An Investigation of Spontaneous Lorentz Violation and Cosmic Inflation

Thesis by

Heywood Tam

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

2010

(Defended May 3, 2010)

Acknowledgements

First, I would like to thank my advisor Sean Carroll. I must admit that I have been extremely fortunate to have an advisor who is so generous, patient, and resourceful. It has always been amazing to me how he can just put aside his work and patiently answer questions and offer advice whenever I show up at his door in search of sage counsel. I thoroughly enjoyed, and have profited much from, the numerous exciting conversations on physics that we have had over the years. His support, guidance, and assistance have played an important role in my growth as a physicist.

In addition to Sean, I am grateful to my other collaborators — Tim Dulaney, Moira Gresham, and Ingunn Wehus — without whom this thesis would not have been possible. Over the years, I have also benefitted much from discussions on cosmology and gravity with others on the 4th floor of Lauritsen, in particular Lotty Ackerman, Matt Johnson, Keith Lee, Arthur Lipstein, Changsoon Park, Jaewon Song, and Chien Yao Tseng. I am particularly indebted to Lotty for her help and support after I joined the Particle Theory Group.

I also want to thank Yan-bei Chen, Brad Filippone, and Marc Kamionkowski for serving on my candidacy and defense committees despite their busy schedules. I would like to thank Sean Carroll, Marc Kamionkowski, and Gil Refael for writing recommendation letters for my postdoc applications. I am grateful to Shirley Hampton and Carol Silberstein for sending out the letters to the long list of universities to which I applied.

In the past few years, while I was not working on cosmology, I had the opportunity to study condensed matter physics under Gil Refael and nonlinear dynamics under Michael Cross. I consider both my advisors, and from them I have learned a great deal of physics. I must also mention that it has been a great joy to collaborate with Tony Lee on synchronization, and Tiamhock Tay on dimerization.

My life at Caltech has been enriched by the presence of the many wonderful friends that I have made here in the last five years. I cherish the numerous engaging conversations that I have had with fellow physicists Zeesh Ahmed, Daphne Chang, Tony Chu, David Doll, Tony Lee, Yousi Ma, Matt Matheny, Piti Ongmongkolkul, Changsoon Park, Jaewon Song, Tiamhock Tay, and Yue Zhou. Outside my circle of physics friends, I was blessed to get to know an exceptional group of friends from the Hong Kong Student Association: Arthur Chan, Chefung Chan, Mannin Chan, Michael Cheng, Jonathan Choi, Kawai Kwok, Lapman Lee, Benjamin Li, Icy Ma, Hoi Yee Nam, Raymond Tsang, Lilian Wong, Stephen Wu, and Wanwan Yang (among others).

Finally, I want to thank my parents for their love and support during these five years that I have been away from home. I am forever grateful for all the sacrifices that they have made for me through the years.

Abstract

In this thesis we re-examine two established ideas in theoretical physics: Lorentz invariance and cosmic inflation.

In the first four chapters, we (i) propose a way to hide large extra dimensions by coupling standard model fields with Lorentz-violating tensor fields with expectation values along the extra dimensions; (ii) examine the stability of theories in which Lorentz invariance is spontaneously broken by fixed-norm 'aether' fields; (iii) investigate the phenomenological properties of the sigma-model aether theory; and (iv) explore the implications of an alternative theory of gravity in which the graviton arises from the Goldstone modes of a two-index symmetric aether field.

In the final chapter, we examine the horizon and flatness problems using the canonical measure (developed by Gibbons, Hawking, and Stewart) on the space of solutions to Einstein's equations. We find that the flatness problem does not exist, while the homogeneity of our universe does represent a substantial fine-tuning. Based on the assumption of unitary evolution (Liouville's theorem), we further dispute the widely accepted claim that inflation makes our universe more natural.

Contents

Acknowledgements **iii**

Abstract **v**

1 Introduction **1**

1.1 Spontaneous Violation of Lorentz Invariance 1

1.1.1 Chapter 2: Aether compactification 1

1.1.2 Chapter 3: Instabilities in the aether 2

1.1.3 Chapter 4: Sigma-model aether 3

1.1.4 Chapter 5: Lorentz violation in Goldstone gravity 4

1.2 Inflation 5

1.2.1 Chapter 6: Unitary evolution and cosmological fine-tuning 5

2 Aether Compactification **8**

2.1 Introduction 8

2.2 Aether 9

2.3 Energy-Momentum and Compactification 11

2.4 Scalars 13

2.5 Gauge Fields 16

2.6 Fermions 17

2.7 Gravity 19

2.8 Conclusions 22

3 Instabilities in the Aether **24**

3.1 Introduction 24

3.2 Models 30

3.2.1 Validity of effective field theory 33
3.3 Boundedness of the Hamiltonian 35
3.3.1 Timelike vector field 39
3.3.2 Spacelike vector field 41
3.3.3 Smooth potential 42
3.3.4 Discussion 44
3.4 Linear Instabilities 46
3.4.1 Timelike vector field 46
3.4.2 Spacelike vector field 49
3.4.3 Stability is not frame-dependent 50
3.5 Negative Energy Modes 51
3.5.1 Spin-1 energies 53
3.5.2 Spin-0 energies 55
3.6 Maxwell and Scalar Theories 57
3.6.1 Maxwell action 59
3.6.2 Scalar action 62
3.7 Conclusions 64

4 Sigma-Model Aether **66**
4.1 Introduction 66
4.2 Excitations in the Presence of Gravity 68
4.3 Experimental Constraints 73
4.4 Cosmological Evolution 75
4.5 Extra Dimensions 77
4.6 Conclusions 79

5 Lorentz Violation in Goldstone Gravity **81**
5.1 Introduction 81
5.2 Goldstone Electromagnetism 85
5.2.1 Photons as Goldstone bosons 85
5.2.2 Radiative corrections and dispersion relations of the Goldstone modes 88
5.3 Goldstone Gravity 90
5.3.1 Gravitons as Goldstone bosons 90

5.3.2 Radiative corrections and dispersion relations 93

5.4 Anisotropic Propagation 96

5.4.1 Dispersion relations 97

5.4.2 Motion of test particles 98

5.4.3 Experimental constraints 101

5.4.4 Corrections to the energy-momentum tensor 102

5.5 Vevs That Do Not Break All Six Generators 103

5.5.1 Gravitons are not necessarily Goldstone 103

5.5.2 An example: Three Goldstone bosons only 105

5.6 Conclusions 109

6 Unitary Evolution and Cosmological Fine-Tuning 111

6.1 Introduction 111

6.2 The Evolution of Our Comoving Patch 115

6.2.1 Autonomy 117

6.2.2 Unitarity 119

6.3 The Canonical Measure 122

6.4 Flatness 128

6.5 Homogeneity 133

6.5.1 Action for a perfect fluid background 134

6.5.2 Action during inflation 136

6.5.3 Computation of the measure 137

6.6 Tracing Perturbations Backwards 138

6.6.1 Relation to Planck-scale cutoffs 139

6.6.2 Evolution of perturbations in a matter-dominated universe 141

6.6.3 Evolution of perturbations in a matter-dominated universe preceded by inflation 143

6.6.4 Results 146

6.7 What is Inflation Good For? 147

6.7.1 The universe is not chosen randomly 150

6.7.2 Inflation as an easy target 152

A Solutions to the Linearized Equations of Motion 155

B Additional Properties of the Goldstone Modes **161**

B.1 Polarizations of Goldstone Modes . . . 161

B.1.1 Time-time . . . 161

B.1.2 Time-space . . . 162

B.1.3 Diagonal space-space . . . 163

B.1.4 Off-diagonal space-space . . . 164

B.2 Proof That Gravitons Can Be Goldstone Bosons . . . 165

B.2.1 The two transverse linear combinations of the six Goldstone modes . 170

B.2.2 The remaining four linear combinations . . . 173

B.3 Proof of the Necessity of Breaking All Six Generators to Get Goldstone Gravitons . . . 176

Bibliography **178**

List of Figures

3.1 Hamiltonian density (vertical axis) when $\beta_1 = 1$, $\beta_* = 1.1$, and $\theta = \psi = \partial_y \phi = \partial_z \phi = 0$ as a function of $\partial_t \phi$ (axis pointing into page) and $\partial_x \phi$ (axis pointing out of page) for various ϕ ranging from zero to $\phi_{crit} = \tanh^{-1} \sqrt{\beta_1/\beta_*}$, the value of ϕ for which the Hamiltonian is flat at $\partial_x \phi = 0$, and beyond. Notice that the Hamiltonian density turns over and becomes negative in the $\partial_t \phi$ direction when $\phi > \phi_{crit}$. 41

4.1 Aether rest frame mode phase velocities squared, v^2, minus the speed of light in units of $8\pi G m^2$ as a function of α. The solid line corresponds to spin-0, the small dashed line to spin-1, and the large dashed line to spin-2. Only for $\alpha = -1$ do none of the modes propagate faster than light ($v^2 - 1 > 0$). 70

4.2 Parameter space allowed (shaded region) by constraints from Cherenkov radiation and PPN. The strongest constraint in the $\alpha < -1$ region is from Eq. (4.18), and for most of the $\alpha > -1$ region the strongest constraint is from the second inequality in Eq. (4.19). The plot on the right is a blowup of the small range of α for which the first constraint in Eq. (4.19) is strongest—when $\alpha = -1$ to within a couple of parts in 100. 75

6.1 The physical system corresponding to our observable universe. Our comoving patch is defined by the interior of the intersection of our past light cone with a cutoff surface, for example the surface of last scattering. This illustration is not geometrically faithful, as the expansion is not linear in time. Despite the change in physical size, we assume that the space of states is of equal size at every moment. 118

6.2 Γ is the phase space of the system. C is the constraint hypersurface of constant Hamiltonian (here chosen to be zero). M is the set of all classical trajectories in C, and Σ is a transverse surface through which each trajectory passes only once. Physical quantities should be independent of the choice of surface. . . . 125

6.3 Log plot of the energy density of perturbation versus the Hubble parameter in a universe that is matter-dominated entirely (lower curve), and a universe that first undergoes inflation and then becomes matter-dominated (upper curve). The wavenumber shown here is $10^{-55}m_{Pl}$. Energy densities in the range of 10^{-122} to $10^{-121}m_{Pl}$ today ($H_0 = 10^{-60}m_{Pl}$) are plotted. For the upper bands, inflation ends at $H_i = 10^{-3}m_{Pl}$. The slow-roll parameter, ϵ, is chosen to be 0.1. This corresponds to $\lambda = 2.24$, and results in approximately 70 e-folds. . 147

6.4 Γ is the space of states for the system defined by our comoving patch. For universes that are approximately Robertson-Walker, it can be foliated into subspaces of states with particular values of the Hubble parameter. Liouville's theorem implies that the choice of Hamiltonian does not affect the volume of a region of phase space as it evolves through time; this is illustrated schematically in the case of two theories with the same number of degrees of freedom, but different scalar-field potentials. Even with a potential that allows for inflation, the fraction of universes that actually inflate is very small. (This depiction is not to scale.) 149

List of Tables

5.1 Comparison between Goldstone Photons and Gravitons 108

Chapter 1

Introduction

1.1 Spontaneous Violation of Lorentz Invariance

Ever since Einstein's theory of special relativity, the Lorentz symmetry — the invariance of the laws of physics under boosts and rotations — has been a guiding principle for physicists in the formulation of physical theories. In recent years, however, there has been a resurgence of interest in the idea of spontaneous violation of Lorentz invariance through tensor fields with non-vanishing vacuum expectation values. The phenomenology of these so-called 'aether' theories is exceedingly rich, and constitutes the subject of investigation in the first four chapters of this thesis.

1.1.1 Chapter 2: Aether compactification

In 1921, Kaluza proposed a unified theory of gravity and electromagnetism by extending general relativity to a five-dimensional spacetime. This raised an imminent question: Why don't we see the extra dimension? Five years later, Klein offered a solution: The extra dimension is compactified on a manifold of a sufficiently small size, so that the Kaluza-Klein excitations become very massive and thus difficult to excite. Since then, extra dimensions have become an essential ingredient in the construction of many physical theories (especially so after string theory has become a central part of mainstream theoretical physics), and considerable effort has been devoted to search for them in experiments.

More recently, the development of braneworld scenarios, in which Standard Model fields are confined to a brane embedded in a larger bulk, opens up the possibility that extra dimensions can actually be much larger (even infinite in size). This way, extra dimensions can be large and yet evade detection, simply because we cannot get there.

In Chapter 2, we propose an alternative way to hide large extra dimensions based on Lorentz-violating tensor fields, without invoking branes. To illustrate the scheme, we consider a scenario in which a single vector 'aether' field acquires a vacuum expectation value along an extra dimension that is compactified on a circle. Interactions with the aether modifies the dispersion relations of other fields, thereby increasing the mass of their Kaluza-Klein excitations, without having to make the extra dimension small. A unique signature of this scenario is the possibility of completely different spacings in the Kaluza-Klein towers of each species of scalars, fermions, and gauge bosons. In general, fermions will experience greater mass enhancement than bosons, while gravitons can be naturally less massive.

Chapter 2 was completed in collaboration with Sean Carroll, and has been published as [1].

1.1.2 Chapter 3: Instabilities in the aether

All realistic physical theories must be stable. This is the motivation behind our analysis in Chapter 3, in which we examine the stability of 'aether' theories in which a fixed-norm vector field acquires a vacuum expectation value and violates Lorentz invariance spontaneously. The potential instability of such theories originates from the fact that the metric has an indefinite signature in a Lorentzian spacetime. In the spirit of effective field theory, we restrict our attention in this chapter to theories in which the kinetic term is at most quadratic in derivatives. The nonzero vev of the aether is enforced by a Lagrange multiplier constraint.

We first examine the boundedness of the Hamiltonian of the theory and find that, for a generic kinetic term, the Hamiltonian is unbounded from below. The sole exception is the sigma model, which has the kinetic term,

$$\mathcal{L}_\sigma = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu). \tag{1.1}$$

If the vector field takes on a timelike vev, the model has a globally bounded Hamiltonian, and is guaranteed to be stable.

The unboundedness of the Hamiltonian alone, however, does not necessarily imply an instability, as the dynamical degrees of freedom might not evolve along the unstable directions. This leads us to perform a linear perturbative analysis about constant background configurations. We find that there are only three choices of kinetic terms for which linear perturbations are non-negative in energy and do not grow exponentially in any frame: the Maxwell ($\mathcal{L}_M = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$), scalar ($\mathcal{L}_S = \frac{1}{2}(\partial_\mu A^\mu)^2$), and sigma-model Lagrangians.

1.1.3 Chapter 4: Sigma-model aether

Chapter 4 is an extension of the previous chapter, and contains an analysis of the phenomenological properties of the timelike sigma model, which is the only stable aether model found in Chapter 3. In the presence of gravity, the theory contains five massless degrees of freedom. If modes are superluminal, then the theory is tightly constrained by limits from gravi-Cherenkov radiation. For a unique choice of parameters, all modes are subluminal, and limits on the PPN preferred frame parameter places strong constraints on the parameters of the theory.

In a Friedmann-Robertson-Walker background dominated by a cosmological fluid, we find that the aether evolves dynamically to be purely timelike in the rest frame of the background fluid. In the presence of a non-expanding extra dimension during a deSitter expansion phase, we show that it is possible for the aether to have a nonzero component along the extra dimension. However, this component will decay away when the cosmological evolution is dominated by a perfect fluid with $w > -1$.

Chapters 3 and 4 were done in collaboration with Sean Carroll, Tim Dulaney, and Moira Gresham, and they are published in [2, 3].

1.1.4 Chapter 5: Lorentz violation in Goldstone gravity

According to Goldstone's theorem, the excitations arising from the spontaneous violation of a continuous symmetry are massless. This opens up the possibility that the photon and graviton, whose masslessness is traditionally associated with gauge and differomorphism invariance, are instead Nambu-Goldstone bosons of spontaneously broken Lorentz invariance. Kraus and Tomboulis examined this possibility in the case of the photon and concluded that this is viable. In Chapter 5, we generalize their analysis onto the case where Lorenz invariance is broken spontaneously by a two-index symmetric tensor. We demonstrate that if the vev breaks all the generators of the Lorentz group, six Goldstone modes emerge, and two linear combinations of which have properties that are identical to the graviton in general relativity at lowest order.

Integrating out the massive degrees of freedom in the theory yields an infinite number of Lorentz-violating radiative correction terms in the low-energy effective Lagrangian. We examine a representative subset of these radiative corrections and find that they modify the dispersion relation of the Goldstone graviton modes such that (i) their phase velocity is anisotropic, and (ii) they oscillate test particles in their vicinity longitudinally (in addition to the transverse motion as predicted by general relativity). If the phase velocity is subluminal, then gravi-Cherenkov radiation becomes possible, and observations of high-energy cosmic rays can be used to constrain the radiative corrections.

The discussion presented in Chapter 5 was completed in conjunction with Sean Carroll and Ingunn Wehus, and has been published in [4].

1.2 Inflation

Cosmic inflation, a period of accelerated expansion in the very early universe, has by now been accepted by most cosmologists as a cornerstone of the standard Big Bang paradigm. It is heralded as an elegant solution to a host of problems in cosmology (the flatness, horizon, and monopole problems). Quantum fluctuations during the inflationary era are believed to seed the large-scale structure of our universe today, and the nearly scale-invariant and Gaussian primordial power spectrum predicted by inflation is found to be in remarkable agreement with a large number of experimental probes.

In the last chapter of this thesis, we challenge the purported role that inflation plays in solving the flatness and horizon problems.

1.2.1 Chapter 6: Unitary evolution and cosmological fine-tuning

Despite its numerous successes, some cosmologists became concerned about the purported ability of inflation to solve the flatness and horizon problems. They came to the realization that, if inflation is highly unnatural itself, it cannot possibly be used to make our fine-tuned current condition more natural. After all, the requirements of having a fairly uniform patch dominated by potential energy over a region larger than the corresponding Hubble length and the slow-roll conditions that guarantee at least 60 e-folds of expansion are very finely tuned conditions. This led Gibbons, Hawking, and Stewart to introduce a canonical measure on the set of classical universes in phase space (the GHS measure) to calculate the probability of inflation. The measure derives from the Hamiltonian (symplectic) structure of general relativity, and has the nice properties that it is (i) independent of the choice of time-slicing, (ii) is always positive, and (iii) respects the underlying symmetry of the theory.

In Chapter 6, we examine the flatness and horizon problems using the GHS measure. To our surprise, we find that in minisuperspace the measure diverges when the curvature vanishes. The moral of the lesson is that caution must be exercised in the discussion of the naturalness—in particular, we should consider initial conditions using a mathematically well-defined measure rather than relying on intuition. Following our analysis of the flatness problem, we generalize our calculation by including perturbations to quantify the fine-tuning involved in the horizon problem, and find that the homogeneity of the observable universe does correspond to a highly fine-tuned condition.

Under the assumption of measure-preserving evolution, we argue that it is impossible for inflation to make our universe more natural. The Liouville theorem forbids inflation (or, in fact, any dynamical process) from evolving a large number of initial conditions to a small number of final states. By formal time-reversibility and entropy arguments, we know that there exists an overwhelming larger set of wildly inhomogeneous initial conditions that can evolve into our current state, as compared with the smooth initial conditions that give rise to inflation. Consequently, if inflation is to offer a satisfactory explanation for why our universe is natural, it must be accompanied by a corresponding theory of initial conditions that favor these smooth states.

Chapter 6 was coauthored with Sean Carroll.

Chapter 2

Aether Compactification

2.1 Introduction

If spacetime has extra dimensions in addition to the four we perceive, they are somehow hidden from us. For a long time, the only known way to achieve this goal was the classic Kaluza-Klein scenario: compactify the dimensions on a manifold of characteristic size $\sim R$. Momentum in the extra dimensions is then quantized in units of R^{-1}, giving rise to a Kaluza-Klein tower of states; if R is sufficiently small, the extra dimensions only become evident at very high energies. More recently, it has become popular to consider scenarios in which Standard Model fields are localized on a brane embedded in a larger bulk [5, 6, 7, 8]. In this picture, the extra dimensions are difficult to perceive because we can't get there.

In this chapter, we consider a new way to keep extra dimensions hidden, or more generally to affect the propagation of fields along directions orthogonal to our macroscopic dimensions: adding Lorentz-violating tensor fields ("aether") with expectation values aligned along the extra directions. Interactions with the aether modify the dispersion relations of other fields, leading (with appropriate choice of parameters) to larger energies associated with extra-dimensional momentum.[1] We should emphasize that we have no underlying rea-

[1] After this work was completed, we became aware of closely related work by Rizzo [9]. He enumerated a complete set of five-dimensional Lorentz-violating operators that preserve Lorentz invariance in 4D, and calculated their effect on the spectrum of the Kaluza-Klein tower. In contrast, our starting point is the

son for choosing any particular values of the relevant parameters; in particular, obtaining very large mass splittings requires unnaturally large parameters. Mass splittings that are different for different species are, however, generic predictions of the model.

This scenario has several novel features. Most importantly, it allows for completely different spacings in the Kaluza-Klein towers of each species. If the couplings are chosen universally, the extra mass given to fermions will be twice that given to bosons. There will also be new degrees of freedom associated with fluctuations of the aether field itself; these are massless Goldstone bosons from the spontaneous breaking of Lorentz invariance, but can be very weakly coupled to ordinary matter. There is a sense in which the effect of the aether field is to distort the background metric, but in a way that is felt differently by different kinds of fields. The extra dimensions can be "large" if the expectation value of the aether field is much larger than the inverse coupling. In contrast to brane-world models, we expect no deviation from Newton's inverse square law even if the extra dimensions are as large as a millimeter, as the gravitational source will be distributed uniformly throughout the extra dimensions rather than confined to a brane. The model has no obvious connection to the hierarchy problem; indeed, hiding large dimensions requires the introduction of a new hierarchy. New physical phenomena associated with the scenario deserve more extensive investigation.

2.2 Aether

For definiteness, consider a five-dimensional flat spacetime with coordinates $x^a = \{x^\mu, x^5\}$ and metric signature $(-++++)$. The fifth dimension is compactified on a circle of radius

expectation value of a dynamical aether field, and its lowest-order couplings to ordinary matter. The modified dispersion relations we derive recover in large measure Rizzo's phenomenological results.

R. The aether is a spacelike five-vector u^a, and we can define a "field strength" tensor

$$V_{ab} = \nabla_a u_b - \nabla_b u_a \,. \tag{2.1}$$

This field is not related to the electromagnetic vector potential A_a or its associated field strength $F_{ab} = \nabla_a A_b - \nabla_b A_a$, nor will the dynamics of u^a respect a U(1) group of gauge transformations. Rather, the aether field will be fixed to have a constant norm, with an action

$$S = M_* \int d^5x \sqrt{-g} \left[-\frac{1}{4} V_{ab} V^{ab} - \lambda(u_a u^a - v^2) + \sum_i \mathcal{L}_i \right] \,. \tag{2.2}$$

The $\mathcal{L}_i$'s are various interaction terms to be considered below, and M_* is an overall scaling parameter. Note that λ is not a fixed parameter, but a Lagrange multiplier enforcing the constraint

$$u^a u_a = v^2 \,. \tag{2.3}$$

We choose conventions such that u^a has dimensions of mass. The equation of motion for u^a, neglecting interactions with other fields for the moment, is

$$\nabla_a V^{ab} + v^{-2} u^b u_c \nabla_d V^{cd} = 0 \,, \tag{2.4}$$

where we have used the equations of motion to solve for λ. Any configuration for which $V_{ab} = 0$ everywhere will solve this equation. In particular, there is a background solution of the form

$$u^a = (0, 0, 0, 0, v) \,, \tag{2.5}$$

so that the aether field points exclusively along the extra direction. We will consider this solution for most of this chapter.

