A MULTI-CHANNEL DOUBLE BOOTSTRAP FOR RHO AND K*

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ABSTRACT

We investigate the features of the bootstrap mechanism in a complex case involving many channels. The ρ meson is generated in the $\pi\pi$, $\pi\omega$, and $K\bar{K}$ channels, while the K* meson is simultaneously formed in the π K and η K channels. A single self-consistent solution to the seven coupled equations which result from the bootstrap conditions exists, and the calculation produces predictions for the ρ and K* masses and for the five coupling constants $\gamma_{\rho\pi\pi}$ (or $\gamma_{\rho\pi\omega}$), $\gamma_{\rho KK}$, $\gamma_{\omega KK}$, $\gamma_{\eta KK^*}$, $\gamma_{\omega KK^*}$. Typical results are M_ρ = 765 Mev, M_{K^*} = 900 Mev, while the prediction for the only experimentally known coupling constant ($\gamma_{\rho\pi\pi}$ or $\gamma_{\rho\pi\omega}$) matches the experimental value. The other coupling constants approximate SU_3 symmetry with the exception of $\gamma_{\eta KK^*}$ and possibly $\gamma_{\omega KK}$. The effects of the ϕ meson, including $\phi - \omega$ mixing, the $\pi\phi$ channel, and ϕ exchange forces, are discussed and estimated. The basic calculational tool is the matrix ND^{-1} technique with the first determinental approximation—the influence of this approximation is evaluated through a comparison of the results using the approximation with those from solving the exact equations for a suitable test problem. The results apparantly justify the use of the approximation. The good results obtained from the calculation demonstrate that the bootstrap idea continues to work even for rather complicated cases, and continues to give sensible answers.

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I. INTRODUCTION

The following material describes a test of the bootstrap hypothesis (1,2,Appendix III)—the idea that all particles are merely bound or resonating states of each other—in a complex case involving many channels. The salient features of the results of most bootstrap calculations are the predictions of masses to a fair degree of accuracy and predictions of widths and/or coupling constants typically two to five times those experimentally observed. In view of the many drastic approximations made it is rewarding that any predictions may be made at all. One of the more serious deficiencies has been the rather cavalier treatment of inelastic processes. It is expected that a more realistic treatment of inelastic effects would reduce the calculated widths toward the observed figures, and a first crude inclusion of inelasticity by Balazs (3) in a bootstrap calculation has demonstrated this. The technique involved replacing the partial wave elastic unitarity relation

$$\text{Im } t = |t|^2 \tag{1.1}$$

for a suitably defined partial wave amplitude t, by the exact

$$\text{Im } t = |t|^2 R \tag{1.2}$$

where R is the ratio of the total cross section to the elastic cross section. Unfortunately, R, which measures the deviation from complete elasticity, is unknown except for the elastic region where R = 1. Balazs took R = 1 for this low energy region, R = 2 for all higher energies. The result is to approximately halve the computed width.

More realistically such problems should be treated as involving many competing channels; however the computational labor required increases rapidly as channels proliferate. Also very serious for the bootstrapper is the profusion of undeterminable paramaters such as masses and coupling constants introduced by the additional channels. One must include enough of the neighboring and low-lying channels to give a fair description of the

physics involved while avoiding these practical mathematical difficulties. An example of a problem amenable to such compromises is the popular ρ bootstrap (4,5). The channels with the lowest thresholds are 2π (276 Mev), 4π (552 Mev), 6π (828 Mev), $\pi\omega$ (923 Mev)(6), $K\bar{K}$ (997 Mev), 8π (1104 Mev), $\eta\rho$ (1310 Mev)(7), $K^*\bar{K}$ (1378 Mev)(8), 10π (1380 Mev),... The channels involving several pairs of pions may be expected to contribute little due to the much smaller phase space available. Thus it is assumed that the major effect of the 4π state arises from situations in which three of the pions are resonating in the ω state. After the removal of the $n\pi$ channels, $n \geq 4$, the remaining channels break up into several distinct groups. First is the $\pi\pi$ channel, widely separated from the next two, $\pi\omega$ and $K\bar{K}$. Then there is a second gap of the order of 300 Mev to the next set, $\eta\rho$, $K^*\bar{K}$,... A reasonable sequence of approximations would first involve a single channel $\pi\pi \to \pi\pi$ calculation. However the $\pi\omega$ and $K\bar{K}$ thresholds are quite close to the mass of the particle we're trying to predict, and these channels are doubtlessly important. All things being equal, for consistency both channels should be included. Then if the algebra involved and computing budgets permit, the next set of channels may be added.

To the present time the three channel ρ bootstrap has not been attempted. Apart from algebraic difficulties the reason is that the problem is not closed, in the sense that all or even most of the paramaters involved are not determined by the self-consistency bootstrap equations. Consider first the ρ bootstraps which are closed, or are nearly so. The single channel problem (9) involves m_π, M_ρ, and $\gamma_{\rho\pi\pi}$.(10). Since M_ρ and $\gamma_{\rho\pi\pi}$ appear as outputs the problem is completely closed (m_π is not determined, but is required only to fix the energy scale). If the $\pi\omega$ channel is now included (9) we introduce the quantities M_ω and $\gamma_{\rho\pi\omega}$. But $\gamma_{\rho\pi\omega}$ appears as output and M_ω may be taken from experiment. The inclusion of the second channel ($\pi\omega$) to the original $\pi\pi$ channel as much as halves the computed ρ width and

greatly improves the predicted ρ mass. A two channel ρ bootstrap including $\pi\pi$ and $K\bar{K}$ is somewhat more unsavory. If done completely it requires as inputs three masses and three coupling constants in addition to the paramaters of the single channel case. For this it gets one additional equation. A truncated version of the problem has been treated (11) ignoring the $K\bar{K} \to K\bar{K}$ forces.

Actually, of course, any reason for excluding either of the $\pi\omega$ or $K\bar{K}$ channels while including the other is fragile on physical grounds, although possibly very compelling for practical reasons. The thresholds are very close and the various couplings, where known, are not drastically different. Isotopic spin crossing constants favor $\pi\omega$ over $K\bar{K}$ to some extent, but little choice may be made between them otherwise (12). The physically reasonable jump from the single channel calculation is to a three channel problem. However if this next step is attempted the happy state of affairs in which the problem is closed abruptly dissolves. In addition to the paramaters of the single channel problem we now require as inputs m_K, M_ω, M_{K^*}, $\gamma_{\rho\pi\omega}$, $\gamma_{\rho KK}$, $\gamma_{\omega KK}$, $\gamma_{\pi KK^*}$, $\gamma_{\omega KK^*}$, while we derive just two additional equations. The problem is no longer closed. If we try to cure this by including the next higher group of communicating channels, $\eta\rho$, $K^*\bar{K}$, etc., the problem is only aggravated. However there is another means to close the bootstrap which is physically reasonable—we simultaneously bootstrap the K*(885 Mev). For this meson the communicating channels with lowest thresholds are πK (631 Mev), ηK (1043 Mev), πK^* (1023 Mev), ρK (1253 Mev), ρK^* (1645 Mev), ωK^* (1660 Mev),...(again we have ignored channels involving more than two particles). Consider bootstrapping the K^* from the πK channel alone. This calculation involves m_π, m_K, M_ρ, M_{K^*}, $\gamma_{\rho\pi\pi}$, $\gamma_{\rho KK}$, $\gamma_{\pi KK^*}$. These quantities are all required in the three channel ρ problem as well, while we obtain two further equations. Therefore a three channel ρ bootstrap and a single channel K^* bootstrap performed simultaneously would very nearly form a closed problem. Unfortunately, however, it

doesn't make much physical sense to bootstrap K* solely from the π K channel—indeed it is only marginally possible. The exchange of ρ in the t channel does produce an attractive force in the I = ½, J = 1 channel, but the exchange of K* itself in the u channel is strongly repulsive. For large enough $\gamma_{\rho\pi\pi}\gamma_{\rho KK}$ a resonant state is possible, but it is quite broad. The other channels may confidently be expected to have a large effect, and we therefore agree to include more of them. The next highest channel is π K*. This shares with the ρ K channel a disagreeable pathology—they are not orthogonal channels, but rather different forms of $\pi\pi$ K. The unsnarling of this difficulty has yet to be demonstrated. We turn therefore to the η K channel. The inclusion of this in the K* bootstrap supplies the forces to make a reasonable two channel bootstrap, as Capps (13) has demonstrated. This now demands m_η and $\gamma_{\eta KK^*}$, but $\gamma_{\eta KK^*}$ appears as output and m_η may be taken from experiment.

All of the preceeding discussion has ignored the existence of the ϕ meson (6) and the ϕ-ω mixing phenomonon (14,15). We discuss the problem of including the $\pi\phi$ channel in Appendix I and show that the effects are likely to be small, even apart from the much higher threshold of the $\pi\phi$ channel. A somewhat different treatment of ϕ-ω mixing is discussed in chapter VI.

Consider then the following "double bootstrap". The ρ is found as the I = 1, $J^{PG} = 1^{-+}$, strangeness 0 resonance in the $\pi\pi$, $\pi\omega$, and $K\bar{K}$ channels. K* is the I = ½, J = 1, strangeness 1 resonance in the π K and η K channels. As inputs to the calculation we require the masses

$$m_\pi, m_K, m_\eta, M_\rho, M_\omega, M_{K^*}$$

and the coupling constants

$$\gamma_{\rho\pi\pi}, \gamma_{\rho\pi\omega}, \gamma_{\pi KK^*}, \gamma_{\eta KK^*}, \gamma_{\rho KK}, \gamma_{\omega KK}, \gamma_{\omega KK^*}.$$

From the self-consistency equations we shall derive seven relations among these paramaters. Therefore six of these thirteen numbers must be fed in at the beginning. Four of these we take to be m_π, m_K, m_η, and M_ω. Of the seven coupling constants

just two, $\gamma_{\rho\pi\pi}$ and $\gamma_{\pi KK^*}$ are known with any precision; $\gamma_{\rho\pi\omega}$ has been estimated by various means (9,16). The most convenient choice for calculation is (cf. chapter V) to take $\gamma_{\rho\pi\omega}$ and $\gamma_{\pi KK^*}$ from experiment and to calculate the remainder. However in the calculation $\gamma_{\rho\pi\omega}$ is varied until the experimental value of $\gamma_{\rho\pi\pi}$ is predicted. This is then equivalent to using $\gamma_{\rho\pi\pi}$ and $\gamma_{\pi KK^*}$ as the input paramaters. Therefore the results of the calculation (assuming that a self-consistent solution is found) will be the prediction of the masses of ρ and K*, and the coupling constants

$$\gamma_{\rho\pi\pi} \text{ (or } \gamma_{\rho\pi\omega}\text{)}, \gamma_{\eta KK^*}, \gamma_{\rho KK}, \gamma_{\omega KK}, \gamma_{\omega KK^*}.$$

It has been implicitly assumed in the foregoing, and here we make it explicit, that this calculation is a bootstrap only in the original restricted sense of producing predictions of particle masses and coupling constants. We do not attempt to bootstrap internal symmetries, the existence of multiplets, the number of dimensions, etc. We explicitly assume SU_2 symmetry (isospin) and SU_3 symmetry is mentioned only for purposes of examining the results for the various coupling constants, most of which are experimentally undetermined.

The basic calculational tool will be the matrix ND^{-1} technique (17). However, if done properly, even after all of the approximations made up 'till now, the procedure results in some staggering number of coupled integral equations. We shall therefore approximate still further and use the first determinental approximation (18). This makes it possible to write in the dependence of various quantities on the coupling constants explicitly and then eliminate some of these by hand until a reasonable number of simultaneous equations remain which may then be amenable to numerical solution. This method is possible at all only due to the happy chance that, experimentally, the locations of the ρ and K* poles in the (energy)^2 plane do not lie on top of any cuts arising from the forcing terms. If this were not the case this approximation technique would predict complex

coupling constants. Of course nothing guarantees that the results of the calculation will place the masses in these safe regions—the demand that they do so imposes some restrictions on the output masses acceptable. It is found (chapter VI) that the predicted ρ mass must fall within the rather narrow window

$$698 \text{ Mev} \leq M_{\rho} \leq 772 \text{ Mev}$$

and likewise

$$740 \text{ Mev} \leq M_{K^*} .$$

Otherwise we get an inconsistency.

In Appendix II we investigate the effects of employing the first determinental approximation by comparing the results obtained using the approximation with those obtained from solving the complete integral equations for a suitable test problem. The results apparantly justify the use of the approximation.

II. KINEMATICS and UNITARITY

Consider reactions involving four arbitrary particles A,B, C,D and their antiparticles. The particle A will be distinguished by its four-momentum p_1, mass m_1, spin s_1, helicity λ_1, intrinsic parity η_1, isotopic spin I_1, and z-projection of isospin i_1; similarly for B,C, and D, and the antiparticles $\bar{A},\bar{B},\bar{C}$, and $\bar{D}$. Figure 1 describes such reactions; note that for maximum symmetry all particles are formally considered incoming, although in any given reaction two of the four-momenta will be time-like and two will be space-like. We define the product of four-vectors a•b as

$$a\cdot b = a_o b_o - \vec{a}\cdot\vec{b} \tag{2.1}$$

and further define the usual Mandlestam variables

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 + p_3)^2 = (p_2 + p_4)^2 \\ u &= (p_1 + p_4)^2 = (p_2 + p_3)^2 \end{aligned} \tag{2.2}$$

A crucial part of the ensuing calculation is the substitution law—the same analytic function of the invariants s,t and/or u describes the six reactions in which alternately s,t, or u is the square of the total center of mass energy; that is, the six reactions

i) $A(p_1) + B(p_2) \to \bar{C}(-p_3) + \bar{D}(-p_4)$	(s channel)
ii) $C(p_3) + D(p_4) \to \bar{A}(-p_1) + \bar{B}(-p_2)$	
iii) $A(p_1) + C(p_3) \to \bar{B}(-p_2) + \bar{D}(-p_4)$	(t channel)
iv) $B(p_2) + D(p_4) \to \bar{A}(-p_1) + \bar{C}(-p_3)$	
v) $A(p_1) + D(p_4) \to \bar{C}(-p_3) + \bar{B}(-p_2)$	(u channel)
vi) $C(p_3) + B(p_2) \to \bar{A}(-p_1) + \bar{D}(-p_4)$	

Consider the first of these. To describe this reaction it is convenient to work in the center of mass system of A and B. We may define the co-ordinate axes such that the outgoing $\bar{C}$ travels along the positive z direction, while A and B lie in the x-z plane. If Θ is the angle of scattering between the direction of the incident A and the outgoing $\bar{C}$, and $x = \cos\theta$,

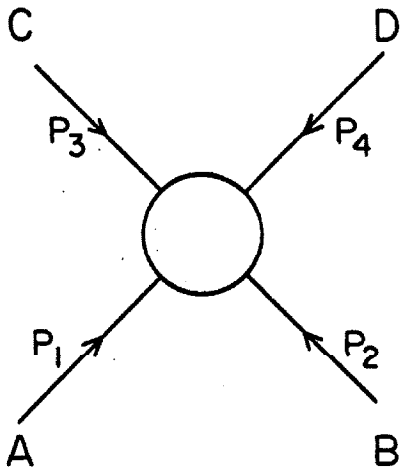

Fig. 1

1. The standard two particle reaction

then in this frame (19)

$$\begin{aligned} p_1 &= (E_1, p(1-x^2)^{\frac{1}{2}}, 0, xp) \\ p_2 &= (E_2, -p(1-x^2)^{\frac{1}{2}}, 0, -xp) \\ p_3 &= (-E_3, 0, 0, -q) \\ p_4 &= (-E_4, 0, 0, q) \end{aligned} \tag{2.3}$$

Here all energies E_i are positive, and p and q are the magnitudes of the initial and final three-momenta, respectively. We have the relations

$$\begin{aligned} E_{1,2} &= (p^2 + m_{1,2}^2)^{\frac{1}{2}} = \frac{s + m_{1,2}^2 - m_{2,1}^2}{(4s)^{\frac{1}{2}}} \\ E_{3,4} &= (q^2 + m_{3,4}^2)^{\frac{1}{2}} = \frac{s + m_{3,4}^2 - m_{4,3}^2}{(4s)^{\frac{1}{2}}} \end{aligned} \tag{2.4}$$

$$\begin{aligned} p^2 &= \left[s - (m_1 - m_2)^2 \right] \left[s - (m_1 + m_2)^2 \right] / 4s \\ q^2 &= \left[s - (m_3 - m_4)^2 \right] \left[s - (m_3 + m_4)^2 \right] / 4s \end{aligned} \tag{2.5}$$

$$\begin{aligned} s &= (E_1 + E_2)^2 = (E_3 + E_4)^2 \\ t &= (E_1 - E_3)^2 - p^2 - q^2 + 2pqx \\ u &= (E_1 - E_4)^2 - p^2 - q^2 - 2pqx \end{aligned} \tag{2.6}$$

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \tag{2.7}$$

The analytic function describing these six reactions is the expectation value of an operator $\hat{T}$ between the various initial and final states. $\hat{T}$ is defined below in terms of the Heisenberg $\hat{S}$ operator which transforms the initial state into the final state. Before considering the definition of $\hat{T}$ we make a short digression concerning the mathematical expression for the term "a sum over intermediate states" which is needed below.

Consider two states consisting of r and s particles respectively:

$$|a\rangle = |p_1, z_1; p_2, z_2; \cdots; p_r, z_r\rangle$$

$$|b\rangle = |q_1,w_1;q_2,w_2;\cdots;q_s,w_s\rangle \tag{2.8}$$

where p_i and q_i are four-momenta and z_i and w_i include all other quantum numbers of the particles (charge, spin, isospin, etc.). We include the masses in z_i and w_i. We define the scalar product $\langle a|b\rangle$ as

$$\langle a|b\rangle = \delta_{rs}\prod_{i=1}^{r}(2\pi)^3\,\delta^{(3)}(\vec{p}_i-\vec{q}_i)\,2E_i\,\delta(z_i,w_i) \tag{2.9}$$

where δ_{rs} is the Kronecker delta and $\delta(z_i,w_i) = 0$ unless all quantum numbers of the two particles being compared are identical, in which case $\delta(z_i,w_i) = 1$. Here E_i is the energy of the i th particle of either $|a\rangle$ or $|b\rangle$. This definition of $\langle a|b\rangle$ is relativistically invariant.

We now introduce the operator

$$\hat{O} = \sum_c |c\rangle\langle c|$$

and insist that

$$\langle a|b\rangle = \sum_c \langle a|c\rangle\langle c|b\rangle \tag{2.10}$$

If $|c\rangle = |k_1,v_1;k_2,v_2;\cdots;k_t,v_t\rangle$ then the requirement that $\hat{O}$ be the identity operator becomes

$$\langle a|b\rangle = \delta_{rs}\prod_{i=1}^{r}(2\pi)^3\,\delta^{(3)}(\vec{p}_i-\vec{q}_i)2E_i\,\delta(z_i,w_i)$$

$$= \sum_c \left[\,\delta_{rt}\prod_{i=1}^{r}(2\pi)^3\,\delta^{(3)}(\vec{p}_i-\vec{k}_i)2E_i\,\delta(z_i,v_i)\,\right]$$

$$\times\left[\,\delta_{ts}\prod_{j=1}^{t}(2\pi)^3\,\delta^{(3)}(\vec{k}_j-\vec{q}_j)2E_j\,\delta(v_j,w_j)\,\right]$$

$$= \delta_{rs}\prod_{i=1}^{r}(2\pi)^3\,\delta^{(3)}(\vec{p}_i-\vec{q}_i)2E_i\,\delta(z_i,w_i)$$

$$\times\sum_c \delta_{rt}\prod_{j=1}^{t}(2\pi)^3\,\delta^{(3)}(\vec{p}_j-\vec{k}_j)2E_j\,\delta(z_j,v_j) \tag{2.11}$$

Therefore

$$\sum_c \delta_{rt} \prod_{j=1}^{t} (2\pi)^3 \, \delta^{(3)}(\vec{p}_j - \vec{k}_j) 2E_j \, \delta(z_j, w_j) = 1 \qquad (2.12)$$

so that

$$\sum_c = \sum_{t=1}^{\infty} \prod_{j=1}^{t} \left[\int \frac{d^3\vec{k}_j}{(2\pi)^3} \frac{1}{2E_j} \sum_{v_j} \right] \qquad (2.13)$$

$$= \sum_{t=1}^{\infty} \prod_{j=1}^{t} \left[\int \frac{d^4k_j}{(2\pi)^4} \, 2\pi \, \delta(k_j^2 - m_j^2) \theta(E_j - m_j) \sum_{v_j} \right] \qquad (2.14)$$

where θ is the step function

$$\theta(x) = \begin{matrix} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{matrix} \qquad (2.15)$$

This defines the operator $\sum_c$; it is manifestly covariant. We shall use this result directly.

We now define the operator $\hat{T}$ in terms of $\hat{S}$ by

$$\langle a | \hat{S} | b \rangle = S_{ab} = \langle a | b \rangle - i(2\pi)^4 \, \delta^{(4)}(P_a - P_b) \langle a | \hat{T} | b \rangle \qquad (2.16)$$

P_a and P_b represent the total four-momenta of the states $|a\rangle$ and $|b\rangle$. Since S_{ab}, $\langle a | b \rangle$, and $\delta^{(4)}(P_a - P_b)$ are relativistically invariant, so is $\langle a | \hat{T} | b \rangle = T_{ab}$. Now $\hat{S}$ is a unitary operator, $\hat{S}^+\hat{S} = \hat{1}$ (we denote the adjoint operator by $^+$). In terms of $\hat{T}$ this reads

$$\frac{1}{2i}(T_{ab} - T_{ab}^+) = -\frac{1}{2}(2\pi)^4 \sum_c \delta^{(4)}(P_a - P_c) \, T_{ac}^+ \, T_{cb} \qquad (2.17)$$

This is the condition forced upon $\hat{T}$ by the unitarity of $\hat{S}$. For $\sum_c$ we understand the operator of equation (2.14).

It is henceforth assumed that equations (2.17) and (2.14), together with the substitution law and the requirements of relativistic invariance suffice to determine T_{ab} completely. In practice the sum in (2.14) is truncated at some (small) value of t (i.e. only few-particle intermediate states are included) and

only the less massive particles are included in $\sum_{\nu_j}$. This reduces the problem to manageable levels and approximate solutions may be found. The details of such a calculation are the subjects of the following chapters.

In the ensuing computation we shall work in the helicity representation of Jacob and Wick (20) in which states are characterized by values of the total helicity. The partial wave expansion takes the form (20)

$$\langle \bar{C}(-p_3, -\lambda_3), \bar{D}(-p_4, -\lambda_4) | \hat{T} | A(p_1, \lambda_1), B(p_2, \lambda_2) \rangle$$

$$= \sum_J (2J+1) \langle \bar{C}(-p_3, -\lambda_3), \bar{D}(-p_4, -\lambda_4) | \hat{T}^J | A(p_1, \lambda_1), B(p_2, \lambda_2) \rangle \times d^J_{\lambda\mu}(\theta) \tag{2.18}$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_4 - \lambda_3$, and θ, as defined earlier, is the scattering angle between A and $\bar{C}$. The inversion of (2.18) reads

$$\langle \bar{C}(-p_3, -\lambda_3), \bar{D}(-p_4, -\lambda_4) | \hat{T}^J | A(p_1, \lambda_1), B(p_2, \lambda_2) \rangle$$

$$= \frac{1}{2} \int_{-1}^{+1} \langle \bar{C}(-p_3, -\lambda_3), \bar{D}(-p_4, -\lambda_4) | \hat{T} | A(p_1, \lambda_1), B(p_2, \lambda_2) \rangle \times d^J_{\lambda\mu}(\theta) \sin\theta \, d\theta \tag{2.19}$$

The $d^J_{\lambda\mu}(\theta)$ are either tabulated in reference (20) or may be computed with the aid of the formulae found there.

Before concluding this chapter it is convenient to demonstrate the following amusing and useful fact: the quantity $pq(E_1 + E_2)(1 - x^2)^{\frac{1}{2}}$ is invariant under any interchange of particles. To show this we use equation (2.6) to solve for 4pqx, eliminating the various E's with the aid of (2.4). We find

$$4pqx = t - u + \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2)}{s} \tag{2.20}$$

Equation (2.5) is next used to find $16p^2q^2$ and the square of (2.20) is subtracted from the resulting expression. Then

$$16p^2q^2 - 16p^2q^2x^2 = s^2 - 2s(m_1^2 + m_2^2 + m_3^2 + m_4^2) + (m_3^2 - m_4^2)^2 + (m_1^2 - m_2^2)^2 + 4(m_1^2 + m_2^2)(m_3^2 + m_4^2) - (t - u)^2 - \frac{2}{s}\left[(m_1^2 + m_2^2)(m_3^2 - m_4^2) + (m_1^2 - m_2^2)(m_3^2+m_4^2) \right] - \frac{2(t - u)}{s}(m_1^2 - m_2^2)(m_3^2 - m_4^2) \qquad (2.21)$$

Part of this may be simplified as follows:

$$Q \equiv s^2 - 2s(m_1^2 + m_2^2 + m_3^2 + m_4^2) + (m_3^2 - m_4^2)^2 + (m_1^2 - m_2^2)^2 + 4(m_1^2 + m_2^2)(m_3^2 + m_4^2) - (t - u)^2$$

$$= - s^2 - t^2 - u^2 - 2st - 2su - 2tu + 4tu + (m_1^2 - m_2^2)^2 + (m_3^2 - m_4^2)^2 + 4(m_1^2 + m_2^2)(m_3^2 + m_4^2) \qquad (2.22)$$

where we have used (2.7). The first six terms are just $-(s + t + u)^2$; again using (2.7) and cancelling terms, we have

$$Q = 4tu + 2(m_1^2m_3^2 + m_1^2m_4^2 + m_2^2m_3^2 + m_2^2m_4^2) - 4(m_1^2m_2^2 + m_3^2m_4^2) \qquad (2.23)$$

We put this back into (2.21), factor out a common $2/s$, rearrange terms, and at length discover

$$16p^2q^2(1 - x^2) = \frac{2}{s}\left[2stu + 4(m_1^2m_2^2m_3^2 + m_1^2m_2^2m_4^2 + m_1^2m_3^2m_4^2 + m_2^2m_3^2m_4^2) - 2s(m_1^2m_2^2 + m_3^2m_4^2) - 2t(m_1^2m_3^2 + m_2^2m_4^2) - 2u(m_1^2m_4^2 + m_2^2m_3^2) \right] \qquad (2.24)$$

Since $(E_1 + E_2)^2 = s$, we finally obtain the result:

$$pq(E_1 + E_2)(1 - x^2)^{\frac{1}{2}} = \frac{1}{2}\left[stu + 2(m_1^2m_2^2m_3^2 + m_1^2m_2^2m_4^2 + m_1^2m_3^2m_4^2 + m_2^2m_3^2m_4^2) - s(m_1^2m_2^2 + m_3^2m_4^2) - t(m_1^2m_3^2 + m_2^2m_4^2) - u(m_1^2m_4^2 + m_2^2m_3^2) \right]^{\frac{1}{2}} \qquad \ldots\ldots(2.25)$$

which, taking into account (2.2), is manifestly covariant under any exchange of particles. This result will be used below in the derivation of the Born forces (chapter IV).

