THE ARITHMETIC AND GEOMETRY OF A CLASS OF ALGEBRAIC SURFACES OF GENERAL TYPE AND GEOMETRIC GENUS ONE

Thesis by

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To Julianne,

who proves everyday that two is

far greater than one.

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pairing” and “purely inseparable,” let me say instead that it always delights and intrigues me, a math nerd who grew up with an accordingly skewed view on life, how she can make the non-mathematical half of the world be just as engaging, beautiful, and fun as the other half.

Abstract

We prove a large monodromy result for a family of complex algebraic surfaces of general type, with invariants $p_g = q = 1$ and $K^2 = 3$, that has been introduced by Catanese and Ciliberto. Unlike other classes of surfaces with $p_g = 1$ studied previously, this monodromy result cannot be proved by showing that the image of period map contains an open ball. Instead we proceed via an analysis of the degenerations of these surfaces and a generalization of Lefschetz's work on the monodromy of hyperplane sections of a smooth projective variety.

As corollaries, we verify three conjectures regarding the Galois representation on the middle ℓ-adic cohomology of such a surface when it is defined over a finitely generated extension of $\mathbb{Q}$, namely semisimplicity, the Tate Conjecture, and the Mumford-Tate Conjecture. We also give an application to the existence of such surfaces over number fields with minimal Picard number. Finally, we use the period map of the given family to give examples of nonspecial subvarieties of certain a Shimura variety of orthogonal type.

Contents

Chapter 1

Introduction

Convention: In this thesis, all fields under consideration will be subfields of $\mathbb{C}$; in particular, any statements of an arithmetic nature (e.g., regarding the Tate Conjecture) will always refer to finitely generated extensions of $\mathbb{Q}$.

In the Enriques classification of algebraic surfaces, the surfaces of general type are far less understood than their counterparts of nonmaximal Kodaira dimension. This state of affairs is not only true geometrically, but arithmetically as well. In particular, given that surfaces of general type are, in some sense, the most common among all surfaces, they offer an important testing ground for a number of well-known, wide open conjectures in arithmetic geometry.

A second class of arithmetically intriguing surfaces are those with geometric genus one. Via the Hodge structure on their middle singular cohomology groups, these surfaces are related to objects that have traditionally received more attention in arithmetic geometry, namely abelian varieties, K3 surfaces, and Shimura varieties. In particular, the relation with abelian varieties, first discovered by Kuga and Satake [KS], is *a priori* only of a transcendental nature, but one expects it to be algebraic in light of the Hodge Conjecture. This expectation opens one to the possibility of being able to transfer known arithmetic results for abelian varieties to surfaces of geometric genus one, an idea first explored by Deligne [Del1].

With this in mind, we focus on a class of surfaces of general type that also have geometric genus one. More specifically, they are surfaces with geometric genus $p_g = 1$, irregularity $q = 1$, self-intersection number $K^2 = 3$ of the canonical divisor, and Albanese fiber genus $g = 3$. These surfaces were introduced and classified over $\mathbb{C}$ by Catanese and Ciliberto [CC].

For this reason, they have been called *Catanese-Ciliberto surfaces of fiber genus three* by Ishida [Ish], but for brevity they will be referred to in this thesis simply as *CC surfaces.*

Catanese and Ciliberto showed that the canonical models of all complex CC surfaces fit into a projective flat family over a smooth irreducible 5-dimensional base. When its canonical model is smooth, we will call a CC surface *admissible.* Since it turns out that the general CC surface has this property, it follows that there is a smooth projective family $\pi_{\mathbb{C}} : \mathcal{X}_{\mathbb{C}} \to \mathcal{S}_{\mathbb{C}}$ containing all admissible complex CC surfaces, where $\mathcal{S}_{\mathbb{C}}$ is a smooth irreducible variety of dimension 5. Our first theorem concerns the monodromy representation of the topological fundamental group $\pi_1(\mathcal{S}(\mathbb{C}), \sigma)$ on the second singular cohomology of the fiber $\mathcal{X}_\sigma$; to state it, we need some notation.

Every CC surface has two numerically independent curves, one being the canonical divisor K and the other a smooth Albanese fiber f. In the family $\pi_{\mathbb{C}} : \mathcal{X}_{\mathbb{C}} \to \mathcal{S}_{\mathbb{C}}$, one can show that the cycle classes of K and f in $H^2(\mathcal{X}_\sigma, \mathbb{Q})(1)$ both come from global sections of the local system $\mathbb{H} := R^2(\pi_{\mathbb{C}}^{\mathrm{an}})_*\mathbb{Q}(1)$. Note also that, as each $\mathcal{X}_\sigma$ is admissible, the class of K is ample. Denote by $\mathbb{V}$ the polarized variation of rational Hodge structure over $\mathcal{S}_{\mathbb{C}}$ that one gets by taking the orthogonal complement in $\mathbb{H}$ of these two global sections with respect to cup product. Let ϕ_σ denote the polarization on $\mathbb{V}_\sigma$. Then the Hodge structure $\mathbb{V}_\sigma$ is of dimension 9 and type $\{(-1,1),(0,0),(1,-1)\}$, with a polarization of signature $(2,7)$.

Our main theorem is a large monodromy result for the family $\pi_{\mathbb{C}} : \mathcal{X}_{\mathbb{C}} \to \mathcal{S}_{\mathbb{C}}$:

Theorem A. *The image of the monodromy representation*

$$\Lambda : \pi_1(\mathcal{S}(\mathbb{C}), \sigma) \to \mathrm{O}(\mathbb{V}_\sigma, \phi_\sigma)$$

is Zariski-dense.

Now suppose one has an admissible CC surface X defined over a finitely generated subfield k of $\mathbb{C}$. The work of Kuga and Satake mentioned above gives a Hodge correspondence between the primitive part of the second cohomology of $X_{\mathbb{C}}$ and that of a certain complex abelian variety $\mathrm{KS}(X_{\mathbb{C}})$, called its Kuga-Satake variety. Using Theorem A, one can show that $\mathrm{KS}(X_{\mathbb{C}})$ has a model A over a finite extension k' of k. After applying a theorem of Polizzi [Pol] and Deligne's "Principle B" [DMOS], it follows that the Hodge correspondence between $X_{\mathbb{C}}$ and $\mathrm{KS}(X_{\mathbb{C}})$ actually arises from an absolute Hodge correspondence between $X_{k'}$ and A. Using this and the work of Faltings [FW], one obtains the following:

Theorem B. *Let X be an admissible CC surface defined over a finitely generated field k with algebraic closure $\bar{k}$. For a prime number ℓ, let*

$$r_\ell : \mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}\big(H^2(X_{\bar{k}}, \mathbb{Q}_\ell)(1)\big)$$

denote the action of the absolute Galois group on the Tate-twisted second ℓ-adic cohomology of $X_{\bar{k}}$. Then the following hold:

(i) The representation r_ℓ is semisimple.

(ii) (Tate Conjecture) Let V_{alg} be the $\mathbb{Q}_\ell$-subspace generated by the image of the cycle class map

$$c_\ell : \mathrm{CH}^1(X_{\bar{k}}) \to H^2(X_{\bar{k}}, \mathbb{Q}_\ell)(1).$$

Then V_{alg} is exactly the subspace of elements in $H^2(X_{\bar{k}}, \mathbb{Q}_\ell)(1)$ that are stablized by an open subgroup of $\mathrm{Gal}(\bar{k}/k)$.

(iii) (Mumford-Tate Conjecture) Let $\mathfrak{G} = \mathrm{MT}(H^2(X_{\mathbb{C}}, \mathbb{Q})(1))$ denote the Mumford-Tate group of the Hodge structure $H^2(X_{\mathbb{C}}, \mathbb{Q})(1)$ and, by comparison, identify $\mathfrak{G}_{\mathbb{Q}_\ell}$ with an algebraic subgroup of $\mathrm{Aut}\big(H^2(X_{\bar{k}}, \mathbb{Q}_\ell)(1)\big)$. Then the image $r_\ell\big(\mathrm{Gal}(\bar{k}/k)\big)$ is a Lie subgroup of $\mathrm{Aut}\big(H^2(X_{\bar{k}}, \mathbb{Q}_\ell)(1)\big)$ and we have

$$\mathrm{Lie}\, r_\ell\big(\mathrm{Gal}(\bar{k}/k)\big) = \mathrm{Lie}\, \mathfrak{G}(\mathbb{Q}_\ell).$$

A second application of Theorem A is to Picard numbers of CC surfaces. One easily shows that all CC surfaces have Picard number $2 \le \rho \le 9$, and Polizzi [Pol] gives examples of complex CC surfaces that show this upper bound is sharp. On the other hand, Theorem A can be used to show that the lower bound is also sharp over $\mathbb{C}$, and this in turn combines with a general theorem of André [And2] to give:

Theorem C. *There exist CC surfaces X over $\bar{\mathbb{Q}}$ with Picard number $\rho(X) = 2$.*

To describe the context of our next theorem, we begin by remarking that the method described above to deduce parts (i) and (ii) of Theorem B from Theorem A has been known to experts for some time (e.g., see [Tat, p.80]). The prototype is the case of K3 surfaces, which follows from work of Deligne [Del1]. André [And1] axiomatizes this strategy and shows how it also implies the Mumford-Tate Conjecture. He shows these axioms apply not only to K3 surfaces, but also to abelian surfaces, a class of surfaces of general type appearing in [Cat, Tod], and cubic fourfolds (where one is concerned with the cohomology group H^4 rather than H^2). In deducing Theorem B from Theorem A, we follow the proof laid out in [And1], but at certain steps we must account for one key difference. In each of the aforementioned cases, the proof of the large monodromy theorem is obtained as a corollary of the following fact: the image of the period map of the family in question contains a Euclidean open ball in the period domain. In the case of CC surfaces, though, where the dimension of moduli is 5 and the dimension of the relevant period domain is 7, this method of proof cannot work.

Fortunately, this inequality of dimensions has a positive aspect as well: if Theorem A can be proved by other means, then one obtains as a corollary an interesting subvariety of a Shimura variety. Indeed, the classifying space for the Hodge structure on the primitive second cohomology of a complex CC surface is a 7-dimensional Hermitian symmetric domain, and so certain quotients yield 7-dimensional Shimura varieties of orthogonal type. Upon taking an appropriate base change $\pi' : \mathcal{X}' \to \mathcal{S}'$ of the family $\pi : \mathcal{X} \to \mathcal{S}$, one obtains a period map

$$\Phi : \mathcal{S}'_{\mathbb{C}} \to \mathcal{V},$$

where $\mathcal{V}$ is a connected component of such a Shimura variety. The image of Φ is an algebraic subvariety of $\mathcal{V}$ by a theorem of Borel [Bor]. Taking its closure $Z := \overline{\Phi(\mathcal{S}'_{\mathbb{C}})}$, one can show that $1 \leq \dim Z \leq 5$. The key property of Z obtained from Theorem A is:

Theorem D. *The subvariety Z is not contained in any proper special subvariety of $\mathcal{V}$. In particular, Z itself is nonspecial.*

Recall that a special subvariety of a Shimura variety is an irreducible component of a Hecke-translated Shimura subvariety. (Alternatively, this is called a subvariety of Hodge type or an irreducible component of a Hirzebruch-Zagier cycle.) Special subvarieties can be combined to give explicit algebraic cycles on the given Shimura variety but, in general, this construction does not give all cycles on the Shimura variety, even modulo homological equivalence (in some smooth compactification). Despite this, it appears to be difficult to explicitly construct examples of nonspecial subvarieties, and herein lies the interest of Theorem D. A similar but more explicit example in this direction is given in Theorem E below.

Let us now say a few words about how Theorem A is proved. The route that we take parallels the classical work of Lefschetz regarding the monodromy of the family of smooth hyperplane sections of a fixed smooth projective variety (see [Lam]). The reason for the parallel is the following. In their classification of complex CC surfaces, Catanese and Ciliberto showed that the admissible CC surfaces X with $\mathrm{Alb}(X) = E$ are exactly the smooth divisors in a certain complete linear system $|\mathfrak{D}|$ on the symmetric cube $E^{(3)}$ of E. Moreover, by construction of their smooth family $\pi_{\mathbb{C}} : \mathcal{X}_{\mathbb{C}} \to \mathcal{S}_{\mathbb{C}}$, there is a 4-dimensional subvariety of $\mathcal{S}_{\mathbb{C}}$ over which the pullback of $\mathcal{X}_{\mathbb{C}}$ is exactly the family of smooth divisors in $|\mathfrak{D}|$. To prove Theorem A, it suffices to instead prove that the monodromy of one of these subfamilies has dense image; more explicitly, it suffices to pick an elliptic curve E and prove that the monodromy of the family of all smooth divisors in $|\mathfrak{D}|$ has dense image.

Since $\mathfrak{D}$ is not very ample on $E^{(3)}$, Lefschetz's theory does not apply directly. Nevertheless, we show that a mild generalization results in a number of hypotheses that, if satisfied, will give the proof. The most difficult of these hypotheses to verify concern the structure of the collection of singular elements in $|\mathfrak{D}|$: they say that this collection must have exactly one irreducible component of codimension one in $|\mathfrak{D}|$ and that the general singular element has a singular locus of just one ordinary double point. We use equations for étale covers of elements of $|\mathfrak{D}|$ given by Ishida [Ish] to show that this holds when E is the following elliptic curve:

$$E_1\colon \quad y^2 = x^3 + x^2 - 59x - 783/4.$$

Specifically, we use the computer program Singular to find one suitably nice pencil $J_1 \subseteq |\mathfrak{D}|$ on $E_1^{(3)}$, and then show that this implies the necessary facts about $|\mathfrak{D}|$. This gives the proof.

Finally, let us comment on this pencil J_1. Using the period map, the smooth elements of J_1 give rise to a curve C in $\mathcal{V}$. One consequence of the generalized Lefschetz theory used in the proof of Theorem A is that pencils in general position in the linear system $|\mathfrak{D}|$ on $E_1^{(3)}$ give 1-dimensional families with large monodromy. In particular, one can show that J_1 is in general position, which allows one to prove:

Theorem E. *The curve C is not contained in any proper special subvariety of $\mathcal{V}$, and hence is not a special subcurve of $\mathcal{V}$. Moreover, C is numerically nontrivial.*

This result is of course very similar to Theorem D, but it warrants separate mention due to the low dimension and explicit nature of J_1. Indeed, the equations of Ishida that describe J_1 make the curve C a much more tractable object than the variety Z. Hopefully there is more to be said about this interesting curve and ones like it.

Here is a brief outline of the thesis.

In §2 we give background on CC surfaces, indicating in particular how the work of Catanese and Ciliberto easily implies similar results for admissible CC surfaces over any algebraically closed subfield of $\mathbb{C}$. We carry out our analysis of the singular elements of $|\mathfrak{D}|$ on $E_1^{(3)}$ in §3 . In §4 we give the description of the generalization of Lefschetz's theory, which we then apply to prove Theorem A. Theorem B is proved in §5 following [And1]. Finally, we prove Theorems C, D and E in §6, all of which are deduced from an intermediate corollary of Theorem A about the generic Mumford-Tate group of $\mathbb{V}$.

We end by noting that the source code and data files for all computer calculations referred to in this thesis (specifically, in Propositions 3.4, 3.5, and 3.7) are available online at [Lyo].

Notations and Conventions: Along with that mentioned at the beginning of this chapter, there are various points where we establish further conventions about the fields under consideration. More precisely, the reader should take note at the beginning of §3, §4.3, and §5.

For a rational Hodge structure W of weight w, we define the Mumford-Tate group $\mathrm{MT}(W)$ as in [PS2, p.30]. That is, if $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ and $h : \mathbb{S} \to \mathrm{GL}(W_{\mathbb{R}})$ defines the Hodge structure on W, then $\mathrm{MT}(W)$ is the largest algebraic subgroup of $\mathrm{GL}(W)$ defined over $\mathbb{Q}$ such that $\mathrm{Im}(h) \subseteq \mathrm{MT}(W)_{\mathbb{R}}$.

For a smooth projective variety X over a field k, let $\rho = \rho(X)$ denote its *geometric* Picard number, i.e., ρ is the rank of the Néron-Severi group $\mathrm{NS}(X_{\bar{k}}) := \mathrm{Pic}(X_{\bar{k}})/\mathrm{Pic}^0(X_{\bar{k}})$ of $X_{\bar{k}}$.

Chapter 2

Preliminaries on CC surfaces

2.1

We begin by giving the definition of CC surfaces in terms of abstract invariants:

Definition. *Let k be a field with algebraic closure $\bar{k}$. A surface X over $\bar{k}$ will be called a* CC surface *if it possesses the following invariants:*

- *geometric genus $p_g = h^0(X, \Omega_X^2) = 1$*
- *irregularity $q = h^1(X, \mathcal{O}_X) = 1$*
- *self-intersection number $K^2 = 3$ of the canonical divisor K*
- *As the Albanese variety $\mathrm{Alb}(X)$ is of dimension $q = 1$, the Albanese map $\mathrm{Alb} : X \to \mathrm{Alb}(X)$ gives a fibration; then the fibers of Alb should have genus $g = 3$.*

We call X admissible *if, additionally, its canonical model is smooth or, equivalently, if K is ample.*

A surface X over k will be called an (admissible) CC surface *if the base change $X_{\bar{k}}$ is an (admissible) CC surface.*

These are surfaces of general type. In [CC], Catanese and Ciliberto [CC] classify all CC surfaces over $\mathbb{C}$. To describe their classification, we need some preliminary notation.

If E is a complex elliptic curve, let $\oplus$ denote the addition law and let $E^{(3)}$ denote its symmetric cube. Then the Abel-Jacobi map

$$\begin{aligned} \mathrm{AJ} \ &: \ E^{(3)} \longrightarrow E \\ &[a, b, c] \mapsto a \oplus b \oplus c \end{aligned}$$

sending an unordered triple of points on E to their sum makes $E^{(3)}$ into a $\mathbb{P}^2$-bundle over E. Denoting by 0 the identity on E, define on $E^{(3)}$ the two divisors

$$\begin{aligned} D_0 &:= \{[0, b, c] \mid b, c \in E\}, \\ F_0 &:= \mathrm{AJ}^{-1}(0) = \{[a, b, c] \mid a \oplus b \oplus c = 0\}. \end{aligned}$$

We set

$$\mathfrak{D} := 4D_0 - F_0. \tag{2.1}$$

Theorem 2.1 (Catanese–Ciliberto [CC])**.** *Let X be a CC surface over $\mathbb{C}$ and let $E = \mathrm{Alb}(X)$. Then there is a morphism $\xi : X \to E^{(3)}$ such that*

(i) the image $\xi(X)$ is isomorphic to the canonical model of X and

(ii) such that $\xi(X)$ to a divisor in the linear system $|\mathfrak{D}|$ on $E^{(3)}$.

