing mechanisms were succesful and several of the small simply connected 4-manifolds $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ $(k \leq 9)$, which were the most challenging 4-manifolds in terms of exhibiting the existence of one exotic smooth structure, were shown to admit *infinitely many* exotic smooth structures. We refer the reader to the papers [56], [21], [4], [11], [12], [3], [22], and [5] for a concise presentation of these ideas and for the current state of affairs in the subject.

Another major success in the 4-dimensional story was the use of these brand new manufactured exotic manifolds to produce a myriad of irreducible 4-manifolds and, thus, fill out a huge part of the symplectic geography plane and its botany counterpart (cf. [29], [21]). The combination of these results with previous efforts ([2], [5], [50]) provides us with a fairly comprehensive understanding of the geography/botany problem for simply connected symplectic 4-manifolds of negative signature. Although the non-negative signature region in the geography/botany plane is still a challenge ([56], [59], [49], [58], [9]), these new techniques have also been useful in the study of such manifolds ([9], [6]).

In this chapter the focus is switched into the non-simply connected realm. The utility of these new techniques is extended in order to address the geography and botany of smooth/symplectic irreducible 4-manifolds with cyclic fundamental group. For all the possible lattice points that correspond to non-spin manifolds of negative signature and a given homeomorphism type, an irreducible symplectic manifold and an infinite family of pairwise non-diffeomorphic non-symplectic irreducible manifolds are manufactured. Such a goal involves building the smallest known 4-manifolds with cyclic fundamental group that are known to admit an irreducible symplectic exotic smooth structure and use them to fill in regions of the plane. In the same fashion, 4-manifolds with non-negative signature are studied. The corresponding coordinates are given within the results for the convenience of the reader. The tools in [22] help us conclude that the manufactured manifolds have infinitely many smooth structures.

The first examples of exotic 4-manifolds with cyclic fundamental group were constructed in [38], [39], [31], [46], [32], and [70]. Efforts towards more general fundamental groups can be found in [10], [12], and in [13].

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1.2 Statements of Results

The results obtained in this chapter follow two directions. First, several symplectic 4-manifolds with cyclic fundamental group and small Euler characteristic are constructed. Second, regions of the geography/botany plane of each fundamental group are filled out.

1.2.1 Notation

The following notation will be used to denote the manufactured symplectic manifolds:

$$X^{\pi_1}_{b_2^+, b_2^-}.$$

The corresponding topological prototypes for which exotic smooth structures are constructed will be

- $b_2^+\mathbb{CP}^2\#b_2^-\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$ and
- $b_2^+\mathbb{CP}^2\#b_2^-\overline{\mathbb{CP}}^2\#S^1\times S^3$.

For example, $X_{1,3}^{\mathbb{Z}_p}$ denotes the symplectic manifold with finite cyclic fundamental group $\mathbb{Z}_p$ and $X_{2,4}^{\mathbb{Z}}$ stands for the one with infinite cyclic fundamental group, both have Euler characteristic $e = 6$ and signature $\sigma = -2$. For the topological prototypes for finite cyclic fundamental groups, we have the following. The piece $\widetilde{L(p,1)\times S^1}$ stands for the surgered product $L(p,1)\times S^1$ of a lens space with the circle; the surgery is performed along $\{pt\}\times S^1$ to kill the loop corresponding to the generator of the infinite cyclic group factor so that π_1 of the surgered manifold comes from the fundamental group of the lens space.

We point out that this notation gives away all the information needed to establish a homeomorphism. In the infinite cyclic fundamental group case, we recall that $b_2(X) = e(X)$; in particular notice that $e(S^1\times S^3) = 0$ since the Euler characteristic is multiplicative. In the finite cyclic case, $e(\widetilde{L(p,1)\times S^1}) = 2$. Thus, these manifolds share the same Euler characteristic $e = 6$ and signature $\sigma = -2$. Thus, $X_{1,3}^{\mathbb{Z}_p}$ is an exotic $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$.

The following definition introduced by A. Akhmedov and B.D. Park in [6] will be used for practical reasons.

Definition 1.1. A smooth 4-manifold X has the ∞-*property* if and only if there exists an irreducible symplectic 4-manifold and infinitely many pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds, all of them homeomorphic to X.

1.2.2 Main results

Theorem 1.2. *Let e and σ denote integers satisfying $2e+3\sigma\geq 0$, and $e+\sigma\equiv 0$ (mod 4). If, in addition,*

$$\sigma \leq -1,$$

then there exists a non-spin irreducible symplectic 4-manifold with cyclic fundamental group (for both choices, finite and infinite) with signature σ and Euler characteristic e.

Expressed in terms of the geography/botany problems, we manufacture irreducible symplectic 4-manifolds with cyclic fundamental group and infinitely many pairwise non-diffemorphic non-symplectic 4-manifolds with cyclic fundamental groups that realize the coordinates

$$(c_1^2, \chi_h) \text{ if } 0 \leq c_1^2 \leq 8\chi - 1.$$

The following result is an extension of the combined efforts of [2] and [5] to the geography/botany problems of cyclic fundamental groups.

Theorem 1.3. *Let (c, χ) be any pair of non-negative integers satisfying*

$$0 \leq c \leq 8\chi - 1.$$

The manifolds

$$(2\chi - 1)\mathbb{CP}^2 \# (10\chi - c - 1)\overline{\mathbb{CP}}^2 \# \widetilde{L(p,1) \times S^1} \text{ and}$$

$$2\chi\mathbb{CP}^2 \# (10\chi - c)\overline{\mathbb{CP}}^2 \# S^1 \times S^3$$

have the ∞-property.

Besides manifolds with negative signature, one is able to fill in other regions. A sample of such results is given below.

The following theorems extend some results in [49] and [22].

Theorem 1.4. *For each integer k, $10 \leq k \leq 18$, there exists an infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible 4-manifolds with the following characteristics.*

- *Only one member is symplectic,*

- *the characteristic numbers for all the members of the family can be chosen from the following three pairs:* $\chi_h = 2$ *and* $c_1^2 = 19 - k$*;* $\chi_h = 3$ *and* $c_1^2 = 19 - k$ *or* $\chi_h = 3$ *and* $c_1^2 = 27 - k$*,*

- *each member of the family contains a symplectic surface* Σ_2 *of genus 2 and self-intersection 0. The fundamental group of the complement of* Σ_2 *in each manifold is isomorphic to the fundamental group of the ambient manifold.*

Theorem 1.5. *For each integer* k*,* $10 \leq k \leq 18$*, there exists an infinite family* $\{X_n\}$ *of pairwise non-diffeomorphic irreducible 4-manifolds with the following characteristics.*

- *Only one member is symplectic,*

- *the characteristic numbers for all the members of the family can be chosen from the following two pairs:* $\chi_h = 4$ *and* $c_1^2 = 33 - k$ *or* $\chi_h = 5$ *and* $c_1^2 = 41 - k$*.*

- *each member of the family contains a symplectic torus* T *of self-intersection 0. The fundamental group of the complement of* T *in each manifold is isomorphic to the fundamental group of the ambient manifold.*

Corollary 1.6. *Let* k *and* q *be integers such that* $10 \leq k \leq 18$ *and* $10 \leq q \leq 20$*. The following 4-manifolds have the* ∞*-property:*

- $4\mathbb{CP}^2\#(1+k)\overline{\mathbb{CP}}^2\#S^1 \times S^3$,

- $6\mathbb{CP}^2\#(3+q)\overline{\mathbb{CP}}^2\#S^1 \times S^3$, $5\mathbb{CP}^2\#(2+q)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times} S^1$,

- $8\mathbb{CP}^2\#(7+k)\overline{\mathbb{CP}}^2\#S^1 \times S^3$, $7\mathbb{CP}^2\#(6+k)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times} S^1$,

- $10\mathbb{CP}^2\#(9+k)\overline{\mathbb{CP}}^2\#S^1\times S^3$, $9\mathbb{CP}^2\#(8+k)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$.

With the next result we start our enterprise into the non-negative signature region of the geography planes.

Theorem 1.7. *Let $k\geq 45$. The manifolds*

$$(2k+2)\mathbb{CP}^2\#(2k+2)\overline{\mathbb{CP}}^2\#S^1\times S^3,$$
$$(2k+1)\mathbb{CP}^2\#(2k+1)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$$

have the ∞-property.

Let $q\geq 49$. The manifolds

$$(2q)\mathbb{CP}^2\#(2q+1)\overline{\mathbb{CP}}^2\#S^1\times S^3,$$
$$(2q-1)\mathbb{CP}^2\#(2q)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$$

By following an idea of Stipsicz [59] employed in [2] and using the recent efforts in [9], [49], and [6], the following points/regions in the plane non-negative signature are shown to be realized.

Theorem 1.8. *There exists a closed minimal symplectic 4-manifold X with cyclic $\pi_1(X)$ for the following choices of characteristic numbers:*

- *$e=94$ and $\sigma=2$ corresponding to $(c_1^2,\chi_h)=(194,24)$,*
- *$e=98$ and $\sigma=2$ corresponding to $(c_1^2,\chi_h)=(202,25)$,*
- *$e=100$ and $\sigma=0$ corresponding to $(c_1^2,\chi_h)=(200,25)$,*
- *$e=100$ and $\sigma=4$ corresponding to $(c_1^2,\chi_h)=(212,26)$,*
- *$e=104$ and $\sigma=4$ corresponding to $(c_1^2,\chi_h)=(220,27)$ or*
- *$e=106$ and $\sigma=2$ corresponding to $(c_1^2,\chi_h)=(218,27)$.*

These manifolds are used to fill in the following regions:

- $(e, \sigma) = (2m + 2, 0)$ and $(c_1^2, \chi_h) = (4m + 4, 1/2(m + 1))$,
- $(e, \sigma) = (2m + 1, 1)$ and $(c_1^2, \chi_h) = (4m + 5, 1/2(m + 1))$, and
- $(e, \sigma) = (2m, 0)$ and $(c_1^2, \chi_h) = (4m + 6, 1/2(m + 1))$.

The following result states the regions in terms of the topological prototypes.

Proposition 1.9. *Let m be an odd positive integer. If $m \geq 49$, then*

- $m\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2 \# \widetilde{L(p, 1) \times S^1}$,
- $(m + 1)\mathbb{CP}^2 \# (m + 1)\overline{\mathbb{CP}}^2 \# S^1 \times S^3$,
- $m\mathbb{CP}^2 \# (m - 1)\overline{\mathbb{CP}}^2 \# \widetilde{L(p, 1) \times S^1}$, *and*
- $(m + 1)\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2 \# S^1 \times S^3$

have the ∞-property.

If $r \geq 47$, then

- $r\mathbb{CP}^2 \# (r - 2)\overline{\mathbb{CP}}^2 \# \widetilde{L(p, 1) \times S^1}$ *and*
- $(r + 1)\mathbb{CP}^2 \# (r - 1)\overline{\mathbb{CP}}^2 \# S^1 \times S^3$

have the ∞-property.

Let s be an odd positive integer. If $s \geq 53$, then

- $s\mathbb{CP}^2 \# (s - 3)\overline{\mathbb{CP}}^2 \# \widetilde{L(p, 1) \times S^1}$ *and*
- $(s + 1)\mathbb{CP}^2 \# (s - 2)\overline{\mathbb{CP}}^2 \# S^1 \times S^3$

have the ∞-property. If $t \geq 51$, then

- $t\mathbb{CP}^2 \# (t-4)\overline{\mathbb{CP}}^2 \# \widetilde{L(p,1) \times S^1}$ *and*

- $(t+1)\mathbb{CP}^2 \# (t-3)\overline{\mathbb{CP}}^2 \# S^1 \times S^3$

have the ∞-property.

These manifolds correspond to the regions

- $(e, \sigma) = (2m-1, 3)$ and $(c_1^2, \chi_h) = (4m+7, 1/2(m+1))$ and
- $(e, \sigma) = (2m-2, 0)$ and $(c_1^2, \chi_h) = (4m+8, 1/2(m+1))$.

Proposition 1.10. *For each odd integer $m \geq 1$ and $10 \leq k \leq 18$, there exists an irreducible symplectic 4-manifold Y with cyclic fundamental group whose characteristic numbers can be chosen amongst the following options:*

1. $\chi(Y) = 25m^2 + 31m + 5$ *and* $c_1^2(Y) = 225m^2 + 248m + 35 - k$;

2. $\chi(Y) = 25m^2 + 31m + 6$ *and* $c_1^2(Y) = 225m^2 + 248m + 43 - k$;

3. $\chi(Y) = 25m^2 + 31m + 6$ *and* $c_1^2(Y) = 225m^2 + 248m + 41 - k$;

4. $\chi(Y) = 25m^2 + 31m + 7$ *and* $c_1^2(Y) = 225m^2 + 248m + 49 - k$;

5. $\chi(Y) = 25m^2 + 31m + 8$ *and* $c_1^2(Y) = 225m^2 + 248m + 57 - k$.

Moreover, the manifolds with the first three choices of coordinates contain a symplectic genus 2 surface Σ of self-intersection zero; the manifolds from the last two choices contain a symplectic torus T of self-intersection zero and $\pi_1(Y - \Sigma) = \pi_1(Y) = \pi_1(Y - T)$.

Proposition 1.11. *Let $n \geq 2$. There exists a symplectic minimal 4-manifold with cyclic fundamental group whose characteristic numbers can be chosen among the following three choices:*

- $e = 75n^2 + 256n + 130$ *and* $\sigma = 25n^2 - 68n - 78$; $(c_1^2, \chi_h) = (225n^2 + 298n + 26, 25n^2 + 94n + 13)$,

- $e = 75n^2 + 256n + 134$ *and* $\sigma = 25n^2 - 68n - 78$ $(c_1^2, \chi_h) = (225n^2 + 298n + 30, 25n^2 + 94n + 14)$ *or*

- $e = 75n^2 + 256n + 136$ *and* $\sigma = 25n^2 - 68n - 80$ $(c_1^2, \chi_h) = (225n^2 + 298n + 32, 25n^2 + 94n + 14)$.

The manifolds corresponding to the given coordinates have the ∞*-property.*

1.3 Background Results on 4-Manifolds

The corresponding topological prototypes used to determine the homeomorphism type of the manufactured manifolds will be a connected sum of $p\mathbb{CP}^2 \# q\overline{\mathbb{CP}}^2$ with a non-simply connected manifold reponsible for the fundamental group. For $\pi_1 = \mathbb{Z}$, we build exotica for $q\mathbb{CP}^2 \# p\overline{\mathbb{CP}}^2 \# S^1 \times S^3$. For $\pi_1 = \mathbb{Z}_p$, then the prototype manifolds would be of the form $q\mathbb{CP}^2 \# p\overline{\mathbb{CP}}^2 \# \widetilde{L(p,1) \times S^1}$.

1.3.1 Homeomorphism Criteria: Case $\pi_1 = \mathbb{Z}_p$

For the finite cyclic fundamental group case, the classification result we will use is given in [33] in the shape of Theorem C.

Theorem 1.12. *(Hambleton, Kreck). Let X be a smooth, closed, oriented 4-manifold with finite cyclic fundamental group. X is classified up to homeomorphism by the fundamental group, the intersection form on $H_2(M;\mathbb{Z})/Tors$ and the ω_2 -type. Moreover, any isometry of the intersection form can be realized by a homeomorphism.*

Since in this scenario we do have 2-torsion, one is to be careful about determining the parity of the intersection form and its ω_2-type. The Enriques surfaces are an example of the sublety of the situation: their intersection form is even, but they are not spin manifolds. In this case, there are three ω_2-types:

1. $\omega_2(\tilde{X}) \neq 0$,
2. $\omega_2(X) = 0$,
3. $\omega_2(\tilde{X}) = 0$, but $\omega_2(X) \neq 0$.

By using the well-know work of Donaldson and of Minkowski-Hasse on the classification of the intersection forms, the previous result can be stated in the following practical terms.

Theorem 1.13. *A smooth, closed, oriented 4-manifold with finite cyclic fundamental and indefinite intersection form is classified up to homeomorphism by the fundamental group, the Betti numbers b_2^+ and b_2^-, the parity of the intersection form, and the ω_2-type.*

However, do notice that for these manifolds, to know the invariants b_2^+ and b_2^- is equivalent to knowing any other two numerical invariants, like e or σ.

Moreover, most of the manufactured manifolds are non-spin; type II does not occur. Deciding the ω_2-type boils down to distinguishing if the universal cover is spin or not.

1.3.2 Homeomorphism Criteria: Case $\pi_1 = \mathbb{Z}$

For a huge region, the following result settles the homeomorphism criteria.

Theorem 1.14. *(Hambleton-Teichner, cf. [36]). If X is a closed, oriented, smooth 4-manifold with infinite cyclic fundamental group and satisfies the inequality*

$$b_2(X) - |\sigma(X)| \geq 6,$$

then X is homeomorphic to the connected sum of $S^1 \times S^3$ with a unique, closed, simply connected 4-manifold. In particular, X is determined up to homeomorphism by its second Betti number $b_2(X)$, its signature $\sigma(X)$, and its ω_2 -type. In particular, X is either spin or non-spin depending on the parity of its intersection form.

However, in more generality we have

Theorem 1.15. *Let X be a closed, orientable 4-manifold with infinite cyclic fundamental group and suppose the intersection form on X is extended from the integers. Then X is homeomorphic to a connected sum of $S^1 \times S^3$ with a simply-connected 4-manifold.*

At this point the condition of a manifold to have an intersection form that is extended from the integers is equivalent to its algebraic numbers complying with the inequality above. It has been conjectured by Hambleton-Teichner that all smoothable 4-manifolds can be topologically decomposed as a connected sum of a simply connected 4-manifold and $S^1 \times S^3$. Because of the equivalence, this is the same as the indefiniteness inequality $b_2 \geq |\sigma| \geq 4$ being all that is needed for the forms to be extended from $\mathbb{Z}$.

1.3.3 Raw Materials

The elements employed in our constructions rely on the constructions of other authors ([5], [2], [10], [11], [12], [22]). In this section we quote the notions, properties and results we used the most for the convenience of the reader.

The following definition was introduced in [2].

Definition 1.16. An ordered triple (X, T_1, T_2) consisting of a symplectic 4-manifold X and two disjointly embedded Lagrangian tori T_1 and T_2 is called a telescoping triple if

1. The tori T_1 and T_2 span a 2-dimensional subspace of $H_2(X;\mathbb{R})$.

2. $\pi_1(X) \cong \mathbb{Z}^2$ and the inclusion induces an isomorphism $\pi_1(X - (T_1 \cup T_2)) \to \pi_1(X)$. In particular, the meridians of the tori are trivial in $\pi_1(X - (T_1 \cup T_2)) \to \pi_1(X)$.

3. The image of the homomorphism induced by the corresponding inclusion $\pi_1(T_1) \to \pi_1(X)$ is a summand $\mathbb{Z} \subset \pi_1(X)$.

4. The homomorphism induced by inclusion $\pi_1(T_2) \to \pi_1(X)$ is an isomorphism.

The telescoping triple is called minimal if X itself is minimal. Some words of explanatino are in order. Notice the importance of the order of the tori. The meridians μ_{T_1}, μ_{T_2} in $\pi_1(X - (T_1 \cup T_2)) \to \pi_1(X)$ are trivial and the relevant fundamental groups are abelian. The push off of an oriented loop $\gamma \subset T_i$ into $X - (T_1 \cup T_2)$ with respect to any (Lagrangian) framing of the normal bundle of T_i represents a well-defined element of $\pi_1(X - (T_1 \cup T_2))$, that is independent of the choices of framing and base-point.

The first condition assures us that the Lagrangian tori T_1 and T_2 are linearly independent in $H_2(X; \mathbb{R})$. This allows for the symplectic form on X to be slightly perturbed so that one of the T_i remains Lagrangian while the other becomes symplectic. It can also be perturbed in such way that both of them become symplectic. If we were to consider a symplectic surface F in X disjoint from T_1 and T_2, the perturbed symplectic form can be chosen so that F remains symplectic.

Removing a surface from a 4-manifold usually introduces new generators into the fundamental group of the resulting manifold. The second condition indicates that the meridians are nullhomotopic in the complement and, thus, the fundamental group of the manifold and the fundamental group of the complement of the tori in the manifold coincide.

Out of two telescoping triples, one is able to produce one as follows.

Proposition 1.17. *(cf. [2]). Let (X, T_1, T_2) and (X', T_1', T_2') be two telescoping triples. Then for an appropriate gluing map the triple*

$$(X \#_{T_2, T_1'} X', T_1, T_2')$$

is again a telescoping triple.
The Euler characteristic and the signature of $X \#_{T_2, T_1'} X'$ are given by $e(X) + e(X')$ and $\sigma(X) + \sigma(X')$.

By Usher's theorem, if both X and X' are minimal the resulting telescoping triple will be minimal too.

For the production of the exotic manifolds with cyclic fundamental groups we have the following.

Proposition 1.18. *Let (X, T_1, T_2) be a telescoping triple. Let l_{T_1} be a Lagrangian push off of a curve on T_1 and m_{T_2} the Lagrangian push off of a curve on T_2 so that l_{T_1} and m_{T_2} generate $\pi_1(X)$.*
The symplectic 4-manifold obtained by performing either $+1$ Luttinger surgery on T_1 along l_{T_1} or $+1$ surgery on T_2 along m_{T_2} has infinite cyclic fundamental group.
By applying a $+1$ Luttinger surgery on T_1 along l_{T_1} and a $+1/p$ Luttinger surgery on T_2 along m_{T_2} a symplectic manifold with finite cyclic fundamental group is obtained.

Proof. We start with the infinite cyclic fundamental group case. Denote by Y the manifold resulting from applying one of the two mentioned surgeries. For the sake of definiteness, say T_1 is the surgered torus and let $T_1 = T$ to simplify notation. A $(0, +1)$ surgery is applied. By definition, the meridians of a telescoping triple are trivial. Therefore, we have

$$\pi_1(Y) = \pi_1(X - T)/N(\mu_T m_T^p l_T^q) = \mathbb{Z} \oplus \mathbb{Z}/N(1 m_T^1 1),$$

where $N(m^1)$ is the normal subgroup generated by m, which is $\mathbb{Z}$. Therefore, $\pi_1(Y) = \mathbb{Z}$ generated by t_1. A surgery on T_1 along l_{T_1} kills t_2 in the fundamental group.
If we apply a $(0, +1/p)$ surgery on T_2 along m_{T_2}, we have $\pi_1 = \mathbb{Z}/N(m_T^p) = \mathbb{Z}_p$.

□

Remark 1. *The fundamental group calculations for the more general torus surgeries are analogous. To check the validity of the claims, it suffices to state*

$$\pi_1 = \pi_1(X - T)/N(\mu_T m_T^p l_T^q).$$

Our basic building blocks are given in the following result.

Theorem 1.19.

- *There exists a minimal telescoping triple (A, T_1, T_2) with $e(A) = 5$, $\sigma(A) = -1$.*
- *For each $g \geq 0$, there exists a minimal telescoping triple (B_g, T_1, T_2) satisfying $e(B_g) = 6 + 4g$, $\sigma(B_g) = -2$.*

- *There exists a minimal telescoping triple* (C, T_1, T_2) *with* $e(C) = 7$, $\sigma(C) = -3$.
- *There exists a minimal telescoping triple* (D, T_1, T_2) *with* $e(D) = 8$, $\sigma(D) = -4$.
- *There exists a minimal telescoping triple* (F, T_1, T_2) *with* $e(F) = 10$, $\sigma(F) = -6$.

These manifolds were constructed in [2] and [66]. By a repeated use of Lemma 2 in [12] and Usher's theorem one proves the following.

Proposition 1.20. *Let* X *be one of the manifolds* A, B_g, C, D, F, *and* T_1, T_2 *the corresponding Lagrangian tori as described in the previous results, with Lagrangian pushoffs* m_{T_i} *and* l_{T_i} *and trivial meridians. Then the symplectic 4-manifolds obtained from* ± 1 *Luttinger surgery on one Lagrangian torus along (accordingly)* m_{T_2} *or* l_{T_1} *are all minimal. The symplectic 4-manifolds obtained from* ± 1 *Luttinger surgery on one Lagrangian tori along (accordingly)* m_{T_2} *or* l_{T_1} *and* $\pm 1/p$ *Luttinger surgery on the other tori along the proper pushoff are all minimal.*

We move on now to mimic the procedure of Lemma 10 in [2] in order to produce a *non-minimal* telescoping triple out of (B, T_1, T_2) that suits perfectly our purposes. The statement is

Lemma 1.21. *The blow-up* $\tilde{B} = B\#16\overline{\mathbb{CP}}^2$ *contains a genus 18 surface* F_{18} *with trivial normal bundle and two Lagrangian tori* $T_1 \times T_2$ *so that the surfaces* F_{18}, T_1, T_2 *are pairwise disjoint,* $(\tilde{B}, F_{18})$ *is relatively minimal and:*

1. $\pi_1(\tilde{B} - (F_{18} \cup T_1 \cup T_2)) = \mathbb{Z}t_1 \oplus \mathbb{Z}t_2$.
2. *The inclusion* $\tilde{B} - (F_{18} \cup T_1 \cup T_2) \subset \tilde{B}$ *induces an isomorphism on fundamental groups. In particular the meridians* $\mu_{F_{18}}$, μ_{T_1}, μ_{T_2} *all vanish in* $\pi_1(\tilde{B} - (F_{18} \cup T_1 \cup T_2))$.
3. *The Lagrangian pushoffs* m_{T_1}, l_{T_1} *of* $\pi_1(T_1)$ *are sent to 1 and* t_2 *respectively in the fundamental group of* $\tilde{B} - (F_{18} \cup T_1 \cup T_2)$.
4. *The Lagrangian pushoffs* m_{T_2}, l_{T_2} *of* $\pi_1(T_2)$ *are sent to* t_1 *and* t_2 *respectively in the fundamental group of* $\tilde{B} - (F_{18} \cup T_1 \cup T_2)$.

5. *There is a standard symplectic generating set* $\{a_1, b_1, a_2, b_2, \ldots, a_{18}, b_{18}\}$ *for* $\pi_1(F_{18})$ *so that the pushoff* $F_{18} \subset \tilde{B} - (F_{18} \cup T_1 \cup T_2)$ *takes* b_{17} *to* t_2 *and* b_{18} *to* t_1*, and all other generators to 1.*

In particular, $(\tilde{B}, T_1, T_2)$ *is a telescoping triple.*

Needless to say, a basic element in these constructions is the computation of fundamental groups. Serious technical issues arise when dealing with fundamental groups and cut-and-paste constructions; keeping track of the base point through the operations is crucial. For example, in order to be able to apply van-Kampen's theorem, the base points must lie on the boundary and great care is required when one is performing fundamental group calculations. The reader is referred to [11], [12], [13], and [22] for more detailed description on this issue. The mechanisms employed in this paper are much softer though, since they depend heavily on those calculations performed in the papers cited before.

1.3.4 Minimality/Irreducibility

The following results allow us to conclude on the irreducibility of the constructed manifolds.

Theorem 1.22. *(Hamilton and Kotschick, [37]). Minimal symplectic 4-manifolds with residually finite fundamental groups are irreducible.*

Free groups and finite cylic groups are a well-known example of residually finite fundamental groups. In particular, the results tell us that the only property we should worry about is minimality. For this purpose, we will make use of the following.