III. *ISOTOPIC SPIN and CROSSING MATRICES*

In this chapter the complications due to isotopic spin are discussed. The result is the evaluation of the isospin "crossing matrices" relating amplitudes of definite isotopic spin in the direct channel to corresponding amplitudes in the t and u channels. At the heart of the calculation is the substitution law. Consider the six reactions indicated by figure 1. If we assign to particles A,B,C,D (as in chapter II all particles are formally taken as incoming) the values I_1,I_2,I_3,I_4 for the isotopic spin, and i_1,i_2,i_3,i_4 for the z-projection of the isotopic spin, then the same function T is the amplitude describing the reactions

i) $A(I_1,i_1) + B(I_2,i_2) \to \bar{C}(I_3,-i_3) + \bar{D}(I_4,-i_4)$ (s channel)

ii) $A(I_1,i_1) + C(I_3,i_3) \to \bar{B}(I_2,-i_2) + \bar{D}(I_4,-i_4)$ (t channel)

iii) $A(I_1,i_1) + D(I_4,i_4) \to \bar{C}(I_3,-i_3) + \bar{B}(I_2,-i_2)$ (u channel)

That is,

$$\begin{aligned} T(ABCD) &= \langle C(I_3,-i_3),D(I_4,-i_4)|\hat{T}|A(I_1,i_1),B(I_2,i_2)\rangle \\ &= \langle \overline{B(I_2,i_2)},D(I_4,-i_4)|\hat{T}|A(I_1,i_1),\overline{\bar{C}(I_3,-i_3)}\rangle \\ &= \langle C(I_3,-i_3),\overline{B(I_2,i_2)}|\hat{T}|A(I_1,i_1),\overline{\bar{D}(I_4,-i_4)}\rangle \end{aligned} \tag{3.1}$$

Here the long bar over a particle denotes the charge conjugated state. For example, if the particle state is related to the isospin state through a phase a (e.g. $|\pi^+\rangle = a_+ |\pi;I=1,I_z=1\rangle$)

$$\begin{aligned} |A(I,i)\rangle &= a_i \,|\, A;I,i\rangle \\ |\bar{A}(I,i')\rangle &= \bar{a}_{i'} \,|\, \bar{A};I,i'\rangle \end{aligned} \tag{3.2}$$

then the charge conjugated state is

$$\begin{aligned} \overline{|A(I,i)\rangle} &= \hat{C}\,|A(I,i)\rangle = a_i\,\hat{C}\,|A;I,i\rangle = a_i\,\phi_A\,|\bar{A};I,-i\rangle \\ &= a_i\bar{a}_{-i}\,\phi_A\,|\bar{A}(I,-i)\rangle \end{aligned} \tag{3.3}$$

where $\hat{C}$ is the charge conjugation operator and ϕ_A is the intrinsic charge conjugation quantum number of the particle A ($\phi_\pi = 1$,

$\phi_\rho = -1$, etc.) Thus $\hat{C}|K^+\rangle = k_+ \hat{C}\,|K;\ I_z=\tfrac{1}{2}\rangle = k_+ \phi_K\,|\bar{K};I_z=-\tfrac{1}{2}\rangle$ $= k_+\bar{k}_-\,\phi_K\,|K^-\rangle = |K^+\rangle$. These phases are important and must be carried along.

We now introduce into (3.1) eigenstates of $\hat{I}$ and $\hat{I}_z$, $|I,I_z\rangle$, and express T(ABCD) in terms of this complete set.

$$\langle \bar{C}(I_3,-i_3),\bar{D}(I_4,-i_4)|\hat{T}|A(I_1,i_1),B(I_2,i_2)\rangle$$

$$= \sum_{I,I_z,I',I'_z} \langle \bar{C}(I_3,-i_3),\bar{D}(I_4,-i_4)\,|\,I,I_z\rangle\langle I,I_z|\hat{T}|I',I'_z\rangle$$

$$\times \langle I',I'_z\,|\,A(I_1,i_1),B(I_2,i_2)\rangle \qquad (3.4)$$

But $\hat{T}$ is diagonal in I and independent of I_z:

$$\langle I,I_z|\hat{T}|I',I'_z\rangle = \delta_{II'}\,\delta_{I_zI'_z}\,\langle I,I_z|\hat{T}|I,I_z\rangle$$

$$= \delta_{II'}\,\delta_{I_zI'_z}\,T_I \qquad (3.5)$$

Moreover T_I of course depends on what particles composed the states of definite I and I_z; we denote this information with the use of a superscript on T indicating the channel, and exhibit the individual isospins of the particles explicitly, e.g.

$$T_I^{(s)}(I_3,I_4;I_1,I_2).$$

Note that the order of the I's in the argument of T_I is important. Then

$$\langle \bar{C}(I_3,-i_3),\bar{D}(I_4,-i_4)|\hat{T}|A(I_1,i_1),B(I_2,i_2)\rangle$$

$$= \sum_I T_I^{(s)}(I_3,I_4;I_1,I_2)\,P_I^{(s)}(I_3,I_4;I_1,I_2) \qquad (3.6)$$

where

$$P_I^{(s)}(I_3,I_4;I_1,I_2) = \langle C(I_3,-i_3),D(I_4,-i_4)|\hat{P}_I|A(I_1,i_1),B(I_2,i_2)\rangle \qquad \ldots.(3.7)$$

is the matrix element in the s channel of the isotopic spin projection operator

$$\hat{P}_I = \sum_{I_z} |I,I_z\rangle\langle I,I_z| \qquad (3.8)$$

The same expansion is carried out in the t and u channels with the result

$$\begin{aligned} T(ABCD) &= \sum_I T_I^{(s)}(I_3,I_4;I_1,I_2)\ P_I^{(s)}(I_3,I_4;I_1,I_2) \\ &= \sum_I T_I^{(t)}(I_2,I_4;I_1,I_3)\ P_I^{(t)}(I_2,I_4;I_1,I_3) \\ &= \sum T_I^{(u)}(I_3,I_2;I_1,I_4)\ P_I^{(u)}(I_3,I_2;I_1,I_4) \end{aligned} \qquad (3.9)$$

These last equations may be inverted to give

$$\begin{aligned} T_I^{(s)}(I_3,I_4;I_1,I_2) &= \sum_{I'} g_{II'}(I_3,I_4;I_1,I_2)\ T_{I'}^{(t)}(I_2,I_4;I_1,I_3) \\ &= \sum_{I'} \bar{g}_{II'}(I_3,I_4;I_1,I_2)\ T_{I'}^{(u)}(I_3,I_2;I_1,I_4) \end{aligned} \qquad \ldots.(3.10)$$

Equation (3.10) then gives the contribution from the exchange of particles in definite isotopic spin states in the t and u channels to the corresponding s channel amplitudes.

We illustrate this process with an example. Consider the reaction in which two iso-vector particles produce a pair of isospinors, e.g. $\pi\pi \to K\bar{K}$, $\pi\rho \to N\bar{N}$, etc.

In the s channel

$$A(1,i_1) + B(1,i_2) \to \bar{C}(\tfrac{1}{2},-i_3) + \bar{D}(\tfrac{1}{2},-i_4). \quad (I=0,1)$$

In the t channel

$$A(1,i_1) + C(\tfrac{1}{2},i_3) \to \bar{B}(1,-i_2) + \bar{D}(\tfrac{1}{2},-i_4). \quad (I=\tfrac{1}{2},\tfrac{3}{2}) \qquad (3.11)$$

In the u channel

$$A(1,i_1) + D(\tfrac{1}{2},i_4) \to \bar{C}(\tfrac{1}{2},-i_3) + \bar{B}(1,-i_2). \quad (I=\tfrac{1}{2},\tfrac{3}{2})$$

The relations between the charge states of the various particles and the corresponding isospin states are taken as:

$$|1,1\rangle = a_+ |A^+\rangle = b_+ |B^+\rangle$$

$$
\begin{aligned}
|1,0\rangle &= a_o|A^o\rangle = b_o|B^o\rangle \\
|1,-1\rangle &= a_-|A^-\rangle = b_-|B^-\rangle \\
|\tfrac{1}{2},\tfrac{1}{2}\rangle &= c_+|C^+\rangle = \bar{c}_+|\bar{C}^+\rangle = d_+|D^+\rangle = \bar{d}_+|\bar{D}^+\rangle \\
|\tfrac{1}{2},-\tfrac{1}{2}\rangle &= c_-|C^-\rangle = \bar{c}_-|\bar{C}^-\rangle = d_-|D^-\rangle = \bar{d}_-|\bar{D}^-\rangle
\end{aligned}
\tag{3.12}
$$

We take all phases real. In the last two equations of (3.12) the + and - represent only the I_z value of the particle and not necessarily the charge; e.g. if C is a kaon, then

$$
|C^+\rangle = |K^+\rangle\ ,\ |C^-\rangle = |K^o\rangle\ ,|\bar{C}^+\rangle = |\bar{K}^o\rangle,\ |\bar{C}^-\rangle = |K^-\rangle\ . \tag{3.13}
$$

The reaction proceeds through definite isotopic spin channels, and the initial and final states are formed with the usual Clebsch-Gordon coupling formulae (we use the fact that $\bar{B} = B$, since we shall apply the results to mesons only).

$$
\left|\tfrac{3}{2},\ \tfrac{3}{2}\right\rangle = a_+c_+\ |A^+C^+\rangle = b_+\bar{d}_+|B^+\bar{D}^+\rangle = b_+\bar{c}_+|\bar{C}^+B^+\rangle = a_+d_+|A^+D^+\rangle
$$

$$
\begin{aligned}
\left|\tfrac{3}{2},\ \tfrac{1}{2}\right\rangle &= a_oc_+(2/3)^{1/2}|A^oC^+\rangle + a_+c_-(1/3)^{1/2}|A^+C^-\rangle \\
&= b_o\bar{d}_+(2/3)^{1/2}|B^o\bar{D}^+\rangle + b_+\bar{d}_-(1/3)^{1/2}|B^+\bar{D}^-\rangle \\
&= b_o\bar{c}_+(2/3)^{1/2}|\bar{C}^+B^o\rangle + b_+\bar{c}_-(1/3)^{1/2}|\bar{C}^-B^+\rangle \\
&= a_od_+(2/3)^{1/2}|A^oD^+\rangle + a_+d_-\ (1/3)^{1/2}|A^+D^-\rangle
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle &= a_oc_-(2/3)^{1/2}|A^oC^-\rangle + a_-c_+(1/3)^{1/2}|A^-C^+\rangle \\
&= b_o\bar{d}_-(2/3)^{1/2}|B^o\bar{D}^-\rangle + b_-\bar{d}_+(1/3)^{1/2}|B^-\bar{D}^+\rangle \\
&= b_o\bar{c}_-(2/3)^{1/2}|\bar{C}^-B^o\rangle + b_-\bar{c}_+(1/3)^{1/2}|\bar{C}^+B^-\rangle \\
&= a_od_-(2/3)^{1/2}|A^oD^-\rangle + a_-d_+(1/3)^{1/2}|A^-D^+\rangle
\end{aligned}
$$

$$
\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle = a_-c_-|A^-C^-\rangle = b_-\bar{d}_-|B^-\bar{D}^-\rangle = b_-\bar{c}_-|\bar{C}^-B^-\rangle = a_-d_-|A^-D^-\rangle
$$

$$
\begin{aligned}
\left|\tfrac{1}{2},\ \tfrac{1}{2}\right\rangle &= -a_oc_+(1/3)^{1/2}|A^oC^+\rangle + a_+c_-(2/3)^{1/2}|A^+C^-\rangle \\
&= -b_o\bar{d}_+(1/3)^{1/2}|B^o\bar{D}^+\rangle + b_+\bar{d}_-(2/3)^{1/2}|B^+\bar{D}^-\rangle
\end{aligned}
$$

$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle = b_o\bar{c}_+(1/3)^{1/2}\left|\bar{C}^+B^o\right\rangle - b_+\bar{c}_-(2/3)^{1/2}\left|\bar{C}^-B^+\right\rangle$$

$$= -a_o d_+(1/3)^{1/2}\left|A^oD^+\right\rangle + a_+ d_-(2/3)^{1/2}\left|A^+D^-\right\rangle$$

$$\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = a_o c_-(1/3)^{1/2}\left|A^oC^-\right\rangle - a_- c_+(2/3)^{1/2}\left|A^-C^+\right\rangle \qquad (3.15)$$

$$= b_o\bar{d}_-(1/3)^{1/2}\left|B^o\bar{D}^-\right\rangle - b_-\bar{d}_+(2/3)^{1/2}\left|B^-\bar{D}^+\right\rangle$$

$$= -b_o\bar{c}_-(1/3)^{1/2}\left|\bar{C}^-B^o\right\rangle + b_-\bar{c}_+(2/3)^{1/2}\left|\bar{C}^+B^-\right\rangle$$

$$= a_o d_-(1/3)^{1/2}\left|A^oD^-\right\rangle - a_- d_+(2/3)^{1/2}\left|A^-D^+\right\rangle$$

$$\left|1,1\right\rangle = a_+b_o(1/2)^{1/2}\left|A^+B^o\right\rangle - a_-b_+(1/2)^{1/2}\left|A^-B^+\right\rangle = \bar{c}_+\bar{d}_+\left|\bar{C}^+\bar{D}^+\right\rangle$$

$$\left|1,0\right\rangle = a_+b_-(1/2)^{1/2}\left|A^+B^-\right\rangle - a_-b_+(1/2)^{1/2}\left|A^-B^+\right\rangle \qquad (3.16)$$

$$= \bar{c}_+\bar{d}_-(1/2)^{1/2}\left|\bar{C}^+\bar{D}^-\right\rangle + \bar{c}_-\bar{d}_+(1/2)^{1/2}\left|\bar{C}^-\bar{D}^+\right\rangle$$

$$\left|1,-1\right\rangle = a_ob_-(1/2)^{1/2}\left|A^oB^-\right\rangle - a_-b_o(1/2)^{1/2}\left|A^-B^o\right\rangle = \bar{c}_-\bar{d}_-\left|\bar{C}^-\bar{D}^-\right\rangle$$

$$\left|0,0\right\rangle = a_+b_-(1/3)^{1/2}\left|A^+B^-\right\rangle + a_-b_+(1/3)^{1/2}\left|A^-B^+\right\rangle - a_ob_o(1/3)^{1/2}\left|A^oB^o\right\rangle$$

$$= \bar{c}_+\bar{d}_-\,(1/2)^{1/2}\left|\bar{C}^+\bar{D}^-\right\rangle - \bar{c}_-\bar{d}_+(1/2)^{1/2}\left|\bar{C}^-\bar{D}^+\right\rangle \qquad (3.17)$$

The next step is to compute the matrix elements of the projection operators between the various physical incoming and outgoing states.

$$P_o^{(s)}(\tfrac{1}{2},\tfrac{1}{2};1,1) = \left\langle C(\tfrac{1}{2},-i_3),D(\tfrac{1}{2},-i_4)\left|\hat{P}_o\right|A(1,i_1),B(1,i_2)\right\rangle$$

$$= (1/6)^{1/2}\Big[\,\bar{c}_+\bar{d}_-a_+b_-(1,-1,-\tfrac{1}{2},\tfrac{1}{2}) + \bar{c}_+\bar{d}_-a_-b_+$$

$$(-1,1,-\tfrac{1}{2},\tfrac{1}{2}) - \bar{c}_+\bar{d}_-a_ob_o(0,0,-\tfrac{1}{2},\tfrac{1}{2}) - \bar{c}_-\bar{d}_+a_+b_-$$

$$(1,-1,\tfrac{1}{2},-\tfrac{1}{2}) - \bar{c}_-\bar{d}_+a_-b_+(-1,1,\tfrac{1}{2},-\tfrac{1}{2})$$

$$+ \bar{c}_-\bar{d}_+a_ob_o(0,0,\tfrac{1}{2},-\tfrac{1}{2})\Big] \qquad (3.18)$$

where (w,x,y,z) denotes the following product of Kronecker deltas: $\delta_{i_1 w}\,\delta_{i_2 x}\,\delta_{i_3 y}\,\delta_{i_4 z}$. In a similar fashion we find

$$P_1^{(s)}(\tfrac{1}{2},\tfrac{1}{2};1,1) = \langle \bar{C}(\tfrac{1}{2},-i_3),\bar{D}(\tfrac{1}{2},-i_4)\,|\,\hat{P}_1\,|\,A(1,i_1),B(1,i_2)\rangle$$

$$= (1/2)^{\frac{1}{2}}\Big[\bar{c}_+\bar{d}_+a_+b_o(1,0,-\tfrac{1}{2},-\tfrac{1}{2}) - \bar{c}_+\bar{d}_+a_ob_+(0,1,-\tfrac{1}{2},-\tfrac{1}{2}) + \bar{c}_-\bar{d}_-a_ob_-(0,-1,\tfrac{1}{2},\tfrac{1}{2}) - \bar{c}_-\bar{d}_-a_-b_o(-1,0,\tfrac{1}{2},\tfrac{1}{2})\Big]$$

$$+ (1/2)\Big[\bar{c}_+\bar{d}_-a_+b_-(1,-1,-\tfrac{1}{2},\tfrac{1}{2}) - \bar{c}_+\bar{d}_-a_-b_+(-1,1,-\tfrac{1}{2},\tfrac{1}{2}) + \bar{c}_-\bar{d}_+a_+b_-(1,-1,\tfrac{1}{2},-\tfrac{1}{2}) - \bar{c}_-\bar{d}_+a_-b_+(-1,1,\tfrac{1}{2},-\tfrac{1}{2})\Big]$$

$$P_{3/2}^{(t)}(1,\tfrac{1}{2};1,\tfrac{1}{2}) = \langle \bar{B}(1,-i_2),\bar{D}(\tfrac{1}{2},-i_4)\,|\,\hat{P}_{3/2}\,|\,A(1,i_1),C(\tfrac{1}{2},i_3)\rangle$$

$$= \phi_B\phi_D\Big[\,b_+\bar{d}_+a_+c_+(1,-1,\tfrac{1}{2},-\tfrac{1}{2}) + b_-\bar{d}_-a_-c_-(-1,1,-\tfrac{1}{2},\tfrac{1}{2})\,\Big]$$

$$+\tfrac{1}{3}\phi_B\phi_D\Big[2b_o\bar{d}_+a_oc_+(0,0,\tfrac{1}{2},-\tfrac{1}{2}) + 2^{\frac{1}{2}}b_o\bar{d}_+a_+c_-(1,0,-\tfrac{1}{2},-\tfrac{1}{2})$$

$$+2^{\frac{1}{2}}b_+\bar{d}_-a_oc_+(0,-1,\tfrac{1}{2},\tfrac{1}{2}) + b_+\bar{d}_-a_+c_-(1,-1,-\tfrac{1}{2},\tfrac{1}{2})$$

$$+2b_o\bar{d}_-a_oc_-(0,0,-\tfrac{1}{2},\tfrac{1}{2}) + 2^{\frac{1}{2}}b_o\bar{d}_-a_-c_+(-1,0,\tfrac{1}{2},\tfrac{1}{2})$$

$$+2^{\frac{1}{2}}b_-\bar{d}_+a_oc_-(0,1,-\tfrac{1}{2},-\tfrac{1}{2}) + b_-\bar{d}_+a_-c_+(-1,1,\tfrac{1}{2},-\tfrac{1}{2})\Big]$$

$$P_{1/2}^{(t)}(1,\tfrac{1}{2};1,\tfrac{1}{2}) = \langle B(1,-i_2),D(\tfrac{1}{2},-i_4)\,|\,\hat{P}_{1/2}\,|\,A(1,i_1),C(\tfrac{1}{2},i_3)\rangle$$

$$=\tfrac{1}{3}\phi_B\phi_D\Big[b_o\bar{d}_+a_oc_+(0,0,\tfrac{1}{2},-\tfrac{1}{2}) - 2^{\frac{1}{2}}b_o\bar{d}_+a_+c_-(1,0,-\tfrac{1}{2},-\tfrac{1}{2})$$

$$-2^{\frac{1}{2}}b_+\bar{d}_-a_oc_+(0,-1,\tfrac{1}{2},\tfrac{1}{2}) + 2b_+\bar{d}_-a_+c_-(1,-1,-\tfrac{1}{2},\tfrac{1}{2})$$

$$+b_o\bar{d}_-a_oc_-(0,0,-\tfrac{1}{2},\tfrac{1}{2}) - 2^{\frac{1}{2}}b_o\bar{d}_-a_-c_+(-1,0,\tfrac{1}{2},\tfrac{1}{2})$$

$$-2^{\frac{1}{2}}b_-\bar{d}_+a_oc_-(0,1,-\tfrac{1}{2},-\tfrac{1}{2}) + 2b_-\bar{d}_+a_-c_+(-1,1,\tfrac{1}{2},-\tfrac{1}{2})\Big]$$

$$P^{(u)}_{3/2}(\tfrac{1}{2},1;1,\tfrac{1}{2}) = \left\langle \bar{C}(\tfrac{1}{2},-i_3),\bar{B}(1,-i_2)\middle|\hat{P}_{3/2}\middle| A(1,i_1),D(\tfrac{1}{2},i_4)\right\rangle$$

$$= \phi_B\phi_C\left[a_+b_+\bar{c}_+d_+(1,-1,-\tfrac{1}{2},\tfrac{1}{2})+a_-d_-b_-\bar{c}_-(-1,1,\tfrac{1}{2},-\tfrac{1}{2})\right]$$

$$+\tfrac{1}{3}\phi_B\phi_C\left[2a_ob_o\bar{c}_+d_+(0,0,-\tfrac{1}{2},\tfrac{1}{2})+2^{\frac{1}{2}}a_+b_o\bar{c}_+d_-(1,0,-\tfrac{1}{2},-\tfrac{1}{2})\right.$$

$$+2^{\frac{1}{2}}a_ob_+\bar{c}_-d_+(0,-1,\tfrac{1}{2},\tfrac{1}{2})+a_+b_+\bar{c}_-d_-(1,-1,\tfrac{1}{2},-\tfrac{1}{2})$$

$$+2a_ob_o\bar{c}_-d_-(0,0,\tfrac{1}{2},-\tfrac{1}{2})+2^{\frac{1}{2}}a_-b_o\bar{c}_-d_+(-1,0,\tfrac{1}{2},\tfrac{1}{2})$$

$$\left.+2^{\frac{1}{2}}a_ob_-\bar{c}_+d_-(0,1,-\tfrac{1}{2},-\tfrac{1}{2})+a_-b_-\bar{c}_+d_+(-1,1,-\tfrac{1}{2},\tfrac{1}{2})\right]$$

$$P^{(u)}_{1/2}(\tfrac{1}{2},1;1,\tfrac{1}{2}) = \left\langle \bar{C}(\tfrac{1}{2},-i_3),\bar{B}(1,-i_2)\middle|\hat{P}_{1/2}\middle| A(1,i_1),D(\tfrac{1}{2},i_4)\right\rangle$$

$$= \tfrac{1}{3}\phi_B\phi_C\left[-a_ob_o\bar{c}_+d_+(0,0,-\tfrac{1}{2},\tfrac{1}{2})+2^{\frac{1}{2}}a_+b_o\bar{c}_+d_-(1,0,-\tfrac{1}{2},-\tfrac{1}{2})\right.$$

$$+2^{\frac{1}{2}}a_ob_+\bar{c}_-d_+(0,-1,\tfrac{1}{2},\tfrac{1}{2})-2a_+b_+\bar{c}_-d_-(1,-1,\tfrac{1}{2},-\tfrac{1}{2})$$

$$-a_ob_o\bar{c}_-d_-(0,0,\tfrac{1}{2},-\tfrac{1}{2})+2^{\frac{1}{2}}a_-b_o\bar{c}_-d_+(-1,0,\tfrac{1}{2},\tfrac{1}{2})$$

$$\left.+2^{\frac{1}{2}}a_ob_-\bar{c}_+d_-(0,1,-\tfrac{1}{2},-\tfrac{1}{2})-2a_-b_-\bar{c}_+d_+(-1,1,-\tfrac{1}{2},\tfrac{1}{2})\right]$$

The final step consists of insisting that the equations

$$\sum_I T^{(s)}_I P^{(s)}_I = \sum_{I'} T^{(t)}_{I'} P^{(t)}_{I'} = \sum_{I''} T^{(u)}_{I''} P^{(u)}_{I''} \qquad (3.19)$$

hold for every possible combination of i_1,i_2,i_3,i_4. We have

$$i_1,i_2 = 1,0,-1 \qquad \text{and} \qquad i_3,i_4 = \tfrac{1}{2},-\tfrac{1}{2}$$

together with the restraint

$$i_1 + i_2 + i_3 + i_4 = 0 \qquad (3.20)$$

There are ten combinations satisfying these requirements: $(0,1,-\tfrac{1}{2},-\tfrac{1}{2}),(0,0,\tfrac{1}{2},-\tfrac{1}{2}),(0,0,-\tfrac{1}{2},\tfrac{1}{2}),(0,-1,\tfrac{1}{2},\tfrac{1}{2}),(1,-1,\tfrac{1}{2},-\tfrac{1}{2}),$ $(1,-1,-\tfrac{1}{2},\tfrac{1}{2}),(1,0,-\tfrac{1}{2},-\tfrac{1}{2}),(-1,1,\tfrac{1}{2},-\tfrac{1}{2}),(-1,1,-\tfrac{1}{2},\tfrac{1}{2}),(-1,0,\tfrac{1}{2},\tfrac{1}{2})$.