Conversely, if E is any complex elliptic curve, then the general element of the linear system $|\mathfrak{D}|$ on $E^{(3)}$ is smooth, and any element of $|\mathfrak{D}|$ with at most rational double points is the canonical model of a CC surface with Albanese variety E.

Catanese and Ciliberto show that

$$h^0(E^{(3)}, \mathcal{O}_{E^{(3)}}(\mathfrak{D})) = 5, \tag{2.2}$$

so that the parameter space for the elements of $|\mathfrak{D}|$ is isomorphic to $\mathbb{P}^4$. We also note that, if E is an elliptic curve over the algebraically closed field k, then $E^{(3)}$, D_0, and F_0 are all defined over k. Thus CC surfaces exist over k as well:

Corollary 2.2. *Let k be an algebraically closed field and let E be an elliptic curve over k. Then the minimal resolution of any element in the linear system $|\mathfrak{D}|$ on $E^{(3)}$ with at most rational double points is a CC surface over k with Albanese variety E. Furthermore, the general element of $|\mathfrak{D}|$ is an admissible CC surface over k.*

We note that the divisors in $|\mathfrak{D}|$ are not hyperplane sections of $E^{(3)}$:

Proposition 2.3. *The line bundle $\mathcal{O}_{E^{(3)}}(\mathfrak{D})$ is ample but not very ample.*

Proof. It suffices to prove this when the base field is $\mathbb{C}$. The fact that $\mathcal{O}_{E^{(3)}}(\mathfrak{D})$ is ample is shown in [CC, Prop. 1.14].

On the other hand, suppose for contradiction that $\mathcal{O}_{E^{(3)}}(\mathfrak{D})$ is very ample. Then it gives an embedding of $E^{(3)}$ as a hypersurface in $\mathbb{P}^4$. One can use [CC, Theorem 1.17] to calculate the Hilbert polynomial of $E^{(3)}$ with respect to this embedding and show that the hypersurface must be of degree 16. But a hypersurface in $\mathbb{P}^4$ of degree 16 has middle Betti number $\frac{15}{16}(15^4 - 1) = 47460$. Since $h^3(E^{(3)}, \mathbb{Q}) = 2$, this is impossible. $\square$

2.2

In [CC], Catanese and Ciliberto describe the construction of a family over a smooth connected base that contains all complex admissible CC surfaces. In fact, their construction works over any algebraically closed k, as we show below.

The basic idea draws from Theorem 2.1, which can be used to construct a family containing all admissible CC surfaces with fixed Albanese variety E; indeed, these are simply the smooth divisors in the linear system $|\mathfrak{D}|$ on the symmetric cube of E, and so one can put them into a family over an open subset of $\mathbb{P}^4$. In the general case below, we work instead with the *relative* symmetric cube of a universal elliptic curve over a modular curve. The parameter space will not be an open subset of $\mathbb{P}^4$, but rather an open subset of a $\mathbb{P}^4$-bundle over this modular curve.

Let Y be an open connected modular curve over k with sufficient level structure to guarantee that we have a universal elliptic curve $\mathcal{E} \to Y$. For instance, if $N > 3$, one can set $Y = Y_1(N)$.

We denote the identity section of $\mathcal{E} \to Y$ by

$$O : Y \to \mathcal{E};$$

this map is a closed immersion, giving a closed subscheme of $\mathcal{E}$ that we also denote by O. Let the *relative* cube of $\mathcal{E}$ be

$$\mathcal{E}^3 := \mathcal{E} \times_Y \mathcal{E} \times_Y \mathcal{E},$$

which has three projection maps $p_i : \mathcal{E}^3 \to \mathcal{E}$. Another map of Y-schemes between $\mathcal{E}^3$ and

$\mathcal{E}$ is given by addition, which denote by

$$\oplus : \mathcal{E}^3 \to \mathcal{E}.$$

The group S_3 acts on $\mathcal{E}^3$ as follows:

$$\sigma \in S_3 \;\longleftrightarrow\; (p_{\sigma 1}, p_{\sigma 2}, p_{\sigma 3}) \in \mathrm{Aut}(\mathcal{E}^3).$$

According to [MFK, Thm 1.10], there exists a quotient variety $\mathcal{E}^{(3)}$ and a finite quotient map $q : \mathcal{E}^3 \to \mathcal{E}^{(3)}$ that one can show has a number of expected properties, including:

- There is a unique Y-scheme structure $p : \mathcal{E}^{(3)} \to Y$ such that the diagram

$$\begin{array}{ccc} \mathcal{E}^3 & \xrightarrow{q} & \mathcal{E}^{(3)} \\ & \searrow & \downarrow p \\ & & Y \end{array}$$

  is commutative.

- For a geometric point $y \in Y(\Omega)$, the fiber $(\mathcal{E}^{(3)})_y$ is the symmetric cube $\mathcal{E}_y^{(3)}$ of $\mathcal{E}_y$.

- The morphism $p : \mathcal{E}^{(3)} \to Y$ is smooth and projective.[1]

Define the following divisors on $\mathcal{E}^3$:

$$\begin{aligned} \mathcal{D}_0' &:= p_1^{-1}(O), \\ \mathcal{F}_0' &:= \oplus^{-1}(O). \end{aligned}$$

Define $\mathcal{D}_0$ (resp., $\mathcal{F}_0$) to be the image of $\mathcal{D}_0'$ (resp., $\mathcal{F}_0'$) under q in $\mathcal{E}^{(3)}$. Then $\mathcal{D}_0$ and $\mathcal{F}_0$ give divisors on $\mathcal{E}^{(3)}$, and we form a third divisor $4\mathcal{D}_0 - \mathcal{F}_0$ with associated invertible sheaf $\mathcal{L} := \mathcal{O}_{\mathcal{E}^{(3)}}(4\mathcal{D}_0 - \mathcal{F}_0)$.

For $y \in Y(\mathbb{C})$, let $\mathcal{L}_y$ denote the pullback of $\mathcal{L}$ to the fiber $\mathcal{E}_y^{(3)}$. If we denote the scheme-theoretic intersections of $\mathcal{D}_0$ and $\mathcal{F}_0$ with the fiber $\mathcal{E}_y^{(3)}$ by $\mathcal{D}_{0,y}$ and $\mathcal{F}_{0,y}$, then

$$\mathcal{L}_y = \mathcal{O}_{\mathcal{E}_y^{(3)}}(4\mathcal{D}_{0,y} - \mathcal{F}_{0,y}). \tag{2.3}$$

[1] In the sense of [Gro, 5.5].

Note that if we write $\mathcal{E}_y = E$ then, in our previous notation, $\mathcal{D}_{0,y}$ is just D_0, $\mathcal{F}_{0,y}$ is F_0, and thus (2.3) says that $\mathcal{L}_y$ on $\mathcal{E}_y^{(3)}$ is $\mathcal{O}_{E^{(3)}}(\mathfrak{D})$ on $E^{(3)}$. Thus by (2.2), we have

$$\dim_{\mathbb{C}} H^0(\mathcal{E}_y^{(3)}, \mathcal{L}_y) = 5.$$

If y actually arises from a point $x \in Y(k)$, so that $\mathcal{E}_y^{(3)}$ is the base change to $\mathbb{C}$ of $\mathcal{E}_x^{(3)}$ and $\mathcal{L}_y$ is the pullback of $\mathcal{L}_x$, this implies that

$$\dim_k H^0(\mathcal{E}_x^{(3)}, \mathcal{L}_x) = 5.$$

Therefore, since p is proper and since $\mathcal{L}$ is flat over Y, the sheaf $\mathcal{W} = p_*\mathcal{L}$ is a locally constant sheaf of rank 5 with the property that

$$H^0(\mathcal{E}_x^{(3)}, \mathcal{L}_x) = \mathcal{W}_x \otimes \kappa(x).$$

Let $\mathcal{S}_0$ be the $\mathbb{P}^4$-bundle $\mathbb{P}(\mathcal{W}) := \operatorname{Proj}(\operatorname{Sym}(\mathcal{W}^\vee))$ over Y. The fiber of $\mathcal{S}_0$ over $x \in Y(k)$ is $\mathbb{P}H^0(\mathcal{E}_x^{(3)}, \mathcal{L}_x)$. We form the fiber product $\mathcal{E}^{(3)} \times_Y \mathcal{S}_0$ and define on it a divisor

$$\mathcal{X}_0 = \left\{ (Q, \sigma) \in \mathcal{E}^{(3)} \times_Y \mathcal{S}_0 \mid \sigma(Q) = 0 \right\}.$$

Denote the projection onto $\mathcal{S}_0$ by $\pi : \mathcal{X}_0 \to \mathcal{S}_0$.

Proposition 2.4. *The morphism $\pi : \mathcal{X}_0 \to \mathcal{S}_0$ is flat and projective.*

Proof. The flatness can be proved locally, so we choose affine open sets $U = \operatorname{Spec} A \subseteq Y$, $V = \operatorname{Spec} B \subseteq \mathcal{S}_0$, and $W = \operatorname{Spec} C \subseteq \mathcal{E}^{(3)}$ such that V and W map into U. More precisely, we assume that $(p_*\mathcal{L})|_U$ is a trivial $\mathcal{O}_U$-module of rank 5, and we let $\{\sigma_0, \ldots, \sigma_4\}$ be a basis of sections of $p_*\mathcal{L}$ over U (which, by definition, are just sections of $\mathcal{L}$ over $\pi^{-1}(U)$). Then $\mathcal{S}_0|_U \simeq U \times \mathbb{P}^4$, with $(\sigma_0^\vee : \ldots : \sigma_4^\vee)$ forming homogeneous coordinates over U. We let $V = \{\sigma_0^\vee \neq 0\}$, so that if $t_i = \sigma_i^\vee / \sigma_0^\vee$, then $B = A[t_1, \ldots, t_4]$. We also suppose that σ_i is represented in W by a single element $f_i \in C$. Then $\mathcal{X}_0$ is represented in $V \times_U W = \operatorname{Spec}(B \otimes_A C)$ by the element $f_0 + f_1 t_1 + \ldots + f_4 t_4 \in B \otimes_A C = C[t_1, \ldots, t_4]$. To prove flatness in this explicit local setting, one can adapt the method of [Mum, §10, Ex. P].

Next, since $p : \mathcal{E}^{(3)} \to Y$ is projective, so is its base change

$$\mathcal{E}^{(3)} \times_Y \mathcal{S}_0 \to \mathcal{S}_0.$$

But π is the composition of a closed immersion and this projection, so it is projective as well. $\square$

Since π is proper, it maps closed sets to closed sets. In particular, the set

$$\{(Q,\sigma) \in \mathcal{X}_0 \mid \sigma \text{ is singular at } Q\} \subseteq \mathcal{X}_0$$

is a closed subset of $\mathcal{X}_0$, and so projects to a closed subset of $\mathcal{S}_0$ that represents the locus of singular fibers of π. We let $\mathcal{S} \subseteq \mathcal{S}_0$ denote its open complement, which represents the locus of smooth fibers of π by Proposition 2.4. By construction, if $y \in Y(k)$ corresponds to the elliptic curve $E = \mathcal{E}_y$, then the fiber of $\mathcal{S}_0$ over y parametrizes the elements of the linear system $|4\mathcal{D}_{0,y} - \mathcal{F}_{0,y}| = |\mathfrak{D}|$ on $\mathcal{E}_y^{(3)} = E^{(3)}$. Hence by Corollary 2.2, $\mathcal{S}$ is nonempty (in fact, has nonempty intersection with each fiber of $\mathcal{S}_0$ over Y). In particular, $\dim \mathcal{S} = 5$. Let $\mathcal{X} = \pi^{-1}(\mathcal{S})$ and by abuse of notation we again use π to denote the projection

$$\pi : \mathcal{X} \to \mathcal{S}.$$

Corollary 2.5. *The morphism $\pi : \mathcal{X} \to \mathcal{S}$ is smooth and projective.*

By construction every fiber of $\pi : \mathcal{X} \to \mathcal{S}$ is an admissible CC surface. The next aim is to prove that any admissible CC surface over k appears in the family $\pi : \mathcal{X} \to \mathcal{S}$. This will follow from the analogous result over $\mathbb{C}$:

Theorem 2.6 (Catanese-Ciliberto)**.** *Form the base change $\pi_{\mathbb{C}} : \mathcal{X}_{\mathbb{C}} \to \mathcal{S}_{\mathbb{C}}$ of π to $\mathbb{C}$. Then every fiber of this family is a complex admissible CC surface and, conversely, any complex admissible CC surface is isomorphic to a fiber of this family.*

Corollary 2.7. *Let $k_0 \subseteq k$ be a subfield such that $\bar{k}_0 = k$. Let X be an admissible CC surface over k_0. Then there is a finite extension k_0' of k_0 such that*

1. *the varieties $\mathcal{X}$, $\mathcal{S}$, and $\pi : \mathcal{X} \to \mathcal{S}$ are all defined over k_0', and*

2. there is a point $s \in \mathcal{S}(k_0')$ such that $X \otimes_{k_0} k_0' \simeq \mathcal{X}_s$.[2]

Proof. By a theorem of Gieseker [Gie], there is a coarse moduli space for CC surfaces that is a quasiprojective variety defined over $\bar{\mathbb{Q}}$, and hence over k. This moduli space is irreducible, as can be seen from the fact that a subfamily of $\mathcal{X}_0 \to \mathcal{S}_0$ contains the canonical models of all complex CC surfaces and from the irreducibility of $\mathcal{S}_0$. Moreover, there is an open subset $\mathcal{M}$ of this moduli space parametrizing those CC surfaces whose canonical model is smooth, i.e., the admissible CC surfaces. By Theorem 2.6, the family $\pi : \mathcal{X} \to \mathcal{S}$ yields a morphism of varieties $f : \mathcal{S} \to \mathcal{M}$ over k such that the base change $f_{\mathbb{C}} : \mathcal{S}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}}$ is surjective on closed points. It follows that $f_{\mathbb{C}}$ is a surjective morphism of topological spaces. Since $\mathcal{M}_{\mathbb{C}} \to \mathcal{M}$ is also surjective, the commutative diagram

$$\begin{array}{ccc} S_{\mathbb{C}} & \xrightarrow{f_{\mathbb{C}}} & \mathcal{M}_{\mathbb{C}} \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & \mathcal{M} \end{array}$$

then implies that $f : S \to \mathcal{M}$ is a surjection.

In particular, there is some $s' \in S(k)$ such that $f(s') = [X \otimes_{k_0} k] \in \mathcal{M}(k)$, which implies that

$$\mathcal{X}_{s'} \simeq X \otimes_{k_0} k \tag{2.4}$$

over k. So we take k_0' to be some finite extension of k_0 over which $\mathcal{X}$, $\mathcal{S}$, and π are defined, and furthermore such that s' arises from a point $s \in \mathcal{S}(k_0')$ and the isomorphism (2.4) arises from a k_0'-isomorphism $X \otimes_{k_0} k_0' \simeq \mathcal{X}_s$ of k_0'-varieties. □

From the construction of the family $\pi : \mathcal{X} \to \mathcal{S}$, we can now conclude:

Corollary 2.8. *Let k be algebraically closed. Then every isomorphism class of admissible CC surfaces over k is realized as a smooth element of the linear system $|\mathfrak{D}|$ on $E^{(3)}$ for some elliptic curve E over k.*

Finally, we record a result of Polizzi that will be of importance in §5:

Theorem 2.9 (Polizzi). *If E is any complex elliptic curve, then there exists on $E^{(3)}$ a smooth divisor in $|\mathfrak{D}|$ having maximal Picard number 9. Furthermore, the Hodge structure*

[2] Note that we are abusing notation here; we should really speak of a model $\mathcal{S}'$ for $\mathcal{S}$ over k_0' and of $s \in \mathcal{S}'(k_0')$.

on its middle cohomology is of CM-type (i.e., its Mumford-Tate group is abelian).

Proof. See [Pol, Prop. 6.18] for the existence of the smooth divisor with Picard number 9. Since the Hodge group of its middle cohomology is a necessarily a subgroup of $SO(2)$, its Mumford-Tate group is abelian. □

Chapter 3

Mildly singular elements in $|\mathfrak{D}|$

Note: In §3, the base field will always be $\mathbb{C}$.

3.1

Let E be a complex elliptic curve and let $|\mathfrak{D}|$ be the complete linear system on $E^{(3)}$ defined in (2.1). Here we describe the techniques used to make certain calculations about the elements of $|\mathfrak{D}|$. These techniques are found in [Ish], which in turn draws from more general situations considered in [Tak], and we refer to either of these sources for more details.

The two fundamental observations that underlie the relevant equations in [Ish] are the following:

1. The Abel-Jacobi map

$$\mathrm{AJ} : E^{(3)} \to E, \quad [a, b, c] \mapsto a \oplus b \oplus c,$$

   makes $E^{(3)}$ into a $\mathbb{P}^2$-bundle $\mathbb{P}(B) \to E$, where B can be taken to be an indecomposable locally free sheaf of rank 3 and degree 1.

2. Let $\tilde{E}$ be an elliptic curve with identity $\tilde{0} \in \tilde{E}$, and let $\varphi : \tilde{E} \to E$ be an isogeny of degree 3. Then $B' := \varphi_* \mathcal{O}_{\tilde{E}}(\tilde{0})$ is an indecomposable locally free sheaf of rank 3 and degree 1 with the property that $\varphi^* B'$ is a direct sum of three invertible sheaves on $\tilde{E}$.

As $E^{(3)} = \mathbb{P}(B)$ and $\mathbb{P}(B')$ are isomorphic, we fix an identification between the two. Defining the $\mathbb{P}^2$-bundle $\tilde{P} := \mathbb{P}(\varphi^* B')$, with projection $\tilde{p} : \tilde{P} \to \tilde{E}$, one obtains a commutative diagram

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\Phi} & E^{(3)} \\ \Big\downarrow{\scriptstyle \tilde{p}} & & \Big\downarrow{\scriptstyle \mathrm{AJ}} \\ \tilde{E} & \xrightarrow{\varphi} & E \end{array}$$

in which $\tilde{P}$ is the fiber product of $\tilde{E}$ and $E^{(3)}$ over E. Thus, if we let $G = \ker \varphi = \{\tilde{0}, C_1, C_2\}$, then both φ and Φ are Galois coverings with group G. More specifically, if $Q \in \tilde{E}$ and $\tau_Q \in \mathrm{Aut}(\tilde{E})$ denotes translation by Q, then $\gamma \in G$ acts on $\tilde{E}$ by τ_γ and on $\tilde{P}$ by the base change $\tilde{\tau}_\gamma$ of τ_γ.

