Generalizations to the Converse of Contraction Mapping Principle

Thesis by

James Sai Wing Wong

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1965
(Submitted May 1, 1964)

Acknowledgements

I wish to express my deep appreciation to Professor H. F. Bohnenblust for his guidance and enlightenment, and for countless hours of patient discussion. I would also like to thank Professors W. A. J. Luxemburg and C. R. DePrima for the inspiration and training I received from them. Finally, I thank my wife, Madeline, for her continuing encouragement and sacrifice.

Abstract

Let X be a non-empty abstract set and $\mathcal{S}$ be a commutative semi-group of operators defined on X into itself. $\mathcal{S}$ is called a contractive semi-group on X if there exists a metric ρ on X such that for each $T \in \mathcal{S}$, $T \neq I$, $\rho(Tx,Ty) \leqslant \lambda(T)\, \rho(x,y)$ for all x, y $\in X$, where $0 \leqslant \lambda(T) < 1$. We find sufficient conditions on $\mathcal{S}$ in order that $\mathcal{S}$ be contractive on X. In the case when $\mathcal{S}$ is generated by a finite number of mutually commuting mappings T_1, T_2, ..., T_n, possessing a common unique fixed point in X, these conditions are automatically satisfied. The resulting statement is the following generalization of the converse of contraction mapping principle: <u>Theorem C</u>. Let X be an abstract set with n mutually commuting mappings $T_1, T_2, \ldots, T_n$ defined on X into itself such that each iteration $T_1^{k_1} \cdots\cdots T_n^{k_n}$ (where $k_1, k_2, \ldots, k_n$ are non-negative integers not all equal to zero) possesses a unique fixed point which is common to every choice of $k_1, k_2, \ldots, k_n$. Then for each $\lambda \in (0,1)$, there exists a complete metric ρ on X such that $\rho(T_i x, T_i y) \leqslant \lambda\, \rho(x,y)$ for $1 \leqslant i \leqslant n$, and for all $x, y \in X$. This result reduces to that of C. Bessaga by taking n = 1. (Rf: C. Bessaga, Colloquim Mathematicum VII (1959), 41-43.)

0. INTRODUCTION

We are here concerned with the classical fixed point theorem of Banach, commonly known as the contraction mapping principle, which states:

Theorem A. (S. Banach [1]) Let T be a mapping of a complete metric space X into itself. If for every pair of elements $x, y \in X$ and some fixed λ, $0 \leqslant \lambda < 1$,

$$\rho(Tx, Ty) \leqslant \lambda \rho(x, y) \qquad (1)$$

Then T has a unique fixed point, and the sequence of iterates $\{T^n x\}$ for each $x \in X$ converge in metric to this unique fixed point.

We call the mapping T satisfying (1) a contraction on X. Theorem A has been used extensively in proving the existence and uniqueness of solutions to various functional equations, particularly integral and differential equations (Kolmogorov and Fomin [2]). It has been applied to prove the convergence of successive approximations of solutions to ordinary differential equations (Luxemburg [3]) and integral equations in L_p-spaces (Willett [4]), to prove the Frobenius-Perron theorem on positive matrices (Samuelson [5]), and to develop many otherwise difficult existence and uniqueness theorems in various function spaces (Thampson [6]). The contraction mapping principle has also been widely used by numerical analysts in the study of convergence and error estimates in well-known function spaces (Schröder [7]). Various generalizations and localizations of Theorem A are given which in one way or other relax the restrictions on the mapping T or the underlying complete metric space X (Edelstein [8], Rakotch [9], Kammerer and Kasriel [10]). These generalizations also find interesting applications in the study of functional equations (Edwards [11]). Moreover, the concept of contractions has been made meaningful in spaces more general than metric spaces, and the corresponding fixed point theorem is proved (Davis [12]).

This thesis is an outgrowth of studies related to the converse of Theorem A. The natural converse statement is the following: " Let X be a complete metric space, and T be a mapping of X into itself such that for each $x \in X$, the sequence of iterates $\{T^n x\}$ converges to a unique fixed point $\omega \in X$. Then there exists a complete metric on X in which T is a contraction. " This is in fact true, even in a stronger sense. The following converse of Theorem A was due to C. Bessaga.

<u>Theorem B.</u> (C. Bessaga [13]) Let X be an abstract set and T be a mapping of X into itself such that for each positive integer $k>0$, the equation $T^k x = x$ holds for some x X implies x = , the unique fix ed point of T. Then for each λ, $0 \leq \lambda < 1$, there exists a complete metric on X such that $\rho(Tx, Ty) \leq \lambda \rho(x, y)$ for all $x, y \in X$.

(A weaker form of Theorem B, in case X is a compact metrizable space, was also given by Janos [14]).

We are interested here in further generalizations of Theorem B. Specifically, we ask whether there exists a metric on X in which mutually commuting mappings $T_1, T_2, \ldots, T_n$ with common unique fixed point are simultaneously contractions . Note that if $T_1, T_2, \ldots, T_n$ are contractions, then every element of the commutative semigroup $\mathcal{S}$ generated by $T_1, T_2, \ldots, T_n$ is again a contraction. Hence we extend the concept of a contraction to the concept of a contractive semigroup. We obtain necessary and sufficient conditions for $\mathcal{S}$ to be contractive in terms of the existence of certain level function on X. Sufficient conditions on $\mathcal{S}$ are also given for $\mathcal{S}$ to be contractive. In case $\mathcal{S}$ is generated by a finite number of mutually commuting mappings with common unique fixed point, these conditions are automatically satisfied. The resulting statement is the following generalization of Theorem B.

