## Chapter 6

## Application to normal collision between two particles

The liquid effect on a particle-wall collision process has been studied by investigating the relation between the coefficient of restitution of a collision and the particle impact Stokes number. When there is another solid sphere existing in the flow field instead of the solid wall, the normal collision process between the two spheres show different particle behaviors since the influence of the target sphere on the surrounding flow field is different especially when the target sphere is not fixed in its initial position. Yang \& Hunt (2006) performed particle-particle collision experiments with two spheres hanging in a glycerol-water mixture. They defined a binary Stokes number by considering the hydrodynamic effect on the two approaching spheres and found that the coefficient of restitution of a particle-particle collision has a similar dependence on the binary Stokes number to the relation between $e$ and $S t$ found in immersed particle-wall collision process.

In the previous chapters, a normal collision process between a solid sphere and a solid wall in a liquid environment has been simulated using the fast immersed boundary method. The proposed contact model that incorporates both the liquid-solid interaction with the wall-effect correction and the solid-solid elastic force predicts the dependence of the coefficient of restitution on the particle impact Stokes number accordant with the empirical trend. A normal collision between two spheres also generates an axisymmetric flow field. The numerical method described in Chapter 3 can be used to solve for the flow field and the coupled particle dynamics as the impact sphere approaches the target. After modifying the liquid-solid interaction and the elastic force terms by considering
the effect of the target sphere, the contact model is applied to simulate a normal collision process between the two spheres. The coefficient of restitution calculated from the simulations are compared with Yang \& Hunt (2006)'s experimental results. Unique particle behaviors for a particle-particle collision process at small Stokes number are represented.

### 6.1 Immersed particle-particle normal collision

To complement previously investigated particle-on-wall immersed collisions, Yang \& Hunt (2006) investigated liquid-immersed head-on particle-particle collision processes, which provide good reference for the current simulations. This section briefly introduces the experiment setup and the definition of the coefficient of restitution and the Stokes numbers for a particle-particle collision process employed in Yang \& Hunt (2006).

The normal particle-particle collision experiment performed by Yang \& Hunt (2006) is shown schematically in figure (6.1). Two spheres were suspended by thin strings as two pendulums in a viscous fluid. A wide range of impact conditions were achieved by changing the release angle, the solid materials and the liquid. A typical result of the trajectories of the two particles is shown in figure (6.2).


Figure 6.1: Schematic experiment setup from Yang \& Hunt (2006)

The impact and rebound velocities of the two spheres, $U_{i 1}, U_{r 1}, U_{i 2}$ and $U_{r 2}$, are taken as the slopes of the fit lines (in a least-squares sense) of the trajectories over a time interval from 10 to 15


Figure 6.2: The trajectories of the two particles from Yang \& Hunt (2006).
ms. The effective coefficient of restitution for such a binary collision is defined as:

$$
\begin{equation*}
e=-\frac{U_{r 1}-U_{r 2}}{U_{i 1}-U_{i 2}} \tag{6.1}
\end{equation*}
$$

in analogy with the conventional definition for a binary dry collision.
The binary particle Stokes number for the colliding sphere was defined as:

$$
\begin{equation*}
S t_{B}=\frac{m^{*}\left(U_{i 1}-U_{i 2}\right)}{6 \pi \mu a^{* 2}} \tag{6.2}
\end{equation*}
$$

in which the particle pair is characterized as a single particle with effective mass $m^{*}=\left(1 / m_{1}+\right.$ $\left.1 / m_{2}\right)^{-1}$ and reduced radius $a^{*}=\left(1 / a_{1}+1 / a_{2}\right)^{-1}$ that moves at an approach velocity $U_{\text {rel }}=$ $\left(U_{i 1}-U_{i 2}\right)$. In analogy with the single-particle Stokes number that is the ratio of the particle inertia to the liquid viscous effect, the binary Stokes number represents the ratio of the available momentum in the solid phase to the viscous dissipation calculated by multiplying an effective viscous force $6 \pi \mu a^{*} U_{\text {rel }}$ with a forcing duration $a^{*} / U_{\text {rel }}$. When the two spheres have the same size, the binary Stokes number can be written as:

$$
\begin{equation*}
S t_{B}=\frac{2}{9} \frac{\rho_{p}^{*}}{\rho_{f}} R e_{\mathrm{rel}} \tag{6.3}
\end{equation*}
$$



