Asymptotics for Orthogonal Polynomials, Exponentially Small Perturbations and Meromorphic Continuations of Herglotz Functions

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Abstract

The thesis consists of a series of results on the theory of orthogonal polynomials on the real line.

1. We establish Szegő asymptotics for matrix-valued measures under the assumption that the absolutely continuous part satisfies Szegő's condition and the mass points satisfy a Blaschke-type condition. This generalizes the scalar analogue of Peherstorfer–Yuditskii [PY01] and the matrix-valued result of Aptekarev–Nikishin [AN83], which handles only a finite number of mass points.

2. We obtain matrix-valued Jost asymptotics for a block Jacobi matrix under an L^1-type condition on parameters, and give a necessary and sufficient condition for an analytic matrix-valued function to be the Jost function of a block Jacobi matrix with exponentially converging parameters. This establishes the matrix-valued analogue of Damanik–Simon [DS06b].

3. The latter results allow us to fully characterize the matrix-valued Weyl–Titchmarsh m-functions of block Jacobi matrices with exponentially converging parameters.

4. We find a necessary and sufficient condition for a finite gap Herglotz function m to be the m-function of a Jacobi matrix with the prescribed "distance" from the isospectral torus $\mathcal{T}_\mathfrak{e}$ of periodic Jacobi matrices associated with a given finite gap set $\mathfrak{e}$ (with all gaps open). The condition is in terms of meromorphic continuations of the function m to a natural Riemann surface $\mathcal{S}_\mathfrak{e}$, and the structure of poles and zeros of m.

5. The results from parts 3 and 4 give certain corollaries on the point perturbations of measures. Namely, we find conditions on when adding or removing a pure point preserves the exponential rate of convergence of Jacobi parameters. The method applies in the matrix-valued case of exponential convergence to the free block Jacobi matrix, and in the scalar case of exponential convergence to a periodic Jacobi matrix. This extends Geronimo's results

from [Ger94].

6. We obtain two results on the equivalence classes of block Jacobi matrices: first, that the Jacobi matrix of type 2 in the Nevai class has A_n coefficients converging to $\mathbf{1}$, and second, that under an L^1-type condition on the Jacobi coefficients, equivalent Jacobi matrices of type 1, 2, and 3 are pairwise asymptotic.

Contents

Chapter 1

Introduction

1.1 Overview

This thesis consists of a number of pairwise closely related results in the theory of orthogonal polynomials. The joint relation is not so easy to characterize though, and the cumbersome title is the consequence of this. Let us give a brief overview of the obtained results and the connections between them.

One of the main topics we will be discussing here is the asymptotic behavior of solutions of a difference equation of the type

$$a_n f_{n+1}(x) + b_n f_n(x) + a_{n-1} f_{n-1}(x) = x f_n(x). \tag{1.1.1}$$

The two settings we will focus on in this paper are the matrix-valued analogue of this and the scalar case when the sequences $\{a_n\}$ and $\{b_n\}$ are periodic. There is a close connection between the two settings, so as we proceed the transition will be smooth.

The difference equation above of course gives rise to a Jacobi operator, which is an operator of the type

$$\mathcal{J} = \begin{pmatrix} b_1 & a_1 & 0 & \\ a_1 & b_2 & a_2 & \ddots \\ 0 & a_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.1.2}$$

acting on an ℓ^2 space. It is a classical fact that the orthonormal polynomials of the spectral measure for this operator satisfies the recursion equation (1.1.1). The asymptotic behavior of these orthonormal polynomials is known as Szegő asymptotics. By now this has been a very well-studied topic for the scalar case. For measures having a single interval as their essential support this was opened with the originating Szegő's paper [Sze20], and closed recently by Damanik and Simon, who found necessary and sufficient conditions for the Szegő asymptotics to hold (see [DS06a] and references therein; see also a discussion in Section 1.3.2 below). For the extensions of the results in terms of more general supports of the measures, see papers by Peherstorfer–Yuditski [PY03], Christiansen–Simon–Zinchenko [CSZa, CSZb], and references therein.

Szegő asymptotics for the matrix-valued case, however, is far less well studied at this point. This will be our first topic of interest here; for the details and further discussion, we refer the reader to Section 1.3.2.

Apart from the orthonormal polynomials, there are of course many other solutions to the recursion (1.1.1). Another natural candidate is the so-called Weyl solution, which is simply the unique (up to a multiplicative constant) decaying one. We say that Jost asymptotics holds if this Weyl solution behaves as the Weyl solution for the free Jacobi matrix (which is, the matrix (1.1.2) with $a_n \equiv 1$, $b_n \equiv 0$; for the relevant definitions see Section 1.3.3). If this is the case, then after a certain normalization, the Weyl solution becomes renamed to Jost.

It turns out that Szegő asymptotics holds at a given point if and only if so does the Jost asymptotics, which provides the link to the subject we discussed earlier. Instead of trying to establish the asymptotics for the most general setting though, we now study the behavior of the Jost solution when the Jacobi matrix $\mathcal{J}$ is exponentially close to being free. By this we mean that the parameters of $\mathcal{J}$ satisfy

$$|1 - a_n^2| + |b_n| \leq CR^{-2n} \tag{1.1.3}$$

for some $R > 1$. It turns out that one can relate, in an if-and-only-if fashion, the rate of exponential convergence R of the parameters and certain analytic properties of the Jost solution. For the scalar case this was done by Damanik and Simon in [DS06b] (see also [GC80]). The matrix-valued analogue of these results is the second topic of our interest here, which is discussed in Section 1.3.3. Jost asymptotics for the matrix-valued case was studied earlier by Geronimo [Ger82], where he established one of the directions of our if-and-only-if result (more details are in Section 1.3.3). For the interested reader, there are closely related results in the theory of orthogonal polynomials on the *unit circle* (as opposed to the *real line* here) and Schrödinger operators: see [NT89, Sim05] and [CS89, New82], respectively.

Hopefully the use of "asymptotics for orthogonal polynomials" and "exponentially small perturbations" in the title has become clear by now. We only need to cover the last part "meromorphic continuations of Herglotz functions".

Given a Jacobi matrix $\mathcal{J}$ as above, let us define the Weyl–Titchmarsh m-function $m(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x-z}$, where μ is the spectral measure of $\mathcal{J}$. This is a meromorphic Herglotz function on $\mathbb{C} \setminus \operatorname{ess\,supp} \mu$ (recall that a Herglotz function is a function satisfying $\operatorname{Im} m(z) \geq 0$ if $\operatorname{Im} z \geq 0$).

Using the above result and the connection between the m-function and the Jost solution, we are able to derive an (if-and-only-if) criterion for a Herglotz function to be the m-function of a Jacobi matrix satisfying (1.1.3). The central condition here is the existence of a meromorphic continuation of m through $\operatorname{ess\,supp} \mu = [-2, 2]$ to some domain of the second sheet of a natural hyperelliptic Riemann surface. This is done in Section 1.3.4. This problem seems not to have been studied before, even in the scalar case, though the methods of Damanik–Simon paper [DS06b] are all that is needed.

The above correspondence of meromorphic continuations of m-functions and exponential perturbations of the free Jacobi matrix can be further extended. Note that the free Jacobi matrix can be viewed as a periodic Jacobi matrix ($a_{n+p} = a_n$, $b_{n+p} = b_n$ for all n) with period $p = 1$. Therefore it is natural to consider exponentially small perturbations of a general p-periodic (scalar) Jacobi matrix and wonder if the m-function behaves in a similar manner. It turns out it does. Note that the essential spectrum of the spectral measure of any such matrix is a finite union of closed intervals

$$\operatorname{ess\,supp} \mu = \mathfrak{e} = \bigcup_{j=1}^{p} [\alpha_j, \beta_j], \quad \alpha_j < \beta_j < \alpha_{j+1}.$$

In Section 1.3.5 we find a necessary and sufficient criterion for a Herglotz function m to be the m-function of a Jacobi matrix $\mathcal{J}$ that satisfies

$$d_n(\mathcal{J}, \mathcal{T}_{\mathfrak{e}}) \leq C R^{-2n},$$

where $\mathcal{T}_{\mathfrak{e}}$ is the isospectral torus of Jacobi matrices with essential spectrum $\mathfrak{e}$, and d_n is an appropriately defined distance analogous to the one in (1.1.3). The central condition is again the existence of a meromorphic continuation of m through the intervals $\mathfrak{e}$ to some domain of a certain hyperelliptic Riemann surface $\mathcal{S}_{\mathfrak{e}}$. Morally, the closer $\mathcal{J}$ is to being periodic, the larger is the domain of meromorphicity of the m-function. As a special case, we are able to characterize the m-functions of eventually periodic Jacobi matrices.

The two main tools we use to prove the results of Section 1.3.5 are the formula of Damanik–Killip–Simon [DKS] that gives the connection to matrix-valued orthogonal polynomials, and the results we obtained in Section 1.3.4. The methods require the condition that the harmonic measures of intervals of $\mathfrak{e}$ are equal. Even though in a sense this is a generic case, this requirement cannot be overcome without completely changing the machinery. This is left as an open question for now.

The characterizations obtained in Sections 1.3.4 and 1.3.5 give us some consequences regarding point perturbations of measures. Namely, assume we are given a Jacobi matrix with the spectral measure μ, and we want to add/remove a pure point to/from the spectrum. The question is — how badly does the Jacobi matrix get changed? We answer these types of questions for the cases of exponentially small perturbations of (matrix-valued) free and (scalar) periodic Jacobi matrices. The perturbed free case, but for the scalar Jacobi matrices only, was considered by Geronimo in [Ger94] (see also Geronimo–Nevai [GN83]). We list the obtained results in Section 1.3.6. Note also that the so-called double commutation method of Gesztesy and Teschl [GT96] gives more or less an explicit formula for the parameters of the new Jacobi matrix. Using this, one might expect to get similar, if not identical, results, but again: only for the perturbations of the scalar free case.

Finally, when studying the (Szegő or Jost) asymptotics for matrix-valued orthogonal polynomials, one has to consider the notion of equivalent and asymptotic Jacobi matrices (see Definitions 1.3.1 and 1.3.5). In Section 1.3.1 we prove two results in this area. The first settles a question of Damanik–Pushnitski–Simon [DPS08] by showing that a so-called type 2 block Jacobi matrix in the Nevai class has converging Jacobi parameters. The second result finds a condition on the Jacobi parameters that ensures that type 1, 2, and 3 Jacobi matrices are pairwise asymptotic.

The results of Section 1.3.1 appear in [Koz10a], and those of Section 1.3.2 in [Koz10b]. The results of Sections 1.3.3–1.3.6 are still in preparation [Koza, Kozb, Kozc].

The organization of the thesis is as follows. We will start with some basics in Section

1.2, just enough to make it possible to state the main results, which will be done in Section 1.3. Chapter 2 contains all of the preliminaries that will be used throughout the proofs. Chapter 3 consists of the actual proofs, as well as the rest of the theorems that did not make it to the main results. Both Section 1.3 and Chapter 3 are divided into six parts, corresponding to the topic breakdown we mentioned in this section.

1.2 Basics

1.2.1 Orthogonal Polynomials on the Real Line

We will introduce some basics of orthogonal polynomials on the real line here. We immediately start with the matrix-valued theory to avoid repetition. The scalar theory is of course a special case $l = 1$. We will mention the differences between the scalar and matrix-valued cases as we proceed.

The proofs of most of the results listed here, along with more details, can be found in the paper by Damanik–Pushnitski–Simon [DPS08].

Let μ be an $l \times l$ matrix-valued Hermitian positive semi-definite finite measure on $\mathbb{R}$ of compact support, normalized by $\mu(\mathbb{R}) = \mathbf{1}$, where $\mathbf{1}$ is the $l \times l$ identity matrix. For any $l \times l$ dimensional matrix functions f, g, define

$$\langle\!\langle f, g \rangle\!\rangle_{L^2(\mu)} = \int f(x)^* d\mu(x) g(x); \tag{1.2.1}$$

$$\langle\!\langle f \rangle\!\rangle^2_{L^2(\mu)} = \langle\!\langle f, f \rangle\!\rangle_{L^2(\mu)}, \tag{1.2.2}$$

where * is the Hermitian conjugation (just complex conjugation if $l = 1$). Here we can regard $\langle\!\langle f \rangle\!\rangle_{L^2(\mu)}$ as the square root of the non-negative definite matrix $\langle\!\langle f, f \rangle\!\rangle_{L^2(\mu)}$.

What we have defined here is the right product of f and g, as opposed to the left product $\int f(x) d\mu(x) g(x)^*$, whose properties are completely analogous.

Measure μ is called non-trivial if $|| \langle\!\langle f \rangle\!\rangle^2_{L^2(\mu)} || > 0$ for all matrix-valued polynomials f. From now on assume μ is non-trivial. Then there exist unique (right) monic polynomials $\mathbf{P}_n^R$ of degree n satisfying

$$\left\langle\!\left\langle \mathbf{P}_n^R, f \right\rangle\!\right\rangle_{L^2(\mu)} = 0 \quad \text{for any polynomial } f \text{ with } \deg f < n.$$

For any choice of unitary $l \times l$ matrices τ_n (we demand $\tau_0 = \mathbf{1}$), the polynomials

$$\mathfrak{p}_n^R = \mathbf{P}_n^R \left\langle\!\left\langle \mathbf{P}_n^R \right\rangle\!\right\rangle_{L^2(\mu)}^{-1} \tau_n \tag{1.2.3}$$

are orthonormal:

$$\left\langle\!\left\langle \mathfrak{p}_n^R, \mathfrak{p}_m^R \right\rangle\!\right\rangle_{L^2(\mu)} = \delta_{n,m}\mathbf{1},$$

where $\delta_{n,m}$ is the Kronecker δ. Using orthogonality one can show that they satisfy the (Jacobi) recurrence relation

$$x\mathfrak{p}_n^R(x) = \mathfrak{p}_{n+1}^R(x)A_{n+1}^* + \mathfrak{p}_n^R(x)B_{n+1} + \mathfrak{p}_{n-1}^R(x)A_n, \quad n = 1, 2, \dots, \tag{1.2.4}$$

where matrices $A_n = \left\langle\!\left\langle \mathfrak{p}_{n-1}^R, x\mathfrak{p}_n^R \right\rangle\!\right\rangle_{L^2(\mu)}$, $B_n = \left\langle\!\left\langle \mathfrak{p}_{n-1}^R, x\mathfrak{p}_{n-1}^R \right\rangle\!\right\rangle_{L^2(\mu)}$ are called Jacobi parameters (with $\mathfrak{p}_{-1}^R = \mathbf{0}$, $A_0 = \mathbf{1}$, the relation holds for $n = 0$ too).

In the exact same fashion, just using the left product instead of right, one can define the left monic orthogonal polynomials $\mathbf{P}_n^L$ and left orthonormal polynomials $\mathfrak{p}_n^L$. It is not hard to see that $\mathbf{P}_n^L(z) = \mathbf{P}_n^R(\bar{z})^*$ and $\mathfrak{p}_n^L(z) = \mathfrak{p}_n^R(\bar{z})^*$.

We will be using the notation $\mathbf{P}_n$, $\mathfrak{p}_n$ for matrix-valued polynomials, while in the case $l = 1$ we will downgrade them to P_n, p_n. This will be useful in Section 3.5 since both of them will be present. Also, whenever we write $\mathfrak{p}_n$ without the sup-index ^{R} or ^{L}, we will mean the right orthonormal polynomial $\mathfrak{p}_n^R$.

Note that if $l = 1$ it is natural to choose $\tau_n = 1$ in (1.2.3). In particular this gives $p_n^R = p_n^L$, the Jacobi parameters become real, and A_n's positive. This choice of τ_n's is not necessarily the best if $l > 1$. See Section 1.3.1 for the further discussion.

We can arrange sequences $\{A_n\}_{n=1}^\infty$, $\{B_n\}_{n=1}^\infty$ (called Jacobi parameters) into an infinite matrix

$$\mathcal{J} = \begin{pmatrix} B_1 & A_1 & \mathbf{0} & \\ A_1^* & B_2 & A_2 & \ddots \\ \mathbf{0} & A_2^* & B_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.2.5}$$

This is called a block Jacobi matrix if $l > 1$. If $l = 1$ then we lose the word "block" and the Jacobi coefficients A_n, B_n become a_n, b_n.

If $A_n \equiv \mathbf{1}$, $B_n \equiv \mathbf{0}$ the corresponding (block) Jacobi matrix is called **free**.

Conversely, any block Jacobi matrix (1.2.5) with invertible $\{A_n\}_{n=1}^\infty$ gives rise to a matrix-valued Hermitian measure μ via the spectral theorem. If $l = 1$ this establishes a one-to-one correspondence between all non-trivial compactly supported measures and bounded Jacobi matrices. If $l > 1$ the same holds, except now the correspondence is with the set of *equivalence classes* of bounded block Jacobi matrices (see Definition 1.3.1). This has the name of Favard's Theorem (see [DPS08] for a proof in the matrix-valued case).

Since we will be considering perturbations of the free case in Sections 1.3.2–1.3.4, the following two classical results will prove to be useful.

Theorem 1.2.1 (Weyl's Theorem)**.** *If* $A_n \to \mathbf{1}$, $B_n \to \mathbf{0}$, *then* $\operatorname{ess\,supp} \mu = [-2, 2]$.

Theorem 1.2.2 (Denisov–Rakhmanov Theorem)**.** *Assume* μ *is a non-trivial* $l \times l$ *matrix-valued measure on* $\mathbb{R}$ *with associated block Jacobi matrix* $\mathcal{J}$ *of type* 3 *such that* $\operatorname{ess\,supp} \mu = [-2, 2]$ *and* $\det \left(\frac{d\mu(x)}{dx} \right) > 0$ *a.e. on* $[-2, 2]$. *Then* $A_n \to \mathbf{1}$, $B_n \to \mathbf{0}$.

The first result is trivial, while the second, in the form given here, is proven in [DKS] (see also [YM01], as well as [Den04, Rak82]).

Define the (Weyl-Titchmarsh) m**-function** of the measure μ to be the meromorphic in $\mathbb{C} \setminus \operatorname{ess\,supp} \mu$ matrix-valued function

$$\mathfrak{m}(z) = \int \frac{d\mu(x)}{x - z}. \tag{1.2.6}$$

Again, we will use the letter m instead of $\mathfrak{m}$ if $l = 1$.

Define $\mathcal{J}^{(1)}$ to be the "once-stripped" Jacobi matrix with Jacobi parameters $\{A_n, B_n\}_{n=2}^\infty$, i.e., the Jacobi matrix of the form (1.2.5) with the first row and column removed. Then the following holds (the matrix-valued version is due to [AN83]):

$$A_1 \mathfrak{m}(z; \mathcal{J}^{(1)}) A_1^* = B_1 - z - \mathfrak{m}(z; \mathcal{J})^{-1}. \tag{1.2.7}$$

1.2.2 Herglotz Functions

Definition 1.2.3. *An analytic in* $\mathbb{C}_+ \equiv \{z : \operatorname{Im} z > 0\}$ $l \times l$ *matrix-valued function* m *is called* ***Herglotz*** *if* $\operatorname{Im} m(z) \geq \mathbf{0}$ *for all* $z \in \mathbb{C}_+$.

Here $\operatorname{Im} T \equiv \frac{T-T^*}{2i}$.

We can also define m on the lower half plane $\mathbb{C}_-$ by reflection $m(z) = m(\bar{z})^*$, so that $\operatorname{Im} m(z) \leq \mathbf{0}$ for all z with $\operatorname{Im} z < 0$. In particular the m-function $\mathfrak{m}$ defined in (1.2.6) is Herglotz.

We will assume from now on that $\det \operatorname{Im} m(z)$ is not identically zero, in which case the inequality in $\operatorname{Im} m(z) \gtrless \mathbf{0}$ is everywhere strict (see [GT00, Lemma 5.3]).

The following result is well-known (see, e.g., [GT00, Thm 5.4]).

Lemma 1.2.4. *Let m be an $l \times l$ matrix-valued Herglotz function. Then there exist an $l \times l$ matrix-valued measure μ on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{1}{1+x^2} d\mu(x) < \infty$, and constant matrices $C = C^*, D \geq \mathbf{0}$ such that*

$$m(z) = C + Dz + \int_{\mathbb{R}} \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x), \quad z \in \mathbb{C}_+. \tag{1.2.8}$$

The absolutely continuous part of μ can be recovered from this representation by

$$f(x) \equiv \frac{d\mu}{dx} = \pi^{-1} \lim_{\varepsilon \downarrow 0} \operatorname{Im} m(x + i\varepsilon), \tag{1.2.9}$$

and the pure point part by

$$\mu(\{\lambda\}) = \lim_{\varepsilon \downarrow 0} \varepsilon \operatorname{Im} m(\lambda + i\varepsilon) = \lim_{\varepsilon \downarrow 0} \varepsilon\, m(\lambda + i\varepsilon). \tag{1.2.10}$$

Definition 1.2.5. *A* ***discrete*** m***-function*** *is a Herglotz function, $m(z)$, which has an analytic continuation from $\mathbb{C}_+$ to $\mathbb{C} \setminus I$ for some bounded interval $I \subset \mathbb{R}$, and satisfies*

$$z \in \mathbb{R} \setminus I \Rightarrow \operatorname{Im} m(z) = \mathbf{0},$$
$$m(z) = z^{-1}\mathbf{1} + O(z^{-2}) \text{ at } \infty.$$

The following is immediate from Lemma 1.2.4.

Lemma 1.2.6. *A function $m(z)$ on $\mathbb{C}_+$ is a discrete m-function if and only if*

$$m(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x-z}$$

for some probability measure μ on $\mathbb{R}$ with bounded support.

1.2.3 Finite Gap Sets and Surface $\mathcal{S}_{\mathfrak{e}}$

In this subsection let us assume that μ is finite and its essential support is a finite union of closed intervals ("**finite gap set**")

$$\operatorname{ess\,supp}\mu = \mathfrak{e} = \bigcup_{j=1}^{g+1}[\alpha_j, \beta_j], \quad \alpha_j < \beta_j < \alpha_{j+1}. \tag{1.2.11}$$

We will be referring to each of $[\alpha_j, \beta_j]$ $(1 \le j \le g+1)$ as "**bands**", and $[\beta_j, \alpha_{j+1}]$ $(1 \le j \le g)$ as "**gaps**". The reason for considering such measures is of course because the spectral measures of compact perturbations of free and periodic Jacobi matrices have their essential spectrum of this type.

Then m is a meromorphic function on $\mathbb{C}\setminus\mathfrak{e}$ and it is natural to ask if m has a meromorphic continuation through $\mathfrak{e}$. Let us introduce a natural Riemann surface that will be used extensively throughout the paper.

Definition 1.2.7. *Assume $\mathfrak{e}$ is a finite gap set* (1.2.11)*. Define $\mathcal{S}_{\mathfrak{e}}$ to be the be the hyperelliptic Riemann surface corresponding to the polynomial $\prod_{j=1}^{g+1}(z-\alpha_j)(z-\beta_j)$.*

We will not give the formal definition, which can be found in any textbook. Informally $\mathcal{S}_{\mathfrak{e}}$ can be described as follows. Denote $\mathcal{S}_+$ and $\mathcal{S}_-$ to be two copies of $\mathbb{C}\cup\{\infty\}$ with a slit along $\mathfrak{e}$ (include $\mathfrak{e}$ as a top edge and exclude it from the lower), and let $\mathcal{S}_{\mathfrak{e}}$ be $\mathcal{S}_+$ and $\mathcal{S}_-$ glued together along $\mathfrak{e}$ in the following way: passing from $\mathbb{C}_+\cap\mathcal{S}_+$ through $\mathfrak{e}$ takes us to $\mathbb{C}_-\cap\mathcal{S}_-$, and from $\mathbb{C}_-\cap\mathcal{S}_+$ to $\mathbb{C}_+\cap\mathcal{S}_-$. Clearly $\mathcal{S}_{\mathfrak{e}}$ is topologically an orientable manifold of genus g.

Let $\pi : \mathcal{S} \to \mathbb{C}\cup\{\infty\}$ be the "projection map" which extends the natural inclusions $\mathcal{S}_+ \hookrightarrow \mathbb{C}\cup\{\infty\}$, $\mathcal{S}_- \hookrightarrow \mathbb{C}\cup\{\infty\}$.

The following notation will be used frequently throughout the paper.

Definition 1.2.8.

- *Denote by z_+ and z_- the two preimages $\pi^{-1}(z)$ of $z \in \mathbb{C}\cup\{\infty\}$ (for $z \in \cup_{j=1}^{g+1}\{\alpha_j,\beta_j\}$, z_+ and z_- coincide).*
- *Let $z^\sharp$ be $\left(\overline{\pi(z)}\right)_-$ if $z\in\mathcal{S}_+$, and $\left(\overline{\pi(z)}\right)_+$ if $z\in\mathcal{S}_-$. In order to make this continuous we make the convention $z^\sharp = z$ for $z\in\pi^{-1}(\mathfrak{e})$.*
- *Let $m^\sharp(z) = m(z^\sharp)^*$.*

1.2.4 Periodic Orthogonal Polynomials on the Real Line

For all the proofs, we refer the reader to [Sim].

A (scalar) Jacobi matrix (1.2.5) is called periodic if there exists an integer $p \geq 1$ such that

$$a_{n+p} = a_n, b_{n+p} = b_n \quad \text{for all } n \tag{1.2.12}$$

(with a_n, b_n instead of A_n, B_n since $l = 1$). One can also talk about two-sided Jacobi matrices, which are operators on $\ell^2(\mathbb{Z})$ of the same tridiagonal form as (1.2.5), where we just extend the indices $\{a_n, b_n\}_{n\in\mathbb{Z}}$ to the whole $\mathbb{Z}$. The same definition of periodicity (1.2.12) applies to a two-sided Jacobi matrix as well. We will commonly use $(a_n, b_n)_{n=1}^{\infty}$, $(a_n, b_n)_{n\in\mathbb{Z}}$ as a notation for these matrices.

For a one- or two-sided periodic Jacobi matrix one can associate a polynomial of degree p with real coefficients

$$\Delta(z) = \operatorname{Tr}\left(\prod_{j=p}^{1} \frac{1}{a_j} \begin{pmatrix} z - b_j & -1 \\ a_j^2 & 0 \end{pmatrix}\right), \tag{1.2.13}$$

which is called the **discriminant** of the matrix. Note that this is just the trace of the update matrix corresponding to the vector $(u_{n+1}, a_n u_n)^T$, where u_n is a solution to the recurrence

$$zu_n = a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1}.$$

It has numerous useful properties, which we list in Section 2.2. The most important for us here is that it determines the spectrum. It turns out that the spectrum of two-sided periodic Jacobi matrix is purely absolutely continuous of multiplicity two, and

$$\sigma((a_n, b_n)_{n\in\mathbb{Z}}) = \Delta^{-1}([-2, 2]).$$

Essential spectrum of one-sided periodic Jacobi matrix is purely absolutely continuous of multiplicity one and we still have

$$\sigma_{ess}((a_n, b_n)_{n=1}^{\infty}) = \Delta^{-1}([-2, 2]).$$

In fact $\Delta^{-1}([-2,2])$ is a finite gap set

$$\mathfrak{e} \equiv \bigcup_{j=1}^{p} [\alpha_j, \beta_j], \quad \alpha_j < \beta_j \le \alpha_{j+1}, \tag{1.2.14}$$

where these intervals are allowed to touch. If some two intervals do touch $\beta_j = \alpha_{j+1}$, then this gap $[\beta_j, \alpha_{j+1}]$ is said to be **closed**, and otherwise it is **open**. Let g be the number of open gaps (in other words, $\mathfrak{e}$ consists precisely of $g+1$ *disjoint* closed intervals), which is consistent with the notation in the previous section.

Finally, $\sigma_{ess}((a_n, b_n)_{n=1}^\infty) \setminus \Delta^{-1}([-2,2])$ may consist of up to g eigenvalues, at most one per each open gap.

Denote $\rho_\mathfrak{e}$ to be the equilibrium measure of $\mathfrak{e}$.

There is an easy criterion for determining when a finite gap set $\mathfrak{e}$ is the (essential) spectrum of some periodic Jacobi matrix.

Lemma 1.2.9. *Let $\mathfrak{e}$ be a finite gap set* (1.2.14)*.*

- *$\mathfrak{e}$ is the (essential) spectrum of some periodic Jacobi matrix if and only if the equilibrium measure of each of the $g+1$ disjoint intervals of $\mathfrak{e}$ is rational.*

- *$\mathfrak{e}$ is the (essential) spectrum of some p-periodic Jacobi matrix with all gaps open if and only if the equilibrium measures of each of the $p = g+1$ disjoint intervals of $\mathfrak{e}$ are equal (and so equal to $1/p$).*

- *If at least one of the $g+1$ disjoint intervals of $\mathfrak{e}$ has irrational equilibrium measure, then one can construct an almost periodic Jacobi matrix with essential spectrum $\mathfrak{e}$.*

We will not go into the the theory of almost periodic Jacobi matrices here ([Sim]).

Now let m be the m-function for a periodic one-sided Jacobi matrix $\mathcal{J} = (a_n, b_n)_{n=1}^\infty$ defined in (1.2.6). Using the recursion (1.2.7) and the fact $\mathcal{J} = \mathcal{J}^{(p)}$, we immediately obtain that m satisfies a certain quadratic equation. After some work one sees that

$$m(z) = \frac{r(z) \pm \sqrt{\Delta^2(z) - 4}}{t(z)}, \tag{1.2.15}$$

where we choose the branch of square root $\sqrt{\Delta^2(z)-4} = \Delta(z) + O(1/\Delta(z))$. Here $r(z), t(z)$ are some polynomials in z. Going back to (1.2.14), one now sees that m has a meromorphic continuation to the full surface $\mathcal{S}_{\mathfrak{e}}$, the genus g hyperelliptic surface constructed in Definition 1.2.7.

Moreover, m has minimal degree $g+1$ (as a topological analytic map $\mathcal{S}_{\mathfrak{e}} \to \mathbb{C} \cup \{\infty\}$) among all meromorphic functions on $\mathcal{S}$ that are not of the form $f \circ \pi$ for some meromorphic f on $\mathbb{C} \cup \{\infty\}$. It turns out there is a one-to-one correspondence between all such minimal meromorphic functions that are Herglotz on $\mathcal{S}_+$ with zero at ∞_+ and a pole at ∞_- and all periodic Jacobi matrices with the same discriminant Δ.

Each such mimimal Herglotz function is completely determined by the location of its poles. There are $g+1$ of them, one at ∞_-, and exactly one on $\pi^{-1}([\beta_j, \alpha_{j+1}])$ for each open gap $[\beta_j, \alpha_{j+1}]$. Note that $\pi^{-1}([\beta_j, \alpha_{j+1}])$ is homeomorphic to a circle S^1. Thus the set of minimal Herglotz functions, and consequently the set of periodic Jacobi matrices with discriminant Δ, is homeomorphic to $(S^1)^g$, a g-dimensional torus.

Definition 1.2.10. *The* ***isospectral torus*** $\mathcal{T}_{\mathfrak{e}}$ *of* $\mathfrak{e}$ *is the set of periodic Jacobi matrices with the same discriminant* Δ *(and consequently, the same essential spectrum).*

We will view $\mathcal{T}_{\mathfrak{e}}$ as a set of $\{(a_n, b_n)_{n=1}^{\infty}\}$ or $\{(a_n, b_n)_{n\in\mathbb{Z}}\}$ depending on the context.

In order to measure closeness of two (one- or two-sided) Jacobi matrices at infinity, let us introduce the following metric (on $\prod_{j=m}^{\infty}(0,R] \times [-R,R]$)

$$d_m((a_n, b_n), (a'_n, b'_n)) = \sum_{k=0}^{\infty} e^{-k}(|a_{m+k} - a'_{m+k}| + |b_{m+k} - b'_{m+k}|),$$

and then

$$d_m((a_n, b_n), \mathcal{T}) = \inf\{d_m((a_n, b_n), (a'_n, b'_n)) \mid (a'_n, b'_n) \in \mathcal{T}\}$$

for any set $\mathcal{T}$. Here (a_n, b_n) can be $(a_n, b_n)_{n=1}^{\infty}$ or $(a_n, b_n)_{n\in\mathbb{Z}}$.

1.3 Main Results

1.3.1 Equivalence Classes of Block Jacobi Matrices

Definition 1.3.1. *Two block Jacobi matrices* $\mathcal{J}$ *and* $\widetilde{\mathcal{J}}$ *are called* ***equivalent*** *if they correspond to the same spectral measure* μ *(but a different choice of* τ_n*'s in* (1.2.3)).

They are equivalent if and only if their Jacobi parameters satisfy

$$\widetilde{A}_n = \sigma_n^* A_n \sigma_{n+1}, \quad \widetilde{B}_n = \sigma_n^* B_n \sigma_n \tag{1.3.1}$$

for unitary σ_n's with $\sigma_1 = \mathbf{1}$ (the connection with τ_j's is $\sigma_n = \tau_{n-1}^* \widetilde{\tau}_{n-1}$). It is easy to see that

$$\widetilde{\mathfrak{p}}_n^R(x) = \mathfrak{p}_n^R(x)\sigma_{n+1}, \tag{1.3.2}$$

where $\widetilde{\mathfrak{p}}_n$ are the orthonormal polynomials for $\widetilde{\mathcal{J}}$ associated with the Jacobi parameters $\{\widetilde{A}_n\}_{n=1}^\infty$, $\{\widetilde{B}_n\}_{n=1}^\infty$.

Definition 1.3.2. *A block Jacobi matrix is of* ***type*** 1 *if* $A_n > 0$ *for all* n*, of* ***type*** 2 *if* $A_1 A_2 \dots A_n > 0$ *for all* n*, and of* ***type*** 3 *if every* A_n *is lower triangular with strictly positive elements on the diagonal.*

Each equivalence class of block Jacobi matrices contains exactly one matrix of type 1, 2, and 3 (follows from the uniqueness of the polar and QR decompositions, see [DPS08] for the proof).

Definition 1.3.3. *We say that* $\mathcal{J}$ *is in the* ***Nevai class*** *if*

$$B_n \to \mathbf{0}, \quad A_n A_n^* \to \mathbf{1}.$$

Note that this definition is invariant within the equivalence class of Jacobi matrices. Our first result is

Theorem 1.3.4. *Assume* $\mathcal{J}$ *belongs to the Nevai class. If* $\mathcal{J}$ *is of type* 1, 2, *or* 3, *then* $A_n \to \mathbf{1}$ *as* $n \to \infty$.

This result was proven in [DPS08] for the type 1 and 3 cases, and was left open for type 2. It is proven here in Section 3.1.

The essence of Theorem 1.3.4 is to show that $\sigma_n^* \sigma_{n+1} \to \mathbf{1}$, where σ_n's are the unitary coefficients from (1.3.1) for $\mathcal{J}$, $\widetilde{\mathcal{J}}$ of type 1, 2 or 3. Note however that we are interested in the asymptotics of the orthonormal polynomials as $n \to \infty$, and because of the relation (1.3.2), it is desirable to know when $\lim_{n\to\infty} \sigma_n$ exists. This explains the need of the following definition.

Definition 1.3.5. *Two equivalent matrices $\mathcal{J}$ and $\widetilde{\mathcal{J}}$ with* (1.3.1) *are called* ***asymptotic*** *to each other if the limit* $\lim_{n\to\infty}\sigma_n$ *exists.*

Clearly this is an equivalence relation on the class of equivalent Jacobi matrices. Note that establishing asymptotics for orthonormal polynomials automatically establishes the corresponding asymptotics for the polynomials corresponding to any Jacobi matrix asymptotic to the original one.

We prove the following theorem.

Theorem 1.3.6. *Assume*

$$\sum_{n=1}^{\infty}[\|\mathbf{1}-A_nA_n^*\|+\|B_n\|]<\infty. \tag{1.3.3}$$

Then the corresponding Jacobi matrices of type 1*,* 2*, and* 3 *are pairwise asymptotic.*

Remarks. 1. The condition (1.3.3) doesn't depend on the choice of the representative of the equivalence class of equivalent matrices.

2. The proof also shows that any equivalent Jacobi matrix, for which eventually each A_n has real eigenvalues, is also asymptotic to type 1, 2, 3.

3. An example of an equivalence class of block Jacobi matrices that fails (1.3.3) and that has type 1 and type 2 nonasymptotic to each other can be found at the end of Section 3.1.1.

1.3.2 Szegő Asymptotics for Matrix-Valued Measures with Countably Many Bound States

We are interested in the asymptotic behavior of $\mathfrak{p}_n$ for the measures μ whose essential support is a single interval. After scaling and translating, we can assume it is $[-2,2]$:

$$\operatorname{ess\,supp}\mu=[-2,2]. \tag{1.3.4}$$

Let $\{E_j\}_{j=1}^N$ be the point masses of μ outside $[-2,2]$ counting multiplicities ($N\le\infty$). Aptekarev and Nikishin in [AN83] show that if the absolutely continuous part $f(x)=\frac{d\mu(x)}{dx}$

satisfies the Szegő condition

$$\int_{-2}^{2} (4-x^2)^{-1/2} \log(\det(f(x)))dx > -\infty, \tag{1.3.5}$$

and N is finite, then there exists $\lim_{n\to\infty} z^n \mathfrak{p}_n(z+z^{-1})$ uniformly on the compacts of $\mathbb{D}$, and the limit function was constructed more or less explicitly. The scalar case $l=1$ (see Peherstorfer–Yuditskii [PY01]; another approach is the combination of Killip–Simon [KS03] and Damanik–Simon [DS06a]: see [Sim, Chapter 3]) suggests that $N=\infty$ should not really spoil the picture as long as the condition

$$\sum_{j=1}^{N} (|E_j|-2)^{1/2} < \infty \tag{1.3.6}$$

holds. In fact this condition is necessary if one expects to have the limit $\lim_{n\to\infty} z^n \mathfrak{p}_n(z+z^{-1})$ to be a Nevanlinna function in $\mathbb{D}$.

Assume that (1.3.4), (1.3.5), and (1.3.6) hold. We prove in Theorem 1.3.7 below that under these assumptions $\lim_{n\to\infty} z^n \mathfrak{p}_n(z+z^{-1})$ exists uniformly in $\mathbb{D}$ and we give a characterization of the limit function. The results are the exact extension to the matrix-valued case of [PY01], and include [AN83, Thm. 2] as its special case ($N<\infty$).

To prove the result, Aptekarev and Nikishin in [AN83] used an induction on the number of the point masses of μ, which does not work if there are infinitely many of them. The approach used here is similar to the one used in [PY01] for the scalar case (which in turn is an extension of the original Szegő's proof for the no-bound states problem, see [Sze20]). Namely, we first construct a Nevanlinna function $L(z)$ (Section 3.2.1), and then consider a certain inner product which, when handled with care, proves that the limit of $z^n \mathfrak{p}_n(z+z^{-1})$ is indeed L (Section 3.2.2).

