Contracts and Markets

Thesis by

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To L, K, Z & T

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Abstract

I merge the standard Principal Agent model with a CAPM-type financial market, to study the interactions of contracts and financial markets. I prove existence of equilibrium in two models, a more general economy allowing for hidden type and action under generic mean variance preferences and a hidden action economy with Markowitz mean-variance preferences. I study economies for which markets have an insurance effect on compensation contracts. I show sufficient conditions for lower variance to obtain in large economies, even with asymmetric information. In this context I show the effect of markets' size on efficiency. I also study moral hazard economies, for which I prove existence of a unique pure strategy equilibrium, and I show that financial markets negatively affect the equilibrium returns of firms. In the final chapter I study the efficiency of securities issued under symmetric information. I find that small markets and low correlation of firms' returns generate inefficiency. I also show that the assumption of symmetry or independence is crucial to obtaining the insurance results in the previous Chapters.

Chapter 1

Contracts and Aftermarkets - Hidden Type

1.1 Introduction

Labor compensation is arguably the most relevant expense for a corporation. In the US, for example, more than 60% of the payment to factors group in the 2008 GDP was in fact to labor. Since the stock of a company is a claim to its profits, the firm's decisions on workers' compensation affect the returns of its stock. In the aggreagate this affects financial markets. By the same token, diversification opportunities offered by markets should influence the design of compensation packages. In view of this consideration it is rather surprising that the economic and financial literature has devoted relatively little attention to these interactions.

Here I address two questions:

- How does the existence of asset markets affect the design of compensation packages?
- How does asymmetric information inside firms affect aggregate risk?

The answer to the former question depends on preferences of individuals and on the distribution of returns of companies. The market provides provides insurance and diversification opportunties to risk averse principals, allowing them to reach a higher utility and eventually changing equilibrium compensation. I show that in large markets with symmetric information and independent returns, the compensation packages offered are *always* less risky. A weaker version of this result applies to the asymmetric information case. The answer to the latter question is that, under the same assumptions as above, the constraints created by asymmetric information will induce securities that are at least as risky as those issued under complete information. Since this will turn out to be the case for every firm in the economy, it will be the case that this type of asymmetric information implies excessive risk also at the aggregate level. I reach these conclusions constructing a model of firms in financial markets. Each firm is formed by an owner and a worker. The skill of workers are initially unobservable. This is the only source of information asymmetry in the model. I address the effect of contracting inside firms on financial markets, and the effect of markets on firms' efficiency.

I do not study the effects of asymmetric information *in* markets, but rather the effects of asymmetric information inside firms *on* markets. This marks the first difference from the General Equilibrium works on insurance markets, starting from the seminal paper of Rothschild and Stiglitz . Another important difference is that, in those papers, the fact that some individuals are risk neutral and act as firms is usually an assumption with the notable exception of Dubey and Geneakoplos, where individuals endogenously form pools to share risk. In this work there are many risk averse investors, who access financial markets to trade away part of the risk they are exposed to. Traditionally, the assumption of risk neutrality of a principal is motivated by the existence of diversification opportunities. The present work also enquires when the usual motivation, the opportunity to trade risks on a financial market, actually provides a justification for the risk neutrality assumption and its implications.

A strand of the finance literature looks at asset pricing in the presence of delegated portfolio management (for a survey, see Stracca, 2003). An example of the approach typical in these papers can be seen in Ou Yang's paper. These studies look at the effects on prices and returns of the classical informational asymmetries phenomena. Moral hazard and adverse selection are largely studied in a CAPM or APT setting, in which a representative principal delegates his investing decisions to an agent. In this literature inefficiencies take the form of deviations from the non-delegated case equilibrium. These deviations can take the form of changes in asset prices and optimal portfolio composition. Besides the different object of interest, the perspective in these works is in a sense opposite of the one taken here. There we have informed parties trading, whereas in the present work it is the uninformed parties accessing markets.

A branch of the general equilibrium literature focuses on the organization of firms and the employment choices of infividuals. Rahman (2005a, 2005b), Ellickson, Grodahl, Scotchmer and Zame (1999, 2001, 2005) and Zame (2008) separately study economies where agents form team or clubs, which can have different functions such as production or consumption. While none of these models includes financial markets, some of them allow for asymmetric information. Their approach is very general, firms and contracts are both endogenously determined. However, this generality makes it hard to derive any predictions on the shape of contracts.

Finally, the works technically closest to mine are those by Magill and Quinzii (2005) and Parlour and Walden (2009) who use models which bear some similarities to this one. As in the present chapter, they take firm formation as an exogenous process, abstracting from labor market considerations, and they allow for contracts inside firms and financial markets across firms. However they use their model to study economies with hidden action. I address the problem of moral hazard in a separate chapter.

The paper is constructed as follows. In Section 2 I introduce the problem and an example. In Section 3 I present the model. In Section 4 I define the notion of equilibrium. In Section 5 I prove existence of equilibrium. In Section 6 I provide sufficient conditions for compensation to be closer to a wage when principals access markets.

1.2 A Simple Example

In this example, I show how markets can affect contracts in a very simple setting. Markets provide diversification for principals. This diversification opportunity makes Principals insure agents more than in a standard P-A model.

There are four individuals, with identical preferences over random variables, $U(X)F\left(\mu_X, \sigma_X^2\right) = \mu_X - \frac{b}{2}\left(\mu_X^2 + \sigma_X^2\right)$ with $b = \frac{1}{4}$. Two of them, the Principals, own an identical technology. Two of them, the Agents, have the skills to operate the technology. Their skills are private information at the contracting stage. Both agents have the same reservation utility of $\bar{u} = 1$.

The skills of agents are identified as average returns. The performance of one agent is stochastically independent from the performance of the other. The mean returns for an agent is given by $\mu_l = 2, \mu_h = 3$ and the variance is $\sigma_l^2 = \sigma_h^2 = 1$.

Each principal designs a menu of two linear contracts to offer to the agent he is going to be randomly matched with. A linear contract is a function of the form $y = \alpha + \beta x$, and is characterized by a pair (α, β) Principals will receive $y_p = -\alpha + (1 - \beta)x$, the agent will receive $y_a = \alpha + \beta x$.

1.2.1 Standard Principal-Agent Model: No Market

The problem of principals is going to be:

$$\max_{\alpha_H,\beta_H,\alpha_L,\beta_L} \frac{1}{2}F\left(\alpha_H+\beta_H\mu_H,\beta_H^2\sigma_H^2\right)+\frac{1}{2}F\left(\alpha_L+\beta_L\mu_L,\beta_L^2\sigma_L^2\right)$$
$$\text{subject to} \quad IR_H, IC_H, IR_L, IC_L$$

The solution to a standard P-A model with linear contracts is to offer a menu of these two contracts:

$$y_L = 1.17$$
$$y_H = -.65+(.64)\,x$$

1.2.2 Principal-Agent with Financial Markets

Now suppose that the Principals can trade their claims on the asset market. The objective function of principals is now different, because it includes the outcome of markets. With mean-variance preferences the asset market equilibrium is determined by a few simple equations, which can be substituted in the objective function.

The outcome of the CAPM market:

- The market portfolio will be characterized by mean and variance
  - $\mu_{MKT}(\alpha_H,\beta_H,\alpha_L,\beta_L)=\alpha_L+\alpha_H+\beta_L t_L+\beta_H t_H$
  - $\sigma^2_{MKT}(\alpha_H,\beta_H,\alpha_L,\beta_L)=\beta_H^2 t_H(1-t_H)+\beta_L^2 t_L(1-t_L)$
- Equilibrium shares will be
  - Market Portfolio:

    $\theta_H=\theta_L=\frac{1}{2}$
  - Riskless asset:

    $\theta_i^P=(-\alpha_i+(1-\beta_i)t_i-\frac{a}{2}(1-\beta_H)^2\sigma_H^2-$

$$\frac{-\alpha_H+(1-\beta_H)t_H-\frac{a}{2}(1-\beta_H)^2\sigma_H^2-\alpha_L+(1-\beta_L)t_L-\frac{a}{2}(1-\beta_L)^2\sigma_L^2}{2}$$

The Principals' problem when Markets are available will be:

$$\max_{\alpha_H,\beta_H,\alpha_L,\beta_L} \frac{1}{2}F\left[\left(q_H-\frac{q_H+q_L}{2}\right)+\frac{1}{2}\mu_{MKT}\left(\alpha_H,\beta_H,\overline{\alpha}_L,\overline{\beta}_L\right),\frac{1}{4}\sigma^2_{MKT}\left(\alpha_H,\beta_H,\overline{\alpha}_L,\overline{\beta}_L\right)\right]$$
$$\frac{1}{2}F\left[\left(q_L-\frac{q_H+q_L}{2}\right)+\frac{1}{2}\mu_{MKT}\left(\alpha_L,\beta_L,\overline{\alpha}_H,\overline{\beta}_H\right),\frac{1}{4}\sigma^2_{MKT}\left(\alpha_L,\beta_L,\overline{\alpha}_H,\overline{\beta}_H\right)\right]$$

subject to

$$IR_H, IC_H, IR_L, IC_L$$

The optimal contracts in this setting will be

$$y_L^{MKT} = 1.17$$
$$y_H^{MKT} = -.43 + (.54)\,x$$

1.2.3 Insurance Effect

An important feature exhibited by this example is that optimal contracts are different when asset markets are present: they are more similar to wages. In this example the returns are independent, so that the market offers diversification opportunitie. In this case these are sufficient for principals to offer safer contracts to agents and achieve a higher expected utility.

However, this is not always going to be the case. Consider an economy identical to the above, but in which agents reservation utility is equal do $\frac{1}{2}$. These are the resulting contracts.

- The equilibrium contracts without markets:

$$y_h = -1.42 + (.7x)$$
$$y_l = .53$$

- The equilibrium contracts with markets:

$$y_h = -1.4 + (.65)x$$
$$y_l = .42 + (\mathbf{.06})x$$

In this case type L gets a riskier contract when markets are present. Why is this the case? In the rest of the paper I will give sufficient conditions for the insurance effect to obtain.

1.3 The Model

1.3.1 Primitives

There are $2N$ individuals. All individuals have identical preferences. They all have quadratic utility on outcomes, in the form $u(x) = x - \frac{b}{2}x^2$. Hence their preferences over random variables can be represented by the function

$$U(X) = E[u(X)]$$
$$E(X) - \frac{b}{2}E\left(X^2\right)$$
$$E(X) - \frac{b}{2}E(X)^2 - \frac{b}{2}Var(X)$$

N individuals are Principals, and N are Agents . There are N firms. Firm n is formed by Principal P_n and Agent A_n. Principal n owns a productive technology X_n, Agent n is a skilled worker, whose skill t_n is drawn from from a distribution F with finite support $\{1, ..., T\}$.

The profits of firm n are described by a random variable $X_{n,t}$. In other words profits depend on technology and skill.

1.3.2 Timeline

Initially firms are identical in terms of internal organization. The Principal owns the firm. Her technology and the Agent's work are necessary for creating profits. Their position differs because while the technology is public, the Agent's skill is his private information.

1. Each principal n will design a mechanism $\mathcal{M}_n = (\mathcal{A}_n, \mathcal{C}_n)$, in which the agent can chooose an action a_n out of set $\mathcal{A}_n$. Depending on the agent's actions in the mechanism, he will be compensated with a contract $C \in \mathcal{C}_n$. A contract comprises of shares of the company $(1 - \beta_n)$ and a transfer $-\alpha_n$. A contract C_n is hence represented by a pair $(\alpha_n, \beta_n) \in \mathbb{R} \times [0, 1]$ and it will be a function of the action taken by the agent $C_n (a_n)$. If he does not work he will get a reservation utility of $\overline{u} \geq 0$ and there will be no profits. From now on, I will slightly abuse notation and indentify an agent's choice of action in a mechanism, with the resulting contract.

2. Once the shares/wage decisions have been taken, production starts, and agents' skills will become public.

3. Finally principals can trade their claims to profits $\alpha_n + \beta_n X_{n,t_n}$ on a market, in which a riskless asset L is also available in zero net supply.

4. After trading takes place, all uncertainty is realized. Compensation and securities pay off.

When	**Who**	**What**	**Knowing What**
0	P_n	M_n	CMN
1	A_n	$C_n \in \mathcal{A}_n$	CMN, $t(n)$, M_n
2	P_n	θ_n	CMN, $\mathbf{t}$

Table 1.1: Timing

1.3.3 Strategies and Payoffs

Let $\mathcal{T}$ be the set of possible realizations of workers' skill values and let $\mathbf{t} = (t_1, ...t_N)$ be its generic element, distributed according to $\mathbf{F} = (F_1, ..., F_N)$

Let $\mathbf{C}$ be the vector of contracts across the economy.

$$\mathbf{C} = \{C_n\}_{n=1}^{N} = \{\alpha_n, \beta_n\}_{n=1}^{N}$$

Let $\theta = (\theta^R|\theta^L)$ be the portfolio held by an agent. θ^R is a N-dimensional vector of positive holdings of the N risky assets, whereas θ^L is the position an investor holds in the riskless asset.

The ex-ante utility from a portfolio θ fixing the matching $\mathbf{t}$ and the contracts C_N, is given by

$$U^3_{p_n}(\theta, \mathbf{C}, \mathbf{t}) = U\left(\theta \cdot (\vec{\mu}(\mathbf{t}, \mathbf{C})|1), \theta^R \mathbf{\Omega}(\mathbf{t}, \mathbf{C})\theta^R\right)$$

Demand θ will depend on available assets (and hence con contracts C and their prices $\mathbf{q}$ (but prices are also a function of contracts).

Agents payoffs depend on the action chosen and on the technology of their principal and on their type t:

$$U^2_{A_n} = U(C_n, t_n)$$

A Principal's *expected* utility when mechanisms are $\mathbf{M}$, the resulting contracts are $\mathbf{C}(\mathbf{M})$ and he holds portfolio θ is

$$U^1_{P_n}(\mathbf{M}, \mathbf{C}(\cdot), \theta) = \int_{\mathcal{T}} U^3_{P_n}(\theta, \mathbf{C}(\mathbf{M}), \mathbf{t})\, \mathbf{F}(\mathbf{t})$$

1.4 Equilibrium

1.4.1 Description

Because individuals take their decisions at each stage looking at the final payoffs, Equilibrium is more easily described starting from the final stage of the game.

Asset Market At this stage the realized skills are observed. There is no asymmetric information. Principals hold one unit of a security equal to their share of returns in their firm, and a riskless asset is available in net zero supply. The solution concept used here is that of Arrow-Debreu Equilibrium. Since securities payoffs are determined by contracts $\mathbf{C}$ and by agents' skills $\mathbf{t}$, the equilibrium portfolios and prices will be function $(\theta, q)(\mathbf{C}, \mathbf{t})$,

Contracting, Agents' turn Each agent A_n observes the mechanism he is offered, M_n, and he knows his own type and the technology of the principal. This is all the payoff relevant information, so every agent is facing a choice between lotteries, and he is not interacting with any other individual, except of course P_n. They pick an action maximizing $U^2_{A_n}(\cdot)$. Their strategies will be functions $a_n(M_n, t_n)$.

Contracting, Principals' turn Each principal offers designs a mechanism, without knowing the skills of any agent. However they correctly forecast the strategies of each agent, and the outcome of asset markets, for any possible mechanism. In other words, they can forecast the equilibrium path for all profiles of mechanisms $\mathbf{M}$ and types $\mathbf{t}$. Principals at this stage play a game against each other. A mixed strategy for P_n is a lottery on possible mechanisms, $\tilde{M}_n \in \Delta(\mathcal{M}_n)$.

The flow of decisions is described schematically below, and the information available at each stage is summarized by the argument of the strategies.

$$\begin{matrix} \tilde{\mathbf{M}} & & C & \\ & \rightarrow \tilde{\mathbf{C}}(\mathbf{M}, \mathbf{t})) \rightarrow & & \rightarrow [(\theta, q)](\mathbf{C}, \mathbf{t}) \\ \mathbf{t} & & \mathbf{t} & \end{matrix}$$

Based on this timeline we can write the utility in the first stage in this form:

$$V_n\left(\tilde{M}_n|\tilde{\mathbf{M}}_n\right) = E_{\tilde{\mathbf{M}}_n}\left[U^1_{P_n}\left(\tilde{M}_n|\tilde{\mathbf{M}}_{-n}, \mathbf{C}\left(\tilde{M}_n|\tilde{\mathbf{M}}_{-n}, \tilde{\mathbf{t}}\right), \theta\left(\mathbf{C}\left(\tilde{M}_n|\tilde{\mathbf{M}}_{-n}, \tilde{\mathbf{t}}\right)\right)\right)\right]$$

Note that $\tilde{\mathbf{t}}$ is a random vector and $U^1_{P_n}$ defined above includes an expectation with respect to its distribution $\mathbf{F}$.