The ten equations which result from substituting these choices for i_1,i_2,i_3,i_4 into (3.19) are found to be consistent if and only if the following relations between the various

particle phases are satisfied:

$$b_+ b_- = -1 \text{ (and } a_+ a_- = -1); \quad c_- \bar{c}_+ = - c_+ \bar{c}_-; \quad d_- \bar{d}_+ = -d_+ \bar{d}_- \qquad (3.21)$$

With these conditions the ten equations reduce to just four different equations:

$$\begin{aligned}
&(\tfrac{1}{2})^{1/2} T_1^{(s)} = c_- \bar{c}_+ \phi_B \phi_C \tfrac{2^{1/2}}{3}(T_{3/2}^{(t)} - T_{1/2}^{(t)}) = d_- \bar{d}_+ \phi_B \phi_D \tfrac{2^{1/2}}{3}(T_{3/2}^{(u)} + T_{1/2}^{(u)}) \\
&(\tfrac{1}{6})^{1/2} T_o^{(s)} = -c_- \bar{c}_+ \phi_B \phi_C (\tfrac{2}{3} T_{3/2}^{(t)} - \tfrac{1}{3} T_{1/2}^{(t)}) = d_- \bar{d}_+ \phi_B \phi_D (\tfrac{2}{3} T_{3/2}^{(u)} - \tfrac{1}{3} T_{1/2}^{(u)}) \\
&-(\tfrac{1}{6})^{1/2} T_o^{(s)} + \tfrac{1}{2} T_1^{(s)} = c_- \bar{c}_+ \phi_B \phi_C T_{3/2}^{(t)} = -d_- \bar{d}_+ \phi_B \phi_D (\tfrac{1}{3} T_{3/2}^{(u)} - \tfrac{2}{3} T_{1/2}^{(u)}) \\
&(\tfrac{1}{6})^{1/2} T_o^{(s)} + \tfrac{1}{2} T_1^{(s)} = -c_- \bar{c}_+ \phi_B \phi_C (\tfrac{1}{3} T_{3/2}^{(t)} + \tfrac{2}{3} T_{1/2}^{(t)}) = d_- \bar{d}_+ \phi_B \phi_D T_{3/2}^{(u)}
\end{aligned} \qquad (3.22)$$

Further, we have

$$(3.22)c + (3.22)d = 2^{1/2} (3.22)a$$

$$(3.22)d - (3.22)c = 2 \quad (3.22)b$$

and just two of the equations are independent, as required. From (3.22)a and (3.22)b we obtain

$$\begin{aligned}
&T_1^{(s)} = c_- \bar{c}_+ \phi_B \phi_C \tfrac{2}{3}(T_{3/2}^{(t)} - T_{1/2}^{(t)}) = d_- \bar{d}_+ \phi_B \phi_D \tfrac{2}{3}(T_{3/2}^{(u)} + T_{1/2}^{(u)}) \\
&T_o^{(s)} = -c_- \bar{c}_+ \phi_B \phi_C 6^{1/2}(\tfrac{2}{3} T_{3/2}^{(t)} + \tfrac{1}{3} T_{1/2}^{(t)}) = d_- \bar{d}_+ \phi_B \phi_D 6^{1/2}(\tfrac{2}{3} T_{3/2}^{(u)} - \tfrac{1}{3} T_{1/2}^{(u)})
\end{aligned}$$

For definiteness we choose the phases as follows:

$-a_+ = a_o = a_- = 1 = -b_+ = b_o = b_-$ (e.g. $|\pi^+\rangle = -|1,1\rangle$)

$-c_+ = c_- = \bar{c}_+ = \bar{c}_- = 1 = d_- = d_+ = -\bar{d}_+ = \bar{d}_-$ (e.g. $|K^+\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle$ $|\bar{K}^o\rangle = -|\tfrac{1}{2},\tfrac{1}{2}\rangle$ —it is assumed in such channels as $K\bar{K}$ that $\bar{C}$ is the outgoing <u>particle</u> (K) and $\bar{D}$ is the outgoing <u>antiparticle</u> ($\bar{K}$)). Then $-c_- \bar{c}_+ = d_- \bar{d}_+ = -1$ and

$$g_{II'}(\tfrac{1}{2},\tfrac{1}{2};1,1) = \phi_B \phi_C \begin{pmatrix} -\frac{6^{1/2}}{3} & -\frac{2(6)^{1/2}}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$$\bar{g}_{II'(\frac{1}{2},\frac{1}{2};1,1)} = \phi_B\phi_D \begin{pmatrix} \frac{6^{\frac{1}{2}}}{3} & -\frac{2(6)^{\frac{1}{2}}}{3} \\ -\frac{2}{3} & -\frac{2}{3} \end{pmatrix}$$

In a similiar fashion we learn

$$g_{II'(1,1;1,1)} = \phi_B\phi_C \begin{pmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

$$\bar{g}_{II'(1,1;1,1)} = \phi_B\phi_D \begin{pmatrix} \frac{1}{3} & -1 & \frac{5}{3} \\ -\frac{1}{3} & \frac{1}{2} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

$$g_{II'(1,\frac{1}{2};1,\frac{1}{2})} = \phi_B\phi_C \begin{pmatrix} \frac{1}{6^{\frac{1}{2}}} & 1 \\ \frac{1}{6^{\frac{1}{2}}} & -\frac{1}{2} \end{pmatrix}$$

$$\bar{g}_{II'(1,\frac{1}{2};1,\frac{1}{2})} = \phi_B\phi_D \begin{pmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$g_{II'(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})} = \phi_B\phi_C \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\bar{g}_{II'(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})} = \phi_B\phi_D \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$g_{II'(1,0;1,0)} = -\frac{1}{3^{\frac{1}{2}}}\delta_{I1}\,\delta_{I'0}\,\phi_B\phi_C \qquad \bar{g}_{II'(1,0;1,0)} = \delta_{I1}\,\delta_{I'0}\phi_B\phi_D$$

$$g_{II'(1,0;1,1)} = -\delta_{I1}\,\delta_{I'1}\,\phi_B\phi_C \qquad g_{II'(1,0;1,1)} = \delta_{I1}\,\delta_{I'1}\phi_B\phi_D$$

$$g_{II},(\tfrac{1}{2},\tfrac{1}{2};1,0) = (\tfrac{2}{3})^{\frac{1}{2}}\ \delta_{I1}\ \delta_{I',\frac{1}{2}}\phi_B\phi_C \qquad \bar{g}_{II},(\tfrac{1}{2},\tfrac{1}{2};1,0) = -(\tfrac{2}{3})^{\frac{1}{2}}\delta_{I1}\delta_{I',\frac{1}{2}}\phi_B\phi_D$$

$$g_{II},(0,\tfrac{1}{2};0,\tfrac{1}{2}) = -\frac{1}{2^{\frac{1}{2}}}\delta_{I\frac{1}{2}}\ \delta_{I',\frac{1}{2}}\ \phi_B\phi_C \qquad \bar{g}_{II},(0,\tfrac{1}{2};0,\tfrac{1}{2}) = \delta_{I\frac{1}{2}}\ \delta_{I',\frac{1}{2}}\ \phi_B\phi_D$$

$$g_{II},(0,\tfrac{1}{2};1,\tfrac{1}{2}) = -(\tfrac{3}{2})^{\frac{1}{2}}\ \delta_{I\frac{1}{2}}\ \delta_{I',1}\ \phi_B\phi_C$$

$$\bar{g}_{II},(0,\tfrac{1}{2};1,\tfrac{1}{2}) = -\ \delta_{I\frac{1}{2}}\ \delta_{I',1}\ \phi_B\phi_D$$

In the matrices the smaller isospin values correspond to the smaller row and column numbers. These results are used in chapter V in which the forces due to exchange of particles are calculated in the Born approximation.

IV. BORN TERMS

The dispersion relation approach to strong interaction physics utilizes Cauchy's Theorem to recover the invariant amplitude describing some process from a knowledge of that amplitude's singularities. Here the latter information is assumed to be given completely by the unitarity condition (2.17) and the substitution law. In this chapter the approximate discontinuities of T obtained by terminating the unitarity sum at t = 1 (the Born terms) are calculated.

In the present bootstrap only the pseudoscalar mesons (PS) and the vector mesons (V) are assumed to exist, we consider only states involving at most two particles, and moreover we totally ignore couplings involving three V's. Then the possible types of states we need consider are PS + PS and PS + V, and there are just three types of reactions: PS + PS → PS + PS, PS + V → PS + PS, PS + V → PS + V. The Born terms for each of these processes are calculated below for arbitrary PS and V and the results later specialized to the particular cases of interest. In all of the following isospin is ignored, since such complications were treated in chapter III, and the problems of symmetrization are likewise deferred.

A. PS + PS → PS + PS

—the s channel—

Consider the reactions shown in figure 2. Here $A,B,\bar{C},\bar{D}$ are PS and V,V',V" are V's of momentum k, helicity λ, and masses $M_V, M_{V'}, M_{V''}$. For the s-channel reaction of figure 2 equations (2.14) and (2.17) give

$$\text{Disc}_s \langle \bar{C}(-p_3),\bar{D}(-p_4)|\hat{T}|A(p_1),B(p_2)\rangle = -\pi \int d^4k\, \delta(k^2 - M_V^2)\theta(k_o - M_V)$$
$$\times \sum_\lambda \langle \bar{C}(-p_3),\bar{D}(-p_4)|\hat{T}^+|V(k,\lambda)\rangle\langle V(k,\lambda)|\hat{T}|A(p_1),B(p)\rangle$$
$$\times\ \delta^{(4)}(p_1 + p_2 - k) \qquad (4.1)$$

where, following Olive (21) we have replaced the left side of (2.17) by the discontinuity of T in the s variable;

$$\text{Disc}_s\, T_{ab}(s) = \frac{T_{ab}(s + i\epsilon) - T_{ab}(s - i\epsilon)}{2i}$$

$$= \frac{T_{ab}(s + i\epsilon) - T^{+}_{ab}(s + i\epsilon)}{2i} \qquad (4.2)$$

In (4.1) it is assumed that the sum over all quantum numbers, $\sum_{\substack{v_j \\ \text{sum.}}}$, has been carried out with the exception of the helicity sum.

The vertex functions, e.g. $\langle V(k,\lambda)|\hat{T}|A(p_1),B(p_2)\rangle$, are strongly limited by the constraints of relativistic invariance. Since V is a vector particle it is described by a polarization four-vector $\epsilon(V;k,\lambda)$, and we have

$$\langle V(k,\lambda)|\hat{T}|A(p_1),B(p_2)\rangle = \epsilon^{*}(V;k,\lambda)\cdot H_{VAB} \qquad (4.3)$$

Here H_{VAB} must be a relativistic four-vector; of the three momenta p_1, p_2, k, only two are independent since $p_1 + p_2 = k$ is imposed by the δ - function in (4.1). Therefore the most general form for H_{VAB} is

$$H_{VAB} = F_{VAB}(p_1 - p_2) + G_{VAB}(p_1 + p_2) \qquad (4.4)$$

with F_{VAB} and G_{VAB} invariant functions. Only one invariant may be formed from p_1, p_2, and k under the restrictions $p_1^2 = m_1^2$, $p_2^2 = m_2^2$, $p_1 + p_2 = k$, which may be taken as $(p_1 + p_2)\cdot(p_1 + p_2)$, or in other words, k^2. Then $F_{VAB} = F_{VAB}(k^2)$, $G_{VAB} = G_{VAB}(k^2)$, and

$$\langle V(k,\lambda)|\hat{T}|A(p_1),B(p_2)\rangle = \epsilon^{*}_{\nu}(V;k,\lambda)\left[F_{VAB}(k^2)(p_1 - p_2)^{\nu} + G_{VAB}(k^2)(p_1 + p_2)^{\nu}\right] \qquad (4.5)$$

It simplifies matters considerably to notice that these expressions will be inserted into (4.1) which contains an integral

including $\delta(k^2 - M_V^2)$. So we may as well evaluate (4.5) at $k^2 = M_V^2$ immediately and save some labor. But on the mass shell we have

$$\epsilon(V;k,\lambda)\cdot k = 0 \tag{4.6}$$

which, since $k = p_1 + p_2$ is real, may be written

$$\epsilon_\nu^*(V;k,\lambda)(p_1 + p_2)^\nu = 0 \tag{4.7}$$

The result is to eliminate the $G_{VAB}(k^2)$ form factor. Moreover, $F_{VAB}(k^2)$ evaluated at $k^2 = M_V^2$ is just some constant; for reasons of convenience and convention we write

$$F_{VAB}(k^2 = M_V^2) = f_{VAB}\gamma_{VAB} \tag{4.8}$$

where f and γ are constants and f_{VAB} is chosen so that in the limit in which SU_3 symmetry is exact all γ's are equal (22) ($f_{\rho\pi\pi} = 2$, $f_{\rho KK} = 1$, $f_{\pi KK^*} = f_{\eta KK^*} = (3/2)^{1/2}$, $f_{KK\omega} = 3^{1/2}$). Then inside the integral of (4.1)

$$\langle V(k,\lambda)|\hat{T}|\,A(p_1),B(p_2)\rangle = f_{VAB}\gamma_{VAB}\epsilon_\nu^*(V;k,\lambda)(p_1 - p_2)^\nu \tag{4.9}$$

In a similar fashion

$$\begin{aligned}\langle \bar{C}(-p_3),\bar{D}(-p_4)|\hat{T}^+|\,V(k,\lambda)\rangle &= \langle V(k,\lambda)|\hat{T}|\,\bar{C}(-p_3),\bar{D}(-p_4)\rangle^* \\ &= \left[f_{V\bar{C}\bar{D}}\gamma_{V\bar{C}\bar{D}}\ \epsilon_\mu^*(V;k,\lambda)(-p_3 + p_4)^\mu\right]^* \\ &= -f_{VCD}\gamma_{VCD}\ \epsilon_\mu(V;k,\lambda)(p_3 - p_4)^\mu\end{aligned} \tag{4.10}$$

is true under the integral; we have used the fact that

$$f_{V\bar{C}\bar{D}}\gamma_{V\bar{C}\bar{D}} = f_{VCD}\gamma_{VCD} \tag{4.11}$$

If we abbreviate $\langle \bar{C}(-p_3),\bar{D}(-p_4)|\hat{T}\,|A(p_1),B(p_2)\rangle$ by $T_{\bar{C}\bar{D},AB}(s,t)$, then we have

$$\text{Disc}_s\, T_{\bar{C}\bar{D},AB}(s,t) = \pi\, f_{VAB} f_{VCD} \gamma_{VAB} \gamma_{VCD} \int d^4k\, \delta(k^2 - M_V^2) \times \theta(k_o - M_V)(p_3 - p_4)^\mu\, P_{\mu\nu}\, (p_1 - p_2)^\nu \quad (4.12)$$

with

$$P_{\mu\nu} = \sum_\lambda \epsilon_\mu(V;k,\lambda)\, \epsilon_\nu^*(V;k,\lambda) \quad (4.13)$$

The most general form for $P_{\mu\nu}$ is evidently $P_{\mu\nu} = a g_{\mu\nu} + b k_\mu k_\nu$, with a and b constants. But from (4.6) we know that

$$k^\mu P_{\mu\nu} = 0 = a k_\nu + b k^2 k_\nu \quad (4.14)$$

so that $b = -a/k^2$. Further

$$\epsilon(V;k,\lambda) \cdot \epsilon^*(V;k,\lambda') = -\delta_{\lambda\lambda'} \quad (4.15)$$

and therefore

$$\sum_\lambda \epsilon(V;k,\lambda)\, \epsilon^*(V;k,\lambda) = -3 = P_{\mu\nu}\, g^{\mu\nu} = a(4 - k^2/k^2) = 3a \quad \ldots.(4.16)$$

Therefore $a = -1$, and

$$P_{\mu\nu} = -g_{\mu\nu} + k_\mu k_\nu / k^2 \quad (4.17)$$

Then

$$(p_3 - p_4)^\mu P_{\mu\nu} (p_1 - p_2)^\nu = -(p_3 - p_4)\cdot(p_1 - p_2) - \frac{(p_3 - p_4)\cdot(p_3 + p_4)(p_1 + p_2)\cdot(p_1 - p_2)}{k^2}$$

$$= -t + u - \frac{(m_3^2 - m_4^2)(m_1^2 - m_2^2)}{M_V^2} \quad (4.18)$$

Therefore

$$\text{Disc}_s\, T_{\bar{C}\bar{D},AB}(s,t) = -\pi\, f_{VAB} f_{VCD} \gamma_{VAB} \gamma_{VCD}\, \delta(s - M_V^2) \times \left[t - u - \frac{(m_1^2 - m_2^2)(m_4^2 - m_3^2)}{M_V^2} \right] \quad (4.19)$$

and finally

$$T_{\bar{C}\bar{D},AB}(s,t) = \frac{f_{VAB}f_{VCD}\gamma_{VAB}\gamma_{VCD}}{s - M_V^2}\left[t - u - \frac{(m_1^2-m_2^2)(m_4^2-m_3^2)}{M_V^2} \right] \quad \ldots.(4.20)$$

or

$$T^{(s)}_{\bar{C}\bar{D},AB}(s,t) = \frac{f_{VAB}f_{VCD}\gamma_{VAB}\gamma_{VCD}(2t + s - M^2)}{s - M_V^2} \quad (4.21)$$

where

$$M^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 + \frac{(m_1^2 - m_2^2)(m_4^2 - m_3^2)}{M_V^2} \quad (4.22)$$

and we have added a superscript to T indicating that this result came from unitarity in the s-channel.

The step from (4.19) to (4.20) is not rigorous, for (4.19) implies (4.20) only to within a ratio $P(s)/P(M_V^2)$, where P is a polynomial of any order. In particular s could be replaced by M_V^2 anywhere in the numerator of (4.20) without changing the validity of (4.19). The full dispersion relation technique utilizes only the discontinuities and this ambiguity does not arise (or at least not until later in the calculation). In the approximation to be made below, however, we use the actual amplitude. We shall simply take the prescription (which leads to the results of Feynmann rules) that for any Born term we make the replacement

$$-\pi\,\delta(z - M^2) \rightarrow \frac{1}{z - M^2} \quad (4.23)$$

where z could be either s,t, or u, when going from the discontinuity of the amplitude in the variable z to the amplitude itself.

We are interested in P waves in the direct channel, since we're looking for vector mesons, $J^P = 1^-$. Therefore the J = 1 projection of (4.21) is taken according to the prescription of

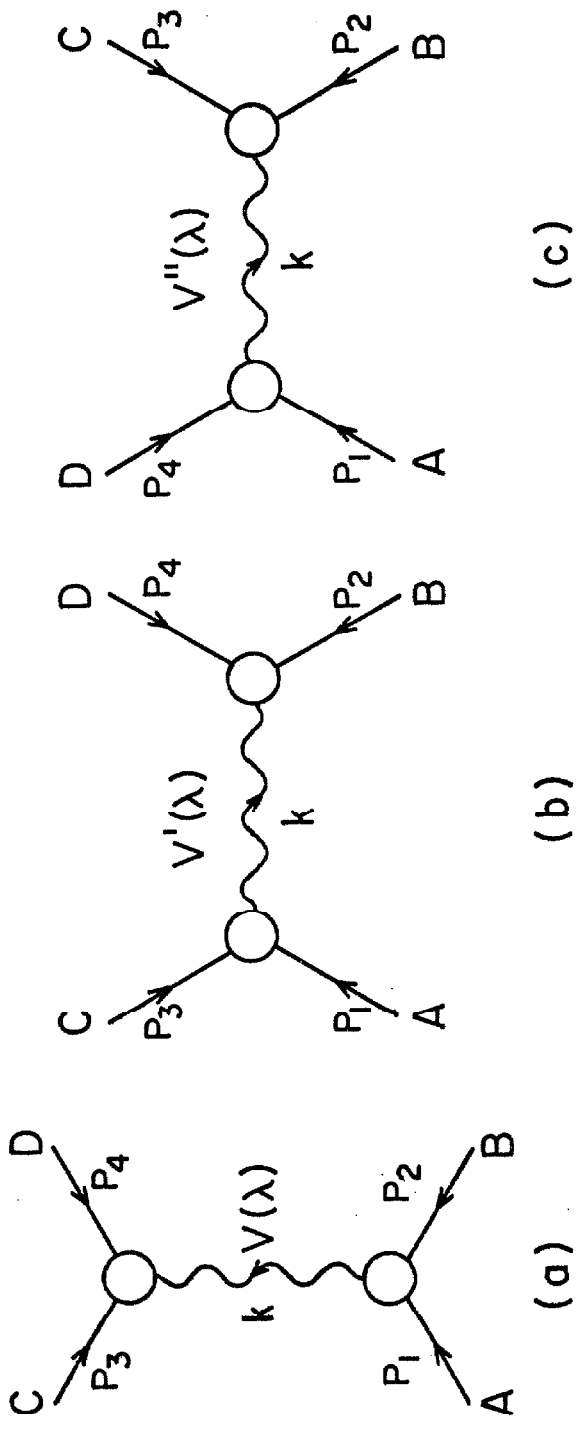

Fig. 2

The PS + PS → PS + PS Reactions

(2.19). Since $A,B,\bar{C},\bar{D}$ are all PS we have

$$T^{(s)J=1}_{\bar{C}\bar{D},AB}(s) = \frac{1}{2}\int_{-1}^{+1} dx\, P_1(x)\, T^{(s)}_{\bar{C}\bar{D},AB}(s,t)$$

$$= \frac{4}{3} f_{VAB} f_{VCD} \gamma_{VAB} \gamma_{VCD} \frac{pq}{s - M_V^2} \qquad (4.24)$$

with p and q as in equation (2.3).

—the t channel—

We now treat the case of a single vector particle exchanged in the t channel, figure 2b. We invoke the substitution law which ultimately says the result is exactly the same as (4.20) if the substitution $B(p_2,m_2) \leftrightarrow C(p_3,m_3)$ is made. Then $s \leftrightarrow t$, $u \leftrightarrow u$, $m_2 \leftrightarrow m_3$, and we find

$$T^{(t)}_{\bar{C}\bar{D},AB}(s,t) = \frac{f_{V'AC} f_{V'BD} \gamma_{V'AC} \gamma_{V'BD}}{t - M_{V'}^2} \left[s - u - \frac{(m_1^2 - m_3^2)(m_4^2 - m_2^2)}{M_{V'}^2} \right]$$

$$= \frac{f_{V'AC} f_{V'BD} \gamma_{V'AC} \gamma_{V'BD} (2s + t - M'^2)}{t - M_{V'}^2} \qquad (4.25)$$

$$M'^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 + \frac{(m_1^2 - m_3^2)(m_4^2 - m_2^2)}{M_{V'}^2} \qquad (4.26)$$

Again the J = 1 wave is projected out:

$$T^{(t)J=1}_{\bar{C}\bar{D},AB}(s) = \frac{f_{V'AC} f_{V'BD} \gamma_{V'AC} \gamma_{V'BD}}{2} (2s + M_{V'}^2 - M'^2) \int_{-1}^{+1} \frac{x\,dx}{t - M_{V'}^2}$$

We write

$$t - M_{V'}^2 = 2pq(x - K) \qquad (4.27)$$

$$K = \frac{s - (m_1^2 + m_2^2 + m_3^2 + m_4^2) + 2M_{V'}^2 + \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2)}{s}}{4pq} \tag{4.28}$$

and learn

$$T^{(t)J=1}_{\bar{C}\bar{D},AB}(s) = \frac{f_{V'AC}f_{V'BD}\gamma_{V'AC}\gamma_{V'BD}}{4} \cdot \left(\frac{2s + M_{V'}^2 - M'^2}{pq}\right) \times \left[2 - K \ln\left|\frac{K+1}{K-1}\right|\right] \tag{4.29}$$

—the u channel—

To find the amplitude corresponding to figure 2c we again use the substitution law. The result is (4.20) with the replacements $s \leftrightarrow u$, $t \leftrightarrow t$, $m_2 \leftrightarrow m_4$.

$$T^{(u)}_{\bar{C}\bar{D},AB}(s,t) = -\frac{f_{V''AD}f_{V''BC}\gamma_{V''AD}\gamma_{V''BC}\ (2s + u - M''^2)}{u - M_{V''}^2} \tag{4.30}$$

$$M''^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 + \frac{(m_1^2 - m_4^2)(m_3^2 - m_2^2)}{M_{V''}^2} \tag{4.31}$$

We put

$$u - M_{V''}^2 = -2pq(x + L) \tag{4.32}$$

$$L = \frac{s - (m_1^2 + m_2^2 + m_3^2 + m_4^2) + 2M_{V''}^2 - \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2)}{s}}{4pq} \tag{4.33}$$

Then

$$T^{(u)J=1}_{\bar{C}\bar{D},AB}(s) = \frac{f_{V''AD}f_{V''BC}\gamma_{V''AD}\gamma_{V''BC}}{4}\ \frac{(2s + M_{V''}^2 - M''^2)}{pq} \times \left[2 - L \ln\left|\frac{L+1}{L-1}\right|\right] \tag{4.34}$$

Finally it must be ensured that the parity of the interacting states is correct. Since we are looking in the $J^P = 1^-$ channels we must have P = -1. According to Jacob and Wick (20) if $\hat{P}$ is the parity operator

$$\hat{P}\,|\,JM;\lambda_1\lambda_2\rangle = \eta_1\eta_2\,(-1)^{J-s_1-s_2}\,|\,JM;-\lambda_1,-\lambda_2\rangle \qquad \ldots.(4.35)$$

where η_i and s_i are the intrinsic parity and spin of the particles. For this first case all particles have $\eta_i = -1$, $s_i = 0$, $\lambda_i = 0$. Then

$$\hat{P}|1\ M;00\rangle = -\,|\,1\ M;00\rangle \qquad (4.36)$$

and the parity is indeed -1. Therefore equations (4.24), (4.29), and (4.34) require no modification. We summarize:

$$PS + PS \rightarrow PS + PS$$

$$\left.\begin{aligned}
T^{(s)}_{\bar{C}\bar{D},AB}J^{P=1^-}(s) &= \frac{4}{3}\,f_{VAB}f_{VCD}\gamma_{VAB}\gamma_{VCD}\,\frac{pq}{s - M_V^2} \\
T^{(t)}_{\bar{C}\bar{D},AB}J^{P=1^-}(s) &= f_{V'AC}f_{V'BD}\gamma_{V'AC}\gamma_{V'BD}\,\frac{2s + M_{V'}^2 - M'^2}{4pq} \\
&\quad \times\left[2 - K\,\ln\left|\frac{K+1}{K-1}\right|\right] \\
T^{(u)}_{\bar{C}\bar{D},AB}J^{P=1^-}(s) &= f_{V''AD}f_{V''BC}\gamma_{V''AD}\gamma_{V''BC}\,\frac{2s + M_{V''}^2 - M''^2}{4pq} \\
&\quad \times\left[2 - L\,\ln\left|\frac{L+1}{L-1}\right|\right]
\end{aligned}\right\} \qquad (4.37)$$

B. $PS + PS \rightarrow PS + V$

—the s channel—

Consider now the process indicated in figure 3a. We

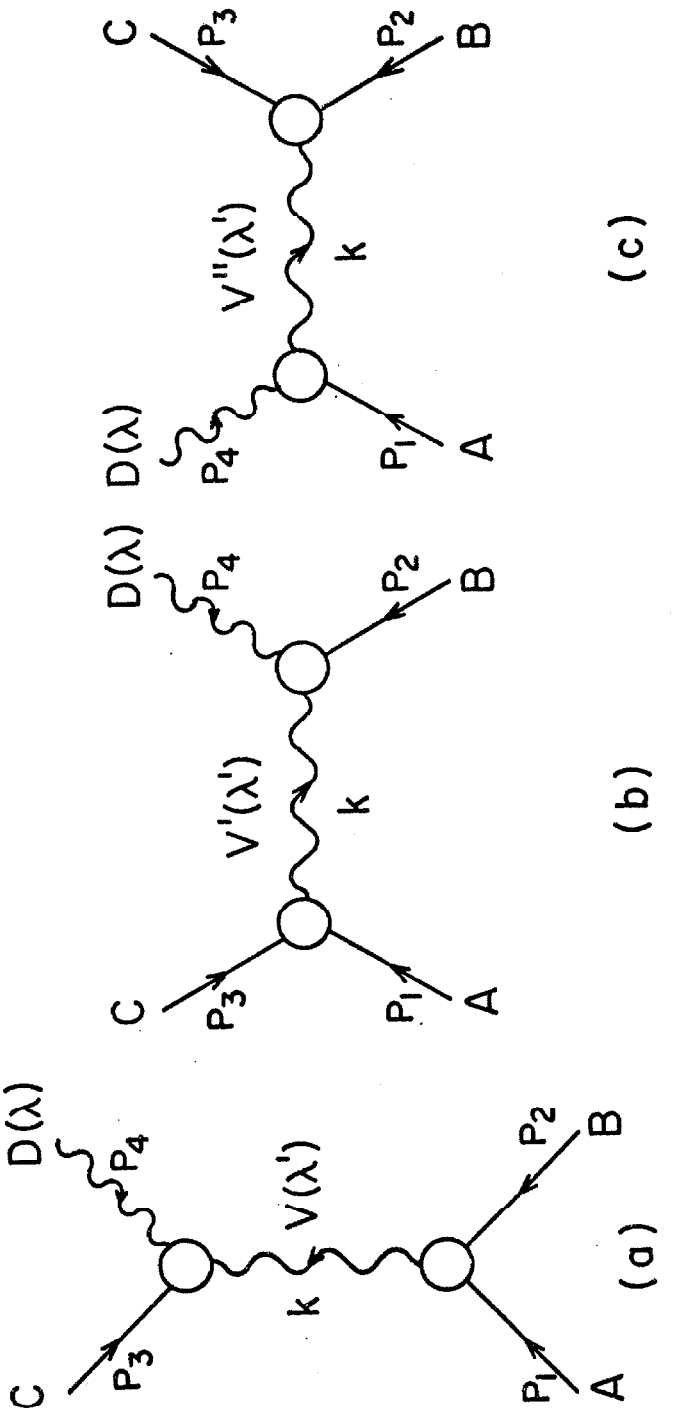

Fig. 3

The PS + PS → PS + V Reactions

assume that $\bar{D}$ is the external vector meson, with helicity $-\lambda$. We have

$$\text{Disc}_s\, T_{\bar{C}\bar{D}(-\lambda),AB}(s,t) = -\pi \int d^4k\, \delta(k^2 - M_V^2)\theta(k_o - M_V)\delta(p_1+p_2-k)$$
$$\times \sum_{\lambda'} \langle \bar{C}(-p_3), \bar{D}(-p_4,-\lambda) | \hat{T}^+ | V(k,\lambda') \rangle$$
$$\times \langle V(k,\lambda') | \hat{T} | A(p_1), B(p_2) \rangle \qquad (4.38)$$