Constraints on four-dimensional Lorentz violation via couplings to Standard Model fields have been extensively studied [10, 11, 12, 13]. The dynamics of the (typically timelike) aether fields themselves and their gravitational effects have also been considered [14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. More recently, attention has turned to the case of spacelike vector fields, especially in the early universe [24, 25].

The particular form of the Lagrangian (2.2) is chosen to ensure stability of the theory; for spacelike vector fields, a generic set of kinetic terms would generally give rise to negative-energy excitations[2]. This specific choice propagates two positive-energy modes: one massless scalar, and one massless pseudoscalar [25]. For purposes of this chapter we will not investigate the fluctuations of u^a in any detail. Although the modes are light, their couplings to Standard Model fields can be suppressed. Nevertheless, we expect that traditional methods of constraining light scalars (such as limits from stellar cooling) will provide interesting bounds on the parameter space of these models.

2.3 Energy-Momentum and Compactification

A crucial property of aether fields is the dependence of their energy density on the spacetime geometry. The energy-momentum tensor takes the form

$$T_{ab} = V_{ac}V_b{}^c - \frac{1}{4}V_{cd}V^{cd}g_{ab} + v^{-2}u_a u_b u_c \nabla_d V^{cd} \,. \tag{2.6}$$

[2]The stability of aether theories turns out to be a very tricky issue; for more details see Chapter 3.

In particular, T_{ab} vanishes when V_{ab} vanishes, as for the constant field configuration in flat space (2.5). The non-vanishing expectation value for the aether field does not by itself produce any energy density. In the context of an extra dimension, this implies that the aether field will not provide a contribution to the effective potential for the radion, so the task of stabilizing the extra dimension must be left to other mechanisms.

When the background spacetime is not Minkowski, however, even a "fixed" aether field can give a non-vanishing energy-momentum tensor. In [17] it was shown that a timelike aether field would produce an energy density proportional to the square of the Hubble constant, while in [24] a spacelike aether field was shown to produce an anisotropic stress. We should therefore check that an otherwise quiescent aether field oriented along an extra dimension does not create energy density when the four-dimensional geometry is curved.

Consider a factorizable geometry with an arbitrary four-dimensional metric and a radion field $b(x^\sigma)$ parameterizing the size of the single extra dimension,

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + b(x)^2 dx_5^2\,, \tag{2.7}$$

where x here stands for the four-dimensional coordinates x^σ. In any such spacetime, there is a background solution

$$u^a = \left(0, 0, 0, 0, \frac{v}{b(x)}\right). \tag{2.8}$$

It is straightforward to verify that this configuration satisfies the equation of motion (2.4), as well as the constraint (2.3), even though V_{ab} does not vanish:

$$V_{\mu 5} = -V_{5\mu} = v\nabla_\mu b\,. \tag{2.9}$$

We can then calculate the energy-momentum tensor associated with the aether:

$$\begin{aligned} T^{(u)}_{\mu\nu} &= \frac{v^2}{b^2}\left(\nabla_\mu b \nabla_\nu b - \frac{1}{2} g_{\mu\nu} \nabla_\sigma b \nabla^\sigma b\right) \\ T^{(u)}_{\mu 5} &= 0 \\ T^{(u)}_{55} &= v^2 \left(\nabla_\sigma \nabla^\sigma b - \frac{1}{2} \nabla_\sigma b \nabla^\sigma b\right) . \end{aligned} \tag{2.10}$$

The important feature is that $T^{(u)}_{ab}$ will vanish when $\nabla_\mu b = 0$. As long as the extra dimension is stabilized and the aether takes on the configuration (2.8), there will be no contributions to the energy-momentum tensor; in particular, neither the expansion of the universe nor the spacetime geometry around a localized gravitating source will be affected.

2.4 Scalars

We now return to flat spacetime ($g_{ab} = \eta_{ab}$) and consider the effects of interactions of the aether on various types of matter fields, beginning with a real scalar ϕ. We impose a $\mathbb{Z}_2$ symmetry, $u^a \to -u^a$. The Lagrangian with the lowest-order coupling is then

$$\mathcal{L}_\phi = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2\mu_\phi^2} u^a u^b \partial_a\phi\partial_b\phi \, , \tag{2.11}$$

with a corresponding equation of motion

$$\partial_a\partial^a\phi - m^2\phi = \mu_\phi^{-2}\partial_a(u^a u^b \partial_b\phi) \, . \tag{2.12}$$

Expanding the scalar in Fourier modes,

$$\phi \propto e^{ik_a x^a} = e^{ik_\mu k^\mu + ik_5 x^5} \, , \tag{2.13}$$

yields a dispersion relation

$$-\, k_\mu k^\mu = m^2 + \left(1 + \alpha_\phi^2\right) k_5^2 \,, \tag{2.14}$$

where

$$\alpha_\phi = v/\mu_\phi \,. \tag{2.15}$$

Note that with our metric signature, $-k_\mu k^\mu = \omega^2 - \vec{k}^2$.

This simple calculation illustrates the effect of the coupling to the spacelike vector field. Compactifying the fifth dimension on a circle of radius R quantizes the momentum in that direction, $k_5 = n/R$. In standard Kaluza-Klein theory, this gives rise to a tower of states of masses $m_{KK}^2 = m^2 + (n/R)^2$. With the addition of the aether field, the mass spacing between different states in the KK tower is enhanced,

$$m_{AC}^2 = m^2 + (1 + \alpha_\phi^2) \left(\frac{n}{R}\right)^2 \,. \tag{2.16}$$

The parameter α_ϕ is a ratio of the aether vev to the mass scale μ_ϕ characterizing the coupling, and could be much larger than unity. If the vev is $v \sim M_{\rm pl}$, and the coupling parameter is $\mu_\phi \sim$ TeV, the masses of the excited modes are enhanced by a factor of 10^{15}. The extra dimension could be as large as $R \sim 1$ mm, and the $n = 1$ state would have a mass of order TeV. Admittedly, we have no compelling reason why there should be such a hierarchy between v and μ_ϕ at this point, other than that it is interesting to contemplate.

We will examine the effects of aether compactification on gravitons below, but it is already possible to see that we should not expect any small-scale deviations from Newton's law, even if the extra dimensions are millimeter-sized. Unlike braneworld compactifications, here the sources are not confined to a thin brane embedded in a large bulk; rather, light fields are zero modes, spread uniformly throughout the extra dimensions. Therefore, the gravitational lines of force do not spread out from the source into the higher-dimensional bulk; the sources are still of *codimension* three in space, and gravity will appear three dimensional. There is correspondingly less motivation for considering macroscopic-sized extra dimensions in this scenario, as they would remain undetectable by tabletop experiments.

One may reasonably ask whether it is appropriate to think of such a scenario as a "large" extra dimension at all, or whether we have simply rescaled the metric in an unusual way. In the Lagrangian (2.11) alone, the effect of the aether field is simply to modify the metric by a disformal transformation, $g^{ab} \rightarrow g^{ab} + u^a u^b$. There is a crucial difference, however, in that the interaction with the aether vector is generically not universal. Different fields will tend to have different mass splittings in their Kaluza-Klein towers. Indeed, we shall see that while the splittings for gauge fields follow the pattern of that for scalars, the splittings for fermions are of order α^4 rather than α^2, and the splittings for gravitons do not involve a mass scale μ at all. Thus, aether compactification is conceptually different from an ordinary extra dimension.

Finally, we point out that if we have not imposed the $\mathbb{Z}_2$ symmetry, the lowest-order coupling becomes $\mu^{-1} u^a \partial_a \phi$. By integration by parts, this is equivalent to $-\mu^{-1}(\partial_a u^a)\phi$, which vanishes given our background solution for u^a in (2.5).

2.5 Gauge Fields

Consider an Abelian gauge field A_a, with field strength tensor F_{ab}. The Lagrangian with the lowest-order coupling to u^a is

$$\mathcal{L}_A = -\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2\mu_A^2}u^a u^b g^{cd} F_{ac}F_{bd}\,, \tag{2.17}$$

with equation of motion

$$\partial_a F^{ab} = \mu_A^{-2}\left(u_c u^b \partial_a F^{ca} - u_c u^a \partial_a F^{cb}\right)\,. \tag{2.18}$$

We can decompose this into $b = 5$ and $b = \nu$ components in the background (2.5):

$$\partial_\mu F^{\mu 5} = 0\,, \tag{2.19}$$

$$\partial_\mu F^{\mu\nu} = -(1+\alpha_A^2)\partial_5 F^{5\nu}\,, \tag{2.20}$$

where

$$\alpha_A = v/\mu_A\,. \tag{2.21}$$

We can take advantage of gauge transformations $A_a \to A_a + \partial_a\lambda$ to set $A_5 = 0$. This leaves some residual gauge freedom; we can still transform $A_\mu \to A_\mu + \partial_\mu\tilde{\lambda}$, as long as $\partial_5\tilde{\lambda} = 0$. In other words, the zero mode retains all of its conventional four-dimensional gauge invariance.

Choose $A_5 = 0$ gauge, and go to Fourier space, $A^\nu \propto \epsilon^\nu e^{ik_\mu x^\mu + ik_5 x^5}$, where ϵ^ν is the polarization vector. Then (2.19) and (2.20) imply

$$k_5 k_\mu \epsilon^\mu = 0\,, \tag{2.22}$$

$$\left[k_\mu k^\mu + (1+\alpha_A^2)k_5^2\right]\epsilon^\nu - k^\nu k_\mu \epsilon^\mu = 0\,. \tag{2.23}$$

When $k_5 = 0$, we obtain the ordinary dispersion relation for a photon. When k_5 is not zero, (2.22) implies $k_\mu \epsilon^\mu = 0$, and the dispersion relation is

$$-k_\mu k^\mu = (1+\alpha_A^2)k_5^2\,. \tag{2.24}$$

Precisely as in the scalar case, the Kaluza-Klein masses are enhanced by a factor $(1+\alpha_A^2)$, although there is no necessary relationship between α_A and α_ϕ. The same reasoning would apply to non-Abelian gauge fields, through a coupling $u^a u^b\,\mathrm{Tr}(G_{ac}G_b{}^c)$.

2.6 Fermions

Next we turn to fermions, taken to be Dirac for simplicity. Given the symmetry $u^a \to -u^a$, we might consider a coupling of the form $u^a u^b \bar\psi\gamma_a\gamma_b\psi$. But because $u^a u^b$ is symmetric in its two indices, this is equivalent to $u^a u^b \bar\psi\gamma_{(a}\gamma_{b)}\psi = u^a u^b \bar\psi g_{ab}\psi = v^2\bar\psi\psi$, so this interaction doesn't violate Lorentz invariance.

The first nontrivial coupling involves one derivative,

$$\mathcal{L}_\psi = i\bar\psi\gamma^a\partial_a\psi - m\bar\psi\psi - \frac{i}{\mu_\psi^2}u^a u^b \bar\psi\gamma_a\partial_b\psi\,, \tag{2.25}$$

leading to an equation of motion

$$i\gamma^a \partial_a \psi - m\psi - \frac{i}{\mu_\psi^2} u^a u^b \gamma_a \partial_b \psi = 0\,. \tag{2.26}$$

Going to Fourier space as before, we ultimately find a dispersion relation

$$-k^a k_a - \frac{2}{\mu_\psi^2}(u^a k_a)^2 - \frac{1}{\mu_\psi^4} u^a u_a (u^b k_b)^2 = m^2\,. \tag{2.27}$$

Plugging in the background (2.5) and defining

$$\alpha_\psi = v/\mu_\psi\,, \tag{2.28}$$

we end up with

$$-k^\mu k_\mu = m^2 + (1+\alpha_\psi^2)^2 k_5^2\,. \tag{2.29}$$

Although the form of this equation is identical to the scalar and gauge-field cases, it is quantitatively different: for large α the enhancement goes as α^4 rather than α^2. If (in the context of some as-yet-unknown underlying theory) all of the mass scales μ are similar, we would expect a much larger mass splitting for fermions in an aether background than for bosons.

Similar to the scalar case, if we do not impose the $\mathbb{Z}_2$ symmetry, we are led to consider the following two lower-order couplings: $u_a \bar{\psi}\gamma^a \psi$ and $\frac{i}{\mu} u^a \bar{\psi}\partial_a \psi$. Following the same procedure as before, the first term leads to the dispersion relation

$$-k_\mu k^\mu = m^2 + v^2 + k_5^2 + 2vk_5 = m^2 + (v+k_5)^2. \tag{2.30}$$

As usual, coupling to u^a enhances the mass spacing of the KK tower, but now the spacing will depend on the direction of the 5th-dimensional momentum as well as its magnitude.

Meanwhile, the second term leads to the dispersion

$$-k_\mu k^\mu = m^2 - 2m\alpha k_5 + (1+\alpha^2)k_5^2 \tag{2.31}$$

$$= (m-\alpha k_5)^2 + k_5^2, \tag{2.32}$$

where $\alpha = v/\mu$. Interestingly, if $(1+\alpha^2)/\alpha < 2mR$, this coupling results in a reduction in m^2 for small n. However, it can be checked that these negative mass corrections are never sufficiently large to lead to tachyons. For n large, the mass spacing is enhanced, as usual.

2.7 Gravity

The aether field can couple nonminimally to gravity through an action

$$S = M_* \int d^5x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R + \alpha_g u^a u^b R_{ab}\right], \tag{2.33}$$

where $M_{\rm pl}$ is the 4-dimensional Planck scale and α_g is dimensionless. The gravitational equation of motion takes the form

$$G_{ab} = \frac{\alpha_g}{2M_{\rm pl}^2}W_{ab}\,, \tag{2.34}$$

where $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ and

$$W_{ab} = R_{cd}u^c u^d g_{ab} + \nabla_c\nabla_a\left(u_b u^c\right) + \nabla_c\nabla_b\left(u_a u^c\right)$$
$$-\nabla_c\nabla_d(u^c u^d)g_{ab} - \nabla_c\nabla^c(u_a u_b)\,. \tag{2.35}$$

Now we consider small fluctuations of the metric,

$$g_{ab} = \eta_{ab} + h_{ab}\,. \tag{2.36}$$

The choice of background field $u^a = (0,0,0,0,v)$ spontaneously breaks diffeomorphism invariance, so not all coordinate transformations are open to us if we want to preserve that form. Under an infinitesimal coordinate transformation parameterized by a vector field, $x^a \to \bar{x}^a = x^a + \xi^a$, the metric fluctuation and aether change by $h_{ab} \to h_{ab} + \partial_a \xi_b + \partial_b \xi_a$ and $u^a \to u^a + \partial_5 \xi^a$. Therefore, we should limit our attention to gauge transformations satisfying $\partial_5 \xi^a = 0$. We can, for example, set $h_{\mu 5} = 0$. We then still have residual gauge freedom in the form of ξ^μ, as long as $\partial_5 \xi^\mu = 0$. This amounts to the usual 4-d gauge freedom for the massless four-dimensional graviton.

Taking advantage of this gauge freedom, we can partly decompose the metric perturbation as

$$\begin{aligned} h_{\mu\nu} &= \bar{h}_{\mu\nu} + \Psi \eta_{\mu\nu}\,, \\ h_{55} &= \Phi\,, \end{aligned} \tag{2.37}$$

where $\eta^{\mu\nu}\bar{h}_{\mu\nu} = 0$. In this decomposition, $\bar{h}_{\mu\nu}$ represents propagating gravitational waves, Ψ represents Newtonian gravitational fields, and Φ is the radion field representing the breathing mode of the extra dimension. The zero mode of this field is a massless scalar coupled to matter with gravitational strength; in a phenomenologically viable model, it would have to be stabilized, presumably by bulk matter fields. The Einstein tensor becomes

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}\Big[-\partial_\lambda\partial^\lambda\bar{h}_{\mu\nu} - \partial_5^2\bar{h}_{\mu\nu} + \partial_\mu\partial^\lambda\bar{h}_{\lambda\nu} \\ &\quad +\partial_\nu\partial^\lambda\bar{h}_{\lambda\mu} - 2\partial_\mu\partial_\nu\Psi - \partial_\mu\partial_\nu\Phi \\ &\quad - \left(\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - 2\partial_\lambda\partial^\lambda\Psi + 3\partial_5^2\Psi - \partial_\lambda\partial^\lambda\Phi\right)\eta_{\mu\nu}\Big] , \qquad (2.38) \\ G_{\mu 5} &= \frac{1}{2}\left(\partial_5\partial^\lambda\bar{h}_{\lambda\mu} - 3\partial_\mu\partial_5\Psi\right) , \qquad (2.39) \\ G_{55} &= \frac{1}{2}\left(-\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} + 3\partial_\lambda\partial^\lambda\Psi\right) , \qquad (2.40) \end{aligned}$$

and (2.35) is

$$\begin{aligned} W_{\mu\nu} &= v^2\left(\partial_5^2\bar{h}_{\mu\nu} - 3\partial_5^2\Psi\eta_{\mu\nu} - \partial_5^2\Phi\eta_{\mu\nu}\right) , \qquad (2.41) \\ W_{\mu 5} &= v^2\partial_\mu\partial_5\Phi\, , \qquad (2.42) \\ W_{55} &= -2v^2(2\partial_5^2\Psi + \partial_5^2\Phi)\, . \qquad (2.43) \end{aligned}$$

We have already argued that there will be no macroscopic deviations from Newton's law on the scale of the extra-dimensional radius R, because the zero-mode fields are distributed uniformly through the extra dimensions. However, we can also inquire about the Kaluza-Klein tower of propagating gravitons. To that end, we set $\Phi = 0 = \Psi$ and consider transverse waves, $\partial^\lambda\bar{h}_{\lambda\mu} = 0$. The gravity equation (2.34) becomes

$$-\frac{1}{2}\partial_c\partial^c\bar{h}_{\mu\nu} = \frac{\alpha_g v^2}{2M_{\rm pl}^2}\partial_5^2\bar{h}_{\mu\nu}\, . \qquad (2.44)$$

This implies a dispersion relation

$$-k_\mu k^\mu = \left(1 + \frac{\alpha_g v^2}{M_{\rm pl}^2}\right) k_5^2 \,. \tag{2.45}$$

As before, there is an altered dispersion relation for modes with bulk momentum. However, the dimensionless coupling α_g appears directly in the Lagrangian, rather than arising as a ratio $\alpha = v/\mu$. It is therefore consistent to imagine scenarios with $\alpha_g \sim 1$, while the other α_i's are substantially larger. In that case, KK gravitons will have masses that are close to the conventional expectation, $m \approx n/R$, even while other fields are much heavier. In the scenario with a single extra dimension, the underlying quantum-gravity scale $M_{QG}^3 = M_* M_P^2$ will still be substantially larger than a TeV, and we do not expect graviton production at colliders; but such a phenomenon might be important in extensions with more than one extra dimension.

2.8 Conclusions

The presence of Lorentz-violating aether fields in extra dimensions introduces novel effects into Kaluza-Klein compactification schemes. Interactions with the aether alter the relationship between the size of the extra dimensions and the mass splittings within the KK towers. With appropriately chosen parameters, modes with extra-dimensional momentum can appear very heavy from a four-dimensional perspective, even with relatively large extra dimensions.

A number of empirical tests of this idea suggest themselves. The most obvious is the possibility of KK towers with substantially different masses for different species. While scalar and gauge-boson mass splittings follow a similar pattern, fermions experience greater enhancement, while gravitons can naturally be less massive. In addition, although we have not considered the prospect carefully in this chapter, oscillations of the aether field itself are potentially detectable. Their couplings will be suppressed by the mass scales μ_i, without being enhanced by the vev v; nevertheless, searches for massless Goldstone bosons should provide interesting constraints on the parameter space.

Our investigation has been phenomenological in nature; we do not have an underlying theory of the aether field, nor any natural expectation for the magnitudes of the parameters v, μ_i, and α_g. The possibility of a hidden millimeter-sized dimension requires a substantial hierarchy, $v/\mu_i \sim 10^{15}$; even in the absence of such large numbers, however, interactions with the aether may lead to subtle yet important effects. It would certainly be interesting to have a deeper understanding of the possible origin of these fields and couplings.

Numerous questions remain to be addressed. We considered a vector field in a single extra dimension, but higher-rank tensors in multiple dimensions should lead to analogous effects. It would also be interesting to study the gravitational effects of the aether fields themselves in non-trivial spacetime backgrounds. The idea of modified extra-dimensional dispersion relations in the presence of Lorentz-violating tensor fields opens up a variety of possibilities that merit further exploration.

Chapter 3

Instabilities in the Aether

3.1 Introduction

The idea of spontaneous violation of Lorentz invariance through tensor fields with non-vanishing expectation values has garnered substantial attention in recent years [12, 14, 15, 16, 17, 18, 19, 25, 26, 27, 28, 29]. Hypothetical interactions between Standard Model fields and Lorentz-violating (LV) tensor fields are tightly constrained by a wide variety of experimental probes, in some cases leading to limits at or above the Planck scale [12, 17, 30, 31, 32, 33, 34].

If these constraints are to be taken seriously, it is necessary to have a sensible theory of the dynamics of the LV tensor fields themselves, at least at the level of low-energy effective field theory. The most straightforward way to construct such a theory is to follow the successful paradigm of scalar field theories with spontaneous symmetry breaking, by introducing a tensor potential that is minimized at some nonzero expectation value, in addition to a kinetic term for the fields. (Alternatively, it can be a derivative of the field that obtains an expectation value, as in ghost condensation models [23, 35, 36].) As an additional simplification, we may consider models in which the nonzero expectation value is enforced by a Lagrange multiplier constraint, rather than by dynamically minimizing a potential; this removes the "longitudinal" mode of the tensor from consideration, and may be thought of as a limit of the potential as the mass near the minimum is taken to infinity. In that case, there will be a vacuum manifold of zero-energy tensor configurations, specified by the constraint.

All such models must confront the tricky question of stability. Ultimately, stability problems stem from the basic fact that the metric has an indefinite signature in a Lorentzian spacetime. Unlike in the case of scalar fields, for tensors it is necessary to use the spacetime metric to define both the kinetic and potential terms for the fields. A generic choice of potential would have field directions in which the energy is unbounded from below, leading to tachyons, while a generic choice of kinetic term would have modes with negative kinetic energies, leading to ghosts. Both phenomena represent instabilities; if the theory has tachyons, small perturbations grow exponentially in time at the linearized level, while if the theory has ghosts, nonlinear interactions create an unlimited number of positive- and negative-energy excitations [37]. There is no simple argument that these unwanted features are necessarily present in any model of LV tensor fields, but the question clearly warrants careful study.

In this chapter we revisit the question of the stability of theories of dynamical Lorentz violation, and argue that most such theories are unstable. In particular, we examine in detail the case of a vector field A_μ with a nonvanishing expectation value, known as the "aether" model or a "bumblebee" model. For generic choices of kinetic term, it is straightforward to show that the Hamiltonian of such a model is unbounded from below, and there exist solutions with bounded initial data that grow exponentially in time.

There are three specific choices of kinetic term for which the analysis is more subtle.

These are the sigma-model kinetic term,

$$\mathcal{L}_K = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu \, , \tag{3.1}$$

which amounts to a set of four scalar fields defined on a target space with a Minkowski metric; the Maxwell kinetic term,

$$\mathcal{L}_K = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \, , \tag{3.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is familiar from electromagnetism; and what we call the "scalar" kinetic term,

$$\mathcal{L}_K = \frac{1}{2}(\partial_\mu A^\mu)^2 \, , \tag{3.3}$$

featuring a single scalar degree of freedom. Our findings may be summarized as follows:

- The sigma-model Lagrangian with the vector field constrained by a Lagrange multiplier to take on a timelike expectation value is the only aether theory for which the Hamiltonian is bounded from below in every frame, ensuring stability. In Chapter 4, we examine the cosmological behavior and observational constraints on this model [3]. If the vector field is spacelike, the Hamiltonian is unbounded and the model is unstable. However, if the constraint in the sigma-model theory is replaced by a smooth potential, allowing the length-changing mode to become a propagating degree of freedom, that mode is necessarily ghostlike (negative kinetic energy) and tachyonic (correct sign mass term), and the Hamiltonian is unbounded below, even in the timelike case. It is therefore unclear whether models of this form can arise in any full theory.