Theorem 1.3.7. *Let μ satisfy (1.3.4), (1.3.5), (1.3.6). Assume $\mathcal{J}$ is of type 2. Then there exists an analytic in $\mathbb{D}$ function L such that*

$$z^n \mathfrak{p}_n\left(z+z^{-1}\right) \to L(z) \quad \textit{uniformly on compacts of } \mathbb{D}; \tag{1.3.7}$$

$$\mathfrak{p}_n(2\cos\theta) = \frac{\left(e^{-in\theta}L(e^{i\theta}) + e^{in\theta}L(e^{-i\theta})\right)}{\sqrt{2}} + o(1) \quad \textit{in } L^2\left(w(\theta)\frac{d\theta}{2\pi}\right) \textit{ sense;} \tag{1.3.8}$$

$$\langle\!\langle \mathfrak{p}_n(x) \rangle\!\rangle_{L^2(\mu_s)} \to \mathbf{0}, \tag{1.3.9}$$

where w is

$$w(\theta) = 2\pi |\sin\theta| \, \frac{d\mu}{dx} \, (2\cos\theta) \, .$$

Remarks. 1. The limit function L is in fact in the matrix-valued Hardy space $\mathbb{H}^2(\mathbb{D})$, and we establish a number of its properties, in particular its multiplicative factorization, see Section 3.2.1.

2. We will show that the asymptotics holds for type 2 Jacobi matrix. Thus by (1.3.2), the polynomials $\widetilde{p}_n$ obey Szegő asymptotics if and only if the limit $\lim_{n\to\infty} \sigma_n$ exists, i.e., if and only if matrix $\widetilde{\mathcal{J}}$ is asymptotic to type 2.

Using results from Section 14 of [DKS] we immediately obtain

Corollary 1.3.8. *Assume the Jacobi parameters of $\mathcal{J}$ satisfy*

$$\sum_{n=1}^{\infty} [\|1 - A_n A_n^*\| + \|B_n\|] < \infty. \tag{1.3.10}$$

Then the associated measure μ satisfies (1.3.4), (1.3.5), (1.3.6), *and so the conclusions of Theorem 1.3.7 hold.*

Remarks. 1. As in Theorem 1.3.7 this establishes Szegő asymptotics for the type 2 Jacobi matrix, as well as for all Jacobi matrices asymptotic to type 2. Therefore by Theorem 1.3.6 Szegő asymptotics holds for matrices of type 1 and 3 (or more generally, for any $\widetilde{\mathcal{J}}$ the $\widetilde{A}_n$-coefficients of which have eventually only real eigenvalues).

2. See also another proof of Corollary 1.3.8 using Jost asymptotics in Theorem 3.3.4.

1.3.3 Jost Asymptotics for Matrix Orthogonal Polynomials

The results of this subsection follow closely the scalar results of Damanik–Simon [DS06b] (see also [Sim05]). Apart from technical complications, the ideas of the proofs are borrowed from the mentioned paper.

We are interested in the $l \times l$ matrix-valued solutions $(f_n(E))_{n=0}^{\infty}$ of

$$f_{n+1}(E) A_n^* + f_n(E)(B_n - E\mathbf{1}) + f_{n-1}(E) A_{n-1} = \mathbf{0}, \quad n = 1, 2, \ldots \tag{1.3.11}$$

By (1.2.4), one solution of this is $f_n(E) = \mathfrak{p}_{n-1}^R(E, \mathcal{J})$.

Definition 1.3.9. *For any two sequences* $(v_n)_{n=0}^{\infty}$, $(w_n)_{n=0}^{\infty}$ *their* ***Wronskian*** *is*

$$W_n(v, w; \mathcal{J}) = v_n A_n w_{n+1} - v_{n+1} A_n^* w_n.$$

If $v_n(E)$ and $w_n(E)$ both solve (1.3.11), then $W_n(v_n(E), w_n(\bar{E})^*)$ is independent of n (see [DPS08]).

In this subsection we will be considering only $\mathcal{J}$ with $\operatorname{ess\,supp} \mu = [-2, 2]$, so it will be convenient to move from $\mathbb{C} \setminus [-2, 2]$ to $\mathbb{D}$ via $z + z^{-1} = E$.

Definition 1.3.10. *The* ***Jost solution***, $\{u_n(z; \mathcal{J})\}_{n=0}^{\infty}$, *is a solution of* (1.3.11) *with*

$$z^{-n} u_n(z; \mathcal{J}) \to \mathbf{1} \tag{1.3.12}$$

as $n \to \infty$, *where* $z + z^{-1} = E$.

In general there may or may not be a solution of (1.3.11) satisfying (1.3.12), though there always exists an ℓ^2 (Weyl's) solution of (1.3.11) for $z \in \mathbb{D}$.

Definition 1.3.11. *If the Jost solution exists (it is then unique, of course), then the* ***Jost function*** *is defined to be*

$$u(z; \mathcal{J}) = W(u_{\cdot}(z; \mathcal{J}), \mathfrak{p}^L_{\cdot-1}(z + z^{-1}; \mathcal{J})) = u_0(z; \mathcal{J}),$$

where $\mathfrak{p}_n^L(z)$ *are left orthonormal polynomials of* $\mathcal{J}$.

The last equality here comes from the constancy of the Wronskian.

In Section 3.3 we establish that the Jost solution and Jost function exist for block Jacobi matrices asymptotic to type 1 under the condition

$$\sum_{n=1}^{\infty} [||B_n|| + ||\mathbf{1} - A_n A_n^*||] < \infty,$$

and establish a number of their properties. See Theorems 3.3.1, 3.3.6, 3.3.10. Theorem 3.3.1 and parts (iv)–(vi) of Theorem 3.3.6 already appeared in [Ger82]. Apart from that, the three new main results here deal with the inverse direction.

Theorem 1.3.12. *Let* u *be an analytic function in a disk* $\mathbb{D}_R = \{z \mid |z| < R\}$ *for some* $R > 1$, *whose only zeros in* $\overline{\mathbb{D}}$ *lie in* $(\overline{\mathbb{D}} \cap \mathbb{R}) \setminus \{0\}$ *with those zeros all simple* (*in the meaning* *that the poles of* $u(z)^{-1}$ *are simple). For each zero* z_j *in* $(\mathbb{D}\cap\mathbb{R})\setminus\{0\}$*, let a nonzero matrix-valued weight* $w_j \geq 0$ *be given so that*

(i) $\sum_j w_j + \frac{2}{\pi}\int_0^\pi \sin^2\theta\,\left[u(e^{i\theta})^* u(e^{i\theta})\right]^{-1} d\theta = \mathbf{1}$

(ii) $\operatorname{Ran} w_j = \ker u(z_j)$ *for all* j*.*

Then there exists a unique measure $d\mu$ *for which* w_j *are the weights and* u *is its Jost function for some choice of Jacobi matrix from the equivalence class corresponding to* $d\mu$*. Any such matrix is of type asymptotic to* $\mathbf{1}$*.*

Now that we established the existence of the measure μ, we can make the following definition, and then state the last two main theorems of the section.

Definition 1.3.13. *Let* u *satisfy the conditions of Thereom 1.3.12. Suppose* u *has a zero at some* $1 > |z_j| > R^{-1}$, $\operatorname{Ran} w_j = \ker u(z_j)$*. The weight* w_j *is said to be* ***canonical*** *if*

$$\frac{z_j}{z_j^{-1} - z_j} w_j\, u(1/\bar{z}_j)^* = -(z_j - z_j^{-1}) \lim_{z\to z_j} (z - z_j) u(z)^{-1}. \tag{1.3.13}$$

Theorem 1.3.14. *If a polynomial* $u(z)$ *obeys*

(i) $u(z)$ *is invertible on* $(\overline{\mathbb{D}}\setminus\mathbb{R})\cup\{0\}$*;*

(ii) all zeros on $\overline{\mathbb{D}}\cap\mathbb{R}$ *are simple;*

(iii) $\sum_j w_j + \frac{2}{\pi}\int_0^\pi \sin^2\theta\,\left[u(e^{i\theta})^* u(e^{i\theta})\right]^{-1} d\theta = \mathbf{1}$ *for some* $w_j \geq 0$, $\operatorname{Ran} w_j = \ker u(z_j)$ *for each zero* z_j *of* u *in* $\mathbb{D}\cap\mathbb{R}$*,*

then u *is the Jost function for a Jacobi matrix with exponentially converging parameters. It has* $\mathbf{1} - A_n A_n^* = B_n = \mathbf{0}$ *for all large* n *if and only if all the weights are canonical.*

Theorem 1.3.15. *Let* $u(z)$ *be analytic in* $\mathbb{D}_R$ *for some* $R > 1$ *and obeys (i), (ii), (iii) from Theorem 1.3.14. Then* u *is the Jost function for a Jacobi matrix with exponentially converging parameters. It has*

$$\limsup_{n\to\infty} (||B_n|| + ||\mathbf{1} - A_n A_n^*||)^{1/2n} \leq R^{-1} \tag{1.3.14}$$

if and only if all weights for z_j *with* $1 > |z_j| > R^{-1}$ *are canonical.*

Remark. By "exponentially converging parameters" it is meant that they satisfy

$$\limsup_{n\to\infty} \left(||B_n|| + ||\mathbf{1} - A_n A_n^*||\right)^{1/2n} \leq r^{-1}$$

for some r (in general $r = \min_j\{|z_j|^{-1}\}$, unless some of the weights are canonical).

1.3.4 Meromorphic Continuations of Matrix Herglotz Functions and Perturbations of the Free Case

We will consider measures μ with essential support one interval. By scaling and translating we can assume that $\operatorname{ess\,supp}\mu = [-2,2]$. Instead of discussing meromorphic continuations of $\mathfrak{m}$ (see (1.2.6)) through $(-2,2)$ to $\mathcal{S}_{[-2,2]}$ (see Definition 1.2.7), it will be convenient to move $\mathbb{C}\setminus[-2,2]$ to $\mathbb{D}$ (and $\mathcal{S}_{[-2,2]}$ to $\mathbb{C}\cup\{\infty\}$) via the inverse of $z \mapsto z + z^{-1}$, and discuss the meromorphic continuations of

$$M(z) = -\mathfrak{m}(z + z^{-1}) \tag{1.3.15}$$

from $\mathbb{D}$ through $\partial\mathbb{D}$. Note that M is also Herglotz in the meaning that $\operatorname{Im} M(z) \gtrless \mathbf{0}$ if $z \in \mathbb{C}_\pm \cap \mathbb{D}$.

The analogue of $\mathfrak{m}^\sharp$ from Definition 1.2.8 is $M^\sharp(z) = M(\bar{z}^{-1})^*$.

We prove the following result.

Theorem 1.3.16. *Let $\mathfrak{m}$ be a discrete $l \times l$ matrix-valued m-function, and M is given by* (1.3.15)*. Let $R > 1$. The following are equivalent:*

(I) The corresponding to $\mathfrak{m}$ Jacobi matrix $\{A_n, B_n\}_{n=1}^\infty$ satisfies

$$\limsup_{n\to\infty} \left(||B_n|| + ||\mathbf{1} - A_n A_n^*||\right)^{1/2n} \leq R^{-1}. \tag{1.3.16}$$

(II) All of the following holds:

(A) M has a meromorphic continuation to $\mathbb{D}_R$.

(B) M has no poles on $\partial\mathbb{D}\setminus\{\pm1\}$, and at most simple poles at ±1.

(C) $(M(z) - M^\sharp(z))^{-1}$ has no poles in $R > |z| > R^{-1}$ except at $z = \pm1$ where there might be simple poles.

(D) If M has a pole at $z_j \in \{z : R^{-1} < |z| < 1\}$ and at z_j^{-1}, then

$$\operatorname{Ran} \operatorname*{Res}_{z=z_j} M(z) \subseteq \ker(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1}, \tag{1.3.17}$$

$$\operatorname{Ran} \operatorname*{Res}_{z=z_j} M(z) \subseteq \left(\operatorname{Ran}\,(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1} M(z_j^{-1})\right)^\perp. \tag{1.3.18}$$

Note that $R = \infty$ is allowed, in which case in (i) the $\limsup$ becomes $\lim$ and equals 0 (the decay of the coefficients is subexponential), while in (ii) M is meromorphic in $\mathbb{C}$. We can also demand that M is actually meromorphic in $\mathbb{C} \cup \{\infty\}$ (which, of course, is the same as saying that M is a rational matrix function), in which case (i) becomes (1.3.19). Therefore we are able to characterize all possible M-functions of eventually-free Jacobi matrices.

Theorem 1.3.17. *Let $\mathfrak{m}$ be a discrete $l \times l$ matrix-valued m-function, and M is given by* (1.3.15)*. The following are equivalent:*

(I) The corresponding to $\mathfrak{m}$ Jacobi matrix $\{A_n, B_n\}_{n=1}^\infty$ satisfies

$$||B_n|| + ||\mathbf{1} - A_n A_n^*|| = \mathbf{0} \quad \text{for all large } n. \tag{1.3.19}$$

(II) All of the following holds:

(A) M is a rational matrix function.

(B) M has no poles on $\partial\mathbb{D} \setminus \{\pm 1\}$, and at most simple poles at ± 1.

(C) $(M(z) - M^\sharp(z))^{-1}$ has no poles in $\mathbb{C} \setminus \{0\}$ except at $z = \pm 1$ where there might be simple poles.

(D) If M has a pole at $z_j \in \mathbb{D}$ and at z_j^{-1}, then

$$\operatorname{Ran} \operatorname*{Res}_{z=z_j} M(z) \subseteq \ker(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1}, \tag{1.3.20}$$

$$\operatorname{Ran} \operatorname*{Res}_{z=z_j} M(z) \subseteq \left(\operatorname{Ran}\,(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1} M(z_j^{-1})\right)^\perp. \tag{1.3.21}$$

Remarks. 1. Condition (1.3.17)/(1.3.20) implies that $(M(z) - M^\sharp(z))^{-1} M(z)$ is analytic at z_j^{-1}, so (1.3.18)/(1.3.21) makes sense.

2. M can have poles of at most order 1 in $\overline{\mathbb{D}}$, however not necessarily so in $\mathbb{C} \setminus \overline{\mathbb{D}}$. In (D), if M has poles of order 1 at both z_j and z_j^{-1} then (1.3.17)–(1.3.18) ((1.3.20)–(1.3.21))

are equivalent to

$$\operatorname{Ran}\widetilde{w}_j \subseteq \operatorname{Ran}(\widetilde{w}_j - z_j^2\widetilde{q}_j),$$
$$\operatorname{Ran}\widetilde{w}_j \cap \operatorname{Ran}\widetilde{q}_j = \varnothing,$$

where $\widetilde{w}_j = -\operatorname{Res}_{z=z_j} M(z)$, $\widetilde{q}_j = \operatorname{Res}_{z=z_j^{-1}} M(z)$ (see Proposition 3.4.1).

3. If $l = 1$ then (D) is equivalent to the condition that M has no simultaneous singularities at points z_j and z_j^{-1} (see Proposition 3.4.1).

4. See also [Ger94, Thm 14] for a somewhat related result on the relation between the exponential decay of Jacobi parameters and properties of the measure μ (for the scalar $l = 1$ case).

5. Conditions (A) and (C) can be restated in terms of the meromorphic continuation of the absolutely continuous density $f(2\cos\theta)$ (as a function of $e^{i\theta} \in \partial\mathbb{D}$) (see Lemma 2.3.1 and the discussion after it). Condition (B) of course just means that there is no point spectrum of μ on $[-2, 2]$. Condition (D) depends on both absolutely continuous and pure point parts of the measure.

1.3.5 Meromorphic Continuations of Finite Gap Herglotz Functions and Periodic Jacobi Matrices

In this subsection we go back to assuming that μ is a (scalar) measure with $\operatorname{ess\,supp}\mu$ finitely many intervals. Recall the $\sharp$-notation from Definition 1.2.8.

Theorem 1.3.18. *Let $\mathfrak{e} = \cup_{j=1}^{p}[\alpha_j, \beta_j]$, $\alpha_j < \beta_j < \alpha_{j+1}$, is such that each $[\alpha_j, \beta_j]$ has equal harmonic measure ("open gaps case").*

Assume $\operatorname{ess\,supp}\mu = \mathfrak{e}$, and let $m(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x-z}$. Let $R > 1$. The following are equivalent:

(i) The associated to μ Jacobi matrix $\mathcal{J}$ satisfies

$$\limsup_{n\to\infty}(d_n(\mathcal{J}, \mathcal{T}_{\mathfrak{e}}))^{1/2n} \le R^{-1},$$

where $\mathcal{T}_{\mathfrak{e}}$ is the isospectral torus corresponding to $\mathfrak{e}$.

(ii) (a) m has a meromorphic continuation to the region $\mathcal{S}_R$,

(b) *m has no poles on $\pi^{-1}(\mathfrak{e})$, except at $\pi^{-1}(\cup_{j=1}^{p}\{\alpha_j,\beta_j\})$ where they are at most simple,*

(c) *$m(z)-m^\sharp(z)$ has no zeros in $\pi^{-1}(E_R)$, except at $\pi^{-1}(\cup_{j=1}^{p}\{\alpha_j,\beta_j\})$ where they are at most simple,*

(d) *If m has a pole at z for $z\in\pi^{-1}(E_R\setminus\mathfrak{e})$ then $z^\sharp$ is not a pole of m.*

Here Δ is the unique polynomial of degree p such that $\mathfrak{e}=\Delta^{-1}[-2,2]$, and $\mathcal{S}_R=\mathcal{S}_+\cup(\mathcal{S}_-\cap E_R)$, where E_R is the union of the interiors of the bounded components of $\Delta^{-1}(x(R\,\partial\mathbb{D}))$, where $x(z)=z+z^{-1}$.

Theorem 1.3.19. *Let $\mathfrak{e}=\cup_{j=1}^{p}[\alpha_j,\beta_j]$, $\alpha_j<\beta_j<\alpha_{j+1}$, is such that each $[\alpha_j,\beta_j]$ has equal equilibrium measure ("open gaps case").*

Assume $\operatorname{ess\,supp}\mu=\mathfrak{e}$, and let $m(z)=\int_{\mathbb{R}}\frac{d\mu(x)}{x-z}$. The following are equivalent:

(i) *The associated to μ Jacobi matrix $\mathcal{J}$ satisfies*

$$d_n(\mathcal{J},\mathcal{T}_{\mathfrak{e}})=0 \qquad \text{for all large } n,$$

where $\mathcal{T}_{\mathfrak{e}}$ is the isospectral torus corresponding to $\mathfrak{e}$.

(ii) *(a)* *m has a meromorphic continuation to $\mathcal{S}$,*

(b) *m has no poles on $\pi^{-1}(\mathfrak{e})$, except at $\pi^{-1}(\cup_{j=1}^{p}\{\alpha_j,\beta_j\})$ where they are at most simple,*

(c) *$m(z)-m^\sharp(z)$ has no zeros in $\mathcal{S}\setminus\{\pm\infty\}$, except at $\pi^{-1}(\cup_{j=1}^{p}\{\alpha_j,\beta_j\})$ where they are at most simple,*

(d) *If m has a pole at z for $z\in\pi^{-1}(\mathbb{C}\setminus\mathfrak{e})$ then $z^\sharp$ is not a pole of m.*

Here Δ as above is the unique polynomial of degree p such that $\mathfrak{e}=\Delta^{-1}[-2,2]$.

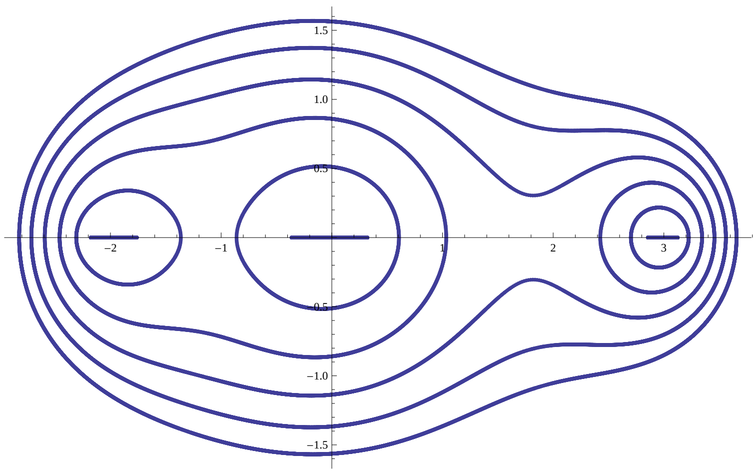
Remarks. 1. Theorems 1.3.18, 1.3.19 for $p=1$ and Theorems 1.3.16, 1.3.17 for $l=1$ are identical.

2. Condition (c) says that $m(z)\neq m^\sharp(z)$ for $z\in\pi^{-1}(E_R\setminus\mathfrak{e})$, that $\operatorname{Im}m(z)\neq 0$ for z in the interior of $\mathfrak{e}$, and that the zero of $m(z)-m^\sharp(z)$ is at most of first order at the edges. Recalling Lemma 1.2.4, the latter two conditions mean that the density $\frac{d\mu}{dx}$ of μ is nonvanishing on $\mathfrak{e}$ except at the edges where it might be square root vanishing (recall

that local coordinates of $\mathcal{S}$ at the edges of $\mathfrak{e}$ are given in terms of $\sqrt{z-z_0}$, not $z-z_0$). Also, by the discussion after Lemma 2.3.1, the conditions (a) and (c) imply that the density $f(x)=\frac{d\mu}{dx}$ has an analytic continuation to $\pi^{-1}(E_R)$ and is non-vanishing except at the band edges. Condition (b) just says that μ has no pure points in $\mathfrak{e}$. However the condition (d) is influenced by both the absolutely continuous density f and the bound states of μ.

3. Instead of demanding (d) to hold for $z\in\pi^{-1}(\mathbb{C}\setminus\mathfrak{e})$ one could demand it also for the points $z\in\pi^{-1}(\mathfrak{e}\setminus\cup_{j=1}^{p}\{\alpha_j,\beta_j\})$, which, given the convention $z^\sharp=z$ $(z\in\pi^{-1}(\mathfrak{e}))$, would simply mean that m has no pole at these points. (b) however also demands that the poles at the band edges are at most simple.

4. Here is an example how E_R evolves as R grows:



Using the results of [Sim, Chapter 5] it is easy to see that E_R are precisely the level sets of the logarithmic potential of the equilibrium measure for $\mathfrak{e}$.

1.3.6 Point Perturbations of Measures

The next theorem shows that under the given conditions removing a pure point is a small perturbation on the Jacobi matrix.

Theorem 1.3.20. *Let* $d\mu(x)=f(x)dx+\sum_{j=1}^{N}w_j\delta(x-E_j)$, $d\widehat{\mu}(x)=f(x)dx+\sum_{j=1}^{N-1}w_j\delta(x-E_j)$, $E_N\notin\operatorname{supp}\widehat{\mu}$.

Let $(A_n)_{n=1}^{\infty}$, $(B_n)_{n=1}^{\infty}$ *be Jacobi parameters for* μ, *and* $(\widehat{A}_n)_{n=1}^{\infty}$, $(\widehat{B}_n)_{n=1}^{\infty}$ *be Jacobi parameters for* $\widehat{\mu}$. *Let* $R>1$. *The following holds true.*

(i) If

$$\limsup_{n\to\infty} \left(||B_n|| + ||\mathbf{1} - A_n A_n^*||\right)^{1/2n} \leq R^{-1}, \tag{1.3.22}$$

then

$$\limsup_{n\to\infty} \left(||\widehat{B}_n|| + ||\mathbf{1} - \widehat{A}_n \widehat{A}_n^*||\right)^{1/2n} \leq R^{-1}. \tag{1.3.23}$$

(ii) If

$$||B_n|| + ||\mathbf{1} - A_n A_n^*|| = \mathbf{0} \quad \textit{for all large } n, \tag{1.3.24}$$

then

$$||\widehat{B}_n|| + ||\mathbf{1} - \widehat{A}_n \widehat{A}_n^*|| = \mathbf{0} \quad \textit{for all large } n. \tag{1.3.25}$$

Thus in this case removing a pure point is a finite rank perturbation.

The next two theorems deal with adding pure points to the measure. We have to consider two different cases $R^{-1} \geq |z_N| > 0$ and $1 > |z_N| > R^{-1}$.

Theorem 1.3.21. *Let* $d\mu(x) = f(x)dx + \sum_{j=1}^{N-1} w_j \delta(x - E_j)$, $d\widehat{\mu}(x) = f(x)dx + \sum_{j=1}^{N} w_j \delta(x - E_j)$, $E_N \notin \operatorname{supp}\mu$. *Let* $R > 1$ *and* $R^{-1} \geq |z_N| > 0$, *where* $z_N + z_N^{-1} = E_N$, $z_N \in \mathbb{D}$.

Let $(A_n)_{n=1}^{\infty}$, $(B_n)_{n=1}^{\infty}$ *be Jacobi parameters for* μ, *and* $(\widehat{A}_n)_{n=1}^{\infty}$, $(\widehat{B}_n)_{n=1}^{\infty}$ *be Jacobi parameters for* $\widehat{\mu}$. *If*

$$\limsup_{n\to\infty} \left(||B_n|| + ||\mathbf{1} - A_n A_n^*||\right)^{1/2n} \leq R^{-1}, \tag{1.3.26}$$

then

$$\limsup_{n\to\infty} \left(||\widehat{B}_n|| + ||\mathbf{1} - \widehat{A}_n \widehat{A}_n^*||\right)^{1/2n} \leq R^{-1}. \tag{1.3.27}$$

Theorem 1.3.22. *Let* $d\mu(x) = f(x)dx + \sum_{j=1}^{N-1} w_j \delta(x - E_j)$, $d\widehat{\mu}(x) = f(x)dx + \sum_{j=1}^{N} w_j \delta(x - E_j)$, $|E_j| > 2$. *Let* $\infty \geq R > 1$ *and* $1 > |z_N| > R^{-1}$, *where* $z_N + z_N^{-1} = E_N$, $z_N \in \mathbb{D}$.

Let $(A_n)_{n=1}^{\infty}$, $(B_n)_{n=1}^{\infty}$ *be Jacobi parameters for* μ, *and* $(\widehat{A}_n)_{n=1}^{\infty}$, $(\widehat{B}_n)_{n=1}^{\infty}$ *be Jacobi parameters for* $\widehat{\mu}$. *Assume*

$$\limsup_{n\to\infty} \left(||B_n|| + ||\mathbf{1} - A_n A_n^*||\right)^{1/2n} \leq R^{-1}. \tag{1.3.28}$$

Then

$$\limsup_{n\to\infty} \left(||\widehat{B}_n|| + ||\mathbf{1} - \widehat{A}_n \widehat{A}_n^*||\right)^{1/2n} \leq |z_N|. \tag{1.3.29}$$

Moreover,

(i) *If $M(z)$ has no pole at z_N^{-1} then*

$$\limsup_{n\to\infty}\left(||\widehat{B}_n|| + ||\mathbf{1} - \widehat{A}_n\widehat{A}_n^*||\right)^{1/2n} = |z_N|, \tag{1.3.30}$$

(ii) *If $M(z)$ has a first order pole at z_N^{-1} with $\operatorname{Res}_{z=z_N^{-1}} M(z) = q_N$, then*

$$\limsup_{n\to\infty}\left(||\widehat{B}_n|| + ||\mathbf{1} - \widehat{A}_n\widehat{A}_n^*||\right)^{1/2n} = \limsup_{n\to\infty}\left(||B_n|| + ||\mathbf{1} - A_nA_n^*||\right)^{1/2n} \le R^{-1} \tag{1.3.31}$$

if and only if $w_N = -(1-z_N^2)Pq_NP$, where P is the orthogonal projection onto an invariant subspace of q_N. If w_N is not of this form, then (1.3.30) *holds.*

(iii) *If $M(z)$ has a pole of order higher than 1 at z_N^{-1}, then*

(a) *If $l = 1$ (i.e., we are in the scalar case), then* (1.3.30) *holds.*

(b) *If $l > 1$, then* (1.3.31) *holds if and only if*

$$\operatorname{Ran} w_N \subseteq \ker(M(z_N^{-1}) - M^\sharp(z_N^{-1}))^{-1}, \tag{1.3.32}$$

$$\operatorname{Ran} w_N \subseteq \operatorname{Ran} \lim_{z\to z_N^{-1}}\left((M(z)-M^\sharp(z))^{-1}M(z) + (M(z)-M^\sharp(z))^{-1}\frac{w_N}{E_N - z - z^{-1}}\right)^\perp \tag{1.3.33}$$

Otherwise, (1.3.30) *holds.*

Remarks. 1. One can also replace (1.3.26)/(1.3.28) and (1.3.27)/(1.3.31) in Theorem 1.3.21/ 1.3.22 with (1.3.24) and (1.3.25), respectively. This is of course a different result from just $R = \infty$ case.

2. In particular these theorems say that if (1.3.22) holds, then adding or removing a pure point is an exponentially small perturbation. Moreover, if (1.3.24) holds, then removing a pure point is a finite rank perturbation, while adding is finite rank only under certain circumstances described in (ii)–(iii).

3. Geronimo [Ger94] (see also [GN83]) proved the scalar analogues of Theorems 1.3.20 and 1.3.21, under the assumption $|E_N| > |E_j|$. On the other hand, he measured the rate of

exponential decay of parameters in a slightly more general way than by (1.3.22).

4. The limits of both summands on the right-hand side of (1.3.33) exist. It does not seem possible to express the condition (1.3.33) in a more explicit and better looking form.

The condition "if $M(z)$ has no pole at z_N^{-1}" in Theorem 1.3.22(i) is something that generically holds, of course (note also, that given that M does not have a pole at z_N, then the pole at z_N^{-1} can only come from meromorphic extension $f(z+z^{-1})$ of the absolutely continuous part $f(2\cos\theta)$, see Lemma 2.3.1). In order to add a mass point at such E_N to the spectrum, while preserving the rate of exponential decay of parameters, one has to modify the absolutely continuous part, as we do in the Theorem 1.3.23 below. Another way of looking at the next result is that we are modifying the Jost function by $\widehat{u}(z) = (z_N - z)u(z)$ (up to a multiplicative constant) to produce a zero at z_N (without producing a pole at z_N^{-1}). The scalar equivalent of the result is Geronimo's [Ger94, Thm 7].

Theorem 1.3.23. *Let* $d\mu(x) = f(x)dx + \sum_{j=1}^{N-1} w_j \delta(x-E_j)$, $d\widehat{\mu}(x) = \frac{1}{E_N - x} f(x)dx + \sum_{j=1}^{N-1} \frac{1}{E_N - E_j} w_j \delta(x-E_j) + w_N \delta(x - E_N)$, *where* $E_N > \max_j\{E_j\}$. *Let* $(A_n)_{n=1}^\infty$, $(B_n)_{n=1}^\infty$ *be Jacobi parameters for* μ, *and* $(\widehat{A}_n)_{n=1}^\infty$, $(\widehat{B}_n)_{n=1}^\infty$ *be Jacobi parameters for* $\widehat{\mu}$.

Assume M *is regular at* z_N^{-1}. *If*

$$\limsup_{n\to\infty} \left(||B_n|| + ||\mathbf{1} - A_n A_n^*||\right)^{1/2n} \le R^{-1} \tag{1.3.34}$$

and

$$w_N = M(z_N) - M(z_N^{-1}), \tag{1.3.35}$$

then

$$\limsup_{n\to\infty} \left(||\widehat{B}_n|| + ||\mathbf{1} - \widehat{A}_n \widehat{A}_n^*||\right)^{1/2n} \le R^{-1}. \tag{1.3.36}$$

Remark. Again, one can replace (1.3.34) and (1.3.36) with (1.3.24) and (1.3.25), respectively. This says that under (1.3.24), we can add a point to the spectrum via a finite rank perturbation.

Note that to keep the weights positive, we need the restriction $E_N > \max_j\{E_j\}$. There is also an implicit restriction that (1.3.35) is positive to ensure $w_N \ge \mathbf{0}$. If $E_N < \min_j\{E_j\}$, then the same construction works if we substitute everywhere $E_N - x$ and $E_N - E_j$ with $x - E_N$ and $E_j - E_N$. If one is willing to allow negative point masses then all these restrictions can be omitted.

This settles the question of adding a point mass at E_N if there is no pole at $M(z_N^{-1})$. The case of order 1 pole is settled in Theorem 1.3.22(ii), of course. Finally, what if $M(z_N^{-1})$ has a pole of order $k \geq 2$? This is equivalent to $u(z_k^{-1})^{-1}$ having a pole of order $k \geq 2$. In the scalar (including scalar periodic) case one can perform the analogous procedure $\widehat{u}(z) = (z_N^{-1} - z)^{-k} u(z)$ (equivalent to multiplying the measure by $(E_N - x)^k$) to get rid of the problem at z_N^{-1} while preserving the weights canonical, and then proceed as in Theorem 1.3.23. In the matrix case we would need to divide out in general by a non-diagonal factor which leads to the nonsymmetric weights $\widehat{w}_j$.

Similar results hold for periodic scalar matrices.

Theorem 1.3.24. *Let $d\mu(x) = f(x)dx + \sum_{j=1}^{N} w_j \delta(x - E_j)$, $d\widehat{\mu}(x) = f(x)dx + \sum_{j=1}^{N-1} w_j \delta(x - E_j)$, $E_N \notin \operatorname{supp}\mu$.*

Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ be the Jacobi parameters for μ, and $(\widehat{a}_n)_{n=1}^{\infty}$, $(\widehat{b}_n)_{n=1}^{\infty}$ be the Jacobi parameters for $\widehat{\mu}$. If

$$\mathcal{J} \text{ is eventually periodic (with all gaps open),}$$

then

$$\widehat{\mathcal{J}} \text{ is eventually periodic (on the same isospectral torus).}$$

Theorem 1.3.25. *Let $d\mu(x) = f(x)dx + \sum_{j=1}^{N-1} w_j \delta(x - E_j)$, $d\widehat{\mu}(x) = f(x)dx + \sum_{j=1}^{N} w_j \delta(x - E_j)$, $E_N \notin \operatorname{supp}\mu$.*

Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ be the Jacobi parameters for μ, and $(\widehat{a}_n)_{n=1}^{\infty}$, $(\widehat{b}_n)_{n=1}^{\infty}$ be the Jacobi parameters for $\widehat{\mu}$. Assume

$$\mathcal{J} \text{ is eventually periodic (with all gaps open).} \tag{1.3.37}$$

Then

$$\limsup_{n\to\infty} \left(d_n(\widehat{\mathcal{J}}, \mathcal{T}_{\mathfrak{e}}) \right)^{1/2n} \leq \frac{|\Delta(E)|}{2} - \sqrt{\frac{|\Delta(E)|^2}{4} - 1}. \tag{1.3.38}$$

Moreover,

(i) If $m(z)$ has no pole at $(E_N)_-$ or has a pole of order ≥ 2 then

$$\limsup_{n\to\infty} \left(d_n(\widehat{\mathcal{J}}, \mathcal{T}_{\mathfrak{e}}) \right)^{1/2n} = \frac{|\Delta(E)|}{2} - \sqrt{\frac{|\Delta(E)|^2}{4} - 1}. \tag{1.3.39}$$

(ii) If $m(z)$ has a pole of order 1 at $(E_N)_-$ with $\operatorname{Res}_{z=(E_N)_-} m(z) = q_N$, then

$$\widehat{\mathcal{J}} \text{ is eventually periodic (on the same isospectral torus)} \tag{1.3.40}$$

if and only if $w_N = q_N$. Otherwise (1.3.39) holds.

Remark. Just as in Theorems (1.3.20)–(1.3.22) one can demand $\limsup_{n\to\infty}(d_n(\mathcal{J}, \mathcal{T}_{\mathfrak{e}}))^{1/2n} \leq R^{-1}$, rather than $\mathcal{J}$ being eventually periodic, and obtain the exact analogue of the results. Similarly, one can perform procedures analogous to the one in Theorem 1.3.23 (see also discussion after the theorem). We omit stating the results since the method should be by now clear.

Chapter 2

Prerequisites

2.1 Matrix-Valued Orthogonal Polynomials on the Real Line

We will need some additional results apart from the basics that we introduced in Section 1.2.

Let us define the second kind polynomials by

$$\mathfrak{q}_n^R(z) = \int_{\mathbb{R}} d\mu(x) \frac{\mathfrak{p}_n^R(z) - \mathfrak{p}_n^R(x)}{z - x}, \quad n = 0, 1, \ldots .$$

It can be shown that $\mathfrak{q}_n^R$ are polynomials of degree $n-1$ and that they satisfy the same recurrence relations (1.2.4). For future reference,

$$\mathfrak{p}_0^R(z) = \mathbf{1}, \quad \mathfrak{p}_1^R(z) = (z - B_1)A_1^{*-1}, \tag{2.1.1}$$

$$\mathfrak{q}_0^R(z) = \mathbf{0}, \quad \mathfrak{q}_1^R(z) = A_1^{*-1}. \tag{2.1.2}$$

Define also $\mathfrak{q}_n^L = \mathfrak{q}_n^R(\bar{z})^*$.

The resolvent of $\mathcal{J}$ has the following block form (see [DPS08, Thm 2.29])

$$(\mathcal{J} - z)^{-1} = \begin{pmatrix} \mathfrak{m} & \mathfrak{q}_1^R + \mathfrak{m}\mathfrak{p}_1^R & \mathfrak{q}_2^R + \mathfrak{m}\mathfrak{p}_2^R & \cdots \\ \mathfrak{q}_1^L + \mathfrak{p}_1^L\mathfrak{m} & \mathfrak{q}_1^L\mathfrak{p}_1^R + \mathfrak{p}_1^L\mathfrak{m}\mathfrak{p}_1^R & \mathfrak{p}_1^L\mathfrak{q}_2^R + \mathfrak{p}_1^L\mathfrak{m}\mathfrak{p}_2^R & \cdots \\ \mathfrak{q}_2^L + \mathfrak{p}_2^L\mathfrak{m} & \mathfrak{q}_2^L\mathfrak{p}_1^R + \mathfrak{p}_2^L\mathfrak{m}\mathfrak{p}_1^R & \mathfrak{q}_2^L\mathfrak{p}_2^R + \mathfrak{p}_2^L\mathfrak{m}\mathfrak{p}_2^R & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} (z), \tag{2.1.3}$$

i.e., its (i,j)-th block entry is $\mathfrak{q}_{i-1}^L\mathfrak{p}_{j-1}^R + \mathfrak{p}_{i-1}^L\mathfrak{m}\mathfrak{p}_{j-1}^R$ if $i \geq j$, and $\mathfrak{p}_{i-1}^L\mathfrak{q}_{j-1}^R + \mathfrak{p}_{i-1}^L\mathfrak{m}\mathfrak{p}_{j-1}^R$ otherwise.

We will use the following result. This is proven in [Den04] for the scalar case, and appears in [DPS08] for the matrix-valued case.

Lemma 2.1.1. *Let $\sigma_{ess}(\mathcal{J}) \subseteq [-2,2]$. Then, for every $\varepsilon > 0$, there exists N such that for $n \geq N$, we have that $\sigma(\mathcal{J}^{(n)}) \subseteq [-2-\varepsilon, 2+\varepsilon]$.*

Assume for the rest of this section, that $\operatorname{ess\,supp} \mu = [-2,2]$, and denote by $\{E_k\}_{k=1}^N$ ($N \leq \infty$) the eigenvalues outside $[-2,2]$.