1.4.2 Definition

An Equilibrium consists of

- A trading strategy θ^*_n for each Principal n and prices $q^* \in \mathbb{R}^{N+1}$ such that $[\theta^*, q^*](\mathbf{C}, \mathbf{t})$ is an Arrow-Debreu Equilibrium for the symmetric information asset market taking place after contracting. Each principal is endowed with one unit of one asset so that the endowment of principal n is $w_n = [0, 0, ..., 1, ..., 0, 0]$ with 1 being in the nth position.

$$\theta_n^*(\mathbf{C}^*, \mathbf{t}) \in arg \max_{\theta \in \mathbb{R}_+^{N+1}} U_{P_n}^3(\theta, \mathbf{C}^*, \mathbf{t})$$

$$s.t.$$

$$\mathbf{q}^*(\mathbf{C}^*, \mathbf{t}) \cdot \theta(\mathbf{C}^*, \mathbf{t}) \leq q_n^*(\mathbf{C}^*, \mathbf{t})$$

$$\sum_{n \in N} \theta_n^* = [\mathbf{1}_N | 0]$$

- For each agent A_n a strategy $C_n^*(M_n^*, t(n))$ such that

$$C_n^*(M_n^*, t(n)) \in arg \max_{C \in \mathcal{C}_n} U_{A_n}^2(C, t_n)$$

- For each principal P_n,a lottery $\tilde{M}_n^*$ of deterministic mechanisms such that

$$\text{supp}\left[\tilde{M}_n^*\right] \subseteq arg \max_{M_n \in \mathcal{M}} V_n^*(M_n|\mathbf{M}_{-n}^*)$$

1.4.3 Result

To prove existence of equilibrium I need the following result, the mechanism design classic, “customized” for this setting in which principals compete with each other..

Let an instance of the previously described economy be G, and the set of its equilibria $E(G)$. Consider now an economy G^Twhich is identical in all respects, but where principals are restricted to offer to agents menus of contracts of size $|T|$, instead of designing general game forms. I am going to show that $\mathcal{E}(G) = \mathcal{E}\left(G^T\right)$. In this way the strategy space of the Principals at the first stage will be finite dimensional.[1] The space of P_n's strategies is going to be $\mathcal{M}_n^{|T|}$.

Proposition 1. *The unrestricted menus economy G and the restricted menu economy G^T, have the same equilibria: $\mathcal{E}(G) = \mathcal{E}\left(G^T\right)$*

Proof. $\mathcal{E}(G) \subseteq \mathcal{E}(G^T)$ Consider an equilibrium $\epsilon = \left(\tilde{\mathbf{M}}, \mathbf{C}, \theta, q\right)$ now construct a strategy profile ϵ^T and show it is an equilibrium for G^T: Let a menu M_n^T of $|T|$ contracts offered in $\epsilon\left(G^T\right)$ be made

[1] In this model, T is finite.

of contracts $\{(y_n)\}_{t\in T}$:

$$y\left(t\right)=C_n\left(M_n,t\right)$$

For agents, it is an optimal action to choose the contract $y\left(t\right)$. Suppose not. Than it would be the case that for some t'

$$U\left(y\left(t'\right),t\right)>U\left(y\left(t\right),t\right)=U\left(C_n\left(M_n,t\right),t\right)$$

By construction $C_n\left(M_n,t'\right)'$ played in M_p yields the same payoff as $y\left(t'\right)$, which contradicts C_n being an equilibrium strategy.

The above described menus are optimal for every principal. Suppose not, then for some principal p there is a $|T|$ sized menu $\tilde{M}'$ such that

$$V_n\left(M'|\tilde{\mathbf{M}}^T_{-n}\right)>V_n\left(\tilde{\mathbf{M}}^T\right)=V_n^*\left(\tilde{\mathbf{M}}\right)$$

The equality follows by construction, and since n could have offered $\tilde{M}'$ in the unrestricted economy, this would contradict ϵ being an equilibrium.

$\mathcal{E}(G^T)\subseteq\mathcal{E}(G)$

Consider an equilibrium e^T: the agents equilibrium strategies $\mathbf{C}^T$ are simply the equilibrium strategies of the real economy, restricted to the domain of $|T|$ sized menus. Moreover, $\tilde{\mathbf{M}}^T$, are equilibrium strategies also in the unrestricted game. Suppose it was not the case, then for some n there is an unrestricted mechanisms lottery $\tilde{M}$ such that

$$V_n^*(\tilde{M}|\tilde{\mathbf{M}}^T_{-n})>V_n^*(\tilde{\mathbf{M}}^T)$$

Note that $V_n(\tilde{M}|\tilde{\mathbf{M}}^T_{-n})$ can be attained by n by offering a lotteries of restricted menus $\tilde{M}'^T$ made of the following contracts y':

$$y_n'(t)=C_n\left(M_n,t\right)$$

This implies

$$V_n^*(\tilde{M}'|\tilde{\mathbf{M}}^{\mathbf{T}}_{-\mathbf{n}})>V_n^*(\tilde{\mathbf{M}}^T)$$

which contradicts $\tilde{\mathbf{M}}^T$ being an equilibrium for G^T □

The utility function of a Principal is continuous in the contracts chosen by each type of Agent so it is continuous in $\mathcal{C}^T$. To make it continuous in the contracts offered by Principals' it has to be that these contracts form Incentive Compatible menus. In other words, menus must come from as subset of $\mathcal{C}^T$ such that C^1 is always chosen by agent 1, C^2 is always chosen by agent 2, and so on. To achieve this the menu in a contracts must satisfy incentive compatibility constraints.

We can be sure that we can restrict attention to these menus because all and only the equilibria of the original unrestricted game are obtained in a game where Principals are allowed to offer only an incentive compatible menu of T contracts. I will call this economy, G^{IC}.

Corollary 2. *The restricted menus economy G and the IC economy G^T, have the same equilibria $\mathcal{E}(G^T) = \mathcal{E}(G^{IC})$*

Proof. $\mathcal{E}(G^T) \subseteq \mathcal{E}(G^{IC})$ The contracts in the direct revelation menus in the proof of Theorem 1 are Incentive Compatible by construction.

$\mathcal{E}(G^{IC}) \subseteq \mathcal{E}(G^T)$ This part of the proof goes as the second part of the proof in Theorem 1: any outcome that a principal can achieve by menus of size T can be achieved by incentive compatibles menu of size T. □

1.5 Existence of Equilibrium

1.5.1 Assumptions

1.5.1.1 Monotonic Preferences

It is well known that Mean-Variance Preferences are not monotonic. This of course can cause problems for the existence of an equilibrium. In a standard CAPM setting, monotonicity of preferences is solved by imposing a bound on the variance aversion of every individual. I will require agents to not be satiated if they owned every return in the economy, regardless of state. I will then show that this implies they will not be satiated in the asset market of the later stage. It is possible to show that if preferences are monotonic for given returns (μ, Ω), they will be monotonic for the returns induced by any contracts in that economy, $(\alpha + \beta\mu, \beta\Omega\beta')$.

Definition 1. *Let X be a generic Random Variable on the state space $S = (s_1, ..., s_S)$ taking values $(x_1, ..., x_S)$. We say that $U(X)$ is monotonic if $\frac{\partial U}{\partial x_i} > 0, \forall i$.*

Lemma 3. *Consider the preferences induced by the utility function*

$$U(X) = E(X) - \frac{b}{2}\left(E(X)^2 + Var(X)\right)$$

They are monotonic on a set of variables $\mathcal{X}$ defined on a finite state space S, if

$$b < \min_{X,s} \frac{1}{X(s)}$$

Proof. The proof amounts to checking (by differentiating) under which conditions on a the utility function is increasing. □

Definition 2. *Preferences are monotonic for the entire economy if* $b < \frac{1}{\sum_n \max_s X_n(s)}$

This definition amounts to stating that if an agent owned the entire returns available to the economy, he would still not be satiated. [2]

Lemma 4. *If preferences are monotonic for the entire economy with assets characterized by returns (μ, Ω), then they will be monotonic on all feasible portfolios for any contracts $(\alpha_n, \beta_n)_{n\in N}$.*

Proof. For preferences to be monotonic for all feasible portfolios it has to be that

$$b < \min_{\theta_n, X_n, s} \frac{1}{\sum_{n\in N} \theta_n X_n(s)}$$

Where the min is taken across portfolios θ such that $\theta_n \in (0, 1)$ and outcomes $X_n(s)$.

So that the lemma can be restated as

$$\max_s \sum_{n\in N} \theta_n (\alpha_n + \beta_n X_n(s)) < \sum_{n\in N} \theta_n \max_s X_n(s)$$

Note that the solution to the maximization on both sides is going to be reached at the aggregate market portfolio ($\theta_n = 1$ for all n) so that the previous is equivalent to

Claim 1

$$\max_s \sum_{n\in N} (\alpha_n + \beta_n X_n(s)) < \sum_{n\in N} \max_s X_n(s)$$

[2] The definition is equivalent to the less stringent $b < \frac{1}{\max_s \sum_n X_n(s)}$ as long as no firms are perfectly correlated $|\rho_{mn}| \neq 1, \forall m, n$

To prove this, note first how

Claim 2

$$\max_s \alpha_n + \beta_n X_n(s) < \max_s X_n(s), \forall n, \alpha_n, \beta_n$$

Suppose by means of contradiction

$$\exists n, \alpha_n, \beta_n \; s.t. \; \max_s -\alpha_n + (1-\beta_n) X_n(s) < 0$$

Which in turn implies

$$0 > -\alpha_n + (1-\beta_n) E(X_n) = E(-\alpha_n + (1-\beta_n) X_n) > EU(-\alpha_n + (1-\beta_n) X_n)$$

However $EU(-\alpha_n + (1-\beta_n))$ has to be greater than zero in equilibrium, so that

$$0 > EU(-\alpha_n + (1-\beta_n) X_n) > 0$$

which is impossible. This proves Claim 2

It follows from Claim 2 that

$$\sum_n \alpha_n + \beta_n \max_s X_n(s) < \sum_n \max_s X_n(s)$$

Finally

$$\sum_{n \in N} \max_s X_n(s) > \sum_{n \in N} \left(\alpha_n + \beta_n \max_s X_n(s) \right) > \max_s \sum_{n \in N} (\alpha_n + \beta_n X_n(s))$$

□

1.5.2 The existence result

Theorem 5. *If preferences are monotonic for the entire economy then there exists an equilibrium in the CAPM contracting economy.*

Proof. I am going to use a well known fixed point result by Glicksberg (1952) to show that there is an equilibrium in the first stage of the game, given that the asset market develops as predicted by the CAPM model.

I need to show that

1. The strategy space $\Delta(\mathcal{M}^{MV})$ is a convex, compact subset of a locally convex Hausdorff space.

2. The best response correspondence of all principals is upper hemi-continuous, convex valued, and nonempty.

For the first part note that the space of Incentive Compatible menus $\mathcal{M}^{MV}$ is a subset of a Euclidean space. It is closed because it is defined by a finite number of weak inequalities, and it is bounded because the larger set of feasible contracts are bounded. Hence it is compact.

The space of lotteries (identified with Borel probability measures) over these Menus is of course convex. It is also compact with respect to the weak* topology. This space of probabilities is a subset of the space of continuous functions $\mathcal{C}(\mathcal{M}^{MV})$, which is locally convex (and Hausdorff) with respect to the weak* topology.[3]

For the second part, convexity of the best response correspondence follows from preferences on random variables being represented by expected utility. I will use Berge's Maximum theorem to show that it is non empty, compact-valued and upper hemi-continous.

To apply the maximum theorem to individuals' best response, it has to be that constraints vary continuously with other principals' strategies, and that the payoff function is continuous in one's own actions.

First note how the constraints correspondence is constant with respect to other principals strategies, and is therefore continuous. Also note how the constraints correspondence maps to the space of Borel probability measures on menus, which is a Hausdorff space as noted above.

We also need to make sure that the payoff function of a principal is continuous in menus. To do this we need to show that

[3] For a treatment of these and other results on weak topologies, and also to see the theorems of Berge and Glicksberg, see Aliprantis, Border (2005)

1. Payoffs at the market stage are a continuous function of the contracts chosen by agents.

2. The contracts chosen by agents are a continuous function of the menus offered.

Claim 1 By Lemma 4, if the preferences are monotonic for (μ, Ω), they are going to be monotonic for the asset markets resulting from all possible contracts C. Under the present assumptions a CAPM equilibrium exists once contracts are chosen.[4] Because in such equilibrium the price of a security can be expressed as $q_n = \alpha_n + \beta_n \mu_n - \frac{b}{N}\left(\beta_n^2 \sigma_n^2 + \Sigma_{m \neq n} \rho_{mn} \beta_m \beta_n \sigma_m \sigma_n\right)$ The indirect utility from a contract profile in the CAPM function is continuous in contracts.

Claim 2 Recall that we can restrict attention to Incentive Compatible menus. If a principal makes a small change to the menu he offers while remaining in this set, every type of agent t , will still find it optimal to pick the contract intended for him, C_t. Hence any small change, will correspond to a small change in the contract picked by each type of agent.

We can conclude that the indirect utility for a principal facing type t is a continuous function of the menus offered.

Taking expectation with respect to **F** over these indirect utilities yields a continuous functional on the domain of lotteries on IC menus.

By the maximum theorem the best response correspondence of each player is now UHC and compact valued, which implies that the game best response is as well.

By Glicksberg's theorem there is a fix point, which is an equilibrium by construction. □

1.6 The Insurance Effect of Markets

Throughout this section I will consider the case of firms whose returns are independent.

1.6.1 Utility from Markets

The point of this section is incorporating the outcome of markets in the principals' utility functions. First consider again what the final holdings are in equilibrium. Every individual will hold the same risky portfolio, an equal fraction $\frac{1}{N}$ of the aggregate endowment, and will spend the rest on the

[4] In the literature briefly reviewed by Nielsen (1990), one can find many sufficient conditions for the existence of CAPM equilibrium, most of them deal with the possibility of satiation of preferences. Things are particularly simple when returns are bounded (which includes this model): monotonicity and local non satiation are guaranteed by a low enough risk aversion

riskless asset (or short it if their remaining endowment is negative). With this in mind the mean and variance of the portfolio held by the agent is readily computed as a function of contracts. For a general principal i we have that

- The holding of riskless asset is $q_i - \frac{1}{N}\sum_{j\in P} q_j$
- The mean of the risky portfolio is $\frac{1}{N}\sum_{j\in N}(\alpha_j + \beta_j\mu_j)$
- The variance of the risky portfolio is $\frac{1}{N^2}\sum_{j\in N}\beta_j^2\sigma_j^2$

Since $q_i = \alpha_i + \beta_i\mu_i - \frac{b}{N}\beta_i^2\sigma_i^2$, we have that the mean of the portfolio simplifies to

$$\alpha_i + \beta_i\mu_i - \frac{b}{N}\beta_i^2\sigma_i^2 + \frac{b}{N^2}\sum_{j\in N}\beta_j^2\sigma_j^2$$

and the variance is of course the variance of the risky part $\frac{1}{N^2}\sum_{j\in N}\beta_j^2\sigma_j^2$

So that if $U(\alpha_i,\beta_i) = F\left(\alpha_i + \beta_i\mu_i, \beta_i^2\sigma_i^2\right)$ is the utility a principal obtains from contract α,β when no markets are available, markets will change this into

$$\begin{aligned} U^M(\alpha_i,\beta_i) &= \\ F^M\left(\alpha_i + \beta_i\mu_i, \beta_i^2\sigma_i^2\right) &= \\ F\left(\alpha_i + \beta_i\mu_i - \frac{b}{N}\beta_i^2\sigma_i^2 + \frac{b}{N^2}\sum_{j\in N}\beta_j^2\sigma_j^2, \frac{1}{N^2}\sum_{j\in N}\beta_j^2\sigma_j^2\right) \end{aligned}$$

Lemma 6. *If the principals are allowed to trade their claims, they will act as if their utility functions were*

$$\begin{aligned} E_{\mathbf{t}}U^M(\alpha_i,\beta_i) &= \\ E_{\mathbf{t}}F^M\left(\alpha_i + \beta_i\mu_{i,t_i}, \beta_i^2\sigma_{i,t_i}^2\right) &= \\ E_{\mathbf{t}}F\left(\alpha_i + \beta_i\mu_{i,t_i} - \frac{b}{N}\beta_i^2\sigma_{i,t_i}^2 + \frac{b}{N^2}\sum_{j\in N}\beta_j^2\sigma_{j,t_j}^2, \frac{1}{N^2}\sum_{j\in N}\beta_j^2\sigma_{j,t_j}^2\right) \end{aligned}$$