Define $L = \mathcal{O}_{E^{(3)}}(\mathfrak{D})$. Then the idea is to transfer the study of sections of L to the study of G-invariant sections of $\Phi^* L$:

Proposition 3.1. *The pullback map*

$$\Phi^* : H^0(E^{(3)}, L) \longrightarrow H^0(\tilde{P}, \Phi^* L)^G. \tag{3.1}$$

is an isomorphism.

The advantage of this is that equations for elements belonging to the right hand side of (3.1) are simpler than those on the left. This is due to the aforementioned fact the locally free sheaf $\varphi^* B'$ splits into a sum of invertible sheaves:

$$\varphi^* B' \simeq \mathcal{O}_{\tilde{E}}(\tilde{0}) \oplus \mathcal{O}_{\tilde{E}}(C_1) \oplus \mathcal{O}_{\tilde{E}}(C_2). \tag{3.2}$$

Thus

$$\begin{aligned} H^0(\tilde{P}, \mathcal{O}_{\tilde{P}}(1)) &\simeq H^0(\tilde{E}, \varphi^* B') \\ &\simeq H^0(\tilde{E}, \mathcal{O}_{\tilde{E}}(\tilde{0})) \oplus H^0(\tilde{E}, \mathcal{O}_{\tilde{E}}(C_1)) \oplus H^0(\tilde{E}, \mathcal{O}_{\tilde{E}}(C_2)). \end{aligned}$$

Let Z_0 (resp., Z_1, Z_2) denote the rational function in $H^0(\tilde{P}, \mathcal{O}_{\tilde{P}}(1))$ that corresponds to the constant function 1 in $H^0(\tilde{E}, \mathcal{O}_{\tilde{E}}(\tilde{0}))$ (resp., $H^0(\tilde{E}, \mathcal{O}_{\tilde{E}}(C_1))$, $H^0(\tilde{E}, \mathcal{O}_{\tilde{E}}(C_2))$). The action of G on these functions is described by

$$\tilde{\tau}^*_{C_1} Z_0 = Z_2, \qquad \tilde{\tau}^*_{C_2} Z_0 = Z_1. \tag{3.3}$$

Following [Tak, p.286], Ishida [Ish, Lemma 1.4] uses the functions Z_0, Z_1, Z_2 to produce five equations that span $H^0(\tilde{P}, \Phi^* L)^G$, which are described as follows. First pick an affine equation $y^2 = w(x)$ for $\tilde{E}$, where w is a monic cubic polynomial with nonzero discriminant. If $C_1 = (\alpha, \beta)$ (and thus $C_2 = (\alpha, -\beta)$), we define three rational functions on $\tilde{E}$ (and, by pullback, on $\tilde{P}$) by

$$\begin{aligned} f(Q) &:= x(Q) - \alpha, \\ g(Q) &:= x(Q \oplus C_2) - \alpha, \\ h(Q) &:= x(Q \oplus C_1) - \alpha. \end{aligned}$$

Setting $\mu = w'(\alpha)$, one can use the facts that $\beta^2 = w(\alpha)$ and that α is a root of the 3-torsion polynomial of $\tilde{E}$ to obtain

$$f = x - \alpha, \tag{3.4}$$

$$g = \frac{4\beta^2(x-\alpha)}{2\beta(y-\beta) - \mu(x-\alpha)}, \tag{3.5}$$

$$h = \frac{4\beta^2(x-\alpha)}{-2\beta(y+\beta) - \mu(x-\alpha)}. \tag{3.6}$$

One also has the relation $fgh = -4\beta^2$. From the definition, the G-action is given by

$$g = \tau_{C_2}^* f, \qquad h = \tau_{C_1}^* f. \tag{3.7}$$

Proposition 3.2 ([Ish, Tak])**.** *A basis for the space $H^0(\tilde{P}, \Phi^* L)^G$ is given by:*

$$\begin{aligned} \Psi_1 &:= fZ_0^4 + gZ_1^4 + hZ_2^4, \\ \Psi_2 &:= Z_0 Z_1 Z_2 (Z_0 + Z_1 + Z_2), \\ \Psi_3 &:= fZ_0^3 Z_2 + gZ_1^3 Z_0 + hZ_2^3 Z_1, \\ \Psi_4 &:= fZ_0^3 Z_1 + gZ_1^3 Z_2 + hZ_2^3 Z_0, \\ \Psi_5 &:= ghZ_1^2 Z_2^2 + fhZ_0^2 Z_2^2 + fgZ_0^2 Z_1^2. \end{aligned}$$

3.2

Next we describe the coordinate charts in which we will work and we adapt the equations of Proposition 3.2 to these coordinates. Let $U := \tilde{E} \setminus \{\tilde{0}, C_1, C_2\}$. Since $y^2 = w(x)$ is an affine Weierstrass equation for $\tilde{E}$ we get

$$\tilde{E} \setminus \{\tilde{0}\} \simeq \operatorname{Spec}\left(\frac{\mathbb{C}[x,y]}{\langle y^2 - w(x)\rangle}\right)$$

and, since $C_1 = (\alpha, \beta)$ and $C_2 = (\alpha, -\beta)$, it follows that

$$U \simeq \operatorname{Spec}\left(\frac{\mathbb{C}[x,y,t]}{\langle y^2 - w(x), (x-\alpha)t - 1\rangle}\right).$$

The definitions of the rational functions Z_0, Z_1, Z_2 on $\tilde{P}$, being a consequence of (3.2), show that $\tilde{P}|_U \simeq U \times \mathbb{P}^2$ via the isomorphism

$$\begin{aligned} \tilde{P}|_U &\xrightarrow{\sim} U \times \mathbb{P}^2 && (3.8)\\ r &\mapsto \big(\tilde{p}(r), \big(Z_0(r) : Z_1(r) : Z_2(r)\big)\big). && (3.9) \end{aligned}$$

Inside $\tilde{P}|_U$ we make the choice of affine open set

$$T := \tilde{P}|_U \cap \{Z_0 \neq 0\}$$

and set $u := Z_1/Z_0, v := Z_2/Z_0$. Then we have

$$T \simeq \operatorname{Spec}\left(\frac{\mathbb{C}[x,y,u,v,t]}{\langle y^2 - w(x), (x-\alpha)t - 1\rangle}\right). \tag{3.10}$$

We will work with these coordinates to establish various results about the sections in $H^0(\tilde{P}, \Phi^*L)^G$ on the dense open set $T \subseteq \tilde{P}$. In doing so, we prefer to work with polynomials in $\mathbb{C}[x,y,u,v,t]$ rather than the original equations Ψ_i. Upon setting $(Z_0 : Z_1 : Z_2) = (1 : u : v)$, we get $\Psi_i \in \mathbb{C}(x,y,u,v,t)$ on T. Hence, by clearing denominators, we can obtain a polynomial basis for $H^0(\tilde{P}, \Phi^*L)^G$ on T. More specifically, if we set

$$\begin{aligned} b_1 &:= 2\beta(y-\beta) - \mu(x-\alpha)\\ b_2 &:= -2\beta(y+\beta) - \mu(x-\alpha), \end{aligned}$$

so that (using (3.4)–(3.6)) $g = -4\beta^2(x-\alpha)/b_1$ and $h = -4\beta^2(x-\alpha)/b_2$, then we can use the relation $fgh = -4\beta^2$ to get $b_1b_2 = -4\beta^2(x-\alpha)^3$. One sees that multiplying each of the Ψ_i by b_1b_2 will clear their common denominator, and (upon removing a common factor of $(x-\alpha)$) this yields the following choice of equations ω_i:

$$\begin{aligned}
\omega_1 &:= b_1b_2 + 4\beta^2 b_2 u^4 + 4\beta^2 b_1 v^4,\\
\omega_2 &:= -4\beta^2(x-\alpha)^2 uv(1+u+v),\\
\omega_3 &:= b_1b_2 v + 4\beta^2 b_2 u^3 + 4\beta^2 b_1 v^3 u,\\
\omega_4 &:= b_1b_2 u + 4\beta^2 b_2 u^3 v + 4\beta^2 b_1 v^3,\\
\omega_5 &:= 4\beta^2(x-\alpha)\left(4\beta^2 u^2v^2 + b_1v^2 + b_2u^2\right).
\end{aligned}$$

The equations ω_i give a basis of $\Gamma(T, \Phi^*L)$ satisfying

$$(\omega_1 : \cdots : \omega_5) = (\Psi_1 : \cdots : \Psi_5)$$

on T.

Next, equations for elements of $H^0(\tilde{P}, \Phi^*L)^G$ near the fiber $\tilde{p}^{-1}(\tilde{0})$ are handled via the method given in [Ish, p.40]. Since Z_0 vanishes on $\tilde{p}^{-1}(\tilde{0})$, $(Z_0 : Z_1 : Z_2)$ do not form relative homogenous coordinates near this fiber. However, if we let $t = x/y$ (not to be confused with the t-coordinate in the affine chart T) then t is a local parameter of $\tilde{E}$ near $\tilde{0}$ and, upon setting $Z_0' = t^{-1}Z_0$, it follows that $(Z_0' : Z_1 : Z_2)$ do form relative homogeneous coordinates near $\tilde{p}^{-1}(\tilde{0})$. Furthermore, the equations

$$\chi_i\big(t, (Z_0' : Z_1 : Z_2)\big) := t^{-1}\Psi_i(tZ_0' : Z_1 : Z_2)$$

for $1 \le i \le 5$ form a basis for $H^0(\tilde{P}, \Phi^*L)^G$ near $\tilde{p}^{-1}(\tilde{0})$. When one expands f, g, h in terms of t (for details, see [Ish, p.39]), these equations become

$$\begin{aligned}
\chi_1 &= 2\beta(Z_1^4 - Z_2^4) + t(Z_0'^4 + \mu Z_1^4 + \mu Z_2^4) + \text{(higher terms)},\\
\chi_2 &= Z_0' Z_1 Z_2 (Z_1 + Z_2) + t Z_0^2 Z_1 Z_2,\\
\chi_3 &= Z_0'^3 Z_2 - 2\beta Z_1 Z_2^3 + t(\mu Z_1 Z_2^3 + 2\beta Z_0' Z_1^3) + \text{(higher terms)},\\
\chi_4 &= Z_0'^3 Z_1 + 2\beta Z_1^3 Z_2 + t(\mu Z_1^3 Z_2 - 2\beta Z_0' Z_2^3) + \text{(higher terms)},\\
\chi_5 &= 2\beta Z_0'^2 (Z_1^2 - Z_2^2) + t(\mu Z_0'^2 Z_2^2 + \mu Z_0'^2 Z_1^2 - 4\beta^2 Z_1^2 Z_2^2) + \text{(higher terms)}.
\end{aligned}$$

These are the equations we will utilize in our study of $H^0(\tilde{P}, \Phi^* L)^G$ near $\tilde{p}^{-1}(\tilde{0})$. Note that outside $\tilde{p}^{-1}(\tilde{0})$ we have

$$(\chi_1 : \cdots : \chi_5) = (\Psi_1 : \cdots : \Psi_5).$$

Finally, we address those points of $\tilde{P}$ that lie outside $T \cup \tilde{p}^{-1}(\tilde{0})$. First note that the action of G interchanges the fibers over $\{\tilde{0}, C_1, C_2\}$, and hence it also preserves their complement $\tilde{P}|_U$.

For the action of G on $r \in \tilde{P}|_U$, let $r = (u, (Z_0 : Z_1 : Z_2))$ as in (3.8). Using $\gamma * r$ to denote the action of $\gamma \in G$ on r, (3.3) gives

$$\begin{aligned}
\tilde{0} * r &= (u, (Z_0 : Z_1 : Z_2))\\
C_1 * r &= (u \oplus C_1, (Z_2 : Z_0 : Z_1))\\
C_2 * r &= (u \oplus C_2, (Z_1 : Z_2 : Z_0)).
\end{aligned} \tag{3.11}$$

Thus all elements of $\tilde{P}|_U$ have G-orbits that intersect T. This observation will allow us to perform the calculations we have in mind only on T, and to then deduce similar information about all of $\tilde{P}|_U$.

Likewise, since G permutes the fibers over $\{\tilde{0}, C_1, C_2\}$, doing calculations for the fiber $\tilde{p}^{-1}(\tilde{0})$ will in fact give us sufficient information about the other two fibers.

One instance of this use of symmetry is in the investigation of local properties of a section in $H^0(E^{(3)}, L)$: for each point $q \in E^{(3)}$, at least one point of the preimage $\Phi^{-1}(q)$ lies in either T or $\tilde{p}^{-1}(\tilde{0})$, and hence it suffices to investigate the the local properties of the lifted section in $H^0(\tilde{P}, \Phi^* L)^G$ only on $T \cup \tilde{p}^{-1}(\tilde{0})$.

Let us illustrate the usefulness of the equations given in [Ish] to answer a question posed in [CC] about $|\mathfrak{D}|$, namely whether it is base-point free.

Proposition 3.3. *There are exactly four base points of $|\mathfrak{D}|$ is smooth, each of which is simple and belongs to the fiber F_0. Furthermore, if $X \in |\mathfrak{D}|$, then the trace of the linear system $|\mathfrak{D}|$ on X also has four simple base points.*

Proof. It is shown in [CC, Lemma 3.3] that any possible base points of $|\mathfrak{D}|$ must be simple. In [Pol, Theorem 3.8], it is shown that there are at most four base points, each of which lies in the fiber F_0. Hence, to prove the proposition, it will suffice to show the existence of at least four base points in F_0.

If $q \in E^{(3)}$ is a base point of $|\mathfrak{D}|$, then all three members of $\Phi^{-1}(q)$ are base points of $\Phi^*|\mathfrak{D}|$. Conversely, if $r \in \tilde{P}$ is a base point of $\Phi^*|\mathfrak{D}|$, then so are all members of the G-orbit of r and $\Phi(r)$ is a base point of $|\mathfrak{D}|$. Since all potential base points of $|\mathfrak{D}|$ must lie in F_0, to prove the first statement it suffices to show that $\Phi^*|\mathfrak{D}|$ has at least four base points in the fiber $\tilde{p}^{-1}(\tilde{0})$.

For this purpose, we use the local equations χ_i near $\tilde{p}^{-1}(\tilde{0})$. The base points in $\tilde{p}^{-1}(\tilde{0})$ form the subvariety

$$Z(\chi_1, \ldots, \chi_5, t) \subseteq \tilde{p}^{-1}(\tilde{0}).$$

If r_1, r_2, r_3 are the roots of $x^3 - 2\beta$ (note that $\beta \neq 0$ since C_1 is 3-torsion), one finds that the four values

$$(1:0:0),\ (r_1 : 1 : -1),\ (r_2 : 1 : -1),\ (r_3 : 1 : -1)$$

for $(Z_0' : Z_1 : Z_2)$ are solutions. This proves the first statement.

For the second statement, the trace of $|\mathfrak{D}|$ on X will have exactly four base points and we must show they are simple. For this purpose, it suffices to show that, at each of the four base points of $|\mathfrak{D}|$, there are two divisors that are smooth at that base point. In fact, as one can check directly using the equations χ_1 and χ_3 and the coordinates for the base points of $\Phi^*|\mathfrak{D}|$ in $\tilde{p}^{-1}(\tilde{0})$ given above, both ψ_1 and ψ_3 are smooth at all four base points. This completes the proof. □

Remark. This gives a second proof that L is not very ample.

3.3

Let E_1 denote the elliptic curve given by the Weierstrass equation

$$E_1 \quad : \quad y^2 = x^3 + x^2 - 59x - 783/4$$

and let $\tilde{E}_1$ denote

$$\tilde{E}_1 \ : \ y^2 = w(x) := x^3 + x^2 + x - 3/4.$$

Then there is an isogeny $\varphi : \tilde{E}_1 \to E_1$ such that $\ker \varphi = \left\{\tilde{0}, C_1, C_2\right\}$, where $C_1 = (\alpha, \beta) := (1, 3/2)$. As in §3.1, we have the commutative diagram

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\Phi} & E_1^{(3)} \\ \Big\downarrow{\scriptstyle \tilde{p}} & & \Big\downarrow{\scriptstyle \mathrm{AJ}} \\ \tilde{E}_1 & \xrightarrow{\varphi} & E_1 \end{array} .$$

Define

$$S_1 := \mathbb{P}H^0(E_1^{(3)}, L) \simeq \mathbb{P}^4.$$

We will investigate the pencil $J_1 \subseteq S_1$ defined parametrically by

$$J_1 := \left\{a\psi_1 + b(\psi_3 - \psi_4) \mid (a : b) \in \mathbb{P}^1\right\}. \tag{3.12}$$

The base locus of J_1 is

$$A_1 := Z(\psi_1, \psi_3 - \psi_4) \subseteq E_1^{(3)}.$$

Proposition 3.4. A_1 *is smooth.*

Proof. Since this is a local question and $\Phi : \tilde{P} \to E_1^{(3)}$ is étale, it suffices to show that

$$\Phi^{-1}(A_1) = Z(\Psi_1, \Psi_3 - \Psi_4) \subseteq \tilde{P}$$

is smooth at the points of T and $\tilde{p}^{-1}(\tilde{0})$ (see the discussion before Proposition 3.3).

In the affine open T (to use the notation in (3.10)), $\Phi^{-1}(A_1)$ corresponds to the ideal

$$I = \left\langle y^2 - w(x), (x - \alpha)t - 1, \omega_1, \omega_3 - \omega_4 \right\rangle$$

in $\mathbb{C}[x, y, u, v, t]$. One forms the 4-by-5 Jacobian matrix of partial derivatives of the generators of I and puts its four 4-by-4 minors into an ideal I_{Jac}. Then one can use Singular to compute that the set of singular points $Z(I, I_{\text{Jac}}) \subset T$ of $\Phi^{-1}(A_1)$ in T is empty.

In the local coordinates near the fiber $\tilde{p}^{-1}(\tilde{0})$, $\Phi^{-1}(A_1)$ is given by $Z(I)$, where

$$I = \langle \chi_1, \chi_3 - \chi_4 \rangle .$$

To find the singular points in $\tilde{p}^{-1}(\tilde{0})$, one makes three calculations, one for each of the cases $Z_0' \neq 0$, $Z_1 \neq 0$, $Z_2 \neq 0$. For instance, for $Z_0' \neq 0$, one sets $Z_0' = 1$ and writes the 2-by-3 Jacobian (for the coordinates t, Z_1, Z_2) coming from the generators of I written above. Placing the 2-by-2 minors into an ideal I_{Jac}, the singularities of $\Phi^{-1}(A_1)$ in $\tilde{p}^{-1}(\tilde{0})$ with $Z_0' \neq 0$ are then given by

$$Z(t, I, I_{\text{Jac}}).$$

One can use Singular to show that this is empty.