Define

$$M(z) = -\mathfrak{m}(z + z^{-1}), \quad z \in \mathbb{D},$$

where $\mathfrak{m}$ is defined in (1.2.6). Using Lemma 1.2.4, one obtains

$$\operatorname{Im} M(e^{i\theta}) = \pi f(2\cos\theta), \quad 0 \leq \theta \leq \pi, \tag{2.1.4}$$

$$\operatorname{Im} M(e^{i\theta}) = -\pi f(2\cos\theta), \quad -\pi \leq \theta \leq 0. \tag{2.1.5}$$

Denote

$$\{z_k\}_{k=1}^N = \left\{ z \in \mathbb{D} \;\middle|\; z = \frac{1}{2}\left(E_k - \sqrt{E_k{}^2 - 4}\right) \right\} = \left\{ z \in \mathbb{D} \mid z + z^{-1} = E_k \right\}, \tag{2.1.6}$$

enumerated in increasing order of their absolute values ($N \leq \infty$). Let us assume each z_k is different, and let n_k be the multiplicity of $z_k + z_k^{-1}$ as the eigenvalue.

We will be using the so-called C_0 Sum Rule from Damanik–Killip–Simon [DKS]. In a slightly changed form, it looks as follows.

Lemma 2.1.2 (Damanik–Killip–Simon [DKS])**.** *Suppose* $\operatorname{ess\,supp} \mu = [-2,2]$ *and* $\{z_k\}_{k=1}^N$ *be as in* (2.1.6)*. Let*

$$\mathcal{Z}(\mathcal{J}) = -\frac{1}{2}\int_{-\pi}^{\pi} \log\det \frac{\operatorname{Im} M(e^{i\theta})}{\sin\theta} \frac{d\theta}{2\pi},$$

$$\mathcal{E}_0(\mathcal{J}) = -\sum_{k=1}^{N} n_k \log|z_k|,$$

$$\mathcal{A}_0(\mathcal{J}) = -\lim_{n\to\infty} \sum_{j=1}^{n} \log\det|A_j|.$$

If any two of $\mathcal{Z}, \mathcal{E}_0, \mathcal{A}_0$ are finite, then so is the third, and

$$\mathcal{Z}(\mathcal{J}) = \mathcal{E}_0(\mathcal{J}) + \mathcal{A}_0(\mathcal{J}).$$

Remarks. 1. Here $|T| \equiv \sqrt{T^*T}$.

2. The minus in the expression for $\mathcal{E}_0(\mathcal{J})$ comes from the fact that we chose $z \in \mathbb{D}$ in (2.1.6) as opposed to $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ in [DKS].

2.2 More on Periodic Orthogonal Polynomials

We combine properties of the discriminant Δ (introduced in Subsection 1.2.4) into lemma.

Lemma 2.2.1. *Let $\mathcal{J}$ be a (one-sided) p-periodic Jacobi matrix, and Δ its discriminant. Then*

(i) $\Delta(z) = p_p(z) - a_p q_{p-1}(z)$, *where p_j, q_j are orthogonal polynomials of the first and second kind of $\mathcal{J}$.*

(ii) $\Delta(z) = \frac{1}{a_1 \dots a_p} \prod_{j=1}^{p} (z - b_j) + O(z^{p-2})$.

(iii)
- $\Delta^{-1}([-2,2]) \subset \mathbb{R}$.
- *Let $x_1^{\pm} \le x_2^{\pm} \le \dots \le x_p^{\pm}$ be the zeros (counting multiplicity) of $\Delta(\lambda) \mp 2$. Then*

$$x_p^+ > x_p^- \ge x_{p-1}^+ > x_{p-1}^- \ge x_{p-2}^+ > x_{p-2}^- \ge \dots.$$

- *$\Delta(\lambda)$ is strictly increasing on each interval (x_{p-2j}^-, x_{p-2j}^+) and strictly decreasing on each interval $(x_{p-2j-1}^+, x_{p-2j-2}^-)$, $j = 0, 1, \dots$. In particular the $p-1$ solutions of $\Delta'(\lambda) = 0$ are all real and lie one per each gap. If a gap is open then the corresponding solution lies in its interior.*

There is a nice connection between the theory of periodic orthogonal polynomials and matrix-valued orthogonal polynomials. Note that applying a polynomial of degree p to the tridiagonal matrix $\mathcal{J}$ gives us $(2p+1)$-diagonal matrix, which can be viewed as a block Jacobi matrix (of type 3) with $p \times p$ matrix-valued entries.

Let S be the right shift operator on $\ell^2(\mathbb{Z})$. Note that $S^p + S^{-p}$ is the free block Jacobi matrix with $p \times p$ block entries.

We will use the following result by Damanik–Killip–Simon, also known by the name "Magic Formula".

Lemma 2.2.2 (Damanik-Killip-Simon [DKS]). *Let $\mathcal{J}_0$ be a p-periodic Jacobi matrix with discriminant $\Delta_{\mathcal{J}_0}$ and isospectral torus $\mathcal{T}_{\mathfrak{e}}$. Let $\mathcal{J}$ be any two-sided Jacobi matrix. Then*

$$\Delta_{\mathcal{J}_0}(\mathcal{J}) = S^p + S^{-p} \quad \Leftrightarrow \quad \mathcal{J} \in \mathcal{T}_{\mathfrak{e}}.$$

Moreover we can "perturb" this result if all gaps are open.

Lemma 2.2.3 (Damanik-Killip-Simon [DKS]). *Let $\mathcal{J}_0$ be a p-periodic Jacobi matrix with discriminant $\Delta_{\mathcal{J}_0}$ and isospectral torus $\mathcal{T}_{\mathfrak{e}}$, such that all gaps of $\mathcal{J}_0$ are open (every interval of $\mathfrak{e}$ has equal equilibrium measure). Let $\mathcal{J}$ be any two-sided Jacobi matrix, and let $\{A_n, B_n\}_{n\in\mathbb{Z}}$ be the $p \times p$ Jacobi parameters of $\Delta_{\mathcal{J}_0}(\mathcal{J})$. Then the following are equivalent:*

(i) $\limsup_{n\to\infty}(d_n(\mathcal{J}, \mathcal{T}_{\mathfrak{e}}))^{1/2n} \leq R^{-1}$.

(I) $\limsup_{n\to\infty}(||\mathbf{1} - A_n A_n^*|| + ||B_n||)^{1/2n} \leq R^{-1}$.

Remark. Since both conditions depend on the behavior of the coefficients at $+\infty$, this result can also be applied to one-sided Jacobi matrices $\mathcal{J}$.

2.3 More on Herglotz Functions

Let m be a Herglotz function (in fact a discrete m-function in our case). Assume the corresponding measure μ has $\operatorname{ess\,supp} \mu = \mathfrak{e}$, a finite gap set. Then m is meromorphic in $(\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e} = \mathcal{S}_+$, and we are interested in conditions under which it has a continuation through the bands of $\mathfrak{e}$. The lemma below clarifies when this happens. The scalar result is due to Greenstein [Gre60], while the matrix-valued can be found in [GT00].

Lemma 2.3.1. *Let m be a matrix-valued Herglotz function with representation* (1.2.8). *Then m can be analytically continued from $\mathcal{S}_+ \cap \mathbb{C}_+$ through an interval $I \subset \mathbb{R}$ if and only if the associated measure μ is purely absolutely continuous on I, and the density $f(x) = \frac{d\mu}{dx}$ is real-analytic on I. In this case, the analytic continuation of m into some domain $\mathcal{D}_-$ of $\mathcal{S}_- \cap \mathbb{C}_-$ is given by*

$$m(z_-) = m(\bar{z}_+)^* + 2\pi i f(\pi(z)), \quad z \in \mathcal{D}_-,$$

where $f(z)$ is the complex-analytic continuation of f to some $\pi(\mathcal{D}_-)$.

Thus one can view any result on the continuation of m as the corresponding result on the continuation of the absolutely continuous part f of μ.

Moreover, now assume that m has some continuation to some neighborhood $\mathcal{D}$ of I in $\mathcal{S}_-$. This means that the two extensions into domains $\mathcal{D}\cap\mathbb{C}_-$ and $\mathcal{D}\cap\mathbb{C}_+$ have to agree on $\mathcal{D}\cap\mathbb{R}$. Note that since $\lim_{\varepsilon\to 0}\operatorname{Im} m(x-i\varepsilon) = -\pi f(x)$, we have $m(z_-) = m(\bar{z}_+)^* - 2\pi i f(\pi(z))$ for $z_- \in \mathcal{D}\cap\mathbb{C}_+$. This means that the continuation of f to $\pi(\mathcal{D}\setminus\mathbb{R})$ has $f(z+i0) = -f(z-i0)$ for any $z \in (\mathcal{D}\cap\mathbb{R})\setminus I$. Therefore m has a continuation to some $\mathcal{D} \subset \mathcal{S}_-$ if and only if f can be continued to $\pi^{-1}(\mathcal{D})$ with $f(z_+) = -f(z_-)$ (in particular f has to be zero or have a pole at the edge). Apart from this, this continuation satisfies $f^\sharp = f$ since f is real on $\pi^{-1}(\mathfrak{e})$.

2.4 Matrix-Valued Functions

Throughout the paper, all meromorphic/analytic matrix functions are assumed to have not identically vanishing determinant.

The order of a pole of an $l\times l$ matrix-valued meromorphic function f is defined to be the minimal $k>0$ such that $\lim_{z\to z_0}(z-z_0)^k f(z)$ is a finite nonzero matrix.

By a zero of a matrix-valued meromorphic function f we call a point at which f^{-1} has a pole.

Denote by $\delta_j \in \mathbb{C}^l$, $1\le j\le l$, the column vector having 1 on the j-th position, and 0 everywhere else.

2.4.1 Smith–McMillan Form

We will make use of the so-called (local) Smith-McMillan form (see, e.g., [BGR90, Thm 3.1.1]).

Lemma 2.4.1. *Let $f(z)$ be an $l\times l$ matrix-valued function meromorphic at z_0 with determinant not identically zero. Then $f(z)$ admits the representation*

$$f(z) = E(z)\operatorname{diag}\left((z-z_0)^{\kappa_1},\dots,(z-z_0)^{\kappa_l}\right)F(z), \tag{2.4.1}$$

where $E(z)$ and $F(z)$ are $l \times l$ matrix-valued functions which are analytic and invertible in a neighborhood of z_0, and $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_l$ are integers (positive, negative, or zero).

This immediately gives us the following corollary.

Lemma 2.4.2. *Let u be an analytic function at z_0 such that z_0 is a zero of $\det u$ of order $k > 0$. Then $\dim \ker u(z_0) = k$ if and only if z_0 is a pole of $u(z)^{-1}$ of order 1.*

If this is the case, then

$$\ker \operatorname*{Res}_{z=z_0} u(z)^{-1} = \operatorname{Ran} u(z_0),$$
$$\operatorname{Ran} \operatorname*{Res}_{z=z_0} u(z)^{-1} = \ker u(z_0).$$

Proof. Both of the conditions in the if-and-only-if statement are equivalent to saying that $\kappa_1 = \ldots = \kappa_k = 1$, $\kappa_{k+1} = \ldots = \kappa_l = 0$ in the Smith-McMillan form of $u(z)$ at z_0. Then note that both $\ker \operatorname{Res}_{z=z_0} u(z)^{-1}$ and $\operatorname{Ran} u(z_0)$ are equal to $E(z_0)\text{span}\,\{\delta_{k+1}, \cdots, \delta_l\}$. Similarly, both $\operatorname{Ran}\ \operatorname{Res}_{z=z_0} u(z)^{-1}$ and $\ker u(z_0)$ are equal to $F(z_0)^{-1}\text{span}\,\{\delta_1, \cdots, \delta_k\}$. $\square$

Definition 2.4.3. *(i) An analytic $\mathbb{C}^l$-valued function $\phi(z)$ with $\phi(z_0) \neq 0$ is called a **left null function** for a meromorphic matrix-valued function f at z_0 of order $k > 0$, if $\phi(z)^T f(z)$ is analytic at z_0 with a zero of order k at z_0.*

*(ii) An analytic $\mathbb{C}^l$-valued function $\psi(z)$ with $\psi(z_0) \neq 0$ is called a **left pole function** for a meromorphic matrix-valued function f at z_0 of order $k > 0$, if there exists an analytic $\mathbb{C}^l$-valued function $\phi(z)$ with $\phi(z_0) \neq 0$ such that $\phi(z)^T f(z) = (z - z_0)^{-k}\psi(z)$.*

Note that ψ is a left pole function for f if and only if ψ is a left null function for f^{-1}. The following is immediate from the definition and will prove to be useful for us.

Lemma 2.4.4. *Let f has a local Smith-McMillan form* (2.4.1) *with $\kappa_1 \geq \ldots \geq \kappa_j > 0$, $0 > \kappa_{l-r+1} \geq \ldots \geq \kappa_l$. Then*

(i) $(E(z)^{-1})^T\delta_1, \ldots (E(z)^{-1})^T\delta_j$ are left null functions for f at z_0 of order $\kappa_1, \ldots, \kappa_j$, respectively.

(ii) $F(z)^T\delta_{l-r+1}, \ldots F(z)^T\delta_l$ are left pole functions for f at z_0 of order $-\kappa_{l-r+1}, \ldots, -\kappa_l$, respectively.

2.4.2 Matrix Outer Functions

Recall that a scalar analytic function G on $\mathbb{D}$ is called outer if it can be recovered from its boundary values $G(e^{i\theta}) \equiv \lim_{r\uparrow 1} G(re^{i\theta})$ by the formula

$$G(z) = c \exp\left\{ \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|G(e^{i\theta})| \frac{d\theta}{2\pi} \right\} \tag{2.4.2}$$

for some constant $|c| = 1$. Note that it is necessary and sufficient $\log|G(e^{i\theta})|$ to be integrable.

The analogue of this is

Lemma 2.4.5 (Wiener–Masani [WM57]). *Suppose $w(\theta)$ is a non-negative matrix-valued function on the unit circle satisfying*

$$\int_{-\pi}^{\pi} \log\det w(\theta) \frac{d\theta}{2\pi} > -\infty.$$

Then there exists a unique matrix-valued $H^2(\mathbb{D})$ function $G(z)$ satisfying

$$G(e^{i\theta})^* G(e^{i\theta}) = w(\theta), \tag{2.4.3}$$

$$G(0)^* = G(0) > 0, \tag{2.4.4}$$

$$\log|\det G(0)| = \int_{-\pi}^{\pi} \log|\det G(e^{i\theta})| \frac{d\theta}{2\pi}. \tag{2.4.5}$$

This is a well-known result of Wiener–Masani [WM57]. The proof of the uniqueness part can be found, e.g., in [DGK78].

Equality (2.4.5) implies (see [Rud87, §17.17]) that $\det G(z)$ is a scalar outer function, which implies (by definition) that $G(z)$ is a matrix-valued outer function. It follows from [Gin64, Thm. 2] (see also [Pot60]) that there exists a Hermitian matrix-valued integrable function $M(\theta)$ such that

$$\operatorname{Tr} M(\theta) = \log|\det G(e^{i\theta})| \tag{2.4.6}$$

and

$$G(z) = \rho \overset{\curvearrowright}{\int_{-\pi}^{\pi}} \exp\left\{ \frac{e^{i\theta}+z}{e^{i\theta}-z} M(\theta) \frac{d\theta}{2\pi} \right\}, \tag{2.4.7}$$

where $\overset{\curvearrowright}{\int}_{-\pi}^{\pi}$ is the Potapov multiplicative integral (see [Pot60])

$$\overset{\curvearrowright}{\int}_{-\pi}^{\pi} \exp\left\{F(\theta)\frac{d\theta}{2\pi}\right\} = \lim_{\Delta\theta_j \to 0} \overset{\curvearrowright}{\prod_{j=0}^{n-1}} e^{F(\phi_j)\Delta\theta_j},$$

$$-\pi = \theta_0 \le \phi_0 \le \theta_1 \le \phi_1 \le \cdots \le \theta_{n-1} \le \phi_{n-1} \le \theta_n = \pi.$$

The arrow above the product sign simply defines the order of the multiplication in the matrix-valued product. ρ in (2.4.7) is a constant unitary matrix which makes the right-hand side of (2.4.7) positive-definite at $z = 0$.

Clearly (2.4.6)–(2.4.7) becomes (2.4.2) if $l = 1$.

2.4.3 Blaschke–Potapov Products

The Blaschke–Potapov elementary factor is a generalization of scalar Blaschke factors (for those familiar with the Potapov theory of J–contractive matrix functions: we are considering the signature matrix J to be just the identity matrix $\mathbf{1}$):

$$B_{z_j,s,U}(z) = U^* \begin{pmatrix} \frac{|z_j|}{z_j}\frac{z_j-z}{1-\bar{z}_j z} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{|z_j|}{z_j}\frac{z_j-z}{1-\bar{z}_j z} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} U, \quad z \in \mathbb{D},$$

where $z_j \in \mathbb{D}$, s is the number of the scalar Blaschke factors on the diagonal ($0 \le s \le l$), and U is a unitary constant matrix. Clearly $B_{z_j,s,U}$ is an analytic in $\mathbb{D}$ function with unitary values on the unit circle.

The well-known result for the convergence of the scalar Blaschke products is still valid for the matrix-valued case: if

$$\sum_{k=1}^{\infty}(1 - |z_k|) < \infty,$$

then the product

$$\overset{\curvearrowright}{\prod_{j=1}^{\infty}} B_{z_j,s_j,U_j}(z)$$

converges uniformly on the compacts of the unit disk (see Potapov [Pot60] and Ginzburg [Gin58] where this is proven even more generally for the operator-valued setting). The limit function is holomorphic in $\mathbb{D}$ with unitary boundary values (see Arov–Simakova [AS76]).

We have freedom here in the choice of the unitary matrices U_j and numbers s_j. We will make use of it in the following lemma.

Lemma 2.4.6. *Let $\{z_k\}_{k=1}^{\infty}$ with $\sum_{k=1}^{\infty}(1-|z_k|)<\infty$ be given, with all z_k pairwise different. For any sequence of subspaces $V_k \subseteq \mathbb{C}^l$, there exists a unique product $B(z) = \overset{\curvearrowright}{\prod}_{j=1}^{\infty} B_{z_j,s_j,U_j}(z)$ for some choice of numbers s_k, $0 \le s_k \le l$, and unitary matrices U_k, that satisfies*

$$\operatorname{Ran} B(z_k) = \ker \operatorname*{Res}_{z=z_k} B(z)^{-1} = V_k \quad \textit{for all } k. \tag{2.4.8}$$

Proof. Easy induction does the job. Let I_m $(0 \le m \le l)$ be the diagonal $l \times l$ matrix with first m diagonal elements 1 and the rest 0, and $B_n(z) = \overset{\curvearrowright}{\prod}_{j=1}^{n} B_{z_j,s_j,U_j}(z)$ be the partial finite product. Assume that we already chose $\{s_k\}_{k=1}^{n-1}$ and $\{U_k\}_{k=1}^{n-1}$ so that $B_{n-1}(z)$ satisfies

$$\ker \operatorname*{Res}_{z=z_k} B_{n-1}(z)^{-1} = V_k, \quad 1 \le k \le n-1.$$

Observe that this implies (2.4.8) holds for $1 \le k \le n-1$ as well. Put $s_n = l - \dim V_n$. Note that

$$B_{z_n,s_n,U_n}(z) = U_n^* \left(\frac{|z_n|}{z_n} \frac{z_n - z}{1 - z\bar{z}_n} I_{s_n} + (I - I_{s_n}) \right) U_n \tag{2.4.9}$$

and

$$\ker \operatorname*{Res}_{z=z_n} B(z)^{-1} = \ker \operatorname*{Res}_{z=z_n} B_n(z)^{-1} = \ker I_{s_n} U_n B_{n-1}(z_n)^{-1}.$$

Note that $B_{n-1}(z_n)$ is invertible (as $z_n \notin \{z_1, \dots, z_{n-1}\}$), so we can put U_n to be any unitary matrix taking the subspace $B_{n-1}(z_n)^{-1} V_n$ to $\operatorname{span}\{\delta_{s_n+1} \cdots \delta_l\}$. Note that the choice of U_n is not unique, but the factor B_{z_n,s_n,U_n} is uniquely defined. $\square$

2.5 Miscellaneous Lemmas

Recall that an infinite product $\prod_{j=1}^{\infty} a_j$ with $a_j \neq 0$ is called absolutely convergent if $\sum_{j=1}^{\infty} |1 - a_j| < \infty$. We will be needing the following easy statements.

Lemma 2.5.1. *(i) If $\prod_{j=1}^{\infty} a_j$ with $a_j \neq 0$ is absolutely convergent then*

$$\sup_{\Lambda \subset \mathbb{N}} \left| \prod_{j \in \Lambda} a_j \right| < \infty.$$

(ii) Let $a_n \to 0$ and $\sum_{j=1}^{\infty} |b_j| < \infty$. Then

$$\sum_{j=0}^{n} a_{n-j} b_j \to 0.$$

Proof. (i) If $\prod_{j=1}^{\infty} a_j$ is absolutely convergent, then so is $\prod_{j=1}^{\infty} |a_j|$, so without loss of generality we can assume $a_j > 0$. Then

$$\prod_{j \in \Lambda} a_j = e^{\sum_{j \in \Lambda} \log a_j} \leq e^{\sum_{j \in \Lambda} |a_j - 1|} \leq e^{\sum_{j=1}^{\infty} |a_j - 1|} < \infty.$$

(ii) For any $\varepsilon > 0$ find N such that $|a_j| < \varepsilon$ for all $j \geq N$. Then for $n > N$:

$$\left| \sum_{j=0}^{n} a_{n-j} b_j \right| \leq \left| \sum_{j=0}^{N} a_{n-j} b_j \right| + \varepsilon \sum_{j=N+1}^{n} |b_j| \leq \left| \sum_{j=0}^{N} a_{n-j} b_j \right| + \varepsilon \sum_{j=1}^{\infty} |b_j|,$$

which implies $\limsup_{n \to \infty} \left| \sum_{j=0}^{n} a_{n-j} b_j \right| \leq \varepsilon \sum_{j=1}^{\infty} |b_j|$, and proves (ii). $\square$

Remark. Note that part (ii) works also for the matrix-valued a's and b's.

Lemma 2.5.2. *There exists a unique $l \times l$ matrix W satisfying*

$$WA = B, \tag{2.5.1}$$

$$\operatorname{Ran} W = \operatorname{Ran} B, \tag{2.5.2}$$

if and only if $\ker A \subseteq \ker B$.

Proof. Straightforward/standard. $\square$

Chapter 3

Proofs

3.1 Equivalence Classes of Block Jacobi Matrices

3.1.1 Proof of Theorems 1.3.4 and 1.3.6

We will be using the following lemma from [Li97]. For self-containment purposes we give a proof of it in the end of the section.

Lemma 3.1.1 (Li [Li97]). *Let ϕ be the map that takes any invertible matrix T to the unitary factor U in its polar decomposition $T = |T|U$, where $|T| = \sqrt{TT^*}$. Then for any invertible $l \times l$ matrices B, D the following holds*

$$||\phi(B) - \phi(BD)||_{HS} \leq \sqrt{||\mathbf{1} - D^{-1}||_{HS}^2 + ||\mathbf{1} - D||_{HS}^2},$$

where $||\cdot||_{HS}$ is the Hilbert–Schmidt norm.

Proof of Theorem 1.3.4. For type 1 and 3, the statement is proven in Damanik–Pushnitski–Simon [DPS08].

Assume $\mathcal{J}$ is of type 2. Denote by $\widehat{\mathcal{J}}$ the type 1 Jacobi matrix equivalent to $\mathcal{J}$. Denote its Jacobi parameters by $\widehat{A}_n, \widehat{B}_n$, and let

$$A_n = \sigma_n^* \widehat{A}_n \sigma_{n+1} \tag{3.1.1}$$

for some unitaries σ_n. Since $\widehat{A}_n \to \mathbf{1}$, we get $A_n = \sigma_n^* \widehat{A}_n \sigma_{n+1} = \left(\sigma_n^* \widehat{A}_n \sigma_n\right) \sigma_n^* \sigma_{n+1}$ converges to $\mathbf{1}$ if and only if $\lim_{n\to\infty} \sigma_n^* \sigma_{n+1} = \mathbf{1}$.

Denote $Q_n = A_1 \dots A_n$, which is a positive-definite matrix. Note that $\widehat{Q}_n = \widehat{A}_1 \dots \widehat{A}_n = A_1 \dots A_n \sigma_{n+1}^* = Q_n \sigma_{n+1}^*$, so $Q_n = |\widehat{Q}_n|$ and $\sigma_{n+1} = \phi(\widehat{Q}_n)^*$. Here ϕ is the same as in Lemma 3.1.1.

Now, $\widehat{A}_{n+1} \to \mathbf{1}$ together with Lemma 3.1.1 implies that $\phi(\widehat{Q}_{n+1}) - \phi(\widehat{Q}_n) = \phi(\widehat{Q}_n \widehat{A}_{n+1}) - \phi(\widehat{Q}_n) \to \mathbf{0}$. Thus, $\sigma_{n+1} - \sigma_n \to \mathbf{0}$, and $\lim_{n\to\infty} \sigma_n^* \sigma_{n+1} = \mathbf{1}$. □

For the type 3 case of Theorem 1.3.6, we will need the following lemma. Recall that the singular values of a matrix A are defined to be the eigenvalues of $|A|$.

Lemma 3.1.2. *There exists a constant c such that for all $l \times l$ matrices A*

$$\sum_{j=1}^{l} (\sigma_j - |\lambda_j|) \le c \sum_{j=1}^{l} (1 - \sigma_j)^2, \tag{3.1.2}$$

where $\{\lambda_j\}_{j=1}^l$ and $\{\sigma_j\}_{j=1}^l$ are the eigenvalues and singular values of A, ordered by $|\lambda_1| \ge \ldots \ge |\lambda_l|$, $\sigma_1 \ge \ldots \ge \sigma_l \ge 0$, where c depends on l only.

Proof. For sufficiently large matrices A the inequality is clear. It also holds for any compact set on which the right-hand side of (3.1.2) does not vanish. Therefore, we only need to worry about neighborhoods of matrices with $\sum_{j=1}^l (1 - \sigma_j)^2 = 0$, that is, unitary matrices.

Consider any matrix A within distance 1/2 from the unitary group. Let $U = \phi(A)$ be the unitary factor in the polar decomposition of A. Since $\phi(A)$ is always the closest unitary to A (see, e.g., [Bha97]), we get

$$\|A - U\| \le 1/2 \quad \text{and} \quad \||A| - \mathbf{1}\| \le 1/2.$$

The second inequality immediately gives $|\sigma_j - 1| \le 1/2$, which in turn implies $||\lambda_j| - 1| \le 1/2$ by (3.1.3) below. The following basic facts are well-known (see [Wey49]):

$$\sigma_1 \ge |\lambda_j| \ge \sigma_l \text{ for any } j; \tag{3.1.3}$$

$$\prod_{j=1}^{l} |\lambda_j| = \prod_{j=1}^{l} \sigma_j. \tag{3.1.4}$$

Let $\varepsilon_j = \sigma_j - 1$, $\delta_j = |\lambda_j| - 1$. Then from (3.1.4),

$$\delta_l = \frac{\prod_{j=1}^{l} \sigma_j}{\prod_{j=1}^{l-1} |\lambda_j|} - 1 = \frac{\prod_{j=1}^{l}(1+\varepsilon_j) - \prod_{j=1}^{l-1}(1+\delta_j)}{\prod_{j=1}^{l-1}(1+\delta_j)},$$

and so

$$\begin{aligned}\sum_{j=1}^{l}(\sigma_j - |\lambda_j|) &= \sum_{j=1}^{l}(\varepsilon_j - \delta_j) \\ &= \frac{\prod\limits_{j=1}^{l-1}(1+\delta_j)\sum\limits_{j=1}^{l}\varepsilon_j - \prod\limits_{j=1}^{l-1}(1+\delta_j)\sum\limits_{j=1}^{l-1}\delta_j - \prod\limits_{j=1}^{l}(1+\varepsilon_j) + \prod\limits_{j=1}^{l-1}(1+\delta_j)}{\prod\limits_{j=1}^{l-1}(1+\delta_j)}.\end{aligned} \tag{3.1.5}$$

The first-order terms (i.e., those involving only one of ε's or δ's) of the numerator cancel out:

$$\sum_{j=1}^{l}\varepsilon_j - \sum_{j=1}^{l-1}\delta_j - \left(1 + \sum_{j=1}^{l}\varepsilon_j\right) + \left(1 + \sum_{j=1}^{l-1}\delta_j\right) = 0.$$

Now note that by (3.1.3), $|\delta_j| \le |\varepsilon_1| + |\varepsilon_l|$. Using this and $|\varepsilon_j\varepsilon_k| \le (\varepsilon_j^2 + \varepsilon_k^2)/2$ we can bound all of the second-order terms (i.e., those with $\varepsilon_j\varepsilon_k$, $\varepsilon_j\delta_k$, and $\delta_j\delta_k$) by $\widetilde{c}\sum_{j=1}^{l}\varepsilon_j^2$, where $\widetilde{c}$ will depend on l only. All of the higher-order terms can be taken care of by using $|\varepsilon_j| < 1, |\delta_j| < 1$ to reduce it to second-order. Finally, the denominator of the right-hand side of (3.1.5) is bounded below by $1/2^l$. Therefore, we obtain

$$\sum_{j=1}^{l}(\sigma_j - |\lambda_j|) \le c\sum_{j=1}^{l}\varepsilon_j^2 = c\sum_{j=1}^{l}(1-\sigma_j)^2,$$

which proves our lemma. □

Lemma 3.1.3. *There exists a constant c so that*

$$\|\mathbf{1} - A\| \le c\|\mathbf{1} - |A|\| \tag{3.1.6}$$

for any $l \times l$ matrix A with real positive eigenvalues, where c depends on l only.

Proof. By the equivalence of norms, we can prove (3.1.6) for the Hilbert–Schmidt norm instead. Let $\lambda_1 \ge \ldots \ge \lambda_l > 0$ be the eigenvalues of A, and let $\sigma_1 \ge \ldots \ge \sigma_l > 0$ be the singular values of A. Note

$$\begin{aligned}\|\mathbf{1}-A\|_{HS}^2 = \mathrm{Tr}\,[(\mathbf{1}-A)(\mathbf{1}-A)^*] &= l - 2\sum_{j=1}^{l}\mathrm{Re}\,\lambda_j + \mathrm{Tr}AA^* \\ &= l - 2\sum_{j=1}^{l}\lambda_j + \sum_{j=1}^{l}\sigma_j^2, \\ \|\mathbf{1}-|A|\|_{HS}^2 = \mathrm{Tr}\left[(\mathbf{1}-|A|)^2\right] &= l - 2\sum_{j=1}^{l}\sigma_j + \sum_{j=1}^{l}\sigma_j^2,\end{aligned}$$

and so $\|\mathbf{1}-A\|_{HS}^2 \leq M\|\mathbf{1}-|A|\|_{HS}^2$ holds if and only if

$$2\sum_{j=1}^{l}(\sigma_j - \lambda_j) \leq (M-1)\sum_{j=1}^{l}(1-\sigma_j)^2.$$

Since $\lambda_j = |\lambda_j|$, the previous lemma proves the result. $\square$

Proof of Theorem 1.3.6. As in Theorem 1.3.4, let $\widehat{A}_n$ be of type 1, and A_n of type 2 with the equivalence (3.1.1). Then keeping the notation of Theorem 1.3.4 and using Lemma 3.1.1, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\|\sigma_n - \sigma_{n+1}\|_{HS} &= \sum_{n=1}^{\infty}\|\phi(\widehat{Q}_{n-1}) - \phi(\widehat{Q}_n)\|_{HS} \\ &\leq \sum_{n=1}^{\infty}\sqrt{||\mathbf{1}-\widehat{A}_n^{-1}||_{HS}^2 + ||\mathbf{1}-\widehat{A}_n||_{HS}^2} \\ &\leq \sum_{n=1}^{\infty}||\mathbf{1}-\widehat{A}_n^{-1}||_{HS} + \sum_{n=1}^{\infty}||\mathbf{1}-\widehat{A}_n||_{HS} \\ &\leq (\sup_n ||\widehat{A}_n||_{HS} + 1)\sum_{n=1}^{\infty}||\mathbf{1}-\widehat{A}_n||_{HS} \\ &\leq (\sup_n ||\widehat{A}_n||_{HS} + 1)\sup_n ||(\mathbf{1}+\widehat{A}_n)^{-1}||_{HS}\sum_{n=1}^{\infty}||\mathbf{1}-\widehat{A}_n^2||_{HS} < \infty,\end{aligned}$$

since $\widehat{A}_n \to \mathbf{1}$, and so $\sup_n ||\widehat{A}_n||_{HS} < \infty$, $\sup_n ||(\mathbf{1}+\widehat{A}_n)^{-1}||_{HS} < \infty$.

This implies that σ_n is Cauchy, and so converges.

An alternative indirect way of proving that type 1 and type 2 are asymptotic to each other is as follows: it is proven in Theorem 1.3.7 that under condition (1.3.3) Szegő asymptotics for the type 2 block Jacobi matrix holds. In Theorem 3.3.4 the same fact is obtained for the type 1 Jacobi matrix. Therefore (1.3.2) implies that the limit $\lim_{n\to\infty}\sigma_n$ exists.

Now assume that $\widehat{A}_n$ is of type 1, and A_n of type 3 with the equivalence (3.1.1). Since all eigenvalues of A_n are real and positive, Lemma 3.1.3 applies, and we get

$$\sum_{n=1}^{\infty}\|\mathbf{1}-A_n\| \le c\sum_{n=1}^{\infty}\|\mathbf{1}-|A_n|\| = c\sum_{n=1}^{\infty}\|\mathbf{1}-\widehat{A}_n\|$$

since $|A_n| = \sigma_n^*\widehat{A}_n\sigma_n$ by (3.1.1). Now $\sum_{n=1}^{\infty}||\mathbf{1}-\widehat{A}_n|| \le \sup_n ||(\mathbf{1}+\widehat{A}_n)^{-1}|| \sum_{n=1}^{\infty}\|\mathbf{1}-\widehat{A}_n^2\| < \infty$, which implies

$$\begin{aligned}\sum_{n=1}^{\infty}\|\sigma_n-\sigma_{n+1}\| = \sum_{n=1}^{\infty}\|\mathbf{1}-\sigma_n^*\sigma_{n+1}\| &\le \sum_{n=1}^{\infty}\|\mathbf{1}-A_n\| + \sum_{n=1}^{\infty}\|A_n-\sigma_n^*\sigma_{n+1}\| \\ &= \sum_{n=1}^{\infty}\|\mathbf{1}-A_n\| + \sum_{n=1}^{\infty}\|\widehat{A}_n-\mathbf{1}\| < \infty.\end{aligned}$$

This shows that σ_n is Cauchy, and so converges. □

Example 1. *Let $D_k = \begin{pmatrix} (k+1)/k & 0 \\ 0 & 1\end{pmatrix}$ for $k \ge 1$. Note that $D_k \to \mathbf{1}$.*

Pick some unitary τ, and define the sequence $\widehat{A}_n$ as follows: $\widehat{A}_1 = \tau^* D_1 \tau$, $\widehat{A}_2 = D_1$, $\widehat{A}_3 = D_1^{-1}$, $\widehat{A}_4 = D_2$, $\widehat{A}_5 = D_3$, $\widehat{A}_6 = D_3^{-1}$, $\widehat{A}_7 = D_2^{-1}$, $\widehat{A}_8 = D_4$, and so on: we define $\widehat{A}_k$'s for $2^j \le k < 2^{j+1}$ in terms of further and further chunks of sequence D_k as

$$\begin{aligned}&\widehat{A}_{2^j} = D_{2^{j-1}}, \ldots, \widehat{A}_{3\cdot 2^{j-1}-1} = D_{2^j-1},\\ &\widehat{A}_{3\cdot 2^{j-1}} = D_{2^j-1}^{-1}, \ldots, \widehat{A}_{2^{j+1}-1} = D_{2^{j-1}}^{-1}.\end{aligned}$$

Note that $\widehat{A}_n > 0$, i.e., the sequence corresponds to a block Jacobi matrix of type 1. Using the notation from Section 2, let $\widehat{Q}_n = \widehat{A}_1 \ldots \widehat{A}_n$. Then

$$\widehat{Q}_{2^j-1} = \widehat{A}_1, \quad \widehat{Q}_{3\cdot 2^{j-1}-1} = \widehat{A}_1 D_{2^{j-1}} \ldots D_{2^j-1} = \widehat{A}_1 D_1,$$

and $\sigma_{2^j} = \phi(\widehat{Q}_{2^j-1})^* = \mathbf{1}$, $\sigma_{3\cdot 2^{j-1}} = \phi(\widehat{Q}_{3\cdot 2^{j-1}-1})^* = \phi(\tau^* D_1 \tau D_1)^*$. Now choose τ such that $\phi(\tau^* D_1 \tau D_1)$ is not positive definite. This gives that $\lim_{n\to\infty}\sigma_n$ doesn't exist, i.e., type 1 and type 2 are not asymptotic to each other.

Of course, the reason is that (1.3.3) fails here: $\sum\|\mathbf{1}-A_nA_n^*\|$ diverges as $\sum\frac{1}{n}$.

