Topics in Topological and Holomorphic Quantum Field Theory

Thesis by

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This thesis is dedicated to my parents.

Acknowledgments

Abstract

We investigate topological quantum field theories (TQFTs) in two, three, and four dimensions, as well as holomorphic quantum field theories (HQFTs) in four dimensions. After a brief overview of the two-dimensional (gauged) A and B models and the corresponding the category of branes, we construct analogous three-dimensional (gauged) A and B models and discuss the two-category of boundary conditions. Compactification allows us to identify the category of line operators in the three-dimensional A and B models with the category of branes in the corresponding two-dimensional A and B models. Furthermore, we use compactification to identify the two-category of surface operators in the four-dimensional GL theory at $t = 1$ and $t = i$ with the two-category of boundary conditions in the corresponding three-dimensional A and B model, respectively.

We construct a four-dimensional HQFT related to $\mathcal{N} = 1$ supersymmetric quantum chromodynamics (SQCD) with gauge group $SU(2)$ and two flavors, as well as a four-dimensional HQFT related to the Seiberg dual chiral model. On closed Kähler surfaces with $h^{2,0} > 0$, we show that the correlation functions of holomorphic SQCD formally compute certain Donaldson invariants. For simply-connected elliptic surfaces (and their blow-ups), we show that the corresponding correlation functions in the holomorphic chiral model explicitly compute these Donaldson invariants.

Contents

Chapter 1

Introduction

Topological quantum field theories (TQFTs) are the latest chapter in the long and fruitful discourse between mathematics and physics. Developed in the late 1980s, the hallmark of these quantum theories is metric independence; correlation functions only depend on the differential and topological structure of the spacetime manifold. TQFTs therefore generate topological invariants on nontrivial spacetime manifolds.[1] Many interesting topological invariants have been formulated in the TQFT framework, including Donaldson invariants of four manifolds [38], Gromov-Witten invariants of symplectic manifolds [39], and the Jones polynomials of knots [40].

The realization of topological invariants in the TQFT framework has been beneficial for both mathematics and physics. Mathematicians were able to leverage robust techniques from quantum field theory to compute, as well as generalize, important topological invariants [6], [41], [43]. Physicists were able to use exact results from mathematics to test dualities between strongly coupled quantum field theories and weakly coupled quantum field theories [34], [17].

On complex manifolds with a Hermitian metric, there is a generalization of TQFTs known as holomorphic quantum field theories (HQFTs) [41], [14]. HQFTs are independent of the underlying Hermitian metric, and therefore generate invariants of the complex manifold. Many interesting invariants have been formulated in the HQFT framework, such as the elliptic genus [41], and the chiral de Rham complex [16], [44] of Calabi-Yau manifolds.

This thesis describes some progress in understanding TQFTs in two, three, and four dimensions, as well as HQFTs in four dimensions. Chapter 2 is brief review of cohomological TQFTs and HQFTs, outlining the construction and properties of these theories. Chapter 3 provides some background on category theory and its application to TQFTs in two, three, and four dimensions. We construct two-dimensional (gauged) A and B models in Chapter 4 and discuss the category of branes. Chapter 5 discusses three-dimensional analogs of the (gauged) A and B models and the 2-categories of branes. We utilize these results in Chapter 6 to describe the two-category of surface operators in the GL

[1] We employ a common abuse of language and refer to invariants at the level of the differential structure as topological invariants.

theory at $t = i$ and $t = 1$. Chapter 7 introduces the holomorphic twist of $\mathcal{N} = 1$ supersymmetric SQCD with $N_c = 2$, $N_f = 2$, as well as the holomorphic twist of the dual chiral model. Correlation functions in both theories are computed and shown to agree in accordance with Seiberg duality.

Chapter 2

Cohomological TQFTs and HQFTs

2.1 Supersymmetry

In the early 1970s, a remarkable new symmetry was proposed relating boson and fermions [11], [36]. The spin-statistics theorem tells us that this "supersymmetry" must change the spin properties of fields, nontrivially extending spacetime Poincaré symmetry. Nahm classified supersymmetric extensions of Poincaré algebra in dimensions greater than two [28], building on Coleman and Mandula's remarkable theorem classifying symmetries of a quantum field theory with nontrivial S-matrix [7]. The generators of supersymmetry Q_α^A transform as spinors of the Lorentz algebra $M_{\mu\nu}$,

$$\left[M_{\mu\nu}, Q_\alpha^A\right] = \left(M_{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta^A,$$

and are invariant with respect to translations P_μ,

$$\left[P_\mu, Q_\alpha^A\right] = 0.$$

The anticommutation relations among the supersymmetry generators have the form

$$\left\{Q_\alpha^A, Q_\beta^B\right\} = \Gamma_{\alpha\beta}^{AB\mu} P_\mu + C_{\alpha\beta}^{ABi} Z_i,$$

where $\Gamma_{\alpha\beta}^{AB\mu}$, $C_{\alpha\beta}^{ABi}$ are numerical coefficients, and Z_i are central charges. The Coleman-Mandula theorem tells us that generators of internal symmetries commute with the Poincaré subalgebra and central charges, so the only possible commutation relations are

$$\left[T_I, Q_\alpha^A\right] = \left(T_I\right)_B^{\ A} Q_\alpha^B.$$

Internal symmetries that act nontrivially on the generators of supersymmetry are known as $\mathcal{R}$-symmetries.

While it is not immediately apparent, supersymmetry offers tremendous phenomenological and theoretical benefits. Supersymmetric extensions of the Standard Model have robust solutions to important phenomenological issues, such as the hierarchy problem [8], gauge coupling unification, and dark matter. On the more theoretical side, supersymmetry gives us a quantitative window into various nonperturbative phenomena in strongly coupled quantum field theories.

2.2 Twisting

Consider a supersymmetric quantum field theory on a nontrivial spacetime manifold. The supersymmetry variation of the action is

$$\delta S = \int_M d^n x \sqrt{g} \big(\nabla_\mu \xi\big)_\alpha G^{\mu\alpha},$$

where the variational parameter ξ_α is a spinor on M, ∇_μ is the covariant derivative, and $G^{\mu\alpha}$ is the supercurrent. Since most manifolds do not have covariantly constant spinors, supersymmetry is generically lost.

In the late 1980s, Witten developed a novel procedure that preserves some supersymmetry on an arbitrary spacetime manifold [38], [39]. This "twisting" procedure replaces the original spin group of the supersymmetric theory with an isomorphic subgroup in the direct product of the spin group and $\mathcal{R}$-symmetry group,

$$Spin(n) \to Spin'(n) \subset Spin(n) \times G_{\mathcal{R}}.$$

For suitable supersymmetric theories, we can choose $Spin'(n)$ so that at least one supercharge Q transforms as a scalar. The corresponding supersymmetry is then present on an arbitrary closed manifold.[1] Since the twisting procedure changes the spin properties of fields, the original action S is no longer Lorentz invariant. However, there exists a Lorentz invariant action $\widetilde{S}$ identical to the original action S on Euclidean spacetime $\mathbb{R}^n$. It is natural to choose $\widetilde{S}$ as the action for the twisted theory.

Johansen generalized the twisting procedure to complex manifolds [14]. The restricted spin group of the supersymmetric theory on a complex manifold (with a Hermitian metric) is replaced with an isomorphic subgroup in the direct product of the restricted spin group and $\mathcal{R}$-symmetry group,

$$U(n) \to U'(n) \subset U(n) \times G_{\mathcal{R}}.$$

For supersymmetric theories with a nontrivial $\mathcal{R}$-symmetry group, we can choose $U'(n)$ so that at least one supercharge Q transforms as a scalar. The corresponding supersymmetry is then present

[1] On manifolds with boundary, there are constraints on the boundary conditions necessary to preserve the symmetry.

on an any closed Kähler manifold. While the original action S is not invariant under $U'(n)$, there exists an invariant action $\widetilde{S}$ identical to the original action S on Euclidean spacetime $\mathbb{C}^n$. It is natural to choose $\widetilde{S}$ as the action for the twisted theory.

2.3 BRST Cohomology and Metric Independence

Let us highlight some properties of the scalar supersymmetry Q in the twisted theory. As with any symmetry generator, correlation functions of Q-exact observables vanish,

$$\langle\{Q, \mathcal{O}\}\rangle = 0.$$

Furthermore, it follows from the twisted super Poincaré algebra that the action of Q on observables is nilpotent in the absence of central charges,

$$\{Q, \{Q, \Phi\}\} = 0.$$

These observations imply that Q acts as a BRST operator, so we can define "physical" observables to lie in the Q-cohomology [38], [39]. We refer to the twisted theory with physical observables living in the Q-cohomology as the cohomological twisted theory.

The cohomological twisted theory is topological if the terms of the action $\widetilde{S}$ involving the spacetime metric g are Q-exact [38], [39],

$$\widetilde{S}(g) = \{Q, V(g)\} + S_d,$$

where S_d only depends on the differential structure of the spacetime manifold. This decomposition ensures that correlation functions of metric-indepedent physical observables $\mathcal{O}$ are invariant with respect to infinitesimal deformations of the metric,

$$\begin{aligned}
\delta_g\langle\mathcal{O}\rangle &= \delta_g \int \mathcal{D}\Phi\ \mathcal{O}\exp(-\widetilde{S}(g)) \\
&= -\int \mathcal{D}\Phi\ \mathcal{O}\{Q, \delta_g V(g)\}\exp(-\widetilde{S}(g)) \\
&= -\int \mathcal{D}\Phi\ \{Q, \mathcal{O}\delta_g V(g)\}\exp(-\widetilde{S}(g)) \\
&= -\langle\{\bar{Q}, \mathcal{O}\delta_g V(g)\}\rangle \\
&= 0.
\end{aligned}$$

Similarly, a cohomological twisted theory is holomorphic if the spacetime Hermitian metric h

only appears in Q-exact terms of the action $\widetilde{S}$ [14],

$$\widetilde{S}(h) = \big\{Q, V(h)\big\} + S_c,$$

where S_c only depends on the complex structure of the spacetime manifold. This decomposition ensures that correlation functions of metric-indepedent physical observables $\mathcal{O}$ are invariant with respect to infinitesimal deformations of the Hermitian metric,

$$\begin{aligned}
\delta_h \langle \mathcal{O} \rangle &= \delta_h \int \mathcal{D}\Phi \; \mathcal{O} \exp(-\widetilde{S}(h)) \\
&= -\int \mathcal{D}\Phi \; \mathcal{O}\big\{Q, \delta_h V(h)\big\} \exp(-\widetilde{S}(h)) \\
&= -\int \mathcal{D}\Phi \; \big\{Q, \mathcal{O}\delta_h V(h)\big\} \exp(-\widetilde{S}(h)) \\
&= -\langle \big\{\bar{Q}, \mathcal{O}\delta_h V(h)\big\} \rangle \\
&= 0.
\end{aligned}$$

Chapter 3

TQFTs and Category Theory

3.1 Categories

In the mid 1940s, Eilenberg and MacLane introduced categories to capture the notion of equivalence found in various mathematical structures (groups, topologies, etc.). A category consists of "objects" and "morphisms". Each morphism f is associated to two objects; a "domain" A and a "codomain" B. The standard notation for morphisms is

$$f : A \to B.$$

The collection of all morphisms with domain A and codomain B is denoted by $\mathrm{Hom}(A, B)$. Morphisms are required to be closed under "composition"; given a morphism $f \in \mathrm{Hom}(B, C)$ and a morphism $g \in \mathrm{Hom}(A, B)$, the "composition" of f and g is a morphism $h \in \mathrm{Hom}(A, C)$. The standard notation for composition is

$$h = f \cdot g.$$

Composition is required to be an associative operation,

$$(f \cdot g) \cdot h = f \cdot (g \cdot h),$$

where $f \in \mathrm{Hom}(C, D)$, $g \in \mathrm{Hom}(B, C)$, and $h \in \mathrm{Hom}(A, B)$. Finally, each object B in the category must have an "identity" morphism $1_B : B \to B$ such that

$$f \cdot 1_B = f,$$

for all $f \in \mathrm{Hom}(B, C)$, and

$$1_B \cdot g = g,$$

and for all $g \in \mathrm{Hom}(A, B)$.

Mappings between categories that preserve the category structure are known as functors. A functor F from category $\mathcal{C}$ to category $\mathcal{C}'$ maps any object A in $\mathcal{C}$ to an object A' in $\mathcal{C}'$. Similarly, F maps any morphism $f : A \to B$ in $\mathcal{C}$ to a morphism $f' : A' \to B'$ in $\mathcal{C}'$ with the appropriate domain and codomain,

$$F(f) : F(A) \to F(B).$$

In addition, the functor F preserves composition of morphisms,

$$F(f \cdot g) = F(f) \cdot F(g),$$

and maps the identity morphisms in $\mathcal{C}$ to the appropriate identity morphisms in $\mathcal{C}'$,

$$F(1_A) = 1_{F(A)}.$$

A 2-category is an extension of a category where we introduce "2-morphisms" that give each collection of morphisms $\mathrm{Hom}(A, B)$ the structure of a category; the objects in each Hom category are morphisms and the morphisms in each Hom category are 2-morphisms. Furthermore, any morphism $f : B \to C$ in the 2-category is a functor between the appropriate Hom categories,

$$f : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C),$$
$$f : \mathrm{Hom}(C, D) \to \mathrm{Hom}(B, D),$$

with the identity morphism going to the identity functor. Finally, we require that the action of morphisms and 2-morphisms commute.[1]

As we shall see in the following sections, the categories and 2-categories that appear in TQFTs have additional structure. One universal feature is that these categories and 2-categories are $\mathbb{C}$-linear; the collection of morphisms $\mathrm{Hom}(A, B)$ and 2-morphisms $\mathrm{Hom}(f, g)$ forms a complex vector space that we will denote by V_{AB} and V_{fg}, respectively. Notice that the composition axiom gives V_{AA} and V_{ff} the structure of an associative algebra, which we shall call the endomorphism algebra.

3.2 The Category of Boundary Conditions in Two-Dimensional TQFTs

In this section, we will show that boundary conditions in a two-dimensional TQFT have the structure of a category, [9]. Objects of this category are boundary conditions, and morphisms are local

[1] We only provide the definition of *strict* 2-categories in this thesis. Generally, one encounters *weak* 2-categories in TQFTs, where the associativity and identity axioms are relaxed to associativity and identity up to isomorphism.

observables $\mathcal{O}_{AB}$ sitting at the junction between an interval with boundary condition A and an interval with boundary condition B, (see Figure 3.1).

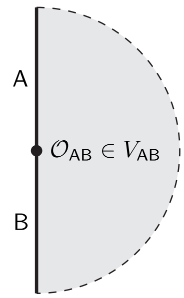

Figure 3.1: Morphisms correspond to local operators at the junction of two boundaries.

Consider two local observables $\mathcal{O}_{AB}$ and $\mathcal{O}_{BC}$ sitting at the appropriate junctions between intervals with boundary conditions A, B, and C (see Figure 3.2). Composition of morphisms comes from contracting the interval B (a trivial operation in a TQFT), fusing the local observables. Since the order of contraction is irrelevant in TQFT, composition is associative. Finally, the trivial local observable realizes the identity morphism for each boundary condition.

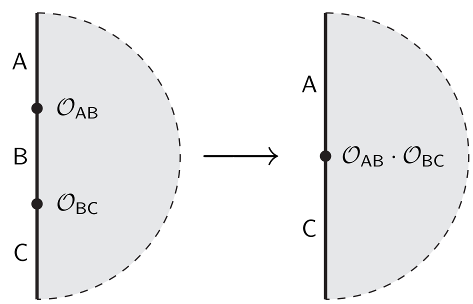

Figure 3.2: Composition of morphisms

Note that conformal transformations about the junction give us a one-to-one correspondence between local observables $\mathcal{O}_{AB}$ and states in the TQFT on $\mathbb{R}_+ \times I$ with boundary conditions A and B on the oriented interval I. So $\mathrm{Hom}(A, B)$ forms a complex vector space V_{AB}. Furthermore, when A and B are the same boundary condition, composition gives $\mathrm{Hom}(A, A)$ the structure of an associative algebra.

3.3 The 2-Category of Two-Dimensional TQFTs

In this section, we will show that the collection of two-dimensional TQFTs has the structure of a 2-category. The objects of the underlying category are two-dimensional TQFTs and morphisms are defect lines A on the boundary between the worldsheet of TQFT $\mathbb{X}$ and the worldsheet of TQFT $\mathbb{Y}$, (see Figure 3.3). Consider defect lines A and B between the worldsheet of TQFTs $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{Z}$

(see Figure 3.4). Composition of morphisms comes from contracting the worldsheet of TQFT $\mathbb{Y}$ (a trivial operation in a TQFT), fusing the defect lines. Since the order of contraction is irrelevant in TQFTs, composition is associative. Finally, the trivial defect line realizes the identity morphism for each two-dimensional TQFT.

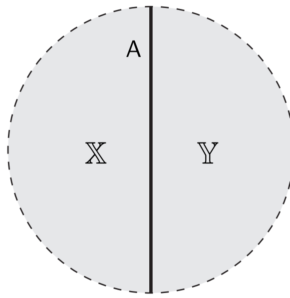

Figure 3.3: Morphisms correspond to defect lines between two TQFTs.

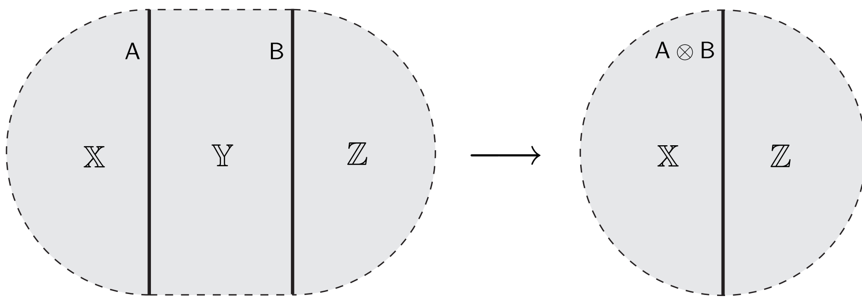

Figure 3.4: Composition of morphisms

From the previous section, we know that the collection of defect lines between a pair of two-dimensional TQFTs forms a category as well. The objects of this category are defect lines and the morphisms are local observables at the junction between two defect lines. The category of two-dimensional TQFTs therefore extends to a 2-category; the objects in this 2-category are two-dimensional TQFTs, the morphisms are defect lines between the worldsheets of two-dimensional TQFTs, and the 2-morphisms are local observables at the junction of defect lines (see Figure 3.5).

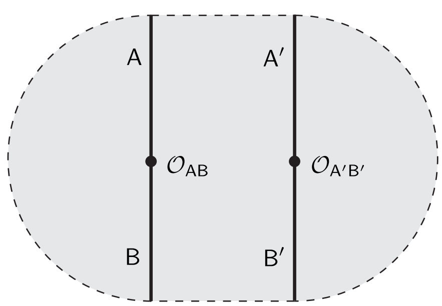

Figure 3.5: Elements in the 2-category of two-dimensional TQFTs

Since the order of contraction is irrelevent in TQFTs, the composition of morphisms and 2-morphisms commute (see Figure (3.6).

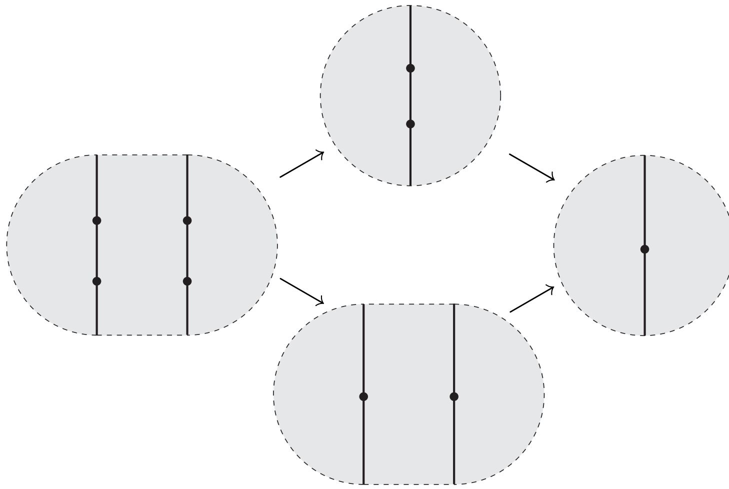

Figure 3.6: Composition of morphisms and 2-morphisms commute.

Notice that defect lines in $\mathrm{Hom}(\mathbb{X},\mathbb{X})$ close under composition, giving this category a "tensor" product,

$$\mathrm{Hom}(\mathbb{X},\mathbb{X}) \otimes \mathrm{Hom}(\mathbb{X},\mathbb{X}) \to \mathrm{Hom}(\mathbb{X},\mathbb{X}).$$

Categories with an additional tensor structure are called a monoidal category.

3.4 The 2-Category of Boundary Conditions in Three-Dimensional TQFTs

Boundary conditions in a three-dimensional TQFT also form a 2-category. The objects of the underlying category are boundary conditions and the morphisms are line observables at the junction between boundary conditions. Consider line observables A and B sitting at the appropriate junction between boundary conditions $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{Z}$. Composition of morphisms comes from contracting the region with boundary condition $\mathbb{Y}$ (a trivial operation in a TQFT), fusing the line observables A and B. Since the order of contraction is irrelevant in TQFTs, composition is associative. Finally, the trivial line observable realizes the identity morphism for each boundary condtion.

From the previous sections, we know that the collection of line observables between a pair of boundary conditions forms a category as well. The objects of this category are line observables and the morphisms are local observables at the junction between two line observables. The category of boundary conditions in a three-dimensional TQFT therefore extends to a 2-category; the objects in this 2-category are boundary conditions, the morphisms are line observables at the junction of boundary conditions, and the 2-morphisms are local observables at the junction of line observables.

By excising a tubular neighborhood about a line, we find a one-to-one correspondence between line observables on $\mathbb{R}^3$ and boundary condition on $S^1 \times \mathbb{R}_+ \times \mathbb{R}$. Furthermore, we can compactify S^1 (a trivial process in a TQFT) to obtain a two-dimensional TQFT with the appropriate boundary condition. The category of line observables in a three-dimensional TQFT is therefore equivalent to the category of boundary conditions in the two-dimensional TQFT obtained from S^1 compactification.[2]

3.5 The 2-Category of Surface Observables in Four-Dimensional TQFTs

In this section, we will show that surface observables in a four-dimensional TQFT form a 2-category. The objects of the underlying category are surface observables and the morphisms are line observables at the boundary between surface observables. Consider line observables A and B sitting at the appropriate boundaries between surface observables $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{Z}$. Composition of morphisms comes from contracting the worldsheet of surface observable $\mathbb{Y}$ (a trivial operation in a TQFT), fusing the line observables A and B. Since the order of contraction is irrelevant in TQFTs, composition is associative. Finally, the trivial line observable realizes the identity morphism for each surface observable.

From the previous sections, we know that the collection of line observables between a pair of surface observables forms a category as well. The objects of this category are line observables and the morphisms are local observables at the junction between two line observables. The category of surface observables in therefore extends to a 2-category; the objects in this 2-category are surface observables, the morphisms are line observables at the boundary of surface observables, and the 2-morphisms are local observables at the junction of line observables.

By excising a tubular neighborhood about a plane, we find a one-to-one correspondence between surface observables on $\mathbb{R}^4$ and boundary conditions on $S^1 \times \mathbb{R}_+ \times \mathbb{R}^2$. Furthermore, we can compactify S^1 (a trivial process in a TQFT) to obtain a three-dimensional TQFT with the appropriate boundary condition. The 2-category of surface observables in a four-dimensional TQFT is therefore equivalent to the 2-category of boundary conditions in the three-dimensional TQFT obtained from S^1 compactification.

[2]Note that some information is lost in the compactification process; the braided structure of the monoidal category of line observables in a three-dimensional TQFT is not captured by the category of boundary conditions in corresponding two-dimensional TQFT.

Chapter 4

Two-Dimensional TQFTs

4.1 $\mathcal{N} = (2,2)$ Nonlinear σ-Model

We begin this chapter with a review of $\mathcal{N} = (2,2)$ nonlinear σ-models. The bosonic fields σ are a map from the worldsheet $\mathbb{R}^2$ into a Kähler target manifold X,

$$\sigma \in \mathrm{Map}\big(\mathbb{R}^2, X\big).$$

The fermionic fields $\psi_\pm$ and $\overline{\psi}_\pm$ are sections of the spin bundle $S_\pm$ on $\mathbb{R}^2$ valued in the pullback of the holomorphic tangent bundle TX and antiholomorphic tangent bundle $\overline{TX}$, respectively,

$$\begin{aligned} \psi_\pm &\in \Gamma\big(\sigma^* TX \otimes S_\pm\big), \\ \overline{\psi}_\pm &\in \Gamma\big(\sigma^* \overline{TX} \otimes S_\pm\big). \end{aligned}$$

The dynamics of the σ-model are governed by the action

$$\begin{aligned} S = \int_{\mathbb{R}^2} d^2x\, \Big(& g_{i\bar{j}} \partial^\mu \sigma^i \partial_\mu \overline{\sigma}^{\bar{j}} - i g_{i\bar{j}} \overline{\psi}_-^{\bar{j}} \big(D_1 + iD_2\big) \psi_-^i \\ & + i g_{i\bar{j}} \overline{\psi}_+^{\bar{j}} \big(D_1 - iD_2\big) \psi_+^i - R_{i\bar{j}k\bar{l}} \psi_+^i \psi_-^k \overline{\psi}_-^{\bar{j}} \overline{\psi}_+^{\bar{l}} \Big), \end{aligned} \tag{4.1}$$

where $g_{i\bar{j}}$ is the Kähler metric on X, $R_{i\bar{j}k\bar{l}}$ is the Riemannian curvature on X, and

$$D_\mu \psi_\pm^i = \partial_\mu \psi_\pm^i + \Gamma^i_{jk} \partial_\mu \sigma^j \psi_\pm^k,$$

with Γ^i_{jk} is the Levi-Civita connection on X.

It is not difficult to see that the action respects the following supersymmetry transformations,

$$\begin{aligned}
\delta\sigma^i &= \xi_+\psi_-^i - \xi_-\psi_+^i, \\
\delta\psi_-^i &= i\overline{\xi}_+(\partial_1 - i\partial_2)\sigma^i + \xi_-\Gamma^i_{jk}\psi_+^j\psi_-^k, \\
\delta\psi_+^i &= i\overline{\xi}_-(\partial_1 + i\partial_2)\sigma^i + \xi_+\Gamma^i_{jk}\psi_+^j\psi_-^k, \\
\delta\overline{\sigma}^i &= \overline{\xi}_-\overline{\psi}_+^{\overline{i}} - \overline{\xi}_+\overline{\psi}_-^{\overline{i}}, \\
\delta\overline{\psi}_-^{\overline{i}} &= -i\xi_+(\partial_1 - i\partial_2)\overline{\sigma}^{\overline{i}} + \overline{\xi}_-\Gamma^{\overline{i}}_{\overline{j}\overline{k}}\overline{\psi}_-^{\overline{j}}\overline{\psi}_+^{\overline{k}}, \\
\delta\overline{\psi}_+^{\overline{i}} &= -i\xi_-(\partial_1 + \partial_2)\overline{\sigma}^{\overline{i}} + \overline{\xi}_+\Gamma^{\overline{i}}_{\overline{j}\overline{k}}\overline{\psi}_-^{\overline{j}}\overline{\psi}_+^{\overline{k}}.
\end{aligned} \tag{4.2}$$

At the classical level, the $\mathcal{N} = (2,2)$ nonlinear σ-model also possesses $U(1)_E$ rotational symmetry, $U(1)_V$ vector $\mathcal{R}$-symmetry, and $U(1)_A$ axial $\mathcal{R}$-symmetry. With respect to these symmetries, the fields and supercharges transforms as shown in the tables below,

Field	$U(1)_E$	$U(1)_V$	$U(1)_A$
σ^i	0	0	0
ψ_-^i	1	−1	1
ψ_+^i	−1	−1	−1
$\overline{\sigma}^{\overline{i}}$	0	0	0
$\overline{\psi}_-^{\overline{i}}$	1	1	−1
$\overline{\psi}_+^{\overline{i}}$	−1	1	1

,

Table 4.1: Charges of fields in $\mathcal{N} = (2,2)$ nonlinear σ-model

	$U(1)_E$	$U(1)_V$	$U(1)_A$
Q_-	1	−1	1
Q_+	−1	−1	−1
$\overline{Q}_-$	1	1	−1
$\overline{Q}_+$	−1	1	1

.

Table 4.2: Charges of $\mathcal{N} = (2,2)$ supercharges

In the quantum theory, the axial $\mathcal{R}$-symmetry is potentially anomalous,

$$\mathrm{Ind}\not{D} = \int_{\mathbb{R}^2} \sigma^* c_1(TX).$$

The anomaly vanishes if and only if $c_1(TX) = 0$, so $U(1)_A$ is a symmetry if and only if the Kähler manifold X is Calabi-Yau.

4.1.1 A-Model

The A-model is constructed by twisting the $U(1)_E$ rotational symmetry by the $U(1)_V$ vector $\mathcal{R}$-symmetry of the $\mathcal{N} = (2,2)$ nonlinear σ-model [39], [41] (see Table 4.1 for the charges of fields in the $\mathcal{N}=(2,2)$ nonlinear σ-model),

$$E \to E + V.$$

Field	$U(1)_E$	$U(1)_A$
σ^i	0	0
χ^i	0	1
$\rho^i_{\bar{z}}$	-2	-1
$\overline{\sigma}^{\bar{i}}$	0	0
$\overline{\rho}^{\bar{i}}_z$	2	-1
$\overline{\chi}^{\bar{i}}$	0	1

.

Table 4.3: Fields in A-model

The bosonic fields σ are a map from the worldsheet Σ into a Kähler target manifold X,

$$\sigma \in \mathrm{Map}\big(\Sigma, X\big).$$

The fermionic fields χ, $\overline{\chi}$ are scalars on Σ valued in the pullback of the holomorphic tangent bundle TX, antiholomorphic tangent bundle $\overline{TX}$, respectively,

$$\chi \in \Gamma\big(\sigma^* TX\big),$$
$$\overline{\chi} \in \Gamma\big(\sigma^* \overline{TX}\big).$$

The fermionic fields ρ, $\overline{\rho}$ are a (0,1)-form, (1,0)-form on Σ valued in the pullback of the holomorphic tangent bundle TX, antiholomorphic tangent bundle $\overline{TX}$, respectively,

$$\rho \in \Gamma\big(\sigma^* TX \otimes \Omega^{0,1}\big),$$
$$\overline{\rho} \in \Gamma\big(\sigma^* \overline{TX} \otimes \Omega^{1,0}\big).$$

We construct the action for the A-model by writing the action of the $\mathcal{N}=(2,2)$ nonlinear σ-model (4.1) covariantly in terms of the twisted fields,

$$\begin{aligned} S_A = \int_\Sigma d^2z \Big(& g_{i\bar{j}} \partial_z \sigma^i \partial_{\bar{z}} \overline{\sigma}^{\bar{j}} + g_{i\bar{j}} \partial_{\bar{z}} \sigma^i \partial_z \overline{\sigma}^{\bar{j}} \\ & + i g_{i\bar{j}} \overline{\rho}^{\bar{j}}_z D_{\bar{z}} \chi^i + i g_{i\bar{j}} \rho^i_{\bar{z}} D_z \overline{\chi}^{\bar{j}} - \frac{1}{2} R_{i\bar{j}k\bar{l}} \rho^i_{\bar{z}} \chi^k \overline{\rho}^{\bar{j}}_z \overline{\chi}^{\bar{l}} \Big). \end{aligned} \tag{4.3}$$

The BRST charge is

$$Q_A = Q_- + \overline{Q}_+,$$

which is a scalar after twisting (see Table 4.2 for the charges of the $\mathcal{N}=(2,2)$ supercharges). The

BRST variations follow from the corresponding supersymmetry transformations (4.2),

$$
\begin{aligned}
&\delta\sigma^i = \xi\chi^i,\\
&\delta\chi^i = 0,\\
&\delta\rho^i_{\bar z} = 2i\xi\partial_{\bar z}\sigma^i + \xi\Gamma^i_{jk}\rho^j_{\bar z}\chi^k,\\
&\delta\overline{\sigma}^i = \xi\overline{\chi}^{\bar i},\\
&\delta\overline{\rho}^{\bar i}_z = 2i\xi\partial_z\overline{\sigma}^{\bar i} + \xi\Gamma^{\bar i}_{\bar j\bar k}\bar\rho^{\bar j}_z\overline{\chi}^{\bar k},\\
&\delta\overline{\chi}^{\bar i} = 0.
\end{aligned}
$$

Notice that the ρ and $\overline{\rho}$ variations are only nilpotent on-shell. It will be convenient to introduce auxiliary bosonic fields P, $\overline{P}$ which are a (0,1)-form, (1,0)-form on Σ valued in the pullback of the holomorphic tangent bundle TX, antiholomorphic tangent bundle $\overline{TX}$, respectively,

$$
\begin{aligned}
&P \in \Gamma\big(\sigma^*TX \otimes \Omega^{0,1}\big),\\
&\overline{P} \in \Gamma\big(\sigma^*\overline{TX} \otimes \Omega^{1,0}\big).
\end{aligned}
$$

We require that P, $\overline{P}$ satisfy the following on-shell constraint,

$$
\begin{aligned}
P^i_{\bar z} &= 2i\partial_{\bar z}\sigma^i + \Gamma^i_{jk}\rho^j_{\bar z}\chi^k,\\
\overline{P}^{\bar i}_z &= 2i\partial_z\overline{\sigma}^{\bar i} + \Gamma^{\bar i}_{\bar j\bar k}\bar\rho^{\bar j}_z\overline{\chi}^{\bar k},
\end{aligned}
\tag{4.4}
$$

so that we can write the BRST variations as

$$
\begin{aligned}
\delta\sigma^i &= \xi\chi^i,\\
\delta\chi^i &= 0,\\
\delta\rho^i_{\bar z} &= \xi P^i_{\bar z},\\
\delta P^i_{\bar z} &= 0,\\
\delta\overline{\sigma}^i &= \xi\overline{\chi}^{\bar i},\\
\delta\overline{\rho}^{\bar i}_z &= \xi\overline{P}^{\bar i}_z,\\
\delta\overline{\chi}^{\bar i} &= 0,\\
\delta\overline{P}^{\bar i}_z &= 0.
\end{aligned}
\tag{4.5}
$$

It is not difficult to construct an action equivalent to original A-model action (4.3) that enforces the on-shell constraints on the auxiliary fields (4.4) and respects the BRST symmetry (4.5),

$$\begin{aligned} S_A = \int_\Sigma d^2z \Big\{ Q_A, \frac{1}{4} g_{i\bar{j}} \rho^i_{\bar{z}} \overline{P}^{\bar{j}}_z + \frac{1}{4} g_{i\bar{j}} P^i_{\bar{z}} \overline{\rho}^{\bar{j}}_z - i g_{i\bar{j}} \overline{\rho}^{\bar{j}}_z \partial_{\bar{z}} \sigma^i - i g_{i\bar{j}} \rho^i_{\bar{z}} \partial_z \overline{\sigma}^{\bar{j}} \\ - \frac{1}{4} \partial_{\bar{k}} g_{i\bar{j}} \overline{\chi}^{\bar{k}} \rho^i_{\bar{z}} \overline{\rho}^{\bar{j}}_z + \frac{1}{4} \partial_k g_{i\bar{j}} \chi^k \rho^i_{\bar{z}} \overline{\rho}^{\bar{j}}_z \Big\} + \int_\Sigma d^2z \left(g_{i\bar{j}} \partial_z \sigma^i \partial_{\bar{z}} \overline{\sigma}^{\bar{j}} - g_{i\bar{j}} \partial_{\bar{z}} \sigma^i \partial_z \overline{\sigma}^{\bar{j}} \right). \end{aligned} \tag{4.6}$$

Notice that this action is BRST exact up to a topological term.