The partial derivatives of these functions are given by

$$\frac{\partial F^M}{\partial \mu_i}\left(\mu_i,\sigma_i^2\right) = \frac{\partial F}{\partial \mu}\left(\mu_i - \frac{b}{N}\sigma_i^2 + \frac{b}{N^2}\sum_{j\in N}\sigma_j^2, \frac{\sum_{j\in N}\sigma_j^2}{N^2}\right)$$

$$\frac{\partial F^M}{\partial \sigma_i^2}\left(\mu_i,\sigma_i^2\right) = \frac{\partial F}{\partial \mu}\left(\mu_i - \frac{b}{N}\sigma_i^2 + \frac{b}{N^2}\sum_{j\in N}\sigma_j^2, \frac{\sum_{j\in N}\sigma_j^2}{N^2}\right)\left(\frac{b}{N^2} - \frac{b}{N}\right) + \frac{\partial F}{\partial \sigma^2}\left(\mu_i - \frac{b}{N}\sigma_i^2 + \frac{b}{N^2}\sum_{j\in N}\sigma_j^2, \frac{\sum_{j\in N}\sigma_j^2}{N^2}\right)\frac{1}{N^2}$$

and so that

$$U^M_{\alpha_i}\left(\alpha_i,\beta_i\right) = F_\mu\left(\mu_i - \frac{b}{N}\sigma_i^2 + \frac{b}{N^2}\sum_{j\in N}\sigma_j^2, \frac{\sum_{j\in N}\sigma_j^2}{N^2}\right)$$

$$U^M_{\beta_i}\left(\alpha_i,\beta_i\right) = \mu_i F_\mu\left(\mu_i - \frac{b}{N}\sigma_i^2 + \frac{b}{N^2}\sum_{j\in N}\sigma_j^2, \frac{\sum_{j\in N}\sigma_j^2}{N^2}\right) + 2\beta_i\sigma_i^2\left[\mu_i F_\mu\left(\mu_i - \frac{b}{N}\sigma_i^2 + \frac{b}{N^2}\sum_{j\in N}\sigma_j^2, \frac{\sum_{j\in N}\sigma_j^2}{N^2}\right)\left(\frac{b}{N^2} - \frac{b}{N}\right) + F_{\sigma^2}\left(\mu_i - \frac{b}{N}\sigma_i^2 + \frac{b}{N^2}\sum_{j\in N}\sigma_j^2, \frac{\sum_{j\in N}\sigma_j^2}{N^2}\right)\frac{1}{N^2}\right]$$

whereas the partial derivatives without markets are

$$U_{\alpha_i}\left(\alpha_i,\beta_i\right) = F_\mu\left(\alpha_i + \beta_i\mu_i, \beta_i^2\sigma_i^2\right)$$

$$U_{\beta_i}\left(\alpha_i,\beta_i\right) = \mu_i F_\mu\left(\alpha_i + \beta_i\mu_i, \beta_i^2\sigma_i^2\right) + 2\beta_i\sigma_i^2 F_{\sigma^2}\left(\alpha_i + \beta_i\mu_i, \beta_i^2\sigma_i^2\right)$$

1.6.2 First Best

Theorem 7. *When information is complete and symmetric, the variance of optimal contracts is smaller in a large market than without markets. There is a number of principals $\overline{N}$ such that*

$$\beta_t^* < \beta_t^* (N), \ \ \forall N \geq \overline{N}$$

Proof. Consider the problem of a principal.

$$\max_{(\alpha_t,\beta_t)_{t=1}^T} \sum_{t=1}^T U(\alpha_t, \beta_t|t)$$
$$s.t. \ \ U(-\alpha_t, 1-\beta_t|t) \geq \overline{u}, \ \ \forall t = 1, ...T$$

Which can be broken into T separate problems

$$\max_{(\alpha_t,\beta_t)} U(\alpha_t, \beta_t|t)$$
$$s.t. \ \ U(-\alpha_t, 1-\beta_t|t) \geq \overline{u}$$

Note that

$$U(\alpha_t, \beta_t|t) = F\left(\alpha_t + \beta_t\mu_t, \beta_t^2\sigma_t^2\right)$$
$$U(-\alpha_t, 1-\beta_t|t) = F\left(-\alpha_t + (1-\beta_t)\mu_t, (1-\beta_t)^2\sigma_t^2\right)$$

Dropping the type subscripts for convenience, we can proceed to solve an individual problem

$$\max_{(\alpha,\beta)} F\left(\alpha + \beta\mu, \beta^2\sigma^2\right)$$
$$s.t. \ \ F\left(-\alpha + (1-\beta)\mu, (1-\beta)^2\sigma^2\right) \geq \overline{u}$$

The first order conditions for this problem amount to

$$\frac{\partial L}{\partial \alpha} = F_{\mu}\left(\alpha+\beta\mu,\beta^{2}\sigma^{2}\right) - \lambda F_{\mu}\left(-\alpha+\left(1-\beta\right)\mu,\left(1-\beta\right)^{2}\sigma^{2}\right) = 0$$

$$\frac{\partial L}{\partial \beta} = \mu F_{\mu}\left(\alpha+\beta\mu,\beta^{2}\sigma^{2}\right) + 2\beta\sigma^{2}F_{\sigma^{2}}\left(\alpha+\beta\mu,\beta^{2}\sigma^{2}\right) +$$

$$-\lambda\left[\mu F_{\mu}\left(-\alpha+\left(1-\beta\right)\mu,\left(1-\beta\right)^{2}\sigma^{2}\right) + 2\left(1-\beta\right)\sigma^{2}F_{\sigma^{2}}\left(-\alpha+\left(1-\beta\right)\mu,\left(1-\beta\right)^{2}\sigma^{2}\right)\right] = 0$$

Solving for the optimal contract we have that

$$\beta^{M*} = \frac{NUM}{DEN}$$

where

$$NUM = F_{\sigma^{2}}\left(-\alpha+\left(1-\beta\right)\mu,\left(1-\beta\right)^{2}\sigma^{2}\right)F_{\mu}\left(\alpha+\beta\mu,\beta^{2}\sigma^{2}\right)$$

$$DEN = F_{\sigma^{2}}\left(-\alpha+\left(1-\beta\right)\mu,\left(1-\beta\right)^{2}\sigma^{2}\right)F_{\mu}\left(\alpha+\beta\mu,\beta^{2}\sigma^{2}\right) + F_{\sigma^{2}}\left(\alpha+\beta\mu,\beta^{2}\sigma^{2}\right)F_{\mu}\left(-\alpha+\left(1-\beta\right)\mu,\left(1-\beta\right)^{2}\sigma^{2}\right)$$

Similarly we can "break" the problem with markets in smaller optimizations like the following.

$$\max_{(\alpha,\beta)} F^{M}\left(\alpha+\beta\mu,\beta^{2}\sigma^{2}\right)$$

$$s.t.\ \ F\left(-\alpha+\left(1-\beta\right)\mu,\left(1-\beta\right)^{2}\sigma^{2}\right) \geq \overline{u}$$

The first order conditions for this problem amount to

$$\frac{\partial L^{M}}{\partial \alpha} = F_{\mu}^{M}\left(\alpha+\beta\mu,\beta^{2}\sigma^{2}\right) - \lambda F_{\mu}^{M}\left(-\alpha+\left(1-\beta\right)\mu,\left(1-\beta\right)^{2}\sigma^{2}\right) = 0$$

$$\frac{\partial L^{M}}{\partial \beta}\mu F_{\mu}^{M}\left(\alpha+\beta\mu,\beta^{2}\sigma^{2}\right) + 2\beta\sigma^{2}F_{\sigma^{2}}^{M}\left(\alpha+\beta\mu,\beta^{2}\sigma^{2}\right) +$$

$$-\lambda\left[\mu F_{\mu}^{M}\left(-\alpha+\left(1-\beta\right)\mu,\left(1-\beta\right)^{2}\sigma^{2}\right) + 2\left(1-\beta\right)\sigma^{2}F_{\sigma^{2}}^{M}\left(-\alpha+\left(1-\beta\right)\mu,\left(1-\beta\right)^{2}\sigma^{2}\right)\right] = 0$$

Solving for the optimal contract we have that

$$\beta^{*} = \frac{NUM_{M}}{DEN_{M}}$$

where

$$NUM_M = F_{\sigma^2}\left(-\alpha + (1-\beta)\mu, (1-\beta)^2\sigma^2\right) F_\mu^M\left(\alpha + \beta\mu, \beta^2\sigma^2\right)$$
$$DEN_M = F_{\sigma^2}\left(-\alpha + (1-\beta)\mu, (1-\beta)^2\sigma^2\right) F_\mu^M\left(\alpha + \beta\mu, \beta^2\sigma^2\right) +$$
$$F_{\sigma^2}^M\left(\alpha + \beta\mu, \beta^2\sigma^2\right) F_\mu\left(-\alpha + (1-\beta)\mu, (1-\beta)^2\sigma^2\right)$$

By the considerations from Section 1.6.1, we note that, since $F_{\sigma^2}^M$ gets arbitrarily small as N gets large ($\frac{b}{N} - \frac{b}{N^2}$ and $\frac{1}{N^2}$ are arbitrarily small for large enough N, so is $F_{\sigma^2}^M$) . β^* is arbitrarily close to 1 in a large economy. Which concludes the proof. □

1.6.3 Ordering Types

To solve a risk sharing principal agent problem with multiple types and asymmetric information, I will need to adapt methods use in the contract theory literature.[5] I will require that, for any firm, the type space is 'ordered' in the sense of Single Crossing Property (ie: any two indifference curves from any two types will cross only once). The following cumbersome notation means exactly this. Given a firm, agents can be ranked (by the slopes of their indifference curves). The ranking need not be the same in every firm.

Let $SCP(n) = (1(n), 2(n), ..., T(n))$ be a function from N to T^N such that $i(n) \neq j(n), \forall i(n), j(n) \in 1, ...T, \forall n$.

Definition 3. *An economy satisfies Single Crossing Property if and only if there is a function SCP, as defined above such that types* $\left((\mu_{1(n)}), \sigma_{1(n)}\right), ..., \left(\mu_{T(n)}, \sigma_{T(n)}\right)\right)$ *satisfy* Single Crossing Property *for all n . That is, if*

$$\frac{U_2(-\alpha, 1-\beta|t(n))}{U_1(-\alpha, 1-\beta|t(n))} > \frac{U_2(-\alpha, 1-\beta|t(n)+1)}{U_1(-\alpha, 1-\beta|t(n)+1)}$$
$$\forall t, p, \alpha, \beta$$

In the case of quadratic utility, SCP amounts to

[5]Similar techniques are used in the literature on second order price discrimination. See for example Maskin and Riley, 1984 and Stiglitz, 1980

$$\mu_t - \frac{b_t \sigma_t^2 (1-\beta)}{1 - b_t(-\alpha) - b_t \mu_t (1-\beta)} > \mu_{t+1} - \frac{b_{t+1} \sigma_{t+1}^2 (1-\beta)}{1 - b_{t+1}(-\alpha) - b_{t+1} \mu_{t+1} (1-\beta)}$$

Proposition 8. *The following cases[6] imply SCP:*

- *Agents have the same mean and different variance*
- *Agents have different mean and the same variance*
- *Agents generate the same outcomes, but the probabilities of success are different.*

The proof amounts to the algebra necessary to verify the definition.

The following two facts will come handy for the proving Theorem 11

Fact 9. $\forall t, (\alpha, \beta), U(-\alpha, 1-\beta|t) > U(-\alpha, 1-\beta|t+1)$

Fact 10. $\forall t < s, \forall (\alpha, \beta), (\alpha', \beta') : \beta \leq \beta'$

$$U(-\alpha, (1-\beta)|\, t) - U(-\alpha', (1-\beta')|\, t) > U(-\alpha, (1-\beta)|\, s) - U(-\alpha', (1-\beta')|\, s)$$

1.6.4 Second Best

Theorem 11. *If types satisfy Single Crossing Property and markets are large enough, the variance of the average contract is smaller when markets are present.*

Proof. The strategy of the proof is to show that -at an optimum- all types shares' ($(1-\beta)$) are bounded by the share of the "best" type, $t = 1$. And that the share of this type become small for large enough market.

A generic principal's problem is given by

$$\max_{(\alpha_t, \beta_t)_{t=1}^T} \sum_{t=1}^{T} U(\alpha_t, \beta_t|t) f(t)$$

$$s.t.\ \ U(-\alpha_t, 1-\beta_t|t) \geq \overline{u}, \ \ \forall t \in \{1, ...T\} \tag{IR}$$

$$U(-\alpha_t, 1-\beta_t|t) \geq U(-\alpha_{t'}, 1-\beta_{t'}|t), \ \ \forall t, t' \in \{1, ..., T\} \tag{IC t t'}$$

[6] It is to be noted that, although it is not formalized in the general model, SCP holds also if agents are distinguished by their risk aversion, and the following results would still apply

SCP and IC constraints imply by usual arguments that yield that $\beta^* (s) \geq \beta^* (t), \forall s < t \in \{1, ...T\}$. This imply that $\beta_1 \leq \beta_t, \forall t$

Now we have to solve for the contract of type 1. To do this I show that the IC_{12} constraint will always be binding, and that this is enough to attain the result.

The first thing to do is to reduce the set of relevant constraints.

Fact 9 implies that

$$U(-\alpha_t, 1 - \beta_t | t) \geq U(-\alpha_t, 1 - \beta_t | T)$$

This together with IR holding for type T and IC holding for type t with respect to the contracts of type T, implies that IR holds for type t.

In other words, if

$$U(-\alpha_T, 1 - \beta_T | T) \geq \overline{u}$$
$$U(-\alpha_t, 1 - \beta_t | t) \geq U(-\alpha_T, 1 - \beta_T | t)$$

we will also have that

$$U(-\alpha_t, 1 - \beta_t | t) \geq \overline{u}$$

We can hence solve the problem without worrying about any of the IR constraints except that of type T .

We can also infer that $IC_{t-1,t}$ will be binding at an optimum for any t. Suppose that it were not binding,

$$U(-\alpha_{t-1}, 1 - \beta_{t-1} | t-1) > U(-\alpha_t, 1 - \beta_t | t-1)$$

Since we are at an optimum it has to be that the IC constraints are satisfied

$$U\left(-\alpha_t, 1-\beta_t|t\right) \geq U\left(-\alpha_k, 1-\beta_k|t\right), \forall k$$

By Fact 10 it has to be that

$$U\left(-\alpha_t, 1-\beta_t|s\right) > U\left(-\alpha_k, 1-\beta_k|s\right), \forall k \geq t, \forall s < t$$

Consider an alternative incentive scheme, which gives less transfer $-\alpha_s$ to all types s lower t. Because their IC constraints for contracts C_k, with $k > t$ ($IC_{s,k}$) are not binding, we are increasing the maximand while remaining in the admissible set of contracts, which contradicts the original scheme being an optimum.

This and SCP imply that constraints $IC_{t,t-1}$ will not be binding.

It also implies that no other IC constraint will bind at the optimum. Fact 9, $IC_{t-1,t}$, and $IC_{t,t+1}$ imply that $IC_{t-1,t+1}$ is satisfied with a strict inequality.

This means that the only relevant constraints for determining the optimal β_1 are in the form

$$U\left(-\alpha_1, 1-\beta_1|1\right) = U\left(-\alpha_2, 1-\beta_2|1\right)$$

I now have to solve a much simpler problem

$$\max_{(\alpha_t,\beta_t)_{t=1}^T} \sum_{t=1}^{T} U\left(\alpha_t, \beta_t|t\right) f\left(t\right)$$

$$s.t. \;\; U\left(-\alpha_T, 1-\beta_T|t\right) = \overline{u} \qquad \text{(IR)}$$

$$U\left(-\alpha_t, 1-\beta_t|t\right) = U\left(-\alpha_{t+1}, 1-\beta_{t+1}|t\right), \;\; \forall t \in \{1, ..., T-1\} \qquad \text{(IC t t +1)}$$

The only first order conditions involving α_1 and β_1 are given by

$$q_1 F_\mu\left(\alpha+\beta\mu,\beta^2\sigma^2\right)-\lambda F_\mu\left(-\alpha+(1-\beta)\mu,(1-\beta)^2\sigma^2\right)=0$$

$$q_1\left(\mu F_\mu\left(\alpha+\beta\mu,\beta^2\sigma^2\right)+2\beta\sigma^2 F_{\sigma^2}\left(\alpha+\beta\mu,\beta^2\sigma^2\right)\right)+$$
$$-\lambda\left[\mu F_\mu\left(-\alpha+(1-\beta)\mu,(1-\beta)^2\sigma^2\right)+2(1-\beta)\sigma^2 F_{\sigma^2}\left(-\alpha+(1-\beta)\mu,(1-\beta)^2\sigma^2\right)\right]=0$$

Which yield an expression for 1 identical to the first best case.

$$\beta_1^*=\frac{NUM_M}{DEN_M}$$

$$NUM_M=F_{\sigma^2}\left(-\alpha+(1-\beta)\mu,(1-\beta)^2\sigma^2\right)F_\mu^M\left(\alpha+\beta\mu,\beta^2\sigma^2\right)$$
$$DEN_M=F_{\sigma^2}\left(-\alpha+(1-\beta)\mu,(1-\beta)^2\sigma^2\right)F_\mu^M\left(\alpha+\beta\mu,\beta^2\sigma^2\right)+$$
$$F_{\sigma^2}^M\left(\alpha+\beta\mu,\beta^2\sigma^2\right)F_\mu\left(-\alpha+(1-\beta)\mu,(1-\beta)^2\sigma^2\right)$$

Solving for β_1 we obtain a formula similar to the first best case. Just as in the first best case we can observe that β^* is arbitrarily close to 1 in a large economy, since $F_{\sigma^2}^M$ can again be made arbitrarily small by choosing an appropriately large N.

This bound allows us to reach the result

Claim: $\sum_{t,n} f(t)\,\beta_n^t \geq \sum_{t,n} f(t)\,\beta_n^t(1)$

Let t be the bijection satisfying SCP for all matches.

The claim is equivalent to

$$\sum_n\sum_t f(t)\,\beta_n^t \geq \sum_n \beta_n^{1(n)}(n)$$

Note that

$$\forall n,t,\ \exists\overline{N}\ s.t.\ \beta_n^{1(n)}(N),\forall N\geq\overline{N}$$

and naturally

$$\forall n,\ \exists \overline{N}\ s.t.\ \beta_n^{1(n)}(N), \forall N \geq \overline{N}, \forall t_n$$

This implies

$$\beta_n^{1(n)} > \sum_t f(t_n)\, \beta_n^t, \forall N \geq \overline{N}$$

and

$$\sum_n \beta_n^{1(n)} > \sum_n \sum_t f(t)\, \beta_n^t, \forall N \geq \overline{N}$$

which concludes the proof.