3.1.2 Proof of Li's Lemma

Proof of Lemma 3.1.1. Let $B = U\Sigma V^*$ and $BD = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^*$ be the singular value decompositions of B and BD (i.e., $U, \widetilde{U}, V, \widetilde{V}$ are unitary, and $\Sigma, \widetilde{\Sigma}$ are positive and diagonal). Denote

$$\begin{aligned} Y &= \widetilde{U}^*(B - BD)V = \widetilde{U}^*U\Sigma - \widetilde{\Sigma}\widetilde{V}^*V, \\ Z &= U^*(B - BD)\widetilde{V} = \Sigma V^*\widetilde{V} - U^*\widetilde{U}\widetilde{\Sigma}. \end{aligned}$$

Then

$$Y - Z^* = (\widetilde{U}^*U - \widetilde{V}^*V)\Sigma + \widetilde{\Sigma}(\widetilde{U}^*U - \widetilde{V}^*V) = X\Sigma + \widetilde{\Sigma}X, \tag{3.1.7}$$

where $X = \widetilde{U}^*U - \widetilde{V}^*V$. On the other hand,

$$\begin{aligned} Y - Z^* &= \widetilde{U}^*(B - BD)V - \widetilde{V}^*(B^* - D^*B^*)U \\ &= \widetilde{\Sigma}\widetilde{V}^*(D^{-1} - \mathbf{1})V - \widetilde{V}^*(\mathbf{1} - D^*)V\Sigma = \widetilde{\Sigma}E - F\Sigma, \end{aligned} \tag{3.1.8}$$

where $E = \widetilde{V}^*(D^{-1} - \mathbf{1})V$, $F = \widetilde{V}^*(\mathbf{1} - D^*)V$. Note that Σ and $\widetilde{\Sigma}$ are diagonal, and therefore, the solution of (3.1.7)=(3.1.8) is

$$x_{ij} = \frac{\widetilde{\sigma}_{ii}e_{ij} - f_{ij}\sigma_{jj}}{\sigma_{jj} + \widetilde{\sigma}_{ii}},$$

where $X \equiv (x_{ij})$, $E \equiv (e_{ij})$, $F \equiv (f_{ij})$, $\Sigma \equiv (\sigma_{ij})$, $\widetilde{\Sigma} \equiv (\widetilde{\sigma}_{ij})$. Note that $\sigma_{jj} > 0$ and $\widetilde{\sigma}_{ii} > 0$, and thus by the Schwarz inequality,

$$|x_{ij}|^2 \leq \frac{\sigma_{jj}^2 + \widetilde{\sigma}_{ii}^2}{(\sigma_{jj} + \widetilde{\sigma}_{ii})^2}(|e_{ij}|^2 + |f_{ij}|^2) \leq |e_{ij}|^2 + |f_{ij}|^2,$$

which implies

$$||X||_{HS}^2 \leq ||E||_{HS}^2 + ||F||_{HS}^2 = ||\mathbf{1} - D^{-1}||_{HS}^2 + ||\mathbf{1} - D||_{HS}^2.$$

Finally, note that $\phi(B) = UV^*$ and $\phi(BD) = \widetilde{U}\widetilde{V}^*$, so $||\phi(B) - \phi(BD)||_{HS} = ||\widetilde{U}XV^*||_{HS} = ||X||_{HS}$, and we are done. $\square$

3.2 Szegő Asymptotics for Matrix-Valued Measures with Countably Many Bound States

We will need a couple of facts about the product we defined in (1.2.1). By $\langle\langle f,g\rangle\rangle_{L^2}$, with the sub-index just $_{L^2}$, we will mean the product with respect to the Lebesgue measure on the real line or the unit circle, depending on the context.

Lemma 3.2.1. *Let $L^2(\mathbf{1}\frac{d\theta}{2\pi})$ be the space of all matrix-valued functions, each entry of which is a scalar $L^2(\frac{d\theta}{2\pi})$-function.*

(a) The following formulae

$$\|f\|_{L^2,1} \equiv \left(\int_{-\pi}^{\pi} \|f(\theta)\|^2 \frac{d\theta}{2\pi}\right)^{1/2},$$
$$\|f\|_{L^2,2} \equiv \left\|\int_{-\pi}^{\pi} f(\theta)^* f(\theta)\frac{d\theta}{2\pi}\right\|^{1/2} = \|\,\langle\langle f,f\rangle\rangle_{L^2}\,\|^{1/2}$$

define two equivalent (semi)norms on $L^2(\mathbf{1}\frac{d\theta}{2\pi})$:

$$\|f\|_{L^2,2} \le \|f\|_{L^2,1} \le l^{1/2}\|f\|_{L^2,2}.$$

(b) For any $f,g \in L^2$,

$$\|\langle\langle f,g\rangle\rangle_{L^2}\| \le l\|f\|_{L^2,2}\|g\|_{L^2,2}.$$

(c) If $f \in L^2(\mathbf{1}\frac{d\theta}{2\pi})$, then its n-th matrix Fourier coefficient $\left\langle\left\langle e^{in\theta}I, f\right\rangle\right\rangle_{L^2} \to \mathbf{0}$ as $n \to \infty$.

Proof. (a) The first inequality is obvious. The second follows from

$$\|f\|^2_{L^2,1} = \int_{-\pi}^{\pi} \|f(\theta)\|^2\frac{d\theta}{2\pi} \le \int_{-\pi}^{\pi} \mathrm{Tr}(f(\theta)^*f(\theta))\frac{d\theta}{2\pi} = \mathrm{Tr}\left(\int_{-\pi}^{\pi} f(\theta)^*f(\theta)\frac{d\theta}{2\pi}\right) \le l\|f\|^2_{L^2,2}.$$

(b) Using Hölder, and the equivalence from (a), we get

$$\|\langle\langle f,g\rangle\rangle_{L^2}\| \le \int_{-\pi}^{\pi} \|g(\theta)\|\,\|f(\theta)\|\frac{d\theta}{2\pi} \le \|f\|_{L^2,1}\|g\|_{L^2,1} \le l\|f\|_{L^2,2}\|g\|_{L^2,2}.$$

(c) Follows by looking at each entry separately. □

We start by constructing L which we hope to be the limiting function.

3.2.1 Construction of the Limit Function L

Let $\mathcal{J}$ be the type 2 Jacobi matrix corresponding to μ, and let $\mathfrak{p}_n$ be the orthonormal polynomials for $\mathcal{J}$.

Let $\nu = \mathrm{Sz}(\mu|_{[-2,2]})$ be the image measure on $\partial\mathbb{D}$ of $\mu|_{[-2,2]}$ under the mapping $\theta \mapsto 2\cos\theta$: $\int_{-\pi}^{\pi} g(2\cos\theta)d\nu(\theta) = \int_{-2}^{2} g(x)d\mu(x)$ for measurable g's. This is what is called the Szegő mapping. Let the Lebesgue decomposition of ν be

$$d\nu(\theta) = w(\theta)\frac{d\theta}{2\pi} + d\nu_s.$$

Then

$$w(\theta) = 2\pi|\sin\theta| f(2\cos\theta), \tag{3.2.1}$$

and so (1.3.5) implies

$$\int_{-\pi}^{\pi} \log\det w(\theta)\frac{d\theta}{2\pi} > -\infty.$$

Therefore Lemma 2.4.5 applies, so there exists a matrix-valued outer $H^2(\mathbb{D})$-function $G(z)$ such that

$$w(\theta) = G(e^{i\theta})^* G(e^{i\theta}), \tag{3.2.2}$$

$$G(0)^* = G(0) > 0, \tag{3.2.3}$$

$$\log|\det G(0)| = \int_{-\pi}^{\pi} \log|\det G(e^{i\theta})|\frac{d\theta}{2\pi}. \tag{3.2.4}$$

Denote w_k to be the weight of μ at $z_k + z_k^{-1}$:

$$w_k = \mu(z_k + z_k^{-1}).$$

Note that the condition (1.3.6) is equivalent to

$$\sum_{k=1}^{N} n_k \log|z_k| > -\infty, \tag{3.2.5}$$

where z_k and n_k are defined in (2.1.6). Now apply Lemma 2.4.6 to obtain the Blaschke–

Potapov product $B(z) = \overset{\frown}{\prod}_{j=1}^{\infty} B_{z_j,s_j,U_j}(z)$ (by (3.2.5) it converges) satisfying

$$\ker \operatorname*{Res}_{z=z_k} \left(B(z)^{-1}G(z)\right) = \ker w_k \quad \text{for all } k. \tag{3.2.6}$$

Indeed, note that $G(z_k)$ is invertible for any k (since $\det G$ is outer, it can't vanish in $\mathbb{D}$), so we can apply Lemma 2.4.6 with $V_k = G(z_k)^{-1} \ker w_k$.

Define for $z \in \mathbb{D}$,

$$L(z) = \frac{1}{\sqrt{2}} G(z)^{-1} B(z) V, \tag{3.2.7}$$

where V is a constant unitary such that $L(0) > 0$.

Let us rewrite the statement of Theorem 1.3.7 in a slightly more general way.

Theorem 3.2.2. *Let μ satisfy* (1.3.4), (1.3.5), (1.3.6)*. Assume $\mathcal{J}$ is of type* 2*, and let $\widetilde{\mathcal{J}}$ be any equivalent to it matrix with Jacobi parameters* (1.3.1) *and orthonormal polynomials $\widetilde{\mathfrak{p}}_n$* (1.3.2)*. Assume $\sigma = \lim_{n\to\infty} \sigma_n$ exists. Then*

$$z^n \widetilde{\mathfrak{p}}_n\left(z + z^{-1}\right) \to L(z)\sigma \quad \textit{uniformly on compacts of } \mathbb{D}; \tag{3.2.8}$$

$$\widetilde{\mathfrak{p}}_n(2\cos\theta) = \frac{1}{\sqrt{2}}\left(e^{-in\theta}L(e^{i\theta}) + e^{in\theta}L(e^{-i\theta}))\right)\sigma + o(1) \quad \textit{in } L^2\left(w(\theta)\frac{d\theta}{2\pi}\right) \textit{ sense;} \tag{3.2.9}$$

$$\langle\!\langle \widetilde{\mathfrak{p}}_n(x) \rangle\!\rangle_{L^2(\mu_s)} \to \mathbf{0}, \tag{3.2.10}$$

where w is defined in (3.2.1)*.*

The limit function L has a factorization (3.2.7)*, where G is the unique $H^2(\mathbb{D})$-function satisfying* (3.2.2)–(3.2.4) (*and thus has the form* (2.4.6)–(2.4.7))*, B is a Blaschke–Potapov product, V is a constant unitary matrix. We have*

$$\ker \operatorname*{Res}_{z=z_k} L(z)^{-1} = \ker w_k \quad \textit{for all } k;$$

$$L(0) > 0.$$

Remarks. 1. We will show that the asymptotics holds for type 2 Jacobi matrix. Thus by (1.3.2), the polynomials $\widetilde{\mathfrak{p}}_n$ obey Szegő asymptotics if and only if the limit $\lim_{n\to\infty} \sigma_n$ exists, so this condition is also necessary.

2. The equivalent way of writing (3.2.9) is

$$G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta) = \frac{1}{\sqrt{2}}\left(e^{-in\theta}B(e^{i\theta}) + e^{in\theta}G(e^{i\theta})G(e^{-i\theta})^{-1}B(e^{-i\theta})\right)V\sigma + o(1) \quad \text{in } L^2\left(\mathbf{1}\frac{d\theta}{2\pi}\right) \text{ sense.}$$

3.2.2 Proof of Theorem 1.3.7

Proof. The beginning of the proof follows closely the proof of the Lemma in [PY01]. Denote

$$s(e^{i\theta}) = G(e^{i\theta})G(e^{-i\theta})^{-1},$$

and consider the following expression. Expanding the product and using (3.2.2), we get

$$\begin{aligned}\mathbf{0} \le &\left\langle\!\!\left\langle G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta) - \frac{1}{\sqrt{2}}\left(e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right)\right\rangle\!\!\right\rangle_{L^2}^2 + \langle\!\langle\widetilde{\mathfrak{p}}_n(x)\rangle\!\rangle_{L^2(\mu_s)}^2 \\ = &\int_{-\pi}^{\pi}\widetilde{\mathfrak{p}}_n(2\cos\theta)^*w(\theta)\widetilde{\mathfrak{p}}_n(2\cos\theta)\frac{d\theta}{2\pi} + \langle\!\langle\widetilde{\mathfrak{p}}_n(x)\rangle\!\rangle_{L^2(\mu_s)}^2 + \frac{1}{2}\left\langle\!\!\left\langle e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right\rangle\!\!\right\rangle_{L^2}^2 \\ &- \sqrt{2}\,\mathrm{Re}\left\langle\!\!\left\langle G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta), e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right\rangle\!\!\right\rangle_{L^2},\end{aligned} \tag{3.2.11}$$

where by $\mathrm{Re}\,T$ we mean $\frac{T+T^*}{2}$.

First of all,

$$\int_{-\pi}^{\pi}\widetilde{\mathfrak{p}}_n(2\cos\theta)^*w(\theta)\widetilde{\mathfrak{p}}_n(2\cos\theta)\frac{d\theta}{2\pi} + \langle\!\langle\widetilde{\mathfrak{p}}_n(x)\rangle\!\rangle_{L^2(\mu_s)}^2 = \langle\!\langle\widetilde{\mathfrak{p}}_n(x)\rangle\!\rangle_{L^2(\mu)}^2 = \mathbf{1}. \tag{3.2.12}$$

Now, observe that

$$\begin{aligned}s(e^{i\theta})^*s(e^{i\theta}) &= G(e^{-i\theta})^{-*}G(e^{i\theta})^*G(e^{i\theta})G(e^{-i\theta})^{-1} \\ &= G(e^{-i\theta})^{-*}w(\theta)G(e^{-i\theta})^{-1} = G(e^{-i\theta})^{-*}w(-\theta)G(e^{-i\theta})^{-1} = \mathbf{1}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2}\left\langle\!\!\left\langle e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right\rangle\!\!\right\rangle_{L^2}^2 &= \mathbf{1} + \mathrm{Re}\left\langle\!\!\left\langle e^{-in\theta}B(e^{i\theta}), e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right\rangle\!\!\right\rangle_{L^2} \\ &= \mathbf{1} + \int_{-\pi}^{\pi}e^{2in\theta}B(e^{i\theta})^*s(e^{i\theta})B(e^{-i\theta})\frac{d\theta}{2\pi} = \mathbf{1} + o(1)\end{aligned} \tag{3.2.13}$$

since the function $k(\theta) = B(e^{i\theta})^* s(e^{i\theta}) B(e^{-i\theta})$ satisfies $\int_{-\pi}^{\pi} k(\theta)^* k(\theta) \frac{d\theta}{2\pi} = \mathbf{1}$, so by parts (a) and (c) of Lemma 3.2.1, its Fourier coefficients converge to the zero matrix.

Note that for any function g on the unit circle we have

$$
\begin{aligned}
&\left\langle\!\!\left\langle G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta) \ , s(e^{i\theta})g(e^{-i\theta}) \right\rangle\!\!\right\rangle_{L^2} \\
&\qquad = \int_{-\pi}^{\pi} \widetilde{\mathfrak{p}}_n(2\cos\theta)^*\, G(e^{i\theta})^*\, G(e^{i\theta}) G(e^{-i\theta})^{-1} g(e^{-i\theta}) \frac{d\theta}{2\pi} \\
&\qquad = \int_{-\pi}^{\pi} \widetilde{\mathfrak{p}}_n(2\cos\theta)^*\, w(\theta) G(e^{-i\theta})^{-1} g(e^{-i\theta}) \frac{d\theta}{2\pi} \\
&\qquad = \int_{-\pi}^{\pi} \widetilde{\mathfrak{p}}_n(2\cos\theta)^*\, w(-\theta) G(e^{-i\theta})^{-1} g(e^{-i\theta}) \frac{d\theta}{2\pi} \\
&\qquad = \int_{-\pi}^{\pi} \widetilde{\mathfrak{p}}_n(2\cos\theta)^*\, G(e^{-i\theta})^*\, g(e^{-i\theta}) \frac{d\theta}{2\pi} \\
&\qquad = \int_{-\pi}^{\pi} \widetilde{\mathfrak{p}}_n(2\cos\theta)^*\, G(e^{i\theta})^*\, g(e^{i\theta}) \frac{d\theta}{2\pi} = \left\langle\!\!\left\langle G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta), g(e^{i\theta}) \right\rangle\!\!\right\rangle_{L^2},
\end{aligned}
$$

so the third term on the right-hand side of (3.2.11) becomes

$$
\begin{aligned}
&\operatorname{Re} \left\langle\!\!\left\langle G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta), e^{-in\theta} B(e^{i\theta}) + e^{in\theta} s(e^{i\theta}) B(e^{-i\theta}) \right\rangle\!\!\right\rangle_{L^2} \\
&\qquad\qquad = 2\operatorname{Re} \left\langle\!\!\left\langle G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta), e^{-in\theta} B(e^{i\theta}) \right\rangle\!\!\right\rangle_{L^2}.
\end{aligned}
\tag{3.2.14}
$$

Lemma 3.2.3. *Let $\widetilde{\mathfrak{p}}_n(x) = r_n x^n + \ldots$ (in other words, $r_n = (\widetilde{A}_1^*)^{-1} \cdots (\widetilde{A}_n^*)^{-1}$). Then r_n are uniformly bounded (with respect to the operator norm).*

Proof. On the one hand, by (3.2.12),

$$
\left\| G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta) \right\|_{L^2,2} \le \| \langle\langle \widetilde{\mathfrak{p}}_n(x) \rangle\rangle_{L^2(\mu)}^2 \|^{1/2} = 1. \tag{3.2.15}
$$

On the other, by Lemma 3.2.1(a) and subharmonicity of $\|h(\cdot)\|^2$,

$$
\begin{aligned}
\left\| G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta) \right\|_{L^2,2} \ge l^{-1/2} \left\| G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta) \right\|_{L^2,1} &= l^{-1/2} \left(\int_{-\pi}^{\pi} \left\| h(e^{i\theta}) \right\|^2 \frac{d\theta}{2\pi} \right)^{1/2} \\
&\ge l^{-1/2} \|h(0)\| = l^{-1/2} \|G(0) r_n\|,
\end{aligned}
$$

where $h(z) \equiv G(z)\left[z^n \widetilde{\mathfrak{p}}_n\left(z + \frac{1}{z}\right)\right]$ is analytic in $\mathbb{D}$. $G(0)$ is invertible, so r_n are uniformly bounded. $\square$

The next lemma will allow us to compute the right-hand side of (3.2.14).

Lemma 3.2.4. *Let r_n be as in the previous lemma. Then*

$$\left\langle\!\left\langle e^{-in\theta}B(e^{i\theta}), G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta)\right\rangle\!\right\rangle_{L^2} = B(0)^{-1}G(0)r_n + o(1). \tag{3.2.16}$$

Proof. The partial products $B_N(z)$ converge to $B(z)$ uniformly on compacts of $\mathbb{D}$. This implies that each Fourier coefficient of $B(e^{i\theta}) - B_N(e^{i\theta})$ goes to 0 as $N \to \infty$. Since $\|B\|_{L^2,2} = \|B_N\|_{L^2,2} = 1$, weak convergence implies the norm convergence $\|B(e^{i\theta}) - B_N(e^{i\theta})\|_{L^2,2} \to 0$. Using (3.2.15) and Lemma 3.2.1(b), we can find $N \in \mathbb{N}$ such that

$$\begin{aligned}\left\|\left\langle\!\left\langle e^{-in\theta}(B(e^{in\theta}) - B_N(e^{-in\theta})), G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta)\right\rangle\!\right\rangle_{L^2}\right\| \\ \le l\|B(e^{i\theta}) - B_N(e^{i\theta})\|_{L^2,2}\|G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta)\|_{L^2,2} < \epsilon\end{aligned} \tag{3.2.17}$$

holds for any $n \in \mathbb{N}$. By Lemma 3.2.3, we can also assume that for this N,

$$\|B(0)^{-1}G(0)r_n - B_N(0)^{-1}G(0)r_n\| < \epsilon \tag{3.2.18}$$

also holds for any n. Now, $B_N(e^{i\theta})^* = B_N(e^{i\theta})^{-1}$, so

$$\begin{aligned}\left\langle\!\left\langle e^{-in\theta}B_N(e^{i\theta})\right.\right. & \left.\left., G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta)\right\rangle\!\right\rangle_{L^2} \\ &= \int_{-\pi}^{\pi} e^{in\theta}B_N(e^{i\theta})^{-1}G(e^{i\theta})\widetilde{\mathfrak{p}}_n(2\cos\theta)\frac{d\theta}{2\pi} \\ &= \int_{\partial\mathbb{D}} B_N(z)^{-1}G(z)\widetilde{\mathfrak{p}}_n\left(z + \frac{1}{z}\right) z^n \frac{dz}{2\pi i z} \\ &= B_N(0)^{-1}G(0)r_n + \sum_{k=1}^{N} \operatorname*{Res}_{z=z_k}\left(B_N(z)^{-1}G(z)\right)\widetilde{\mathfrak{p}}_n(E_k)z_k^{n-1}.\end{aligned} \tag{3.2.19}$$

By the construction, (3.2.6) holds, which implies $\ker \operatorname{Res}_{z=z_k}\left(B_N(z)^{-1}G(z)\right) = \ker w_k = \ker w_k^{1/2}$, which allows us to write $\operatorname{Res}_{z=z_k}\left(B_N(z)^{-1}G(z)\right) = S_k w_k^{1/2}$ for some matrix S_k. Thus,

$$\left\|\sum_{k=1}^{N} \operatorname*{Res}_{z=z_k}\left(B_N(z)^{-1}G(z)\right)\widetilde{\mathfrak{p}}_n(E_k)z_k^{n-1}\right\| \le \sup_{1\le k\le N}\|S_k\|\sum_{k=1}^{N}\|w_k^{1/2}\widetilde{\mathfrak{p}}_n(E_k)\|\,|z_k|^{n-1}. \tag{3.2.20}$$

But $\|w_k^{1/2}\widetilde{\mathfrak{p}}_n(E_k)\| = (\|\widetilde{\mathfrak{p}}_n(E_k)^* w_k \widetilde{\mathfrak{p}}_n(E_k)\|)^{1/2} \le \|\left\langle\!\left\langle\widetilde{\mathfrak{p}}_n(x)\right\rangle\!\right\rangle_{L^2(\mu)}\| = 1$. Since N was fixed, this proves that the right-hand side of (3.2.20) goes to 0 when $n \to \infty$. Combining (3.2.17), (3.2.18), (3.2.19), and (3.2.20), we obtain (3.2.16). □

Now, plugging (3.2.12), (3.2.13), (3.2.14), and (3.2.16) into (3.2.11), we obtain

$$\mathbf{0} \leq 2\mathbf{1} - 2\sqrt{2}\,\mathrm{Re}\left(B(0)^{-1}G(0)r_n\right) + o(1). \tag{3.2.21}$$

Observe that (3.2.21) holds for any initial choice of unitaries σ_n in (1.3.1). Let $\mathfrak{p}_n(x) = \kappa_n x^n + \dots$ (in other words, $\kappa_n = (A_1^*)^{-1} \cdots (A_n^*)^{-1} > 0$). Then (1.3.2) gives $r_n = \kappa_n \sigma_{n+1}$. For each n, pick unitary σ_{n+1} such that $B(0)^{-1}G(0)r_n = B(0)^{-1}G(0)\kappa_n\sigma_{n+1}$ is positive-definite. Then (3.2.21) gives

$$\sqrt{2}\,B(0)^{-1}G(0)\kappa_n\sigma_{n+1} \leq \mathbf{1} + o(1). \tag{3.2.22}$$

Denote $H_n \equiv \sqrt{2}\,B(0)^{-1}G(0)r_n > \mathbf{0}$. Let $\{\eta_s^{(n)}\}_{s=1}^l$ be the eigenvalues of H_n in non-increasing order. $\eta_s^{(n)} > 0$ for any n, s. Then (3.2.22) implies

$$\limsup_{n\to\infty} \eta_s^{(n)} \leq 1 \tag{3.2.23}$$

for each $s = 1, \dots, l$.

On the other hand, let us compute the determinant of H_n. By (2.4.9) and (2.4.5),

$$\log\det B(0)^{-1}G(0) = -\sum_k n_k \log|z_k| + \int_{-\pi}^{\pi} \log|\det G(e^{i\theta})|\frac{d\theta}{2\pi},$$

and by Lemma 2.1.2,

$$\begin{aligned}
\lim_{n\to\infty}\sum_{j=1}^n \log\det|\widetilde{A}_j| + \sum_k n_k \log|z_k| &= \frac{1}{2}\int_{-\pi}^{\pi} \log\det\frac{\mathrm{Im}\,M(e^{i\theta})}{\sin\theta}\frac{d\theta}{2\pi} \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \log\det\frac{\pi f(2\cos\theta)}{|\sin\theta|}\frac{d\theta}{2\pi} = \frac{1}{2}\int_{-\pi}^{\pi} \log\det\frac{w(\theta)}{2\sin^2\theta}\frac{d\theta}{2\pi} \\
&= \int_{-\pi}^{\pi} \log|\det G(e^{i\theta})|\frac{d\theta}{2\pi} - \frac{l}{4\pi}\int_{-\pi}^{\pi} \log\left(2\sin^2\theta\right)d\theta \\
&= \int_{-\pi}^{\pi} \log|\det G(e^{i\theta})|\frac{d\theta}{2\pi} + \frac{l}{2}\log 2.
\end{aligned} \tag{3.2.24}$$

Now note that

$$\log\det r_n = -\sum_{j=1}^{n} \log\det \widetilde{A}_j^* = -\sum_{j=1}^{n} \log\det |\widetilde{A}_j| + \log\det \rho_n$$

for some unitary matrix ρ_n. However, $H_n > \mathbf{0}$, so $\det \rho_n$ must be 1 as otherwise $\log\det H_n = \frac{l}{2}\log 2 + \log\det B(0)^{-1}G(0) + \log\det r_n$ cannot be real. Thus we obtain

$$\begin{aligned}\log\det H_n &= \frac{l}{2}\log 2 + \log\det B(0)^{-1}G(0) + \log\det r_n \\ &= \frac{l}{2}\log 2 - \sum_k n_k \log|z_k| + \int_{-\pi}^{\pi} \log|\det G(e^{i\theta})| \frac{d\theta}{2\pi} - \sum_{j=1}^{n} \log\det|\widetilde{A}_j| \to 0\end{aligned}$$

by (3.2.24). Thus $\lim_{n\to\infty} \det H_n = 1$. Together with (3.2.23) this implies $\lim_{n\to\infty} \eta_s^{(n)} = 1$ for each s, and so $H_n \to \mathbf{1}$. This proves $\kappa_n \sigma_{n+1} \to 2^{-1/2} G(0)^{-1} B(0)$. But $|\kappa_n \sigma_{n+1}| = \kappa_n$ (here temporarily $|T| \equiv \sqrt{TT^*}$ instead of $\sqrt{T^*T}$), so

$$\kappa_n \to 2^{-1/2} \left|G(0)^{-1}B(0)\right| = L(0).$$

Also, $\sigma_n \to V^*$.

Thus for the chosen σ's, the right-hand side of (3.2.21) goes to the zero matrix. This implies

$$\left\| G(e^{i\theta}) \widetilde{\mathfrak{p}}_n(2\cos\theta) - \frac{1}{\sqrt{2}} \left(e^{-in\theta} B(e^{i\theta}) + e^{in\theta} s(e^{i\theta}) B(e^{-i\theta}) \right) \right\|_{L^2,2} \to 0$$

and

$$\langle\!\langle \widetilde{\mathfrak{p}}_n(x) \rangle\!\rangle_{L^2(\mu_s)}^2 \to \mathbf{0}.$$

Taking into account that $\mathfrak{p}_n(x) = \widetilde{\mathfrak{p}}_n(x) \sigma_{n+1}^*$ and $\sigma_n \to V^*$, we get

$$\left\| G(e^{i\theta}) \mathfrak{p}_n(2\cos\theta) - \frac{1}{\sqrt{2}} \left(e^{-in\theta} B(e^{i\theta}) + e^{in\theta} s(e^{i\theta}) B(e^{-i\theta}) \right) V \right\|_{L^2,2} \to 0$$

and

$$\langle\!\langle \mathfrak{p}_n(x) \rangle\!\rangle_{L^2(\mu_s)}^2 \to \mathbf{0}.$$

This proves (3.2.9)–(3.2.10) for the type 2 case. To prove (3.2.8), by Lemma 3.2.1(b),

$$\begin{aligned}\left\| \left\langle\!\!\left\langle \frac{e^{-in\theta}}{1-e^{i\theta}\bar{z}}\mathbf{1}, G(e^{i\theta})\mathfrak{p}_n(2\cos\theta) - \frac{1}{\sqrt{2}}\left(e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right)V \right\rangle\!\!\right\rangle_{L^2} \right\| \\ \leq \frac{l}{\sqrt{1-|z|^2}} \left\| G(e^{i\theta})\mathfrak{p}_n(2\cos\theta) - \frac{1}{\sqrt{2}}\left(e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right)V \right\|_{L^2,2} \to 0\end{aligned} \tag{3.2.25}$$

uniformly on compacts of $\mathbb{D}$. On the other hand,

$$\begin{aligned}&\left\langle\!\!\left\langle \frac{e^{-in\theta}}{1-e^{i\theta}\bar{z}}\mathbf{1}, G(e^{i\theta})\mathfrak{p}_n(2\cos\theta) - \frac{1}{\sqrt{2}}e^{-in\theta}B(e^{i\theta})V \right\rangle\!\!\right\rangle_{L^2} \\ &\qquad = \int_{-\pi}^{\pi} \frac{e^{in\theta}}{1-e^{-i\theta}z}\left(G(e^{i\theta})\mathfrak{p}_n(2\cos\theta) - \frac{1}{\sqrt{2}}e^{-in\theta}B(e^{i\theta})V\right)\frac{d\theta}{2\pi} \\ &\qquad = z^n G(z)\mathfrak{p}_n(z+z^{-1}) - \frac{1}{\sqrt{2}}B(z)V,\end{aligned} \tag{3.2.26}$$

and

$$\left\langle\!\!\left\langle \frac{e^{-in\theta}}{1-e^{i\theta}\bar{z}}\mathbf{1}, \frac{1}{\sqrt{2}}\left(e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right)V \right\rangle\!\!\right\rangle_{L^2} \to \mathbf{0} \quad \text{uniformly on compacts of } \mathbb{D} \tag{3.2.27}$$

by Lemma 3.2.1(c). Together, (3.2.25), (3.2.26) and (3.2.27) give

$$z^n\mathfrak{p}_n\left(z+z^{-1}\right) \to L(z) \quad \text{uniformly on compacts of } \mathbb{D}.$$

Thus we proved (3.2.8)–(3.2.10) for the type 2 case. The result for any $\widetilde{\mathcal{J}}$ asymptotic to type 2 follows immediately from $\widetilde{\mathfrak{p}}_n(x) = \mathfrak{p}_n(x)\sigma_{n+1}$. $\square$

3.3 Jost Asymptotics for Matrix-Valued Orthogonal Polynomials

In this section we will be using notation

$$\mathcal{J}^{(k)} = \begin{pmatrix} B_{k+1} & A_{k+1} & \mathbf{0} & \\ A_{k+1}^* & B_{k+2} & A_{k+2} & \ddots \\ \mathbf{0} & A_{k+2}^* & B_{k+3} & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad \widetilde{\mathcal{J}}_k = \begin{pmatrix} B_1 & A_1 & \mathbf{0} & & & & & & \\ A_1^* & B_2 & A_2 & & & & & & \\ \mathbf{0} & A_2^* & \ddots & \ddots & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & A_{k-1}^* & B_k & A_k & \mathbf{0} & & \\ & & & \mathbf{0} & A_k^* & \mathbf{0} & \mathbf{1} & & \\ & & & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \ddots & \\ & & & & & & \ddots & \ddots & \end{pmatrix}.$$

Recall that we introduced the M-functions $M(z) = -\mathfrak{m}(z+z^{-1})$. Denote $M^{(k)}(z)$ to be the M-function corresponding to $\mathcal{J}^{(k)}$ (in particular $M^{(0)} = M$). Then the relation (1.2.7) takes form

$$A_{n+1} M^{(n+1)}(z) A_{n+1}^* = \left(z + \frac{1}{z}\right)\mathbf{1} - B_{n+1} - M^{(n)}(z)^{-1} \tag{3.3.1}$$

for $z \in \mathbb{D}$, $n \geq 0$.

Since $M^{(n)}(z)/z = \mathbf{1} + O(z)$ at $z = 0$, this gives

$$\left(\frac{M^{(n)}(z)}{z}\right)^{-1} = \mathbf{1} - B_{n+1} z - (A_{n+1}A_{n+1}^* - \mathbf{1})z^2 + O(z^3). \tag{3.3.2}$$

3.3.1 Jost Function via the Geronimo–Case Equations

3.3.1.1 Jost function for eventually free Jacobi matrices

First we will show existence and derive some properties of the Jost solution and the Jost function for the matrices $\widetilde{\mathcal{J}}_k$. Clearly we can construct a unique solution $u_n(z;\widetilde{\mathcal{J}}_k)$ which solves (1.3.11) for $\widetilde{\mathcal{J}}_k$ and satisfies $u_n(z;\widetilde{\mathcal{J}}_k) = z^n \mathbf{1}$ if $n \geq k+1$, where $z + z^{-1} = E$.

Since $u_k(z;\widetilde{\mathcal{J}}_k) = z^k A_k^{-1}$, taking the Wronskian at $n = k$, we find,

$$u(z;\widetilde{\mathcal{J}}_k) = z^k \mathfrak{p}_k^L(z+z^{-1};\widetilde{\mathcal{J}}_k) - z^{k+1} A_k^* \mathfrak{p}_{k-1}^L(z+z^{-1};\widetilde{\mathcal{J}}_k).$$

This suggests to define

$$g_n(z) = z^n \left(\mathfrak{p}_n^L\left(z+z^{-1};\mathcal{J}\right) - zA_n^*\mathfrak{p}_{n-1}^L\left(z+z^{-1};\mathcal{J}\right)\right) \tag{3.3.3}$$

and

$$c_n(z) = z^n\mathfrak{p}_n^L\left(z+z^{-1};\mathcal{J}\right). \tag{3.3.4}$$

Clearly g_n is a polynomial in z of degree at most $2n$, and c_n of degree exactly $2n$. The equation (3.3.3) can be written as

$$g_n(z) = c_n(z) - z^2A_n^*c_{n-1}(z). \tag{3.3.5}$$

Since $\mathfrak{p}_n^L(z;\mathcal{J}) = \mathfrak{p}_n^L(z;\widetilde{\mathcal{J}}_k)$ for $n \le k$, we have

$$g_n(z) = u(z;\widetilde{\mathcal{J}}_n). \tag{3.3.6}$$

Multiplying by z^{n+1} the recursion relation for left orthogonal polynomials (we will start writing $\mathfrak{p}_n(z)$ instead of $\mathfrak{p}_n(z;\mathcal{J})$ when $\mathcal{J}$ is clear from the context)

$$A_{n+1}\mathfrak{p}_{n+1}^L\left(z+\frac{1}{z}\right) + \left(B_{n+1} - \left(z+\frac{1}{z}\right)\mathbf{1}\right)\mathfrak{p}_n^L\left(z+\frac{1}{z}\right) + A_n^*\mathfrak{p}_{n-1}^L\left(z+\frac{1}{z}\right) = \mathbf{0}$$

and using (3.3.5), we get

$$A_{n+1}c_{n+1}(z) = \left(z^2\mathbf{1} - zB_{n+1}\right)c_n(z) + g_n(z). \tag{3.3.7}$$

Combining (3.3.5) and (3.3.7), we obtain

$$A_{n+1}g_{n+1}(z) = \left(z^2\left(\mathbf{1} - A_{n+1}A_{n+1}^*\right) - zB_{n+1}\right)c_n(z) + g_n(z). \tag{3.3.8}$$

The recursion equations (3.3.7) and (3.3.8) with the initial conditions $g_0(z) = c_0(z) = \mathbf{1}$ are called the Geronimo-Case equations. They can also be written in the form

$$\begin{pmatrix} c_{n+1} \\ g_{n+1} \end{pmatrix} = V_{n+1}\begin{pmatrix} c_n \\ g_n \end{pmatrix}, \tag{3.3.9}$$

where V_n is the $2l \times 2l$ matrix

$$V_n(z) = \begin{pmatrix} A_n^{-1} & \mathbf{0} \\ \mathbf{0} & A_n^{-1} \end{pmatrix} \begin{pmatrix} z^2\mathbf{1} - zB_n & \mathbf{1} \\ z^2(\mathbf{1} - A_nA_n^*) - zB_n & \mathbf{1} \end{pmatrix}. \tag{3.3.10}$$

Since $u = g_n$ if $A_k = \mathbf{1}, B_k = \mathbf{0}$ for $k \geq n+1$, it is straightforward to see the following theorem holds.

Theorem 3.3.1. *Let $A_kA_k^* - \mathbf{1} = B_k = \mathbf{0}$ for $k \geq n+1$ (i.e., $\mathcal{J} = \widetilde{\mathcal{J}}_n$), then $u(z;\mathcal{J})$ is a polynomial. Moreover:*

- *if $A_nA_n^* \neq \mathbf{1}$, then* $\deg(u) = 2n$*;*
- *if $A_nA_n^* = \mathbf{1}$, but $B_n \neq \mathbf{0}$, then* $\deg(u) = 2n - 1$.

Proof. By (3.3.6), $u(z;\mathcal{J}) = g_n(z)$, and then (3.3.8) gives

$$u(z;\mathcal{J}) = A_n^{-1}\left[\left(z^2\left(\mathbf{1} - A_nA_n^*\right) - zB_n\right)c_{n-1}(z) + g_{n-1}(z)\right].$$

Since $\deg g_k \leq 2k$ and $\deg c_k = 2k$, we obtain each statement of the theorem by induction. $\square$

3.3.1.2 The general case

Just as in [DS06b], we will be making one of the three successively stronger hypotheses on the Jacobi coefficients:

$$\sum_{n=1}^{\infty}\left[||B_n|| + ||\mathbf{1} - A_nA_n^*||\right] < \infty \tag{$A1$}$$

$$\sum_{n=1}^{\infty} n\left[||B_n|| + ||\mathbf{1} - A_nA_n^*||\right] < \infty \tag{$A2$}$$

$$||B_n|| + ||\mathbf{1} - A_nA_n^*|| \leq CR^{-2n} \quad \text{for some } R > 1 \tag{$A3$}$$

and study properties of the Jost function for each case.

Note that we have the following:

Lemma 3.3.2. *If the Jacobi parameters satisfy (A1), and $\mathcal{J}$ is of type asymptotic to* 1*, then the product $\overset{\curvearrowright}{\prod}_{n=1}^{\infty} A_n$ converges, and the limit is an invertible matrix. Moreover, $\prod_{n=1}^{\infty}||A_n^{-1}|| < \infty$ and $\prod_{n=1}^{\infty}||A_n|| < \infty$, and the products converge absolutely.*

Proof. Assume $\mathcal{J}$ is of type 1, i.e., $A_n = A_n^* > 0$. Then $\prod_{n=1}^{\infty} ||A_n^{-1}|| < \infty$ follows from

$$\begin{aligned}\sum_{n=1}^{\infty} |1 - ||A_n^{-1}||| &\le \sum_{n=1}^{\infty} ||\mathbf{1} - A_n^{-1}|| \le \sum_{n=1}^{\infty} ||A_n^{-1}|| \, ||\mathbf{1} - A_n|| \\ &\le \sup_j ||A_j^{-1}|| \sum_{n=1}^{\infty} ||\mathbf{1} - A_n^2|| \, ||(\mathbf{1} + A_n)^{-1}|| \qquad (3.3.11) \\ &\le c \sum_{n=1}^{\infty} ||\mathbf{1} - A_n^2|| < \infty,\end{aligned}$$

where we can bound $||A_n^{-1}||$ and $||(\mathbf{1} + A_n)^{-1}||$ uniformly since $\mathcal{J}$ is in the Nevai class, so $A_n \to \mathbf{1}$, so $(\mathbf{1} + A_n)^{-1} \to \frac{1}{2}\mathbf{1}$.

The bound for $\sum_{n=1}^{\infty} |1 - \|A_n\||$ is analogous.