- In the Maxwell case, the Hamiltonian is unbounded below; however, a perturbative analysis does not reveal any explicit instabilities in the form of tachyons or ghosts. The timelike mode of the vector acts as a Lagrange multiplier, and there are fewer propagating degrees of freedom at the linear level (a "spin-1" mode propagates, but not a "spin-0" mode). Nevertheless, singularities can arise in evolution from generic initial data: for a spacelike vector, for example, the field evolves to a configuration in which the fixed-norm constraint cannot be satisfied (or perhaps just to a point where the effective field theory breaks down). In the timelike case, a certain subset of initial data is well-behaved, but, provided the vector field couples only to conserved currents, the theory reduces precisely to conventional electromagnetism, with no observable violations of Lorentz invariance. It is unclear whether there exists a subset of initial data that leads to observable violations of Lorentz invariance while avoiding problems in smooth time evolution.

- The scalar case is superficially similar to the Maxwell case, in that the Hamiltonian is unbounded below, but a perturbative analysis does not reveal any instabilities. Again, there are fewer degrees of freedom at the linear level; in this case, the spin-1 mode does not propagate. There is a scalar degree of freedom, but it does not correspond to a propagating mode at the level of perturbation theory (the dispersion relation is conventional, but the energy vanishes to quadratic order in the perturbations). For the timelike aether field, obstacles arise in the time evolution that are similar to those of a spacelike vector in the Maxwell case; for a spacelike aether field with a scalar action, the behavior is less clear.

- For any other choice of kinetic term, aether theories are always unstable.

Interestingly, these three choices of aether dynamics are precisely those for which there is a unique propagation speed for all dynamical modes; this is the same condition required to ensure that the Generalized Second Law is respected by a Lorentz-violating theory [38, 39].

One reason why our findings concerning stability seem more restrictive than those of some previous analyses is that we insist on perturbative stability in all Lorentz frames, which is necessary in theories where the form of the Hamiltonian is frame-dependent. In a Lorentz-invariant field theory, it suffices to pick a Lorentz frame and examine the behavior of small fluctuations; if they grow exponentially, the model is unstable, while if they oscillate, the model is stable. In Lorentz-violating theories, in contrast, such an analysis might miss an instability in one frame that is manifest at the linear level in some other frame [32, 40, 41]. This can be traced to the fact that a perturbation that is "small" in one frame (the value of the perturbation is bounded everywhere along some initial spacelike slice), but grows exponentially with time as measured in that frame, will appear "large" (unbounded on every spacelike slice) in some other frame.

As an explicit example, consider a model of a timelike vector with a background configuration $\bar{A}_\mu = (m, 0, 0, 0)$, and perturbations $\delta a^\mu = \epsilon^\mu e^{-i\omega t} e^{i\vec{k}\cdot\vec{x}}$, where ϵ^μ is some constant polarization vector. In this frame, we will see that the dispersion relation takes the form

$$\omega^2 = v^2 \vec{k}^2 \,. \tag{3.4}$$

Clearly, the frequency ω will be real for every real wave vector $\vec{k}$, and such modes simply oscillate rather than growing in time. It is tempting to conclude that models of this form are perturbatively stable for any value of v. However, we will see below that when $v > 1$, there exist other frames (boosted with respect to the original) in which $\vec{k}$ can be real but

ω is necessarily complex, indicating an instability. These correspond to wave vectors for which, evaluated in the original frame, both ω and $\vec{k}$ are complex. Modes with complex spatial wave vectors are not considered to be "perturbations," since the fields blow up at spatial infinity. However, in the presence of Lorentz violation, a complex spatial wave vector in one frame may correspond to a real spatial wave vector in a boosted frame. We will show that instabilities can arise from initial data defined on a constant-time hypersurface (in a boosted frame) constructed solely from modes with real spatial wave vectors. Such modes are bounded at spatial infinity (in that frame), and could be superimposed to form wave packets with compact support. Since the notion of stability is not frame dependent, the existence of at least one such frame indicates that the theory is unstable, even if there is no linear instability in the aether rest frame.

Several prior investigations have considered the question of stability in theories with LV vector fields. Lim [18] calculated the Hamiltonian for small perturbations around a constant timelike vector field in the rest frame, and derived restrictions on the coefficients of the kinetic terms. Bluhm et al. [42] also examined the timelike case with a Lagrange multiplier constraint, and showed that the Maxwell kinetic term led to stable dynamics on a certain branch of the solution space if the vector was coupled to a conserved current. It was also found, in [42], that most LV vector field theories have Hamiltonians that are unbounded below. Boundedness of the Hamiltonian was also considered in [43]. In the context of effective field theory, Gripaios [44] analyzed small fluctuations of LV vector fields about a flat background. Dulaney, Gresham, and Wise [25] showed that only the Maxwell choice was stable to small perturbations in the spacelike case assuming the energy of the linearized modes was nonzero.[1] Elliot, Moore, and Stoica [31] showed that the sigma-model

[1]This effectively eliminates the scalar case.

kinetic term is stable in the presence of a constraint, but not with a potential.

In the next section, we define notation and fully specify the models we are considering. We then turn to an analysis of the Hamiltonians for such models, and show that they are always unbounded below unless the kinetic term takes on the sigma-model form and the vector field is timelike. This result does not by itself indicate an instability, as there may not be any dynamical degree of freedom that actually evolves along the unstable direction. Therefore, in the following section we look carefully at linear stability around constant configurations, and isolate modes that grow exponentially with time. In the section after that we show that the models that are not already unstable at the linear level end up having ghosts, with the exception of the Maxwell and scalar cases. We then examine some features of those two theories in particular.

3.2 Models

We will consider a dynamical vector field A_μ propagating in Minkowski spacetime with signature $(-+++)$. The action takes the form

$$S_A = \int d^4x \, (\mathcal{L}_K + \mathcal{L}_V) \ , \tag{3.5}$$

where $\mathcal{L}_K$ is the kinetic Lagrange density and $\mathcal{L}_V$ is (minus) the potential. A general kinetic term that is quadratic in derivatives of the field can be written[2]

$$\mathcal{L}_K = -\beta_1(\partial_\mu A_\nu)(\partial^\mu A^\nu) - \beta_2(\partial_\mu A^\mu)^2 - \beta_3(\partial_\mu A_\nu)(\partial^\nu A^\mu) - \beta_4 \frac{A^\mu A^\nu}{m^2}(\partial_\mu A_\rho)(\partial_\nu A^\rho) \, . \tag{3.7}$$

[2] In terms of the coefficients, c_i, defined in [16] and used in many other publications on aether theories,

$$\beta_i = \frac{c_i}{16\pi G m^2} \tag{3.6}$$

where G is the gravitational constant.

In flat spacetime, setting the fields to constant values at infinity, we can integrate by parts to write an equivalent Lagrange density as

$$\mathcal{L}_K = -\frac{1}{2}\beta_1 F_{\mu\nu}F^{\mu\nu} - \beta_*(\partial_\mu A^\mu)^2 - \beta_4 \frac{A^\mu A^\nu}{m^2}(\partial_\mu A_\rho)(\partial_\nu A^\rho)\,, \tag{3.8}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and we have defined

$$\beta_* = \beta_1 + \beta_2 + \beta_3\,. \tag{3.9}$$

In terms of these variables, the models specified above with no linear instabilities or negative-energy ghosts are:

- Sigma model: $\beta_1 = \beta_*$,
- Maxwell: $\beta_* = 0$, and
- Scalar: $\beta_1 = 0$,

in all cases with $\beta_4 = 0$.

The vector field will obtain a nonvanishing vacuum expectation value from the potential. For most of the chapter we will take the potential to be a Lagrange multipler constraint that strictly fixes the norm of the vector:

$$\mathcal{L}_V = \lambda(A^\mu A_\mu \pm m^2)\,, \tag{3.10}$$

where λ is a Lagrange multiplier whose variation enforces the constraint

$$A^\mu A_\mu = \mp m^2\,. \tag{3.11}$$

If the upper sign is chosen, the vector will be timelike, and it will be spacelike for the lower sign. Later we will examine how things change when the constraint is replaced by a smooth potential of the form $\mathcal{L}_V = -V(A_\mu) \propto \xi(A_\mu A^\mu \pm m^2)^2$. It will turn out that the theory defined with a smooth potential is only stable in the limit as $\xi \to \infty$. In any case, unless we specify otherwise, we assume that the norm of the vector is determined by the constraint (3.11).

We are left with an action

$$S_A = \int d^4x \left[-\frac{1}{2}\beta_1 F_{\mu\nu}F^{\mu\nu} - \beta_*(\partial_\mu A^\mu)^2 - \beta_4 \frac{A^\mu A^\nu}{m^2}(\partial_\mu A_\rho)(\partial_\nu A^\rho) + \lambda(A^\mu A_\mu \pm m^2) \right] . \tag{3.12}$$

The Euler-Lagrange equation obtained by varying with respect to A_μ is

$$\beta_1 \partial_\mu F^{\mu\nu} + \beta_* \partial^\nu \partial_\mu A^\mu + \beta_4 G^\nu = -\lambda A^\nu , \tag{3.13}$$

where we have defined

$$G^\nu = \frac{1}{m^2}\left[A^\lambda(\partial_\lambda A^\sigma)F_\sigma{}^\nu + A^\sigma(\partial_\lambda A^\lambda \partial_\sigma A^\nu + A^\lambda \partial_\lambda \partial_\sigma A^\nu) \right] . \tag{3.14}$$

Since the fixed-norm condition (3.11) is a constraint, we can consistently plug it back into the equations of motion. Multiplying (3.13) by A_ν and using the constraint, we can solve for the Lagrange multiplier,

$$\lambda = \pm\frac{1}{m^2}\left(\beta_1 \partial_\mu F^{\mu\nu} + \beta_* \partial^\nu \partial_\mu A^\mu + \beta_4 G^\nu\right) A_\nu . \tag{3.15}$$

Inserting this back into (3.13), we can write the equation of motion as a system of three independent equations:

$$Q_\rho \equiv \left(\eta_{\rho\nu} \pm \frac{A_\rho A_\nu}{m^2}\right)\left(\beta_1 \partial_\mu F^{\mu\nu} + \beta_* \partial^\nu \partial_\mu A^\mu + \beta_4 G^\nu\right) = 0. \tag{3.16}$$

The tensor $\eta_{\rho\nu} \pm m^{-2} A_\rho A_\nu$ acts to take what would be the equation of motion in the absence of the constraint, and project it into the hyperplane orthogonal to A_μ. There are only three independent equations because $A^\rho Q_\rho$ vanishes identically, given the fixed norm constraint.

3.2.1 Validity of effective field theory

As in this chapter we will restrict our attention to classical field theory, it is important to check that any purported instabilities are found in a regime where a low-energy effective field theory should be valid. The low-energy degrees of freedom in our models are Goldstone bosons resulting from the breaking of Lorentz invariance. The effective Lagrangian will consist of an infinite series of terms of progressively higher order in derivatives of the fields, suppressed by appropriate powers of some ultraviolet mass scale M. If we were dealing with the theory of a scalar field Φ, the low-energy effective theory would be valid when the canonical kinetic term $(\partial\Phi)^2$ was large compared to a higher-derivative term such as

$$\frac{1}{M^2}(\partial^2\Phi)^2 \,. \tag{3.17}$$

For fluctuations with wavevector $k^\mu = (\omega, \vec{k})$, we have $\partial\Phi \sim k\Phi$, and the lowest-order terms accurately describe the dynamics whenever $|\vec{k}| < M$. A fluctuation that has a low momentum in one frame can, of course, have a high momentum in some other frame, but the converse is also true; the set of perturbations that can be safely considered "low-energy" looks the same in any frame.

With a Lorentz-violating vector field, the situation is altered. In addition to higher-derivative terms of the form $M^{-2}(\partial^2 A)^2$, the possibility of extra factors of the vector expectation value leads us to consider terms such as

$$\mathcal{L}_4 = \frac{1}{M^8} A^6 (\partial^2 A)^2 \,. \tag{3.18}$$

The number of such higher dimension operators in the effective field theory is greatly reduced because $A_\mu A^\mu = -m^2$ and, therefore, $A_\mu \partial_\nu A^\mu = 0$. It can be shown that an independent operator with n derivatives includes at most $2n$ vector fields, so that the term highlighted here has the largest number of A's with four derivatives. We expect that the ultraviolet cutoff M is of order the vector norm, $M \approx m$. Hence, when we consider a background timelike vector field in its rest frame,

$$\bar{A}_\mu = (m, 0, 0, 0) \,, \tag{3.19}$$

the $\mathcal{L}_4$ term reduces to $m^{-2}(\partial^2 A)^2$, and the effective field theory is valid for modes with $k < m$, just as in the scalar case.

But now consider a highly boosted frame, with

$$\bar{A}_\mu = (m \cosh \eta, m \sinh \eta, 0, 0) \,. \tag{3.20}$$

At large η, individual components of A will scale as $e^{|\eta|}$, and the higher-derivative term schematically becomes

$$\mathcal{L}_4 \sim \frac{1}{m^2} e^{6|\eta|} (\partial^2 A)^2 \,. \tag{3.21}$$

For modes with spatial wave vector $k = |\vec{k}|$ (as measured in this boosted frame), we are therefore comparing $m^{-2}e^{6|\eta|}k^4$ with the canonical term k^2. The lowest-order terms therefore only dominate for wave vectors with

$$k < e^{-3|\eta|}m\,. \tag{3.22}$$

In the presence of Lorentz violation, therefore, the realm of validity of the effective field theory may be considerably diminished in highly boosted frames. We will be careful in what follows to restrict our conclusions to those that can be reached by only considering perturbations that are accurately described by the two-derivative terms. The instabilities we uncover are infrared phenomena, which cannot be cured by changing the behavior of the theory in the ultraviolet. We have been careful to include all of the lowest order terms in the effective field theory expansion—the terms in (3.8).

3.3 Boundedness of the Hamiltonian

We would like to establish whether there are any values of the parameters β_1, β_*, and β_4 for which the aether model described above is physically reasonable. In practice, we take this to mean that there exist background configurations that are stable under small perturbations. It seems hard to justify taking an unstable background as a starting point for phenomenological investigations of experimental constraints, as we would expect the field to evolve on microscopic timescales away from its starting point.

"Stability" of a background solution X_0 to a set of classical equations of motion means that, for any small neighborhood U_0 of X_0 in the phase space, there is another neighborhood U_1 of X_0 such that the time evolution of any point in U_0 remains in U_1 for all times. More informally, small perturbations oscillate around the original background, rather than growing with time. A standard way of demonstrating stability is to show that the Hamiltonian is a local minimum at the background under consideration. Since the Hamiltonian is conserved under time evolution, the allowed evolution of a small perturbation will be bounded to a small neighborhood of that minimum, ensuring stability. Note that the converse does not necessarily hold; the presence of other conserved quantities can be enough to ensure stability even if the Hamiltonian is not bounded from below.

One might worry about invoking the Hamiltonian in a theory where Lorentz invariance has been spontaneously violated. Indeed, as we shall see, the form of the Hamiltonian for small perturbations will depend on the Lorentz frame in which they are expressed. To search for possible linear instabilities, it is necessary to consider the behavior of small perturbations in every Lorentz frame.

The Hamiltonian density, derived from the action (3.12) via a Legendre transformation, is

$$\mathcal{H} = \frac{\partial \mathcal{L}_A}{\partial(\partial_0 A_\mu)}\partial_0 A_\mu - \mathcal{L}_A \tag{3.23}$$

$$= \frac{\beta_1}{2}F_{ij}^2 + \beta_1(\partial_0 A_i)^2 - \beta_1(\partial_i A_0)^2 + \beta_*(\partial_i A_i)^2 - \beta_*(\partial_0 A_0)^2$$
$$+ \beta_4 \frac{A^j A^k}{m^2}(\partial_j A_\rho)(\partial_k A^\rho) - \beta_4 \frac{A^0 A^0}{m^2}(\partial_0 A_\rho)(\partial_0 A^\rho), \tag{3.24}$$

where Latin indices i, j run over $\{1, 2, 3\}$. The total Hamiltonian corresponding to this density is

$$
\begin{aligned}
H &= \int d^3x \mathcal{H} \\
&= \int d^3x \big(\beta_1(\partial_\mu A_i \partial_\mu A_i - \partial_\mu A_0 \partial_\mu A_0) + (\beta_1 - \beta_*)[(\partial_0 A_0)^2 - (\partial_i A_i)^2] \\
&\quad + \beta_4 \frac{A_j A_k}{m^2}(\partial_j A_\rho)(\partial_k A^\rho) - \beta_4 \frac{A_0 A_0}{m^2}(\partial_0 A_\rho)(\partial_0 A^\rho)\big) \,. \qquad (3.25)
\end{aligned}
$$

We have integrated by parts and assumed that $\partial_i A_j$ vanishes at spatial infinity; repeated lowered indices are summed (without any factors of the metric). Note that this Hamiltonian is identical to that of a theory with a smooth (positive semi-definite) potential instead of a Lagrange multiplier term, evaluated at field configurations for which the potential is minimized. Therefore, if the Hamiltonian is unbounded when the fixed-norm constraint is enforced by a Lagrange multiplier, it will also be unbounded in the case of a smooth potential.

There are only three dynamical degrees of freedom, so we may reparameterize A_μ such that the fixed-norm constraint is automatically enforced and the allowed three-dimensional subspace is manifest. We define a boost variable ϕ and angular variables θ and ψ, so that we can write

$$A_0 \equiv m \cosh \phi \qquad (3.26)$$

$$A_i \equiv m \sinh \phi f_i(\theta, \psi) \qquad (3.27)$$

in the timelike case with $A_\mu A^\mu = -m^2$, and

$$A_0 \equiv m \sinh\phi \tag{3.28}$$

$$A_i \equiv m \cosh\phi f_i(\theta,\psi) \tag{3.29}$$

in the spacelike case with $A_\mu A^\mu = +m^2$. In these expressions,

$$f_1 \equiv \cos\theta\cos\psi \tag{3.30}$$

$$f_2 \equiv \cos\theta\sin\psi \tag{3.31}$$

$$f_3 \equiv \sin\theta\,, \tag{3.32}$$

so that $f_i f_i = 1$. In terms of this parameterization, the Hamiltonian density for a timelike aether field becomes

$$\frac{\mathcal{H}^{(t)}}{m^2} = \beta_1 \sinh^2\phi \partial_\mu f_i \partial_\mu f_i + \beta_1 \partial_\mu\phi\partial_\mu\phi + (\beta_1 - \beta_*)\left[(\partial_0\phi)^2 \sinh^2\phi - (\cosh\phi f_i \partial_i\phi + \sinh\phi \partial_i f_i)^2\right]$$
$$+ \beta_4 \sinh^2\phi\left[(f_i\partial_i\phi)^2 + \sinh^2\phi (f_i\partial_i f_l)(f_j\partial_j f_l)\right] - \beta_4\cosh^2\phi\left[(\partial_0\phi)^2 + \sinh^2\phi(\partial_0 f_i)^2\right], \tag{3.33}$$

while for the spacelike case we have

$$\frac{\mathcal{H}^{(s)}}{m^2} = \beta_1 \cosh^2\phi \partial_\mu f_i \partial_\mu f_i - \beta_1 \partial_\mu\phi\partial_\mu\phi + (\beta_1 - \beta_*)\left[(\partial_0\phi)^2 \cosh^2\phi - (\sinh\phi f_i \partial_i\phi + \cosh\phi \partial_i f_i)^2\right]$$
$$- \beta_4 \cosh^2\phi\left[(f_i\partial_i\phi)^2 - \cosh^2\phi (f_i\partial_i f_l)(f_j\partial_j f_l)\right] + \beta_4\sinh^2\phi\left[(\partial_0\phi)^2 - \cosh^2\phi(\partial_0 f_i)^2\right]. \tag{3.34}$$

Expressed in terms of the variables ϕ, θ, ψ, the Hamiltonian is a function of initial data that automatically respects the fixed-norm constraint. We assume that the derivatives

$\partial_\mu A_\nu(t_0, \vec{x})$ vanish at spatial infinity.

3.3.1 Timelike vector field

We can now determine which values of the parameters $\{\beta_1, \beta_*, \beta_4\}$ lead to Hamiltonians that are bounded below, starting with the case of a timelike aether field. We can examine the various possible cases in turn.

- **Case One:** $\beta_1 = \beta_*$ **and** $\beta_4 = 0$**.**

  This is the sigma-model kinetic term (3.1). In this case the Hamiltonian density simplifies to

  $$\mathcal{H}^{(t)} = m^2\beta_1(\sinh^2\phi\partial_\mu f_i\partial_\mu f_i + \partial_\mu\phi\partial_\mu\phi)\,. \tag{3.35}$$

  It is manifestly non-negative when $\beta_1 > 0$, and non-positive when $\beta_1 < 0$. The sigma-model choice $\beta_1 = \beta_* > 0$ therefore results in a theory that is stable. (See also §6.2 of [28].)

- **Case Two:** $\beta_1 < 0$ **and** $\beta_4 = 0$**.**

  In this case, consider configurations with $(\partial_0 f_i) \neq 0$, $(\partial_i f_j) = 0$, $\partial_\mu\phi = 0$, $\sinh^2\phi \gg 1$. Then we have

  $$\mathcal{H}^{(t)} \sim m^2\beta_1\sinh^2\phi(\partial_0 f_i)^2. \tag{3.36}$$

  For $\beta_1 < 0$, the Hamiltonian can be arbitrarily negative for any value of β_*.

- **Case Three:** $\beta_1 \geq 0$**,** $\beta_* < \beta_1$**, and** $\beta_4 = 0$**.**

  We consider configurations with $\partial_\mu f_i = 0$, $f_i\partial_i\phi \neq 0$, $\partial_0\phi = 0$, $\cosh^2\phi \gg 1$, which gives

  $$\mathcal{H}^{(t)} \sim m^2(\beta_* - \beta_1)\cosh^2\phi(f_i\partial_i\phi)^2. \tag{3.37}$$

Again, this can be arbitrarily negative.

- **Case Four:** $\beta_1 \geq 0$**,** $\beta_* > \beta_1$**, and** $\beta_4 = 0$**.**

  Now we consider configurations with $\partial_\mu f_i = 0$, $f_i \partial_i \phi = 0$, $\partial_0 \phi \neq 0$, $\sinh^2 \phi \gg 1$. Then,

$$\mathcal{H}^{(t)} \sim m^2(\beta_1 - \beta_*) \sinh^2 \phi (\partial_0 \phi)^2, \tag{3.38}$$

  which can be arbitrarily negative.

- **Case Five:** $\beta_4 \neq 0$**.**

  Now we consider configurations with $\partial_\mu f_i \neq 0$, $\partial_\mu \phi = 0$ and $\sinh^2 \phi \gg 1$. Then,

$$\mathcal{H}^{(t)} \sim m^2 \beta_4 \left[\sinh^4 \phi (f_i \partial_i f_l)(f_k \partial_k f_l) - \sinh^2 \phi \cosh^2 \phi (\partial_0 f_i)^2 \right] , \tag{3.39}$$

  which can be arbitrarily negative for any nonzero β_4 and for any values of β_1 and β_*.

For any case other than the sigma-model choice $\beta_1 = \beta_*$, it is therefore straightforward to find configurations with arbitrarily negative values of the Hamiltonian.