□

1.6.5 Effects on Welfare and on Asset Markets

As it is usually the case, asymmetric information entails a loss of efficiency.

Corollary 12. *Under the assumptions of Theorem 11, the equilibrium is inefficient compared to the complete information case.*

Proof. It follows from the fact that IR_T and $IC_{T-1,T}$ are both binding that *at least* the contract of type T is Pareto dominated by the complete information solution of Theorem 7.

□

Let $\Delta_t(N) = \beta^{t*}(N) - \beta^{t1}(N)$, where $\beta^{t1}(N)$ is the first best solution. I will use this as a measure of inefficiency.

Corollary 13. *For any principal-agent pair, there is a number $\overline{N}$, such that any market with $N \geq \overline{N}$ traders induces more efficient contracts, $\Delta_t(N) < \Delta_t(1)$*

Proof. The proof again leverages on the fact that both first and second best contracts are arbitrarily close to 1 for a large enough markets. The claim is trivially satisfied for $\beta^{1*}(N)$ (the second best solution, with markets), since it is a Pareto Efficient Contract by construction. For $\beta^{t*}(N), t > 1$ consider $\Delta_t(N) = \beta^{t*}(N) - \beta^{t1}(N)$, where $\beta^{t1}(N)$ is the first best solution with markets. Since $\beta^{t*}(N)$ and $\beta^{t1}(N)$ both go to one as N gets large, there must be a $\overline{N}$ such that $\Delta_t(N) < \Delta_t(1)$ for all $N > \overline{N}$. □

1.7 Conclusion

This chapter integrates a model of principal-agent interaction with asset markets. Principals and Agents are randomly matched. Each pair produces random returns, whose distribution is known only to the agent at the contracting stage. Every Principal offers a menu of contracts to the Agent he is matched with, and the Agents make their pick. What marks the difference from the standard contracting model is that Principals have access to an asset market on which they trade their shares of returns.

I present a general framework and define a notion of equilibrium. I prove a revelation principle result, to simplify the study of firms' structure and prove existence of equilibrium. Under standard assumptions of contract theory, I study the interactions of financial markets on contracts. The existence of markets, induces less risky compensation for agents. In a large market diversification opportunities multiply, and contracts become less and less risky.

Contracting inside firms induces inefficient aggregate risk in an economy, however the size of this inefficiency is reduced by a large enough market. As noted this is a limiting result and it leaves the open question of the behavior in small markets.

Two extensions of this model seem particularly interesting. The first one is studying the effect of correlation of firms returns (systemic risk) on the screening problem and the loss of welfare. The second one is studying the effect of financial markets on the labour market, in particular on the matching of workers' with firms.

Chapter 2

Contracts and Aftermarkets - Hidden Action

2.1 Introduction

It is well known that financial markets can favor the efficient allocation of resources to production and the sharing of risk. Less is known about their effects on incentives, for those who have limited access to markets. In this chapter I propose a stylized model of firms and financial markets, to capture these effects. The concern here is how different access to markets can affect the incentives to production.

This chapter relates to two branches of economic theory. The literature on Walrasian economies in presence of Moral Hazard issues, and the literature on endogenous securities. I will discuss the fit of my work in these literatures, but first I am going to relate my work to two recent papers, which are particularly relevant to the discussion.

Parlour and Walden (2008) construct a model based on CAPM where workers are also the dispersed owners of the firms in the economy. Their assumptions and their main object of interest are opposite of mine. By assuming workers are the only traders, and cleverly constructing a simple shock that cannot be traded away on markets, they mostly analyze the effects of moral hazard on financial markets. I am on the other hand more interested in the effect of financial markets on contracts. For this reason I cannot take firms as abstract risk neutral principals at the contracting stage. In fact Risk Neutrality corresponds only to a special case (firms with independent returns).

Magill and Quinzii (2005) ask a different although related question. What set of securities is needed, for risk sharing and incentives to coexist? They study several cases, distinguished by the amount of information available for contracting, and for each case characterize the set of securities which allow (constrained) efficiency. In this chapter securities are determined in equilibrium, but they are chosen out of a very simple set. In the future it would be worth describing the endogenous choice of complex securities, to study the efficiency of financial engineering, in terms of incentives and risk sharing, but this is out of the scope of this chapter.

There is by now a vast literature on moral hazard in general equilibrium settings. Helpman and Laffont (1975) and Prescott and Townsend (1984) are among the first to tackle the topic. Their works are concerned with efficiency in exchange and production economies, in which individuals can exert a costly, unobservable payoff relevant action. These papers are the first in a long, but not large, series of works, which extend the study of efficiency to more general economies. My work is different in that I include financial markets, and I trade off some generality for more precise comparative statics results.

A strand of the asset pricing literature looks at asset pricing in the presence of delegated portfolio management (for a survey, see Stracca, 2003). An example of the approach typical in these papers can be seen in Ou Yang's paper. These studies look at the effects on prices and returns of the classical informational asymmetries phenomena. Moral hazard and adverse selection are largely studied in a CAPM or APT setting, in which a representative principal delegates his investing decisions to an agent. In this literature inefficiencies take the form of deviations from the non-delegated case equilibrium. These deviations can take the form of changes in asset prices and optimal portfolio composition. Besides the different object of interest, the perspective in these works is in a sense opposite of the one taken here. There we have informed parties trading, whereas in the present work it is the uninformed parties accessing markets.

In Section 2 I present the model. In Section 3 I define the notion of equilibrium. In section 4 I show existence and uniqueness. In section 5 I analyze the effect of markets on production. Section 6 concludes. The appendix discusses issues concerning Markowitz preferences, when used in a dynamic contest.

2.2 The Model

This stylized model is meant to capture certain features of interactions within firms and across firms in financial markets. It is helpful to introduce first the model of a single firm, and then proceed to define how they interact.

2.2.1 Primitives

A generic firm i is constituted by a principal-agent pair p_i, a_i. Each of these firms i generates random returns. All individuals have identical Mean Variance Preferences over random variables in the familiar form [1]

$$U(X) = \mu(X) - \frac{b}{2}\sigma^2(X)$$

Agents have a reservation utility, $\overline{u}$. Agents choose a costly action (which can be interpreted as the effort put into production) e_i from the interval $E = [\underline{e}, \overline{e}]$. The cost of effort is a continuous, strictly increasing real valued function $c(e)$.

The returns of a firm $\tilde{X}_i(e_i)$ depend stochastically on the effort level of the agent employed. Because of the assumptions on preferences, I can restrict my attention to the mean and the variance of the firm's returns, and express them as a function of effort levels.

$$\mu_i(e_i)$$
$$\sigma^2(e_i)$$

A *contract* C_i is a contingent agreement on how to split returns between the Principal and the Agent forming firm i. I impose the restriction that these sharing rules be affine. If X is the random variable describing the profits of the firm, an admissible rule describing the principal's and agent's

[1] Note that all the analysis in this paper would apply to the CARA/normal framework, except that conditions for existence would be less demanding.

share must be of the form:

$$X_{a_i} = -\alpha + (1-\beta)X$$
$$X_{p_i} = \alpha + \beta X$$

with $\alpha \in \mathbf{R}$ and $\beta \in [0,1]$.

A contract C_i amounts to a pair α_i, β_i. In a binary setting this imposes the restriction that both shares be increasing in firm's returns.

Now consider that there are N firms. Let $e = [e_i(C_i)]_{1\leq i\leq N}$ be the vector of efforts, and $C = [C_i]_{1\leq i\leq N}$ be the vector of contracts in each firm .

The joint distribution of the vector of firms' profits $\mathbf{X}(e) = [X_i(e_i)]_{1\leq i\leq N}$ firms' profits will be uniquely determined by the effort levels across firms.

$$\mu(e) = \begin{bmatrix} \mu_1(e_1) \\ \vdots \\ \mu_N(e_N) \end{bmatrix}$$
$$\Omega_{jk}(e) = \rho_{jk}(e_j, e_k)\,\sigma_j(e_j)\,\sigma_k(e_k)$$

I will denote this vector and matrix as $\mu(e)$ and $\Omega(e)$.

Principals will have access to a financial market where they can trade their claims to profits and a riskless asset available in zero net supply.

The risky assets available can be described by a random vector, $\mathbf{X_p}(e, C) = [X_{p_i}(e_i, C_i)]_{1\leq i\leq N} = [\alpha_i + \beta X_i(e_i)]_{1\leq i\leq N}$

Because of our assumption on preferences, from now on I will simply identify the principals' shares with the vector of means $\mu(C, e) = \alpha + \beta\mu(e)$ and the variance-covariance matrix $\Omega(C, e) = \beta'\Omega(e)\beta$.

2.2.2 Timeline

I will consider here the case of "hidden action". In other words, the two parties cannot write contracts on the actual effort level. The conditions for existence trivially imply the conditions for existence when effort is observable.

CMN will denote the primitives of the game that are common knowledge at every stage, which are:

- The vector of means and the variance-covariance matrix of payoffs as a function of effort levels

$$\mu(\cdot)$$
$$\Omega(\cdot\cdot)$$

- The preferences of individuals (the variance aversion parameter b).
- The reservation utility of agents $\overline{u}$.

The economy reaches its equilibrium in 3 stages.

1. Each Principal p_i designs a contract $C_i = (\alpha_i, \beta_i)$. $\mathcal{C}_i$ is the set of all possible contracts for firm i. At this stage every principal knows CMN.

   The profile of offered contracts is $C = \{C_1, ..., C_N\} \in \mathcal{C} = \Pi_{i=1}^{N}\mathcal{C}_i$

2. Each agent a_i chooses his effort level, based on the contract C_i.

   A strategy of an agent in firm i is a function of contracts mapping to possible effort levels.

$$e_i : \mathcal{C}_i \to E$$
$$e_i : C_i \mapsto e_i(C_i)$$

   The strategy profile of all agents can be written as a vector of functions $e = [e_i(C_i)]_{1\leq i\leq N}$ of contracts offered.

3. *Before* uncertainty is realized, principals trade their claims to returns on an asset market, where a riskless asset L is available in zero supply. At this stage principals can observe all contracts and make conjectures on the level of effort and hence the distribution of returns of all firms. Every principal j will form some beliefs γ_i^j on the effort level e_i . $\gamma_i^j(e_i|C_i)$ is the

cumulative distribution function induced by the beliefs on firm i under contract C_i . Principal j holds θ_j^i shares of firm i and the price of the stock of firm i is q_i

This table summarizes the choices each individual faces at a given time, and the information available to them.

When	**Who**	**What**	**Knowing What**
0	p_i	C_i	CMN
1	a_i	e_i	CMN, C_i
2	p_i	θ_i	CMN, C

Table 2.1: Timing

2.2.3 Payoffs

Let θ be the portfolio held by an agent. Let $\theta = (\theta_R|\theta_L)$ Where θ_R is an N-dimensional vector of positive holdings of the N risky assets, whereas θ_L is the position an investor holds in the riskless asset.

The ex-ante utility from a portfolio θ fixing the contracts C and effort choices e, is given by

$$\theta \cdot (\mu(C,e)|1) - \frac{b}{2}\theta_R' \mathbf{\Omega}(C,e)\theta_R$$

However principals don't observe the efforts e so they evaluate utility of portfolios based on contracts C and the beliefs they induce $\gamma(\cdot|C)$

$$U_{p_i}(C,\theta,\gamma) =$$

$$\theta \cdot \int_E \mu(C,e|1)\, d\gamma(e|C) - \frac{b}{2}\int_E \left(\theta_R'\Omega(C,e)\theta_R + \left(\mu(C,e) - \int_E \mu(C,e)\right)^2\right) d\gamma(e|C)$$

Note how the squared term is always positive. This will pose a problem, but in equilibrium beliefs will be degenerate and the term will be equal to zero.

Demand θ will depend on available assets and their prices (but prices are also a function of contracts).

Agents payoffs depend on the effort chosen and the contract in place.

$$U_{a_i}\left(C_i, e_i\right) = \mu_i\left(C_i, e_i\right) - \frac{b}{2}\sigma_i^2\left(C_i, e_i\right) - c\left(e_i\right)$$

2.3 Equilibrium

2.3.1 Description

Because individuals take their decisions at each stage looking at the final payoffs, Equilibrium is more easily described starting from the final stage of the game.

Asset Market Principals hold one unit of a security equal to their share of returns in their firm, and they all have the same information. Because all individuals have the same beliefs the solution concept used here is that of Arrow-Debreu Equilibrium. The equilibrium portfolios and prices will be based on expected payoffs induced by C. They will be a function $(\theta, q)\left(C, \gamma\left(e|C\right)\right)$.

Contracting, the Agents' turn Each agent a_i observes the contract he is offered, C_i, and he knows his own type and the technology of the principal. This is all the payoff relevant information, so every agent is facing a choice between lotteries, and he is not playing against other players. They simply choose an effort level maximizing $U_{a_i}(\cdot)$. As noted their strategies will be functions $e_i\left(C_i\right)$.

Contracting, the Principals' turn Each principal designs a contract. They correctly conjecture the action of each agent, and the outcome of asset markets, given contracts. They can forecast the equilibrium path for all possible strategy profiles. Hence, this stage can be seen as a game principals play against each other. I will focus on equilibria in pure strategy equilibria (and show their existence).

The flow of decisions is described schematically below, and the information available at each stage is summarized by the argument of the strategies.

$$C \rightarrow e\left(C\right) \rightarrow \theta\left(C, q\left(C\right), \gamma\left(e|C\right)\right)$$

The utility in the first stage can be written as:

$$V_{p_i}(C) = U_{p_i}(C, \theta(\gamma(e|C)), \gamma(e|C))$$

2.3.2 Definition

An Equilibrium consists of

- A trading strategy θ_i^* for each Principal p_i and prices $q^* \in \mathbb{R}^N$ such that $[\theta^*, q^*](C, \gamma)$ is an Arrow-Debreu Equilibrium for the asset market when contracts are C . Each principal is endowed with one unit of one asset so that the endowment of principal p_i is $w_i = [0, 0, ..., 1, ..., 0, 0]$ with 1 being in the i-th position.

$$\theta_i^*(C, \gamma(e|C)) \in arg \max_{\theta_i \in \mathbb{R}_+^{N+1}} U_{p_i}(C, \theta_i, \gamma(e|C))$$

such that

$$q^*(C, \gamma(e|C)) \cdot \theta_i(C, \gamma(e|C)) \leq q^*(C, \gamma(e|C)) \cdot w_i$$

$$\sum_{i \in N} \theta_i^* = [\mathbf{1}_N | 0]$$

- Beliefs $\gamma^*(e|C)$ such that

$$supp(\gamma_i^*) \subseteq arg \max_{\tilde{e}_i \in E} U_{a_i}(C_i, \tilde{e}_i)$$

- For each agent a_i a strategy $e_i^*(C_i)$ such that

$$e_i(C_i) \in arg \max_{e_i \in E} U_{a_i}(C_i, e_i)$$

- For each principal p_i, a contract C_i^* such that

$$\begin{aligned} C_i^* &\in arg \max_{C_i \in \mathcal{C}_i} V_i^*\left(C_i, C_{-i}^*\right) \\ &= U_{p_i}\left(\left(C_i, C_{-i}^*\right), \gamma^*\left(e|C_i, C_{-i}^*\right), \theta^*\left(\left(C_i, C_{-i}^*\right), \gamma^*\left(e|C_i, C_{-i}^*\right)\right)\right) \end{aligned}$$

2.4 Existence of Equilibrium

2.4.1 Assumptions

2.4.1.1 Monotonic Preferences

It is well known that Mean-Variance Preferences need not be monotonic. This could pose problems for the existence of equilibrium. In a standard CAPM setting, monotonicity of preferences is solved by imposing a bound on the variance aversion of every individual. Because I restrict attention to linear contracts, it is possible to show that, if preferences are monotonic for given returns, they will be monotonic for any prevailing contracts.