The calculations for $Z_1 \neq 0, Z_2 \neq 0$ give similar results. Thus $\Phi^{-1}(A_1)$ has no singularities in $\tilde{p}^{-1}(\tilde{0})$ or T. □

Next, the total space of the pencil J_1 is

$$\mathcal{Y}_1 := \text{Bl}_{A_1}(E_1^{(3)}) = \left\{ (q, s) \in E_1^{(3)} \times J_1 \mid s(q) = 0 \right\}.$$

We let the second projection be $p_1 : \mathcal{Y}_1 \to J_1$, which has fibers

$$\mathcal{Y}_{1,s} = Z(s) \times \{s\} \simeq Z(s) \subseteq E_1^{(3)}$$

for each $s \in J_1$.

Proposition 3.5. *There are exactly* 42 *values of* $s \in J_1$ *such that* $\mathcal{Y}_{1,s}$ *is singular. Moreover, each of these singular fibers contains exactly one singular point.*

Proof. We start by finding all values of $(a : b) \in \mathbb{P}^1$ such that $a\psi_1 + b(\psi_3 - \psi_4)$ is singular. Again, since this is a local question, we can instead study the equations $a\Psi_1 + b(\Psi_3 - \Psi_4)$. More precisely, using the shorthand $\Psi(a : b)$ to denote $a\Psi_1 + b(\Psi_3 - \Psi_4)$, we will show there are exactly 42 values of $(a : b)$ such that $\Psi(a : b)$ is singular, and that each such $\Psi(a : b)$ has exactly three singularities.

First we claim there are 42 values of $(a : b)$ such that $\Psi(a : b)$ has at least one singularity in the open affine T. In fact, one knows $\Psi_1 = \Psi(1 : 0)$ is smooth by [Ish, Example 2.2], so we may assume that $b = 1$. Setting

$$\begin{aligned} e_1 &= y^2 - w(x) \\ e_2 &= (x - \alpha)t - 1 \\ e_3 &= a\omega_1 + (\omega_3 - \omega_4), \end{aligned}$$

the zero set of $\Psi(a : 1)$ in T corresponds to the ideal $\langle e_1, e_2, e_3 \rangle$ inside $\mathbb{C}[x, y, u, v, t]$. One finds the singular locus of $\Psi(a : 1)$ in T to be given by the ideal

$$I = \left\langle e_1, e_2, e_3, \frac{\partial e_1}{\partial x}\frac{\partial e_3}{\partial y} - \frac{\partial e_1}{\partial y}\frac{\partial e_3}{\partial x}, \frac{\partial e_3}{\partial u}, \frac{\partial e_3}{\partial v} \right\rangle .$$

In order to find all a such that we have a proper inclusion $I \subsetneq \mathbb{C}[x, y, u, v, t]$, let us instead regard I as an ideal in the ring $\mathbb{C}[x, y, u, v, t, a]$. Then the values of a in question are the roots of a generator of the principal ideal $I \cap \mathbb{C}[a]$. One can use Singular to show that $I \cap \mathbb{C}[a]$ is generated by a polynomial of degree 42 with distinct roots. This proves the claim.

A computation in Singular shows that the ring $\mathbb{C}[x, y, u, v, t, a]/I$ has Krull dimension 0 and has dimension 126 as a vector space over $\mathbb{C}$. Thus there are at most 126 distinct points in

$$Z(I) \subseteq \mathrm{Spec}\big(\mathbb{C}[x, y, u, v, t, a]\big).$$

On the other hand, one can numerically approximate the coordinates of all points in Z(I) to find 126 distinct solutions. As one would expect, upon inspection of these numerical coordinates, one finds:

(i) For each of the 42 values of a, there are three singularities of $\Psi(a : 1)$ in T.

(ii) The u and v coordinates of each singularity are nonzero.

From (ii) we conclude that, for these 42 values of a, all singularities of $\Psi(a : 1)$ in $\tilde{P}|_U$ actually lie in T or, equivalently, that $\Psi(a : 1)$ has no singularities in

$$\tilde{P}|_U \cap \{Z_0 = 0\} .$$

Indeed, if there were such a singularity, then by (3.11) its G-orbit would produce a singular point of the G-symmetric equation $\Psi(a:1)$ in T that satisfied $u=0$ or $v=0$; this is ruled out by (ii).

The same argument rules out the existence of *any* value of a such that $\Psi(a:1)$ has a singularity in

$$\tilde{P}|_U \cap \{Z_0 = 0\}:$$

if it existed, then by G-symmetry it would have appeared already in the list of the 42 values of a above and been in contradiction with (ii).

Finally, we check there is no value of a for which $\Psi(a:1)$ has singularities in the fiber $\tilde{p}^{-1}(\tilde{0})$. As in Proposition 3.4, we check the three cases $Z_0' \neq 0$, $Z_1 \neq 0$, and $Z_2 \neq 0$. For instance, in the case of $Z_0' \neq 0$, we first set $Z_0' = 1$ in $a\chi_1 + (\chi_3 - \chi_4)$; let us denote this substitution by $p_0 \in \mathbb{C}[a, Z_1, Z_2][[t]]$. Then there exists some a such that $\Psi(a:1)$ has a singular point over $\tilde{p}^{-1}(\tilde{0})$ with $Z_0' \neq 1$ if and only if

$$Z\left(t, p_0, \frac{\partial p_0}{\partial t}, \frac{\partial p_0}{\partial Z_1}, \frac{\partial p_0}{\partial Z_2}\right)$$

is nonempty. One can check using Singular that it is in fact empty. Similar conclusions hold for the cases $Z_1 \neq 0$ and $Z_2 \neq 0$. Thus, for all a, $\Psi(a:1)$ is smooth over $\tilde{p}^{-1}(\tilde{0})$ and hence, by G-symmetry, it is also smooth in the fibers over C_1 and C_2 as well.

In summary, we have found there are exactly 42 values of $(a:b)$ such that $\Psi(a:b)$ is singular, and have shown that each such $\Psi(a:b)$ contains exactly three singularities. □

Proposition 3.6. *Suppose that $\mathcal{Y}_{1,s}$ is singular for some $s \in J_1$. Then the singular locus of $\mathcal{Y}_{1,s}$ is one ordinary double point.*

Proof. Let $s \in J_1$ be such that $\mathcal{Y}_{1,s}$ is singular. By Proposition 3.5, there is only one singularity $q(s) \in \mathcal{Y}_{1,s} \simeq Z(s)$, and we denote its Milnor number as $\mu(s)$. Recall that $\mu(s)$ is a positive integer and that $\mu(s) = 1$ if and only if $q(s)$ is an ordinary double point of $\mathcal{Y}_{1,s}$.

By Proposition 3.4, $\mathcal{Y}_1$ is nonsingular and so according to [Ful, 14.1.5(d)] there is a certain zero-cycle γ on $\mathcal{Y}_1$ satisfying each of the following:

1. γ is supported on the set of critical points of $p_1 : \mathcal{Y}_1 \to J_1$.

2. Let a critical point of p_1 correspond to the isolated singularity $q(s) \in \mathcal{Y}_{1,s}$, $s \in J_1$.

Then the restriction of γ to $\{q(s)\}$ is the zero-cycle $\mu(s)q(s)$.

3. Letting e denote the topological Euler characteristic, one has

$$\deg(\gamma) = e(J_1)e(X) - e(\mathcal{Y}_1), \tag{3.13}$$

where X is a typical fiber of $p_1 : \mathcal{Y}_1 \to J_1$.

Thus we obtain

$$e(J_1)e(X) - e(\mathcal{Y}_1) = \sum_{\substack{s\in J_1, \\ \mathcal{Y}_{1,s} \text{ singular}}} \mu(s). \tag{3.14}$$

We have $e(J_1) = 2$, as $J_1 \simeq \mathbb{P}^1$, and, as X is an admissible CC surface, Noether's formula gives $e(X) = 9$. Since $\mathcal{Y}_1 = \mathrm{Bl}_{A_1}(E_1^{(3)})$, it follows from [GH, pp.605–606] that $e(\mathcal{Y}_1) = e(E_1^{(3)}) + e(A_1)$. We have $e(E_1^{(3)}) = 0$ [Mac] and, regarding A_1 as a smooth curve on X, the adjunction formula gives

$$-e(A_1) = 2g(A_1) - 2 = A_1.(A_1 + K),$$

where K is a canonical divisor on X. If $\iota : X \to E_1^{(3)}$ denotes the embedding then, up to numerical equivalence, we have $A_1 = \iota^*\mathfrak{D} = \iota^*(4D_0 - F_0)$ and $K = \iota^*(D_0)$. Since X is numerically equivalent to $4D_0 - F_0$ in $E_1^{(3)}$, we calculate

$$\begin{aligned} A_1.(A_1 + K) &= \iota^*\big((4D_0 - F_0).(5D_0 - F_0)\big) \\ &= \iota^*(20D_0^2 - 9D_0.F_0) \\ &= (4D_0 - F_0).(20D_0^2 - 9D_0.F_0) \\ &= 80D_0^3 - 56D_0^2.F_0 \\ &= 24. \end{aligned}$$

Thus $e(\mathcal{Y}_1) = -24$ and we get

$$e(J_1)e(X) - e(\mathcal{Y}_1) = 2 \cdot 9 - (-24) = 42.$$

Therefore (3.14) becomes

$$\sum_{\substack{s\in J_1,\\ \mathcal{Y}_{1,s}\text{ singular}}} \mu(s) = 42,$$

and since there are 42 singular values of $s \in J_1$, we conclude that $\mu(s) = 1$ for all s. □

3.4

Recall that

$$S_1 = \mathbb{P}H^0(E_1^{(3)}, L) \simeq \mathbb{P}^4.$$

Define

$$R_1 := \{s \in S_1 \mid Z(s) \text{ is singular}\} \subseteq S_1. \tag{3.15}$$

By the fact that $|\mathfrak{D}|$ has only simple base points [CC, Lemma 3.3], R_1 is a proper Zariski-closed subset of S_1. Endow R_1 with its unique structure as a reduced subscheme of S_1.

There is a rational map $\eta : E_1^{(3)} \to R_1$ that, if it is defined somewhere, is given by

$$\begin{array}{rccl} \eta & : & E_1^{(3)} & \dashrightarrow & R_1 \\ & & q & \mapsto & (\text{the unique } s \text{ such that } Z(s) \text{ is singular at } q). \end{array}$$

To express this map more algebraically, pick a point $q \in E_1^{(3)}$ and let (x_1, x_2, x_3) be local coordinates of $E_1^{(3)}$ at q. Then the divisor

$$Z(a_1\psi_1 + \ldots + a_5\psi_5)$$

(passes through and) has a singularity at q if and only if the column vector $(a_1, \ldots, a_5)^t$ belongs to the kernel of the matrix

$$M(q) := \begin{bmatrix} \psi_1(q) & \ldots & \psi_5(q) \\ (\partial\psi_1/\partial x_1)(q) & \ldots & (\partial\psi_5/\partial x_1)(q) \\ (\partial\psi_1/\partial x_2)(q) & \ldots & (\partial\psi_5/\partial x_2)(q) \\ (\partial\psi_1/\partial x_3)(q) & \ldots & (\partial\psi_5/\partial x_3)(q) \end{bmatrix}.$$

Thus $\eta(q)$ is defined if and only if $M(q)$ has rank 4, in which case the projectivized kernel $\eta(q)$ can be described easily in terms of the five 4-by-4 minors of $M(q)$. Indeed, if $m_i(M(q))$

denotes the 4-by-4 minor of $M(q)$ obtained by omitting the ith column, then

$$\eta(q) = \sum_{i=1}^{5} (-1)^i m_i(M(q))\psi_i \in R_1. \tag{3.16}$$

The following shows that η is defined on a dense subset of $E_1^{(3)}$:

Proposition 3.7. *Choose any $s \in J_1$ such that $\mathcal{Y}_{1,s} \simeq Z(s)$ is singular, and let $q \in Z(s)$ denote the unique singularity. Then* $\operatorname{rank}(M(q)) = 4$*, so that η is defined at q and $\eta(q) = s$.*

Proof. To verify the local property of the matrix $M(q)$ having full rank, it suffices to show instead that, for $r \in \Phi^{-1}(q) \in \tilde{P}$, the matrix

$$\tilde{M}(r) := \begin{bmatrix} \omega_1(r) & \dots & \omega_5(r) \\ (\partial\omega_1/\partial y_1)(r) & \dots & (\partial\omega_5/\partial y_1)(r) \\ (\partial\omega_1/\partial y_2)(r) & \dots & (\partial\omega_5/\partial y_2)(r) \\ (\partial\omega_1/\partial y_3)(r) & \dots & (\partial\omega_5/\partial y_3)(r) \end{bmatrix}, \tag{3.17}$$

has full rank, where (y_1, y_2, y_3) are some local coordinates of $\tilde{P}$ near r.

Suppose that $\Phi^{-1}(Z(s))$ is singular at the point $r \in \tilde{P}$. We recall from the proof of Proposition 3.5 that in fact $r \in T \subseteq \tilde{P}$, and thus we may write coordinates $r = (x_0, y_0, u_0, v_0, t_0)$. In fact, we are only concerned with local information at r, we may ignore the t-coordinate and write $r = (x_0, y_0, u_0, v_0)$ as a point in

$$\operatorname{Spec}\left(\frac{\mathbb{C}[x, y, u, v]}{\langle y^2 - w(x)\rangle}\right).$$

As discussed in the proof of Proposition 3.5, one can use SINGULAR to represent these coordinates represented numerically. Upon doing so, a fact one finds is that $y_0 \neq 0$. This implies that if we set

$$\bar{x} = x - x_0, \quad \bar{y} = y - y_0, \quad \bar{u} = u - u_0, \quad \bar{v} = v - v_0,$$

then $\bar{x}$ is a local parameter of $\tilde{E}$ near (x_0, y_0). Hence $\{\bar{x}, \bar{u}, \bar{v}\}$ is a set of local parameters for $\tilde{P}$ near r. Since

$$\bar{y} = \frac{w'(x_0)}{2y_0}\bar{x} + (\text{higher powers of } \bar{x}),$$

one can use this substitution to get expressions for the ω_i in terms of the parameters $\{\bar{x}, \bar{u}, \bar{v}\}$. Then one checks numerically (e.g., using a program such as MATHEMATICA) that the minor $m_1(\tilde{M}(r))$ is nonzero. Therefore, $\tilde{M}(r)$ has full rank and so does $M(q)$.

The equality $\eta(q) = s$ then follows by definition. □

From the rational map $\eta : E_1^{(3)} \to R_1$, we obtain one irreducible component of R_1: let

$$\hat{R}_1 := \overline{\eta(E_1^{(3)})} \subseteq R_1$$

denote the Zariski-closure of the image of η.

Theorem 3.8. *The following hold:*

(a) $\dim \hat{R}_1 = 3$.

(b) The only irreducible component of R_1 *having dimension* 3 *is* $\hat{R}_1$.

(c) $\deg \hat{R}_1 = 42$.

(d) There is a Zariski-dense open subset $U \subseteq \hat{R}_1$ *such that, for all* $s \in U$*, the singular locus of* $Z(s)$ *is exactly one ordinary double point.*

Proof. Suppose that $\dim \hat{R}_1 < 3 = \dim E_1^{(3)}$. Then all fibers of η have dimension at least 1. In particular, if $Z(s)$ is singular for some $s \in J_1$, then the preimage $\eta^{-1}(s)$ (which is well-defined by Proposition 3.7) must be a 1-dimensional subvariety $Z(s)$ consisting entirely of singular points. But $Z(s)$ has only one singularity by Proposition 3.5, a contradiction. This proves (a).

Before proceeding, we claim the following: inside every 3-dimensional irreducible component of R_1, there is a dense open subset whose points represent divisors having only isolated singularities. To see this, define the variety

$$Z_1 := \left\{ (q, s) \in E_1^{(3)} \times R_1 \;\middle|\; q \in \operatorname{Sing}(Z(s)) \right\}.$$

By definition of R_1, the second projection $\operatorname{pr}_2 : Z_1 \to R_1$ is surjective and $\operatorname{pr}_2^{-1}(s) \simeq \operatorname{Sing}(Z(s))$. Thus the claim will be proved if we show that each 3-dimensional component of R_1 contains one point s such that $\operatorname{Sing}(Z(s))$ has dimension zero. But this follows from Proposition 3.5, since the line J_1 must intersect all 3-dimensional components of R_1.

Now consider the set Q_{ψ_1} of all lines through $\psi_1 \in S_1 \setminus R_1$, to which we give the structure of $\mathbb{P}^3$. Note that $J_1 \in Q_{\psi_1}$ and thus by Propositions 3.3 and 3.4, there is a dense open subset of $V \subseteq Q_{\psi_1}$ such that if $J \in V$ then

(i) J is in general position with respect to the 3-dimensional irreducible components of R_1,

(ii) the base locus of J is smooth, and

(iii) if $s \in J$, then $Z(s)$ contains at most isolated singularities.

By Propositions 3.5 and 3.7, we know that $\deg \hat{R}_1 \geq 42$. Thus to prove the rest of the theorem, it will suffice to show the following: if $J \in V$, then there are exactly 42 values of $s \in J$ such that $Z(s)$ is singular and, for each such s, $\mathrm{Sing}(Z(s))$ is exactly one ordinary double point. But given that J has properties (ii) and (iii), we can reach this conclusion by using the same type of intersection theory calculation done for J_1 in the proof of Proposition 3.6. □

Remark. Although not needed in the sequel, one can prove that the analog of Theorem 3.8 holds for all but perhaps finitely many isomorphism classes of elliptic curves E. More precisely, we have the following, which we state without proof:

Let E be any elliptic curve, let $S = \mathbb{P}H^0(E^{(3)}, L)$, and let $R \subseteq S$ denote the closed subvariety of S that represents the singular elements of $|\mathfrak{D}|$. Then $\dim R = 3$ and one irreducible 3-dimensional component $\hat{R}$ of R is a hypersurface of degree 42 in S.

Furthermore, for all but at most finitely many isomorphism classes of E, this component $\hat{R}$ is the only 3-dimensional component of R and it has is a dense open subset $U \subseteq \hat{R}$ such that, for all $s \in U$, $\mathrm{Sing}(Z(s))$ is one ordinary double point.