If we again agree to imply the value inside the integral, we use equation (4.9):

$$\langle V(k,\lambda') | \hat{T} | A(p_1), B(p_2) \rangle = f_{VAB}\gamma_{VAB}\, \epsilon^*_\mu(V;k,\lambda')(p_1 - p_2)^\mu .$$

We again argue from relativistic invariance to find the PS-V-V coupling:

$$\langle \bar{C}(-p_3), \bar{D}(-p_4,-\lambda) | \hat{T}^+ | V(k,\lambda') \rangle = \langle V(k,\lambda') | \hat{T} | \bar{C}(-p_3), \bar{D}(-p_4,-\lambda) \rangle^*$$
$$= \left[\epsilon^*_\nu(V;k,\lambda')\, R^{\nu\mu}_{VCD}\, \epsilon_\mu(\bar{D};-p_4,-\lambda) \right]^* \qquad \ldots.(4.39)$$

To form the tensor R_{VCD} we have just the vectors p_3 and p_4, since k is not independent. Further, R_{VCD} must be odd under space inversion in order that the vertex function be a world scalar function; the unique choice is

$$R^{\nu\mu}_{VCD} = F_{VCD}\, \epsilon^{\tau\kappa\nu\mu} p_{3_\tau} p_{4_\kappa} \qquad (4.40)$$

Here F_{VCD} is an invariant function of k^2 with the dimensions of $(\text{energy})^{-1}$. We use this evaluated at $k^2 = M_V^2$ and write

$$F_{VCD}(k^2 = M_V^2) = -\frac{g_{VCD}\gamma_{VCD}}{m_\pi} \qquad (4.41)$$

where g_{VCD} performs the same function as the f_{VCD} defined earlier ($g_{\rho\pi\omega} = 1$, $g_{\omega KK^*} = 1/(2(3)^{1/2})$). The scale factor m_π, common to all couplings, is chosen to make the various γ's as

nearly equal as possible when symmetry breaking is allowed (cf. chapter VI). We then find

$$\text{Disc}_s\, T_{\bar{C}\bar{D}(-\lambda),AB}(s,t) = \frac{\pi\, f_{VAB} g_{VCD} \gamma_{VAB} \gamma_{VCD}}{m_\pi}\, \delta(s - M_V^2)\, Z \qquad (4.42)$$

$$Z = \epsilon^*_\mu(\bar{D};-p_4,-\lambda)\epsilon^{\tau\kappa\nu\mu}\, p_{3_\tau} p_{4_\kappa} (-g_{\nu\sigma} + k_\nu k_\sigma / M_V^2)(p_1 - p_2)^\sigma \qquad (4.43)$$

$$= 2\epsilon_{\tau\kappa\nu\mu} p_3^\tau\, p_4^\kappa\, p_2^\nu \epsilon^{*\mu}(\bar{D};-p_4,-\lambda) \qquad (4.44)$$

It is next necessary to determine $\epsilon^*(\bar{D};-p_4,-\lambda)$. For $-\vec{p}_4$ along the +z axis, $-p_4 = (E_4,0,0,q)$, the conditions

$$\epsilon(\bar{D};-p_4,\lambda)\cdot\epsilon^*(\bar{D};-p_4,\lambda') = -\delta_{\lambda\lambda'}$$
$$\epsilon(\bar{D};-p_4,\lambda)\cdot p_4 = 0 \qquad (4.45)$$
$$\hat{S}_z\, \epsilon(\bar{D};-p_4,\lambda) = \lambda\epsilon(\bar{D};-p_4,\lambda)$$

where $\hat{S}_z$ is the operator for the third component of spin, suffice to determine $\epsilon(\bar{D};-p_4,\lambda)$ to within a phase. In order to ensure time reversal invariance we take the following phase convention: in this frame where $-\vec{p}_4$ is along the +z axis

$$i\epsilon(\bar{D};-p_4,\lambda) = \delta_{\lambda\pm1} 2^{-1/2}(0,-\lambda,-i,0) + \delta_{\lambda 0}\frac{1}{m_4}(q,0,0,E_4) \qquad (4.46)$$

Now in the case of interest, $-\vec{p}_4$ is not along the +z axis, but rather along the -z axis; according to Jacob and Wick (20) we must make the transformation

$$i\epsilon(\bar{D};-p_4,\lambda) \to e^{-i\pi J_y}\, i\epsilon(\bar{D};-p_4,\lambda) \qquad (4.47)$$

Since also we want the results for helicity $-\lambda$ instead of λ, we finally take

$$\epsilon(\bar{D};-p_4,-\lambda) = -\delta_{\lambda\pm1}\, i2^{-1/2}(0,-\lambda,-i,0) - \delta_{\lambda 0}\frac{i}{m_4}(q,0,0,-E_4) \qquad (4.48)$$

We may now evaluate Z of equation (4.44) with the help of the equations (2.3); we find

$$Z = -2\,\delta_{\lambda\pm1}\, pq(E_1 + E_2)\left[\frac{1 - x^2}{2}\right]^{1/2} \qquad (4.49)$$

Therefore

$$\text{Disc}_s\, T_{\bar{C}\bar{D}(-\lambda),AB}(s,t) = -\frac{\pi\, f_{VAB} g_{VCD} \gamma_{VAB} \gamma_{VCD}}{m_\pi}\, \delta_{\lambda\pm 1}\, (E_1 + E_2) \times pq\left[2(1-x^2)\right]^{\frac{1}{2}}\, \delta(s - M_V^2) \quad (4.50)$$

and with the prescription (4.23) we deduce

$$T^{(s)}_{\bar{C}\bar{D}(-\lambda),AB}(s,t) = \frac{f_{VAB} g_{VCD} \gamma_{VAB} \gamma_{VCD}\, (E_1+E_2) pq\, 2^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}\, \delta_{\lambda\pm 1}}{m_\pi (s - M_V^2)} \quad \ldots.(4.51)$$

Now the J = 1 partial wave is projected out:

$$T^{(s)J=1}_{\bar{C}\bar{D}(-\lambda),AB}(s) = \frac{1}{2}\int_{-1}^{+1} dx\; d^1_{0\lambda}(x)\, T^{(s)}_{\bar{C}\bar{D}(-\lambda),AB}(s,t) \quad (4.52)$$

$$d^1_{0\lambda}(x) = \lambda \delta_{\lambda\pm 1}\, 2^{-\frac{1}{2}}(1-x^2)^{\frac{1}{2}} + x\, \delta_{\lambda 0} \quad (4.53)$$

and the result is

$$T^{(s)J=1}_{\bar{C}\bar{D}(-\lambda),AB}(s) = \frac{2}{3}\, f_{VAB} g_{VCD} \gamma_{VAB} \gamma_{VCD}\, \frac{pq(E_1 + E_2)}{m_\pi (s - M_V^2)} \quad (4.54)$$

—the t channel—

For the exchange term represented by figure 3b we invoke as usual the substitution law. The amplitude for this reaction is given by (4.51) with the substitution $B(p_2,m_2) \leftrightarrow C(p_3,m_3)$. In chapter II it was shown that the quantity

$$pq(E_1 + E_2)(1 - x^2)^{\frac{1}{2}}$$

is invariant under particle exchange. Therefore

$$T^{(t)}_{\bar{C}\bar{D}(-\lambda),AB}(s,t) = \frac{f_{V'AC} g_{V'BD} \gamma_{V'AC} \gamma_{V'BD}\, (E_1+E_2) pq\, 2^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}} \delta_{\lambda\pm 1}}{m_\pi (t - M_{V'}^2)} \quad \ldots.(4.55)$$

Again we want the J = 1 projection of this amplitude. We use (4.53) and the analogue of (4.52) for $T^{(t)}$ and find

$$T^{(t)J=1}_{\bar{C}\bar{D}(-\lambda),AB}(s) = -\frac{1}{4} f_{V'AC} g_{V'BD} \gamma_{V'AC} \gamma_{V'BD} \frac{\lambda(E_1 + E_2)}{m_\pi}$$

$$\times \left[2K - (K^2 - 1) \ln \left| \frac{K + 1}{K - 1} \right| \right] \qquad (4.56)$$

with K given by (4.28).

—the u channel—

Here we use (4.51) with $B(p_2) \leftrightarrow D(p_4, \lambda)$

$$T^{(u)}_{\bar{C}\bar{D}(-\lambda),AB}(s,t) = \frac{f_{V''BC} g_{V''AD} \gamma_{V''AD} \gamma_{V''BC} (E_1+E_2) pq \, 2^{1/2} (1-x^2)^{1/2} \delta_{\lambda \neq 1}}{m_\pi (u - M^2_{V''})} \qquad ....(4.57)$$

and

$$T^{(u)J=1}_{\bar{C}\bar{D}(-\lambda),AB}(s) = -\frac{1}{4} f_{V''BC} g_{V''AD} \gamma_{V''AD} \gamma_{V''BC} \frac{\lambda(E_1 + E_2)}{m_\pi}$$

$$\times \left[2L - (L^2 - 1) \ln \left| \frac{L + 1}{L - 1} \right| \right] \qquad (4.58)$$

Again we must ensure that the proper eigenstates of the parity operator are used. For the PS + V state equation (4.35) gives

$$\hat{P} \, | 1 \, M; 0 \, \lambda \rangle = | 1 \, M; 0 \, -\lambda \rangle \qquad (4.59)$$

Therefore the proper state is

$$\frac{| 1 \, M; 0 \lambda \rangle - | 1 \, M; 0 \, -\lambda \rangle}{2^{1/2}} \qquad (4.60)$$

and

$$\langle PS, V \, | \, \hat{T} \, | \, PS, PS \rangle = \frac{1}{2^{1/2}} \left[\langle PS, V(\lambda) | \hat{T} | PS, PS \rangle - \langle PS, V(-\lambda) | \hat{T} | PS, PS \rangle \right] \qquad (4.61)$$

Since $\langle PS, V(\lambda) | \hat{T} | PS, PS \rangle$ is proportional to λ (equations (4.54),(4.56),(4.58)) the net result is to multiply these terms by $2^{1/2}$. In summary,

$$PS + PS \to PS + V(-\lambda)$$

$$T^{(s)J^P=1^-}_{\bar{C}\bar{D}(-\lambda),AB}(s) = \frac{2\cdot 2^{1/2}}{3} f_{VAB} g_{VCD} \gamma_{VAB} \gamma_{VCD} \frac{\lambda p q (E_1 + E_2)}{m_\pi (s - M_V^2)}$$

$$T^{(t)J^P=1^-}_{\bar{C}\bar{D}(-\lambda),AB}(s) = -\frac{2^{1/2}}{4} f_{V'AC} g_{V'BD} \gamma_{V'AC} \gamma_{V'BD} \frac{\lambda (E_1 + E_2)}{m_\pi} \times \left[2K - (K^2 - 1) \ln \left| \frac{K+1}{K-1} \right| \right]$$

$$T^{(u)J^P=1^-}_{\bar{C}\bar{D}(-\lambda),AB}(s) = -\frac{2^{1/2}}{4} f_{V''BC} g_{V''AD} \gamma_{V''AD} \gamma_{V''BC} \frac{\lambda (E_1 + E_2)}{m_\pi} \times \left[2L - (L^2 - 1) \ln \left| \frac{L+1}{L-1} \right| \right]$$

(4.62)

Also we find for the reaction

$$PS + V(\lambda) \to PS + PS$$

$$T^{(s)J^P=1^-}_{\bar{C}\bar{D},AB(\lambda)}(s) = -\frac{2\cdot 2^{1/2}}{3} f_{VCD} g_{VAB} \gamma_{VCD} \gamma_{VAB} \frac{\lambda p q (E_1 + E_2)}{m_\pi (s - M_V^2)}$$

$$T^{(t)J^P=1^-}_{\bar{C}\bar{D},AB(\lambda)}(s) = \frac{2^{1/2}}{4} f_{V'AC} g_{V'BD} \gamma_{V'AC} \gamma_{V'BD} \frac{\lambda (E_1 + E_2)}{m_\pi} \times \left[2K - (K^2 - 1) \ln \left| \frac{K+1}{K-1} \right| \right]$$

$$T^{(u)J^P=1^-}_{\bar{C}\bar{D},AB(\lambda)}(s) = \frac{2^{1/2}}{4} f_{V''AD} g_{V''BC} \gamma_{V''AD} \gamma_{V''BC} \frac{\lambda (E_1 + E_2)}{m_\pi} \times \left[2L - (L^2 - 1) \ln \left| \frac{L+1}{L-1} \right| \right]$$

(4.63)

C. $PS + V \to PS + V$

—the s channel—

We now consider the reaction of figure 4a. It is assumed that B and $\bar{D}$ are the external vector particles of helicities λ and $-\lambda'$, respectively. We begin with unitarity:

$$\begin{aligned}\text{Disc}_s \langle \bar{C}(-p_3),\bar{D}(-p_4,-\lambda')|\hat{T}|A(p_1),B(p_2,\lambda)\rangle \\ = -\pi\int d^4k\ \delta(k^2 - M_V^2)\theta(k_o - M_V)\ \delta(p_1 + p_2 - k) \\ \times \sum_{\lambda''}\langle \bar{C}(-p_3),\bar{D}(-p_4,-\lambda')|\hat{T}^+|V(k,\lambda'')\rangle \\ \langle V(k,\lambda'')|\hat{T}|A(p_1),B(p_2,\lambda)\rangle \qquad (4.64)\end{aligned}$$

For the vertex functions we use the form of (4.39),(4.40), and (4.41). We find

$$\text{Disc}_s\ T_{\bar{C}\bar{D}(-\lambda'),AB(\lambda)}(s,t) = -\frac{\pi\, g_{VAB}g_{VCD}\gamma_{VAB}\gamma_{VCD}}{m_\pi^2}\,\delta(s-M_V^2)Z \qquad (4.65)$$

where

$$\begin{aligned}Z &= \epsilon_\kappa^*(\bar{D};-p_4,-\lambda')p_{3_\alpha}p_{4_\beta}\epsilon^{\alpha\beta\gamma\kappa}(-g_{\gamma\tau} + k_\gamma k_\tau/M_V^2)\epsilon^{\rho\sigma\tau\omega}p_{1_\rho}p_{2_\sigma} \\ &\qquad \times \epsilon_\omega(B;p_2,\lambda) \qquad (4.66)\\ &= -\epsilon_\kappa^*(\bar{D};-p_4,-\lambda')p_{3_\alpha}p_{4_\beta}\epsilon^{\gamma\alpha\beta\kappa}\epsilon_{\gamma\rho\sigma\omega}p_1^\rho p_2^\sigma\epsilon^\omega(B,p_2,\lambda) \qquad (4.67)\end{aligned}$$

We now use the identity

$$-\epsilon^{\gamma\alpha\beta\kappa}\epsilon_{\gamma\rho\sigma\omega} = \begin{vmatrix}\delta_\rho^\alpha & \delta_\rho^\beta & \delta_\rho^\kappa \\ \delta_\sigma^\alpha & \delta_\sigma^\beta & \delta_\sigma^\kappa \\ \delta_\omega^\alpha & \delta_\omega^\beta & \delta_\omega^\kappa\end{vmatrix} \qquad (4.68)$$

and find that

$$\begin{aligned}Z = (p_3\cdot p_1)\left[(p_4\cdot p_2)(\epsilon_4^*\cdot\epsilon_2) - (p_4\cdot\epsilon_2)(\epsilon_4^*\cdot p_2)\right] \\ - (p_3\cdot p_2)\left[(p_4\cdot p_1)(\epsilon_4^*\cdot\epsilon_2) - (p_4\cdot\epsilon_2)(\epsilon_4^*\cdot p_1)\right] \qquad (4.69)\\ + (p_3\cdot\epsilon_2)\left[(p_4\cdot p_1)(\epsilon_4^*\cdot p_2) - (p_4\cdot p_2)(\epsilon_4^*\cdot p_1)\right]\end{aligned}$$

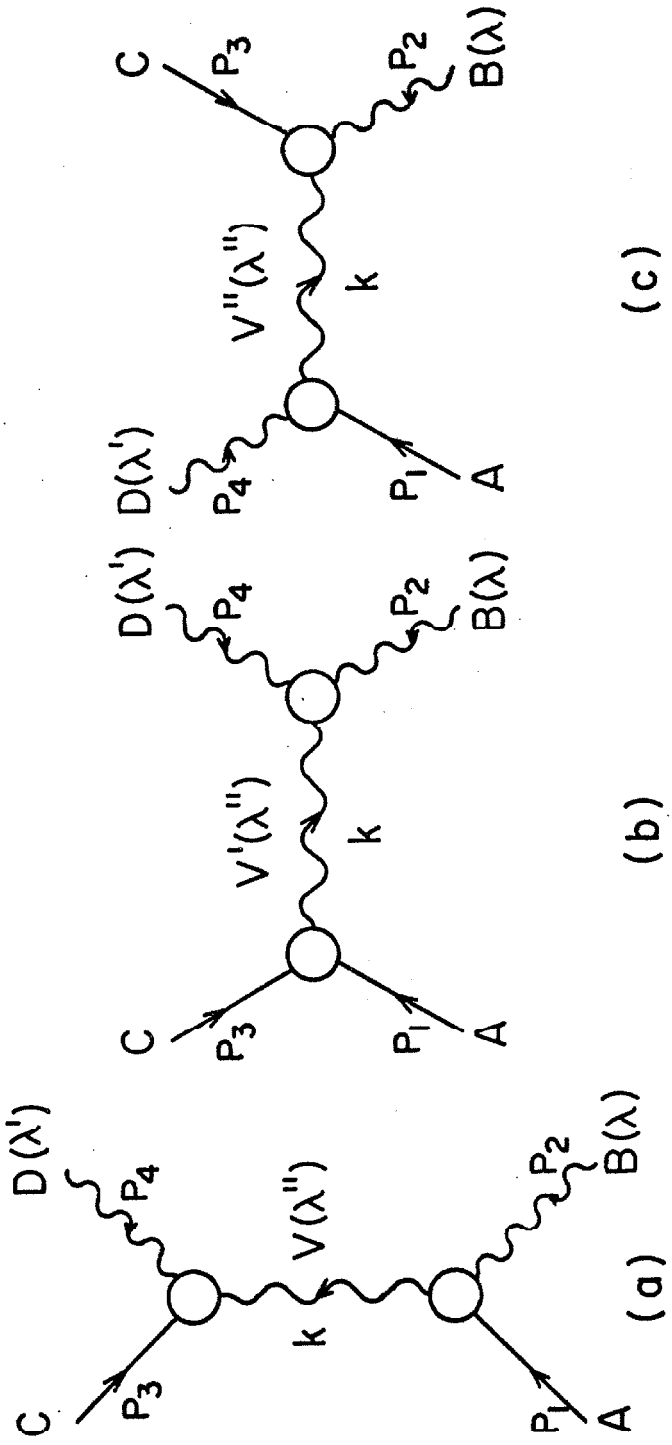

Fig. 4

The PS + V → PS + V Reactions

where $\epsilon^*(D;-p_4,-\lambda')$ and $\epsilon(B;p_2,\lambda)$ have been replaced by ϵ_4^* and ϵ_2, respectively. We know ϵ_4^* from (4.48):

$$\epsilon_4^* = \delta_{\lambda' \pm 1} i2^{-\frac{1}{2}}(0,-\lambda',+i,0) + \delta_{\lambda' 0} \frac{i}{m_4}(q,0,0,-E_4) \tag{4.70}$$

To find ϵ_2 we note that if $p_2 = (E_2,0,0,p)$ we would have

$$i\epsilon_2 = \delta_{\lambda \pm 1} 2^{-\frac{1}{2}}(0,-\lambda,-i,0) + \delta_{\lambda 0} \frac{1}{m_2}(p,0,0,E_2) \tag{4.71}$$

But since $p_2 = (E_2,(1-x^2)^{\frac{1}{2}}p,0,-px)$ we have

$$\epsilon_2 \to e^{-iJ_j(\pi + \theta)} \epsilon_2 \tag{4.72}$$

with $e^{-iJ_y\theta}$ given by

$$e^{-iJ_j\theta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & (1-x^2)^{\frac{1}{2}} \\ 0 & 0 & 1 & 0 \\ 0 & -(1-x^2)^{\frac{1}{2}} & 0 & x \end{pmatrix} \tag{4.73}$$

so that

$$\epsilon_2 = -\delta_{\lambda \pm 1} i2^{-\frac{1}{2}}(0,\lambda x,-i,-\lambda(1-x^2)^{\frac{1}{2}}) - \delta_{\lambda 0} \frac{i}{m_2}(p,-E_2(1-x^2)^{\frac{1}{2}},$$

$$0,-E_2) \tag{4.74}$$

With these expressions for ϵ_4^* and ϵ_2, equations (2.3) are used to discover that

$$Z = \frac{1}{2} pqs(x - \lambda\lambda')\delta_{\lambda \pm 1}\delta_{\lambda' \pm 1} + z \tag{4.75}$$

where z involves the terms not containing $\delta_{\lambda \pm 1}\delta_{\lambda' \pm 1}$. Then

$$T^{(s)}_{\bar{C}\bar{D}(-\lambda'),AB(\lambda)}(s,t) = \frac{g_{VAB}g_{VCD}\gamma_{VAB}\gamma_{VCD}}{m_\pi^2(s - M_V^2)} \left[\frac{pqs}{2}(x - \lambda\lambda')\delta_{\lambda \pm 1}\delta_{\lambda' \pm 1} + z \right] \tag{4.76}$$

The J = 1 partial wave is projected according to (2.19) with

$$d^1_{-\lambda\lambda'}(x) = \frac{1 - \lambda\lambda' x}{2}\,\delta_{\lambda\pm 1}\,\delta_{\lambda'\pm 1} + d' \tag{4.77}$$

where d' contains terms not proportional to $\delta_{\lambda\pm1}\,\delta_{\lambda'\pm1}$. Then

$$T^{(s)J=1}_{\bar{C}\bar{D}(-\lambda'),AB(\lambda)}(s) = -\lambda\lambda'\,\frac{g_{VAB}g_{VCD}\gamma_{VAB}\gamma_{VCD}\;pqs}{3m_\pi^2(s - M_V^2)} + T' \tag{4.78}$$

Again T' contains those terms not involving $\delta_{\lambda\pm1}\,\delta_{\lambda'\pm1}$.

—the t channel—

The t channel reaction, figure 4b, involves a V-V-V coupling which we have agreed to ignore. This channel is not considered further.

—the u channel—

The contribution from the u channel reaction, figure 4c, could be calculated from the substitution law, as in earlier cases. However it appears in this case to be simpler to calculate it directly from unitarity.

$$\begin{aligned}
&\mathrm{Disc}_u \langle \bar{C}(-p_3),\bar{B}(-p_2,-\lambda)|\hat{T}|\,A(p_1),D(p_4,\lambda')\rangle \\
&\quad = -\pi\int d^4k\;\delta(k^2 - M_{V''}^2)\theta(k_o - M_{V''})\;\delta(p_1 + p_4 - k) \\
&\quad\quad \times \sum_{\lambda''} \langle \bar{C}(-p_3),\bar{B}(-p_2,-\lambda)|\hat{T}^+|\,V''(k,\lambda'')\rangle \\
&\quad\quad \times \langle V''(k,\lambda'')|\hat{T}|\,A(p_1),D(p_4,\lambda')\rangle \qquad (4.79) \\
&\quad = -\frac{\pi\, g_{V''CB}g_{V''AD}\gamma_{V''BC}\gamma_{V''AD}}{m_\pi^2}\,\delta(u - M_{V''}^2)\;Z' \qquad (4.80)
\end{aligned}$$

$$Z' = -\epsilon^{\nu\alpha\beta\mu}\,\epsilon_{\nu\tau\kappa\rho}\,p_{3\alpha}\,p_{2\beta}\,p_1^\tau\,p_4^\kappa\,\epsilon_{2\mu}^*\,\epsilon_4^\rho \tag{4.81}$$

where

$$\begin{aligned}
\epsilon_2^* &= \epsilon^*(\bar{B};-p_2,-\lambda) \\
&= \delta_{\lambda\pm1}\,i2^{-1/2}(0,\lambda x,i,-\lambda(1-x^2)^{1/2}) + \delta_{\lambda 0}\,\frac{i}{m_2}(p,-E_2(1-x^2)^{1/2},0, \\
&\qquad\qquad -E2x)
\end{aligned}$$

$$\epsilon_4 = \epsilon(D;p_4,\lambda') = -\delta_{\lambda',\pm 1}\, i2^{-1/2}(0,-\lambda',-i,0) - \delta_{\lambda' 0}\,\frac{i}{m_4}(q,0,0,E_4) \tag{4.82}$$

After a short computation we find

$$Z' = \frac{pq}{2}(ax^2 + bx + c)\delta_{\lambda,\pm 1}\,\delta_{\lambda',\pm 1} + z' \tag{4.83}$$

where

$$\begin{aligned} a &= -pq - \lambda\lambda' E_2E_4 \\ b &= E_1E_3 + E_2E_4 - \lambda\lambda'\left(\frac{p}{q}E_3E_4 + \frac{q}{p}E_1E_2\right) \\ c &= pq + \frac{p}{q}E_3E_4 + \frac{q}{p}E_1E_2 - \lambda\lambda' E_1E_3 \end{aligned} \tag{4.84}$$

and z' contains terms not involving $\delta_{\lambda,\pm 1}\,\delta_{\lambda',\pm 1}$. After projecting out the $J = 1$ partial wave the result is

$$T^{(u)J=1}_{C\bar{D}(-\chi),AB(\lambda)}(s) = \lambda\lambda'\,\frac{g_{V''AD}g_{V''BC}\gamma_{V''AD}\gamma_{V''BC}}{8m_\pi^2}\left[-\frac{1}{3}pq + Q + \frac{1}{2}(E_1E_3 - LQ)\ln\left|\frac{L+1}{L-1}\right|\right] + T'' \tag{4.85}$$

where

$$Q = pq + 2E_1E_2\frac{q}{p} + 2E_3E_4\frac{p}{q} - L(2E_2E_4 + E_1E_3) - L^2pq \tag{4.86}$$

and T'' is that part of T not proportional to $\lambda\lambda'$.