Definition 4. *Let X be a generic Random Variable on the state space $S = (s_1, ..., s_M)$ taking values $(x_1, ..., x_M)$. $U(X)$ is monotonic if $\frac{\partial U}{\partial x_i} > 0, \forall i$.*

Lemma 14. *Consider the preferences induced by the utility function*

$$U(X) = E(X) - \frac{b}{2}Var(X)$$

They are monotonic on a set of variables $\mathcal{X}$ defined on a finite state space S, if

$$b < \min_{X,s} \frac{1}{|x_s - \mu_X|}$$

Proof. The proof amounts to checking (by differentiating) under which conditions on b the utility function is increasing. □

Lemma 15. *If preferences are monotonic for all feasible portfolios in an economy with assets characterized by returns (μ, Ω), then they will be monotonic on all feasible portfolios for any contracts $(\alpha_i, \beta_i)_{i \in N}$.*

Proof. For preferences to be monotonic for all feasible portfolios it has to be that

$$\begin{aligned} b <& \frac{1}{|\max \sum_{i \in N} \theta_i x_i - \sum_{i \in N} \theta_i \mu_i|} \\ =& \frac{1}{|\max \sum_{i \in N} \theta_i \left(x_i - \mu_i\right)|}. \end{aligned}$$

Where the max is taken across portfolios θ such that $\theta_i \in (0,1)$ and outcomes $x_i \in supp\,(X_i)$. Note that

$$\begin{aligned}&\sum_{i\in N}\theta_i\left[\left(-\alpha_i+(1-\beta_i)\right]x_i-\left[-\alpha_i+(1-\beta_i)\,\mu_i\right)\right]\\ =&\sum_{i\in N}\theta_i\left((1-\beta_i)\,x_i-(1-\beta_i)\,\mu_i\right)\\ =&\sum_{i\in N}\theta_i\,(1-\beta_i)\,(x_i-\mu_i)\end{aligned}$$

I claim that

$$\max|\sum_{i\in N}\theta_i\,(1-\beta_i)\,(x_i-\mu_i)\,|\leq\max|\sum_{i\in N}\theta_i\,(x_i-\mu_i)\,|$$

Note that the solution to the maximization on both sides is going to be reached at the aggregate market portfolio so that the previous is equivalent to

$$\max|\sum_{i\in N}(1-\beta_i)\,(x_i-\mu_i)\,|\leq\max|\sum_{i\in N}(x_i-\mu_i)\,|$$

Since it will also be the case that at the maximum all the x_i's chosen will be greater (or smaller) than the μ_i's so that

$$\max\sum_{i\in N}(1-\beta_i)\,|\,(x_i-\mu_i)\,|\leq\max\sum_{i\in N}|\,(x_i-\mu_i)\,|$$

Observing that $\beta_i \in [0,1]$ concludes the proof □

Assumption 1. *For a given set of assets, for every individual i, their risk tolerance parameter b_i lies on the interval $\left(0,\overline{b}\right)$, where* $\overline{b}=\min_{X,s}\frac{1}{|x_s-\mu_X|}$

This ensures that everyone's preferences will be monotonic.

2.4.1.2 Cost and Productivity of Effort

The purpose of the following assumptions is making sure that the only randomness in the economy is due to the uncertainty of firms' returns. Markowitz preferences are dynamically inconsistent and introducing further randomizations (such as some individual playing a mixed strategy) is undesirable in many ways. [2] Moreover, if only one effort choice is optimal for an agent, then principals can correctly infer the equilibrium effort by observing contracts. To achieve this I need each agent's objective function to be concave in effort for any possible contract (α, β).

Assumption 2. *1. The cost function of an agent $c(e)$ is strictly increasing and strictly convex.*

$$\frac{\partial c}{\partial e} > 0, \frac{\partial^2 c}{\partial e^2} > 0$$

2. The effect of effort on the mean distribution of returns is such that utility is strictly increasing and concave in effort, the effect on the variance is strictly decreasing and convex for all $\left(0, \overline{b}\right)$

$$\frac{\partial \mu}{\partial e} > 0, \frac{\partial^2 \mu}{\partial e^2} \leq 0$$
$$\frac{\partial \sigma^2}{\partial e} < 0, \frac{\partial^2 \sigma^2}{\partial e^2} > 0$$

3. Moreover, I require the effect of effort on the variance to be bounded relative to the effect on mean returns.

$$\mu_e > |\sigma_e^2|$$
$$|\mu_{ee}| > |\sigma_{ee}^2|$$

Note how the first set of conditions implies that

$$\frac{\partial \mu_{X(e)}}{\partial e} - \frac{b}{2} \frac{\partial \sigma_{X(e)}^2}{\partial e} > 0$$
$$\frac{\partial^2 \mu_{X(e)}}{\partial e^2} - \frac{b}{2} \frac{\partial^2 \sigma_{X(e)}^2}{\partial e^2} < 0$$

Lemma 16. *Under Assumptions 1 and 2, the solution to the agent's problem, $e^* (\beta)$ is*

- *unique*
- *continuous*

[2] See Appendix

- *decreasing*
- *concave*

Proof. Note how $e^*(\alpha, \beta)$ solves

$$\max_e -\alpha + (1-\beta)\,\mu(e) - \frac{b}{2}\,(1-\beta)^2\,\sigma^2(e) - c(e)$$

The First Order Conditions of these problems amount to

$$(1-\beta)\,\mu_e(e) - \frac{b}{2}\,(1-\beta)^2\,\sigma_e^2(e) - c_e(e) = 0$$

Uniqueness. It follows from Assumption 2 that this derivative is a strictly decreasing function on $E = [\underline{e}, \overline{e}]$. To see this note that by, Assumption 2, concavity is guaranteed for any risk tolerance parameter in the interval $\left(0, \overline{b}\right)$. This implies that the assumptions for derivatives will also be true for

$$\mu_e - \frac{b}{2}\,(1-\beta)\,\sigma_e^2$$

for β between 0 and 1. This and the fact that cost is convex imply uniquness of the solution.

Continuity. for β in $[0, 1)$ follows from the implicit function theorem. When $\beta = 1$ the optimal effort is $\underline{e}$.

Monotonicity and *Concavity.* Applying the implicit function theorem to the FOCs we obtain that

$$\frac{\partial e^*}{\partial \beta} = \frac{\mu_e - b\,(1-\beta)\,\sigma_e^2}{(1-\beta)\,\mu_{ee} - \frac{b}{2}\,(1-\beta)^2\,\sigma_{ee}^2 - c_{ee}} < 0$$

$$\frac{\partial^2 e^*}{\partial \beta^2} = \frac{\frac{b^2}{2}\,(1-\beta)^2\,\sigma_e^2\sigma_{ee}^2 + \mu_{ee}\mu_e - b\,(1-\beta)\,\sigma_{ee}^2\mu_e - bc_{ee}\sigma_e^2}{\left((1-\beta)\,\mu_{ee} - \frac{b}{2}\,(1-\beta)^2\,\sigma_{ee}^2 - c_{ee}\right)^2} < 0$$

□

Corollary 17. *In equilibrium, principals correctly conjecture the equilibrium effort of agents.*

$$\begin{aligned}\gamma_i^*\left(e_i|C_i\right) &= 1, e_i \geq e_i^*\left(C_i\right)\\ &= 0, e_i < e_i^*\left(C_i\right)\end{aligned}$$

This follows immediately from the definition of equilibrium and Lemma 16. By doing this, I am removing one potential layer of randomization. The objective function of a principal at the first stage will be.

$$U_{p_i}\left(C,\theta,\gamma^*\right) = \theta \cdot \left(\mu\left(C, e^*\left(C\right)\right)|1\right) - \frac{b}{2}\theta_R'\mathbf{\Omega}\left(C, e^*\left(C\right)\right)\theta_R$$

Lemma 18. *The best response of a principal is single valued.*

Proof. The proof amounts to showing that every principal is maximizing a strictly concave function on a convex set. Once the reaction function of the agent in incorporated in his individual rationality constraint, the resulting set need not be convex. However, by substituting the IR constraint in the objective function, I define a simple maximization problem on β in $[0,1]$. By Assumption 2.3, the resulting maximand is a strictly concave function.

Let's start by studying the sign of the second derivative with respect to β of the objective function of a generic principal i. To make the proof more readable I abuse notation and suppress all the i subscripts.

$$\begin{aligned}
&2e_\beta\left(\mu_e - 2\left(\frac{b}{N}-\frac{b}{2N^2}\right)\beta\sigma_e^2 - 2\left(\frac{b}{N}-\frac{b}{N^2}\right)\left(\sum_{j\neq i}\rho_{ij}\beta_j\sigma_j\right)\sigma_e\right)+\\
&e_\beta^2\left(\beta\mu_{ee} - \left(\frac{b}{N}-\frac{b}{2N^2}\right)\beta^2\sigma_{ee}^2 - \left(\frac{b}{N}-\frac{b}{N^2}\right)\left(\sum_{j\neq i}\rho_{ij}\beta_j\sigma_j\right)\beta\sigma_{ee}\right)+\\
&e_{\beta\beta}\left(\beta\mu_e - \left(\frac{b}{N}-\frac{b}{2N^2}\right)\beta^2\sigma_e^2 - \left(\frac{b}{N}-\frac{b}{N^2}\right)\left(\sum_{j\neq i}\rho_{ij}\beta_j\sigma_j\right)\beta\sigma_e\right)+\\
&-2\left(\frac{b}{N}-\frac{b}{2N^2}\right)\sigma^2
\end{aligned}$$

If $\left(\sum_{j\neq i}\rho_{ij}\beta_j\sigma_j\right)$ is positive, it follows from Assumptions 2 and 1 that the addend on each line is negative. If the coefficient is negative we can observe that $\left(\frac{b}{N}-\frac{b}{N^2}\right)\left(\sum_{j\neq i}\rho_{ij}\beta_j\sigma_j\right)$ is smaller than 1, by Assumption 1. This together with Assumption 2.3 implies that all the addends are negative as desired.

Let's now study the second derivative of the IR constraint reducing it to

$$\begin{aligned} e_\beta^2\left[(1-\beta)\,\mu_{ee}-\frac{b}{2}(1-\beta)^2\sigma_{ee}^2-c_{ee}\right]+ \\ e_{\beta\beta}\left[(1-\beta)\,\mu_{e}-\frac{b}{2}(1-\beta)^2\sigma_{e}^2-c_{e}\right]+ \\ -2e_\beta\left[\mu_e-b\,(1-\beta)\,\sigma_e^2\right] \end{aligned}$$

By Lemma 16

$$e_\beta=\frac{\mu_e-b\,(1-\beta)\,\sigma_e^2}{(1-\beta)\,\mu_{ee}-\frac{b}{2}(1-\beta)^2\sigma_{ee}^2-c_{ee}}$$

Substituting the first term with $e_\beta\left[\mu_e-b\,(1-\beta)\,\sigma_e^2\right]$

$$\begin{aligned} e_{\beta\beta}\left[(1-\beta)\,\mu_{e}-\frac{b}{2}(1-\beta)^2\sigma_{e}^2-c_{e}\right]+ \\ -e_\beta\left[\mu_e-b\,(1-\beta)\,\sigma_e^2\right] \end{aligned}$$

The last term is problematic because it is positive.

Substituting for α in the objective function, we have $V(\beta)$ defined on $[0,1]$ The derivatives of this function will be given by

$$\frac{\partial^k V}{\partial\beta^k}=\frac{\partial^k U}{\partial\beta^k}+\frac{\partial^k IR}{\partial\beta^k}$$

Adding the last term of $\frac{\partial^2 IR}{\partial\beta^2}$ to the next to last term of $\frac{\partial^2 U}{\partial\beta^2}$

$$e_\beta \left(2 \left[\mu_e - 2 \left(\frac{b}{N} - \frac{b}{2N^2} \right) \beta \sigma_e^2 - 2 \left(\frac{b}{N} - \frac{b}{N^2} \right) \left(\sum_{j \neq i} \rho_{ij} \beta_j \sigma_j \right) \sigma_e \right] - \left[\mu_e - b \left(1 - \beta\right) \sigma_e^2 \right] \right) =$$

$$e_\beta \left(\mu_e - 4 \left[\left(\frac{1}{N} - \frac{1}{2N^2} \right) \beta + (1 - \beta) \right] b \sigma_e^2 - 2 \left(\frac{b}{N} - \frac{b}{N^2} \right) \left(\sum_{j \neq i} \rho_{ij} \beta_j \sigma_j \right) \sigma_e \right)$$

The term multiplying e_β needs to be positive. Inspecting the expression, it is clear that the "worse" possible scenario is when $\beta = 0$ and $\left(\sum_{j \neq i} \rho_{ij} \beta_j \sigma_j \right)$ takes the lowest possible value (the largest negative value). As noted above $\left(\frac{b}{N} - \frac{b}{N^2} \right) \left(\sum_{j \neq i} \rho_{ij} \beta_j \sigma_j \right)$ is smaller than 1.

A sufficient condition for this expression to be always positive is

$$\mu_e + b \sigma_e^2 + b \sigma_e > 0$$

But this is implied by Assumption 2.3 . The second derivative of V is hence negative, which concludes the proof. □

2.4.2 Existence

Theorem 19. *If the mean-variance preferences are monotonic for an asset market economy characterized by the mean vector μ and variance-covariance matrix Ω then there exists an equilibrium in the CAPM contracting economy.*

Proof. Because the Markowitz preferences are not expected utility -and they also fail to satisfy the 'betweenness axiom' (see Dekel, Safra, Segal (1991) -the best response correspondences of individuals would not be convexified by allowing for lotteries. This makes using the fixed point theorems by Glicksberg or Kakuthani impossible. I will instead show that the game can be reduced to a one-shot game in which principals have a *continuous* best response function *function* and apply Brouwer's fixed point theorem.

Principals' Turn: Asset Market. Principal utility will be given by the CAPM analytical solution.

$$U_{p_i} = q_i - \frac{\sum_{i \in N} q_i}{N} + \frac{\sum_{i \in N} \mu_i}{N} - \frac{b}{2} \frac{\mathbf{1'}\Omega\mathbf{1}}{N^2}$$

As noted earlier prices and returns q, μ, Ω are all continuous functions of contracts C and efforts e .

Agent's Turn: Effort Choice. By Lemma 16 effort levels e are continuous functions of contracts C, hence the final utility of a principal can be described as a continuous function of contracts C .

Principals' Turn: Contract Design. By Lemma 18 the Principals' problem is equivalent to a strictly concave optimization so that their best response function is always unique. By the maximum theorem it is also continuous.

The strategy space $\mathcal{C}$ is a rectangle, which is a convex, compact subset of $\mathbb{R}^{2N}$.

This satisfies the hypotheses of Brower fixed point theorem. There exist a fix point C^* which determines uniquely equilibrium efforts, beliefs, prices and portfolios. $C^*, e^*, \gamma^*, \theta^*, q^*$ form an equilibrium by construction.

□

2.5 The Insurance Effect of Markets - Moral Hazard

Consider the following special case. Individuals are identical in the sense that they all have Markowitz type of Mean-Variance preferences, and they all have the same risk tolerance coefficient. Technologies have different variances (denoted by σ_i^2 and different correlation coefficients ρ_{ij}). The mean returns of firms are determined by the effort of agents e_i, specifically $\mu_i(e_i) = e_i$. Cost is quadratic $c(e_i) = \frac{c}{2}e_i^2$

2.5.1 First Best Equilibrium

For the purpose of having a benchmark for optimal risk sharing and optimal effort, let's have a look at the first best (observable action) case.

2.5.1.1 No Markets

A Principal and an Agent agree on an action and a (random) payment.

$$\max_{\alpha,\beta,e} \quad \alpha + \beta e - \frac{b}{2}\beta^2\sigma^2$$
$$\text{such that} \quad -\alpha + (1-\beta)\,e - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}e^2 \geq \overline{u} \tag{IR}$$

The optimal solution, action and contract is

$$e^* = \frac{1}{c}$$
$$(\alpha^*, \beta^*) = \left(-\frac{b}{8}\sigma^2 + \overline{u}, \frac{1}{2}\right)$$

Proof. Substitute for α in the objective function to obtain

$$\max_{\beta,e} \quad U(\beta, e) = (1-\beta)\,e - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}e_i^2 - \overline{u} + \beta e - \frac{b}{2}\beta^2\sigma^2$$

which yields the first order conditions

$$\frac{\partial U}{\partial \beta} = a\,(1-\beta)\,\sigma^2 - a\beta\sigma^2 = 0$$
$$\frac{\partial U}{\partial e} = 1 - ce = 0$$

Which give the solution. □

2.5.1.2 Financial Markets

Now I consider the case of many firms. To do this I solve the optimization problem of an arbitrary principal, who has now access to a financial market. The form of the utility function follows from similar considerations as in the hidden type case seen in Chapter 1..

Lemma 20. *Principal i behaves as if his utility function were*

$$
\begin{aligned}
&\alpha_i + \beta_i e_i + \\
&+ \left(\frac{b}{2N^2} - \frac{b}{N} \right) \beta_i^2 \sigma_i^2 + \\
&+ \left(\frac{b}{N^2} - \frac{b}{N} \right) \sum_{j \neq i} \rho_{ij} \beta_i \beta_j \sigma_i \sigma_j
\end{aligned}
$$

Proof. Using the CAPM pricing formula, the riskless share of principal from firm i

$$
\begin{aligned}
&\alpha_i + \beta_i \mu_i - \frac{b}{N} \left(\beta_i^2 \sigma_i^2 + \sum_{l \neq i} \rho_{il} \beta_i \beta_l \sigma_i \sigma_l \right) + \\
&- \frac{\sum_{j \in N} \alpha_j + \beta_j \mu_j}{N} + \frac{b}{N^2} \sum_{j \in N} \left(\beta_j^2 \sigma_j^2 + \sum_{k \neq j} \rho_{jk} \beta_j \beta_k \sigma_j \sigma_k \right)
\end{aligned}
$$

Each principal holds a fraction of the aggregate portfolio, in particular he holds a random variable from which he gets utility

$$
\frac{\sum_{j \in N} \alpha_j + \beta_j \mu_j}{N} - \frac{b}{N^2} \left(\sum_{j \in N} \beta_j^2 \sigma_j^2 + 2 \sum_{k \neq j} \rho_{jk} \beta_j \beta_k \sigma_j \sigma_k \right)
$$

Because utility is linear in mean, and the riskless asset has variance zero, adding the two and simplifying obtains the claim.