Note that we also showed that $\sum_{n=1}^{\infty} ||\mathbf{1} - A_n|| < \infty$. It is proven in [Tre99] that given this, the limit $\overset{\curvearrowright}{\prod}_{n=1}^{\infty} A_n$ exists and is invertible.

Now let $\widetilde{\mathcal{J}}$ be any matrix satisfying (A1) asymptotic to type 1, satisfying (1.3.1). Then $\overset{\curvearrowright}{\prod}_{n=1}^{N} \widetilde{A}_n = \overset{\curvearrowright}{\prod}_{n=1}^{N} A_n \sigma_{N+1}$ also has an invertible limit. □

Define g_n and c_n by (3.3.7) and (3.3.8) with the initial conditions $g_0(z) = c_0(z) = \mathbf{1}$.

Lemma 3.3.3. *Assume $\mathcal{J}$ is of type* 1.

(i) Let (A1) hold. Then uniformly on compacts K of $\overline{\mathbb{D}} \setminus \{\pm 1\} \equiv \mathbb{E}$,

$$\sup_{n \in \mathbb{N}, z \in K} ||c_n(z)|| + ||g_n(z)|| < \infty. \qquad (3.3.12)$$

(ii) Let (A2) hold. Then

$$\sup_{n \in \mathbb{N}, z \in \overline{\mathbb{D}}} ||g_n(z)|| < \infty, \qquad (3.3.13)$$

$$\sup_{n \in \mathbb{N}, z \in \overline{\mathbb{D}}} \frac{||c_n(z)||}{1 + n} < \infty. \qquad (3.3.14)$$

(iii) Let (A3) hold. Let K be any compact subset of $z \in \{z \mid |z| < R\} \equiv \mathbb{D}_R$ with $r = \sup_{z \in K} |z| > 1$. There exists some constant C such that for all $z \in K$

$$||c_n(z)|| + ||g_n(z)|| \le C \left[\max(1, r)\right]^{2n}. \qquad (3.3.15)$$

In each of these cases the limit

$$g_\infty(z) = \lim_{n\to\infty} g_n(z)$$

exists, uniformly on compacts of the corresponding region: $\mathbb{E}$ *for* (A1), $\overline{\mathbb{D}}$ *for* (A2), *and* $\mathbb{D}_R$ *for* (A3). g_∞ *is continuous there, and analytic in the interior.*

Proof. (i) Define the norm $\left\|\begin{pmatrix} A \\ B \end{pmatrix}\right\| = ||A|| + ||B||$ for any $l \times l$ matrices A, B, and let $||V||_{in}$ for any $2l \times 2l$ matrix V be the induced operator norm. Taking (3.3.9) into account, the estimates (3.3.12) and (3.3.15) will be proved if we show the corresponding results for $||V_n(z)\dots V_1(z)||_{in}$. Observe that for $z \neq \pm 1$,

$$\begin{pmatrix} z^2\mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = L(z)\begin{pmatrix} z^2\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} L(z)^{-1},$$

where

$$L(z) = \begin{pmatrix} \mathbf{1} & \frac{1}{1-z^2}\mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad L(z)^{-1} = \begin{pmatrix} \mathbf{1} & -\frac{1}{1-z^2}\mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

So denoting

$$F_n = L(z)^{-1}\begin{pmatrix} -zB_n & \mathbf{0} \\ z^2(\mathbf{1} - A_nA_n^*) - zB_n & \mathbf{0} \end{pmatrix} L(z),$$

we obtain from (3.3.10),

$$\begin{aligned} V_n = \begin{pmatrix} A_n^{-1} & \mathbf{0} \\ \mathbf{0} & A_n^{-1} \end{pmatrix} L(z)\left[\begin{pmatrix} z^2\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} + F_n\right] L(z)^{-1} \\ = L(z)\begin{pmatrix} A_n^{-1} & \mathbf{0} \\ \mathbf{0} & A_n^{-1} \end{pmatrix}\left[\begin{pmatrix} z^2\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} + F_n\right] L(z)^{-1} \end{aligned}$$

since $L(z)$ and $\begin{pmatrix} A_n^{-1} & \mathbf{0} \\ \mathbf{0} & A_n^{-1} \end{pmatrix}$ commute.

Then we get that for any z, $z \neq \pm 1$,

$$\|V_n \dots V_1\|_{in} \leq \|L(z)\|_{in} \, \|L(z)^{-1}\|_{in} \, [\max(1, |z|)]^{2n} \times \\ \times \prod_{j=1}^{n} ||A_j^{-1}|| \prod_{j=1}^{n} \left(1 + \|L(z)\|_{in} \, \|L(z)^{-1}\|_{in} \left(||B_j|| + ||\mathbf{1} - A_j A_j^*||\right)\right). \quad (3.3.16)$$

By Lemma 3.3.2, we can bound $\prod_{j=1}^{n} ||A_j^{-1}||$.

For any compact K of $\mathbb{E}$, $\sup_{z \in K} \|L(z)\|_{in} \, \|L(z)^{-1}\|_{in} < \infty$, so taking supremum in (3.3.16) over $z \in K$ and using (A1) we obtain

$$\sup_{n \in \mathbb{N}, z \in K} ||c_n(z)|| + ||g_n(z)|| = M < \infty$$

for some constant M.

(ii) Note that by Lemma 2.5.1(i), we have

$$\sup_{\Lambda \subset \mathbb{N}} \prod_{j \in \Lambda} ||A_j^{-1}|| = p < \infty.$$

Let us show inductively that

$$||g_n(z)|| \leq \prod_{j=1}^{n} ||A_j^{-1}|| \prod_{j=1}^{n} \left[1 + j(||B_j|| + ||\mathbf{1} - A_j A_j^*||)\right]$$

and

$$||c_n(z)|| \leq (n+1) \prod_{j=1}^{n} ||A_j^{-1}|| \prod_{j=1}^{n} \left[1 + j(||B_j|| + ||\mathbf{1} - A_j A_j^*||)\right].$$

For $n = 0$ the inequalities are trivial. Now, if these inequalities hold for n then using (3.3.7) and (3.3.8):

$$||g_{n+1}(z)|| \leq ||A_{n+1}^{-1}|| \prod_{j=1}^{n} \left[(n+1)(||B_{n+1}|| + ||\mathbf{1} - A_{n+1} A_{n+1}^*||) + 1\right] \times \\ \times \prod_{j=1}^{n} ||A_j^{-1}|| \prod_{j=1}^{n} \left[1 + j(||B_j|| + ||\mathbf{1} - A_j A_j^*||)\right]$$

and

$$
\begin{aligned}
||c_{n+1}(z)|| \leq ||A_{n+1}^{-1}|| \prod_{j=1}^{n} [(n+1)(1+||B_{n+1}||)+1] \times \\
\times \prod_{j=1}^{n} ||A_j^{-1}|| \prod_{j=1}^{n} \left[1+j(||B_j||+||\mathbf{1}-A_jA_j^*||)\right] \\
\leq (n+2) \prod_{j=1}^{n+1} ||A_j^{-1}|| \prod_{j=1}^{n+1} \left[1+j(||B_j||+||\mathbf{1}-A_jA_j^*||)\right].
\end{aligned}
$$

By Lemma 3.3.2, $\prod_{n=1}^{\infty} ||A_n^{-1}||$ is absolutely convergent, so $(A2)$ implies (3.3.13) and (3.3.14).

(iii) Since $||g_n||$ and $||c_n||$ are subharmonic functions, by the maximum principle we need to prove the estimate (3.3.15) for the circle $|z| = r$. This follows immediately from (3.3.16). Note that this property does not really require $(A3)$, just $(A1)$ (the existence of the limit however will).

Now to show the convergence of g_n, note that by (3.3.8),

$$
\begin{aligned}
||g_{n+1}(z) - g_n(z)|| = ||A_{n+1}^{-1}\left(z^2\left(\mathbf{1}-A_{n+1}A_{n+1}^*\right) - zB_{n+1}\right)c_n(z) + \left(A_{n+1}^{-1}-\mathbf{1}\right)g_n(z)|| \\
\leq \left[\sup_j ||A_j^{-1}||\, [\max(1,r)]^{2n}\, (||B_n|| + ||\mathbf{1}-A_nA_n^*||) + ||\mathbf{1}-A_{n+1}^{-1}||\right] \times \\
\times \sup_{n\in\mathbb{N}, z\in K} (||c_n(z)|| + ||g_n(z)||). \quad (3.3.17)
\end{aligned}
$$

Since we are in the type 1 situation, we can use the same reasoning as in (3.3.11) to get $||\mathbf{1} - A_{n+1}^{-1}|| \leq c||\mathbf{1} - A_{n+1}A_{n+1}^*||$, and then (3.3.17), together with the estimates in (i), (ii), and (iii), gives $\sum_{n=0}^{\infty} ||g_{n+1}(z) - g_n(z)|| < \infty$ uniformly on compacts of $\mathbb{E}$, $\overline{\mathbb{D}}$, $\mathbb{D}_R$, respectively. This proves the existence and analyticity/continuity properties of g_∞. □

As a consequence we obtain Szegő asymptotics of the orthonormal polynomials in the unit disk (compare with Corollary 1.3.8).

Theorem 3.3.4. *Assume $(A1)$ holds, i.e., $\sum_{n=1}^{\infty}[||B_n|| + ||\mathbf{1} - A_nA_n^*||] < \infty$, and let $\mathcal{J}$ be of type 1. Then uniformly on compacts of $\mathbb{D}$ the limit*

$$
\lim_{n\to\infty} z^n \mathfrak{p}_n^L\left(z+z^{-1}\right) \tag{3.3.18}
$$

exists, and is equal to $\frac{1}{1-z^2}\, g_\infty(z)$.

Proof. Note that by Lemma 2.5.1, $\prod_{n=1}^{\infty} ||A_n^{-1}||$ is absolutely convergent, so by Lemma 2.5.1(i), we have

$$\sup_{\Lambda\subset\mathbb{N}} \prod_{j\in\Lambda} ||A_j^{-1}|| = p < \infty. \tag{3.3.19}$$

Let K be any compact of $\mathbb{D}$, and $M = \sup_{n\in\mathbb{N},z\in K} ||c_n(z)|| + ||g_n(z)||$. By the Geronimo–Case equations,

$$||c_n - A_n^{-1} g_{n-1} - z^2 A_n^{-1} c_{n-1}|| \leq M \|A_n^{-1}\| \|B_n\| \leq Mp\|B_n\|.$$

Repeating this, we get

$$\begin{aligned} &||c_n - A_n^{-1} g_{n-1} - z^2 A_n^{-1} A_{n-1}^{-1} g_{n-2} - z^4 A_n^{-1} A_{n-1}^{-1} c_{n-2}|| \\ &\qquad \leq Mp\|B_n\| + |z|^2 M \|A_n^{-1}\| \|A_{n-1}^{-1}\| \|B_{n-1}\| \\ &\qquad \leq Mp\|B_n\| + |z|^2 Mp\|B_{n-1}\|. \end{aligned} \tag{3.3.20}$$

Iterating it further, we get

$$||c_n - f_n|| \leq Mp \sum_{j=1}^{n} |z|^{2(n-j)} \|B_j\|, \tag{3.3.21}$$

where

$$f_n = A_n^{-1} g_{n-1} + z^2 A_n^{-1} A_{n-1}^{-1} g_{n-2} + \ldots + z^{2(n-1)} A_n^{-1} A_{n-1}^{-1} \ldots A_1^{-1} g_0 + z^{2n} A_n^{-1} A_{n-1}^{-1} \ldots A_1^{-1} c_0.$$

By Lemma 2.5.1(ii) the right-hand side of (3.3.21) goes to zero. Finally, note that

$$\begin{aligned} \left\| \overset{\curvearrowright}{\prod_{k=n+1}^{\infty}} A_k\, g_\infty \frac{1-z^{2n}}{1-z^2} - f_n \right\| &\leq p \sum_{j=0}^{n-1} |z|^{2(n-1-j)} \left\| \overset{\curvearrowright}{\prod_{k=n+1}^{\infty}} A_k\, g_\infty - A_n^{-1} A_{n-1}^{-1} \ldots A_{j+1}^{-1} g_j \right\| \\ &\leq p^2 \sum_{j=0}^{n-1} |z|^{2(n-1-j)} \left\| \overset{\curvearrowright}{\prod_{k=1}^{\infty}} A_k\, g_\infty - A_1 \ldots A_j g_j \right\|. \end{aligned} \tag{3.3.22}$$

By Lemma 3.3.2, the product $\overset{\curvearrowright}{\prod_{k=1}^{\infty}} A_k$ converges, and by Lemma 2.5.1(ii) the right-hand side of (3.3.22) goes to zero. Easy to see that the convergence in (3.3.21) and (3.3.22) is actually uniform. Thus we established $\lim_{n\to\infty} c_n = \frac{1}{1-z^2} g_\infty$. $\square$

Remark. Another way of showing this is to use the analogous arguments to [Sim, Lemma 3.7.5] to show that Szegő asymptotics (i.e., (3.3.18)) at $z \in \mathbb{D}$ holds if and only if the Jost asymptotics does (i.e., (3.3.28)), so that Theorem 3.3.6 implies Theorem 3.3.4.

Denote the limit function $g_\infty(z)$ of Lemma 3.3.3 as $u(z; \mathcal{J})$ and call it the Jost function (in Theorem 3.3.6 below we will show that it is indeed the case). Lemma 3.3.3 establishes the existence of the Jost function for the type 1 situation only. The next theorem says that the Jost function exists if and only if the Jacobi matrix is asymptotic to type 1.

Theorem 3.3.5. *Let $\mathcal{J}$ with Jacobi parameters $(A_n)_{n=1}^\infty$, $(B_n)_{n=1}^\infty$ be of type 1 and satisfy (A1). Let $\widetilde{\mathcal{J}}$ with Jacobi parameters $(\widetilde{A}_n)_{n=1}^\infty$, $(\widetilde{B}_n)_{n=1}^\infty$ be equivalent to $\mathcal{J}$, i.e,*

$$\widetilde{A}_n = \sigma_n^* A_n \sigma_{n+1}, \tag{3.3.23}$$

$$\widetilde{B}_n = \sigma_n^* B_n \sigma_n \tag{3.3.24}$$

for some unitary $\mathbf{1} = \sigma_1, \sigma_2, \sigma_3, \ldots$ Then the Jost function for $\widetilde{\mathcal{J}}$ exists if and only if $\lim_{n\to\infty} \sigma_n$ exists, in which case

$$u(z; \widetilde{\mathcal{J}}) = \lim_{n\to\infty} \sigma_n^* \, u(z; \mathcal{J}) \sigma_1. \tag{3.3.25}$$

Proof. We prove inductively that $\widetilde{g}_n = \sigma_{n+1}^* g_n \sigma_1$ and $\widetilde{c}_n = \sigma_{n+1}^* c_n \sigma_1$. For $n = 0$ this is trivial, and assuming this holds for n, we prove it for $n+1$:

$$\begin{aligned}
\widetilde{g}_{n+1}(z) &= \widetilde{A}_{n+1}^{-1} \left[\widetilde{g}_n(z) + \left(z^2 \left(\mathbf{1} - \widetilde{A}_{n+1} \widetilde{A}_{n+1}^* \right) - z \widetilde{B}_{n+1} \right) \widetilde{c}_n(z) \right] \\
&= \sigma_{n+2}^* A_{n+1}^{-1} \sigma_{n+1} \left[\sigma_{n+1}^* g_n(z) \right. \\
&\qquad\qquad \left. + \left(z^2 \left(\mathbf{1} - \sigma_{n+1}^* A_{n+1} A_{n+1}^* \sigma_{n+1} \right) - z \sigma_{n+1}^* B_{n+1} \sigma_{n+1} \right) \sigma_{n+1}^* c_n(z) \right] \sigma_1 \\
&= \sigma_{n+2}^* A_{n+1}^{-1} \left[\left(z^2 \left(\mathbf{1} - A_{n+1} A_{n+1}^* \right) - z B_{n+1} \right) c_n(z) + g_n(z) \right] \sigma_1 = \sigma_{n+2}^* g_{n+1}(z) \sigma_1,
\end{aligned}$$

and similarly for $\widetilde{c}_{n+1} = \sigma_{n+2}^* c_{n+1} \sigma_1$. The limit $\lim_{n\to\infty} g_n(z)$ exists by Lemma 3.3.3, so $\lim_{n\to\infty} \widetilde{g}_n(z)$ exists if and only if exists the limit $\lim_{n\to\infty} \sigma_n$, in which case $u(z; \widetilde{\mathcal{J}}) = \lim_{n\to\infty} \sigma_n^* \, u(z; \mathcal{J}) \sigma_1$. □

Assume $\mathcal{J}$ is a Jacobi matrix asymptotic to type 1, and let its Jacobi parameters satisfy (A1), (A2), or (A3). Then so do the parameters of $\mathcal{J}^{(k)}$ for all k, and thus $u(z; \mathcal{J}^{(k)})$ exists in $\mathbb{E}$, $\overline{\mathbb{D}}$, $\mathbb{D}_R$, respectively (which will be called "the appropriate region" in what follows). We define the Jost solution (in Theorem 3.3.6 below we will show it is indeed the Jost solution defined earlier) by

$$u_n(z;\mathcal{J}) = z^n u(z;\mathcal{J}^{(n)})A_n^{-1}. \tag{3.3.26}$$

Observe that by (the arguments of) Theorem 3.3.5, the Jost solutions of equivalent Jacobi matrices are related via

$$u_k(z;\widetilde{\mathcal{J}}) = \lim_{n\to\infty} \sigma_n^* u_k(z;\mathcal{J})\sigma_k.$$

Recall that $\mathfrak{m}(z) = \int \frac{1}{x-z} d\mu(x)$ and $M(z) = -\mathfrak{m}(z+z^{-1};\mathcal{J})$. For each discrete eigenvalue E_j of $\mathcal{J}$ outside $[-2,2]$, let $z_j \in \mathbb{D}$ be such that $z_j + z_j{}^{-1} = E_j$, and denote $\widetilde{w}_j = -\lim_{z\to z_j}(z-z_j)M(z)$, $w_j = \mu(E_j) = -\lim_{E\to E_j}(E-E_j)\mathfrak{m}(E) = (z_j^{-1}-z_j)z_j^{-1}\widetilde{w}_j$ $(w_j, \widetilde{w}_j \geq \mathbf{0})$.

In the next theorem and until Section 3.5 by $g^\sharp(z)$ we denote the function $g(1/\bar{z})^*$.

Theorem 3.3.6. *Assume $\mathcal{J}$ is a Jacobi matrix asymptotic to type* 1, *and let its Jacobi parameters satisfy* (A1), (A2), *or* (A3).

(i) *$u_n(z;\mathcal{J})$ in the appropriate region satisfies*

$$u_{n+1}(z;\mathcal{J})A_n^* + u_n(z;\mathcal{J})(B_n - (z+z^{-1})\mathbf{1}) + u_{n-1}(z;\mathcal{J})A_{n-1} = \mathbf{0}, \quad n = 1,2,\dots. \tag{3.3.27}$$

(ii) *In the appropriate region,*

$$\lim_{n\to\infty} z^{-n}u_n(z;\mathcal{J}) = \mathbf{1}. \tag{3.3.28}$$

(iii) *For $z \in \mathbb{D}$,*

$$u(z;\mathcal{J}^{(1)}) = z^{-1}u(z;\mathcal{J})M(z;\mathcal{J})A_1. \tag{3.3.29}$$

(iv) *The only zeros of $u(z;\mathcal{J})$ in $\mathbb{D}$ are at real points z_j with $z_j + z_j^{-1} \equiv E_j$ a discrete eigenvalue of $\mathcal{J}$. Each pole of $u(z;\mathcal{J})^{-1}$ in $\mathbb{D}$ is of order* 1, *and the order of z_j as a zero of $\det u(z;\mathcal{J})$ equals to the multiplicity of E_j as an eigenvalue of $\mathcal{J}$. Moreover,*

$$\ker u(z_j;\mathcal{J}) = \operatorname{Ran} w_j = \operatorname{Ran} \widetilde{w}_j. \tag{3.3.30}$$

(v) The only poles of $u(z;\mathcal{J})^{-1}$ in $\partial\mathbb{D}$ are possible ones at ± 1, in which case they are of order 1.

(vi) $M(z;\mathcal{J})$ has a continuation from $\mathbb{D}$ to $\overline{\mathbb{D}}\setminus\{\pm 1\}$, which is everywhere finite and invertible on $\partial\mathbb{D}\setminus\{\pm 1\}$, and

$$\operatorname{Im} M(e^{i\theta}) = \sin\theta \left[u(e^{i\theta};\mathcal{J})^* u(e^{i\theta};\mathcal{J})\right]^{-1}. \tag{3.3.31}$$

(vii) The following recurrence holds:

$$u(z;\mathcal{J}^{(2)}) = z^{-1}u(z;\mathcal{J}^{(1)})A_1^{-1}((z+z^{-1})\mathbf{1}-B_1)A_1^{*-1}A_2 - z^{-2}u(z;\mathcal{J})A_1^{*-1}A_2.$$

Now assume $(A3)$ holds.

(viii) If $(A3)$ holds, then M can be extended meromorphically to $\{z \mid |z| < R\}$, and

$$M(z) = M^\sharp(z) + (z - z^{-1})\left[u^\sharp(z;\mathcal{J})u(z;\mathcal{J})\right]^{-1}, \quad R^{-1} < |z| < R. \tag{3.3.32}$$

(ix) For each z_j with $R^{-1} < |z_j| < 1$,

$$\widetilde{w}_j u(1/\bar{z}_j;\mathcal{J})^* = -(z_j - z_j^{-1}) \operatorname*{Res}_{z=z_j} u(z;\mathcal{J})^{-1}, \tag{3.3.33}$$

in particular,

$$\ker u(1/\bar{z}_j;\mathcal{J})^* \subseteq \ker \operatorname*{Res}_{z=z_j} u(z;\mathcal{J})^{-1} = \operatorname{Ran} u(z_j;\mathcal{J}). \tag{3.3.34}$$

Remarks. 1. Part (vi) shows that if $(A1)$ holds then there is no point spectrum in $[-2,2]$.

2. Part (vii) shows that if $u(z;\mathcal{J})$ and $u(z;\mathcal{J}^{(1)})$ are analytic, then so is $u(z;\mathcal{J}^{(n)})$ for any n. This is why the inductive argument for the inverse direction works.

Proof. (i) Note that since $\widetilde{u}(z;\mathcal{J}_l) = g_l(z;\mathcal{J}) \to u(x;\mathcal{J})$, it suffices to show (3.3.27) for $\mathcal{J} \equiv \widetilde{\mathcal{J}}_l$.

Let $v_n(z;\widetilde{\mathcal{J}}_l)$ be the "old" definition of Jost solution, i.e., the solution of (3.3.27) for $\mathcal{J} \equiv \widetilde{\mathcal{J}}_l$ such that $v_n(z;\widetilde{\mathcal{J}}_l) = z^n$ for large n. Note that by (3.3.6) $v_0(z;\widetilde{\mathcal{J}}_l) = g_l(z;\widetilde{\mathcal{J}}_l) = \lim_{k\to\infty} g_k(z;\widetilde{\mathcal{J}}_l) = u_0(z;\widetilde{\mathcal{J}}_l)$, where the middle equality comes from (3.3.8).

Since $\mathcal{J}^{(k)}$ shifts indices by k, and $z^n = z^{-k}(z^{n+k})$, we have for all $n \geq 1$ and $k \geq 1$,

$$v_n\left(z; \left[\widetilde{\mathcal{J}_l}\right]^{(k)}\right) = z^{-k} v_{n+k}\left(z; \widetilde{\mathcal{J}_l}\right).$$

For $n = 0$, the difference equation (3.3.27) then gives

$$v_0\left(z; \left[\widetilde{\mathcal{J}_l}\right]^{(k)}\right) = z^{-k} v_k\left(z; \widetilde{\mathcal{J}_l}\right) A_k,$$

and so

$$v_k\left(z; \widetilde{\mathcal{J}_l}\right) = z^k v_0\left(z; \left[\widetilde{\mathcal{J}_l}\right]^{(k)}\right) A_k^{-1} = z^k u_0\left(z; \left[\widetilde{\mathcal{J}_l}\right]^{(k)}\right) A_k^{-1} \equiv u_k\left(z; \widetilde{\mathcal{J}_l}\right).$$

(ii) It follows from (3.3.17) that

$$\begin{aligned} ||u(z;\mathcal{J}^{(n)}) - \mathbf{1}|| &\leq \sum_{j=0}^{\infty} ||g_{j+1}(z;\mathcal{J}^{(n)}) - g_j(z;\mathcal{J}^{(n)})|| \\ &\leq \sup_k ||A_k^{-1}|| \sup_{k\in\mathbb{N}, z\in K} \left(||c_k(z;\mathcal{J}^{(n)})|| + ||g_k(z;\mathcal{J}^{(n)})||\right) \times \\ &\quad \times \sum_{j=n+1}^{\infty} \left[[\max(1,r)]^{2n} \left(||B_j|| + ||\mathbf{1} - A_j A_j^*||\right) + ||\mathbf{1} - A_{j+1}^{-1}||\right]. \end{aligned} \tag{3.3.35}$$

Now, assuming $\mathcal{J}$ is of type 1, we can bound $||\mathbf{1} - A_j^{-1}|| \leq c||\mathbf{1} - A_j A_j^*||$, and then Lemma 3.3.3 gives the convergence of the right hand side of (3.3.35).

If $\widetilde{\mathcal{J}}$ is of type asymptotic to 1, then by Theorem 3.3.5 we get

$$\lim_{k\to\infty} z^{-k} u_k(z; \widetilde{\mathcal{J}}) = \lim_{k\to\infty} \lim_{n\to\infty} \sigma_n^* \, z^{-k} u_k(z; \mathcal{J}) \sigma_k = \lim_{n\to\infty} \sigma_n^* \lim_{k\to\infty} \sigma_k = \mathbf{1}.$$

(iii) By [DPS08, Thm 2.16(iii)], we get $u_1(z;\mathcal{J}) = -u_0(z;\mathcal{J})\mathfrak{m}(z + z^{-1};\mathcal{J})$, hence

$$u(z;\mathcal{J}^{(1)}) = z^{-1} u_1(z;\mathcal{J}) A_1 = z^{-1} u(z;\mathcal{J}) M(z;\mathcal{J}) A_1.$$

(iv) Observe that if $M(z;\mathcal{J})$ is regular at z, then $u(z;\mathcal{J})$ is invertible at z. Otherwise we can pick an eigenvector f with $f^* u(z;\mathcal{J}) = \mathbf{0}$ and see that $f^* u_1(z;\mathcal{J}) = f^* u(z;\mathcal{J}) M(z;\mathcal{J}) = \mathbf{0}$, and then $f^* u_n(z;\mathcal{J}) = \mathbf{0}$ for all n from (3.3.27). This would contradict (ii).

Thus the only possible zeros are at z_j's with $z_j + z_j^{-1} = E_j$ being an eigenvalue of $\mathcal{J}$. Let q_k be the multiplicity of E_j as an eigenvalue of $\mathcal{J}^{(k)}$. By Lemma 2.1.1, $\sigma(\mathcal{J}^{(N)}) \subset [-2-\epsilon, 2+\epsilon]$ for sufficiently big N, so $q_n = 0$ for all $n \geq N$. Since $q_N = 0$, $M(z; \mathcal{J}^{(N)})$ is regular at z_j, and then the arguments above show that $u(z; \mathcal{J}^{(N)})$ is invertible at z_j. Now let us prove the statement about zeros of the determinant inductively assuming we know it for $N, N-1, \ldots, n+1$. By [DPS08, Thm 2.28], $\det M(z; \mathcal{J}^{(n)})$ has zero of order $q_{n+1} - q_n$ at $z = z_j$, and then (3.3.29) gives $\det u(z; \mathcal{J}^{(n)}) = z^n \det u(z; \mathcal{J}^{(n+1)}) \det M(z; \mathcal{J}^{(n)})^{-1} \det A_{n+1}^{-1}$ has zero of order $q_{n+1} - (q_{n+1} - q_n) = q_n$ at $z = z_j$. Thus $\det u(z; \mathcal{J})$ has zero of order q_0 at $z = z_j$.

Hence $\dim \ker u(z_j; \mathcal{J}) \leq q_0$. However,

$$\mathbf{0} = \lim_{z \to z_j} (z - z_j) u(z; \mathcal{J}^{(1)}) = z_j^{-1} u(z_j; \mathcal{J}) \lim_{z \to z_j} (z - z_j) M(z; \mathcal{J}) A_1 = z_j^{-1} u(z_j; \mathcal{J}) \widetilde{w}_j A_1,$$

which implies $\operatorname{Ran} \widetilde{w}_j \subseteq \ker u(z_j; \mathcal{J})$. Then $q_0 = \dim \operatorname{Ran} \widetilde{w}_j \leq \dim \ker u(z_j; \mathcal{J}) \leq q_0$, which means $\operatorname{Ran} \widetilde{w}_j = \ker u(z_j; \mathcal{J})$. $\operatorname{Ran} \widetilde{w}_j = \operatorname{Ran} w_j$ is obvious.

Since $\dim \ker u(z_j; \mathcal{J}) = q_0$ and $\det u(z; \mathcal{J})$ has zero of order q_0 at $z = z_j$, by Lemma 2.4.2 the order of the pole of $u(z; \mathcal{J})^{-1}$ at $z = z_j$ cannot be bigger than 1.

(v) If $z \in \partial\mathbb{D}$, then $u_n(z; \mathcal{J})$ and $u_n(z^{-1}; \mathcal{J})$ solve the same Jacobi equation, and so the Wronskian $W_n(u_\cdot(z; \mathcal{J}); u_\cdot(\bar{z}^{-1}; \mathcal{J})^*)$ is constant. By (ii), the Wronskian at infinity is $\lim_{n\to\infty} u_n(z) A_n u_{n+1}(z)^* - u_{n+1}(z) A_n^* u_n(z)^* = (z^{-1} - z)\mathbf{1}$, while evaluating it at zero gives

$$u_0(z) u_1(z)^* - u_1(z) u_0(z)^* = (z^{-1} - z)\mathbf{1},$$

or

$$\operatorname{Im}\left[u_1(e^{i\theta}) u_0(e^{i\theta})^*\right] = \sin\theta\, \mathbf{1}. \tag{3.3.36}$$

This implies that for $\theta \neq 0$, $u_0(e^{i\theta}; \mathcal{J})$ is invertible.

To prove that the poles at ± 1 are at most of order 1, just note that using (3.3.31) (which is proven in (vi)), the absolutely continuous part of μ is

$$f(2\cos\theta) = \pi^{-1} \left|\operatorname{Im} M(e^{i\theta})\right| = \pi^{-1} |\sin\theta| \left[u(e^{i\theta})^* u(e^{i\theta})\right]^{-1},$$

and then in order for

$$\int_{-2}^{2} f(x)dx = 2\int_0^\pi \sin\theta\, f(2\cos\theta)d\theta = \frac{2}{\pi}\int_0^\pi \sin^2\theta\, \left[u(e^{i\theta})^* u(e^{i\theta})\right]^{-1} d\theta$$

to be finite, we must have that the pole of $u(z)^{-1}$ at ± 1 is at most of order 1.

(vi) By $u_1(z;\mathcal{J}) = u(z;\mathcal{J})M(z;\mathcal{J})$, for $\theta \neq 0$,

$$\begin{aligned}\operatorname{Im} M(e^{i\theta}) &= \operatorname{Im} u(e^{i\theta})^{-1}u_1(e^{i\theta}) = \operatorname{Im}\left(u(e^{i\theta})^{-1}u_1(e^{i\theta})\left[u(e^{i\theta})^* u(e^{i\theta})^{*-1}\right]\right)\\ &= u(e^{i\theta})^{-1}\operatorname{Im}\left[u_1(e^{i\theta})u(e^{i\theta})^*\right]u(e^{i\theta})^{*-1} = \sin\theta\,\left[u(e^{i\theta})^* u(e^{i\theta})\right]^{-1}\end{aligned} \tag{3.3.37}$$

by (3.3.36).

(vii) This part follows immediately from (3.3.26) and (i). One can also obtain this using (iii) and (3.3.1) only.

(viii) By (iii), M is meromorphic in the region where u's are analytic. Note that (3.3.32) at $z = e^{i\theta}$ is (3.3.31). Thus if we define $\widehat{M}(z) = M^\sharp(z) + (z - z^{-1})\left[u^\sharp(z;\mathcal{J})u(z;\mathcal{J})\right]^{-1}$ for $1 < |z| < R$, then $M(z) = \widehat{M}(z)$ on $\partial\mathbb{D}$, and (3.3.32) follows by analytic continuation.

(ix) Note that $\mathcal{J}^{(1)}$ also satisfies $(A3)$, and so $u(z;\mathcal{J}^{(1)})$ is analytic in $\mathbb{D}_R$. Combining (3.3.29) and (3.3.32) we obtain

$$u(z;\mathcal{J}^{(1)}) = z^{-1}u(z;\mathcal{J})\left[M^\sharp(z) + (z - z^{-1})\left[u^\sharp(z;\mathcal{J})u(z;\mathcal{J})\right]^{-1}\right]A_1, \quad R^{-1} < |z| < R.$$

Analyticity of $u(z;\mathcal{J}^{(1)})$ at z_j^{-1} means that the residues must cancel out:

$$\begin{aligned}\mathbf{0} &= \lim_{z\to z_j^{-1}}(z - z_j^{-1})u(z;\mathcal{J})M^\sharp(z) + \lim_{z\to z_j^{-1}}(z - z_j^{-1})(z - z^{-1})\left[u^\sharp(z;\mathcal{J})\right]^{-1}\\ &= u(z_j^{-1};\mathcal{J})\lim_{z\to z_j}(z^{-1} - z_j^{-1})M(\bar{z})^* + (z_j^{-1} - z_j)\lim_{z\to z_j}(z^{-1} - z_j^{-1})\left[u(\bar{z};\mathcal{J})^*\right]^{-1}\\ &= \frac{1}{z_j^2}u(z_j^{-1};\mathcal{J})\widetilde{w}_j^* + \frac{1}{z_j^2}(z_j - z_j^{-1})[\lim_{z\to z_j}(z - z_j)u(z;\mathcal{J})^{-1}]^*,\end{aligned}$$

which gives (3.3.33).

The rightmost equality of (3.3.34) comes from Lemma 2.4.2. The containment part of (3.3.34) follows immediately from (3.3.33). □

We also see

Lemma 3.3.7. *Assume $\mathcal{J}$ is a Jacobi matrix asymptotic to type* 1*, and let its Jacobi parameters satisfy* (*A*1)*,* (*A*2)*, or* (*A*3)*. Then uniformly on the compacts of the appropriate region,*

$$u(z;\mathcal{J}^{(n)}) \to \mathbf{1},$$
$$M(z;\mathcal{J}^{(n)}) \to z\mathbf{1},$$

where $u^{(n)}$ and $M^{(n)}$ are the Jost function and the M-function, respectively, for the n times stripped operator $\mathcal{J}^{(n)}$.

Proof. Note that $M^{(n)}(z) = zu(z;\mathcal{J}^{(n)})^{-1}u(z;\mathcal{J}^{(n+1)})A_{n+1}^{-1} = A_n^{-1}u_n(z;\mathcal{J})^{-1}u_{n+1}(z;\mathcal{J})$. But $A_n \to \mathbf{1}$ and $z^{-n}u_n(z) \to \mathbf{1}$ uniformly on compacts of the appropriate region by (3.3.28). This and (3.3.26) give the result. □

To end this section, we get the following result for free as a corollary from Theorems 3.3.6, 3.3.4, and Corollary 1.3.8. The scalar analogue is proven in Killip–Simon [KS03, Thm 9.14].

Theorem 3.3.8. *Let $\mathcal{J}$ be of type asymptotic to type* 1 *and satisfies* (*A*1)*. Then $u(z;\mathcal{J})$ has the following factorization:*

$$u(z;\mathcal{J}) = UB(z)O(z),$$

where U is a constant unitary matrix, $B(z)$ is a matrix-valued Blaschke-Potapov product with zeros at $\{z_j\}$, and $O(z)$ is a matrix-valued outer function, uniquely defined from the conditions

$$\begin{aligned} O(e^{i\theta})^* O(e^{i\theta}) &= \sin\theta \left(\operatorname{Im} M(e^{i\theta})\right)^{-1}, \\ O(0) &= O(0)^* > \mathbf{0}, \\ \log\left|\det O(e^{i\theta})\right| &= \int_{-\pi}^{\pi} \log\left|\det O(e^{i\theta})\right| \frac{d\theta}{2\pi}. \end{aligned} \tag{3.3.38}$$

In particular, u has trivial singular inner part.

Remarks. 1. That the outer factor O can be uniquely defined from the conditions (3.3.38), as long as (3.3.39) holds, is Lemma 2.4.5.

2. O has an integral representation (2.4.6)–(2.4.7) in terms of Potapov multiplicative integral.

Proof. By Theorem 3.3.4 $u(z;\mathcal{J}) = (1-z^2)L(z)$, where $L(z) = \lim_{n\to\infty} z^n \mathfrak{p}_n(z+z^{-1})$. By Corollary 1.3.8, $L(z)$ is an $H^2(\mathbb{D})$ function with no singular inner part. Since $1-z^2$ is a bounded outer function, u is an $H^2(\mathbb{D})$ function with no singular inner part as well.

By (3.3.31), $u(e^{i\theta};\mathcal{J})^* u(e^{i\theta};\mathcal{J}) = \sin\theta \left(\operatorname{Im} M(e^{i\theta})\right)^{-1}$, and so (3.3.38) has to hold. Note that

$$\int_{-\pi}^{\pi} \log\det\left[\sin\theta(\operatorname{Im} M(e^{i\theta}))^{-1}\right] \frac{d\theta}{2\pi} > -\infty \tag{3.3.39}$$

is equivalent to

$$\left|\int_{-2}^{2} (4-x^2)^{-1/2} \log\det f(x)dx\right| \frac{d\theta}{2\pi} < \infty, \tag{3.3.40}$$

which is indeed finite given $(A1)$ (see [DKS, Section 14]). □

3.3.2 The Inverse Direction

Now we start with an analytic function u and seek to construct a measure such that u is its Jost function. We do this in Subsection 3.3.2.1. In the proof of Theorem 1.3.12 however, we appeal to the results later in the section. Note that this theorem is never used in Subsections 3.3.2.2 and 3.3.2.3 (i.e., we are never assuming that u is actually the Jost function for μ). In Subsections 3.3.2.2 and 3.3.2.3 we derive the exponential decay of the Jacobi parameters of μ, proving Theorems 1.3.14 and 1.3.15. Subsection 3.3.2.4 is just a restatement of the results in terms of the so-called perturbation determinants.