<u>Theorem C.</u> ([15]) Let X be an abstract set with n mutually commuting mappings $T_1, T_2, \ldots, T_n$ defined on X into itself such that each iteration $T_1^{k_1} \cdots\cdots T_n^{k_n}$ (where $k_1, k_2, \ldots, k_n$ are non-negative integers not all equal to zero) possesses a unique fixed point which is common to every choice of $k_1, k_2, \ldots, k_n$. Then for each $\lambda \in (0,1)$, there exists a complete metric ρ on X such that $\rho(T_i x, T_i y) \leqslant \lambda \rho(x,y)$ for $1 \leqslant i \leqslant n$, and for all $x, y \in X$.

In section 1, we introduce basic notations and terminologies whic are necessary for all later discussions. Section 2 introduces the concept of a contractive semi-group $\mathcal{S}$, and presents a necessary and sufficient condition for $\mathcal{S}$ to be contractive. The main theorem is proved in section 3 where sufficient conditions are imposed on

$\mathcal{S}$ to insure that it be contractive. This result is then applied in section 4 to prove Theorem C. Finally, we make several remarks which lead to questions for further research.

References throughout this thesis are given by a number in brackets indicating a particular article or book in question. A complete list of references arranged in the order of appearance is given at the end of this thesis. We indicate the end of a proof by the abbreviated notation $\square$.

1. DEFINITIONA AND NOTATIONS

Let X be a non-empty abstract set and $\mathcal{S}$ be a commutative semi-group of operators on X into itself, containing the identity I. $\mathcal{S}$ is said to be contractive (completely contractive) semi-group on X if there exists a metric (complete metric) ρ on X such that for each $S \in \mathcal{S}$, $\rho(Sx,Sy) \leqslant \lambda(S)\rho(x,y)$ for all $x,y \in X$ where $0 \leqslant \lambda(S) < 1$ for $S \neq I$ and $\lambda(I) = 1$. We say that $\mathcal{S}$ is a uniformly contractive (uniformly completely contractive) semi-group on X if there exists a real number λ such that $\lambda(S) \leqslant \lambda < 1$ for all $S \in \mathcal{S}$, $S \neq I$. In all later discussions we say $\mathcal{S}$ contractive, completely contractive, or respectively uniformly contractive for short. In order to aviod dealing with the trivial contractive semi-group $\mathcal{S} = \{I\}$, we assume that $\mathcal{S}$ contains at least one other element T, $T \neq I$.

$X_1 \subseteq X$ is called an $\mathcal{S}$-invariant set if $\mathcal{S}X_1 \subseteq X_1$. Obviously X and the empty set $\emptyset$ are $\mathcal{S}$-invariant sets. Consider the set $[a] = \mathcal{S}\{a\} = \{x : x = Ta \text{ for some } T \in \mathcal{S}\}$. Clearly $\mathcal{S}[a] \subseteq [a]$, and hence $[a]$ is an $\mathcal{S}$-invariant set. It is the smallest $\mathcal{S}$-invariant set containing a. Note that arbitrary unions and intersections preserve the $\mathcal{S}$-invariance. Similarly, $\mathcal{S}_1 \subseteq \mathcal{S}$ is called an invariant set if $\mathcal{S}\mathcal{S}_1 \subseteq \mathcal{S}_1$. An invariant set may be considered as an $\mathcal{S}$-invaraint set in $\mathcal{S}$. Hence remarks on $\mathcal{S}$-invariant sets hold similarly for invariant sets.

A function λ is called <u>contractive on $\mathcal{S}$</u> if $0 \leqslant \lambda(S) < 1$ for all $S \in \mathcal{S}$, $S \neq I$, and $\lambda(I) = 1$. The function λ is called <u>uniformly contractive on $\mathcal{S}$</u>, if there exists a λ such that $\lambda(S) \leqslant \lambda < 1$ for all $S \in \mathcal{S}$, $S \neq I$. A function ρ is called a <u>level function with respect to λ</u> if :

(i) its domain of definition Y, is an $\mathcal{S}$- invariant set;

(ii) $0 \leqslant \rho(x) < \infty$ for all $x \in Y$;

(iii) $\rho(Tx) \leqslant \lambda(T)\rho(x)$ for all $x \in Y$, where λ is contractive on $\mathcal{S}$;

(iv) $\rho(x_1) = \rho(x_2) = 0$ implies $x_1 = x_2$;

We call the function σ <u>a length function on $\mathcal{S}$</u> if it satisfies the conditions:

(i) $0 \leqslant \sigma(S) < 1$ for $S \neq I$ and $\sigma(I) = 1$;

(ii) $\sigma(ST) \leqslant \sigma(S)\sigma(T)$;

(iii) $\sigma(S_1) = \sigma(S_2) = 0$ implies $S_1 = S_2$.

A length function on $\mathcal{S}$ is certainly contractive on $\mathcal{S}$ and hence it may be regarded as level function on $\mathcal{S}$.

We adapt the following terminologies for arbitrary partially ordered sets. Let P be an arbitrary partially ordered set. Any two elements $x, y \in P$ are called <u>comparable</u> if either $x \leqslant y$ or $y \leqslant x$ holds, otherwise they are called <u>non-comparable</u>. By a <u>transverse set</u> we mean a subset of P whose elements are pairwise mutually non-comparable. A subset J of P is called an <u>ideal</u> if $x \in P$ and $x \geqslant y$ for some $y \in J$ implies $x \in J$. An ideal J is called <u>principal</u> if it is of the form $\{ x :$

$x \in P,\ x \geqslant y \}$ for some fixed element $y \in P$ and it is denoted by $\langle y \rangle$. The element y is uniquely determined by J and is called the generator of the principal ideal J. (Refer to G. Birkoff [16] for other terminologies on partially ordered sets not explained here.)

2. CONTRACTIVE SEMI-GROUPS

We first propose to prove a necessary and sufficient condition for $\mathcal{S}$ to be contractive.

Theorem 1. If $\mathcal{S}$ is contractive on X, then there exists a level function on the full set X.