Figure 6.3: Comparison of the inter-particle and the particle-wall immersed collisions. The figure is taken from Yang (2006).
where $\rho_{p}^{*}=\left(1 / \rho_{1}+1 / \rho_{2}\right)^{-1}$ is an effective solid density and $R e_{\text {rel }}=D U_{\text {rel }} / \nu$ is the relative Reynolds number. The relation between the binary Stokes number and the relative Reynolds number is consistent with the conventional definition, $S t=\left(\rho_{p} / \rho_{l}\right) R e / 9$.

The functional relation between the effective coefficient of restitution and the binary Stokes number obtained from their experiments follows the empirical trend found for particle-wall collisions, as shown in figure (6.3). Collisions between different solid materials including steel, glass and delrin were marked with different indicators. Detailed discussion of this figure can be found in Yang (2006).

### 6.2 Modification of the contact model

To simulate a normal collision between two particles as described in the previous section with the current numerical method, the pendulum motion in the vertical direction is neglected so that the two spheres are assumed to move horizontally along the line across their centers and the surrounding flow
is axisymmetric. The assumption is appropriate since the simulation focuses on a collision process only from 30 ms before contact to 30 ms after contact. The displacement in the vertical direction during this time period is less than $0.015 D$ as found from the experimental data of Yang (2006).

The sphere motion in the horizontal direction is described by the equation:

$$
\begin{equation*}
m_{p, m} \frac{d \tilde{V}_{m}}{d \tilde{t}}=\tilde{f}_{\mathrm{sL}, m}+\tilde{f}_{\mathrm{Ss}, m} \tag{6.4}
\end{equation*}
$$

where $m=1$ and $m=2$ stand for impact sphere and target sphere, respectively; $\tilde{f}_{\mathrm{sL}, m}$ and $\tilde{f}_{\mathrm{ss}, m}$ are the liquid-solid interaction and solid-solid elastic force terms proposed for the contact model in the form of equation (4.14) and equation (4.19) as:

$$
\begin{gather*}
\tilde{f}_{\mathrm{SL}}=H\left(\frac{\delta}{\delta_{\mathrm{SL}}}\right) \tilde{f}_{\mathrm{THR}}+\left[1-H\left(\frac{\delta}{\delta_{\mathrm{SL}}}\right)\right] \tilde{f}_{\mathrm{SIM}},  \tag{6.5}\\
\tilde{f}_{\mathrm{SS}}=F\left(\frac{\delta}{\delta_{\mathrm{SS}}}\right) e_{d} W_{o} . \tag{6.6}
\end{gather*}
$$

However, when the fixed target wall is replaced by a movable sphere, the hydrodynamic forces and the elastic force change correspondingly. To apply the above contact model, the analytical expressions for $\tilde{f}_{\text {THR }}$ and $W_{o}$ are modified to accounting for the geometry and movability effect of the target sphere.

### 6.2.1 Liquid-solid interaction with the target sphere effect

In a particle-particle collision process, the existence of a target spherical particle affects the liquidsolid interaction force exerted on the impact sphere. However, the influence is different from the wall effect since the surrounding fluid can pass around the target particle and the movability of the target particle further complicates the coupled evolution of the flow field. Thus, Yang's formulas for the hydrodynamic forces with a wall effect, equations (4.4), (4.8) and (4.11) in Chapter 4 are modified as:

$$
\begin{equation*}
\tilde{f}_{D, m}=-6 \pi \mu a_{m} U_{m} \lambda\left(\delta^{* *}, R e_{m, h}^{*}\right), \tag{6.7}
\end{equation*}
$$

$$
\begin{gather*}
\tilde{f}_{A M, m}=-\frac{1}{2} m_{l, m}\left[1+3 W\left(\delta^{* *}\right)\right] \frac{d U_{m}}{d t}+\frac{3}{4} m_{l, m} U_{m} \frac{U_{\mathrm{rel}}}{a^{*}} \frac{d W\left(\delta^{* *}\right)}{d \delta^{* *}}  \tag{6.8}\\
\tilde{f}_{H, m}=-6 \pi \mu a_{m} K_{H}\left(\delta^{* *}\right)^{3 / 2} \int_{0}^{t} \frac{d U_{m}}{d \tau} K(t-\tau) d \tau \tag{6.9}
\end{gather*}
$$