Throughout this section let u be an analytic function in $\mathbb{D}_R$ for some $R > 1$ satisfying the conditions of Theorem 1.3.12. Note that by (2.1.4)–(2.1.5) and (3.3.31) the absolutely continuous part $f(x)$ of μ is forced to be $f(2\cos\theta) = \pi^{-1}|\sin\theta| \left[u(e^{i\theta})^* u(e^{i\theta})\right]^{-1}$, and its singular part to be pure point with some weights w_j at $E_j = z_j + z_j^{-1}$, where z_j are zeros of u in $\mathbb{D}$. By Theorem 3.3.6(iv), w_j must satisfy the condition (ii) of Theorem 1.3.12. Assuming also (i), this μ is a probability measure. Its M-function satisfies (2.1.4), so

$$\operatorname{Im} M(e^{i\theta}) = \sin\theta \left[u(e^{i\theta})^* u(e^{i\theta})\right]^{-1} \tag{3.3.41}$$

holds. Just as in the proof of Theorem 3.3.6(viii), we can extend M meromorphically to $\mathbb{D}_R$ and see that

$$M(z) = M^\sharp(z) + (z - z^{-1}) \left[u^\sharp(z) u(z)\right]^{-1}, \quad R^{-1} < |z| < R.$$

Let $\mathcal{J}$ with Jacobi parameters $(A_n)_{n=1}^{\infty}$, $(B_n)_{n=1}^{\infty}$ be the type 1 Jacobi matrix for $d\mu$. Define inductively

$$u^{(n+1)}(z) = z^{-1}u^{(n)}(z)M^{(n)}(z)A_{n+1}; \tag{3.3.42}$$

$$A_{n+1}M^{(n+1)}(z)A_{n+1}^{*} = \left(z+\frac{1}{z}\right)\mathbf{1} - B_{n+1} - M^{(n)}(z)^{-1}. \tag{3.3.43}$$

Then $M^{(n)}$ is the M-function for $\mathcal{J}^{(n)}$ and, by an easy induction,

$$M^{(n)}(z) = M^{(n)\sharp}(z) + (z - z^{-1})\left[u^{(n)\sharp}(z)u^{(n)}(z)\right]^{-1}, \quad R^{-1} < |z| < R, \tag{3.3.44}$$

holds.

3.3.2.1 Proof of Theorem 1.3.12

For reader's convenience let us restate the theorem.

Theorem 3.3.9. *Let u be an analytic function in a disk $\mathbb{D}_R$ for some $R > 1$, whose only zeros in $\overline{\mathbb{D}}$ lie in $(\overline{\mathbb{D}}\cap\mathbb{R})\setminus\{0\}$ with those zeros all simple. For each zero z_j in $(\mathbb{D}\cap\mathbb{R})\setminus\{0\}$, let a nonzero matrix-valued weight $w_j \geq 0$ be given so that*

(i) $\sum_j w_j + \frac{2}{\pi}\int_0^{\pi}\sin^2\theta\,\left[u(e^{i\theta})^{*}u(e^{i\theta})\right]^{-1}d\theta = \mathbf{1}$,

(ii) $\operatorname{Ran} w_j = \ker u(z_j)$ *for all j.*

Then there exists a unique measure $d\mu$ for which w_j are the weights and u is its Jost function for some choice of Jacobi matrix from the equivalence class corresponding to $d\mu$. Any such matrix is of type asymptotic to 1.

Remark. It is clear that any two matrices having u as its Jost function are asymptotic to each other, and moreover, related by $\widetilde{\mathcal{J}} = U\mathcal{J}U^{-1}$, where U is an $l\times l$ block diagonal unitary $U = \sigma_1\oplus\sigma_2\oplus\sigma_3\oplus\ldots$, where σ_n are unitary with $\sigma_1 = \mathbf{1}$ and $\lim_{n\to\infty}\sigma_n = \mathbf{1}$ (which is a stronger condition than just being asymptotic).

Proof. The results of this section show that $||B_n||$ and $||\mathbf{1} - A_nA_n^{*}||$ decay exponentially (with the rate r^{-2n}, where r could be only slightly larger than 1). Thus the Jost function $\widetilde{u}$ exists and is analytic in $\mathbb{D}_r$. Consider

$$g(z) = \widetilde{u}(z)u(z)^{-1}. \tag{3.3.45}$$

We want to prove g is analytic and nonvanishing. Since u^{-1} has a first order pole at z_j, $\widetilde{u}u^{-1}$ is analytic at z_j if and only if

$$\widetilde{u}(z_j) \operatorname*{Res}_{z=z_j} u(z)^{-1} = \mathbf{0}, \tag{3.3.46}$$

which is equivalent to the condition $\operatorname{Ran} \operatorname{Res}_{z=z_j} u(z)^{-1} \subseteq \ker \widetilde{u}(z_j)$. However by Lemma 2.4.2, $\operatorname{Ran} \operatorname{Res}_{z=z_j} u(z)^{-1} = \ker u(z_j)$, which equals to $\operatorname{Ran} w_j$ by the condition (ii). By Theorem 3.3.6(iv), $\operatorname{Ran} w_j = \ker \widetilde{u}(z_j)$, and (3.3.46) follows.

$g(z)$ is analytic at ± 1 by the following arguments. By (3.3.31) and (3.3.41),

$$u(\pm 1)^* u(\pm 1) = \widetilde{u}(\pm 1)^* \widetilde{u}(\pm 1).$$

This implies $\ker u(\pm 1) = \ker \widetilde{u}(\pm 1)$ (since $\ker T = \ker T^*T$), and then identical arguments as for z_j's show that $g(z)$ is analytic at ± 1.

Thus we have proved g is analytic on a neighborhood of $\overline{\mathbb{D}}$, and switching the roles of u and $\widetilde{u}$, we obtain that g is also non-vanishing there.

Now,

$$\begin{aligned} g(z)^* g(z) = [u(z)^{-1}]^* \widetilde{u}(z)^* \widetilde{u}(z) u(z)^{-1} &= \sin\theta\, [u(z)^{-1}]^* [\operatorname{Im} M(e^{i\theta})]^{-1} u(z)^{-1} \\ &= [u(z)^{-1}]^* u(z)^* u(z) u(z)^{-1} = \mathbf{1}. \end{aligned}$$

So $g(z)^* g(z)$ is analytic and invertible on $\overline{\mathbb{D}}$ and unitary on $\partial\mathbb{D}$, which implies (e.g., by the Schwarz reflection) that $g(z) \equiv v_0$ for some constant unitary v_0. Thus, $u(z) = v_0^* \widetilde{u}(z)$. Then Theorem 3.3.5 implies that u is the Jost function for the Jacobi matrix with parameters $(A_1 v_0, v_0^* A_2 v_0, v_0^* A_3 v_0, \ldots)$, $(B_1, v_0^* B_2 v_0, v_0^* B_3 v_0, \ldots)$. $\square$

3.3.2.2 Proof of Theorems 1.3.14 and 1.3.15 for the case of no bound states

In this subsection we prove Theorems 1.3.14 and 1.3.15 for the case when μ has no bound states. Thus these theorems take the following form.

Theorem 3.3.10. *Let $u(z)$ be a polynomial obeying*

(i) $u(z)$ is nondegenerate on $\overline{\mathbb{D}} \setminus \{\pm 1\}$;

(ii) if ± 1 are zeros, they are simple;

(iii) $\frac{2}{\pi}\int_0^\pi \sin^2\theta \left[u(e^{i\theta})^* u(e^{i\theta})\right]^{-1} d\theta = \mathbf{1}$.

Then u is the Jost function of a Jabobi matrix with

$$\mathbf{1} - A_n A_n^* = B_n = \mathbf{0} \quad \textit{for all large } n. \tag{3.3.47}$$

Theorem 3.3.11. *Let $u(z)$ be analytic in $\mathbb{D}_R$ for some $R > 1$ and obeys (i)–(iii) from Theorem 3.3.10, then u is the Jost function of a Jacobi matrix with*

$$\limsup_{n\to\infty} \left(||B_n|| + ||\mathbf{1} - A_n A_n^*||\right)^{1/2n} \leq R^{-1}. \tag{3.3.48}$$

Remark. We denoted $(A_n)_{n=1}^\infty$, $(B_n)_{n=1}^\infty$ to be the type 1 Jacobi coefficients for $d\mu$. u will be the Jost function for a different Jacobi matrix (asymptotic to it). However (3.3.47) and (3.3.48) are invariant within the class of equivalent Jacobi matrices.

Note that (3.3.42) and (3.3.43) define $u^{(n)}$ and $M^{(n)}$, which are in general meromorphic functions in $\mathbb{D}_R$. We will show below that $u^{(n)}$ are actually analytic. Let us first prove the following lemma.

Lemma 3.3.12. *Let $u^{(n)}$ and $M^{(n)}$ be given by* (3.3.42) *and* (3.3.43). *Then $u^{(n)}$ has no zeros on $\partial\mathbb{D}$ except possibly at $\{\pm 1\}$, in which case they are simple.*

Proof. Since (3.3.44) holds, we obtain

$$f^{(n)}(2\cos\theta) = \pi^{-1}\left|\operatorname{Im} M^{(n)}(e^{i\theta})\right| = \pi^{-1}|\sin\theta| \left[u^{(n)}(e^{i\theta})^* u^{(n)}(e^{i\theta})\right]^{-1},$$

where $f^{(n)}$ is the density of the spectral measure $\mu^{(n)}$ of $\mathcal{J}^{(n)}$. Since $\int_{-\pi}^{\pi} |\sin\theta| f^{(n)}(2\cos\theta)d\theta \leq \mu^{(n)}(\mathbb{R}) \leq \mathbf{1}$, we get the result. $\square$

Now we can obtain analyticity of $u^{(n)}$ for $n \geq 1$.

Theorem 3.3.13. *If u is analytic in $\mathbb{D}_R$ and nonvanishing on $\overline{\mathbb{D}}\setminus\{\pm 1\}$ with at most simple zeros at ± 1, then the same is true of each $u^{(n)}$.*

Proof. We use induction on n. The inductive hypothesis will be to assume

(a) $u^{(n)}$ is analytic in $\mathbb{D}_R$,

(b) $u^{(n)}$ is invertible on $\overline{\mathbb{D}} \setminus \{\pm 1\}$,

(c) $u^{(n)}$ has at most simple zeros at ± 1,

(d) $M^{(n)}$ has no poles in $\overline{\mathbb{D}} \setminus \{\pm 1\}$,

(e) $M^{(n)}$ has at most simple poles at ± 1,

(f) $(M^{(n)})^{-1}$ has no poles in $\overline{\mathbb{D}} \setminus \{\pm 1\}$,

(g) $(M^{(n-1)})^{-1}$ has at most simple poles at ± 1.

Let us check the base case $n = 0$. (a)–(c) are given. That M has no poles in $\mathbb{D}$ follows from the fact that μ has no eigenvalues outside $[-2, 2]$, and no poles of M on $\partial\mathbb{D} \setminus \{\pm 1\}$ corresponds to the absence of the point spectrum in $(-2, 2)$. Also, no point spectrum at ± 2 implies $\lim_{\varepsilon \downarrow 0} \varepsilon \mathfrak{m}(\pm 2 + i\varepsilon) = 0$ which translates to $\lim_{z \to \pm 1}(z \mp 1)^2 M(z) = 0$. Thus we established (d) and (e).

Observe that M cannot have zeros on $(-1, 0) \cup (0, 1)$ since this would correspond to $\int_{-2}^{2} \frac{d\mu(x)}{x-z}$ being noninvertible at some $|z| > 2$. On $\{z \in \mathbb{D} \mid \operatorname{Im} z > 0\}$ we have $\operatorname{Im} M(z) > \mathbf{0}$, so M is invertible. Same for $\{z \in \mathbb{D} \mid \operatorname{Im} z < 0\}$. Finally, M is also invertible on $\partial\mathbb{D} \setminus \{\pm 1\}$ since $\operatorname{Im} M$ is invertible there by (3.3.41). Thus M^{-1} has no poles in $\overline{\mathbb{D}} \setminus \{\pm 1\}$, i.e., (f) holds.

(g) is vacuous for $n = 0$.

Now assume that (a)–(g) hold for n, and let us show they hold for $n+1$ as well. By (d) $M^{(n)}$ is meromorphic on $\mathbb{D}_R$ with poles possible only in $\{z \mid 1 < |z| < R\} \cup \{\pm 1\}$. Using

$$M^{(n)}(z) = M^{(n)\sharp}(z) + (z - z^{-1}) \left[u^{(n)\sharp}(z) u^{(n)}(z)\right]^{-1}, \quad R^{-1} < |z| < R, \tag{3.3.49}$$

we see the following:

(i) $M^{(n)}$ has a pole at z_k, $1 < |z_k| < R$, only if $u^{(n)}(z_k)$ is not invertible, since $u^{(n)\sharp}(z_k)$ is invertible by (b) and $M^{(n)\sharp}(z_k)$ is regular by (d). Then (3.3.42) and (3.3.49) imply

$$u^{(n+1)}(z_k) = z_k^{-1} u^{(n)}(z_k) M^{(n)\sharp}(z_k) A_{n+1} + (1 - z_k^{-2})[u^{(n)\sharp}(z_k)]^{-1} A_{n+1}$$

is regular.

(ii) Assume $M^{(n)}$ has a pole at ± 1. By (c) and (e), $u^{(n)}$ and $M^{(n)}$ have at most order 1 poles at ± 1, so let

$$\operatorname*{Res}_{z=1} M^{(n)}(z) = T, \tag{3.3.50}$$

$$\operatorname*{Res}_{z=1} u^{(n)}(z)^{-1} = C. \tag{3.3.51}$$

From the definition of $M^{(n)}$, the matrix T must be Hermitian. Easy to see,

$$\operatorname*{Res}_{z=1} M^{(n)\sharp}(z) = -T^* = -T,$$
$$\operatorname*{Res}_{z=1} u^{(n)\sharp}(z)^{-1} = -C^*,$$

and then computing residues of both sides of (3.3.49) gives

$$2T = -2CC^*. \tag{3.3.52}$$

Now, by (3.3.42),

$$\operatorname*{Res}_{z=1} u^{(n+1)}(z) = \lim_{z\to 1}(z-1)u^{(n+1)}(z) = u^{(n)}(1)TA_{n+1} = -u^{(n)}(1)CC^*A_{n+1} = \mathbf{0},$$

since $\operatorname{Ran} C = \ker u^{(n)}(1)$ (by Lemma 2.4.2). Hence $u^{(n+1)}$ is regular at $z = \pm 1$.

This proves part (a) of the inductive step.

$u^{(n+1)}$ is invertible on $\overline{\mathbb{D}} \setminus \{\pm 1\}$ since $u^{(n)}$ is invertible and $(M^{(n)})^{-1}$ has no poles (by (b) and (f)). This establishes (b).

(c) is obtained in Lemma 3.3.12.

(d) for $n+1$ follows from (3.3.43) and (f) for n.

(f) for $n+1$ follows by the exact same arguments as for $n = 0$ before.

(g) follows from $M^{(n)}(z)^{-1} = z^{-1}A_{n+1}u^{(n+1)}(z)^{-1}u^{(n)}(z)$ and Lemma 3.3.12.

Finally, (e) follows from (3.3.43) since we just established that $M^{(n)}(z)^{-1}$ has at most simple poles at ± 1. $\square$

Note that $\operatorname{ess\,supp} \mu = [-2, 2]$ with $\det f(x) > 0$ on $(-2, 2)$, and so Denisov–Rakhmanov theorem (Lemma 1.2.2) implies that $\mathcal{J}$ is in the Nevai class. By Theorem 1.3.4 we obtain $A_n \to \mathbf{1}, B_n \to \mathbf{0}$. This means that $\mathcal{J}^{(n)}$ converges in norm to the free block Jacobi matrix, which implies that resolvents converge:

$$M^{(n)}(z) \to z\mathbf{1} \quad \text{uniformly on compacts of } \mathbb{D}. \tag{3.3.53}$$

Now combine (3.3.42) and (3.3.44) to get

$$u^{(n+1)}(z) = (1 - z^{-2})(u^{(n)\sharp}(z))^{-1}A_{n+1} + z^{-2}u^{(n)}(z)N_n^\sharp(z)A_{n+1}, \tag{3.3.54}$$

where $N_n(z) = M^{(n)}(z)/z$, $N_n^\sharp(z) = zM^\sharp(z)$.

Let us fix any R_1 with $1 < R_1 < R$. Given any $L^2(\mathbf{1}\frac{d\theta}{2\pi})$ function on $R_1\partial\mathbb{D}$, define

$$|||f|||_{R_1} = \left(\int_{-\pi}^{\pi} \left|\left|(P_+f)(R_1e^{i\theta})\right|\right|^2 \frac{d\theta}{2\pi}\right)^{1/2},$$

where P_+ is the projection in $L^2(\mathbf{1}\frac{d\theta}{2\pi})$ onto $\{e^{in\theta}\}_{n=1}^\infty$, and $||\cdot||$ is the Hilbert-Schmidt norm till the end of this section. In particular, if f is analytic in $\mathbb{D}_R$,

$$|||f|||_{R_1} = \left(\int_{-\pi}^{\pi} \left|\left|f(R_1e^{i\theta}) - f(0)\right|\right|^2 \frac{d\theta}{2\pi}\right)^{1/2}.$$

Now note that since $(u^{(n)\sharp})^{-1}$ is analytic in $(\mathbb{C}\cup\{\infty\})\backslash\mathbb{D}$, $P_+((1-z^{-2})(u^{(n)\sharp}(z))^{-1}A_{n+1}) = \mathbf{0}$. For the same reasons, $P_+(z^{-2}u^{(n)}(0)N_n^\sharp(z)A_{n+1}) = \mathbf{0}$. Thus

$$P_+(u^{(n+1)}) = P_+\left(z^{-2}(u^{(n)}(z) - u^{(n)}(0))N_n^\sharp(z)\right)A_{n+1}.$$

Since P_+ is a projection on L^2, using submultiplicativity of the Hilbert-Schmidt norm we get

$$|||u^{(n+1)}|||_{R_1} \le R_1^{-2}|||u^{(n)}|||_{R_1}\,||A_{n+1}|| \sup_{|z|=R_1} ||N_n^\sharp(z)||,$$

which by induction gives

$$|||u^{(n+1)}|||_{R_1} \le R_1^{-2n}|||u|||_{R_1}\left[\prod_{j=1}^{n} ||A_{j+1}|| \sup_{|z|=R_1} ||N_j^\sharp(z)||\right]. \tag{3.3.55}$$

Now since $||A_j|| \to \mathbf{1}$ and $\sup_{|z|=R_1} ||N_j^\sharp(z)|| \le \sup_{|z|\le R_1^{-1}} ||M^{(j)}(z)/z|| \to \mathbf{1}$ by (3.3.53), we get that for any $\varepsilon > 0$ there exists a constant c_ε such that

$$\left[\prod_{j=1}^{n} ||A_{j+1}|| \sup_{|z|=R_1^{-1}} ||N_j(z)||\right] \leq c_\varepsilon (1+\varepsilon)^{2n},$$

and so

$$|||u^{(n+1)}|||_{R_1} \leq C_\varepsilon (R_1 - \varepsilon)^{-2n} \tag{3.3.56}$$

for some new constant C_ε.

Proof of Theorem 3.3.10. Since u is a polynomial, then taking n and R_1 sufficiently large in (3.3.55), one can see that $|||u^{(n)}|||_{R_1} = 0$, which implies $u^{(n)}(z) = u^{(n)}(0)$. Then by the condition (iii) of the theorem, $u^{(n)}(z) = 1$, and so $f^{(n)}(2\cos\theta) = \pi^{-1}|\sin\theta|$ is free, that is, $\mathbf{1} - A_n A_n^* = B_n = \mathbf{0}$ for all large n. □

Remark. One can be more careful and relate the degree of u to the maximal n where $\mathbf{1} - A_n A_n^* = B_n = \mathbf{0}$ is violated, just as in Theorem 3.3.1.

Proof of Theorem 3.3.11. Define $s_n(z) = u^{(n)}(z)u^{(n)}(0)^{-1} - \mathbf{1}$. Note that by Szegő asymptotics (Theorem 1.3.7), the limit $z^n \mathfrak{p}_n(z + z^{-1})$ exists. In particular at $z = 0$ this gives that there exists $\lim_{n\to\infty} A_1 \dots A_n \equiv K$, with K invertible. Then $u^{(n)}(0) = u(0)A_1 \dots A_n \to u(0)K$ is bounded in norm from above and below away from 0. Then

$$|||s_n|||_{R_1} \leq |||u^{(n)}|||_{R_1} \, ||u^{(n)}(0)^{-1}|| \leq C_\varepsilon (R_1 - \varepsilon)^{-2n}$$

for some new constant C_ε. Using Cauchy formula, one easily obtains from this

$$||s_n(z)|| \leq \widetilde{C}_\varepsilon (R_1 - \varepsilon)^{-2n} \quad \text{uniformly in } \mathbb{D}_{R_1 - 2\varepsilon}. \tag{3.3.57}$$

Now note that by (3.3.42)

$$\frac{M^{(n)}(z)}{z} = u^{(n)}(z)^{-1} u^{(n+1)}(z) A_{n+1}^{-1} = u^{(n)}(0)^{-1} (\mathbf{1} + s_n(z))^{-1} (\mathbf{1} + s_{n+1}(z)) u^{(n)}(0),$$

and so

$$\sup_{|z|\le 1/2} \left|\left| \frac{M^{(n)}(z)}{z} - \mathbf{1} \right|\right| \le \sup_{|z|\le 1/2} \left|\left| u^{(n)}(0)^{-1}(\mathbf{1}+s_n(z))^{-1}u^{(n)}(0) - \mathbf{1} \right|\right|$$
$$+ \sup_{|z|\le 1/2} \left|\left| u^{(n)}(0)^{-1}(\mathbf{1}+s_n(z))^{-1}s_{n+1}(z)u^{(n)}(0) \right|\right|.$$

The second term can be made exponentially small simply by using (3.3.57), while the first is

$$\begin{aligned}
\left|\left| u^{(n)}(0)^{-1}(\mathbf{1}+s_n(z))^{-1}u^{(n)}(0) - \mathbf{1} \right|\right| &= \left|\left| u^{(n)}(0)^{-1}\sum_{j=0}^{\infty} s_n(z)^j u^{(n)}(0) - \mathbf{1} \right|\right| \\
&= \left|\left| u^{(n)}(0)^{-1}\sum_{j=1}^{\infty} s_n(z)^j u^{(n)}(0) \right|\right| \\
&\le ||u^{(n)}(0)^{-1}||\, ||u^{(n)}(0)|| \frac{||s_n(z)||}{1-||s_n(z)||}
\end{aligned}$$

which is also uniformly exponentially small. Thus

$$\sup_{|z|\le 1/2} \left|\left| \frac{M^{(n)}(z)}{z} - \mathbf{1} \right|\right| \le \widehat{C}_\varepsilon (R_1-\varepsilon)^{-2n}.$$

Using this, (3.3.2), and the Cauchy formula, we obtain

$$||B_n|| + ||\mathbf{1} - A_n A_n^*|| \le \widehat{C}_\varepsilon (R_1-\varepsilon)^{-2n}.$$

Since $R_1 < R$ and $\varepsilon > 0$ were arbitrary, we obtain (3.3.48). □

Note that instead of $1/2$ we could have taken any constant smaller than $R_1 - \varepsilon$ here. Therefore we have shown that $M^{(n)}(z) \to z\mathbf{1}$ uniformly on compacts of $\mathbb{D}_R$.

3.3.2.3 Proof of Theorems 1.3.14 and 1.3.15 for the general case

Recall Definition 1.3.13 of canonical weight: w_j is canonical if

$$\widetilde{w}_j\, u(1/\bar{z}_j)^* = -(z_j - z_j^{-1}) \lim_{z\to z_j} (z-z_j)u(z)^{-1}, \tag{3.3.58}$$

where as before $w_j = (z_j^{-1} - z_j)z_j^{-1}\widetilde{w}_j$. As clear from the calculation in Theorem 3.3.6(ix), the weight is canonical if and only if $u^{(1)}(z)$ is regular at z_j^{-1}.

Lemma 3.3.14. *Assume $u(z)$ and $u^{(1)}(z)$ are analytic in $\mathbb{D}_R$. Then for any $n \geq 2$, $u^{(n)}(z)$ is analytic in $\mathbb{D}_R$.*

Proof. Note that part (vii) of Theorem 3.3.6 can be proved using only (3.3.29) and (3.3.1). Therefore (3.3.42) and (3.3.43) allow us to conclude that

$$u^{(n+2)}(z) = z^{-1}u^{(n+1)}(z)A_{n+1}^{-1}\left((z+z^{-1})\mathbf{1} - B_{n+1}\right){A_{n+1}^*}^{-1}A_{n+2} - z^{-2}u^{(n)}(z){A_{n+1}^*}^{-1}A_{n+2},$$

which proves our statement (easy to see that $z = 0$ in fact is not causing any troubles here). $\square$

Remark. What this lemma says is that if all the weights of u are canonical, then they are automatically canonical for every $u^{(n)}$.

For the inductive step in this case we will need the following result.

Lemma 3.3.15. *If u and M satisfy*

(a) $\ker u(\xi) = \operatorname{Ran}\operatorname{Res}_{z=\xi} M(z)$ *for all $\xi \in \mathbb{D}$;*

(b) all poles of u^{-1} in $\overline{\mathbb{D}} \cap \mathbb{R}$ are simple,

then the same is true for all $u^{(n)}$ and $M^{(n)}$.

Proof. Assume both conditions hold for $u^{(n)}$ and $M^{(n)}$.

Take any $\xi \in \mathbb{D}$. Note that in the Smith–McMillan form (Lemma 2.4.1) of $u^{(n)}$ at $z = \xi$ each power κ_j of $(z-\xi)^{\kappa_j}$ must be 0 or 1 by (b). Thus

$$u^{(n)}(z) = E(z)\begin{pmatrix} (z-\xi)\mathbf{1}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{l-s} \end{pmatrix} F(z),$$

where $\mathbf{1}_j$ is the $j \times j$ identity matrix. Now since $M^{(n+1)}$ can have only first order poles in $\mathbb{D}$, it means that $M^{(n)}$ can have only first order zeros/poles in $\mathbb{D}$. Then the Smith–McMillan form of $(M^{(n)})^{-1}$ at ξ is

$$M^{(n)}(z)^{-1} = G(z)\begin{pmatrix} (z-\xi)\mathbf{1}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{z-\xi}\mathbf{1}_{l-p-q} \end{pmatrix} H(z).$$

Observe that $E(z), F(z), G(z), H(z)$ are analytic and invertible in a neighborhood of ξ. Now note that

$$\ker u^{(n)}(\xi) = F(\xi)^{-1}\mathrm{span}\{\delta_1, \ldots, \delta_s\},$$

and

$$\mathrm{Ran}\,\operatorname*{Res}_{z=\xi} M^{(n)}(z) = H(\xi)^{-1}\mathrm{span}\{\delta_1, \ldots, \delta_p\}.$$

Then the condition (a) implies that $s = p$, and that $\mathrm{span}\{\delta_1, \ldots, \delta_p\}$ is an invariant subspace of the matrix $V \equiv H(\xi)F(\xi)^{-1}$. Thus

$$V = \begin{pmatrix} V_{11} & V_{12} \\ \mathbf{0} & V_{22} \end{pmatrix},$$

where V_{11} is an (invertible) $p \times p$ matrix, V_{22} is an (invertible) $(l-p) \times (l-p)$ matrix, and V_{12} is an $s \times (l-p)$ matrix.

By (a) $u^{(n+1)}(z)$ is analytic at ξ. Now consider $u^{(n+1)}(z)^{-1}$ at $z = \xi$. We want to show the following limit is finite:

$$\lim_{z\to\xi}(z-\xi)u^{(n+1)}(z)^{-1} = A_{n+1}^{-1}\lim_{z\to\xi}(z-\xi)M^{(n)}(z)^{-1}u^{(n)}(z)^{-1}$$
$$= A_{n+1}^{-1}G(\xi)\lim_{z\to\xi}(z-\xi)\begin{pmatrix} (z-\xi)\mathbf{1}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{z-\xi}\mathbf{1}_{l-p-q} \end{pmatrix} V \begin{pmatrix} \frac{1}{z-\xi}\mathbf{1}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{l-p-q} \end{pmatrix} E(\xi)^{-1}. \tag{3.3.59}$$

But

$$\begin{pmatrix} (z-\xi)\mathbf{1}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{l-p} \end{pmatrix}\begin{pmatrix} V_{11} & V_{12} \\ \mathbf{0} & V_{22} \end{pmatrix}\begin{pmatrix} \frac{1}{z-\xi}\mathbf{1}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{l-p} \end{pmatrix} = \begin{pmatrix} V_{11} & (z-\xi)V_{12} \\ \mathbf{0} & V_{22} \end{pmatrix},$$

which means that (3.3.59) is equal to

$$A_{n+1}^{-1}G(\xi)\lim_{z\to\xi}(z-\xi)\begin{pmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \frac{1}{z-\xi}\mathbf{1}_{l-p-q}\end{pmatrix}\widetilde{V}\begin{pmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{l-p-q}\end{pmatrix}E(\xi)^{-1}$$

$$= A_{n+1}^{-1}G(\xi)\begin{pmatrix}\mathbf{0}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{l-p-q}\end{pmatrix}\widetilde{V}E(\xi)^{-1}, \quad (3.3.60)$$

where $\widetilde{V}=\begin{pmatrix}V_{11} & \mathbf{0}\\ \mathbf{0} & V_{22}\end{pmatrix}$. This establishes (b) for $u^{(n+1)}$ for $\xi\in\mathbb{D}\cap\mathbb{R}$. The fact that ±1 is at most first order pole of $(u^{(n+1)})^{-1}$ is already proved in Lemma 3.3.12.

To show that (a) holds for $u^{(n+1)}$, note that by Lemma 2.4.2 (which applies since we already know that $(u^{(n+1)})^{-1}$ has at most simple pole),

$$\ker u^{(n+1)}(\xi)=\ker u^{(n)}(\xi)M^{(n)}(\xi)A_{n+1}=\operatorname{Ran}\operatorname*{Res}_{z=\xi}A_{n+1}^{-1}\left(M^{(n)}(z)^{-1}u^{(n)}(z)^{-1}\right),$$

and by (3.3.43),

$$\operatorname{Ran}\operatorname*{Res}_{z=\xi}M^{(n+1)}(z)=\operatorname{Ran}\operatorname*{Res}_{z=\xi}A_{n+1}^{-1}M^{(n)}(z)^{-1}.$$

By the calculations (3.3.59)–(3.3.60) above, it is easy to see that both of these spaces are equal to

$$\operatorname{Ran}A_{n+1}^{-1}G(\xi)\begin{pmatrix}\mathbf{0}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{l-p-q}\end{pmatrix}.$$

$\square$

This gives us the analogue of Theorem 3.3.13.

Lemma 3.3.16. *If u is analytic in $\mathbb{D}_R$, satisfies (a)–(b) of Lemma 3.3.15, and all the weights with $1>|z_j|>R^{-1}$ are canonical, then the same is true of each $u^{(n)}$.*

Proof. The arguments of Theorem 3.3.13, together with the result of Lemma 3.3.15, give the result. Note that condition (a) ensures analyticity of $u^{(1)}$ at z_j, and canonic weights ensure analyticity of $u^{(1)}$ at z_j^{-1}. The weights for $u^{(n)}$ for $n \geq 1$ are canonical by Lemma 3.3.14. □

Proof of Theorem 1.3.14. If some of the weights are not canonical then $u^{(1)}$ is not entire, and so $\mathbf{1} - A_n A_n^* = B_n = \mathbf{0}$ cannot hold for all large n.

Now assume all the weights are canonical. Then all $u^{(n)}$'s are entire by Lemma 3.3.16. For r sufficiently large, (3.3.54) implies

$$\sup_{|z|\leq r} ||u^{(n+1)}(z)|| \leq O(1) \left(1 + r^{-2} \sup_{|z|\leq r} ||u^{(n)}(z)|| \right),$$

which inductively shows that if u is a polynomial then $u^{(n)}$ is a polynomial with

$$\deg u^{(n)} \leq \max\{0, \deg u - 2n\}.$$

Then $u^{(N)}$ is a constant for some large N. By Lemma 3.3.15, $M^{(N)}$ has no poles, and so (3.3.44) implies that $u^{(N)}$ satisfies the condition (iii) of Theorem 3.3.10 (as well as conditions (i) and (ii), of course). This implies $\mathbf{1} - A_n A_n^* = B_n = \mathbf{0}$ for all large n. □

Proof of Theorem 1.3.15. If some of the weights with $1 > |z_j| > R^{-1}$ are not canonical then $u^{(1)}$ is not analytic at $\{z_j^{-1}\}$, and so $\limsup_{n\to\infty} (||B_n|| + ||\mathbf{1} - A_n A_n^*||)^{1/2n} \leq R^{-1}$ cannot hold.

Assume now that all the weights with $1 > |z_j| > R^{-1}$ are canonical. Then all $u^{(n)}$'s are entire by Lemma 3.3.16.

Now let us fix R_1 and R_2 with $1 < R_2 < R_1 < R$. By Lemma 2.1.1 there exists N such that zeros of $u^{(n)}$ in $\mathbb{D}$ all lie in $\{z \in \mathbb{C} : R_2^{-1} < |z| < 1\}$ for every $n \geq N$. This means that $(u^{(n)\sharp})^{-1}$ and $N_n^\sharp$ are analytic in $(\mathbb{C} \cup \{\infty\}) \setminus \mathbb{D}_{R_2}$, where N_n is defined in (3.3.45). Now the arguments after (3.3.45) work without changes and prove that (3.3.56) holds. This estimate was the only ingredient that was used in the proof of Theorem 3.3.11. This proves Theorem 1.3.15 for the general case. □

3.3.2.4 Results in terms of the perturbation determinant

Assuming the Jost function exists, define the **perturbation determinant** by

$$L(z) = u(z)u(0)^{-1}.$$

Clearly, $L(0) = \mathbf{1}$. Note that by (3.3.29) and $u^{(n)}(0) \to \mathbf{1}$ ((3.3.56)) we have

$$u(0) = \overset{\curvearrowleft}{\prod_{n=1}^{n}} A_n^{-1}.$$

We can reformulate Theorems 3.3.10 and 3.3.11 as follows.

Theorem 3.3.17. *Let $L(z)$ be a polynomial obeying*

(i) $L(z)$ is nondegenerate on $\overline{\mathbb{D}} \setminus \{\pm 1\}$;

(ii) if ± 1 are zeros, they are simple;

(iii) $L(0) = \mathbf{1}$.

Then L is the perturbation determinant for some Jacobi matrix (asymptotic to type 1), and each such matrix obeys $\mathbf{1} - A_n A_n^* = B_n = \mathbf{0}$ for all large n.

Theorem 3.3.18. *Let $L(z)$ be analytic in $\{z \mid |z| < R\}$ for some $R > 1$ and obeys (i)–(iii) from Theorem 3.3.17, then L is the perturbation determinant for some Jacobi matrix (asymptotic to type 1), and each such matrix has*

$$\limsup_{n\to\infty} \left(||B_n|| + ||\mathbf{1} - A_n A_n^*||\right)^{1/2n} \le R^{-1}.$$

Remarks. 1. It is clear from the proof that the corresponding measure in the above two theorems (as well as in the two theorems below) is not uniquely defined, but all possible $d\gamma$'s are related by $d\gamma_1 = v^* d\gamma_2 v$ for constant unitaries v.

2. In other words, every two Jacobi matrices having the same perturbation determinant are related by $\widetilde{\mathcal{J}} = U\mathcal{J}U^{-1}$, where U is an $l \times l$ block diagonal unitary $U = \sigma_1 \oplus \sigma_2 \oplus \sigma_3 \oplus \ldots$, where σ_n are unitary with $\lim_{n\to\infty} \sigma_n = \mathbf{1}$, and σ_1 is allowed to be different from $\mathbf{1}$.

Proofs. Pick any unitary σ and let $u(z) = L(z)\sqrt{H}\sigma$, where

$$H = \frac{2}{\pi}\int_0^\pi \sin^2\theta \left[L(e^{i\theta})^* L(e^{i\theta})\right]^{-1} d\theta \geq \mathbf{0}.$$

Then

$$\frac{2}{\pi}\int_0^\pi \sin^2\theta \left[u(e^{i\theta})^* u(e^{i\theta})\right]^{-1} d\theta = \mathbf{1},$$

and so Theorems 3.3.10, 3.3.11 apply. $\square$

Now assume there are bound states.

Lemma 2.5.2 implies that if $\ker f(1/\bar{z}_j)^* \subseteq \ker \operatorname{Res}_{z=z_j} f(z)^{-1}$, then there exists a unique matrix $\widetilde{w}_j$ solving

$$\widetilde{w}_j f(1/\bar{z}_j)^* = -(z_j - z_j^{-1}) \operatorname*{Res}_{z=z_j} f(z)^{-1}, \tag{3.3.61}$$

$$\operatorname{Ran} \widetilde{w}_j = \operatorname{Ran} \operatorname*{Res}_{z=z_j} f(z)^{-1} \tag{3.3.62}$$

(compare it with (3.3.34) and (3.3.30)). Observe that if the zeros f at z_j's are simple then by Lemma 2.4.2 $\operatorname{Ran}\operatorname{Res}_{z=z_j} f(z)^{-1} = \ker f(z_j)$ and $\ker \operatorname{Res}_{z=z_j} f(z)^{-1} = \operatorname{Ran} f(z_j)$. Hence we obtain the following results.

Theorem 3.3.19. *A polynomial $L(z)$ is the perturbation determinant for some Jacobi matrix with $\mathbf{1} - A_n A_n^* = B_n = \mathbf{0}$ for all large n if and only if it obeys*

(i) *$L(z)$ is nondegenerate on $(\overline{\mathbb{D}} \setminus \mathbb{R}) \cup \{0\}$;*

(ii) *all zeros on $\overline{\mathbb{D}} \cap \mathbb{R}$ are simple;*

(iii) $\ker L(1/\bar{z}_j)^* \subseteq \operatorname{Ran} L(z_j)$ *for each zero z_j in $\mathbb{D}$, and the unique solution corresponding to (3.3.61)–(3.3.62) is Hermitian and nonnegative;*

(iv) $L(0) = \mathbf{1}$.