Local observables in the A-model are elements in the BRST cohomology on smooth functionals of scalar fields, σ^i, $\overline{\sigma}^{\bar{j}}$, χ^i, and $\overline{\chi}^{\bar{i}}$. It is not difficult to see that the BRST cohomology is isomorphic to the de Rham cohomology of X, with χ mapping to (1,0)-forms on X, $\overline{\chi}$ mapping to (0,1)-forms on X, and Q_A mapping to the exterior derivative,

$$\begin{aligned} \chi^i &\longleftrightarrow dz^i, \\ \overline{\chi}^{\bar{i}} &\longleftrightarrow dz^{\bar{i}}, \\ Q_A &\longleftrightarrow d. \end{aligned}$$

Now consider a worldsheet Σ with boundary $\partial\Sigma$. Taking the variation of the action, we have the following boundary terms,

$$\begin{aligned} \delta S^b_A = \int_{\partial\Sigma} \Big(g_{i\bar{j}} \delta\sigma^i \wedge \star d\overline{\sigma}^{\bar{j}} + g_{i\bar{j}} \delta\sigma^{\bar{j}} \wedge \star d\sigma^i \\ + i g_{i\bar{j}} \overline{\rho}^{\bar{j}} \left(\delta\chi^i + \Gamma^i_{jk} \delta\sigma^j \chi^k \right) + i g_{i\bar{j}} \rho^i \left(\delta\overline{\chi}^{\bar{j}} + \Gamma^{\bar{j}}_{\bar{k}\bar{l}} \delta\overline{\sigma}^{\bar{k}} \overline{\chi}^{\bar{l}} \right) \Big). \end{aligned} \tag{4.7}$$

Let L be the submanifold of X determining allowed classical variations of σ on the boundary,

$$\sigma|_{\partial\Sigma} \in \mathrm{Map}\big(\partial\Sigma, L\big). \tag{4.8}$$

We can determine the allowed classical variations of χ and $\bar{\chi}$ using BRST symmetry,

$$\begin{aligned} \big(\chi + \bar{\chi}\big)|_{\partial\Sigma} \in \Gamma\big(\sigma^* TL\big), \\ -\big(J\chi + J\bar{\chi}\big)|_{\partial\Sigma} \in \Gamma\big(\sigma^* TL\big). \end{aligned} \tag{4.9}$$

where J is the complex structure on X. Given the allowed classical variations (4.8), (4.9), and the boundary variational terms (4.7), we see that the following boundary conditions are necessary for the classical theory to be well-defined,

$$\begin{aligned}\star d\sigma|_{\partial\Sigma} &\in \Gamma\big(\sigma^* NL \otimes \Omega^1_{\partial\Sigma}\big),\\ \big(\rho+\overline{\rho}\big)|_{\partial\Sigma} &\in \Gamma\big(\sigma^* NL \otimes \Omega^1_{\partial\Sigma}\big),\\ -\big(J\rho+J\overline{\rho}\big)|_{\partial\Sigma} &\in \Gamma\big(\sigma^* NL \otimes \Omega^1_{\partial\Sigma}\big).\end{aligned}$$

Finally, we require that the normal component of the BRST current vanish on the boundary to preserve BRST symmetry,

$$\begin{aligned}\star j|_{\partial\Sigma} &= g_{i\bar{j}}\big(\chi^i \wedge \star\partial\overline{\sigma}^{\bar{j}} + \star\overline{\partial}\sigma^i \wedge \overline{\chi}^{\bar{j}}\big)|_{\partial\Sigma}\\ &= \frac{1}{2} g_{i\bar{j}}\big(\chi^i \wedge \star d\overline{\sigma}^{\bar{j}} + \star d\sigma^i \wedge \overline{\chi}^{\bar{j}}\big)|_{\partial\Sigma} - \frac{i}{2} g_{i\bar{j}}\big(\chi^i \wedge d\overline{\sigma}^{\bar{j}} - d\sigma^i \wedge \overline{\chi}^{\bar{j}}\big)|_{\partial\Sigma} = 0.\end{aligned}$$

This constraint is met if and only if the Kähler form ω vanishes when restricted to the tangent bundle of TL and the normal bundle NL,

$$\begin{aligned}\omega|_{TL} &= 0,\\ \omega|_{NL} &= 0.\end{aligned}$$

The submanifold L must be isotropic and coisotropic, and therefore a Lagrangian submanifold of X.

As we discussed in Chapter 3, boundary conditions of a 2d TQFT form a category. Extending our analysis to including a gauge field on L and taking into account potential anomalies in the $U(1)_A$ axial $\mathcal{R}$-symmetry, it is possible to show that objects in this category include flat vector bundles on special Lagrangian submanifolds of X with a unitary connection [42], [29]. The Fukaya category of X, $\mathcal{F}(X)$ encapsulates precisely this information [23], however it is known that $\mathcal{F}(X)$ is a proper subcategory of the full category of A-branes [15].

4.1.2 B-Model

The B-model is constructed by twisting the $U(1)_E$ rotational symmetry by the $U(1)_A$ axial $\mathcal{R}$-symmetry of the $\mathcal{N} = (2,2)$ nonlinear σ-model with a Calabi-Yau target manifold [41] (see Table 4.1 for the charges of fields in the $\mathcal{N} = (2,2)$ nonlinear σ-model),

$$E \to E + A.$$

The bosonic fields σ are maps from the worldsheet Σ into a Calabi-Yau target manifold X,

$$\sigma \in \mathrm{Map}\big(\Sigma, X\big).$$

Field	$U(1)_E$	$U(1)_V$
σ^i	0	0
ρ^i_z	2	-1
$\rho^i_{\bar{z}}$	-2	-1
$\overline{\sigma}^{\bar{i}}$	0	0
$\overline{\psi}^{\bar{i}}$	0	1
$\overline{\chi}^{\bar{i}}$	0	1

.

Table 4.4: Fields in B-model

The fermionic field ρ is a 1-form on Σ valued in the pullback of the holomorphic tangent bundle TX,

$$\rho \in \Gamma\big(\sigma^* TX \otimes \Omega^1\big).$$

The fermionic fields $\overline{\psi}$ and $\overline{\chi}$ are scalars on Σ valued in the pullback of the antiholomorphic tangent bundle $\overline{TX}$,

$$\overline{\psi} \in \Gamma\big(\sigma^* \overline{TX}\big),$$
$$\overline{\chi} \in \Gamma\big(\sigma^* \overline{TX}\big).$$

We construct the action for the B-model by writing the action of the $\mathcal{N} = (2,2)$ nonlinear σ-model (4.1) covariantly in terms of the twisted fields,

$$S_B = \int_\Sigma d^2z \Big(g_{i\bar{j}} \partial_z \sigma^i \partial_{\bar{z}} \overline{\sigma}^{\bar{j}} + g_{i\bar{j}} \partial_{\bar{z}} \sigma^i \partial_z \overline{\sigma}^{\bar{j}} - i g_{i\bar{j}} \overline{\psi}^{\bar{j}} D_{\bar{z}} \rho^i_z - i g_{i\bar{j}} \overline{\chi}^{\bar{j}} D_z \rho^i_{\bar{z}} + \frac{1}{2} R_{i\bar{j}k\bar{l}} \rho^i_{\bar{z}} \rho^k_z \overline{\psi}^{\bar{j}} \overline{\chi}^{\bar{l}} \Big).$$

The BRST charge is

$$Q_B = \overline{Q}_- + \overline{Q}_+,$$

which is a scalar after twisting (see Table 4.2 for the charges of the $\mathcal{N} = (2,2)$ supercharges). The BRST variations follow from the corresponding supersymmetry transformations (4.2),

$$\delta \sigma^i = 0,$$
$$\delta \rho^i_z = 2i\xi \partial_z \sigma^i,$$
$$\delta \rho^i_{\bar{z}} = 2i\xi \partial_{\bar{z}} \sigma^i,$$
$$\delta \overline{\sigma}^{\bar{i}} = -\xi \big(\overline{\psi}^{\bar{i}} + \overline{\chi}^{\bar{i}} \big),$$
$$\delta \overline{\psi}^{\bar{i}} = -\xi \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \overline{\psi}^{\bar{j}} \overline{\chi}^{\bar{k}},$$
$$\delta \overline{\chi}^{\bar{i}} = \xi \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \overline{\psi}^{\bar{j}} \overline{\chi}^{\bar{k}}.$$

These BRST variations suggest that we make the following field redefinitions,

$$\overline{\eta}^{\bar{i}} = -(\overline{\psi}^{\bar{i}} + \overline{\chi}^{\bar{i}}),$$
$$\overline{\theta}_i = g_{i\bar{j}}(\overline{\psi}^{\bar{j}} - \overline{\chi}^{\bar{j}}),$$

where the fermionic fields $\overline{\eta}$ and $\overline{\theta}$ are scalars on Σ valued in the pullback of the antiholomorphic tangent bundle $\overline{TX}$ and the holomorphic cotangent bundle T^*X, respectively,

$$\overline{\eta} \in \Gamma\big(\sigma^*\overline{TX}\big),$$
$$\overline{\theta} \in \Gamma\big(\sigma^*T^*X\big).$$

The action written in terms of $\overline{\eta}$ and $\overline{\theta}$ is

$$\begin{aligned} S_B = \int_\Sigma d^2z\, \Big(& g_{i\bar{j}}\partial_z\sigma^i\partial_{\bar{z}}\overline{\sigma}^{\bar{j}} + g_{i\bar{j}}\partial_{\bar{z}}\sigma^i\partial_z\overline{\sigma}^{\bar{j}} + \frac{i}{2}g_{i\bar{j}}\rho^i_z D_{\bar{z}}\overline{\eta}^{\bar{j}} \\ & - \frac{i}{2}\overline{\theta}_i D_{\bar{z}}\rho^i_z + \frac{i}{2}g_{i\bar{j}}\rho^i_{\bar{z}}D_z\overline{\eta}^{\bar{j}} + \frac{i}{2}\overline{\theta}_i D_z\rho^i_{\bar{z}} - \frac{1}{4}R^l_{i\bar{j}k}\overline{\theta}_l\overline{\eta}^{\bar{j}}\rho^i_{\bar{z}}\rho^k_z\Big), \end{aligned} \tag{4.10}$$

and the BRST variations are

$$\begin{aligned} \delta\sigma^i &= 0, \\ \delta\rho^i_z &= 2i\xi\partial_z\sigma^i, \\ \delta\rho^i_{\bar{z}} &= 2i\xi\partial_{\bar{z}}\sigma^i, \\ \delta\overline{\sigma}^i &= \xi\overline{\eta}^{\bar{i}}, \\ \delta\overline{\eta}^{\bar{i}} &= 0, \\ \delta\overline{\theta}_i &= 0. \end{aligned} \tag{4.11}$$

It is not difficult to see that the B-model action is BRST exact up to metric independent terms,

$$\begin{aligned} S_B = \int_\Sigma d^2z\, \Big\{ & \delta\Big(- \frac{i}{2}g_{i\bar{j}}\rho^i_z\partial_{\bar{z}}\overline{\sigma}^{\bar{j}} - \frac{i}{2}g_{i\bar{j}}\rho^i_{\bar{z}}\partial_z\overline{\sigma}^{\bar{j}}\Big) \\ & - \frac{i}{2}\overline{\theta}_i D_{\bar{z}}\rho^i_z + \frac{i}{2}\overline{\theta}_i D_z\rho^i_{\bar{z}} - \frac{1}{4}R^l_{i\bar{j}k}\overline{\theta}_l\overline{\eta}^{\bar{j}}\rho^i_{\bar{z}}\rho^k_z\Big\}. \end{aligned}$$

Local observables in the B-model are elements in the BRST cohomology on smooth functionals of scalar fields, σ^i, $\overline{\sigma}^{\bar{i}}$, $\overline{\eta}^{\bar{i}}$, and $\overline{\theta}_i$. It is not difficult to see that the BRST cohomology is isomorphic to the Dolbeault cohomology on antiholomorphic forms valued in exterior powers of the holomorphic tangent bundle, with $\overline{\eta}$ mapping to (0,1)-forms on X, $\overline{\theta}$ mapping to holomorphic vectors fields on X, and Q_B mapping to the Dolbeault operator,

$$\overline{\eta}^{\bar{i}} \longleftrightarrow d\overline{z}^{\bar{i}},$$
$$\overline{\theta}^{\bar{i}} \longleftrightarrow \frac{\partial}{\partial z^i},$$
$$Q_B \longleftrightarrow \overline{\partial}.$$

Line observables can be realized as a one-dimensional theory coupled to the B-model [1], [18]. Assuming that the line operators preserves the $U(1)_V$ vector $\mathcal{R}$-symmetry, the state space V in the quantum mechanical theory has a $\mathbb{Z}$-grading. The coupling between the B-model and quantum mechanical theory determines a smooth $\mathbb{Z}$-graded vector bundle E on X with fiber V. The generator of BRST symmetry determines an antiholomorphic, flat superconnection $\overline{\nabla} = \overline{\partial} + A$ on E,

$$\overline{\nabla}^2 s = 0,$$

for all $s \in \Gamma(E)$. Conversely, we can construct a line observable given an antiholomorphic, flat superconnection $\overline{\nabla} = \overline{\partial} + A$ on a $\mathbb{Z}$-graded smooth vector bundle E over X,

$$\mathcal{O}(\gamma) = \operatorname{Tr} \mathcal{P} \exp \left\{ \int_\gamma i \frac{\partial A}{\partial \overline{\eta}^{\bar{i}}} d\overline{\sigma}^{\bar{i}} - \frac{1}{2} \frac{\partial A}{\partial \sigma^i} \rho^i \right\}.$$

Flatness of the superconnection ensures that this line observable lives in the BRST cohomology.

As we discussed in Chapter 3, line observables of a 2d TQFT form a category. A theorem by Block [5] identifies antiholomorphic, flat superconnections on smooth $\mathbb{Z}$-graded vector bundle over X with objects in the bounded, derived category of coherent sheaves on X, $\mathcal{D}^b(X)$. There is considerable evidence that $\mathcal{D}^b(X)$ is the full category of line operators in the B-model [9].

4.1.3 Mirror Symmetry

For many Calabi-Yau manifolds X, there exists a "mirror" Calabi-Yau manifold $\widetilde{X}$ such that the $\mathcal{N} = (2,2)$ nonlinear σ-model with target manifold X is dual to the $\mathcal{N} = (2,2)$ nonlinear σ-model with target manifold $\widetilde{X}$, [24]. The mirror map acts as an involution on the (super)charges,

$$A \longleftrightarrow V,$$
$$Q_- \longleftrightarrow \overline{Q}_-.$$

This mirror map restricts to a duality between the A-model on X ($\widetilde{X}$) and the B-model on $\widetilde{X}$ (X). The isomorphism of local observables implies that the Hodge number on X and $\widetilde{X}$ have the following relation,

$$h^{p,q}(X) = h^{n-p,q}(\widetilde{X}).$$

The isomorphism of line observables implies Kontsevich's homological mirror symmetry conjecture [23] (with a suitable generalization of the Fukaya category),

$$\mathcal{D}^b(X) = \mathcal{F}^0(\widetilde{X}),$$
$$\mathcal{D}^b(\widetilde{X}) = \mathcal{F}^0(X).$$

4.2 $\mathcal{N} = (2,2)$ SYM Theory

In this section, we review $\mathcal{N} = 2$ supersymmetric Yang-Mills theories. This theory consists of an adjoint complex scalar field ϕ, four adjoint Weyl spinors $\lambda_\pm$, $\overline{\lambda}_\pm$, and an gauge field A_μ with the following action,

$$\begin{aligned} S = \frac{1}{g^2}\int_{\mathbb{R}^2} d^2x \, \mathrm{Tr}\Big(& F_{12}^2 - i\overline{\lambda}_-\big(D_1 + iD_2\big)\lambda_- + i\overline{\lambda}_+\big(D_1 - iD_2\big)\lambda_+ \\ & + D^\mu\overline{\phi}D_\mu\phi + \overline{\lambda}_-\big[\phi,\lambda_+\big] + \overline{\lambda}_+\big[\overline{\phi},\lambda_-\big] + \frac{1}{4}\big[\phi,\overline{\phi}\big]^2\Big) \end{aligned} \tag{4.12}$$

where D is the gauge covariant derivative,

$$D_\mu\Phi = \partial_\mu\Phi + i\big[A_\mu,\Phi\big].$$

It is not difficult to see that the action respects the following supersymmetry transformations,

$$\begin{aligned} \delta A_1 &= \frac{i}{2}\overline{\xi}_+\lambda_+ + \frac{i}{2}\xi_+\overline{\lambda}_+ - \frac{i}{2}\overline{\xi}_-\lambda_- - \frac{i}{2}\xi_-\overline{\lambda}_-, \\ \delta A_2 &= \frac{1}{2}\overline{\xi}_+\lambda_+ + \frac{1}{2}\xi_+\overline{\lambda}_+ + \frac{1}{2}\overline{\xi}_-\lambda_- + \frac{1}{2}\xi_-\overline{\lambda}_-, \\ \delta\lambda_- &= -i\xi_-F_{12} - \xi_+\big(D_1 - iD_2\big)\phi - \frac{i}{2}\xi_-\big[\phi,\overline{\phi}\big], \\ \delta\lambda_+ &= i\xi_+F_{12} + \xi_-\big(D_1 + iD_2\big)\overline{\phi} + \frac{i}{2}\xi_+\big[\phi,\overline{\phi}\big], \\ \delta\overline{\lambda}_- &= -i\overline{\xi}_-F_{12} - \overline{\xi}_+\big(D_1 - iD_2\big)\overline{\phi} + \frac{i}{2}\overline{\xi}_-\big[\phi,\overline{\phi}\big], \\ \delta\overline{\lambda}_+ &= i\overline{\xi}_+F_{12} + \overline{\xi}_-\big(D_1 + iD_2\big)\phi - \frac{i}{2}\overline{\xi}_+\big[\phi,\overline{\phi}\big], \\ \delta\phi &= -i\overline{\xi}_+\lambda_- - i\xi_-\overline{\lambda}_+, \\ \delta\overline{\phi} &= -i\xi_+\overline{\lambda}_- - i\overline{\xi}_-\lambda_+. \end{aligned} \tag{4.13}$$

This theory possesses $U(1)_E$ rotational symmetry, $U(1)_V$ vector $\mathcal{R}$-symmetry, and $U(1)_A$ axial $\mathcal{R}$-symmetry. With respect to these symmetries, the supercharges transform as shown in Table 4.2 and the fields transforms as shown in the table below,

Field	$U(1)_E$	$U(1)_V$	$U(1)_A$
A_z	2	0	0
$A_{\bar{z}}$	-2	0	0
λ_-	1	1	1
λ_+	-1	1	-1
$\overline{\lambda}_-$	1	-1	-1
$\overline{\lambda}_+$	-1	-1	1
ϕ	0	0	2
$\overline{\phi}$	0	0	-2

,

Table 4.5: Charges of fields in $\mathcal{N} = (2, 2)$ SYM theory

4.2.1 A-Type Gauge Theory

The A-type gauge theory is constructed by twisting the $U(1)_E$ rotational symmetry by the $U(1)_V$ vector $\mathcal{R}$-symmetry of the $\mathcal{N} = (2, 2)$ SYM theory (see Table 4.5 for the charges of field in the $\mathcal{N} = (2, 2)$ SYM theory),

$$E \rightarrow E + V.$$

Field	$U(1)_E$	$U(1)_A$
A_z	2	0
$A_{\bar{z}}$	-2	0
ψ_z	2	1
η	0	-1
$\chi_{z\bar{z}}$	0	-1
$\psi_{\bar{z}}$	-2	1
ϕ	0	2
$\overline{\phi}$	0	-2

,

Table 4.6: Fields in A-type gauge theory

This theory consists of an adjoint complex scalar field ϕ, an adjoint fermionic scalar field η, an adjoint fermionic 1-form ψ, an adjoint fermionic 2-form χ, and a gauge field A. The BRST charge is

$$Q_A = Q_- + \overline{Q}_+,$$

which is a scalar after twisting (see Table 4.2 for the charges of the $\mathcal{N} = (2, 2)$ supercharges). The

BRST variations follow from the corresponding supersymmetry transformations (4.13),

$$\begin{aligned} \delta A &= \xi\psi, \\ \delta\eta &= \xi\left[\phi,\overline{\phi}\right], \\ \delta\psi &= i\xi D\phi, \\ \delta\chi &= \xi F, \\ \delta\phi &= 0, \\ \delta\overline{\phi} &= \xi\eta. \end{aligned}$$

Notice that the variations are nilpotent (up to gauge transformations) except for χ,

$$\begin{aligned} \delta^2 A &= i\xi_1\xi_2 D\phi, \\ \delta^2\eta &= \xi_1\xi_2\left[\phi,\eta\right], \\ \delta^2\psi &= \xi_1\xi_2\left[\phi,\psi\right], \\ \delta^2\chi &= \xi_1\xi_2 D\psi, \\ \delta^2\phi &= 0, \\ \delta^2\overline{\phi} &= \xi_1\xi_2\left[\phi,\overline{\phi}\right]. \end{aligned}$$

It will be convenient to introduce an auxiliary bosonic 2-form P in the adjoint representation. We require that P satisfy the following on-shell constraint,

$$P = F, \tag{4.14}$$

so that we can write the BRST variations as

$$\begin{aligned} \delta A &= \xi\psi, \\ \delta\eta &= \xi\left[\phi,\overline{\phi}\right], \\ \delta\psi &= i\xi D\phi, \\ \delta\chi &= \xi P, \\ \delta\phi &= 0, \\ \delta\overline{\phi} &= \xi\eta, \\ \delta P &= \xi\left[\phi,\chi\right]. \end{aligned} \tag{4.15}$$

It is not difficult to construct an action equivalent to original A-model action that enforces the on-shell constraints on the auxiliary fields (4.14) and respects the BRST symmetry (4.19),

$$S_A = \frac{1}{g^2}\int_\Sigma \Big\{Q_A, i\psi\wedge\star D\bar{\phi} + 4\chi\wedge\star\big(P - 2F - \star\frac{1}{2}[\phi,\bar{\phi}]\big)\Big\}. \tag{4.16}$$

We now construct the category of branes for the A-type gauge theory with gauge group $U(1)$ [21]. Neumann boundary conditions require that $\star F$ and $\star d\phi$ vanish along the boundary, while A and ϕ are free along the boundary. BRST-invariance requires $\star\chi$ and the restriction of $\star\psi$ vanish on the boundary, while η and the restriction of ψ remain unconstrained. The algebra of BRST-invariant observables on the Neumann boundary is spanned by powers of ϕ, the algebra $\mathcal{O}$ of holomorphic functions on $\mathbb{C}$. One can construct a more general boundary condition by placing additional degrees of freedom on the boundary which live in a vector space V graded by the $U(1)_A$ charge. The BRST operator gives rise to a degree-1 differential $T : V \to V$ which may depend polynomially on ϕ. Thus we may attach a brane to any free DG-module $M = (V\otimes\mathcal{O}, T)$ over the graded algebra $\mathcal{O}$. The space of morphisms between any two such branes $M_1 = (V_1\otimes\mathcal{O}, T_1)$ and $M_2 = (V_2\otimes\mathcal{O}, T_2)$ is the cohomology of the complex $\mathrm{Hom}_{\mathcal{O}}(M_1, M_2)$, which agrees with the space of morphisms in the category $D^b(\mathbb{C})$.

In the A-type 2d gauge theory one may also consider the Dirichlet boundary condition which sets $\phi = 0$ on the boundary and requires the restriction of the gauge field to be trivial. One might guess that it corresponds to the skyscraper sheaf at the origin of $\mathbb{C}$, and indeed one can verify that the space of morphisms from any of the branes considered above to the Dirichlet brane agrees with the space of morphisms from the corresponding complex of vector bundles on $\mathbb{C}$ to the skyscraper sheaf.

Another way to approach the problem is construct an isomorphism between the A-type gauge theory and the B-model with target $\mathbb{C}$. From the physical viewpoint, an isomorphism between two-dimensional TQFTs $\mathbb{X}$ and $\mathbb{Y}$ is an invertible topological defect line A between them. In the present case, there is a unique candidate for such a defect line. Recall that a B-model with target $\mathbb{C}$ has a bosonic scalar σ, a fermionic 1-form ρ, and fermionic 0-forms θ and ξ. The BRST transformations read

$$\begin{aligned}
\delta\phi &= 0,\\
\delta\bar{\phi} &= \eta,\\
\delta\eta &= 0,\\
\delta\theta &= 0,\\
\delta\rho &= d\phi.
\end{aligned}$$

The field σ has ghost number 2, the fields ρ has ghost number 1, and the fields η and θ have ghost number -1. The action of the B-model is

$$S = -\frac{1}{2}\delta \int_{M_2} \rho \wedge \star d\bar{\phi} + \int_{M_2} \theta \wedge d\rho.$$

Obviously, the ghost-number 2 bosons σ and ϕ must be identified on the defect line, up to a numerical factor which can be read of the action. Similarly, the fermionic 1-forms ψ and ρ must be identified, as well as the fermionic 0-forms β and η. Finally, one must identify $\star\chi$ and θ. BRST invariance then requires $\star F$ to vanish on the boundary, which means that the gauge field obeys the Neumann boundary condition.

Note that this defect line is essentially the trivial defect line for the fermionic fields and σ. Since the zero-energy sector of the bosonic $U(1)$ gauge theory is trivial, the invertibility of the defect line is almost obvious. Let us show this more formally. First, consider two parallel defect lines with a sliver of the A-type 2d gauge theory between them. The sliver has the shape $\mathbb{R} \times I$, where $\mathbb{R}$ parameterizes the direction along the defect lines. The statement that the product of two defect lines is the trivial defect line in the B-model is equivalent to the statement that the $U(1)$ gauge theory on an interval with Neumann boundary conditions on both ends has a unique ground state. This is obviously true, because the space-like component of A in the sliver can be gauged away by a time-independent gauge transformation, and therefore the physical phase space of the $U(1)$ gauge theory on an interval is a point.

Second, consider the opposite situation where a sliver of the B-model is sandwiched between two defect lines. We would like to show that this is equivalent to the trivial defect line in the A-type 2d gauge theory. The sliver has the shape $S^1 \times I$. For simplicity we will assume that the worldsheet with a sliver removed consists of two connected components. Each component is an oriented manifold with a boundary isomorphic to S^1, and the path-integral of the A-type 2d gauge theory defines a vector in the Hilbert space V corresponding to S^1. Any topological defect line in the A-type gauge theory defines an element in $V^* \otimes V \simeq \mathrm{End}(V)$. We would like to show that the B-model sliver corresponds to the identity element in $V^* \otimes V$. First we note that V can be identified with the tensor product of the Hilbert space of the zero-energy gauge degrees of freedom and the Hilbert space of the zero-energy degrees of freedom of σ and the fermions. As mentioned above, the defect line separating the A-type 2d gauge theory and the B-model acts as the trivial defect line on σ and the fermions, so in this sector the statement is obvious. As for the gauge sector, the corresponding Hilbert space of zero-energy states is one-dimensional, so the B-model sliver is proportional to the identity operator. The argument of the preceding paragraph shows that its trace is one, so the sliver must be the identity operator.

4.2.2 B-Type Gauge Theory

The B-type gauge theory is constructed by twisting the $U(1)_E$ rotational symmetry by the $U(1)_A$ axial $\mathcal{R}$-symmetry of the $\mathcal{N} = (2,2)$ SYM theory [21] (see Table 4.5 for the charges of the fields in the $\mathcal{N} = (2,2)$ SYM theory),

$$E \to E + A.$$

Field	$U(1)_E$	$U(1)_V$
A_z	2	0
$A_{\bar{z}}$	-2	0
ψ_z	2	1
$\psi_{\bar{z}}$	-2	1
β	0	-1
$\zeta_{z\bar{z}}$	0	-1
ϕ_z	2	0
$\overline{\phi}_{\bar{z}}$	-2	0

Table 4.7: Fields in B-type gauge theory

This theory consists of an adjoint bosonic 1-form ϕ, an adjoint fermionic scalar β, an adjoint fermionic 1-form ψ, an adjoint fermionic 2-form ζ, and a gauge field A. We construct the action for the B-type gauge theory by writing the action of the $\mathcal{N} = (2,2)$ SYM theory (4.12) covariantly in terms of the twisted fields,

$$S = -\frac{1}{g^2}\int_\Sigma \mathrm{Tr}\left(\mathcal{F}\wedge\star\bar{\mathcal{F}} + D\star\phi\wedge\star D\star\phi - 2i\zeta\wedge\star\mathcal{D}\psi - 2\beta\wedge\mathcal{D}\star\psi\right), \tag{4.17}$$

where D is the covariant derivative with respect to the connection A, $\mathcal{D}$ is the covariant derivative with respect to the complexified connection $\mathcal{A} = A + i\phi$, and $\mathcal{F}$ is the curvature of the complexified connection. The BRST charge is

$$Q_B = \overline{Q}_- + \overline{Q}_+,$$

which is a scalar after twisting (see Table 4.2 for the charges of the $\mathcal{N} = (2,2)$ supercharges). The BRST variations follow from the corresponding supersymmetry transformations (4.13),

$$\begin{aligned}
\delta A &= \epsilon\psi,\\
\delta\psi &= 0,\\
\delta\beta &= i\star\mathcal{D}\star\phi,\\
\delta\zeta &= -i\mathcal{F},\\
\delta\phi &= i\epsilon\psi.
\end{aligned}$$

Notice that the β variations are only nilpotent on-shell,

$$\delta^2\beta = - \star \mathcal{D} \star \psi.$$

It will be convenient to introduce an auxiliary bosonic scalar P which satisfies the on-shell constraint

$$P = \star D \star \phi, \tag{4.18}$$

so that we can write the BRST variations as

$$\begin{aligned} \delta A &= \epsilon\psi, \\ \delta\psi &= 0, \\ \delta\beta &= iP, \\ \delta\zeta &= -i\mathcal{F}, \\ \delta\phi &= i\epsilon\psi, \\ \delta P &= 0. \end{aligned} \tag{4.19}$$

It is not difficult to construct an action equivalent to the original B-type gauge theory action (4.17) that enforces the on-shell constraint (4.18) and respect the BRST symmetry (4.19)

$$S = -\frac{1}{g^2}\int_\Sigma \Big\{ Q_B, \mathrm{Tr}\left(i\zeta \wedge \star\bar{\mathcal{F}} + i\beta \wedge \star\big(P - 2 \star D \star \phi\big)\right)\Big\}.$$

Notice that this action is BRST exact up to a topological term.

Local observables are typically BRST-invariant, gauge-invariant scalar functions of fields. There are no nontrivial local observables of this kind in the B-type gauge theory. However, there are nontrivial BRST-invariant local disorder operators which are defined by allowing certain singularities in the fields. For example, one can require the connection $\mathcal{A}$ to have a nontrivial holonomy around the insertion point. Such local operators are analogous to Gukov-Witten surface operators in 4d gauge theory. More systematically, to determine what kind of local operators are allowed one can reduce the 2d gauge theory on a circle and study the space of the states of the resulting 1d TQFT. In the present case, this 1d TQFT is a gauged sigma-model with target $G_{\mathbb{C}}$. From the 2d viewpoint, the target space parameterizes the holonomy of $\mathcal{A}$. BRST-invariant wave-functions are holomorphic functions on $G_{\mathbb{C}}$ invariant with respect to conjugation, characters of $G_{\mathbb{C}}$. More generally, one may consider non-normalizable wavefunctions, such as delta-functions supported on closed $G_{\mathbb{C}}$-invariant complex submanifolds of $G_{\mathbb{C}}$. For example, the identity operator can be thought of as a delta-function supported at the identity element, while Gukov-Witten-type local operators are delta-functions supported on closed conjugacy classes in $G_{\mathbb{C}}$.

There are also BRST-invariant and gauge-invariant line observables, the most obvious of which are Wilson line operators for the complex BRST-invariant connection $\mathcal{A}$. To define them, one needs to pick a finite-dimensional graded representation V of G and consider the holonomy of $\mathcal{A}$ in the representation V.