Nevertheless, a perturbative analysis of the Hamiltonian would not necessarily discoverthat it was unbounded. The reason for this is shown in Fig. 3.1, which shows the Hamiltonian density for the theory with $\beta_1 = 1$, $\beta_* = 1.1$, in a restricted subspace where $\partial_y \phi = \partial_z \phi = 0$ and $\theta = \phi = 0$, leaving only ϕ, $\partial_t \phi$, and $\partial_x \phi$ as independent variables. We have plotted $\mathcal{H}$ as a function of $\partial_t \phi$ and $\partial_x \phi$ for four different values of ϕ. When ϕ is sufficiently small, so that the vector is close to being purely timelike, the point $\partial_t \phi = \partial_x \phi = 0$ is a local minimum. Consequently, perturbations about constant configurations with small ϕ would appear stable. But for large values of ϕ, the unboundedness of the Hamiltonian becomes apparent. This phenomenon will arise again when we consider the evolution of

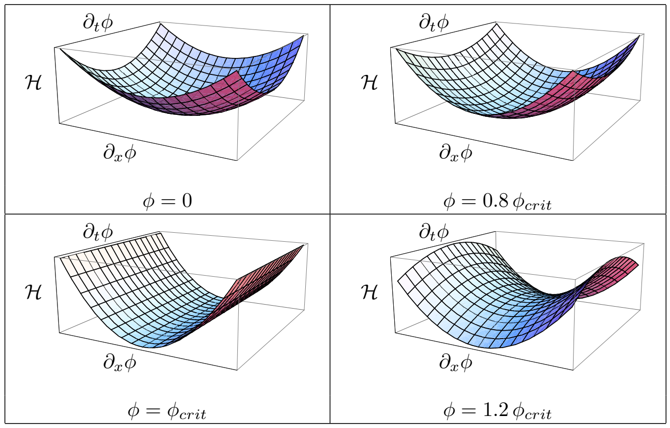

Figure 3.1: Hamiltonian density (vertical axis) when $\beta_1 = 1$, $\beta_* = 1.1$, and $\theta = \psi = \partial_y\phi = \partial_z\phi = 0$ as a function of $\partial_t\phi$ (axis pointing into page) and $\partial_x\phi$ (axis pointing out of page) for various ϕ ranging from zero to $\phi_{crit} = \tanh^{-1}\sqrt{\beta_1/\beta_*}$, the value of ϕ for which the Hamiltonian is flat at $\partial_x\phi = 0$, and beyond. Notice that the Hamiltonian density turns over and becomes negative in the $\partial_t\phi$ direction when $\phi > \phi_{crit}$.

small perturbations in the next section. At the end of this section, we will explain why such regions of large ϕ are still in the regime of validity of the effective field theory expansion.

3.3.2 Spacelike vector field

We now perform an equivalent analysis for an aether field with a spacelike expectation value. In this case all of the possibilities lead to Hamiltonians (3.34) that are unbounded below, and the case $\beta_1 = \beta_* > 0$ is not picked out.

- **Case One:** $\beta_1 < 0$ **and** $\beta_4 = 0$**.**

  Taking $(\partial_\mu\phi) = 0$, $\partial_j f_i = 0$, $\partial_0 f_i \neq 0$, we find

$$\mathcal{H}^{(s)} \sim m^2\beta_1 \cosh^2\phi(\partial_0 f_i)^2. \qquad (3.40)$$

- **Case Two:** $\beta_1 > 0$**,** $\beta_* \leq \beta_1$**, and** $\beta_4 = 0$**.**

Now we consider $\partial_\mu f_i = 0$, $\partial_i \phi \neq 0$, $\partial_0 \phi = 0$, giving

$$\mathcal{H}^{(s)} \sim m^2 \left[-\beta_1 \partial_i \phi \partial_i \phi + (\beta_* - \beta_1) \sinh^2 \phi (f_i \partial_i \phi)^2 \right]. \tag{3.41}$$

- **Case Three:** $\beta_1 \geq 0$**,** $\beta_* > \beta_1$**, and** $\beta_4 = 0$**.**

  In this case we examine $(\partial_0 \phi) \neq 0$, $\partial_\mu f_i = 0$, $\partial_i \phi = 0$, which leads to

$$\mathcal{H}^{(s)} \sim m^2 (\beta_1 - \beta_*) \cosh^2 \phi (\partial_0 \phi)^2. \tag{3.42}$$

- **Case Four:** $\beta_4 \neq 0$**.**

  Now we consider configurations with $\partial_\mu f_i \neq 0$, $\partial_\mu \phi = 0$ and $\sinh^2 \phi \gg 1$. Then,

$$\mathcal{H}^{(s)} \sim m^2 \beta_4 \left(\cosh^4 \phi (f_i \partial_i f_l)(f_k \partial_k f_l) - \cosh^2 \phi \sinh^2 \phi (\partial_0 f_i)^2 \right). \tag{3.43}$$

In every case, it is clear that we can find initial data for a spacelike vector field that makes the Hamiltonian as negative as we please, for all possible β_1, β_4, and β_*.

3.3.3 Smooth potential

The usual interpretation of a Lagrange multiplier constraint is that it is the low-energy limit of smooth potentials when the massive degrees of freedom associated with excitations away from the minimum cannot be excited. We now investigate whether these degrees of freedom can destabilize the theory. Consider the most general, dimension four, positive semi-definite smooth potential that has a minimum when the vector field takes a timelike vacuum expectation value,

$$V = \frac{\xi}{4}(A_\mu A^\mu + m^2)^2, \tag{3.44}$$

where ξ is a positive dimensionless parameter. The precise form of the potential should not affect the results as long as the potential is non-negative and has the global minimum at $A_\mu A^\mu = -m^2$.

We have seen that the Hamiltonian is unbounded from below unless the kinetic term takes the sigma-model form, $(\partial_\mu A_\nu)(\partial^\mu A^\nu)$. Thus we take the Lagrangian to be

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) - \frac{\xi}{4}(A_\mu A^\mu + m^2)^2. \tag{3.45}$$

Consider some fixed timelike vacuum $\bar{A}_\mu$ satisfying $\bar{A}_\mu \bar{A}^\mu = -m^2$. We may decompose the aether field into a scaling of the norm, represented by a scalar Φ, and an orthogonal displacement, represented by vector B_μ satisfying $\bar{A}_\mu B^\mu = 0$. We thus have

$$A_\mu = \bar{A}_\mu - \frac{\bar{A}_\mu \Phi}{m} + B_\mu\,, \tag{3.46}$$

where

$$B_\mu = \left(\eta_{\mu\nu} + \frac{\bar{A}_\mu \bar{A}_\nu}{m^2}\right) A^\nu \quad \text{and} \quad \Phi = \frac{\bar{A}_\mu A^\mu}{m} + m. \tag{3.47}$$

With this parameterization, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}(\partial_\mu B_\nu)(\partial^\mu B^\nu) - \frac{\xi}{4}(2m\Phi + B_\mu B^\mu - \Phi^2)^2. \tag{3.48}$$

The field Φ automatically has a wrong sign kinetic term, and, at the linear level, propagates with a dispersion relation of the form

$$\omega_\Phi^2 = \vec{k}^2 - 2\xi m^2. \tag{3.49}$$

We see that in the case of a smooth potential, there exists a ghostlike mode (wrong-sign kinetic term) that is also tachyonic with spacelike wave vector and a group velocity that generically exceeds the speed of light. It is easy to see that sufficiently long-wavelength perturbations will exhibit exponential growth. The existence of a ghost when the norm of the vector field is not strictly fixed was shown in [31].

In the limit as ξ goes to infinity, the equations of motion enforce a fixed-norm constraint and the ghostlike and tachyonic degree of freedom freezes. The theory is equivalent to one of a Lagrange multiplier if the limit is taken appropriately.

3.3.4 Discussion

To summarize, we have found that the action in (3.12) leads to a Hamiltonian that is globally bounded from below only in the case of a timelike sigma-model Lagrangian, corresponding to $\beta_1 = \beta_* > 0$ and $\beta_4 = 0$. Furthermore, we have verified (as was shown in [31]) that if the Lagrange multiplier term is replaced by a smooth, positive semi-definite potential, then a tachyonic ghost propagates and the theory is destabilized.

If the Hamiltonian is bounded below, the theory is stable, but the converse is not necessarily true. The sigma-model theory is the only one for which this criterion suffices to guarantee stability. In the next section, we will examine the linear stability of these models by considering the growth of perturbations. Although some models are stable at the linear level, we will see in the following section that most of these have negative-energy ghosts, and are therefore unstable once interactions are included. The only exceptions, both ghost-free and linearly stable, are the Maxwell (3.2) and scalar (3.3) models.

We showed in the previous section that, unless $\beta_* - \beta_1$ and β_4 are exactly zero, the Hamiltonian is unbounded from below. However, the effective field theory breaks down before arbitrarily negative values of the Hamiltonian can be reached; when $\beta_* \neq \beta_1$ and/or $\beta_4 \neq 0$, in regions of phase space in which $\mathcal{H} < 0$ (schematically),

$$\mathcal{H} \sim -m^2 e^{4|\phi|}(\partial\Theta)^2 \qquad \text{where} \qquad \Theta \in \{\phi, \theta, \psi\}. \tag{3.50}$$

The effective field theory breaks down when kinetic terms with four derivatives (the terms of next highest order in the effective field theory expansion) are on the order of terms with two derivatives, or, in the angle parameterization, when

$$m^2 e^{4|\phi|}(\partial\Theta)^2 \sim e^{8|\phi|}(\partial\Theta)^4. \tag{3.51}$$

In other words, the effective field theory is only valid when

$$e^{2|\phi|}|\partial\Theta| < m. \tag{3.52}$$

In principle, terms in the effective action with four or more derivatives could add positive contributions to the Hamiltonian to make it bounded from below. However, our analysis shows that the Hamiltonian (in models other than the timelike sigma model with fixed norm) is necessarily concave down around the set of configurations with constant aether fields. If higher-derivative terms intervene to stabilize the Hamiltonian, the true vacuum would not have $H = 0$. Theories could also be deemed stable if there are additional symmetries that lead to conserved currents (other than energy-momentum density) or to a reduced number of physical degrees of freedom.

Regardless of the presence of terms beyond leading order in the effective field theory expansion, due to the presence of the ghost-like and tachyonic mode (found in the previous section), there is an unavoidable problem with perturbations when the field moves in a smooth, positive semi-definite potential. This exponential instability will be present regardless of higher-order terms in the EFT expansion because it occurs for very long-wavelength modes (at least around constant-field backgrounds).

3.4 Linear Instabilities

We have found that the Hamiltonian of a generic aether model is unbounded below. In this section, we investigate whether there exist actual physical instabilities at the linear level—i.e., whether small perturbations grow exponentially with time. It will be necessary to consider the behavior of small fluctuations in every Lorentz frame,[3] not only in the aether rest frame [32, 40, 41]. We find a range of parameters β_i for which the theories are tachyon-free; these correspond (unsurprisingly) to dispersion relations for which the phase velocity satisfies $0 \leq v^2 \leq 1$. In §3.5 we consider the existence of ghosts.

3.4.1 Timelike vector field

Suppose Lorentz invariance is spontaneously broken so that there is a preferred rest frame, and imagine that perturbations of some field in that frame have the following dispersion

[3]The theory of perturbations about a constant background is equivalent to a theory with explicit Lorentz violation because the first-order Lagrange density includes the term, $\lambda\bar{A}^{\mu}\delta A_{\mu}$, where $\bar{A}^{\mu}$ is effectively some constant coefficient.

relation:

$$v^{-2}\omega^2 = \vec{k} \cdot \vec{k}. \tag{3.53}$$

This can be written in frame-invariant notation as

$$(v^{-2} - 1)(t^\mu k_\mu)^2 = k_\mu k^\mu, \tag{3.54}$$

where t^μ is a timelike Lorentz vector that characterizes the 4-velocity of the preferred rest frame. So, in the rest frame, $t^\mu = \{1, 0, 0, 0\}$. Indeed, in the Appendix A, we find dispersion relations for the aether modes of exactly the form in (3.54) with $t^\mu = \bar{A}^\mu/m$ and (A.27)

$$v^2 = \frac{\beta_1}{\beta_1 - \beta_4} \tag{3.55}$$

and (A.28)

$$v^2 = \frac{\beta_*}{\beta_1 - \beta_4}. \tag{3.56}$$

Now consider the dispersion relation for perturbations of the field in another ("primed") frame. Let's solve for $k'_0 = \omega'$, the frequency of perturbations in the new frame. Expanded out, the dispersion relation reads

$$\omega'^2(1 + (v^{-2} - 1)(t'^0)^2) + 2\omega'(v^{-2} - 1)t'^0 t'^i k'_i - \vec{k}' \cdot \vec{k}' + (v^{-2} - 1)(t'^i k'_i)^2 = 0 \tag{3.57}$$

where $i \in \{1, 2, 3\}$. The solution for ω' is:

$$\omega' = \frac{-(v^{-2} - 1)t'^0 t'^i k'_i \pm \sqrt{D_{(t)}}}{1 + (v^{-2} - 1)(t'^0)^2}\,, \tag{3.58}$$

where

$$D_{(t)} = \vec{k}' \cdot \vec{k}' + (v^{-2} - 1)\left((t'^0)^2 \vec{k}' \cdot \vec{k}' - (t'^i k'_i)^2\right). \tag{3.59}$$

In general, $t'^0 = \cosh\eta$ and $t'^i = \sinh\eta\, \hat{n}^i$, where $\hat{n}_i \hat{n}^i = 1$ and $\eta = \cosh^{-1}\gamma$ is a boost parameter. We therefore have

$$D_{(t)} = \vec{k}' \cdot \vec{k}' \left\{1 + (v^{-2} - 1)\left[\cosh^2\eta - \sinh^2\eta\,(\hat{n} \cdot \hat{k}')^2\right]\right\}, \tag{3.60}$$

where $\hat{k}' = \vec{k}'/|\vec{k}'|$. Thus $D_{(t)}$ is clearly greater than zero if $v \leq 1$. However, if $v > 1$ then $D_{(t)}$ can be negative for very large boosts if $\vec{k}'$ is not parallel to the boost direction.

The sign of the discriminant $D_{(t)}$ determines whether the frequency ω' is real- or complex-valued. We have shown that when the phase velocity v of some field excitation is greater than the speed of light in a preferred rest frame, then there is a (highly boosted) frame in which the excitation looks unstable—that is, the frequency of the field excitation can be imaginary. More specifically, plane waves traveling along the boost direction with boost parameter $\gamma = \cosh\eta$ have a growing amplitude if $\gamma^2 > 1/(1 - v^{-2}) > 0$.

In Appendix A, we find dispersion relations of the form in (3.54) for the various massless excitations about a constant timelike background ($t^\mu = \bar{A}^\mu/m$). Requiring stability and thus $0 \leq v^2 \leq 1$ leads to the inequalities,

$$0 \leq \frac{\beta_1}{\beta_1 - \beta_4} \leq 1 \tag{3.61}$$

and

$$0 \leq \frac{\beta_*}{\beta_1 - \beta_4} \leq 1\,. \tag{3.62}$$

Models satisfying these relations are stable with respect to linear perturbations in any

Lorentz frame.

3.4.2 Spacelike vector field

We show in Appendix A that fluctuations about a spacelike, fixed-norm, vector field background have dispersion relations of the form

$$(v^2 - 1)(s^\mu k_\mu)^2 = -k_\mu k^\mu, \tag{3.63}$$

with $s^\mu = \bar{A}^\mu/m$ and (A.27)

$$v^2 = \frac{\beta_1 + \beta_4}{\beta_1} \tag{3.64}$$

and (A.28)

$$v^2 = \frac{\beta_1 + \beta_4}{\beta_*}. \tag{3.65}$$

In frames where $s^\mu = \{0, \hat{s}\}$, v is the phase velocity in the $\hat{s}$ direction.

Consider solving for $k'_0 = \omega'$ in an arbitrary ("primed") frame. The solution is as in (3.58), but with $v^{-2} \to 2 - v^2$ and $t'^\mu \to s'^\mu$. Thus,

$$\omega' = \frac{(v^2 - 1)s'^0 s'^i k'_i \pm \sqrt{D_{(s)}}}{1 + (1 - v^2)(s'^0)^2}\,, \tag{3.66}$$

where

$$D_{(s)} = \vec{k}' \cdot \vec{k}' - (v^2 - 1)\left[(s'^0)^2 \vec{k}' \cdot \vec{k}' - (s'^i k'_i)^2\right]. \tag{3.67}$$

In general, $s'^0 = \sinh\eta$ and $s'^i = \cosh\eta\,\hat{n}^i$ where $\hat{n}_i \hat{n}^i = 1$ and $\eta = \cosh^{-1}\gamma$ is a boost parameter. So,

$$D_{(s)} = \vec{k}' \cdot \vec{k}' \left\{1 - (v^2 - 1)\left[\sinh^2\eta - \cosh^2\eta\,(\hat{n} \cdot \hat{k}')^2\right]\right\}, \tag{3.68}$$

which can be rewritten,

$$D_{(s)} = \vec{k}' \cdot \vec{k}' \left\{ v^2 + (1 - v^2) \cosh^2 \eta \left[1 - (\hat{n} \cdot \hat{k}')^2 \right] \right\} . \tag{3.69}$$

It is clear that $D_{(s)}$ is non-negative for all values of η if and only if $0 \leq v^2 \leq 1$. The theory will be unstable unless $0 \leq v^2 \leq 1$.

The dispersion relations of the form (3.63) for the massless excitations about the spacelike background are given in Appendix A. The requirement that $0 \leq v^2 \leq 1$ implies

$$0 \leq \frac{\beta_1 + \beta_4}{\beta_1} \leq 1 \tag{3.70}$$

and

$$0 \leq \frac{\beta_1 + \beta_4}{\beta_*} \leq 1 \, . \tag{3.71}$$

Models of spacelike aether fields will only be stable with respect to linear perturbations if these relations are statisfied.

The requirements (3.62) or (3.71) do not apply in the Maxwell case (when $\beta_* = 0 = \beta_4$), and those of (3.61) or (3.70) do not apply in the scalar case (when $\beta_1 = 0 = \beta_4$), since the corresponding degrees of freedom in each case do not propagate.

3.4.3 Stability is not frame-dependent

The excitations about a constant background are massless (i.e., the frequency is proportional to the magnitude of the spatial wave vector), but they generally do not propagate along the light cone. In fact, when $v > 1$, the wave vector is timelike even though the cone along which excitations propagate is strictly outside the light cone. We have shown that such excitations blow up in some frame. The exponential instability occurs for observers in boosted frames. In these frames, portions of constant-time hypersurfaces are actually inside the cone along which excitations propagate.

Why do we see the instability in only *some* frames when performing a linear stability analysis? Consider boosting the wave four-vectors of such excitations with complex-valued frequencies and real-valued spatial wave vectors back to the rest frame. Then, in the rest frame, both the frequency and the spatial wave vector will have nonzero imaginary parts. Such solutions with complex-valued $\vec{k}$ require initial data that grow at spatial infinity and are therefore not really "perturbations" of the background. But even though the aether field defines a rest frame, there is no restriction against considering small perturbations defined on a constant-time hypersurface in any frame. Well-behaved initial data can be decomposed into modes with real spatial wave vectors; if any such modes lead to runaway growth, the theory is unstable.

3.5 Negative Energy Modes

We found above that manifest perturbative stability in all frames requires $0 \leq v^2 \leq 1$. In Appendix A, we show that there are two kinds of propagating modes, except when $\beta_* = \beta_4 = 0$ or when $\beta_1 = \beta_4 = 0$. Based on the dispersion relations for these modes, the $0 \leq v^2 \leq 1$ stability requirements translated into the inequalities for β_*, β_1, and β_4 in (3.61)–(3.62) for timelike aether and (3.70)–(3.71) for spacelike aether. We shall henceforth assume that these inequalities hold and, therefore, that ω and $\vec{k}$ for each mode are real in every frame. We will now show that, even when these requirements are satisfied and the theories are linearly stable, there will be negative-energy ghosts that imply instabilities at the nonlinear level (except for the sigma model, Maxwell, and scalar cases).

For timelike vector fields, with respect to the aether rest frame, the various modes correspond to two spin-1 degrees of freedom and one spin-0 degree of freedom. Based on their similarity in form to the timelike aether rest frame modes, we will label these modes once and for all as "spin-1" or "spin-0", even though these classifications are only technically correct for timelike fields in the aether rest frame.

The solutions to the first-order equations of motion for perturbations δA_μ about an arbitrary, constant, background $\bar{A}_\mu$ satisfying $\bar{A}^\mu \bar{A}_\mu \pm m^2 = 0$ are (see Appendix A):

$$\delta A_\mu = \int d^4k \, q_\mu(k) e^{ik_\mu x^\mu}, \qquad q_\mu(k) = q^*_\mu(-k) \tag{3.72}$$

where either,

$$q_\mu(k) = i\alpha^\nu k^\rho \frac{\bar{A}^\sigma}{m} \epsilon_{\mu\nu\rho\sigma} \;\text{ and }\; \beta_1 k_\mu k^\mu + \beta_4 \frac{(\bar{A}_\mu k^\mu)^2}{m^2} = 0 \;\text{ and }\; \alpha^\nu \bar{A}_\nu = 0 \qquad \text{(spin-1)} \tag{3.73}$$

where α^ν are real-valued constants or

$$q_\mu = i\alpha \left(\eta_{\mu\nu} \pm \frac{\bar{A}_\mu \bar{A}_\nu}{m^2} \right) k^\nu \qquad \text{and} \qquad \left(\beta_* \eta_{\mu\nu} + (\beta_4 \pm (\beta_* - \beta_1)) \frac{\bar{A}_\mu \bar{A}_\nu}{m^2} \right) k^\mu k^\nu = 0 \qquad \text{(spin-0)} \tag{3.74}$$

where α is a real-valued constant.

Note that when $\beta_1 = \beta_4 = 0$, corresponding to the scalar form of (3.3), the spin-1 dispersion relation is satisfied trivially, because the spin-1 mode does not propagate in this case. Similarly, when $\beta_* = \beta_4 = 0$, the kinetic term takes on the Maxwell form in (3.2) and the spin-0 dispersion relation becomes $\bar{A}_\mu k^\mu = 0$; the spin-0 mode does not propagate in that case.

The Hamiltonian (3.25) for either of these modes is

$$H = \int d^3k \, \left\{ \left[\beta_1(\omega^2 + \vec{k}\cdot\vec{k}) + \beta_4(-(\bar{a}^0\omega)^2 + (\bar{a}^i k_i)^2) \right] q^\mu q^*_\mu + (\beta_1 - \beta_*)(\omega^2 q^*_0 q_0 + k_i q^*_i k_j q_j) \right\} \, , \tag{3.75}$$

where $k_0 = \omega = \omega(\vec{k})$ is given by the solution to a dispersion relation and where $\bar{a}^\mu \equiv \bar{A}^\mu/m$. One can show that, as long as β_1 and β_4 satisfy the conditions (3.61) or (3.70) that guarantee real frequencies ω in all frames, we will have

$$q^*_\mu q^\mu \geq 0 \tag{3.76}$$

for all timelike and spacelike vector perturbations. We will now proceed to evaluate the Hamiltonian for each mode in different theories.

3.5.1 Spin-1 energies

In this section we consider nonvanishing β_4, and show that the spin-1 mode can carry negative energy even when the conditions for linear stability are satisfied.

Timelike vector field. Without loss of generality, set

$$\bar{A}_\mu = m(\cosh\eta, \sinh\eta\,\hat{n}), \tag{3.77}$$

where $\hat{n}\cdot\hat{n} = 1$. The energy of the spin-1 mode in the timelike case is given by

$$H = \int d^3k (\vec{k}\cdot\vec{k}) q^*_\mu q^\mu \left[\frac{2X \mp \beta_4 \sinh(2\eta)(\hat{n}\cdot\hat{k})\sqrt{X}}{\beta_1 - \beta_4\cosh^2\eta} \right] , \tag{3.78}$$

where

$$X = \beta_1 \left\{ \beta_1 + \beta_4 \left[(\hat{n} \cdot \hat{k})^2 \sinh^2 \eta - \cosh^2 \eta \right] \right\} . \tag{3.79}$$

Looking specifically at modes for which $\hat{n} \cdot \hat{k} = +1$, we find

$$H = \int d^3k (\vec{k} \cdot \vec{k}) q^*_\mu q^\mu \left[\frac{2\beta_1(\beta_1 - \beta_4) \mp \beta_4 \sinh(2\eta) \sqrt{\beta_1(\beta_1 - \beta_4)}}{\beta_1 - \beta_4 \cosh^2 \eta} \right] . \tag{3.80}$$

The energy of such a spin-1 perturbation can be negative when $|\beta_4 \sinh(2\eta)| > 2\sqrt{\beta_1(\beta_1 - \beta_4)}$. Thus it is possible to have negative energy perturbations whenever $\beta_4 \neq 0$. Perturbations with wave numbers perpendicular to the boost direction have positive semi-definite energies.