Theorem 3.3.20. *Let $L(z)$ be analytic in $\{z \mid |z| < R\}$ for some $R > 1$. $L(z)$ is the perturbation determinant for some Jacobi matrix with* $\limsup_{n\to\infty} (||B_n|| + ||\mathbf{1} - A_n A_n^*||)^{1/2n} \leq R^{-1}$ *if and only if it obeys (i), (ii), (iv), and (iii) for every z_j with $1 > |z_j| > R^{-1}$.*

Proofs. Denote v_j to be the nonnegative solutions of (3.3.61)–(3.3.62) corresponding to $1 > |z_j| > R^{-1}$. For the rest of z_j's pick any nonnegative v_j. Let $\widetilde{w}_j = \sigma^* H^{-1/2} v_j H^{-1/2}\sigma \ge \mathbf{0}$, $w_j = (z_j^{-1} - z_j) z_j^{-1} \widetilde{w}_j$, and $u(z) = L(z)\sqrt{H}\sigma$, where σ is any unitary matrix, and

$$H = \sum_j (z_j^{-1} - z_j) z_j^{-1} v_j + \frac{2}{\pi}\int_0^\pi \sin^2\theta \left[L(e^{i\theta})^* L(e^{i\theta})\right]^{-1} d\theta \ge \mathbf{0}.$$

Then

$$\sum_j w_j + \frac{2}{\pi}\int_0^\pi \sin^2\theta \left[u(e^{i\theta})^* u(e^{i\theta})\right]^{-1} d\theta = \sigma^* H^{-1/2} H H^{-1/2}\sigma = \mathbf{1}.$$

Moreover, $\widetilde{w}_j$ solves (3.3.61)–(3.3.62) with f replaced by u for every $1 > |z_j| > R^{-1}$. This means that the condition (iii) of Theorem 1.3.14/1.3.15 holds, and all the weights for z_j with $1 > |z_j| > R^{-1}$ are canonical. Thus Theorems 1.3.14/1.3.15 apply and we are done. $\square$

3.4 Meromorphic Continuations of Matrix Herglotz Functions and Perturbations of the Free Case

3.4.1 Proof of Theorems 1.3.16 and 1.3.17

Proof of Theorem 1.3.16. (I)$\Rightarrow$(II) Assume (I) holds. (A) follows from Theorem 3.3.6 (viii). (B) follows from Theorem 3.3.6 (vi) and (v). (C) is immediate from (3.3.32).

Now let us show (D). First of all, it is a straightforward calculation to see that for any F with a first order pole,

$$\operatorname*{Res}_{z=\bar{z}_0^{-1}} F^\sharp(z) = -\frac{1}{\bar{z}_0^2}(\operatorname*{Res}_{z=z_0} F(z))^*. \tag{3.4.1}$$

Since $u(z;\mathcal{J})$ is analytic at z_j^{-1}, then using (3.3.32),

$$\mathbf{0} = \operatorname*{Res}_{z=z_j^{-1}} u(z;\mathcal{J}) = (z_j^{-1} - z_j) \operatorname*{Res}_{z=z_j^{-1}} u^\sharp(z;\mathcal{J})^{-1}(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1},$$

which implies

$$\operatorname{Ran}(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1} \subseteq \ker \operatorname*{Res}_{z=z_j^{-1}} u^\sharp(z;\mathcal{J})^{-1}. \tag{3.4.2}$$

Now, $\ker \operatorname{Res}_{z=z_j^{-1}} u^\sharp(z;\mathcal{J})^{-1} = \ker \operatorname{Res}_{z=z_j} u(z;\mathcal{J})^{-1*} = \operatorname{Ran} u(z_j;\mathcal{J})^* = \ker u(z_j;\mathcal{J})^\perp = \operatorname{Ran} \widetilde{w}_j{}^\perp = \operatorname{Ran}\operatorname{Res}_{z=z_j} M(z)^\perp$, and $\operatorname{Ran}(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1} = \ker(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1}$ since M is Hermitian on the real line. This gives (1.3.17).

Note that $(M(z) - M^\sharp(z))^{-1}M(z) = \mathbf{1} + (M(z) - M^\sharp(z))^{-1}M^\sharp(z)$ is analytic at z_j^{-1} since $(M(z) - M^\sharp(z))^{-1}M^\sharp(z)$ is analytic at z_j^{-1} by (1.3.17).

Now, by (3.3.29), $u(z;\mathcal{J})M(z)$ must be analytic at z_j^{-1}. Then using (3.3.32),

$$\mathbf{0} = \operatorname*{Res}_{z=z_j^{-1}} u(z;\mathcal{J})M(z) = (z_j^{-1} - z_j) \operatorname*{Res}_{z=z_j^{-1}} u^\sharp(z;\mathcal{J})^{-1}(M(z) - M^\sharp(z))^{-1}M(z),$$

which implies $\operatorname{Ran}(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1}M(z_j^{-1}) \subseteq \ker \operatorname{Res}_{z=z_j^{-1}} u^\sharp(z;\mathcal{J})^{-1} = \operatorname{Ran} u^\sharp(z_j^{-1};\mathcal{J}) = \operatorname{Ran} \widetilde{w}_j{}^\perp$, which is (1.3.18).

(II)⇒(I) Now assume (A)–(D) holds. Because of (A), M has only finitely many poles $\{z_j\}$ in $\mathbb{D}$, all of which are real and simple since M is Herglotz (see [GT00]). Let $\widetilde{w}_j = -\operatorname{Res}_{z=z_j} M(z)$.

Now we construct a function u as described in Theorem 3.3.10 and the remarks after it. First, there exists an outer function O satisfying (3.3.38) by the Wiener–Masani theorem (Lemma 2.4.5) since Szegő's condition (3.3.39) trivially holds. Then form a matrix-valued Blashcke product $B = \prod_j B_{z_j,s_j,U_j}$ with $s_j = \dim \operatorname{Ran} \widetilde{w}_j$, where we pick unitary matrices U_j so that $\ker B(z_j)O(z_j) = \operatorname{Ran} \widetilde{w}_j$ (this can be done inductively just as in Lemma 2.4.6). Now put $u(z) = B(z)O(z)$, which is an $\mathbb{H}^2(\mathbb{D})$-function.

Define

$$\widehat{u}(z) = (z - z^{-1})u^\sharp(z)^{-1}(M(z) - M^\sharp(z))^{-1}, \quad 1 < |z| < R. \tag{3.4.3}$$

Since by contruction $u(e^{i\theta})^*u(e^{i\theta}) = \sin\theta(\operatorname{Im} M(e^{i\theta}))^{-1}$, we have $\widehat{u}(e^{i\theta}) = u(e^{i\theta})$, where the values of $u, \widehat{u}$ on $\partial\mathbb{D}$ are meant in the sense of nontangential limits. Now note by (C), $\sin\theta(\operatorname{Im} M(e^{i\theta}))^{-1}$ is continuous, and therefore $\sup_{z\in\partial\mathbb{D}} ||u(z)|| < \infty$. By the Smirnov maximum principle for matrix-valued functions (see [Gin67]), $\sup_{z\in\mathbb{D}} ||u(z)|| \le \sup_{z\in\partial\mathbb{D}} ||u(z)|| < \infty$, i.e., u is bounded on $\mathbb{D}$. Note that u^{-1} is bounded on a neighborhood of any point of $\partial\mathbb{D} \setminus \{\pm 1\}$, and then so is $\widehat{u}$ by (3.4.3). Therefore Schwarz reflection principle allows us to conclude that $\widehat{u}$ is a meromorphic continuation of u. Since u is bounded on $\overline{\mathbb{D}}$, ± 1 must be removable singularities.

Note that by (B), $M(z) - M^\sharp(z)$ in regular on $\partial\mathbb{D} \setminus \{\pm 1\}$ with at most simple poles at ± 1. Therefore (3.4.3) proves that u has no zeros on $\partial\mathbb{D} \setminus \{\pm 1\}$ with at most simple zeros at ± 1.

Thus u satisfies all of the conditions of Theorem 1.3.12 (with $w_j = (z_j^{-1} - z_j)z_j^{-1}\widetilde{w}_j$), and it's clear that the unique measure μ of Theorem 1.3.12 is the measure corresponding to M. In order to apply Theorem 1.3.15 we need to show that u is analytic (rather than just meromorphic) in $\mathbb{D}_R$, and that the weights for those z_j with $1 > |z_j| > R^{-1}$ are canonical.

(3.4.3) shows that singularities of u can only happen at z_j^{-1}, in which case they are simple poles. Note that (1.3.17) can be rewritten as

$$\begin{aligned}\operatorname{Ran}(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1} &= \left(\ker(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1}\right)^\perp \supseteq \operatorname{Ran}\widetilde{w}_j{}^\perp = \ker u(z_j)^\perp \\ &= \operatorname{Ran} u(z_j) = \operatorname{Ran} u^\sharp(z_j^{-1}) = \ker \operatorname*{Res}_{z=z_j^{-1}} u^\sharp(z)^{-1},\end{aligned} \tag{3.4.4}$$

where in the second-to-last equality we used (3.4.1). This and (3.4.3) imply

$$\operatorname*{Res}_{z=z_j^{-1}} u(z) = (z_j - z_j^{-1}) \operatorname*{Res}_{z=z_j^{-1}} u^\sharp(z)^{-1}(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1} = \mathbf{0},$$

i.e., there is no pole at z_j, i.e., u is analytic in $\mathbb{D}_R$.

By the remark after (3.3.58) we will establish that all the weights are canonical if we show that $u^{(1)}(z) = z^{-1}u(z)M(z)A_{n+1}$ is analytic at z_j^{-1}. This is what (1.3.18) is for.

First of all, note that $\operatorname{Ran}\operatorname{Res}_{z=z_j^{-1}} M^\sharp(z) = \operatorname{Ran}\operatorname{Res}_{z=z_j} M(z)$ (just use (3.4.1) and $\widetilde{w}_j = \widetilde{w}_j^*$), so (1.3.17) implies that $(M(z) - M^\sharp(z))^{-1}M(z) = \mathbf{1} + (M(z) - M^\sharp(z))^{-1}M^\sharp(z)$ is analytic at z_j^{-1}. This justifies that the use of the expression in (1.3.18). Now note that (1.3.18) can be rewritten as

$$\operatorname{Ran}(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1}M(z_j^{-1}) \subseteq \operatorname{Ran}\widetilde{w}_j{}^\perp = \ker \operatorname*{Res}_{z=z_j^{-1}} u^\sharp(z)^{-1}, \tag{3.4.5}$$

which implies that $u^\sharp(z_j^{-1})^{-1}(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1}M(z_j^{-1})$ is analytic. By (3.4.3) this is $u(z_j^{-1})M(z_j^{-1})$.

Theorem 1.3.15 applies, giving (1.3.16). □

Proof of Theorem 1.3.17. That (I) implies (II) is clear from (3.3.29) and the fact that u and $u^{(1)}$ are polynomials.

Assume (II) holds. Then, going through the proof of the previous theorem, note that u is entire and by (3.4.3) grows at most polynomially. Therefore it is a polynomial, and so Theorem 1.3.14 applies. □

In the remarks after Theorems 1.3.16 and 1.3.17 we mentioned that condition (D) can be restated in a better-looking form in some special cases. Let us prove it here.

Proposition 3.4.1. • *If M has a pole of the first order at z_j^{-1} then (D) is equivalent to*

$$\operatorname{Ran} \widetilde{w}_j \subseteq \operatorname{Ran} (\widetilde{w}_j - z_j^2 \widetilde{q}_j), \tag{3.4.6}$$

$$\operatorname{Ran} \widetilde{w}_j \cap \operatorname{Ran} \widetilde{q}_j = \varnothing, \tag{3.4.7}$$

where $\widetilde{w}_j = -\operatorname{Res}_{z=z_j} M(z)$, $\widetilde{q}_j = \operatorname{Res}_{z=z_j^{-1}} M(z)$.

- *If $l = 1$, then (D) is equivalent to the condition that M has no simultaneous singularities at points z_j and z_j^{-1}.*

Proof. If M has a first order pole at z_j^{-1}, then we can apply Lemma 2.4.2 to the analytic function $(M - M^\sharp)^{-1}$ and see that (1.3.17) can be rewritten as

$$\operatorname{Ran} \widetilde{w}_j \subseteq \operatorname{Ran} \operatorname*{Res}_{z=z_j^{-1}} M(z) - M^\sharp(z) = \operatorname{Ran} (\widetilde{q}_j - \frac{1}{z_j^2} \widetilde{w}_j), \tag{3.4.8}$$

where we used (3.4.1) and the fact that and $\widetilde{w}_j$ and $\widetilde{q}_j$ are Hermitian.

Now note that (1.3.18) is equivalent to

$$\mathbf{0} = \operatorname*{Res}_{z=z_j^{-1}} u(z;\mathcal{J})M(z) = u(z_j^{-1};\mathcal{J}) \operatorname*{Res}_{z=z_j^{-1}} M(z),$$

which means

$$\begin{aligned}\operatorname{Ran} \widetilde{q}_j \subseteq \ker u(z_j^{-1};\mathcal{J}) &= \operatorname{Ran} \operatorname*{Res}_{z=z_j^{-1}} u(z;\mathcal{J})^{-1} = \operatorname{Ran} \operatorname*{Res}_{z=z_j^{-1}} (M(z) - M^\sharp(z)) u^\sharp(z;\mathcal{J}) \\ &= \operatorname{Ran} (\widetilde{q}_j - \frac{1}{z_j^2} \widetilde{w}_j) u(z_j;\mathcal{J})^* = \operatorname{Ran} \widetilde{q}_j u(z_j;\mathcal{J})^*\end{aligned} \tag{3.4.9}$$

where we successively used here: Lemma 2.4.2, (3.3.32), (3.4.1), and (3.3.30). Finally, note that (3.4.9) is equivalent to $\ker \widetilde{q}_j \subseteq \ker u(z_j;\mathcal{J})\widetilde{q}_j$, i.e., $\operatorname{Ran} \widetilde{q}_j \cap \ker u(z_j;\mathcal{J}) = \varnothing$, which is (3.4.7) by (3.3.30).

Now let $l = 1$, and assume M pole of order 1 at $z_j \in \mathbb{D}$ (it cannot have higher order poles there), and of order $k \geq 1$ at z_j^{-1}. Then $\lim_{z \to z_j^{-1}} (1 - \frac{M^\sharp(z)}{M(z)})$ is finite, so $\lim_{z \to z_j^{-1}} (1 -$

$\frac{M^\sharp(z)}{M(z)})^{-1}$ is nonzero (and it actually cannot be infinite by (1.3.17)). Therefore the right-hand side of (1.3.18) becomes $\left(\text{Ran}\lim_{z\to z_j^{-1}}(1-\frac{M^\sharp(z)}{M(z)})^{-1}\right)^{\perp}=\{0\}$. But the left-hand side is $\mathbb{C}$, a contradiction. $\square$

3.5 Meromorphic Continuations of Finite Gap Herglotz Functions and Periodic Jacobi Matrices

3.5.1 Notation

Denote by $\mathcal{S}=\mathcal{S}_{\mathfrak{e}}$ to be the (genus $p-1$) Riemann surface corresponding to $\mathcal{J}$, and by $\mathcal{R}=\mathcal{S}_{[-2,2]}$ the (genus 0) Riemann surface corresponding to $\Delta(\mathcal{J})$ (i.e., the hyperelliptic surface corresponding to z^2-4). We will denote both projections $\mathcal{S}\to\mathbb{C}\cup\{\infty\}$ and $\mathcal{R}\to\mathbb{C}\cup\{\infty\}$ by the same symbol π.

Denote $\mathcal{S}_R=\mathcal{S}_+\cup(\mathcal{S}_-\cap E_R)$, where E_R is the union of the interiors of the bounded components of $\Delta^{-1}(x(R\,\partial\mathbb{D}))$, where $x(z)=z+z^{-1}$. Also, $\mathcal{R}_R=\mathcal{R}_+\cup(\mathcal{R}_-\cap F_R)$, where F_R is the interior of the bounded component of $x(R\,\partial\mathbb{D})$ (ellipse).

Let m be the meromorphic in $\mathcal{S}_+\subset\mathcal{S}_{\mathfrak{e}}$ m-function of $\mathcal{J}$, and $\mathfrak{m}_\Delta$ to be the meromorphic in $\mathcal{R}_+$ m-function of the block Jacobi matrix $\Delta(\mathcal{J})$ with $p\times p$ matrix entries. Let μ and μ_Δ be the spectral measures for $\mathcal{J}$ and $\Delta(\mathcal{J})$.

As in Definition 1.2.8, let $z^\sharp$ be $\left(\overline{\pi(z)}\right)_-$ if $z\in\mathcal{S}_+$ and $\left(\overline{\pi(z)}\right)_+$ if $z\in\mathcal{S}_-$ with the convention $z^\sharp=z$ for $z\in\pi^{-1}(\mathfrak{e})$. Similarly let $\lambda^\sharp$ be $\left(\overline{\pi(\lambda)}\right)_-$ if $\lambda\in\mathcal{R}_+$ and $\left(\overline{\pi(z)}\right)_+$ if $\lambda\in\mathcal{R}_-$ with the convention $\lambda^\sharp=\lambda$ for $\lambda\in\pi^{-1}([-2,2])$. Let $m^\sharp(z)=\overline{m(z^\sharp)}$ and $\mathfrak{m}_\Delta^\sharp(\lambda)=\mathfrak{m}_\Delta(\lambda^\sharp)^*$.

Let $\{\gamma_j\}_{j=1}^{p-1}$ be the $p-1$ real solutions $\Delta'(z)=0$ (they are indeed all real by Lemma 2.2.1). Denote also $\zeta_j=\Delta(\gamma_j)$, and $\{\xi_j\}_{j=1}^N$ to be all of the preimages $\Delta^{-1}(\zeta_j)$ (so the set $\{\xi_j\}_{j=1}^N$ contains all γ_j's and finitely many of other points).

It will be convenient to define $\mathcal{R}_1\subset\mathcal{R}$ as the union of $\mathcal{R}_+\cap\mathbb{C}_+$, $\mathcal{R}_-\cap\mathbb{C}_-$ and the interval $[-2,2]\subset\mathcal{R}_+$ between them. Similarly let $\mathcal{R}_2\subset\mathcal{R}$ be the union of $\mathcal{R}_-\cap\mathbb{C}_+$, $\mathcal{R}_+\cap\mathbb{C}_-$ and the interval $[-2,2]\subset\mathcal{R}_-$ between them. Clearly $\mathcal{R}_1$, $\mathcal{R}_2$ are simply-connected subsets of $\mathcal{R}$ that have only ± 2 as common points.

Denote the p inverse functions of Δ by f_j defined in $\mathbb{C}_+\cup\mathbb{C}_-\cup[-2,2]$ (to avoid the critical points of Δ, which are all in $(-\infty,-2)\cup(2,\infty)$): $\Delta(z)=\lambda\Rightarrow z=f_j(\lambda)$. In fact let us also define $\widetilde{f}_j$ to be the function on $\mathcal{R}_1 \cup \mathcal{R}_2$ to $\mathcal{S}$ defined from the conditions

$$\begin{aligned} \pi \circ \widetilde{f}_j &= f_j, \\ \widetilde{f}_j(\lambda) &\in \mathcal{S}_+ \quad \text{if } \lambda \in \mathcal{R}_+, \\ \widetilde{f}_j(\lambda) &\in \mathcal{S}_- \quad \text{if } \lambda \in \mathcal{R}_-. \end{aligned}$$

Finally, extend $\widetilde{f}_j$ to $(-\infty,-2)\cup(2,\infty)$ on $\mathcal{R}_+$ and $\mathcal{R}_-$ by demanding it to be continuous "from above" (i.e., $\widetilde{f}_j(z_0) = \lim_{\mathcal{R}_+\cap\mathbb{C}_+\ni z\to z_0} \widetilde{f}_j(z)$ for $z_0 \in \mathcal{R}_+ \cap [(-\infty,-2)\cup(2,\infty)]$ and $\widetilde{f}_j(z_0) = \lim_{\mathcal{R}_-\cap\mathbb{C}_+\ni z\to z_0} \widetilde{f}_j(z)$ for $z_0 \in \mathcal{R}_- \cap [(-\infty,-2)\cup(2,\infty)]$). By doing this we are ensured that all p of the preimages (counting multiplicities) $\Delta^{-1}(\lambda)$ are counted in by $\pi(\widetilde{f}_j(\lambda))$, $1 \le j \le p$ for any λ.

Define $\widetilde{\Delta}$ from $\mathcal{S}$ to $\mathcal{R}$ in the analogous way:

$$\begin{aligned} \pi \circ \widetilde{\Delta} &= \Delta, \\ \widetilde{\Delta}(z) &\in \mathcal{R}_+ \quad \text{if } z \in \mathcal{S}_+, \\ \widetilde{\Delta}(z) &\in \mathcal{R}_- \quad \text{if } z \in \mathcal{S}_-. \end{aligned}$$

3.5.2 Lemmas

Lemma 3.5.1. *For $\lambda \in \mathcal{R}_+$,*

$$m(\widetilde{f}_l(\lambda)) = \left((\Delta(\mathcal{J}) - \lambda)^{-1} \prod_{j\neq l} (\mathcal{J} - f_j(\lambda))\, \delta_1, \delta_1 \right). \tag{3.5.1}$$

Proof. Since $(x - f_l(\lambda))^{-1} = (\Delta(x) - \lambda)^{-1} \prod_{j\neq l}(x - f_j(\lambda))$, we obtain

$$(\mathcal{J} - \widetilde{f}_l(\lambda))^{-1} = (\Delta(\mathcal{J}) - \lambda)^{-1} \prod_{j\neq l} (\mathcal{J} - f_j(\lambda)) \quad \text{for } \lambda \in \mathcal{R}_+ \tag{3.5.2}$$

(note also that $\prod_{j\neq l}(\mathcal{J} - f_j(\lambda))$ is a finite-banded matrix, so the multiplication on the right-hand side is well-defined). Recalling (2.1.3), we obtain the result. □

Note that (3.5.2) allows one to extend m using the $\mathfrak{m}_\Delta$, but not vice versa since we cannot invert the operator on the right. There is a trick that will help us, though.

Lemma 3.5.2. *For $\lambda \in \mathcal{R}_+$,*

$$\begin{aligned} &\mathfrak{m}_\Delta(\lambda)(S_{11} + \mathfrak{p}_1^R(\lambda) S_{21}) + A_1^{-1*} S_{21} \\ &= \sum_{j=1}^p \begin{pmatrix} q_0 + p_0 m & q_1 + p_1 m & \cdots & q_{p-1} + p_{p-1} m \\ q_1 + p_1 m & q_1 p_1 + p_1^2 m & \cdots & q_{p-1} p_1 + p_{p-1} p_1 m \\ \vdots & \vdots & \ddots & \vdots \\ q_{p-1} + p_{p-1} m & q_{p-1} p_1 + p_{p-1} p_1 m & \cdots & q_{p-1} p_{p-1} + p_{p-1}^2 m \end{pmatrix} (\widetilde{f}_j(\lambda)), \end{aligned} \tag{3.5.3}$$

where S_{ij} is the (i,j)-th $p \times p$ block entry of $\Delta'(\mathcal{J})$, and p_j, q_j are the first and second kind polynomials for $\mathcal{J}$.

Proof. Sum the equalities (3.5.2) from $l = 1$ to p:

$$\sum_{l=1}^p (\mathcal{J} - \widetilde{f}_l(\lambda))^{-1} = (\Delta(\mathcal{J}) - \lambda)^{-1} \sum_{l=1}^p \prod_{j \neq l} (\mathcal{J} - f_j(\lambda)). \tag{3.5.4}$$

Note that

$$\sum_{l=1}^p \prod_{j \neq l} (x - f_j(\lambda)) = \Delta'(x) \tag{3.5.5}$$

(to see this, just differentiate $\prod_{j=1}^p (x - f_j(\lambda)) = \Delta(x) - \lambda$ with respect to x). Therefore $\sum_{l=1}^p \prod_{j \neq l} (\mathcal{J} - f_j(\lambda)) = \Delta'(\mathcal{J})$. Recall (2.1.3), and write out the first $p \times p$ block of the left-hand side of (3.5.4) and of the product on the right-hand side of (3.5.4). We obtain

$$\text{RHS of (3.5.3)} = \mathfrak{m}_\Delta(\lambda) S_{11} + (\mathfrak{q}_1^R(\lambda) + \mathfrak{m}_\Delta(\lambda) \mathfrak{p}_1^R(\lambda)) S_{21}.$$

Since $\mathfrak{q}_1^R(\lambda) = A_1^{*-1}$, we obtain (3.5.3). $\square$

Lemma 3.5.3. *If m and $\mathfrak{m}_\Delta$ have meromorphic continuations to $\mathcal{S}_R$ and $\mathcal{R}_R$, respectively,* *then for* $\lambda \in \pi^{-1}(F_R)$,

$$\left[\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda)\right](S_{11} + \mathfrak{p}_1(\lambda)S_{21}) = \sum_{j=1}^{p} \left[m(\widetilde{f}_j(\lambda)) - m^\sharp(\widetilde{f}_j(\lambda))\right] \times$$
$$\times \begin{pmatrix} 1 & p_1(f_j(\lambda)) & \cdots & p_{p-1}(f_j(\lambda)) \\ p_1(f_j(\lambda)) & p_1^2(f_j(\lambda)) & \cdots & p_1(f_j(\lambda))p_{p-1}(f_j(\lambda)) \\ \vdots & \vdots & \ddots & \vdots \\ p_{p-1}(f_j(\lambda)) & p_1(f_j(\lambda))p_{p-1}(f_j(\lambda)) & \cdots & p^2_{p-1}(f_j(\lambda)) \end{pmatrix}. \quad (3.5.6)$$

Proof. Immediate from the previous lemma. □

Lemma 3.5.4. *The following holds:*

$$\det(S_{11} + \mathfrak{p}_1(\lambda)S_{21}) = c_1 \prod_{j=1}^{p-1}(\lambda - \Delta(\gamma_j)), \quad (3.5.7)$$

$$\ker(S_{11} + \mathfrak{p}_1(\lambda)S_{21}) = \operatorname{span}\{v_1(\lambda), \cdots, v_p(\lambda)\}^\perp, \quad (3.5.8)$$

where $v_j(\lambda) = (1, p_1(f_j(\lambda)), \cdots, p_{p-1}(f_j(\lambda)))^*$. *In particular* $S_{11} + \mathfrak{p}_1(\lambda)S_{21}$ *is singular if and only if* $\lambda = \Delta(\gamma_j)$, $j = 1, \cdots, p-1$, *and these zeros are simple.*

Proof. Note that by (2.1.1), $S_{11} + \mathfrak{p}_1(\lambda)S_{21} = S_{11} + (\lambda\mathbf{1} - B_1)A_1^{*-1}S_{21} = S_{11} + (\lambda\mathbf{1} - T_{11})T_{21}^{-1}S_{21}$, where S_{ij} and T_{ij} are the $p \times p$ blocks of $\Delta'(\mathcal{J})$ and $\Delta(\mathcal{J})$, respectively.

Take any $\mu \in \mathbb{C}$, and let $\widehat{u}(\mu) = (1, p_1(\mu), \ldots, p_j(\mu), \ldots)^*$, $u_1(\mu) = (1, p_1(\mu), \ldots, p_{p-1}(\mu))^*$, $u_2(\mu) = (p_p(\mu), p_{p+1}(\mu), \ldots, p_{2p-1}(\mu))^*$. Then $\widehat{u}^*\mathcal{J} = \mu\widehat{u}^*$ in the formal sense (since $\widehat{u} \notin \ell^2$). This gives $\widehat{u}^*\Delta(\mathcal{J}) = \Delta(\mu)\widehat{u}^*$ and $\widehat{u}^*\Delta'(\mathcal{J}) = \Delta'(\mu)\widehat{u}^*$ in the formal sense. This implies

$$u_1^*T_{11} + u_2^*T_{21} = \Delta(\mu)u_1^* \Rightarrow u_1^*T_{11}T_{21}^{-1} + u_2^* = \Delta(\mu)u_1^*T_{21}^{-1},$$
$$u_1^*S_{11} + u_2^*S_{21} = \Delta'(\mu)u_1^*,$$

which gives that

$$u_1^*[S_{11} + (\lambda - T_{11})T_{21}^{-1}S_{21}] = u_1^*[\Delta'(\mu) + (\lambda - \Delta(\mu))T_{21}^{-1}S_{21}].$$

This shows that $u_1(\mu)$ is an eigenvector of $[S_{11} + (\lambda - T_{11})T_{21}^{-1}S_{21}]^*$ if $\lambda = \Delta(\mu)$, and it is actually in the kernel if $\mu = \gamma_j$. Note that $T_{21}^{-1}S_{21}$ is a matrix with 0's on and below the main diagonal with positive elements right above it, which implies that the degree of the polynomial $\det(S_{11} + (\lambda\mathbf{1} - T_{11})T_{21}^{-1}S_{21})$ is $p - 1$. This establishes (3.5.7).

Now note that we just showed that each $v_j(\lambda)$, $1 \le j \le p$, is an eigenvector of $[S_{11} + (\lambda - T_{11})T_{21}^{-1}S_{21}]^*$. Then

$$\ker(S_{11} + \mathfrak{p}_1(\lambda)S_{21}) = \left(\operatorname{Ran}\,[S_{11} + (\lambda - T_{11})T_{21}^{-1}S_{21}]^*\right)^{\perp} = \operatorname{span}\{v_1(\lambda), \cdots, v_p(\lambda)\}^{\perp}.$$

In the last equality the inclusion $\subseteq$ follows from the fact that eigenvectors lie in the range, and the inclusion $\supseteq$ follows by counting the dimensions (note that the system of vectors $\{(1, p_1(z_j), \cdots, p_{p-1}(z_j))\}_{j=1}^{k}$ is linearly independent if and only if all the points z_j are distinct: easy use of Vandermonde and the fact that p_n is of degree n).

Finally, the zeros at $\Delta(\gamma_j)$ are simple by Lemma 2.4.2. $\square$

Remark. It is clear from the proof that Lemma 3.5.4 is a just a special case of the following fact: for any polynomials r_1, r_2 of degrees $k_1 > k_2$, $\det(S_{11}+(\lambda\mathbf{1}-T_{11})T_{21}^{-1}S_{21}) = c\prod_{j=1}^{k_2}(\lambda - r_1(\zeta_j))$, where S_{ij} and T_{ij} are the $k_1 \times k_1$ blocks of $r_2(\mathcal{J})$ and $r_1(\mathcal{J})$, respectively, and ζ_j are the zeros of r_2.

Lemma 3.5.5. *The following holds:*

$$\det(\textit{RHS of } (3.5.6)) = c_2 \prod_{j=1}^{p-1}(\lambda - \Delta(\gamma_j)) \prod_{j=1}^{p}\left(m(\widetilde{f}_j(\lambda)) - m^{\sharp}(\widetilde{f}_j(\lambda))\right).$$

If $\lambda = \Delta(\gamma_k)$ and all of $m(\widetilde{f}_j(\lambda)), m^{\sharp}(\widetilde{f}_j(\lambda))$ are regular and (pairwise) not equal, then the zero of the RHS of (3.5.6) at λ is simple and its kernel is equal to $\operatorname{span}\{v_1(\lambda), \cdots, v_p(\lambda)\}^{\perp}$, where $v_j(\lambda) = (1, p_1(f_j(\lambda)), \cdots, p_{p-1}(f_j(\lambda)))^*$.

Proof. Let $\alpha_j = m(\widetilde{f}_j(\lambda)) - m^{\sharp}(\widetilde{f}_j(\lambda))$. The determinant on the RHS of (3.5.6) can be computed as follows:

$$\det\left(\sum_{j=1}^{n}\alpha_j\,[p_{k-1}(f_j(\lambda))p_{s-1}(f_j(\lambda))]_{k,s=1}^{p}\right) = \det\left(\left[\sum_{j=1}^{n}\alpha_j p_{k-1}(f_j(\lambda))p_{s-1}(f_j(\lambda))\right]_{k,s=1}^{p}\right)$$
$$= \det\left([\alpha_j p_{k-1}(f_j(\lambda))]_{k,j=1}^{p}\,[p_{s-1}(f_j(\lambda))]_{j,s=1}^{p}\right) = \left(\det\,[p_{s-1}(f_j(\lambda))]_{j,s=1}^{p}\right)^2 \prod_{j=1}^{p}\alpha_j. \quad (3.5.9)$$

Since p_j is of degree j, it is easy to see that $\det[p_{s-1}(f_j(\lambda))]_{j,s=1}^p$ is just reduced to the Vandermonde determinant, and so (3.5.9) equals to

$$\prod_{j=1}^p \alpha_j \prod_{j<s} (f_j(\lambda) - f_s(\lambda))^2 = \prod_{j=1}^p \alpha_j \prod_{j=1}^p \prod_{\substack{s=1 \\ s\neq j}}^p (f_j(\lambda) - f_s(\lambda)).$$

Now observe that $\prod_{s=1 s\neq j}^p (f_j(\lambda) - f_s(\lambda)) = \Delta'(f_j(\lambda))$ by (3.5.5), and so the last expression equals to

$$\prod_{j=1}^p \alpha_j \prod_{j=1}^p \Delta'(f_j(\lambda)) = c_2 \prod_{j=1}^p \alpha_j \prod_{j=1}^p \prod_{s=1}^{p-1} (f_j(\lambda) - \gamma_s) = c_2 \prod_{j=1}^p \alpha_j \prod_{s=1}^{p-1} (\lambda - \Delta(\gamma_s)),$$

where c_2 is the leading coefficient of Δ'.

That any vector orthogonal to $\{v_1(\lambda), \cdots, v_p(\lambda)\}$ must be in the kernel is clear since the j-th row of the matrix in (3.5.6) is obtained from its first row by multiplication by p_{j-1}. Then counting the dimensions we convince ourselves that the kernel is indeed $\operatorname{span}\{v_1(\lambda), \cdots, v_p(\lambda)\}^\perp$. Each zero is simple by Lemma 2.4.2. $\square$

Damanik–Killip–Simon derive the following explicit formula relating the determinant of the density $\frac{d\mu_\Delta}{dx}$ of μ_Δ and the density $\frac{d\mu}{dx}$ of μ (see [DKS, Prop 11.1]). In our notation it looks as follows:

$$\det\left[\frac{d\mu_\Delta(\lambda)}{d\lambda}\right] = \frac{1}{\alpha_p^p} \prod_{j=1}^p a_j^{2p-2j} \prod_{j=1}^p \frac{d\mu}{dx}(f_j(\lambda)),$$

where α_p is the leading coefficient of Δ. The next lemma then looks natural. Note that if we take $\lambda \in \mathfrak{e}$ in the lemma, we obtain the formula above.

Lemma 3.5.6. *If m and m_Δ have meromorphic continuations to $\mathcal{S}_R$ and $\mathcal{R}_R$, respectively, then for $\lambda \in \pi^{-1}(F_R)$,*

$$\det\left(\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}_\Delta^\sharp(\lambda)\right) = c \prod_{j=1}^p \left(m(\widetilde{f}_j(\lambda)) - m^\sharp(\widetilde{f}_j(\lambda))\right), \qquad (3.5.10)$$

where $c = \frac{1}{\alpha_p^p} \prod_{j=1}^p a_j^{2p-2j}$, where α_p is the leading coefficient of Δ.

Proof. The previous three lemmas gives the result up to a multiplicative constant. The value of the constant must of course be equal to the constant obtained by Damanik, Killip, and Simon [DKS, Prop 11.1]. $\square$

The following lemma is a bit messy to prove, but will make our life so much easier.

Lemma 3.5.7. *Let $a_0 > 0, b_0 \in \mathbb{R}$, and let $\mathcal{J}^{(-1)} = (a_n, b_n)_{n=0}^{\infty}$ be the Jacobi matrix obtained from $\mathcal{J} = (a_n, b_n)_{n=1}^{\infty}$ by adding one column and one row with the corresponding parameters a_0, b_0. Let m and $m^{(-1)}$ be the m-functions of $\mathcal{J}$ and $\mathcal{J}^{(-1)}$. If m satisfies (ii) of Theorem 1.3.18 (of Theorem 1.3.19), then so does $m^{(-1)}$.*

Moreover, for any $\varepsilon > 0$ one can add finitely many $\{a_j, b_j\}_{j=-k+1}^{0}$ to form the Jacobi matrix $\mathcal{J}^{(-k)} = (a_n, b_n)_{n=-k+1}^{\infty}$ satisfying

(†) $m^{(-k)}$ does not have poles at any $(\xi_j)_\pm$ and band edges;

(‡) for any two poles z_1, z_2 of $m^{(-k)}$ in $\mathcal{S}_{R-\varepsilon}$, $\widetilde{\Delta}(z_1) \neq \widetilde{\Delta}(z_2)$, $\widetilde{\Delta}(z_1^\sharp) \neq \widetilde{\Delta}(z_2)$.

Remark. It seems $k = 1$ should be sufficient, but would overcomplicate the proof for no reason.

Proof. By the recursion

$$a_0^2 m(z) = -z + b_0 - {m^{(-1)}(z)}^{-1} \tag{3.5.11}$$

we can extend $m^{(-1)}$ to the same domain as m (so $m^{(-1)}$ satisfies (ii)(a) of Theorem 1.3.18), and then

$$m(z) - m^\sharp(z) = \frac{m^{(-1)}(z) - m^{(-1)\sharp}(z)}{m^{(-1)}(z) m^{(-1)\sharp}(z)} \quad \text{for } z \in \pi^{-1}(E_R). \tag{3.5.12}$$

Assume $m^{(-1)}(z)$ has a pole at an interior point of $\mathfrak{e}$. Then (3.5.11) implies that that m is real at this point, which violates (ii)(c). Assume $m^{(-1)}(z)$ has a pole of order $k \geq 2$ at a band edge. Then $m^{(-1)\sharp}(z)$ has the same order pole at this point, and then (3.5.12) implies that $m - m^\sharp$ has a zero of order at least $2k - k \geq 2$, contradicting (ii)(c) for m. Thus $m^{(-1)}$ satisfies (ii)(b).

Assume $m^{(-1)}(z)$ and $m^{(-1)\sharp}(z)$ are both finite and $m^{(-1)}(z) - m^{(-1)\sharp}(z) = 0$, for some z not at a band edge. Then (3.5.12) implies that m violates (ii)(c) or (ii)(d) of Theorem 1.3.18, a contradiction. Thus $m^{(-1)}$ satisfies (ii)(c) for z not at a band edge.

Assume $m^{(-1)}$ is finite and nonzero at a band edge. Then so is $m^{(-1)\sharp}$, and then (3.5.12) shows that $m^{(-1)} - m^{(-1)\sharp}$ has at most first order pole there. Now let $m^{(-1)}$ have a zero of order $k \geq 1$ there. Then (3.5.11) shows that necessarily $k = 1$. But then $m^\sharp$ has order 1 pole at this band edge with the leading coefficient being negative of that of m ($\sqrt{\Delta^2 - 4}$ changes sign when we change sheets). This shows that $m^{(-1)} - m^{(-1)\sharp}$ has nonzero leading coefficient near the first order term, i.e., has first order zero too. Lastly, assume $m^{(-1)}$ has a pole at a band edge. We showed that then this pole is simple. Again, $m^\sharp$ has a first order pole with the leading coefficient being negative to that of m, and so $m - m^\sharp$ still has a pole, and therefore does not vanish. Thus $m^{(-1)}$ satisfies (ii)(c).

Finally, assume $m^{(-1)}(z)$ and $m^{(-1)\sharp}(z)$ both have a pole at $z \in \pi^{-1}(E_R)$. Then by (3.5.11) $m(z) = m^\sharp(z) = \frac{b_0-\pi(z)}{a_0^2}$, which means that m violates (ii)(c). Therefore $m^{(-1)}$ satisfies (ii)(d).