Proof. X is certainly an $\mathcal{S}$-invariant set. Since $\mathcal{S}$ is contractive on X, for each $T \in \mathcal{S}$, $T \neq I$, and for each $x \in X$, we have $\rho(T^p x, x) \leqslant \frac{\rho(Tx, x)}{1 - \lambda(T)}$ for all non-negative integers p. For each $T \in \mathcal{S}$, $T \neq I$, $\{\rho(T^n x, x)\}$ is a Cauchy sequence. Denote the limit of this sequence by $\rho_T(x)$. We claim that this limit is independent of T, i.e. for each pair S, $T \in \mathcal{S}$, S, $T \neq I$, $\rho_T(x) = \rho_S(x)$. Note that for S, T $\neq I$,

$$\begin{aligned} |\rho(T^n x, x) - \rho(S^n x, x)| &\leqslant \rho(S^n x, T^n x) \\ &\leqslant \rho(S^n x, S^n T^n x) + \rho(S^n T^n x, T^n x) \\ &\leqslant \lambda^n(S)\,\rho(x, T^n x) + \lambda^n(T)\,\rho(S^n x, x) \\ &\leqslant \lambda^n(S)\frac{\rho(x, Tx)}{1 - \lambda(T)} + \lambda^n(T)\frac{\rho(Sx, x)}{1 - \lambda(S)} . \end{aligned}$$

Since the right hand side tends to zero as n tends to infinity, this shows that $\rho_T(x) = \rho_S(x)$ as desired. We may now denote the common limit by $\rho(x)$, i.e. $\rho(x) = \lim_{n \to \infty} \rho(T^n x, x)$ where $T \in \mathcal{S}$, $T \neq I$. Obviously $0 \leqslant \rho(x) < \infty$. Furthermore,

$$\rho(Tx) = \lim_{n \to \infty} \rho(T^n x, Tx) \leqslant \lambda(T) \lim_{n \to \infty} \rho(T^{n-1} x, x)$$

$$= \lambda(T)\,\rho(x)\,.$$

Finally, for each pair x_1, $x_2 \in X$, we have:

$$\begin{aligned} \rho(x_1, x_2) &\leqslant \rho(x_1, T^n x_1) + \rho(T^n x_1, T^n x_2) + \rho(T^n x_2, x_2) \\ &\leqslant \rho(x_1, T^n x_1) + \lambda^n(T) \quad (x_1, x_2) + \rho(T^n x_2, x_2) \end{aligned}$$

In particular by choosing $T \neq I$, hence $0 \leqslant \lambda(T) < 1$, and letting n tends to infinity we obtain that $\rho(x_1) = \rho(x_2) = 0$ implies $x_1 = x_2$. ∎

The existence of a level function on the full set X is not only necessary, as shown, but also sufficient:

<u>Theorem 2</u>. If there exists a level function on the full set X, then $\mathcal{S}$ is contractive.

<u>Proof.</u> Let $\rho(x)$ be a level function with respect to a certain contractive function λ. Define a metric $\tilde{\rho}$ on X by:

$$\tilde{\rho}(x, y) = \begin{cases} \rho(x) + \rho(y) & \text{if } x \neq y; \\ 0 & \text{if } x = y. \end{cases} \tag{2}$$

Clearly $\tilde{\rho}$ is a metric on X, and for each $T \neq I$, we have

$$\tilde{\rho}(Tx, Ty) \leqslant \lambda(T)\,\tilde{\rho}(x, y)\,,$$

since for $x \neq y$,

$$\begin{aligned} \tilde{\rho}(Tx, Ty) &\leqslant \rho(Tx) + \rho(Ty) \\ &\leqslant \lambda(T)\rho(x) + \lambda(T)\rho(y) \\ &= \lambda(T)\,\tilde{\rho}(x, y)\,, \end{aligned}$$

and for $x = y$ the above inequality is obvious. ∎

The definition of uniformly contractive semi-group depends on the existence of certain uniformly contractive function λ. Note that the definition is actually independent of λ. Suppose $\mathcal{S}$ is uniformly contractive with respect to some uniformly contractive function λ. Then by Theorem 1 there exists a level function $\rho(x)$ on X satisfying $\rho(Tx) \leqslant \lambda \rho(x)$ for all $x \in X$, and all $T \in \mathcal{S}$, $T \neq I$. For any real number α, $0 < \alpha < \infty$, define $\rho_\alpha(x) = [\rho(x)]^\alpha$. This new function satisfies $\rho_\alpha(Tx) \leqslant \lambda^\alpha \rho_\alpha(x)$ and is easily seen to be again a level function. For any $\mu \in (0,1)$ we can choose ρ_α such that $\lambda^\alpha = \mu$, and hence by Theorem 2, $\mathcal{S}$ is uniformly contractive with respect to μ.

Repeating the arguments in Theorems 1 and 2 with λ replacing $\lambda(T)$ throughout, we obtain:

Theorem 3. $\mathcal{S}$ is uniformly contractive if and only if there exists a level function defined on the full set X with respect to a uniformly contractive function.

Theorem 4. $\mathcal{S}$ is completely contractive if and only if $\mathcal{S}$ is contractive and there exists an element $\omega \in X$ such that $S\omega = \omega$ for some $S \in \mathcal{S}$, $S \neq I$.

Proof. Let $\mathcal{S}$ be completely contractive. Choose $S \in \mathcal{S}$, $S \neq I$. There exists by the contraction mapping principle an element $\omega \in X$, such that $S\omega = \omega$. Conversely, assume that $\mathcal{S}$ is contractive and that there exists $\omega \in X$ such that $S\omega = \omega$ for some $S \in \mathcal{S}$, $S \neq I$. Construct a level function ρ on X as defined in Theorem 1. The value $\rho(\omega)$ of this level function at ω must be zero, since $\lambda(S) \neq 1$, and $\rho(\omega) = \rho(S\omega) \leqslant \lambda(S)\rho(\omega)$.