Theorem 1.23. *(Usher, [69]). Let* $X = Y \#_{\Sigma \equiv \Sigma} Y'$ *be the symplectic sum where the surfaces have genus greater than zero.*

1. *If either* $Y - \Sigma$ *or* $Y' - \Sigma'$ *contains an embedded symplectic sphere of square -1, then* X *is not minimal.*

2. *If one of the summands, say Y for definiteness, admits the structure of an S^2-bundle over a surface of genus g such that Σ is a section of this S^2-bundle, then X is minimal if and only if Y' is minimal.*

3. *In all other cases, X is minimal.*

Thus, to assure that the manufactured manifolds are minimal it suffices to exclude the first two cases. For such a purpose, by taking a look at the building blocks of the symplectic sums, it is usual to blow-up points to obtain the symplectic surface of self-intersection 0 used for the construction. The exceptional spheres introduced by the blow-up process are the only -1 spheres. They are the only threats for our manifolds not being minimal. To be assured that the first scenario of Usher's theorem is not possible, we need to check that every exceptional sphere does indeed intersect transversally at one point on the surface.

When working on a symplectic context, there is another useful method to eliminate the first two cases of Usher's theorem. The result appears as Corollary 3 in [44], here we stated as a theorem due to its role.

Theorem 1.24. *(Li). Let X be a symplectic 4-manifold that is not rational or ruled. Then every smoothly embedded -1 sphere is homologous to a symplectic -1 curve up to sign. If X is the blow-up of a minimal symplectic 4-manifold with $E_1, \cdots, E_n$ represented by exceptional curves, then the E_i are the only classes represented by a smoothly embedded -1 sphere. Therefore, any orientation preserving diffeomorphism maps E_i to some E_j up to sign.*

1.4 Strategy

The blueprint to the manufacturing process of symplectic irreducible 4-manifolds with cyclic fundamental group has two paths. The first one has already been observed by other authors (cf. [13], [12], [2], and [5]) and we proceed to explain it. When one is aiming at building a simply connected minimal symplectic 4-manifold using Luttinger surgeries, the process can be interrupted before applying the last ± 1 Luttinger surgery. The fundamental group of the resulting manifold will be infinite cyclic. We can then go ahead and apply a $\pm 1/p$ Luttinger surgery on the Lagrangian tori that is still left unused and produce a manifold with finite cyclic fundamental group.

The second path consists of starting with a simply connected irreducible symplectic 4-manifold X that contains a symplectic torus or a symplectic surface of genus 2, both of self-intersection zero and both having simply connected complement inside X. We can build the proper symplectic sum with one of the raw materials presented in section 3 along the corresponding symplectic surface. Then, we apply Luttinger surgeries to manipulate the manifolds' π_1 as we need to obtain cyclic fundamental groups.

After pinning down a topological prototype for the constructed manifolds, the required torus surgeries are applied to produce an infinite family of pairwise non-diffeomorphic, non-symplectic manifolds sharing the same homeomorphism type. Since the manufacturing process is strongly related for both types of cyclic groups, our proofs will carry on both cases at the same time.

Remark 2. The No-2-Torsion Hypothesis. *One might be able to argue without any reference to the SW invariants that any given symplectic irreducible manifold constructed is an exotic copy of its corresponding topological prototype. However, in order to establish the existence of infinitely many exotic smooth structures one does need these invariants. As it is explained in [22], the Morgan-Mrowka-Szabo formula [41] is employed to distinguish the Seiberg-Witten invariants and, by doing so, conclude that the members of the infinite family* $\{X_n\}$ *of irreducible manifolds which were obtained by torus surgeries are pairwise non-diffeomorphic (see [22], the remark preceeding Corollary 14 in [12] and [3]).*

This involves a one-to-one correspondence between the set of spin$^{\mathbb{C}}$ *structures on the manifold and the characteristic elements of* H^2*. Our constructions build exotic manifolds for every single finite cyclic fundamental group. To establish the* ∞*-property on the manufactured manifolds, we assume that their fundamental groups do lack 2-torsion.*

We start our endeavor and exemplify the first path of our strategy. The method to fill the region $(c_1^2, \chi_h) = (5+8k, 1+k)$ (for $k \geq 2$) for cyclic fundamental groups was suggested in [2]. We follow their proof closely and we adapt it to our needs.

Start by defining two minimal simply connected symplectic 4-manifolds:

$$X_- := X^1_{3,5} \text{ and } X_+ := X^1_{1,3}$$

with $e(X_-) = 10$, $\sigma(X_-) = -2$ and $e(X_+) = 6$, $\sigma(X_+) = -2$.

Each of these manifolds contains a symplectic surface F of genus 2 and trivial normal bundle, as well as a symplectic torus H_1 of square -1. Out of these submanifolds a symplectic genus 3 surface F_3 of square 1 is obtained by symplectically resolving the union $H_1 \cup F$. One gets rid of the self-intersection in the sense that one considers now the proper transform $\tilde{F}_3$ of F_3 in $\tilde{X}_\pm$: the blow up $X_\pm$ at a point on F_3 provides us with a symplectic surface $\tilde{F}_3$ of genus three and self-intersection 0. The minimality of $X_\pm$ assures that the meridian of $\tilde{F}_3$ intersects the exceptional sphere, then $\pi_1(\tilde{X}_\pm - \tilde{F}_3) = 1$.

Consider now the product $F_3 \times G$ of a genus 3 surface with a genus g surface and its product symplectic form. This is the step in the manufacturing process where manipulates the the fundamental groups to obtain the desired manifold. For our purposes, we will only perform $2g-1$ Luttinger surgeries on the following $2g-1$ disjoint Lagrangian tori along the corresponding curves

$$Y_1 \times A_j \text{ along } l_{Y_1 \times A_j} = a_j \text{ and}$$
$$Y_2 \times B_j \text{ along } l_{Y_2 \times B_j} = b_j,$$

where $j = 1, \ldots, g$ by leaving (say for definiteness purposes) $Y_2 \times B_g$ alone, i.e., not performing this surgery. By doing so, one obtains a manifold Z_g. The fundamental group of Z_g is given by theorem 1 of [2] to be the group generated by the $6+2g$ loops $x_1, y_1, x_2, y_2, x_3, y_3$ (from the $\pi_1(F_3)$) and $a_1, b_1, \ldots, a_g, b_g$ (from $\pi_1(G)$) and the relations

$$[x_1, b_j] = a_j \text{ for } j = 1, \ldots g \text{ and}$$
$$[x_2, a_j] = b_j \text{ for } j = 1, \ldots g-1.$$

Build the symplectic sum $\tilde{Q}_{\pm,g} := \tilde{X}_{\pm}\#_{\tilde{F}_3=F}Z_g$. Its fundamental group is infinite cyclic: notice that $\pi_1(\tilde{X}-\tilde{F}_3)=1$, so this block kills the generators x_i and y_i during the symplectic sum. The relations from the Luttinger surgeries kill a_j and b_j except for b_g. Therefore, $\pi_1(\tilde{Q}_{\pm,g}) =< b_g >$.

One can now perform a $+1/p$ -Luttinger surgery along the remaining $Y_2 \times B_g$ along $l_{Y_2\times B_g}$ to produce a manifold $Q_{\pm,g}$ with fundamental group $\mathbb{Z}_p$.

Since $\tilde{X}_{\pm}$ is relatively minimal by Li's theorem, the only hypothesis needed to apply Usher's theorem and conclude that $Q_{\pm,g}$ is minimal as well is $g \geq 1$. One then can go on and compute

$$e(Q_{-,g}) = 11 + 8g,\ \sigma(Q_{-,g}) = -3,$$
$$e(Q_{+,g}) = 7 + 8g,\ \sigma(Q_{+,g}) = -3.$$

If k is even, rename $X_{1+2k,4+2k} = Q_{+,k/2}$; if k is odd, set $X_{1+2k,4+2k} = Q_{-,(k-1)/2}$.

This procedure manufactures the manifolds of the result we now state.

Theorem 1.25. *Let $k \geq 2$. The manifolds*

$$(1+2k)\mathbb{CP}^2\#(4+2k)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times} S^1 \text{ and}$$

$$(2+2k)\mathbb{CP}^2\#(5+2k)\overline{\mathbb{CP}}^2\#S^3\times S^1$$

have the ∞-property.

Remark 3. *The last Luttinger surgery applied in our constructions kills a loop carrying a generator of the fundamental group. At the cost of leaving the setting of symplectic manifolds, one could apply a more general torus surgery instead. The resulting core torus from the surgery is nullhomologous in the manufactured manifold. It serves as a dial to change the smooth structure at will ([23], [22]). One can then proceed to use the Morgan-Mrowka-Szabo formula to prove that the irreducible members of the infinite family produced by the torus surgery are pairwise non-diffeomorphic. We refer the reader to [23], [22], and [41]* *for details. A concise explanation is given in the remark above Corollary 14 in [12].*

On the fundamental group calculations related to the infinite family of exotic manifolds homeomorphic to some topological prototype: for an exotic manifold X, the fundamental group of X_n differs only from the one of X by replacing a single relation of the form $b = [a^{-1}, d]$ by $b = [a^{-1}, d]^n$. Thus the only thing needed is to check that raising the power of the commutator in such a relation does not affect the fundamental group calculations (see [3] for more details).

As a consequence of the telescoping triples presented in last section, one obtains the following result.

Proposition 1.26. *Let $k \in \{2, 3, 4, 5, 7, 8\}$. The manifolds*

$$\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2 \# \widetilde{L(p,1) \times} S^1$$

have the ∞-property.

The telescoping triples are rather practical black boxes when one is filling regions in the geography plane. However, they offer no information on how the construction for an exotic manifold goes. We proceed to give a more detailed description of such a process for manifolds that can be obtained out of the construction process for exotic $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$'s carried out in [5].

Example 1.27. $(c_1^2, \chi_h) = (3, 1)$: In [5] the following symplectic sum was used

$$(T^4 \# \overline{\mathbb{CP}}^2) \#_{\Sigma_2} (T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2)$$

and applied two surgeries on the 4-torus blown up once to obtain the mentioned simply connected exotic manifold. Do notice that the symplectic sum already kills two generators without any help from the surgeries. Now, we want to skip a surgery to obtain a manifold with an infinite cyclic fundamental group. We proceed to exhibit the fundamental group calculations of the process.

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the generators of the fundamental group of the 4-torus blown up once. Then, they all satisfy $[\alpha_i, \alpha_j] = 1$. Let a_i', b_i' i = 1, 2 be the generators of the genus 2 surface $\Sigma_2' \subset T^4 \# \overline{\mathbb{CP}}^2$. Assume that the inclusion induces a homomorphism on the fundamental groups that map the generators as follows:

$$a_1' \mapsto \alpha_1, b_1' \mapsto \alpha_2, a_2' \mapsto \alpha_3^2, b_2 \mapsto \alpha_4.$$

We will apply one Luttinger surgery on this block. From our first example above, we have learned that the generator α_3 is to be killed, otherwise one obtains a finite cyclic fundamental group of order 2. Thus, we will apply $(\alpha_2' \times \alpha_3', \alpha_3', -p)$ and introduce the relation $\alpha_3 = [\alpha_1^{-1}, \alpha_4^{-1}]$ to kill α_3. The surgery $(\alpha_2'' \times \alpha_4', \alpha_4', -m/r)$ used in [5] to produce simply connected manifolds will *not* be applied. Denote by S the surgered manifold.

Consider now the other building block. Let c, d be the generators of $\pi_1(S^2 \times T^2 \# 4\overline{\mathbb{CP}}^2)$ satisfying $[c, d] = 1$ and a_i, b_i the generators of Σ_2. Assume the inclusion $\Sigma_2' \subset S^2 \times T^2 \# 4\overline{\mathbb{CP}}^2$ induces a map on the fundamental groups that map the generators as follows:

$$a_1 \mapsto c, b_1 \mapsto d, a_2 \mapsto c^{-1}, b_2 \mapsto d^{-1}.$$

We remark that both genus 2 surfaces intersect an exceptional sphere inside the corresponding block and thus both meridians are nullhomotopic. Assume the orientation reversing diffeomorphism $\partial(nbh(\Sigma_2')) \to \partial(nbh(\Sigma_2))$ induces a homomorphism on the fundamental groups which maps the generators of π_1 as follows:

$$a_i' \mapsto a_i,\ b_i' \mapsto b_i \text{ for } i = 1, 2.$$

We build the symplectic sum $S = Y \#_{\Sigma_2' = \Sigma_2} T^2 \times S^2 \# 4\overline{\mathbb{CP}}^2$. The presentation of $\pi_1(S)$ is

$$< \alpha_1, \alpha_2, \alpha_3, \alpha_4, c, d | \alpha_3 = [\alpha_1^{-1}, \alpha_4^{-1}], [\alpha_1, \alpha_2] = [\alpha_1, \alpha_3] = [\alpha_2, \alpha_3] = [\alpha_2, \alpha_4], [c, d], c = \alpha_1, d = \alpha_2, \alpha_3^2 = c^{-1}, \alpha_4 = d^{-1} >.$$

So we have $\alpha_4 = \alpha_2^{-1}$ and substituting it in $\alpha_3 = [\alpha_1^{-1}, \alpha_4^{-1}]$ implies that $\alpha_3 = 1 = c$ since α_1 and α_2 commute. This establishes that the only surviving generator is $\alpha_4 = d^{-1}$. Rename $S = X^{\mathbb{Z}}_{2,7}$.

One can then apply $(\alpha_2'' \times \alpha_4', \alpha_4', -1/p)$ Luttinger surgery on the unused Lagrangian torus to obtain $X^{\mathbb{Z}_p}_{1,6}$.

Since $X^{\mathbb{Z}_p}_{1,6}$ has an odd intersection form and its universal cover has signature $\sigma = -5p$, it follows by Hambleton-Kreck's criteria that it is homeomorphic to $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2 \# \widetilde{L(p,1) \times S^1}$. However, its minimality implies that they are not diffeomorphic.

To build the next example we use Theorem 11 and Proposition 12 of [12]. In the previous sense, these manifolds come out of the process to find exotic $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$'s.

Proposition 1.28. *There exist irreducible symplectic 4-manifolds $X^{\mathbb{Z}}_{2,4}$ and $X^{\mathbb{Z}_p}_{1,3}$. The manifold $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \# \widetilde{L(p,1) \times S^1}$ has the ∞-property.*

One starts with the symplectic sum of $T^2 \times \Sigma_2$ and $T^4 \# 2\overline{\mathbb{CP}}^2$ along a genus 2 surface. Call this manifold Z. We will surger Z to obtain the minimal symplectic $X^{\mathbb{Z}}_{2,4}$, $X^{\mathbb{Z}_p}_{1,3}$ and an infinite family of non-symplectic pairwise non-diffeomorphic minimal 4-manifolds for each one.

Then we will prove that the manufactured manifolds with finite cyclic fundamental group have the claimed underlying topological prototype by establishing the existence of a homeomorphism using Hambleton-Kreck's criteria. We follow Baldridge-Kirk's notation.

Proof. The chosen surgeries and the relations they introduce into the fundamental group are

1. $(T_1', m_1', +1)$ − − − − − − − − − − − − − − $b_1 = [a_2^{-1}, a_1^{-1}]$,
2. $(T_1, l_1, -1)$ − − − − − − − − − − − − − − $a_1 = [b_1^{-1}, y^{-1}]$,
3. $(T_2', l_2', +1)$ − − − − − − − − − − − − − − $b_2 = [b_1, a_2]$,
4. $(T_3, m_3, -1)$ − − − − − − − − − − − − − − $x^{-1} = [b_2^{-1}, y^{-1}]$,

5. $(T_4, l_4, -1/p)$ – – – – – – – – – – – – – – – $a_2^p = [x^{-1}, b_2]$,

6. $(T_2, m_2, -k/1)$ – – – – – – – – – – – – – – – $y = [x^{-1}, b_1]^k$.

The two relations introduced by the first two surgeries take down a_1 and b_1: $a_1 = [b_1^{-1}, y^{-1}] = [[a_2^{-1}, a_1^{-1}], y^{-1}] = 1$ and by Theorem 11 in [12], y commutes with both a_1's. This results in $a_1 = 1$, which implies $b_1 = 1$. The relation introduced by the surgery on T_2' along l_2' and the fact that $b_1 = 1$ kill b_2.

The fourth surgery (along T_3) takes out x^{-1} and the fifth surgery sets $a_2^p = 1$. We kill the last surviving generator by surgering T_2 along m_2. This establishes

$$\pi_1 = \mathbb{Z}_p$$

and $X_{1,3}^{\mathbb{Z}_p}$ has been produced. If we apply a $p = 1$ surgery instead and kill the other generator, we can obtain a manifold with $\pi_1 = \mathbb{Z}$, i.e., this different path manufactures $X_{2,4}^{\mathbb{Z}}$.

Since the surgeries respect the Euler characteristic and the signature we have that $e = 6$ and $\sigma = -2$ and both have an odd intersection form.

Now we proceed on to seeing that we have chosen the correct topological prototype for the homeomorphism type. We have that $b_2^+ = 1$ and $b_2^- = 3$. Since the intersection form of the manifold is odd, type II is ruled out. We claim that the manifolds are of type I indeed. To rule out type III, we observe that the universal cover has Euler characteristic $6p$ and signature $-2p$. For simplicity, assume $p \neq 0$ mod 8, then by Rohlin's theorem the universal cover will not be spin. Thus, these manifolds are of ω_2-type I. The homeomorphism follows from the quoted result of Hambleton-Kreck. □

The last examples realize the pairs $(c_1^2, \chi) = (6, 1)$ and $(3, 1)$.

Remark 4. *P. Kirk and S. Baldridge obtained a similar result for* $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \# \widetilde{L(p,1) \times S^1}$ *(cf. [42]).*

The possible choices of Luttinger surgeries to skip in order to obtain a 4-manifold with infinite cyclic fundamental group are not unique. For example, when one extends the third instance of Theorem 1 in [5], instead of skipping $(a_2' \times c_1', c_1', +1)$ in the $Y_n(m)$ summand of $X_n(m)$ as the authors did in order to obtain simply connected manifolds, we will skip

$(a'_2 \times c'_2, a'_2, -1)$ to get the needed 4-manifolds with infinite cyclic fundamental group. We then proceed to aply $(a'_2 \times c'_2, a'_2, -1/p)$ to conclude the finite cyclic fundamental group case.

Proposition 1.29. *Let $n \geq 3$. The following 4-manifolds have the ∞-property.*

- $(2n-1)\mathbb{CP}^2 \# 2n\overline{\mathbb{CP}}^2 \# \widetilde{L(p,1) \times} S^1$,
- $2n\mathbb{CP}^2 \# (2n+1)\overline{\mathbb{CP}}^2 \# S^1 \times S^3$.

Proof. Let Z' be the irreducible symplectic 4-manifold constructed in [5]. It contains a genus 2 symplectic surface Σ'_2 of self-intersection 0 and $\pi_1(Z' - \Sigma'_2)$ is a quotient of the group $< \alpha_1, \alpha_2, \alpha_3, \alpha_4 | \alpha_3 = [\alpha_1^{-1}, \alpha_4^{-1}], [\alpha_1, \alpha_3] = 1, [\alpha_2, \alpha_3] = [\alpha_2, \alpha_4] = 1 >$.

In Section 2 of [5], an infinite family of pairwise non-diffeomorphic irreducible 4-manifolds which has the same cohomology ring as $(2n-3)$ is constructed by applying $2n+3$ Luttinger surgeries and a single m torus surgery on the product $\Sigma_2 \times \Sigma_n$ of a genus 2 surface with a genus n surface. Let $Y_n(m)'$ be the 4-manifold obtained by applying only $2n+2$ Luttinger surgeries and an m torus surgery. Using the notation of [22] and [5], we choose to not apply $(a'_2 \times c'_2, a'_2, -1)$; this means than in $\pi_1(Y_n(m)')$ all the fundamental group relations given in [5] for $Y_n(m)$ still hold except for $[b_2^{-1}, d_2^{-1}] = a_2$. There is a genus 2 symplectic surface $\Sigma_2 \subset Y_n(m)'$ of self-intersection 0.

Take the fiber sum

$$S_n(m) = Y_n(m)' \#_\Phi Z'$$

using a diffeomorphism $\Phi : \partial(N_{\Sigma_2}) \to \partial(N_{\Sigma'_2})$. Notice that $S_n(m)$ is symplectic if $m = 1$. Let a'_i, b'_i be the standard generators of $\pi_1(\Sigma'_2)$ and a_i, b_i be the generators of $\pi_1(\Sigma_2)$; thus, the fundamental group of $Y_n(m)'$ is generated by $a_1, b_1, a_2, b_2, c_1, d_1, \ldots, c_n, d_n$. Assume that $\Phi_\star$ maps the generators of π_1 as follows:

$$a_i \mapsto a'_i,\ b_i \mapsto b'_i.$$

The group $\pi_1(S_n(m))$ is a quotient of the group $\pi(Y_n(m) - N_{\Sigma_2}) \star \pi_1(Z' - N_{\Sigma'_2}) / < a_1 = \alpha_1, b_1 = \alpha_2, b_2 = \alpha_4, \mu(\Sigma_2) = \mu(\Sigma'_2) >$. Notice that the existence of a -1 sphere in Z' which intersects the genus 2 surface allows us to build a nullhomotopy for the meridian of Σ_2'. We proceed to show that all but one generators in $\pi_1(S_n(m)$ are trivial.

Using the relations given in Section 2 of [5] we have that $a_1 = [b_1^{-1}, d_1^{-1}] = [b_1^{-1}, [c_1^{-1}, b_2]^{-1}] = [b_1^{-1}, [b_2, c_1^{-1}]]$. Moreover, from the fundamental group of the building block Z' we know that $[\alpha_2, \alpha_4] = 1$ and since b_2 is identified with α_2 and b_4 with α_4, we have that $[b_1, b_2] = 1$. Since b_1 commutes with c_1, then $a_1 = 1$.

Once we have killed a_1, one can get rid of b_1, b_2, c_1, c_2, d_1 and d_2 by using the first seven surgeries in (4) of Section 2 in [5], and we conclude that $c_n = 1 = d_n$ for $n \geq 3$ by using the last $2(n-2)$ Luttinger surgeries of (4). This implies that $\alpha_1 = 1 = \alpha_2 = \alpha_4$ and $\alpha_3 = 1 = [a_1^{-1}, b_2^{-1}] = 1$. Thus, the infinite family of irreducible pairwise non-diffeomorphic 4-manifolds have $\pi_1(S_n(m)) =< a_2 >= \mathbb{Z}$. Notice that the Lagrangian torus $a_2 \times c_2$ is still unused. We can now go and apply $(a_2' \times c_2', a_2', -1p)$ Luttinger surgery to $S_n(m)$ and obtain an infinite family $\{X_n(m)\}$ of pairwise non-diffeomorphic irreducible 4-manifolds with $\pi_1 = \mathbb{Z}_p$; $X_n(1)$ is symplectic.

We can compute

$$e(S_n(m)) = e(X_n(m)) = e(Y_n(m)') + e(Z') - 2e(\Sigma_2) = 4n + 1,$$
$$\sigma(S_n(m)) = \sigma(X_n(m)) = \sigma(Y_n(m)') + \sigma(Z') = -1.$$

From Hambleton-Teichner's result, we conclude that $S_n(m)$ is homeomorphic to $2n\mathbb{CP}^2\#(2n+1)\overline{\mathbb{CP}}^2\#S^1\times S^3$ when $n \geq 2$. By Hambleton-Kreck's result, we conclude that $S_n(m)$ is homeomorphic to $(2n-1)\mathbb{CP}^2\#2n\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$ when $p \neq 0 \bmod 16$.

$\square$

Example 1.30.

- $(c, \chi) = (13, 2)$: by skipping $(a_2' \times c', c', +1/p)$ one obtains $X_{4,7}^{\mathbb{Z}}$. This manifold is non-spin, it has characteristic numbers $e = 11$ and $\sigma = -3$. The theorem of Hambleton-Teichner implies that it is homeomorphic to

$$4\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2\#S^1 \times S^3.$$

We proceed to apply the surgery $(a_2' \times c', c', +1/p)$ on the unused Lagrangian tori with the given Lagrangian framing in order to obtain $X_{3,6}^{\mathbb{Z}_p}$. This manifold is non-spin as well. Its universal cover has signature $\sigma = -3p$, which by Rohlin's theorem, implies it is nonspin as well.Hambleton-Kreck's result says that $X_{3,6}^{\mathbb{Z}_p}$ is homeomorphic to

$$3\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2 \# \widetilde{L(p,1) \times} S^1.$$

- $(c, \chi) = (11, 2)$: By skipping a surgery one obtains $X_{4,9}^{\mathbb{Z}}$ and by applying $(\alpha_2'' \times \alpha_4', \alpha_4', -1/p)$ one obtains $X_{3,8}^{\mathbb{Z}_p}$.

- $(c, \chi) = (9, 2)$: $X_{4,11}^{\mathbb{Z}}$ and by applying $(\alpha_2'' \times \alpha_4', \alpha_4', -1/p)$ one obtains $X_{3,10}^{\mathbb{Z}_p}$.

The choice of Luttinger surgery to skip is not unique. In the next section we exemplify these phenomena. It is unknown if the resulting manifolds, independently of the chosen Luttinger surgeries, are diffeomorphic.

1.4.1 Constructing Manifolds via Telescoping Triples

In this section, the telescoping triples (X, T_1, T_2) built in [2] will be employed on the manufacturing procedure of the exotic manifolds. The fundamental group of the manifold X is $\mathbb{Z}t_1 \oplus \mathbb{Z}t_2$. One is able to think as the Lagrangian push-off m_{T_2} being responsible for the $\mathbb{Z}t_1$ factor and the Lagrangian push-off l_{T_1} responsible for the $\mathbb{Z}t_2$ factor. To produce an infinite family $\{X_n\}$ of irreducible 4-manifolds with infinite cyclic fundamental group, it suffices to apply a single torus surgery: either $(T_1, l_{T_1}, +n/1)$ and obtain $\pi_1 = \mathbb{Z}t_1$. One could apply $(T_2, m_{T_2}, +n/1)$ as well and get $\pi_1 = \mathbb{Z}t_2$. The family produced in both cases has a unique symplectic member for $n = 1$.

In order to obtain 4-manifolds with finite cyclic fundamental group of order p, one needs to apply two surgeries. Start by applying $(T_1, l_{T_1}, +1/p)$ and then $(T_2, m_{T_2}, +n/1)$. The first surgery is a Luttinger surgery and it provides us with a manifold with fundamental group $\mathbb{Z}t_1 \oplus \mathbb{Z}_p t_2$. The second surgery is a general torus surgery. It has two duties: kill the

$\mathbb{Z}t_1$ factor in π_1 and to produce an infinite family of $\{X_n\}$ of irreducible 4-manifolds where X_1 is the only member having a symplectic structure.

One obtains several organical results by applying the previous recipe to the telescoping triples. By using Corollary 9 in [2], one obtains that all manifolds in the points $(c_1^2, \chi_h) = (6+8g, 1+g)$ (for a non-negative integer g) of the plane have the ∞-property. In this case the manifolds $X^{\mathbb{Z}_p}_{1+2g,3+2g}$ and the infinite family $\{X_n\}$ come out of the manifold B_g.

Proposition 1.31. *Let $g \geq 0$ and $q \geq 1$. The manifolds*

$$(1+2g)\mathbb{CP}^2\#(3+2g)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$$

and

$$(2+2q)\mathbb{CP}^2\#(4+2q)\overline{\mathbb{CP}}^2\#S^3\times S^1$$

have the ∞-property.

We now need to take care of the topological prototype. The two surgeries have already provided us with the dial to change the smooth structure at will.