Spacelike vector field. Without loss of generality, for the spacelike case we set

$$\bar{A}_\mu = m(\sinh \eta, \cosh \eta \, \hat{n}) , \tag{3.81}$$

where $\hat{n} \cdot \hat{n} = 1$. The energy of the spin-1 mode in this case is given by

$$H = \int d^3k (\vec{k} \cdot \vec{k}) q^*_\mu q^\mu \left[\frac{2X \mp \beta_4 \sinh(2\eta)(\hat{n} \cdot \hat{k})\sqrt{X}}{\beta_1 - \beta_4 \sinh^2 \eta} \right] , \tag{3.82}$$

where

$$X = \beta_1 \left\{ \beta_1 + \beta_4 \left[(\hat{n} \cdot \hat{k})^2 \cosh^2 \eta - \sinh^2 \eta \right] \right\} . \tag{3.83}$$

Looking at modes for which $\hat{n} \cdot \hat{k} = +1$, we find

$$H = \int d^3k (\vec{k} \cdot \vec{k}) q^*_\mu q^\mu \left[\frac{2\beta_1(\beta_1 + \beta_4) \mp \beta_4 \sinh(2\eta) \sqrt{\beta_1(\beta_1 + \beta_4)}}{\beta_1 - \beta_4 \sinh^2 \eta} \right] . \tag{3.84}$$

Thus, the energy of perturbations can be negative when $|\beta_4 \sinh(2\eta)| > 2\sqrt{\beta_1(\beta_1+\beta_4)}$. Thus it is possible to have negative energy perturbations whenever $\beta_4 \neq 0$. Perturbations with wave numbers perpendicular to the boost direction have positive semi-definite energies. In either the timelike or spacelike case, models with $\beta_4 \neq 0$ feature spin-1 modes that can be ghostlike.

We note that the effective field theory is valid when $k < e^{-3|\eta|}m$, as detailed in §3.2.1. But even if η is very large, the effective field theory is still valid for very long wavelength perturbations, and therefore such long wavelength modes with negative energies lead to genuine instabilities.

3.5.2 Spin-0 energies

We now assume the inequalities required for linear stability, (3.62) or (3.71), and also that $\beta_4 = 0$. We showed above that, otherwise, there are growing modes in some frame or there are propagating spin-1 modes that have negative energy in some frame. When $\beta_* \neq 0$, the energy of the spin-0 mode in (3.74) is given by

$$H = 2\beta_1\alpha^2 \int d^3k\, (\bar{a}_\rho k^\rho)^2 \left(\omega^2(\vec{k})\left[\pm 1 - (1-\beta_1/\beta_*)\bar{a}_0^2\right] + \omega(\vec{k})\,\bar{a}_0(1-\beta_1/\beta_*)\bar{a}_i k_i\right) \quad (3.85)$$

for $\bar{A}_\mu \bar{A}^\mu \pm m^2 = 0$ and $\bar{a}_\mu \equiv \bar{A}_\mu/m$.

Timelike vector field. We will now show that the quadratic order Hamiltonian can be negative when the background is timelike and the kinetic term does not take one of the special forms (sigma model, Maxwell, or scalar). Without loss of generality we set $\bar{a}_0 = \cosh\eta$ and $\bar{a}_i = \sinh\eta\,\hat{n}_i$, where $\hat{n}\cdot\hat{n} = 1$. Then plugging the freqency $\omega(\vec{k})$, as defined by the spin-0 dispersion relation, into the Hamiltonian (3.85) gives

$$H = \beta_1 \alpha^2 \int d^3k \, (\bar{a}_\rho k^\rho)^2 \left[\frac{2X \pm (1 - \beta_1/\beta_*) \sinh 2\eta (\hat{n} \cdot \hat{k}) \sqrt{X}}{1 + (\beta_1/\beta_* - 1) \cosh^2 \eta} \right], \tag{3.86}$$

where

$$X = 1 + (\beta_1/\beta_* - 1)[\cosh^2 \eta - (\hat{n} \cdot \hat{k})^2 \sinh^2 \eta]. \tag{3.87}$$

If $\hat{n} \cdot \hat{k} \neq 0$, the energy can be negative. In particular, if $\hat{n} \cdot \hat{k} = 1$ we have

$$H = \beta_1 \alpha^2 \int d^3k \, (\bar{a}_\rho k^\rho)^2 \left[2 \frac{\beta_1/\beta_* \pm (1 - \beta_1/\beta_*) \sinh 2\eta \sqrt{\beta_1/\beta_*}}{1 + (\beta_1/\beta_* - 1) \cosh^2 \eta} \right]. \tag{3.88}$$

Given that $\beta_1/\beta_* - 1 \geq 0$, H can be negative when $|\sinh 2\eta| > 2\sqrt{\beta_1/\beta_*}/(\beta_1/\beta_* - 1)$.

We have thus shown that, for timelike backgrounds, there are modes that in some frame have negative energies and/or growing amplitudes as long as $\beta_1 \neq \beta_*$, $\beta_1 \neq 0$, and $\beta_* \neq 0$. Therefore, the only possibly stable theories of timelike aether fields are the special cases mentioned earlier: the sigma-model ($\beta_1 = \beta_*$), Maxwell ($\beta_* = 0$), and scalar ($\beta_1 = 0$) kinetic terms.

Spacelike vector field. For the spacelike case, without loss of generality we set $\bar{a}_0 = \sinh \eta$ and $\bar{a}_i = \cosh \eta \, \hat{n}_i$, where $\hat{n} \cdot \hat{n} = 1$. Once again, plugging the frequency $\omega(k)$ into the Hamiltonian (3.85) gives

$$H = \beta_1 \alpha^2 \int d^3k \, (\bar{a}_\rho k^\rho)^2 \left[\frac{-2X \pm (1 - \beta_1/\beta_*) \sinh 2\eta (\hat{n} \cdot \hat{k}) \sqrt{X}}{1 + (1 - \beta_1/\beta_*) \sinh^2 \eta} \right], \tag{3.89}$$

where

$$X = 1 + (1 - \beta_1/\beta_*) \left[\sinh^2 \eta - (\hat{n} \cdot \hat{k})^2 \cosh^2 \eta \right]. \tag{3.90}$$

Upon inspection, one can see that there are values of $\hat{n} \cdot \hat{k}$ and η that make H negative, except when $\beta_* = 0$ (Maxwell) or $\beta_1 = 0$ (scalar). Again, the Hamiltonian density is less than zero for modes with wavelengths sufficiently long ($k < e^{-3|\eta|}m$), so the effective theory is valid.

3.6 Maxwell and Scalar Theories

We have shown that the only version of the aether theory (3.12) for which the Hamiltonian is bounded below is the timelike sigma-model theory $\mathcal{L}_K = -(1/2)(\partial_\mu A_\nu)(\partial^\mu A^\nu)$, corresponding to the choices $\beta_1 = \beta_*$, $\beta_4 = 0$, with the fixed-norm condition imposed by a Lagrange multiplier constraint. (Here and below, we rescale the field to canonically normalize the kinetic terms.) However, when we looked for explicit instabilities in the form of tachyons or ghosts in the last two sections, we found two other models for which such pathologies are absent: the Maxwell Lagragian

$$\mathcal{L}_K = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{3.91}$$

corresponding to $\beta_* = 0 = \beta_4$, and the scalar Lagrangian

$$\mathcal{L}_K = \frac{1}{2}(\partial_\mu A^\mu)^2\,, \tag{3.92}$$

corresponding to $\beta_1 = 0 = \beta_4$. In both of these cases, we found that the Hamiltonian is unbounded below,[4] but a configuration with a small positive energy does not appear to run away into an unbounded region of phase space characterized by large negative and positive balancing contributions to the total energy.

These two models are also distinguished in another way: there are fewer than three

[4]Boundedness of the Hamiltonian was considered in [45].

propagating degrees of freedom at first order in perturbations in the Maxwell and scalar Lagrangian cases, while there are three in all others. This is closely tied to the absence of perturbative instabilities; the ultimate cause of those instabilities can be traced to the difficulty in making all of the degrees of freedom simultaneously well-behaved. The drop in number of degrees of freedom stems from the fact that A_0 lacks time derivatives in the Maxwell Lagrangian and that the A_i lack time derivatives in the scalar Lagrangian. In other words, some of the vector components are themselves Lagrange multipliers in these special cases.

Only two perturbative degrees of freedom—the spin-1 modes—propagate in the Maxwell case (cf. (3.73) and (3.74) when $\beta_* = 0 = \beta_4$). The "mode" in (3.74) is a gauge degree of freedom; at first order in perturbations the Lagrangian has a gauge-like symmetry under $\delta A_\mu \to \delta A_\mu + \partial_\mu \phi(x)$ where $\bar{A}^\mu \partial_\mu \phi = 0$. As expected of a gauge degree of freedom, the spin-0 mode has zero energy and does not propagate. Meanwhile, the spin-1 perturbations propagate as well-behaved plane waves and have positive energy. We note that the Dirac method for counting degrees of freedom in constrained dynamical systems implies that there are *three* degrees of freedom [42].[5] The additional degree of freedom, not apparent at the linear level, could conceivably cause an instability; this mode does not propagate because it is gauge-like at the linear level, but there is no gauge symmetry in the full theory.

In the scalar case, there are no propagating spin-1 degrees of freedom. The spin-0 degree of freedom has a nontrivial dispersion relation but no energy density (cf. (3.73), (3.74), (3.86), and (3.89) when $\beta_1 = 0 = \beta_4$) at leading order in the perturbations. Essentially, the fixed-norm constraint is incompatible with what would be a single propagating scalar mode in this model; the theory is still dynamical, but perturbation theory fails to

[5] For a discussion of constrained dynamical systems see [46].

capture its dynamical content.

Each of these models displays some idiosyncratic features, which we now consider in turn.

3.6.1 Maxwell action

The equation of motion for the Maxwell Lagrangian with a fixed-norm constraint is

$$\partial_\mu F^{\mu\nu} = -2\lambda A^\nu \, . \tag{3.93}$$

Setting $A_\mu A^\mu = \mp m^2$, the Lagrange multiplier is given by

$$\lambda = \pm \frac{1}{2m^2} A_\nu \partial_\mu F^{\mu\nu} \, . \tag{3.94}$$

For timelike aether fields, the sign of λ is preserved along timelike trajectories since, when the kinetic term takes the special Maxwell form, there is a conserved current (in addition to energy-momentum density) due to the Bianchi identity[6]:

$$0 = \partial_\nu(\partial_\mu F^{\mu\nu}) = -2\partial_\nu(\lambda A^\nu). \tag{3.95}$$

In particular, the condition that $\lambda = 0$ is conserved along timelike A^ν [15, 42]. In the presence of interactions this will continue to be true only if the coupling to external sources takes the form of an interaction with a conserved current, $A_\mu J^\mu$ with $\partial_\mu J^\mu = 0$.

If we take the timelike Maxwell theory coupled to a conserved current and restrict to initial data satisfying $\lambda = 0$ at every point in space, the theory reduces precisely to

[6] If $\lambda > 0$ initially, then it must pass through $\lambda = 0$ to reach $\lambda < 0$—but $\lambda = 0$ is conserved along timelike trajectories, so λ can at best stop at $\lambda = 0$.

Maxwell electrodynamics—not only in the equation of motion, but also in the energy-momentum tensor. We can therefore be confident that this theory, restricted to this subset of initial data, is perfectly well-behaved, simply because it is identical to conventional electromagnetism in a nonlinear gauge [21, 43, 47].

In the case of a spacelike vector expectation value, there is an explicit obstruction to finding smooth time evolution for generic initial data. In this case, the constraint equations are

$$-A_0^2 + A_i A_i = m^2 \qquad \text{and} \qquad \partial_i \partial^i A_0 - \partial_0 \partial_i A^i = -2\lambda A_0. \tag{3.96}$$

Suppose spatially homogeneous initial conditions for the A_i are given. Without loss of generality, we can align axes such that

$$A_\mu(t_0) = (A_0(t_0), 0, 0, A_3(t_0)), \tag{3.97}$$

where $-A_0^2 + A_3^2 = m^2$. If $A_i A_i \neq m^2$, the equations of motion are

$$\partial_\mu F^\mu{}_\nu = 0. \tag{3.98}$$

The $\nu = 3$ equation reads

$$\partial_\mu F^\mu{}_3 = -\frac{\partial^2 A_3}{\partial t^2} = 0, \tag{3.99}$$

whose solutions are given by

$$A_3(t) = A_3(t_0) + C(t - t_0), \tag{3.100}$$

where C is determined by initial conditions. A_0 is determined by the fixed-norm constraint

$A_0 = \pm\sqrt{A_3^2 - m^2}$. If $C \neq 0$, A_0 will eventually evolve to zero. Beyond this point, A_3 keeps decreasing, and the fixed-norm condition requires that A_0 be imaginary, which is unacceptable since A_μ is a real-valued vector field. Note that this never happens in the timelike case, as there always exists some real A_0 that satisfies the constraint for any value of A_3. The problem is that A_3 evolves into the ball $A_i^2 < m^2$, which is catastrophic for the spacelike, but not the timelike, case. An analogous problem arises even when the Lagrange multiplier constraint is replaced by a smooth potential.

It is possible that this obstruction to a well-defined evolution will be regulated by terms of higher order in the effective field theory. Using the fixed-norm constraint and solving for A_0, the derivative is

$$\partial_\mu A_0 = \frac{A_i}{\sqrt{A_j A_j - m^2}} \partial_\mu A_i. \tag{3.101}$$

As $A_j A_j$ approaches m^2, with finite derivatives of the spatial components, the derivative of the A_0 component becomes unbounded. If higher-order terms in the effective action have time derivatives of the component A_0, these terms could become relevant to the vector field's dynamical evolution, indicating that we have left the realm of validity of the low-energy effective field theory we are considering.

We are left with the question of how to interpret the timelike Maxwell theory with intial data for which $\lambda \neq 0$. If we restrict our attention to initial data for which $\lambda < 0$ everywhere, then the evolution of the A_i would be determined and the Hamiltonian would be positive.

We have

$$H = \frac{1}{2}\int d^3x\,\left(\frac{1}{2}F_{ij}^2 + (\partial_0 A_i)^2 - (\partial_i A_0)^2\right) \tag{3.102}$$

$$= \frac{1}{2}\int d^3x\,\left(\frac{1}{2}F_{ij}^2 + F_{0i}F_{0i} - 2(\partial_i A_0)F_{i0}\right) \tag{3.103}$$

$$= \frac{1}{2}\int d^3x\,\left(\frac{1}{2}F_{ij}^2 + F_{0i}F_{0i} + 2A_0\partial_i F_{i0}\right) \tag{3.104}$$

$$= \frac{1}{2}\int d^3x\,\left(\frac{1}{2}F_{ij}^2 + F_{0i}F_{0i} - 4\lambda A_0^2\right), \tag{3.105}$$

which is manifestly positive when $\lambda < 0$. However, it is not clear why we should be restricted to this form of initial data, nor whether even this restriction is enough to ensure stability beyond perturbation theory.

The status of this model in both the spacelike and timelike cases remains unclear. However, there are indications of further problems. For the spacelike case, Peloso *et. al.* find a linear instability for perturbations with wave numbers on the order of the Hubble parameter in an exponentially expanding cosmology [48, 49]. For the timelike case, Seifert found a gravitational instability in the presence of a spherically symmetric source [50].

3.6.2 Scalar action

The equation of motion for the scalar Lagrangian with a fixed-norm constraint is

$$\partial^\nu \partial_\mu A^\mu = 2\lambda A^\nu. \tag{3.106}$$

Using the fixed-norm constraint ($A_\mu A^\mu = \mp m^2$), we can solve for the Lagrange multiplier field,

$$\lambda = \mp\frac{1}{2m^2}A_\nu \partial^\nu \partial_\mu A^\mu. \tag{3.107}$$

In contrast with the Maxwell theory, in the scalar theory it is the timelike case for which we can demonstrate obstacles to smooth evolution, while the spacelike case is less clear. (The Hamiltonian is bounded below, but there are no perturbative instabilities or known obstacles to smooth evolution.)

When the vector field is timelike, we have four constraint equations in the scalar case,

$$A_0^2 - A_i A_i = m^2 \qquad \text{and} \qquad \partial_i(\partial_\mu A^\mu) = 2\lambda A_i. \tag{3.108}$$

Suppose we give homogeneous initial conditions such that $A_0(t_0) > m$. Align axes such that,

$$A_\mu(t_0) = \left(A_0(t_0), 0, 0, A_3(t_0)\right), \tag{3.109}$$

where $A_3(t_0)^2 = A_0(t_0)^2 - m^2$. Note that, since $A_3(t_0) \neq 0$, we have that $\lambda = 0$ from the $\nu = 3$ equation of motion. The $\nu = 0$ equation of motion therefore gives,

$$\frac{d^2 A_0}{dt^2} = 0. \tag{3.110}$$

We see that the timelike component of the vector field has the time-evolution,

$$A_0(t) = A_0(t_0) + C(t - t_0). \tag{3.111}$$

For generic homogeneous initial conditions, $C \neq 0$. In this case, A_0 will not have a smooth time evolution since A_0 will saturate the fixed-norm constraint, and beyond this point A_0 will continue to decrease in magnitude. To satisfy the fixed-norm constraint, the spatial components of the vector field A_i would need to be imaginary, which is unacceptable since A_μ is a real-valued vector field. This problem never occurs for the spacelike case since there always exist real values of A_i that satisfy the constraint for any A_0.

Again, it is possible that this obstruction to a well-defined evolution will be regulated by terms of higher order in the effective field theory. The time derivative of A_3 is

$$\partial_\mu A_3 = \frac{A_0}{\sqrt{A_0 A_0 - m^2}} \partial_\mu A_0. \tag{3.112}$$

As $A_0 A_0$ approaches m^2, with finite derivatives of A_0, the derivative of the spatial component A_3 becomes unbounded. If higher-order terms in the effective action have time derivatives of the components A_i, these terms could become relevant to the vector field's dynamical evolution, indicating that we have left the realm of validity of the low-energy effective field theory we are considering.

Whether or not a theory with a scalar kinetic term and fixed expectation value is viable remains uncertain.

3.7 Conclusions

In this chapter, we addressed the issue of stability in theories in which Lorentz invariance is spontaneously broken by a dynamical fixed-norm vector field with an action

$$S = \int d^4x \left(-\frac{1}{2}\beta_1 F_{\mu\nu}F^{\mu\nu} - \beta_*(\partial_\mu A^\mu)^2 - \beta_4 \frac{A^\mu A^\nu}{m^2}(\partial_\mu A_\rho)(\partial_\nu A^\rho) + \lambda(A^\mu A_\mu \pm m^2) \right), \tag{3.113}$$

where λ is a Lagrange multiplier that strictly enforces the fixed-norm constraint. In the spirit of effective field theory, we limited our attention to only kinetic terms that are quadratic in derivatives, and took care to ensure that our discussion applies to regimes in which an effective field theory expansion is valid.

We examined the boundedness of the Hamiltonian of the theory and showed that, for generic choices of kinetic term, the Hamiltonian is unbounded from below. Thus for a generic kinetic term, we have shown that a constant fixed-norm background is not the true vacuum of the theory. The only exception is the timelike sigma-model Lagrangian ($\beta_1 = \beta_*$, $\beta_4 = 0$ and $A^\mu A_\mu = -m^2$), in which case the Hamiltonian is positive-definite, ensuring stability. However, if the vector field instead acquires its vacuum expectation value by minimizing a smooth potential, we demonstrated (as was done previously in [31]) that the theory is plagued by the existence of a tachyonic ghost, and the Hamiltonian is unbounded from below. The timelike fixed-norm sigma-model theory nevertheless serves as a viable starting point for phenomenological investigations of Lorentz invariance; we explore some of this phenomenology in Chapter 4.

We next examined the dispersion relations and energies of first-order perturbations about constant background configurations. We showed that, in addition to the sigma-model case, there are only two other choices of kinetic term for which perturbations have non-negative energies and do not grow exponentially in any frame: the Maxwell ($\beta_* = \beta_4 = 0$) and scalar ($\beta_1 = \beta_4 = 0$) Lagrangians. In either case, the theory has fewer than three propagating degrees of freedom at the linear level, as some of the vector components in the action lack time derivatives and act as additional Lagrange multipliers. A subset of the phase space for the Maxwell theory with a timelike aether field is well-defined and stable, but is identical to ordinary electromagnetism. For the Maxwell theory with a spacelike aether field, or the scalar theory with a timelike field, we can find explicit obstructions to smooth time evolution. It remains unclear whether the timelike Maxwell theory or the spacelike scalar theory can exhibit true violation of Lorentz invariance while remaining well-behaved.

Chapter 4

Sigma-Model Aether

4.1 Introduction

Models of fixed-norm vector fields, sometimes called "aether" theories, serve a useful purpose as a phenomenological framework in which to investigate violations of Lorentz invariance at low energies [12, 14, 15, 16, 17, 18, 19]. For a recent review, see [34]. In Chapter 3, we argue that almost all such models are plagued by instabilities. For related work on stability in aether theories, see [25, 29, 30, 31, 40, 42, 45, 48].

There is one version of the aether theory that is stable under small perturbations and in which the Hamiltonian is globally bounded when only two-derivative terms are included in the action. This model is defined by a kinetic Lagrange density of the form

$$\mathcal{L}_\sigma^{\text{kinetic}} = -\frac{1}{2}(\nabla_\mu A_\nu)(\nabla^\mu A^\nu)\,, \tag{4.1}$$

where A_μ is a dynamical four-vector aether field. (The spacelike version has an unbounded Hamiltonian and is unstable.) We refer to the theory defined by this action as "sigma-model aether," due to its resemblance to a theory of scalar fields propagating on a fixed manifold with an internal metric, familiar from studies of spontaneous symmetry breaking.

The aether theory is not identical to such a σ-model—in particular in curved space where covariant derivatives act on the vector—but the nomenclature is convenient.

Even though this theory is stable, it has an important drawback. It is conventional in aether models to give the vector field an expectation value by means of a Lagrange multiplier, which enforces the fixed-norm constraint

$$A_\mu A^\mu = -m^2 \,. \tag{4.2}$$

We take m^2 to be positive and use a metric signature $(- + ++)$, so that this defines a timelike vector field. Despite the convenience of this formulation, it seems likely that a more complete version of the theory would arise as a limit of a theory in which the expectation value is fixed by minimizing a smooth potential of the form $V(A_\mu) = \xi(A_\mu A^\mu + m^2)^2$. As we showed in [2], any such theory would be plagued by ghosts and tachyons. As far as we can tell, therefore, the sigma-model aether theory cannot be derived from models with a smooth potential.

Nevertheless, as it is the only globally well-behaved example of any of the aether theories, examining the dynamics and experimental constraints on this model is worthwhile. We undertake such an investigation in this chapter.

First we examine the degrees of freedom in this theory, taking into account the mixing with the gravitational field. There are three different massless modes, of spins 0, 1, and 2 in the aether rest frame.[1] Demanding that none of the modes propagate faster than light fixes a unique value for the coupling of the vector field to the Ricci tensor. We use experimental constraints on the preferred frame parameters $\alpha_{1,2}$ in the Parameterized Post-Newtonian

[1]The lack of rotational symmetry in frames other than the aether rest frame make classification of modes by spin in such frames impossible. But the aether rest frame has rotational symmetry, which allows for the spin classification with respect to this frame.