Define a new metric $\tilde{\rho}$ on X be (2). Let $\{x_n\}$ be a Cauchy sequence in X with respect to this new metric. If $\rho(x_n)$ tends to zero as n tends to infinity, then the sequence $\{x_n\}$ has the limit ω, since $\tilde{\rho}(x_n, \omega) \leqslant \rho(x_n) + \rho(\omega) = \rho(x_n)$. On the other hand if $\rho(x_n)$ does not tend to zero then there exists a subsequence $\{y_n\} \subseteq \{x_n\}$ such that $\rho(y_n) \geqslant \delta > 0$. By assumption there exists a $N \geqslant 0$ such that $\tilde{\rho}(y_n, y_m) < \delta$ for $n, m \geqslant N$. This implies $y_n = y_m$ for all $n, m \geqslant N$, and the subsequence $\{y_n\}$ has a limit namely y_N. As a Cauchy sequence, the full sequence has the same limit. $\square$

Theorem 4 shows that completely contractive semi-groups are essentially contractive semi-groups. For any non-completely contractive semi-group $\mathcal{S}$ we may always add the point ω to X and define $T\omega = \omega$ for all $T \in \mathcal{S}$ to make it completely contractive.

<u>Theorem 5</u>. If $\mathcal{S}$ is contractive on X, then there exists a length function defined on $\mathcal{S}$.

<u>Proof</u>. Define $\sigma(S) = \sup_{x \neq y} \dfrac{\rho(Sx, Sy)}{\rho(x, y)}$. If $S \neq I$, then $0 \leqslant \sigma(S) \leqslant \lambda(S) < 1$. If $S = I$, then $\sigma(S) = 1$. For any $S, T \in \mathcal{S}$ and $x \neq y$, we have $\rho(STx, STy) \leqslant \sigma(S)\rho(Tx, Ty)$ for $Tx \neq Ty$. But this inequality obviously holds even if $Tx = Ty$. Moreover $\rho(STx, STy) \leqslant \sigma(S)\sigma(T)$

$\rho(x,y)$. Hence by dividing through with $\rho(x,y)$ we easily conclude $\sigma(ST) \leqslant \sigma(S)\,\sigma(T)$. Finally, assume $\sigma(S_1) = \sigma(S_2) = 0$, then for all $x, y \in X$, $x \neq y$, $\rho(S_1x, S_1y) = \rho(S_2x, S_2y) = 0$. This implies the existence of ω_1, ω_2 such that $S_1x = \omega_1$, and $S_2x = \omega_2$ for all $x \in X$. Now for any $x \in X$, we have

$$\begin{aligned}\rho(S_1x, S_2x) &\leqslant \rho(S_1x, S_1S_2x) + \rho(S_1S_2x, S_2x)\\ &= \rho(\omega_1, \omega_1) + \rho(\omega_2, \omega_2) = 0.\end{aligned}$$

Thus $\sigma(S_1) = \sigma(S_2) = 0$ implies that $S_1 = S_2$. $\Box$

We remark the interesting fact that if $\mathcal{S}$ is contractive (uniformly contractive) on X then $\mathcal{S}$ is also contractive (uniformly contractive) on itself. The converse is obviously not true.

3. THE MAIN THEOREM

Let X be a non-empty abstract set and $\mathcal{S}$ be a commutative semi-group of operators on X into itself, containing the identity I and at least one other element T, $T \neq I$. We ask ourselves the question: what conditions should $\mathcal{S}$ satisfy in order that it be contractive or respectively uniformly contractive on X? Theorem 5 gives some necessary conditions for any contractive semi-group in terms of a length function σ on $\mathcal{S}$. The following result provides a set of sufficient conditions on $\mathcal{S}$.

Theorem 6. Let X and $\mathcal{S}$ be given as above. If $\mathcal{S}$ satisfies the conditions:

(a) that for each $T \in \mathcal{S}$, $T \neq I$, $Tx_1 = x_1$ and $Tx_2 = x_2$ imply $x_1 = x_2$;

(b) that there exists a length function σ defined on $\mathcal{S}$;

(c) that for any given invariant set $\mathcal{J} \subseteq \mathcal{S}$, there exists a finite set $\mathcal{B} \subseteq \mathcal{J}$ such that for each $T \in \mathcal{J}$, there corresponds a $U \in \mathcal{B}$ satisfying $T = US$ for some $S \in \mathcal{S}$ and $\sigma(T) = \sigma(U)\sigma(S)$;

then $\mathcal{S}$ is contractive.

(The conditions (a) and (b) have been proved to be necessary and are easily seen to be independent of each other.)

Lemma 1. Let X and $\mathcal{S}$ be given as in Theorem 6. Suppose that ρ_1, ρ_2 are two level functions defined with respect to the same contractive function λ and let X_1, X_2 be their respective domains of definition. If there exist positive constants c_1, c_2 such that $c_1 \rho_1(x) \leqslant c_2 \rho_2(x)$ for all $x \in X_1 \cap X_2$, then ρ_1 can be extended to $X_1 \cup X_2$.

Proof. Define the function ρ on $X_1 \cup X_2$ by

$$\rho(x) = \begin{cases} \rho_1(x) & \text{if } x \in X_1, \\ \dfrac{c_2}{c_1} \rho_2(x) & \text{if } x \in X_2,\ x \notin X_1. \end{cases}$$

Conditions (i), (ii) of the definition of a level function are obviously satisfied. Condition (iii) is also obvious if $x \in X_1$ or $Tx \notin X_1$. Suppose now that $x \in X_2$, $x \notin X_1$, and $Tx \in X_1$, then $\lambda(T)\rho(x) = \lambda(T) \frac{c_2}{c_1} \rho_2(x) \geqslant \frac{c_2}{c_1} \rho_2(Tx) \geqslant \rho_1(Tx) = \rho(Tx)$. Finally, to prove that $\rho(x_1) = \rho(x_2) = 0$ implies $x_1 = x_2$, we need to prove only that $\rho_1(x_1) = \rho_2(x_2) = 0$ implies $x_1 = x_2$. Choose $T \in \mathcal{S}$, $T \neq I$, then $\rho_1(Tx_1) \leqslant \lambda(T) \rho_1(x_1) = 0$ and hence $Tx_1 = x_1$. Similarly $Tx_2 = x_2$. By assumption (a), we conclude $x_1 = x_2$. $\square$

Lemma 2. Let X and $\mathcal{S}$ be given as in Theorem 6. Given $a \in X$ then there exists a level function on $[a]$ with respect to the length function σ.