where the subindex $m=1,2$ stands for the impact sphere and the target sphere; $\delta^{* *}=h / a^{*}$ is the gap between the two spheres non-dimensionalized by the reduced radius $a^{*}=\left(1 / a_{1}+1 / a_{2}\right)^{-1}$, which replaces $\delta^{*}=h / a$ used in the original wall correction term; $U_{\text {rel }}=U_{1}-U_{2}$ is the relative velocity of the two spheres ; $R e_{m, h}^{*}=h U_{\text {rel }} / \nu$ is the gap Reynolds number. The correction terms $\lambda\left(\delta^{* *}, R e_{m, h}^{*}\right)$, $K_{H}\left(\delta^{* *}\right), W\left(\delta^{* *}\right)$, and $d W\left(\delta^{* *}\right) / d \delta^{* *}$ depend on $\delta^{* *}$ in the same manner as given in equations (4.3), (4.12), (4.6) and (4.7). The effect of the target sphere converges to the wall correction form when the radius $a_{2}$ increases to infinitely large.

The analytical result is compared with a simulated result in which the trajectories of the particles are prescribed with an experimental result from Yang (2006). As shown in figure (6.4), the analytical and the simulated results match each other for moderate gap and deviate with diminishing gap after $\delta<0.05 D$, which is similar to the result found in a particle-wall collision. Thus the modified formulas for the hydrodynamic forces are adopted in the liquid-solid interaction term as:

$$
\begin{equation*}
\tilde{f}_{\mathrm{SL}, m}=H\left(\frac{\delta}{\delta_{\mathrm{SL}}}\right) \tilde{f}_{\mathrm{THR}, m}+\left[1-H\left(\frac{\delta}{\delta_{\mathrm{SL}}}\right)\right] \tilde{f}_{\mathrm{SIM}, m} \tag{6.10}
\end{equation*}
$$

where $\tilde{f}_{\mathrm{THR}, m}=\tilde{f}_{D, m}+\tilde{f}_{A M, m}+\tilde{f}_{H, m}$ and $\tilde{f}_{\mathrm{SIM}, m}$ is the simulated force on the corresponding particle with indicator $m$. The same Heaviside function defined by equation (4.15) as shown in figure (4.9) is employed to blend the theoretical result and the simulated results in a manner similar to that described in Chapter 4. The non-dimensional parameter $\delta_{\mathrm{SL}}$ is conservatively taken as $0.1 D$.

Thus, the hydrodynamic forces exerted on the two spheres are calculated directly from the simulation when the gap between the two spheres is still large; the analytical results start to be counted as the impact sphere approaches the target and $\tilde{f}_{\mathrm{SL}, m}=\frac{1}{2} \tilde{f}_{\mathrm{THR}, m}+\frac{1}{2} \tilde{f}_{\mathrm{SIM}, m}$ at $\delta=0.1 D ;$ when the two sphere are about to collide, the hydrodynamic forces are taken as the values calculated from the analytical expressions.


Figure 6.4: Comparison between the analytical and simulated results for the hydrodynamic force in a particle-particle collision

### 6.2.2 Elastic effect between two spheres

To modify the elastic force $W_{o}$ in the solid-solid interaction term (equation 4.19), the Hertz elastic contact theory is considered for a normal contact between two elastic spheres as shown in figure (6.5). The pressure distribution on the contact area is:


Figure 6.5: Schematic of a contact between two elastic spheres

$$
\begin{equation*}
P=P_{o}\left(1-\frac{r^{2}}{b^{2}}\right)^{1 / 2} \tag{6.11}
\end{equation*}
$$

with $P_{o}=\frac{2}{\pi} E^{*}\left(\frac{\varepsilon}{a^{*}}\right)^{1 / 2}$, where $b=\sqrt{a^{*} \varepsilon}$ is the radius of contact area, $a^{*}$ is the reduced radius and $\varepsilon$ is the indent depth. The impact force can be calculated by integrating the pressure over the contact area:

$$
\begin{equation*}
W=\int_{0}^{b} P_{o}\left(1-\frac{r^{2}}{b^{2}}\right)^{1 / 2} 2 \pi r d r=\frac{4}{3} E^{*} b^{2}\left(\frac{\varepsilon}{a^{*}}\right)^{1 / 2}=\frac{4}{3} E^{*} \sqrt{a^{*}} \varepsilon^{3 / 2} . \tag{6.12}
\end{equation*}
$$