□

$$
\begin{aligned}
\max \quad & U_i^{MKT}(\alpha_i, \beta_i) = \\
& \alpha_i + \beta_i e_i + \\
& + \left(\frac{b}{2N^2} - \frac{b}{N}\right)\beta_i^2 \sigma_i^2 + \\
& + \left(\frac{b}{N^2} - \frac{b}{N}\right)\sum_{j \neq i} \rho_{ij}\beta_i\beta_j\sigma_i\sigma_j \\
\text{s.t.} \quad & -\alpha_i + (1-\beta_i)\, e_i - \frac{b}{2}(1-\beta_i)^2 \sigma^2 - \frac{c}{2} e_i^2 \geq \overline{u} \qquad \text{(IR)}
\end{aligned}
$$

The optimal solution, action and contract is

$$
\begin{aligned}
e_i^{MKT} =& \frac{1}{c} \\
\beta_i^{MKT} =& \frac{\sigma_i^2 - \left(\frac{N-1}{N^2}\right)\sum_{j\neq i}\rho_{ij}\beta_j\sigma_i\sigma_j}{\sigma_i^2 + \left(\frac{2N-1}{N^2}\right)\sigma_i^2}
\end{aligned}
$$

Proof. Substitute for α_i in the objective function to obtain

$$
\begin{aligned}
\max_{\beta_i, e_i} U_i^{MKT} & (1-\beta_i)\, e_i - \frac{b}{2}(1-\beta_i)^2\sigma^2 - \frac{c}{2}e_i^2 - \overline{u} + \beta_i e_i + \\
& + \left(\frac{b}{2N^2} - \frac{b}{N}\right)\beta_i^2\sigma_i^2 + \\
& + \left(\frac{b}{N^2} - \frac{b}{N}\right)\sum_{j\neq i}\rho_{ij}\beta_j\sigma_i\sigma_j
\end{aligned}
$$

which yields the first order conditions

$$
\begin{aligned}
\frac{\partial U_i^{MKT}}{\partial \beta} &= a\,(1-\beta)\,\sigma_i^2 - \left(\frac{b}{N^2} - \frac{a2}{N}\right)\beta_i\sigma_i^2 - \left(\frac{b}{N^2} - \frac{b}{N}\right)\sum_{j\neq i}\rho_{ij}\beta_i\beta_j\sigma_j = 0 \\
\frac{\partial U_i^{MKT}}{\partial e} &= 1 - ce = 0
\end{aligned}
$$

Which give the solution. □

The key observation is that the optimal action stays the same, but the first best contract changes.

As noted in the case of hidden type economies, the effect of markets is that principal acts as if they were less risk averse. This changes the optimal risk sharing.

To better understand the effects of markets, let's focus on the special case in which all technologies are identical and so are their correlation coefficients.

$$\sigma_i = \sigma_j = \sigma, \forall i, j$$
$$\rho_{ij} = \rho, \forall i, j$$

The symmetric solution in this case is

$$\beta = \frac{1 - \frac{(N-1)^2}{N^2} \rho\beta}{1 + \frac{2N-1}{N^2}}$$

So that in equilibrium

$$\beta^{MKT} = \frac{N^2}{N^2 + 2N - 1 + \rho\left(N^2 - 2N + 1\right)}$$

Note how, unless there is perfect correlation, in equilibrium, a principal always take more risk than in the no the market case $\left(\beta_i = \frac{1}{2}\right)$. In fact the equilibrium contracts will be identical to the no market case only if technologies are perfectly correlated $(\rho = 1)$.

$$\forall \rho < 1,$$
$$\beta^{MKT} < \frac{N^2}{N^2 + 2N - 1 + N^2 - 2N + 1} = \frac{1}{2}$$

2.5.2 Second Best Equilibrium

2.5.2.1 No Markets

Let's now turn to the more interesting case of unobservable actions. How do markets affect the equilibrium actions and returns? Here the decisions on risk sharing and effort are interdependent.

Again let's first look at a firm "in isolation"

$$\max_{\alpha,\beta,e} \ \alpha + \beta e - \frac{b}{2}\beta^2\sigma^2$$

$$\text{such that} \quad -\alpha + (1-\beta)e - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}e^2 \geq \overline{u} \qquad \text{(IR)}$$

$$e \in arg\max_{\tilde{e}} \quad -\alpha + (1-\beta)\tilde{e} - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}\tilde{e}^2 \qquad \text{(IC)}$$

To simplify the problem let's start by solving the problem of an agent facing a given contract (α, β) .

$$\max_{\tilde{e}} \quad -\alpha + (1-\beta)\tilde{e} - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}\tilde{e}^2$$

$$\max_{e} \quad -\alpha + (1-\beta)e - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{c}{2}e^2$$

Because Individual Rationality can be optimally attained with the transfer α, the optimal action e can be obtained from the first order condition of the agent problem.

$$(1-\beta) - ce = o$$

Plugging $e^* = \frac{1-\beta}{c}$ back into the Principal's objective function and the agent's IR constraint yields the following problem

$$\max_{\alpha,\beta} \ \alpha + \beta\left(\frac{1-\beta}{c}\right) - \frac{b}{2}\beta^2\sigma^2$$

$$\text{such that} \quad -\alpha + \frac{(1-\beta)^2}{c} - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{(1-\beta)^2}{2c} \geq \overline{u} \qquad \text{(IR)}$$

The solution to this problem is

$$\beta = \frac{b\sigma^2}{2b\sigma^2 + \frac{1}{c}}$$
$$e = \frac{bc\sigma^2 + \frac{1}{c}}{2b\sigma^2 + 1}$$

Proof. Again substituting α results in

$$\max_{\alpha,\beta} \beta\left(\frac{1-\beta}{c}\right) - \frac{b}{2}\beta^2\sigma^2 + \frac{(1-\beta)^2}{c} - \frac{b}{2}(1-\beta)^2\sigma^2 - \frac{(1-\beta)^2}{2c} - \overline{u}$$

Differentiating with respect to β gives the first order condition

$$-2b\beta\sigma^2 + b\sigma^2 - \frac{\beta}{c} = 0$$

The solution follows immediately from this and the fact that $e = \frac{1-\beta}{c}$ □

Note how $\beta < \frac{1}{2}$. Risk sharing is distorted to give the proper incentive to the agent. Note that $\beta > 0$, so that $e < \frac{1}{c}$. Because there is a trade-off between incentivizing the agent and optimally sharing risk between two risk averse individuals, the first best cannot be achieved.

I am now going to analyze how markets affect this tradeoff.

2.5.2.2 Financial Markets

Consider the market with many principals from the previous section, except now the action in each firm is unobservable.

$$
\begin{aligned}
\max \quad & U_i^{MKT}(\alpha_i,\beta_i) = \\
& \alpha_i + \beta_i e_i + \\
& + \left(\frac{b}{2N^2} - \frac{b}{N}\right)\beta_i^2\sigma_i^2 + \\
& + \left(\frac{b}{N^2} - \frac{b}{N}\right)\sum_{j\neq i}\rho_{ij}\beta_j\sigma_i\sigma_j \\
\text{s.t.} \quad & -\alpha_i + (1-\beta_i)\,e_i - \frac{b}{2}(1-\beta_i)^2\sigma^2 - \frac{c}{2}e_i^2 \geq \overline{u} \qquad \text{(IR)} \\
& e_i \in arg\max_{\tilde{e}} \quad -\alpha + (1-\beta)\tilde{e} - \frac{b}{2} + (1-\beta)^2\sigma^2 - \frac{c}{2}\tilde{e}^2 \qquad \text{(IC)}
\end{aligned}
$$

The solution is now

$$
\beta_i^{MKT} = \frac{b\left(\sigma_i^2 - \left(\frac{N-1}{N^2}\right)\sum_{j\neq i}\rho_{ij}\beta_j\sigma_i\sigma_j\right)}{b\left(\sigma_i^2 + \left(\frac{2N-1}{N^2}\right)\sigma_i^2\right) + \frac{1}{c}}
$$

In the symmetric case

$$
\beta^{MKT} = \frac{b\sigma^2 N^2}{b\sigma^2\left(N^2 + 2N - 1 + \rho\,(N-1)^2\right) + \frac{N^2}{c}}
$$

The solution differs from the first best case because of the $\frac{N^2}{c}$ term added to the denominator. So the equilibrium securities will be less risky than in the first best case. Moreover as observed for the non market case, the equilibrium effort is lower than at the optimum. Because β^{MKT} is again increasing in N, the equilibrium effort and hence returns are decreasing N.

In a fully symmetric case this result holds for every firm. This will not be the case if we drop the assumption of symmetry. However, the result will still hold in the aggregate in an economy where firms marginal distributions are identical, but not their conditionals. In other words I allow for different ρ_{ij} coefficients in the variance covariance matrix. This means that while every firm is identical when there are no markets, they are *ex-ante* different in terms of diversification opportunities they face (and offer). This can be interpreted as the existence of different "sectors".

Proposition 21. *Consider an economy with identical marginal distributions characterized by mean e_i, variance σ^2, cost of effort $\frac{c}{2}e^2$. The aggregate output with markets is lower than the aggregate output without markets.*

$$\Sigma_{i=1}^{N} e_i^* \geq \Sigma_{i=1}^{N} e_i^M$$

with the inequality holding strictly unless $\rho_{ij} = 1, \forall i, j$

Proof. Claim 1.

$$\Sigma_{i=1}^{N} e_i^* \geq \Sigma_{i=1}^{N} e_i^M \iff \Sigma_{i=1}^{N} \beta_i^* \leq \Sigma_{i=1}^{N} \beta_i^M$$

To see this note that

$$\begin{aligned} \Sigma_{i=1}^{N} e_i^* \geq \Sigma_{i=1}^{N} e_i^M &\iff \\ \Sigma_{i=1}^{N} \frac{(1-\beta_i^*)}{c} \geq \Sigma_{i=1}^{N} \frac{\left(1-\beta_i^M\right)}{c} &\iff \\ \Sigma_{i=1}^{N} (1-\beta_i^*) \geq \Sigma_{i=1}^{N} \left(1-\beta_i^M\right) &\iff \\ \Sigma_{i=1}^{N} \beta_i^* \leq \Sigma_{i=1}^{N} \beta_i^M & \end{aligned}$$

After estabilishing the above inequality. Observe that under the assumptions

$$\begin{aligned} \beta_i^* &= \frac{1}{2 + \frac{1}{cb\sigma^2}} \\ \beta_i^M &= \frac{1 - \frac{N-1}{N^2} \Sigma_{j \neq i} \rho_{ij} \beta_j^M}{1 + \frac{1}{cb\sigma^2} + \frac{2n-1}{N^2}} \end{aligned}$$

I need to prove that

$$\Sigma_{i=1}^{N} \frac{1}{2 + \frac{1}{cb\sigma^2}} \leq \Sigma_{i=1}^{N} \frac{1 - \frac{N-1}{N^2} \Sigma_{j \neq i} \rho_{ij} \beta_j^M}{1 + \frac{1}{cb\sigma^2} + \frac{2N-1}{N^2}}$$

Claim 2.

$$\Sigma_{i=1}^{N}\beta_i^* \leq \Sigma_{i=1}^{N}\beta_i^M \iff$$
$$\Sigma_{i=1}^{N}\beta_i^* \geq \Sigma_{i=1}^{N}\Sigma_{j\neq i}\frac{1}{N-1}\rho_{ij}\beta_j^M$$

Note that the denominator of β_i^M can be rewritten as $2+\frac{1}{cb\sigma^2}-\frac{(N-1)^2}{N^2}$. The first inequality amounts to

$$\frac{N}{2+\frac{1}{cb\sigma^2}} \leq \frac{N}{2+\frac{1}{cb\sigma^2}-\left(\frac{N-1}{N}\right)^2} - \frac{\frac{N-1}{N^2}\Sigma_{i=1}^{N}\Sigma_{j\neq i}\rho_{ij}\beta_j^M}{2+\frac{1}{cb\sigma^2}-\left(\frac{N-1}{N}\right)^2} \iff$$
$$\frac{N\left(N-1\right)}{\left(2+\frac{1}{cb\sigma^2}-\left(\frac{N-1}{N}\right)^2\right)\left(2+\frac{1}{cb\sigma^2}\right)} - \frac{\frac{N-1}{N^2}\Sigma_{i=1}^{N}\Sigma_{j\neq i}\rho_{ij}\beta_j^M}{2+\frac{1}{cb\sigma^2}-\left(\frac{N-1}{N}\right)^2} \geq 0 \iff$$

Simplifying the last inequality

$$N\left(N-1\right)-\left(2+\frac{1}{cb\sigma^2}\right)\Sigma_{i=1}^{N}\Sigma_{j\neq i}\rho_{ij}\beta_j^M$$

Claim 2 immediately follows.

To conclude the proof, suppose by means of contradiction that the proposition did not hold, by Claim 1, $\Sigma_{i=1}^{N}\beta_i^* > \Sigma_{i=1}^{N}\beta_i^M$. By claim 2 this implies that $\Sigma_{i=1}^{N}\beta_i^* < \Sigma_{i=1}^{N}\Sigma_{j\neq i}\frac{1}{N-1}\rho_{ij}\beta_j^M$

Then it should be that

$$\Sigma_{i=1}^{N}\Sigma_{j\neq i}\frac{1}{N-1}\rho_{ij}\beta_j^M > \Sigma_{i=1}^{N}\beta_i^M$$

This can be rewritten as

$$\Sigma_{i=1}^{N}\left(1-\Sigma_{j\neq i}\frac{1}{N-1}\rho_{ij}\right)\beta_i^M < 0$$

which is impossible since $\rho_{ij} < 1, \forall i, j$

$\square$

Financial Markets change the terms of risk sharing inside firms. Unlike the first best case, this has an effect on the equilibrium action, because it is exactly risk providing incentives for the agent to exert some positive effort. As the relative terms of risk sharing change, we move further away from the case in which the agent is risk neutral and optimal effort is obtained.

In this case, it seems markets reward firms for variance against returns, but that is not exactly the case. Markets reward firms for providing opportunities for hedging and diversification in specific states of the world.

To get a better intuition for this result, suppose there is a state space $S = \{SUN, RAIN\}$

Consider a principal owning the an ice cream factory. Her firm returns can take two values, $\overline{x} + e > \underline{x} + e$ respectively, in state SUN and in state $RAIN$. The security she is selling to the market will return $\alpha + \beta\overline{x} + \beta e$ and $\alpha + \beta\underline{x} + \beta e$. When designing an incentive contract she is playing with 3 variables α, β, e. When markets are not available, the underlying state representation does not matter, the solution α^*, β^*, e^* will be some tradeoff between risk sharing and incentives (increasing β decreases e and viceversa . However consider now what happens if there is another firm, whose securities return $\alpha' + \beta'\overline{x} + \beta e'$ when the state is $RAIN$ and $\alpha' + \beta'\underline{x} + \beta e'$ when the state is SUN. Because both principals are risk averse and there is now more return available in state $RAIN$, the owner of the ice cream factory will want to design a security generating more return in state SUN. Let's consider what happens to returns in each state when she increases the

3 components of her contract

$$\Delta\alpha$$
$$RAIN : \Delta\alpha$$
$$SUN : \Delta\alpha$$
$$\Delta\beta$$
$$RAIN : \Delta\beta\underline{x} + \Delta\beta e$$
$$SUN : \Delta\beta\overline{x} + \Delta\beta e$$
$$\Delta e$$
$$RAIN : \Delta e\beta$$
$$SUN : \Delta e\beta$$

Note how β is the only component of the contract which has a different effect in different states, and specifically it provides more exactly where needed: in state SUN. On the other hand, the agent, who does not access markets, values equally returns in equally likely states (risk aversion would make returns in $RAIN$ more valuable, if anything). Because markets change the relative value of returns in difference states for principals, the solution under markets will exhibit different risk sharing $\beta^{MKT} > \beta^*$ and consequently lower equilibrium effort $e^{MKT} < e^*$.

Understanding this effect allows one to construct an example in which contracts and firms' output are affected in the opposite way as described above.

Consider an economy with two types of firms (50 of each kind). The low variance firms have variance .25, the high variance firms have variance 1. They are perfectly positively correlated $\rho_{ij} = 1, \forall i, j$. Risk aversion is .2, cost of effort is .5 . The average output without markets is given by $\frac{106}{56}$. Numerical evaluation yields an average output 2.11 .

In chapter 3 I discuss extensively how this particular distortion of risk sharing can occur even in a simple first best setting. The key intuition is that the low variance firm has a comparative advantage in providing returns in the low state, where the other firms are obtaining an even lower performance.

2.6 Conclusion

This chapter integrates a model of principal-agent interaction with asset markets. Each pair/firm produces random returns, whose distribution depends on agents' efforts. Every Principal offers a contract to the Agent he is matched with, and the Agents choose their costly action. What marks the difference from the standard contracting model is that Principals have access to an asset market on which they trade their shares of returns. This assumption is meant to capture the limited degree of access to financial markets, available to the average worker, who cannot entirely insure his labor risk.

I present a unified framework for which I define a notion of equilibrium and prove its existence and uniqueness. Under standard assumptions of contract theory, I study the interactions of financial markets on contracts.

On one hand, Moral Hazard inside firms induces suboptimal aggregate risk in an economy. On the other hand, introducing markets for principals, has ambiguous effects. If the marginal distributions of returns of firms are identical, markets will induce lower production levels. I construct an example, where asymmetry of marginal distributions and high degree of correlations induces riskier compensation packages in certain sectors and a positive effect on aggregate output.