Proof. From the characteristic numbers of these manifolds we get $b_2^+ = 1+2g$ and $b_2^{-1} = 3+2g$. They all have an odd intersection form. We claim they are of ω_2-type I). Their universal cover is not spin by Rokhlin's theorem (this argument leaves out the $n = 0$ (mod 8) cases). Thus Hambleton-Kreck's and Hambleton-Teichner's criteria respectively say that they are homeomorphic to the chosen topological prototype.

$\square$

1.4.2 More Examples

In [12], a minimal symplectic 4-manifold X_1 with fundamental group $\mathbb{Z}$ was constructed. This manifold provides a smaller substitute for $E(1)$ to be used in symplectic sums when only one generator is desired to be killed. By gluing either X_1 or $E(1)$ with the manifolds from the telescoping triples given in [2] along T_1 or along T_2 accordingly, one is able to fill in several points and regions as follows.

To make it visually clear, the constructions are collected into the next table. The columns are arranged as follows. The first one indicates the corresponding symplectic sum with either X_1 or $E(1)$. The second and third collumns display the manifold of infinite cyclic fundamental group that we obtain right out of the symplectic sum, and its finite cyclic fundamental group brother that one obtains after applying $+1/p$ Luttinger surgery on T_2. The chosen notation immediately gives away the Euler characteristic and signature of the manifolds. The last column indicates the coordinates on the plane.

Table 1.1: Putting the pieces together

Symplectic Sum	$\pi_1 = \mathbb{Z}$	$\pi_1 = \mathbb{Z}_p$	(c_1^2, χ_h)
$A\#_{T_1=T}X_1$	$X_{4,7}^{\mathbb{Z}}$	$X_{3,6}^{\mathbb{Z}_p}$	$(13, 2)$
$B_g\#_{T_1=T}X_1$	$X_{4+2g,8+2g}^{\mathbb{Z}}$	$X_{3+2g,7+2g}^{\mathbb{Z}_p}$	$(12+8g, 2+g)$
$C\#_{T_1=T}X_1$	$X_{4,9}^{\mathbb{Z}}$	$X_{3,8}^{\mathbb{Z}_p}$	$(11, 2)$
$D\#_{T_1=T}X_1$	$X_{4,10}^{\mathbb{Z}}$	$X_{3,9}^{\mathbb{Z}_p}$	$(10, 2)$
$F\#_{T_1=T}X_1$	$X_{4,12}^{\mathbb{Z}}$	$X_{3,11}^{\mathbb{Z}_p}$	$(8, 2)$
$A\#_{T_1=T}E(1)$	$X_{4,13}^{\mathbb{Z}}$	$X_{3,12}^{\mathbb{Z}_p}$	$(7, 2)$
$B_g\#_{T_1=T}E(1)$	$X_{4+2g,14+2g}^{\mathbb{Z}}$	$X_{3+2g,13+2g}^{\mathbb{Z}_p}$	$(6+8g, 2)$
$C\#_{T_1=T}E(1)$	$X_{4,15}^{\mathbb{Z}}$	$X_{3,14}^{\mathbb{Z}_p}$	$(5, 2)$
$D\#_{T_1=T}E(1)$	$X_{4,16}^{\mathbb{Z}}$	$X_{3,15}^{\mathbb{Z}_p}$	$(4, 2)$
$F\#_{T_1=T}E(1)$	$X_{4,18}^{\mathbb{Z}}$	$X_{3,17}^{\mathbb{Z}_p}$	$(2, 2)$

The gathering of these constructions yields

Proposition 1.32. *Let $k \in \{5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 17\}$. The manifolds*

- $3\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$
- $4\mathbb{CP}^2\#(k+1)\overline{\mathbb{CP}}^2\#S^1\times S^3$

have the ∞-property.

Filling in the points $(c_1^2, \chi_h) = (19-k, 2)$. In the same spirit we have

Lemma 1.33. *Let $g \geq 1$. The manifolds*

- $(3+2g)\mathbb{CP}^2\#(7+2g)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$,

- $(4+2g)\mathbb{CP}^2\#(8+2g)\overline{\mathbb{CP}}^2\#S^1\times S^3$

have the ∞*-property.*

Thus filling the points $(c_1^2, \chi_h) = (12+8g, 2+g)$.

For the remaining part of the section, we will take the second path we mentioned at the beginning to produce more manifolds. Here we build on the efforts of other authors ([49], [6]) set on the simply connected case. We remind the reader that the second path has as a starting point a simply connected irreducible symplectic manifold X which contains a symplectic torus of self-intersection 0 (or a symplectic surface of self-intersection 0 in general) with a simply connected complement. One builds the symplectic sum with a minimal symplectic 4-manifold with non-trivial fundamental group and applies Luttinger surgeries to it in order to obtain a manifold with cyclic fundamental group.

Proposition 1.34. *Let* Y *be a minimal symplectic 4-manifold which contains a symplectic torus* T *of self intersection 0. Assume* $\pi_1(Y) = 1 = \pi_1(Y - T)$*. Then there exists an infinite family of pairwise non-diffeomorphic irreducible 4-manifolds* $\{X_n\}$ *which only has one symplectic member. Moreover, all of its members can be chosen to have as characteristic numbers one of the following three choices:*

1. $e = e(Y)$ *and* $\sigma = \sigma(Y)$*;*
2. $e = e(Y) + 4$ *and* $\sigma = \sigma(Y)$*;*
3. $e = e(Y) + 6$ *and* $\sigma = \sigma(Y) = -2$.

The next procedure will be followed to obtain the claimed manifolds. First, build the symplectic sum of Y and a minimal non-simply connected 4-manifold $\tilde{X}$. The manifold $\tilde{X}$ must have the required characteristic numbers, enough Lagrangian tori with geometrically dual tori to surger that are disjoint from the surface involved in the symplectic sum, and the map $\pi_1(T) \to \pi_1(\tilde{X})$ must not be surjective. By modifying our chosen $\tilde{X}$, one obtains the possible characteristic numbers.

The symplectic sum kills some of the generators. To obtain our cyclic groups, the rest of the generators will be dealt with by the surgeries. The existence of the infinity family will follow from applying a general torus surgery at the corresponding point and Corollary 3 in [22].

Proof. Assume we want to build a manifold with characteristic numbers as in 2. of our statement. Consider the minimal symplectic manifold Z of proposition in Section 3. This manifold has $e = 4$ and $\sigma = 0$. Since $S_8 \subset Z$ is a homologically essential Lagrangian torus, the symplectic sum can be perturbed so that S_8 becomes symplectic, while all the other tori stay Lagrangian. Consider the symplectic sum V of Y and Z along the tori T and S_8. The manifold V is minimal by Usher's theorem. Its characteristic numbers are $e(V) = e(Y) + e(Z) = e(Y) + 4$ and $\sigma(V) = \Sigma(Y) + \sigma(Z) = \sigma(Y)$.

The fundamental group of V is generated by $x_1, y_1, x_2, y_2, a_1, b_1, a_2, b_2, g_1, \ldots, g_n$. In this notation the g_i's represent the meridians of the torus. Although we have a specific commutator representing the meridian, we will not use it for what follows; thus our choice of notation. Furthermore, since $\pi_1(X - T) = 1$ and the meridians are identified, we have that the symplectic sum kills all the these g_i's as well as two generators y_2 and b_2. In particular, the normal subgroup of $\pi_1(V)$ generated by the meridian and the corresponding relations is trivial. We proceed to show how the needed generators are killed via surgeries.

We start by applying $(S_3, l_3, +1)$, which introduces the relation $[b_2^{-1}, y_1^{-1}] = a_2^{-1}$. Thus, a_2 is killed. The relations introduced by the surgeries $(S_7, m_7, +1)$, $(S_6, l_6, +1)$, $(S_1, m_1, +1)$ and $(S_2, l_2, +n/1)$ kill x_2, b_1, x_1 and a_1 respectively. If one stops at this point, a manifold with infinite cyclic fundamental group generated by y_1 is obtained.

In order to produce minimal symplectic manifolds with finite cyclic fundamental group, one applies $(S_4, m_4, +1/p)$. Now the generator y_1 is subject to the relation $y_1^p = 1$. Notice that all the surgered Lagrangian tori have geometrically dual tori. We can apply Fintushel-Park-Stern's corollary to conclude that the manifolds in the family $\{X_n\}$ are pairwise non-diffeomorphic. Hamilton-Kotschick's result imply that the manifolds are irreducible.

The other two cases are similar. Assume that we want to build a manifold with characteristic numbers as in 3. of our claim. Consider the manifold B of Section 3 built in [12]. Perturb the symplectic sum so that T_1 becomes symplectic while T_2 remains Lagrangian. Build the symplectic sum of B and Y along T_1 and T. The symplectic sum has $\pi_1 = \mathbb{Z}t_1$, $e = e(Y) + 6$, $\sigma = \sigma(Y) - 2$ and it is an irreducible symplectic manifold. By applying $(T_2, m_{T_2}, +1/p)$, one obtains the finite cyclic fundamental group manifolds. To produce the manifolds with $e = e(Y)$ and $\sigma = \sigma(Y)$, one glues in a copy of T^4 (see [11]).

□

One can alter the above procedure and glue along genus 2 surfaces instead of tori to obtain a similar proposition with the appropriate increase in the characteristic numbers.

Proposition 1.35. *Let Y be a minimal symplectic 4-manifold which contains a symplectic surface Σ_2 of genus 2 and self intersection 0. Assume $\pi_1(Y) = 1 = \pi_1(Y - \Sigma_2)$. Then there exists an infinite family of pairwise non-diffeomorphic irreducible 4-manifolds $\{X_n\}$ which only has one symplectic member. Moreover, all of its members can be chosen to have as characteristic numbers one of the following two choices:*

1. $e = e(Y) + 10$ *and* $\sigma = \sigma(Y) - 2$;

2. $e = e(Y) + 14$ *and* $\sigma = \sigma(Y) - 2$;

Proof. The proof is similar to the last proposition. One glues B and Y along F and Σ_2 to obtain a symplectic 4-manifold with $\pi_1 = \mathbb{Z}t_1$; by applying $(T_2, l_{T_2}, +1/p)$ one obtains 4-manifolds with $\pi_1 = \mathbb{Z}_p$. The corresponding torus surgery produces the infinite family. For the choice of manifold for 2. in our claim, see Lemma 17 in [2].

□

The utility of these two results can be noted right away since they imply that the following manifolds have the ∞-property.

- $(b_2^+(Y)+1)\mathbb{CP}^2\#(b_2^-(Y)+1)\overline{\mathbb{CP}}^2\#S^1\times S^3$,

- $(b_2^+(Y)+3)\mathbb{CP}^2\#(b_2^-(Y)+3)\overline{\mathbb{CP}}^2\#S^1\times S^3$,
  $(b_2^+(Y)+2)\mathbb{CP}^2\#(b_2^-(Y)+2)\overline{\mathbb{CP}}^2\#L\widetilde{(p,1)\times} S^3$,

- $(b_2^+(Y)+3)\mathbb{CP}^2\#(b_2^-(Y)+5)\overline{\mathbb{CP}}^2\#S^1\times S^3$,
  $(b_2^+(Y)+2)\mathbb{CP}^2\#(b_2^-(Y)+4)\overline{\mathbb{CP}}^2\#L\widetilde{(p,1)\times} S^3$,

- $(b_2^+(Y)+5)\mathbb{CP}^2\#(b_2^-(Y)+7)\overline{\mathbb{CP}}^2\#S^1\times S^3$,
  $(b_2^+(Y)+4)\mathbb{CP}^2\#(b_2^-(Y)+6)\overline{\mathbb{CP}}^2\#L\widetilde{(p,1)\times} S^3$,

- $(b_2^+(Y)+7)\mathbb{CP}^2\#(b_2^-(Y)+9)\overline{\mathbb{CP}}^2\#S^1\times S^3$,
  $(b_2^+(Y)+6)\mathbb{CP}^2\#(b_2^-(Y)+8)\overline{\mathbb{CP}}^2\#L\widetilde{(p,1)\times} S^3$.

This procedure allows us to construct some more manifolds if we specify the chosen simply connected blocks used in the symplectic sums. If we combine the procedure with Lemma 2.1 and/or Proposition 2.1 in [49] accordingly, we fill out another wide region. If we build the symplectic sum along tori, we have:

Theorem 1.36. *For each integer k, $10 \leq k \leq 18$, there exists an infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible 4-manifolds with the following characteristics.*

- *Only one member is symplectic,*
- *the characteristic numbers for all the members of the family can be chosen from the following three pairs: $\chi_h = 2$ and $c_1^2 = 19-k$; $\chi_h = 3$ and $c_1^2 = 19-k$ or $\chi_h = 3$ and $c_1^2 = 27-k$.*
- *each member of the family contains a symplectic surface Σ_2 of genus 2 and self-intersection 0. The fundamental group of the complement of Σ_2 in each manifold is isomorphic to the fundamental group of the ambient manifold.*

If instead of using the tori found in the manifold built by J. Park, we used the symplectic genus 2 surfaces we obtain

Theorem 1.37. *For each integer k, $10 \leq k \leq 18$, there exists an infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible 4-manifolds with the following characteristics.*

- *Only one member is symplectic,*
- *the characteristic numbers for all the members of the family can be chosen from the following two pairs: $\chi_h = 4$ and $c_1^2 = 33 - k$, or $\chi_h = 5$ and $c_1^2 = 41 - k$,*
- *each member of the family contains a symplectic torus T of self-intersection 0. The fundamental group of the complement of T in each manifold is isomorphic to the fundamental group of the ambient manifold.*

Corollary 1.38. *Let k and q be integers such that $10 \leq k \leq 18$ and $10 \leq 20$. The following 4-manifolds have the ∞-property:*

- $4\mathbb{CP}^2\#(1+k)\overline{\mathbb{CP}}^2\#S^1 \times S^3$,
- $6\mathbb{CP}^2\#(3+q)\overline{\mathbb{CP}}^2\#S^1 \times S^3$, $5\mathbb{CP}^2\#(2+q)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$,
- $8\mathbb{CP}^2\#(7+k)\overline{\mathbb{CP}}^2\#S^1 \times S^3$, $7\mathbb{CP}^2\#(6+k)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$,
- $10\mathbb{CP}^2\#(9+k)\overline{\mathbb{CP}}^2\#S^1 \times S^3$, $9\mathbb{CP}^2\#(8+k)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$.

Other choices of simply connected 4-manifolds that can be used as building blocks (and are found in the literature) produce the following manifolds.

Example 1.39.

- Consider the symplectic manifold $E'(k) = E(k)_{2,3}$ obtained from the elliptic surface $E(k)$ by performing two log transforms of order 2 and 3. It contains a torus T with trivial normal bundle and $\pi_1(E'(k) - T) = 1$. By the procedure suggested in the previous proposition one is able to build two manifolds: one with $\pi_1 = \mathbb{Z}$ and other with $\pi_1 = \mathbb{Z}_p$, both with $e = 12k$ and $\sigma = -8k$.

One gets that the constructed manifolds have $b_2^+ = 2k$, $b_2^- = 10k$ and odd intersection form. So, the manifolds

$$2k\mathbb{CP}^2 \# 10k\overline{\mathbb{CP}}^2 \# S^1 \times S^3 \text{ and } (2k-1)\mathbb{CP}^2 \# (10k-1)\overline{\mathbb{CP}}^2 \# L\widetilde{(p,1) \times} S^1$$

have the ∞-property.

Other choices of simply connected manifolds produce more manifolds which will be needed later.

- Using the minimal manifold built by R.E. Gompf $S_{1,1}$ ([28]) with $e = 23$ and $\sigma = -15$, we conclude that

$$4\mathbb{CP}^2 \# 18\overline{\mathbb{CP}}^2 \# S^1 \times S^3 \text{ and } 3\mathbb{CP}^2 \# 17\overline{\mathbb{CP}}^2 \# L\widetilde{(p,1) \times} S^1$$

have the ∞-property.

- Other manifolds built by R.E. Gompf in [28] that will come in handy later on are the folowing. Applying the proposition to the manifold $R_{2,1}$, which has $e = 21$ and $\sigma = -13$, we obtain that the manifolds

$$4\mathbb{CP}^2 \# 17\overline{\mathbb{CP}}^2 \# S^1 \times S^3 \text{ and } 3\mathbb{CP}^2 \# 16\overline{\mathbb{CP}}^2 \# L\widetilde{(p,1) \times} S^1$$

have the ∞-property.

The same procedure for $R_{2,2}$ with $e = 19$ and $\sigma = -11$ leads to the conclusion that

$$4\mathbb{CP}^2 \# 15\overline{\mathbb{CP}}^2 \# S^1 \times S^3 \text{ and } 3\mathbb{CP}^2 \# 14\overline{\mathbb{CP}}^2 \# L\widetilde{(p,1) \times} S^1$$

have the ∞-property as well.

- B. D. Park constructed a minimal simply connected symplectic 4-manifold with $e = 17$ and $\sigma = -9$ which contains a torus with the requires characteristics (cf [52]). Out of his manifold one shows that the manifolds

$$4\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2\#S^1 \times S^3 \text{ and } 3\mathbb{CP}^2\#12\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$$

have the ∞-property.

1.4.3 Examples with Odd Signature

Proposition 1.40. *There exist irreducible 4-manifolds $X^{\mathbb{Z}}_{6,11}$ and $X^{\mathbb{Z}_p}_{5,10}$ with $e = 17$ and $\sigma = -5$ that are homeomorphic (respectively) to $6\mathbb{CP}\#11\overline{\mathbb{CP}}^2\#S^1 \times S^3$ and $5\mathbb{CP}\#10\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$.*

Proof. Consider the manifold B of the telescoping triple from Theorem 7 in [2]. It contains a genus 2 symplectic surface F of square zero and a geometrically dual symplectic torus H_1 with square -1. One produces a genus 3 symplectic surface $F_3 \subset B$ of square 1 by symplectically resolving $F \cup H_1$.

We get rid of the its self-intersection by the standard procedure. Blow up B at a point on F_3 to obtain $\tilde{B}$ and consider the proper transform $\tilde{F_3}$ of F. So we have a square zero symplectic surface of genus 3 $\tilde{F_3} \subset \tilde{B} = B\#\overline{\mathbb{CP}}^2$.

Li's theorem tells us that $\tilde{F_3}$ intersects the exceptional sphere in $\tilde{B}$. Therefore, we can find a nullhomotopy for its meridian through the sphere and obtain an isomorphism

$$\pi_1(\tilde{B} - \tilde{F}) \to \pi_1(B).$$

Rename $\tilde{B} = A$ and $\tilde{F_3} = F_3$. Lemma 10 in [2] says that $\pi_1(A - (F_3 \cup T_1 \cup T_2)) = \mathbb{Z}t_1 \oplus \mathbb{Z}t_2$.

Now produce a genus 3 surface inside $Y = T^2 \times F_2$ by taking the union of the geometrically dual symplectic surfaces $T^2 \times \{p\}$ and $\{q\} \times F_2$ and symplectically resolving it. Note that the inclusion induces a surjective homomorphism $\pi_1(F_3) \to \pi_1(Y)$.

We obtain a symplectic genus 3 surface F_3 with self-intersection 2. Get rid of the self-interection as it was done before to obtain $\tilde{F_3} \subset \tilde{Y} = T^2 \times F_2 \# 2\overline{\mathbb{CP}}^2$.

Once again the meridian of $\tilde{F_3}$ is nullhomotopic. Consider the symplectic sum

$$S = A \#_{F_3 = \tilde{F_3}} \tilde{Y}.$$

By the lemma constructed in the telescoping triples section, we know that the generators $a_1, b_1, a_2, b_2, a_3, b_3$ of $\pi_1(F_3)$ are taken to 1 except for $b_2 \mapsto t_2$ and $b_3 \mapsto t_1$. One can use the Lagrangian push-offs suggested in the same lemma and apply the surgeries $(T_2, m_{T_2}, +1)$ and and $(T_1, l_{T_1}, +1/p)$ to produce a manifold with $\pi_1 = \mathbb{Z}_p$. One would leave the last one out to obtain $\pi_1 = \mathbb{Z}$.

Usher's theorem says that S is a minimal manifold and its characteristic numbers can be computed to be $e(S) = 17$ and $\sigma(S) = -5$. The homeomorphism is settled by either Hambleton-Kreck's theorem or by Hambleton-Teichner. Rename S accordingly. $\square$

This settles the point $(c_1^2, \chi_h) = (19, 3)$. One can go ahead and play with the building blocks in the previous process to address the points $(17, 3)$ and $(15, 3)$.

Proposition 1.41. *There exist irreducible 4-manifolds $X_{6,13}^{\mathbb{Z}}$ and $X_{5,11}^{\mathbb{Z}_p}$ with $e = 19$ and $\sigma = -7$ that are homeomorphic (respectively) to $6\mathbb{CP}\#13\overline{\mathbb{CP}}^2\#S^1 \times S^3$ and $5\mathbb{CP}\#11\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$.*

There exist irreducible 4-manifolds $X_{6,15}^{\mathbb{Z}}$ and $X_{5,14}^{\mathbb{Z}_p}$ with $e = 21$ and $\sigma = -9$ that are homeomorphic (respectively) to $6\mathbb{CP}\#11\overline{\mathbb{CP}}^2\#S^1 \times S^3$ and $5\mathbb{CP}\#10\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$.

For the manifolds in the first part of the proposition, consider the product of two tori $Z = T^2 \times T^2$ and build a genus 3 symplectic surfaces as follows. Take three distinct points $p_1, p_2, p_3 \in T^2$ to indicate the three symplectic surfaces $T^2 \times \{p_1\}$, $T^2 \times \{p_2\}$and $\{p_3\} \times T^2$. By symplectically resolving their union, one obtains a genus 3 symplectic surface $F_3' \subset Z$ of square 4. The homomorphism $\pi_1(F_3') \to \pi_1(Z)$ induced by inclusion is surjective. One proceeds to blow up Z at four points along F_3' to obtain a surface with trivial normal bundle. The proper transform $\tilde{F}_3' \subset \tilde{Z} = Z\#4\overline{\mathbb{CP}}^2$ is such a surface.

For the manifolds in the second part of the previous proposition, consider $Z = T^2 \times S^2$ and choose three different points $p_1, p_2, p_3 \in S^2$ and $\{q\} \in T^2$. We could use them to point out the four symplectic surfaces $T^2 \times \{p_i\}$ $(i = 1, 2, , 3)$ and $\{q\} \in S^2$. Consider their union and symplectically resolve it to obtain a genus 3 symplectic surface of square 6 $F_3' \subset Z$. Once again, the homomorphism induced by inclusion $\pi_1(F_3') \to \pi_1(Z)$ is surjective. Blow up Z at six points along F_3' and consider the proper transform $\tilde{F}_3' \subset \tilde{Z} = Z\#6\overline{\mathbb{CP}}^2$, which is now the genus 3 symplectic surface with trivial normal bundle needed to build the symplectic sum.

Corollary 1.42. *The above manifolds have the ∞-property.*

1.5 Region

1.5.1 Main Region

In this section we address the question of the existence of an irreducible symplectic 4-manifold and infinitely many pairwise non-diffeomorphic non-symplectic 4-manifolds having finite cyclic fundamental group that realize the coordinates:

$$(e, \sigma) \text{ when } 2e + 3\sigma \geq 0,\ e + \sigma =\equiv 0 \pmod 4 \text{ and } \sigma \leq -1.$$

In other terms, we wish to construct irreducible manifolds with finite cyclic fundamental group realizing all pairs of integers

$$(c_1^2, \chi_h) \text{ when } 0 \leq c_1^2 \leq 8\chi_h - 1.$$

The plan of attack to establish that these manifolds have the ∞-property is to generalize the main result of [2] (Theorem B and Theorem 22); this settles the region with signature at most -2. Then, we fill in the gaps by generalizing the results contained in [5] to extend the region up to signature at most -1.

Note that under the chosen coordinates, a 4-manifold with $c_1^2 = 8\chi + k$ has signature k; therefore, the line $c_1^2 = 8\chi - 1$ corresponds to manifolds with $\sigma = -2$.

Theorem 1.43. *For any pair* (c, χ) *of non-negative integers satisfying*

$$0 \leq c \leq 8\chi - 2$$

there exists a minimal symplectic 4-manifold with finite cyclic fundamental group $Y = X_{2\chi-1,10\chi-c-1}$ *with odd intersection form and*

$$c = c_1^2(Y) \text{ and } \chi = \chi_h(Y).$$

Hence Y *is homeomorphic but not diffeomorphic to*

$$(2\chi - 1)\mathbb{CP}^2 \# (10\chi - c - 1)\overline{\mathbb{CP}}^2 \# \widetilde{L(p,1) \times} S^1.$$

Proof. The beginning of the proof consists of manufacturing the manifolds that realize all the pairs. This task is divided in two with respect to the parity of c. Let us start by considering c to be even. Set $(m, n) = (1/2c, \chi)$.
By the quoted lemma, we have integers b, c, d, g, and k so that

$$m = d + 2c + 3b + 4g \text{ and } n = b + c + d + g + k \text{ and } g > 0 \text{ implies } b \geq 1.$$

The pairs are realized via symplectic sums where the raw materials are the manifolds B, B_g, D, F from the telescoping triples and the manifold $E'(k)$ built in [2]. The relation between the arithmetic setting on the characteristic numbers and the number of manifolds needed for the correct mix is

1. b copies of B if $g = 0$ and $b - 1$ copies of B_g when $g \geq 1$,
2. c copies of D, and
3. d copies of F.

Each one of the manifolds B, D and F belongs to a telescoping triple, thus they contain two essential Lagrangian tori. We will chain them together along these tori via symplectic sums to create a symplectic manifold Z, which will be minimal by Usher's theorem. Notice that Proposition 3 of [2] assures us that at each step of the process, the result is a telescoping triple.

If $g = 0$, then

$$Z := B\#_s \cdots B\#_s D\#_s \cdots D\#_s D\#_s F\#_s \cdots \#_s F.$$

The notation chosen in [2] indicates the symplectic sum along the appropriate tori. This might involve that the symplectic forms on the manifolds need to be perturbed so that a Lagrangian tori becomes symplectic. This is possible since the building blocks are all members of a telescoping triple. The unused Lagrangian tori will be relabelled T_1 and T_2 and we construct indeed a telescoping triple (Z, T_1, T_2).

The characteristic numbers of Z are computed to be

$$e(Z) = be(B) + ce(D) + de(F) = 6b + 8c + 10d. \text{ and } \sigma(B) = -2(b + 2c + 3d).$$

If $g \geq 1$, we take

$$Z := B_g\#_s B\#_s \cdots B\#_s D\#_s \cdots D\#_s D\#_s F\#_s \cdots \#_s F$$

with characteristic numbers

$$e(Z) = 4g + 6b + 8c + 10d \text{ and } \sigma(Z) = -2(b + 2c + 3d).$$

At this point we point out that the building blocks B_g, D, and F all contain a surface of odd square which is disjoint from the Lagrangian tori used to perform the symplectic sum and the following surgeries on Z. Therefore, all the manifolds coming out of performing either symplectic sums with Z and a manifold with odd intersection form and/or surgeries on Z will have odd intersection forms.

We carry on with the process of realizing the the given pairs with irreducible manifolds with the desired fundamental group for. We divide the enterprise by cases.

Case $k = 0$: Apply $(T_2, m_{T_2}, +1/p)$ Luttinger surgery to Z to obtain an intermediate manifold Z_0 with $\pi_1(Z_0) = \mathbb{Z}_p \oplus \mathbb{Z}$. Then apply $(T_1, l_{T_1}, +n/1)$ torus surgery to kill the $\mathbb{Z}$ factor on the fundamental group and produce an infinite family $\{Y_n\}$ whose minimal members all have finite cyclic fundamental group and only X_1 has a symplectic structure.