Chapter 4

Large monodromy

4.1

Let us recall some notation from §2. Let k be an algebraically closed subfield of $\mathbb{C}$. We have the symmetric cube $\mathcal{E}^{(3)} \to Y$ of the universal elliptic curve $\mathcal{E} \to Y$ over the modular curve Y over k, which has two divisors $\mathcal{F}_0$ and $\mathcal{D}_0$. We have the open subset $\mathcal{S}$ of a projective bundle $\mathcal{S}_0$ over Y and the divisor $\mathcal{X} \subseteq \mathcal{E}^{(3)} \times_Y \mathcal{S}$, whose projection to $\mathcal{S}$ gives the smooth family $\pi : \mathcal{X} \to \mathcal{S}$ that contains all admissible CC surfaces over k.

We form two divisors ${}^F\mathcal{X}$ and ${}^D\mathcal{X}$ on $\mathcal{X}$, arising from the following pullback diagram:

$$
\begin{array}{ccccc}
{}^F\mathcal{X} & \longrightarrow & \mathcal{F}_0 \times_Y \mathcal{S} & & \\
\downarrow & & \downarrow & \searrow & \\
\mathcal{X} & \longrightarrow & \mathcal{E}^{(3)} \times_Y \mathcal{S} & \longrightarrow & \mathcal{S}. \\
\uparrow & & \uparrow & \nearrow & \\
{}^D\mathcal{X} & \longrightarrow & \mathcal{D}_0 \times_Y \mathcal{S} & &
\end{array}
$$

On a fiber $\mathcal{X}_s$, $s \in \mathcal{S}(k)$, ${}^F\mathcal{X}$ and ${}^D\mathcal{X}$ cut out an Albanese fiber f and the canonical divisor K.

Lemma 4.1. *Let X be any CC surface and let f be an Albanese fiber and let K be its canonical divisor. Then f and K are numerically independent.*

Proof. Suppose that $af + bK$ is numerically equivalent to zero. Then as $f^2 = 0$ and $f.K = 4$ (coming from the adjunction formula), we must have $a = b = 0$. □

Corollary 4.2. *For every CC surface X over k, the Picard number $\rho(X)$ satisifies $\rho(X) \geq 2$.*

Now we focus on the situation over $\mathbb{C}$, bringing in singular cohomology and Hodge theory. Let $\bar{\mathcal{X}}$ be a smooth compactification of $\mathcal{X}$ over k and let ${}^F\bar{\mathcal{X}}$ and ${}^D\bar{\mathcal{X}}$ denote the Zariski-closure of ${}^F\mathcal{X}$ and ${}^D\mathcal{X}$ in $\bar{\mathcal{X}}$. Let $[{}^F\bar{\mathcal{X}}_{\mathbb{C}}], [{}^D\bar{\mathcal{X}}_{\mathbb{C}}] \in H^2(\bar{\mathcal{X}}_{\mathbb{C}}, \mathbb{Z})$ denote their cycle classes in the singular cohomology of $\bar{\mathcal{X}}_{\mathbb{C}}$.

Define the local system of abelian groups $\mathbb{H}_{\mathbb{Z}} := R^2(\pi_{\mathbb{C}}^{\mathrm{an}})_*\mathbb{Z}(1)/(\mathrm{torsion})$ on $\mathcal{S}_{\mathbb{C}}$ and let $\mathbb{H} := \mathbb{H}_{\mathbb{Z}} \otimes \mathbb{Q}$. Using the Leray spectral sequence and the map $H^2(\bar{\mathcal{X}}_{\mathbb{C}}, \mathbb{Q}) \to H^2(\mathcal{X}_{\mathbb{C}}, \mathbb{Q})$, one has a map

$$H^2(\bar{\mathcal{X}}_{\mathbb{C}}, \mathbb{Q}) \to H^0(\mathcal{S}_{\mathbb{C}}, \mathbb{H}).$$

(This map is surjective by Deligne's Theorem of the Fixed Part, though we will not need this.) In particular, the classes $[{}^F\bar{\mathcal{X}}_{\mathbb{C}}]$ and $[{}^D\bar{\mathcal{X}}_{\mathbb{C}}]$ give two global sections η_1, η_2 of the local system $\mathbb{H}$.

Proposition 4.3. *The global sections $\eta_1, \eta_2 \in H^0(\mathcal{S}_{\mathbb{C}}, \mathbb{H})$ are linearly independent.*

Proof. It suffices to show the restrictions of these sections to the fiber $H^2(\mathcal{X}_s, \mathbb{Q})$ of $\mathbb{H}$ at a point $s \in \mathcal{S}(\mathbb{C})$ are independent. But by construction, these restrictions are just the cycle classes of an Albanese fiber and the canonical divisor on $\mathcal{X}_s$, so this follows from Lemma 4.1. □

Let

$$\phi : \mathbb{H}_{\mathbb{Z}} \otimes \mathbb{H}_{\mathbb{Z}} \to R^4(\pi_{\mathbb{C}}^{\mathrm{an}})_*\mathbb{Z}(2) \simeq \mathbb{Z}$$

be the cup product form. Noting that the global sections η_1, η_2 of $\mathbb{H}$ actually come from global sections of $\mathbb{H}_{\mathbb{Z}}$, let $\mathbb{V}_{\mathbb{Z}}$ be the orthogonal complement under ϕ of the rank 2 local subsystem of $\mathbb{H}_{\mathbb{Z}}$ generated by $\{\eta_1, \eta_2\}$. Let $\mathbb{V} = \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q}$. Then $\mathbb{V}$ underlies a variation of Hodge structure of weight zero and (by Noether's formula) rank 9 with Hodge numbers $h^{1,-1} = h^{-1,1} = 1, h^{0,0} = 7$. Moreover, one of the global sections η_i of $\mathbb{H}_{\mathbb{Z}}$ restricts to the class of the canonical divisor in each fiber $\mathbb{H}_{\mathbb{Z},s} = H^2(\mathcal{X}_s, \mathbb{Z})(1)$, which is ample since each $\mathcal{X}_s$ is an admissible CC surface. Thus the cup product form $\phi : \mathbb{V} \otimes \mathbb{V} \to \mathbb{Q}$ makes $\mathbb{V}$ into a polarized variation of Hodge structure on $\mathcal{S}_{\mathbb{C}}$.

Pick a point $\sigma \in \mathcal{S}(\mathbb{C})$. The local system underlying $\mathbb{V}$ is equivalent to the monodromy representation

$$\Lambda : \pi_1(\mathcal{S}_{\mathbb{C}}, \sigma) \to \mathrm{O}(\mathbb{V}_\sigma, \phi_\sigma).$$

Then the assertion of Theorem A, whose proof is the goal of §4, is that the image of Λ is Zariski-dense.

4.2

While Theorem A refers to the entire family $\pi_{\mathbb{C}} : \mathcal{X}_{\mathbb{C}} \to \mathcal{S}_{\mathbb{C}}$, it can be deduced from a similar result for a subfamily. Recall that in the construction of $\pi : \mathcal{X} \to \mathcal{S}$ in §2.2, we started with a $\mathbb{P}^4$-bundle $\mathcal{S}_0$ over the modular curve Y. Choose a subvariety $J \hookrightarrow (\mathcal{S}_0)_{\mathbb{C}}$ and let $J^* := J \cap \mathcal{S}_{\mathbb{C}}$, with $\iota : J^* \hookrightarrow \mathcal{S}_{\mathbb{C}}$ the embedding. If we consider the pullback $\iota^*\mathbb{V}$ to J^* and have a closed point $\sigma \in J^* \subseteq \mathcal{S}_{\mathbb{C}}$, then the two monodromy representations arising from $\mathbb{V}$ and $\iota^*\mathbb{V}$ are related via the diagram

$$\begin{array}{ccc} \pi_1(J^*, \sigma) & \longrightarrow & \mathrm{O}\big((\iota^*\mathbb{V})_\sigma, \phi_\sigma\big) \\ \downarrow{\scriptstyle \iota_*} & & \| \\ \pi_1(\mathcal{S}_{\mathbb{C}}, \sigma) & \longrightarrow & \mathrm{O}(\mathbb{V}_\sigma, \phi_\sigma). \end{array}$$

Thus if the image of $\pi_1(J^*, \sigma)$ is Zariski-dense in $\mathrm{O}(\mathbb{V}_\sigma, \phi_\sigma)$, then so is the image of $\pi_1(\mathcal{S}_{\mathbb{C}}, \sigma)$.

In §4.4, we will make the following choice for J. Suppose the point $y \in Y(\mathbb{C})$ corresponds to the elliptic curve E_1 defined in §3.3. Then $\mathcal{S}_{0,y} \simeq S_1 = \mathbb{P}H^0(E_1^{(3)}, L)$. Under this identification, $J \subset \mathcal{S}_{0,y}$ will chosen to be a general line representing a pencil in $|\mathfrak{D}|$, and hence the pullback of $\mathcal{X} \to \mathcal{S}$ to J^* will be the restriction of the total space $\mathcal{Y} \to J$ of the pencil J to the smooth fibers. Thus our aim is to prove:

Theorem 4.4. *For $\sigma \in J^*$, the Zariski-closure of the image of the monodromy representation*

$$\lambda : \pi_1(J^*, \sigma) \to \mathrm{O}(\mathbb{V}_\sigma, \phi_\sigma) \tag{4.1}$$

is $\mathrm{O}(\mathbb{V}_\sigma, \phi_\sigma)$.

In summary, we are reduced to investigating the monodromy a sufficiently nice pencil of divisors in the complete linear system $|\mathfrak{D}|$ on the smooth projective variety $E_1^{(3)}$. While this is reminiscent of the classical theory of Lefschetz concerning hyperplane sections of a smooth projective variety, this theory does not apply directly to our case, since $\mathfrak{D}$ is not very ample by Proposition 2.3. But as we show below, a sufficiently general version of Lefschetz's theory will suit our purposes.

4.3

Note: In §4.3, the base field will always be $\mathbb{C}$. Moreover, while in many instances purposefully similar, the notation in §4.3 will be independent of all previous chapters.

Here we describe a mild generalization, presumably known to experts, of Lefschetz's classical work on the monodromy representation of the family of hyperplane sections of a smooth projective variety. Our exposition draws from the two accounts [Lam] and [PS1, §3]; one can also consult [DK, Expose XVII]. See the end of this chapter for the specific choices that give the original context of Lefschetz.

Let W be a smooth projective variety of dimension $n+1$ and let X be a smooth subvariety of W of codimension 1, with $i : X \hookrightarrow W$ denoting the inclusion. Letting $\mathrm{PD} : H^n(X, \mathbb{Q}) \to H_n(X, \mathbb{Q})$ denote the Poincaré duality isomorphism, we define two subspaces of $H^n(X, \mathbb{Q})$:

$$\begin{aligned} I &= \mathrm{im}\big(i^* : H^n(W, \mathbb{Q}) \to H^n(X, \mathbb{Q})\big), \\ V &= \mathrm{PD}^{-1}\Big(\ker\big(i_* : H_n(X, \mathbb{Q}) \to H_n(W, \mathbb{Q})\big)\Big). \end{aligned}$$

Let ϕ denote the nondegenerate bilinear form on $H^n(X, \mathbb{Q})$ arising from cup product. Then unraveling the definitions shows that, with respect to ϕ, we have $V^\perp = I$.

Here are the first two assumptions we will make:

(L1) The maps

$$i^* : H^k(W, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

are isomorphic for $0 \leq k \leq n-1$ and injective for $k = n$.

(L2) The map

$$\cdot \cup [X] : H^n(W, \mathbb{Q}) \to H^{n+2}(W, \mathbb{Q})$$

given by taking the cup product with the cycle class $[X] \in H^2(W, \mathbb{Q})$ of X is an isomorphism.

Lemma 4.5. *The assumptions (L1) and (L2) imply the vector space decomposition*

$$H^n(X, \mathbb{Q}) = I \oplus V. \tag{4.2}$$

Proof. See [PS1, Prop. 3.6]. □

Next consider the space

$$S := \mathbb{P}H^0(W, \mathcal{O}_W(X)).$$

We will assume:

(L3) $\dim S \geq 2$.

Given this assumption, we may fix a pencil $J \subseteq S$, i.e., $J \simeq \mathbb{P}^1$ is a projective line in S. The base locus of J is

$$A := \bigcap_{s \in J} Z(s) \subseteq W$$

and the total space of J is

$$\mathcal{Y} := \mathrm{Bl}_A(W) = \{(s, w) \in J \times W \mid s(w) = 0\}.$$

Let $p : \mathcal{Y} \to J$ denote the first projection. We assume that J satisfies the following three assumptions:

(L4) If σ denotes the equation of X, then $\sigma \in J$. Thus $\mathcal{Y}_\sigma \simeq Z(\sigma) = X$.

(L5) A is smooth.

(L6) If $\mathcal{Y}_s$ is singular for $s \in J$, then its singular locus consists exactly of one ordinary double point.

Since X is smooth, (L4) implies that the number of singular fibers of $p : \mathcal{Y} \to J$ is finite; choose an indexing $s_1, \ldots, s_r$ of those $s \in J$ such that $\mathcal{Y}_s$ is singular. The assumption (L5) implies that $\mathcal{Y} = \mathrm{Bl}_A(W)$ is smooth. Rephrasing (L6) in terms of critical values and nondegenerate critical points of the map $p : \mathcal{Y} \to J$, we conclude:

Proposition 4.6. *With the assumptions (L4)–(L6), $p : \mathcal{Y} \to J$ is a holomorphic map from an $(n+1)$-dimensional compact complex manifold to the complex projective line, having r critical values and r nondegenerate critical points.*

It is exactly in the setup of Proposition 4.6 that classical Picard-Lefschetz theory is applicable, a detailed reference for which is [Lam, §5,6]. Let us summarize what we need of this theory. Let

$$J^* := J \setminus \{s_1, \dots, s_r\}$$

denote those $s \in J$ such that $\mathcal{Y}_s$ is nonsingular. Then we choose generators for $\pi_1(J^*, \sigma)$ as follows. For each $1 \le i \le r$, we let κ_i be a path that starts at the base point σ and travels to a point s_i^* near s_i; let c_i be a loop based at s_i^* that travels once around s_i in the counterclockwise direction. We set $\gamma_i = \kappa_i \cdot c_i \cdot \kappa_i^{-1}$ and assume that each of the r paths γ_i do not cross in any point other than the base point σ. Then the γ_i generate $\pi_1(J^*, \sigma)$.

The nth cohomology groups of the fibers $\mathcal{Y}_s$ piece together to give the local system $R^n(p_*^{\mathrm{an}})\mathbb{Q}$ on J^* that respects the cup product ϕ; we let

$$\lambda_0 : \pi_1(J^*, \sigma) \to \mathrm{Aut}(H^n(X, \mathbb{Q}), \phi) = \begin{cases} \mathrm{O}(H^n(X, \mathbb{Q}), \phi) & \text{if } n \text{ even} \\ \mathrm{Sp}(H^n(X, \mathbb{Q}), \phi) & \text{if } n \text{ odd} \end{cases} \tag{4.3}$$

denote the corresponding monodromy representation at σ. For each $1 \le i \le r$, Picard-Lefschetz theory yields a so-called *vanishing cycle*, which is a class in $H_n(X, \mathbb{Q})$. Considering their Poincaré duals gives a collection of cocycles in $H^n(X, \mathbb{Q})$ that we will denote by δ_i, $1 \le i \le r$. It can be shown that V is generated by the collection $\{\delta_i\}_i$. Let $T_i := \lambda_0(\gamma_i)$; then these $\{T_i\}_i$ generate the image of λ_0.

Theorem 4.7 (Picard-Lefschetz formula)**.** *With the assumptions (L4)–(L6), we have*

$$T_i(x) = x + (-1)^{\frac{(n+1)(n+2)}{2}} \phi(x, \delta_i)\delta_i \tag{4.4}$$

for all $x \in H^n(X, \mathbb{Q})$ and

$$\phi(\delta_i, \delta_i) = \begin{cases} 0, & n \text{ odd} \\ 2 \cdot (-1)^{n/2}, & n \text{ even.} \end{cases} \tag{4.5}$$

Proof. See [Lam, §6] (where the theorem is phrased homologically). □

Using these formulas, one can conclude:

Corollary 4.8. *With the assumptions (L1)–(L6), we have*

$$H^n(X, \mathbb{Q})^{\pi_1(J^*, \sigma)} = I.$$

Furthermore, V is a subrepresentation of $H^n(X,\mathbb{Q})$, i.e., the direct sum decomposition (4.2) also holds as $\pi_1(J^*,\sigma)$-modules.

Our final set of assumptions concerns the collection of singular divisors in the linear system $|X|$. With $S = \mathbb{P}H^0(W, \mathcal{O}_W(X))$ as above, let

$$R := \{ s \in S \mid Z(s) \subseteq W \text{ is singular } \}.$$

Then R is a proper closed subset of S, and we give R its unique reduced subscheme structure. Our final assumptions are:

(L7) The subvariety R has codimension 1 in S.

(L8) J is in general position with respect to R (meaning it only intersects the codimension 1 components of R, and does so transversally).

(L9) R has exactly one irreducible component $\hat{R}$ of codimension 1.

Theorem 4.9. *Let*

$$\lambda : \pi_1(J^*,\sigma) \to \operatorname{Aut}(V,\phi)$$

denote the subrepresentation V. With the assumptions (L1)–(L9), λ is absolutely irreducible.

Proof. A proof follows along the lines of that given in [Lam, §7] of the classical case. The main ingredients are the Picard-Lefschetz formula (4.4) and the fact that one can show the operators $\{\lambda(\gamma_i)\}_i$ are pairwise conjugate in the image of λ. By abuse of notation, let us also use T_i to denote $\lambda(\gamma_i)$. We indicate only those steps in the proof that need modification in our more general situation, and leave the reader to consult [Lam] for full details.

By an easy modification of the argument in [Lam, (7.4.1)], the two assumptions (L7) and (L8) imply that the inclusion $J^* \hookrightarrow S^*$ induces a surjection of fundamental groups

$$\pi_1(J^*,\sigma) \twoheadrightarrow \pi_1(S^*,\sigma). \tag{4.6}$$

The necessary modification is as follows. Given our current definitions of S, R, and σ, let Q_σ be the set of lines in S passing through σ (which can be identified with any hyperplane in $S \setminus \{\sigma\}$) and let $\ell_\sigma : S \setminus \{\sigma\} \to Q_\sigma$ be the morphism sending a point to the line through itself and σ. Finally, let $C \subseteq Q_\sigma$ be the image under ℓ_σ of all points in R that are nonsmooth (as points in R) or that lie off of $\hat{R}$; in other words, the elements of $Q_\sigma \setminus C$ represent lines that are in general position with respect to R. Then the argument in [Lam] makes the extra assumption that all components of R have codimension 1, which we cannot assume in our case. In either case, though, C is still a proper closed subvariety of Q_σ and, since this key point holds, the argument in [Lam] still applies to prove (4.6).