Proof. Define for $x \in [a]$, $\rho(x) = \inf_{T \in \mathcal{A}_x} \sigma(T)$, where $\mathcal{A}_x = \{ T: Ta = x \}$. Since $x \in [a]$ implies that $\mathcal{A}_x$ is non-empty we have $0 \leqslant \rho(x) < \infty$. Next we note that $\rho(Sx) = \inf_{U \in \mathcal{A}_{Sx}} \sigma(U) \leqslant \inf_{T \in \mathcal{A}_x} \sigma(ST)$

$\leqslant \sigma(S) \inf_{T \in \mathcal{A}_x} \sigma(T) = \sigma(S)\rho(x)$. Finally, let $x \in [a]$, $\rho(x) = 0$. Consider the invariant set $\mathcal{J}_x = \mathcal{S}\mathcal{A}_x$ and denote by $\mathcal{B}_x$ the finite set corresponding to $\mathcal{J}_x$ according to assumption (c). Suppose $\mathcal{A}_x \neq \mathcal{B}_x$, then there exists a $T_1 \in \mathcal{A}_x$ such that $T_1 = U_1 S_1$ where $U_1 \in \mathcal{B}_x$, $S_1 \in \mathcal{S}$ and $S_1 \neq I$. Since $U_1 \in \mathcal{B}_x \subseteq \mathcal{J}_x$, there exists $T_2 \in \mathcal{A}_x$ such that $U_1 = T_2 S_2$ where $S_2 \in \mathcal{S}$. Since $\sigma(S_1 S_2) \leqslant \sigma(S_1)\sigma(S_2) \leqslant \sigma(S_1) < 1$, so $S_1 S_2 \neq I$. Now $x = T_1 a = U_1 S_1 a = T_2 S_1 S_2 a = S_1 S_2 x$. For any $T \in \mathcal{S}$, note that $TS_1 S_2 x = Tx = S_1 S_2 (Tx)$. Hence by assumption (a), $Tx = x$ for all $T \in \mathcal{S}$. The same conclusion can be reached in the case when $\mathcal{A}_x = \mathcal{B}_x$. Indeed, since $\mathcal{B}_x$ is finite then there exists a $S \in \mathcal{A}_x$ such that $\sigma(S) = 0$. For any $T \in \mathcal{S}$, $\sigma(ST) \leqslant \sigma(S)\sigma(T) = 0$. Thus $ST = S$. Therefore, $x = Sa = STa = Tx$. If now $\rho(x_1) = \rho(x_2) = 0$ then $Tx_1 = x_1$ and $Tx_2 = x_2$ for all $T \in \mathcal{S}$. In particular by choosing $T \neq I$, we conclude from assumption (a) that $x_1 = x_2$. $\square$

<u>Proof of Theorem 6.</u> Let X_1 be an invariant set in X and ρ_1 be a level function on X_1 defined with respect to the length function σ. Suppose $a \notin X_1$, we claim that ρ_1 can be extended to $X_1 \cup [a]$. Denote by ρ_2 the level function defined on $[a]$ according to Lemma 2. Consider the set $\mathcal{J} = \{T : T \in \mathcal{S}, Ta \in X_1 \cap [a]\}$. Clearly, $\mathcal{J}$ is an invariant set in $\mathcal{S}$. Let $\mathcal{B}$ be the finite set corresponding to $\mathcal{J}$ according to assumption (c). In addition let $\mathcal{B}' = \{U : U \in \mathcal{B}, \rho_1(Ua) \neq 0\}$ and $\mathcal{B}'' = \{U : U \in \mathcal{B}, \rho_2(Ua) \neq 0\}$. Define $c_1 = \operatorname{Min}_{U \in \mathcal{B}''} \rho_2(Ua)$ and $c_2 = \operatorname{Max}_{U \in \mathcal{B}'} \rho_1(Ua)$. Choose $x \in X_1 \cap [a]$ and

consider the sets $\mathcal{A}_x$, $\mathcal{J}_x$ and $\mathcal{B}_x$ as introduced in Lemma 2. We first note that if $\mathcal{B}' = \emptyset$ then for each $T \in \mathcal{A}_x$, $T = U\,S$ for some $U \in \mathcal{B}$, we have $\rho_1(x) = \rho_1(Ta) = \rho_1(U\,S\,a) \leqslant \sigma(S)\,\rho_1(U\,a) = 0$. Now $\rho_1(x) = 0$ implies that $Sx = x$ for all $S \in \mathcal{S}$, since $\rho_1(Sx) \leqslant \lambda(S)\rho_1(x)$ for all $S \neq I$. Pick any $S \neq I$, we conclude from $\rho_2(x) = \rho_2(Sx) \leqslant \lambda(S)\,\rho_2(x)$ that $\rho_2(x) = 0$. Hence in this case the inequality $c_1\,\rho_1(x) \leqslant c_2\,\rho_2(x)$ holds in a trivial way. Similar conclusion holds if $\mathcal{B}'' = \emptyset$. We may now assume that $\mathcal{B}' \neq \emptyset$ and $\mathcal{B}'' \neq \emptyset$. It is readily seen from above that $\rho_1(x) = 0$ if and only if $\rho_2(x) = 0$. So, we may also assume that $\rho_2(x) \neq 0$. Suppose $\mathcal{A}_x \neq \mathcal{B}_x$, then by repeating the same argument as in Lemma 2, we conclude $Tx = x$ for all $T \in \mathcal{S}$; in particular if $T \neq I$, then $\rho_2(x) = \rho_2(Tx) \leqslant \sigma(T)\,\rho_2(x)$ implies $\rho_2(x) = 0$. Again, the desired inequality holds trivially. Suppose now that $\mathcal{A}_x = \mathcal{B}_x$, then we may choose $T \in \mathcal{A}_x$, such that $\rho_2(x) = \sigma(T)$. Note that $\rho_2(x) \neq 0$ implies $\rho_2(Ua) \neq 0$, and from the definition of ρ_2, $\rho_2(a) = 1$. For otherwise there exists $V \neq I$ such that $Va = a$, and thus $Sa = a$ for all $S \in \mathcal{S}$. In particular $a = Ta = x \in X_1$, contradicting $a \notin X_1$. Thus,