The category of branes for this 2d TQFT is the category of finite-dimensional graded representations of G [21]. To see this, consider the Neumann boundary condition for the gauge field, that is, leave the restriction of $\mathcal{A}$ to the boundary free and require the restriction of $\star\phi$ to vanish. BRST-invariance then requires ζ and the restriction of $\star\psi$ to vanish on the boundary. Since the gauge field $\mathcal{A}$ on the boundary is unconstrained and BRST-invariant, we may couple to it an arbitrary finite-dimensional graded representation V of G. That is, we may include into the path-integral the holonomy of $\mathcal{A}$ in the representation V. Thus boundary conditions are naturally labeled by representations of G. Given any two irreducible representations V_1 and V_2 one can form a junction between them only if V_1 and V_2 are isomorphic (because there are no nontrivial BRST-invariant local operators on the Neumann boundary). Further, if $V_1 \simeq V_2$, the space of morphisms between them is $\mathrm{Hom}_G(V_1, V_2)$ (for the same reason).

4.3 Gauged B-Model

We now couple the B-type gauge theory to a B-model [21]. Let X be a Calabi-Yau manifold that admits a G-action preserving the Calabi-Yau structure. The infinitesimal action of G is described by a holomorphic vector field V^I with values in $\mathfrak{g}^*$ (the dual of the Lie algebra of G). Consider the following modification to the BRST transformations of the σ-model fields,

$$\begin{aligned}
&\delta\sigma^I = 0,\\
&\delta\bar{\sigma}^{\bar{I}} = \xi\bar{\eta}^{\bar{I}},\\
&\delta\bar{\eta}^{\bar{I}} = 0,\\
&\delta\bar{\theta}_I = 0,\\
&\delta\rho^I = d\sigma^I + V^I(\mathcal{A}) = \mathcal{D}\sigma^I.
\end{aligned}$$

This is a covariantized version of the usual B-model BRST transformations. The appearance of the covariant derivative $\mathcal{D}\sigma^I$ means that σ is now interpreted as a section of a fiber bundle over M_2 with typical fiber X which is associated to a principal G-bundle $\mathcal{P}$ over M_2. Since the connection $\mathcal{A}$ is BRST-invariant, these BRST transformations still satisfy $\delta^2 = 0$.

To construct a BRST-invariant action we take the usual action of the B-model and covariantize all derivatives. The covariantized action is not BRST-invariant, but this can be corrected for by adding a new term proportional to $\theta_I V^I(\zeta)$, where ζ is the fermionic $\mathrm{Ad}(\mathcal{P})$-valued 2-form which is part of the B-type 2d gauge theory. The full matter action is

$$S = \int_{M_2} \delta\left(g_{I\bar{J}}\rho^I \wedge \star\bar{\mathcal{D}}\sigma^{\bar{J}}\right) + \int_{M_2}\left(-i\theta_I V^I(\zeta) + \theta_I \mathcal{D}\rho^I + \frac{1}{2}R^I_{JK\bar{L}}\theta_I\rho^J\rho^K\eta^{\bar{L}}\right).$$

Here $g_{I\bar{J}}$ is the Kähler metric, R is its curvature tensor, and the covariant derivative of ρ includes both the Levi-Civita connection and the gauge connection:

$$\mathcal{D}\rho^I = d\rho^I + \Gamma^I_{JK}d\sigma^J\rho^K + \nabla_J V^I(\mathcal{A})\rho^J, \quad \nabla_J V^I = \partial_J V^I + \Gamma^I_{JK}V^K.$$

The covariant derivative $\bar{\mathcal{D}}\sigma^{\bar{J}}$ is defined so as to make the bosonic part of the action positive-definite:

$$\bar{\mathcal{D}}\sigma^{\bar{J}} = d\sigma^{\bar{J}} - V^{\bar{J}}(\bar{\mathcal{A}}), \quad \bar{\mathcal{A}} = A - i\phi = -\mathcal{A}^\dagger.$$

Since the category of branes for the B-model with target X is $D^b(Coh(X))$, a natural guess for the category of branes for the gauged B-model is $D^b_{G_{\mathbb{C}}}(Coh(X))$. We will now describe a construction of the boundary action corresponding to an equivariant complex of holomorphic vector bundles on X. Let E be a graded complex vector bundle over X with a holomorphic structure $\bar{\partial}^E : E \to E\otimes\Omega^{0,\bullet}(X)$, $(\bar{\partial}^E)^2 = 0$, and a holomorphic degree-1 endomorphism $T : E \to E$, $\bar{\partial}^E T = 0$ satisfying $T^2 = 0$. To write down a concrete boundary action we will assume that we are also given a Hermitian metric on each graded component of E, so that $\bar{\partial}^E$ gives rise to a connection ∇^E on E. We will denote the corresponding connection 1-form by ω and its curvature by F^E. We assume that we are given a lift of the G-action on X to a G-action on the total space of E which is fiberwise-linear and compatible with $\bar{\partial}^E$, T, and the Hermitian metric. Infinitesimally, the Lie algebra $\mathfrak{g}$ acts on a section s of E as follows:

$$(f, s) \mapsto f(s) = V^I(f)\nabla^E_I s + V^{\bar{I}}(f)\nabla^E_{\bar{I}} s + R(f)s, \quad f \in \mathfrak{g}.$$

Here $\nabla^E = d + \omega$, and R is a degree-0 bundle morphism $R : E \to E \otimes \mathfrak{g}^*$. The condition that the G-action commutes with ∇^E implies

$$\nabla^E R = \iota_V F^E.$$

The condition that the G-action commutes with T implies

$$V^I \nabla^E_I T + [R, T] = 0.$$

Consider now the following field-dependent connection 1-form on the pull-back bundle $\sigma^* E$:

$$\mathcal{N} = \omega_I d\sigma^I + \omega_{\bar{I}} d\sigma^{\bar{I}} - R(\mathcal{A}) + \rho^I \eta^{\bar{J}} F^E_{I\bar{J}} + \rho^I \nabla^E_I T.$$

With some work one can check that its BRST variation satisfies

$$\delta\mathcal{N} = d(\omega_{\bar{I}}\eta^{\bar{I}} + T) + [\mathcal{N}, \omega_{\bar{I}}\eta^{\bar{I}} + T].$$

Therefore the supertrace of its holonomy is BRST-invariant and can be used as a boundary weight factor in the path-integral associated. By definition the boundary action is minus the logarithm of the boundary weight factor.

Let us consider a ghost-number zero boundary observable $\mathcal{O}$ in the presence of a such a weight factor. It is an element of $\mathrm{End}(E)$ depending on the fields σ, η and of total degree zero. More invariantly, we may think of it as a section of $\mathrm{End}(E) \otimes \Omega^{0,\bullet}(X)$. The BRST-variation of the boundary weight factor in the presence of $\mathcal{O}$ is proportional to

$$\eta^{\bar{I}}\nabla^{E}_{\bar{I}}\mathcal{O} + [T, \mathcal{O}].$$

Hence BRST-invariant boundary observables are sections of $\mathrm{End}(E)\otimes\Omega^{0,\bullet}(X)$ which are annihilated by $\bar{\partial}^E$ and commute with T. Further, a BRST-invariant $\mathcal{O}$ it is gauge-invariant iff it satisfies

$$V^I\nabla^E_I\mathcal{O} + [R, \mathcal{O}] = 0$$

Together these conditions mean that $\mathcal{O}$ represents an endomorphism of the equivariant complex $(E, \bar{\partial}^E, T)$ regarded as an object of $D^b_{G_{\mathbb{C}}}(Coh(X))$. It is also easy to see that such an observable $\mathcal{O}$ is a BRST-variation of a gauge-invariant observable iff it is homotopic to zero. In some cases this implies that the category of branes in the gauged B-model of the kind we have constructed is equivalent to $D^b_{G_{\mathbb{C}}}(Coh(X))$. This happens if any G-equivariant coherent sheaf on X has a G-equivariant resolution by G-equivariant holomorphic vector bundles. Such an X is said to have a G-resolution property. An example of such X is $\mathbb{C}^n$ with a linear action of G, or more generally a smooth affine variety with an affine action of G. Note that for a general complex manifold X the resolution property may fail even if G is trivial. But for trivial G the cure is known: one has to replace complexes of holomorphic vector bundles with more general DG-modules over the Dolbeault DG-algebra of X [5]. These more general DG-modules also arise naturally from the physical viewpoint [1], [18]. We expect that for any complex Lie group G with a complex-analytic action on X a G-equivariant coherent sheaf on X has a G-equivariant resolution by these more general DG-modules. This would imply that the category of B-branes for the gauged B-model is equivalent to $D^b_{G_{\mathbb{C}}}(Coh(X))$ [21].

Chapter 5

Three-Dimensional TQFTs

5.1 A-Model

We begin this chapter by constructing a three-dimensional analog of the A-model [22]. The bosonic field σ is a map from spacetime M into a Riemannian manifold X,

$$\sigma \in \mathrm{Map}\big(M, X\big).$$

The bosonic field τ is a 1-form on M valued in the pullback of the tangent bundle TX,

$$\tau \in \Gamma\big(\sigma^* TX \otimes \Omega^1\big).$$

The fermionic fields η and β are scalars on M valued in the pullback of the tangent bundle TX,

$$\eta \in \Gamma\big(\sigma^* TX\big),$$
$$\beta \in \Gamma\big(\sigma^* TX\big).$$

The fermionic fields ψ and χ are 1-forms on M valued in the pullback of the tangent bundle TX,

$$\psi \in \Gamma\big(\sigma^* TX \otimes \Omega^1\big),$$
$$\chi \in \Gamma\big(\sigma^* TX \otimes \Omega^1\big).$$

It is convenient to introduce an auxiliary a scalar P and auxiliary 1-form $\tilde{P}$ valued in the pullback of the tangent bundle TX,

$$P \in \Gamma\big(\sigma^* TX\big),$$
$$\tilde{P} \in \Gamma\big(\sigma^* TX \otimes \Omega^1\big).$$

The BRST variations of these fields are

$$\begin{aligned}
\delta\sigma^i &= \xi\eta^i, \\
\delta\tau^i &= \xi\big(\psi^i - \Gamma^i_{jk}\eta^j\tau^k\big), \\
\delta\eta^i &= 0, \\
\delta\psi^i &= \xi\big(\frac{1}{2}R^i_{jkl}\tau^j\eta^k\eta^l - \Gamma^i_{jk}\eta^j\psi^k\big), \\
\delta\beta^i &= \xi\big(P^i - \Gamma^i_{jk}\eta^j\beta^k\big), \\
\delta\chi^i &= \xi\big(\tilde{P}^i - \Gamma^i_{jk}\eta^j\chi^k\big), \\
\delta P^i &= \xi\big(\frac{1}{2}R^i_{jkl}\beta^j\eta^k\eta^l - \Gamma^i_{jk}\eta^j P^k\big), \\
\delta\tilde{P}^i &= \xi\big(\frac{1}{2}R^i_{jkl}\chi^j\eta^k\eta^l - \Gamma^i_{jk}\eta^j\tilde{P}^k\big),
\end{aligned}$$

and the action for the A-model is

$$\tilde{S} = -\int_M \left\{Q, g_{ij}\chi^i \wedge \star\left(\tilde{P}^j - 2\left(d\sigma^j - \star D\tau^j\right)\right) + g_{ij}\beta^i \wedge \star\left(P^j - 2\star D\star\tau^j\right)\right\},$$

where g_{ij} is the Riemannian metric on X, $\star$ denotes the Hodge star on M, and

$$\begin{aligned}
D\tau^i &= d\tau^i + \Gamma^i_{jk}d\sigma^j \wedge \tau^k, \\
D\star\tau^i &= d\star\tau^i + \Gamma^i_{jk}d\sigma^j \wedge \star\tau^k,
\end{aligned}$$

with Γ^i_{jk} the Levi-Civita connection on X. In addition to BRST symmetry, there is a $U(1)$ ghost number symmetry where η, ψ have charge 1, β, χ have charge -1, and the bosonic fields are uncharged.

Local observables in the A-model are elements in the BRST cohomology on smooth functionals of scalar fields, σ and η. It is not difficult to see that the BRST cohomology is isomorphic to the de Rham cohomology of X, with η mapping to 1-forms on X and Q mapping to the exterior derivative,

$$\begin{aligned}
\eta^i &\longleftrightarrow dx^i, \\
Q &\longleftrightarrow d.
\end{aligned}$$

This theory admits line observables as well. The most obvious are obtained by picking a vector bundle on non-simply-connected Riemannian manifolds X with a flat connection and considering the holonomy of the pull-back of this connection via the map σ. We refer to such line operators as Wilson lines.

Finally, we consider boundary conditions of the A-model. The most obvious boundary condition is to require the restriction of τ to the boundary to vanish and to impose the free boundary condition on σ and the normal component of τ_3. If the boundary is given by the equation $x^3 = 0$ and the metric near the boundary is taken to be Euclidean, these boundary conditions read

$$\tau_1^i = \tau_2^i = 0, \quad \partial_3 \tau_3^i = \partial_3 \sigma^i = 0.$$

These conditions on bosons are compatible with BPS equations and therefore are a candidate for a BRST-invariant boundary condition. The conditions on fermions are then uniquely determined: on the boundary we must have

$$\psi_1^i = \psi_2^i = 0, \quad \beta = \chi_3^i = 0,$$

with all other fermions unconstrained. We will call this the N boundary, to indicate that σ satisfies the Neumann condition.

A complementary boundary condition is to require σ to map ∂M to a particular point on X, the Dirichlet boundary condition. BRST invariance uniquely determines the boundary conditions on all other fields. Namely, we must have

$$\tau_3^i = 0, \quad \partial_3 \tau_1^i = \partial_3 \tau_2^i = 0, \quad \eta^i = \psi_3^i = 0, \quad \chi_1^i = \chi_2^i = 0.$$

We will call this the D boundary, to indicate that σ satisfies the Dirichlet condition.

We may also consider boundary conditions intermediate between N and D conditions. Let us pick a closed submanifold $Y \subset X$ and require σ to map ∂M to Y. We also impose the Neumann condition $\partial_3 \sigma^i = 0$ on the components of σ normal to Y. BRST-invariance then uniquely determines the boundary conditions for all other fields. In particular, the components of τ_1 and τ_2 normal to Y and components of τ_3 tangent to Y satisfy the Neumann condition, while the components of τ_1 and τ_2 tangent to Y and components of τ_3 normal to Y satisfy the Dirichlet condition. Thus we get one boundary conditions for each submanifold Y of X.

Boundary conditions for the 2d A-model can be deformed by a flat abelian gauge field. Similar possibility exists in 3d: one may add to the action a boundary term of the form

$$i \int_{\partial M} \sigma^* B, \tag{5.1}$$

where B is a closed 2-form on the submanifold Y. This is in fact the most general deformation possible. To classify boundary deformations systematically, one considers a BRST-invariant boundary observable $\mathcal{O}$ with ghost number two. A deformation of the action can be obtained by integrating over ∂M the descendant $\mathcal{O}^{(2)}$, which is a 2-form of ghost number zero satisfying

$$\delta \mathcal{O}^{(2)} = d\mathcal{O}^{(1)}, \quad \delta \mathcal{O}^{(1)} = d\mathcal{O}.$$

In our case boundary observables are BRST-invariant functions of σ and η. Since σ on the boundary lies in Y and η is tangent to Y, one may identify the space of boundary observables with closed differential forms on Y. Ghost-number two observables are precisely closed 2-forms on Y, and the corresponding deformation of the action is of the form (5.1).

In the 2d case one can consider adding boundary degrees of freedom, leading to flat vector bundles over Y (which is Lagrangian in the 2d case). Similarly, one can consider adding boundary degrees of freedom in the 3d A-model. Such boundary degrees of freedom are described by a 2d TQFT "fibered" over Y. For example, one may take a family of 2d A-models parameterized by points of Y. We leave the construction of the corresponding boundary action for future work.

As an example, consider the N condition. This condition sets the components of τ tangent to the boundary to zero. Thus the only bosonic fields in the effective 2d TQFT will be σ^i and $\tau_3^i = \lambda^i$. The BPS equations reduce to

$$d\sigma^i = *D\lambda^i.$$

This equation looks very much like a holomorphic instanton equation, suggesting that the effective 2d TQFT is an A-model. In fact, one can rewrite the above equation as a condition for a map $\Phi = (\sigma, \lambda)$ from the worldsheet Σ to TX to be pseudoholomorphic, provided we choose a suitable almost-complex structure on TX. This is the almost-complex structure defined by the condition that its $+i$ eigenspace is spanned by tangent vectors of the form

$$\frac{\partial}{\partial\sigma^i} - \Gamma^j_{ki}\lambda^k\frac{\partial}{\partial\lambda^j} + i\frac{\partial}{\partial\lambda^i}.$$

In matrix form the almost-complex structure is

$$J = \begin{pmatrix} \Gamma\lambda & 1 \\ -1-(\Gamma\lambda)^2 & -\Gamma\lambda \end{pmatrix},$$

where $\Gamma\lambda$ is a matrix with elements $\Gamma^i_{jk}\lambda^k$. This almost-complex structure is not integrable, in general.

We conclude that the effective 2d TQFT is the A-model with target $TX \simeq T^*X$. This means that the category of boundary line operators on the N boundary is equivalent to the Fukaya-Floer category of T^*X. Wilson lines correspond to the case when this Lagrangian submanifold is X itself embedded into T^*X as the zero section.

5.2 Rozansky-Witten Model

We review the Rozansky-Witten model in this section [30], which is a three-dimensional analog of the B-model. The bosonic fields σ are maps from spacetime M into a hyperkähler target manifold

X,

$$\sigma \in \mathrm{Map}\big(M, X\big).$$

The fermionic fields χ are a 1-form on M valued in the pullback of the holomorphic tangent bundle TX,

$$\chi \in \Gamma\big(\sigma^* TX \otimes \Omega^1\big),$$

and fermionic fields $\overline{\eta}$ are a scalar on M valued in the pullback of the antiholomorphic tangent bundle $\overline{TX}$,

$$\overline{\eta} \in \Gamma\big(\sigma^* \overline{TX}\big).$$

The action for the Rozanksy-Witten model is

$$\begin{aligned} S_{RW} = \int_M \Big(& g_{i\bar{j}} d\sigma^i \wedge \star d\overline{\sigma}^{\bar{j}} - g_{i\bar{j}} \chi^i \wedge \star D\overline{\eta}^{\bar{j}} + \frac{1}{2}\Omega_{ij}\chi^i \wedge D\chi^j \\ & + \frac{1}{6}\Omega_{ij} R^j_{kl\bar{m}} \chi^i \wedge \chi^k \wedge \chi^l \wedge \overline{\eta}^{\bar{m}} \Big), \end{aligned} \tag{5.2}$$

where $g_{i\bar{j}}$ is the Kähler metric on X, Ω_{ij} is the holomorphic symplectic form on X, $R^j_{kl\bar{m}}$ is the Riemannian curvature on X, and

$$\begin{aligned} D\overline{\eta}^{\bar{i}} &= d\overline{\eta}^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}} d\overline{\sigma}^{\bar{j}} \overline{\eta}^{\bar{k}}, \\ D\chi^i &= d\chi^i + \Gamma^i_{jk} d\sigma^j \wedge \chi^k, \end{aligned}$$

with Γ^i_{jk}, $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$ the Levi-Civita connection on X. It is not difficult to see that this action respects the following BRST transformations,

$$\begin{aligned} \delta\sigma^i &= 0, \\ \delta\overline{\sigma}^{\bar{i}} &= \xi\overline{\eta}^{\bar{i}}, \\ \delta\chi^i &= d\sigma^i, \\ \delta\overline{\eta}^{\bar{i}} &= 0. \end{aligned} \tag{5.3}$$

Notice that the action is BRST exact up to a metric independent term,

$$S_{RW} = \int_M \Big\{ Q, g_{i\bar{j}} \chi^i \wedge \star d\overline{\sigma}^{\bar{j}} \Big\} + \frac{1}{2}\Omega_{ij}\chi^i \wedge \nabla\chi^j + \frac{1}{6}\Omega_{ij} R^j_{kl\bar{m}} \chi^i \wedge \chi^k \wedge \chi^l \wedge \overline{\eta}^{\bar{m}} \Big).$$

Local observables in the Rozansky-Witten model are elements in the BRST cohomology on smooth functionals of scalar fields, σ^i, $\overline{\sigma}^{\bar{i}}$, and $\overline{\eta}^{\bar{i}}$. It is not difficult to see that the BRST cohomology is isomorphic to the Dolbeault cohomology on antiholomorphic forms, with $\overline{\eta}$ mapping to (0,1)-forms on X and Q mapping to the Dolbeault operator,

$$\overline{\eta}^{\bar{i}} \longleftrightarrow d\overline{z}^{\bar{i}},$$
$$Q \longleftrightarrow \overline{\partial}.$$

Line observables in the Rozansky-Witten model are in one-to-one correspondence with boundary condition of the two-dimensional theory obtained by compactification on a circle [18]. The compactification of the Rozansky-Witten model on a circle is simply the two-dimensional B-model, therefore the category of line observables is the bounded, derived category of coherent sheaves on X, $\mathcal{D}^b(X)$ [18].

Classical boundary conditions in the Rozansky-Witten model correspond to complex Lagrangian submanifolds of X [18]. Additional boundary conditions come from coupling a two-dimensional B-model to the boundary. This class of boundary conditions are labelled by a Calabi-Yau fibration over a complex Lagrangian submanifold. The complete 2-category of boundary conditions is discussed in detail in [18].

5.3 B-Type Gauge Theory

In this section we describe the B-type gauge theory in three dimensions, [3]. The theory consists of a gauge field A, an adjoint bosonic 1-form ϕ, an adjoint fermionic 2-form ζ, an adjoint fermionic 1-form λ, and two adjoint fermionic scalars ρ, $\tilde{\rho}$. Furthermore, it is convenient to introduce an auxiliary adjoint, bosonic scalar field P. The action for this theory is

$$\tilde{S} = -\frac{1}{2e^2}\int_{M_3} \mathrm{Tr}\Big(\mathcal{F}\wedge\star\bar{\mathcal{F}} - P\wedge\star\big(P - 2d_A^\star\phi\big) - 2i\zeta\wedge\star d_{\bar{\mathcal{A}}}\lambda - 2\tilde{\rho}\wedge\star d_{\mathcal{A}}^\star\lambda - 2e^2\rho\wedge d_{\mathcal{A}}\zeta\Big)$$

where $\mathcal{A}$, $\bar{\mathcal{A}}$ are the complexified connections $A \pm i\phi$ and $\mathcal{F}$, $\bar{\mathcal{F}}$ are the corresponding field strengths. Notice that there this theory possesses a $U(1)$ ghost number symmetry where ρ, λ have charge 1, ζ, $\tilde{\rho}$ have charge -1, and the bosonic fields are uncharged. The BRST variation of the fields are

$$\delta A = \xi\lambda,$$
$$\delta\phi = i\xi\lambda,$$
$$\delta\lambda = 0,$$
$$\delta\zeta = -i\xi\mathcal{F},$$
$$\delta\rho = 0,$$
$$\delta\tilde{\rho} = i\xi P,$$
$$\delta P = 0.$$

The action is BRST exact up to a metric independent term,

$$\tilde{S} = -\frac{1}{2e^2}\int_M \Big\{Q, \mathrm{Tr}\Big(i\zeta\wedge\star\bar{\mathcal{F}} + i\tilde{\rho}\wedge\star\big(P - 2d_A^\star\phi\big)\Big) + \int_M \mathrm{Tr}\big(\rho\wedge d_{\mathcal{A}}\zeta\big)\Big\}.$$

Local observables in this TQFT are gauge invariant functions of ρ, which correspond to elements in the exterior algebra $\Lambda^\bullet(\mathfrak{g})$ invariant with respect to the adjoint action. It is known that the algebra of G-invariant elements in $\Lambda^\bullet(\mathfrak{g})$ is isomorphic to the de Rham cohomology of the Lie group G, so the spectrum of local observables in simply the de Rham cohomology of the gauge group.

Line observables can be realized as a one-dimensional TQFT coupled to the B-type gauge theory. Assuming that the line observable preserves the $U(1)$ ghost number symmetry, the state space in the quantum mechanical theory has a $\mathbb{Z}$ grading. Endomorphisms of V are naturally graded as well. Let us denote the degree one endomorphism that generates the BRST symmetry in this theory as $T(\Phi)\in\mathrm{End}(V)$ and the degree zero endomorphisms that generate the gauge symmetry as $R_a(\Phi)\in\mathrm{End}(V)$, where Φ represents the fields in the topological gauge theory. Since ρ is the only BRST invariant scalar field in the TQFT, it is natural to assume that T and R_a are simply functions of ρ. Nilpotence of BRST generator implies that

$$T(\rho)^2 = 0.$$

Furthermore, since the gauge symmetry preserves grading, R_a is ρ-independent. Finally, since the gauge symmetry δ_g and BRST symmetry δ commute, T and R_a must satisfy the following relation,

$$\begin{aligned} 0 &= \big[\delta_g(f), \delta\big] \\ &= \big[f,\rho\big]^a \frac{\partial T}{\partial\rho^a} + \big[f^a R_a, T\big] \end{aligned}$$

where $f\in\mathfrak{g}$. To construct the line observable associated to the triple (V,T,R), we apply the descent procedure to T to get a connection 1-form on the graded vector bundle with fiber V. By definition, the descendant connection $\mathcal{N}$ is defined by the equation

$$\delta\mathcal{N} = dT + [\mathcal{N}, T].$$

Using these relations and the fermionic equations of motion, we find

$$\mathcal{N} = \frac{i}{2e^2}\star\bar{\mathcal{F}}^a\frac{\partial T}{\partial\rho^a} + \mathcal{A}^a R_a. \tag{5.4}$$

The supertrace of the holonomy of $\mathcal{N}$ along a curve γ in M is therefore a BRST invariant, gauge invariant loop operator in the topological gauge theory. The holonomy itself defines a line operator

[21].

Line operators in any 3d TQFT form a braided monoidal category. The subcategory formed by line operators described above is the G-equivariant derived category of DG-modules over the DG-algebra $\Lambda^\bullet(\mathfrak{g})$ (with zero differential). To see this, consider a local operator inserted at the junction of two Wilson lines corresponding to the triples (V_1, T_1, R_1) and (V_2, T_2, R_2). Since we are looking for BRST-invariant operators, one may assume that it is a function $\mathcal{O}$ of ρ valued in $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, or in other words an element of $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2 \otimes \Lambda^\bullet(\mathfrak{g}))$. The BRST-operator acts on $\mathcal{O}$ by

$$\delta\mathcal{O} = T_2\mathcal{O} \pm \mathcal{O}T_1$$

where the sign is plus or minus depending on whether the total degree of $\mathcal{O}$ is odd or even. Gauge transformations act on $\mathcal{O}$ in the obvious way and commute with the BRST operator. The space of morphisms between the line operators is the cohomology of δ on the G-invariant part of $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2 \otimes \Lambda^\bullet(\mathfrak{g}))$.

The monoidal structure is obvious on the classical level and given by the tensor product. There can be no quantum corrections to this result since the gauge coupling e^2 is an irrelevant parameter. The braiding is trivial for the same reason.

There exist yet more general line operators. To see this, we may use the dimensional reduction trick and identify the category of line operators in the 3d theory with the category of branes in the 2d theory obtained by compactifying the 3d theory on a circle. One can show that reduction gives a B-model with target $G_{\mathbb{C}}$ coupled to a B-type gauge theory with gauge group G [21]. From the 3d viewpoint, $G_{\mathbb{C}}$ parameterizes the holonomy of the connection $\mathcal{A}$ along the compactification circle. The gauge group G acts on $G_{\mathbb{C}}$ by conjugation. As explained in Section 4.3, the category of branes for this TQFT is the equivariant derived category of coherent sheaves $D^b_{G_{\mathbb{C}}}(Coh(G_{\mathbb{C}}))$. Line operators considered above correspond to coherent sheaves supported at the identity element of $G_{\mathbb{C}}$. Physically, this follows from the fact that the gauge field $\mathcal{A}$ is nonsingular for such line operators, and therefore the holonomy along the circle linking the line operator must be trivial. More generally, one may also consider Gukov-Witten-type line operators for which the conjugacy class of the holonomy of $\mathcal{A}$ is fixed.

5.4 Gauged A-Model

Suppose now that X admits an action of a compact Lie group G. We will now show how to couple the 3d A-model with target X to the A-type 3d gauge theory with gauge group G [22]. The latter theory is the dimensional reduction of the four-dimensional Donaldson-Witten theory [38]. Its bosonic fields are a gauge field A, a scalar field ζ in the adjoint representation of G, and a complex scalar field σ

(also in the adjoint representation). Its fermionic fields are a pair of 1-forms λ and $\tilde{\lambda}$ and a pair of 0-forms ρ and $\tilde{\rho}$. The BRST transformations of these fields before coupling to topological matter are

$$\delta A = \lambda, \tag{5.5}$$

$$\delta\lambda = -d_A\sigma, \tag{5.6}$$

$$\delta\zeta = \rho, \tag{5.7}$$

$$\delta\rho = [\sigma, \zeta], \tag{5.8}$$

$$\delta\sigma = 0, \tag{5.9}$$

$$\delta\bar{\sigma} = \tilde{\rho}, \tag{5.10}$$

$$\delta\tilde{\rho} = [\sigma, \bar{\sigma}], \tag{5.11}$$

$$\delta\tilde{\lambda} = \star F - d_A\zeta, \tag{5.12}$$

where d_A is the covariant derivative with respect to A and $\bar{\sigma} = -\sigma^\dagger$. These BRST transformations satisfy $\delta^2 = \delta_g(\sigma)$ modulo fermionic equations of motion, where $\delta_g(\sigma)$ is the gauge transformation with the parameter σ. To write down an action it is convenient to introduce an auxiliary bosonic 1-form H and redefine

$$\delta\tilde{\lambda} = H, \quad \delta H = [\sigma, \lambda].$$

The action is then chosen so that the equations of motion for H set $H = \star F - d_A\zeta$. A suitable action is

$$S_{gauge} = -\frac{1}{2e^2}\delta\int_M \mathrm{Tr}\left[\tilde{\lambda}\wedge\star(H - 2(\star F - d_A\zeta)) + \lambda\wedge\star d_A\bar{\sigma}\right].$$

The group G is assumed to act by isometries on the target manifold X of the 3d A-model. Infinitesimally this action is described by a vector field $V = V^i(\phi)\partial_i$ on X with values in the dual of the Lie algebra $\mathfrak{g}$ of G. By definition, an infinitesimal gauge transformation of ϕ^i corresponding to an element $a \in \mathfrak{g}$ is

$$\delta_g(a)\phi^i = V^i(a).$$

Gauge transformations of fields taking values in ϕ^*TX involve derivatives of V^i, for example:

$$\delta_g(a)\eta^i = \eta^k\nabla_k V^i - \Gamma^i_{jk}V^j(a)\eta^k = \eta^k\partial_k V^i(a).$$

Gauge-covariant derivatives of fields are defined accordingly,

$$D\phi^i = d\phi^i + V^i(A), \quad D\eta^i = d\eta^i + \Gamma^i_{jk}d\phi^j\eta^k + \eta^k\partial_k V^i(A),$$

where A is the gauge field.

To couple the 3d A-model to the A-type 3d gauge theory we modify the BRST transformations for matter fields so that $\delta^2 = \delta_g(\sigma)$ on all fields. The modified transformations are

$$\delta\phi^i = \eta^i, \tag{5.13}$$

$$\delta\eta^i = V^i(\sigma), \tag{5.14}$$

$$\delta\tau^i = \psi^i - \Gamma^i_{jk}\eta^j\tau^k, \tag{5.15}$$

$$\delta\psi^i = \frac{1}{2}R^i_{klj}\eta^l\eta^j\tau^k + \tau^k\nabla_k V^i(\sigma) - \Gamma^i_{jk}\eta^j\psi^k, \tag{5.16}$$

$$\delta\beta^i = P^i - \Gamma^i_{jk}\eta^j\beta^k, \tag{5.17}$$

$$\delta P^i = \frac{1}{2}R^i_{klj}\eta^l\eta^j\beta^k - \Gamma^i_{jk}\eta^j P^k + \beta^k\nabla_k V^i(\sigma), \tag{5.18}$$

$$\delta\chi^i = \tilde{P}^i - \Gamma^i_{jk}\eta^j\chi^k, \tag{5.19}$$

$$\delta\tilde{P}^i = \frac{1}{2}R^i_{klj}\eta^l\eta^j\chi^k - \Gamma^i_{jk}\eta^j\tilde{P}^k. + \chi^k\nabla_k V^i(\sigma). \tag{5.20}$$

The action of the gauged 3d A-model is the sum of S_{gauge}, a BRST-exact matter action

$$\tilde{S}' = -\delta\int_M \left[g_{ij}\chi^i\wedge\star\left(\tilde{P}^j - 2\left(D\phi^j - \star D\tau^j\right)\right) + g_{ij}\beta^i\wedge\star\left(P^j - 2D^\star\tau^j\right)\right],$$

and a topological term

$$S'_{top} = \int_M d(g_{ij}\tau^i D\phi^j) = \int_M g_{ij}D\tau^i D\phi^j - \int_M g_{ij}\tau^i V^j(F),$$

where $F = dA + A\wedge A$.