Next, given that the family $\mathcal{Y} \to J$ over J is just the base change via $J \hookrightarrow S$ of a similar family over S (whose fibers are the elements of the linear system $|X|$), we know that λ_0 factors as

$$\begin{array}{ccc} \pi_1(J^*, \sigma) & \twoheadrightarrow & \pi_1(S^*, \sigma) \\ & \searrow^{\lambda_0} & \downarrow \\ & & \mathrm{Aut}(H^n(X, \mathbb{Q}), \phi). \end{array}$$

Furthermore, the surjection $\pi_1(J^*, \sigma) \twoheadrightarrow \pi_1(S^*, \sigma)$ shows that V is also a $\pi_1(S^*, \sigma)$-module, and so λ factors as

$$\begin{array}{ccc} \pi_1(J^*, \sigma) & \twoheadrightarrow & \pi_1(S^*, \sigma) \\ & \searrow^{\lambda} & \downarrow \\ & & \mathrm{Aut}(V, \phi). \end{array}$$

Hence, to show that the operators $\{T_i\}_i \subseteq \mathrm{Aut}(V, \phi)$ are pairwise conjugate, it suffices to show that the images of the paths $\{\gamma_i\}_i$ are pairwise conjugate in $\pi_1(S^*, \sigma)$. But this is implied by the assumptions (L3), (L7), and (L9), thanks to an easy modification of the argument in [Lam, (7.5.1)]. The modification one must make is similar to that above, i.e., we must take note of the possibility of lower dimensional components of R. Define the subvariety

$$Z := \ell_\sigma^{-1}(C) \cap R \subseteq R,$$

consisting of those points $s \in R$ such that $\ell_\sigma(s) \in Q_\sigma$ is not in general position with respect to S; in particular, $R \setminus Z = \hat{R} \setminus Z \subseteq \hat{R}$ by virtue of (L9). By virtue of (L3) and (L7), R has dimension at least 1. As in the case of [Lam, (7.5.1)], $\dim Z < \dim R = \dim \hat{R}$, and so one can find a path through the dense open subset $R \setminus Z$ of $\hat{R}$ between any two of the points in $\{s_i\}_i \subseteq J \cap \hat{R}$. The rest of the argument in [Lam] applies to prove that the $\{\gamma_i\}_i$ are pairwise conjugate in $\pi_1(S^*, \sigma)$.

Now that we know the operators $\{T_i\}_i$ are pairwise conjugate, the rest of the proof follows exactly as in [Lam]. In particular, one byproduct of the proof that we record for future reference is the following. If we choose a pair $1 \leq i, j \leq r$ and $g \in \pi_1(L^*, \sigma)$ such that $\lambda(g_{ij})^{-1} \cdot T_i \cdot \lambda(g_{ij}) = T_j$, then

$$\lambda(g_{ij})\delta_j = \pm\delta_i. \tag{4.7}$$

□

For a strengthened form of Theorem 4.9, we exploit the following general lemma of Deligne:

Lemma 4.10 (Deligne)**.** *Let U be a finite-dimensional complex vector space equipped with a nondegenerate bilinear form ϕ_0, let M be an algebraic subgroup of* $\mathrm{Aut}(U, \phi_0)$*, and let $\mathcal{O}$ be an orbit of M that generates U.*

- *(a) Suppose that ϕ_0 is alternating and that M is the smallest algebraic subgroup of* $\mathrm{Aut}(U, \phi_0) = \mathrm{Sp}(U, \phi_0)$ *which contains all of the transvections*

$$u \mapsto u + \phi_0(u, \delta)\delta$$

  as δ ranges over $\mathcal{O}$. Then $M = \mathrm{Sp}(U, \phi_0)$.

- *(s) Suppose that ϕ_0 is symmetric, that all elements $\delta \in \mathcal{O}$ satisfy $\phi_0(\delta, \delta) = 2$, and that M is the smallest algebraic subgroup of* $\mathrm{Aut}(U, \phi_0) = \mathrm{O}(U, \phi_0)$ *that contains all of the reflections*

$$u \mapsto u - \phi_0(u, \delta)\delta$$

  as δ ranges over $\mathcal{O}$. Then either M is finite or $M = \mathrm{O}(U, \phi_0)$.

Proof. See [Del2, Lemmas $4.4.2^a, 4.4.2^s$]. □

Corollary 4.11. *If (L1) through (L9) hold, we have the following:*

- *(a) If n is odd, the Zariski-closure of the image of λ is the full group* $\mathrm{Aut}(V, \phi) = Sp(V, \phi)$*.*
- *(s) If n is even, the Zariski-closure of the image of λ is either finite or the full group* $\mathrm{Aut}(V, \phi) = O(V, \phi)$*.*

Proof. Let $\mathcal{O}$ be the orbit of δ_1. By (4.7) we see that $\mathcal{O}$ contains either δ_i or $-\delta_i$ for all i, and hence $\mathcal{O}$ generates V. Let

$$\phi_0 := \begin{cases} \phi & \text{if } n \equiv 0, 3 \pmod 4 \\ -\phi & \text{if } n \equiv 1, 2 \pmod 4. \end{cases}$$

Let $U := V \otimes_{\mathbb{Q}} \mathbb{C}$ and extend ϕ_0 from V to U.

Suppose that n is odd, so that ϕ_0 is alternating, and let M be defined as in Lemma 4.10(a); we must show that M is the Zariski-closure $N \subseteq \mathrm{Sp}(U, \phi_0)$ of the image $\lambda(\pi_1(J^*, \sigma))$. In one direction, we have $M \subseteq N$; indeed, the generators of M are the transvections

$$v \mapsto v + \phi_0(v, \lambda(g)\delta_1)\lambda(g)\delta_1 \tag{4.8}$$

for $g \in \pi_1(J^*, \sigma)$, and (4.8) is equal to $\lambda_0(g\gamma_1 g^{-1}) \in N$. In the other direction, (4.7) shows that M must contain all of the transvections T_i. Since the collection $\{T_i\}_i$ generates $\lambda(\pi_1(J^*, \sigma))$, and hence N, we see that $N \subseteq M$.

Thus we have shown that the smallest algebraic subgroup of $\mathrm{Sp}(U, \phi_0)$ defined over $\mathbb{C}$ that contains the image of λ is $\mathrm{Sp}(U, \phi_0)$ itself. It follows that the Zariski-closure of the image of λ in (the rational algebraic group) $\mathrm{Sp}(V, \phi_0)$ is $\mathrm{Sp}(V, \phi_0)$ itself. This completes the proof of part (a), since $\mathrm{Sp}(V, \phi_0) = \mathrm{Sp}(V, \phi)$.

When n is even, we have $\phi_0(\delta_1, \delta_1) = 2$ and thus all $\delta \in \mathcal{O}$ satisfy $\phi_0(\delta, \delta) = 2$. The rest of the proof of part (s) is similar to part (a). □

Remark. We note that the classical case considered by Lefschetz is the following. Let X be a smooth very ample divisor on W and let $W \hookrightarrow \mathbb{P}^N$ be an embedding given by the complete linear system $|X|$. Then X is a smooth hyperplane section of W relative to this embedding, $S = (\mathbb{P}^N)^\vee$ is the dual projective space, and $R \subseteq S$ is the dual variety of W. One knows that R is always irreducible, but it need not always be a hypersurface. (But this is "usually" the case, see [Tev, Theorem 1.18].) If R is a hypersurface, then J is a pencil that intersects R transversally in $\deg R = r$ points.

4.4

We now return to the setting of complex CC surfaces. Following the outline in §4.3, we will verify that the assumptions (L1) through (L9) hold in the case when

- $W = E_1^{(3)}$ (and hence $n = 2$),
- $S = S_1$ and $R = R_1$,
- $\sigma \in S$ any point in $S \setminus R$, so that $X = Z(\sigma)$ is smooth.
- $J \subseteq S$ is a line through σ that is in general position with respect to R, that has smooth base locus, and such that $\mathrm{Sing}(Z(s))$ is one ordinary double point for all $s \in J \cap R$; by Theorem 3.8 (as well as its proof), such a J exists.

Immediately we obtain:

Proposition 4.12. *(L3) through (L9) hold.*

Proof. First, (L3) holds as $h^0(E_1^{(3)}, L) = 5$. Since $\sigma \in J$ by definition, (L4) holds. By definition of J, (L5), (L6), and (L8) hold. Finally, (L7) is given by Theorem 3.8(a) and (L9) by Theorem 3.8(b). □

Proposition 4.13. *(L1) holds.*

Proof. By [Mac], we have

$$h^1(E_1^{(3)}, \mathbb{Q}) = h^2(E_1^{(3)}, \mathbb{Q}) = 2.$$

As the case $k = 0$ is clear, we consider the case $k = 1$. By Theorem 2.1 we have $E_1 \simeq \mathrm{Alb}(X)$. Thus we have the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & E_1^{(3)} \\ \downarrow{\scriptstyle \mathrm{Alb}} & & \downarrow \\ \mathrm{Alb}(X) & \xrightarrow{\sim} & E_1. \end{array}$$

Since $\mathrm{Alb}^* : H^1(\mathrm{Alb}(X), \mathbb{Q}) \to H^1(X, \mathbb{Q})$ is an isomorphism, this implies $H^1(E_1, \mathbb{Q}) \to H^1(E_1^{(3)}, \mathbb{Q})$ is injective. Furthermore, $h^1(E_1, \mathbb{Q}) = h^1(E_1^{(3)}, \mathbb{Q}) = 2$, implying that $H^1(E_1, \mathbb{Q}) \to H^1(E_1^{(3)}, \mathbb{Q})$ is actually an isomorphism. Therefore, so must $H^1(E_1^{(3)}, \mathbb{Q}) \to H^1(X, \mathbb{Q})$ be an isomorphism.

For the case $k = 2$, one can show by a proof similar to that of Proposition 4.3 that the two divisor classes $[D_0], [F_0] \in H^2(E^{(3)}, \mathbb{Q})$ have linearly independent images in $H^2(X, \mathbb{Q})$. This completes the proof. □

Proposition 4.14. *(L2) holds.*

Proof. By Proposition 2.3, the line bundle $L^{\otimes j}$ is very ample for large j. Thus, by the Hard Lefschetz Theorem, the map

$$\cdot \cup [jX] : H^2(E_1^{(3)}, \mathbb{Q}) \to H^4(E_1^{(3)}, \mathbb{Q})$$

is an isomorphism. Hence the same is true with $[X]$ in place of $[jX]$, proving that (L2) holds. □

Having verified properties (L1) through (L9), Theorem 4.9 and Corollary 4.11 give:

Corollary 4.15. *The monodromy representation λ in (4.1) is absolutely irreducible. Furthermore, the image of λ is either finite or Zariski-dense in* $\mathrm{O}(\mathbb{V}_s, \phi_s)$.

Proof of Theorem A. Recall that we may instead prove Theorem 4.4 from the end of §4.2. Let M denote the Zariski-closure of the image of λ and assume that M is finite. Since the action of M on $\mathbb{V}_s$ is absolutely irreducible by Corollary 4.15, this implies that any M-invariant nonzero bilinear form $\mathbb{V}_s$ is either positive or negative definite. But ϕ_s, which is nonzero and M-invariant, has signature $(2, 7)$. This is a contradiction. □

Chapter 5

Applications to Galois representations

Note: Throughout §5, k will be a finitely generated subfield of $\mathbb{C}$ and $\bar{k}$ will be its algebraic closure in $\mathbb{C}$. *We will frequently replace k by a finite extension when necessary, sometimes without mention.* This will not be problematic since it is sufficient to prove the statements in Theorem B over some finite extension of the original field of definition.

Our proof of Theorem B follows the axiomatic approach given by André [And1], and we refer the reader there for full details. However, a modification of André's axioms is required in the present situation (see the end of §5.1), and we discuss more carefully those points of the argument that are potentially affected by this modification.

5.1

Let X be an admissible CC surface defined over k and assume that the canonical divisor K and an Albanese fiber f are both defined over k as well. Let

$$\xi_1 = [K]_B,\ \xi_2 = [f]_B \in H_{\mathbb{Z}} := H^2(X_{\mathbb{C}}, \mathbb{Z})(1)/(\text{torsion})$$

be their cycle classes. Then $\{\xi_1, \xi_2\}$ generates a rank 2 subgroup of $H_{\mathbb{Z}}$ by Lemma 4.1. If

$$\theta : H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to H^4(X_{\mathbb{C}}, \mathbb{Z})(2) \simeq \mathbb{Z}$$

is the bilinear form given by the cup product, we define the orthogonal complement

$$V_{\mathbb{Z}} := \{\xi_1, \xi_2\}^{\perp} \subseteq H_{\mathbb{Z}}. \tag{5.1}$$

Because K is ample, $(V_{\mathbb{Z}}, \theta)$ is an integral polarized Hodge structure. Let $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$.

We will take the smooth projective family $\pi : \mathcal{X} \to \mathcal{S}$ constructed in §2.2, with $\mathcal{S}$ smooth and geometrically irreducible, to be defined over k and to possess the following properties:

(F1) There is a point $s \in \mathcal{S}(k)$ such that X is isomorphic to $\mathcal{X}_s$ over k. (Corollary 2.7) After fixing such an isomorphism, we may assume that $X = \mathcal{X}_s$.

(F2) The elements ξ_1, ξ_2 extend to global sections η_1, η_2 of $\mathbb{H}_{\mathbb{Z}} = R^2(\pi_{\mathbb{C}}^{\mathrm{an}})_*\mathbb{Z}(1)$, the first of which restricts to an ample divisor class on every fiber. Recall the cup product form

$$\phi : \mathbb{H}_{\mathbb{Z}} \otimes \mathbb{H}_{\mathbb{Z}} \longrightarrow R^4(\pi_{\mathbb{C}}^{\mathrm{an}})_*\mathbb{Z}(2) \simeq \mathbb{Z}.$$

The orthogonal complement $\mathbb{V}_{\mathbb{Z}}$ of the global sections η_1, η_2 is a polarized variation of Hodge structure of weight zero with $h^{-1,1} = h^{1,-1} = 1, h^{0,0} = 7$, and $h^{p,q} = 0$ otherwise. (See §4.1.) Let $\mathbb{V} = \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q}$. Define $\sigma = s_{\mathbb{C}} \in \mathcal{S}(\mathbb{C})$. Then $(\mathbb{V}_{\mathbb{Z},\sigma}, \phi_\sigma) = (V_{\mathbb{Z}}, \theta)$.

(F3) There exists $\mu \in \mathcal{S}(\mathbb{C})$ such that $\mathbb{V}_\mu$ is a Hodge structure of CM-type. (Theorem 2.9)

(F4) The image of the monodromy representation

$$\Lambda : \pi_1(\mathcal{S}_{\mathbb{C}}, \sigma) \to \mathrm{O}(\mathbb{V}_\sigma, \phi_\sigma) = \mathrm{O}(V, \theta)$$

is Zariski-dense. (Theorem A)

(F5) For all $\tau \in \mathcal{S}(\mathbb{C})$, the elements of $\mathbb{V}_\tau \subseteq H^2(\mathcal{X}_\tau, \mathbb{Q})(1)$ of type $(0,0)$ are algebraic, i.e., they belong to the subspace generated by the cycle classes of divisors on $\mathcal{X}_\tau$. (Lefschetz (1,1)-Theorem)

These properties are similar to the axioms being considered in [And1, p.207], but they differ in two ways. The first is that in (F2) we focus on a subvariation of the full primitive cohomology of the family, albeit one whose complement in $\mathbb{H} = R^2(\pi_{\mathbb{C}}^{\mathrm{an}})_*\mathbb{Q}(1)$ is still algebraic. The second and more significant difference is that the image of the period map of the variation $\mathbb{V}$ over $\mathcal{S}_{\mathbb{C}}$ cannot possibly contain an open subset of the period domain (which has dimension 7). We replace this instead with (F3) and (F4).

5.2

In the course of proving Theorem B, we will work with motives for absolute Hodge cycles, the main reference for which is [DMOS]. (We could have instead chosen to work with the stronger notion of *motivated cycles*, as in [And1]. See [And1, §1.5] and, more generally, [And2] for more details about this.) In particular, given two motives $\mathscr{W}_1$ and $\mathscr{W}_2$ over a subfield F of $\mathbb{C}$ with Betti realizations W_1 and W_2, to say that a Hodge correspondence $c : W_1 \to W_2$ is absolute Hodge over F is to say that c is the Betti realization of a morphism $\mathscr{W}_1 \to \mathscr{W}_2$.

Here are two motives that will appear below. There is a Hodge cycle $\pi_2 \in H^4(X \times X, \mathbb{Q})(2)$ that is absolute Hodge over k and such that the Betti realization (under our fixed embedding $k \hookrightarrow \mathbb{C}$) of the effective motive $\mathscr{H}^2(X) := (X, \pi_2)$ is $H^2(X, \mathbb{Q})$ [DMOS, p.28, Ex. 2.1(b)]. Let

$$\delta = \pi_2 - [K \times K]_B - [f \times f]_B \in H^4(X \times X, \mathbb{Q})(2).$$

Since K and f are defined over k, δ is also absolute Hodge over k. We define the motive

$$\mathscr{V}_k := (X, \delta, 1). \tag{5.2}$$

The Betti realization of $\mathscr{V}_k$ is $V \subseteq H^2(X, \mathbb{Q})(1)$. We will denote the base change of this motive to $\bar{k}$ (resp., $\mathbb{C}$) by $\mathscr{V}_{\bar{k}}$ (resp., $\mathscr{V}_{\mathbb{C}}$).

The idea of the proof of Theorem B is to show that the motive $\mathscr{V}_k$ is in $\mathrm{Mot}_{\mathrm{AV}}(k)$, the Tannakian category generated by the motives of abelian varieties over k (Theorem 5.3), and then to exploit the work of Faltings [FW] on abelian varieties.

5.3

For the purposes of both this chapter and §6, we will describe elements of the Kuga-Satake-Deligne construction, which associates complex abelian schemes to certain variations of

Hodge structure, including $\mathbb{V}_{\mathbb{Z}}$. For more details, we refer to [Del1, And1]. Regarding the original construction of Kuga and Satake (i.e., the special case when the variation is over a point), one can consult [KS, vG].