$$\begin{aligned} c_1\,\rho_1(x) &= c_1\,\rho_1(USa) \leqslant c_1\,\sigma(S)\,\rho_1(Ua) \\ &\leqslant \rho_2(Ua)\sigma(S)\,\rho_1(Ua) \leqslant \sigma(S)\sigma(U)\,\rho_2(a)\,\rho_1(Ua) \\ &\leqslant c_2\sigma(T)\,\rho_2(a) = c_2\sigma(T) = c_2\,\rho_2(x). \end{aligned}$$

Apply Lemma 1 to extend ρ_1 over $X_1 \cup [a]$.

Let Φ be the family of all level functions defined with respect to the length function σ. Φ is non-empty for it contains the level function on the empty set $\emptyset$. Let X_ρ be the domain of definition corresponding to $\rho \in \Phi$. We say $\rho_1 \leqslant \rho_2$ if (i) $X_{\rho_1} \subseteq X_{\rho_2}$, (ii) $\rho_1 = \rho_2$ on X_{ρ_1}. Clearly $\leqslant$ defines a partial ordering on Φ. Suppose now that Ψ is a totally ordered subset of Φ. Define a level function $\hat{\rho}$ on $\bigcup_{\rho \in \Psi} X_\rho$ by $\hat{\rho}(x) = \rho(x)$ if $x \in X_\rho$ for some $\rho \in \Psi$. Since Ψ is totally ordered, this definition of $\hat{\rho}$ is unambiguous. $\hat{\rho}$ is clearly an upper bound for Ψ and thus Φ satisfies the hypothesis of Zorn's lemma. Therefore there must exists a maximal element $\rho_M \in \Phi$. We claim that $X = X_{\rho_M}$. For otherwise there exists $a \in X$, $a \notin X_{\rho_M}$ and we may extend ρ_M to $X_{\rho_M} \cup [a]$, contradicting the maximality of ρ_M. Knowing the existence of a level function on the full set X, we conclude by Theorem 2 that $\mathcal{S}$ is contractive. $\square$

We remark that if the length function σ on $\mathcal{S}$ is in addition uniformly contractive on $\mathcal{S}$, then Theorem 6 together with Theorem 3 imply that $\mathcal{S}$ is uniformly contractive on X.

4. SEMI-GROUPS GENERATED BY A FINITE NUMBER OF ELEMENTS

Let X be a non-empty abstract set and $T_1, T_2, \ldots, T_n$ be mutually commuting mappings defined on X into itself. Denote by $\mathcal{S}$ the commutative semi-group containing the identity which is generated by $T_1, T_2, \ldots, T_n$. Obviously, we may restrict ourselves to the case where all the T_i's are different from the identity. In this case, there exists a set of necessary and sufficient conditions for $\mathcal{S}$ to be uniformly contractive. In particular, assumptions (a) and (b) of Theorem 6 are both necessary and sufficient. In fact, we can prove that $\mathcal{S}$ is uniformly contractive under assumption (a) and only part of assumption (b).

<u>Theorem 7.</u> Let X and $\mathcal{S}$ be given as above. *If* $\mathcal{S}$ *satisfies* the conditions:

(a) that for each $T \in \mathcal{S}$, $T \neq I$, $Tx_1 = x_1$ and $Tx_2 = x_2$ imply $x_1 = x_2$;

(b) that for each pair $S, T \in \mathcal{S}$, $ST = I$ implies $S = T = I$;

then $\mathcal{S}$ is uniformly contractive.

<u>Corollary.</u> Let X and $\mathcal{S}$ be given as in Theorem 7. Suppose there exists an element $\omega \in X$, such that for some $S \in \mathcal{S}$, $S \neq I$, $S\omega = \omega$. Then $\mathcal{S}$ *is uniformly completely contractive.*

We will prove this result by applying Theorem 5. For the sake of convenience, we first introduce some useful notations. Let Q denote the set of n-tuples ($k_1, k_2, \ldots, k_n$) where the k_i's are non-negative integers. Define $\varphi(k) = \sum_{i=1}^{i=n} k_i$ for all $k \in Q$. Note that $\varphi(k)$ is finite for all $k \in Q$. We also define a partial ordering on Q by : $p \leqslant q$ if and only if $p_i \leqslant q_i$ for all i, and p = q if and only if $p_i = q_i$, for all i, $1 \leqslant i \leqslant n$. Obviously, Q forms a semi-group with respect to vector addition, and the mapping $k \longrightarrow T^k = T_1^{k_1} \ldots\ldots T_n^{k_n}$ defines a homomorphism of Q into $\mathcal{S}$.

We next prove two algebraic lemmas that are important in the proof of Theorem 7.

Lemma 3. Every transverse set $M \subseteq Q$ is finite.