5.5 Gauged Rozansky-Witten Model

In this section, we couple the B-type gauge theory to a Rozansky-Witten model whose target X has a G-action compatible with the hyperkähler structure [21]. Let V_a, $a = 1, 2, \cdots, \dim G$, be the vector fields on X corresponding to the generators of the G-action. Let μ_+, μ_-, and μ_3 be the moment maps corresponding to the holomorphic symplectic form Ω, the antiholomorphic symplectic form $\bar{\Omega}$, and the Kähler form J, respectively,

$$d\mu_{+a} = -i_{V_a}(\Omega), \tag{5.21}$$

$$d\mu_{-a} = -i_{V_a}(\bar{\Omega}), \tag{5.22}$$

$$d\mu_{3a} = i_{V_a}(J). \tag{5.23}$$

where $i_V(\omega)$ is the interior product of the form ω with the vector V. The BRST variation of the fields are

$$
\begin{aligned}
\delta A &= \xi\lambda, & \delta\sigma^I &= 0,\\
\delta\phi &= i\xi\lambda, & \delta\sigma^{\bar{I}} &= \xi\eta^{\bar{I}},\\
\delta\lambda &= 0, & \delta\eta^{\bar{I}} &= 0,\\
\delta\zeta &= -i\xi\mathcal{F}, & \delta\chi^I &= \xi\mathcal{D}\sigma^I,\\
\delta\rho &= i\xi\mu_+, & &\\
\delta\tilde{\rho} &= i\xi P, & &\\
\delta P &= 0, & &
\end{aligned}
\tag{5.24}
$$

where $\mathcal{D}\sigma^I = d\sigma^I + \mathcal{A}^a V_a^I$. The action for this gauged Rozansky-Witten model is

$$
S = \int_{M_3} \big(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4\big), \tag{5.25}
$$

with

$$
\mathcal{L}_1 = -\frac{1}{2e^2}\delta\,\mathrm{Tr}\Big(i\zeta\wedge\star\bar{\mathcal{F}} + i\tilde{\rho}\wedge\star\big(P - 2d_A^*\phi - 2e^2\mu_3\big)\Big), \tag{5.26}
$$

$$
\mathcal{L}_2 = \delta\Big(g_{I\bar{J}}\chi^I\wedge\star\bar{\mathcal{D}}\sigma^{\bar{J}}\Big), \tag{5.27}
$$

$$
\mathcal{L}_3 = \delta\Big(\frac{i}{2}e^2\mathrm{Tr}\big(\rho\wedge\star\mu_-\big)\Big), \tag{5.28}
$$

$$
\begin{aligned}
\mathcal{L}_4 = {}& \mathrm{Tr}\big(\rho\wedge d_{\mathcal{A}}\zeta\big) + i\mathrm{Tr}\big(\Omega_{IJ}\chi^I\wedge\zeta V^J\big) + \frac{1}{2}\Omega_{IJ}\chi^I\wedge\mathcal{D}\chi^J\\
&+ \frac{1}{6}\Omega_{IJ}\mathcal{R}^J_{KL\bar{M}}\chi^I\wedge\chi^K\wedge\chi^L\wedge\eta^{\bar{M}},
\end{aligned}
\tag{5.29}
$$

where $\mathcal{D}\chi^I = d\chi^I + \Gamma^I_{JK}d\phi^J\wedge\chi^K + \mathcal{A}^a\partial_J V_a^I\wedge\chi^J + \mathcal{A}^a\Gamma^I_{JK}V_a^K\wedge\chi^J$.

Local observables in the gauged Rozansky-Witten model are BRST and gauge invariant functions of ρ^a, σ^I, $\sigma^{\bar{I}}$, and $\eta^{\bar{I}}$, which correspond to elements in the cohomology of $\Lambda(\mathfrak{g}^*)\otimes\Omega^{0,\bullet}(X)$ with respect to the following nilpotent operator,

$$
\delta = i\mu_+^a T_a + (-1)^\ell\bar{\partial}_X, \tag{5.30}
$$

where T_a are a basis for $\mathfrak{g}$ and ℓ is the degree of the element in $\Lambda(\mathfrak{g}^*)$.

Chapter 6

Four-Dimensional TQFTs

6.1 $\mathcal{N} = 2$ Linear σ-Model

We begin this chapter by discussing $\mathcal{N} = 2$ linear σ-models. The bosonic fields ϕ are maps from Euclidean spacetime, $\mathbb{R}^4$, into the hyperkähler target manifold $\mathbb{H}^N$ (which is isomorphic to $\mathbb{C}^{2N}$ after a choice of complex structure),

$$\phi \in \mathrm{Map}\big(\mathbb{R}^4, \mathbb{H}^N\big).$$

The fermionic fields ψ and $\overline{\psi}$ are sections of the spin bundle S_+ and S_- on $\mathbb{R}^4$ valued in the pullback of the holomorphic tangent bundle $T\mathbb{H}^N$ and antiholomorphic tangent bundle $\overline{T\mathbb{H}^N}$, respectively,

$$\begin{aligned} \psi &\in \Gamma\big(\phi^* T\mathbb{H}^N \otimes S_+\big), \\ \overline{\psi} &\in \Gamma\big(\phi^* \overline{T\mathbb{H}^N} \otimes S_-\big). \end{aligned}$$

The dynamics of the σ-model are governed by the action

$$S = \int_{\mathbb{R}^4} d^4x \left(\delta_{i\bar{j}} \partial^\mu \phi^i \partial_\mu \overline{\phi}^{\bar{j}} + i \delta_{i\bar{j}} \overline{\psi}^{\bar{j}} \bar{\sigma}^\mu \partial_\mu \psi^i \right).$$

The left action of quarternions on $\mathbb{H}^N$ corresponds to $SU(2)_\mathcal{R}$ $\mathcal{R}$-symmetry, while the right action of quarternions on $\mathbb{H}^N$ gives rise to an additional $SU(2)_X$ symmetry. Let us introduce the following notation to make the $SU(2)_\mathcal{R} \times SU(2)_X$ action on $\mathbb{H}^N$ manifest,

$$\begin{aligned} \phi^I_{11'} &= \phi^{2I-1}, & \psi^I_{1'} &= \psi^{2I-1}, \\ \phi^I_{12'} &= \phi^{2I}, & \psi^I_{2'} &= \psi^{2I}, \\ \phi^I_{21'} &= -\overline{\phi}^{\overline{2I}}, & \overline{\psi}^{1'I} &= \overline{\psi}^{\overline{2I-1}}, \\ \phi^I_{22'} &= \overline{\phi}^{\overline{2I-1}}, & \overline{\psi}^{2'I} &= \overline{\psi}^{\overline{2I}}, \end{aligned}$$

where $SU(2)_{\mathcal{R}}$ acts on the unprimed index and $SU(2)_X$ acts on the primed index. Using this notation, we can write the action in a form that is manifestly $SU(2)_{\mathcal{R}} \times SU(2)_X$ invariant,

$$S = \int_{\mathbb{R}^4} d^4x \left(\frac{1}{2} \delta_{IJ} \epsilon^{ab} \epsilon^{a'b'} \partial^\mu \phi^I_{aa'} \partial_\mu \phi^J_{bb'} + i \delta_{IJ} \overline{\psi}^{a'I} \bar{\sigma}^\mu \partial_\mu \psi^J_{a'} \right). \tag{6.1}$$

It is not difficult to see that the action respects the following supersymmetry transformations,

$$\begin{aligned} \delta \phi^I_{aa'} &= \sqrt{2} \xi_a \psi^I_{a'} + \sqrt{2} \epsilon_{ab} \epsilon_{a'b'} \bar{\xi}^b \overline{\psi}^{b'I}, \\ \delta \psi^I_{a'} &= i\sqrt{2} \sigma^\mu \overline{\xi}^a \partial_\mu \phi^I_{aa'}, \\ \delta \overline{\psi}^{a'I} &= i\sqrt{2} \epsilon^{ab} \epsilon^{a'b'} \bar{\sigma}^\mu \xi_a \partial_\mu \phi^I_{bb'}. \end{aligned} \tag{6.2}$$

With respect to the $SU(2)_L \times SU(2)_R$ rotational symmetry, $SU(2)_{\mathcal{R}}$ $\mathcal{R}$-symmetry, and $SU(2)_X$ symmetry, the fields and supercharges transforms as shown in the tables below.

Field	$SU(2)_L$	$SU(2)_R$	$SU(2)_{\mathcal{R}}$	$SU(2)_X$
$\phi^I_{aa'}$	1	1	2	2
$\psi^I_{\alpha a'}$	2	1	1	2
$\overline{\psi}^{\dot{\alpha} a' I}$	1	2	1	2

Table 6.1: Charges of fields in $\mathcal{N} = 2$ linear σ-model

Field	$SU(2)_L$	$SU(2)_R$	$SU(2)_{\mathcal{R}}$	$SU(2)_X$
$Q_{\alpha a}$	2	1	2	1
$\overline{Q}^{\dot{\alpha} a}$	1	2	2	1

Table 6.2: Charges of $\mathcal{N} = 2$ supercharges

6.1.1 Linear A-Model

The linear A-model is constructed by twisting the $SU(2)_L$ rotational symmetry of the $\mathcal{N} = 2$ linear σ-model by the diagonal subgroup of the $SU(2)_{\mathcal{R}} \times SU(2)_X$ symmetry (see Table 6.1 for the charges of fields in the $\mathcal{N} = 2$ linear σ-model),

$$SU(2)_L \to SU(2)_{L'} \triangleleft SU(2)_R \times SU(2)_{\mathcal{R}} \times SU(2)_X$$

where $G \triangleleft G \times G \times G$ is the diagonal subgroup.

Field	$SU(2)_L$	$SU(2)_R$
σ^I	1	1
$\tau^{I-}_{\mu\nu}$	3	1
η^I	1	1
$\psi^{I-}_{\mu\nu}$	3	1
χ^I_μ	2	2

Table 6.3: Fields in the linear A-model

The bosonic field σ is a map from spacetime M into the Riemannian manifold $\mathbb{R}^N$,

$$\sigma \in \mathrm{Map}\big(M, \mathbb{R}^N\big).$$

The bosonic field τ is an antiselfdual 2-form on M valued in the pullback of the tangent bundle $T\mathbb{R}^N$,

$$\tau \in \Gamma\big(\sigma^* T\mathbb{R}^N \otimes \Omega^{2-}\big).$$

The fermionic fields η, ψ, and χ are a scalar, antiselfdual 2-form, and 1-form on M, respectively, valued in the pullback of the tangent bundle $T\mathbb{R}^N$,

$$\eta \in \Gamma\big(\sigma^* T\mathbb{R}^N\big),$$
$$\psi \in \Gamma\big(\sigma^* T\mathbb{R}^N \otimes \Omega^{2-}\big),$$
$$\chi \in \Gamma\big(\sigma^* T\mathbb{R}^N \otimes \Omega^{1}\big).$$

We construct the action for the linear A-model by writing the action of the $\mathcal{N} = 2$ linear σ-model (6.1) covariantly in terms of the twisted fields,

$$S = \int_M d^4x \left(\frac{1}{4}\delta_{IJ}\partial^\mu \sigma^I \partial_\mu \sigma^J + \delta_{IJ}\partial^\nu \tau^{I-}_{\mu\nu}\partial_\lambda \tau^{\mu\lambda J-} + \frac{i}{2}\delta_{IJ}\chi^I_\mu \partial^\mu \eta^J - i\delta_{IJ}\chi_\mu \partial_\nu \psi^{\mu\nu J-} \right). \tag{6.3}$$

The BRST charge is

$$Q_A = \epsilon^{\alpha a} Q_{\alpha a}$$

which is a scalar after twisting (see Table 6.2 for the charges of the $\mathcal{N} = 2$ supercharges). The BRST variations follow from the corresponding supersymmetry transformations (6.2),

$$\delta\sigma^I = \sqrt{2}\xi\eta^I,$$
$$\delta\tau^{I-}_{\mu\nu} = \sqrt{2}\xi\psi^{I-}_{\mu\nu},$$
$$\delta\eta^I = 0,$$
$$\delta\psi^{I-}_{\mu\nu} = 0,$$
$$\delta\chi^I_\mu = i\sqrt{2}\xi\partial_\mu\sigma^I - 2i\sqrt{2}\xi\partial^\nu\tau^{I-}_{\mu\nu}.$$

Notice that the χ variation is only nilpotent on-shell. It will be convenient to introduce an auxiliary bosonic field P which is a 1-form on M valued in the pullback of the tangent bundle $T\mathbb{R}^N$,

$$P \in \Gamma\big(\sigma^* T\mathbb{R}^N \otimes \Omega^1\big).$$

We require that P satisfies the following constraint on-shell,

$$P^I_\mu = i\partial_\mu \sigma^I - 2i\partial^\nu \tau^{I-}_{\mu\nu}. \tag{6.4}$$

so that we can write the BRST variations as

$$\begin{aligned}
\delta\sigma^I &= \sqrt{2}\xi\eta^I, \\
\delta\tau^{I-}_{\mu\nu} &= \sqrt{2}\xi\psi^{I-}_{\mu\nu}, \\
\delta\eta^I &= 0, \\
\delta\psi^{I-}_{\mu\nu} &= 0, \\
\delta\chi^I_\mu &= \sqrt{2}\xi P^I_\mu, \\
\delta P^I_\mu &= 0.
\end{aligned} \tag{6.5}$$

It is not difficult to construct an action equivalent to the original linear A-model action (6.3) that enforces the appropriate constraint on the auxiliary field (6.4) and respects the BRST symmetry (6.5),

$$\begin{aligned}
S = \int_M d^4x \bigg(&\frac{1}{4}\delta_{IJ} P^{\mu I} P^J_\mu - \frac{1}{2}\delta_{IJ} P^{\mu I}\big(i\partial_\mu\sigma^J - 2i\partial^\nu\tau^{J-}_{\mu\nu}\big) \\
&+ \frac{i}{2}\delta_{IJ}\chi^I_\mu \partial^\mu \eta^J - i\delta_{IJ}\chi_\mu \partial_\nu \psi^{\mu\nu J-}\bigg).
\end{aligned} \tag{6.6}$$

Note that the action for the linear A-model is Q-exact,

$$S = \int_M d^4x \Big\{Q, \frac{1}{4\sqrt{2}}\delta_{IJ}\chi^{\mu I} P^J_\mu - \frac{1}{2\sqrt{2}}\delta_{IJ}\chi^{\mu I}\big(i\partial_\mu\sigma^J - 2i\partial^\nu\tau^{J-}_{\mu\nu}\big)\Big\}. \tag{6.7}$$

6.1.2 A-Model

The linear A-model has a covariant extension to arbitrary Riemannian target manifolds. The bosonic field σ is a map from spacetime M into a Riemannian manifold X,

$$\sigma \in \mathrm{Map}\big(M, X\big).$$

The bosonic field τ is an antiselfdual 2-form on M valued in the pullback of the tangent bundle TX,

$$\tau \in \Gamma\big(\sigma^* TX \otimes \Omega^{2-}\big).$$

The fermionic fields η, ψ, and χ are a scalar, antiselfdual 2-form, and 1-form on M, respectively, valued in the pullback of the tangent bundle TX,

$$\begin{aligned}
&\eta \in \Gamma\big(\sigma^* TX\big),\\
&\psi \in \Gamma\big(\sigma^* TX \otimes \Omega^{2-}\big),\\
&\chi \in \Gamma\big(\sigma^* TX \otimes \Omega^{1}\big).
\end{aligned}$$

We construct the action for the A-model by making the linear A-model action (6.6) covariant and adding the appropriate curvature terms,

$$\begin{aligned}
S = \int_M d^4x \bigg(&\frac{1}{4} g_{IJ} P^{\mu I} P^J_\mu - \frac{1}{2} g_{IJ} P^{\mu I}\big(i\partial_\mu \sigma^J - 2iD^\nu \tau^{J-}_{\mu\nu}\big) + \frac{i}{2} g_{IJ}\chi^I_\mu D^\mu \eta^J \\
&- i g_{IJ}\chi^I_\mu D_\nu \psi^{\mu\nu J-} - \frac{1}{8} g_{IJ} R^J_{KLM}\chi^I_\mu\big(\chi^{\mu K}\eta^L\eta^M - 8i\tau^{\mu\nu K-}\partial_\nu\sigma^L\eta^M\big)\bigg),
\end{aligned} \tag{6.8}$$

where

$$\begin{aligned}
D^\nu \tau^{I-}_{\mu\nu} &= \partial^\nu \tau^{J-}_{\mu\nu} + \Gamma^I_{JK}\partial^\nu\sigma^J\tau^{K-}_{\mu\nu},\\
D^\mu \eta^I &= \partial^\mu \eta^I + \Gamma^I_{JK}\partial^\mu\sigma^J\eta^K,\\
D^\nu \psi^{I-}_{\mu\nu} &= \partial^\nu \psi^{J-}_{\mu\nu} + \Gamma^I_{JK}\partial^\nu\sigma^J\psi^{K-}_{\mu\nu}.
\end{aligned}$$

The appropriate extension of the BRST variations are

$$\begin{aligned}
\delta\sigma^I &= \xi\eta^I,\\
\delta\tau^{I-}_{\mu\nu} &= \xi\big(\psi^{I-}_{\mu\nu} - \Gamma^I_{JK}\eta^J\tau^{K-}_{\mu\nu}\big),\\
\delta\eta^I &= 0,\\
\delta\psi^{I-}_{\mu\nu} &= \xi\Big(\frac{1}{2}R^I_{JKL}\tau^{J-}_{\mu\nu}\eta^K\eta^L - \Gamma^I_{JK}\eta^J\psi^{K-}_{\mu\nu}\Big),\\
\delta\chi^I_\mu &= \xi\big(P^I_\mu - \Gamma^I_{JK}\eta^J\chi^K_\mu\big),\\
\delta P^I_\mu &= \xi\big(\frac{1}{2}R^I_{JKL}\chi^J_\mu\eta^K\eta^L - \Gamma^I_{JK}\eta^J P^K_\mu\big),
\end{aligned} \tag{6.9}$$

where we have scaled the fields relative to the linear A-model to remove awkward factors of $\sqrt{2}$. Note that the action for the A-model is Q-exact,

$$S = \int_M d^4x \Big\{Q, \frac{1}{4} g_{IJ}\chi^{\mu I} P^J_\mu - \frac{1}{2} g_{IJ}\chi^{\mu I}\big(i\partial_\mu\sigma^J - 2iD^\nu\tau^{J-}_{\mu\nu}\big)\Big\}. \tag{6.10}$$

6.2 $\mathcal{N} = 4$ SYM Theory

In this section we review $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This theory consists of adjoint complex scalar fields ϕ_{AB}, adjoint Weyl spinors $\lambda_{\alpha A}$ and a gauge field A_μ with the following action,

$$S = \int_{\mathbb{R}^4} \mathrm{Tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}^A\bar{\sigma}^\mu D_\mu\lambda_A - \frac{1}{4}D^\mu\phi^{\dagger AB}D_\mu\phi_{AB} + ig\frac{1}{\sqrt{2}}\phi^{\dagger AB}\big[\lambda_A, \lambda_B\big] + ig\frac{1}{\sqrt{2}}\phi_{AB}\big[\bar{\lambda}^A, \bar{\lambda}^B\big] + \frac{1}{16}g^2\big[\phi_{AB}, \phi_{CD}\big]\big[\phi^{\dagger AB}, \phi^{\dagger CD}\big]\Big)$$

where the $\mathcal{R}$-symmetry index $A = 1, \cdots, 4$. The complex scalar fields are antisymmetric in the $\mathcal{R}$-symmetry indices,

$$\phi_{AB} = -\phi_{BA}$$

and satisfy the following reality condition,

$$\phi^{\dagger AB} = \frac{1}{2}\epsilon^{ABCD}\phi_{CD}.$$

It is not difficult to see that the action respects the following supersymmetry transformations,

$$\begin{aligned}
\delta A_\mu &= i\xi_A\sigma_\mu\bar{\lambda}^A + i\bar{\xi}^A\bar{\sigma}_\mu\lambda_A \\
\delta\lambda_A &= \sigma^{\mu\nu}\xi_A F_{\mu\nu} + ig\big[\phi_{AB}, \phi^{\dagger BC}\big]\xi_C - i\sqrt{2}\sigma^\mu\bar{\xi}^B D_\mu\phi_{AB} \\
\delta\bar{\lambda}^A &= \bar{\sigma}^{\mu\nu}\bar{\xi}^A F_{\mu\nu} + ig\big[\phi^{\dagger AB}, \phi_{BC}\big]\bar{\xi}^C - i\sqrt{2}\bar{\sigma}^\mu\xi_B D_\mu\phi^{\dagger AB} \\
\delta\phi_{AB} &= \sqrt{2}\xi_A\lambda_B - \sqrt{2}\xi_B\lambda_A + \sqrt{2}\epsilon_{ABCD}\bar{\xi}^C\bar{\lambda}^D \\
\delta\phi^{\dagger AB} &= \sqrt{2}\bar{\xi}^A\bar{\lambda}^B - \sqrt{2}\bar{\xi}^B\bar{\lambda}^A + \sqrt{2}\epsilon^{ABCD}\xi_C\lambda_D.
\end{aligned}$$

This theory has $SU(4)_\mathcal{R}$ $\mathcal{R}$-symmetry as well as $SU(2)_L \times SU(2)_R$ spin symmetry. With respect these global symmetries, the supercharges transform as shown in Table 6.4 while the fields transform as shown in Table 6.5 .

	$SU(4)_\mathcal{R}$	$SU(2)_L$	$SU(2)_R$
$Q^A{}_\alpha$	$\bar{\mathbf{4}}$	$\mathbf{2}$	$\mathbf{1}$
$\bar{Q}_A{}^{\dot\alpha}$	$\mathbf{4}$	$\mathbf{1}$	$\mathbf{2}$

Table 6.4: $\mathcal{N} = 4$ supercharges and their transformation properties

6.3 Geometric Langlands Theory

The bosonic fields in the GL-twisted 4d theory are a gauge field A_μ (a connection on a principal G-bundle $\mathcal{P}$ over a 4-manifold M_4), a 1-form $\phi_\mu dx^\mu$ with values in $\mathrm{Ad}(\mathcal{P})$, and a 0-form σ with

Field	$SU(4)_{\mathcal{R}}$	$SU(2)_L$	$SU(2)_R$
A_μ	**1**	**2**	**2**
$\lambda_{A\alpha}$	**4**	**2**	**1**
$\bar{\lambda}^{A\dot{\alpha}}$	$\bar{\mathbf{4}}$	**1**	**2**
ϕ_{AB}	**6**	**1**	**1**

Table 6.5: Fields in $\mathcal{N} = 4$ SYM theory and their transformation properties

values in the complexification of $\mathrm{Ad}(\mathcal{P})$. The conventions are the same as in [17]; in particular, real adjoint-valued fields are regarded as anti-Hermitian, and the covariant derivative in the adjoint representation takes the form $d_A = d + [A, \cdot]$. The fermionic fields are a pair of $\mathrm{Ad}(\mathcal{P})_{\mathbb{C}}$-valued 1-forms ψ and $\tilde{\psi}$, a pair of $\mathrm{Ad}(\mathcal{P})_{\mathbb{C}}$-valued 0-forms η and $\tilde{\eta}$, and an $\mathrm{Ad}(\mathcal{P})_{\mathbb{C}}$-valued 2-form χ. The fields A and ϕ have ghost number 0, the fields ψ and $\tilde{\psi}$ have ghost number 1, the fields $\eta, \tilde{\eta}$, and χ have ghost number -1, and the field σ has ghost number 2. The BRST transformations are

$$
\begin{aligned}
&\delta A = i(\psi + t\tilde{\psi}),\\
&\delta\phi = i(t\psi - \tilde{\psi}),\\
&\delta\sigma = 0,\\
&\delta\bar{\sigma} = i(\eta + t\tilde{\eta}),\\
&\delta\psi = d_A\sigma + t[\phi, \sigma]\\
&\delta\tilde{\psi} = td_A\sigma - [\phi, \sigma],\\
&\delta\eta = td_A^*\phi + [\bar{\sigma}, \sigma],\\
&\delta\tilde{\eta} = -d_A^*\phi + t[\bar{\sigma}, \sigma],\\
&\delta\chi = \frac{1+t}{2}(F - \frac{1}{2}[\phi, \phi] + *d_A\phi) + \frac{1-t}{2}(*(F - \frac{1}{2}[\phi, \phi]) - d_A\phi).
\end{aligned}
$$

Here t takes values in $\mathbb{C}\bigcup\{\infty\}$, $\bar{\sigma} = -\sigma^\dagger$, $*$ is the 4d Hodge star operator, and $d^* = *d*$. For $t \neq \pm i$ the action can be wriiten as a BRST-exact term plus a term which depends only on the topology of the bundle $\mathcal{P}$:

$$S = \delta \int_{M_4} V - \frac{\Psi}{4\pi i} \int_{M_4} \mathrm{Tr}\, F \wedge F, \tag{6.11}$$

where

$$\Psi = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \frac{t^2 - 1}{t^2 + 1}.$$

Here θ is the theta-angle of the 4d gauge theory and e^2 is the gauge coupling. The explicit form of V can be found in [17].

The simplest surface operators have been introduced by Gukov and Witten [12]. They are disorder operators corresponding to a codimension-2 singularity in the fields of the form

$$A = \alpha d\theta, \quad \phi = \beta \frac{dr}{r} - \gamma d\theta.$$

Here α is an element of a maximal torus T of G, and β, γ are elements of the Lie algebra $\mathfrak{t}$ of T. For simplicity, let us assume that the triple (α, β, γ) breaks G down to T. Gauge transformations which preserve T form the Weyl group $\mathcal{W}$; the triplet (α, β, γ) is defined up to the action of $\mathcal{W}$ on $\mathsf{T} \times \mathfrak{t} \times \mathfrak{t}$. All fields other than A and ϕ are nonsingular.

The surface operator depends on an additional parameter η taking values in the torus $\mathrm{Hom}(\Lambda_{\mathrm{cochar}}, U(1))$. Here $\Lambda_{\mathrm{cochar}}$ is the lattice of magnetic charges $\mathrm{Hom}(U(1), \mathsf{T})$. Equivalently, as explained in [12], η can be thought of as taking values in ${}^L\mathsf{T}$, the maximal torus of the Langlands-dual group. The parameter η arises as follows. First, note that the above singularity in the fields breaks the gauge group down to T. Thus if D is the codimension-2 submanifold on which the surface operator is supported, the restriction of the gauge field to D has a first Chern class $c_1|_D$ taking values in $\Lambda_{\mathrm{cochar}}$. Given η we can insert into the path-integral a phase factor

$$\eta(c_1(D)).$$

This factor depends only on the behavior of the gauge field on D and can be regarded as an η-dependent modification of the surface operator defined above.

Gukov-Witten surface operators are BRST-invariant for arbitrary t, but their properties depend on t. We will see below that there are many other surface operators. In what follows we will focus on the cases $t = i$, $t = 1$, and $t = 0$. The first two cases are exchanged by S-duality (at zero θ-angle) and play a prominent role in the physical approach to the Geometric Langlands Program [17]. The last case is self-dual and is the most natural starting point for understanding Quantum Geometric Langlands Duality [19].

6.3.1 Compactification at $t = i$ on a Circle

In this section we show that the GL-twisted theory at $t = i$ compactified on a circle is equivalent to a gauged version of the Rozansky-Witten model. For nonabelian gauge group, the precise determination of the target space of this model is rather subtle: naively, one can perform the compactification by simply requiring all fields to be independent of the coordinate x^4 of the circle, and reducing the field $A_4 + i\phi_4$ of the GL-twisted theory to a $\mathfrak{g}_{\mathbb{C}}$-valued scalar τ in three dimensions. As we will see in detail, in this case one obtains a gauged Rozansky-Witten sigma model with target $T^*\mathfrak{g}_{\mathbb{C}}$, where the gauge group acts on the base by the adjoint representation and the fiber by the coadjoint representation. However, the coordinate τ for the base is subject to global identifications, due to the possibility of performing x^4-dependent gauge transformations with nontrivial holonomy around the compactification circle. Therefore, we should regard τ as merely a local coordinate on the true target space of the theory, which we conjecture to be the cotangent bundle $T^*G_{\mathbb{C}}$.

Let us see how the naive reduction works in detail. The bosonic fields in the GL-twisted theory are a 4d gauge field A, an adjoint-valued 1-form ϕ and an adjoint-valued complex 0-form σ. The fermionic fields are a pair of 1-forms ψ and $\tilde{\psi}$, a pair of 0-forms η and $\tilde{\eta}$, and a 2-form χ, all adjoint-valued.

It was observed by Marcus [25] that, precisely at $t = \pm i$, the action can be expressed as the sum of a BRST-exact piece and a BRST-inexact fermionic piece (by contrast with the situation for $t \neq \pm i$, where the only BRST-inexact term is a purely bosonic term depending on the topology of the gauge bundle). It is convenient to work with the complexified connections $\mathcal{A} = A + i\phi$ and $\bar{\mathcal{A}} = A - i\phi$ as well as the covariant derivatives $d_{\mathcal{A}}$, $d_{\bar{\mathcal{A}}}$ and curvatures $\mathcal{F}$, $\bar{\mathcal{F}}$ with respect to these connections. We set the theta angle to zero, and place the theory on a four manifold M_4. The action for GL-twisted theory at $t = i$ reads

$$S = \int_{M_4} (\mathcal{L}_1 + \mathcal{L}_2)$$

where

$$\begin{aligned}
\mathcal{L}_1 &= -\frac{1}{2e^2}\,\delta\,\mathrm{Tr}\Big\{(\chi^+ - i\chi^-)\wedge *\bar{\mathcal{F}} + d_{\bar{\mathcal{A}}}\bar{\sigma}\wedge *(\psi - i\tilde{\psi}) \\
&\qquad + \frac{i}{2}(\eta - i\tilde{\eta})\wedge *\Big(i[\bar{\sigma},\sigma] - d_A^*\phi\Big)\Big\}, \\
\mathcal{L}_2 &= \frac{i}{e^2}\,\mathrm{Tr}\Big\{(\chi^+ - i\chi^-)\wedge\Big(d_{\mathcal{A}}(\psi - i\tilde{\psi}) - [\chi^+ - i\chi^-, \sigma]\Big)\Big\}.
\end{aligned}$$

Here, $*$ is the 4d Hodge star and $d_A^*\phi = *d_A * \phi$.

We now take M_4 to be the product manifold $M_3 \times S^1$ with product metric, where M_3 is a three manifold and the coordinate x^4 ranges from 0 to 2π (the circumference of the S^1). We require fields to be independent of x^4, thereby obtaining an effective 3d theory on M_3. It is useful to label the fields of this 3d theory as follows:

$$\begin{aligned}
A^{(3d)} &= A|_{M_3}, & \sigma^{(3d)} &= \sqrt{2}\sigma,\\
\phi^{(3d)} &= \phi|_{M_3}, & \tau &= (A_4 + i\phi_4),\\
\mathcal{A}^{(3d)} &= (A+i\phi)|_{M_3} & \eta^{\bar\sigma} &= \sqrt{2}i\left(\eta + i\tilde\eta\right),\\
\bar{\mathcal{A}}^{(3d)} &= (A-i\phi)|_{M_3} & \eta^{\bar\tau} &= 2i\left(\psi_4 + i\tilde\psi_4\right),\\
\lambda &= i\left(\psi + i\tilde\psi\right)|_{M_3}, & \chi^\sigma &= \frac{1}{\sqrt{2}}\left(\psi - i\tilde\psi\right)|_{M_3},\\
\zeta &= -i\left(\chi^+ - i\chi^-\right)|_{M_3}, & \star\chi^\tau &= *\left(\chi^+ - i\chi^-\right)|_{M_3},\\
\rho &= \frac{1}{\sqrt{2}}\left(\psi_4 - i\tilde\psi_4\right), & &\\
\tilde\rho &= \frac{1}{2}\left(\eta - i\tilde\eta\right). & &
\end{aligned}$$

Henceforth, we drop the superscripts $(3d)$ and take $d_{\mathcal{A}}$ to refer to covariant derivatives with respect to these 3d fields. We have written $\star$ for the 3d Hodge star and $d_A^\star\phi = \star d_A \star \phi$. In summary, we have the following bosons: a 3d gauge field A, a 1-form ϕ, and a pair of complex 0-forms σ and τ, all adjoint-valued. We have the following fermions: a pair of 1-forms χ^τ and χ^σ, a 1-form λ, a pair of 0-forms $\eta^{\bar\tau}$ and $\eta^{\bar\sigma}$, another pair of 0-forms ρ and $\tilde\rho$, and a 2-form ζ, all adjoint-valued.

In addition, it is useful to introduce an auxiliary, adjoint-valued 0-form P in order to make the BRST variations nilpotent off-shell; P-dependent terms in the action are chosen to ensure that its equation of motion is

$$P = d_A^\star\phi - \frac{i}{2}([\bar\sigma,\sigma] + [\bar\tau,\tau]).$$

The dimensional reduction of the BRST variations are as follows

$$\begin{aligned}
\delta A &= \lambda, & \delta\sigma &= 0,\\
\delta\phi &= i\lambda, & \delta\tau &= 0,\\
\delta\lambda &= 0, & \delta\bar\sigma &= \eta^{\bar\sigma}\\
\delta\zeta &= -i\mathcal{F}, & \delta\bar\tau &= \eta^{\bar\tau}\\
\delta\rho &= [\tau,\sigma], & \delta\eta^{\bar\sigma} &= 0,\\
\delta\tilde\rho &= iP, & \delta\eta^{\bar\tau} &= 0,\\
\delta P &= 0, & \delta\chi^\sigma &= d_{\mathcal{A}}\sigma,\\
& & \delta\chi^\tau &= d_{\mathcal{A}}\tau.
\end{aligned}$$

We have $\delta^2 = 0$ identically, without need to resort to a gauge transformation. After an overall rescaling, the dimensional reduction of the action is as follows

$$S = \int_{M_3} (\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4)$$

where

$$\begin{aligned}
\mathcal{L}_1 &= -\frac{1}{2e^2}\,\delta\,\mathrm{Tr}\Big\{i\zeta\wedge\star\bar{\mathcal{F}} + i\tilde{\rho}\wedge\star\big(P - 2d_A^\star\,\phi + i[\bar{\sigma},\sigma] + i[\bar{\tau},\tau]\big)\Big\}\\
\mathcal{L}_2 &= -\frac{1}{2e^2}\,\delta\,\mathrm{Tr}\Big\{\chi^\tau\wedge\star d_{\bar{\mathcal{A}}}\bar{\tau} + \chi^\sigma\wedge\star d_{\bar{\mathcal{A}}}\bar{\sigma}\Big\}\\
\mathcal{L}_3 &= -\frac{1}{2e^2}\,\delta\,\mathrm{Tr}\Big\{\rho\wedge\star[\bar{\tau},\bar{\sigma}]\Big\},\\
\mathcal{L}_4 &= \frac{\sqrt{2}}{e^2}\,\mathrm{Tr}\Big\{i\chi^\tau\wedge d_{\mathcal{A}}\chi^\sigma - \zeta\wedge\Big(\mathrm{d}_{\mathcal{A}}\,\rho + [\chi^\sigma,\tau] - [\chi^\tau,\sigma]\Big)\Big\}.
\end{aligned}$$

The BRST-inexact piece $\mathcal{L}_4$ is metric-independent, as befits a topological field theory. We have the correct field content for a gauged Rozansky-Witten sigma model. The target space is parameterized by $\mathfrak{g}_{\mathbb{C}}$-valued scalars σ and τ, both of which are acted on by the gauge group in the adjoint representation. Since σ has ghost number 2, we may identify the target space with $T^*[2]\mathfrak{g}_{\mathbb{C}}$, where [2] indicates that the fiber coordinate sits in cohomological degree 2 (here, we are using the negative-definite quadratic form Tr to coordinatize the fiber $\mathfrak{g}_{\mathbb{C}}^*$ by a $\mathfrak{g}_{\mathbb{C}}$-valued scalar). The G-invariant symplectic form on the target space can be read off the term $\mathcal{L}_4$ of the action and is proportional to

$$\Omega = \mathrm{Tr}\, d\sigma d\tau.$$

Additionally, the G-invariant Kähler form on the target space can be inferred the term $\mathcal{L}_2$ of the action and is proportional to

$$J = \frac{i}{2}\mathrm{Tr}(d\sigma d\bar{\sigma} + d\tau d\bar{\tau}).$$

The moment maps μ_3, μ_+, μ_- of the G-action with respect to symplectic forms J, Ω, and $\bar{\Omega}$ are proportional to the following quadratic functions of the coordinates:

$$\begin{aligned}
\mu_3 &= -\frac{i}{2}([\bar{\sigma},\sigma] + [\bar{\tau},\tau])\\
\mu_+ &= i[\sigma,\tau]\\
\mu_- &= -i[\bar{\sigma},\bar{\tau}].
\end{aligned}$$

After rescaling the fields ρ, σ, τ, $\eta^{\bar{\sigma}}$, $\eta^{\bar{\tau}}$, χ^σ, and χ^τ by factors of e^2, and adjusting the relative normalization of the BRST-inexact and BRST-exact terms (which normalization does not affect the properties of the theory), one finds that the action and variations above reproduce those of a gauged Rozansky-Witten model.