Interesting directions for the future include making welfare comparisons with the case where workers can (at least partially) access financial markets, and making the decision of entering markets endogenous in a non trivial way (making access costly).

2.7 Appendix

2.7.1 Preferences

Markowitz preferences cannot be represented by expected utility. That is they cannot be represented by a linear functional in the space of mixtures. As a result they do not satisfy the independence axiom or -equivalently, but more significantly in this context- they are not "dynamically consistent". Furthermore they pose a threat to equilibrium existence.

Consider two random variables X_1 and X_2. We will have that

$$U(X_1) = E(X_1) - \frac{b}{2}Var(X_1)$$
$$U(X_2) = E(X_2) - \frac{b}{2}Var(X_2)$$

Now consider a mixture with probability p of X_1 and X_2. which we will call Y If individuals' preferences were "linear", we would have that.

$$U_{EU}(Y) = pU(X_1) + (1-p)U(X_2)$$

Note that if a "linear" decision maker is indifferent between two random variables, he will also be indifferent between any mixture of them. Now compare with a Markowitz Individual

$$U(Y) > E(Y) - \frac{a}{2}Var(Y) =$$
$$pU(X_1) + (1-p)U(X_2) - \frac{a}{2}\left(pE\left(X_1 - E(Y)\right)^2 + (1-p)\left(X_1 - E(Y)\right)^2\right)$$

He would choose any of the two RV he is indifferent to above *any* mixture of the two. This implies that his best response correspondence is not convex in the space of mixtures, unless it is a singleton, which makes it impossible to use Kakutani's fixed point Theorem.[3] To address this issue, I make sure that the model of Chapter 2 and 3 admits an equilibrium in pure strategies.

As noted in the literature, it is possible to construct sequences of choices in between which some uncertainty is resolved, such that the agent makes plans he would not stick to. In this paper, this would lead to contracts which are not efficient as soon as they are signed even in a first best setting with symmetric information.

Here is an example of this issue. Consider a standard principal agent problem with two types of agents. The type of the agent is known as soon as principals and agents are matched

If Mean-Variance Preferences had a linear representation, the principal's problem would look like this

[3] This also implies that these preferences fail to satisfy Between-ness as in Dekel (1986). This renders also other methods for proving existence ineffective

$$\max_{(\alpha_1,\beta_1,\alpha_2,\beta_2)} \rho\left[\alpha_1 + \beta_1\mu_1 - \frac{a}{2}\beta_1^2\sigma_1^2\right] + \\ (1-\rho)\left[\alpha_2 + \beta_2\mu_2 - \frac{a}{2}\beta_2^2\sigma_2^2\right] \\ s.t.\ \ IR_t : -\alpha_t + (1-\beta_t)\mu_t - \frac{a}{2}(1-\beta_t)^2\sigma_t^2 \geq \overline{u}$$

However the actual problem will look different because *the variance of a mixture is not the convex combination of variances*:

$$\max_{(\alpha_1,\beta_1,\alpha_2,\beta_2)} \rho\left[\alpha_1 + \beta_1\mu_1 - \frac{a}{2}\left(\beta_1^2\sigma_1^2 + (\alpha_1 + \beta_1\mu_1)^2\right)\right] + \\ (1-\rho)\left[\alpha_2 + \beta_2\mu_2 - \frac{a}{2}\left(\beta_2^2\sigma_2^2 + (\alpha_2 + \beta_2\mu_2)^2\right)\right] + \\ \frac{a}{2}\left[\rho\left(\alpha_1 + \beta_1\mu_1\right) + (1-\rho)\left(\alpha_2 + \beta_2\mu_2\right)\right]^2 \\ s.t.\ \ IR_t : -\alpha_t + (1-\beta_t)\mu_t - \frac{a}{2}(1-\beta_t)^2\sigma_t^2 \geq \overline{u}$$

To see the dynamic inconsistency, let's turn this into a numeric example.

$$\mu_1 = 3, \sigma_1^2 = 1 \\ \mu_2 = 2, \sigma_2^2 = \frac{1}{2} \\ \rho = \frac{1}{2}$$

The random portion of optimal contracts in the first case are given by $\beta_1 = \frac{1}{2}, \beta_2 = \frac{1}{2}$, which are exactly the same that the principal would offer to each type if he knew which type is the agent he was dealing with.

The Markowitz principal's solution is instead $\beta_1 = .467, \beta_2 = .539$, to which he would prefer $\frac{1}{2}, \frac{1}{2}$ as soon as uncertainty is resolved. Note that this Pareto improving renegotiation is not induced by voluntary information revelation from the agent, but simply by uncertainty resolving. Similarly, but more complicated, examples could be constructed for the model in this paper. [4]

[4] To read more on the topic of dynamic non-expected utility preferences, Machina (1989)

Chapter 3

A Few Observations on Security Design with Symmetric Information

3.1 Introduction

In this chapter I study symmetric information economies. I am going to discuss several issues arising from the different risk sharing possibilities introduced by financial markets. I will study economies without any informational frictions, in which inefficiencies will arise from strategic behavior.

In Section 3.2 I study the extent to which the results obtained under complete information in Chapter 1 and 2 can be generalized. It turns out that some of the assumptions made in those chapters are crucial to obtain an insurance effect. In particular the role of uncorrelated returns and the symmetry of firms will become clearer.

In Section 3.3 I will analyze the behavior of principals in designing securities, when they do not act fully competitively. I will study how the number of traders affects efficiency of securities issued.

In Section 3.4 I look at a large market, I consider N identical productive units, but I depart from the assumptions I made so far, by allowing one firm to be a merger of several productive units. I will describe the effects of market power on both large and small firms.

3.2 Markets Make Contracts Riskier

In Chapters 1 and 2 first best risk sharing follows a simple pattern. More diversification opportunities for principals imply less risky contracts for agents who do not access markets. However I will show here that need not be the case under more general assumptions. The insurance results of Chapter 1 were possible because of the independence assumption. In Chapter 2 they were possible because of the symmetry assumption (every firm had the same mean and the same variance). The following examples show that enough positive correlation among firms returns, and enough difference in the variance of returns will induce some firms to pay workers with more risky contracts than they would without markets. Surprisingly this can be the case even in a large economy.

3.2.1 A Small Market

Consider a model similar to that of Chapter 2. A firm constitutes of a principal and an agent, but no action is needed for production. Principals will only need to compensate the agent to the point in which their participation constraint is satisfied. This will be achieved with a contract which induces optimal risk sharing plus some transfer.

The problem of a Principal who does not have access to markets is given by

$$\max_{\alpha,\beta} \ \alpha + \beta\mu - \frac{b}{2}\beta^2\sigma^2,$$

$$\text{such that} \quad -\alpha + (1-\beta)\mu - \frac{b}{2}(1-\beta)^2\sigma^2 \geq \overline{u}. \tag{IR}$$

The optimal contract will induce symmetric risk sharing in equal parts. The only notable difference from the First Best section of Chapter 2 is the lack of action e and its cost. This does not affect the optimal β as the first order conditions are the same as in Chapter 2.

Let us first see what happens in a small market. Consider two firms. Both firms will have mean returns μ. One firm has variance $\overline{\sigma}^2$ and the other has variance $\underline{\sigma}^2$, with $\overline{\sigma}^2 > \underline{\sigma}^2$.

Proposition 22. *There are values of $\rho, \overline{\sigma}, \underline{\sigma}$, with $\rho \in (0,1], \overline{\sigma} > \underline{\sigma}$ defining two firms economies for which the low risk firm offers riskier contracts when markets are introduced.*

Proof. The problem of the Principal in the low risk firm.

$$\begin{aligned}
&\max \quad U^{MKT}\left(\underline{\alpha}, \underline{\beta}\right) \\
&\qquad =\underline{\alpha} + \underline{\beta}\mu - \frac{3}{8}b\underline{\beta}^2\underline{\sigma}^2 - \frac{1}{4}b\rho\underline{\beta}\overline{\beta}\underline{\sigma}\overline{\sigma}, \\
&\text{such that} \quad -\underline{\alpha} + \left(1-\underline{\beta}\right)\mu - \frac{b}{2}\left(1-\underline{\beta}\right)^2\underline{\sigma}^2 2 \geq \overline{u}. \qquad \text{(IR)}
\end{aligned}$$

And the problem of the principal in the high risk firm.

$$\begin{aligned}
&\max \quad U^{MKT}\left(\overline{\alpha}, \overline{\beta}\right) \\
&\qquad =\overline{\alpha} + \overline{\beta}\mu - \frac{3}{8}b\overline{\beta}^2\overline{\sigma}^2 - \frac{1}{4}b\rho\overline{\beta}\underline{\beta}\overline{\sigma}\underline{\sigma}, \\
&\text{such that} \quad -\overline{\alpha} + \left(1-\overline{\beta}\right)\mu - \frac{b}{2}\left(1-\overline{\beta}\right)^2\overline{\sigma}^2 2 \geq \overline{u}. \qquad \text{(IR)}
\end{aligned}$$

From Chapter 2 we can derive the optimal contracts given the other firm contracts.

$$\begin{aligned}
\underline{\beta} &= \frac{4\underline{\sigma} - \rho\overline{\beta}\overline{\sigma}}{7\underline{\sigma}}, \\
\overline{\beta} &= \frac{4\overline{\sigma} - \rho\underline{\beta}\underline{\sigma}}{7\overline{\sigma}}.
\end{aligned}$$

And solve for equilibrium contracts.

$$\begin{aligned}
\underline{\beta}^* &= \frac{28\underline{\sigma} - 4\rho\overline{\sigma}}{49\underline{\sigma} + \rho^2\underline{\sigma}}, \\
\overline{\beta}^* &= \frac{28\overline{\sigma} - 4\rho\underline{\sigma}}{49\overline{\sigma} + \rho^2\overline{\sigma}}.
\end{aligned}$$

Note how $\underline{\beta}^* < \frac{1}{2}$ if $\underline{\sigma} < \frac{2\rho}{7-\rho^2}\overline{\sigma}$. □

Similarly to Chapter 1 we have a case in which asset markets induce a riskier contract. One could be tempted to conclude that Markowitz preferences suffer from the same issues as quadratic expected utility does, as noted in Chapter 1 in markets with a small number of agents. It will turn out it is not the case, and in fact large markets can actually exacerbate this phenomenon.

3.2.2 A Large Market

Consider an economy with N firms. All firms have the same mean returns μ. A fraction γ of firms have variance $\overline{\sigma}^2$ and $1-\gamma$ have variance $\underline{\sigma}^2$, with $\overline{\sigma}^2 > \underline{\sigma}^2$.[1] In this simple setup it is easy to relax one or both the assumptions from the other chapters. ρ controls the degree of correlation across firms. The difference in the variances controls how similar are the marginal distribution of returns for each firm.

Proposition 23. *There are values of $\gamma, \rho, \overline{\sigma}, \underline{\sigma}$ with $\gamma \in (0,1), \rho \in (0,1], \overline{\sigma} > \underline{\sigma}$ such that in a large economy low risk firms offer riskier contracts when markets are introduced.*

Proof. A generic Principal's problem

$$\begin{aligned} \max \quad & U_i^{MKT}(\alpha_i, \beta_i) \\ & = \alpha_i + \beta_i \mu \\ & \quad + \left(\frac{b}{2N^2} - \frac{b}{N}\right) \beta_i^2 \sigma_i^2 \\ & \quad + \left(\frac{b}{N^2} - \frac{b}{N}\right) \sum_{j \neq i} \rho_{ij} \beta_i \beta_j \sigma_i \sigma_j, \\ \text{such that} \quad & -\alpha_i + (1-\beta_i)\mu - \frac{b}{2}(1-\beta_i)^2 \sigma^2 \geq \overline{u}. \end{aligned} \tag{IR}$$

From which we get the following condition for optimal contracts

$$\beta_i^{MKT} = \frac{\sigma_i^2 - \left(\frac{N-1}{N^2}\right) \sum_{j \neq i} \rho_{ij} \beta_j \sigma_i \sigma_j}{\sigma_i^2 + \left(\frac{2N-1}{N^2}\right) \sigma_i^2}, \forall i.$$

Looking for a symmetric equilibrium we can write the optimal contracts as functions of other contracts for both types of firms.

[1]Of course γ should be rational and γN should be always an integer. This can be easily obtained by defining an initial economy by two integers g and h with g firms with high variance and h firms with low variance, and then replicating them. However, in this section I am interested in the behavior of large economies in which the two approaches yield the same results.

$$\underline{\beta} = \frac{\underline{\sigma}^2 - \frac{N-1}{N^2}\left[(N-1)(1-\gamma)\rho\underline{\beta}\underline{\sigma}^2 + (N-1)\gamma\rho\overline{\beta}\overline{\sigma}\underline{\sigma}\right]}{\underline{\sigma}^2 + \frac{2N-1}{N^2}\underline{\sigma}^2},$$
$$\overline{\beta} = \frac{\overline{\sigma}^2 - \frac{N-1}{N^2}\left[(N-1)\gamma\rho\overline{\beta}\overline{\sigma}^2 + (N-1)(1-\gamma)\rho\underline{\beta}\underline{\sigma}\overline{\sigma}\right]}{\overline{\sigma}^2 + \frac{2N-1}{N^2}\overline{\sigma}^2}$$

I am considering large economies so I will look at the contracts as $N \to \infty$, [2]

$$\underline{\beta} = \frac{\underline{\sigma} - (1-\gamma)\rho\underline{\beta}\underline{\sigma} - \gamma\rho\overline{\beta}\overline{\sigma}}{\underline{\sigma}},$$
$$\overline{\beta} = \frac{\overline{\sigma} - \gamma\rho\overline{\beta}\overline{\sigma} - (1-\gamma)\rho\underline{\beta}\underline{\sigma}}{\overline{\sigma}}.$$

Solving the system we first obtain contracts for one type as a function of the other type.

$$\underline{\beta} = \frac{\underline{\sigma} - \gamma\rho\overline{\beta}\overline{\sigma}}{\underline{\sigma} + (1-\gamma)\rho\underline{\sigma}},$$
$$\overline{\beta} = \frac{\overline{\sigma} - (1-\gamma)\rho\underline{\beta}\underline{\sigma}}{\overline{\sigma} + \gamma\rho\overline{\sigma}}.$$

And finally the equilibrium contracts

$$\underline{\beta} = \frac{\underline{\sigma} - \gamma\rho(\overline{\sigma} - \underline{\sigma})}{\underline{\sigma}(1+\rho)},$$
$$\overline{\beta} = \frac{\overline{\sigma} + (1-\gamma)\rho(\overline{\sigma} - \underline{\sigma})}{\overline{\sigma}(1+\rho)}.$$

We can now observe that

$$\underline{\beta} < \frac{1}{2} \iff \underline{\sigma} < \frac{2\gamma\rho}{2\gamma + (1-\rho)}\overline{\sigma}.$$

[2]Because the problem is strictly convex, and the assumptions of the Maximum Theorem are satisfied, best responses are continuos functions of others strategies and of the asset market equilibrium (which is also a continuous function of strategies). Hence taking limits on N and then solving for equilibrium corresponds to taking the limit of the sequence of equilbria.

Note how this is never satisfied if $\rho = 0$ and always satisfied when $\rho = 1$ (since we assumed all along that $\overline{\sigma} > \underline{\sigma}$). □

An interesting fact is that in the limit, every worker gets exactly the same compensation package (ie: the same random variable).

Corollary 24. $-\underline{\alpha} + \left(1 - \underline{\beta}\right) \underline{X} = -\overline{\alpha} + \left(1 - \overline{\beta}\right) \overline{X}$

Proof.

$$\begin{aligned} \left(1 - \underline{\beta}\right) \underline{\sigma} =& \frac{\rho\underline{\sigma} + \gamma\left(\overline{\sigma} - \underline{\sigma}\right)}{1+\rho} = \frac{\rho\underline{\sigma} + \gamma\rho\left(\overline{\sigma} - \underline{\sigma}\right)}{1+\rho}, \\ \left(1 - \overline{\beta}\right) \overline{\sigma} =& \frac{\rho\overline{\sigma} - \left(1-\gamma\right)\left(\overline{\sigma} - \underline{\sigma}\right)}{1+\rho} = \frac{\rho\underline{\sigma} + \gamma\rho\left(\overline{\sigma} - \underline{\sigma}\right)}{1+\rho}. \end{aligned}$$

We can conclude that agents obtain contract with the same standard deviation, and hence variance. Because agents are pushed to their IR constraint, if contracts have the same variance, they will also have the same mean. Because we are working with binary random variables it is enough to show that the compensation packages have the same mean and variance, to show that they are identical. □

We have seen in Chapter 1 that small markets do not necessarily induce less risky contracts, what is striking is that this can be the case also for large markets. By setting $\gamma = \frac{1}{2}$ we have a replica economy of the initial two firms example. We can see that for certain parameters the contracts in the large economy will be even riskier than in the two firms case.