Case: $k \geq 1$ and one of b, c, d is positive. Take the symplectic sum

$$S := Z\#_{T_1=T} E'(k)$$

of the manifold of the telescoping triple (Z, T_1, T_2) with the symplectic manifold obtained from the elliptic surface of odd intersection form by applying log transformations $E'(k) := E(k)_{2,3}$ along T_1 and a generic fiber of $E'(k)$. We claim that $\pi_1(S) = \mathbb{Z}$. Since $\pi_1(E'(k)) = 1 = \pi_1(E'(k) - T)$ and Z is part of a telescoping triple, the Seifert-van Kampen theorem shows that the symplectic sum only killed one generator of $\pi_1(Z - T_1)$, thus the fundamental group of S is infinite cyclic. We can then apply a $+1/p$ Luttinger surgery to obtain our desired symplectic minimal manifold.

Case: $k \geq 1$ and all b, c, d are zero. Consider the symplectic sum S of the symplectic manifold $E'(k)$ and T^4 along a symplectic torus of self-intersection zero. This produces a minimal symplectic manifold with $\pi_1(S) = \mathbb{Z} \oplus \mathbb{Z}$ and the same characteristic numbers since $e(T^4) = 0 = \sigma(T^4)$. Notice that the four Lagrangian tori in T^4 can be pushed off and remain Lagrangian within the standard Weinstein neighborhoods while they lie in the complement of some small tubular neighborhoods of the two symplectic tori. One can now apply the usual procedure to obtain the desired fundamental group.

At this point, we would also like to mention that minimal elliptic surfaces $E(n, 0)_{p,q}$ with $1 \leq p \leq q$ for which $\pi_1 = \mathbb{Z}_{gcd(p,q)}$ have already been constructed (see Theorem 8.3.12 in [29]).

This concludes part one of the proof.

Suppose that c is odd and consider the region $7 \leq c \leq 8\chi - 11$. Let us reparametrize the region by setting $(c', \chi') = (c - 7, \chi - 2)$. Now the region looks like $0 \leq c' \leq 8\chi' - 2$ and c' is even. Consider the manifold Z of the telescoping triple constructed to realize the pair (c', χ'). Perturb the symplectic form of Z so that T_1 becomes symplectic while T_2 stays Lagrangian. This is possible since the Lagrangian tori are linearly independent in $H_2(Z; \mathbb{R})$ (cf. [28]).

Consider the simply connected, minimal, symplectic 4-manifold $S_{1,1}$ (Lemma 5.5 in [28]) with $e = 23$ and $\sigma = -15$. It contains a symplectic torus F_1 with $\pi_1(S_{1,1} - F_1) = 1$. Construct the symplectic sum S of Z with $S_{1,1}$ along T_1 and F_1. Just like above, one concludes that $\pi_1(S) = \mathbb{Z}$. We apply $(T_2, l_{T_2}, +1/p)$ Luttinger surgery on S to produce a minimal symplectic manifold $X_{c,\chi}$ with $\pi_1 = \mathbb{Z}_p$. Since $c_1^2(S_{1,1}) = 1$ and $\chi_h(S_{1,1}) = 2$, we have that $X_{c,\chi}$ realizes the following coordinates in the geography plane:

$$(c_1^2, \chi_h) = (c, \chi).$$

Let us work now on the region $7 \leq c \leq 8\chi - 11$ while still assuming c to be odd. The process is analogous to the previous paragraph, with only a small change in the ingredients of the construction. Now we re-parametrize by $(c', \chi') = (c - 7, \chi - 2)$; so one has $0 \leq c' \leq 8\chi' - 2$ and c' even. Consider the manifold Z constructed for the corresponding pair (c', χ').

Now consider the simply connected, minimal, symplectic 4-manifold $X^1_{3,12}$ built by B.D. Park in [52]. It has $e = 17$, $\sigma = -9$ and contains a symplectic torus $T_{2,4}$ with simply connected complement. Take the symplectic sum

$$S := Z \#_{T_1 = T_{2,4}} X^1_{3,12}$$

along $T_{2,4}$ and T_1 (and NOT T_2 like it was done in [2]). By using Seifert-van Kampen theorem we conclude that $\pi_1(S) = \mathbb{Z}$. By applying $(T_2, l_{T_2}, +1/p)$ we obtain a minimal symplectic 4-manifold with $\pi_1 = \mathbb{Z}_p$ that realizes the pair (c, χ). This last assertion is true since $c_1^2(X^1_{3,12}) = 7$ and $\chi(X^1_{3,12}) = 2$.

In order to realize all pairs (c, χ) with c odd and within the region $21 \leq c \leq 8\chi - 5$, one proceeds as above but instead of gluing in $X^1_{3,12}$, one uses the manifold $P_{5,8}$ constructed in [2]. It has $\pi_1 = \mathbb{Z}$, $e = 14$ and $\sigma = -3$, or $c_1^2 = 21$ and $\chi_h = 3$.

The region $21 \leq c = 8\chi - 3$ is expressed by the pairs $(c, \chi) = (5 + 8k, 1 + k)$ for any $k \geq 2$ and it was already covered using telescoping triples in a previous section.

Concerning the homeomorphism types of the constructed manifolds, we mention the following. The manifolds constructed in all these regions have odd intersection forms and we know their b_2^+ and b_2^{-1} numbers. For the lines in the plane corresponding to odd signatures and those that are not multiples of 16, one concludes immediately that they all have type

I and that the homeomorphism type is as claimed.

The region corresponding to the manifolds $(2\chi-1)\mathbb{CP}^2\#2\chi\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$ and $2\chi\mathbb{CP}^2\#(2+1)\chi\overline{\mathbb{CP}}^2\#S^1\times S^3$ was filled in the previous section. To conclude the proof, one needs to apply the proper homeomorphism criteria.

□

The procedure of the proof leaves out several points of the geography plane. We point them out now and sketch how they are filled.

Remark 5.

- $(c_1^2,\chi)=(1,1)$ *corresponding to* $X^{\mathbb{Z}}_{2,9}$ *and* $X^{\mathbb{Z}_p}_{1,8}$, *comes out of the telescoping triples.*
- $(c_1^2,\chi)=(3,1)$ *corresponding to* $X^{\mathbb{Z}}_{2,7}$ *and* $X^{\mathbb{Z}_p}_{1,6}$, *was built in the previous section as an example.*
- $(c_1^2,\chi)=(5,1)$ *corresponding to* $X^{\mathbb{Z}}_{2,5}$ *and* $X^{\mathbb{Z}_p}_{1,4}$, *comes out of the telescoping triples.*
- $(c_1^2,\chi)=(1,2)$ *corresponding to* $X^{\mathbb{Z}}_{4,19}$ *and* $X^{\mathbb{Z}_p}_{3,18}$;

  these manifolds come out of the symplectic sum of the manifold $S_{1,1}$ *(Lemma 5.5 in [Go]) constructed by Gompf and a copy of* $T^2\times\Sigma_2$ *(see [2] for the fundamental group calculations). One then surgers the symplectic sum accordingly.*

  To fill in the next point, one can build and surger the symplectic sum of the minimal symplectic 4-manifold homeomorphic to $3\mathbb{CP}^2\#12\overline{\mathbb{CP}}^2$ *containing a torus of self-intersection 0 and simply connected complement built by B.D. Park in [50].*
- $(c_1^2,\chi)=(7,2)$ *corresponding to* $X^{\mathbb{Z}}_{4,13}$ *and* $X^{\mathbb{Z}_p}_{3,12}$,

  The following three points were filled in in the previous section.
- $(c_1^2,\chi)=(15,3)$ *corresponding to* $X^{\mathbb{Z}}_{6,15}$ *and* $X^{\mathbb{Z}_p}_{5,14}$,
- $(c_1^2,\chi)=(17,3)$ *corresponding to* $X^{\mathbb{Z}}_{6,13}$ *and* $X^{\mathbb{Z}_p}_{5,12}$,

- $(c_1^2, \chi) = (19, 3)$ *corresponding to* $X^{\mathbb{Z}}_{6,11}$ *and* $X^{\mathbb{Z}_p}_{5,10}$.

1.5.2 Signature Greater or Equal than -1

As done in [2], one can use an idea of Stipsicz (cf. [59]) to fill in the following regions for $\sigma = 0$ and $\sigma = -1$.

Theorem 1.44. *For all the integers $k \geq 45$, there exists a minimal symplectic 4-manifold $X_{2k+1,2k+1}$ with Euler characteristic $4k+4$, signature $\sigma = 0$, and $\pi_1 = \mathbb{Z}_p$.*
For all the integers $q \geq 49$, there exists a minimal symplectic 4-manifold $X_{2q-1,2q}$ with Euler characteristic $4q+1$, signature $\sigma = -1$, and $\pi_1 = \mathbb{Z}_p$.

This result fills in the points of the form $(c_1^2, \chi_h) = (8k+8, k+1)$ for $k \geq 45$ and $(8q-1, q)$ for $q \geq 49$. The prototype manifolds for these guys are (accordingly):

$$(2k+1)\mathbb{CP}^2\#(2k+1)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times} S^1 \text{ and}$$
$$(2q-1)\mathbb{CP}^2\#(2q)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times} S^1.$$

Proof. Consider the telescoping triple (B, T_1, T_2). The manifold B contains a symplectic surface F of genus 2 and trivial normal bundle and a geometrically dual surface G of genus 2 and trivial normal bundle as well. The union $F \cup G$ is disjoint from the Lagrangian tori $T_1 \cup T_2$. Perform $+1$ Luttinger surgery on T_1 along l_{T_1} to kill t_2. Let R be the resulting minimal symplectic manifold. Proceed to perturb the symplectic form on R so that T_2 becomes symplectic. Concerning the fundamental group, we have $\pi_1(R - T_2) = \pi_1(R) = \mathbb{Z}t_1$ and the map induced by inclusion $\pi_1(T_2) \to \pi_1(R)$ is surjective.

Consider the symplectic sum Y of the irreducible symplectic manifold homeomorphic to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ with $T^2 \times \Sigma_2$ along a genus 2 surface. The manifold Y has fundamental group $\mathbb{Z} \oplus \mathbb{Z}$ and contains the Lagrangian tori T_1, T_2, T_3, T_4 (see Theorem 18, [11]); it has $e = 10$ and $\sigma = -2$. One obtains a symplectic manifold with infinite cyclic fundamental group by applying -1 Luttinger surgery on T_1 along m_1. One can then apply $-1/p$ Luttinger surgery on T_2 along m_2 to obtain a manifold with finite cyclic fundamental group of order p. Denote by X the minimal symplectic manifold with cyclic fundamental group. There are two symplectic tori T_3 and T_4 left unused. They both have trivial normal bundles and trivial meridians in $X - (T_3 \cup T_4)$ so by a correct choice of gluing map for the symplectic sum, the embedding $T_3 \to X$ is chosen so that $\pi_1(T_3) \to \pi_1(X)$ maps one generator to the identity and the other to the generator of $\pi_1(X)$.

We build the symplectic sum

$$Q = X\#_{T_3=T_2}R.$$

Notice that the surfaces F and G persist in Q as symplectic surfaces of square zero and are geometrically dual. The fundamental group of Q is cyclic, it has a single generator and the relation it inherits from $\pi_1(X)$. The characteristic numbers can be computed to be $e(Q) = 16$ and $\sigma(Q) = -4$; it follows from them that Q is not a rational nor a ruled surface. The symplectic torus T_4 coming from the X piece has the quality that its meridian is trivial. So, the inclusion $Q - T_4 \subset Q$ induces an isomorphism on fundamental groups.

One can go through the same procedure using the telescoping triple $\tilde{B}$ of the lemma located in the fourth section of this paper instead of B and build $\tilde{B}\#_{T_2=T_3}X$. This amounts to describing how the genus 18 surface and self-intersection zero is obtained out of B. Since its construction was described previously, we skip it and carry on with the proof. Denote the result of symplectic summing $\tilde{B}$ and $X$by $\tilde{Q}$. By Li's theorem, every exceptional sphere in $\tilde{Q}$ intersects F_{18} and $\pi_1(\tilde{Q} - F_{18}) = \pi_1(\tilde{Q}) = \mathbb{Z} \oplus \mathbb{Z}$.

Let us consider the Lefschetz fibration $H \to K$ over a surface K of genus 2 constructed in [59] (Lemma 2.1). The characteristic numbers of the fibration are $e = 75$ and $\sigma = 25$. It has a symplectic section of square -1 and the fibers are genus 16 surfaces. This fibration will be used as a building block. To argue that it is minimal, we notice that H is an algebraic surface; the BMY inequality (see [29] for details) implies that it is holomorphically minimal. By a result of Hamilton and Kotschick (see [37]) it is minimal from a symplectic point of view as well. Since it lies on the BMY lie, H is not rational nor ruled.

We proceed to construct a genus 18 surface of self-intersection zero from the fiber and section of this Lefschetz fibration. Consider the union of a fiber and section. We have a surface of genus and with double points. By resolving symplectically, we obtain a genus 18 symplectic surface of square 1 $\Sigma' \subset H$. The exact sequence

$$\pi_1(\Sigma') \to \pi_1(H) \to \pi_1(K)$$

implies that the homomorphism $\pi_1(\Sigma') \to \pi_1(H)$ is surjective.

Now let us get rid of the $[\Sigma'] = 1$ point. Blow-up H once along Σ' and consider the proper transform $\tilde{\Sigma}' \subset \tilde{H} = H\#\overline{\mathbb{CP}}^2$. Since H was neither taional nor ruled, Li's theorem implies that every exceptional sphere in $\tilde{H}$ intersects $\tilde{\Sigma}'$. Because the meridian of the surface intersects the exceptional sphere, the necessary nullhomotopy can be built and we have that $\pi_1(\tilde{H} - \tilde{\Sigma}') \to \pi_1(\tilde{H})$ is an isomorphism and $\phi : \pi_1(\tilde{\Sigma}') \to \pi_1(\tilde{H})$ is surjective.

Consider the symplectic sum

$$S = \tilde{Q}\#_{\tilde{\Sigma}=\tilde{\Sigma}'}\tilde{H}.$$

The surjectivity of ϕ implies that the map $\pi_1(\tilde{Q}) \to \pi_1(S)$ is surjective too. We need to establish that this last homomorphism is actually an isomorphism. For this we observe the following: Let b_i be a generator for $\pi_1(\tilde{H})$. If we consider the fiber of this element under ϕ^{-1} and compose it with the map $\pi_1(F_{18}) \to \pi_1(S)$ as was indicated in the lemma, we see that x_i is not trivial only in the cases when the inverse image gets mapped either to b_{18}. Therefore $\pi_1(S)$ is cyclic.

The characteristic numbers are $e(S) = 176$ and $\sigma(S) = 4$, i.e., $c_1^2(S) = 364$ and $\chi_h(S) = 45$. Out of these numbers one can conclude (in the abscence of 2-torsion, as usual), that these manifolds have odd intersection forms. Another way of noticing this fact is to observe that the manifolds used in the construction have a torus of self-intersection -1. Furthermore, notice that the torus T_4 has not been used yet and the map $\pi_1(T_4) \to \pi_1(S)$ induced by inclusion is trivial.

Now we apply Theorem 23 of [2] and its extension in [5] to produce the minimal symplectic 4-manifolds with cyclic fundamental group and odd intersection form

$$X^{\mathbb{Z}_p}_{89+2\chi,85+10\chi+c} \text{ and } X^{\mathbb{Z}}_{90+2\chi,86+10\chi+c}$$

with characteristic numbers $c_1^2 = 364 + c$ and $\chi_h = 45 + \chi$ for any (c, χ) in the region $0 \leq c \leq 8\chi - 1$ when c is even. To be able to appreciate better the zero signature quality of the manifolds produced, substitute $c = 8\chi - 4$ for any $\chi \geq 1$:

$$X^{\mathbb{Z}_p}_{89+2\chi,89+2\chi} \text{ and } X^{\mathbb{Z}}_{90+2\chi,90+2\chi}.$$

To produce minimal symplectic 4-manifolds with signature -1 we proceed as follows. Apply Luttinger surgery on B to kill one $\mathbb{Z}$-factor of the fundamental group. Call the resulting manifold $\hat{B}_1$ $(\pi_1(\hat{B}_1) = \mathbb{Z})$. Build the symplectic sum

$$W = \hat{B}_1 \#_{T_1=T} P_{1+2k,4+2k}$$

with the manifold $P_{1+2k,4+2k}$ of Remark 1 in [2]. The homomorphism $\pi_1(T) \to \pi_1(P_{1+2k,4+2k})$ is surjective, $\pi_1(P_{1+2k,4+2k} - T) \to \pi_1(P_{1+2k,4+2k})$ is an isomorphism and $\pi_1(T) \to \pi_1(\hat{B}_1)$ has image a cyclic summand. The gluing map in the symplectic sum W can be chosen in such manner that the map $\pi_1(T_2) \to \pi_1(W) = \mathbb{Z}$ sends one generator to the identity and the other to the generator of $\pi_1(W) = \mathbb{Z}$.

Construct now the symplectic sum $Z = W \#_{T_2=T_4} S$, where S is the manifold constructed above. Then Z has cyclic fundamental group and by renaming Z accordingly to the fundamental group we produce

$$X^{\mathbb{Z}_p}_{93+2k,94+2k} \text{ and } X^{\mathbb{Z}}_{94+2k,95+2k}.$$

Their characteristic numbers are $e = 189 + 4k$ and $\sigma = -1$ for any $k \geq 2$.

The results of Hambleton-Kreck and Hambleton-Teichner settle the homeomorphism type of these manifolds.

$\square$

1.5.3 Non-negative Signature

The last result in the previous section fills in a big region for manifolds with zero signature. In this section, we proceed to fill in regions of the plane that correspond to non-spin manifolds with both zero and positive signature. We address both infinite cyclic and finite cyclic fundamental groups in every result. Our main sources to do so are the results in [59], [9], [49], and [6].

Let us start by using Proposition 8 in [9] to fill in the following regions.

Proposition 1.45. *Let $n \geq 2$. There exists a symplectic minimal 4-manifold with cyclic fundamental group whose characteristic numbers can be chosen from the following three choices:*

- $e = 75n^2 + 256n + 130$ *and* $\sigma = 25n^2 - 68n - 78$; $(c_1^2, \chi_h) = (225n^2 + 298n + 26, 25n^2 + 94n + 13)$,
- $e = 75n^2 + 256n + 134$ *and* $\sigma = 25n^2 - 68n - 78$ $(c_1^2, \chi_h) = (225n^2 + 298n + 30, 25n^2 + 94n + 14)$, *or*
- $e = 75n^2 + 256n + 136$ *and* $\sigma = 25n^2 - 68n - 80$ $(c_1^2, \chi_h) = (225n^2 + 298n + 32, 25n^2 + 94n + 14)$.

Proof. The manifold $W(n)$ constructed in [9] has a symplectic torus T_2 with trivial normal bundle and $\pi_1(W(n) - T) = 1$. We build the symplectic sum of $W(n)$ and a manifold from Proposition 50 above along the corresponding tori. The possible characteristic numbers come from the three choices given in Proposition 35.

□

From this proposition one concludes that the manifolds

- $(50n^2+94n+26)\mathbb{CP}^2\#(25n^2+162n+104)\overline{\mathbb{CP}}^2\#S^1\times S^3$, $(50n^2+94n+25)\mathbb{CP}^2\#(25n^2+162n+103)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$; and

- $(50n^2+94n+28)\mathbb{CP}^2\#(25n^2+162n+104+q)\overline{\mathbb{CP}}^2\#S^1\times S^3$, $(50n^2+94n+27)\mathbb{CP}^2\#(25n^2+162n+103+q)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$ for $q\in\{2,4\}$

have the ∞-property.

Now we combine the building blocks of given in our proposition of Section 3 above and Proposition 2.1 in [49] to obtain the following result.

Proposition 1.46. *For each odd integer $m\geq 1$ and $10\leq k\leq 18$, there exists an irreducible symplectic 4-manifold Y with cyclic fundamental group whose characteristic numbers can be chosen from the following options:*

1. $\chi(Y)=25m^2+31m+5$ *and* $c_1^2(Y)=225m^2+248m+35-k$;
2. $\chi(Y)=25m^2+31m+6$ *and* $c_1^2(Y)=225m^2+248m+43-k$;
3. $\chi(Y)=25m^2+31m+6$ *and* $c_1^2(Y)=225m^2+248m+41-k$;
4. $\chi(Y)=25m^2+31m+7$ *and* $c_1^2(Y)=225m^2+248m+49-k$;
5. $\chi(Y)=25m^2+31m+8$ *and* $c_1^2(Y)=225m^2+248m+57-k$.

Moreover, the manifolds with the first three choices of coordinates contain a symplectic genus 2 surface Σ of self-intersection zero; the manifolds from the last two choices contain a symplectic torus T of self-intersection zero and $\pi_1(Y-\Sigma)=\pi_1(Y)=\pi_1(Y-T)$.

The characteristic numbers of Proposition 2.1 [49] are:

$$e=74m^2+124m+25+k,$$
$$\sigma=25m^2-5-k.$$

The following manifolds have the ∞-property and the symplectic member of each corresponding infinite family contains a symplectic genus 2 surface of self-intersection 0.

- $(50m^2+62m+10)\mathbb{CP}^2\#(25m^2+62m+15+k)\overline{\mathbb{CP}}^2\#S^1\times S^3$; $(50m^2+62m+9)\mathbb{CP}^2\#(25m^2+62m+14+k)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$.

- $(50m^2+62m+12)\mathbb{CP}^2\#(25m^2+62m+17+k)\overline{\mathbb{CP}}^2\#S^1\times S^3$; $(50m^2+62m+11)\mathbb{CP}^2\#(25m^2+62m+16+k)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$.

- $(50m^2+62m+12)\mathbb{CP}^2\#(25m^2+62m+19+k)\overline{\mathbb{CP}}^2\#S^1\times S^3$; $(50m^2+62m+11)\mathbb{CP}^2\#(25m^2+62m+18+k)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$.

The symplectic member of the infinite families with the following topological prototypes contains a symplectic torus of self-intersection 0.

- $(50m^2+62m+14)\mathbb{CP}^2\#(25m^2+62m+21+k)\overline{\mathbb{CP}}^2\#S^1\times S^3$; $(50m^2+62m+13)\mathbb{CP}^2\#(25m^2+62m+20+k)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$.

- $(50m^2+62m+16)\mathbb{CP}^2\#(25m^2+62m+23+k)\overline{\mathbb{CP}}^2\#S^1\times S^3$; $(50m^2+62m+15)\mathbb{CP}^2\#(25m^2+62m+22+k)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$.

Theorem 1.47. *There exists a closed, minimal, symplectic 4-manifold X with cyclic $\pi_1(X)$ for the following choices of characteristic numbers:*

- $e = 94$ *and* $\sigma = 2$*;* $(c_1^2, \chi_h) = (194, 24)$,
- $e = 98$ *and* $\sigma = 2$ $(c_1^2, \chi_h) = (202, 25)$,
- $e = 100$ *and* $\sigma = 0$*;* $(c_1^2, \chi_h) = (200, 25)$,
- $e = 100$ *and* $\sigma = 4$*;* $(c_1^2, \chi_h) = (212, 26)$,
- $e = 104$ *and* $\sigma = 4$*;* $(c_1^2, \chi_h) = (220, 27)$, *or*
- $e = 106$ *and* $\sigma = 2$*;* $(c_1^2, \chi_h) = (218, 27)$.

Proof. Theorem 4.1 in [5] builds a manifold M with $e(M) = 94$, $\sigma(M) = 2$ which contains a symplectic torus T with self-intersection and $\pi_1(M - T) = 1$. Build the symplectic sum of this manifolds with one of the manifolds from proposition of Section 3.3 above. The different choices of characteristic numbers correspond to using the manifold N from Theorem 4.2 in [5] instead and different choices of manifolds that can be involved in the symplectic sum. To check that the different symplectic sums have the other claimed properties is straightforward.

□

Remark 6. *All the manifolds above contain a symplectic torus of self-intersection zero. Going through the proofs of Theorems 4.1 and 4.2 in [5], one sees that the manifolds M and N are obtained by building the symplectic sum of $Y_n(1)$ and the total space X_2 of a genus 7 fibration over a surface of genus 2 ($n = 7$ to produce M and $n = 9$ to produce N). In both cases, the building block $Y_n(1)$ has plenty of such tori. For example, $Y_9(1)$ contains 14 pairs of geometrically dual Lagrangian tori that are all disjoint from the genus 9 surface used to build the symplectic sum $N = Y_9(1)\#_{\Sigma_9}X_2$. One could go ahead and use one of these 32 Lagrangian tori to obtain the claimed torus T by perturbing the symplectic form on N so that T becomes symplectic.*

Moreover, the homomorphism $\pi_1(T) \to \pi_1(S)$ (where S is one of the manifolds from the last two theorems) factorizes through the respective $\pi_1(Y_n(1)')$. In particular, the images of the generators of $\pi_1(T)$ are trivial.

One concludes that the following manifolds enjoy the ∞ -property:

- $48\mathbb{CP}^2\#46\overline{\mathbb{CP}}^2\#S^1 \times S^3$; $47\mathbb{CP}^2\#45\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)} \times S^1$;

- $49\mathbb{CP}^2\#47\overline{\mathbb{CP}}^2\#S^1 \times S^3$; $48\mathbb{CP}^2\#46\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)} \times S^1$;

- $50\mathbb{CP}^2\#50\overline{\mathbb{CP}}^2\#S^1 \times S^3$; $49\mathbb{CP}^2\#49\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)} \times S^1$;

- $52\mathbb{CP}^2\#48\overline{\mathbb{CP}}^2\#S^1 \times S^3$; $41\mathbb{CP}^2\#47\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$;

- $54\mathbb{CP}^2\#50\overline{\mathbb{CP}}^2\#S^1 \times S^3$; $53\mathbb{CP}^2\#49\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$;

- $54\mathbb{CP}^2\#52\overline{\mathbb{CP}}^2\#S^1 \times S^3$; $53\mathbb{CP}^2\#51\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$.

We proceed to use these manifolds to fill in regions of the plane corresponding to non-negative signature. Theorem 4.1 in [5] is used to produce 4-manifolds with signature $\sigma = 0, 1, 2$.

Proposition 1.48. *Let m be an odd positive integer. If $m \geq 49$, then*

- $m\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$,

- $(m+1)\mathbb{CP}^2\#(m+1)\overline{\mathbb{CP}}^2\#S^1 \times S^3$ *(with characteristic numbers* $(e, \sigma) = (2m+2, 0)$ *and* $(c_1^2, \chi_h) = (4m+4, 1/2(m+1))$,

- $m\mathbb{CP}^2\#(m-1)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$,

- $(m+1)\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2\#S^1 \times S^3$ *(with characteristic numbers* $(e, \sigma) = (2m+1, 1)$ *and* $(c_1^2, \chi_h) = (4m+5, 1/2(m+1))$,

have the ∞-property. If $m \geq 47$, then

- $m\mathbb{CP}^2\#(m-2)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times S^1}$ *and*

- $(m+1)\mathbb{CP}^2\#(m-1)\overline{\mathbb{CP}}^2\#S^1 \times S^3$ *(with characteristic numbers* $(e, \sigma) = (2m, 0)$ *and* $(c_1^2, \chi_h) = (4m+6, 1/2(m+1))$.

have the ∞-property.