6.3.2 Surface Operators at $t = i$ with an Abelian Gauge Group

As explained in [12], at $t = i$ varying the parameters β and η changes the surface operator only by BRST-exact terms. Thus Gukov-Witten operators depend on a single complex parameter $\alpha - i\gamma$. But there exist much more general surface operators. To study them systematically, it is convenient to use the fact that surface operators in the 4d TQFT are in 1-1 correspondence with boundary conditions in the 3d TQFT compactified on a circle. The advantage of the 3d viewpoint is that the problem of classification of boundary conditions is more familiar. In particular, for $t = i$ the 3d TQFT that one gets is a gauged version of the Rozansky-Witten model, so we can use many of the results of [18] where boundary conditions for the Rozansky-Witten model have been studied.

In this section we consider the case $G = U(1)$. Reduction to 3d amounts to declaring all fields to be independent of the x^4 direction which is periodic with period 2π. The reduced theory has the following bosonic fields: a 3d gauge field A, a 1-form ϕ, a complex 0-form σ, and a pair of 0-forms A_4 and ϕ_4. More properly, one should work with a $U(1)$-valued scalar $\exp(-2\pi A_4)$ which represents the holonomy of the gauge field along the compact direction. This field is invariant with respect to x^4-dependent gauge transformations

$$A_4 \mapsto A_4 + im, \quad m \in \mathbb{Z}.$$

The fermionic fields are 1-forms $\psi, \tilde{\psi}, \chi, \tilde{\chi}$, and 0-forms $\eta, \tilde{\eta}, \psi_4, \tilde{\psi}_4$.

At $t = i$ it is convenient to combine A_4 and ϕ_4 into a complex 0-form $\tau = A_4 + i\phi_4$, or more properly into a gauge-invariant $\mathbb{C}^*$-valued scalar $\exp(-2\pi\tau)$. Then τ and σ are BRST-invariant. We also define the complex 3d gauge field $\mathcal{A} = A + i\phi$ which is BRST-invariant and the corresponding curvature $\mathcal{F} = d\mathcal{A}$. The BRST transformations of other fields are

$$\begin{aligned}
\delta(A - i\phi) &= 2i(\psi + i\tilde{\psi}), \\
\delta\bar{\sigma} &= i(\eta + i\tilde{\eta}), \\
\delta\tau &= -2i(\psi_4 + i\tilde{\psi}_4), \\
\delta\psi &= d\sigma \\
\delta\tilde{\psi} &= id\sigma, \\
\delta\psi_4 &= 0, \\
\delta\tilde{\psi}_4 &= 0, \\
\delta\eta &= id^\star\phi, \\
\delta\tilde{\eta} &= -d^\star\phi, \\
\delta\chi &= \mathcal{F}, \\
\delta\tilde{\chi} &= d\tau.
\end{aligned}$$

Here $d^\star = \star d \star$ and $\star$ denotes the 3d Hodge star operator.

These BRST-transformations are nilpotent off-shell. One can make them nilpotent on-shell by introducing a suitable auxiliary field, as discussed in the Section 6.3.1. The 3d action contains both a BRST-exact metric-dependent term and a BRST-closed metric-independent term. Its explicit form is given in the Section 6.3.1.

The analysis of boundary conditions in the 3d theory is greatly facilitated by the observation that this 3d theory decomposes into two independent sectors, the Rozansky-Witten model with target $T^*\mathbb{C}^*$ and a topological $U(1)$ gauge theory. Let us discuss these two 3d TQFTs in turn.

The fields of this model are a subset of the fields of the 3d theory listed above. The bosonic ones are the $\mathbb{C}^*$-valued scalar $h = \exp(-2\pi\tau)$ and the $\mathbb{C}$-valued scalar σ. The fermionic ones are the 0-forms $\psi_4 + i\tilde{\psi}_4, \eta + i\tilde{\eta}$ and the 1-forms $\psi - i\tilde{\psi}, \tilde{\chi}$. The RW model can be defined for any complex symplectic target space X, and $T^*\mathbb{C}^*$ is a special case with the symplectic form $d\tau \wedge d\sigma$. It is shown in Section 6.3.1 that the correct 3d action arises from the 4d action of the GL-twisted theory upon reduction.

For a general X the RW model has $\mathbb{Z}_2$ ghost number symmetry, but as explained in [18] when X is a cotangent bundle one can promote it to a $U(1)$ ghost number symmetry by letting the fiber coordinates have ghost number two. This agrees with the fact that σ has ghost number two already in the 4d theory. To emphasize that the fiber coordinate has ghost number two we will denote the target manifold $T^*[2]\mathbb{C}^*$.

According to [18] the simplest boundary conditions in the RW model correspond to complex Lagrangian submanifolds of X. If we want to preserve ghost number symmetry, these Lagrangian submanifolds must be invariant with respect to the rescaling $\sigma \mapsto \lambda^2\sigma$, $\lambda \in \mathbb{C}^*$. This requires the Lagrangian submanifold of $T^*\mathbb{C}^*$ to be the conormal bundle of a complex submanifold in $\mathbb{C}^*$. This means that a (closed) $\mathbb{C}^*$-invariant complex Lagrangian submanifold is either the zero section $\sigma = 0$ or one of the fibers of the cotangent bundle given by $\tau = \tau_0$. The zero section boundary condition plays a special role and will be denoted $\mathbb{X}_0$ in this subsection.

More general boundary conditions correspond to families of B-models or Landau-Ginzburg models parameterized by points in a complex Lagrangian submanifold. As mentioned in [18] and explained in more detail in [20] it is sufficient to restrict oneself to the case when the Lagrangian submanifold is the zero section $\sigma = 0$. One can describe these boundary conditions more algebraically as follows. Recall that the category of boundary line operators on the boundary $\mathbb{X}_0$ is a monoidal category which we denote $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$. Given any boundary condition $\mathbb{X}$ one may consider the category $\mathbb{V}_{\mathbb{X}\mathbb{X}_0}$ of boundary defect lines which may separate $\mathbb{X}$ from $\mathbb{X}_0$. This category is a module category over the monoidal category $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$. It was proposed in [18] that this module category completely characterizes the boundary condition $\mathbb{X}$. Concretely, in the case of the RW model with target $T^*[2]\mathbb{C}^*$ the category of boundary line operators $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$ is equivalent to $D^b(Coh(\mathbb{C}^*))$. One way to see it is to reduce the RW model on an interval with the boundary condition $\mathbb{X}_0$ on both boundaries. The resulting 2d TQFT is a B-model with target $\mathbb{C}^*$, and its category of branes may be identified with $D^b(Coh(\mathbb{C}^*))$. The 2d viewpoint does not allow one to determine the monoidal structure, but one can show that it is given by the usual derived tensor product [18, 20].

It was further argued in [18, 20] that the 2-category of boundary conditions for the RW model with target $T^*[2]\mathbb{C}^*$ is equivalent to the derived 2-category of module categories over $D^b(Coh(\mathbb{C}^*))$. That is, it is the 2-category of derived categorical sheaves over $\mathbb{C}^*$ as defined by B. Toen and G. Vezzosi [32]. This provides an algebraic description of boundary line operators and their OPEs for all boundary conditions.

There are two different topological gauge theories in 3d which can be obtained by twisting $N = 4$ $d = 3$ super-Yang-Mills theory. The first one is the dimensional reduction of the Donaldson-Witten twist of $N = 2$ $d = 4$ super-Yang-Mills theory. The second one is intrinsic to 3d and has been first discussed by Blau and Thompson [3]. We will refer to them as A-type and B-type topological gauge theories respectively. The reason for this terminology is that the BPS equations in the former theory are elliptic, as in the usual A-model, while in the latter theory they are overdetermined, as in the usual B-model. The definition and some properties of the B-type 3d gauge theory (for a general gauge group) are described in the Section 5.3. In this subsection we only deal with the abelian case.

Consider the 3d bosonic fields A, ϕ and the fermionic fields $\psi + i\tilde{\psi}, \chi, \eta - i\tilde{\eta}, \psi_4 - i\tilde{\psi}_4$. It is easy to check that their BRST transformations at $t = i$ are exactly the same as for the B-type 3d gauge theory. The action has a BRST-exact metric-dependent piece and a BRST-closed metric-independent piece:

$$S = -\frac{1}{2e^2}\,\delta\int_{M_3}\left(\chi\wedge\star\mathcal{F} - \frac{i}{2}(\eta - i\tilde{\eta})\wedge\star d^\star\phi\right) + \frac{1}{2e^2}\int_{M_3}(\psi_4 - i\tilde{\psi}_4)d\chi.$$

In principle we should gauge-fix the theory and modify the BRST operator appropriately; we leave this as an exercise for the reader.

As in any gauge theory, the most natural boundary conditions are the Dirichlet and Neumann ones. The Dirichlet condition requires the restriction of $A + i\phi$ to the boundary to be trivial. In addition, one requires ϕ_3 (the component of ϕ orthogonal to the boundary) to satisfy the Neumann condition $\partial_3\phi_3 = 0$. BRST-invariance then fixes the boundary conditions for fermions: the restriction of the forms $\psi + i\tilde{\psi}$, χ and $\eta - i\tilde{\eta}$ to the boundary must vanish, The Neumann boundary condition leaves the restriction of $\mathcal{A}$ to the boundary unconstrained but requires the restriction of the 1-form $\star\mathcal{F} = \star d\mathcal{A}$ to vanish. In addition ϕ_3 must satisfy the Dirichlet boundary condition, i.e. it must take a prescribed value on the boundary. In the Neumann case BRST-invariance requires the restrictions of the fermions $\star\chi$, $\psi_3 + i\tilde{\psi}_3$ and $\psi_4 - i\tilde{\psi}_4$ to vanish. Note that in the Dirichlet case the gauge group is completely broken at the boundary, while in the Neumann case it is unbroken.

The Dirichlet condition does not have any parameters, while the Neumann condition seems to depend on a single real parameter β, the boundary value of ϕ_3. On the quantum level there is another parameter: we can add to the action a boundary topological term

$$\theta\int_{\partial M_3}\frac{\mathcal{F}}{2\pi}.$$

In fact, both parameters are irrelevant, in the sense that topological correlators do not depend on them. The irrelevance of the parameter θ follows from the fact that the above topological term is BRST-exact and equal to

$$\frac{\theta}{2\pi}\delta\int_{\partial M_3}\chi.$$

To see the irrelevance of the parameter β, note that to shift β we need to add to the action a boundary term proportional to

$$\int_{\partial M_3}\partial_3\phi_3.$$

Since ϕ_1 and ϕ_2 vanish on the boundary, this is also BRST-exact and proportional to

$$\delta\int_{\partial M_3}(\eta - i\tilde{\eta}).$$

Following the same line of thought as in [18], one can try to describe the 2-category of boundary conditions in this theory by picking a distinguished boundary condition $\mathbb{X}_0$ and characterizing any other boundary condition $\mathbb{X}$ by the category $\mathbb{V}_{\mathbb{X}\mathbb{X}_0}$ of defect line operators between $\mathbb{X}$ and $\mathbb{X}_0$. That is, one attaches to any boundary condition $\mathbb{X}$ a module category $\mathbb{V}_{\mathbb{X}\mathbb{X}_0}$ over the monoidal category $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$.

An obvious guess for the distinguished boundary condition is the free (Neumann) one since it leaves the gauge group unbroken. To determine the category of boundary line operators $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$ for this boundary condition, one may reduce the 3d theory on an interval and study the category of branes in the resulting 2d TQFT. In the Neumann case, reduction on an interval gives the following result: the bosonic fields are the gauge field A and the 1-form ϕ, the fermionic ones are the 0-form $\eta - i\tilde{\eta}$, the 1-form $\psi + i\tilde{\psi}$ and the 2-form χ. This is the field content of a B-type topological gauge theory in 2d, see Section 4.2.2. It is easy to check that the BRST transformations of these fields are also the same as in the B-type 2d gauge theory. The category of branes for this 2d TQFT is the category of graded finite-dimensional representations of $G = U(1)$, see Section 4.2.2 for details. This is because the only boundary degrees of freedom one can attach are described by a vector space which carries a representation of the gauge group. The monoidal structure cannot be determined from 2d considerations, but it easy to see that it is given by the usual tensor product. Indeed, a brane corresponding to a representation space V is obtained by inserting the holonomy of the complex connection $\mathcal{A} = A + i\phi$ in the representation V into the path-integral. From the 3d viewpoint this means that the corresponding boundary line operator is the Wilson line operator for $\mathcal{A}$ in the representation V. On the classical level, the fusion of two Wilson line operators in representations V_1 and V_2 gives the Wilson line in representation $V_1 \otimes V_2$, and clearly there can be no quantum corrections to this result (the gauge coupling e^2 is an irrelevant parameter).

To summarize, the monoidal category $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$ is the category of graded finite-dimensional representations of $\mathbb{C}^*$, or equivalently the equivariant derived category of coherent sheaves over a point which we denote $D^b_{\mathbb{C}^*}(Coh(\bullet))$. We propose that the 2-category of boundary conditions is equivalent to the 2-category of module categories over $D^b_{\mathbb{C}^*}(Coh(\bullet))$. To give a concrete class of examples of such a module category, consider a Calabi-Yau manifold Y with a $\mathbb{C}^*$ action. The corresponding B-model can be coupled to the boundary gauge field and provides a natural set of topological boundary degrees of freedom for the 3d gauge theory. The corresponding category of boundary-changing line operators is the $\mathbb{C}^*$-equivariant bounded derived category of Y which is obviously a module category over $D^b_{\mathbb{C}^*}(Coh(\bullet))$.

It is fairly obvious how to combine the two models. The most basic boundary condition in the full theory is $\sigma = 0$ in the RW sector and the free (Neumann) condition in the gauge sector. We will call this the distinguished boundary condition. The bosonic fields which are free on the boundary are the $\mathbb{C}^*$-valued scalar $h = \exp(-2\pi\tau)$ and the restriction of the complex gauge field $\mathcal{A} = A + i\phi$. More general boundary conditions involve a boundary B-model or a boundary Landau-Ginzburg model fibered over $\mathbb{C}^*$ and admitting a $\mathbb{C}^*$-action. The fibration over $\mathbb{C}^*$ determines the coupling to the boundary value of τ, while the $\mathbb{C}^*$-action determines the coupling to the boundary gauge field

$\mathcal{A}$.

As in the RW model, we can give a more algebraic definition of the set of all boundary conditions in the full theory. This description is also useful because it suggests how to define the 2-category structure of the set of boundary conditions. We consider the monoidal category of boundary line operators for the distinguished boundary condition. This is the category of branes for the 2d TQFT obtained by reducing the gauged RW model on an interval. Since the reduction of the B-type 3d gauge theory gives the B-type 2d gauge theory, and the reduction of the RW model gives the B-model with target $\mathbb{C}^*$, the effective 2d TQFT is the gauged B-model with target $\mathbb{C}^*$, where the gauge group $U(1)$ acts trivially. As described in Section 4.3, the corresponding category of branes is equivalent to $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}^*))$. The monoidal structure cannot be determined from the 2d considerations, but it is easy to see (given the results for the RW model and the B-type gauge theory in 3d) that it is given by the derived tensor product.

Every boundary condition gives rise to a module category over this monoidal category. It is natural to conjecture that the converse is also true, every reasonable module category over this monoidal category can be thought of as a boundary condition for the full 3d TQFT. For example, we may consider a family of Calabi-Yau manifolds parameterized by points of $\mathbb{C}^*$ such that each model in the family has a $\mathbb{C}^*$ symmetry. The corresponding module category is the $\mathbb{C}^*$-equivariant derived category of the total space of the fibration. This gives us a conjectural description of the 2-category of surface operators in the parent 4d gauge theory.

Let us describe how Gukov-Witten surface operators fit into this picture. Such operators depend on a complex parameter $h_0 = \exp(-2\pi(\alpha - i\gamma))$ taking values in $\mathbb{C}^*$. From the 3d viewpoint, h_0 determines the boundary value of the scalar $h = \exp(-2\pi\tau)$ in the RW sector. The other scalar σ is left free. Thus the boundary conditions for the RW sector correspond to a Lagrangian submanifold of $T^*[2]\mathbb{C}^*$ given by $h = h_0$ (the fiber over the point h_0). The gauge sector boundary conditions are of Neumann type and have no nontrivial parameters.There are no boundary degrees of freedom. From our algebraic viewpoint we may describe this as follows. In the usual RW theory the fiber over $h = h_0$ corresponds to a skyscraper sheaf of DG-categories over $\mathbb{C}^*$ whose "stalk" over h_0 is the category of bounded complexes of vector spaces. We may denote it $D^b(Coh(\bullet))$. Including the gauge degrees of freedom means working with a sheaf of categories with a $\mathbb{C}^*$ action. Thus we simply consider a skyscraper sheaf of categories over $\mathbb{C}^*$ whose "stalk" over h_0 is the category of $\mathbb{C}^*$-equivariant complexes of vector spaces $D^b_{\mathbb{C}^*}(Coh(\bullet))$. The monoidal category $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}^*))$ acts on it in a fairly obvious manner: one simply tensors an object of $D^b_{\mathbb{C}^*}(Coh(\bullet))$ with the (derived) restriction of an object of $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}^*))$ to the point $h = h_0$.

The category of morphisms between two different skycraper sheaves of categories is trivial (the set of objects is empty). This corresponds to the fact that two different Gukov-Witten surface operators cannot join along a boundary-changing line operator. But the category of line operators sitting on a particular Gukov-Witten surface operator (the endomorphism category of a Gukov-Witten surface operator) is nontrivial. Its most obvious objects are Wilson lines for the complexified gauge field $\mathcal{A}$, which are obviously BRST-invariant. Such operators are labeled by irreducible representations of $\mathbb{C}^*$. One might guess therefore that the category of surface line operators is simply the category of representations of $\mathbb{C}^*$, or perhaps the category of $\mathbb{C}^*$-equivariant complexes of vector spaces which we denoted $D^b_{\mathbb{C}^*}(Coh(\bullet))$ above. However, this naive guess is wrong, which can be seen by inspecting BRST-invariant local operators which can be inserted into such a Wilson line operator. From the abstract viewpoint they form an algebra (the endomorphism algebra of an object in the category of line operators). It is clear that any power of the field σ gives such an operator, so the algebra of local operators on a line operator is the algebra of polynomial functions of a single variable of ghost number 2. In what follows we will denote the line parameterized by σ by $\mathbb{C}[2]$ to indicate that σ sits in degree 2; thus $\mathbb{C}[2]$ is a purely even graded manifold. On the other hand, the algebra of endomorphisms of an irreducible representation of $\mathbb{C}^*$ is simply $\mathbb{C}$.

To determine what the category of line operators is it is convenient to take the 2d viewpoint and reduce the 3d theory on an interval with the Gukov-Witten-type boundary condition on both ends. Let x^3 denote the coordinate on the interval. Gukov-Witten boundary conditions eliminate the complex scalar h (which is now locked at the value h_0) and the field ϕ_3 but keep the complex scalar σ and the gauge field $\mathcal{A}$. Thus the effective 2d theory also has two sectors: the B-model with target $\mathbb{C}[2]$ and the B-type 2d topological gauge theory. According to Section 4.3, the corresponding category of branes is equivalent to the $\mathbb{C}^*$-equivariant derived category of $\mathbb{C}[2]$: its objects can be regarded as $\mathbb{C}^*$-equivariant complexes of holomorphic vector bundles on $\mathbb{C}[2]$ (with a trivial $\mathbb{C}^*$ action on $\mathbb{C}[2]$).

This answer is independent of the parameter $h_0 = \exp(-2\pi(\alpha - i\gamma))$ of the Gukov-Witten surface operator. In particular, we can choose the trivial surface operator $h_0 = 1$, in which case we should get the category of bulk line operators in the GL-twisted theory at $t = i$.

It is not difficult to see that this answer for the category of bulk line operators agrees with the computation of the endomorphism algebra of a Wilson line explained above. Indeed, an insertion of a Wilson line does not put any constraints on σ and does not add any degrees of freedom, and therefore should correspond to a trivial line bundle over $\mathbb{C}[2]$. Its fiber carries a representation of $\mathbb{C}^*$ determined by the charge of the Wilson line. The endomorphism algebra of such an object of $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}[2]))$ is simply the algebra of polynomial functions on $\mathbb{C}[2]$.

It is now clear that the category of line operators contains objects other than Wilson lines. For example, we may consider a skyscraper sheaf at the origin of $\mathbb{C}[2]$, whose stalk at the origin is a complex line V carrying some representation of $\mathbb{C}^*$. There are two different way to define the corresponding line operator. First, we may consider a free resolution of the skyscraper:

$$V[-2] \otimes \mathcal{O} \to V \otimes \mathcal{O},$$

where $V[-2]$ means V placed in ghost degree -2, $\mathcal{O}$ is the algebra of polynomial functions on $\mathbb{C}[2]$, and the cochain map is multiplication by σ. The shift by -2 is needed so that the cochain map has total degree 1. The existence of such a resolution means that we can realize the "skyscraper" line operator as a "bound state" of two Wilson lines both associated with the representation V but placed in different cohomological degrees. The corresponding bulk line operator is obtained using the formulas of Section 4.3, where the target of the gauged B-model is taken to be $\mathbb{C}[2]$, the vector bundle E on $\mathbb{C}[2]$ is trivial and of rank 2, with graded components in degrees 1 and 0, and the bundle morphism T from the former to the latter component is multiplication by σ. In accordance with the Section 4.3, we consider a superconnection on $\sigma^* E$ of the form

$$\mathcal{N} = \begin{pmatrix} n\mathcal{A} & 0 \\ \frac{1}{2}(\psi - i\tilde{\psi}) & n\mathcal{A} \end{pmatrix},$$

where $n \in \mathbb{Z}$ is the weight with which $\mathbb{C}^*$ acts on V. The bulk line operator corresponding to the skyscraper sheaf at the origin of $\mathbb{C}[2]$ is the holonomy of this superconnection along the insertion line ℓ.

Another (equivalent) way is to take seriously the fact that the skyscraper sheaf is localized at $\sigma = 0$ and require the field σ to vanish at the insertion line ℓ. To make this well-defined, one needs to excise a small tubular neighborhood of ℓ and impose a suitable boundary condition on the resulting boundary. This condition must set $\sigma = 0$ and leave the components of $\mathcal{A}$ tangent to the boundary ℓ unconstrained. BRST-invariance determines uniquely the boundary conditions for all other fields.

6.3.3 Surface Operators at $t = i$ with a Nonabelian Gauge Group

To generalize the preceding discussion to the nonabelian case we need to understand the 3d TQFT which is obtained by compactifying the 4d gauge theory on a circle. This is less straightforward than in the abelian case, because requiring the fields to be independent of the x^4 coordinate is not a gauge-invariant condition. One can try to avoid dealing with this issue by first fixing a gauge such that A_4 does not depend on x^4. This works in the neighborhood of $A_4 = 0$, when the holonomy of A along S^1 is close to 1. But in general the condition that A_4 is x^4-independent does not fix the freedom to make x^4-dependent gauge transformations. For example, suppose A_4 is proportional to an element $\mu \in \mathfrak{g}$ which satisfies

$$\exp(2\pi\mu) = 1.$$

Such μ are precisely those which lie in the G-orbits of the cocharacter lattice of G. Then the gauge transformation

$$g(x^4) = \exp(\mu x^4)$$

shifts A_4 by μ:

$$A_4 \mapsto A_4 + \mu.$$

Such a gauge transformation in general makes other fields x^4-dependent.

It is shown in Section 6.3.1 that the naive reduction procedure which requires all fields to be independent of x^4 gives the gauged Rozansky-Witten model with target $T^*[2]\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}^*_{\mathbb{C}}[2]$, where the gauge group G acts on the base $\mathfrak{g}$ and the fiber $\mathfrak{g}^*[2]$ via the adjoint and coadjoint representations, respectively. The symplectic form is the canonical form on the cotangent bundle. The true target space of the reduced model is $T^*[2]G_{\mathbb{C}}$ which contains an open neighborhood of the origin in $T^*[2]\mathfrak{g}_{\mathbb{C}}$ as an open subset. We conjecture that the 3d theory is the gauged Rozansky-Witten model with target $T^*[2]G_{\mathbb{C}}$, basically because it is the only obvious possibility.

Let us consider some boundary conditions in the gauged Rozansky-Witten model with target $T^*[2]G_{\mathbb{C}}$. The most natural boundary condition in the gauge sector is the Neumann condition, which preserves full gauge-invariance on the boundary. In the matter sector one has to pick a G-invariant complex Lagrangian submanifold of $T^*[2]G_{\mathbb{C}}$ which is invariant with respect to the rescaling of the fiber. Such a Lagrangian submanifold can be constructed by picking a $G_{\mathbb{C}}$-invariant closed complex submanifold of $G_{\mathbb{C}}$ and taking its conormal bundle. For example, one can take the whole $G_{\mathbb{C}}$, and then the Lagrangian submanifold is given by $\sigma = 0$. We will call the resulting boundary condition in the gauged RW model the distinguished boundary condition. It is an analogue of the NN condition in the abelian case.

Another natural choice of a G-invariant Lagrangian submanifold is the conormal bundle of a complex conjugacy class in $G_{\mathbb{C}}$. In order for the submanifold to be closed take the conjugacy class to be semisimple. This boundary condition is a nonabelian analogue of the ND condition. The corresponding surface operator is a semisimple Gukov-Witten-type surface operator. Indeed, fixing a semisimple conjugacy class of $\exp(-2\pi(A_4 + i\phi_4))$ is the same as fixing a semisimple conjugacy class of the limiting holonomy of the complex connection $A + i\phi$ in the 4d gauge theory. More generally, if the conjugacy class is not closed, one needs to consider the conormal bundle of its closure.

It is easy to analyze boundary line operators for these boundary conditions. Reducing the 3d theory on an interval with the distinguished boundary conditions we get a B-type 2d gauge theory coupled to a B-model with target $G_{\mathbb{C}}$. The gauge group acts on $G_{\mathbb{C}}$ by conjugation. According to Section 4.3, the corresponding category of branes is equivalent to $D^b_{G_{\mathbb{C}}}(Coh(G_{\mathbb{C}}))$. The monoidal structure cannot be deduced from the 2d considerations, but the same analysis as in the usual RW model shows that it is given by the derived tensor product.

In the Gukov-Witten case we need to fix a semisimple complex conjugacy class $\mathcal{C}$ in $G_{\mathbb{C}}$. Let $N^*\mathcal{C}$ denote the total space of its conormal bundle in $T^*G_{\mathbb{C}}$. Concretely, it is the space of pairs (g,σ), where $g\in\mathcal{C}$ and $\sigma\in\mathfrak{g}_{\mathbb{C}}$ satisfies $\mathrm{Tr}\,\sigma\, g^{-1}\delta g=0$ for any δg tangent to $\mathcal{C}$ at g. The fiber coordinate σ has cohomological degree 2; to indicate this we will denote the corresponding graded complex manifold $N^*[2]\mathcal{C}$. Reduction on an interval in the Gukov-Witten case gives a B-type 2d gauge theory coupled to a B-model whose target is $N^*[2]\mathcal{C}$. Its category of branes is $D^b_{G_{\mathbb{C}}}(Coh(N^*[2]\mathcal{C}))$. The monoidal structure is given by the derived tensor product.

It is interesting to consider the special case of a Gukov-Witten surface operator corresponding to the trivial conjugacy class in $G_{\mathbb{C}}$ (the identity). This is the trivial surface operator, so the category of 3d boundary line operators in this case can be identified with the category of bulk line operators in the 4d TQFT. The conormal bundle of the identity element is simply the dual of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$; the group $G_{\mathbb{C}}$ acts on it by the adjoint representation. Thus the category of 4d bulk line operators is equivalent to $D^b_{G_{\mathbb{C}}}(Coh(\mathfrak{g}^*_{\mathbb{C}}[2]))$. In other words, it is the $G_{\mathbb{C}}$-equivariant derived category of the graded algebra $\oplus_p \mathrm{Sym}^p\mathfrak{g}$ where the p^{th} component sits in cohomological degree $2p$.

In view of this result it is interesting to consider local operators sitting at the junction of two Wilson loops in representations V_1 and V_2 of G. The corresponding objects of the category $D^b_{G_{\mathbb{C}}}(Coh(\mathfrak{g}^*_{\mathbb{C}}[2]))$ are free modules over $\mathfrak{A}=\oplus_p\mathrm{Sym}^p\mathfrak{g}[2]$ of the form $V_1\otimes_{\mathbb{C}}\mathfrak{A}$ and $V_2\otimes\mathfrak{A}$, with the obvious $G_{\mathbb{C}}$ action. The space of morphisms between them is the space of $G_{\mathbb{C}}$-invariants in the infinite-dimensional graded representation

$$V_1^*\otimes V_2\otimes\mathfrak{A}.$$

Indeed, a BRST-invariant and gauge-invariant junction of two Wilson lines should be an operator in representation $V_1^*\otimes V_2$ constructed out of the complex scalar σ taking values in $\mathfrak{g}_{\mathbb{C}}$. The space of such operators in ghost number $2p$ is $\mathrm{Hom}_G(\mathrm{Sym}^p\mathfrak{g}, V_1^*\otimes V_2)$, where Hom_G denotes the space of morphisms in the category of representations of G. Summing over all p we get the above answer.

As in the abelian case, the above examples do not exhaust the set of objects in the 2-category of surface operators. For example, in [45] more complicated surface operators have been considered which involve higher-order poles for the complex connection $\mathcal{A}=A+i\phi$. By analogy with the Rozansky-Witten model we propose that the most general surface operator at $t=i$ (or equivalently, the most general boundary condition in the 3d theory) can be defined as a module category of the monoidal category of boundary line operators for the distinguished boundary condition $\mathbb{X}_0$. As explained above, this monoidal category is $D^b_{G_{\mathbb{C}}}(Coh(G_{\mathbb{C}}))$. Concretely, this means that the most general surface operator can be obtained by fibering a family of 2d TQFTs over $G_{\mathbb{C}}$, so that the $G_{\mathbb{C}}$ action on the base (by conjugation) lifts to a $G_{\mathbb{C}}$ action on the whole family. For example, one may consider a complex manifolds X which is a fibration over $G_{\mathbb{C}}$, so that fibers are Calabi-Yau manifolds, and one is given a lift of the $G_{\mathbb{C}}$ action on the base (by conjugation) to a $G_{\mathbb{C}}$ action on the total space.

Given any surface operator $\mathbb{X}$, one may construct a module category over $D^b_{G_{\mathbb{C}}}(Coh(G_{\mathbb{C}}))$ by looking at the category of line operators sitting at the junction of $\mathbb{X}$ and the distinguished surface operator $\mathbb{X}_0$. This category is the category of branes in the 2d TQFT obtained by compactifying the 3d TQFT on an interval, with the boundary conditions on the two ends given by $\mathbb{X}$ and $\mathbb{X}_0$. Equivalently, one may compactify the 4d TQFT on a twice-punctured 2-sphere, with surface operators $\mathbb{X}$ and $\mathbb{X}_0$ inserted at the two punctures.

For example, if we consider a surface operator defined, as in [45], by a prescribed singularity in the complex connection $\mathcal{A}$, and take into account that the distinguished surface operator is defined by allowing the holonomy of $\mathcal{A}$ to be free, we see that the space of vacua of the effective 2d TQFT is the moduli space of connections on a punctured disc with the prescribed singularity at the origin. Let us denote this moduli space $\mathcal{M}$. If in the definition of $\mathcal{M}$ we divide by the group of gauge transformations which reduce to the identity at some chosen point on the boundary of the disk, then $\mathcal{M}$ is acted upon by $G_{\mathbb{C}}$ and is fibered over $G_{\mathbb{C}}$ (the holonomy of $\mathcal{A}$ along the boundary of the disk). It looks plausible that the effective 2d TQFT is the B-model with target $\mathcal{M}$ coupled to a B-type gauge theory with gauge group $G_{\mathbb{C}}$. Its category of branes is a module category over $D^b_{G_{\mathbb{C}}}(Coh(G_{\mathbb{C}}))$.