Proposition 25. $\forall\rho > 1, \exists\overline{\sigma}, \underline{\sigma}$ *such that*

$$\underline{\beta}\left(2\right) > \lim_{N\to\infty} \underline{\beta}\left(N\right).$$

Proof. The claim amounts to showing that there are variances such that

$$\frac{28\underline{\sigma} - 4\rho\overline{\sigma}}{49\underline{\sigma} + \rho^2\underline{\sigma}} > \frac{\underline{\sigma} - \frac{1}{2}\rho\left(\overline{\sigma} - \underline{\sigma}\right)}{\underline{\sigma}\left(1+\rho\right)}.$$

Some algebra shows that this boils down to

$$\underline{\sigma} < \left(\frac{\frac{1}{2}\rho^2 - 4\rho + \frac{41}{2}}{21 + \frac{1}{2}\rho^3 + \rho^2 - \frac{7}{2}\rho} \right) \rho\overline{\sigma}.$$

The fraction on the right side is strictly positive, which allows us to conclude that we can find parameters satisfying our claim if and only if $\rho > 0$. □

How do markets (small and large) induce principals to behave as if they were more "risk averse"? A good intuition for this comes from thinking of two perfectly correlated technologies ($\rho = 1$) with different variances. In this case we can think of two states of the world L and H, with the returns of both types of firms being higher in H. The low variance firms have higher returns in L and lower in H than high variance firms do. Because of this they have an advantage in providing returns in state L. Markets make sure that this advantage is exploited by giving incentives to these firms to issue lower variance securities, compared to the share of profits they would keep for themselves when markets are not present. When correlation is not perfect, this effect still persists but to a lesser degree.

3.3 Number of Traders and Efficiency

It is a relevant question, which I did not tackle so far, weather the securities issued are efficient in the case of symmetric information. When Principals design securities their utility functions take into account the effect they have on the value of other securities, which will ultimately affect their holding of the riskless asset. I will study this problem within a model of identical firms in which I will let the number of firms vary.

In the symmetric examples from Chapter 2, all principals are identical ex-ante *and* after the contracting stage equilibrium. They end up with identical securities, and hence with 0 holdings of risky assets. I am going to show that outcome is non efficient.

$$\begin{aligned}
\max \quad & U_i^{MKT}(\alpha_i, \beta_i) \\
& = \alpha_i + \beta_i \mu_i \\
& + \left(\frac{b}{2N^2} - \frac{b}{N}\right) \beta_i^2 \sigma_i^2 \\
& + \left(\frac{b}{N^2} - \frac{b}{N}\right) \sum_{j \neq i} \rho_{ij} \beta_i \beta_j \sigma_i \sigma_j \\
\text{such that} \quad & -\alpha_i + (1 - \beta_i)\mu_i - \frac{b}{2}(1 - \beta_i)^2 \sigma^2 \geq \overline{u}. \qquad \text{(IR)}
\end{aligned}$$

The optimal contract is

$$\beta_i^{MKT} = \frac{\sigma_i^2 - \left(\frac{N-1}{N^2}\right) \sum_{j \neq i} \rho_{ij} \beta_j \sigma_i \sigma_j}{\sigma_i^2 + \left(\frac{2N-1}{N^2}\right) \sigma_i^2}.$$

In the symmetric case $\sigma_i = \sigma_j = \sigma, \rho_{ij} = \rho, \forall i, j$

The first order conditions for a symmetric equilibrium is in this case

$$\beta = \frac{1 - \frac{(N-1)^2}{N^2} \rho \beta}{1 + \frac{2N-1}{N^2}}.$$

From which the equilibrium contracts

$$\beta^{MKT} = \frac{N^2}{N^2 + 2N - 1 + \rho\left(N^2 - 2N + 1\right)}.$$

To see this, consider a different game, in which each principal designs a piece of the aggregate endowment to be equally split, and no riskless asset transfers take place.[3]

[3] A straightforward way to generalize this to the asymmetric case, would be to let individuals design securities and then give them unconditionally the transfers, and the fractions of aggreagte portfolio, they would have had received had they traded on the market

$$\max \quad U_i(\alpha_i, \beta_i)$$

$$\frac{1}{N}\sum_{i=1}^{N}(\alpha_i + \beta_i\mu_i) - \left(\frac{b}{2N^2}\right)\left(\sum_{i=1}^{N}\beta_i^2\sigma_i^2 + 2\sum_{j\neq i}\rho_{ij}\beta_i\beta_j\sigma_i\sigma_j\right),$$

$$\text{such that} \quad -\alpha_i + (1-\beta_i)\mu_i - \frac{b}{2}(1-\beta_i)^2\sigma^2 \geq \overline{u}. \tag{IR}$$

β^E is the solution to this problem.

$$\beta_i^E = \frac{N\sigma_i - \sum_{j\neq i}\rho_{ij}\beta_j\sigma_j}{(N+1)\sigma_i}.$$

Note how, once we fix other players strategies, we can already observe how the best response induces riskier share for a principal than in the market case. This is because in the market setting a principal takes into account the effect of her actions on market prices. She takes the pricing mechanism as given, but because she can determine the characteristics of her security, she can influence prices of all securities, and hence other traders' endowments. Again let's focus on the symmetric case, in which case the equilibrium will be

$$\beta_E = \frac{N}{(N+1)+\rho(N-1)}.$$

Lemma 26. *The contracts induced by β_E give every individual a higher utility than those induced by β_M*

Proof. Without loss of generality assume that $\overline{u} = 0$

Claim

$$\alpha_E + \beta_E\mu - \frac{b}{2}\frac{N(1-\rho)+\rho N^2}{N^2}\beta_E^2\sigma^2 \geq$$
$$\alpha_M + \beta_M\mu - \frac{b}{2}\frac{N(1-\rho)+\rho N^2}{N^2}\beta_M^2\sigma^2.$$

We can substitute the α 's from the reservation constraints to obtain

$$\left(1-\beta_{E}\right)\mu-\frac{b}{2}\left(1-\beta_{E}\right)^{2}\sigma^{2}+\beta_{E}\mu-\frac{b}{2}\frac{N\left(1-\rho\right)+\rho N^{2}}{N^{2}}\beta_{E}^{2}\sigma^{2}\geq$$
$$\left(1-\beta_{M}\right)\mu-\frac{b}{2}\left(1-\beta_{M}\right)^{2}\sigma^{2}+\beta_{M}\mu-\frac{b}{2}\frac{N\left(1-\rho\right)+\rho N^{2}}{N^{2}}\beta_{M}^{2}\sigma^{2}.$$

Rearranging and expanding we obtain

$$-\frac{b}{2}\left(\beta_{E}^{2}-2\beta_{E}\right)\sigma^{2}-\frac{b}{2}\frac{N\left(1-\rho\right)+\rho N^{2}}{N^{2}}\beta_{E}\sigma^{2}\geq$$
$$-\frac{b}{2}\left(\beta_{M}^{2}-2\beta_{M}\right)\sigma^{2}-\frac{b}{2}\frac{N\left(1-\rho\right)+\rho N^{2}}{N^{2}}\beta_{M}\sigma^{2}.$$

$$b\sigma^{2}\left[\left(\beta_{E}-\beta_{M}\right)-\frac{\left(1-\rho\right)N+\rho N^{2}+N^{2}}{2N^{2}}\left(\beta_{E}^{2}-\beta_{M}^{2}\right)\right]\geq 0.$$

We can divide by $\beta_E-\beta_M$ and simplify N in the fraction above to obtain

$$b\sigma^{2}\left[1-\frac{\left(1-\rho\right)+\left(1+\rho\right)N}{2N}\left(\beta_{E}+\beta_{M}\right)\right]\geq 0.$$

which boils down to

$$2N-\left[\left(1-\rho\right)+\left(N+1+\rho\left(N-1\right)\right)\right]\left(\beta_{E}+\beta_{M}\right)\geq 0.$$

Some tedious algebra shows that $\left(\beta_E+\beta_M\right)$ is equal to

$$\frac{2\left(1+\rho\right)N^{2}+3\left(1-\rho\right)N-\left(1+\rho\right)}{\left(\left(N+1\right)+\rho\left(N-1\right)\right)\left(N^{2}+2N-1+\rho\left(N^{2}-2N+1\right)\right)}.$$

So that we have

$$b\sigma^2\left[2N-\frac{2\left(1+\rho\right)N^2+3\left(1-\rho\right)N-\left(1+\rho\right)}{N^2+2N-1+\rho\left(N^2-2N+1\right)}\right]\geq 0.$$

Multiplying and observing that the denominator is positive, we have left to show that the following polynomial is positive

$$2\left(1+\rho\right)N^3-6\rho N^2-5\left(1-\rho\right)+\left(1+\rho\right).$$

This has to be the case because it is positive for $N=2$, and it is an increasing function of N (its derivative is in fact $6\left(1+\rho\right)N^2-12\rho N-5\left(1-\rho\right)$). □

Good news come from the fact that $\lim_{N\to\infty}\beta_E=\lim_{N\to\infty}\beta_M=\frac{1}{1+\rho}$. As N gets large the difference between the efficient allocation and the market outcome gets smaller.

Unfortunately, It is also straightforward to see that $\beta_E>\beta_M \iff \rho<1, \forall N<\infty$. This means that markets do better when we have little use for them, when firms returns are perfectly correlated.

3.4 Large firms

In this section I study the equilibrium property of economies in which one firm is large relative to the market. I consider N identical productive units, each with returns characterized by mean μ, variance σ^2, and pairwise correlation coefficient ρ. Consider a large number of firms N. One principal owns γN productive units, and the remaining $\left(1-\gamma\right)N$ constitute separate firms.[4]

The first tedious step is to write the quantities relevant for the principal utility functions. Firms are going to be of two types: one large firm and many small firms, I will look for a symmetric equilibrium, in which all small firms issue the same contracts/securities. The large firm will issue a security characterized by α,β and the small firms' by α_i,β_i (or α_j,β_j when I need to distinguish among them).

The mean and variance of the aggregate endowment

[4]Again, it would be proper to have K units owned by a firm and the remaining $N-K$ being individual firms. This however would lead to a more complicated notation whose effects become negligible as N and K get large.

$$\alpha + \beta\mu + \Sigma\alpha_i + \beta_i\mu_i = \gamma N \left(\alpha + \beta\mu\right) + \left(1-\gamma\right) N \left(\alpha_i + \beta_i \mu\right),$$

$$[\beta|\boldsymbol{\beta}]\, \boldsymbol{\Omega}\, [\beta|\boldsymbol{\beta}]' = \gamma N \beta^2\sigma^2 + (1-\gamma) N \beta_i \sigma^2 + 2\gamma^2 N^2 \rho\beta^2\sigma^2 + 2(1-\gamma)^2 N^2 \rho \beta_i^2 \sigma^2 + 2\gamma(1-\gamma) N^2 \rho\beta\beta_i\sigma^2.$$

The price of the security issued by the large firm is

$$q_\gamma = \gamma N (\alpha + \beta\mu) - \frac{b}{(1-\gamma) N} \left[\gamma N \beta^2 + 2\gamma^2 N^2 \rho \beta^2 + \gamma (1-\gamma) N^2 \rho\beta\beta_i\right] \sigma^2.$$

The price of a security issued by a small firm i is

$$q_i = \alpha_i + \beta_i \mu_i - \frac{b}{(1-\gamma) N} \left[\beta_i^2 + (1-\gamma) N \rho \beta_i^2 + \gamma N \rho\beta\beta_i\right] \sigma^2.$$

Note that the covariance of the productive units within the large firm are counted twice. This is because they are part of the variance of the security issued, and this is what marks the difference with the smaller firms. The large firm partially internalizes the effects of contracting inside the single unit on asset prices.

The average price needed to compute the riskless asset holding is

$$\begin{aligned}\frac{q_\gamma + \Sigma q_i}{N} =& \frac{1}{(1-\gamma) N} \left[\gamma N (\alpha + \beta\mu) + (1-\gamma)\mu\right] \\ &- \frac{1}{(1-\gamma)^2 N^2} \left[\gamma \left(N\beta^2 + 2\gamma N^2 \rho\beta^2 + (1-\gamma) N^2 \rho\beta\beta_i\right) + (1-\gamma)\left(N\beta_i^2 + (1-\gamma) N^2 \rho \beta_i^2 + \gamma N^2 \rho\beta\beta_i\right)\right].\end{aligned}$$

With this in mind we can construct the utility function of the principal of the large firm and drop the parts where β does not appear as they are not relevant to the solution.

$$
\begin{aligned}
U_{\gamma}(\alpha,\beta) &= \gamma N(\alpha+\beta\mu) \\
-\left(\frac{b}{(1-\gamma)N}-\frac{b}{2(1-\gamma)^2N^2}\right)&\left(\gamma N\beta^2+2\gamma^2N^2\rho\beta^2\right)\sigma^2 \\
-\left(\frac{b}{(1-\gamma)N}-\frac{b}{(1-\gamma)^2N^2}\right)&\left(\gamma(1-\gamma)N^2\rho\beta\beta_i\right)\sigma^2.
\end{aligned}
$$

For optimization purposes, we can divide this function by γN and use this objective function.

$$
\begin{aligned}
U(\alpha,\beta)&(\alpha+\beta\mu) \\
-\left(\frac{b}{(1-\gamma)N}-\frac{b}{2(1-\gamma)^2N^2}\right)&\left(\beta^2+2\gamma N\rho\beta^2\right)\sigma^2 \\
-\left(\frac{b}{(1-\gamma)N}-\frac{b}{(1-\gamma)^2N^2}\right)&\left((1-\gamma)N\rho\beta\beta_i\right)\sigma^2.
\end{aligned}
$$

This is the utility function for a principal in a generic small firm i, also trimmed of parts irrelevant to the optimization problem.

$$
\begin{aligned}
U_i(\alpha_i,\beta_i) &= \alpha_i+\beta_i\mu \\
-\left(\frac{b}{(1-\gamma)N}-\frac{b}{2(1-\gamma)^2N^2}\right)&\beta_i^2\sigma^2 \\
-\left(\frac{b}{(1-\gamma)N}-\frac{b}{(1-\gamma)^2N^2}\right)&\left(\gamma N\rho\beta\beta_i+(1-\gamma)N\rho\beta_i\beta_j\right).
\end{aligned}
$$

From the usual procedure of substituting α and α_i from the agents' IR constraints, we obtain the optimal contracts as a function of others'.

$$
\begin{aligned}
\beta^* &= \frac{1-\left(1-\frac{1}{(1-\gamma)N}\right)\rho\beta_i}{1+2\left(\frac{1}{(1-\gamma)N}-\frac{1}{2(1-\gamma)^2N^2}\right)(1+\gamma N\rho)}, \\
\beta_i^* &= \frac{N^2-(N-1)\left[\gamma N\rho\beta+(1-\gamma)N\rho\beta_i\right]}{N^2+2N-1}.
\end{aligned}
$$

As N gets large we have

$$\beta = \frac{(1-\gamma)(1-\rho\beta_i)}{(1-\gamma)+2\gamma\rho},$$
$$\beta_i = \frac{1-\gamma\rho\beta}{1+(1-\gamma)\rho}.$$

Solve for β yields

$$\beta^* = \frac{(1-\gamma\rho)}{1+(1-\gamma)\rho+\gamma\rho^2+\frac{2\gamma}{1-\gamma}\rho}.$$

We can now draw some conclusion on the contracts.

Proposition 27. *The large firms offers riskier contracts than in the symmetric case. Small firms offer less risky contracts.*

$$\beta < \beta^{symm} < \beta_i.$$

Proof. We know from Chapter 2 or Section 3.3 that β^{symm} is equal to $\frac{1}{1+\rho}$.

Part 1. $\beta < \frac{1}{1+\rho}$.

We want to show that

$$\frac{(1-\gamma\rho)}{1+(1-\gamma)\rho+\gamma\rho^2+\frac{2\gamma}{1-\gamma}\rho} < \frac{1}{1+\rho}.$$

expanding this yields $\frac{2\gamma}{1-\gamma} > 0$, which is true $\forall\gamma \in (0,1)$.

Part 2. $\beta_i > \frac{1}{1+\rho}$.

Since $\beta_i = \frac{1-\gamma\rho\beta}{1+(1-\gamma)\rho}$ and $\beta < \frac{1}{1+\rho}$ it has to be the case that

$$\beta_i > \frac{1-\frac{\gamma\rho}{1+\rho}}{1+(1-\gamma)\rho} = \frac{1+\rho-\gamma\rho}{(1+\rho)(1+(1-\gamma)\rho)} = \frac{1+(1-\gamma)\rho}{(1+\rho)(1+(1-\gamma)\rho)} = \frac{1}{1+\rho}.$$

This concludes the proof. □

I conjecture that similarly to the case of a small number of trader- this equilibrium is inefficient and Pareto-dominated by an allocation with the securities induced by $\beta = \frac{1}{1+\rho}$ and the same transfers as the riskless asset holdings of this equilibrium.

3.5 Conclusions

In this chapter, I study some implications on the design of securities and contracts.

In Section 1 I look at what seems an"anomaly". Even when markets are large, diversification opportunities do not necessarily translate to those who do not access financial markets. In Section 2 I look at inefficiencies induced by a small market on security design. A reassuring feature of this inefficiency is that its size becomes negligibly small with market size. A less reassuring feature is that it is larger when more diversification opportunities are available, which is exactly when we need financial markets. In Section 3 I consider the case of a firm that is large relative to the market, and the properties of equilibrium securities. I finalize the study of efficiency in this setting.

This chapter tackles important questions on financial markets. More work needs to be done in terms of generalizing the phenomena exhibited here, to better understand how relevant they are to the real economy. Another important question, also in light of recent economic event, is weather firms (or financial institutions) choose an efficient amount of diversification in equilibrium. This would amount to making the ρ_i's coefficients endogenous. This is better suited to a model in which there is competition on the "agents" side, a direction I intend to pursue.

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