Finally we ensure the proper parity eigenvalue using the state of (4.60).

$$\langle PS,V\,|\,\hat{T}\,|\,PS,V\rangle = \frac{1}{2}\Big[\langle PS,V(\lambda')|\,\hat{T}\,|PS,V(\lambda)\rangle + \langle PS,V(-\lambda')|\,\hat{T}\,|PS,V(-\lambda)\rangle - \langle PS,V(\lambda')|\,\hat{T}\,|PS,V(-\lambda)\rangle - \langle PS,V(-\lambda')|\,\hat{T}\,|PS,V(\lambda)\rangle\Big] \tag{4.87}$$

For $T^{J=1}$ of the form

$$T^{J=1} = T_1\lambda\lambda' + T_2\lambda + T_3\lambda' + T_4 \tag{4.88}$$

the net result is

$$T^{J^P=1^-} = 2\,\lambda\lambda'\,T_1 \tag{4.89}$$

all other terms cancelling. Therefore in summary

$$\left.\begin{aligned} T^{(s)J^P=1^-}_{\bar{C}\bar{D}(-\lambda'),AB(\lambda)}(s) &= -\frac{2\lambda\lambda'\, g_{VAB}g_{VCD}\gamma_{VAB}\gamma_{VCD}\,pqs}{3m_\pi^2\,(s-M_V^2)} \\ T^{(u)J^P=1^-}_{\bar{C}\bar{D}(-\lambda'),AB(\lambda)}(s) &= \frac{\lambda\lambda'\, g_{V''AD}g_{V''BC}\gamma_{V''AD}\gamma_{V''BC}}{4m_\pi^2}\left[-\frac{1}{3}\,pq\right. \\ &\quad \left. + Q + \frac{1}{2}(E_1E_3 - LQ)\ln\left|\frac{L+1}{L-1}\right|\right] \end{aligned}\right\} \tag{4.90}$$

In the following chapter equations (4.37),(4.62),(4.63), and (4.90) will be specialized to the cases of interest in the present bootstrap.

V. DERIVATION OF THE SELF-CONSISTENT EQUATIONS

The invariant amplitude T_{ij} defined by (2.16) satisfies the unitarity condition (2.17). If the complete unitarity sum is approximated by the term with t = 2 we find for this contribution in the s channel

$$\text{Disc}_s\, T_{ij} = -\frac{1}{32\pi^2}\, q_k\, s^{-\frac{1}{2}}\theta_k \int d\Omega_k\; T^+_{ik}\, T_{kj} \tag{5.1}$$

where $\theta_k = \theta(s - s_k)$, s_k is the physical threshold energy in the k th channel, $d\Omega_k$ is the infinitesimal scattering solid angle in this channel, and summation over repeated indices is implied. For the moment we ignore the complications arising from the presence of identical particles. If we further project the partial waves according to (2.18) and (2.19) the result is

$$\text{Disc}_s\, T^J_{ij} = -\frac{1}{8\pi}\, s^{-\frac{1}{2}}\, T^{+J}_{ik}\, q_k\, \theta_k\, T^J_{kj} \tag{5.2}$$

In this approximation all T^J_{ij} refer to two particle states. The cut of T given by (5.2) will be called the "right-hand cut". All other cuts of T in the variable s will be collectively termed "left-hand cuts". We define a new amplitude $\bar{T}$ by

$$\begin{aligned} \bar{T}_{ij} &= \bar{q}_i^{\,m}\, \bar{q}_j^{\,n}\, T^{J^P=1^-}_{ij} \\ &= N_{ik}\, D^{-1}_{kj} \end{aligned} \tag{5.3}$$

where $\bar{q}_i = q_i/m_\pi$, and m and n are for the moment undetermined; they are chosen so that the decomposition of $\bar{T}$ into ND^{-1} gives N only the left-hand cuts of $\bar{T}$ and D only the right-hand cut. Then

$$\text{Disc}_s\, \bar{T}_{ij} = -\frac{1}{8\pi}\, m_\pi\, s^{-\frac{1}{2}}\, \bar{T}^+_{ik}\, \bar{q}_k^{\,l-m-n}\, \bar{T}_{kj} \tag{5.4}$$

and at threshold

$$\bar{T}_{ij} \sim \bar{q}_i^{\,m+1}\, \bar{q}_j^{\,n+1} \tag{5.5}$$

Note that T and $\bar{T}$ are dimensionless. We then have

$$N_{ij} = \frac{1}{\pi}\int_L \frac{\mathrm{Disc}_L\, \bar{T}_{ik}(s')\, D_{kj}(s')ds'}{s' - s}$$

$$D_{ij} = \delta_{ij} + \frac{(s - s_o)m_\pi}{8\pi^2}\int_{s_i}^{\infty} \frac{\bar{q}_i^{\,1-m-n}(s')N_{ij}(s')ds'}{(s' - s)(s' - s_o)\, s'^{1/2}} \qquad (5.6)$$

Here L denotes the union of all left-hand cuts and $\mathrm{Disc}_L\, \bar{T}$ is the sum of the discontinuities of $\bar{T}$ on all these cuts. The subtraction point s_o in D_{ij} may depend on i and j, and it is assumed that no further subtractions are required. In order to reduce the problem to tractable dimensions we must make still further approximations. The circular cuts of $\bar{T}$ arising from mass differences are dismissed without further apology. In addition we use the first determinental approximation (18) and set $D_{kj} \approx \delta_{kj}$ in the first of equations (5.6). Therefore

$$N_{ij} \approx \bar{T}^L_{ij} \qquad (5.7)$$

where $\bar{T}^L$ is the result of dispersing $\mathrm{Disc}_L\, \bar{T}$. For $\bar{T}^L$ we take the t and u channel Born terms calculated in chapter IV. We write $(\gamma\gamma')_{ij}$ as an abbreviation for the product of the two coupling constants, whichever they may be, appearing in the expression for $\bar{T}^L_{ij}$, and further write

$$\bar{T}^L_{ij} = (\gamma\gamma')_{ij}\, \bar{q}_i^{\,m+1}\, \bar{q}_j^{\,n+1}\, f_{ij}. \qquad (5.8)$$

f_{ij} is dimensionless and smooth at threshold. For the second of equations (5.6) we therefore find

$$D_{ij} = \delta_{ij} + (\gamma\gamma')_{ij}\, g_{ij} \qquad (5.9)$$

$$g_{ij} = \frac{s - s_o}{8\pi^2}\int_{s_i}^{\infty} \frac{\bar{q}_i^{\,2-n}(s')\, \bar{q}_j^{\,1+n}(s')f_{ij}(s')\, ds'}{(s' - s)(s' - s_o)(s/m_\pi^2)^{1/2}} \qquad (5.10)$$

which is put into dimensionless form

$$g_{ij} = \frac{x - x_o}{16\pi^2} \int_{x_i}^{\infty} \frac{v_i^{\frac{2-n}{2}}(x')\, v_j^{\frac{1+n}{2}}(x')\, f_{ij}(x')\, dx'}{(x' - x)(x' - x_o)\, x'^{1/2}} \tag{5.11}$$

when the substitution $v_i = q_i^2/m_\pi^2 = \bar{q}_i^2$, $x = s/4m_\pi^2$, etc. is made. Therefore (5.3) becomes

$$\bar{T}_{ij} = (\gamma\gamma')_{ik}\, \bar{q}_i^{m+1}\, \bar{q}_j^{n+1}\, f_{ik}\, (I + \bar{g})^{-1}_{kj} \tag{5.12}$$

where I is the unit matrix and

$$\bar{g}_{ij} = (\gamma\gamma')_{ij}\, g_{ij} \tag{5.13}$$

Equivalent to this is

$$t_{ij} = (\gamma\gamma')_{ik}\, (\bar{q}_k/\bar{q}_j)^{n+1}\, f_{ik}\, (I + \bar{g})^{-1}_{kj} \tag{5.14}$$

where

$$t_{ij} = \bar{T}_{ij}/\bar{q}_i^{m+1}\, \bar{q}_j^{n+1} \tag{5.15}$$

is <u>smooth</u> at threshold. To complete the self-consistency equations, we demand that (5.14) hold in the region of the resonance, $s = M_R^2$. At such points T approximates the form of the s-channel Born terms calculated in chapter IV. We insert the observed width and write

$$t_{ij}(s \approx M_R^2) = \frac{(\gamma\gamma')_{ij}\, r_{ij}}{s - M_R^2 + iM_R\Gamma_R} \tag{5.16}$$

which is put in dimensionless form

$$t_{ij}(x \approx x_R) = \frac{(\gamma\gamma')_{ij}\, \bar{r}_{ij}}{x - x_R + i(M_R\Gamma_R/4m_\pi^2)} \tag{5.17}$$

while from (5.14)

$$t_{ij}(x \approx x_R) = (\gamma\gamma')_{ik}\, (\bar{q}_k/\bar{q}_j)^{n+1}\, f_{ik}\, \bar{D}_{kj}/D \tag{5.18}$$

where $\bar{D}$ is the cofactor matrix of D^{-1}:

$$D^{-1}_{kj} = \bar{D}_{kj} / D \tag{5.19}$$

and D is the determinant of D_{ij}. Near $x = x_R$ we write

$$D \approx D_r(x_R) + (x - x_R)D'_r(x_R) + i\, D_i(x_R) \tag{5.20}$$

where the subscripts r and i denote the real and imaginary parts, respectively. In order that (5.17) and (5.18) be consistent it is apparent that at $x = x_R$ we require

$$D_r = 0$$

$$\frac{(\gamma\gamma')_{ik}\, f_{ik}(\bar{q}_k / \bar{q}_j)^{n+1}\, \bar{D}_{kj}}{D'_r} = (\gamma\gamma')_{ij}\, \bar{r}_{ij} \tag{5.21}$$

Equations (5.21) are the self-consistent equations for any general reaction. In an exact calculation the second of these equations is satisfied for $i \neq j$ if it is satisfied for the diagonal terms and the first equation is also true. In this calculation in which the first determinental approximation is employed the off-diagonal equations will not be satisfied, and they cannot be used as further self-consistent equations. After a solution is found the divergence of the off-diagonal terms from the value given by (5.21) will be a measure of the crudeness of the approximation (23).

We must now choose the values of m and n appearing in (5.3). These should be selected so that the preceeding equations —which are certainly true above the largest s_k—may be analytically continued to lower s values. In the first determinental method the threshold behavior is satisfied automatically, and cannot be used to limit m and n. However the integrals g_{ij}, equation (5.11), indicate that $\frac{l+n}{2}$ should be an integer in order to avoid complex values arising in the cases where $s_i < s_j$ from integration over a region where v_j is negative. The common choices are n = +1,-1. For n = +1 we are writing ND^{-1} for $(q_i q_j)^{\frac{1}{2}} e^{i\delta} \sin\delta$; for n = -1 we assume that the amplitude

$(q_i q_j)^{-3/2}\, e^{i\delta} \sin\delta$ is reducible to ND^{-1}. The first of these behaves like $q_i^2\, q_j^2$ at threshold, the second is smooth. There is a problem with the choice n = +1; barring accidents from D^{-1}, equation (5.14) predicts poles at the j th threshold for t_{ij} instead of the smooth behavior known there. For this reason we choose n = -1.

We now specialize the foregoing results to the particular problem in mind. Consider first the K* bootstrap. From (5.3) and (5.8) for n = -1 we have

$$(\gamma\gamma')_{ij}\, f_{ij} = T^L_{ij}\ (m_\pi^2 / q_i q_j) \tag{5.22}$$

The left-hand cuts of $T_{ij}(K^*)$ we approximate as arising from the logarithms of equations (4.37) which refer to the exchange forces of figure 5. The isotopic spin crossing matrix elements for projection into the s channel I = ½ state for the reactions 5a through 5d are $1, -\frac{1}{3}, 1, 1$ respectively. We call π K channel one, η K channel two, and eventually discover

$$(\gamma\gamma')_{ij} f_{ij}(K^*) = \begin{pmatrix} \gamma_1\gamma_5\, f_{11} + \gamma_3^2\, \bar{f}_{11} & \gamma_3\gamma_4\, f_{12} \\ \gamma_3\gamma_4\, f_{12} & \gamma_4^2\, f_{22} \end{pmatrix} \tag{5.23}$$

where

$$f_{11} = 2\, \frac{2x + R_2 - \frac{1 + R_1}{2}}{v_1^2} \left[2 - B_1 \ln \left| \frac{B_1 + 1}{B_1 - 1} \right| \right]$$

$$\bar{f}_{11} = -\frac{1}{2}\, \frac{2x + R_3 - \frac{1+R_1}{2} - \frac{(R_1 - 1)^2}{16R_3}}{v_1^2} \left[2 - C_1 \ln \left| \frac{C_1 + 1}{C_1 - 1} \right| \right]$$

$$f_{22} = \frac{3}{2}\, \frac{2x + R_3 - \frac{R_1+R_4}{2} - \frac{(R_1 - R_4)^2}{16R_3}}{v_2^2} \left[2 - C_3 \ln \left| \frac{C_3 + 1}{C_3 - 1} \right| \right]$$

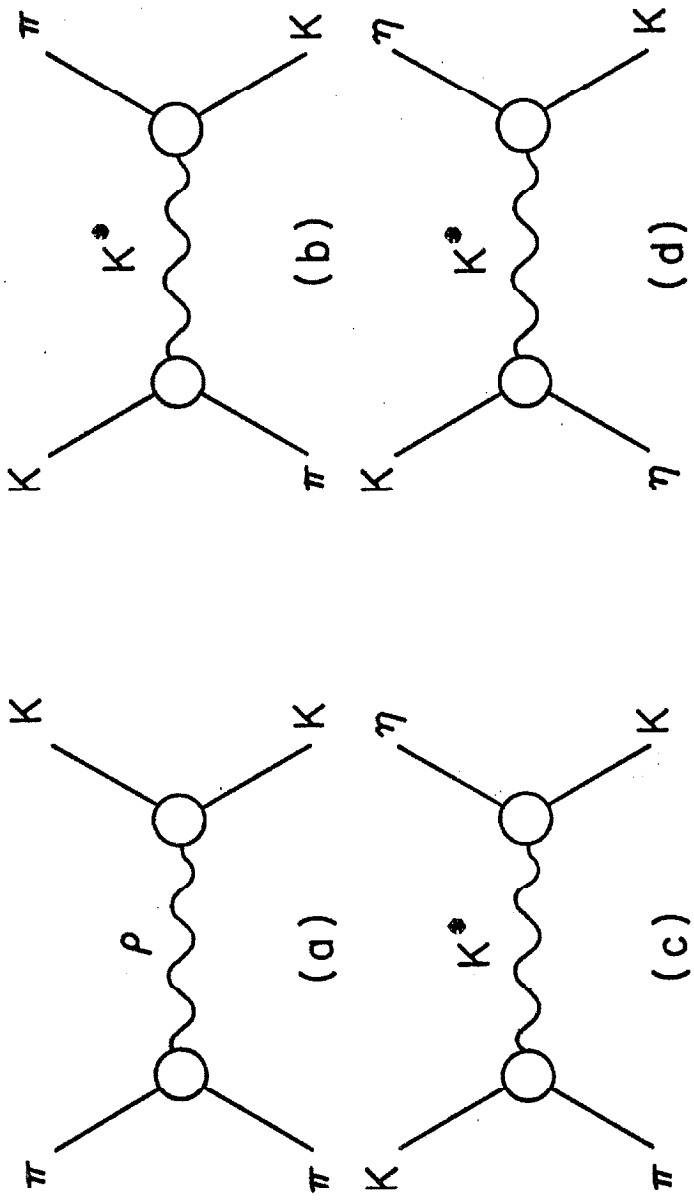

Fig. 5

Exchange diagrams for the K* bootstrap

$$f_{12} = \frac{3}{2} \frac{2x + R_3 - (\frac{1}{4} + \frac{1}{2} R_1 + \frac{1}{4} R_4) - \frac{1}{16 R_3}(1 - R_1)(R_4 - R_1)}{v_1 \, v_2}$$

$$x \left[2 - C_3 \ln \left| \frac{C_2 + 1}{C_2 - 1} \right| \right]$$

$x = s/ 4m_\pi^2$; $R_1 = m_K^2 / m_\pi^2$; $R_2 = M_\rho^2 / 4m_\pi^2$; $R_3 = M_{K^*}^2 / 4m_\pi^2$;

$R_4 = m_\eta^2 / m_\pi^2$; $R_5 = M_\omega^2 / 4m_\pi^2$;

$$v_1 = x - \frac{1}{2}(1 + R_1) + \frac{1}{16x}(1 - R_1)^2$$

$$v_2 = x - \frac{1}{2}(R_4 + R_1) + \frac{1}{16x}(R_4 - R_1)^2$$

$$B_1 = 1 + \frac{2R_2}{v_1} ; \quad C_1 = 1 + \frac{2R_3}{v_1} - \frac{1}{8xv_1}(1 - R_1)^2;$$

$$C_2 = \frac{x - (\frac{1}{4} + \frac{1}{2} R_1 + \frac{1}{4} R_4) - \frac{1}{16x}(1 - R_1)(R_4 - R_1) + 2R_3}{(v_1 \, v_2)^{\frac{1}{2}}}$$

$$C_3 = 1 + \frac{2R_3}{v_2} - \frac{1}{8xv_2}(R_1 - R_4)^2$$

and

$\gamma_1 = \gamma_{\rho\pi\pi}$; $\gamma_2 = \gamma_{\rho\pi\omega}$; $\gamma_3 = \gamma_{\pi KK^*}$; $\gamma_4 = \gamma_{\eta KK^*}$;

$\gamma_5 = \gamma_{\rho KK}$; $\gamma_6 = \gamma_{\omega KK}$; $\gamma_7 = \gamma_{\omega KK^*}$

Then we have

$$D_{ij}(K^*) = \begin{pmatrix} 1 + \gamma_1 \gamma_5 \, g_{11} + \gamma_3^2 \, \bar{g}_{11} & \gamma_3 \gamma_4 \, g_{12} \\ \gamma_3 \gamma_4 \, g_{21} & 1 + \gamma_4^2 \, g_{22} \end{pmatrix} \qquad (5.24)$$

with
$$g_{ij} = \frac{x - x_o}{16\pi^2}\int_{x_i}^{\infty} \frac{v_i^{3/2}(x')\, f_{ij}(x')\, dx'}{(x' - x)(x' - x_o)\, x'^{1/2}} \tag{5.25}$$

If we desire the real part of D_{ij} the principle value of the integral is implied if $x > x_i$. The thresholds x_i are given by:

$$x_1 = \frac{1}{4}\,(1 + R_1^{1/2})^2 \; ; \qquad x_2 = \tfrac{1}{4}(R_4^{1/2} + R_1^{1/2})^2 \tag{5.26}$$

For the right-hand side of (5.21) we find from (4.37)

$$(\gamma\gamma')_{ij}\, \bar{r}_{ij}(K^*) = \begin{pmatrix} \frac{1}{2}\gamma_3^2 & \frac{1}{2}\,\gamma_3\gamma_4 \\ \frac{1}{2}\gamma_3\gamma_4 & \frac{1}{2}\,\gamma_4^2 \end{pmatrix} \tag{5.27}$$

For the K* case equations (5.21) give three relations between the unknowns $\gamma_1\gamma_5$, γ_3, γ_4^2 , M_ρ, and M_{K^*} (all PS masses are taken from experiment). We also take $\gamma_3 = \gamma_{\pi KK^*}$ from experiment and use the three equations to solve for $\gamma_1\gamma_5$, γ_4^2 , and M_{K^*} in terms of M_ρ . We arrive at the equations

$$\gamma_4^2 = \frac{-B \pm (B^2 - AC)^{1/2}}{A}$$

$$\gamma_1\gamma_5 = \frac{\dfrac{\gamma_3^2\gamma_4^2\, g_{21}g_{12}}{1 + \gamma_4^2\, g_{22}} - \gamma_3^2\, \bar{g}_{11} - 1}{g_{11}} \tag{5.28}$$

where

$$A = e\, g_{22}^2 + g_{22}(g - f) + bd$$

$$B = e\, g_{22} + \tfrac{1}{2}\,(a\, g_{22} + g - f - bc)$$

$$C = e + a$$

$$a = \gamma_3^2\, g_{12}\, f_{12} \; ; \qquad b = \gamma_3^2\, g_{21}\, g_{12} \; ; \qquad c = f_{22}$$

$$d = \bar{r}_{22}\, g'_{22}\ ; \qquad e = \bar{r}_{22}\left[\gamma_3^2\, \bar{g}'_{11} - \frac{g'_{11}}{g_{11}}\,(1 + \gamma_3^2\, \bar{g}_{11})\right]$$

$$f = \bar{r}_{22}\gamma_3^2\,(g'_{12}\, g_{21} + g_{12}\, g'_{21})\ ; \quad g = \bar{r}_{22}\,\frac{g'_{11}}{g_{11}}\gamma_3^2\, g_{21}\, g_{12} \tag{5.29}$$

In the above ' denotes differentiation with respect to $x = s/4m_\pi^2$. The third equation reads

$$(\gamma_1\gamma_5\, f_{11} + \gamma_3^2\, \bar{f}_{11})(1 + \gamma_4^2\, g_{22}) - \gamma_3^2\gamma_4^2\, f_{12}\, g_{21} - \bar{r}_{11}\gamma_3^2\Big[$$

$$(\gamma_1\gamma_5\, g'_{11} + \gamma_3^2\, \bar{g}'_{11})(1 + \gamma_4^2\, g_{22}) + (1 + \gamma_1\gamma_5 g_{11} + \gamma_3^2\, \bar{g}_{11})$$

$$\times\ (\gamma_4^2\, g'_{22}) - \gamma_3^2\gamma_4^2\,(g'_{12}\, g_{21} + g_{12}\, g'_{21})\Big] = 0 \tag{5.30}$$

The solution of equations (5.28), (5.29) and (5.30) proceeds as follows: a value of M_ρ is chosen and for this M_ρ a series of M_{K^*} values is taken. For each such pair (M_ρ, M_{K^*}) the quantities of (5.29) and calculated, γ_4^2 and $\gamma_1\gamma_5$ are found from (5.28) and inserted into (5.30). For a given M_ρ a value of M_{K^*} is eventually found such that (5.30) is satisfied. For this M_ρ the values of M_{K^*}, γ_4^2, and $\gamma_1\gamma_5$ are therefore all known which satisfy the first set of self-consistency equations. The procedure is then repeated for other M_ρ values until a table of $\gamma_1\gamma_5$, γ_4^2, and M_{K^*} vs. M_ρ is formed. These values are then inserted into the self-consistency equations arising from the ρ bootstrap to be developed below. It is noted that in solving for γ_4^2 an ambiguity has developed in which two values are possible. Of course the conditions that γ_4^2 be neither complex nor negative eliminate some solutions, but in general both solutions must be carried along to the ρ bootstrap where presumably only one will be such as to satisfy the remaining equations.

The ρ bootstrap, involving as it does three channels, is somewhat more complicated in details although the procedure is

similar to that described above. We call $\pi\pi$, $\pi\omega$, $K\bar{K}$ the first, second, and third channels respectively. To take into account the identity of particles in channel one we must modify the t matrix according to the prescription

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2} t_{11} & 2^{-\frac{1}{2}} t_{12} & 2^{-\frac{1}{2}} t_{13} \\ 2^{-\frac{1}{2}} t_{21} & t_{22} & t_{23} \\ 2^{-\frac{1}{2}} t_{31} & t_{32} & t_{33} \end{pmatrix}$$

For the pole residues corresponding to figure 6(a - f) we have

$$(\gamma\gamma')_{ij}\, \bar{r}_{ij}(\rho) = \begin{pmatrix} \frac{2}{3}\gamma_1^2 & \frac{2}{3}\gamma_1\gamma_2 R_2^{\frac{1}{2}} & \frac{2^{\frac{1}{2}}}{3}\gamma_1\gamma_5 \\ \frac{2}{3}\gamma_1\gamma_2 R_2^{\frac{1}{2}} & \frac{2}{3}\gamma_2^2 R_2 & \frac{2^{\frac{1}{2}}}{3}\gamma_2\gamma_5 R_2^{\frac{1}{2}} \\ \frac{2^{\frac{1}{2}}}{3}\gamma_1\gamma_5 & \frac{2^{\frac{1}{2}}}{3}\gamma_2\gamma_5 R_2^{\frac{1}{2}} & \frac{1}{3}\gamma_5^2 \end{pmatrix} \quad (5.31)$$

We have used $\lambda = -1, \lambda' = +1$. The final result of the calculation is independent of this choice so long as $\lambda\lambda' = -1$. The contributions to the left-hand cuts arise from the processes shown in figure 7(a - g) and the various time reversed processes. The isospin crossing constants for projection into the I = 1 s channel state are (24) $\frac{1}{2} + \frac{1}{2}$, $1 + 1$, $-\frac{2}{3} - \frac{2}{3}$, $1, (\frac{2}{3})^{\frac{1}{2}} + (\frac{2}{3})^{\frac{1}{2}}$, $-\frac{1}{2}, \frac{1}{2}$ respectively. The result is

$$(\gamma\gamma')_{ij}\, f_{ij}(\rho) = \begin{pmatrix} \gamma_1^2 F_{11} & \gamma_1\gamma_2 F_{12} & \gamma_3^2 F_{13} \\ \gamma_1\gamma_2 F_{21} & \gamma_2^2 F_{22} & \gamma_3\gamma_7 F_{23} \\ \gamma_3^2 F_{31} & \gamma_3\gamma_7 F_{32} & \gamma_5^2 F_{33} + \gamma_6^2 \bar{F}_{33} \end{pmatrix} \quad (5.32)$$

where

$$F_{11} = 2\,\frac{2x + R_2 - 1}{v_3^2}\left[2 - B_3 \ln\left|\frac{B_3 + 1}{B_3 - 1}\right|\right]$$