6.3.4 Surface Operators at $t = 1$ with an Abelian Gauge Group

For $t = 1$ the 4d TQFT compactified on a circle also decomposes into two sectors (gauge and matter), but the analysis of boundary conditions is less straightforward because neither sector has been studied previously. For this reason we will restrict ourselves to the abelian case, which is fairly elementary.

The gauge sector consists of a 3d gauge field A, a real bosonic 0-form ϕ_4, a complex bosonic 0-form σ, a fermionic 2-form $\chi = \frac{1}{2}\chi_{ij}dx^i dx^j$, a fermionic 1-form $\psi + \tilde{\psi}$, and fermionic 0-forms

$\psi_4 - \tilde{\psi}_4, \eta + \tilde{\eta}$. Their BRST transformations read

$$\begin{aligned}
\delta A &= i(\psi + \tilde{\psi}), \\
\delta \phi_4 &= i(\psi_4 - \tilde{\psi}_4), \\
\delta \sigma &= 0, \\
\delta \bar{\sigma} &= i(\eta + \tilde{\eta}), \\
\delta(\psi + \tilde{\psi}) &= 2d\sigma, \\
\delta(\psi_4 - \tilde{\psi}_4) &= 0, \\
\delta(\eta + \tilde{\eta}) &= 0, \\
\delta \chi &= F + \star d\phi_4.
\end{aligned}$$

On-shell they satisfy $\delta^2 = 2i\delta_g(\sigma)$, where $\delta_g(\sigma)$ is a gauge transformation with the parameter σ. The action is BRST-exact:

$$S_{gauge} = -\frac{1}{2e^2}\,\delta \int_{M_3} \left(\chi \wedge \star(F + \star d\phi_4) + \frac{1}{2}(\psi + \tilde{\psi}) \wedge \star d\bar{\sigma} \right).$$

The matter sector consists of a real periodic scalar A_4, a real bosonic 1-form ϕ, fermionic 1-forms $\psi - \tilde{\psi}, \rho = \chi_{i4} dx^i$, and fermionic 0-forms $\psi_4 + \tilde{\psi}_4, \eta - \tilde{\eta}$. Their BRST transformations read

$$\begin{aligned}
\delta A_4 &= i(\psi_4 + \tilde{\psi}_4), \\
\delta \phi &= i(\psi - \tilde{\psi}), \\
\delta(\psi - \tilde{\psi}) &= 0, \\
\delta(\psi_4 + \tilde{\psi}_4) &= 0, \\
\delta \rho &= dA_4 + \star d\phi, \\
\delta(\eta - \tilde{\eta}) &= 2d^\star \phi.
\end{aligned}$$

On-shell they satisfy $\delta^2 = 0$. The matter action is also BRST-exact:

$$S_{matter} = -\frac{1}{2e^2}\,\delta \int_{M_3} \left(\rho \wedge \star(dA_4 + \star d\phi) + \frac{1}{2}(\eta - \tilde{\eta}) \wedge \star d^\star \phi \right).$$

The gauge sector is the dimensional reduction of the Donaldson-Witten 4d TQFT [38] down to 3d. Above we have called this theory an A-type gauge theory. However, this by itself does not teach us very much, since boundary conditions in this theory have not been discussed previously. Without adding boundary degrees of freedom, the only choices are the Dirichlet and Neumann boundary conditions for gauge fields, with BRST-invariance fixing the conditions on all other fields.

Let us begin with the Dirichlet condition which says that the restriction of A to the boundary is trivial. Since $\delta^2 = 2i\delta_g(\sigma)$, this makes sense only if σ also vanishes on the boundary. BRST-invariance then requires $\eta + \tilde{\eta}$ and the restriction of the 1-form $\psi + \tilde{\psi}$ to vanish. The fermionic equations of motion then require the restriction of χ to vanish, and the BRST-invariance implies that ϕ_4 must satisfy the Neumann condition $\partial_3\phi_4 = 0$, where we assumed that the boundary is given by $x^3 = 0$.

The Dirichlet boundary condition has the property that it has no nontrivial local BRST-invariant boundary observables. Indeed, the only nonvanishing BRST-invariant 0-form is $\psi_4 - \tilde{\psi}_4$, but it is BRST-exact. To analyze boundary line operators, we use the dimensional reduction trick and compactify the 3d theory on an interval with the Dirichlet boundary conditions. The only bosonic fields in the effective 2d theory are the constant mode of ϕ_4 and the holonomy of A along the interval parameterized by x^3. That is, the bosonic fields are a real scalar and a periodic real scalar. The effective 2d TQFT is therefore a sigma-model with target $\mathbb{R} \times S^1$. In fact, it can be regarded as an A-model with target T^*S^1. The easiest way to see this is to note that the path-integral of the 3d theory localizes on configurations given by solutions of the Bogomolny equations

$$F + \star d\phi_4 = 0.$$

Upon setting all fields to zero except A_3 and ϕ_4 and assuming that they are independent of x^3, this equation becomes

$$dA_3 + \star d\phi_4 = 0,$$

where $\star$ is the 2d the Hodge star operator. This is an elliptic equation which can be interpreted as the holomorphic instanton equation, provided we declare $A_3 + i\phi_4$ to be a complex coordinate on the target. Since the action of the 4d theory is BRST-exact, so is the action of the 2d model. This agrees with the well-known fact that the action of an A-model is BRST-exact if the symplectic form on the target space is exact.

The category of line operators on the Dirichlet boundary is therefore the Fukaya-Floer category of T^*S^1 whose simplest objects are Lagrangian submanifolds equipped with unitary vector bundles with flat connections. Since this category arises as the endomorphism category of an object in a 2-category, it must have a monoidal structure, which is not visible from the purely 2d viewpoint. In fact, we do not expect the Fukaya-Floer category of a general symplectic manifold to have a natural monoidal structure. We will argue below that the monoidal structure is induced by the mirror symmetry which establishes the equivalence of the Fukaya-Floer category of T^*S^1 with $D^b(Coh(\mathbb{C}^*))$ and the monoidal structure on the latter category. For now we just note that the base S^1 has a distinguished point corresponding to the trivial holonomy of A on the interval. The fiber over this point is a Lagrangian submanifold in T^*S^1 and is the identity object with respect to the monoidal structure. The distinguished point allows us to identity S^1 with the group manifold $U(1)$.

Now let us consider the Neumann condition for the 3d gauge field A. This means that the gauge symmetry is unbroken on the boundary and the restriction of the 1-form $\star F$ vanishes. Then the Bogomolny equation requires ϕ_4 to have the Dirichlet boundary condition $\phi_4 = a = const$, and by BRST-invariance $\psi_4 - \tilde{\psi}_4$ must vanish at $x^3 = 0$. Fermionic equations of motion imply then that $\psi_3 + \tilde{\psi}_3$ vanishes as well, and since $\delta(\psi_3 + \tilde{\psi}_3) = 2\partial_3\sigma$, the field σ satisfies the Neumann condition. Finally, the restriction of the 1-form $\star\chi$ to the boundary must vanish, in order for the fermionic boundary conditions to be consistent. Indeed, if x^1 is regarded as the time direction, then $(\star\chi)_2$ is canonically conjugate to $\psi_3 + \tilde{\psi}_3$, so if one of them vanishes, so should the other. Similarly, if x^2 is regarded as time, then $(\star\chi)_1$ is canonically conjugate to $\psi_3 + \tilde{\psi}_3$ and therefore must vanish too.

In the Neumann case the space of BRST-invariant local observables on the boundary is spanned by powers of the field σ. To determine the category of boundary line operators one has to reduce the 3d gauge theory on an interval with the Neumann boundary conditions. The bosonic fields of the effective 2d theory are the 2d gauge field and the constant mode of the scalar σ, the fermionic ones are the 0-form $\eta + \tilde{\eta}$, the 1-form $\psi + \tilde{\psi}$, and the 2-form χ. Their BRST transformations are

$$\begin{aligned}
\delta A &= i(\psi + \tilde{\psi}), \\
\delta\sigma &= 0, \\
\delta\bar{\sigma} &= i(\eta + \tilde{\eta}), \\
\delta(\eta + \tilde{\eta}) &= 0, \\
\delta(\psi + \tilde{\psi}) &= 2d\sigma, \\
\delta\chi &= F.
\end{aligned}$$

This 2d TQFT can be obtained from the usual $N = (2, 2)$ $d = 2$ supersymmetric gauge theory by means of a twist which makes use of the $U(1)_V$ R-symmetry. Since this is the same R-symmetry as that used for constructing an A-type sigma-model, we might call this TQFT an A-type 2d gauge theory. As far as we know, its boundary conditions have not been analyzed in the literature previously. It is shown in Section 4.2.1 that its category of branes is equivalent to the bounded derived category of coherent sheaves on the graded line $\mathbb{C}[2]$. [1] Again, the 3d origin of this category means that it must have monoidal structure. Here it is given by the usual derived tensor product of complexes of coherent sheaves. The trivial line bundle on $\mathbb{C}[2]$ is the identity object. From the 3d viewpoint, it corresponds to the "invisible" line operator on the boundary.

As mentioned above, the Neumann condition depends on a real parameter a, the boundary value of the scalar ϕ_4. On the quantum level there is another parameter which takes values in $\mathbb{R}/2\pi\mathbb{Z}$. It

[1] This category is equivalent to the $U(1)$-equivariant constructible derived category of sheaves over a point [2].

enters as the coefficient of a topological term in the boundary action:

$$\theta \int_{x^3=0} \frac{F}{2\pi}.$$

Thus overall the Neumann condition in the gauge sector has the parameter space $\mathbb{R} \times \mathrm{S}^1 \simeq \mathbb{C}^*$.

We may impose either Dirichlet or Neumann condition on the periodic scalar A_4. Let us discuss these two possibilities in turn.

If A_4 satisfies the Dirichlet condition, then BRST-invariance requires the 1-form ϕ to satisfy the Neumann condition. This means that the components of ϕ tangent to the boundary are free and satisfy $\partial_3\phi_1 = \partial_3\phi_2 = 0$, while the component ϕ_3 takes a fixed value $\phi_3 = a$ on the boundary. BRST-invariance also requires the following fermions to vanish on the boundary: $\psi_4 + \tilde{\psi}_4$, $\psi_3 - \tilde{\psi}_3$, ρ_1, ρ_2. The real parameter a together with the boundary value of A_4 combine into a parameter taking values in $S^1 \times \mathbb{R}$. These parameters are actually irrelevant, in the sense that topological correlators do not depend on them. To see this, note that shifting the boundary value of A_4 can be achieved by adding a boundary term to the action of the form

$$\int_{x^3=0} \partial_3 A_4 d^2x = \int_{x^3=0} \left(\delta\rho_3 - (\partial_1\phi_2 - \partial_2\phi_1)\right) d^2x$$

We see that up to a total derivative this boundary term is BRST-exact, hence does not affect the correlators. A similar argument can be made for the boundary value of ϕ_3.

The reduction on an interval with the Dirichlet boundary conditions gives rise to a 2d TQFT whose only bosonic field is a real 1-form ϕ. Such a 2d TQFT has not been considered previously, but it is closely related to an A-model with target $T^*\mathbb{R}$. To see this, consider an $N = (2,2)$ supersymmetric sigma-model with target $\mathbb{C}$ (with the standard flat metric). This model has a $U(1)$ symmetry which acts on the target space coordinate Z by

$$Z \mapsto e^{i\alpha} Z.$$

One can add a multiple of the corresponding $U(1)$ current to the standard R-current, thereby defining a new R-current. When performing the A-twist, we can choose this modified R-current instead of the standard one. If Z has charge two with respect to the modified R-symmetry, after twist Re Z and Im Z will become components of a 1-form. We will call the resulting 2d TQFT the modified A-model.

Apart from the bosonic 1-form ϕ, the modified A-model has a fermionic 1-form $\psi - \tilde{\psi}$ and a pair of fermionic 0-forms $\eta - \tilde{\eta}$ and ρ (the latter comes from the component ρ_3 of the 1-form ρ in 3d).

Their BRST transformations are

$$\delta\phi = i(\psi - \tilde{\psi}),$$
$$\delta(\psi - \tilde{\psi}) = 0,$$
$$\delta(\eta - \tilde{\eta}) = 2d^\star\phi,$$
$$\delta\rho = \star d\phi.$$

Here $\star$ is the 2d Hodge star operator, and $d^\star = \star d \star$.

To understand the category of boundary line operators in 3d, we need to describe the category of boundary conditions for the modified A-model. This is fairly straightforward. A natural class of boundary conditions is obtained by imposing on the boundary

$$(a\phi + b \star \phi)\,|_{\partial M_2} = 0.$$

The special cases $b = 0$ and $a = 0$ correspond to the 2d Dirichlet and Neumann conditions. Since the theory obviously has a symmetry rotating ϕ into $\star\phi$, it is sufficient to consider the Neumann condition $\star\phi| = 0$. BRST-invariance requires the restriction of $\star\psi$ and ρ to vanish on such a boundary. It is easy to see that there are no nontrivial BRST-invariant boundary observables (the only BRST-invariant fermion ψ is BRST-exact), so there is no possibility to couple boundary degrees of freedom in a nontrivial way. This implies that the category of boundary conditions is the same as for a trivial 2d TQFT, the category of complexes of finite-dimensional vector spaces. We may denote it $D^b(Coh(\bullet))$.

There is an important subtlety here related to the fact that the scalar A_4 is periodic with period 1. When reducing on an interval, this means that there are "winding sectors", where

$$\int dx^3 \partial_3 A_4 = n, \quad n \in \mathbb{Z}.$$

This winding is constant along a connected component of the boundary and does not affect the 2d theory in any way. We may incorporate it by introducing an additional integer label on each boundary component which serves as a conserved boundary charge. This is mathematically equivalent to saying that the category of boundary conditions is the category of $\mathbb{C}^*$-equivariant coherent sheaves over a point $D^b_{\mathbb{C}^*}(Coh(\bullet))$. Objects of this category are complexes of finite-dimensional vector spaces with a $\mathbb{C}^*$-action, such that the differentials in the complex commute with the $\mathbb{C}^*$ action. Morphisms are required to preserve the $\mathbb{C}^*$-action, having zero $\mathbb{C}^*$-charge.

If A_4 satisfies the Neumann condition $\partial_3 A_4 = 0$, then BRST-invariance requires ϕ to satisfy the Dirichlet condition. That is, the restriction of ϕ to the boundary must vanish, and ϕ_3 must satisfy

$\partial_3\phi_3 = 0$. This boundary condition does not have any parameters.

The reduction on an interval gives rise to the A-model with the bosonic fields A_4 and ϕ_3. This can be seen, for example, by looking at the 3d BPS equation $dA_4 + \star d\phi = 0$ and restricting to field configurations where $\phi_1 = \phi_2 = 0$ and A_4 and ϕ_3 are independent of x^3. For such field configuration the BPS equation becomes the holomorphic instanton equation with target $S^1 \times \mathbb{R} \simeq \mathbb{C}^*$. From the symplectic viewpoint, $\mathbb{C}^*$ with its standard Kähler form is isomorphic to T^*S^1. Thus the category of boundary line operators in this case is the Fukaya-Floer category of T^*S^1. Since this category arises as the category of boundary line operators in the 3d TQFT, it must have a monoidal structure. Although the category appears to be the same as in the gauge sector with the Dirichlet boundary condition, we will see that the monoidal structure is completely different and is induced by the equivalence between (a version of) the Fukaya-Floer category of T^*S^1 and the constructible derived category of S^1 [27]. In particular, the identity object (the invisible boundary line operator) is different and corresponds to the zero section of T^*S^1 with a trivial rank-1 local system. This illustrates the fact that a monoidal structure on branes in a 2d TQFT depends on the way this 2d TQFT is realized as a compactification of a 3d TQFT on an interval.

6.3.5 Electric-Magnetic Duality

We are now ready to describe how the 4d electric-magnetic duality acts on various boundary conditions described above. Since for both gauge and matter sectors one can have either Dirichlet or Neumann conditions, there are four possibilities to consider.

From the 3d viewpoint, 4d electric-magnetic duality amounts to dualizing the 3d gauge field A into a periodic scalar, and simultaneously dualizing the periodic scalar A_4 into a 3d gauge field. It is easy to see that electric-magnetic duality applied to the A-type gauge theory gives the Rozansky-Witten model with target $T^*[2]\mathbb{C}^*$, it maps the A-type gauge sector to the B-type matter sector. Similarly, it maps the A-type matter sector into the B-type gauge theory (with gauge group $U(1)$). In other words, electric-magnetic duality reduces to particle-vortex duality done twice.

The dual of the Neumann condition for a periodic scalar is the Dirichlet condition for the gauge field, and vice-versa. We will use this well-known fact repeatedly in what follows.

The first possibility is the Dirichlet condition in both gauge and matter sectors at $t = 1$. The Dirichlet condition in the A-type gauge sector maps into a boundary condition in the Rozansky-Witten model with target $T^*[2]\mathbb{C}^*$ which sets $\sigma = 0$ on the boundary and leaves the complex scalar τ free to fluctuate. The Dirichlet condition in the A-type matter sector is mapped to the Neumann condition in the B-type gauge theory. Note that the Dirichlet condition in the A-type matter sector has two real parameters taking values in S^1 and $\mathbb{R}$. The former one is mapped to a boundary theta-angle, a boundary term in the action of the form

$$\theta \int_{x^3=0} \frac{F}{2\pi} = \theta \int_{x^3=0} \frac{\mathcal{F}}{2\pi}.$$

The latter parameter is the boundary value of the field ϕ_3. Both of these parameters are irrelevant, as discussed in Section 4.

As discussed above, the category of boundary line operators in the A-type 3d gauge theory is the Fukaya-Floer category of $T^*U(1)$. On the other hand, the category of boundary line operators in the Rozansky-Witten model is $D^b(Coh(\mathbb{C}^*))$, as explained in [18]. These categories are equivalent, by the usual 2d mirror symmetry.

Let us recall how 2d mirror symmetry acts on some objects in this case. The trivial line bundle on $\mathbb{C}^*$ is mapped to the fiber over a distinguished point of the base S^1. This distinguished point allows us to identify S^1 with the group manifold $U(1)$. More generally, we may consider a holomorphic line bundle on $\mathbb{C}^*$ with a $\bar{\partial}$-connection of the form

$$\bar{\partial} + i\lambda \frac{d\bar{z}}{\bar{z}}, \quad \lambda \in \mathbb{C}.$$

We will denote such a line bundle $\mathcal{L}_\lambda$. Gauge transformations can be used to eliminate the imaginary part of λ. They also can shift the real part of λ by an arbitrary integer. Thus we may regard the parameter λ as taking values in $\mathbb{R}/\mathbb{Z} \simeq S^1$. Mirror symmetry maps $\mathcal{L}_\lambda$ to a Lagrangian submanifold in $T^*U(1)$ which is a fiber over the point $\exp(2\pi i\lambda) \in U(1)$.

Applying mirror symmetry to the obvious monoidal structure on $D^b(Coh(\mathbb{C}^*))$ given by the derived tensor product we get a monoidal structure on the Fukaya category of $T^*U(1)$. The trivial holomorphic line bundle on $\mathbb{C}^*$, which serves as the identity object in $D^b(Coh(\mathbb{C}^*))$, is mapped to the Lagrangian fiber over the identity element of $U(1)$. If we consider two Lagrangian fibers over the points $\exp(2\pi i\lambda_1), \exp(2\pi i\lambda_2) \in U(1)$, their mirrors are line bundles $\mathcal{L}_{\lambda_1}$ and $\mathcal{L}_{\lambda_2}$. Their tensor product is a line bundle $\mathcal{L}_{\lambda_1+\lambda_2}$ whose mirror is the Lagrangian fiber over the point $\exp(2\pi i(\lambda_1 + \lambda_2)) \in U(1)$. Clearly, this rule for tensoring objects of the Fukaya category makes use of the group structure of $U(1)$. It is a convolution-type tensor product.

Another natural class of Lagrangian submanifolds to consider are constant sections of $T^*U(1)$, submanifolds given by the equation $\phi_4 = const$. These submanifolds are circles and may carry a nontrivial flat connection. Thus such A-branes are labeled by points of $\mathbb{R} \times U(1) \simeq \mathbb{C}^*$. The mirror objects are skyscraper sheaves on $\mathbb{C}^*$. The derived tensor product of two skyscrapers supported at different points is obviously the zero object. The derived tensor product of a skyscraper with itself can be shown to be isomorphic to the sum of the skyscraper and the skyscraper shifted by -1. That is, it is a skyscraper sheaf over the same point whose stalk is a graded vector space $\mathbb{C}[-1] \oplus \mathbb{C}$. Applying mirror symmetry, we see that the tensor product of a section of $T^*U(1)$ with itself must be the sum of two copies of the same section, but with the Maslov grading of one of them shifted by -1. We do not know how to reproduce this result without appealing to mirror symmetry, computing the product of boundary line operators in the A-type gauge theory.

As discussed above, the category of boundary line operators in the A-type matter sector is the category of branes in a somewhat unusual 2d TQFT which is a modification of the A-model with target $T^*\mathbb{R}$. It was argued above that this category is equivalent to $D^b_{\mathbb{C}^*}(Coh(\bullet))$. This agrees with the B-side, where the reduction on an interval gives a B-type 2d gauge theory.

Putting the gauge and matter sectors together, we see that the DD boundary condition on the A-side is mapped to what we called the distinguished boundary condition on the B-side. The category of boundary line operators for such a boundary condition is the $\mathbb{C}^*$-equivariant derived category of coherent sheaves $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}^*))$ with its obvious monoidal structure. On the A-side we get a graded version of the Fukaya-Floer category of $T^*U(1)$ where a flat vector bundle over a Lagrangian submanifold has an additional integer grading and morphisms are required to have degree zero with respect to it. This grading arises from the winding number of the periodic scalar A_4.

We can also interpret the duality in 4d terms. Indeed, it is easy to see that the DD boundary condition on the A-side arises from a 4d Dirichlet boundary condition at $t = 1$, while its dual on the B-side arises from the 4d Neumann condition at $t = i$. Thus electric-magnetic duality exchanges Dirichlet and Neumann boundary conditions in 4d, as expected. The surface operators corresponding to such 4d boundary conditions can be interpreted as follows: we excise a tubular neighborhood of the support of the surface operator and impose the 4d boundary condition on the resulting boundary. In a TQFT, such a procedure gives a surface operator (there is no need to take the limit where the thickness of the tubular neighborhood goes to zero).

This condition is the distinguished boundary condition on the A-side, since the gauge group is unbroken on the boundary, and the periodic scalar A_4 is free to explore the whole circle. It is mapped by electric-magnetic duality to the Dirichlet boundary condition for the B-type gauge theory and the boundary condition in the RW model with target $\mathbb{C}^*$ which fixes the $\mathbb{C}^*$-valued scalar τ and leaves σ free. Note that both the Neumann boundary condition in the A-type gauge theory and the corresponding boundary condition in the RW model have a parameter taking values in $\mathbb{C}^* \simeq \mathbb{R} \times U(1)$.

Let us compare the categories of boundary line operators. The category of boundary line operators in the A-type gauge theory is the bounded derived category of coherent sheaves $D^b(Coh(\mathbb{C}[2]))$. The category of boundary line operators in the RW model is also $D^b(Coh(\mathbb{C}[2]))$. The category of boundary line operators in the A-type matter sector is the Fukaya-Floer category of T^*S^1. The category of boundary line operators in the B-type gauge sector is $D^b(Coh(\mathbb{C}^*))$. Their equivalence is a special case of the usual 2d mirror symmetry.

But there is more: we expect that the categories of boundary line operators are equivalent as monoidal categories. This is easy to see directly for the RW model with target $\mathbb{C}^*$ and A-type gauge theory with gauge group $U(1)$. Indeed, in both cases typical objects in the category of boundary line operators are complexes of holomorphic vector bundles which can be represented by Wilson line operators on the boundary for some superconnection on the pull-back vector bundle. In the classical approximation, fusing two such boundary line operators corresponds to the tensor product of complexes, and there can be no quantum corrections to this result.

It is more complicated to compare the monoidal structures for the other pair of dual theories (B-type gauge theory and A-type matter). We will not attempt to do an independent computation on the A-side but instead describe the monoidal structure on the B-side and then explain what it corresponds to on the A-side.

Note that since $\mathbb{C}^*$ is a complex Lie group, the category $D^b(Coh(\mathbb{C}^*))$ has two natural monoidal structures: the derived tensor product, and the convolution-type product. The former one does not make use of the group structure, while the latter one does. The identity object of the former one is the sheaf of holomorphic functions on $\mathbb{C}^*$, while for the latter structure it is the skyscraper sheaf at the identity point $1 \in \mathbb{C}^*$. It is the latter monoidal structure which describes the fusion of boundary line operators on the B-side. Indeed, the 3d meaning of the coordinate on $\mathbb{C}^*$ is the holonomy of the connection $A + i\phi$ along a small semi-circle with both ends on the boundary and centered at the boundary line operator (see Figure 6.1). Skyscraper sheaves correspond to boundary line operators

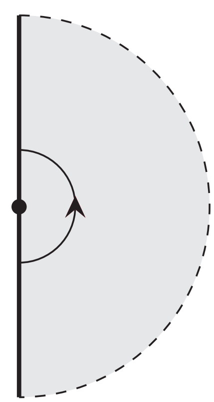

Figure 6.1: A skyscraper sheaf corresponds to a boundary line operator for which the holonomy of $A + i\phi$ along a small semi-circle around it is fixed. The dot marks the location of the boundary line operator, which we view here in cross section.

for which this holonomy is fixed. In particular, the skyscraper sheaf at $1 \in \mathbb{C}^*$ corresponds to the “invisible” boundary line operator for which this holonomy is trivial. By definition, this is the identity object in the monoidal category of boundary line operators.

Mirror symmetry maps a skyscraper sheaf on $\mathbb{C}^*$ to a Lagrangian submanifold of T^*S^1 which is a graph of a closed 1-form α on S^1. Topologically this submanifold is a circle and is equipped with a trivial line bundle with a flat unitary connection. The moduli space of such an object is $\mathbb{C}^*$: for

$\lambda \in \mathbb{C}^*$ the phase of λ determines the holonomy of the unitary connection, while the absolute value determines the integral of α on S^1. Thus the identity object on the B-side is mirror to the zero section of T^*S^1 with a trivial flat connection. To describe the monoidal structure on the A-side it is best to recall a theorem of Nadler [26] according to which (a version of) the Fukaya-Floer category of T^*X is equivalent to the constructible derived category of X. Recall that a constructible sheaf on a real manifold X is a sheaf which is locally constant on the strata of a Whitney stratification of X; such sheaves can be regarded as generalizations of flat connections. Objects of the constructible derived category are bounded complexes of sheaves whose cohomology sheaves are constructible. The constructible derived category has an obvious monoidal structure arising from the tensor product of complexes of sheaves. The sheaf of locally constant functions is the identity object with respect to this monoidal structure. According to [27, 26], this object corresponds to the zero section of T^*S^1 with a trivial flat connection. This suggests that the monoidal structure on the A-side is given by the tensor product on the constructible derived category. It is easy to check that this is compatible with the way mirror symmetry acts on the skyscraper sheaves on $\mathbb{C}^*$.

We can try put the gauge and matter sectors together. On the B-side, we have the B-model with target $\mathbb{C}^* \times \mathbb{C}[2]$ whose category of branes is $D^b(Coh(\mathbb{C}^* \times \mathbb{C}[2]))$. On the A-side, we have an A-model with target T^*S^1 tensored with an A-type 2d gauge theory with gauge group $U(1)$. One could guess that the corresponding category of branes is a $U(1)$-equivariant version of the Fukaya-Floer category of T^*S^1. More generally, one could guess that the category of branes in an A-model with target T^*X tensored with the A-type 2d $U(1)$ gauge theory is a $U(1)$-equivariant version of the Fukaya-Floer category of T^*X. It is not clear to us how to define such an equivariant Fukaya-Floer category mathematically. Given the results of [27, 26], a natural guess is the equivariant constructible derived category of sheaves on X. As a check, note that when X is a point, the $U(1)$-equivariant constructible derived category is equivalent to $D^b(Coh(\mathbb{C}[2]))$ [2]. As mentioned above and explained in Section 4.2.1, this is indeed the category of branes for the A-type 2d gauge theory. The monoidal structure seems to be the standard one (derived tensor product). On the B-side, on the other hand, the monoidal structure is a combination of the tensor product of coherent sheaves on $\mathbb{C}[2]$ and the convolution product on $\mathbb{C}^*$.

Next consider the boundary condition on the A-side which is a combination of the Dirichlet condition in the gauge sector and the Neumann condition for A_4 in the matter sector. It is dual to the Dirichlet condition for the B-type gauge sector and a boundary condition for the RW model with target $T^*[2]\mathbb{C}^*$ which sets $\sigma = 0$ and leaves the complex scalar $\tau = A_4 + i\phi_4$ free to fluctuate.

On the B-side reduction on an interval gives a B-model with target $\mathbb{C}^* \times \mathbb{C}^*$, therefore the category of boundary line operators is $D^b(Coh(\mathbb{C}^* \times \mathbb{C}^*))$. On the A-side reduction gives an A-model with target $T^*U(1) \times T^*U(1)$, therefore the category of boundary line operators is the Fukaya-Floer category. The two categories are equivalent by the usual 2d mirror symmetry. The monoidal structure is easiest to determine on the B-side. It is neither the derived tensor product, nor the convolution, but a combination of both. This happens because the two copies of $\mathbb{C}^*$ have a very different origin: one of them arises from a 3d B-type gauge theory, and the other one arises from the Rozansky-Witten model with target $T^*[2]\mathbb{C}^*$.

Finally we consider the boundary condition on the A-side which is a combination of the Neumann condition in the gauge sector and the Dirichlet condition for A_4. This is the case which corresponds to the Gukov-Witten surface operator at $t = 1$. Indeed, the Dirichlet conditions for A_4, ϕ_4, and ϕ_3 mean that the holonomy of A is fixed, while the 1-form ϕ has a singularity of the form

$$\beta \frac{dr}{r} - \gamma d\theta,$$

where $-\gamma$ is the boundary value of ϕ_4 and β is the boundary value of ϕ_3. The boundary value of A_4 is the Gukov-Witten parameter α. The Neumann condition in the gauge sector also depends on the boundary theta-angle which corresponds to the Gukov-Witten parameter η. As explained above, the boundary values of A_4 and ϕ_3 are actually irrelevant. This agrees with the results of [12], where it is shown that at $t = 1$ the parameters α and β are irrelevant. Thus the true parameter space of the surface operator on the A-side is $\mathbb{C}^*$.

Electric-magnetic duality maps the DD condition to the Neumann condition for the B-type gauge theory and the boundary condition in the RW model which fixes τ and leaves σ free to fluctuate. The latter boundary condition depends on the boundary value of the field $\tau = A_4 + i\phi_4$. From the 4d viewpoint this boundary value encodes the Gukov-Witten parameters α and γ. These are the relevant parameters at $t = i$, as explained in [12]. The Neumann boundary condition in the B-type gauge theory also has two parameters (the boundary value of ϕ_3 and the boundary theta-angle) which correspond to the Gukov-Witten parameters β and η. But as explained above and from a different viewpoint in [12], these parameters are irrelevant at $t = i$.

Let us compare the categories of 3d boundary line operators, which from the 4d viewpoint are interpreted as categories of line operators sitting on Gukov-Witten surface operators. On the B-side reduction on an interval gives a B-model with target $\mathbb{C}[2]$ tensored with a B-type 2d gauge theory, therefore the category of boundary line operators is $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}[2]))$. On the A-side reduction on an interval gives an A-type 2d gauge theory tensored with a modified A-model with target $T^*\mathbb{R}$. Its category of branes is a modification of the category of boundary conditions for the A-type 2d gauge theory where the space of boundary degrees of freedom has additional integer grading coming from the winding of the periodic scalar A_4, and morphisms are required to have degree zero with respect to it. Since branes in the A-type 2d gauge theory can be identified with objects of $D^b(Coh(\mathbb{C}[2]))$, the category of boundary conditions in the combined system is equivalent to $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}[2]))$, in agreement with what we got on the B-side.

By analogy with the Rozansky-Witten model, one may conjecture that the 2-category of surface operators at $t = 0$ can be described in terms of module categories over the monoidal category of boundary line operators for the distinguished boundary condition (the NN condition). We have argued above that this monoidal category is the $U(1)$-equivariant constructible derived category of S^1, where the $U(1)$ action on S^1 is trivial. It is probably better to think about it as a sheaf of $U(1)$-equivariant monoidal DG-categories over S^1. To each surface operator we may associate a sheaf of $U(1)$-equivariant module categories over this sheaf of $U(1)$-equivariant monoidal categories, and we conjecture that this map is an equivalence of 2-categories. Gukov-Witten-type operators correspond to skyscraper sheaves on S^1.

Electric-magnetic duality then implies that there is an equivalence between this 2-category and the 2-category of coherent $\mathbb{C}^*$-equivariant derived categorical sheaves over $\mathbb{C}^*$.

6.3.6 Surface Operators at $t = 0$ with an Abelian Gauge Group

The 3d theory again decomposes into the gauge and matter sectors. Let us start with the gauge sector. The bosonic fields are a gauge field A, a periodic scalar A_4, and a complex scalar σ. The fermionic fields are two 0-forms η and ψ_4, a 1-form ψ and a 2-form χ_+. Thus subscript $+$ indicates that χ_+ originates from the self-dual part of the 2-form χ in four dimensions.The BRST transformations are

$$\begin{aligned}
\delta A &= i\psi, \\
\delta A_4 &= i\psi_4, \\
\delta\psi &= d\sigma, \\
\delta\psi_4 &= 0, \\
\delta\sigma &= 0, \\
\delta\bar{\sigma} &= i\eta, \\
\delta\eta &= 0, \\
\delta\chi_+ &= F + \star dA_4.
\end{aligned}$$

The field content and BRST transformations are the same as in the A-type 3d gauge theory, the main difference being that the bosonic scalar A_4 is periodic. The action of the gauge sector contains, apart from a BRST-exact term, a topological term

$$S_{top} = -\frac{2\pi}{e^2}\int_{M_3} F \wedge dA_4. \qquad (6.12)$$

Note that it is the periodicity of A_4 that makes this topological term nontrivial in general.