Proof. We already know that $48\mathbb{CP}^2\#46\overline{\mathbb{CP}}^2\#S^1\times S^3$ and $47\mathbb{CP}^2\#45\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$ have the ∞-property. We apply Akhmedov-Park's result ([5]) to them. Since there is no margin for confusion, we deal with the infinite cyclic case and the finite cyclic case together. Let X be either one of these two manifolds: $\chi_h(X)=24$ and $c_1^2=194$. By Theorem 1 in [6], there exists a minimal symplectic 4-manifold Y with $\chi_h(Y)=\chi+24$ and $c+194$. By Hambleton-Teichner's criteria in the infinite cyclic fundamental group and by Hambleton-Kreck's criteria in the finite cyclic fundamental group case, such Y is homeomorphic to

- if $\pi_1(Y)=\mathbb{Z}$: $(2\chi+48)\mathbb{CP}^2\#(10\chi-c+46)\overline{\mathbb{CP}}^2\#S^1\times S^3$ or
- if $\pi_1(X)=\mathbb{Z}_p$: $(2\chi+47)\mathbb{CP}^2\#(10\chi-c+45)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$.

By setting the constants from Akhmedov-Park's theorem to be $c=8\chi-s$, where $s\in\{0,1,2\}$, we produce an irreducible symplectic 4-manifold Y homeomorphic to

- if $\pi_1(Y)=\mathbb{Z}$: $(2\chi+48)\mathbb{CP}^2\#(2\chi+46+s)\overline{\mathbb{CP}}^2\#S^1\times S^3$ or
- if $\pi_1(X)=\mathbb{Z}_p$: $(2\chi+47)\mathbb{CP}^2\#(2\chi+45+s)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$.

A torus surgery on a nullhomologous torus in Y as explained in [22] produces infinite families of pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds homeomorphic $(2\chi+48)\mathbb{CP}^2\#(2\chi+46+s)\overline{\mathbb{CP}}^2\#S^1\times S^3$ if $\pi_1(Y)=\mathbb{Z}$ or homeomorphic to $(2\chi+47)\mathbb{CP}^2\#(2\chi+45+s)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$ if $\pi_1(Y)=\mathbb{Z}_p$.

□

Similarly, a result concerning a large region of non-spin 4-manifolds with cyclic fundamental group and signature $\sigma=3,4$ is obtained.

Proposition 1.49. *Let m be an odd positive integer. If $m\geq 53$, then*

- $m\mathbb{CP}^2\#(m-3)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$ *and*

- $(m+1)\mathbb{CP}^2\#(m-2)\overline{\mathbb{CP}}^2\#S^1\times S^3$ *(with characteristic numbers $(e,\sigma)=(2m-1,3)$ and $(c_1^2,\chi_h)=(4m+7,1/2(m+1))$*

have the ∞-property. If $m\geq 51$, then

- $m\mathbb{CP}^2\#(m-4)\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$ *and*

- $(m+1)\mathbb{CP}^2\#(m-3)\overline{\mathbb{CP}}^2\#S^1\times S^3$ *(with characteristic numbers* $(e,\sigma)=(2m-2,0)$ *and* $(c_1^2,\chi_h)=(4m+8,1/2(m+1))$

have the ∞*-property.*

Chapter 2

Abelian non-cyclic π_1

In this chapter we construct several irreducible 4-manifolds, both small and arbitrarily large, with abelian non-cyclic fundamental group. The manufacturing procedure allows us to fill in numerous points in the geography plane of symplectic manifolds with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_p$ and $\mathbb{Z}_q \oplus \mathbb{Z}_p$ ($gcd(p, q) \neq 1$). We then study the botany of these points for $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p$.

2.1 Introduction

The main results are:

Theorem 2.1. *Let G be either $\mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}_p$, or $\mathbb{Z}_q \oplus \mathbb{Z}_p$. Let $n \geq 1$ and $m \geq 1$. For each of the following pairs of integers*

1. $(c, \chi) = (7n, n)$,
2. $(c, \chi) = (5n, n)$,
3. $(c, \chi) = (4n, n)$,
4. $(c, \chi) = (2n, n)$,
5. $(c, \chi) = ((6 + 8g)n, (1 + g)n)$ *(for $g \geq 0$)*,
6. $(c, \chi) = (7n + (6 + 8g)m, n + (1 + g)m)$,
7. $(c, \chi) = (7n + 5m, n + m)$,
8. $(c, \chi) = (7n + 4m, n + m)$,

9. $(c, \chi) = (7n + 2m, n + m)$,

10. $(c, \chi) = ((6 + 8g)n + 5m, (1 + g)n + m)$ *(for $g \geq 0$)*,

11. $(c, \chi) = ((6 + 8g)n + 4m, (1 + g)n + m)$ *(for $g \geq 0$)*,

12. $(c, \chi) = ((6 + 8g)n + 2m, (1 + g)n + m)$ *(for $g \geq 0$)*,

13. $(c, \chi) = (5n + 4m, n + m)$,

14. $(c, \chi) = (5n + 2m, n + m)$, *and*

15. $(c, \chi) = (4n + 2m, n + m)$,

there exists a symplectic irreducible 4-manifold X with

$$\pi_1(X) = G \text{ and } (c_1^2(X), \chi_h(X)) = (c, \chi).$$

Proposition 2.2. *Fix $\pi_1(X) = \mathbb{Z}_p \oplus \mathbb{Z}_p$, where p is a prime number greater than two. Let (c, χ) be any pair of integers given in Theorem 2.1 such that $n + m \geq 2$. There exists an infinite family $\{X_n\}$ of homeomorphic, pairwise non-diffeomorphic irreducible smooth non-symplectic 4-manifolds realizing the coordinates (c, χ).*

The characteristic numbers are given in terms of $\chi_h = 1/4(e + \sigma)$ and $c_1^2 = 2e + 3\sigma$, where e is the Euler characteristic of the manifold X and σ its signature.

The geography problem for abelian fundamental groups of small rank has already been previously studied with great success. In R.E. Gompf's gorgeous paper [28] where the symplectic sum operation was introduced, infinitely many minimal symplectic 4-manifolds with $b_2^+ > 1$ were constructed. R.E. Gompf also constructed a new family of symplectic spin 4-manifolds with any prescribed fundamental group. In [12], [13], and [14], more and smaller symplectic manifolds were constructed.

Other construction techniques have also been implemented. For the group $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_p$, examples with big Euler characteristic were constructed using genus 2 Lefschetz fibrations in [47] and [55]. Results studying the symplectic geography for prescribed fundamental groups appeared in [14] and [12]. Concerning the botany, J. Park in [49] constructed infinitely many smooth structures on big 4-manifolds with finitely generated fundamental group.

The addition of Luttinger surgery (cf. [45], [8]) into the manufacturing procedure has provided clean constructions to study rather effectively the geography of simply connected 4-manifolds (cf. [12], [2], [5]). On the botany part, the technique of using a nullhomologous torus as a dial in order to change the smooth structure developed in [23] and [22] has proven to be a succesful tool to study the lack of smooth uniqueness. In this paper, we apply these efforts to manifolds with the three given fundamental groups.

Our results provide manifolds with both $12\chi - c$ small and arbitrarily large. Most of the points filled in by Theorem 2.1 were not yet considered elsewhere. For example, the point $(7, 1)$ corresponds to the smallest manifold built up to now. A blunt overlap occurs for the points $(6 + 8g, 1 + g)$, $(5, 1)$, and $(4, 1)$, which have been filled in already by constructions given in [12] and [13]; we are using their constructions to build larger manifolds, thus filling in considerably many more points. The existence of at least two smooth structures on complex surfaces with finite non-cyclic fundamental groups was first studied in [33]. Proposition 2 takes advantage of the recent techniques and offers a myriad of new exotic irreducible 4-manifolds with finite abelian, yet non-cyclic fundamental group hosting infinitely many smooth structures; it includes the smallest manifold with such π_1 known to possess this quality.

The assumption $gcd(p, q) \neq 1$ serves the sole purpose of emphasizing that the results that appear here are disjoint from the cyclic case studied in [65] (Chapter 1). We feel the results presented here deserve their own space and they should not be buried in a long paper for several reasons. Amongst them is the employment of the homeomorphism criteria for finite groups of odd order (cf. [33]) given in Section 2.6.3.

The structure of the chapter is as follows. The geography is addressed first; Section 2.2 starts by describing the ingredients we will use to build the manifolds of Theorem 2.1. The manufacturing procedure starts later on in this section. The results that allow us to conclude irreducibility are presented in Section 2.3. The fourth section takes care of the fundamental group calculations. The fifth section gathers up our efforts into the proof of

Theorem 1. The last part of the paper goes into the botany, where Section 2.6 takes on the existence of the exotic smooth structures claimed in Proposition 2.2.

2.2 Raw Materials

The following definition was introduced in [2].

Definition 2.3. An ordered triple (X, T_1, T_2) consisting of a symplectic 4-manifold X and two disjointly embedded Lagrangian tori T_1 and T_2 is called a *telescoping triple* if:

1. The tori T_1 and T_2 span a 2-dimensional subspace of $H_2(X; \mathbb{R})$.

2. $\pi_1(X) \cong \mathbb{Z}^2$ and the inclusion induces an isomorphism $\pi_1(X - (T_1 \cup T_2)) \to \pi_1(X)$. In particular, the meridians of the tori are trivial in $\pi_1(X - (T_1 \cup T_2))$.

3. The image of the homomorphism induced by the corresponding inclusion $\pi_1(T_1) \to \pi_1(X)$ is a summand $\mathbb{Z} \subset \pi_1(X)$.

4. The homomorphism induced by inclusion $\pi_1(T_2) \to \pi_1(X)$ is an isomorphism.

The telescoping triple is called *minimal* if X itself is minimal. Notice the importance of the order of the tori. The meridians μ_{T_1}, μ_{T_2} in $\pi_1(X - (T_1 \cup T_2))$ are trivial and the relevant fundamental groups are abelian. The push-off of an oriented loop $\gamma \subset T_i$ into $X - (T_1 \cup T_2)$ with respect to any (Lagrangian) framing of the normal bundle of T_i represents a well-defined element of $\pi_1(X - (T_1 \cup T_2))$ which is independent of the choices of framing and base-point.

The first condition assures us that the Lagrangian tori T_1 and T_2 are linearly independent in $H_2(X; \mathbb{R})$. This allows for the symplectic form on X to be slightly perturbed so that one of the T_i remains Lagrangian while the other becomes symplectic. The symplectic form can also be perturbed in such way that both tori become symplectic. If we were to consider a symplectic surface F in X disjoint from T_1 and T_2, the perturbed symplectic form can be chosen so that F remains symplectic.

Removing a surface from a 4-manifold usually introduces new generators into the fundamental group of the resulting manifold. The second condition indicates that the meridians are nullhomotopic in the complement and, thus, the fundamental group of the manifold and the fundamental group of the complement of the tori in the manifold coincide.

Out of two telescoping triples, one is able to produce another telescoping triple as follows. If both X and X' are symplectic manifolds, then the symplectic sum along the symplectic tori $X\#_{T_2,T_1'}X'$ has a symplectic structure ([28]). If both X and X' are minimal, then the resulting telescoping triple is minimal too (by Usher's theorem cf. [69]).

Proposition 2.4. *(cf. [2]). Let (X, T_1, T_2) and (X', T_1', T_2') be two telescoping triples. Then for an appropriate gluing map the triple*

$$(X\#_{T_2,T_1'}X', T_1, T_2')$$

is again a telescoping triple. The Euler characteristic and the signature of $X\#_{T_2,T_1'}X'$ are given by $e(X) + e(X')$ and $\sigma(X) + \sigma(X')$.

We refer the reader to Theorems 20 and 13 and to Proposition 12 in [12] for the proof and for more details. The building blocks we will use are gathered together in the following theorem.

Theorem 2.5.

- *There exists a minimal telescoping triple (A, T_1, T_2) with $e(A) = 5$, $\sigma(A) = -1$.*
- *For each $g \geq 0$, there exists a minimal telescoping triple (B_g, T_1, T_2) satisfying $e(B_g) = 6 + 4g$, $\sigma(B_g) = -2$.*
- *There exists a minimal telescoping triple (C, T_1, T_2) with $e(C) = 7$, $\sigma(C) = -3$.*
- *There exists a minimal telescoping triple (D, T_1, T_2) with $e(D) = 8$, $\sigma(D) = -4$.*
- *There exists a minimal telescoping triple (F, T_1, T_2) with $e(F) = 10$, $\sigma(F) = -6$.*

The manifolds B_g, D, and F were already built in [2]. They are taken out of the constructions given in [12] by the following mechanism. The main goal of [12] is to construct simply connected 4-manifolds by applying Luttinger surgery to symplectic sums. If one is careful about the fundamental group calculations, the procedure can be interrupted by NOT performing two surgeries, and thus obtain a symplectic manifold with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}$. Furthermore, the skipped surgeries have to be chosen carefully so that the unused Lagrangian tori comply with the requirements and the pieces can then be aligned into a telescoping triple.

To finish the proof of Theorem 2.5, we construct (A, T_1, T_2) and (C, T_1, T_2) by applying this mechanism to the constructions in [5]. This is done in the following two lemmas, where we follow the notation of [5].

Lemma 2.6. *There exists a telescoping triple* (A, T_1, T_2) *with* $e(C) = 5$ *and* $\sigma(C) = -1$.

Proof. This telescoping triple is obtained out of the construction of an exotic irreducible symplectic $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$ given in [5]. The two surgeries to be skipped are $(a_2' \times c', c', +1/p)$ and $(b_1' \times c'', b_1', -1)$ (the notation is explained in [23]). Rename the corresponding tori T_1 and T_2. This procedure manufactures a minimal symplectic manifold A. Notice that the tori are linearly independent in $H_2(A; \mathbb{R})$. We need to check that such a manifold has indeed $\pi_1 = \mathbb{Z}^2$ and that it contains the required tori.

Let us begin with the fundamental group calculations. By combining the relations coming from the surgeries $(a_1' \times c', a_1', -1)$ and $(a_2'' \times d', d', +1)$ that were performed on the $\Sigma_2 \times T^2$ block (see [5] for details) we have $\alpha_1 = a_1 = [b_1^{-1}, d^{-1}] = [b_1^{-1}, [b_2, c^{-1}]^{-1}] = [b_1^{-1}, [c^{-1}, b_2]] = 1$. One concludes this commutator is trivial by observing how the generators are identified during the gluing and using the commutators $[\alpha_2, \alpha_4] = 1$ and $[b_1, c] = 1$. Substituting this in the relations coming from the surgeries applied to the building block $T^4 \# \overline{\mathbb{CP}}^2$, we obtain $\alpha_3 = a_2 = 1$ and $\alpha_4 = b_2 = 1$. By looking at the relations from the other building block we see $d = 1$. Note that the meridians of the surfaces along which the gluing is performed are trivial. Thus only two commuting generators survive in the group presentation.

We check that the meridian of the first torus is $\mu_{T_1} = [d^{-1}, b_2^{-1}] = 1$ and its Lagrangian push-offs are $m_{T_1} = c$ and $l_{T_1} = a_2 = 1$. For the torus T_2 one sees $\mu_{T_2} = [a_1^{-1}, d] = 1$ and its Lagrangian push-offs are $m_{T_2} = c$ and $l_{T_2} = b_1$. So, $\pi_1(A - (T_1 \cup T_2))$ is generated by the commuting elements b_1 and c. By the Mayer-Vietoris sequence we see $H_1(A - (T_1 \cup T_2)) = \mathbb{Z}^2$. Thus $\pi_1(A - (T_1 \cup T_2) = \mathbb{Z}b_1 \oplus \mathbb{Z}c$. We conclude (A, T_1, T_2) is a telescoping triple.

$\square$

Lemma 2.7. *There exists a telescoping triple (C, T_1, T_2) with $e(C) = 7$ and $\sigma(C) = -3$.*

Proof. We follow the construction of an exotic irreducible symplectic $\mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$ given in [5]. The surgeries $(\alpha_2' \times \alpha_3'', \alpha_2', -1)$ in the $T^4\#2\overline{\mathbb{CP}}^2$ block and $(\alpha_2'' \times \alpha_4', \alpha_4', -1)$ in the $T^4\#\overline{\mathbb{CP}}^2$ block will NOT be performed. Call these tori T_2 and T_1 respectively and the resulting manifold C. Notice that they are linearly independent in $H_2(C; \mathbb{R})$.

We apply $(\alpha_1' \times \alpha_3', \alpha_1', -1)$ on the $T^4\#2\overline{\mathbb{CP}}^2$. This introduces the relation $\alpha_1 = [\alpha_2^{-1}, \alpha_4^{-1}]$. Using the commutator $[\alpha_2, \alpha_4] = 1$, one sees $\alpha_1 = 1$. The relation $\alpha_3 = [\alpha_1^{-1}, \alpha_4^{-1}]$ obtained by applying a Luttinger surgery on the $T^4\#\overline{\mathbb{CP}}^2$ building block implies $\alpha_3 = 1$. The surfaces of genus 2 along which the symplectic sum is performed have trivial meridians.

The meridian of T_1 is $\mu_{T_1} = [a_1^{-1}, \alpha_4] = 1$ and its Lagrangian push-offs are $m_{T_1} = \alpha_2$ and $l_{T_1} = \alpha_3 = 1$. The meridian of T_2 is given by $\mu_{T_2} = [\alpha_1, \alpha_3^{-1}] = 1$ and its Lagrangian push-offs are $m_{T_2} = \alpha_4$ and $l_{T_2} = \alpha_2$. We have that $\pi_1(C - (T_1 \cup T_2))$ is generated by the commuting elements α_2 and α_4. The Mayer-Vietoris sequence computes $H_1(C - (T_1 \cup T_2)) = \mathbb{Z}^2$, thus $\pi_1(C - (T_1 \cup T_2)) = \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_4$. Thus, (C, T_1, T_2) is a telescoping triple. $\square$

Remark 7. *One is able to realize the point $(c_1^2, \chi_h) = (3, 1)$ for the fundamental groups $\pi_1 = \mathbb{Z}^2$ and $\pi_1 = \mathbb{Z}$ during the manufacturing process of an exotic irreducible symplectic $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$. Consider the symplectic sum of $T^4\#\overline{\mathbb{CP}}^2$ and $T^2 \times S^2\#4\overline{\mathbb{CP}}^2$ along a genus 2 surface given in [5]. The resulting minimal symplectic 4-manifold has a fundamental group with the following presentation*

$$< \alpha_1, \alpha_2, \alpha_3 | [\alpha_1, \alpha_2] = 1, [\alpha_2, \alpha_3] = 1, \alpha_1^{-1} = \alpha_3^2 > \cong \mathbb{Z} \oplus \mathbb{Z}.$$

If we apply the surgery $(\alpha_2'' \times \alpha_4', \alpha_4', -1)$, *the relation* $\alpha_4 = [\alpha_1, \alpha_3^{-1}]$ *is introduced to the fundamental group presentation and we obtain a manifold with fundamental group*

$$\pi_1 =< \alpha_1, \alpha_3 | \alpha_1^{-1} = \alpha_3^2 >\cong \mathbb{Z}.$$

If we apply the surgery $(\alpha_2' \times \alpha_3', \alpha_3', -1)$, *the relation* $\alpha_3 = [\alpha_1^{-1}, \alpha_4^{-1}]$ *is introduced to the fundamental group presentation and we obtain a manifold with fundamental group* $\pi_1 =< \alpha_2 >= \mathbb{Z}$.

One can go on and build more telescoping triples out of these five by using Proposition 4. We proceed to do so now. Let us start by setting some useful notation. Let (X, T_1, T_2) be a telescoping triple. We will denote by $X_n := \#n(X)$ the manifold obtained by building the symplectic sum (cf. [28]) of n copies of X along the proper tori.

Proposition 2.8. *For each* $n \geq 1$ *and* $m \geq 1$, *the following minimal telescoping triples with the given characteristic numbers exist:*

1. (A_n, T_1, T_2) *satisfying* $e(A_n) = 5n$ *and* $\sigma(A_n) = -n$.
2. (C_n, T_1, T_2) *satisfying* $e(C_n) = 7n$ *and* $\sigma(C_n) = -3n$.
3. (D_n, T_1, T_2) *satisfying* $e(D_n) = 8n$ *and* $\sigma(D_n) = -4n$.
4. (F_n, T_1, T_2) *satisfying* $e(F_n) = 10n$ *and* $\sigma(F_n) = -6n$.
5. $(\#n(B_g), T_1, T_2)$ *satisfying* $e(\#n(B_g)) = (6 + 4g)n$ *and* $\sigma(\#n(B_g)) = -2n$.
6. $(A_n\#m(B_g), T_1, T_2)$ *satisfying* $e(A_n\#m(B_g)) = 5n + (6 + 4g)m$ *and* $\sigma(A_n\#m(B_g)) = -n - 2m$.
7. $(A_n\#C_m, T_1, T_2)$ *satisfying* $e(A_n\#C_m) = 5n + 7m$ *and* $\sigma(A_n\#C_m) = -n - 3m$.
8. $(A_n\#D_m, T_1, T_2)$ *satisfying* $e(A_n\#D_m) = 5n + 8m$ *and* $\sigma(A_n\#D_m) = -n - 4m$.
9. $(A_n\#F_m, T_1, T_2)$ *satisfying* $e(A_n\#F_m) = 5n + 10m$ *and* $\sigma(A_n\#F_m) = -n - 6m$.
10. $(\#n(B_g)\#C_m, T_1, T_2)$ *satisfying* $e(\#n(B_g)\#C_m) = (6+4g)n+7m$ *and* $\sigma(\#n(B_g)\#C_m) = -2n - 3m$.

11. $(\#n(B_g)\#D_m, T_1, T_2)$ *satisfying* $e(\#n(B_g)\#D_m) = (6{+}4g)n{+}8m$ *and* $\sigma(n(B_g)\#D_m) = -2n - 4m$.

12. $(\#n(B_g)\#F_m, T_1, T_2)$ *satisfying* $e(\#n(B_g)\#F_m) = (6{+}4g)n{+}10m$ *and* $\sigma(n(B_g)\#F_m) = -2n - 6m$.

13. $(C_n\#D_m, T_1, T_2)$ *satisfying* $e(C_n\#D_m) = 7n + 8m$ *and* $\sigma(C_n\#D_m) = -3n - 4m$.

14. $(C_n\#F_m, T_1, T_2)$ *satisfying* $e(C_n\#F_m) = 7n + 10m$ *and* $\sigma(C_n\#F_m) = -3n - 6m$.

15. $(D_n\#F_m, T_1, T_2)$ *satisfying* $e(D_n\#F_m) = 8n + 10m$ *and* $\sigma(D_n\#F_m) = -4n - 6m$.

The claim about minimality is proved in the next section.

2.3 Minimality and Irreducibility

The following result allows us to conclude the irreducibility of the constructed minimal 4-manifolds.

Theorem 2.9. *(Hamilton and Kotschick, [37]). Minimal symplectic 4-manifolds with residually finite fundamental groups are irreducible.*

Finite groups and free groups are well-known examples of residually finite groups. Since the direct products of residually finite groups are residually finite groups themselves, the previous result implies that all we need to worry about is producing minimal manifolds in order to conclude on their irreducibility. This endeavor follows from Usher's theorem.

Theorem 2.10. *(Usher, [69]). Let* $X = Y\#_{\Sigma\equiv\Sigma}Y'$ *be the symplectic sum where the surfaces have genus greater than zero.*

1. *If either* $Y - \Sigma$ *or* $Y' - \Sigma'$ *contains an embedded symplectic sphere of square -1, then* X *is not minimal.*

2. *If one of the summands, say Y for definiteness, admits the structure of an S^2-bundle over a surface of genus g such that Σ is a section of this S^2-bundle, then X is minimal if and only if Y' is minimal.*

3. *In all other cases, X is minimal.*

This theorem implies that the manifolds of Proposition 8 are minimal.

2.4 Luttinger Surgery and its Effects on π_1

Let T be a Lagrangian torus inside a symplectic 4-manifold M. *Luttinger surgery* (cf. [45], [8]) is the surgical procedure of taking out a tubular neighborhood of the torus nbh(T) in M and gluing it back in, in such way that the resulting manifold admits a symplectic structure. The symplectic form is unchanged away from a neighborhood of T. We proceed to give an overview of the process before we get into the fundamental group calculations.

The Darboux-Weinstein theorem (cf. [15]) implies the existence of a parametrization of a tubular neighborhood $T \times D^2 \to nbh(T) \subset M$ such that the image of $T \times \{d\}$ is Lagrangian for all $d \in D^2$. Let $d \in D - \{0\}$. The parametrization of the tubular neighborhood provides us with a particular type of push-off $F_d : T \times \{d\} \subset M - T$ called the *Lagrangian push-off* or *Lagrangian framing.* Let $\gamma \subset T$ be an embedded curve. Its image $F_d(\gamma)$ under the Lagrangian push-off is called the *Lagrangian push-off* of γ. These curves are used to parametrize the Luttinger surgery.

A *meridian* of T is a curve isotopic to $\{t\} \times \partial D^2 \subset \partial(nbd(T))$ and it is denoted by μ_t. Consider two embedded curves in T which intersect transversally in one point and consider their Lagrangian push-offs m_T and l_T. The group $H_1(\partial(nbd(T)) = H_1(T^3)$ is generated by μ_T, m_t, and l_T. We take advantage of the commutativity of $\pi_1(T^3)$ and choose a base-point t on $\partial(nbh(T))$, so that we can refer unambiguously to $\mu_T, m_T, l_T \in \pi_1(\partial(nbd(T)), t)$.

Under this notation, a general torus surgery is the process of removing a tubular neighborhood of T in M and gluing it back in such a way that the curve representing $\mu_T^k m_T^p l_T^q$

bounds a disk for some triple of integers k, p, and q. In order to obtain a symplectic manifold after the surgery, we need to set $k = \pm 1$ (cf. [12]).

When the base-point x of M is chosen off the boundary of the tubular neighborhood of T, the based loops μ_T, m_t and l_T are to be joined by the same path in $M - T$. By doing so, these curves define elements of $\pi_1(M - T, x)$. The 4-manifold Y resulting from Luttinger surgery on M has fundamental group

$$\pi_1(M - T)/N(\mu_T m_T^p l_T^q)$$

where $N(\mu_T m_T^p l_T^q)$ denotes the normal subgroup generated by $\mu_T m_T^p l_T^q$.

We proceed now with the fundamental group calculations needed to prove Theorem 1. To do so, we plug into the previous general picture the information we have for the telescoping triples. Let (X, T_1, T_2) be a telescoping triple. The fundamental group of X has the presentation $< t_1, t_2 | [t_1, t_2] = 1 >$. Let us apply $+1/p$ Luttinger surgery on T_1 along l_{T_1} and call Y_1 the resulting manifold. Since the meridian μ_{T_1} is trivial we have

$$\pi_1(Y_1) = \pi_1(X - T)/N(\mu_T m_{T_1}^0 l_{T_1}^p) = \mathbb{Z} \oplus \mathbb{Z}/N(1 \cdot 1 \cdot l_{T_1}^p).$$

Thus, $\pi_1(Y_1) =< t_1, t_2 | [t_1, t_2] = 1, t_2^p = 1 >$.

Let us apply now $+1/q$ Luttinger surgery on T_2 along m_{T_2} and call the resulting manifold Y_2 the resulting manifold. Since the meridian μ_{T_2} is trivial we have

$$\pi_1(Y_2) = \mathbb{Z} \oplus \mathbb{Z}_p/N(1 \cdot m_{T_2}^q \cdot 1).$$

Thus, $\pi_1(Y_2) =< t_1, t_2 | [t_1, t_2] = 1, t_1^q = 1 = t_2^p >$.

The reader might have already noticed the symmetry of these calculations.