(PPN) expansion to limit the magnitude of the vacuum expectation value, m. The spin-2 mode can propagate subluminally for some values of the vector field/Ricci tensor coupling; in such cases very tight restrictions on the vacuum expectation value, m, due to limits from vacuum Cherenkov radiation from gravitons come into play.

Finally, we consider the cosmological evolution of the vector field in two different backgrounds. We study the evolution of the timelike vector field in a general flat-Friedmann-Robertson-Walker (FRW) background and find that the vector field tends to align to be orthogonal to constant density hypersurfaces. In a background consisting of a timelike dimension, three expanding spatial dimensions, and one compact (non-expanding) extra-dimension, we find that the vector field can evolve to have a nonzero projection in the direction of the compact extra-dimension if the large dimensions are de Sitter-like. We take this as evidence that a timelike vector field with the Lagrangian that satisfies the aforementioned theoretical and experimental constraints would not lead to any significant departure from statistical isotropy.

4.2 Excitations in the Presence of Gravity

We would like to understand the experimental constraints on, and cosmological evolution of, the sigma-model aether theory. For both of these questions, it is important to consider the effects of gravity. But whereas the flat-space model with a kinetic Lagrangian of the form (4.1) is unique, in curved space there is the possibility of an explicit coupling to curvature. The full action we consider is

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{2}(\nabla_\mu A_\nu)(\nabla^\mu A^\nu) + \frac{\alpha}{2}R_{\mu\nu}A^\mu A^\nu + \frac{\lambda}{2}(A_\mu A^\mu + m^2)\right] . \quad (4.3)$$

Here, λ is the Lagrange multiplier that enforces the fixed-norm constraint (4.2), α is a dimensionless coupling, $R_{\mu\nu}$ is the Ricci tensor and R is the curvature scalar. Note that, given the fixed-norm constraint, there are no other scalar operators that could be formed solely from A_μ and the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$. By integrating by parts and using $R_{\mu\nu}A^\mu A^\nu = A^\nu[\nabla_\mu, \nabla_\nu]A^\mu$, this curvature coupling could equivalently be written purely in terms of covariant derivatives of A_μ; the form (4.3) has the advantage of emphasizing that the new term has no effects in flat spacetime.

In [2] we showed that the sigma-model aether theory was stable in the presence of small perturbations in flat spacetime; the possibility of mixing with gravitons implies that we should check once more in curved spacetime. The equations of motion for the vector field are,

$$-\nabla_\mu \nabla^\mu A^\nu = \lambda A^\nu + \alpha R^{\mu\nu} A_\mu, \tag{4.4}$$

along with the fixed norm constraint from the equation of motion for λ. Assuming the fixed norm constraint, the equations of motion can be written in the form

$$\left(g^{\sigma\nu} + \frac{1}{m^2} A^\sigma A^\nu\right)\left(\nabla_\rho \nabla^\rho A_\sigma + \alpha R_{\rho\sigma} A^\rho\right) = 0. \tag{4.5}$$

The tensor $(g^{\sigma\nu} + A_\rho A_\nu/m^2)$ acts to take what would be the equation of motion in the absence of the constraint, and project it into the hyperplane orthogonal to A_μ.

The Einstein-aether system has a total of five degrees of freedom, all of which propagate as massless fields: one spin-2 graviton, one spin-1 excitation, and one spin-0 excitation. Each of these dispersion relations can be written (in the short-wavelength limit) in frame-invariant notation as,

$$k_\mu k^\mu = \left(\frac{1-v^2}{v^2}\right)\left(\frac{\bar{A}_\mu k^\mu}{m}\right)^2, \tag{4.6}$$

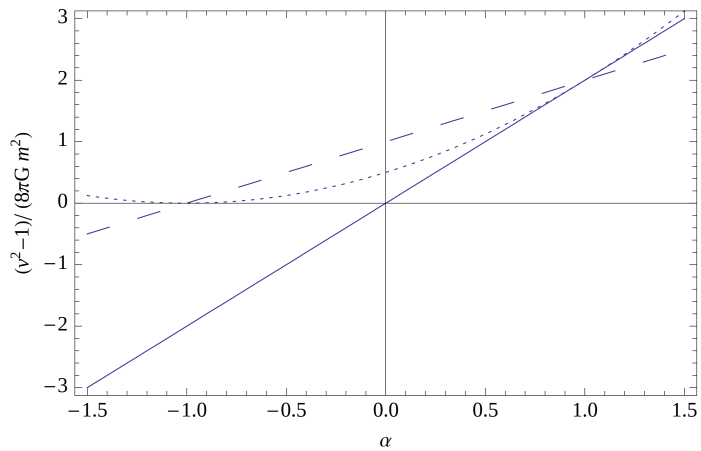

Figure 4.1: Aether rest frame mode phase velocities squared, v^2, minus the speed of light in units of $8\pi Gm^2$ as a function of α. The solid line corresponds to spin-0, the small dashed line to spin-1, and the large dashed line to spin-2. Only for $\alpha = -1$ do none of the modes propagate faster than light ($v^2 - 1 > 0$).

where v is the phase velocity in the aether rest frame. The squared phase velocities of the gravity-aether modes are [16],

$$v^2 = \frac{1}{1 - 8\pi Gm^2(1+\alpha)} \approx 1 + 8\pi Gm^2(1+\alpha) \qquad \text{(spin-2)} \tag{4.7}$$

$$v^2 = \frac{2 - 8\pi Gm^2(1+\alpha)(1-\alpha)}{2\,(1 - 8\pi Gm^2(1+\alpha))} \approx 1 + 4\pi Gm^2(1+\alpha)^2 \qquad \text{(spin-1)} \tag{4.8}$$

$$v^2 = \frac{2 - 8\pi Gm^2}{(1 - 8\pi Gm^2(1+\alpha))\,(2 + 8\pi Gm^2(1-2\alpha))} \approx 1 + 16\pi Gm^2\alpha \qquad \text{(spin-0)} \tag{4.9}$$

where G is the gravitational constant appearing in Einstein's action. The approximate equalities hold assuming $8\pi Gm^2 \ll 1$.[2]

These squared mode phase velocities minus the squared speed of light are plotted in Fig. 4.1 as a function of α. It is clear that the only value of α for which none of the modes

[2]The relationship between the parameters in Eq. (4.3) (α, m^2) and those in [16] (c_1, c_2, c_3, c_4) is:

$$c_1 = 8\pi Gm^2, \qquad -c_2 = c_3 = \alpha 8\pi Gm^2, \qquad c_4 = 0. \tag{4.10}$$

propagate superluminally ($v^2 > 1$) is

$$\alpha = -1\,. \tag{4.11}$$

We therefore have a *unique* version of a Lorentz-violating aether theory for which the Hamiltonian is bounded below (in flat space) and that is free of superluminal modes when coupled to gravity: the sigma-model kinetic term with an expectation value fixed by a Lagrange-multiplier constraint and a coupling to curvature of the form in (4.3) with $\alpha = -1$. In what follows, we will generally allow α to remain as a free parameter when considering experimental limits, keeping in mind that models with $\alpha \neq -1$ are plagued by superluminal modes. We will find that the experimental limits on m are actually weakest when $\alpha = -1$.

Before moving on, however, we should note that the existence of superluminal phase velocities does not constitute *prima facie* evidence that the theory is ill-behaved. There are two reasons for suspecting that superluminal propagation is bad. First, in [2], we showed that such models were associated with perturbative instabilities: there is always a frame in which small perturbations grow exponentially with time. Second, acausal propagation around a closed loop in spacetime could potentially occur if the background aether field were not constant through space [17, 31]. But in the presence of gravity, these arguments are not decisive. There now exists a scale beyond which we expect the theory to break down: namely, length scales on the order of M_{pl}^{-1}. Perhaps there is some length scale involved in boosting to a frame where the instability is apparent (or, equivalently, in approaching a trajectory that is a closed timelike curve) that is order M_{pl}^{-1}.

Again, in a background flat spacetime with a background timelike aether field $\bar{A}_\mu =$ constant, the dispersion relations have the generic form

$$(v^{-2}-1)(t^{\mu}k_{\mu})^2 = k_{\mu}k^{\mu}, \tag{4.12}$$

where $t_{\mu} = \bar{A}_{\mu}/m$ characterizes the 4-velocity of the preferred rest frame. The velocity v^2 is given by Eqs. (4.7)–(4.9). In a boosted frame, where $t^{\mu} = (-\cosh\eta, \sinh\eta\,\hat{n})$, the frequency is given by

$$\frac{\omega}{|\vec{k}|} = \frac{-(1-v^{-2})\sinh\eta\cosh\eta(\hat{k}\cdot\hat{n}) \pm \sqrt{1-(1-v^{-2})(\cosh^2\eta - \sinh^2\eta(\hat{n}\cdot\hat{k})^2)}}{1-(1-v^{-2})\cosh^2\eta}. \tag{4.13}$$

Let us parametrize the boost in the standard way as,

$$\cosh^2\eta = \frac{1}{1-\beta^2} \qquad 0 \le \beta^2 < 1. \tag{4.14}$$

Then

$$\frac{\omega}{|\vec{k}|} = \frac{-(1-v^{-2})\beta(\hat{k}\cdot\hat{n}) \pm \sqrt{1-\beta^2}\sqrt{v^{-2}-\beta^2+\beta^2(1-v^{-2})(\hat{n}\cdot\hat{k})^2}}{v^{-2}-\beta^2}. \tag{4.15}$$

There is a pole in the frequency at $\beta^2 = v^{-2}$. The pole is physical if $v > 1$ and, (in the limit as $\hat{n}\cdot\hat{k} \to 0$) as β passes through the pole ($\beta^2 \to \beta^2 > v^{-2}$), the frequency acquires a nonzero imaginary part, which corresponds to growing mode amplitudes. (The frequency becomes imaginary at some $\beta^2 < 1$ as long as $\hat{n}\cdot\hat{k} \neq 1$.) The time scale on which the mode grows is set by $1/Im(\omega)$. In frames with a boost factor greater than the inverse rest-frame mode speed, $\beta > v^{-1}$, the time scale on which mode amplitudes grow is maximal for modes with wave vectors perpendicular to the boost direction ($\hat{n} \cdot \hat{k} = 0$) and is given by

$$T_{MAX}(\beta) = \frac{1}{|Im(\omega)|} = |\vec{k}|^{-1} \frac{\sqrt{\beta^2 - v^{-2}}}{\sqrt{1-\beta^2}} \qquad \text{when} \qquad v^2 > 1. \tag{4.16}$$

We generically expect the linearized gravity analysis that led to the propagation speeds in Eqs. (4.6)–(4.9) to be valid for wave vectors that are much greater in magnitude than the energy scale set by other energy density in the space-time—generally, the Hubble scale, H. Thus the analysis makes sense for $|\vec{k}|^{-1} \ll H^{-1}$ and (as long as $1 - \beta^2$ is not infinitesimal) there will be instabilities on time scales less than the inverse Hubble scale and (unless $\beta^2 - v^{-2}$ is infinitesimal) greater than M_{Pl}^{-1}.

Thus, not only could superluminal propagation speeds lead to closed timelike curves and violations of causality, but the existence of instabilities on an unremarkable range of less-than-Hubble-radius time scales in boosted frames indicates that such superluminal propagation speeds lead to instabilities. If $v > 1$, it appears as if instabilities can be accessed without crossing some scale threshold beyond which we'd expect the model to break down.

4.3 Experimental Constraints

We now apply existing experimental limits to the sigma-model aether theory, keeping for the moment α as well as m^2 as free parameters. Direct coupling of the aether field to standard model fields fits into the framework of the "Lorentz-violating extension" of the standard model considered in [12]. Such couplings are very tighly constrained by various experiments (for a discussion of experimental constraints, see [32]). The relevant limit from

Cherenkov radiation in [31] translates to,[3]

$$-8\pi G m^2(1+\alpha) < 1\times 10^{-15}. \tag{4.18}$$

Limits on PPN parameters give some of the strongest constraints on α and m^2 when $\alpha \approx -1$ (since the constraint in Eq. (4.18) is automatically satisfied). The preferred frame parameters must satisfy $|\alpha_1| < 10^{-4}$ and $|\alpha_2| < 10^{-7}$ [33]. We have the limits [34],

$$|\alpha_1| \approx |4\alpha^2(8\pi G_N m^2)| < 10^{-4} \quad \text{and} \quad |\alpha_2| \approx |(\alpha+1)(8\pi G_N m^2)| < 10^{-7}, \tag{4.19}$$

where G_N is the gravitational constant as measured in our solar system or table-top experiments. This gravitational constant is related to the parameter in the action G by,

$$G_N = \frac{G}{1-4\pi G m^2}. \tag{4.20}$$

If we require that all modes have phase speeds v that satisfy $v^2 \leq 1$, then we must have $\alpha = -1$ and

$$8\pi G_N m^2 < 10^{-4}. \tag{4.21}$$

All relevant constraints (allowing modes to have larger than unity phase velocities) are summarized in Fig. 4.2. Constraints from big bang nucleosynthesis [17] are significantly weaker than the PPN and Cherenkov constraint above.

[3] [31] uses the same parameters as in [16, 34], thus the translation between our parameters and the parameters used in [16, 34, 31] is (as stated in a previous footnote),

$$c_1 = 8\pi G m^2, \qquad -c_2 = c_3 = \alpha 8\pi G m^2, \qquad c_4 = 0. \tag{4.17}$$

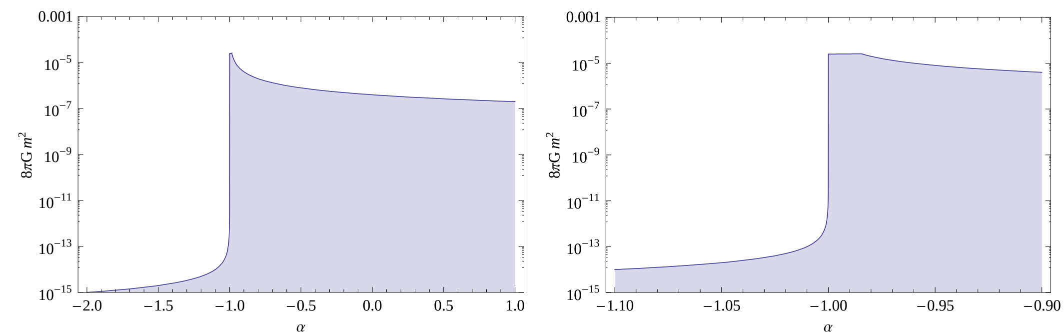

Figure 4.2: Parameter space allowed (shaded region) by constraints from Cherenkov radiation and PPN. The strongest constraint in the $\alpha < -1$ region is from Eq. (4.18), and for most of the $\alpha > -1$ region the strongest constraint is from the second inequality in Eq. (4.19). The plot on the right is a blowup of the small range of α for which the first constraint in Eq. (4.19) is strongest—when $\alpha = -1$ to within a couple of parts in 100.

4.4 Cosmological Evolution

We now turn to the evolution of the sigma-model aether field in a cosmological background. It is usually assumed in the literature that the aether preferred frame coincides with the cosmological rest frame—i.e., that in Robertson-Walker coordinates, a timelike aether field has zero spatial components, or a spacelike aether field has zero time component. Under this assumption, there has been some analysis of cosmological evolution in the presence of aether fields [24, 25, 51, 52]. Cosmological alignment in a de Sitter background was considered in [53]. Evolution of vector field perturbations in a more general context, including the effect on primordial power spectra, was considered in [18, 54].

Here, we relax the aforementioned assumption. We determine the dynamical evolution of the aether alignment with respect to constant density hypersurfaces of flat-FRW backgrounds, assuming that the aether field has a negligible effect on the form of the background geometry. Unlike Minkowski space, a Robertson-Walker metric features a preferred frame in which the density of the cosmological fluid is the same everywhere. We will show that a homogeneous timelike vector field tends to align in the presence of a homogeneous cosmological fluid such that its rest frame coincides with the rest frame of the cosmological fluid.

Take the background spacetime to be that of a flat Friedmann-Robertson-Walker (FRW) cosmology,

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\,. \tag{4.22}$$

We take the equation state of the cosmological fluid to be $p_{fluid} = w\rho_{fluid}$. The Friedmann equation then implies $a(t) = t^{2/3(1+w)}$ for $w \neq -1$, and $a(t) = e^{Ht}$ (with H constant) for $w = -1$. We assume that m^2/M_P^2 is small, so that the back reaction of the vector field on the FRW geometry will be small, and the evolution of the vector field will be well approximated by its evolution in the FRW background.

Suppose the vector field is homogeneous. This is a reasonable assumption given that the background spacetime is homogeneous and therefore should only affect the time evolution of the vector field. We may use the rotational invariance of the FRW background to choose coordinates such that the x-axis is aligned with the spatial part of the vector field. Then, without loss of generality, $A_0 = m\cosh(\phi(t))$ and $A_x = ma(t)\sinh(\phi(t))$. In this case the equations of motion reduce to,

$$\phi''(t) + 3H(t)\phi'(t) + (H^2(t) + \alpha H'(t))\sinh(2\phi(t)) = 0, \tag{4.23}$$

where $H(t) = a'(t)/a(t)$. Expanding to first order in the angle ϕ we have for $w \neq -1$,

$$\phi'' + \left(\frac{2}{(1+w)t}\right)\phi' + \left(\frac{8 - 12\alpha(1+w)}{9(1+w)^2t^2}\right)\phi = 0. \tag{4.24}$$

It is a simple exercise to show that ϕ behaves as a damped oscillator for all $-1 < w < 1$

and $\alpha < \frac{2}{3(1+w)}$. For the case of a constant Hubble parameter ($w = -1$),

$$\phi(t) = Ae^{-Ht} + Be^{-2Ht}. \tag{4.25}$$

One can see even for large $\phi(t)$ that $|\phi(t)|$ generically decreases when $-1 < w < 1$ and $\alpha < \frac{2}{3(1+w)}$ because, since $\sinh(\phi) = -\sinh(-\phi)$, the essential features of the full equation mirror those of the linearized equation.

We conclude that a timelike vector field will generically tend to align to be purely timelike in the rest frame of the cosmological fluid, thereby restoring isotropy of the cosmological background. We do not examine the case of a spacelike aether field, since that is perturbatively unstable.

4.5 Extra Dimensions

Consider now the evolution of the vector field in a background spacetime with metric,

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) + dr^2. \tag{4.26}$$

This metric is the local distance measure for a spacetime in which the infinite spatial dimensions expand as a usual flat FRW metric, for general equation of state parameter w as discussed in the previous section, and a compact extra dimension with coordinate r does not expand. A scenario in which a spacelike aether is aligned completely along the compact fifth extra-dimension was considered in [1].

The equations of motion are once again,

$$(g^{\sigma\nu} + A^\sigma A^\nu/m^2)(\nabla_\rho \nabla^\rho A_\sigma + \alpha R_{\rho\sigma} A^\rho) = 0. \tag{4.27}$$

and $A_\mu A^\mu = -m^2$. Consider homogeneous configurations where, without loss of generality,

$$A_0 = m\cosh\phi(t),\;\; A_x = a(t)m\sinh\phi(t)\cos\theta(t),\;\; A_y = A_z = 0,\;\; \text{and}\;\; A_r = m\sinh\phi(t)\sin\theta(t). \tag{4.28}$$

The $\nu = 0$ equation of motion (Eq. (4.27)) reads,

$$\left(\frac{1}{2}(5-\cos 2\theta)(H^2(1+\alpha)+\alpha H') - 2\alpha H^2\cos^2\theta - (\theta')^2\right)\sinh 2\phi + 6H\phi' + 2\phi'' = 0. \tag{4.29}$$

When $\theta'^2 \ll H^2$, we can treat θ as being essentially constant and then the above equation determines the evolution of ϕ. Numerical simulations indicate that ϕ decays to zero, whatever the value of θ, if $-1 < \alpha < \frac{2}{3(1+w)}$. One can see the decay of ϕ (given the bounds on α) explicitly by expanding about $\phi = 0$ and $\theta =$ constant when ϕ is small.

If H is constant (i.e., the non-compact dimensions are de Sitter-like) and the vector field is aligned entirely along the spacelike dimension and the compact dimension (so $\theta = \pi/2$), then the equation of motion for $\phi(t)$ is,

$$\phi''(t) + 3H\phi'(t) + \frac{3}{2}(1+\alpha)H^2\sinh(2\phi(t)) = 0, \tag{4.30}$$

the solution to which is

$$\phi(t) = A_+ e^{-\alpha_+ Ht/2} + A_- e^{-\alpha_- Ht/2}, \qquad \text{where} \qquad \alpha_\pm = 3\left(1 \pm \sqrt{1-\frac{4}{3}(1+\alpha)}\right) \tag{4.31}$$

when $|\phi(t)| \ll 1$. If $1+\alpha > 0$ then ϕ decays to zero. If $\alpha = -1$, ϕ decays to a (generically nonzero) constant, and ϕ can grow with time if $\alpha < -1$. It is interesting to see that, for the case where no perturbative modes propagate superluminally—the case where $\alpha = -1$—

the fixed-norm vector field can evolve during a de Sitter expansion phase so that it has a nonzero component in the compact fifth dimension while otherwise aligning so that isotropy is restored in the rest frame of the cosmological fluid. However, when the Universe enters a phase of expansion where $a(t) = t^{2/3(1+w)}$ and w is strictly greater than -1 (and less than 1), then the component of the vector field in the fifth dimension will decay away.

4.6 Conclusions

We investigated the dynamics of and limits on parameters in a theory with a fixed-norm timelike vector field whose kinetic term takes the form of a sigma-model. We argued in Chapter 3 that such sigma-model theories are the only aether models with two-derivative kinetic terms and a fixed-norm vector field for which the Hamiltonian is bounded below.

In the presence of gravity, the action for sigma-model aether is:

$$S_A = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{2}(\nabla_\mu A_\nu)(\nabla^\mu A^\nu) + \frac{\alpha}{2}R_{\mu\nu}A^\mu A^\nu + \frac{\lambda}{2}(A_\mu A^\mu + m^2)\right]. \quad (4.32)$$

We showed that the five massless degrees of freedom in the linearized theory will not propagate faster than light only if $\alpha = -1$ and we argued that faster-than-light degrees of freedom generically lead to instabilities on less-than-Hubble-length time scales. In this special case $\alpha = -1$, the vacuum expectation value, m^2, must be less than about $10^{-4}M_p^2$, where M_p is the Planck mass, in order to comply with limits on the PPN preferred frame parameter, α_2. Relaxing the $\alpha = -1$ assumption, we summarized the strongest limits on the parameters $\{\alpha, m\}$ (from gravitational Cherenkov radiation and the PPN preferred frame parameters) in Fig. 4.2.

We also showed that the aether field tends to dynamically align such that it is orthogonal to constant density hypersurfaces for the theoretically and experimentally relevant portion of the parameter space. The dynamics forces the rest frame of the aether and that of the perfect fluid dominating the cosmological evolution to coincide. Finally, we showed that the dynamics allows for the possibility of a nonzero spatial component in a non-expanding fifth dimension during a de Sitter era. Even a spatial component in a non-expanding fifth dimension will decay away during non-de Sitter eras, e.g., in a matter- or radiation-dominated era. We take this as evidence that aether fields with well-behaved (semi)classical dynamics will not lead to any significant departure from isotropy.

Chapter 5

Lorentz Violation in Goldstone Gravity

5.1 Introduction

The existence of massless particles is conventionally explained by the requirement to preserve gauge symmetries. In the case of electromagnetism, the masslessness of the photon is required so that local $U(1)$ gauge invariance is maintained; in the case of general relativity, the masslessness of the graviton has its origin in diffeomorphism invariance.