The above topological term term comes from the dimensional reduction of a topological term in 4d

$$-\frac{1}{2e^2}\int F\wedge F.$$

Here we assumed that the 4d theta-angle vanishes and that the coordinate x^4 has period 2π.

In the matter sector the only bosonic fields are a 0-form ϕ_4 and a 1-form ϕ. The fermionic fields are a pair of 0-forms $\tilde{\eta}$ and $\tilde{\psi}_4$, a 1-form $\tilde{\psi}$, and a 2-form χ_- which arises from the anti-self-dual part of the 2-form χ in four dimensions. The matter content and BRST transformations are the same as for the $t=1$ matter sector, except that the periodic scalar A_4 is replaced with a non-periodic scalar ϕ_4. The matter action is BRST-exact.

As for $t=1$, we may consider either Dirichlet or Neumann conditions for the gauge field, and then BRST-invariance determines the rest. The category of boundary line operators is determined by compactifying the theory on an interval with the appropriate boundary conditions and analyzing branes in the resulting 2d TQFT.

In the Neumann case the effective 2d TQFT is the A-type 2d gauge theory, just as for $t=1$. As explained above, its category of boundary conditions is equivalent to $D^b(Coh(\mathbb{C}[2]))$.

In the Dirichlet case the effective 2d TQFT is a topological sigma-model with two bosonic fields, A_4 and the holonomy of the 3d gauge field A along the interval. Both are periodic scalars, so the target of the sigma-model is T^2. The BPS equations reduce to a holomorphic instanton equation

$$dA_3+\star dA_4=0,$$

which means that we are dealing with an A-model with target T^2. Its category of branes is the Fukaya-Floer category of T^2, which is fairly nontrivial (and by mirror symmetry equivalent to the bounded derived category of coherent sheaves on an elliptic curve). The A-model depends on the symplectic form on T^2 which can be read off the topological piece of the action (6.12). Setting A_1 and A_2 to zero and reducing on an interval of length 2π it becomes

$$\frac{-4\pi^2}{e^2}\int_{M_2} dA_3\wedge dA_4.$$

We may regard this expression as an integral of the pull-back of a symplectic 2-form

$$\frac{4\pi^2}{e^2}dx\wedge dy$$

on the 2-torus with periodic coordinates x, y, both with period one. The symplectic area of this 2-torus is $4\pi^2/e^2$.

We do not know how to describe the monoidal structure on this category arising from the fusion of boundary line operators.

As for $t = 1$, we may consider either the Dirichlet or Neumann conditions for the scalars ϕ_3 and ϕ_4 (BRST-invariance requires them to be of the same type). In the Dirichlet case reduction on an interval gives the modified A-model whose only bosonic field is a real 1-form ϕ in two dimensions. As discussed above, the category of branes is the same as for a trivial TQFT. It is equivalent to $D^b(Coh(\bullet))$. Unlike in the $t = 1$ case, there are no "winding sectors", since the scalars ϕ_3 and ϕ_4 are not periodic. So the category of boundary line operators in this case is $D^b(Coh(\bullet))$, with its standard monoidal structure.

If ϕ_3 and ϕ_4 satisfy the Neumann condition, then the restriction of the 1-form ϕ to the 2d boundary must vanish. Reducing on an interval, we get an A-model whose only bosonic fields are ϕ_3 and ϕ_4, namely an A-model with target $T^*\mathbb{R}$. Its category of branes is the Fukaya-Floer category of $T^*\mathbb{R}$. Since this should be thought as the category of boundary line operators in a 3d TQFT, it should have a monoidal structure. Since the only difference compared to the $t = 1$ matter sector is the noncompactness of ϕ_4, we expect that after we apply the equivalence of [26], this monoidal structure becomes the standard monoidal structure on the constructible derived category of $\mathbb{R}$.

The DD boundary condition corresponds to a surface operator such that the 1-form ϕ has a fixed singularity of the form

$$\beta\frac{dr}{r} - \gamma d\theta,$$

while the holonomy of the gauge field A is allowed to fluctuate, and the scalar field σ vanishes at the insertion surface. To define such an operator properly, one has to excise a tubular neighborhood of the insertion surface and impose suitable conditions on the newly created boundary.

Since the matter sector in the Dirichlet case does not have interesting boundary conditions, the category of boundary line operators is the same as in the gauge sector, the Fukaya-Floer category of T^2 with the symplectic area $\mathfrak{S} = 4\pi^2/e^2$. From the 4d viewpoint, this is the category of line operators on the surface operator.

Electric-magnetic duality maps the DD condition to itself. Indeed, it does not affect the matter sector, while in the gauge sector it maps the periodic scalar A_4 into a gauge field and maps the gauge field to a periodic scalar. Since in the DD case A_4 satisifies the Neumann condition, the dual gauge field satisfies the Dirichlet condition. Contrariwise, the Dirichlet condition for the gauge field is mapped by duality to the Dirichlet condition for the new periodic scalar. The only effect of duality is to replace e^2 with $4\pi^2/e^2$. Therefore the symplectic area of the T^2 is also inverted:

$$\mathfrak{S} \mapsto \mathfrak{S}' = \frac{4\pi^2}{\mathfrak{S}}.$$

The Fukaya-Floer categories of two tori whose symplectic areas are related as above are equivalent by the usual T-duality. Moreover, we expect that the monoidal structure (which we have not determined!) is preserved by T-duality.

The NN condition corresponds to the surface operator such that A has a fixed singularity of the form

$$\alpha d\theta,$$

while the singularity for the 1-form ϕ is allowed to fluctuate. To define such a surface operator properly, one has to impose suitable conditions on a boundary of a tubular neighborhood of the insertion surface.

Upon reduction on an interval with NN boundary conditions on both ends, we get a 2d TQFT which is a product of an A-type 2d gauge theory and an A-model with target $T^*\mathbb{R}$. Its category of branes is an equivariant version of the Fukaya-Floer category of $T^*\mathbb{R}$. It was conjectured above that it is equivalent to the equivariant constructible derived category of $\mathbb{R}$, with the standard monoidal structure (derived tensor product).

Electric-magnetic duality maps the NN condition to itself, for the same reason as in the DD case. It acts trivially on the category of line operators, because the bosonic fields which survive the reduction on an interval (that is, σ, ϕ_3, and ϕ_4) are not involved in the duality.

The DN condition corresponds to a surface operator such that both A and ϕ are allowed to have fluctuating singularities, while σ has to vanish at the surface operator. Upon reduction on an interval with DN boundary conditions on both ends, we get a product of an A-model with target T^2 and an A-model with target $T^*\mathbb{R}$. Its category of branes is the Fukaya-Floer category of $T^2 \times T^*\mathbb{R}$. Electric-magnetic duality maps the DN condition to itself. Its action on the category of line operators amounts to a T-duality on T^2 (duality acts trivially on the matter sector). The monoidal structure (which we have not determined) must be preserved by T-duality.

This case corresponds to the Gukov-Witten surface operator where the holonomy of A is fixed, and the 1-form ϕ has a fixed singularity of the form

$$\beta \frac{dr}{r} - \gamma d\theta.$$

Reduction on an interval with ND boundary conditions gives a 2d TQFT which is a product of an A-type 2d gauge theory and a modified A-model whose only bosonic field is a real 1-form. Since there are no interesting boundary conditions in the latter theory, the category of boundary conditions in this case is the same as in the former theory. That is, it is the $U(1)$-equivariant constructible derived category of sheaves over a point, or equivalently $D^b(Coh(\mathbb{C}[2]))$ [2]. This is therefore the category of line operators sitting on the Gukov-Witten surface operator. The monoidal structure is the standard one (derived tensor product).

In particular, since the trivial surface operator is a special case of the Gukov-Witten surface operator, we conclude that the category of bulk line operators in the GL-twisted theory at $t = 0$ is $D^b(Coh(\mathbb{C}[2]))$. In 4d terms, this can be interpreted as saying that all bulk line operators can be constructed by taking a sum of several copies of the trivial line operator and deforming it using the descendants of the BRST-invariant field σ and its powers. This agrees with the results of [19], where it was argued that neither Wilson nor 't Hooft line operators are allowed at $t = 0$.

Electric-magnetic duality maps the ND condition to itself. It acts trivially on the category of line operators since the field σ is not involved in the duality.

Chapter 7

Four-Dimensional HQFTs

7.1 $\mathcal{N} = 1$ SQCD with $N_c = 2$ and $N_f = 2$

We begin this chapter with a review of $\mathcal{N} = 1$ SQCD with gauge group $SU(2)$ and two flavors. This theory consists of four squarks q^a, quarks ψ^a_α, and auxiliary fields G^a in the fundamental representation of $SU(2)$, a gaugino λ_α in the adjoint representation of $SU(2)$, and an $SU(2)$ gauge field A_μ with the following action,

$$\begin{aligned} S = \int_{\mathbb{R}^4} d^4x \, \Big\{ & \frac{1}{g^2} \mathrm{tr}\Big(-\frac{1}{2} F^{\mu\nu} F_{\mu\nu} - 2i\lambda^\dagger \bar{\sigma}^\mu \nabla_\mu \lambda \Big) + \frac{i\theta}{32\pi^2} \mathrm{tr}\Big(\epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} \Big) \\ & - \nabla^\mu q^\dagger_a \nabla_\mu q^a - i\psi^\dagger_a \bar{\sigma}^\mu \nabla_\mu \psi^a + G^\dagger_a G^a + i\sqrt{2}\Big(q^\dagger_a \lambda \psi^a - \psi^\dagger_a \lambda^\dagger q^a \Big) - \frac{1}{8} g^2 \big(q^\dagger_a \overrightarrow{T} q^a \big)^2 \Big\} \end{aligned} \tag{7.1}$$

where $\overrightarrow{T}$ are the Pauli matrices.

Viewing Euclidean spacetime as a complex manifold with Kähler metric, this theory has a $U(1)_\mathcal{R}$ $\mathcal{R}$-symmetry, $U(1)_X \times SU(2)_X$ spin symmetry, $SU(4)_F$ chiral symmetry, and $U(1)_A$ axial symmetry. With respect to these global symmetries, the supercharges transform as shown in Table 7.1 while the fields and their conjugate transform as shown in Table 7.2.

	$U(1)_\mathcal{R}$	$U(1)_X$	$SU(2)_X$
Q_α	1	0	**2**
$\bar{Q}^1$	-1	-1	**1**
$\bar{Q}^2$	-1	1	**1**

Table 7.1: $\mathcal{N} = 1$ supercharges and their transformation properties

Field	$U(1)_{\mathcal{R}}$	$U(1)_X$	$SU(2)_X$	$SU(4)_F$	$U(1)_A$
q^a	-1	0	$\mathbf{1}$	$\mathbf{4}$	1
ψ^a_α	0	0	$\mathbf{2}$	$\mathbf{4}$	1
G^a	1	0	$\mathbf{1}$	$\mathbf{4}$	1
$q^\dagger_a$	1	0	$\mathbf{1}$	$\bar{\mathbf{4}}$	-1
$\psi^{\dagger 1}_a$	0	-1	$\mathbf{1}$	$\bar{\mathbf{4}}$	-1
$\psi^{\dagger 2}_a$	0	1	$\mathbf{1}$	$\bar{\mathbf{4}}$	-1
$G^\dagger_a$	-1	0	$\mathbf{1}$	$\bar{\mathbf{4}}$	-1
A_i	0	-1	$\mathbf{2}$	$\mathbf{1}$	0
$A_{\bar{i}}$	0	1	$\mathbf{2}$	$\mathbf{1}$	0
λ_α	-1	0	$\mathbf{2}$	$\mathbf{1}$	0
$\lambda^{\dagger 1}$	1	-1	$\mathbf{1}$	$\mathbf{1}$	0
$\lambda^{\dagger 2}$	1	1	$\mathbf{1}$	$\mathbf{1}$	0

Table 7.2: Fields in $\mathcal{N} = 1$ SQCD and their transformation properties

7.2 Holomorphic SQCD with $N_c = 2$ and $N_f = 2$

Holomorphic SQCD is constructed by twisting the $U(1)_X$ spin symmetry of $\mathcal{N} = 1$ SQCD by a combination $\mathcal{R}'$ of the $U(1)_{\mathcal{R}}$ $\mathcal{R}$-symmetry and $U(1)_A$ axial symmetry,

$$X \to X + \mathcal{R}'. \tag{7.2}$$

The $U(1)_{\mathcal{R}}$ symmetry alone is unsuitable for twisting; it suffers from gauge and gravitational anomalies. The following combination of $U(1)_{\mathcal{R}}$ and $U(1)_A$ symmetry is free of gauge anomalies,

$$\mathcal{R}' = \mathcal{R} + A. \tag{7.3}$$

Furthermore, we can eliminate the gravitational anomalies with a suitable choice of path integral measure.

The easiest way to construct an appropriate path integral measure is to augment SQCD by an $\mathcal{N} = 1$ SYM theory with gauge group $U(1)^5$. The $U(1)_{\mathcal{R}'}$ symmetry of the augmented theory is anomaly free. Since SQCD and the SYM theory do not interact, correlation functions of the combined theory factor into SQCD correlation functions and SYM correlation functions. We can therefore construct an anomaly-free measure for SQCD from any nontrivial correlation function in the SYM theory. The action for the SYM theory is

$$S' = \int_{\mathbb{R}^4} d^4x \left\{ \frac{1}{g'^2} \mathrm{tr}\Big(-\frac{1}{4} F'^{\mu\nu} F'^{I}_{\mu\nu} - i\lambda'^\dagger \bar{\sigma}^\mu \partial_\mu \lambda' \Big) + \frac{i\theta'}{64\pi^2} \mathrm{tr}\Big(\epsilon^{\mu\nu\sigma\rho} F'_{\mu\nu} F'_{\sigma\rho} \Big) \right\}, \tag{7.4}$$

where the gaugino λ'_α is in the adjoint representation of $U(1)^5$, and A'_μ is a $U(1)^5$ gauge field. With respect to the global symmetries, these fields and their conjugate transform as shown in Table 7.3.

Field	$U(1)_{\mathcal{R}}$	$U(1)_X$	$SU(2)_X$	$SU(4)_F$	$U(1)_A$
A'_i	0	-1	$\mathbf{2}$	$\mathbf{1}$	0
$A'_{\bar{i}}$	0	1	$\mathbf{2}$	$\mathbf{1}$	0
λ'_α	-1	0	$\mathbf{2}$	$\mathbf{1}$	0
$\lambda'^{\dagger 1}$	1	-1	$\mathbf{1}$	$\mathbf{1}$	0
$\lambda'^{\dagger 2}$	1	1	$\mathbf{1}$	$\mathbf{1}$	0

Table 7.3: Fields in SYM theory and their transformation properties

Performing the holomorphic twist (7.2) with the anomaly free $U(1)_{\mathcal{R}'}$ symmetry of augmented theory, we find that the twisted fields transform as shown in Table 7.4.

Field	$U(1)_{\mathcal{R}}$	$U(1)_X$	$SU(2)_X$	$SU(4)_F$	$U(1)_A$
q^a	-1	0	$\mathbf{1}$	$\mathbf{4}$	1
$\chi^a_{\bar{i}}$	0	1	$\mathbf{2}$	$\mathbf{4}$	1
$G^a_{\bar{1}\bar{2}}$	1	2	$\mathbf{1}$	$\mathbf{4}$	1
$q^\dagger_a$	1	0	$\mathbf{1}$	$\bar{\mathbf{4}}$	-1
$\rho^\dagger_{a12}$	0	-2	$\mathbf{1}$	$\bar{\mathbf{4}}$	-1
$\sigma^\dagger_a$	0	0	$\mathbf{1}$	$\bar{\mathbf{4}}$	-1
$G^\dagger_{a12}$	-1	-2	$\mathbf{1}$	$\bar{\mathbf{4}}$	-1
A_i	0	-1	$\mathbf{2}$	$\mathbf{1}$	0
$A_{\bar{i}}$	0	1	$\mathbf{2}$	$\mathbf{1}$	0
ζ_i	-1	-1	$\mathbf{2}$	$\mathbf{1}$	0
$\varphi^\dagger$	1	0	$\mathbf{1}$	$\mathbf{1}$	0
$\omega^\dagger_{\bar{1}\bar{2}}$	1	2	$\mathbf{1}$	$\mathbf{1}$	0
H	0	0	$\mathbf{1}$	$\mathbf{1}$	0
A'_i	0	-1	$\mathbf{2}$	$\mathbf{1}$	0
$A'_{\bar{i}}$	0	1	$\mathbf{2}$	$\mathbf{1}$	0
ζ'_i	-1	-1	$\mathbf{2}$	$\mathbf{1}$	0
$\varphi'^\dagger$	1	0	$\mathbf{1}$	$\mathbf{1}$	0
$\omega'^\dagger_{\bar{1}\bar{2}}$	1	2	$\mathbf{1}$	$\mathbf{1}$	0
H'	0	0	$\mathbf{1}$	$\mathbf{1}$	0

Table 7.4: Fields in holomorphic SQCD + SYM and their transformation properties

Writing the action for the theory covariantly in terms of the twisted fields, we have

$$\begin{aligned}
\tilde{S} = \int_X d^4z \Bigg\{ &\frac{1}{g^2}\mathrm{tr}\Big(-h\big(h^{i\bar{j}}F_{i\bar{j}}\big)^2 + 4F_{12}F_{\bar{1}\bar{2}} + 2ihh^{i\bar{j}}\varphi^\dagger\nabla_{\bar{j}}\zeta_i + i\omega^\dagger_{\bar{1}\bar{2}}\big(\nabla_1\zeta_2 - \nabla_2\zeta_1\big)\Big) \\
&+ \frac{\tau}{2\pi i}\mathrm{tr}\Big(F_{1\bar{1}}F_{2\bar{2}} - F_{12}F_{\bar{1}\bar{2}} + F_{1\bar{2}}F_{\bar{1}2}\Big) + 2hh^{i\bar{j}}\nabla_i q^\dagger_a\nabla_{\bar{j}}q^a + ihh^{i\bar{j}}q^\dagger_a F_{i\bar{j}}q^a \\
&- ihh^{i\bar{j}}\nabla_i\sigma^\dagger_a\chi^a_{\bar{j}} - \frac{i}{2}\rho^\dagger_{a12}\big(\nabla_{\bar{1}}\chi^a_{\bar{2}} - \nabla_{\bar{2}}\chi^a_{\bar{1}}\big) - \frac{1}{4}G^\dagger_{a12}G^a_{\bar{1}\bar{2}} - i\frac{1}{\sqrt{2}}hh^{i\bar{j}}q^\dagger_a\zeta_i\chi^a_{\bar{j}} \\
&- i\sqrt{2}h\sigma^\dagger_a\varphi^\dagger q^a + i\frac{1}{2\sqrt{2}}\rho^\dagger_{a12}\omega^\dagger_{\bar{1}\bar{2}}q^a + \frac{1}{8}hg^2\big(q^\dagger_a\overrightarrow{T}q^a\big)^2\Bigg\}, \\
\tilde{S}' = \int_X d^4z \Bigg\{ &\frac{1}{g'^2}\mathrm{tr}\Big(-\frac{1}{2}h\big(h^{i\bar{j}}F'_{i\bar{j}}\big)^2 + 2F'_{12}F'_{\bar{1}\bar{2}} + ihh^{i\bar{j}}\varphi'^\dagger\partial_{\bar{j}}\zeta'_i + \frac{i}{2}\omega'^\dagger_{\bar{1}\bar{2}}\big(\partial_1\zeta'_2 - \partial_2\zeta'_1\big)\Big) \\
&+ \frac{\tau'}{4\pi i}\mathrm{tr}\Big(F'_{1\bar{1}}F'_{2\bar{2}} - F'_{12}F'_{\bar{1}\bar{2}} + F'_{1\bar{2}}F'_{\bar{1}2}\Big)\Bigg\}
\end{aligned} \tag{7.5}$$

where $h_{i\bar{j}}$ is the Kähler metric, $h = \det h_{i\bar{j}}$, and $\tau = \frac{4\pi i}{g^2} - \frac{\theta}{2\pi}$, $\tau' = \frac{4\pi i}{g'^2} - \frac{\theta'}{2\pi}$ are the complexified gauge coupling constants. The supersymmetry variations of the twisted fields (with respect to the scalar supercharge) are

$$\begin{aligned}
\delta q^a &= 0, & \delta A_i &= i\bar{\xi}\zeta_i, & \delta A'_i &= i\bar{\xi}\zeta'_i, \\
\delta\chi^a_{\bar{i}} &= 2i\sqrt{2}\bar{\xi}\nabla_{\bar{i}}q^a, & \delta A_{\bar{i}} &= 0, & \delta A'_{\bar{i}} &= 0, \\
\delta G^a_{\bar{1}\bar{2}} &= 2i\bar{\xi}\omega^\dagger_{\bar{1}\bar{2}}q^a & \delta\zeta_i &= 0, & \delta\zeta'_i &= 0, \\
&\quad - 2i\sqrt{2}\bar{\xi}\epsilon^{\bar{i}\bar{j}}\nabla_{\bar{i}}\chi^a_{\bar{j}}, & \delta\varphi^\dagger &= -\bar{\xi}h^{i\bar{j}}F_{i\bar{j}} + \frac{i}{4}\bar{\xi}g^2q^\dagger_a\overrightarrow{T}q^a\cdot\overrightarrow{T}, & \delta\varphi'^\dagger &= -\bar{\xi}h^{i\bar{j}}F'_{i\bar{j}}, \\
\delta q^\dagger_a &= \sqrt{2}\bar{\xi}\sigma^\dagger_a, & \delta\omega^\dagger_{\bar{1}\bar{2}} &= -4\bar{\xi}F_{\bar{1}\bar{2}}, & \delta\omega'^\dagger_{\bar{1}\bar{2}} &= -4\bar{\xi}F'_{\bar{1}\bar{2}}, \\
\delta\rho^\dagger_{a12} &= \sqrt{2}\bar{\xi}G^\dagger_{a12}, \\
\delta\sigma^\dagger_a &= 0, \\
\delta G^\dagger_{a12} &= 0.
\end{aligned} \tag{7.6}$$

Notice that the $\varphi^\dagger$ and $\varphi'^\dagger$ variations have two undesirable properties; they explicitly depend on the Hermitian metric and they are not off-shell nilpotent. We remedy these problems by introducing auxiliary fields H, H' in the adjoint of $SU(2)$, $U(1)^5$ which take the following values on-shell,

$$\begin{aligned}
H &= ih^{i\bar{j}}F_{i\bar{j}} + \frac{1}{4}g^2q^\dagger_a\overrightarrow{T}q^a\cdot\overrightarrow{T}, \\
H' &= ih^{i\bar{j}}F'_{i\bar{j}}.
\end{aligned} \tag{7.7}$$

With these auxiliary fields, we can write the supersymmetry variations of $\varphi^\dagger$ and $\varphi'^\dagger$ in a metric independent form,

$$\begin{aligned} \delta\varphi^\dagger &= i\bar{\xi}H, \\ \delta\varphi'^\dagger &= i\bar{\xi}H'. \end{aligned} \tag{7.8}$$

For these variations to be nilpotent, we require that

$$\begin{aligned} \delta H &= 0, \\ \delta H' &= 0. \end{aligned} \tag{7.9}$$

It is not difficult to construct an action equivalent to (7.5) that enforces (7.7) and respects the supersymmetry variations (7.6), (7.8), (7.9),

$$\begin{aligned} \tilde{S} = \int_X d^4z \Big\{ &\frac{1}{g^2}\mathrm{tr}\Big(-hH^2 + 2hH\big(ih^{i\bar{j}}F_{i\bar{j}} + \frac{1}{4}g^2 q_a^\dagger \overrightarrow{T} q^a \cdot \overrightarrow{T}\big) + 4F_{12}F_{\bar{1}\bar{2}} + 2ihh^{i\bar{j}}\varphi^\dagger\nabla_{\bar{j}}\zeta_i \\ &+ i\omega^\dagger_{\bar{1}\bar{2}}\big(\nabla_1\zeta_2 - \nabla_2\zeta_1\big)\Big) + \frac{\tau}{2\pi i}\mathrm{tr}\Big(F_{1\bar{1}}F_{2\bar{2}} - F_{12}F_{\bar{1}\bar{2}} + F_{1\bar{2}}F_{\bar{1}2}\Big) + 2hh^{i\bar{j}}\nabla_i q_a^\dagger \nabla_{\bar{j}} q^a \\ &- ihh^{i\bar{j}}\nabla_i\sigma_a^\dagger\chi^a_{\bar{j}} - \frac{i}{2}\rho^\dagger_{a12}\big(\nabla_{\bar{1}}\chi^a_{\bar{2}} - \nabla_{\bar{2}}\chi^a_{\bar{1}}\big) - \frac{1}{4}G^\dagger_{a12}G^a_{\bar{1}\bar{2}} - i\frac{1}{\sqrt{2}}hh^{i\bar{j}}q_a^\dagger\zeta_i\chi^a_{\bar{j}} \\ &- i\sqrt{2}h\sigma_a^\dagger\varphi^\dagger q^a + i\frac{1}{2\sqrt{2}}\rho^\dagger_{a12}\omega^\dagger_{\bar{1}\bar{2}}q^a\Big\}, \\ \tilde{S}' = \int_X d^4z \Big\{ &\frac{1}{g'^2}\mathrm{tr}\Big(-\frac{1}{2}hH'^2 + hH'\big(ih^{i\bar{j}}F'_{i\bar{j}}\big) + 2F'_{12}F'_{\bar{1}\bar{2}} + ihh^{i\bar{j}}\varphi'^\dagger\partial_{\bar{j}}\zeta'_i \\ &+ \frac{i}{2}\omega'^\dagger_{\bar{1}\bar{2}}\big(\partial_1\zeta'_2 - \partial_2\zeta'_1\big)\Big) + \frac{\tau'}{4\pi i}\mathrm{tr}\Big(F'_{1\bar{1}}F'_{2\bar{2}} - F'_{12}F'_{\bar{1}\bar{2}} + F'_{1\bar{2}}F'_{\bar{1}2}\Big)\Big\}. \end{aligned} \tag{7.10}$$

Using the supersymmetry variations, we can show that the twisted action is $\bar{Q}$-exact up to a topological term,

$$\begin{aligned} \tilde{S} = \int_X d^4z \Big\{ &\bar{Q},\ \frac{1}{g^2}\mathrm{tr}\Big(ih\varphi^\dagger H - 2ih\varphi^\dagger\big(ih^{i\bar{j}}F_{i\bar{j}} + \frac{1}{4}g^2 q_a^\dagger\overrightarrow{T}q^a\cdot\overrightarrow{T}\big) - F_{12}\omega^\dagger_{\bar{1}\bar{2}}\Big) \\ &- \frac{i}{\sqrt{2}}hh^{i\bar{j}}\nabla_i q_a^\dagger\chi^a_{\bar{j}} - \frac{1}{4\sqrt{2}}\rho^\dagger_{a12}G^a_{\bar{1}\bar{2}}\Big\} + \frac{\tau}{4\pi i}\int_X \mathrm{tr}\big(F\wedge F\big) \\ \tilde{S}' = \int_X d^4z \Big\{ &\bar{Q},\frac{1}{2g'^2}\mathrm{tr}\Big(ih\varphi'^\dagger H' - 2ih\varphi'^\dagger\big(ih^{i\bar{j}}F'_{i\bar{j}}\big) - F'_{12}\omega'^\dagger_{\bar{1}\bar{2}}\Big)\Big\} + \frac{\tau'}{8\pi i}\int_X \mathrm{tr}\big(F'\wedge F'\big). \end{aligned} \tag{7.11}$$

It follows that correlators of metric independent physical observables (i.e., supersymmetric observables) are invariant with respect to infinitesimal deformations of the Kähler metric,

$$\begin{aligned} \delta_h\langle\mathcal{O}\rangle &= -\int \mathcal{D}\Phi\ \mathcal{O}\big(\delta_h\widetilde{S} + \delta_h\widetilde{S}'\big)\exp(-\widetilde{S} - \widetilde{S}') \\ &= -\int \mathcal{D}\Phi\ \mathcal{O}\big\{\bar{Q},\cdots\big\}\exp(-\widetilde{S} - \widetilde{S}') \\ &= -\int \mathcal{D}\Phi\ \big\{\bar{Q},\mathcal{O}\cdots\big\}\exp(-\widetilde{S} - \widetilde{S}') = 0. \end{aligned} \tag{7.12}$$

7.2.1 Observables

Having constructed the holomorphic theory, we must now find physical observables on closed Kähler sufaces.[1] From the supersymmetry variations, we see that the gauge invariant physical observables in the SQCD sector of the theory are the glueball, mesons, and their descendents,

$$\begin{aligned}
\mathcal{G}_{(0)}(\pi) &= \int_X \mathrm{tr}\big(\zeta \wedge \zeta\big) \wedge \pi, \\
\mathcal{G}_{(1)}(\eta) &= \int_X \mathrm{tr}\big(F \wedge \zeta\big) \wedge \eta, \\
\mathcal{G}_{(2)}(e) &= \int_X \mathrm{tr}\big(F \wedge F\big) \wedge e, \\
\mathcal{M}^{ab}_{(0)}(\upsilon) &= \int_X \big\langle q^a, q^b \big\rangle \wedge \upsilon, \\
\mathcal{M}^{ab}_{(1)}(\theta) &= \int_X \Big\{\big\langle \chi^a, q^b \big\rangle + \big\langle q^a, \chi^b \big\rangle\Big\} \wedge \theta, \\
\mathcal{M}^{ab}_{(2)}(f) &= \int_X \Big\{\big\langle G^a, q^b \big\rangle - \big\langle \chi^a, \chi^b \big\rangle + \big\langle q^a, G^b \big\rangle\Big\} \wedge f,
\end{aligned} \tag{7.13}$$

where π, η, e are Dolbeault harmonic $(0,2)$, $(0,1)$, and $(0,0)$ forms, υ, θ, f are Dolbeault harmonic $(2,2)$, $(2,1)$, and $(2,0)$ forms, and $\langle\ ,\ \rangle : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ is the skew-symmetric map

$$\big\langle v, w \big\rangle = \epsilon^{ij} v_i w_j. \tag{7.14}$$

The gauge invariant physical observables in SYM sector of the theory are the left chiral gluinos and their descendents,

$$\begin{aligned}
\mathcal{G}'^{I}_{(0)}(\varrho) &= \int_X \zeta'^{I} \wedge \varrho, \\
\mathcal{G}'^{I}_{(1)}(\vartheta) &= \int_X F'^{I} \wedge \vartheta,
\end{aligned} \tag{7.15}$$

where $I = 1, 2, \cdots, 5$ indexes the $U(1)$ factors of the gauge group and ϱ, ϑ are Dolbeault harmonic $(1,2)$, $(1,1)$ forms. If we restrict the path integral to the trivial $U(1)^5$ bundle and eliminate the constant mode of the auxiliary field H', there are two additional gauge invariant physical observables,

$$\begin{aligned}
\Phi'^{I}(\upsilon) &= \int_X \varphi'^{\dagger I} \wedge \upsilon, \\
\Omega'^{I}(f) &= \int_X \omega'^{\dagger I} \wedge f.
\end{aligned} \tag{7.16}$$

Notice that $\mathcal{G}'^{I}_{(1)}(\vartheta)$ vanishes after imposing this restriction, since

$$\int_X F'^{I} \wedge \vartheta = 0 \tag{7.17}$$

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[1] The restriction to closed manifolds is necessary for the descent procedure.

for the trivial $U(1)^5$ bundle. While such a modification to the path integral seems peculiar, the additional observables are necessary for nontrivial correlation functions.

7.2.2 Correlation Functions

Our final task is to compute correlation functions in the holomorphic theory. We begin by identifying certain selection rules for nonvanishing correlation functions. Notice that there is a one-to-one correspondence between fermionic zero modes of the twisted SYM action and gauge invariant physical observables in the SYM sector (after restricting the path integral to the trivial $U(1)^5$ bundle and eliminating the constant mode of H'). Nontrivial correlation functions in the holomorphic theory therefore have the form

$$\Big\langle \prod_{I=1}^{5} \Big\{ \prod_{\varrho\in\mathcal{B}^{(1,2)}} \mathcal{G}'^{I}_{(0)}(\varrho) \prod_{\upsilon\in\mathcal{B}^{(2,2)}} \Phi'^{I}(\upsilon) \prod_{f\in\mathcal{B}^{(2,0)}} \Omega'^{I}(f) \Big\} \mathcal{O}_{SQCD} \Big\rangle, \tag{7.18}$$

where $\mathcal{B}^{(p,q)}$ is a basis for $H^{(p,q)}(X)$ and $\mathcal{O}_{SQCD}$ is an observable in the SQCD sector.

Additional selection rules come from the $U(1)_{\mathcal{R}}$, $SU(4)_F$, and $U(1)_A$ symmetries. Recall that the path integral measure is invariant under the $SU(4)_F$ symmetry, but has nontrivial $U(1)_{\mathcal{R}}$ and $U(1)_A$ charge,

$$\mathcal{R}(\mathcal{D}\Phi) = -8\chi + 4n, \tag{7.19}$$

$$A(\mathcal{D}\Phi) = +8\chi - 4n, \tag{7.20}$$

where χ is the holomorphic Euler characteristic of X and n is the instanton number of the $SU(2)$ bundle. So correlation functions of the form (7.18) vanish unless $\mathcal{O}_{SQCD}$ is $SU(4)_F$ invariant with the following $U(1)_{\mathcal{R}}$ and $U(1)_A$ charge,

$$\begin{aligned} \mathcal{R}(\mathcal{O}_{SQCD}) &= 3\chi - 4n, \\ A(\mathcal{O}_{SQCD}) &= -8\chi + 4n. \end{aligned} \tag{7.21}$$

We now evaluate correlation functions satisfying these selection rules. The path integral over twisted SYM fields simply introduces normalization factors necessary for the twisted SQCD theory to have a well-defined path integral measure. To evaluate the path integral over twisted SQCD fields, we employ the following localization technique: the saddle point approximation about fixed points of any Grassmann symmetry is exact [41]. From the variations (7.6), (7.8), and (7.9), we see that the bosonic fixed points with respect to supersymmetry are field configurations satisfying the following equations,

$$\nabla_{\bar{i}} q^a = 0, \tag{7.22}$$

$$ih^{i\bar{j}} F_{i\bar{j}} + \frac{1}{4} g^2 q_a^\dagger \overrightarrow{T} q^a \cdot \overrightarrow{T} = 0, \tag{7.23}$$

$$F_{\bar{1}\bar{2}} = 0, \tag{7.24}$$

where we have eliminated the auxiliary fields $G^a_{\bar{1}\bar{2}}$, $G^\dagger_{a12}$, and H. Unfortunately, the moduli space of solution to these equations is not well understood.