Proposition 2.11. *Let (X, T_1, T_2) be a minimal telescoping triple. Let l_{T_1} be a Lagrangian push-off of a curve on T_1 and m_{T_2} the Lagrangian push-off of a curve on T_2 so that l_{T_1} and m_{T_2} generate $\pi_1(X)$.*

- *The minimal symplectic 4-manifold obtained by performing either $+1/p$ Luttinger surgery on T_1 along l_{T_1} or $+1/p$ surgery on T_2 along m_{T_2} has fundamental group isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_p$.*

- *The minimal symplectic 4-manifold obtained by performing $+1/p$ Luttinger surgery on T_1 along l_{T_1} and $+1/q$ surgery on T_2 along m_{T_2} has fundamental group isomorphic to $\mathbb{Z}_q \oplus \mathbb{Z}_p$.*

The proof is omitted. It is based on a repeated use of Lemma 2 in [12] and Usher's theorem (cf. [69]). The reader is suggested to look at the proofs of Theorems 8, 10 and 13 of [12] for a blueprint to the proof.

2.5 Proof of Theorem 2.1

Proof. The possible choices for characteristic numbers in Theorem 1 are in a one-to-one correspondence with the telescoping triples of Proposition 2.8. The enumeration indicates that, in order to produce the manifold in Theorem 2.1 with one of the choices for characteristic numbers claimed in item # (k), we start with the telescoping triple of item # (k) in Proposition 2.8 ($k \in \{1, 2, 3, 4, 5, \ldots, 14, 15\}$). Let $S := (X, T_1, T_2)$ be the chosen minimal telescoping triple. The manifolds of Theorem 2.1 are produced by applying Luttinger surgery to S according to the choice of characteristic numbers. By Proposition 2.11 we know that out of S one produces two symplectic manifolds: Y_1 with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_p$ and Y_2 with $\pi_1 = \mathbb{Z}_q \oplus \mathbb{Z}_p$. Since Luttinger surgery does not change the Euler characteristic nor the signature, the resulting manifolds Y_1 and Y_2 share the same characteristic numbers as X.

Proposition 2.11 states that Y_1 and Y_2 are minimal. By Hamilton-Kotschick's result, both of them are irreducible. The calculation of the characteristic numbers of Y_1 and Y_2 is straightforward. Since our chosen S was arbitrary, this finishes the proof.

$\square$

2.6 Exotic Smooth Structures on 4-Manifolds with Abelian Finite Non-Cyclic π_1

The purpose of this section is to put on display the exotic smooth structures for the constructed manifolds having $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p$, i.e., to prove Proposition 2.2.

2.6.1 Smooth Topological Prototype

We proceed to construct the underlying smooth manifold on which infinitely many exotic smooth structures will be displayed. Start with the product of a lens space and a circle: $L(p,1) \times S^1$. Its Euler characteristic is zero as well as its signature. Consider the map

$$L(p,1) \times S^1 \to L(p,1) \times S^1$$
$$\{pt\} \times \alpha \mapsto \{pt\} \times \alpha^p$$

We perform surgery on $L(p,1) \times S^1$: cut out the loop α^p and glue in a disc in order to kill the corresponding generator

$$L\widetilde{(p,1) \times} S^1 := L(p,1) \times S^1 - (S^1 \times D^3) \cup S^2 \times D^2.$$

The resulting manifold has zero signature and Euler characteristic two. By the Seifert-Van Kampen theorem, one concludes $\pi_1(L\widetilde{(p,1) \times} S^1) = \mathbb{Z}_p \oplus \mathbb{Z}_p$.

Since we are aiming at non-spin manifolds, our topological prototypes will have the shape

$$b_2^+ \mathbb{CP}^2 \# b_2^- \overline{\mathbb{CP}}^2 \# \text{Ł}\widetilde{(p,1) \times} S^1$$

but spin 4-manifolds with $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p$ are also built in such a straightforward manner.

2.6.2 An infinite family $\{X_n\}$

We apply now the procedure described in [23] and [22] to produce infinitely many distinct smooth structures on any of our topological prototypes. Let X_0 be the manifold obtained by applying $+1/p$ Luttinger surgery on T_2 along l_{T_2} to any of the manifolds from the telescoping triples previously constructed. Since X_0 is a minimal symplectic manifold with $b_2^+ = 2$, its Seiberg-Witten invariant is non-trivial by [62].

The infinite family $\{X_n\}$ is obtained by applying a $+n/p$ torus surgery to X_0 on T_1 along m_{T_1}. Notice that now $k = n$ according to our notation of Section 4; only the case $k = 1 = n$ produces a symplectic manifold. We take a closer look at the process to see that we comply with the hypothesis of Corollary 2 in [22].

The boundary of the tubular neighborhood of T_1 in X_0 is a 3-torus whose fundamental group is generated by the loops μ_{T_1}, m_{T_1}, and l_{T_1}. Notice that in $\pi_1(X_0 - T_1)$, the meridian is trivial $\mu_{T_1} = 1$, $m_{T_1} = x$ and $l_{T_1} = 1$, where x is a generator in $\pi_1(X_0) = \mathbb{Z}_p \oplus \mathbb{Z}x$. The manifolds in the family $\{X_n\}$ can be described as the result of applying to X_0 an n/p surgery on T_1 along m_{T_1}, and so $\mu_{T_1}^n m_{T_1} = x^p$ is killed.

Let X be the manifold obtained from $X_0 - T_1$ by gluing a thick torus $T^2 \times D^2$ in a manner that $\gamma = S^1 \times \{1\} \times \{1\}$ is sent to l_{T_1}, $\lambda = \{1\} \times S^1 \times \{1\}$ is sent to μ_{T_1}, and $\mu_X = \{(1,1)\} \times \partial D^2$ is sent to $m_{T_1}^{-p}$. If $n \neq 1$, the manifold X will not be symplectic, but in any case $\pi_1(X) = \mathbb{Z}_p \oplus \mathbb{Z}_p$. Denote by $\Lambda \subset X_0$ the core torus of the surgery.

Notice that given the identifications on the loops during the surgery, $\lambda = \mu_{T_1} = 1$, thus it is nullhomotopic in $X_0 - T_1 = X - \Lambda$; in particular, λ is nullhomologous. The torus surgery kills one generator of H_1 and two generators of H_2; Λ is a nullhomologous torus. One obtains a manifold X_n by applying n surgery on Λ along λ with $\pi_1(X_n) = \mathbb{Z}_p \oplus \mathbb{Z}_p$. The manifold X_0 can be recovered from X by applying a $0/1$ surgery on Λ along λ.

By Corollary 2 in [22], we produce an infinite family $\{X_n\}$ of pairwise non-diffeomorphic 4-manifolds. These manifolds will have the same cohomology ring as the corresponding topological prototype. Thus we have the following lemma.

Lemma 2.12. *There exists an infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds with $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p$ sharing the same Euler characteristic, signature, and type as a given topological prototype constructed in the previous subsection.*

2.6.3 Homeomorphism Criteria

Now we need to see that the manifolds produced share indeed the same underlying topological prototype. Ian Hambleton and Matthias Kreck proved the needed homeomorphism criteria in [33] (Theorem B). They showed that topological 4-manifolds with odd order fundamental group and large Euler characteristic are classified up to homeomorphism by explicit invariants.

The precise statement of their result includes a lower bound for the Euler characteristic in terms of an integer number $d(\pi)$, which depends on the fundamental group of the manifold. We proceed to explain the notation employed.

Let $\pi_1 = \pi$ be a finite group and let $d(\pi)$ be the minimal $\mathbb{Z}$-rank for the abelian group $\Omega^3\mathbb{Z} \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}$. One minimizes over all representatives of $\Omega^3\mathbb{Z}$, the kernel of a projective resolution of length three (cf. [34]) of $\mathbb{Z}$ over the group ring $\mathbb{Z}[\pi]$. In particular, $\Omega^3\mathbb{Z}$ is a submodule of $\pi_2(X)$. The minimal representative is given by $\pi_2(K)$, where K is a 2-complex with the given π_1.

The result we will use in order to conclude on the homeomorphism type of our manifolds is the following:

Theorem 2.13. *(Hambleton-Kreck, cf [33]). Let M be a closed oriented manifold of dimension four, and let $\pi_1(X) = \pi$ be a finite group of odd order. When $\omega_2(\tilde{X}) = 0$ (resp. $\omega_2(\tilde{X}) \neq 0$), assume that*

$$b_2(X) - |\sigma(X)| > 2d(\pi),$$

(resp. $> 2d(\pi)+2$). Then M is classified up to homeomorphism by the signature, Euler characteristic, type, Kirby-Siebenmann invariant, and fundamental class in $H_4(\pi, \mathbb{Z})/Out(\pi)$.

Notice that since $p \geq 3$ is assumed to be a prime number, π_1 has odd order and no 2-torsion. Therefore, the type of the manifold is indicated by the parity of its intersection form over $\mathbb{Z}$. All of our manufactured manifolds are non-spin; since they are smooth, the

Kirby-Siebenmann invariant vanishes.

For the finite groups $\pi = \mathbb{Z}_p \oplus \mathbb{Z}_p$, we claim

$$d(\pi) = 1.$$

We are indebted to Matthias Kreck for explaining to us the argument [43]. Assume $\pi = \pi_1$ is a finite group and let K be a 2-complex with fundamental group π_1. The minimal Euler characteristic of a K is given by $d(\pi) + 1$. We claim $d(\pi) = 1$.

Consider the map from K to the Eilenberg-MacLane space $K(\pi, 1)$ which induces an isomorphism on π_1. Then the induced map on $H_2(K; \mathbb{Z}_p)$ is surjective. Thus, the Euler characteristic of K is greater or equal than 3 - 2 + 1. This implies $d(\pi)$ is greater or equal than 1.

To conclude now $d(\pi) = 1$, consider the standard presentation of $Z_p \oplus \mathbb{Z}_p$ given by

$$< x, y | x^p = 1, y^p = 1, [x, y] = 1 >.$$

The 2-complex realizing this presentation has Euler characteristic $2 = d(\pi) + 1$. Therefore, $d(\pi) = 1$ as claimed.

In order to conclude on the homeomorphism type of our manufactured manifolds, we only need to know the numerical invariants b_2^+ and b_2^- which need to satisfy

$$b_2(X) - |\sigma(X)| > 4.$$

2.6.4 Proof of Proposition 2.2

The proof of Proposition 2.2 is now clear if one rewrites it in the following form. First, in order for our manifolds to satisfy the inequality in the previous paragraph, the integers n and m from Theorem 1 need to be as follows. If $m = 0$ (similar for $n = 0$) or it does not appear in the statement, then $n \geq 2$ ($m \geq 2$). Thus we have

Proposition 2.14. *Assume $n + m \geq 2$. The manifolds*

$$b_2^+\mathbb{CP}^2\#b_2^-\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times} S^1$$

with the following coordinates admit infinitely many exotic irreducible smooth structures, only one of which is symplectic.

1. $(b_2^+, b_2^-) = (2n-1, 3n-1)$,

2. $(b_2^+, b_2^-) = (2n-1, 5n-1)$,

3. $(b_2^+, b_2^-) = (2n-1, 6n-1)$,

4. $(b_2^+, b_2^-) = (2n-1, 8n-1)$,

5. $(b_2^+, b_2^-) = ((2+2g)n-1, (4+2g)n-1)$,

6. $(b_2^+, b_2^-) = (2n+(2+2g)m-1, 3n+(4+2g)m-1)$,

7. $(b_2^+, b_2^-) = (2n+2m-1, 3n+5m-1)$,

8. $(b_2^+, b_2^-) = (2n+2m-1, 3n+6m-1)$,

9. $(b_2^+, b_2^-) = (2n+2m-1, 3n+8m-1)$,

10. $(b_2^+, b_2^-) = ((2+2g)n+2m-1, (4+2g)n+5m-1)$,

11. $(b_2^+, b_2^-) = ((2+2g)n+2m-1, (4+2g)n+6m-1)$,

12. $(b_2^+, b_2^-) = ((2+2g)n+2m-1, (4+2g)n+8m-1)$,

13. $(b_2^+, b_2^-) = (2n + 2m - 1, 5n + 6m - 1)$,

14. $(b_2^+, b_2^-) = (2n + 2m - 1, 5n + 8m - 1)$,

15. $(b_2^+, b_2^-) = (2n + 2m - 1, 6n + 8m - 1)$.

Proof. The infinite families are provided by Lemma 2.12. Choosing the topological prototype accordingly to the coordinates, by Theorem 2.13 and the discussion that follows we conclude on the homeomorphism type. Notice that the enumeration of the coordinates presented in Proposition 2.14 correspond exactly to the ones in Theorem 2.1. $\square$

Chapter 3

Spin geography and botany

In this paper we study the geography and botany of symplectic spin 4-manifolds with abelian fundamental group. Building on the constructions in [50] and [48], the techniques employed allow us to give alternative proofs and extend their results to the non-simply connected realm. New testing ground for a conjecture concerning 4-manifolds with even b_2^+ (in the spirit of [17]) is provided.

3.1 Introduction

The geography and botany of irreducible spin simply connected 4-manifolds have been succesfully studied in [19], [59], [28], [50], [49], and [48], so that most of the existence questions have been settled. The recent addition of Luttinger surgery (cf. [45], [8]) to the repertoire of symplectic constructions was extremely powerful. Not only did it allow an impressive development in our understanding of simply connected 4-manifolds ([2], [12], [5]), but also had as a natural consequence the study of the geography for other fundamental groups ([12], [65], [66]).

The progress concerning the botany has not been any less poignant. R. Fintushel and R. Stern's work on surgery on nullhomologous tori ([23], [22]) unveiled a myriad of exotic smooth structures that were previously out of reach through an elegant geometric-topological mechanism. The same authors in joint work with B.D. Park (cf. [22]) exploited a duality between Luttinger surgery and its counterpart on nullhomologous tori that enabled the hand-in-hand study of the symplectic geography and its botany used by many authors these days, this note included.

In order to put the results of this chapter into context, we give a rough outline of the current knowledge on the geography of symplectic spin 4-manifolds with $\pi_1 = 1$. In [50], B.D. Park and S. Szabó proved that every allowed homeomorphism type located in the region $0 \leq c_1^2 < 8\chi_h$ and with odd b_2^+ is realized by a simply connected spin irreducible symplectic 4-manifold (Theorem 1.1, [50]). J. Park obtained a similar yet much broader result (Theorem 1.1, [48]) which also encompassed spin symplectic simply connected 4-manifolds of zero and positive signature. In particular, he cleverly used a complex spin surface built by C. Persson, C. Peters and G. Xiao in [53] to produce an infinite number of exotic smooth structures on $(2n+1)(S^2 \times S^2)$ for a rather large number n.

Our first result concerns the geography of spin manifolds with negative signature. It provides an extension of B.D. Park and Z. Szabó 's result to non-trivial abelian fundamental groups. In the simply connected case, we also offer an alternative proof to their theorem.

Theorem 3.1. *Let $s \geq 1$ and let G be either $1, \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q$ (and assume $n \geq 2$) or $\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_p, \mathbb{Z} \oplus \mathbb{Z}$ (and $n \geq 1$). For each of the following pairs of integers*

$$(c, \chi) = (8n - 8, 2s + n - 1),$$

there exists an irreducible symplectic spin 4-manifold X with

$$\pi_1(X) = G \text{ and } (c_1^2(X), \chi_h(X)) = (c, \chi).$$

Concerning 4-manifolds with non-negative signature, by following closely J. Park's main construction in [48] one obtains the following result.

Theorem 3.2. *Let G be as above. Except for finitely many lattice points, every pair (c, χ) lying in the region $8\chi \leq c \leq 8.76\chi$ is realized by an irreducible symplectic spin 4-manifold with*

$$\pi_1(X) = G \text{ and } (c_1^2(X), \chi_h(X)) = (c, \chi).$$

Concerning their botany, we have the following two results.

Proposition 3.3. *Fix $\pi_1(X) = 1, \mathbb{Z}_p, \mathbb{Z}_q \oplus \mathbb{Z}_q$ or $\mathbb{Z}$, where q is a prime number greater than two. Let (c, χ) be any pair of integers given in Theorem 3.1 and/or in Theorem*

3.2. There exists an infinite family $\{X_n\}$ of homeomorphic, pairwise non-diffeomorphic irreducible smooth non-symplectic 4-manifolds realizing the coordinates (c, χ).

À la J. Park, for the manifolds with zero signature of Theorem 3.2 we have

Corollary 3.4. *There exists an integer N such that $\forall n \geq N$ the manifolds*

- $(2n+1)(S^2 \times S^2)\#L\widetilde{(p,1) \times} S^1$,
- $(2n+1)(S^2 \times S^2)\#L\widehat{(p,1) \times} S^1$, *and*
- $(2n)(S^2 \times S^2)\#S^1 \times S^3$

have infinitely many exotic irreducible smooth structures. Only one of these smooth structures admits a symplectic structure.

Here the piece $L\widetilde{(p,1) \times} S^1$ stands for the surgered product $L(p,1) \times S^1$ of a lens space with the circle; the surgery is performed along $\{pt\} \times \alpha$ to kill the loop corresponding to the generator of the infinite cyclic group factor so that $\pi_1 = \mathbb{Z}_p$ of the surgered manifold comes from the fundamental group of the lens space. If instead, we cut out a loop $\{pt\} \times \alpha^p$ and glue in a disc to kill the corresponding generator $(S^2 \times D^2)$, then we obtain a 4-manifold with $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p$. Such manifold is denoted by $L\widehat{(p,1) \times} S^1$.

The last contribution to be described concerns new testing ground for a conjecture about the non-existence of irreducible smooth 4-manifolds with even b_2^+.

Corollary 3.5. *(Compare with [17]) Let X be*

$$K3\#S^2 \times S^2\#S^3 \times S^1 \text{ or}$$
$$H(7k'-1)\#S^2 \times S^2\#S^3 \times S^1.$$

There exists an infinite family $\{X_n\}$ of irreducible pairwise non-diffeomorphic 4-manifolds, all of them sharing the homeomorphism type of X.

The chapter is organized as follows. Section 3.2 provides the reader with a description of the building blocks and the tools that are employed in our constructions. This section includes the two crucial lemmas for our results as well. In Section 3.3 we employ them to prove Theorem 3.1 and half of Proposition 3.3. A description of J. Park' s construction is given in the fourth section, as well as a proof of Theorem 3.2, Corollary 3.4, and the remaining part of the proof of Proposition 3.3. The last section contains new testing ground for a conjecture concerning simply connected manifolds with even b_2^+, including a myriad of new manifolds sharing certain similarities with the one built by R. Fintushel and R. Stern in [17].

3.2 Tools and Raw materials

3.2.1 Symplectic sums

In his beautiful paper [28], R.E. Gompf introduced *the symplectic sum*, a procedure to build symplectic 4-manifolds that has become essential in our understanding of symplectic 4-manifolds. The following result gathers the properties we will use.

Lemma 3.6. *(Gompf, [28]). Let X and Y be spin symplectic 4-manifolds, each containing a symplectic surface Σ_g of genus g and self-intersection 0. Then the symplectic sum $X\#_{\Sigma_g}Y$ is a spin symplectic irreducible manifold with coordinates*

$$c_1^2(X\#_{\Sigma_g}Y) = c_1^2(X) + c_1^2(Y) + 8(g-1) \text{ and}$$
$$\chi_h(X\#_{\Sigma_g}Y) = \chi_h(X) + \chi_h(Y) + (g-1).$$

The reader is reminded that a spin symplectic 4-manifold is mechanically irreducible, since its Seiberg-Witten invariant is non-trivial (cf. [63], [62]) and it can not be the blow-up of another manifold, otherwise it would not be spin.

3.2.2 Luttinger surgery and nullhomologous tori

Carving a torus out of a 4-manifold and then gluing it back in differently is a standard topological procedure to unveil exotic smooth structures. Recently this idea as been exploited succesfully in three directions. First, perform such an operation symplectically by adding Luttinger surgery to the palette of constructions of symplectic manifolds; second, use it to construct not only simply connected symplectic manifolds, but also manifolds with several fundamental groups; and last but not least, use a (nullhomologous) torus that canonically comes out of these surgeries as a dial to change the smooth structure at will. We proceed to give an overview of this well-oiled machinery. For specific details on the construction the reader is advised to consult the references given below.

Let $T \subset X$ be a torus of self-intersection zero, thus having a tubular neighborhood $N_T \cong T^2 \times D^2$. Let α and β be the generators of $\pi_1(T)$ and consider the meridian μ_T of T in X and the pushoffs S^1_α, S^1_β in $\partial N_T = T^3$; these are loops homologous in N_T to α and β respectively. The manifold obtained from X by performing a p/q - surgery on T along β is defined as

$$X_{T,\beta}(q/p) = X - N_T \cup_\phi T^2 \times D^2,$$

where the gluing map $\phi : T^2 \times \partial D^2 \to \partial(X - N_T)$ satisfies $\phi_*([\partial D^2]) = p[S^1_\beta] + q[\mu_T]$ in $H_1(\partial(X - N_T)); \mathbb{Z})$. Denote core torus $S^1 \times S^1 \times \{0\} \subset X_{T,\beta}(q/p)$ by $T_{q/p}$. The surgery reduces b_1 by one and b_2 by two. The fundamental group of the resulting manifold is given by $\pi_1(X_{T,\beta}(q/p))$.

If X is symplectic and T Lagrangian, then performing a $1/p$ surgery on the prefered Lagrangian framing of N_T results in $X_{T,\beta}(1/p)$ being symplectic (cf. [8]). Concerning the botany, the paper [22] introduced a procedure to use the nullhomologous torus $T_{q/p}$ to manufacture infinitely many exotic smooth structures starting with a manifold with non-trivial Seiberg-Witten invariant (for example, the symplectic manifold where $T_{q/p}$ was obtained from), by applying a more general $n/1$ - surgery on $T_{q/p}$ (see [22] or the discussion following Theorem 13 in [12] for more details). This manufactures an infinite family $\{X_n\}$ of pairwise non-diffeomorphic non-symplectic 4-manifolds.

If X is assumed to be spin, one can endow $X_{T,\beta}(q/p)$ with a spin structure by choosing a suitable bundle automorphism $T^2 \times D^2 \to T^2 \times D^2$ as follows. Fix a spin structure on $X - N_T$ and one on $T^2 \times D^2$. Their difference is given by an element in $H^1(T^2 \times D^2; \mathbb{Z}_2) \cong H^1(T^2; \mathbb{Z}_2)$. This element, on the other hand, can be readily seen to be the image of an appropriate bundle automorphism under the coefficient homomorphism $H^1(T^2; \mathbb{Z}) \to H^1(T^2; \mathbb{Z}_2)$. Thus, identifying two spin structures on $T^2 \times D^2$ coming from $X - N_T$ and from $T^2 \times D^2$, yields a spin structure for $X_{T,\beta}(q/p)$ itself.

We use the remaining part of the section to introduce the building blocks in our constructions.

3.2.3 Surgered T^4

This building block will allow us to manipulate the fundamental group of our constructions without adding anything to the Euler characteristic or to the signature. Let $\pi_1(T^4)$ be generated by x, y, a, b. Removing a surface from a 4-manifold would normally introduce more generators to the fundamental group of the complement. In [12], S. Baldridge and P. Kirk showed that the fundamental group of the complement of two Lagrangian tori T_1 and T_2 inside the 4-torus is generated by four elements, just like $\pi_1(T^4)$ itself.

Proposition 3.7. *(Baldridge-Kirk, cf. [12]) The fundamental group of $T^4-(T_1 \cup T_2)$ is generated by the loops x, y, a, b and the relations $[x, a] = [y, a] = 1$ hold. The meridians of the tori and the two Lagrangian pushoffs of their generators are given by the following formulae:*

$$\mu_1 = [b^{-1}, y^{-1}], m_1 = x, l_1 = a \text{ and}$$
$$\mu_2 = [x^{-1}, b], m_2 = y, l_2 = bab^{-1}.$$

As a corollary of their efforts one obtains the following lemma.

Lemma 3.8. *Let X be a simply connected spin symplectic 4-manifold containing a symplectic torus such that $\pi_1(X-T) = 1$. There exists a spin symplectic 4-manifold with Chern numbers $\chi_h(Z) = \chi_h(X)$ and $c_1^2(Z) = c_1^2(X)$. The fundamental group of Z can be chosen to be*

1. $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}$,
2. $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_q$,
3. $\pi_1 = \mathbb{Z}$

Proof. Let $T_1 \subset T^4$ be as above. Perturb the symplectic form on T^4 such that T_1 becomes symplectic while T_2 stays Lagrangian. The torus T_1 carries the generators x and b. Take the symplectic sum $Y := T^4 \#_{T_1=T} X$. Since the meridian of T in $X - T$ is trivial, the relation $[y, b] = 1$ holds in the fundamental group of this newly constructed manifold. Therefore, the symplectic sum results in a manifold Y with $\pi_1(Y) = \mathbb{Z}y \oplus \mathbb{Z}b$. We can now proceed to apply $1/q$ Luttinger surgery to T_2 to produce a manifold with $\pi = \mathbb{Z}_p \oplus \mathbb{Z}b$; for $q = 1$ we have $\pi_1 = \mathbb{Z}$ and for $q > 1$, $\pi_1 = \mathbb{Z}_q \oplus \mathbb{Z}b$.

□

3.2.4 Cohomology $(2n-3)(S^2 \times S^2)$

In [22], R. Fintushel, B.D. Park, and R. Stern built an infinite family of irreducible pairwise non-diffeomorphic spin 4-manifolds with the same integer cohomology ring as $S^2 \times S^2$. Then, A. Akhmedov and B.D. Park generalized the construction in [5], by producing an infinite family of irreducible pairwise non-diffeomorphic spin 4-manifolds $\{Y_n(m) | m = 1, 2, 3, \ldots\}$ with only one symplectic member which has the same integer cohomology ring as $(2n-3)(S^2 \times S^2)$ with $n \geq 2$. The characteristic numbers of these manifolds are $e = 4n - 4$ and $\sigma = 0$; equivalent, $\chi_h = n - 1$ and $c_1^2 = 8n - 8$.

These manifolds are constructed by applying $2n + 3$ Luttinger surgeries and one torus surgery to $\Sigma_2 \times \Sigma_n$ (the product of a genus 2 surface with a genus n surface). Let a_i, b_i, c_j and d_j ($i = 1, 2$, $j = 1, \ldots, n$) be the standard generators of $\pi_1(\Sigma_2)$ and $\pi_1(\Sigma_n)$ respectively. The following relations hold in $\pi_1(Y_n(m))$. We refer the reader to [5] for further details.

$$
\begin{gathered}
[b_1^{-1}, d_1^{-1}] = a_1, [a_1^{-1}, d_1] = b_1, [b_2^{-1}, d_2^{-1}] = a_2, [a_2^{-1}, d_2] = b_2, \\
[d_1^{-1}, b_2^{-1}] = c_1, [c_1^{-1}, b_2] = d_1, [d_2^{-1}, b_1^{-1}] = c_2, [c_2^{-1}, b_1] = d_2, \\
[a_1, c_1] = 1, [a_1, c_2] = 1, [a_1, d_2] = 1, [b_1, c_1] = 1, \\
[a_2, c_1] = 1, [a_2, c_2] = 1, [a_2, d_1] = 1, [b_2, c_2] = 1, \\
[a_1, b_1][a_2, b_2] = 1,\ [c_1, d_1][c_2, d_2] = 1,
\end{gathered}
$$

and

$$
\begin{gathered}
[a_1^{-1}, d_3^{-1}] = c_3, [a_2^{-1}, c_3^{-1}] = d_3, \cdots, [a_1^{-1}, d_n^{-1}] = c_n, [a_2^{-1}, c_n^{-1}] = d_n, \\
[b_1, c_3] = 1, [b_2, d_3] = 1, \cdots, [b_1, c_n] = 1, [b_2, d_n] = 1, \\
\textstyle\prod_{j=2}^{n} [c_j, d_j] = 1.
\end{gathered}
$$

These manifolds are our basic building block for manipulating the fundamental group. We employ them to obtain the following.