Let us prove the "moreover" part now. Note that all the poles of $m^{(-1)}$ occur at the points where $a_0^2 m(z) = b_0 - z$. Denote the finite number of distinct poles of m in $\mathcal{S}_{R-\varepsilon}$ by $\{z_j\}_{j=1}^K$. Let $M_1 = \max_j |z_j|$. Choose small $\delta > 0$ such that the δ-neighborhoods $U_\delta(z_j)$ of these points are disjoint and inside $\mathcal{S}_{R-\varepsilon}$. Choose $M_2 > 0$ to be larger than the supremum of $|m(z)|$ over all z in $\mathcal{S}_{R-\varepsilon}$ not in these neighborhoods. Fix any small $\sqrt{\delta/M_2} > a_0 > 0$. Now let $b_0(t) = M_1 + M_2 + t$ for $t \ge 0$. For each such $a_0, b_0(t)$ let $m^{(-1)}(a_0, b_0(t))$ be the m-function of $\mathcal{J}^{(-1)} = (a_n, b_n)_{n=0}^\infty$. Note that if z is not in one of $U_\delta(z_j)$ or $U_\delta(b_0(t)_+)$, then z cannot be a pole of $m^{(-1)}(a_0, b_0(t))$. Indeed, for such z, $|a_0^2 m(z)| \le \delta < |b_0 - z|$. Note that $a_0^2 m(z) + z$ around each z_n is locally k_j-to-1 (where $k_j \ge 1$ is the order of the pole at z_j). Therefore assuming t is large enough, we will have precisely k_j distinct solutions to $a_0^2 m(z) + z = b_0(t)$ (they are distinct since z_j itself cannot be a solution), i.e., there are precisely k_j distinct first order poles of $m^{(-1)}(a_0, b_0(t))$ in each $U_\delta(z_j)$. Finally, for large enough t there will be exactly one solution to $a_0^2 m(z) = b_0(t) - z$ in $U_\delta(b_0(t)_+)$ (note that for large t, $b_0(t)$ is not in E_R, and we can ignore $U_\delta(b_0(t)_-)$). Indeed, m is monotonically increasing to zero as $\mathbb{R} \ni z \to +\infty$. Therefore for large t, $a_0^2 m(z) = b_0(t) - z$ will have exactly one real solution in $U_\delta(b_0(t))$. Since any pole of $m^{(-1)}$ on $\mathcal{S}_+$ must be real, we do not have to worry about nonreal poles in $U_\delta(b_0(t))$.

Thus there are precisely $1 + \sum_{j=1}^K k_j$ first order poles of $m^{(-1)}(a_0, b_0(t))$ in $\mathcal{S}_{R-\varepsilon}$, which are distinct for any t large enough. Denote the locations of these poles by $z_j(t)$ (note each $z_j(t)$ is a continuous function).

The restriction (†) requires only $b_0(t) \ne a_0^2 m((\xi_j)_\pm) + \xi_j$, and $b_0(t) \ne a_0^2 m(\alpha_j) + \alpha_j$, $b_0(t) \ne a_0^2 m(\beta_j) + \beta_j$, which excludes only finite number of allowable $b_0(t)$. Choosing any other $b_0(t)$ therefore produces $\mathcal{J}^{(-1)}$ satisfying (†) and having only first order poles in $\mathcal{S}_{R-\varepsilon}$. Thus without loss of generality we may assume that m already satisfies (†) and has only first order poles in $\mathcal{S}_{R-\varepsilon}$.

In particular, $k_j = 1$. Now there will be precisely $K+1$ poles of $m^{(-1)}$, one in each $U_\delta(z_j)$, $1 \le j \le K$, and one in $U_\delta(b_0(t))$.

Assume that δ is small enough so that $\widetilde{\Delta}(U_\delta(z_j)) \cap \widetilde{\Delta}(U_\delta(z_n)) = \varnothing$ provided $\widetilde{\Delta}(z_j) \neq \widetilde{\Delta}(z_n)$, and $\widetilde{\Delta}(U_\delta(z_j)^\sharp) \cap \widetilde{\Delta}(U_\delta(z_n)) = \varnothing$ provided $\widetilde{\Delta}(z_j^\sharp) \neq \widetilde{\Delta}(z_n)$. Then if (‡) holds for m for some z_j, z_n, then it will still hold for the corresponding poles of $m^{(-1)}$.

Now assume that (‡) does not hold for m, say for the poles z_1 and z_2. Without loss of generality we may assume $\widetilde{\Delta}(z_1) = \widetilde{\Delta}(z_2) \equiv \lambda_0$ (the case $\widetilde{\Delta}(z_1^\sharp) = \widetilde{\Delta}(z_2)$ can be treated in the same way). If we pick the coefficients a_0 and $b_0(t)$ so that $m^{(-1)}(a_0, b_0(t))$ satisfies (‡) for the corresponding poles in $U_\delta(z_1)$ and $U_\delta(z_2)$, then we can keep repeating this procedure to get rid of all "resonances".

Suppose that no matter what t is, $t \to \infty$, the condition (‡) fails for $m^{(-1)}(a_0, b_0(t))$ at $z_1(t)$ and $z_2(t)$, where $z_1(t)$, $z_2(t)$ are the unique solutions of $a_0^2 m(z) = b_0(t) - z$ in $U_\delta(z_1)$, $U_\delta(z_2)$, respectively. This implies $\widetilde{\Delta}(z_1(t)) = \widetilde{\Delta}(z_2(t)) =: \lambda(t)$. This means that we can choose *different* branches f_1, f_2 of Δ^{-1} (note that we are avoiding critical points since (†) holds) such that

$$a_0^2(m(\widetilde{f}_1(\lambda(t))) - m(\widetilde{f}_2(\lambda(t)))) = f_2(\lambda(t)) - f_1(\lambda(t)).$$

By analytic continuation we in fact obtain

$$a_0^2(m(\widetilde{f}_1(\lambda)) - m(\widetilde{f}_2(\lambda))) = f_2(\lambda) - f_1(\lambda).$$

for all λ in a neighborhood of λ_0.

This may in fact happen. However choose now any $0 < a_0' < a_0$. Either we get rid of the resonance for this a_0' and some $b_0(t)$, or we again obtain

$${a_0'}^2(m(\widetilde{f}_1(\lambda)) - m(\widetilde{f}_2(\lambda))) = f_2(\lambda) - f_1(\lambda)$$

for all λ in a neighborhood of λ_0. The last two equalities imply $f_2(\lambda) - f_1(\lambda) \equiv 0$ giving the contradiction. $\square$

For the other direction we will use the following result, which is the analogue of Lemma 3.3.7.

Lemma 3.5.8. *Assume* $\limsup_{n\to\infty}(d_n(\mathcal{J},\mathcal{J}^0))^{1/2n} \le R^{-1}$, *where* $\mathcal{J}_0$ *is a (one-sided) p-periodic Jacobi matrix in* $\mathcal{T}_{\mathfrak{e}}$. *Let* $m^{(n)}$ *be the m-function of* $\mathcal{J}^{(n)}$ *and* m^0 *be the m-function of* $\mathcal{J}^0$. *Then* $m^{(np)}(z) \to m^0(z)$ *as* $n \to \infty$ *for any* $z \in \mathcal{S}_R$.

Remark. In fact, the convergence is uniform on compacts *with respect to the spherical distance* on the Riemann sphere.

Proof. Note that $\mathcal{J}^{(np)} \to \mathcal{J}^0$ in norm since $(\mathcal{J}^0)^{(np)} = \mathcal{J}^0$. This also gives us $\Delta(\mathcal{J}^{(np)}) \to \Delta(\mathcal{J}^0)$.

Let us write $m(\mathcal{J})$ to mean the m-function of a Jacobi matrix $\mathcal{J}$ evaluated at a point $z \in \mathcal{S}$, the dependence on which we will omit for convenience. Let us also write $\mathfrak{m}_\Delta(\mathcal{J})$ to mean the (matrix-valued) m-function of a block Jacobi matrix $\mathcal{J}$ evaluated at $\widetilde{\Delta}(z) \in \mathcal{R}$.

Let also m^0 be the m-function of $\mathcal{J}^0$, evaluated at z, and $\mathfrak{m}_\Delta^0$ be the m-function of the free block Jacobi matrix $(\frac{-\lambda\pm\sqrt{\lambda^2-4}}{2}\mathbf{1})$, evaluated at $\widetilde{\Delta}(z)$.

Let us write (3.5.1) as $m(\mathcal{J}) = g(\mathfrak{m}_\Delta(\Delta(\mathcal{J})), \{a_j\}_{j=1}^N, \{b_j\}_{j=1}^N)$, where g is a continuous function that takes one $p \times p$ matrix-valued parameter and $2N$ real parameters (here $\{a_j\}_{j=1}^N, \{b_j\}_{j=1}^N$ are the first Jacobi parameters of the matrix $\mathcal{J}$). Indeed, the right-hand side of (3.5.1) depends on $\mathfrak{m}_\Delta(\Delta(\mathcal{J}))$, the first orthogonal polynomial $\mathfrak{p}_1$ of $\Delta(\mathcal{J})$ and on the first column of the product $\prod_{j\ne l}(\mathcal{J}-f_j(\Delta(z)))$. The latter two objects are smooth functions (in fact polynomials) of first N Jacobi parameters $\{a_j\}_{j=1}^N, \{b_j\}_{j=1}^N$ of $\mathcal{J}$, for N sufficiently large ($N = 2p$ should suffice). This proves that if $\mathcal{J}_k \to \mathcal{J}$ and $\mathfrak{m}_\Delta(\Delta(\mathcal{J}_k)) \to \mathfrak{m}_\Delta(\Delta(\mathcal{J}))$ then $m(\mathcal{J}_k) \to m(\mathcal{J})$.

By Lemma 3.3.7 we have $\mathfrak{m}_\Delta(\Delta(\mathcal{J})^{(n)}) \to \mathfrak{m}_\Delta^0$ for $\lambda \in \mathcal{R}_R$. Note that $\Delta(\mathcal{J}^{(np)}) \ne \Delta(\mathcal{J})^{(n)}$. However, $\Delta(\mathcal{J}^{(np)})$ and $\Delta(\mathcal{J})^{(n)}$ differ only in the first block entry, which implies $\Delta(\mathcal{J}^{(np)})^{(1)} = \Delta(\mathcal{J})^{(n+1)}$. Thus

$$\mathfrak{m}_\Delta(\Delta(\mathcal{J}^{(np)})^{(1)}) = \mathfrak{m}_\Delta(\Delta(\mathcal{J})^{(n+1)}) \to \mathfrak{m}_\Delta^0.$$

Taking $\mathcal{J} = \mathcal{J}^0$ in the last expression produces $\mathfrak{m}_\Delta(\Delta(\mathcal{J}^0)^{(1)}) = \mathfrak{m}_\Delta^0$. Therefore

$$\mathfrak{m}_\Delta(\Delta(\mathcal{J}^{(np)})^{(1)}) \to \mathfrak{m}_\Delta(\Delta(\mathcal{J}^0)^{(1)}).$$

Now use (3.5.11): the first Jacobi parameters of $\Delta(\mathcal{J}^{(np)})$ converge to the first Jacobi parameters of $\Delta(\mathcal{J}^0)$, which implies that $\mathfrak{m}_\Delta(\Delta(\mathcal{J}^{(np)})) \to \mathfrak{m}_\Delta(\Delta(\mathcal{J}^0))$ if $\Delta(z)$ is a regular point of $\mathfrak{m}_\Delta(\Delta(\mathcal{J}^0))$. This gives us $m(\mathcal{J}^{(np)}) \to m(\mathcal{J}^0)$ by continuity of g, for all z such that $\Delta(z)$ is regular for $\mathfrak{m}_\Delta(\Delta(\mathcal{J}^0))$.

In fact, note that the convergence $\mathfrak{m}_\Delta(\Delta(\mathcal{J}^{(np)})^{(1)}) \to \mathfrak{m}_\Delta(\Delta(\mathcal{J}^0)^{(1)})$ is given by Lemma 3.3.7 to be uniform on compacts ($\mathfrak{m}_\Delta(\Delta(\mathcal{J}^0)^{(1)})$ is analytic). Therefore

$$m(\mathcal{J}^{(np)}) = g\left(\left((B_0^{(np)} - \Delta(z) - A_0^{(np)}\mathfrak{m}_\Delta(\Delta(\mathcal{J}^{(np)})^{(1)})A_0^{(np)*}\right)^{-1}, \{a_j\}_{j=1}^N, \{b_j\}_{j=1}^N\right)$$

is just some rational function of finitely many uniformly convergent analytic functions. It means that $m(\mathcal{J}^{(np)})$ is a sequence of meromorphic functions that converges to $m(\mathcal{J}^0)$ uniformly (on compacts) with respect to the spherical distance. In particular if $m(\mathcal{J}^0)$ has a pole at z, then $m(\mathcal{J}^{(np)}) \to \infty$. □

3.5.3 Proof of Theorems 1.3.18 and 1.3.19

Proof of Theorem 1.3.18. (ii)⇒(i) Passing from m to $m^{(-1)}$ in Lemma 3.5.7, we may assume that m itself satisfies (†) and (‡).

We want to apply Theorem 1.3.16 to $\Delta(\mathcal{J})$.

(II)(A) holds by (ii)(a) and Lemma 3.5.2, and analytic continuation. Indeed, for any $\lambda \in \mathcal{R}_R$, $\widetilde{f}_j(\lambda) \in \mathcal{S}_R$, so all we need to check is continuity along $(-\infty,-2] \cup [2,\infty)$ in $\mathcal{R}_-$. Let $\eta \in (2,\infty) \setminus \{\zeta_j\}$, where ζ_j are the images of zeros of Δ', and consider

$$\lim_{\mathcal{R}_-\cap\mathbb{C}_+\ni\lambda\to\eta_-} \mathfrak{m}_\Delta(\lambda) - \lim_{\mathcal{R}_-\cap\mathbb{C}_-\ni\lambda\to\eta_-} \mathfrak{m}_\Delta(\lambda). \tag{3.5.13}$$

Note that even though $\lim_{\mathcal{R}_-\cap\mathbb{C}_+\lambda\to\eta_-} \widetilde{f}_j(\lambda)$ is not equal to $\lim_{\mathcal{R}_-\cap\mathbb{C}_-\ni\lambda\to\eta_-} \widetilde{f}_j(\lambda)$ in general, we however still have

$$\left\{\lim_{\mathcal{R}_-\cap\mathbb{C}_+\lambda\to\eta_-} \widetilde{f}_j(\lambda)\right\}_{1\le j\le p} = \left\{\lim_{\mathcal{R}_-\cap\mathbb{C}_-\ni\lambda\to\eta_-} \widetilde{f}_j(\lambda)\right\}_{1\le j\le p} = \{(f_j(\eta))_-\}_{1\le j\le p} \tag{3.5.14}$$

as sets (these points just get permuted). Then (3.5.3) shows that (3.5.13) is zero. Finally, there cannot be essential singularities at $\{\zeta_j\}$ and $\{\pm 2\}$ since limits of $\mathfrak{m}_\Delta$ at these points from each of the half-planes exist, so by Casorati–Weierstrass the singularities must be removable or poles. Therefore (II)(A) holds.

(ii)(b) implies that $\Delta(\mathcal{J})$ has no pure points on $\mathfrak{e}$, which gives (II)(B).

Assume (II)(C) does not hold, and there is a pole of $\left[\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}_\Delta^\sharp(\lambda)\right]^{-1}$ at $\lambda_0 \in$

$\pi^{-1}(E_R \setminus \cup_{j=1}^{p}\{\alpha_j, \beta_j\})$. By symmetry, we can assume $\lambda_0 \in \mathcal{R}_+$. Without loss of generality, let $\lambda_0 \in \mathcal{R}_+ \cap (\mathbb{C}_+ \cup \mathbb{R})$. By (3.5.6), the poles of $\left[\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda)\right]^{-1}$ may come only from the inverse of the right-hand side of (3.5.6).

Assume first that $\widetilde{f}_j(\lambda_0)$ are all regular points for m and $m^\sharp$. Then the RHS of (3.5.6) is regular at λ_0, so its determinant must be zero. By Lemma 3.5.5 and (ii)(b), $\lambda_0 = \Delta(\gamma_j)$ for some j.

Then

$$\left[\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda)\right]^{-1} = (S_{11} + \mathfrak{p}_1(\lambda)S_{21})F(\lambda)^{-1},$$

where F is the right-hand side of (3.5.6). By Lemma 3.5.4, F^{-1} has a simple pole at λ_0, so

$$\operatorname*{Res}_{\lambda=\lambda_0}\left[\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda)\right]^{-1} = (S_{11} + \mathfrak{p}_1(\lambda_0)S_{21}) \operatorname*{Res}_{\lambda=\lambda_0} F(\lambda)^{-1}.$$

But using Lemmas 2.4.2, 3.5.5, and 3.5.4 we get $\operatorname{Ran}\operatorname{Res}_{\lambda=\lambda_0} F(\lambda)^{-1} = \ker F(\lambda_0) = \ker(S_{11}+\mathfrak{p}_1(\lambda_0)S_{21})$, which implies $\operatorname{Res}_{\lambda=\lambda_0}\left[\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda)\right]^{-1} = \mathbf{0}$, i.e., $\left[\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda)\right]^{-1}$ is regular at λ_0.

Now assume that $z_0 = \widetilde{f}_k(\lambda_0)$ for some k is a pole for m or $m^\sharp$ (without loss of generality, let it be pole for $m^\sharp$). By Lemma 3.5.7(‡), every other $m(\widetilde{f}_j(\lambda_0)), m^\sharp(\widetilde{f}_j(\lambda_0))$ is regular. By Lemma 3.5.7(†), $\lambda_0 \neq \Delta(\gamma_j)$ for any j. Therefore $S_{11} + \mathfrak{p}_1(\lambda_0)S_{21}$ is invertible. Let $n \geq 1$ be the order of the pole of $m^\sharp$ at z_0. By Lemma 3.5.3 $\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda)$ has a pole of order k at λ_0 (use the fact that $\Delta'(z_0) \neq 0$). Let its Smith-McMillan form be

$$\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda) = E(\lambda)\operatorname{diag}\left((\lambda-\lambda_0)^{\kappa_1}, \ldots, (z-z_0)^{\kappa_l}\right)F(\lambda)$$

with $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_l = -k$. By Lemma 3.5.6 (and (ii)(b)), $\det(\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda))$ has also a pole of order k. Therefore $\kappa_1 + \ldots + \kappa_{l-1} = 0$. Note that $(\lambda-\lambda_0)^k[\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda)]$ has rank 1 by (3.5.6) (as each matrix $[p_{k-1}(f_j(\lambda))p_{s-1}(f_j(\lambda))]_{k,s=1}^p$ is of rank 1 and $S_{11} + \mathfrak{p}_1(\lambda_0)S_{21}$ is invertible). Therefore $\kappa_{l-1} > -k$.

Assume $0 > \kappa_{l-1} > -k$. Then by Lemma 2.4.4 there exists an analytic $\mathbb{C}^l$-valued function ϕ_{l-1} such that $\phi_{l-1}(\lambda_0) \neq 0$ and $(\lambda-\lambda_0)^{-\kappa_{l-1}}\phi_{l-1}(\lambda)^T(\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda)) = \psi_{l-1}(\lambda)$ is analytic at λ_0 with $\psi_{l-1}(\lambda_0) \neq 0$. Now plug this into (3.5.6). We claim that in fact

$$\lim_{z\to z_0}(\lambda-\lambda_0)^{-\kappa_{l-1}}\phi_{l-1}(\lambda)^T(\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}^\sharp_\Delta(\lambda)) = 0.$$

The reason is that $m(\widetilde{f}_k(\lambda_0)) - m^\sharp(\widetilde{f}_k(\lambda_0))$ has a pole of order $k > -\kappa_{l-1}$, which forces $\phi_{l-1}(\lambda_0)^T$ to be in the kernel of $[p_{k-1}(z_0)p_{s-1}(z_0)]_{k,s=1}^p$. But any other $m(\widetilde{f}_j(\lambda_0)) - m^\sharp(\widetilde{f}_j(\lambda_0))$ $(j \neq k)$ is regular, so each of those terms vanish too. Therefore we conclude $\psi_{l-1}(\lambda_0) = 0$, a contradiction.

We showed that $\kappa_{l-1} \geq 0$. Since $\kappa_1 + \ldots + \kappa_{l-1} = 0$ and $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_{l-1}$, we obtain $\kappa_1 = \kappa_2 = \ldots = \kappa_{l-1} = 0$, which implies that $\left[\mathfrak{m}_\Delta(\lambda_0) - \mathfrak{m}_\Delta^\sharp(\lambda_0)\right]^{-1}$ is regular.

Finally we need to show that there are at most simple poles of $\left[\mathfrak{m}_\Delta(\lambda) - \mathfrak{m}_\Delta^\sharp(\lambda)\right]^{-1}$ at $\lambda_0 = \pm 1$. Let k be the number of the simple zeros of $m(\widetilde{f}_j(\lambda_0)) - m^\sharp(\widetilde{f}_j(\lambda_0))$, and let the corresponding indices be $j_1, \ldots, j_k$. There are no poles of m at $\widetilde{f}_j(\lambda_0)$ by (†), so $m - m^\sharp$ is analytic there. Repeating the arguments of Lemma 3.5.5, one sees that kernel of the right-hand side of (3.5.6) is $\operatorname{span}[\{v_1, \cdots, v_p\} \setminus \{v_{j_1}, \cdots, v_{j_k}\}]^\perp$, where $v_j = (1, p_1(f_j(\lambda_0)), \cdots, p_{p-1}(f_j(\lambda_0)))^*$. Since v_j are linearly independent, we see that the dimension of the kernel is precisely k. Since the determinant of the right-hand side has a zero of order k at λ_0 (Lemma 3.5.5), we conclude that its inverse has a simple pole (Lemma 2.4.2).

This establishes that $\mathfrak{m}_\Delta$ satisfies (II)(C).

Finally, let us check (II)(D). Assume $\mathfrak{m}_\Delta$ has a pole at $(\lambda_0)_+$ and $(\lambda_0)_-$ for some $\lambda \in \mathbb{C} \setminus [-2, 2]$. By the part (†) of Lemma 3.5.7, $\lambda_0 \neq \Delta(\gamma_j)$ for any j. This implies that $S_{11} + \mathfrak{p}_1(\lambda_0)S_{21}$ is invertible, and so the pole of $\mathfrak{m}_\Delta((\lambda_0)_+)$ must have come from a pole of $m(f_j((\lambda_0)_+))$ or $m^\sharp(f_j((\lambda_0)_+))$ for some j. Similarly, the pole of $\mathfrak{m}_\Delta((\lambda_0)_-)$ comes from a pole of $m(f_k((\lambda_0)_-))$ or $m^\sharp(f_k((\lambda_0)_-))$ for some k. This violates the condition (‡) of Lemma 3.5.7.

Thus we are in position to apply Theorem 1.3.16. Therefore $\Delta(\mathcal{J})$ satisfies (I) which implies (i) by Lemma 2.2.3.

(i)⇒(ii) The condition (i) implies (I) holds for $\Delta(\mathcal{J})$ by Lemma 2.2.3, which in turn implies (II)(A)–(D) hold by Theorem 1.3.16.

(ii)(a) holds by Lemma 3.5.1 and analytic continuation. Indeed, for each l, $1 \leq l \leq p$, it allows us to meromorphically extend m to (the interior of) the region $\widetilde{f}_l(F_E)$ of $\mathcal{S}_-$. The only thing we need to check is that the extension is continuous on the boundaries of these regions, i.e., on $(\Delta^{-1}((-\infty, 2] \cup [2, \infty)))_-$. Take some z_0 there, different from the band edges and ξ_j's, and let $\lambda_0 = \widetilde{\Delta}(z_0)$. Without loss of generality let us assume that

$$\lim_{\mathcal{R}_-\cap\mathbb{C}_+\ni\lambda\to\lambda_0} \widetilde{f}_1(\lambda) = \lim_{\mathcal{R}_-\cap\mathbb{C}_-\ni\lambda\to\lambda_0} \widetilde{f}_2(\lambda).$$

Then using (3.5.14), we obtain

$$\left\{\lim_{\mathcal{R}_-\cap\mathbb{C}_+\lambda\to\eta_-} \widetilde{f}_j(\lambda)\right\}_{1\le j\le p, j\ne 1} = \left\{\lim_{\mathcal{R}_-\cap\mathbb{C}_-\ni\lambda\to\eta_-} \widetilde{f}_j(\lambda)\right\}_{1\le j\le p, j\ne 2} = \{(f_j(\eta))_-\}_{1\le j\le p} \setminus \{z_0\}.$$

Then (3.5.1) (and the fact that $\mathcal{J}-x_j$ commute for different j's) shows that m has the same limit at z_0 when approaching from regions $\widetilde{f}_1(F_E)$ and $\widetilde{f}_2(F_E)$. Finally, the singularities at ξ_j and band edges must be either removable or poles again by Casorati–Weierstrass arguments. Thus we established (ii)(a).

If (ii)(b) did not hold, then eigenvalues of $\mathcal{J}$ in $\mathfrak{e}$ would produce eigenvalues of $\Delta(\mathcal{J})$ in $[-2,2]$, which would contradict (II)(B).

Now let us show (ii)(c) and (ii)(d). Observe that (i) implies that that there exists a periodic Jacobi matrix $\mathcal{J}^0$ in $\mathcal{T}_\mathfrak{e}$ such that $d_n(\mathcal{J},\mathcal{J}^0)\to 0$. Thus $\limsup_{n\to\infty}(d_n(\mathcal{J},\mathcal{J}^0))^{1/2n} \le R^{-1}$ and we can apply Lemma 3.5.8. Now if (ii)(c) or (ii)(d) fails at a point $z\in\pi^{-1}(E_R\setminus\mathfrak{e})$ then $m(z) = m^\sharp(z)$ (where we allow $\infty=\infty$) implies $m^{(n)}(z) = m^{(n)\sharp}(z)$ for every n by (3.5.11). This implies $m^0(z) = \lim_{n\to\infty} m^{(n)}(z) = \lim_{n\to\infty} m^{(n)\sharp}(z) = m^{0\sharp}(z)$, a contradiction.

Now assume that $\operatorname{Im} m(z) = 0$ for some z in the interior of $\mathfrak{e}$. Then by (3.5.11), every $m^{(n)}$ is also real at this point. This implies that $m^0(z)$ is real, which is impossible (e.g., by (1.2.15)).

Finally assume that $(m(z)-m^\sharp(z))^{-1}$ has a pole of order $k\ge 2$ at some band edge $z_0\in\pi^{-1}(\cup_{j=1}^p\{\alpha_j,\beta_j\})$. At z_0, $(S_{11}+\mathfrak{p}_1^R(\Delta(z_0))S_{21})$ is invertible by Lemma 3.5.4, and then Lemma 3.5.2 implies that $\mathfrak{m}_\Delta$ has a pole of order $k\ge 2$ at ± 2. This contradicts to the condition (II)(C) of Theorem 1.3.16. □

Proof of Theorem 1.3.19. (i)⇒(ii) The condition that $d_n(\mathcal{J},\mathcal{T}_\mathfrak{e}) = 0$ for all large n implies that $\Delta(\mathcal{J})$ is eventually free by the Magic Formula. Then Theorem 1.3.17 implies that m_Δ has a meromorphic continuation to the whole surface $\mathcal{R}$. Then Lemma 2.4.2 allows us to extend m to the whole $\mathcal{S}$ as well. Parts (b), (c), and (d) are already proven in the previous theorem.

(ii)⇒(i) The result is obtained by following the proof of the previous theorem, but applying Theorem 1.3.17 instead of Theorem 1.3.16 (note that $\mathfrak{m}_\Delta$ has full meromorphic continuation to $\mathcal{R}$ by (ii)(a) and Lemma 3.5.2). □

3.6 Point Perturbations of Measures

3.6.1 Perturbations of the Matrix-Valued Free Case

Remark. It is clear, that the Jost functions in Theorem 1.3.20 are related by $\widehat{u}(z) = B^{-1}_{z_N,s_N,U_N}(z)u(z)(1-w_N)^{-1/2}$, where B_{z_N,s_N,U_N} is an appropriately constructed elementary Blaschke-Potapov factor, and $\widehat{u}(z) = B_{z_N,s_N,U_N}(z)u(z)(1+w_N)^{-1/2}$ in Theorems 1.3.21 and 1.3.22. Thus instead of using Theorem 1.3.16, one can use Theorem 1.3.14 to get a (simpler) proof of Theorems 1.3.20 and 1.3.21.

Proof of Theorem 1.3.20. (i) Applying Theorem 1.3.16, we get that M satisfies the conditions (A), (B), (C), (D). Note that the M-function of the Jacobi matrix $(\widehat{A}_n)_{n=1}^\infty$, $(\widehat{B}_n)_{n=1}^\infty$ equals $\widehat{M}(z) = M(z) + \frac{1}{E_N - z - z^{-1}} w_N$. Thus the condition (A) of Theorem 1.3.16 for M implies the condition (A) for $\widehat{M}$. (B) is clear. Note also that $(\widehat{M}(z) - \widehat{M}^\sharp(z))^{-1} = (M(z) - M^\sharp(z))^{-1}$, so (C) is automatic. Finally, let us check the condition (D) for $\widehat{M}$ at a point z_j (for $1 \le j \le N-1$ since z_N is not a pole of $\widehat{M}$). (1.3.20) is immediate since

$$\operatorname{Ran} \operatorname*{Res}_{z=z_j} \widehat{M}(z) = \operatorname{Ran} \operatorname*{Res}_{z=z_j} M(z). \tag{3.6.1}$$

To prove (1.3.21) observe that

$$\begin{aligned}
&\operatorname{Ran} (\widehat{M}(z_j^{-1}) - \widehat{M}^\sharp(z_j^{-1}))^{-1} \widehat{M}(z_j^{-1}) \\
&= \operatorname{Ran} \left((M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1} M(z_j^{-1}) + (M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1} \frac{1}{E_N - z_j - z_j^{-1}} w_N \right) \\
&\qquad = \operatorname{Ran} (M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1} M(z_j^{-1}),
\end{aligned} \tag{3.6.2}$$

where the last equality comes from

$$\operatorname{Ran}\left((M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1}\frac{1}{E_N - z_j - z_j^{-1}}w_N\right) \subseteq \operatorname{Ran}(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1}$$
$$\subseteq \operatorname{Ran}(M(z_j^{-1}) - M^\sharp(z_j^{-1}))^{-1}M(z_j^{-1}), \quad (3.6.3)$$

where we used (3.4.4) and (3.4.5) in the last inclusion. Therefore (3.6.1) and (3.6.2) imply (1.3.18) for $\widehat{M}$, and so Theorem 1.3.16 gives the result.

(ii) follows by the exact same argument using Theorem 1.3.17 instead of Theorem 1.3.16.

□

Proof of Theorem 1.3.21. Now $\widehat{M}(z) = M(z) - \frac{1}{E_N - z - z^{-1}}w_N$ and exactly the same arguments work. □

Proof of Theorem 1.3.22. Again, $\widehat{M}(z) = M(z) - \frac{1}{E_N - z - z^{-1}}w_N$, and the only possible issue is checking the condition (II)(D) of Theorem 1.3.16 for $\widehat{M}$ at $z = z_N$.

(i) We establish (1.3.30) if we show that (1.3.17)–(1.3.18) fails at z_N. If M has no pole at z_N^{-1}, then $\widehat{M}(z) - \widehat{M}^\sharp(z) = M(z) - M^\sharp(z)$ is regular at z_N^{-1}, and so $\ker(\widehat{M}(z_j^{-1}) - \widehat{M}^\sharp(z_j^{-1}))^{-1} = \{0\}$. Therefore if (1.3.17) holds then $w_N = \mathbf{0}$, a contradiction.

(ii) Assume (1.3.31) holds. Then z_N is canonical (for $\widehat{M}$), and so (3.4.6)–(3.4.7) hold for $\widehat{M}$ at $z = z_N^{-1}$. One easily sees that $\widetilde{w}_N \equiv -\operatorname{Res}_{z=z_j}\widehat{M}(z) = \frac{z_N^2}{1-z_N^2}w_N$, and $\widetilde{q}_N \equiv \operatorname{Res}_{z=z_N^{-1}}\widehat{M}(z) = q_N + \frac{1}{1-z_N^2}w_N$. Therefore (3.4.6)–(3.4.7) amount to $\operatorname{Ran} w_N \subseteq \operatorname{Ran} q_N$ and $\operatorname{Ran} w_N \cap \operatorname{Ran}(q_N + w_N/(1 - z_N^2)) = \varnothing$. Now write $-(1 - z_N^2)q_N = (-(1 - z_N^2)q_N - w_N) + w_N$ and pre- and post-multiply by P, the orthogonal projection onto $\operatorname{Ran} w_N$, to obtain $-(1 - z_N^2)Pq_NP = Pw_NP = w_N$. Clearly $\operatorname{Ran} w_N$ is invariant for q_N.

Conversely, if w_N is not of this form, then $\operatorname{Ran} w_N \subseteq \operatorname{Ran} q_N$ and $\operatorname{Ran} w_N \cap \operatorname{Ran}(q_N + w_N/(1 - z_N^2)) = \varnothing$ cannot hold, so z_N is not canonical for $\widehat{M}$. This implies (1.3.30).

(iii) Part (b) is just (1.3.17)–(1.3.18) for $\widehat{M}$.

Now if $l = 1$, then by Proposition 3.4.1, the equations (1.3.17)–(1.3.18) are equivalent to saying that there are no singularities at z_N and z_N^{-1}. But if the order of the pole of M at z_N^{-1} is bigger than 1, then $\widehat{M}$ also has a pole at z_N^{-1}. $\widehat{M}$ has a first order pole at z_N as well (as all E_j are assumed to be distinct). Therefore z_N cannot be canonical for $\widehat{M}$. □

Proof of Theorem 1.3.23. Let us use the Jost functions approach, rather than the M-functions one.

Let us renormalize $d\widehat{\mu}(x)$ to be

$$d\widehat{\mu}(x) = \frac{1}{E_N - x} Hf(x)Hdx + \sum_{j=1}^{N-1} \frac{1}{E_N - E_j} Hw_jH\delta(x - E_j) + Hw_NH\delta(x - E_N),$$

where $H \geq \mathbf{0}$ is

$$H = \left(\int_{-2}^{2} \frac{1}{E_N - x} f(x)dx + \sum_{j=1}^{N-1} \frac{1}{E_N - E_j} w_j + w_N \right)^{-1/2},$$

so that $d\widehat{\mu}$ has the total weight $\mathbf{1}$ (this does not change the Jacobi parameters).

Note that M has no singularity at z_N (as $\mu(\{E_N\}) = \mathbf{0}$), and recall the equality

$$M(z) = M^\sharp(z) + (z - z^{-1}) \left[u^\sharp(z)u(z) \right]^{-1}.$$

This implies that M being regular at z_N^{-1} is equivalent to $u(z_N^{-1})$ being invertible.

Using Theorem 1.3.12, one sees that the Jost fucntion for $\widehat{\mu}$ is $\widehat{u}(z) = \frac{z_N - z}{\sqrt{z_N}} u(z)H^{-1}$. Indeed, we just need to check the boundary values recover the absolutely continuous part of $\widehat{\mu}$ correctly:

$$\begin{aligned}
\frac{\sin\theta}{\pi} \widehat{u}(e^{i\theta})^{-1}\widehat{u}(e^{i\theta})^{*-1} &= \frac{\sin\theta}{\pi} \frac{z_N}{(z_N - e^{i\theta})(z_N - e^{-i\theta})} Hu(e^{i\theta})^{-1}u(e^{i\theta})^{*-1}H \\
&= \frac{z_N}{(z_N - e^{i\theta})(z_N - e^{-i\theta})} Hf(2\cos\theta)H \\
&= \frac{1}{E_N - 2\cos\theta} Hf(2\cos\theta)H = \widehat{f}(2\cos\theta).
\end{aligned}$$

In order to apply Theorem 1.3.15, we just need to show that z_j's are canonical for $1 \leq j \leq N$. $\operatorname{Ran} \widehat{w}_j = \ker \widehat{u}(z_j)$ is straightforward. Now we need to check the equality (1.3.13) for $\widehat{u}$ and $\widetilde{\widehat{w}}_j \equiv \frac{z_j^2}{1-z_j^2} \widehat{\mu}(\{E_j\})$. Substituting the expressions for $\widehat{u}$ and $\widetilde{\widehat{w}}_j$, and using the fact that (1.3.13) holds for u and $\widetilde{w}_j$ we get canonicity of z_j for $1 \leq j \leq N-1$. For z_N, we see that (1.3.13) is equivalent to

$$\frac{z_N^2}{1 - z_N^2} Hw_NH \frac{z_N - 1/z_N}{\sqrt{z_N}} H^{-1}u(z_N^{-1})^* = -(z_N - z_N^{-1})\sqrt{z_N}Hu(z_N)^{-1}, \tag{3.6.4}$$

which reduces to

$$w_N = (z_N - z_N^{-1})u(z_N)^{-1}u(z_N^{-1})^{*-1},$$

since $u(z_N^{-1})$ is invertible by assumption. Finally, note that the last expression is (1.3.35) by (3.3.32). □

3.6.2 Perturbations of the Scalar Periodic Case

Proof of Theorem 1.3.24. Since $\mathcal{J}$ is eventually periodic, m satisfies (ii)(a)–(d) of Theorem 1.3.19. Then note that $\widehat{m}(z) = m(z) - \frac{w_N}{E_N - z}$ and $\widehat{m}(z) - \widehat{m}^\sharp(z) = m(z) - m^\sharp(z)$, which implies that $\widehat{m}$ satisfies (ii)(a)–(c). (ii)(d) is also satisfied since m had a pole at $(E_N)_+$, which means there was no pole at $(E_N)_-$. Therefore $\widehat{m}$ has a pole at $(E_N)_-$ and not at $(E_N)_+$. □

Proof of Theorem 1.3.25. If m has no pole at $(E_N)_-$ then $\widehat{m}(z) = m(z) + \frac{w_N}{E_N - z}$ has poles at both $(E_N)_+$ and $(E_N)_-$, which means that (ii)(d) fails for $\widehat{m}$ at $z = E_N$. Therefore the largest radius R that we can take in Theorem 1.3.18 can be found from $|\Delta(E)| = R + R^{-1}$ (since E_N is real). This gives that the right-hand side of (1.3.38) is R^{-1}, which is $\frac{|\Delta(E)|}{2} - \sqrt{\frac{|\Delta(E)|^2}{4} - 1}$.

Similarly, if m has a pole of order ≥ 2, then the poles cannot cancel out, and $\widehat{m}(z) = m(z) + \frac{w_N}{E_N - z}$ has poles at both $(E_N)_+$ and $(E_N)_-$.

Finally, if m has a pole of order 1, then the condition (d) holds at $z = E_N$ if and only if the residues of m and $\frac{w_N}{E_N - z}$ at $(E_N)_-$ cancel out, i.e., $q_N = w_N$. □

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