First we recall the original construction of Kuga and Satake, as recast by Deligne. Let $W_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module of rank $N+2$, let $W = W_{\mathbb{Z}} \otimes \mathbb{Q}$, and $W_{\mathbb{R}} = W_{\mathbb{Z}} \otimes \mathbb{R}$. Suppose that $W_{\mathbb{Z}}$ has a Hodge structure of weight 0 such that $h^{1,-1} = h^{-1,1} = 1$ and $h^{0,0} = N$ (and thus $h^{p,q} = 0$ otherwise). Furthermore, let

$$\phi : W_{\mathbb{Z}} \otimes W_{\mathbb{Z}} \to \mathbb{Z}$$

be a polarization of the Hodge structure $W_{\mathbb{Z}}$. Then this is equivalent to a morphism of real algebraic groups

$$h : \mathbb{S} \to \mathrm{SO}(W_{\mathbb{R}}, \phi_{\mathbb{R}})$$

of a certain type. One has the short exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GSpin}(W_{\mathbb{R}}) \longrightarrow \mathrm{SO}(W_{\mathbb{R}}, \phi_{\mathbb{R}}) \longrightarrow 1$$

and one would like to lift h to a morphism $\tilde{h} : \mathbb{S} \to \mathrm{GSpin}(W_{\mathbb{R}})$. One can do so uniquely by imposing the following condition on $\tilde{h}$: the diagram

$$\begin{array}{ccccc} \mathbb{G}_m & \xrightarrow{w} & \mathbb{S} & \xrightarrow{t} & \mathbb{G}_m \\ \| & & \downarrow \tilde{h} & & \| \\ \mathbb{G}_m & \longrightarrow & \mathrm{GSpin}(W_{\mathbb{R}}) & \xrightarrow{N^{-1}} & \mathbb{G}_m . \end{array}$$

must commute, where $w : \mathbb{G}_m \to \mathbb{S}$ denotes the weight homomorphism, $t(z) = (z\bar{z})^{-2}$, and N is the spinor norm. Conversely, one can recover h from $\tilde{h}$.

The morphism $\tilde{h}$ now gives rise to two different polarizable rational Hodge structures on the even Clifford algebra $C^+(W_{\mathbb{Z}})$, both as a consequence of the inclusion $\mathrm{GSpin}(W) \hookrightarrow C^+(W)^*$:

1. Via the adjoint action of $\mathrm{GSpin}(W)$

$$g *_{ad} e := geg^{-1}$$

we obtain a polarizable weight zero Hodge structure $C^+(W_{\mathbb{Z}})_{ad}$ whose nonzero Hodge numbers belong to $\{(-1,1),(0,0),(1,-1)\}$.

2. Via the action of $\mathrm{GSpin}(W)$ by left multiplication

$$g *_s e := ge$$

we obtain a polarizable weight one Hodge structure $C^+(W_{\mathbb{Z}})_s$ whose nonzero Hodge numbers belong to $\{(1,0),(0,1)\}$. Hence, up to isomorphism, there is a unique complex abelian variety $\mathrm{KS}(W_{\mathbb{Z}})$ of dimension 2^N, called the *Kuga-Satake variety* of $W_{\mathbb{Z}}$, such that

$$H^1(\mathrm{KS}(W_{\mathbb{Z}}),\mathbb{Z}) \simeq C^+(W_{\mathbb{Z}})_s$$

as Hodge structures.

Moreover, if we denote by C^+ the ring $C^+(W_{\mathbb{Z}})$ then $C^+(W_{\mathbb{Z}})_s$ is a right C^+-module (by the action of right multiplication). This action is compatible with the Hodge structure on $C^+(W_{\mathbb{Z}})_s$, and one can show that there is an isomorphism of Hodge structures

$$\mathrm{End}_{C^+}(C^+(W_{\mathbb{Z}})_s) \simeq C^+(W_{\mathbb{Z}})_{ad}.$$

Deligne shows how to relativize this construction. Rather than describe the general picture, let us invoke what we need for our situation, all of which is a consequence of (F2). Recall the notation in §5.1. Fix n large enough so that both of the arithmetic groups

$$\Gamma := \{a \in \mathrm{SO}(V_{\mathbb{Z}}) \mid a \equiv 1 \pmod{n}\} \quad (5.3)$$

$$\tilde{\Gamma} := \left\{A \in \mathrm{GSpin}(V)(\mathbb{Q}) \mid A \equiv 1 \ (\mathrm{mod}\ n) \text{ in } C^+(V_{\mathbb{Z}})\right\} \quad (5.4)$$

are torsion-free and isomorphic under the map $\mathrm{GSpin}(V)(\mathbb{Q}) \to \mathrm{SO}(V)(\mathbb{Q})$. Then one can find a connected finite étale covering $v : \mathcal{S}' \to \mathcal{S}$, which we may assume to be defined over k, such that the following holds: the monodromy representation underlying $(\mathbb{V}'_{\mathbb{Z}}, \phi') := v^*(\mathbb{V}_{\mathbb{Z}}, \phi)$ has image contained in Γ (after making appropriate identifications between $V_{\mathbb{Z}}$ and a fiber of $\mathbb{V}'_{\mathbb{Z}}$ as in (5.7) below). Furthermore, letting C^+ be the ring $C^+(V_{\mathbb{Z}})$, we obtain

(a) a complex abelian scheme $a : \mathcal{A} \to \mathcal{S}'_{\mathbb{C}}$,

(b) an embedding $\mu : C^+ \hookrightarrow \mathrm{End}_{\mathcal{S}'_{\mathbb{C}}}(\mathcal{A})$, and

(c) if $(\mathbb{V}'_{\mathbb{Z}}, \phi') := v^*(\mathbb{V}_{\mathbb{Z}}, \phi)$, an isomorphism of integral variations of Hodge structure

$$u_{\mathbb{Z}} : C^+(\mathbb{V}'_{\mathbb{Z}}) \xrightarrow{\sim} \underline{\mathrm{End}}_{C^+}(R^1 a_*^{\mathrm{an}} \mathbb{Z}) \tag{5.5}$$

that is also an isomorphism of local systems of rings. Tensoring with $\mathbb{Q}$, we get an isomorphism of rational variations of Hodge structure

$$u : C^+(\mathbb{V}') \xrightarrow{\sim} \underline{\mathrm{End}}_{C^+}(R^1 a_*^{\mathrm{an}} \mathbb{Q}). \tag{5.6}$$

Let $\pi' : \mathcal{X}' \to \mathcal{S}'$ denote the pullback of $\pi : \mathcal{X} \to \mathcal{S}$ via v, let $s' \in \mathcal{S}'(k)$ be a preimage of the point $s \in \mathcal{S}(k)$ in (F1), and let $\sigma' = s'_{\mathbb{C}}$. Then we make identifications $X = \mathcal{X}_s = \mathcal{X}'_{s'}$,

$$(V_{\mathbb{Z}}, \theta) = (\mathbb{V}_\sigma, \phi_\sigma) = (\mathbb{V}'_{\sigma'}, \phi'_{\sigma'}), \tag{5.7}$$

a complex abelian variety $\mathcal{A}_{\sigma'}$, and (via (5.7)) an isomorphism of weight zero Hodge structures

$$u_{\sigma'} : C^+(V) \xrightarrow{\sim} \mathrm{End}_{C^+}(H^1(\mathcal{A}_{\sigma'}, \mathbb{Q})). \tag{5.8}$$

More generally, for any $\tau' \in \mathcal{S}'(\mathbb{C})$, we have an isomorphism of Hodge structures

$$u_{\tau'} : C^+(\mathbb{V}'_{\tau'}) \xrightarrow{\sim} \mathrm{End}_{C^+}(H^1(\mathcal{A}_{\tau'}, \mathbb{Q}))$$

and one can show that $\mathcal{A}_{\tau'} \simeq \mathrm{KS}(\mathbb{V}'_{\tau'})$.

5.4

We now use Theorem A to demonstrate finer properties of the complex abelian variety $\mathcal{A}_{\sigma'}$ and the Hodge correspondence $u_{\sigma'}$.

Proposition 5.1. *The Hodge correspondence $u_{\sigma'}$ in (5.8) is absolute Hodge over $\mathbb{C}$.*

Proof. Let $\mu' \in \mathcal{S}'(\mathbb{C})$ be a preimage of the point $\mu \in \mathcal{S}(\mathbb{C})$ in (F3). Then $\mathbb{V}'_{\mu'}$ is a CM Hodge structure, which implies that $\mathbb{V}'_{\mu'}$ is the Hodge realization of an object in the Tannakian category generated by the motives of CM abelian varieties (see [Sch, §6.1]). Therefore $u_{\mu'}$

is a Hodge correspondence between (subquotients of) Hodge structures of complex abelian varieties. By [DMOS, Thm 2.11], $u_{\mu'}$ is therefore absolute Hodge and, by Principle B [DMOS, Thm 2.12], this implies that $u_{\tau'}$ is absolute Hodge for all $\tau' \in \mathcal{S}'(\mathbb{C})$, including $\tau' = \sigma'$. □

Remark. Instead of using Principle B and the CM Hodge structure of Polizzi, one can alternatively prove Proposition 5.1 by using Theorem A, following the arguments in [And1, Prop. 6.2.1].

Proposition 5.2. *There is an absolute Hodge correspondence $\gamma_{\mathbb{C}}$ over $\mathbb{C}$*

$$\gamma_{\mathbb{C}} : V \hookrightarrow \mathrm{End}_{C^+}(H^1(\mathcal{A}_{\sigma'}, \mathbb{Q})). \tag{5.9}$$

Proof. We modify the argument in [And1, 6.2.2, 6.4.1]. First one chooses an embedding $V \otimes \det V \hookrightarrow C^+(V)$ that is $\mathrm{O}(V)$-invariant. As the motivic Galois group of $\mathscr{V}_{\mathbb{C}}$ is also a subgroup of $\mathrm{O}(V)$, it also fixes this embedding, meaning the embedding is an absolute Hodge correspondence. By composition with $u_{\sigma'}$, we obtain an absolute Hodge correspondence

$$V \otimes \det V \hookrightarrow \mathrm{End}_{C^+}(H^1(\mathcal{A}_{\sigma'}, \mathbb{Q})).$$

To finish, one must show that $\det V$ is the trivial Hodge structure. For this, one realizes $\det V$ as the fiber over $\sigma' \in \mathcal{S}'(\mathbb{C})$ of the variation of Hodge structure $\det \mathbb{V}'$. Then one applies a deformation argument, which involves showing that the Hodge structure $\det \mathbb{V}'_{\mu'}$ at the special point $\mu' \in \mathcal{S}'(\mathbb{C})$ in (F3) is trivial (using the existence of algebraic classes in $\mathbb{V}'_{\mu'}$). For full details, see [And1]. □

Theorem 5.3. *There is an abelian variety A over k such that $A_{\mathbb{C}} \simeq \mathcal{A}_{\sigma'}$, and the absolute Hodge correspondences $u_{\sigma'}$ in (5.8) and $\gamma_{\mathbb{C}}$ in (5.9) descend to absolute Hodge correspondences over k.*

Proof. The proof of the existence of A follows that in [And1, §5.5], which in turn is a stronger version of [Del1, Prop. 6.5]. Without reproducing the entire proof, we will take care to make clear the role played by (F4), i.e., the Zariski-density of the monodromy.

Consider the collection $\mathcal{C}_1 = (\mathcal{S}', \pi', \mathcal{X}', a, \mathcal{A}, \mu)$; as this collection is defined by a finite number of equations, we may (after replacing k by a finite extension) find a smooth connected variety T over k such that $\mathcal{C}_1$ descends to a collection $\mathcal{C}_2 = (\mathcal{S}_2, \pi_2, \mathcal{X}_2, a_2, \mathcal{A}_2, \mu_2)$ over the function field $k(T)$ of T. Note that as $\pi' : \mathcal{X}' \to \mathcal{S}'$ is defined over k, $\pi_2 : \mathcal{X}_2 \to \mathcal{S}_2$ is obtained simply by base change from k to $k(T)$:

$$\mathcal{S}_2 = \mathcal{S}'_{k(T)}, \quad \mathcal{X}_2 = \mathcal{X}'_{k(T)}, \quad \pi_2 = \pi'_{k(T)}.$$

In fact, upon replacing T if necessary, the collection $\mathcal{C}_2$ is the generic fiber of a collection $C_3 = (\mathcal{S}_3, \pi_3, \mathcal{X}_3, a_3, \mathcal{A}_3, \mu_3)$ defined over T. Just as before, the first three objects in $\mathcal{C}_3$ are obtained by base change from k to T:

$$\mathcal{S}_3 = \mathcal{S}'_T, \quad \mathcal{X}_3 = \mathcal{X}'_T, \quad \pi_2 = \pi'_T.$$

To achieve the existence of A, we will first show the existence of an isomorphism of local systems of rings over $(\mathcal{S}_3)_{\mathbb{C}}$ similar to $u_{\mathbb{Z}}$ in (5.5). This is not automatic, since the transcendental isomorphism $u_{\mathbb{Z}}$ does not necessarily "descend" to T along with the collection $\mathcal{C}_1$. Rather we will use the Zariski-density of the monodromy representation

$$\Lambda' : \pi_1(\mathcal{S}'_{\mathbb{C}}, \sigma') \to \mathrm{SO}(V, \theta)$$

that underlies $\mathbb{V}'$ (using the identification (5.7)) to arrive at such an isomorphism.

For a prime number ℓ, one uses comparison to obtain from $u_{\mathbb{Z}}$ an isomorphism of $\mathbb{Z}_\ell$-sheaves of algebras

$$u_\ell : C^+({\mathbb{V}'}_{\text{ét}}) \xrightarrow{\sim} \underline{\mathrm{End}}_{C^+}(R^1 a_* \mathbb{Z}_\ell)$$

in the étale topology on $\mathcal{S}'_{\mathbb{C}}$; here ${\mathbb{V}'}_{\text{ét}}$ is the subsheaf of the $\mathbb{Z}_\ell$-sheaf $R^2(\pi'_{\mathbb{C}})_* \mathbb{Z}_\ell(1)$ obtained by removing the global sections arising from the cycle classes of K and f (i.e., the construction is exactly similar to that of $\mathbb{V}'_{\mathbb{Z}}$). We claim that u_ℓ is unique. To show this, it suffices to show that the isomorphism $u_{\mathbb{Z}}$ is unique. If there were two such isomorphisms, one would obtain an automorphism of the local system $C^+(\mathbb{V}')$, i.e., a $\pi_1(\mathcal{S}'_{\mathbb{C}}, \sigma')$-invariant automorphism of the fiber $C^+(V)$. But by the Zariski-density of the monodromy representation Λ', it follows that such an automorphism necessarily commutes with a dense subgroup of $\mathrm{Spin}(V)$ in its action on $C^+(V)$ by left multiplication. Deligne [Del1, Lemma 3.5] shows that this implies the automorphism of $C^+(V)$ is the identity. Hence $u_{\mathbb{Z}}$ is unique and so is

u_ℓ.

Without using this uniqueness, u_ℓ automatically descends *a priori* only to an isomorphism of étale sheaves over $\mathcal{S}_2 \otimes_{k(T)} \overline{k(T)}$, where $\overline{k(T)}$ is the algebraic closure of $k(T)$. In other words, when one restricts u_ℓ to a fiber over a point of $\mathcal{S}_2$, it is invariant under the action of π_1^{geom}. But in fact the action of the group π_1^{arith} at the point sends the restriction of u_ℓ to another π_1^{geom}-invariant isomorphism of $\mathbb{Z}_\ell$-algebras; by the aforementioned uniqueness of u_ℓ, this means that π_1^{arith} fixes this restriction, meaning u_ℓ descends to an isomorphism of étale sheaves over $k(T)$.

Hence, after perhaps replacing T again, u_ℓ is the generic fiber of a larger isomorphism of étale sheaves over T; this in turn gives rise to the analytic isomorphism of local systems of rings

$$C^+(\mathbb{V}'_T) \xrightarrow{\sim} \underline{\mathrm{End}}_{C^+}(R^1 a_3^{\mathrm{an}} \mathbb{Z}) \tag{5.10}$$

over $(\mathcal{S}_3)_\mathbb{C} = \mathcal{S}'_\mathbb{C} \times_\mathbb{C} T_\mathbb{C}$; here $\mathbb{V}'_T$ denotes the pullback of $\mathbb{V}'$ from $\mathcal{S}'_\mathbb{C}$ to $\mathcal{S}'_\mathbb{C} \times_\mathbb{C} T_\mathbb{C}$. Then, as in [And1, Lemma 5.5.1], one uses (5.10) to prove that any specialization of the abelian scheme $\mathcal{A}_3 \to T$ to a point along $s' \times_k T \subseteq T$ in fact gives a model for the abelian variety $\mathcal{A}'_{\sigma'}$ over the residue field of that point. In particular, choosing a k-valued point of T, we get a model A for $\mathcal{A}'_{\sigma'}$ over k, completing the first part of the theorem.

Finally, by definition, the absolute Hodge cycles on a variety Y over $\mathbb{C}$ are defined to be the base change of the absolute Hodge cycles for a model Y_0 of Y over any algebraically closed field k_0 of finite transcendence degree. (This definition is independent of the choice of (k_0, Y_0), see [DMOS, Prop. 2.9].) Hence the absolute Hodge correspondence $\gamma_\mathbb{C}$ must come from an absolute Hodge correspondence over $\bar{k}$

$$\gamma_{\bar{k}} : \mathscr{V}_{\bar{k}} \hookrightarrow \mathscr{E}nd_{C^+}(\mathscr{H}^1(A_{\bar{k}})),$$

where the right hand side represents the object in $\mathrm{Mot}_{\mathrm{AV}}(\bar{k})$ whose Betti realization is $\mathrm{End}_{C^+}(H^1(A_\mathbb{C}, \mathbb{Q}))$ (which is well-defined by [DMOS]).

On the other hand, $\gamma_{\bar{k}}$ is fixed by an open subgroup of $\mathrm{Gal}(\bar{k}/k)$, which we may assume to be the whole group $\mathrm{Gal}(\bar{k}/k)$ itself. By definition, this means that $\gamma_{\bar{k}}$ comes from an absolute Hodge correspondence over k

$$\gamma_k : \mathscr{V}_k \hookrightarrow \mathscr{E}nd_{C^+}(\mathscr{H}^1(A)). \tag{5.11}$$

One shows in a similar manner that $u_{\sigma'}$ descends to an absolute Hodge correspondence

$$\mathscr{C}^+(\mathscr{V}_k) \xrightarrow{\sim} \mathscr{E}nd_{C^+}(\mathscr{H}^1(A)) \tag{5.12}$$

over k. □

5.5

Proof of Theorem B. Note that it suffices to prove each of the three statements after replacing k by a finite extension. Thus we may assume that both the canonical divisor K and an Albanese fiber f of X are defined over k, that axioms (F1) through (F5) hold, and that Theorem 5.3 holds.