Lemma 3.9. *Let X be a simply connected spin symplectic 4-manifold containing a symplectic torus such that $\pi_1(X - T) = 1$. Then for all $n \geq 1$ there exists a spin symplectic 4-manifold with Chern numbers $\chi_h(Z) = \chi_h(X) + n - 1$ and $c_1^2(Z) = c_1^2(X) + 8n - 8$. The fundamental group of Z can be chosen to be*

1. $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}$,
2. $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_q$,
3. $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_q$,
4. $\pi_1 = \mathbb{Z}_p$,
5. $\pi_1 = \mathbb{Z}$, *or*
6. $\pi_1 = 1$.

Furthermore, Z contains a Lagrangian torus such that the inclusion induced homomorphism $\pi_1(Z - T) \to \pi_1(Z)$ is an isomorphism.

Proof. Consider the case $n = 2$. Let S be the manifold obtained by applying 5 Luttinger ± 1 -surgeries to $\Sigma_2 \times \Sigma_2$. The surgeries that are not to be performed are $(a_1' \times c_1', a_1', -1)$, $(a_2' \times c_2', a_2', -1)$, and $(a_2'' \times d_1', d_1', +1)$. Call these three tori T_1, T_2, and T_3 respectively. In $\pi_1(S)$ all the relations from $\pi_1(Y_2(1))$ hold except for $[b_1^{-1}, d_1^{-1}] = a_1$, $[b_2^{-1}, d_2^{-1}] = a_2$, and $[c_2^{-1}, b_1] = d_2$.

Build the symplectic sum of X and S along the corresponding torus in X and T_1 in S and call the resulting manifold $S_{\mathbb{Z}\oplus\mathbb{Z}}$. The meridian of T_1, $[b_1^{-1}, d_1^{-1}] = a_1$ is killed during the symplectic sum and the surviving relations show that $\pi_1(S_{\mathbb{Z}\oplus\mathbb{Z}} - T_2 \cup T_3)$ is generated by the two commuting elements a_2 and d_1. The Mayer-Vietoris sequence shows that $H_1(S_{\mathbb{Z}\oplus\mathbb{Z}} - T_2 \cup T_3); \mathbb{Z}) = \mathbb{Z}^2$, thus $\pi_1(S_{\mathbb{Z}\oplus\mathbb{Z}}) = \mathbb{Z}a_2 \oplus \mathbb{Z}d_1$. It is straight-forward to check $e(S_{\mathbb{Z}\oplus\mathbb{Z}}) = e(X) + 4$ and $\sigma(S_{\mathbb{Z}\oplus\mathbb{Z}}) = \sigma(X)$.

Notice that the geometrically dual torus T' to T_1 is contained in $S_{\mathbb{Z}\oplus\mathbb{Z}}$ and its meridian is trivial in the complement. This implies $\pi_1(S_{\mathbb{Z}\oplus\mathbb{Z}} - T') \cong \pi_1(S_{\mathbb{Z}\oplus\mathbb{Z}}) = \mathbb{Z}^2$. Thus, item (1) of the lemma has been produced.

Applying $(a_2' \times c_2', a_2', -1/q)$, aka $-1/q$ Luttinger surgery to $S_{\mathbb{Z}\oplus\mathbb{Z}}$ on T_2 along a_2' produces item (2). By applying $(a_2'' \times d_1', d_1', +1/p)$ to the resulting manifold one produces item (3) ($p > 1$) and item (4) ($p = 1$). Applying $(a_2'' \times d_1', d_1', +1)$ to $S_{\mathbb{Z}\oplus\mathbb{Z}}$ produces item (5), while item (6) on the list is produced by applying both surgeries $(a_2'' \times d_1', d_1', +1)$ and $(a_2' \times c_2', a_2', -1)$ to $S_{\mathbb{Z}\oplus\mathbb{Z}}$.

The cases $n \geq 3$ follow almost verbatim the procedure described above substituting $\Sigma_2 \times \Sigma_2$ with $\Sigma_2 \times \Sigma_n$. The details are left to the reader. We do point out that the bigger n is, the more Lagrangian tori the resulting manifold contains. For example, the surgered $\Sigma_2 \times \Sigma_5$ contains 12 Lagrangian tori while the surgered $\Sigma_2 \times \Sigma_7$ has 20 Lagrangian tori; all of them have trivial meridian. $\square$

Remark 8. *Concerning the production of an infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible smooth manifolds we have the following. Properly applying a torus surgery on a nullhomologous torus as sketched at the end of 2.2 or in [22] produces the desired family. To conclude on their homeomorphism type, one must check that these manifolds have the desired fundamental group; we already know their Chern invariants remained unchanged after the surgery. For this purpose it suffices to see that the effect such surgery has on the presentation of the fundamental groups is to replace a relation of the form $[a, b] = c^p$ by $[a, b]^n = c^p$ for a given p and n and generators a, b. Given that in the proofs of Lemma 9 and Lemma 10 we concluded that the original relation is trivial, then raising it to any power will result in a trivial relation as well. Hence, we will make no distinctions in future sections about the computations of π_1 of the infinite families.*

3.2.5 Horikawa surfaces

The complex surfaces satisfying $c_1^2 = 2\chi_h - 6$ are commonly known as Horikawa surfaces and are denoted by $H(4k-1)$. They are constructed as branched covers of the Hirzebruch surface $\mathbb{F}_{2m}$ along disconnected curves and we point out that a simply connected Horikawa surface is spin if and only if k is even. The Chern invariants of the specific manifolds we will be using, $H(8k'-1)$, are given by $(c_1^2, \chi_h) = (16k'-8, 8k'-1)$. Moreover, $H(8k'-1)$ contains an embedded Lagrangian torus which intersects a 2-sphere transversally at one point (cf. [19], [59]).

3.2.6 A spin surface of positive signature

In [53], U. Persson, C. Peters and G. Xiao constructed a simply connected spin complex surface Y of positive signature which contains a holomorphic curve Σ_g of genus g and trivial self-intersection. Furthermore, the meridian of this surface in the complement is trivial since Y also contains an embedded 2-sphere $\mathbb{CP}^1$ intersecting Σ_g transversely at a point. Its Chern invariants are approximately $\chi_h(Y) \approx 6857x^2$ and $c_1^2(Y) \approx 60068x^2$.

3.2.7 (Knot surgered) Elliptic minimal surfaces

Our last building block is also a classical element in the construction of 4-manifolds and we only remind the reader of its properties relevant to our purposes. Let $E(2s)$ denote the underlying smooth 4-manifold of the simply connected minimal elliptic surface without multiple fibers and with geometric genus $p_g = 2s-1$ (cf. [27] or Prop. 3.1.11 in [**?**]). Its Chern numbers are given by $c_1^2 = 0$ and $\chi_h = 2s$. Notice that in particular $E(2)$ is a $K3$ surface. In Section 3 and Section 4, it is easy to see where the manifold $E(2s)$ can be replaced by a knot surgered exotic version $E(2s)_K$ [19].

3.3 Negative signature

3.3.1 Examples with $\sigma = -16s$ for $s > 0$

Proposition 3.10. *Let $s \geq 1$. For $\pi = 1, \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q$ assume $n \geq 2$ and for $\pi = \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_p$ and $\mathbb{Z} \oplus \mathbb{Z}$ assume $n \geq 1$. There exists a spin irreducible symplectic manifold X satisfying $c_1^2 = 8n-8$, $\chi_h = n+2s-1$ and $\pi_1(X) = \pi$.*

Proof. The proposition follows from employing $X = E(2s)_K$ in Lemma 3.8 and Lemma 3.9. $\square$

By applying the corresponding homeomorphism criteria, we conclude that the manifolds constructed are homeomorphic to the following topological prototypes:

- $\pi = 1$: $E(2s)\#(2n-2)(S^2 \times S^2)$ (cf. [26]).

- $\pi = \mathbb{Z}_p$: $E(2s)\#(2n-2)(S^2 \times S^2)\#L\widetilde{(p,1) \times} S^1$ (cf. [34]).

- $\pi = \mathbb{Z}_q \oplus \mathbb{Z}_q$: $E(2s)\#(2n-2)(S^2 \times S^2)\#L\widehat{(p,1) \times} S^1$ (cf. [34]).

- $\pi = \mathbb{Z}$: $E(2s)\#(2n-1)(S^2 \times S^2)\#S^3 \times S^1$ (cf. [36]).

Thus, considering Remark 3.1 we have the following.

Corollary 3.11. *The manifolds*

- $E(2s)\#(2n-2)(S^2 \times S^2)$,
- $E(2s)\#(2n-2)(S^2 \times S^2)\#L\widetilde{(p,1) \times} S^1$,
- $E(2s)\#(2n-2)(S^2 \times S^2)\#L\widehat{(p,1) \times} S^1$ *and*
- $E(2s)\#(2n-1)(S^2 \times S^2)\#S^3 \times S^1$.

admit infinitely many exotic irreducible smooth structures.

3.3.2 Examples with $\sigma = -16s - 16$ for $s \geq 0$

Proposition 3.12. *Let $s \geq 0$. For $\pi = 1, \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q$ assume $b \geq 2$ and for $\pi = \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_p$ and $\mathbb{Z} \oplus \mathbb{Z}$ assume $n \geq 1$. There exists a spin irreducible symplectic manifold X satisfying $(c_1^2, \chi_h) = (8n-8, 2s+n+1)$ and $\pi_1(X) = \pi$.*

Proof. The proposition follows by using $X = K3_k \#_{T^2} E(2s)$, where $K3_k$ stands for an irreducible exotic $K3$ surfaces produced by knot surgery, in Lemma 3.8 and Lemma 3.9. □

For these manifolds, we obtained the following.

Corollary 3.13. *The manifolds*

- $E(2(s+1))\#(2n-2)(S^2 \times S^2)$,
- $E(2(s+1))\#(2n-2)(S^2 \times S^2)\#L\widetilde{(p,1) \times} S^1$,
- $E(2(s+1))\#(2n-2)(S^2 \times S^2)\#L\widehat{(p,1) \times} S^1$, *and*
- $E(2(s+1))\#(2n-1)(S^2 \times S^2)\#S^3 \times S^1$

admit infinitely many exotic irreducible smooth structures.

3.3.3 Examples with $\sigma = -48k'$ for $k' > 0$

Employing the Horikawa surfaces $H(8k'-1)$ and $H(7)\#_{T=T}\#H(8k'-1)$ in Lemma 3.8 and Lemma 3.9 yields the following proposition.

Proposition 3.14. *Let $k' > 0$. For $\pi = 1, \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q$ assume $n \geq 2$ and for $\pi = \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_p$ and $\mathbb{Z} \oplus \mathbb{Z}$ assume $n \geq 1$. There exists a spin irreducible symplectic manifold X satisfying*

- $c_1^2(X) = 16k' + 8n - 16, \chi_h(X) = 8k' + n - 2$, *or*
- $c_1^2(X) = 16k' + 8n + 88, \chi_h(X) = 8k' + n + 53$

and $\pi_1(X) = \pi$.

Corollary 3.15. *The manifolds*

- $H(8k'-1)\#(2n-2)(S^2 \times S^2)$,
- $H(8k'-1)\#(2n-2)(S^2 \times S^2)\#L\widetilde{(p,1) \times} S^1$,
- $H(8k'-1)\#(2n-2)(S^2 \times S^2)\#L\widehat{(p,1) \times} S^1$, *and*
- $H(8k'-1)\#(2n-1)(S^2 \times S^2)\#S^3 \times S^1$

admit infinitely many exotic irreducible smooth structures.

3.4 Nonnegative signature

3.4.1 J. Park's construction

In [48], J. Park used the spin complex surface described in 3.2.6 above to realize all but finitely many allowed points in the region $0 \leq c_1^2 \leq 8.74\chi_h$ for trivial fundamental group.

Given that we already filled in the points of negative signature above, we now follow his construction in [48] almost verbatim in order to address the region $8 \leq c_1^2 \leq 8.76\chi_h$. We start by describing the argument and main building blocks in [48].

Consider a simply connected spin symplectic 4-manifold Z which contains a symplectic torus T in a cusp neighborhood N and symplectic surface Σ_g of genus g and zero self-intersection, Σ_g disjoint from N. The Chern invariants of this manifold are $c_1^2(Z) = 8g^2 - 16g + 8$ and $\chi_h(Z) = 2g^2 - g + 1$. In particular its signature is given by $\sigma(Z) = -8g^2 + 8g$. Now take the spin complex surface introduced in 2.6 above and build the symplectic sum

$$X := \overbrace{Y \#_{\Sigma_g} \cdots \#_{\Sigma_g} Y}^{k} \#_{\Sigma_g} Z.$$

Assume the integer k is such that X has positive signature. Furthermore, $\pi_1(X) = 1$ since all the pieces are simply connected and the meridian of Σ_g in $Y - \Sigma_g$ is trivial. The Chern numbers can be calculated to be $c_1^2(X) = kc_1^2(Y) + c(Z) + 8k(g-1)$ and $\chi_h(Y) = k\chi_h(Y) + \chi_h(Z) + k(g-1)$, thus by considering large enough integers k and x, one has

$$\frac{c_1^2(X)}{\chi_h(X)} = \frac{kc_1^2(Y)+c(Z)+8k(g-1)}{k\chi_h(Y)+\chi_h(Z)+k(g-1)} \approx \frac{c_1^2(Y)}{\chi_h(Y)} \approx \frac{60068x^2}{6857x^2} = 8.76009\cdots$$

J. Park then fixes k and x big enough such that $\frac{c_1^2(X)}{\chi_h(X)} > 8.76$ holds. At this point one should notice that X contains a symplectic torus of self-intersection zero lying on the building block Z. In fact, one can also find such tori in the Y blocks. To finish his argument, he then proceeds to define a line $c = f(\chi)$ by

$$f(\chi) = \frac{c(X)}{\chi(X)} \cdot (\chi - c(X)/2 - 6) + c(X)$$

and whose slope $\frac{c(X)}{\chi(X)} = \frac{c_1^2(X)}{\chi_h(X)}$ is greater than 8.76. Finally, build the simply connected manifold $W := \overbrace{X \#_{T^2} X \#_{T^2} \#_{T^2} \cdots \#_{T^2} X}^{m} \#_{T^2} V$ (where the block V can be chosen from $H(8k'-1)\#_{T^2} E(2s), H(7)\#_{T^2} H(8k'-1)\#_{T^2} E(2s)$ or a simply connected manifold constructed in this paper in Proposition 15). Then, for some integer m, for every allowed lattice point (c, χ) in the first quadrant of the geography plane that complies with $c = f(\chi)$, there exists an irreducible symplectic simply connected spin 4-manifold W with

$(c, \chi) = (c_1^2(W), \chi_h(W))$.

Given that W has a torus T of self-intersection zero and of trivial meridian in $W - T$, Lemma 3.9 and Lemma 3.10 imply the following (Theorem 3.2):

Proposition 3.16. *Let $\pi = 1, \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_p$, and $\mathbb{Z} \oplus \mathbb{Z}$. Except for finitely many lattice points, for every allowed pair (c, χ) lying in the region*

$$8\chi \leq c \leq 8.76\chi,$$

there exists a spin irreducible symplectic manifold X satisfying

$$\pi_1(X) = G \text{ and } (c_1^2(X), \chi_h(X) = (c, \chi).$$

Concerning the manifolds with negative signature from the previous proposition, we have the following.

Corollary 3.17. *There exists an integer N such that $\forall\ n \geq N$ the manifolds*

- $(2n+1)(S^2 \times S^2)$,
- $(2n+1)(S^2 \times S^2)\#L(\widetilde{p, 1) \times} S^1$,
- $(2n+1)(S^2 \times S^2)\#L(\widehat{p, 1) \times} S^1$, *and*
- $(2n+2)(S^2 \times S^2)\#S^1 \times S^3$

have infinitely many exotic irreducible smooth structures.

3.5 Testing ground for a conjecture concerning manifolds with even b_2^+

3.5.1 Recovering an example of Fintushel and Stern

In [17], R. Fintushel and R. Stern constructed a manifold X homeomorphic to $K3\#S^2 \times S^2\#S^3 \times S^1$ and, using Donaldson's invariants, concluded it was exotic. By surging out the loop carryng $\pi_1(X)$, one obtains a smooth simply connected manifold $\overline{K}$ with $b_2^+ = 4$. Given the exotic nature of X, one could suspect $\overline{K}$ to be exotic as well. They proved this is not the case, $\overline{K}$ is actually diffeomorphic to $K\#S^2 \times S^2$ for some 4-manifold K, thus providing circumstantial evidence for the conjecture that there does not exist an irreducible 4-manifold with even b_2^+.

The usage of the new techniques produces a myriad of new candidates.

Proposition 3.18. *There exists an infinite family $\{X_n\}$ with one symplectic member of irreducible pairwise non-diffeomorphic 4-manifolds, all of them homeomorphic to*

$$K3\#S^2 \times S^2\#S^3 \times S^1.$$

3.5.2 More candidates

Just as above, out of Proposition 3.15 one can produce more manifolds with even b_2^+ by surgerying away the fundamental group of the following infinite family.

Proposition 3.19. *There exists an infinite family $\{X_n\}$ with one symplectic member of irreducible pairwise non-diffeomorphic 4-manifolds, all of them homeomorphic to*

$$H(8k' - 1)\#S^2 \times S^2\#S^3 \times S^1.$$

Remark 9. *For all the manifolds with non-trivial fundamental group produced in this paper, one can surger away a loop carrying a generator of π_1 at the cost of adding two to the Euler characteristic and repeat this operation until one obtains a manifold with $\pi_1 = 1$. In any case, this procedure always results in a manifold with even b_2^+.*

As testing ground for the mentioned conjecture, we ask the following

Question 1. *Are the simply connected 4-manifolds obtained by surgerying away the fundamental group from X_n irreducible?*

For the answer, the reader is invited to Chapter 5.

Chapter 4

Interaction between the two realms

4.1 Introduction

Two samples of the interplay between exotic 4-manifolds with trivial and with non-trivial fundamental group are given in this short note. First, we put on display exotic simply connected 4-manifolds as universal covers of exotic manifolds with finite cyclic fundamental group. On the other direction, we use a recent technique of Fintushel and Stern to unveil exotic smooth structures on manifolds with non-trivial π_1 out of standard versions of simply connected ones. This provides more evidence for a conjecture of Fintushel and Stern. We proceed to put the first situation in perspective.

Consider the infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible smooth manifolds with finite cyclic π_1, ω_2-type I) (non-spin) with Euler characteristic 6, and signature -2 produced in [65] by using the tools from [11] and the techniques of [22]. From Theorem C in [33] we know the homeomorphism type of these manifolds is given by the topological prototype

$$\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}.$$

By fixing an integer $p \geq 2$ and taking a look at its universal cover $\tilde{X}$, we see that it is non-spin, has Euler characteristic $6p$, and signature $-2p$. Thus, by Freedman's theorem (cf. [26])

$$\tilde{X} \cong_{C^0} (2p-1)\mathbb{CP}^2\#(4p-1)\overline{\mathbb{CP}}^2.$$

At this point it is natural to ask whether $\tilde{X}$ is standard or exotic.

If $\tilde{X}$ were standard, then the action of the cyclic group would be exotic. In [25] exotic smooth actions on a myriad of simply-connected 4-manifolds were constructed. The first result of this note addresses another phenomena. That is, we exhibit exotic irreducible smooth structures on p-covers of exotic irreducible 4-manifolds with finite cyclic fundamental group.

With more generality, the circumstances of our first result are as follows. Let $\{X_n\}$ be a family of pairwise homeomorphic, yet non-diffeomorphic irreducible smooth 4-manifolds with finite cyclic fundamental and non-trivial Seiberg-Witten invariants; assume at least one member is symplectic (cf. [65]). Let $X \in \{X_n\}$. We have

Theorem 4.1. *Let $\pi : \tilde{X} \to X$ be a p-cover. The universal cover $\tilde{X}$ admits an exotic irreducible symplectic smooth structure. In the case of double covers, $\tilde{X}$ admits an exotic irreducible non-symplectic smooth structure as well.*

Our second result is greatly indebted to the recently introduced technique of Fintushel and Stern [24]. For the groups $\pi_1 = \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z} \oplus \mathbb{Z}_p, \mathbb{Z} \oplus \mathbb{Z}$, and $\mathbb{Z}$, we have

Theorem 4.2. *By applying surgeries on nullhomologous tori in $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ (with $k = 2, \ldots, 7$ and 9) one obtains*

- *an infinite family of minimal exotic $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2 \# \widetilde{L(p,1) \times S^1}$,*
- *an infinite family of pairwise non-diffeomorphic minimal 4-manifolds sharing the same $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p$ and the homology of $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2 \# \widehat{L(p,1) \times S^1}$,*
- *an infinite family of pairwise non-diffeomorphic minimal 4-manifolds sharing the same π_1 and the homology of $2\mathbb{CP}^2 \# (k+1)\overline{\mathbb{CP}}^2 \# L(p,1) \times S^1$,*
- *an infinite family of pairwise non-diffeomorphic minimal 4-manifolds sharing the same π_1 and the homology of $2\mathbb{CP}^2 \# (k+1)\overline{\mathbb{CP}}^2 \# T^2 \times S^2$, and*
- *an infinite family of pairwise non-diffeomorphic minimal 4-manifolds sharing the same π_1 and the homology of $2\mathbb{CP}^2 \# (k+1)\overline{\mathbb{CP}}^2 \# S^1 \times S^3$.*

Here, $L(\widetilde{p,1) \times} S^1$ and $L(\widehat{p,1) \times} S^1$ stand for the 4-manifolds obtained by taking the product of the lens space $L(p,1)$ and S^1, and then surging the loop that generates the $\mathbb{Z}$-factor in the fundamental group accordingly.

This chapter is organized as follows. Section 4.2 is devoted to the construction of exotic covers. The third section provides an outline to construct exotic non-simply connected 4-manifolds out of standard simply connected ones.

4.2 Exotic p-covers

Let $X^{\mathbb{Z}_p}_{b_2^+, b_2^-}$ denote any of the irreducible smooth manifolds with finite cyclic π_1 constructed in [65] (do notice that our arguments work in more generality). We have that its homeomorphism type is $b_2^+\mathbb{CP}^2 \# b_2^- \overline{\mathbb{CP}}^2 \# L(\widetilde{p,1) \times} S^1$. Its p-cover $\tilde{X}_{b_2^+ + 2, 2b_2^- + 1}$ is homeomorphic to

$$(p(b_2^+ + 1) - 1)\mathbb{CP}^2 \# (p(b_2^- + 1) - 1)\overline{\mathbb{CP}}^2$$

by Freedman's Theorem (cf. [26]).

For every p-cover we have the following.

Lemma 4.3. *Let X be the symplectic member of the family $\{X_n\}$. Then $\tilde{X}$ is an exotic symplectic*

$$(p(b_2^+ + 1) - 1)\mathbb{CP}^2 \# (p(b_2^- + 1) - 1)\overline{\mathbb{CP}}^2.$$

Proof. Let $\pi : \tilde{X} \to X$ be a covering and let ω be a symplectic form on X. Then $\pi^*\omega$ is a symplectic form on $\tilde{X}$. Taubes' theorems (cf. [64], [63]) now implies that $SW_{\tilde{X}} \neq 0$. $\square$

4.2.1 SW invariants on double covers

The main ingredient in the proof of Theorem 4.1 is the following formula for the Seiberg-Witten invariants of a double cover $\pi : \tilde{X} \to X$.

Theorem 4.4. *(Ruan and Wang, cf. [54]) Suppose that $\pi : \tilde{X} \to X$ is an unramified double cover. It is clear that there is a well-defined pull back spin^c structure of X such that $d_L = 0$, $c_1(L)$ is non-torsion. Then the Seiberg-Witten invariants satisfy the following relation:*

$$SW(\tilde{\xi}) = \sum_{\gamma\in K} SW(\xi \otimes \gamma) \ mod\ 2,$$

where K is the set of isomorphism classes of complex lines bundles on X which pull back to the trivial bundle on $\tilde{X}$.

4.2.2 Proof of Theorem 4.1

The first result in the introduction is a corollary of Ruan-Wang's formula and Lemma 4.3.

Proof. From $\{X_n\}$, take the irreducible symplectic member X and an irreducible non-symplectic X_1. By Theorem C in [33], both X and X_1 have

$$b_2^+\mathbb{CP}^2\# b_2^-\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times} S^1$$

as a topological prototype. Consider the universal covers $\tilde{X}$ and $\tilde{X}_1$. By Freedman's Theorem (cf. [26]), we know these manifolds are homeomorphic to

$$(p(b_2^+ + 1) - 1)\mathbb{CP}^2\#(p(b_2^- + 1) - 1)\overline{\mathbb{CP}}^2.$$

Lemma 4.3 allows us to conclude the existence of the exotic symplectic copies for a p-cover $\pi : \tilde{X} \to X$.

In the case $p = 2$, Ruan-Wang's result implies the existence of an exotic

$$(2b_2^+ + 1)\mathbb{CP}^2\#(2b_2^- + 1)\overline{\mathbb{CP}}^2$$

which does not admit a symplectic structure. □

4.3 Pinwheels and nullhomologous surgery

The surgery techniques on nullhomologous tori introduced by Fintushel and Stern in [24] enable the construction of exotic 4-manifolds with abelian fundamental group from standard simply connected manifolds by applying surgeries on nullhomologous tori. This procedure was already envisioned in their previous work ([23]). In particular, Theorem 4.2 is a corollary of the recent tools introduced in [24] to find such tori, and of the blueprint to manipulate the fundamental group calculations exemplified below in 4.3.2 for the manifolds constructed out of $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$. A contribution of the recent preprint [24] that is worth noticing is that it unifies the recent constructions of small exotic 4-manifolds.

4.3.1 Scheme of the construction

The reader is advised to look at [24] for details and for further references. A crucial point in their technique is to find the nullhomologous tori inside the standard versions of a manifold. Fintushel and Stern start with an standard $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ with k as in our theorem above. Then, by the help of actions of T^2 on that given manifold, they endowed it with a pinwheel structure where the surfaces of the gluings are spheres. They proceed to find a new pinwheel structure by ambiently pushing 2-handles in the starting pinwheel presentation. The surfaces of the gluings are now tori and the components of the new pinwheel contain the needed nullhomologous tori.

By applying surgeries on these tori, they construct a non-simply connected (symplectic) 4-manifold with nontrivial Seiberg-Witten invariants X_1 which contains tori carrying the generators of its fundamental group. We point out that this symplectic manifold appears to be the model manifold in the reverse-engineering technique of [12], [3], and [22]. For example, for the procedure on $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ the symplectic manifold constructed appears to be $Sym^2(\Sigma_3)$, just like in [22]. This, however, remains unproven.

The manifold X_1 is now surgered along these tori, thus producing an infinite family of pairwise nondiffeomorphic manifolds with $b_1 = 0$. One concludes the resulting manifolds are simply connected by looking at the resulting fundamental group presentations. The composition of the surgeries from the standard $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ with the ones applied to the manifold X_1 gives a direct construction of exotic 4-manifolds out of standard versions. As it was mentioned in [24], this technique can be applied to many other manifolds.

In the next part of this chapter, we show how to employ these techniques to produce non-simply connected exotic 4-manifolds.