In 1963, Bjorken proposed an alternative viewpoint: the photon can be a Goldstone boson associated with the spontaneous breaking of Lorentz invariance [55, 56, 57, 58, 59]. The idea was subsequently generalized and applied to the case of gravity by Phillips and others [60, 61, 62, 63, 64].

In ordinary Maxwell electrodynamics, gauge invariance reduces the four components of the vector potential A_μ down to the two propagating degrees of freedom of a massless spin-1 particle. Gauge invariance forbids a potential $V(A_\mu)$, which keeps the photon massless and prohibits a longitudinal mode, and it also forbids kinetic terms such as $(\partial_\mu A^\mu)^2$, which would allow a spin-0 mode to propagate. In the Goldstone approach, there is no gauge invariance, and the vector field acquires a vev via a potential. Regardless of the form of the vev, there are always three massless Goldstone excitations, all of which would propagate for a generic choice of kinetic term. To avoid the extra degree of freedom, we can choose the Maxwell kinetic term, even though it is not required by gauge invariance. Then two linear combinations of the Goldstone modes have exactly the same properties as the photon in electromagnetism. The remaining longitudinal mode is auxiliary, and does not propagate, so that the theory is indistinguishable from electromagnetism in the low energy limit. (In the presence of Lorentz violation, Goldstone's theorem no longer ensures one propagating mode for each broken symmetry generator.) This identification can be overturned by radiative corrections, since there is no gauge invariance to protect the form of the propagator.

The graviton case is similar, except that now it is a symmetric two-index tensor that acquires a vev. A propagating massless spin-2 particle has two degrees of freedom. Because the Lorentz group has six generators, there are sufficient degrees of freedom in the Goldstone bosons to reproduce the graviton. However, we will see that this is not automatic, as in the photon case; whether we get the correct Goldstone modes to recover the transverse-traceless oscillations of conventional gravitons will depend on the choice of vev. The case where all six generators are broken was examined by Kraus and Tomboulis in [62], where they also discussed how such a modified theory of gravity can possibly evade the cosmological constant problem.

Recently, Kostelecky and Potting examined in detail the scenario in which a symmetric two-index tensor acquires a vev via a potential [65]. With a kinetic term quadratic in derivatives and preserving diffeomorphism invariance, they found that, just as in the photon case, two linear combinations of the resulting six Goldstone bosons obey the linearized Einstein's equations in a special gauge (which they termed the 'cardinal' gauge), while the remaining four linear combinations do not propagate. Together with four additional massive modes, they account for the ten degrees of freedom contained in the two-index symmetric tensor. By requiring self-consistent coupling to the energy-momentum tensor, they also demonstrated that the theory can be used to construct a nonlinear theory via a bootstrap procedure (analogous to the way in which general relativity is obtained from the linearized theory). We expect the massive modes to be near the Planck scale, outside the low-energy theory, so the nonlinear theory is equivalent to general relativity with conventional coupling to matter.

Kraus and Tomboulis [62] pointed out that these massive modes nevertheless have a crucial effect: integrating them out introduces an infinite number of radiative-correction terms to the low-energy Lagrangian, which can change the theory in important ways. Since these corrections are not controlled by gauge invariance, in general they will modify the dispersion relations of the Goldstone modes. At higher order, therefore, the Goldstone bosons arising from Lorentz violation can, in principle, be distinguished from the graviton in linearized general relativity.

In this chapter, we examine some of these correction terms and study their effects on the properties of the Goldstone bosons. (The terms we consider are those that are most straightforward to analyze, but their impacts should be generic.) We find that, for a general vev, these terms modify the dispersion relations of the Goldstone modes in such a way that their speed of propagation is anisotropic. If the speed is subluminal in some directions, gravi-Cherenkov radiation by cosmic rays becomes possible. Observations of high-energy cosmic rays thus allow us to constrain these higher-order radiative corrections. These corrections also effect the polarization tensors of the conventional gravitons, leading to longitudinal oscillations in the motion of test particles, in addition to the conventional transverse $+$ and $\times$ patterns predicted in general relativity. This could lead to novel experimental tests of the theory, although we do not know of any constraints on this phenomenon from currently available data.

Another difference between Goldstone gravity and general relativity is that the former predicts the existence of other massless particles in addition to the two conventional massless spin-2 polarizations. This is reminiscent of the photon case, in which a longitudinal mode (in addition to the two transverse modes) becomes dynamical in the presence of the radiative corrections induced by integrating out the massive modes. Analogously, we expect that there should be four longitudinal Goldstone bosons that can become dynamical. The polarization tensors of these modes can be written as a sum of eight symmetric tensors constructed from k_μ and the vev. By imposing the four cardinal gauge conditions, we can relate these eight coefficients, leaving four independent parameters for the four Goldstone modes.

In the next section, we briefly review the case of Goldstone photons, including the effects of radiative corrections as emphasized in [62]. We then carry out an analogous analysis for gravitons, showing how radiative corrections bring to life new massless modes. In Section 5.4 we concentrate on the two modes of the graviton, demonstrating that they propagate anisotropically in the presence of a generic vev and considering some experimental limits on the corresponding parameters. In Section 5.5 we examine models where the vev doesn't completely break the Lorentz group, and gravitons are only partially constructed from Goldstone bosons, or they originate from residual diffeomorphism invariance. Appendix B describes the relationship between different patterns of symmetry breaking and the modes corresponding to gravitons.

5.2 Goldstone Electromagnetism

5.2.1 Photons as Goldstone bosons

Before we delve into the graviton case, we first discuss the scenario in which the photon arises as a Goldstone boson of spontaneous Lorentz violation, commonly known as the 'bumblebee' model [62, 66]. We will see below that the graviton case mirrors the vector case.

We consider the Lagrangian for a vector field A_μ,

$$\mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - V(\bar{A}_\mu, a_\mu), \tag{5.1}$$

where $A_\mu = \bar{A}_\mu + a_\mu$ and $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the corresponding field-strength tensor. The potential V gives A_μ a vev $\bar{A}_\mu$ (with $\partial_\mu \bar{A}_\nu = 0$), thereby violating Lorentz invariance spontaneously. For a thorough analysis of the case for which $\bar{A}_\mu$ is spacelike, see [62].

We consider here the usual Maxwell kinetic term, which by itself preserves gauge invariance, as our aim is to have a theory that reproduces electromagnetism at lowest order. The stability of theories with more generic kinetic terms was considered in [2].

The Goldstone boson fields can be constructed from the vev by the action of spacetime-dependent infinitesimal Lorentz transformations,

$$a_\mu = -\Theta_\mu{}^\nu(x) \bar{A}_\nu. \tag{5.2}$$

Here, $\Theta_{\mu\nu}$ is an antisymmetric tensor of the form

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ -\beta_1 & 0 & \theta_3 & -\theta_2 \\ -\beta_2 & -\theta_3 & 0 & \theta_1 \\ -\beta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}, \tag{5.3}$$

where $\beta_i = \bar{\beta}_i e^{ik_\alpha x^\alpha}$ are the infinitesimal rapidities corresponding to boosts, and $\theta_i = \bar{\theta} e^{ik_\alpha x^\alpha}$ are the infinitesimal angles corresponding to rotations. Note that the three Goldstone modes a_μ are orthogonal to the vev $\bar{A}_\mu$. The remaining length-changing mode (parallel to $\bar{A}_\mu$) is massive.

We can consider vevs $\bar{A}_\mu$ that are timelike or spacelike. When it is timelike, without loss of generality we can boost to a frame in which only $A_0 \neq 0$. This breaks the original $SO(3,1)$ to $SO(3)$, preserving rotational invariance. From Eq. (5.2), the three Goldstone bosons come from the three broken boost generators, and are given by

$$a_\mu = -\Theta_{\mu 0} \tag{5.4}$$

$$= \begin{pmatrix} 0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}. \tag{5.5}$$

Each choice of the vev corresponds to a particular gauge in electromagnetism. Having a timelike vev is equivalent to the Coulomb gauge, in which we set the scalar potential to zero ($A_0 = 0$). That is, the physics of the theory is completely equivalent to that of free Maxwell electrodynamics, but with a particular gauge condition imposed. This gauge choice is compatible with the transverse condition ($k_\mu A^\mu = 0$) that we usually impose in electromagnetism. Together these are consistent with the Lorenz gauge, making a timelike vev a natural gauge choice to describe a free photon. For example, if we want to describe a photon moving in the x_i direction, we can just set A_i to zero.

If instead $\bar{A}_\mu$ is spacelike, we can rotate axes such that only $\bar{A}_3 \neq 0$. This reduces the $SO(3,1)$ symmetry that we begin with to $SO(2,1)$, resulting in three Goldstone modes (one boost and two rotations):

$$a_\mu = -\Theta_{\mu 3} \tag{5.6}$$

$$= \begin{pmatrix} \beta_3 \\ \theta_2 \\ -\theta_1 \\ 0 \end{pmatrix}. \tag{5.7}$$

Having a spacelike vev is equivalent to imposing the axial gauge ($\vec{s} \cdot \vec{a} = 0$), where $\vec{s}$ is a unit spatial vector. In order to describe a photon that propagates in a direction orthogonal to $\bar{A}_\mu$, a_μ is necessarily unbounded somewhere at spatial infinity. There is thus a question whether the Lorentz-violating theory, as an effective field theory, is capable of describing a photon in the axial gauge. Since the field value can be large, we should, in the spirit of effective field theory, retain higher-order kinetic terms in the Lagrangian. We won't pursue this issue in this chapter.

5.2.2 Radiative corrections and dispersion relations of the Goldstone modes

As we have seen, the vev $\bar{A}_\mu$ always leads to three Goldstone bosons, which can be classified into two transverse modes and one longitudinal mode. The transverse modes satisfy the condition $k^\mu a_\mu = 0$. With the kinetic term in (5.1), they satisfy the dispersion relation $k^\mu k_\mu = 0$, and thus propagate isotropically at the speed of light. Hence, they have the right properties to be identified as the photon.

The remaining longitudinal degree of freedom is orthogonal to the two transverse modes. This allows us to specify its polarization as

$$\epsilon_\mu^{(longitudinal)} = k_\mu - \frac{(\bar{A}^\alpha k_\alpha)}{A^\beta A_\beta}\bar{A}_\mu. \tag{5.8}$$

At lowest order, this longitudinal mode does not propagate, and corresponds to the pure-gauge mode in electromagnetism.

As we will see later, this way of decomposing the Goldstone modes into transverse and longitudinal degrees of freedom will be highly similar in the graviton case. Expressing the longitudinal mode in the basis k_μ and $\bar{A}_\mu$ makes it automatically orthogonal to the transverse modes.

As was pointed out in [62], we expect that there would be higher-order radiative correction terms induced in the low-energy effective Lagrangian as we integrate out the massive fluctuations of A_μ. These terms will in general modify the dispersion relations of the Goldstone bosons. If we restrict our attention to only two derivatives, there are seven such terms, which are listed in [62] and which take the form:

$$
\begin{aligned}
& f_1(A^2)\partial_\mu A_\nu \partial^\mu A^\nu \\
& f_2(A^2)\partial_\mu A_\nu \partial^\nu A^\mu \\
& f_3(A^2)A^\mu A^\alpha \partial_\mu A_\nu \partial_\alpha A^\nu \\
& f_4(A^2)A^\nu A^\alpha \partial_\mu A_\nu \partial_\alpha A^\mu \\
& f_5(A^2)A^\nu A^\alpha \partial_\mu A_\nu \partial^\mu A_\alpha \\
& f_6(A^2)A^\mu A^\nu A^\alpha \partial_\mu \partial_\nu A_\alpha \\
& f_7(A^2)A^\mu A^\nu A^\alpha A^\beta \partial_\mu A_\nu \partial_\alpha A_\beta \,,
\end{aligned}
\tag{5.9}
$$

where $f_i(A^2)$ are scalar functions of $A^\mu A_\mu$. This list exhausts all possible such terms, since $A^\mu A_\mu$ is a constant. The situation will be different in the two-index case, where infinitely many such terms can be generated in the effective Lagrangian, as we will discuss later.

If we assume that $\bar{A}_\mu a^\mu$ is small, the first three terms in (5.9) dominate over the rest. They modify the dispersion relations of the two transverse Goldstone bosons to

$$(1 + d_1)k^\mu k_\mu + d_2(\bar{A}^\mu k_\mu)^2 = 0, \tag{5.10}$$

where d_1 and d_2 are undetermined coefficients and are presumably small. The additional term implies that the phase velocity of the two transverse modes is anisotropic.

Meanwhile, in the presence of these radiative corrections, the longitudinal mode becomes dynamical and has the dispersion relation

$$k^\mu k_\mu + d_3(\bar{A}_\mu k^\mu)^2 = 0, \tag{5.11}$$

where d_3 is an undetermined coefficient.

5.3 Goldstone Gravity

5.3.1 Gravitons as Goldstone bosons

The analysis of spontaneous Lorentz violation via a symmetric two-index tensor is in many ways similar to the vector case that we previously discussed. In particular, we will focus on a model called 'cardinal gravity', introduced recently by Kostelecky and Potting [65]. They showed that when a two-index symmetric tensor acquires a vev which breaks all six generators of the Lorentz group in Minkowski spacetime, two linear combinations of the resulting Goldstone modes have properties that are identical to those of the graviton in a special (cardinal) gauge in linearized general relativity. We have included our own version of this argument in Appendix B.

As in the photon case, higher-order radiative correction terms resembling (5.9) will generically appear in the low-energy effective Lagrangian as we integrate out the four massive modes to extract their contribution to the low energy physics. In the two-index case, there are infinitely many such terms. In this chapter, we will focus on a representative subset of these terms, and examine their resulting Lorentz-violating effects on the Goldstone modes. For example, in the presence of these higher-order terms, two linear combinations of the six Goldstone modes that are to be identified as the graviton will now propagate at different phase velocities in different directions. In addition, the four remaining linear combinations that are originally auxiliary will now become dynamical, just like the longitudinal mode in the vector case.

We begin with the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}[(\partial_\mu \tilde{h}^{\mu\nu})(\partial_\nu \tilde{h}) - (\partial_\mu \tilde{h}^{\rho\sigma})(\partial_\rho \tilde{h}^{\mu}_{\sigma}) \\ &+\frac{1}{2}\eta^{\mu\nu}(\partial_\mu \tilde{h}^{\rho\sigma})(\partial_\nu \tilde{h}_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \tilde{h})(\partial_\nu \tilde{h})] \\ &+(\text{radiative corrections}) - V(\tilde{h}_{\mu\nu}\tilde{h}^{\mu\nu}),\end{aligned} \tag{5.12}$$

where $\tilde{h}^{\mu\nu}$ is a symmetric two-index tensor field defined on a spacetime with Minkowski metric $\eta_{\mu\nu}$. In analogy to the electromagnetic case, we have chosen the linearized Einstein-Hilbert kinetic term, which by itself preserves diffeomorphism invariance ($\tilde{h}_{\mu\nu} \rightarrow \tilde{h}_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}$, for some vector ξ^μ).

As in the vector case, the field $\tilde{h}_{\mu\nu}$ acquires a vev $H_{\mu\nu}$ via the potential V. The Goldstone modes that result are given by acting spacetime-dependent infinitesimal Lorentz transformations on this vev [66, 67]:

$$h_{\mu\nu} = -\Theta_\mu{}^\alpha H_{\alpha\nu} - \Theta_\nu{}^\alpha H_{\mu\alpha}, \tag{5.13}$$

where $\tilde{h}_{\mu\nu} = H_{\mu\nu} + h_{\mu\nu}$ and $\Theta_{\mu\nu}$ is as defined in (5.3). Unless stated otherwise, from now on we assume that $H_{\mu\nu}$ breaks all six generators of the Lorentz group, and thus gives rise to six potential Goldstone bosons.

Note that in the form of (5.13), the Goldstone bosons automatically fulfill four conditions, dubbed 'cardinal' by Kostelecky and Potting in [65]:

$$\eta^{\mu\nu}\epsilon_{\mu\nu} = 0 \tag{5.14}$$

$$H^{\mu\nu}\epsilon_{\mu\nu} = 0 \tag{5.15}$$

$$H^{\mu}{}_{\alpha}H^{\nu\alpha}\epsilon_{\mu\nu} = 0 \tag{5.16}$$

$$H^{\mu\alpha}H_{\alpha\beta}H^{\beta\nu}\epsilon_{\mu\nu} = 0, \tag{5.17}$$

where $h_{\mu\nu} = \epsilon_{\mu\nu}e^{ik_\alpha x^\alpha}$. Since we could diagonalize $H_{\mu\nu}$ via an appropriate orthogonal transformation, there can be at most four such independent constraints, one for each eigenvalue. Contracting $\epsilon_{\mu\nu}$ with terms of higher order in $H_{\mu\nu}$ (e.g., $H^{\mu\alpha}H_{\alpha\beta}H^{\beta\gamma}H_{\gamma\nu}$) also yields zero, but the resulting constraints are not independent.

The cardinal conditions are very similar to that ($\bar{A}_\mu a^\mu = 0$) in the vector case, but now there are four orthogonality conditions instead of one. They can be viewed as 'directions' along which the massive modes reside (just as the length-changing mode of the vector is parallel to the vev). There are thus in general four massive degrees of freedom in the theory.

Kostelecky and Potting demonstrated that the cardinal gauge is attainable for generic k^μ in general relativity. In Appendix B we derive necessary and sufficient conditions under which the cardinal gauge is a valid gauge choice.

Starting with the ten independent components in $h_{\mu\nu}$, imposing the four cardinal gauge conditions reduces that to six, which is exactly the right number to accomodate the six Goldstone modes. The situation becomes more complicated when the vev does not break all six generators. In that case, there are fewer Goldstone bosons, as well as fewer gauge conditions. However, there might also be residual diffeomorphism invariance. The theory can contain massless excitations that originate from spontaneous Lorentz violation and/or diffeomorphism invariance.

As in the photon case, it is most convenient to decompose the six Goldstone modes into two linear combinations that are transverse, and four other orthogonal linear combinations. The two transverse modes obey the linearized Einstein's equations and have the dispersion relation

$$k^\mu k_\mu = 0, \tag{5.18}$$

corresponding to masseless particles propagating isotropically at the speed of light. These can therefore be identified as the graviton. The remaining four modes are auxiliary and do not propagate. At this order, the theory is thus equivalent to linearized general relativity in the cardinal gauge, if we treat the massive modes as absent at low energies.

5.3.2 Radiative corrections and dispersion relations

Corrections to the effective field theory arise from integrating out the massive modes. As in the photon case, the resulting radiative-correction terms induce additional Lorentz-violating effects when $\tilde{h}_{\mu\nu}$ acquires a vev. As before, we restrict our attention to only terms that are quadratic in derivatives of $h_{\mu\nu}$. We will demonstrate that these terms will modify the dispersion relations of the two transverse linear combinations that correspond to the graviton. We also argue that, just as the longitudinal mode in the vector case, the four remaining Goldstone modes become dynamical.

There are four types of kinetic terms that are independent of $H_{\mu\nu}$. The terms and their corresponding contributions to the equation of motion are as follows:

$$\begin{aligned}
\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} &\rightarrow 2\Box h_{\mu\nu} && (5.19)\\
\partial_\mu h^{\mu\nu}\partial_\nu h &\rightarrow \partial_\mu\partial_\nu h + \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} && (5.20)\\
\partial_\mu h^{\rho\sigma}\partial_\rho h^\mu{}_\sigma &\rightarrow 2\partial_{(\mu|}\partial_\sigma h^\sigma{}_{|\nu)} && (5.21)\\
\partial_\mu h\partial^\mu h &\rightarrow 2\eta_{\mu\nu}\Box h. && (5.22)
\end{aligned}$$

Each of these terms already appears in the Lagrangian (5.12), with specific numerical coefficients. The corrections will change the value of these coefficients, generically leading to violations of diffeomorphism invariance.

At linear order in $H_{\mu\nu}$ we have the following possible kinetic terms, and their contributions to the equation of motion:

$$H^{\alpha\beta}\partial_\alpha h_{\rho\sigma}\partial_\beta h^{\rho\sigma} \rightarrow 2H^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} \tag{5.23}$$

$$H^{\alpha\beta}\partial^\rho h_{\alpha\rho}\partial^\sigma h_{\beta\sigma} \rightarrow 2H_{(\mu|}{}^{\beta}\partial_{|\nu)}\partial^\sigma h_{\beta\sigma} \tag{5.24}$$

$$H^{\alpha\beta}\partial_\sigma h_{\alpha\beta}\partial_\rho h^{\rho\sigma} \rightarrow H_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + H^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} \tag{5.25}$$

$$H^{\alpha}{}_{\beta}\partial_\rho h_{\alpha\sigma}\partial^\rho h^{\beta\sigma} \rightarrow 2H_{(\mu|}{}^{\sigma}\Box h_{\sigma|\nu)} \tag{5.26}$$

$$H^{\alpha\beta}\partial_\rho h_{\alpha\beta}\partial^\rho h \rightarrow H_{\mu\nu}\Box h + (H^{\alpha\beta}\Box h_{\alpha\beta})\eta_{\mu\nu} \tag{5.27}$$

$$H^{\alpha\beta}\partial_\alpha h_{\beta\sigma}\partial_\rho h^{\rho\sigma} \rightarrow H_{(\mu|}{}^{\alpha}\partial_\alpha\partial^\beta h_{\beta|\nu)} + H^{\alpha\beta}\partial_{(\mu|}\partial_\alpha h_{\beta|\nu)} \tag{5.28}$$

$$H^{\alpha\beta}\partial_\alpha h_{\beta\rho}\partial^\rho h \rightarrow H_{(\mu|}{}^{\alpha}\partial_\alpha\partial_{|\nu)} h + (H^{\alpha\beta}\partial^\sigma\partial_\alpha h_{\beta\sigma})\eta_{\mu\nu} \tag{5.29}$$

$$H^{\alpha\beta}\partial_\alpha h\partial_\beta h \rightarrow 2H^{\alpha\beta}\partial_\alpha\partial_\beta h\eta_{\mu\nu}. \tag{5.30}$$

Unlike the photon case, there are infinitely many radiative correction terms that can be generated at higher orders in the vev. Assuming that $H_{\mu\nu}$ is in general small compared to the background metric, we will focus only on those that either do not depend on, or those linear in, $H_{\mu\nu}$. We will later discuss a possible experimental test to constrain $H_{\mu\nu}$.

We first consider the four auxiliary modes. In the form of (5.13), they obey the four cardinal gauge conditions (5.14). They are also orthogonal to the two transverse degrees of freedom that correspond to the graviton,

$$\epsilon^{(aux)}_{\mu\nu}\epsilon^{\mu\nu}_{(trans)} = 0. \tag{5.31}$$

Together, these are six conditions that reduce the ten independent components of $\epsilon_{\mu\nu}^{(aux)}$ to four degrees of freedom. In analogy to (5.8) in the photon case, we can express these four modes in terms of the wave vector and the vev as

$$\begin{aligned} \epsilon_{\mu\nu}^{(aux)} &= b_1 \eta_{\mu\nu} + b_2 H_{\mu\nu} + b_3 H_\mu{}^\alpha H_{\alpha\nu} \\ &\quad + b_4 H_{\mu\alpha} H^{\alpha\beta} H_{\beta\nu} + b_5 k_\mu k_\nu \\ &\quad + b_6 H_{(\mu|\alpha} k^\alpha k_{|\nu)} + b_7 H_{(\mu|\alpha} H^{\alpha\beta} k_\beta k_{|\nu)} \\ &\quad + b_8 H_{(\mu|\alpha} H^{\alpha\beta} H_{\beta\gamma} k^\gamma k_{|\nu)}, \end{aligned} \tag{5.32}$$

where the eight coefficients b_i are constrained by imposing the four cardinal gauge conditions (5.14) – (5.17). This leaves four independent coefficients for the four modes.