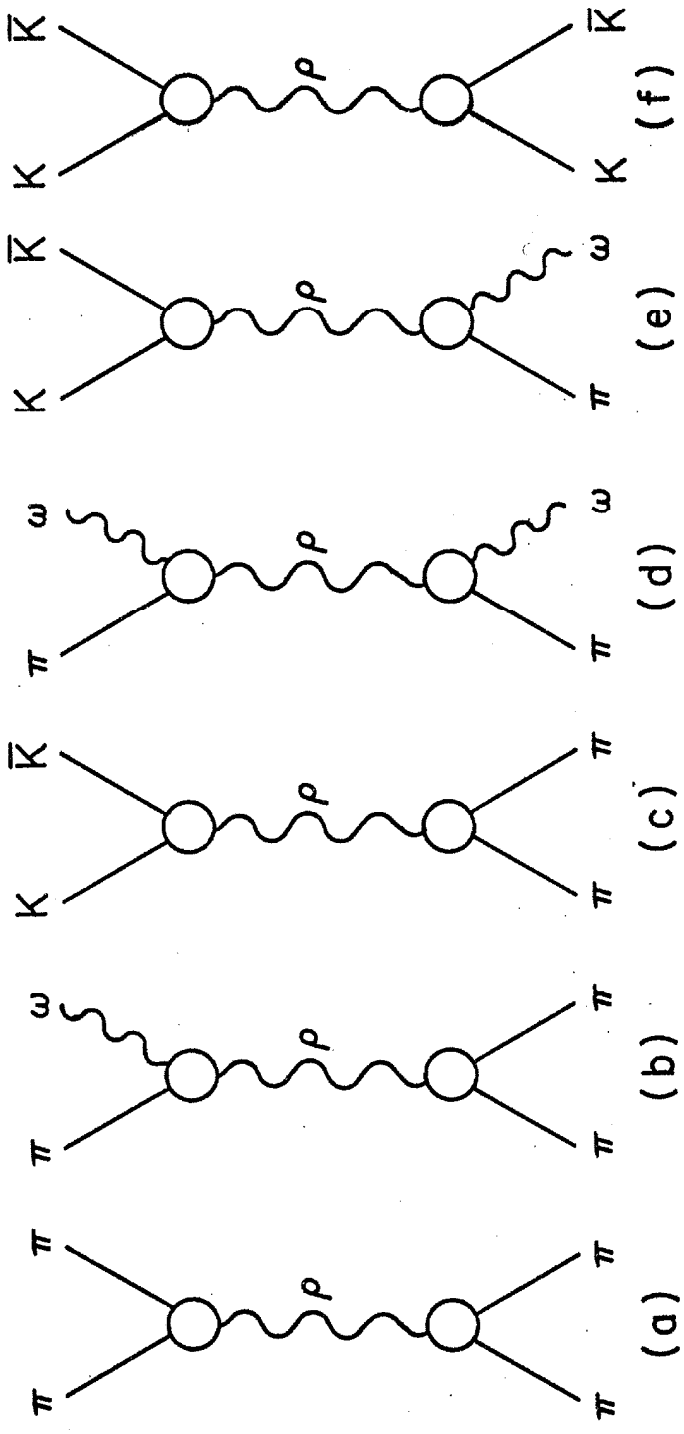

Fig. 6

Pole diagrams for the ρ bootstrap

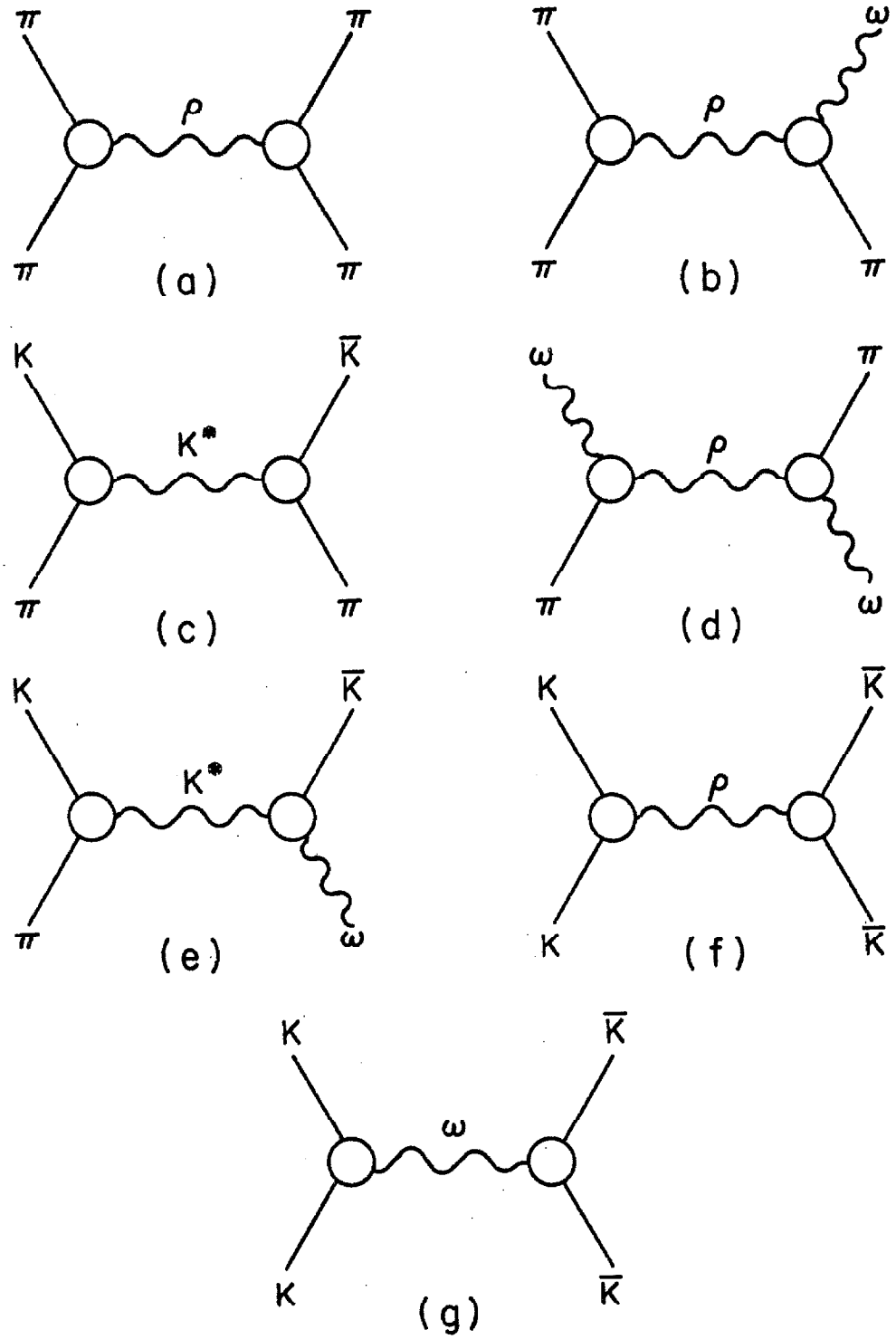

Fig. 7

Exchange diagrams for the ρ bootstrap

$$F_{12} = F_{21} = -2\left[\frac{x}{v_3 v_5}\right]^{1/2}\left[2B_7 - (B_7^2 - 1)\ln\left|\frac{B_7 + 1}{B_7 - 1}\right|\right]$$

$$F_{13} = F_{31} = -2^{1/2}\,\frac{2x + R_3 - \frac{1}{2}(1 + R_1) + \frac{1}{16R_3}(1 - R_1)^2}{v_3 v_4}$$

$$x\left[2 - B_4 \ln\left|\frac{B_4 + 1}{B_4 - 1}\right|\right]$$

$$F_{22} = -\frac{1}{4v_5}\left[-\frac{1}{3}v_5 + \bar{b}_o + \frac{1}{2}(v_5 + 1 - C_4\bar{b}_o)\ln\left|\frac{C_4 + 1}{C_4 - 1}\right|\right]$$

$$F_{23} = F_{32} = -\frac{1}{3\cdot 2^{1/2}}\left[\frac{3x}{v_4 v_5}\right]^{1/2}\left[2B_8 - (B_8^2 - 1)\ln\left|\frac{B_8 + 1}{B_8 - 1}\right|\right]$$

$$\bar{F}_{33} = \frac{3}{2}\,\frac{2x + R_5 - R_1}{v_4^2}\left[2 - B_6 \ln\left|\frac{B_6 + 1}{B_6 - 1}\right|\right]$$

$$F_{33} = -\frac{1}{2}\,\frac{2x + R_2 - R_1}{v_4^2}\left[2 - B_5 \ln\left|\frac{B_5 + 1}{B_5 - 1}\right|\right]$$

$$v_3 = x - 1\ ;\quad v_4 = x - R_1\ ;\quad v_5 = x - 2(R_5 + \tfrac{1}{4}) + \frac{1}{x}(R_5 - \tfrac{1}{4})^2$$

$$B_3 = 1 + \frac{2R_2}{v_3}\ ;\quad B_4 = \frac{x - \frac{1}{2}(1 + R_1) + 2R_3}{(v_3 v_4)^{1/2}}\ ;\quad B_5 = 1 + \frac{2R_2}{v_4}$$

$$B_6 = 1 + \frac{2R_5}{v_4}\ ;\quad B_7 = \frac{x - \frac{3}{4} - R_5 + 2R_2}{(v_3 v_5)^{1/2}}$$

$$B_8 = \frac{x - (\frac{1}{4} + \frac{1}{2}R_1 + R_5) + 2R_3}{(v_4 v_5)^{1/2}}\ ;\quad C_4 = \frac{x - 2(R_5 + \frac{1}{4}) + 2R_2 - \frac{1}{x}(R_5 - \frac{1}{4})^2}{v_5}$$

$$\bar{b}_o = a_1 - C_4 a_2 + C_4^2 a_3$$

$$a_1 = v_5 + 4x - \frac{4}{x}(R_5 - \frac{1}{4})^2 \; ; \; a_2 = 3v_5 + 8R_5 + 1 \; ; \; a_3 = -v_5$$

The equations for the f_{ij} and F_{ij} must be analytically continued in the case that the B's or C's appearing in the logarithms become complex. Since these variables are of the form $h(s)/q_i q_j$ they become complex only for off-diagonal terms, and only between the i th and j th thresholds. For the PS+PS→PS+PS reaction we have

$$2 - B \ln\left|\frac{B+1}{B-1}\right| = \int_{-1}^{+1} \frac{x\,dx}{x - B} \rightarrow 2 + \frac{2B^2}{(-B^2)^{1/2}} \tan^{-1}(1/(-B^2)^{1/2}) \qquad \ldots.(5.33)$$

for $B^2 < 0$, and the same holds for $B \rightarrow C$. For the PS + PS → PS + V amplitudes

$$2B - (B^2 - 1)\ln\left|\frac{B+1}{B-1}\right| = \int_{-1}^{+1} \frac{x^2 - 1}{x - B} \rightarrow 2q_i q_j \left(\frac{B}{q_i q_j}\right) x\left[1 + \frac{B^2 - 1}{(-B^2)^{1/2}} \tan^{-1}(1/(-B^2)^{1/2})\right] \qquad (5.34)$$

if $B^2 < 0$. Again the same is true for $B \rightarrow C$. These expressions are sufficient to ensure the reality of the t amplitude except where

$$-1 < \frac{B+1}{B-1} < +1 \quad \text{and} \quad -1 < \frac{C+1}{C-1} < +1.$$

The self-consistency equations for the ρ problem yield four equations for the unknowns M_ρ, γ_1, γ_6, γ_7 (γ_2 and γ_3 are taken from experiment, while $\gamma_1\gamma_5$, γ_4^2, and M_{K^*} are known as functions of M_ρ from the K* bootstrap). The method of solution is as follows: γ_6 and γ_7 are eliminated analytically and determined in terms of M_ρ and γ_1. This requires two equations. For a given value of M_ρ a series of γ_1 values is taken until values are found such that the remaining two equations are satisfied. In general this will happen for two distinct values of γ_1. A new M_ρ is chosen and the process repeated until eventually a value of M_ρ is found for which a single

γ_1 suffices to satisfy both remaining equations. This is the self-consistent solution.

Accordingly we proceed to eliminate γ_6 and γ_7 from the problem. Equation (5.32) assumes the form

$$(\gamma\gamma')_{ij}\, f_{ij}(\rho) = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5\gamma_7 \\ a_3 & a_5\gamma_7 & a_6\gamma_6^2 + a_7 \end{pmatrix} \tag{5.35}$$

with

$$a_1 = \gamma_1^2 F_{11}\ ;\quad a_2 = \gamma_1\gamma_2 F_{12}\ ;\quad a_3 = \gamma_3^2 F_{13}\ ;\quad a_4 = \gamma_2^2 F_{22}\ ;$$

$$a_5 = \gamma_3 F_{23}\ ;\quad a_6 = \bar{F}_{33}\ ;\quad a_7 = \gamma_5^2 F_{33}. \tag{5.36}$$

all known in terms of $M\rho$ and γ_1. Further

$$D_{ij}(\rho) = \begin{pmatrix} b_1 & b_2 & b_3 \\ \bar{b}_2 & b_4 & b_5\gamma_7 \\ \bar{b}_3 & \bar{b}_5\gamma_7 & b_6\gamma_6^2 + b_7 \end{pmatrix} \tag{5.37}$$

where

$$b_1 = 1 + \gamma_1^2 G_{11}\ ;\quad b_2 = \gamma_1\gamma_2 G_{12}\ ;\quad b_3 = \gamma_3^2 G_{13}\ ;$$

$$\bar{b}_2 = \gamma_1\gamma_2 G_{21}\ ;\quad b_4 = 1 + \gamma_2^2 G_{22}\ ;\quad b_5 = \gamma_3 G_{23} \tag{5.38}$$

$$\bar{b}_3 = \gamma_3^2 G_{31}\ ;\quad \bar{b}_5 = \gamma_3 G_{32}\ ;\quad b_6 = \bar{G}_{33}\ ;\quad b_7 = 1 + \gamma_5^2 G_{33}$$

and

$$G_{ij} = \frac{x - x_o}{16\pi^2} \int_{x_{i'}}^{\Lambda_{ij}} \frac{v_{i'}^{3/2}(x')\, F_{i'j'}(x')\, dx'}{(x' - x)(x' - x_o)\, x'^{1/2}} \tag{5.39}$$

For $x > x_{i'}$, the principle value is taken in order to find the real part. Here i' and j' are given by

$$i',j' = \begin{cases} 3 & i,j = 1 \\ 5 & i,j = 2 \\ 4 & i,j = 3 \end{cases} \tag{5.40}$$

in order to make the transition from the labelling of the channels to the labelling of the momenta. The threshold x_i, are given by

$$x_3 = 1 \; ; \quad x_4 = R_1 \; ; \; x_5 = (\tfrac{1}{2} + R_5^{\frac{1}{2}})^2 \qquad (5.41)$$

The cutoff Λ_{ij} is needed only for G_{22}, and the manner in which the solution depends on Λ_{ij} is explored in the succeeding chapter. The requirement that $D_r(\rho)\big|_{x = R_2} = 0$ becomes

$$A_1 + A_2\gamma_6^2 + A_3\gamma_7 + A_4\gamma_7^2 = 0 \qquad (5.42)$$

with

$$\begin{aligned} A_1 &= b_7(b_1b_4 - b_2\bar{b}_2) - b_3\bar{b}_3b_4 \\ A_2 &= b_6(b_1b_4 - b_2\bar{b}_2) \\ A_3 &= \bar{b}_5\bar{b}_2b_3 + \bar{b}_3b_2b_5 \\ A_4 &= - b_1b_5\bar{b}_5 \end{aligned} \qquad (5.43)$$

The residue self-consistency equation for the 11 channel reads

$$A_5 + A_6\gamma_6^2 + A_7\gamma_7 + A_8\gamma_7^2 = 0 \qquad (5.44)$$

where

$$\begin{aligned} A_5 &= a_1b_4b_7 - a_2\bar{b}_2b_7 - a_3b_4\bar{b}_3 - \gamma_1^2\,\bar{r}_{11}(\rho)\,A_1' \\ A_6 &= a_1b_4b_6 - a_2\bar{b}_2b_6 - \gamma_1^2\,\bar{r}_{11}(\rho)\,A_2' \\ A_7 &= a_2\bar{b}_3b_5 + a_3\bar{b}_2\bar{b}_5 - \gamma_1^2\,\bar{r}_{11}(\rho)\,A_3' \\ A_8 &= - a_1b_5\bar{b}_5 - \gamma_1^2\,\bar{r}_{11}(\rho)\,A_4' \end{aligned} \qquad (5.45)$$

Here ' denotes differentiation with respect to $x = s/4m_\pi^2$. Equations (5.42) and (5.44) suffice to determine γ_6^2 and γ_7. The two final self-consistency equations are:

$$\begin{aligned} &(a_4b_1b_7 - a_2b_2b_7 - a_4b_3\bar{b}_3) + \gamma_6^2(a_4b_1b_6 - a_2b_2b_6) \\ &+\gamma_7(a_2b_3\bar{b}_5 + a_5\bar{b}_3b_2) + \gamma_7^2(-a_5b_1\bar{b}_5) - \gamma_2^2\,\bar{r}_{22}(\rho)D'(\rho)_{x=R_2} = 0 \end{aligned}$$

....(5.46)

and

$$(a_7(b_1b_4 - b_2\bar{b}_2) - a_3b_3b_4) + \gamma_6^2\, a_6(b_1b_4 - b_2\bar{b}_2)$$
$$+\gamma_7(a_5\bar{b}_2b_3 + a_3b_2b_5) + \gamma_7^2(-a_5b_1b_5) - \gamma_5^2\, r_{33}(\rho)D'(\rho)\Big|_{x=R_2} = 0$$
$$\ldots\ldots(5.47)$$

Ambiguity is introduced into the problem by the two roots of the quadratic equation for γ_7 deriving from (5.42) and (5.44) although presumably only one solution will satisfy all of the self-consistent equations. In the following section the solution to the system of equations presented here is described and the self-consistent solutions displayed.

VI. RESULTS

The system of equations described in the preceeding section must now be solved. All numerical calculations were carried out on the CIT-IBM-7094 computor system. The majority of the calculation consists in the evaluation of the integrals g_{ij}, G_{ij}, and their derivatives g'_{ij}, G'_{ij}. These integrals are once subtracted; unlike the exact solutions to the integral equations the solutions found with the approximations made here are not independent of the subtraction points. Accordingly the choice must be made with some care and then varied to determine the sensitivity of the solutions to these paramaters. It is customary to perform the subtraction in the region of the left-hand cuts. We next describe the location of these cuts for the various forcing functions.

All left-hand cuts are supposed to originate from the presence of the terms

$$\ln\left|\frac{K+1}{K-1}\right| , \quad \ln\left|\frac{L+1}{L-1}\right|$$

in the Born amplitudes (we ignore the circular cuts due to mass differences). K and L are defined by (4.28) and (4.33). The cuts arise in the regions of s for which $|K|<1$ and/or $|L|<1$. This translates to the statement that cuts arise for values of s such that

$$s^2 + s\left[M'^2 - (m_1^2 + m_2^2 + m_3^2 + m_4^2) - \frac{(m_1^2 - m_3^2)(m_4^2 - m_2^2)}{M'^2}\right]$$

$$+ (m_1^2 - m_2^2)(m_3^2 - m_4^2) + \frac{m_1^2 m_4^2 - m_2^2 m_3^2}{M'^2}(m_1^2 - m_2^2 - m_3^2 + m_4^2) \lesseqgtr 0 \qquad \ldots\ldots(6.1)$$

where the inequality is assigned $\lesseqgtr 0$ for $s \gtrless 0$. Equation (6.1) is the statement $|K|<1$; the condition for $|L|<1$ is obtained from (6.1) by the substitutions $M' \to M''$, $m_4 \leftrightarrow m_3$.

Consider first the K* bootstrap and equation (6.1). There

arise cuts on the real axis for reactions of figures 5b and 5d. For 5b the cut exists for

$$-\infty < s < 2(m_\pi^2 + m_K^2) - 2M_{K^*}^2 \tag{6.2}$$

and

$$0 < s < \frac{1}{M_{K^*}^2}(m_\pi^2 - m_K^2)^2$$

In order that the predicted K* pole escape these cuts it is necessary that the resultant K* mass satisfy

$$M_{K^*} > (m_K^2 - m_\pi^2)^{1/2} = 475 \text{ Mev} \tag{6.3}$$

Reaction 5d produces a cut for

$$-\infty < s < 0$$

and

$$\frac{1}{M_{K^*}^2}(m_\eta^2 - m_K^2) < s < 2(m_\eta^2 + m_K^2) - M_{K^*}^2 \tag{6.4}$$

In order to avoid this cut we require

$$M_{K^*}^2 > (m_K^2 + m_\eta^2)^{1/2} = 740 \text{ Mev} \tag{6.5}$$

The other processes produce cuts off the real axis; the ends of the cuts are

$$\begin{aligned} &\text{fig. 5a:} \quad x_c = -.4 \pm 2.9i \\ &\text{fig. 5c:} \quad x_c = .06 \pm 2.0i \end{aligned} \tag{6.6}$$

where $x_c = s_c / 4m_\pi^2$, and s_c denotes the end of the cuts (the values for which $|K| = 1$ or $|L| = 1$.) The subtraction point for the various K* integrals is now chosen as $x_o(K^*) = 0.0$ for all K* integrals. This is near the thresholds for all cuts; the dependence of the final solutions for varying $x_o(K^*)$ is described below.

Now examine the ρ bootstrap. The cut arising from figure 7a runs for

$$-\infty < s < 4m_\pi^2 - M_\rho^2 \tag{6.7}$$

If the experimental values are used this is for $x < -6.7$. From 7b comes a cut off the real axis whose ends are at $x_c = .55 \pm 1.3i$. The cut from 7c is on the real axis for negative s if

$M_{K^*} > m_\pi + m_K = 632$ Mev. Since we already require $M_{K^*} > 740$ Mev this requirement is met. The cut runs for

$$-\infty < s < 2(m_\pi^2 + m_K^2) - M_{K^*}^2 - \frac{1}{M_{K^*}^2}(m_\pi^2 - m_K^2)^2 \qquad (6.8)$$

$$\text{or} \quad -\infty < x < -4.4$$

when experimental masses are inserted. From 7d arise cuts for

$$-\infty < s < 0$$

and $$\frac{1}{M_\rho^2}(M_\omega^2 - m_\pi^2)^2 < s < 2(m_\pi^2 + M_\omega^2) - M_\rho^2 \qquad (6.9)$$

In order that the ρ pole avoid these cuts we require that

$$M_\rho < (M_\omega^2 - m_\pi^2)^{1/2} = 772 \text{ Mev} \qquad (6.10)$$

Further off axis cuts arise from figure 7e, whose ends are at $x_c = 2.42 \pm 3.22i$. Finally, a real cut comes from 7f for

$$0 < s < 4m_K^2 - M_\rho^2 \qquad (6.11)$$

and

$$0 < s < 4m_K^2 - M_\omega^2$$

In order to avoid these cuts it is sufficient to demand that

$$M_\rho > 2^{1/2} m_K = 698 \text{ Mev} \qquad (6.12)$$

and

$$M_\rho > (4m_K^2 - M_\omega^2)^{1/2} = 600 \text{ Mev}$$

Again we shall choose a single subtraction point for all integrals. The standard choice will be $x_o(\rho) = -5.0$. Effects of variations of this paramater are studied below.

As mentioned above the calculation conveniently splits into two distinct parts. In the first, the K* bootstrap equations are used to find $\gamma_1\gamma_5$, γ_4^2, and M_{K^*} as functions of M_ρ. There are two sets of solutions, corresponding to the two roots of the quadratic equation for γ_4^2 (equation (5.28)). The data adopted as the "standard case", from which variations of paramaters are made, are

$m_\pi = 138.1$; $m_K = 493.8$; $m_\eta = 550.0$; $M_\omega = 785.0$;
$\gamma_2 = 1.37 (\gamma^2_{\rho\pi\omega}/4\pi = .15)$; $\gamma_3 = 4.67$ ($\gamma^2_{\pi KK^*}/4\pi = 1.74$, $\Gamma_{K^*} = 50$ Mev); $x_o(K^*) = 0.0$; $x_o(\rho) = -5.0$;
$\Lambda_{22} = 40$; $\Lambda_{ij} = \infty$, $i,j \neq 2$ (6.13)

The results of this first phase (phase I) of the standard case are shown in figure 8. The range of ρ masses is 600 Mev to 900 Mev, and the acceptable window for ρ masses of 698 Mev to 772 Mev is indicated by the arrows. The Root 1 solutions arise from taking the + sign in front of the radical of equation (5.28), Root 2 involves the - sign. It is noted that Root 1 would be much the desired solution since the couplings are much smaller than those of Root 2. However each solution must be taken over to phase II, the ρ bootstrap. Here a second ambiguity appears, again owing to the two roots of a quadratic equation; we call these two solutions root 1 and root 2.

Consider first the Root 2-root 1 case. Within the permissible M_ρ window it is impossible to find a simultaneous solution to equations (5.46) and (5.47). The same is true for the Root 2-root 2 case. This situation continues to hold as $x_o(K^*)$, $x_o(\rho)$, and Λ_{ij} are allowed to vary. Thus Root 2 leads to no solutions.

We next investigate the more promising Root 1 solutions. For the Root 1-root 2 case it is impossible to find values of M_ρ and γ_1 such that equation (5.46) is ever satisfied, much less simultaneously with equation (5.47) (root 1 involves taking the + sign in front of the radical resulting from eliminating γ_6^2 from equations (5.42) and (5.44) and solving for γ_7; root 2 takes the - sign). Therefore the only remaining candidates for the complete self-consistent solution are those of the Root 1-root 1 variety. One of them works. A convenient method illustrating the solution is a plot of the γ_1 values which satisfy (5.46)—curve 1 of figure 9—and which satisfy (5.47)—curve 2 of figure 9—vs. M_ρ. The results are shown in figure