4.3.2 Process for $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$

Our starting point is the manifolds in [24], and we follow closely their calculations on π_1 based on the analysis done in [12]. We provide a blueprint on how the fundamental group calculations should be manipulated to produce manifolds with a desired fundamental group. We do so by proving Theorem 4.2 for the instance $k = 2$. The other calculations follow almost verbatim.

Proof. The fundamental group of X_1 is generated by a_i, b_i and y_0 with $i = 0, 1, 2$. We are aiming at constructing an intermediate manifold $\tilde{X}_1$ with $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}$. We proceed as follows.

In the block $\hat{A}$ perform the surgery that introduces the relation $b_0[a_0^{-1}, b_1] = 1$.

In the block A perform the surgery that introduces the relation $b_1[a_1^{-1}, b_2] = 1$.

The combination of these two relations and the commutativity of a_0 with a_1 and with b_2 implies $b_0 = 1$.

Now move to the block I_0; the generator y_0 is killed by applying the surgery introducing the relation $y_0[a_2^{-1}, b_0] = 1$. This implies that $b_2 = y_0\xi = 1$. The generator b_1 is killed by the surgery done on the A block.

Notice that the surviving generators a_0 and a_1 commute. We have thus obtained a symplectic 4-manifold $\tilde{X}_1$ with $\pi_1 = \mathbb{Z}a_0 \oplus \mathbb{Z}a_1$. To produce an infinite family with the same fundamental group and the same homology as our intermediate manifold $\tilde{X} = 1$, it suffices to apply the surgery on the block I_0 which introduces the relation $y_0[a_2^{-1}, b_0]^n = 1$. This manufactures an infinite family of pairwise non-diffeomorphic manifolds, which are all candidates to be exotic $2\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2\#T^2 \times S^2$.

The infinite family $\{Z_n\}$ of pairwise non-diffeomorphic candidates for exotic $2\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2\#S^1 \times S^3$ (see [36] for the homeomorphism criteria) are obtained as follows. Perform in the A block the surgery responsible for introducing the relation $a_1[b_2^{-1}, b_1^{-1}]^n = 1$. This kills the generator a_1 and all the manifolds obtained have $\pi_1 = \mathbb{Z}a_0$.

If for this last surgery $n = 1$ and one applies in the $\hat{A}$ block the surgery that introduces the relation $a_0^p[b_1^{-1}, b_0^{-1}]^m = 1$, we obtain an infinite family $\{Y_n\}$ of pairwise non-diffeomorphic 4-manifolds, all homeomorphic (cf. [33]) to

$$\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2\#\widetilde{L(p,1) \times} S^1.$$

Taking the intermediate manifold $\tilde{X}_1$ and applying the surgeries

- $A : a_1^p[b_2^{-1}, b_1^{-1}] = 1$ and
- $\hat{A} : a_0^p[b_1^{-1}, b_0^{-1}]^r = 1$,

one obtains an infinite family of pairwise non-diffeomorphic manifolds candidates to be homeomorphic to $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2 \# L(p, \widehat{1) \times} S^1$. $\square$

Chapter 5

On the Manifolds of Sections 4.3 and 4.5, and Future Research Projects

Regarding the manifolds of Question 1 at the end of Chapter 4, R.E. Gompf has suggested to us [30] that all of them are irreducible. For the sake of closure to the raised question, we proceed to explain his argument by taking a closer look at the construction. Let $\pi_1(T^4) = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}a \oplus \mathbb{Z}b$. Abusing notation, let the Lagrangian tori T_1 and T_2 be spanned by the curves x, a and y, b and the symplectic tori T_3 and T_4 spanned by a, b and x, y, respectively. Recall how we produced a manifold X in the infinite family $\{X_n\}$ of Section 4.5. First, build the fiber sum of a manifold Q containing a torus of self-intersection T (and trivial meridian in $\pi_1(Q-T) = 1 = \pi_1(Q)$) with T^4 along T_1, apply a torus surgery on T_2 and then kill the surviving generator by doing an ordinary surgery on the loop b. Denote the manifold obtained this way by Z.

The surgered loop b lies on T_3, and the surgery transforms such torus into an embedded sphere S with self-intersection 0. We will see that S is actually a factor of an $S^2 \times S^2$ summand in the resulting manifold Z. Start by observing that the meridian of S is nullhomotopic in $X - S$. Indeed, S has T_4 for a dual torus; since the loop x spanning T_4 becomes nullhomotopic after the symplectic sum, we obtain an immersed sphere S' by surging T_4. This immersed sphere S' intersects S in a single point, offering a nullhomotopy for the meridian of S as it was claimed.

By carving out S, one obtains a manifold Y containing a standardly embedded circle C

on which surgery gives the starting X back. This exhibits X as $Y\#S^2\times S^2$, thus irreducible. Notice that the argument depends only on the T^4 block of our constructions and it applies to all the manifolds considered in Section 4.5 of Chapter 4 above. Thus, we have

Lemma 5.1. *(Gompf, [30]) The 4-manifolds with* $\pi_1 = 1$ *constructed in Section 4.5 above are reducible.*

5.1 The shape of things to come

5.1.1 2-knots

Not everything is lost. As a side-effect, we came across an infinite family of nullhomologous knots with infinite cyclic knot group inside 4-manifolds like our Q above, which are topologically equivalent but have nondiffeomorphic complements. This is currently work in progress [67].

5.1.2 Homeomorphism criteria

In the recent paper [35], I. Hambleton, M. Kreck and P. Teichner have established a homeomorphism clasification for closed oriented topological 4-manifolds with solvable Baumslag-Solitar fundamental groups based on their ω_2-type, the equivariant intersection form and the Kirby-Siebenmann invariant. This includes $\mathbb{Z}\oplus\mathbb{Z}$ and, therefore, the manifolds constructed above. It is an interesting task to find numerical invariants to conclude a homeomorphism criteria. This involves studying algebraic K-theoretical methods to come up with numerical invariants (Euler characteristic, signature, type). This is work in process [68].

Chapter 6

Work in Progress: Project 1

We end the thesis by stating two on-going projects. The second one is joint work with Jonathan Yazinski.

6.1 Irreducible spin 4-manifolds with abelian π_1 and $\sigma = 0$

Assuming the existence of an exotic symplectic $S^2 \times S^2$ which contains a symplectic surface of genus 2 and self-intersection zero, in this short note we address the existence and (lack of) uniqueness of irreducible spin symplectic smooth 4-manifolds. The tools employed allow us to study manifolds with several non-trivial abelian fundamental groups, and address the botany in some of these cases. Our results use an exotic $S^2 \times S^2$, whose construction was outlined in the recent *Preliminary report* [7].

In that paper, the authors claim that by modifying their construction in [7], they were able to build to build exotic copies of the connected sums $(2k - 1)(S^2 \times S^2)$. The proofs employed in this chapter are of a completely different nature; we make use of auxiliary building blocks and we do not need to modify the construction in [7].

Moreover, the agenda of this paper is to exploit the recent construction techniques to study 4-manifolds with abelian fundamental group of small rank, and not only simply connected 4-manifolds.

Our main result regarding the symplectic geography is the following.

Theorem 6.1. *Let $g \geq 0$ and let G be either $1, \mathbb{Z}_p, \mathbb{Z}_p \oplus \mathbb{Z}_q, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_p$, or $\mathbb{Z} \oplus \mathbb{Z}$. For each of the following pairs of integers*

$$(c, \chi) = (8 + 8g, 1 + g),$$

there exist an irreducible symplectic spin 4-manifold X with

$$\pi_1(X) = G \text{ and } (c_1^2(X), \chi_h(X)) = (c, \chi).$$

Concerning the botany, we have

Proposition 6.2.

- *For every $k \geq 2$ integer, there exists an infinite family $\{Z_n\}$ of irreducible pairwise nondiffeomorphic manifolds, all of them homeomorphic to*

$$(2k-1)(S^2 \times S^2).$$

- *For every $k \geq 1$ integer, there exists an infinite family $\{Y_n\}$ of irreducible pairwise nondiffeomorphic manifolds, all of them homeomorphic to*

$$(2k-1)(S^2 \times S^2)\#L\widetilde{(p,1) \times} S^1.$$

- *For every $k \geq 1$ integer, there exists an infinite family $\{W_n\}$ of irreducible pairwise nondiffeomorphic manifolds, all of them homeomorphic to*

$$(2k-1)(S^2 \times S^2)\#L\widehat{(q,1) \times} S^1.$$

- *For every $k \geq 2$ integer, there exists an infinite family $\{V_n\}$ of irreducible pairwise nondiffeomorphic manifolds, all of them homeomorphic to*

$$(2k)(S^2 \times S^2)\#S^3 \times S^1.$$

This chapter is work in progress. It is organized as follows. Section 6.2 deals with the construction of symplectic irreducible manifolds homeomorphic to the connected sum of $(2k-1)$ copies of $S^2 \times S^2$. In Section 6.3, we study symplectic manifolds with more general fundamental groups. This section contains our main technical tool when building manifolds with abelian fundamental groups of small rank (Proposition 6.5) and a description of the topological prototypes used to pin down the homeomorphism types for the myriad of manifolds produced. The exposition of the work in progress finishes with Section 6.4, where our claim regarding the botany is proven.

6.2 Symplectic geography of simply connected spin 4-manifolds with signature zero

6.2.1 Warm up example: a symplectic $3(S^2 \times S^2)$

Consider the manifold M built in [7]. According to A. Akhmedov and B.D. Park, this symplectic manifold contains a symplectic surface of genus 2: the quotient $q(\Sigma_2 \times \{\omega_0\})$. Denote it by Σ_2. Furthermore, this Σ_2 intersects transversally the genus two surface $q(\{z_0\} \times \Sigma_3)$. Thus, the meridian of Σ_2 is dictated by the product of commutators coming from $q(\{z_0\} \times \Sigma_3)$, which were killed during the Luttinger surgeries ((9) in [7]). This implies $\pi_1(M - \Sigma_2) = 1$.

Now consider the spin manifold Q_2 constructed in [28], tagged as Building Block 5.8 in R.E. Gompf's paper. It has zero Euler characteristic and zero signature. This Q_2 contains a genus 2 symplectic surface Σ and $\pi_1(Q_2 - \Sigma)/ < \pi_1(\Sigma'') >= 1$ (see Lemma 5.9 in [28]), where Σ'' is a parallel copy of the surface Σ in $Q_2 - \Sigma$. Now, build the symplectic sum ([28])

$$Z := M\#_{\Sigma_2=\Sigma}Q_2.$$

It follows from Seifert-Van Kampen's theorem that $\pi_1(Z) = 1$. One computes directly $e(Z) = e(M) + 4 = 8$ and $\sigma(Z) = 0$. Thus, by applying Freedman's theorem (cf [26]) to our Z, we conclude the following.

Lemma 6.3. *There exists an irreducible symplectic 4-manifold homeomorphic to* $3(S^2 \times S^2)$.

6.2.2 Exotic symplectic $(2k-1)(S^2 \times S^2)$

We proceed now to iterate the usage of Gompf's manifold Q_2 in the previous construction in order to address the symplectic geography completely.

Proposition 6.4. *Let $k \geq 2$ be an integer. There exists an irreducible symplectic 4-manifold homeomorphic to $(2k-1)(S^2 \times S^2)$*

Proof. Take n copies of Q_2, $\{Q_2^{(1)}, \ldots, Q_2^{(n)}\}$, and inside each of them consider a genus 2 symplectic surface $\Sigma(j)$. Now, inside the manifold M, let $\{\Sigma^1, \cdots, \Sigma^n\}$ be n parallel copies of the symplectic surface of genus 2, Σ. Take one of these surfaces, say Σ^j, and build the symplectic sum of M with each $Q_2^{(j)}$ along $\Sigma^j = \Sigma(j)$. Then continue to build the symplectic sum, one by one, of a copy of a parallel surface in M with a copy of Q_2 along the corresponding $\Sigma(i)$. We get

$$Z_n := M\#_{n\Sigma_g} n(Q_2) = \bigsqcup_j (Q_2^{(j)} - \Sigma(j)) \bigcup_j (M - \cup\Sigma^j),$$

where the block $\bigsqcup_j (Q_2^{(j)} - \Sigma(j))$ stands for the disjoint union of the copies. Notice that for all j, Σ^j gets identified with $\Sigma(j)$; the choice of gluing map can be supressed in our definiton of Z_n by Remark 8.1.3 in [29]. The characteristic numbers of Z_n are $e(Z_n) = e(M) + 4n = 4 + 4n$ and $\sigma(Z_n) = 0$. We claim $\pi_1(Z_n) = 1$.

Indeed, the inclusion $\Sigma'' \hookrightarrow M - \bigcup_j \Sigma^j$ induces the trivial map on π_1. Thus, all loops contained in the building block $Q_2^{(j)} - \Sigma(j)$ are trivial in $\pi_1(Z_n)$. Moreover, the meridians of the surfaces $\Sigma(j)$ normally generate $\pi_1(M - \cup_j \Sigma^j)$, and they can be pushed off into $Q_2^{(j)} - \Sigma(j)$. Therefore, Seifert-Van Kampen's theorem says $\pi_1(Z_n) = 1$ as was claimed. Rename $Z_n = Z_k$.

By Freedman's theorem (cf [26]), the manifold Z_k is homeomorphic to $(2k-1)(S^2 \times S^2)$. □

Remark 10. *Note that the proof for the case $k = 1$ was outlined in [7]. In the same preliminary report, the authors claim that by modifying their construction they are able to prove the claim for $k \geq 2$. Proposition 5.4 is disjoint from their results from two perspetives. First, our proofs are different. Second, we are interested in abelian fundamental groups, and not only simply connected manifolds.*

6.3 More abelian π_1's

In what follows we turn our attention to the symplectic geography of spin 4-manifolds whose fundamental group is amongst the following choices:

- $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}$,
- $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_p$,
- $\pi_1 = \mathbb{Z}$,
- $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_q$,
- $\pi_1 = \mathbb{Z}_p$ and
- $\pi_1 = 1$.

6.3.1 Technical tool

Using symplectic sums ([28]) and Luttinger surgeries ([45], [8]) we produce our main tool in the study of the geography.

Proposition 6.5. *Let X be a symplectic simply connected manifold containing a symplectic surface of genus 2 of self-intersection zero, Σ. Assume $\pi_1(X - \Sigma) = 1$. Let $g \geq 0$ and assume $\pi_1(X - \Sigma)$. There exists an irreducible spin symplectic 4-manifold with characteristic numbers $e(Z) = e(X) + 4g$ and $\sigma(Z) = \sigma(X)$. The fundamental group of Z can be chosen to be*

1. $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}$,
2. $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}_q$,
3. $\pi_1 = \mathbb{Z}_p \oplus \mathbb{Z}_q$,
4. $\pi_1 = \mathbb{Z}_p$,
5. $\pi_1 = \mathbb{Z}$ *or*
6. $\pi_1 = 1$.

The cases $\pi_1 = 1, \mathbb{Z}_p$ with $g = 0$ were claimed in [7]. We proceed to prove the rest of our assertion.

Proof. For $g = 0$, the proposition follows from [7]; for example, one obtains a manifold with infinite cyclic fundamental group by *not* performing one of the surgeries. Now, let $g = 1$. Take the product of $T^2 \times \Sigma_2$ of a torus and a genus 2 surface with the product symplectic form, and build the symplectic sum ([28])

$$S_1 := X\#_{\Sigma=\Sigma_2} T^2 \times \Sigma_2.$$

By Proposition 7 of [12] (using the notation there), an application of Seifert-Van Kampen's theorem concludes $\pi_1(S_1) = \mathbb{Z}x \oplus \mathbb{Z}y$. Notice that in S_1 we have two Lagrangian tori carrying the generators x and y each. Applying a -1/q Luttinger surgery on T_1 along $m_1 = x$ produces a symplectic spin manifold with $\pi_1 = \mathbb{Z}_q \oplus \mathbb{Z}$ ([45], [8]). If to that manifold one applies a -1/p Luttinger surgery, one obtains a symplectic spin manifold with $\pi_1 = \mathbb{Z}_q \oplus \mathbb{Z}_p$ (if p = 1, we obtained a manifold with infinite cyclic fundamental group. If we apply to S a -1 Luttinger surgery on T_1 along m_1 and a -1/p Luttinger surgery, we obtain a manifold with finite cyclic fundamental group of order p; if $p = 1$, then the resulting manifold is simply connected.

The instances corresponding to $g \geq 2$, one builds

$$S_g := X\#_{\Sigma=\Sigma_2} \widetilde{\Sigma_2 \times \Sigma_g},$$

where the block $\widetilde{\Sigma_2 \times \Sigma_g}$ stands for the surgered product of a surface of genus 2 and a surface of genus g (see [22] (for $g = 2$) and [4] (for $g \geq 3$) regarding the details on the fundamental groups needed for our computations).

We remind the reader that a spin symplectic 4-manifold is irreducible. Indeed, by Taubes' results ([67], [64]) the Seiberg-Witten invariants of such manifold are nontrivial, and it is not the blow-up of another manifold, since it is spin. Therefore, it is minimal. Irreducibility now follows from [37].

$\square$

Theorem 6.1 follows now as a corollary of Proposition 6.5 and the work done in Section 6.2.

Remark 11. *Concerning the production of an infinite family $\{X_n\}$ of pairwise non-diffeomorphic irreducible smooth manifolds we have the following. Properly applying a torus surgery on a nullhomologous torus (see [22] or the remark that follows Theorem 13 in [12]) produces the desired family. To conclude on their homeomorphism type, one must check that these manifolds have the desired fundamental group; we already know their characteristic numbers remained unchanged after the surgery. For this purpose, it suffices to see that the effect* *such surgery has on the presentation of the fundamental groups is to replace a relation of the form* $[a,b]=c^p$ *by* $[a,b]^n=c^p$ *for a given* p *and* n *and generators* a,b. *Given that in the proof of Proposition 5 we concluded that the original relation is trivial, then raising it to any power will result in a trivial relation as well. Hence, we make no further distinctions about the computations of* π_1 *of the infinite families.*

Remark 12. *During the computations involved in the proof of the previous proposition, one notices that many other fundamental groups can be obtained during the procedure.*

6.3.2 Smooth topological prototypes

In order to fix a homeomorphism type for the exotic manifolds with non-trivial π_1 built here, we will employ the following smooth topological prototypes:

- $\pi_1=\mathbb{Z}:(b_2^+ +1)(S^2\times S^2)\#S^3\times S^1$,
- $\pi_1=\mathbb{Z}_p: b_2^+(S^2\times S^2)\#L(\widetilde{p,1)\times} S^1$ and
- $\pi_1=\mathbb{Z}_q\oplus\mathbb{Z}_q: b_2^+(S^2\times S^2)\#L(\widehat{p,1)\times} S^1$.

The common characteristic of these smooth manifolds is that the last block carries all the fundamental group. To construct it, take the product of a Lens space and a circle: $L(p,1)\times S^1$. The Euler characteristic of this manifold is zero, as well as its signature. Consider the map

$$L(p,1)\times S^1\to L(p,1)\times S^1$$
$$\{pt\}\times\alpha\mapsto\{pt\}\times\alpha^p$$

We perform surgery on $L(p,1)\times S^1$: cut out the loop α^p and glue in a disc $(S^2\times D^2)$ in order to kill the corresponding generator

$$L(\widehat{p,1)\times} S^1:=L(p,1)\times S^1-(S^1\times D^3)\cup S^2\times D^2.$$

The resulting manifold has zero signature and Euler characteristic two. By the Seifert-Van Kampen theorem, one concludes $\pi_1(L(\widehat{p,1)\times} S^1)=\mathbb{Z}_p\oplus\mathbb{Z}_p$ and $\pi_1(L(\widetilde{p,1)\times} S^1)=\mathbb{Z}_p$.

6.4 Botany

We now proceed to build a myriad of irreducible smooth structures on the topological prototypes built above. From now on, we assume

$$\pi_1(L(\widehat{q, 1) \times} S^1) = \mathbb{Z}_q \oplus \mathbb{Z}_q,$$

where $q \geq 3$ is an odd integer. Regarding the lack of a unique smooth structure we have the following result (Proposition 6.2 above).

Proposition 6.6.

- *For every $k \geq 2$ integer, there exists an infinite family $\{Z_n\}$ of irreducible pairwise nondiffeomorphic manifolds, all of them homeomorphic to*

$$(2k-1)(S^2 \times S^2).$$

- *For every $k \geq 1$ integer, there exists an infinite family $\{Y_n\}$ of irreducible pairwise nondiffeomorphic manifolds, all of them homeomorphic to*

$$(2k-1)(S^2 \times S^2)\#L(\widetilde{p, 1) \times} S^1.$$

- *For every $k \geq 1$ integer, there exists an infinite family $\{W_n\}$ of irreducible pairwise nondiffeomorphic manifolds, all of them homeomorphic to*

$$(2k-1)(S^2 \times S^2)\#L(\widehat{q, 1) \times} S^1.$$

- *For every $k \geq 2$ integer, there exists an infinite family $\{V_n\}$ of irreducible pairwise nondiffeomorphic manifolds, all of them homeomorphic to*

$$(2k)(S^2 \times S^2)\#S^3 \times S^1.$$

In each of these families, one member is symplectic.

Proof. The infinite families $\{Z_n\}, \{Y_n\}, \{W_n\}$ and $\{V_n\}$ were constructed in Proposition 6.5 and Remark 6.2. We need to conclude on the homeomorphism types. The simply connected case follows from Freedman's theorem ([26]). The homeomorphism criteria for the manifolds with $\pi_1 = \mathbb{Z}_p$ and $\pi_1 = \mathbb{Z}_q \oplus \mathbb{Z}_q$ is given by Hambleton-Kreck's theorems in [33] (for a proof of the fact $d(\pi) = 1$ for the noncyclic π_1 case see [66]). The infinite cyclic fundamental group case follows from Hambleton-Teichner's result in [36]. $\square$

Chapter 7

Work in Progress: Project 2

7.1 Exotic smooth structures on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$

Jonathan Yazinski and I are currently pursuing an idea which aims at proving the following.

Theorem 7.1. *There exists an infinite family $\{X_n\}$ of irreducible pairwise nondiffeomorphic 4-manifolds, all of them homeomoprhic to $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$.*

We wish to employ the techniques in [22]. At the moment, we are facing critical issues when trying to determine that our construction is not diffeomorphic to the standard manifold via Seiberg-Witten theory. We do believe, for several reasons and even if the initial goal is not accomplished, that this idea is worth pursuing. We explain one interesting reason in particular.

If we are indeed able to apply the full program of [22], a slight modification in our construction results in the following

Corollary 7.2. *There exists an infinite family $\{C_n\}$ of irreducible pairwise nondiffeomorphic 4-manifolds, all of them homeomorphic to $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2\#\widetilde{L(p,1)\times S^1}$.*

The block $\widetilde{L(p,1)\times S^1}$ stands for the result of surgering the product of a Lens space and a circle in such way that it has $\pi_1 = \mathbb{Z}_p$, Euler characteristic two and zero signature. By looking at the universal cover of these manifolds we conclude

Corollary 7.3. *The manifold $3\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$ admits an exotic smooth structure.*

This is exciting work in progress.

7.2 Raw Materials

We employ two building blocks. First, we consider an S^2-bundle over a torus containing a section of square 1 and a section of square -1. Call this manifold X. Our second building block is the product of a torus and a genus 2 surface $Y := T^2 \times \Sigma_2$ endowed with the product symplectic form. The symplectic manifold Y contains a symplectic surface of genus 2 and self-intersection zero $\{pt\} \times \Sigma_2$ which we denote by F.

Our model manifold is

$$M := X \#_{\Sigma = F} Y,$$

the fiber sum of the S^2-bundle and Y along surfaces of genus 2. A direct calculation shows that the Euler characteristic of M is 4 and its signature 0. We proceed to explain how the required surface of genus 2 inside X is constructed.

7.2.1 A Genus 2 Surface inside the Bundle

We construct a genus 2 surface inside X by tubing two sections together. We proceed as follows. Consider an involution σ on S^2 whose fixed point set is S^1. The bundle X contains a section of square 1, T_1, and a section of square -1, T_{-1}.

Use the involution on the 2-sphere to specify an involution on X; in particular we have $\sigma(T_1) = \sigma(T_{-1})$. Consider a path ξ in X which connects the sections T_1 and T_{-1} and which is invariant under σ. Now pick a tube contained in a neighborhood and glue T_1 with T_{-1} with it. This results in a surface Σ of genus 2 and self-intersection 0. In particular notice we have $\sigma(\Sigma) = \Sigma$.

Regarding the way the tubing is performed, notice one would like for it to produce the desired representative for the computations of Section 3. Carefulness in doing so is required. We have two choices for the tubing.

7.3 SW Computations/Issues

Our construction fits the recent procedure theme of [22]. However, unlike the recent constructions of exotic smooth structures, our model manifold is *not* symplectic. This forces the computations of the Seiberg-Witten invariants to be more involved.

For the full implementation of the techniques of [22], one wishes to see that our model manifold has non-vanishing Seiberg-Witten invariants.

The key ingredient for this is Corollary 3.3 in [40]. In order to apply this result, we need to use a Wall crossing formula argument on our building block X (the bundle), to see that one of the chambers have non-vanishing invariants. We proceed to explain the issues we are tangled in at the moment.

7.3.1 SW invariants of $T^2 \times \Sigma_2$

The Seiberg-Witten invariants of this building block have been computed following [51]. Viewing a Seiberg-Witten invariant as an element of the group ring $\mathbb{Z}\left[H^2\left(T^2 \times \Sigma_2;\mathbb{Z}\right)\right]$, we have that

$$SW_{T^2\times\Sigma_2} = \left(PD\left[T^2 \times \{*\}\right]^{-1} - PD\left[T^2 \times \{*\}\right]\right)^2$$

7.3.2 SW invariants of X

The bundle admits a metric of positive scalar curvature. We need to look at the chambers; the one with small η will have vanishing invariant. We would like a general Wall-crossing formula argument to conclude the non-triviality for the invariants of this building block.

A basis for $H_2(X)$ is given by $\{T_1, T_{-1}\}$. It is straight-forward to see that in this basis

$$[\Sigma] = [T_1] + [T_{-1}].$$

Now consider a fiber S^2 of the bundle and, just to simplify notation, denote $T := T_1$. These surfaces provide us with a new basis for $H_2(X)$, $\{[T], [S^2]\}$. We claim that in this new basis

$$[\Sigma] = 2[T] - [S^2].$$

Indeed, we can convert the first basis into the second one by using $[\Sigma]\cdot[T] = [T]\cdot[T] + [T_{-1}]\cdot[T] = 1+0 = 1$ and $[\Sigma]\cdot[S^2] = [T_1]\cdot[S^2]+[T_{-1}][S^2] = 1+1 = 2$. Thus, $[\Sigma] = 2[T]-[S^2]$

as claimed.

We will apply the Wall-crossing formula to show that $SW_X(l_1) \neq 0$ in one of the chambers. First we check that the manifold X does satisfy the required hypothesis.

In order to compute that $SW_X(l_1) \neq 0$, we need to compute the Σ^*-negative Seiberg-Witten invariant, in the sense of [40]. Notice that the Seiberg-Witten invariants will vanish in the chamber with generic $\eta \approx 0$ and g, since X admits a metric of positive scalar curvature [**?**]. We can compute that this chamber is the Σ^*-positive chamber, and so $SW_X^{\Sigma}(l_1) \neq 0$.

7.3.3 Tubing the sections and representatives

Tubing T_1 and T_{-1} has to be done carefully, having in mind we would like to have $[T_1]+[T_{-1}]$ has the representative. One selection of tubing results in a sum of these homology classes and other selection results in their difference. We are pursuing both paths.

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