This result is the same as the impact force for a elastic sphere contacting with a solid wall given in equation (4.16) except that the radius of the sphere $a$ in (4.16) is replaced by the reduced radius $a^{*}$. Thus, the maximum elastic force between the two spheres is solved using the same approach as discussed in Chapter 4:

$$
\begin{equation*}
W_{o 2 p}=\frac{4}{3} E^{*}\left(\frac{15 m^{*} U_{\mathrm{rel}}^{2}}{16 E^{*} a^{* 3}}\right)^{3 / 5} \tag{6.13}
\end{equation*}
$$

where the reduced mass, $m^{*}$, reduced modulus $E^{*}$ and reduced radius $a^{*}$ are used together with the relative velocity $U_{\text {rel }}$.

The solid-solid interaction term in the contact model for a particle-particle collision is modified
to the form:

$$
\begin{equation*}
\tilde{f}_{\mathrm{SS}}=F\left(\frac{\delta}{\delta_{\mathrm{SS}}}\right) e_{d} W_{o 2 p} \tag{6.14}
\end{equation*}
$$

where, the function $F\left(\frac{\delta}{\delta_{\text {ss }}}\right)$ is defined the same as equation (4.20) in particle-wall collision model as shown in figure (4.11).

Thus, when the gap between the two spheres is greater than $\delta_{\mathrm{SS}}$, the elastic interaction between the two spheres is irrelevant; when the gap decreases below the threshold value, $\delta_{\mathrm{SS}}$, an elastic-like force increases from zero exponentially to the dry collision value for $\delta=0$. The non-dimensional parameter $\delta_{\mathrm{SS}}$ is determined in the following section.

### 6.3 Simulation and results

With the above contact model, the normal collision process between two sphere with different binary Stokes numbers are simulated. The resulting trajectories of the impact and rebound particles are compared with the experimental data of Yang \& Hunt (2006) so that the non-dimensional parameter $\delta_{\text {SS }}$ is calibrated by choosing the value that presents the best fit to the experimental trajectory. The effective coefficient of restitution calculated from the simulations with different solid materials and different initial conditions is compared with the experimental results. The special dynamic behaviors of the two particles in a collision process with small binary Stokes number are obtained accordant with the observation in the experiments.

### 6.3.1 Simulation setup

Yang \& Hunt (2006) measured the trajectories of the impact and target particles in their immersed head-on collision experiments. For an example, the result of a collision event named 'gg3462' from Yang \& Hunt (2006) is plotted in figure (6.6).

To compare a simulation for a normal particle-particle collision to the experimental results, the simulation is set up as follows. Three-level multi-grid computation domains are employed with the two particles with identical radius placed along the axis of symmetry. The first-level domain is


Figure 6.6: The trajectories of the impact and target spheres from Yang \& Hunt (2006).


Figure 6.7: First level computation domain for a particle-particle collision.
shown in figure (6.7). According to the available experimental data, the impact sphere starts from an initial velocity, $V_{I o}$; the target sphere has a zero initial velocity. The motion equation of the two spheres is:

$$
\begin{equation*}
m_{p, m} \frac{d \tilde{V}_{m}}{d \tilde{t}}=\tilde{f}_{\mathrm{SL}, m}+\tilde{f}_{\mathrm{SS}, m} \tag{6.15}
\end{equation*}
$$

where $m=1,2, \tilde{f}_{\mathrm{SL}, m}$ and $\tilde{f}_{\mathrm{SS}, m}$ are given in equation (6.10) and (6.14) as the modified contact model described in the previous section. The equation is non-dimensionalized with $L_{o}=D$, the diameter of the spheres, and $t_{o}=D / V_{I o}$ as:

$$
\begin{equation*}
\tau_{m} \frac{d V_{m}}{d t}=f_{\mathrm{SL}, m}+f_{\mathrm{SS}, m} \tag{6.16}
\end{equation*}
$$

where $\tau_{m}=\rho_{p . m} / \rho_{l}$ is the density ratio. The Navier-Stokes equations for the flow field nondimensionalized with $L_{o}=D$ and $t_{o}=D / V_{I o}$ have a non-dimensional parameter, $\operatorname{Re}=V_{I o} D / \nu$, that is determined by the initial condition and the material properties given in the experimental data. For the case given in figure (6.6), the two spheres are both glass spheres and the diameter of the spheres is 12.7 mm corresponding to 148 pixel in the image. The liquid properties are given as $\mu=0.004 \mathrm{~Pa} \cdot \mathrm{~s}$ and $\rho_{l}=1112 \mathrm{~kg} / \mathrm{m}^{3}$. The initial velocity is $0.110 \mathrm{~m} / \mathrm{s}$. Thus, the input Reynolds number for the simulation is $R e=249$. The elastic properties of the spheres can be found in Yang (2006). The dimensionless time step in the simulation is 0.001 and the CFL number is less than 0.5 for every time step.

The trajectories calculated from the simulation vary with different $\delta_{\mathrm{SS}}$. For the three values of $\delta_{\mathrm{SS}}$ used in the simulations, $\delta_{\mathrm{SS}}=0.017,0.024$ and 0.030 , a larger value of $\delta_{\mathrm{SS}}$ produces a smaller relative velocity between the impact and target spheres after collisions. After comparing with the experimental results, this dimensionless parameter is calibrated as $\delta_{\mathrm{SS}}=0.024$, which produces the best fit as shown in figure (6.8). Compared with $\delta_{\mathrm{SS}}=0.017$ for a particle-wall immersed collision process, the threshold value $\delta_{\text {SS }}$ for a inter-particle collision process is larger. That can be explained by the fact that the minimum distance achieved during a particle-particle collision process is larger than the value for a particle-wall collision because of the geometry difference and the movability of


Figure 6.8: Trajectories from the simulations and the experiment
the target.

### 6.3.2 Unique behaviors of two spheres colliding in a liquid

Yang \& Hunt (2006) observed unique behaviors of the two colliding spheres in a liquid environment when the binary Stokes number is small. The target sphere moves prior to contact when $S t_{B}$ ranges from $2 \sim 10$ since the pressure front building up in the interstitial liquid layer transmits the momentum of the impact sphere to the target sphere. After contact, the two spheres stick together and move with identical velocity, which Yang and Hunt called 'group velocity', $U_{G}$. Such a precollision target motion does not occur in a dry collision process in which the surrounding medium effect is negligible.

The current simulation can represent the pre-collision target motion and the after-collision group motion as shown in figure (6.9). Two simulations are run to compare the different liquid effect as the liquid viscosity changes. The two glass spheres are placed at the same initial positions with $0.2 D$ distance between each other in the two cases and the impact sphere has the same initial velocity. In figure $6.9(\mathrm{a})$, the liquid viscosity is $\mu=0.089 \mathrm{~Pa} \cdot \mathrm{~s}$ and the impact particle Reynolds number is
12. The target sphere has a slight displacement before the impact sphere reaches it. After contact, the two spheres move with a group velocity $U_{G}=18 \mathrm{~mm} / \mathrm{s}$. When increasing the liquid viscosity to $\mu=0.480 \mathrm{~Pa} \cdot \mathrm{~s}$, the velocity of the target sphere increases as the impact sphere approaches. The momentum of the impact sphere is transmitted to the target by the interstitial liquid. Finally, the two spheres move at a group velocity $U_{G}=4.5 \mathrm{~mm} / \mathrm{s}$ even there is no contact and the distance between the two spheres stays at 0.07 D . The binary Stokes number for the two cases are $S t_{B}=10$ for figure $6.9(\mathrm{a})$ and $S t_{B}=0$ for $6.9(\mathrm{~b})$. The coefficient of restitution for these two cases are both zero although the spheres are still moving after collision.


Figure 6.9: Group motion of the two spheres after contact. The solid line and dashed line represent the trajectories of the impact and target spheres, respectively.