Proof. The proof is by induction on the dimension n. The statement obviously is true for n = 1. Let M be a transverse set in Q and let $p = (p_1, p_2, \ldots, p_n)$ be a fixed chosen element of M. Note that M may be written as $\bigcup_{i=1}^{n} \bigcup_{r=0}^{p_i} M_{ir}$, where $M_{ir} = \{ q: q \in M, q_i = r \}$. Let $Q' = \{ p: p = (p_1, p_2, \ldots, p_{n-1}) \}$ be the set of all (n-1)-tuples of non-negative integers endowed with the same partial ordering as that of Q. Denote by Q_{ir} the set $\{ q: q \in Q, q_i = r \}$. For each i and each r, the mapping $f_{ir}(q_1, q_2, \ldots, q_{n-1}) = (q_1, q_2, \ldots, q_{i-1}, r, q_i, \ldots \ldots, q_{n-1})$ defines an isomorphism with respect to ordering between Q' and Q_{ir}. Note that $M_{ir} = M \cap Q_{ir}$ which is clearly a transverse set. Hence its image $f_{ir}^{-1}(M_{ir})$ in Q' is again transverse, and by the induction hypothesis is finite. Since each M_{ir} is finite, therefore M is finite. $\square$

<u>Lemma 4.</u> Let P be an arbitrary partially ordered set which satisfies the descending chain condition. Then every ideal $J \subseteq P$ may be written as a set union of a family of principle ideals whose generators form a transverse set.

<u>Proof.</u> For any ideal $J \subseteq P$, we consider the transverse set R $= \{ x: x \in J, \text{ and if } y \in J \text{ and } y \leqslant x \text{ then } x = y \}$. We claim that $J = \bigcup_{x \in R} \langle x \rangle$. Let $x \in J$ and $x \notin R$. We may pick an element $x_1 \in J$ such that $x > x_1$. If $x_1 \in R$, then $x \in \langle x_1 \rangle$. Otherwise we may continue to pick another element $x_2 \in J$ such that $x_1 > x_2$. Let $\{ x_i : i = 1, 2, 3, \ldots \}$ be the set of elements in J constructed inductively as above. By the descending chain condition, it has a minimal element, say y. Suppose now $z \in J$ and $z \leqslant y$. Since y is minimal, we must have $y = z$. Hence $y \in R$, and $x \in \langle y \rangle$. $\Box$

We remark that the set R is uniquely determined by J. To see this we assume that the ideal J may be written as :

$$J = \bigcup_{x \in R} \langle x \rangle = \bigcup_{x' \in R'} \langle x' \rangle$$

where R and R' are both transverse in X. Let $x \in R$ then there exists a $x' \in R'$ such that $x \in \langle x' \rangle$, i.e. $x \geqslant x'$. On the other hand, for $x' \in R'$, there exists a $x'' \in R$, such that $x' \in \langle x'' \rangle$, i.e. $x' \geqslant x''$. Hence we have $x \geqslant x' \geqslant x''$. Since $x, x'' \in R$, and R is transverse, therefore $x' = x \in R'$. Thus $R \subseteq R'$. Similarly, we obtain $R' \subseteq R$.

<u>Proof of Theorem 7.</u> Assumption (a) of Theorem 6 is satisfied by

hypothesis. For each $S \in \mathcal{S}$, let $K(S) = \{p : p \in Q,\ T^p = S\}$. Choose any $\lambda \in (0,1)$, and define $\sigma(S) = \inf_{k \in K(S)} \lambda^{\varphi(k)}$. Clearly $0 \leq \sigma(S) \leq \lambda < 1$ for all $S \neq I$, and $\sigma(I) = 1$. Moreover $\sigma(ST) = \inf_{k \in K(ST)} \lambda^{\varphi(k)} \leq$

$\inf_{p \in K(S)} \inf_{q \in K(T)} \lambda^{\varphi(p+q)} = \inf_{p \in K(S)} \lambda^{\varphi(p)} \inf_{q \in K(T)} \lambda^{\varphi(q)} = \sigma(S)\sigma(T)$. Finally, to show that $\sigma(S_1) = \sigma(S_2) = 0$ implies $S_1 = S_2$, we first note that if for any $S \in \mathcal{S}$, $K(S)$ is transverse, then $\sigma(S) = \operatorname{Min}_{k \in K(S)} \lambda^{\varphi(k)} > 0$, (since $K(S)$ is finite by Lemma 3). On the other hand, if $K(S)$ is not transverse, then there exist $p, q \in K(S)$, $p > q$. Now $T^{p-q}(Sx) = Sx$ for all $x \in X$, and $T^{p-q} \neq I$. Again by assumption (a), we have $Sx = \theta$ for all $x \in X$. Hence $\sigma(S) = 0$ implies that $Sx = \theta$ for all $x \in X$. Now suppose $\sigma(S_1) = \sigma(S_2) = 0$, then there exist θ_1, θ_2 such that $S_1 x = \theta_1$ and $S_2 x = \theta_2$ for all $x \in X$. Since $\theta_1 = S_1 S_2 \theta_1 = S_2 S_1 \theta_1 = \theta_2$, so $S_1 = S_2$.

We next show that assumption (c) of Theorem 6 is also satisfied. Let $\mathcal{J} \subseteq \mathcal{S}$ be an invariant set. Consider the set $J = \{p : p \in Q,\ T^p \in \mathcal{J}\}$ which is clearly an ideal in Q. By Lemma 4, there exists a transverse set B such that $J = \bigcup_{p \in B} \langle p \rangle$. Let $\mathcal{B} = \{ T^p :$ for some $p \in B \}$. Obviously $\mathcal{B} \subseteq \mathcal{J}$ and $\mathcal{B}$ is finite. For $T \in \mathcal{J}$, $\sigma(T) = 0$, choose any $U \in \mathcal{B}$, and observe $UT = T$ and $\sigma(T) = \sigma(T)\sigma(U)$. On the other hand, if $\sigma(T) \neq 0$, we may pick $p \in K(T)$ such that $\sigma(T) = \lambda^{\varphi(p)}$. Let $r \in B$ such that $p \geq r$. Thus, $T^r \in \mathcal{B}$. Note that $\sigma(T) = \lambda^{\varphi(p)} = \lambda^{\varphi(r) + \varphi(p-r)} \geq \sigma(U)\sigma(S)$, where $S = T^{p-r} \in \mathcal{S}$. We hence conclude $\sigma(T) = \sigma(U)\sigma(S)$ since the reverse inequality always holds. Now the set $\mathcal{B} \subseteq \mathcal{G}$ satisfies the condition required by assumption (c) of Theorem 6. Apply Theorem 6 and Theorem 3, we conclude that $\mathcal{S}$ is uniformly contractive. The corollary follows immediately from Theorem 4. $\square$