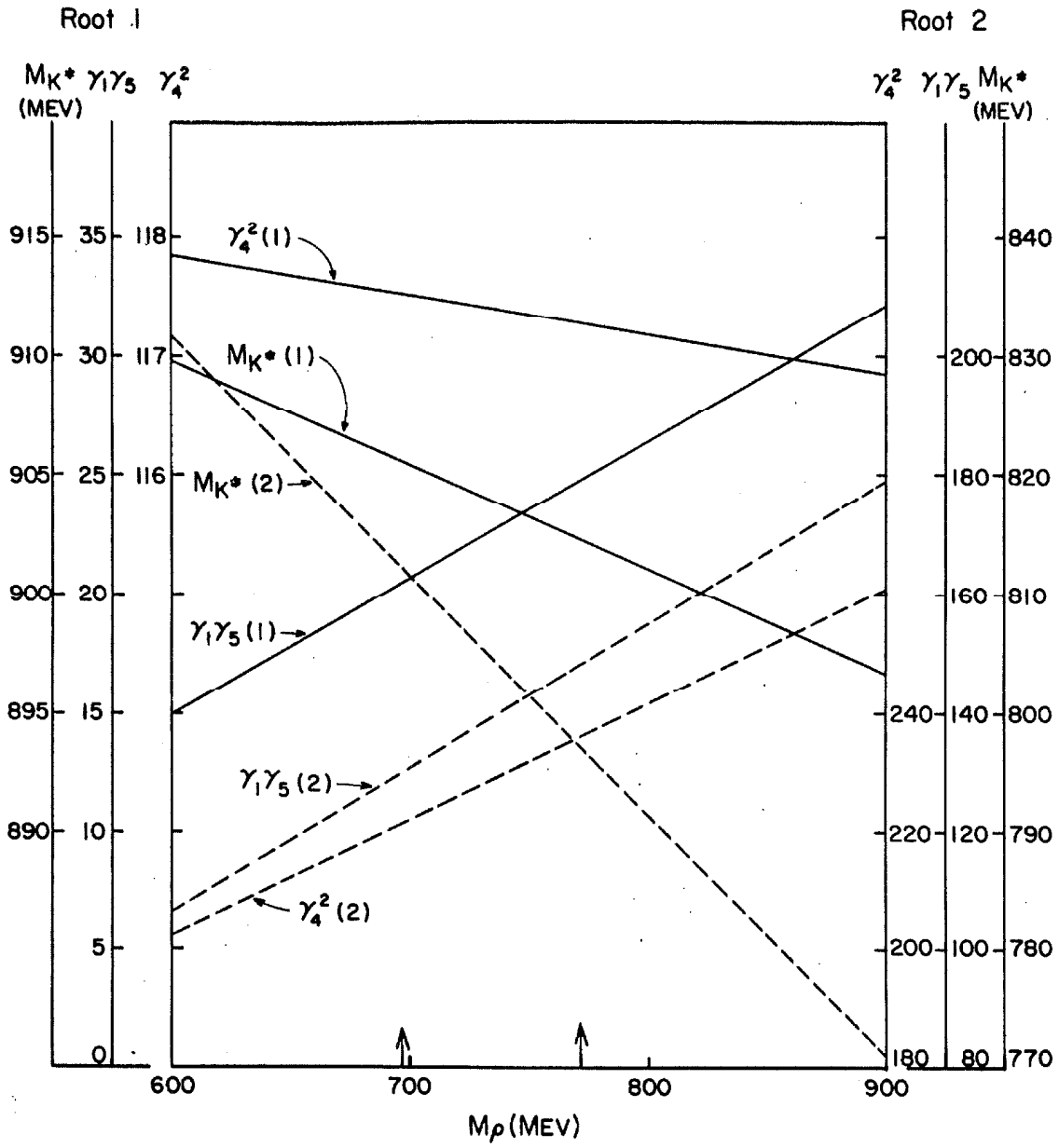

Fig. 8

Results of Phase I for the standard conditions

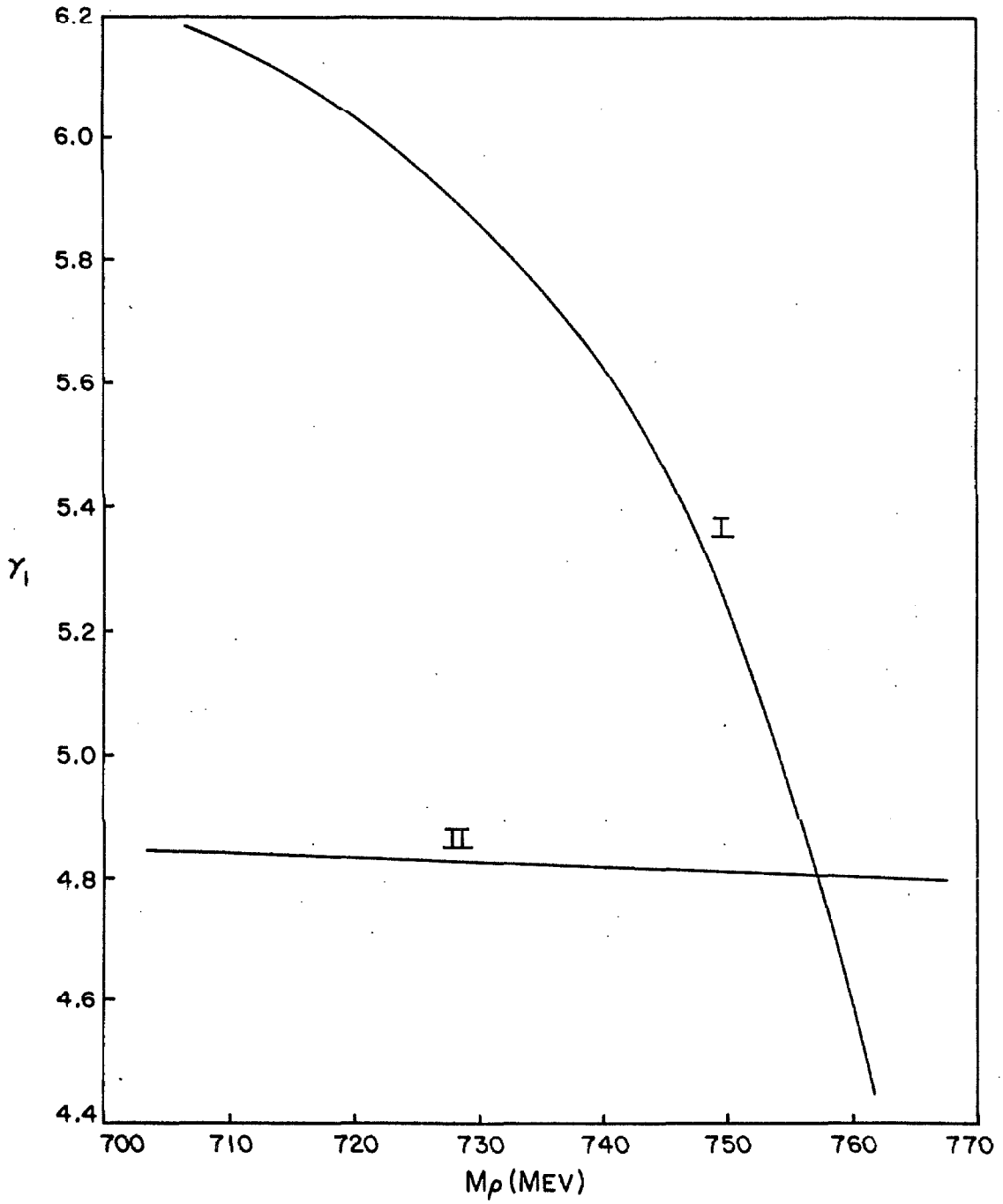

Fig. 9

Results of Phase II for the standard conditions

9. The two curves do in fact intersect and the self-consistent values predicted for the standard case are:

$$
\begin{aligned}
&M_\rho = 756 \text{ Mev} && \gamma^2_{\eta KK^*}/4\pi = 9.31 \\
&M_{K^*} = 903 \text{ Mev} && \gamma^2_{\rho KK}/4\pi = 1.95 \\
&\gamma^2_{\rho\pi\pi}/4\pi = 1.83 && \gamma^2_{\omega KK}/4\pi = 9.33 \\
&\gamma^2_{\rho\pi\omega}/4\pi = .15 \text{ (fed in)} && \gamma^2_{\omega KK^*}/4\pi = .716 \\
&\gamma^2_{\pi KK^*}/4\pi = 1.74 \text{ (fed in)} &&
\end{aligned}
\qquad (6.14)
$$

For the definition of coupling constants employed here, the experimental value of $\gamma^2_{\rho\pi\pi}/4\pi$ corresponding to a ρ width of 100 Mev is approximately 1.0. It is noted that there is appreciable breaking of SU_3 symmetry predicted. It is recalled that for exact SU_3 symmetry we would have

$$\gamma_{\rho\pi\pi} = \gamma_{\pi KK^*} = \gamma_{\rho KK} = \gamma_{\omega KK} = \gamma_{\eta KK^*}$$

and

$$\gamma_{\rho\pi\omega} = \gamma_{\omega KK^*}$$

The results obtained above show large deviations by $\gamma_{\omega KK}$ and $\gamma_{\eta KK^*}$, approximately 2.3 times the $\gamma_{\rho KK}$, $\gamma_{\pi KK^*}$, and $\gamma_{\rho\pi\pi}$ values (however see Appendix I). For the PS-V-V couplings $\gamma_{\omega KK^*}$ is approximately 2.2 times $\gamma_{\rho\pi\omega}$. However this is for a rather small $\gamma^2_{\rho\pi\omega}/4\pi$ value of .15, the self-consistent result of reference (9). If instead the value of $\gamma^2_{\rho\pi\omega}/4\pi$ of .35, obtained from the theory of reference (16) with a full width for the ω of 9 Mev, is used instead the solution (6.14) changes to

$$
\begin{aligned}
&M_\rho = 720 \text{ Mev} && \gamma^2_{\eta KK^*}/4\pi = 9.31 \\
&M_{K^*} = 905 \text{ Mev} && \gamma^2_{\rho KK}/4\pi = 2.81 \\
&\gamma^2_{\rho\pi\pi}/4\pi = 1.05 && \gamma^2_{\omega KK}/4\pi = 10.7 \\
&\gamma^2_{\rho\pi\omega}/4\pi = .35 \text{ (fed in)} && \gamma^2_{\omega KK^*}/4\pi = .576 \\
&\gamma^2_{\pi KK^*}/4\pi = 1.74 \text{ (fed in)} &&
\end{aligned}
\qquad (6.15)
$$

The PS-V-V splitting is now much smaller, and in addition the value of $\gamma^2_{\rho\pi\pi}/4\pi$ is equal to the experimental value! Thus if we had chosen to employ $\gamma_{\rho\pi\pi}$ and $\gamma_{\pi KK^*}$ as the input quantities instead of $\gamma_{\rho\pi\omega}$ and $\gamma_{\pi KK^*}$ we would have obtained the results of (6.15). It should be remarked that $\gamma_{\rho\pi\omega}$ and $\gamma_{\omega KK^*}$ are comparable only because the same scale factor (m_π) was used for each in the definition of the vertex function, equation (4.41). If we had used some other convention, for instance that of using the mass of the PS involved instead of m_π for all couplings the value of $\gamma_{\omega KK^*}$ would be increased by a factor $m_K/m_\pi = 3.57$.

If we translate the results of Capps' K* bootstrap into the notation used here (25) we find that when $\gamma_{\pi KK^*}$ assumes its experimental value Capps predicts

$$\gamma^2_{\eta KK^*}/4\pi = 8.68$$

$$\gamma_{\rho\pi\pi}\gamma_{\rho KK}/4\pi = 2.00$$

while this calculation gives

$$\gamma^2_{\eta KK^*}/4\pi = 9.31$$

$$\gamma_{\rho\pi\pi}\gamma_{\rho KK}/4\pi = 1.90$$

The results are very close considering the very different techniques. The main reason for this is no doubt because the self-consistent results found here are so close to the experimental values used by Capps (26).

We next investigate the effects of varying some of the paramaters of (6.13). Consider first the cutoff Λ_{22}. This is the only cutoff necessary, as all integrals except G_{22} converge. The standard choice is $\Lambda_{22} = 40$, in units of $4m_\pi^2$, or $\Lambda_{22} = (1750\ \text{Mev})^2$. Calculations were performed with $\Lambda_{22} = 200$ ($(3900\text{Mev})^2$) and $\Lambda_{22} = 1000$($(8750\ \text{Mev})^2$). The solutions agree in all respects within 2%, even though the value of the cutoff integral G_{22} changed by $\sim$570%. The results are sufficiently insensitive to the cutoff.

It is next asked how sensitive the results are to cutoffs on integrals other than G_{22}, that is, to the unknown high energy regions. To test this we place the same cutoff Λ on <u>all</u> integrals and inspect the results for Λ other than infinity. The effects are significant (Table I). The trend toward drastically increased coupling constants, especially $\gamma_{\eta KK^*}$, is evident. Physically this stems from the fact that the cutoff reduces the size of the forces and the coupling constants must be increased to make up the difference. The $\gamma_{\eta KK^*}$ has a particularly difficult time since it supplies most of the forces to bind the K* (in the region of interest the attractive ρ exchange outweighs the repulsive K* exchange in the π K, indicating that a single channel K* bootstrap is possible). The same trend toward larger couplings was found by Zachariasen and Zemach (9).

We next investigate the effects of placement of the subtraction points. Consider first the position $x_o(K^*)$ used in the K* bootstrap. The standard case used $x_o(K^*) = 0$. We now vary $x_o(K^*)$ between +3 and -3. The results of Phase I for such values of $x_o(K^*)$ are shown in figure 10 with M_ρ as a paramater (we show only the Root 1 solution). The effects are appreciable. As the subtraction point becomes more positive the coupling constants must become larger in order to allow D to move from +1 at the subtraction point to 0 at M_{K^*}, a shorter distance as $x_o(K^*)$ becomes more positive. This is compounded by the curious effect that M_{K^*} decreases with increasing $x_o(K^*)$, instead of increasing.

We next restore $x_o(K^*)$ to its standard value and allow $x_o(\rho)$ to vary from its standard value of -5.0 in the negative direction. The values of $\gamma^2_{\rho\pi\omega}/4\pi$ and $\gamma^2_{\pi KK^*}/4\pi$ are fixed, $\gamma^2_{\eta KK^*}/4\pi$ is very insensitive to M_ρ and therefore to $x_o(\rho)$, and therefore retains its standard value of 9.3. M_{K^*} is also insensitive to M_ρ and remains approximately 903 ± 3 Mev. In a manner similar to the K* case the various coupling constants

TABLE I.

	$\Lambda = 1000$ (8.75 Gev)^2	$\Lambda = 200$ (3.9 Gev)^2
M_ρ	760 Mev	768 Mev
M_{K^*}	903 Mev	902 Mev
$\gamma_1^2/4\pi$	2.23	3.2
$\gamma_2^2/4\pi$	.15 (fed in)	.15
$\gamma_3^2/4\pi$	1.74 (fed in)	1.74
$\gamma_4^2/4\pi$	10.7	15.4
$\gamma_5^2/4\pi$	1.71	1.81
$\gamma_6^2/4\pi$	10.6	13.9
$\gamma_7^2/4\pi$	1.02	1.68

Table I. Variation of the self-consistent solution with changing the cutoff

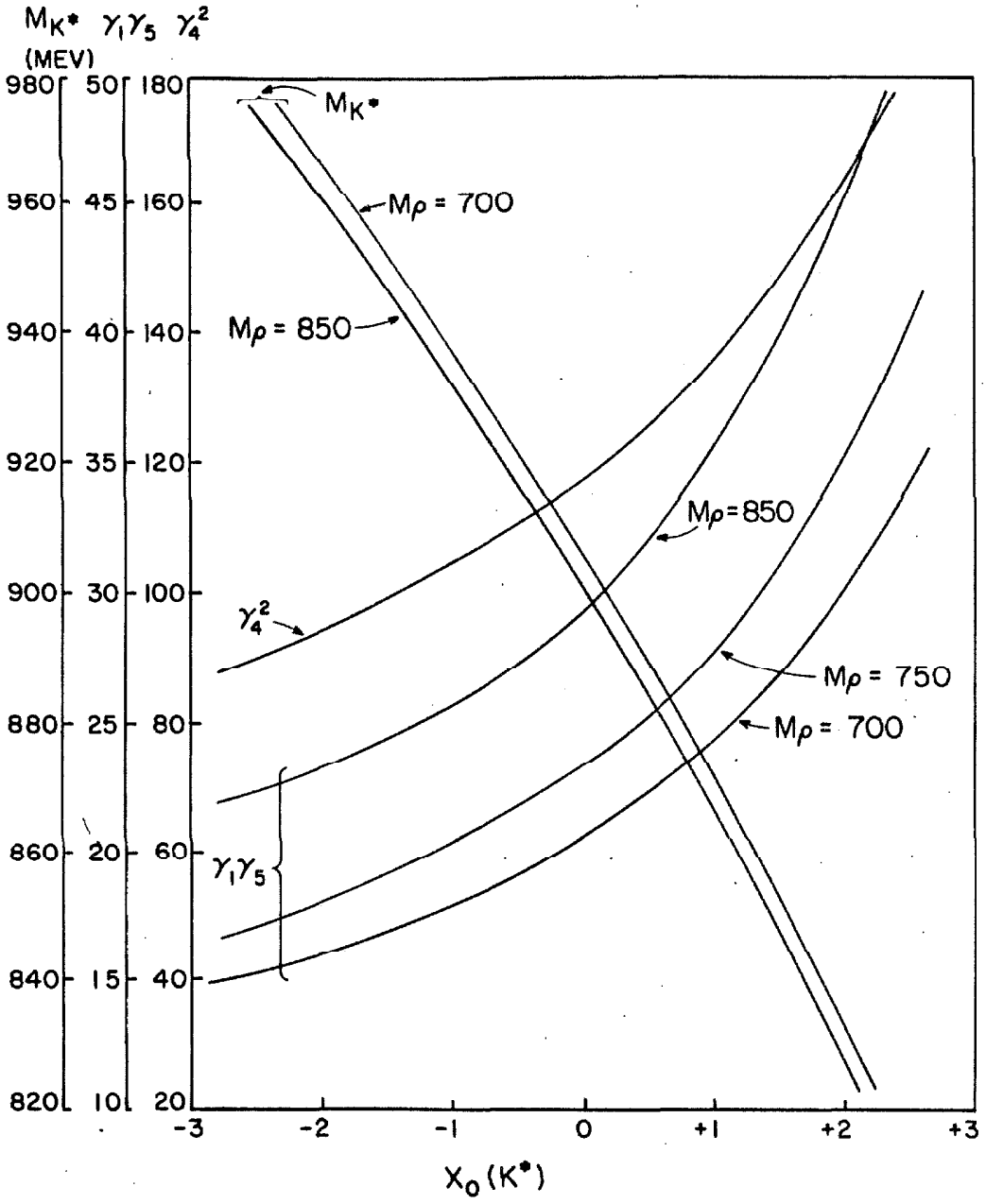

Fig. 10

Dependence of Phase I results on $x_o(K^*)$

decrease with decreasing $x_o(\rho)$. This is not true for $\gamma^2_{\rho KK}$ because even though decreasing $x_o(\rho)$ yields smaller values of M_ρ (as expected) and therefore smaller values of $\gamma_1\gamma_5 = \gamma_{\rho\pi\pi}\gamma_{\rho KK}$ (cf. figure 8), the self-consistent result for $\gamma_{\rho\pi\pi}$ decreases even faster. Thus as $x_o(\rho)$ moves to -10.0, the self-consistent paramaters become M_ρ = 750 Mev, $\gamma^2_{\rho\pi\pi}/4\pi$ = 1.21, $\gamma^2_{\rho KK}/4\pi$ = 2.9, $\gamma^2_{\omega KK}/4\pi$ = 7.6, $\gamma^2_{\omega KK^*}/4\pi$ = .59; for $x_o(\rho)$ = -20.0 the results are M_ρ = 735 Mev, $\gamma^2_{\rho\pi\pi}/4\pi$ = .7, $\gamma^2_{\rho KK}/4\pi$ = 4.6; $\gamma^2_{\omega KK}/4\pi$ = 6.2, $\gamma^2_{\omega KK^*}/4\pi$ = .4. These results are similar to those obtained by Zachariasen and Zemach (9).

The model described above has ignored the existence of the ϕ meson at 1020 Mev and the ramifications of ϕ -ω mixing (14, 15). We investigate the effects of this omission by the crude expedient of changing the input mass of the ω from 785 Mev to 930 Mev, the mass predicted by the Gell-Mann,Okubo mass formula (22,27). We find that the self-consistent value for M_ρ slides up toward the upper edge of the permissible window $(M^2_\omega - m^2_\pi)^{1/2}$; we obtain M_ρ = 900 Mev. The coupling constants are quite insensitive to the change; $\gamma^2_{\rho KK}/4\pi$ becomes approximately 3.6 while none of the remaining γ's changes from the values of (6.14) by more than 5%, and M_{K^*} drops slightly to 896 Mev.

We finally examine the matter of the unsymmetrical nature of the resulting solutions. As a measure of this we compute the ratio M_{ij}/M_{ji}, where we call the left-hand side of equation (5.21)b the matrix M. For the K* bootstrap under the standard conditions we find

$$M_{21}/M_{12} = 1.65$$

For the ρ bootstrap we have

$$M_{12}/M_{21} = 2.$$

$$M_{13}/M_{31} = 1.5$$

$$M_{32}/M_{23} = 2.5$$

For the two channel K* bootstrap the non-symmetry is typical

of that found in other two-channel bootstraps. That found in the three channel ρ bootstrap is somewhat greater; similar calculations do not exist, so comparison is impossible. The approximation method of reference (23) avoids this particular sickness.

In conclusion we remark that the agreement between $\gamma^2_{\rho\pi\pi}/4\pi$ and the ρ and K* masses predicted here and found by experiment is encouraging. Particularly gratifying is the fact that $\gamma_{\rho\pi\pi}$ is no longer impossibly large. The other predicted coupling constants, experimentally unknown and likely to remain so for the forseeable future, tend toward the SU_3 values with only minor splittings, with the notable exceptions of $\gamma_{\eta KK^*}$ and $\gamma_{\omega KK}$. The numerical values evidently cannot be taken seriously in view of the dependence on the various subtraction points and the intractable problem of the high energy behavior of the forcing terms. The former problem may be alleviated with improved approximation schemes; the latter will be with us for some indefinite (long) time. Yet despite these problems the striking fact remains that the bootstrap solutions continue to exist even in such complex situations as that treated here, and continue to give sensible answers.

APPENDIX I.

Here we investigate the modifications to the calculation required if we accept the existence of the ϕ meson and the complications of $\phi - \omega$ mixing. This influences only the ρ portion of the double bootstrap, since ω (and hence ϕ) do not enter into the two channel K* bootstrap. We therefore consider a four channel ρ bootstrap consisting of the channels $\pi\pi$, $\pi\omega$, $\pi\phi$, $K\bar{K}$ for channels 1,2,3,4 respectively. The numerator function has the form

$$N = \begin{pmatrix} F_1 & F_2 & \bar{F}_2 & F_3 \\ F_2 & F_4 & F_7 & F_5 \\ \bar{F}_2 & F_7 & \bar{F}_4 & \bar{F}_5 \\ F_3 & F_5 & \bar{F}_5 & F_6 \end{pmatrix} = \langle i \,|\, \hat{T}^L \,|\, j \rangle \qquad \text{(a.1)}$$

where $\bar{F}_i$ is the same as F_i with the replacements $M_\omega \to M_\phi$, $\gamma_{\omega AB} \to \gamma_{\phi AB}$. The function F_6 now includes terms involving ρ, ω, and ϕ exchange. We write

$$F_4 = \gamma^2_{\rho\pi\omega}\, f(M_\omega, M_\omega) \qquad \text{(a.2)}$$

with $f(M_A, M_B)$ the Born term (equation (4.90)) describing the u channel contribution to the π A $\to \pi$ B amplitude. Then

$$\bar{F}_4 = \gamma^2_{\rho\pi\phi}\, f(M_\phi, M_\phi) \qquad F_7 = \gamma_{\rho\pi\phi}\,\gamma_{\rho\pi\omega}\, f(M_\omega, M_\phi) \qquad \text{(a.3)}$$

We define two new orthogonal states,

$$|\pi\, A\rangle = a\,|\pi\omega\rangle + b\,|\pi\phi\rangle \qquad |\pi\, B\rangle = c\,|\pi\omega\rangle + d\,|\pi\phi\rangle \qquad \text{(a.4)}$$

$$\langle \pi\, A \,|\, \pi\, B\rangle = 0 = ac + bd; \quad a^2 + b^2 = 1 = c^2 + d^2$$

These are three conditions on a,b,c,d. We shall impose a fourth condition, that $\langle \pi\, A \,|\, \hat{T}^L \,|\, \pi\, A\rangle = 0$. This takes the form

$$a^2 F_4 + b^2 \bar{F}_4 + 2ab\, F_7 = 0 \qquad \text{(a.5)}$$

We define α by

$$\alpha^2 = \bar{F}_4/F_4 = \frac{\gamma^2_{\rho\pi\phi}}{\gamma^2_{\rho\pi\omega}} \frac{f(M_\phi, M_\phi)}{f(M_\omega, M_\omega)} \qquad (a.6)$$

The function $f(M_A, M_B)$ does not strongly depend on M_A and M_B. It is a fair approximation ($\sim$15%) to write

$$f^2(M_\phi, M_\omega) = f(M_\omega, M_\omega)\, f(M_\phi, M_\phi) \qquad (a.7)$$

Then

$$F_7 = \alpha F_4. \qquad (a.8)$$

With these approximations (a.5) may be solved to give

$$a = \alpha (\alpha^2 + 1)^{-1/2} \qquad b = -(\alpha^2 + 1)^{-1/2}$$

and (a.4) then implies (a.9)

$$c = (\alpha^2 + 1)^{-1/2} \qquad d = \alpha (\alpha^2 + 1)^{-1/2}$$

The masses of A and B are

$$M_A = (\alpha^2 M_\omega + M_\phi)(\alpha^2 + 1)^{-1/2}$$
$$M_B = (M_\omega + \alpha^2 M_\phi)(\alpha^2 + 1)^{-1/2} \qquad (a.10)$$

Note that $\langle \pi\, A | \hat{T}^L | \pi\, B \rangle = 0$ automatically.

For the remaining channels we have

$$\langle \pi\,\pi\, | \hat{T}^L |_\pi\, A \rangle = \frac{\alpha F_2 - \bar{F}_2}{\alpha^2 + 1}$$
$$\langle K\, \bar{K}\, | \hat{T}^L |_\pi\, A \rangle = \frac{\alpha F_5 - \bar{F}_5}{\alpha^2 + 1} \qquad (a.11)$$

$$\begin{aligned} F_2 &= \gamma_{\rho\pi\pi}\gamma_{\rho\pi\omega}\, g(m_\pi, m_\pi, m_\pi, M_\omega) \\ \bar{F}_2 &= \gamma_{\rho\pi\pi}\gamma_{\rho\pi\phi}\, g(m_\pi, m_\pi, m_\pi, M_\phi) \\ F_5 &= \gamma_{\pi K K^*}\gamma_{\omega K K^*}\, g(m_K, m_K, m_\pi, M_\omega) \\ \bar{F}_5 &= \gamma_{\pi K K^*}\gamma_{\phi K K^*}\, g(m_K, m_K, m_\pi, M_\phi) \end{aligned} \qquad (a.12)$$

where $g(m_A, m_B, m_C, M_D)$ is the t or u Born term force for the process A + B → C + D. Again the mass dependence of g is weak

and we can make a pretty fair approximation that

$$g(m_\pi, m_\pi, m_\pi, M_\omega)\, f(M_\phi, M_\phi) = g(m_\pi, m_\pi, m_\pi, M_\phi)\, f(M_\omega, M_\omega) \qquad \text{....(a.13)}$$

so that $\bar{F}_2 = \alpha F_2$ and $\langle \pi\pi | \hat{T}^L | \pi A \rangle = 0$. For the $K\bar{K}$ channel a similar relation holds and we find

$$\bar{F}_5 = \frac{\gamma_{\phi K K^*}}{\gamma_{\omega K K^*}} \frac{\gamma_{\rho\pi\omega}}{\gamma_{\phi\pi\rho}} \alpha F_5 \qquad \text{(a.14)}$$

About $\gamma_{\phi KK^*}$ and $\gamma_{\omega KK^*}$ we know nothing. We therefore make the obvious assumption that SU_3 works and obtain $\bar{F}_5 = \alpha F_5$. Then $\langle K \bar{K} | \hat{T}^L | \pi A \rangle$ also vanishes.

With the approximations made above the πA channel decouples completely and we are left with a three channel $\pi\pi$, πB, $K\bar{K}$ problem, the mass of the B being given by (a.10).

To find the value of α we write

$$\begin{aligned} |\phi\rangle &= \cos\theta |\omega_8\rangle - \sin\theta |\omega_1\rangle \\ |\omega\rangle &= \sin\theta |\omega_8\rangle + \cos\theta |\omega_1\rangle \end{aligned} \qquad \text{(a.15)}$$

where ω_1 is the SU_3 unitary singlet and ω_8 is the I = Y = 0 member of the SU_3 unitary octet. If a and b are the singlet and octet (D - type)(22) coupling constants, respectively, then

$$\frac{\gamma_{\rho\pi\phi}}{\gamma_{\rho\pi\omega}} = \frac{b\cos\theta - a\sin\theta}{a\cos\theta + b\sin\theta} \qquad \text{(a.16)}$$

The quantities θ and a/b have been variously evaluated by many authors (15-19). For any of these solutions $\gamma_{\rho\pi\phi}/\gamma_{\rho\pi\omega} \lesssim .15$. The ratio $f(M_\phi, M_\phi)/f(M_\omega, M_\omega) < 2$ for all values of the energy. Therefore $\alpha^2 < .04$ and the state B is almost pure ω. The effects of the $\pi\phi$ channel are seen to be quite small.

More important is the effect of ϕ exchange in the $K\bar{K} \to K\bar{K}$ amplitude. However, since the form of the Born terms for the ϕ and ω are so similar, to a very good approximation we can lump the ϕ and ω exchange terms together with an effective

coupling constant $(\gamma^2_{\omega KK} + \gamma^2_{\phi KK})$. All calculations go through as before; it is only necessary to interpret the $\gamma^2_{\omega KK}$ found in the bootstrap as actually being $\gamma^2_{\omega KK} + \gamma^2_{\phi KK}$. It is interesting that if $\gamma_{\phi KK} \sim \gamma_{\omega KK}$ this predicts a value of

$$\gamma^2_{\omega KK}/4\pi = 2.3$$

which is much closer to the SU_3 solution that the solution of (6.14).

APPENDIX II.

The calculations performed in this paper have employed the first determinental, or first iterative, solutions to the coupled integral equations for N and D, equations (5.6). This approximation suffers from a variety of diseases (non-symmetrical solutions, dependence on subtraction points, the possibility of complex couplings, etc.). It is therefore of considerable interest to compare the results obtained using the approximation with the results deriving from the exact solution of equations (5.6). Fulco, Shaw, and Wong (12) (FSW) have performed calculations for the ρ resonance using the full integral equation solution. They do not perform a bootstrap, but rather simply calculate the cross section for $\pi\pi \to \pi\pi$ in the one, two, and three channel cases with Born term forces using experimental values for the input paramaters—the experimentally undetermined couplings are taken from the predictions of SU_3 symmetry.