For closed Kähler surfaces with $h^{2,0}(X) > 0$, we can arrive at a simpler moduli space by adding twisted mass terms for the matter fields,

$$\begin{aligned} \widetilde{S}_m &= \int_X d^4z \Big\{ -\frac{1}{4} m_{ab12} \Big(\langle G^a_{\bar{1}\bar{2}}, q^b \rangle + \langle q^a, G^b_{\bar{1}\bar{2}} \rangle - \langle \chi^a_{\bar{1}}, \chi^b_{\bar{2}} \rangle + \langle \chi^a_{\bar{2}}, \chi^b_{\bar{1}} \rangle \Big) \Big\} \\ \widetilde{S}_{m^\dagger} &= \int_X d^4z \Big\{ -\frac{1}{4} m^{\dagger ab}_{\bar{1}\bar{2}} \Big(\langle G^\dagger_{a12}, q^\dagger_b \rangle + \langle q^\dagger_a, G^\dagger_{b12} \rangle + \langle \sigma^\dagger_a, \rho^\dagger_{b12} \rangle - \langle \rho^\dagger_{a12}, \sigma^\dagger_a \rangle \Big) \Big\}, \end{aligned} \tag{7.25}$$

where $m_{ab} \in H^{2,0}(X)$ is the twisted mass. Notice that these terms respect the holomorphic nature of the theory; they preserve supersymmetry and they do not introduce any dependence on the Kähler structure. The twisted mass terms have two other important properties; $\widetilde{S}_m$ has $U(1)_A$ charge 2 and $\widetilde{S}_{m^\dagger}$ is $\bar{Q}$-exact,

$$\widetilde{S}_{m^\dagger} = \int_X d^4z \Big\{ \bar{Q}, -\frac{1}{4\sqrt{2}} m^{\dagger ab}_{\bar{1}\bar{2}} \Big(\langle \rho^\dagger_{a12}, q^\dagger_b \rangle + \langle q^\dagger_a, \rho^\dagger_{b12} \rangle \Big) \Big\}. \tag{7.26}$$

Treating the addition of these terms as an insertion of the observable $e^{-\widetilde{S}_m} e^{-\widetilde{S}_{m^\dagger}}$, the properties above show that correlation functions satisfying the selection rules are unchanged by the twisted mass terms. These terms do however change the moduli space of bosonic fixed points, adding an additional constraint equation,

$$m_{ab12} q^b = 0. \tag{7.27}$$

Choosing the twisted mass to be nondegenerate outside the canonical divisor, the moduli space of solutions to equations (7.22), (7.23), (7.24), and (7.27) is simply the moduli space of instantons $\mathcal{M}$,

$$\breve{q}^a = 0, \tag{7.28}$$

$$h^{i\bar{j}} \breve{F}_{i\bar{j}} = 0, \tag{7.29}$$

$$\breve{F}_{\bar{1}\bar{2}} = 0. \tag{7.30}$$

Taking the supersymmetry variation of equations (7.29), (7.30), we see that fermionic moduli correspond to (0,1) forms on $\mathcal{M}$,

$$h^{i\bar{j}}\nabla_{\bar{j}}\breve{\zeta}_i = 0, \tag{7.31}$$

$$\nabla_1\breve{\zeta}_2 - \nabla_2\breve{\zeta}_1 = 0. \tag{7.32}$$

So correlation functions reduce to superintegrals over the antiholomorphic cotangent bundle of $\mathcal{M}$. Notice that $\bar{Q}$ corresponds to the Dolbeault operator $\bar{\partial}$ on $\mathcal{M}$.

For an $SU(2)$ bundle with instanton number n, the complex dimension of the moduli space is

$$\dim_{\mathbb{C}} \mathcal{M} = 4n - 3\chi. \tag{7.33}$$

Comparing this expression with the form of the $U(1)_A$ and $U(1)_{\mathcal{R}}$ gauge anomalies, we see that observables with $U(1)_A$ charge p and $U(1)_{\mathcal{R}}$ charge $-q$ become Dolbeault harmonic (p,q) forms on $\mathcal{M}$. Performing the Gaussian path integral over quadratic fluctuations and the path integral over fermionic moduli, we find that the saddle point approximation realizes the Donaldson map $\mu : H^{p,q}(X) \to H^{p,q}(\mathcal{M})$. Correlation functions in the holomorphic theory on closed Kähler surfaces with $h^{2,0} > 0$ are therefore Donaldson invariants. The relevant Donaldson invariants have been computed for simply-connected elliptic surfaces (and their blow-ups) [10]. Using these results, we find that all correlation functions vanish on surfaces with $h^{2,0} > 1$. For simply-connected elliptic surfaces with $h^{2,0} = 1$ (and their blow-ups), the nonvanishing correlation functions (up to normalization of the observables) are

$$\begin{aligned}
&\Big\langle \prod_{I=1}^{5} \Big\{\Phi'^I(\upsilon)\Omega'^I(f)\Big\}\Big(\mathcal{G}_{(0)}(\pi)\Big)^5 \Big(\epsilon_{abcd}\mathcal{M}^{ab}_{(0)}(\upsilon)\mathcal{M}^{cd}_{(0)}(\upsilon)\Big)^{l_1}\Big(\epsilon_{abcd}\mathcal{G}_{(0)}(\pi)\mathcal{M}^{ab}_{(2)}(f)\mathcal{M}^{cd}_{(0)}(\upsilon)\Big)^{l_2} \\
&\qquad \Big(\epsilon_{abcd}\mathcal{G}_{(0)}(\pi)\mathcal{M}^{ab}_{(2)}(f)\mathcal{G}_{(0)}(\pi)\mathcal{M}^{cd}_{(2)}(f)\Big)^{l_3}\Big(\mathcal{G}_{(2)}(e)\Big)^{l_4}\Big\rangle \\
&\qquad = \Big(\int_X \upsilon\Big)^{(2l_1+l_2+5)}(l_2+2l_3+5)!\Big(\int_X \pi\wedge f\Big)^{(l_2+2l_3+5)} e^{l_4}(l_1+l_2+l_3+4)^{l_4} \times \\
&\qquad\qquad \Lambda^{4(l_1+l_2+l_3+4)},
\end{aligned} \tag{7.34}$$

where Λ is the strong coupling scale of SQCD,

$$\Lambda^4 = e^{2\pi i\tau}. \tag{7.35}$$

7.3 $\mathcal{N}=1$ Chiral Model

In this section, we review the $\mathcal{N}=1$ nonlinear σ-model into the following hypersurface,

$$\mathcal{H}=\Big\{z^A\in\mathbb{C}^6:\sum_{A=1}^{6}(z^A)^2=\Lambda^4\Big\},\tag{7.36}$$

where Λ is a complex parameter. It will be convenient to realize this theory as a linear σ-model with the constraint (7.36) imposed by auxiliary fields. This linear σ-model consists of six mesons ϕ^A, mesinos ψ^A, and auxiliary fields F^A, as well as an auxiliary scalar field ϕ^X, auxiliary Weyl spinor ψ^X, and auxiliary scalar field F^X. The action for the chiral model is

$$\begin{aligned}S=\int_{\mathbb{R}^4}d^4x\Big\{&\delta_{AB}\Big(-\partial^\mu\phi^{\dagger A}\partial_\mu\phi^B-i\psi^{\dagger A}\bar{\sigma}^\mu\partial_\mu\psi^B+F^{\dagger A}F^B\Big)+F^A\partial_A W+F^X\partial_X W\\&-\frac{1}{2}\psi^A\psi^B\partial_A\partial_B W-\psi^A\psi^X\partial_A\partial_X W+F^{\dagger A}\bar{\partial}_A W^\dagger+F^{\dagger X}\bar{\partial}_X W^\dagger\\&-\frac{1}{2}\psi^{\dagger A}\psi^{\dagger B}\bar{\partial}_A\bar{\partial}_B W^\dagger-\psi^{\dagger A}\psi^{\dagger X}\bar{\partial}_A\bar{\partial}_X W^\dagger\Big\},\end{aligned}\tag{7.37}$$

where the superpotential W has the form

$$W=\frac{1}{\Lambda^4}\phi^X\Big(\delta_{AB}\phi^A\phi^B-\Lambda^4\Big).\tag{7.38}$$

Viewing Euclidean spacetime as a complex manifold with Kähler metric, this theory has a $U(1)_\mathcal{R}$ $\mathcal{R}$-symmetry, $U(1)_X\times SU(2)_X$ spin symmetry, $SO(6)_F$ flavor symmetry and $U(1)_A$ axial symmetry. With respect to these global symmetries, the supercharges transform as shown in Table 7.5 while the fields and their conjugate transform as shown in Table 7.6.

	$U(1)_\mathcal{R}$	$U(1)_X$	$SU(2)_X$
Q_α	1	0	**2**
$\bar{Q}^1$	-1	-1	**1**
$\bar{Q}^2$	-1	1	**1**

Table 7.5: $\mathcal{N}=1$ supercharges and their transformation properties

7.4 Holomorphic Chiral Model

The holomorphic chiral model is constructed by twisting the $U(1)_X$ spin symmetry of the $\mathcal{N}=1$ model by the $U(1)_\mathcal{R}$ $\mathcal{R}$-symmetry,

$$X\to X+\mathcal{R}.\tag{7.39}$$

Field	$U(1)_{\mathcal{R}}$	$U(1)_X$	$SU(2)_X$	$SO(6)_F$
ϕ^A	0	0	$\mathbf{1}$	$\mathbf{6}$
ψ^A_α	1	0	$\mathbf{2}$	$\mathbf{6}$
F^A	2	0	$\mathbf{1}$	$\mathbf{6}$
$\phi^{\dagger A}$	0	0	$\mathbf{1}$	$\mathbf{6}$
$\psi^{\dagger A1}$	-1	-1	$\mathbf{1}$	$\mathbf{6}$
$\psi^{\dagger A2}$	-1	1	$\mathbf{1}$	$\mathbf{6}$
$F^{\dagger A}$	-2	0	$\mathbf{1}$	$\mathbf{6}$
ϕ^X	-2	0	$\mathbf{1}$	$\mathbf{1}$
ψ^X_α	-1	0	$\mathbf{2}$	$\mathbf{1}$
F^X	0	0	$\mathbf{1}$	$\mathbf{1}$
$\phi^{\dagger X}$	2	0	$\mathbf{1}$	$\mathbf{1}$
$\psi^{\dagger 1X}$	1	-1	$\mathbf{1}$	$\mathbf{1}$
$\psi^{\dagger 2X}$	1	1	$\mathbf{1}$	$\mathbf{1}$
$F^{\dagger X}$	0	0	$\mathbf{1}$	$\mathbf{1}$

Table 7.6: Fields in the chiral model and their transformation properties

The $U(1)_{\mathcal{R}}$ symmetry seemingly suffers from gravitational anomalies, however with a suitable choice of path integral measure we can eliminate these anomalies. The simplest way to construct such a measure is to augment the chiral model by an $\mathcal{N} = 1$ SYM theory with gauge group $U(1)^5$. The $U(1)_{\mathcal{R}}$ symmetry of the augmented theory is anomaly free. Since the chiral model and the SYM theory do not interact, correlation functions of the combined theory factor into chiral model correlation functions and SYM correlation functions. We can therefore construct an anomaly free measure for the chiral model from any nontrivial correlation function in the SYM theory. The action for the SYM theory is

$$S' = \int_{\mathbb{R}^4} d^4x \left\{ \frac{1}{g'^2} \mathrm{tr}\Big(-\frac{1}{4} F'^{\mu\nu} F'^{I}_{\mu\nu} - i\lambda'^\dagger \bar{\sigma}^\mu \partial_\mu \lambda' \Big) + \frac{i\theta'}{64\pi^2} \mathrm{tr}\Big(\epsilon^{\mu\nu\sigma\rho} F'_{\mu\nu} F'_{\sigma\rho} \Big) \right\}, \tag{7.40}$$

where the gaugino λ'_α is in the adjoint representation of $U(1)^5$, and A'_μ is a $U(1)^5$ gauge field. With respect to the global symmetries, these fields and their conjugate transform as shown in Table 7.7.

Field	$U(1)_{\mathcal{R}}$	$U(1)_X$	$SU(2)_X$	$SU(4)_F$	$U(1)_A$
A'_i	0	-1	$\mathbf{2}$	$\mathbf{1}$	0
$A'_{\bar{i}}$	0	1	$\mathbf{2}$	$\mathbf{1}$	0
λ'_α	-1	0	$\mathbf{2}$	$\mathbf{1}$	0
$\lambda'^{\dagger 1}$	1	-1	$\mathbf{1}$	$\mathbf{1}$	0
$\lambda'^{\dagger 2}$	1	1	$\mathbf{1}$	$\mathbf{1}$	0

Table 7.7: Fields in SYM theory and their transformation properties

Performing the holomorphic twist (7.39) of the augmented theory, we find that the twisted fields transform as shown in Table 7.8.

Field	$U(1)_{\mathcal{R}}$	$U(1)_X$	$SU(2)_X$	$SO(6)_F$
ϕ^A	0	0	$\mathbf{1}$	$\mathbf{6}$
$\chi^A_{\bar{i}}$	1	1	$\mathbf{2}$	$\mathbf{6}$
$F^A_{\bar{1}\bar{2}}$	2	2	$\mathbf{1}$	$\mathbf{6}$
$\phi^{\dagger A}$	0	0	$\mathbf{1}$	$\mathbf{6}$
$\rho^{\dagger A}_{12}$	-1	-2	$\mathbf{1}$	$\mathbf{6}$
$\sigma^{\dagger A}$	-1	0	$\mathbf{1}$	$\mathbf{6}$
$F^{\dagger A}_{12}$	-2	-2	$\mathbf{1}$	$\mathbf{6}$
ϕ^X_{12}	-2	-2	$\mathbf{1}$	$\mathbf{1}$
$\chi^X_{12\bar{i}}$	-1	-1	$\mathbf{2}$	$\mathbf{1}$
$F^X_{12\bar{1}\bar{2}}$	0	0	$\mathbf{1}$	$\mathbf{1}$
$\phi^{\dagger X}_{\bar{1}\bar{2}}$	2	2	$\mathbf{1}$	$\bar{\mathbf{1}}$
$\rho^{\dagger X}_{12\bar{1}\bar{2}}$	1	0	$\mathbf{1}$	$\bar{\mathbf{1}}$
$\sigma^{\dagger X}_{\bar{1}\bar{2}}$	1	2	$\mathbf{1}$	$\bar{\mathbf{1}}$
$F^{\dagger X}_{12\bar{1}\bar{2}}$	0	0	$\mathbf{1}$	$\bar{\mathbf{1}}$
A'_i	0	-1	$\mathbf{2}$	$\mathbf{1}$
$A'_{\bar{i}}$	0	1	$\mathbf{2}$	$\mathbf{1}$
ζ'_i	-1	-1	$\mathbf{2}$	$\mathbf{1}$
$\varphi'^{\dagger}$	1	0	$\mathbf{1}$	$\mathbf{1}$
$\omega'^{\dagger}_{\bar{1}\bar{2}}$	1	2	$\mathbf{1}$	$\mathbf{1}$
H'	0	0	$\mathbf{1}$	$\mathbf{1}$

Table 7.8: Fields in the holomorphic chiral model + SYM and their transformation properties

Writing the action covariantly in terms of the twisted fields, we have

$$
\begin{aligned}
\widetilde{S} = \int_X d^4z \Big\{ & \delta_{AB}\Big(2hh^{i\bar{j}}\partial_i\phi^{\dagger A}\partial_{\bar{j}}\phi^B - ihh^{i\bar{j}}\partial_i\sigma^{\dagger A}\chi^B_{\bar{j}} - \frac{i}{2}\rho^{\dagger A}_{12}\big(\partial_{\bar{1}}\chi^B_{\bar{2}} - \partial_{\bar{2}}\chi^B_{\bar{1}}\big) \\
& - \frac{1}{4}F^{\dagger A}_{12}F^B_{\bar{1}\bar{2}}\Big) - \frac{1}{4}F^A_{\bar{1}\bar{2}}\big[\partial_A W\big]_{12} - \frac{1}{4}F^X_{12\bar{1}\bar{2}}\partial_X W + \frac{1}{4}\chi^A_{\bar{1}}\chi^B_{\bar{2}}\big[\partial_A\partial_B W\big]_{12} \\
& + \frac{1}{4}\chi^A_{\bar{1}}\chi^X_{12\bar{2}}\partial_A\partial_X W - \frac{1}{4}\chi^A_{\bar{2}}\chi^X_{12\bar{1}}\partial_A\partial_X W - \frac{1}{4}F^{\dagger A}_{12}\big[\bar{\partial}_A W^\dagger\big]_{\bar{1}\bar{2}} - \frac{1}{4}F^{\dagger X}_{12\bar{1}\bar{2}}\bar{\partial}_X W^\dagger \\
& - \frac{1}{4}\sigma^{\dagger A}\rho^{\dagger B}_{12}\big[\bar{\partial}_A\bar{\partial}_B W^\dagger\big]_{\bar{1}\bar{2}} - \frac{1}{4}\sigma^{\dagger A}\rho^{\dagger X}_{12\bar{1}\bar{2}}\bar{\partial}_A\bar{\partial}_X W^\dagger + \frac{1}{4}\rho^{\dagger A}_{12}\sigma^{\dagger X}_{\bar{1}\bar{2}}\bar{\partial}_A\bar{\partial}_X W^\dagger \Big\}, \\
\tilde{S}' = \int_X d^4z \Big\{ & \frac{1}{g'^2}\mathrm{tr}\Big(-\frac{1}{2}hH'^2 + hH'\big(ih^{i\bar{j}}F'_{i\bar{j}}\big) + 2F'_{12}F'_{\bar{1}\bar{2}} + ihh^{i\bar{j}}\varphi'^{\dagger}\partial_{\bar{j}}\zeta'_i \\
& + \frac{i}{2}\omega'^{\dagger}_{\bar{1}\bar{2}}\big(\partial_1\zeta'_2 - \partial_2\zeta'_1\big)\Big) + \frac{\tau'}{4\pi i}\mathrm{tr}\Big(F'_{1\bar{1}}F'_{2\bar{2}} - F'_{12}F'_{\bar{1}\bar{2}} + F'_{1\bar{2}}F'_{\bar{1}2}\Big)\Big\},
\end{aligned}
\tag{7.41}
$$

where $h_{i\bar{j}}$ is the Kähler metric, $h = \det h_{i\bar{j}}$, and $\tau' = \frac{4\pi i}{g'^2} - \frac{\theta'}{2\pi}$ is the complexified gauge coupling constant. The supersymmetry variations of the twisted fields (with respect to the scalar supercharge) are

$$\begin{aligned}
&\delta\phi^A = 0, && \delta\phi^X_{12} = 0, && \delta A'_i = i\bar{\xi}\zeta'_i,\\
&\delta\chi^A_{\bar{i}} = 2i\sqrt{2}\bar{\xi}\bar{\partial}_{\bar{i}}\phi^A, && \delta\chi^X_{12\bar{i}} = 2i\sqrt{2}\bar{\xi}\bar{\partial}_{\bar{i}}\phi^X_{12}, && \delta A'_{\bar{i}} = 0,\\
&\delta F^A_{\bar{1}\bar{2}} = -2i\sqrt{2}\bar{\xi}\epsilon^{\bar{i}\bar{j}}\bar{\partial}_{\bar{i}}\chi^A_{\bar{j}}, && \delta F^X_{12\bar{1}\bar{2}} = -2i\sqrt{2}\bar{\xi}\epsilon^{\bar{i}\bar{j}}\bar{\partial}_{\bar{i}}\chi^X_{12\bar{j}}, && \delta\zeta'_i = 0,\\
&\delta\phi^{\dagger A} = \sqrt{2}\bar{\xi}\sigma^{\dagger A}, && \delta\phi^{\dagger X}_{\bar{1}\bar{2}} = \sqrt{2}\bar{\xi}\sigma^{\dagger X}_{\bar{1}\bar{2}}, && \delta\varphi'^{\dagger} = i\bar{\xi}H',\\
&\delta\rho^{\dagger A}_{12} = \sqrt{2}\bar{\xi}F^{\dagger A}_{12}, && \delta\rho^{\dagger X}_{12\bar{1}\bar{2}} = \sqrt{2}\bar{\xi}F^{\dagger X}_{12\bar{1}\bar{2}}, && \delta\omega'^{\dagger}_{\bar{1}\bar{2}} = -4\bar{\xi}F'_{\bar{1}\bar{2}},\\
&\delta\sigma^{\dagger A} = 0 && \delta\sigma^{\dagger X}_{\bar{1}\bar{2}} = 0, && \delta H' = 0.\\
&\delta F^{\dagger A}_{12} = 0, && \delta F^{\dagger X}_{12\bar{1}\bar{2}} = 0.
\end{aligned} \tag{7.42}$$

Notice that the terms in the twisted action that depend on the Hermitian metric are $\bar{Q}$-exact,

$$\begin{aligned}
\widetilde{S} = \int_X d^4z \bigg\{ &-\frac{1}{4}F^A_{\bar{1}\bar{2}}\big[\partial_A W\big]_{12} - \frac{1}{4}F^X_{12\bar{1}\bar{2}}\partial_X W + \frac{1}{4}\chi^A_{\bar{1}}\chi^B_{\bar{2}}\big[\partial_A\partial_B W\big]_{12}\\
&+\frac{1}{4}\chi^A_{\bar{1}}\chi^X_{12\bar{2}}\partial_A\partial_X W - \frac{1}{4}\chi^A_{\bar{2}}\chi^X_{12\bar{1}}\partial_A\partial_X W + \Big\{\bar{Q}, \delta_{AB}\Big(-\frac{i}{\sqrt{2}}hh^{i\bar{j}}\partial_i\phi^{\dagger A}\chi^B_{\bar{j}}\\
&-\frac{1}{4\sqrt{2}}\rho^{\dagger A}_{12}F^B_{\bar{1}\bar{2}}\Big) - \frac{1}{4\sqrt{2}}\rho^{\dagger A}_{12}\big[\bar{\partial}_A W^{\dagger}\big]_{\bar{1}\bar{2}} - \frac{1}{4\sqrt{2}}\rho^{\dagger X}_{12\bar{1}\bar{2}}\bar{\partial}_X W^{\dagger}\Big\}\bigg\}.\\
\tilde{S}' = \int_X d^4z &\left\{\bar{Q}, \frac{1}{2g'^2}\mathrm{tr}\Big(ih\varphi'^{\dagger}H' - 2ih\varphi'^{\dagger}\big(ih^{i\bar{j}}F'_{i\bar{j}}\big) - F'_{12}\omega'^{\dagger}_{\bar{1}\bar{2}}\Big)\right\} + \frac{\tau'}{8\pi i}\int_X \mathrm{tr}\big(F'\wedge F'\big).
\end{aligned} \tag{7.43}$$

It follows that correlators of metric independent physical observables (i.e., supersymmetric observables) are invariant with respect to infinitesimal deformations of the Kähler metric,

$$\begin{aligned}
\delta_h\langle\mathcal{O}\rangle &= -\int \mathcal{D}\Phi\,\mathcal{O}\big(\delta_h\widetilde{S} + \delta_h\widetilde{S}'\big)\exp(-\widetilde{S} - \widetilde{S}')\\
&= -\int \mathcal{D}\Phi\,\mathcal{O}\big\{\bar{Q}, \cdots\big\}\exp(-\widetilde{S} - \widetilde{S}')\\
&= -\int \mathcal{D}\Phi\,\big\{\bar{Q}, \mathcal{O}\cdots\big\}\exp(-\widetilde{S} - \widetilde{S}') = 0.
\end{aligned} \tag{7.44}$$

7.4.1 Observables

Our next task is to construct physical observables on closed Kähler sufaces.[2] From the supersymmetry variations, we see that the physical observables in the chiral model sector are the mesons, the auxiliary scalar ϕ^X, and their descendents,

[2]The restriction to closed manifolds is necessary for the descent procedure.

$$\begin{aligned}
\mathcal{M}^A_{(0)}(\upsilon) &= \int_X \phi^A \wedge \upsilon, \\
\mathcal{M}^A_{(1)}(\theta) &= \int_X \chi^A \wedge \theta, \\
\mathcal{M}^A_{(2)}(f) &= \int_X F^A \wedge f, \\
\mathcal{X}_{(0)}(\pi) &= \int_X \phi^X \wedge \pi, \\
\mathcal{X}_{(1)}(\eta) &= \int_X \chi^X \wedge \eta, \\
\mathcal{X}_{(2)}(e) &= \int_X F^X \wedge e,
\end{aligned} \tag{7.45}$$

where υ, θ, f are Dolbeault harmonic $(2,2)$, $(2,1)$, and $(2,0)$ forms and π, η, e are Dolbeault harmonic $(0,2)$, $(0,1)$, and $(0,0)$ forms.

The gauge invariant physical observables in SYM sector of the theory (restrict the path integral to the trivial $U(1)^5$ bundle and eliminating the constant mode of the auxiliary field H') are the gluinos,

$$\begin{aligned}
\mathcal{G}'^I_{(0)}(\varrho) &= \int_X \zeta'^I \wedge \varrho, \\
\Phi'^I(\upsilon) &= \int_X \varphi'^{\dagger I} \wedge \upsilon, \\
\Omega'^I(f) &= \int_X \omega'^{\dagger I} \wedge f,
\end{aligned} \tag{7.46}$$

where $I = 1, 2, \cdots, 5$ indexes the $U(1)$ factors of the gauge group and ϱ is a Dolbeault harmonic $(1,2)$ form.

7.4.2 Correlation Functions

Before computing correlation functions in the holomorphic theory, let's identify some selection rules for nonvanishing correlation functions. Notice that there is a one-to-one correspondence between fermionic zero modes of the twisted SYM action and gauge invariant physical observables in the SYM sector (restricting the path integral to the trivial $U(1)^5$ bundle and eliminating the constant mode of H'). Nontrivial correlation functions in the holomorphic theory therefore have the form

$$\Big\langle \prod_{I=1}^{5} \Big\{ \prod_{\varrho \in \mathcal{B}^{(1,2)}} \mathcal{G}'^I_{(0)}(\varrho) \prod_{\upsilon \in \mathcal{B}^{(2,2)}} \Phi'^I(\upsilon) \prod_{f \in \mathcal{B}^{(2,0)}} \Omega'^I(f) \Big\} \mathcal{O}_{CM} \Big\rangle, \tag{7.47}$$

where $\mathcal{B}^{(p,q)}$ is a basis for $H^{(p,q)}(X)$ and $\mathcal{O}_{CM}$ is an observable in the chiral model sector. Additional selection rules come from the $U(1)_{\mathcal{R}}$ and $SO(6)_F$ symmetries; correlation functions of the form (7.47) vanish unless $\mathcal{O}_{CM}$ is $SO(6)_F$ invariant with the following $U(1)_{\mathcal{R}}$ charge,

$$\mathcal{R}(\mathcal{O}_{CM}) = -5\chi, \tag{7.48}$$

where χ is the holomorphic Euler characteristic of X.

Now let's evaluate correlation functions satisfying these selection rules. The path integral over twisted SYM fields simply introduces normalization factors necessary for the twisted chiral model to have a well-defined path integral measure. To evaluate of path integral over twisted chiral model fields, we use the following localization technique: the saddle point approximation about fixed points of any Grassmann symmetry is exact [41]. From the variations (7.42), we see that fixed points with respect to supersymmetry are field configurations satisfying the following equations,

$$\begin{aligned}\partial_{\bar{i}}\phi^A &= 0,\\ \partial_{\bar{i}}\phi^X_{12} &= 0,\\ \left[\partial_A W\right]_{12} &= 0,\\ \partial_X W &= 0.\end{aligned} \tag{7.49}$$

where we have eliminated the auxiliary fields $F^A_{\bar{1}\bar{2}}$, $F^{\dagger A}_{12}$, $F^X_{12\bar{1}\bar{2}}$, and $F^{\dagger X}_{12\bar{1}\bar{2}}$. The moduli space of solutions to these equations is the hypersurface $\mathcal{H}$ (7.36),

$$\breve{\phi}^X_{12} = 0, \tag{7.50}$$

$$\breve{\phi}^A \in \mathcal{H}. \tag{7.51}$$

For Kähler surfaces with $h^{2,0}(X) > 0$, we can simplify this moduli space by making the following perturbation to the superpotential,

$$\delta W = \delta_{AB} m^A \phi^B. \tag{7.52}$$

where $m^A \in H^{2,0}(X)$. This perturbation introduces are two additional terms to the action,

$$\delta\widetilde{S}_m = -\int_X d^4z\, \frac{1}{4}\delta_{AB} m^A_{12} F^B_{\bar{1}\bar{2}}, \tag{7.53}$$

$$\delta\widetilde{S}_{m^\dagger} = -\int_X d^4z\, \frac{1}{4}\delta_{AB} m^{\dagger A}_{\bar{1}\bar{2}} F^{\dagger B}_{12}. \tag{7.54}$$

Notice that the $\delta\widetilde{S}_m$ has $U(1)_{\mathcal{R}}$ charge 2 and $\delta\widetilde{S}_{m^\dagger}$ is $\bar{Q}$-exact,

$$\delta\widetilde{S}_{m^\dagger} = -\int_X d^4z \left\{\bar{Q}, \frac{1}{4\sqrt{2}}\delta_{AB} m^{\dagger A}_{\bar{1}\bar{2}} \rho^{\dagger B}_{12}\right\}. \tag{7.55}$$

Treating these additional terms as an insertion of the observable $e^{-\widetilde{S}_m} e^{-\widetilde{S}_{m^\dagger}}$, the properties above show that correlation functions satisfying the selection rules are unchanged. This perturbation does however change the moduli space of fixed points. For closed Kähler surfaces with $h^{2,0}(X) > 1$, there are generically no solutions to the perturbed saddle point equations. So all correlation functions in the holomorphic theory vanish on such surfaces. For closed Kähler surfaces with $h^{2,0}(X) = 1$, solutions to the perturbed saddle point equations are

$$\begin{aligned} \breve{\phi}^A_{\pm} &= \pm \frac{\Lambda^2 m_0^A}{\sqrt{\delta_{AB} m_0^A m_0^B}}, \\ \breve{\phi}^X_{12\pm} &= \mp \frac{\Lambda^2 \sqrt{\delta_{AB} m_0^A m_0^B}}{2} f^0_{12}, \end{aligned} \tag{7.56}$$

where $f^0 \in H^{2,0}(X)$ is the normalized basis element and $m^A = m_0^A f^0$. Performing the Gaussian path integral over quadratic fluctuations and summing the contribution from each saddle point, we find that the nonvanishing correlation functions (up to normalization of the observables) on simply-connected Kähler surfaces with $h^{2,0}(X) = 1$ are

$$\begin{aligned} &\Big\langle \prod_{I=1}^{5} \Big\{ \Phi'^I(\upsilon)\Omega'^I(f) \Big\} \Big(\mathcal{X}_{(0)}(\pi) \Big)^5 \Big(\delta_{AB} \mathcal{M}^A_{(0)}(\upsilon) \mathcal{M}^B_{(0)}(\upsilon) \Big)^{l_1} \Big(\delta_{AB} \mathcal{X}_{(0)}(\pi) \mathcal{M}^A_{(2)}(f) \mathcal{M}^B_{(0)}(\upsilon) \Big)^{l_2} \\ &\quad \Big(\delta_{AB} \mathcal{X}_{(0)}(\pi) \mathcal{M}^A_{(2)}(f) \mathcal{X}_{(0)}(\pi) \mathcal{M}^B_{(2)}(f) \Big)^{l_3} \Big(\mathcal{X}_{(2)}(e) \Big)^{l_4} \Big\rangle \\ &\qquad = \Big(\int_X \upsilon \Big)^{(2l_1+l_2+5)} (l_2 + 2l_3 + 5)! \Big(\int_X \pi \wedge f \Big)^{(l_2+2l_3+5)} e^{l_4} (l_1 + l_2 + l_3 + 4)^{l_4} \times \\ &\qquad\quad \Lambda^{4(l_1+l_2+l_3+6)}. \end{aligned} \tag{7.57}$$

7.5 Seiberg Duality

Seiberg proposed that the low-energy effective description of $\mathcal{N} = 1$ SQCD with gauge group $SU(2)$ and two flavors is the $\mathcal{N} = 1$ chiral model discussed in Section 7.3 [31]. This implies that holomorphic SQCD with gauge group $SU(2)$ and two flavors is equivalent to the holomorphic chiral model. The mapping of observables under this duality is

$$\begin{aligned} \mathcal{G}_{(0)}(\pi) &\longleftrightarrow \mathcal{X}_{(0)}(\pi), \\ \mathcal{G}_{(1)}(\eta) &\longleftrightarrow \mathcal{X}_{(1)}(\eta), \\ \mathcal{G}_{(2)}(e) &\longleftrightarrow \mathcal{X}_{(2)}(e), \\ \mathcal{M}^{ab}_{(0)}(\upsilon) &\longleftrightarrow \mathcal{M}^A_{(0)}(\upsilon), \\ \mathcal{M}^{ab}_{(1)}(\theta) &\longleftrightarrow \mathcal{M}^A_{(1)}(\theta), \\ \mathcal{M}^{ab}_{(2)}(f) &\longleftrightarrow \mathcal{M}^A_{(2)}(f). \end{aligned}$$

Normalizing the measure of the holomorphic chiral model by $\Lambda^{-4\chi}$, we find that correlation functions in these theories agree precisely under the duality map, providing a highly nontrivial test of Seiberg's conjecture.

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