5. REMARKS

We first remark that Theorem 7 cannot be extended to the corresponding case where $\mathcal{S}$ is generated by a countably infinite number of mappings. To see this, we consider the following example : Let $X = [0, \infty)$ and $T_i x = x + \frac{1}{i}$, $i = 1, 2, 3, \ldots$. Clearly X and the commutative semi-group $\mathcal{S}$ generated by all the T_i's satisfy the hypothesis of Theorem 7. But $\mathcal{S}$ is not uniformly contractive. Assume the contrary, then by Theorem 3 there exists a level function ρ on X such that $\rho(Tx) \leqslant \lambda \rho(x)$ for all $x \in X$ and all $T \in \mathcal{S}$, $T \neq I$, where $0 \leqslant \lambda \leqslant 1$. Since $\infty \notin X$, therefore $\rho(x) \neq 0$ for all $x \in X$. For any m we may write $\rho(T_2 x) = \rho(T_{2m}^m x) \leqslant \lambda^m \rho(x)$. Letting m tend to infinity, we obtain a desired contradiction. Nevertheless in this case $\mathcal{S}$ is contractive on X. Indeed, $\rho(x) = \lambda^x$, for any $\lambda \in (0, 1)$, is a level function on X. (Note that in this case the contractive function is clearly not uniform.)

Let X be a metrilizable space and T be a mapping of X into itself such that for each positive integer k, the equation $T^k x = x$ holds for some $x \in X$ implies $x = \omega$, the unique fixed point of T. We now ask ourselves the question : does there exist a metric in which T is a contraction and which at the same time reproduces the original topology ? The answer is negative even in case X is compact. Suppose X be any compact metrilizable space, and T be a mapping of X into itself which possesses a unique inverse. In this case, we claim that there does not exist a metric on X which satisfies the above-mentioned requirements unless X is only a singleton set. Assume the contrary, i.e. there exists a metric ρ on X such that $\rho(Tx, Ty) \leqslant \lambda \rho(x,y)$ for all $x, y \in X$ and ρ induces a topology same as that given on X. Since X is compact in the original topology, so it also is compact in the metric topology induced by ρ. Denote by D the diameter of X with respect to ρ, i.e. $D = \sup_{x,y \in X} \rho(x,y)$. Choose $x, y \in X$, such that $x \neq y$. Note that $\rho(x,y) \leqslant \lambda^n \rho(T^{-n}x, T^{-n}y) \leqslant \lambda^n D$. Letting n tends to infinity, we arrive at the desired contradiction.

As a postscript, we list here several questions for further investigation.

(i) What is a set of necessary and sufficient conditions for $\mathcal{S}$ to be contractive ? (This is not known even in case $\mathcal{S}$ is generated by a countably infinite number of mappings.)

(ii) Assumptions (a) and (b) of Theorem 6 are shown to be necessary for $\mathcal{S}$ to be contractive, ---are they also sufficient ?

(iii) Let X be any compact metrilizable space and T a mapping of X into itself satisfying the condition imposed in the previous paragraph. What additional conditions are sufficient to insure the existence of a metric which will reproduce the original topology and at the same time make the mapping T a contraction? (This is not known even in case $X = [0, 1]$.)

REFERENCES

1. S. Banach," Sur les operations dans les ensembles abstraits et leur application aux equations integrals ", Fundamenta Mathematica 3 (1922), 133-181.

2. A. N. Kolmogorov and S. V. Fomin, Elements of the theory of functions and functional analysis . Vol 1 (Metric and Normed Spaces) Graylock Press (1957), New York.

3. W. A. J. Luxemburg, " On the convergence of successive approximations in the theory of ordinary differential equations II ", Indag. Math. 20 (1958), 540-546.

4. D. W. Willett, " Non-linear vector integral equations as contraction mappings ", Arch. for Rat. Mech. and Anal., 15 (1964), 79-86.

5. H. Samuelson," On the Perron-Frobenius theorem ", Michigan Math. J. 4 (1957), 57-59.

6. A. C. Thomp son, " On certain contraction mappings in a partially ordered vector space.", Proc. Amer. Math. Soc. 14 (1963), 438-443.

7. J. Schröder," A unified theory for estimation and iteration in metric spaces and partially ordered spaces ", MRC No.237, Univ. of Wiscousin, (1961).

8. M. Edelstein," An extension of Banach's contraction principle ", Proc. Amer. Math. Soc. 12(1961), 7-10.

9. E. Rakotch," Note on ϵ-contractive mappings ", Bulletin of Research Council of Israel, 10B (1961), 53-58.

10. W. J. Kammerer and R. H. Kasriel," On contractive mappings in uniform spaces ", Proc. Amer. Math. Soc. 15(1964), 288-290.

11. R. E. Edwards," A fixed point theorem with application to convolution equations ", J. of Australian Math. Soc. III (1963), 385-395.

12. A. S. Davis," Fixpoint theorem for contractions of a well chained topological space ", Proc. Amer. Math. Soc. 14 (1963), 981-985

13. C. Bessaga," On the converse of the Banach Fixed-point principle ", Coll. Math. VII (1959), 41-43.

14. L. Janos," Converse to contraction mapping principle ", Amer. Math. Soc. Notices 11 (1964), 224.

15. J. S. W. Wong," A generalization of the converse of contraction mapping principle ", Amer. Math. Soc. Notices 11 (1964), 385.

16. G. Birkoff, Lattice Theory, Amer. Math. Soc. Coll. Publications, XXV (1948).