We perform a similar calculation using the same input forces, but calculating the output using the first determinental approximation. Following FSW we place a cutoff on all integrals and adjust it until we obtain the ρ resonance at 760 Mev. We then examine the resulting width of the resonance. Since this is not a self-consistent calculation the calculated width is quite different from that assumed for the input. The results of FSW are compared with the first determinental type solutions in figure 11, for the one channel ($\pi\pi$) and two channel ($\pi\pi, \pi\omega$) cases. It must be remembered that the first determinental solutions involve a second paramater besides the cutoff, namely the subtraction point. The results are not independent of the choice for $x_0(\rho)$. The curves in figure 11 employ $x_0(\rho) = -5$. For $x_0(\rho)$ between -2. and -10. the width varies between approximately 225 Mev and 300 Mev, respectively. This is about half the width found by FSW.

For the single channel case it is found to be impossible to

force the ρ mass to be 760 Mev, for any values of $x_0(\rho)$ and the cutoff, using the approximation. This is different from the results of FSW—apparently the full integral equation solutions effectively increase the single channel forces.

To check whether the narrow width found here was due to the larger cutoff used ($\sim 65\ m_\pi$ vs. $\sim 37\ m_\pi$ for FSW) we performed the calculation using the cutoff of $37\ m_\pi$. This result is also plotted in figure 11. The resonance centers at ~840 Mev with a full width of ~320 Mev; the width has increased, but is still below the results of FSW.

Finally we again follow FSW and plot the results of the single channel calculation ($\pi\ \pi$) using the cutoff found for the two channel ($\pi\ \pi, \pi\omega$) problem. This gives an indication of the relative importance of the two channels (figure 11). It appears that the $\pi\ \pi$ channel is practically irrelevant to the shape of the cross section—although the $\pi\ \pi$ channel supplies all of the attraction to bind the ρ, at the position of the resonance it contributes something like 2% of the total cross section. This is similar to the findings of FSW.

The two sets of solutions, that of FSW and that developed here, are undeniably different, but not unrecognizably so. The essential features of both solutions are the same and the details differ only by factors like two. In view of the fact that no numbers deriving from present bootstraps can be trusted to within such factors anyway, the use of the first determinental approximation appears permissible. With future improvements in the presently intractable portions of the theory (e.g. finding the correct forcing terms) this luxury may be no longer justified, but for the present its use is invaluable. In complex problems such as the bootstrap described here it is imperative to find some way to eliminate some of the paramaters algebraically—otherwise the computor bill generated while finding a self-consistent solution involving, say, seven paramaters as is the case here, is likely to exceed any preassigned positive number.

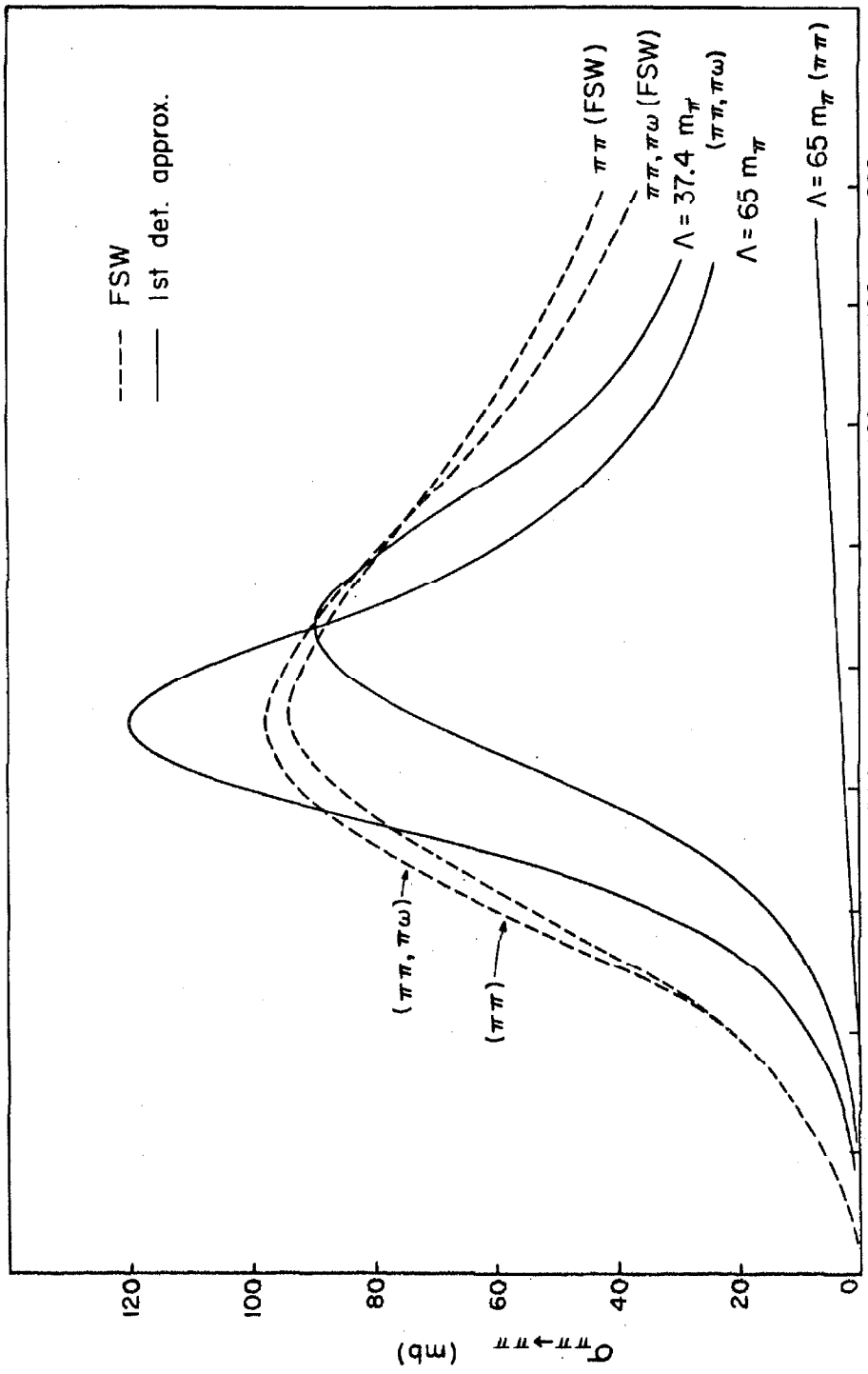

Fig. 11

A Comparison of first determinental approximation solutions with exact integral equation solutions.

APPENDIX III. INTRODUCTION TO THE BOOTSTRAP IDEA

You boil it in sawdust: You salt it in glue
You condense it with locusts and tape
Still keeping one principle object in view—
To preserve its symmetrical shape.
Fit the Fifth, The Hunting of the Snark
Lewis Carroll.

This appendix will attempt to impart at least a cursory knowledge of the bootstrap hypothesis and philosophy to the non-afficianado of the field. The bootstrap idea is an outstanding example of the tendency toward generalizing a successful idea, or at least an idea not yet unsuccessful, as far as possible until the breaking point is reached. The bootstrap began life as a comment concerning a specific reaction, was generalized to include all strongly interacting particles, was generalized further by enlarging its sphere of relevance from just masses and coupling constants to include the internal symmetries of particles, such as isotopic spin, parity, SU_3 symmetry, it is currently concerning itself with electromagnetic and weak interactions, and the end is not yet in sight. The avid bootstrapper of the present may claim that the bootstrap theory, or perhaps the theory which is presumed to be developing from the bootstrap idea, explains why the strongly interacting particles which exist do exist, why they have the quantum numbers that they do, and why particles which do not exist really couldn't exist. If he is very enthusiastic he may claim that in addition it explains how weak interactions and electromagnetic effects can develop, why we live in a space of 3 + 1 dimensions, and why this is the best of all possible worlds. At present none of these claims may be demonstrated—neither may they be refuted. At the present state of the art only such crude calculations as may be carried out give suggestive inferences that there may be something to at

least part of these ideas. The present theoretical situation is so bad that any technique which attains even a very rough success must be treated with a certain amount of deference. And when the same technique obtains similar successes, although individually tiny, in a rapidly growing number of cases without a single failure or contradiction the support for the idea must grow. A theory is deemed successful, not when it has passed all possible tests, but when it has passed some large and impressive number and has failed none. The bootstrap hypothesis thus far has survived a number of tests and to date no outright failure has been reported.

The bootstrap hypothesis is basically a dynamical theory; in its present formulation it is firmly united to dispersion theory, or the S-matrix theory of strong interactions, which in turn is based on principles abstracted from field theory. It was soon realized that physics basically consists of scattering experiments; accordingly great emphasis was placed on formal scattering theory. The difficulties which arose in these calculations were all traceable to relativistic effects. The equation $E = mc^2$ is justly celebrated, but the effects it describes have virtually stopped theoretical particle physics for the past 30 years. The equivalence of mass and energy can be a great boon to the experimentalist who wishes to investigate some particle—he need only buy a sufficiently large accelerator and make his own particles to look at (there are some who might describe this account as a mild oversimplification). To the theorist wanting to describe some reaction the effect is more insidious. The theory instructs him to consider all energies, but as the energy increases more particles may be produced, and their effects must also be considered. The result is that any given reaction is perforce coupled with an infinitude of others. This fact effectively eliminates any possibility of performing an exact calculation, and in practice the infinite chain must be broken somewhere and partial systems considered. Both field

theory and dispersion theory break the chain in more or less the same fashion and consider generally the same sets of coupled reactions. They differ chiefly in the manner in which they arrive at physical predictions which may be compared with experiment. However the different steps are apparantly non-critical, for both procedures seem invariably to arrive at the same results. The bootstrap hypothesis may be the first means of differentiating the two theories. The bootstrap idea is a natural outgrowth of dispersion theory over the past several years, and is heavily couched in its language. The field-theorist-turned-bootstrapper could possibly arrive at the same results as the S-matricist, but the distortions to the original concepts of field theory would probably make the result unrecognizable. The field theorist claims that if we postulate a certain number of elementary fields *which interact through couplings of various, given, strengths,* then in principle all the rest of physics follows. The bootstrapper supposes that nothing is given and still all the rest of physics follows. The fashion in which this comes about is discussed below, and we now turn our attention to the S-matrix.

With the recognition that physics may be explained almost entirely in terms of scattering reactions between various coupled states, Heisenberg (28) introduced the matrix $\hat{S}$ which transforms the initial set of states into the final set. Möller (29) later defined the operator $\hat{S} - \hat{1}$, and factored out an energy-momentum delta function from all of its matrix elements, leaving a matrix $\hat{T}$. Knowledge of $\hat{T}$ or $\hat{S}$ is sufficient to describe the cross section or any other feature of a given reaction. Certain general properties of $\hat{S}$ and $\hat{T}$ could be imposed. First, of course, the theory must be Lorentz-invariant. Second, the S matrix is unitary, $\hat{S}^{+}\hat{S} = \hat{1} = \hat{S}\,\hat{S}^{+}$. This condition leads to singularities of S in some of the momentum variables. From field theoretic calculations it was found for all of the processes investigated that S was analytic except for these singularities, and no singularities were found due to any other mechanism than

the unitarity condition. At times the connection is not obvious, for some of the singularities may be due to the existence of other singularities which arise directly from unitarity. The connection was made very early of an association of the forces acting between particles with the singularities of the S-matrix. In field theory, the forces between particles arise from the exchange of various objects between the particles. These exchange reactions produce singularities in the S-matrix, the position and strength of which are functions of the exchanged particle's quantum numbers. But then these singularities were found to be exactly consistent with the unitarity condition. The S-matricist adroitly reverses this entire outline. Here the unitarity condition is postulated as basic, this implies singularities, which may then be associated with particles whose quantum numbers are determined by the features of the singularity. The original hope of the S-matricist was this: given certain fundamental singularities and the requirements of unitarity and analyticity of the S-matrix, the existence of other singularities is implied, which may then be interpreted as manifestations of other particles whose characteristics are completely determined by the features of the associated singularity. That is, the consequences of the unitarity and analyaticity conditions are so difficult to determine that the rather brilliant suggestion was made that the conditions determine practically all of physics. A great deal of work in this direction has been done in the past several years.

Only rather recently, however, has a splendid new fillip been added, the pièce de resistance of the whole affair. This consists in the assertion that the singularities produced from the action of the unitarity and analyticity conditions with the original given singularities may themselves in various other reactions imply, directly or indirectly, the existence of the very singularities which were postulated at the outset of the

computation. This is the bootstrap idea, which closes the calculational circle. For physical understanding, replace the word "singularity" in all of the above, by the words "mass" or "particle". That is, because the S-matrix is a Lorentz-invariant analytic function of all momentum variables with only those singularities required by unitarity, particle A may help produce particle B, which helps produce C, which is involved in the formation of D,..., while some of B,C,D,... are instrumental in the formation of A. It is the closing of this circle which is the heart of the bootstrap idea.

Consider an example of how such a situation may come about. In order to explain the electromagnetic structure of the nucleon Frazer and Fulco (30) postulated the existence of a meson which should be a resonance in the $\pi\pi$ system with I = 1, J = 1. Such a meson was eventually found (31) and was called ρ. From the theoretical side attacks were made to discover the source of the forces which bound the two pions together to form ρ. After a number of failures Chew and Mandlestam (1) hit upon a successful idea. Forces are supposed to arise from particle exchange—the exchanges lead to singularities in the S-matrix; the residues or discontinuities of these singularities depend both on the quantum numbers of the exchanged particle and on the quantum numbers of the state being formed. For example, two pions can be in a state with I = 0,1,2, and J = 0,1,2,3,... Both the size and sign of the singularity produced by the exchange of a particle of given quantum numbers varies with the choice of I and J. The size of the singulairty refers to the magnitude of the force, the sign determines whether it is attractive or repulsive. For an exchanged particle with I = 1,J = 1, the resulting force is large and attractive in the I = 1, J = 1 state of two pions. Read differently, a large attractive force in the I = 1, J = 1 system of two pions, possibly large enough to make a resonance possible, arises from the exchange of a dipion resonance with I = 1, J = 1.

Thus it was natural to suggest a bootstrap mechanism for the ρ, in which a large portion of the forces necessary to bind two pions together to make a ρ comes from ρ exchange. In the first bootstrap calculation (9) the ρ was given a certain mass and coupling strength to two pions, the force due to ρ exchange was calculated in some approximation, and a dipion resonance in the $I = 1$, $J = 1$ channel was found corresponding to a certain mass and coupling constant. It was then demanded that the output values of the mass and coupling constant be identical with those originally postulated. A solution to this self-consistency problem was found to exist. All such calculations are necessarily abominably crude, but the resulting predictions are not unrecognizably different from the experimental values.

The next step was the extension of the bootstrap hypothesis to include all strongly interacting particles (2). In the ρ bootstrap outlined above, the ρ properly bootstraps itself, but the pions were treated as fundamental elementary particles. It is now proposed that we could calculate the properties of pions in some other bootstrap—this would involve many particles, including ρ. These other particles would also be bootstrapped in terms of still other particles. Eventually one reaches a closed system in which all of the particles are determined in terms of all of the others. Unfortunately it appears that this complete, closed set consists of the entire universe. It is obviously necessary to break the logical chain somewhere and do partial calculations, taking some of the input values from experiment or other sources. An example of a bootstrap of greater complexity that is more nearly closed is the reciprical bootstrap of the nucleon N and the famous π N (3,3) resonance, N*. Here the exchange forces are such that N exchange dominates the forces producing the N*, while N* exchange dominates the forces making N. Here again the pion features are fed in at the beginning and are not determined in the actual computation. In addition there

are many more forces which have not been included, and inelastic effects are very important.

This last effect, inelasticity, is another consequence of relativity. The theory instructs us to consider the reaction $\pi\pi \to K\bar{K}$ whenever we talk about the $\pi\pi \to \pi\pi$ reaction (or the $K\bar{K} \to K\bar{K}$ reaction). Similarly $\pi\pi \to N\bar{N}$, $\pi\pi \to \bar{N}N^*$, $\pi\pi \to \pi\omega$, etc., all should be included. In practice approximations may be made by ignoring the more massive states—thus $\pi\pi \to N\bar{N}$ is presumed to be less important that $\pi\pi \to K\bar{K}$ because the greater mass of $N\bar{N}$ over $K\bar{K}$ violates energy conservation, quantum-mechanically-wise, to a greater extent. In a similar fashion exchange forces from massive objects count less than those from lighter objects because the latter forces have longer ranges—the range of a force is inversely proportional to the mass of the exchanged particle. These features permit a systematic approximation technique. One includes first the least massive particles and in order to improve a calculation one later agrees to include heavier and heavier systems. This is not a drawback peculair to bootstraps, it is the barrier found by quantum-mechanicians thirty years ago. The problems are enormously complicated, and calculations are crude at best. This is why competing theories cannot be readily evaluated, *for almost nothing may be calculated with* anything.

It is a firm principle, however, that better results should follow from more complete calculations. In addition, outright failure on a theory in a given case is sufficient grounds for assassination. These points are the justification for continued computations in new and more complex situations. Thus bootstrap solutions should improve as more of the universe is considered; the particular relevance of the calculation performed in this thesis is the test of the bootstrap calculation in a case considerably more complex than any hitherto treated. Further, the bootstrap must not predict the existence of particles which in

fact do not exist. Ideally the bootstrap solution to the analytic S-matrix should be unique and an exact image of the real world. So far the idea has been successful.

Consider now the particular bootstrap performed in this thesis. We assume the validity of S-matrix theory, and assume that all singularities of the S-matrix arise from the unitarity condition. In common with any problem of strong interaction physics we must limit the universe that we investigate. Basically we are attempting to improve the ρ bootstrap by including more inelastic effects. For reasons set down rather more fully in chapter I we agree to consider the $\pi\pi$, $\pi\omega$, and $K\bar{K}$ states simultaneously. Each of these states couples to the others and the ρ is found as an $I = 1$, $J = 1$ resonance in each. The forces between the particles are presumed to arise solely from exchanges of single particles—this may not be too bad an approximation, for the single particles which we exchange are without exception resonances in the multi-particle systems which could be exchanged, and therefore presumably include the dominant effects. This approximation is the standard one for any bootstrap calculation, and for virtually all of the remaining dynamical calculations, field theory or dispersion theory. The forces calculated in this fashion are called Born forces. We do not have enough equations to determine all of the features of all of the particles which are involved, so we alleviate this drawback by two methods. First we simultaneously bootstrap the K* meson. For reasons outlined in the first chapter this offers two further equations. The K* bootstrap consists of just two channels, π K and η K . Even after this ploy some quantities are left undetermined. The complete calculation involves thirteen numbers, while we find just seven equations. We therefore take six of the numbers from experiment.

In the course of the calculation we have need to solve a non-linear integral equation. This we solve by the ND^{-1}

technique (17). This already confesses an inadequacy of handling states involving more than two particles. What effects this omission, common to any calculation in strong interaction physics without exception may have, are by all practitioners of the art devoutly hoped to be small. No one knows if this is true. Even admitting the validity of this particular approximation, we further abuse the problem by making an approximate solution to the integral equations, called the first determinental approximation (18). This device, or something like it, is mandatory for a problem of this complexity as the computation would otherwise be hopelessly beyond the abilities of a modern computor to perform, or a physics department to pay for. The price paid for this ability to complete the solution is judged in Appendix II to be acceptably small.

After all of these approximations any results obtained may justly be regarded with a healthy degree of scepticism. Yet striking agreement with experimental results can be reached, much better than can possibly be expected in view of the paucity of accurate procedures. This has been taken optimistically as evidence that there is really something to the entire scheme. The whole difficulty, of course, is that the bootstrap is a dynamical theory, and as yet no one has any idea how to do dynamical calculations with any degree of validity. There is at present a large and growing number of theorists who have nothing at all to do with dynamics, they revere instead the noble visage of Symmetry (large S is henceforth abandoned.) We consider this matter further.

The study of symmetries has been most profitable in the history of physics. As an aid to calculation and as a means of classification it has found great utility. There are some examples which are understood in terms of general principles. That physics is independent of translations in space-time is reflected in the conservation of total four-momentum. Invariance with respect to rotations in three-space leads to angular

momentum conservation. Other symmetries, nonuniversal, are also famous. We include here isotopic (or rather isobaric) spin, and more recently the broken but recognizable symmetry of the SU_3 group (unitary symmetry). A great deal of physics may be deduced from such symmetries—new elements were predicted from Mendelyeev's periodic table, new particles and branching ratios can be obtained from the consequences of isotopic spin conservation, while the prediction of the strangeness -2 particle Ω^- is a great triumph of unitary symmetry. Higher and higher symmetries, more and more broken, are continually being suggested and evaluated.

From the point of view of field theory these symmetries are given, non-calculable facts. This has also been the opinion of S-matricists up to the past few years. But in a completely bootstrapped universe in which all particles are assumed to be completely determined by self-consistency equations, there should not be any arbitrary freedoms. In a world where masses are not arbitrary, given quantities it appears to make little sense to impose a priori that the masses of several different particles should be equal, even if what it is that they are equal to is calculable self-consistently. No one has demonstrated that such internal symmetries as isospin and unitary symmetry really do follow from the bootstrap conditions, but several attempts have at least made the idea plausible. Parity, too, has been investigated from the bootstrap point of view. In sum, we are again left with the usual situation in which no one can prove that the idea works, but it is not implausible, and it has certainly not been shown to be false. No matter how successful and productive a symmetry scheme may be it is most desirous that a satisfactory basis for the existence of that symmetry be found.

The bootstrap hypothesis has now been subjected to a variety of tests of more and more generality. The idea that all strongly interacting particles should have their origin in a mechanism

similar to that producing the ρ is exceptionally appealing in view of the proliferation of newly discovered (or predicted) particles. The matter of deciding which of these are elementary and which are not has become so difficult that a most reasonable and welcome suggestion is that none of them are; all are on an equivalent footing. It is possible and plausible that internal symmetries may be completely determined by the bootstrap conditions. An indication that electromagnetic and weak interaction effects are not beyond the range of the bootstrap has been made by Dashen and Frautschi (32). That everything, or even that most of physics should follow from such esoteric and non-intuitive laws as form the basis for S-matrix theory, analyticity and unitarity, may be philosophically displeasing. There are, however, many alternatives. The easiest and most obvious escape is to say that the bootstrap idea is wrong, or simply an identity, and physics really doesn't follow from the bootstrap idea. But if the bootstrap idea is thought to be correct, one can still say that there are really different principles for its foundation or, alternately, that the principles of S-matrix theory are based in turn on some deeper physical principles. If none of these attempts works it is still always possible to note that the principle of least action isn't really very intuitive either, yet people have accepted it as a perfectly good foundation for a healthy portion of physics. If people did philosophise to Hamilton they're forgotten now. Moral: if a principle works, it's a good one.

REFERENCES

1. G. F. Chew and S. Mandlestam, Phys. Rev. 119,467 (1960), and Nuovo Cimento 19,752 (1961)

2. G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 7, 394 (1961)

3. L. A. P. Balazs, Phys. Rev. 132,867 (1963)

4. The experimental situation of the ρ meson is described by G. Button, G. R. Kalbfleisch, G. R. Lynch, B.C. Maglic', A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. 126, 1858 (1962)

5. There are far too many calculations of the ρ bootstrap to list. We shall cite below only those about which we make specific comments.

6. The experimental status of the ω meson is reviewed by N. Gelfand, D. Miller, M. Nussbaum, J. Ratau, J. Schultz, et. al., Phys. Rev. Letters 11,436 (1963)

7. The η meson is described by M. Chretien, F. Bulos, H. R. Crouch, Jr., R. E. Lanou, Jr., J. T. Massimo, et. al., Phys. Rev. Letters 9, 127 (1962)

8. The K* is described by M. H. Alston et al., Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN (CERN, Geneva, 1962),pp 291-294; also R. Armenteros et al., ibid.,pp 295-297.

9. F. Zachariasen and C. Zemach, Phys. Rev. 128,849 (1962)

10. We use m to denote a pseudoscalar meson mass, M to denote that of a vector meson, and γ_{ijk} to denote the coupling constant between the i,j, and k particles.

11. B. Dui, J. L. Gervais, and H. R. Rubenstein, Nuovo Cimento 31,341 (1963)

12. Since the completion of this work a calculation by Fulco, Shaw, and Wong has indicated the importance of the $\pi\omega$ channel relative to the $K\bar{K}$ channel. J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. (to be published). See also appendix II.

13. Richard H. Capps, Phys. Rev. 131,1307 (1963)

14. S. Okubo, Physics Letters 5,165 (1963)
J. J. Sakurai, Phys. Rev. 132,434 (1963)
S. L. Glashow, Phys. Rev. Letters 11,48 (1963)
Roger F. Dashen and David H. Sharp, Phys. Rev. 133,B1585 (1964)

15. Y. S. Kim, S. Oneda, and J. C. Pati, Phys. Rev. 135,B1076 (1964)

16. M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters 8,261 (1962)

17. J. D. Bjørken, Phys. Rev. Letters 4,473 (1960)

18. M. Baker, Ann. Phys. 4,271 (1958). This is the same as the first iterative solution, and is described in more detail in chapter V.

19. We use units such that $\hbar = c = 1$.

20. W. Jacob and G. C. Wick, Ann. Phys. 7,404 (1959)

21. D. I. Olive, Nuovo Cimento 26,73 (1962)

22. M. Gell-Mann, California Institute of Technology Syncyrotron Report CTSL-20 (unpublished). The couplings employed in this work refer to couplings between states of definite isotopic spin. For example the $\gamma_{\rho\pi\pi}$ used here is $\sqrt{2}$ times that defined by the effective Hamiltonian of reference 9, $2\gamma_{\rho\pi\pi}\ i\epsilon_{\alpha\beta i}\ (p + q)$.

23. The approximation scheme of A. W. Martin, Phys. Rev. 135, B967 (1964) might be used here with advantage as its solutions are both symmetrical and independent of the subtration point.

24. When two numbers are presented the first refers to the t channel, the second to the u channel. Single numbers refer to either the t or the u channel, whichever is appropriate.

25. Capps' couplings are related to those used here by

$$2^{\frac{1}{2}}\gamma_{\rho\pi\pi}\,\gamma_{\rho\,KK}(\text{Capps}) = \frac{1}{2}\gamma_{\rho\pi\pi}\,\gamma_{\rho KK}(\text{here})$$

$$\gamma^2_{\pi KK^*}(\text{Capps}) = \gamma^2_{\eta\,KK^*}(\text{Capps}) = \frac{3}{8}\gamma^2_{\pi_{KK^*}}\ (\text{here}) = \frac{3}{8}\gamma^2_{\eta KK^*}(\text{here})$$

26. It may also be noted that different subtraction points were taken (Capps uses $x_o(K^*) = +.35$ instead of $x_o(K^*) = 0.$ used here). See figure 10. In addition Capps approximates the various Born terms in a different manner than that done here.

27. S. Okubo, Progr. Theoret. Phys. (Kyoto) 27,949 (1962)

28. W. Heisenberg, Z. Physik, 120, 513,673 (1943)

29. C. Möller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd., 23, 1 (1945)

30. W. Frazer and J. Fulco, Phys. Rev. 117, 1609 (1960)

31. Reference 4 and the various references cited therein

32. Roger F. Dashen and Steven C. Frautschi, Phys. Rev. (to be published).