### 6.3.3 The effective coefficient of restitution

Simulations with different material properties and initial conditions are run to obtain normal headon collisions with a wide range of binary Stokes numbers. The limitation of the numerical method restricts the particle Reynolds number to be less than 300 to obtain an axisymmetric flow filed. The effective coefficient of restitution is obtained based on the slopes of the trajectories as

$$
e=-\frac{U_{r 1}-U_{r 2}}{U_{i 1}-U_{i 2}}
$$

where the slopes are obtained as shown in figure (6.2) with time interval 10 ms . The binary Stokes number is calculated as the definition in equation (6.2) as:

$$
S t_{B}=\frac{m^{*}\left(U_{i 1}-U_{i 2}\right)}{6 \pi \mu a^{* 2}}=\frac{2}{9} \frac{\rho_{p}^{*}}{\rho_{f}} R e_{\mathrm{rel}}
$$

where

$$
R e_{\mathrm{rel}}=\frac{\rho_{l} D U_{\mathrm{rel}}}{\mu}
$$

with the relative velocity $U_{\text {rel }}=U_{i, 1}-U_{i, 2}$ determining the difference between the slopes of the two impact trajectories. The input parameters for the simulation and the results of $e$ and $S t_{B}$ are listed in Table (6.1). Steel and glass spheres are used as impact or target sphere in the different runs so that the reduced density $\rho_{p}^{*}=\left(1 / \rho_{1}+1 / \rho_{2}\right)^{-1}$ has different value as the result of the combination of different materials.

| run | $\mu(\mathrm{cP})$ | $\rho_{l}\left(\mathrm{~g} \cdot \mathrm{~cm}^{-3}\right)$ | $\rho_{p}^{*}\left(\mathrm{~g} \cdot \mathrm{~cm}^{-3}\right)$ | $V_{I o}\left(\mathrm{~cm} \cdot \mathrm{~s}^{-1}\right)$ | $R e_{I o}$ | $S t_{B}$ | $e$ | comment |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 4.1 | 1.11 | 7.78 | 8.4 | 264 | 197 | 0.87 | steel-steel |
| 2 | 4.1 | 1.11 | 3.83 | 9.6 | 294 | 104 | 0.75 | steel-glass |
| 3 | 4.1 | 1.11 | 2.54 | 8.0 | 249 | 59 | 0.68 | glass-glass |
| 4 | 43.0 | 1.20 | 7.78 | 11.3 | 33 | 22 | 0.33 | steel-steel |
| 5 | 43.0 | 1.20 | 3.83 | 8.7 | 26 | 7.5 | 0.00 | steel-glass |
| 6 | 43.0 | 1.20 | 2.54 | 7.0 | 22 | 2 | 0.00 | glass-glass |

Table 6.1: Simulations for head-on collision between two spheres.


Figure 6.10: Coefficient of restitution as a function of binary Stokes number

The calculated coefficient of restitution is presented as a function of the binary Stokes number and compared with the experimental data of Yang \& Hunt (2006), as shown in figure (6.10). For binary Stokes numbers ranging from $50 \sim 200$, the effective coefficient of restitution varies from 0.68 to 0.87 within the range of the experimental results. At $S t_{B}=22$, the coefficient of restitution is $e=0.33$ much less than the computations for larger $S t_{B}$. When the binary Stokes number is less than 10 , the relative velocity between the target and impact spheres is zero so that the effective coefficient of restitution is zero. The dependence of $e$ on $S t_{B}$ is consistent with the empirical trend. The experimental data scatter over a wide rang for $5<S t_{B}<30$, which was explained in Yang \& Hunt (2006) as a sphere impacting at lower Stokes numbers is sensitive to the sphere surface roughness, the roundness and the disturbances in the ambient flow yielding a larger error bar. The uncertainty of experiments in calculating velocity leads to errors in calculating e and St and for low Stokes numbers, e increases with large slope with increasing St so that even small error makes the data scatter in a wide range. The current simulation results provide a good estimation for the effective coefficient of restitution for immersed collisions within this lower binary Stokes number range.

Thus, it is concluded that the modified contact model appropriately incorporates the geometry
and movability effect of the target sphere so that it captures the essential contact mechanism for particle-particle collisions in a viscous liquid environment.