It follows that the representation $H^2(X_{\bar{k}}, \mathbb{Q}_\ell)(1)$ contains a 2-dimensional trivial subrepresentation and its orthogonal complement V_ℓ. This decomposition of vector spaces over $\mathbb{Q}_\ell$ is the ℓ-adic realization of the decomposition of motives

$$\mathscr{H}^2(X)(1) = \mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathscr{V}_k \tag{5.13}$$

where $\mathbf{1}_k$ denotes the trivial motive over k. Thus, upon taking the ℓ-adic realization of (5.13), one sees that the aforementioned direct sum of vector spaces over $\mathbb{Q}_\ell$ is also a decomposition of $\mathrm{Gal}(\bar{k}/k)$-modules. The truth of part (i) now follows from the semisimplicity of V_ℓ, which in turn follows from the absolute Hodge correspondence γ_k (5.11) proved Theorem 5.3 and Faltings' proof of the semisimplicity conjecture for abelian varieties [FW].

For part (ii), one must show the elements of V_ℓ fixed by an open subgroup of $\mathrm{Gal}(\bar{k}/k)$ are algebraic. This follows from the absolute Hodge correspondence γ_k, Faltings' proof of Tate's isogeny conjecture for abelian varieties, and (F5). See [And1, §7.2] for details.

Finally, for part (iii), we note that $\mathfrak{G}$ is the identity component of the motivic Galois group $\mathfrak{H}$ of the motive $\mathscr{H}^2(X)(1)$. Thus it suffices to prove that the Lie group $r_\ell(\mathrm{Gal}(\bar{k}/k))$ is an open Lie subgroup of $\mathfrak{H}(\mathbb{Q}_\ell)$. By (5.13), $\mathfrak{H}$ is the product of two copies of the trivial group and the motivic Galois group of $\mathscr{V}_k$. The arguments in [And1, §7.3,7.4] allow one to conclude that the image of $\mathrm{Gal}(\bar{k}/k)$ in $\mathrm{GL}(V_\ell)$ is an open Lie subgroup of the $\mathbb{Q}_\ell$-points of the motivic Galois group of $\mathscr{V}_k$, which implies part (iii). (We remark that this is the point in the proof where one uses the absolute Hodge correspondence (5.12) from Theorem

5.3.) □

Chapter 6

Applications to Picard numbers and the period map

6.1

The starting point for this chapter is a discussion of the Mumford-Tate group of the Hodge structure of a fiber of the variation $\mathbb{V}$ on $\mathcal{S}_{\mathbb{C}}$ at a very general point. This group is intimately related to the monodromy representation of the underlying local system of $\mathbb{V}$.

Proposition 6.1. *If $\tau \in \mathcal{S}(\mathbb{C})$ is very general (i.e., if τ lies outside of a certain countable collection of proper closed subvarieties of $\mathcal{S}_{\mathbb{C}}$), then*

$$\mathrm{MT}(\mathbb{V}_\tau) = \mathrm{SO}(\mathbb{V}_\tau, \phi_\tau).$$

Proof. Note that since $\mathbb{V}_\tau$ is of weight zero with polarization ϕ_τ, we necessarily have $\mathrm{MT}(\mathbb{V}_\tau) \subseteq \mathrm{SO}(\mathbb{V}_\tau, \phi_\tau)$. On the other hand, since τ is very general, one knows that the connected component of the closure of image of the monodromy representation is contained in $\mathrm{MT}(\mathbb{V}_\tau)$ [PS2, Prop. 10.14]. Thus, by Theorem A, we have the opposite inclusion $\mathrm{SO}(\mathbb{V}_\tau, \phi_\tau) \subseteq \mathrm{MT}(\mathbb{V}_\tau)$. □

Corollary 6.2. *Let $A_\tau := \mathrm{KS}(\mathbb{V}_{\mathbb{Z},\tau})$ be the Kuga-Satake variety of $\mathbb{V}_{\mathbb{Z},\tau}$. If τ is very general, then*

$$\mathrm{MT}(A_\tau) := \mathrm{MT}(H^1(A_\tau, \mathbb{Q})) = \mathrm{GSpin}(\mathbb{V}_\tau).$$

Proof. Following the notation in §5.3, let $W_{\mathbb{Z}} = \mathbb{V}_{\mathbb{Z},\tau}$. Then $h : \mathbb{S} \to \mathrm{SO}(W_{\mathbb{R}}, \phi_{\mathbb{R}})$ gives the Hodge structure on $W_{\mathbb{Z}}$ and $\tilde{h} : \mathbb{S} \to \mathrm{GSpin}(W_{\mathbb{R}})$ gives the Hodge structure on $H^1(A_\tau, \mathbb{Z})$, so we automatically have $\mathrm{MT}(A_\tau) \subseteq \mathrm{GSpin}(W)$. Furthermore, since $h(\mathbb{S})$ is dense in

$\mathrm{SO}(W_{\mathbb{R}}, \phi_{\mathbb{R}})$, one sees that $\tilde{h}(\mathbb{S})$ must have dense intersection with $\mathrm{Spin}(W_{\mathbb{R}})$. Finally, $\mathbb{G}_m \subseteq \tilde{h}(\mathbb{S})$ by definition, so one concludes that $\tilde{h}(\mathbb{S})$ is dense in $\mathrm{GSpin}(W_{\mathbb{R}})$. □

6.2

Proof of Theorem C. As the family $\mathcal{X} \to \mathcal{S}$ is defined over $\bar{\mathbb{Q}}$, it suffices to show the existence of some $t \in \mathcal{S}(\bar{\mathbb{Q}})$ such that $\rho(\mathcal{X}_t) = 2$.

First consider a complex admissible CC surface $\mathcal{X}_\tau$. Let $T_\tau \subseteq \mathbb{V}_\tau$ denote the subspace of transcendental classes in $H^2(\mathcal{X}_\tau, \mathbb{Q})(1)$ (i.e., the orthogonal complement of the algebraic classes). Then a general result of Zarhin [Zar] concerning surfaces with $p_g = 1$ shows that

$$\mathrm{MT}(\mathbb{V}_\tau) = \mathrm{SO}(T_\tau, \phi_\tau) \subseteq \mathrm{SO}(\mathbb{V}_\tau, \phi_\tau).$$

Thus if $\tau \in \mathcal{S}(\mathbb{C})$ is very general, then we must have $\mathbb{V}_\tau = T_\tau$ by Proposition 6.1, which implies that $\rho(\mathcal{X}_\tau) = 2$. This shows the existence over $\mathbb{C}$.

Given that the countable collection $\mathcal{S}(\bar{\mathbb{Q}})$ could potentially lie in the complement of the collection of very general points in $\mathcal{S}(\mathbb{C})$ referred to in Proposition 6.1, one needs a stronger result to show the existence over $\bar{\mathbb{Q}}$. For this we use [And2, Thm 5.2(3)], which (as formulated in [MP]) says the following: if $\eta \in \mathcal{S}$ denotes the generic point, then there exists a point $t \in \mathcal{S}(\bar{\mathbb{Q}})$ such that $\rho(\mathcal{X}_\eta) = \rho(\mathcal{X}_t)$. (Recall our convention that ρ always denotes the *geometric* Picard number.)

Thus it remains to establish that $\rho(\mathcal{X}_\eta) = 2$. Choose $\tau \in \mathcal{S}(\mathbb{C})$ such that $\rho(\mathcal{X}_\tau) = 2$. Then using basic arguments about Néron-Severi groups (specifically, see [MP, Prop. 3.1, 3.6]), we have

$$2 = \rho(\mathcal{X}_\tau) \geq \rho(\mathcal{X}_\eta \otimes \mathbb{C}) = \rho(\mathcal{X}_\eta) \geq 2,$$

so that $\rho(\mathcal{X}_\eta) = 2$ as desired. □

6.3

Fix a complex admissible CC surface Y. In fact, for convenience, we will take $Y = X_{\mathbb{C}}$ from §5.1 and will keep the notation $(V_{\mathbb{Z}}, \theta)$. Let D be the set of all homomorphisms $h : \mathbb{S} \to \mathrm{SO}(V_{\mathbb{R}}, \theta_{\mathbb{R}})$ that make $(V_{\mathbb{Z}}, \theta)$ into a polarized Hodge structure of weight zero with

Hodge numbers $h^{-1,1} = h^{1,-1} = 1$, $h^{0,0} = 7$. By the Kuga-Satake construction in §5.3, D is in canonical bijection with the set $\tilde{D}$ of all homomorphisms $\tilde{h} : \mathbb{S} \to \mathrm{GSpin}(V_{\mathbb{R}})$ such that (i) $\tilde{h}(\mathbb{G}_m) = \mathbb{G}_m$ and (ii) the composition

$$\mathbb{S} \xrightarrow{\tilde{h}} \mathrm{GSpin}(V_{\mathbb{R}}) \longrightarrow \mathrm{SO}(V_{\mathbb{R}}, \theta_{\mathbb{R}})$$

belongs to D. Moreover, all elements of D form a single orbit under the action of conjugation by elements of $\mathrm{SO}(V,\theta)(\mathbb{R})$, and since $\mathrm{GSpin}(V)(\mathbb{R}) \twoheadrightarrow \mathrm{SO}(V,\theta)(\mathbb{R})$ it follows that all elements of $\tilde{D}$ form a single orbit under conjugation by $\mathrm{GSpin}(V)(\mathbb{R})$. In fact, the pair $(\mathrm{GSpin}(V), \tilde{D})$ is a Shimura datum.

If we give D the complex structure coming from its status as the classifying space for the specified type of polarized Hodge structures on $(V_{\mathbb{Z}}, \theta)$, then (under the identification $D \leftrightarrow \tilde{D}$) this is the same as the complex structure on $\tilde{D}$ coming from the status of $(\mathrm{GSpin}(V), \tilde{D})$ as a Shimura datum. We can write $D = D^+ \cup D^-$ as the disjoint union of two Hermitian symmetric domains. If Γ and $\tilde{\Gamma}$ are as in (5.3) and (5.4), then the quotient $\mathcal{V} := \Gamma\backslash D^+ = \tilde{\Gamma}\backslash\tilde{D}^+$ has a canonical structure of quasiprojective variety [BB]. This variety $\mathcal{V}$ is a connected component of a 7-dimensional Shimura variety $\mathrm{Sh}_K(\mathrm{GSpin}(V), \tilde{D})$ of orthogonal type, for a compact open subgroup K of $\mathrm{GSpin}(V)(\mathbb{A}_f)$ such that $\tilde{\Gamma} = K \cap \mathrm{GSpin}(\mathbb{Q})$. A similar situation holds for D^-.

We recall from §5.3 the connected finite étale cover $v : \mathcal{S}' \to \mathcal{S}$ such that the pullback variation of Hodge structure $\mathbb{V}'$ on $\mathcal{S}'_{\mathbb{C}}$ has monodromy with image in Γ. We use our earlier identification $Y = X_{\mathbb{C}} = \mathcal{X}'_{\sigma'}$ and $(V,\theta) = (\mathbb{V}'_{\sigma'}, \phi'_{\sigma'})$. These choices induce a period map from $\mathcal{S}'_{\mathbb{C}}$ to $\Gamma\backslash D$; since $\mathcal{S}'$ is connected, we may assume this period map takes the form

$$\Phi : \mathcal{S}'_{\mathbb{C}} \to \mathcal{V}.$$

Initially Φ is a map in the analytic category, but a theorem of Borel [Bor] shows that it is in fact a map of algebraic varieties.

Also living over $\mathcal{S}'_{\mathbb{C}}$ is the abelian scheme $a : \mathcal{A} \to \mathcal{S}'_{\mathbb{C}}$ and the associated variation $R^1 a_*^{\mathrm{an}}\mathbb{Q}$. By construction, the monodromy of $R^1 a_*^{\mathrm{an}}\mathbb{Q}$ has image in $\tilde{\Gamma}$, which (along with our choice of μ' and $Y \tilde{\to} \mathcal{X}'_{\mu'}$) induces a period map

$$\tilde{\Phi} : \mathcal{S}'_{\mathbb{C}} \to \mathcal{V}$$

that is again algebraic. Also by construction, we have $\Phi = \tilde{\Phi}$.

We are interested $Z := \overline{\Phi(\mathcal{S}'_{\mathbb{C}})}$, the closure of the image of Φ. Note that Z is necessarily connected. As a first remark, we have:

Proposition 6.3. *We have* $1 \leq \dim Z \leq 5$.

Proof. First we note that Z is not a single point. Indeed, there are at least two different Hodge structures in the family $\pi_{\mathbb{C}} : \mathcal{X}_{\mathbb{C}} \to \mathcal{S}_{\mathbb{C}}$, since there are admissible CC surfaces with Picard number 9 (Theorem 2.9) and Picard number 2 (Theorem C). This gives the first inequality, and the second follows from the fact that $\dim \mathcal{S}' = 5$. □

We wish to investigate the smallest *special subvariety* of $\mathcal{V}$ that contains Z, which is the subject of Theorem D. The special subvarieties of $\mathcal{V}$, which are also called *subvarieties of Hodge type*, are defined to be the irreducible components in $\mathcal{V}$ of Hecke-translated Shimura subvarieties. For further properties of special subvarieties we refer to [Moo, Yaf].

Proof of Theorem D. We use the identification $\mathcal{V} = \tilde{\Gamma}\backslash\tilde{D}^+$ and $\Phi = \tilde{\Phi} : \mathcal{S}'_{\mathbb{C}} \to \mathcal{V}$.

To find the smallest special subvariety of $\mathcal{V}$ containing Z involves the use of the generic Mumford-Tate group of $R^1 a_*^{\mathrm{an}}\mathbb{Q}$. (See [Yaf, p.386].) This is a subgroup $\mathrm{MT}(\mathcal{A})$ of $\mathrm{GSpin}(V)$ that is canonically identified via parallel translation with $\mathrm{MT}(\mathcal{A}_{\tau'})$ for any very general point $\tau' \in \mathcal{S}'(\mathbb{C})$ (i.e., $\mathrm{MT}(\mathcal{A})$ is independent of the choice very general point τ' and the choice of path between σ' and τ'). The inclusion $\mathrm{MT}(\mathcal{A}) \hookrightarrow \mathrm{GSpin}(V)$ induces a morphism of Shimura data

$$\mathrm{Sh}(\mathrm{MT}(\mathcal{A}), \tilde{D}_{\mathrm{MT}}) \to \mathrm{Sh}(\mathrm{GSpin}(V), \tilde{D}),$$

where $\tilde{D}_{\mathrm{MT}}$ is the orbit under conjugation by $\mathrm{MT}(\mathcal{A})(\mathbb{R})$ of the chosen point $h_{\sigma'} : \mathbb{S} \to \mathrm{GSpin}(V)$ in $\tilde{D}$. This induces

$$\mathrm{Sh}_{K\cap \mathrm{MT}(\mathcal{A})(\mathbb{A}_f)}(\mathrm{MT}(\mathcal{A}), \tilde{D}_{\mathrm{MT}}) \to \mathrm{Sh}_K(\mathrm{GSpin}(V), \tilde{D}),$$

and the image of this map is the smallest special subvariety containing Z.

By Corollary 6.2 we have $\mathrm{MT}(\mathcal{A}_{\tau'}) = \mathrm{GSpin}(\mathbb{V}'_{\tau'})$ if τ' is very general. Applying parallel translation, this implies that $\mathrm{MT}(\mathcal{A}) = \mathrm{GSpin}(V)$, and thus the morphism of Shimura data above is the identity morphism. This gives the first statement of the theorem.

For the second statement, we note that $\dim Z \leq 5 < 7 = \dim \mathcal{V}$. Thus Z cannot possibly equal $\mathcal{V}$ and so, by the first statement, is not special. □

Finally, recall the explicit pencil $J_1 \subseteq |\mathfrak{D}|$ on $E_1^{(3)}$ defined in (3.12) that was used in the proof of Theorem 3.8. Let J_1^* denote the open subset where the fibers are smooth. By construction of $\mathcal{S}_{\mathbb{C}}$, one can identify J_1^* with a curve in $\mathcal{S}_{\mathbb{C}}$, and thus by pullback obtain a curve in $\mathcal{S}'_{\mathbb{C}}$. We denote by C the closure of the image of this curve in $\mathcal{V}$ under the period map Φ. This subvariety C of $\mathcal{V}$ is the subject of Theorem E.

Proof of Theorem E. Let $J \subseteq |\mathfrak{D}|$ be a pencil and let J^* be its smooth locus. Recall that, by the generalized Lefschetz theory described in §4.3 and applied in §4.4, if J is in general position in $|\mathfrak{D}|$ then the total space over J^* is a family of admissible CC surfaces with large monodromy.

Now let us show that J_1 is in general position, i.e., let us show that J_1 intersects R_1 (defined in (3.15)) only at smooth points of its 3-dimensional component $\hat{R}_1$ and does so transversally. The first point follows from deformation theory, since the singularities of all singular fibers of J_1 are ordinary double points (Proposition 3.6). The second point follows from the fact that J_1 has 42 singular fibers, which is equal to the degree of $\hat{R}_1$ in $|\mathfrak{D}|$ (Proposition 3.5, Theorem 3.8). Therefore, by using the same methods as in the proof of Theorems C and D, we conclude that C is not contained in any proper special subvariety of $\mathcal{V}$ and that J_1 must contain a smooth element with Picard number 2.

Next we claim that the smooth element $Z(\psi_3 - \psi_4)$ in J_1 has Picard number at least 5. One can show this by looking at the Albanese fibration of this surface, which (as can be seen by looking at the equation) is reducible over the nontrivial 2-torsion points of E. Components of these singular fibers can be used to show the Picard number of $Z(\psi_3 - \psi_4)$ is at least 5.

From this claim we conclude two things. First, since two elements of J_1^* have different Picard numbers, the result of Zarhin [Zar] shows the Hodge structures of the elements of J_1^* are not all the same. Thus the subvariety C is not a point. Second, the numerical equivalence class of C is nonzero, since C necessarily intersects any codimension one special subvariety of $\mathcal{V}$ containing the image of $\psi_3 - \psi_4 \in J_1$ in a finite nonzero number of points. □

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