The basis polarization tensors $\epsilon_{\mu\nu}^{(aux)}$ are chosen so that the conditions (5.31) are automatically satisfied. At lowest order, these four modes do not propagate (as is demonstrated in Appendix B). However, in the presence of the radiative correction terms, we expect that they become dynamical, similar to the longitudinal mode in the vector case. There will now be a contribution from (5.19), which adds the term $k^\mu k_\mu$ to their dispersion relation. We do not pursue the calculation of the dispersion relations of these auxiliary modes in this chapter. The method to do so can be found in [62], in which the dispersion of the longitudinal mode in the photon case is computed.

5.4 Anisotropic Propagation

Now we consider the effects of the radiative correction terms on the two transverse propagating linear combinations, which will be the main focus of this chapter. We will not be considering all of the terms, however, as the task of diagonalizing the resulting equations of motion is highly nontrivial. Rather, we focus on a number of representative terms and see what are some of the Lorentz-violating effects typical in this theory. This will provide a guide on how we can experimentally differentiate the theory from general relativity, given that the two theories are identical at lowest order.

5.4.1 Dispersion relations

Of the four terms (5.19) → (5.22), only the first term modifies the dispersion relation, which becomes

$$(1+c_1)k^\mu k_\mu = 0, \tag{5.33}$$

where c_1 is some undetermined constant. In the absence of other terms in the dispersion relation, this correction is immaterial. We can divide by $1 + c_1$ and obtain the usual $k^\mu k_\mu = 0$, so excitations propagate isotropically along the light cones.

If we also incorporate the radiative corrections that are linear in $H_{\mu\nu}$, the equations of motion are still easily diagonalizable except for Eq. (5.26) and Eq. (5.28). We will thus focus on the effects of the other six terms. The polarization tensors of the transverse Goldstone modes remain unchanged, but their dispersion relations are now modified:

$$k_\mu k^\mu - c_2 H^{\mu\nu} k_\mu k_\nu = 0, \tag{5.34}$$

where c_2 is some undetermined coefficient that is expected to be small.

As in (5.10), the effect of the additional term in the dispersion relation is to make the phase velocity of the transverse modes become anisotropic for a generic vev. The phase velocity is given by the ratio of the frequency ω and magnitude of the momentum k,

$$v = \frac{\omega}{|\vec{k}|} = 1 - \frac{c_2}{2} n_\mu H^{\mu\nu} n_\nu, \tag{5.35}$$

where $n_\mu = (1, \vec{n})$ and $\vec{n} = \vec{k}/|\vec{k}|$.

Note that in the case where $H_{\mu\nu}$ can be written as $t_\mu t_\nu$, where t^μ is timelike, we can always boost to a frame in which the speed of the graviton is isotropic, and the dispersion relation has the form

$$\omega^2 + v^2 \vec{k} \cdot \vec{k} = 0, \tag{5.36}$$

where the propagation speed is different from the speed of light. $H_{\mu\nu} = t_\mu t_\nu$ thus defines a preferred rest frame, in which $t_\mu = (1, 0, 0, 0)$.

5.4.2 Motion of test particles

We now want to investigate how the modified dispersion affects the motion of test particles in the presence of the transverse Goldstone modes. Consider nearby particles with separation vector S^μ. The geodesic deviation equation of the test particles is

$$\frac{D^2}{d\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho S^\sigma, \tag{5.37}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor, τ is the proper time, and U^μ is the four-velocity of the test particles. The notation $\frac{D}{d\tau} = \frac{dx^\mu}{d\tau} \nabla_\mu$ denotes the directional covariant derivative.

To first order, we can set $U^\mu = (1, 0, 0, 0)$. Likewise, we can replace the Riemann tensor by its linearized version and the proper time τ by t. Eq. (5.37) then becomes

$$\frac{\partial^2}{\partial t^2} S^\mu = {R^{(1)}}^\mu{}_{00\sigma} S^\sigma, \tag{5.38}$$

where

$$R^{(1)}_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}). \tag{5.39}$$

For simplicity, we assume that the transverse modes propagate in the z direction, so that $k^\mu = (\omega, 0, 0, k)$. Note that $\omega \neq k$, since the dispersion is no longer $k^\mu k_\mu = 0$. As is shown in Appendix B (B.24), the polarization tensor of the two transverse modes is

$$p_{\mu\nu} = \begin{pmatrix} p_{00} & p_{10} & p_{20} & -p_{00} \\ p_{10} & h_+ & h_\times & -p_{10} \\ p_{20} & h_\times & -h_+ & -p_{20} \\ -p_{00} & -p_{10} & -p_{20} & p_{00} \end{pmatrix}, \tag{5.40}$$

where $h_{\mu\nu} = p_{\mu\nu} e^{ik_\alpha x^\alpha}$. The constants p_{00}, p_{10}, and p_{20} can be determined by imposing the cardinal gauge conditions.[1] Because we do not start from a diffeomorphism-invariant formulation, we do not have the gauge freedom to set these coefficients to zero.

In Fourier space, Eq. (5.38) becomes

$$\begin{aligned} \omega^2 \delta S^\mu &= \frac{1}{2}(\omega^2 p^\mu{}_\sigma + k_\sigma k^\mu p_{00} + \\ &\quad k_\sigma \omega p^\mu{}_0 + \omega k^\mu p_{0\sigma}) S^\sigma(0), \end{aligned} \tag{5.41}$$

[1] In Appendix B, we give an explicit formula for p_{00}, p_{10}, and p_{20} in terms of components of $H_{\mu\nu}$. The constants as they appear in (5.40) are of the form $k_{(\mu}\xi_{|\nu)}$. They are therefore just gauge modes, so they are not physically observable if the theory is diffeomorphism invariant (as in general relativity). In Goldstone gravity, however, diffeomorphism invariance is broken, so the cardinal gauge mode components p_{01} and p_{02} in (5.44) and (5.45) can actually effect the motion of test particles, once radiative corrections are included.

where $S^\mu(x^\mu) = S^\mu(0) + \delta S^\mu(x^\mu)$, and $S^\mu(0) = S^\mu(t=0, \vec{x} = \vec{0})$ is the initial position of the test particle.

With $h_{\mu\nu} \propto e^{ik_\mu x^\mu}$, the zeroth component of Eq. (5.41) reads

$$\begin{aligned} \omega^2 \delta S^0 &= \frac{1}{2}(\omega^2 {p^0}_\sigma + k_\sigma k^0 p_{00} \\ &\quad + k_\sigma \omega {p^0}_0 + \omega k^0 p_{0\sigma}) S^\sigma(0) \\ &= 0, \end{aligned} \tag{5.42}$$

which is identically zero. There is no deflection in the time direction, as expected.

For $\mu = 1$, we have

$$\begin{aligned} \omega^2 \delta S^1 &= \frac{1}{2}(\omega^2 h_+ S^1(0) + \omega^2 h_\times S^2(0) \\ &\quad + (-\omega^2 + k\omega) p_{01} S^3(0)). \end{aligned} \tag{5.43}$$

If the dispersion relation is simply $k^\mu k_\mu = 0$, the last term is zero. However, with the modification $cH^{\mu\nu}k_\mu k_\nu$ in the dispersion, $\omega \neq k$, and

$$\begin{aligned} \delta S^1 &= \frac{1}{2}\big[h_+ S^1(0) + h_\times S^2(0) \\ &\quad - \frac{c_2}{2}(H_{33} + H_{00} + 2H_{03}) p_{01} S^3(0)\big]. \end{aligned} \tag{5.44}$$

Following the same procedure, the $\mu = 2$ equation reads

$$\begin{aligned} \delta S^2 &= \frac{1}{2}(h_\times S^1(0) - h_+ S^2(0) \\ &\quad - \frac{c_2}{2}(H_{33} + H_{00} + 2H_{03}) p_{02} S^3(0)). \end{aligned} \tag{5.45}$$

The first two terms in (5.44) and (5.45) correspond to the usual + and × polarizations. However, both δS^1 and δS^2 are now also functions of the longitudinal separation $S^3(0)$.

Similar to Eq. (5.42), the $\mu = 3$ equation is normally identically zero. However, because of the modified dispersion, we have

$$\delta S^3 = -\frac{c_2}{2}(H_{00} + H_{33} + 2H_{03})(p_{01}S^1(0) + p_{02}S^2(0)). \tag{5.46}$$

Thus, the test particles will also undergo longitudinal oscillations. Notice that the amplitude of the oscillation is a function of the transverse position of the test particles. Hence the motion is not uniform along z.

Similar to the graviton in general relativity, the two transverse modes have two polarizations (conveniently labelled + and × here). The novel feature is that now both polarizations are accompanied by transverse oscillations that depend on longitudinal separation, and longitudinal oscillations that depend on transverse separation.

5.4.3 Experimental constraints

If the speed of gravity $v_{graviton}$ is less than the speed of light, ultra-high energy cosmic rays will be able to emit 'gravi-Cherenkov radiation'. This is analogous to the way in which a light source emits Cherenkov radiation in a medium. The fact that we observe ultra-high-energy cosmic rays puts a limit on the effectiveness of gravi-Cherenkov radiation, thereby placing a stringent lower bound on the propagation speed of the Goldstone modes (if they are to be interpreted as the graviton). We will use this to constrain the magnitude of the correction term $c_2 H^{\mu\nu} k_\mu k_\nu$ in the graviton dispersion relation (5.34).

In [68], it was found that, if gravi-Cherenkov radiation occurs, the maximum travelling time of a cosmic ray is

$$t_{max} = \frac{M_{Pl}^2}{(n-1)^2 p^3}, \tag{5.47}$$

where p is the final momentum (when detected on Earth) and $n = v_{cosmic}/v_{graviton}$ is the refractive index.

Using estimates in [68], this translates to

$$n - 1 \approx \frac{c_2}{2} n_\mu H^{\mu\nu} n_\nu < 2 \times 10^{-15}. \tag{5.48}$$

The speed of the Goldstone graviton can thus only be very slightly less than the speed of light.

5.4.4 Corrections to the energy-momentum tensor

The correction to the dispersion relation also has an effect on the energy-momentum tensor of the transverse Goldstone modes.

We define the energy-momentum tensor to be

$$t_{\mu\nu} = -\frac{1}{8\pi G}\left(R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta^{\rho\sigma} R^{(2)}_{\rho\sigma}[h^{(1)}]\eta_{\mu\nu}\right), \tag{5.49}$$

where $R^{(i)}_{\mu\nu}[h^{(j)}]$ is the parts of the expanded Ricci tensor that are i^{th}-order in the metric perturbation, while h^j is the j^{th}-order expansion of the field $h_{\mu\nu}$. Hence, $R^{(i)}_{\mu\nu}[h^{(j)}]$ is of order $h^{(i\times j)}$.

As $t_{\mu\nu}$ is not diffeomorphism invariant, we should average over several wavelengths to obtain a reasonable measure of the energy-momentum. Imposing the cardinal conditions obeyed by the transverse Goldstone modes, Eq. (5.49) simplifies to

$$t_{\mu\nu}^{(0)} = \frac{1}{64\pi G} k_\mu k_\nu \epsilon_{\rho\sigma}^{(trans)} \epsilon^{\rho\sigma}_{(trans)}. \tag{5.50}$$

With the modification to the dispersion relation of the gravitons, k_μ changes as $k_\mu \to k_\mu + \frac{c_2}{2} H_{\mu\nu} k^\nu$ up to first order. The energy-momentum tensor (5.49) becomes

$$t_{\mu\nu} = t_{\mu\nu}^{(0)} + \frac{c_2 \pi}{16G} (h_+^2 + h_\times^2) H_{\mu\alpha} k^\alpha H_{\nu\beta} k^\beta. \tag{5.51}$$

The flux of energy and momentum carried by the transverse Goldstone modes are therefore anisotropic, depending on $H_{\mu\nu}$. This makes sense, as the modes propagate at different phase velocities in different directions.

It has been estimated that the energy flux due to a typical supernova explosion at cosmological distances is approximately $10^{-19} erg/cm^2/s$. Given the experimental constraint from gravi-Cherenkov radiation on the size of $c_2 H_{\mu\nu}$, the corrections are undetectable with current technologies.

5.5 Vevs That Do Not Break All Six Generators

5.5.1 Gravitons are not necessarily Goldstone

For vector fields, an expectation value along with the Maxwell kinetic term naturally leads to photon-like Goldstone modes, regardless of the form of the vev. We start out with four degrees of freedom in the vector A_μ. The direction parallel to the vev is a massive mode, while the three orthogonal directions are the massless Goldstone excitations. We can further form two linear combinations of the Goldstone modes, such that they are transverse and obey the dispersion relation $k^\mu k_\mu = 0$. The longitudinal mode does not propagate.

A similar story holds in the graviton case, as long as all six generators of the Lorentz group are broken, giving rise to six Goldstone bosons. (See Table 1 for a comparison with the photon case.) In this case, diffeomorphism invariance is also completely broken, and the counting proceeds analogously. We start with ten degrees of freedom in $h_{\mu\nu}$. The four cardinal gauge conditions define four 'directions' along which the massive modes live. This leaves six degrees of freedom for the six Goldstone bosons. Imposing the four transverse conditions $k^\mu h_{\mu\nu} = 0$ leaves us with two linear combinations that obey the dispersion relation $k^\mu k_\mu = 0$. The remaining four longitudinal modes are auxiliary and do not propagate.

This particularly straightforward case is the one that we have been focusing on so far. In this section, we will explore what happens when not all six generators are broken by the vev. In this case, there can be residual diffeomorphism invariance in the theory. The Lorentz-violating theory might still contain two massless modes to be interpreted as the graviton, which now originate from diffemorphism invariance rather than Lorentz violation (so they are more like the gravitons in general relativity). This can never happen in the photon case, because the vev always completely breaks gauge invariance.

5.5.2 An example: Three Goldstone bosons only

We now wish to examine in detail a theory whose vev gives rise to three Goldstone modes only. Consider the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}[(\partial_\mu \tilde{h}^{\mu\nu})(\partial_\nu \tilde{h}) - (\partial_\mu \tilde{h}^{\rho\sigma})(\partial_\rho \tilde{h}^{\mu}_{\sigma}) \\ &\quad + \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \tilde{h}^{\rho\sigma})(\partial_\nu \tilde{h}_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \tilde{h})(\partial_\nu \tilde{h})] \\ &\quad + \lambda(\tilde{h}^{\mu\nu}\tilde{h}_{\mu\nu} - m^2), \end{aligned} \tag{5.52}$$

where, for simplicity, we choose the potential to be a Lagrange multiplier instead of a smooth potential. This fixes the length of $\tilde{h}_{\mu\nu} = H_{\mu\nu} + h_{\mu\nu}$. The corresponding equations of motion are

$$Q_{\mu\nu\rho\sigma}G^{\rho\sigma} = 0, \tag{5.53}$$

where

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}(\partial_\sigma\partial_\nu h^{\sigma}{}_{\mu} + \partial_\sigma\partial_\mu h^{\sigma}{}_{\nu} - \partial_\mu\partial_\nu h \\ &\quad - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\Box h) \end{aligned} \tag{5.54}$$

is the usual linearized Einstein tensor, and $Q_{\mu\nu\rho\sigma} = \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{m^2}H_{\mu\nu}H_{\rho\sigma}$ is a projection operator. Thus, (5.53) is essentially Einstein's equations projected onto the hypersurface orthogonal to $H^{\mu\nu}$. Note that we do not consider radiative corrections in this section.

Since the equations are linear, it is more convenient to switch to Fourier space ($\partial_\mu \to ik_\mu$), turning the differential equations into algebraic ones, which can then be written as a

9×9 matrix equation. Assume that $m^2 > 0$ in (5.52), one possible vev that minimizes the potential is

$$H_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.55}$$

which leads to three Goldstone modes (three boosts):

$$h_{\mu\nu}^{(Goldstone)} = \begin{pmatrix} 0 & -\beta_1 & -\beta_2 & -\beta_3 \\ -\beta_1 & 0 & 0 & 0 \\ -\beta_2 & 0 & 0 & 0 \\ -\beta_3 & 0 & 0 & 0 \end{pmatrix}. \tag{5.56}$$

As we demonstrate in Appendix B (where we give the most general polarization tensor of a graviton propagating in the z direction), it is impossible for a graviton to have all vanishing spatial components. Thus, no linear combinations of these three Goldstone modes in (5.56) can possibly behave like the graviton.

Nonetheless, the theory does contain two massless degrees of freedom, as we now demonstrate by directly solving the equations of motion. The first-order fixed-norm constraint $H_{\mu\nu}h^{\mu\nu} = 0$ (essentially the second cardinal gauge condition) implies that $h_{00} = 0$. The linearized equations of motion in momentum space are then

$$\begin{pmatrix} k_2^2+k_3^2 & -k_1k_2 & -k_1k_3 & 0 & k_0k_2 & k_0k_3 & -2k_0k_1 & 0 & -2k_0k_1 \\ -k_1k_2 & k_1^2+k_3^2 & -k_2k_3 & -2k_0k_2 & k_0k_1 & 0 & 0 & k_0k_3 & -2k_0k_2 \\ -k_1k_3 & -k_2k_3 & k_1^2+k_2^2 & -2k_0k_3 & 0 & k_0k_1 & -2k_0k_3 & k_0k_2 & 0 \\ 0 & -2k_0k_2 & -2k_0k_3 & 0 & 0 & 0 & 2(-k_0^2+k_3^2) & -2k_2k_3 & 2(-k_0^2+k_2^2) \\ k_0k_2 & k_0k_1 & 0 & 0 & k_0^2-k_3^2 & k_2k_3 & 0 & k_1k_3 & -2k_1k_2 \\ k_0k_3 & 0 & k_0k_1 & 0 & k_2k_3 & k_0^2-k_1^2 & -2k_1k_3 & k_1k_2 & 0 \\ -2k_0k_1 & 0 & -2k_0k_3 & 2(-k_0^2+k_3^2) & 0 & -2k_1k_3 & 0 & 0 & 2(-k_0^2+k_1^2) \\ 0 & k_0k_3 & k_0k_2 & -2k_2k_3 & k_1k_3 & k_1k_2 & 0 & k_0^2-k_1^2 & 0 \\ -2k_0k_1 & -2k_0k_2 & 0 & 2(-k_0^2+k_2^2) & -2k_1k_2 & 0 & 2(-k_0^2+k_1^2) & 0 & 0 \end{pmatrix} \begin{pmatrix} h_{01} \\ h_{02} \\ h_{03} \\ h_{11}/2 \\ h_{12} \\ h_{13} \\ h_{22}/2 \\ h_{23} \\ h_{33}/2 \end{pmatrix} = 0. \tag{5.57}$$

Without loss of generality (since rotational invariance is preserved), we align axes such that $k^\mu = (\omega, 0, 0, k)$. The equations of motion (5.57) have three zero eigenvalues, which is consistent with the fact that there are three residual gauge degrees of freedom. Meanwhile, there are two eigenvalues $\omega^2 - k^2$, and setting them to zero yields the dispersion relation $-\omega^2 + k^2 = k^\mu k_\mu = 0$. The corresponding eigenvectors have polarization tensors

$$p_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.58}$$

and

$$p_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.59}$$

These are exactly the + and × polarizations of a graviton propagating in the z direction in general relativity. Thus, the theory does contain two massless gravitons, but they do not arise as Goldstone bosons of spontaneous Lorentz violation.

	Photon	Graviton
Number of Goldstone Modes	3	6
Equivalent Gauge Conditions	Temporal or Axial	Cardinal
Number of Gauge Conditions/Massive Modes	1	4
Number of Transverse Modes	2	2
Number of Longitudinal Modes	1	4
Kinetic Term	Maxwell	Einstein-Hilbert

Table 5.1: Comparison between Goldstone Photons and Gravitons

The origin of these massless excitations are more appropriately associated with residual diffeomorphism invariance. With the chosen ground state (5.55), the Lagrangian remains invariant under the transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ for three independent functions ξ_i (corresponding to the three zero eigenvalues of the equations of motion). This guarantees the lack of mass terms for the components h_+ and $h_\times$ in the Lagrangian.

Furthermore, the simple vev (5.55) gives only two, rather than four, cardinal gauge conditions. There are thus fewer massive 'directions' in spacetime. Of the four conditions in (5.14), only two are independent. Since $H_{\mu\nu} \propto H_{\mu\rho} H^{\rho}{}_{\nu} \propto H_{\mu\rho} H^{\rho\sigma} H_{\sigma\nu}$, the last two gauge conditions in (5.14) are equivalent to the second. There are thus two, rather than four, massive modes.

Let's compare this theory with the one that we have been considering in earlier sections. Before, the vev broke both Lorentz invariance and diffeomorphism completely. There are four cardinal gauge conditions, which implies that there are four massive modes. The remaining six degrees of freedom correspond to the six broken generators of the Lorentz group. Two linear combinations of the six propagate, while the remaining four are auxiliary. Together, they add up to the ten degrees of freedom in $h_{\mu\nu}$.

In contrast, the theory that we consider in this section has a vev that breaks diffeomorphism invariance only partially. There are three remaining pure gauge modes. Because the vev preserves rotational invariance, only the three boost generators are broken, resulting in three Goldstone modes; none of them propagates, however. There are also only two massive modes, as the vev gives rise to only two independent cardinal gauge conditions. Together with the remaining two massless excitations that are identical to the graviton in general relativity, they account for the ten degrees of freedom that we started with in $h_{\mu\nu}$.

The possibilities are thus far richer in the graviton case than the photon case. In the former, there are three possibilities: the vev can break three, five, or six generators of the Lorentz group (We only discuss the first and the last case in this chapter.) When there are fewer than six Goldstone bosons, it is possible that the theory has residual diffeomorphism invariance, which can also result in massless excitations with the right properties to be interpreted as the graviton.

5.6 Conclusions

Recently, Kostelecky and Potting [65] examined in detail a scenario in which a symmetric two-index tensor acquires a vev via a potential. Two linear combinations of the six resulting Goldstone modes are dynamical and have properties identical to those of the graviton in general relativity. Because they originate in spontaneous symmetry breaking, this would provide a natural explanation for why the graviton is massless, without the need to invoke gauge invariance.

It was pointed out in [62] that, if we view the theory as an effective field theory, we should integrate out the massive modes, which would generate an infinite number of radiative correction terms in the low-energy effective Lagrangian. These terms are covariant in form, but involve the vev $H_{\mu\nu}$, thereby inducing additional Lorentz-violating effects. In this chapter, we examined the phenomenological properties of a subset of these radiative correction terms. In particular, we showed that they modify the dispersion relation of the two dynamical degrees of freedom, which becomes

$$k_\mu k^\mu - c_2 H^{\mu\nu} k_\mu k_\nu = 0. \tag{5.60}$$

This implies that the phase velocity of the dynamical modes is in general anisotropic. Another interesting consequence of the modified dispersion (5.60) is that test particles in their vicinity would be deflected differently from those near the graviton in general relativity. They would undergo both transverse and longitudinal oscillations that depend on the longitudinal and transverse separation, respectively.

We also investigated the relationship between different forms of the vev $H_{\mu\nu}$ and the corresponding Goldstone modes. Unlike in the photon case, for gravity there exist vevs for which there are not enough Goldstone modes to construct the conventional graviton — the gravitons may exist, but not as broken-symmetry generators acting on the vev.

Our analysis of the radiative-correction terms is by no means complete. For one thing, we have left out their effects on the four remaining Goldstone modes that become dynamical when they are present. Also, we only discussed terms that are linear in $H_{\mu\nu}$ and ignored higher-order corrections, which we believe to be sub-dominant, since Lorentz invariance has been verified to great accuracy at low energies. However, it is conceivable that the higher-order corrections would lead to interesting effects in addition to those that are discussed in this chapter, so they certainly merit further investigation. Finally, it would also be worthwhile to check whether the presence of the radiative corrections destabilize the theory.

Chapter 6

Unitary Evolution and Cosmological Fine-Tuning

6.1 Introduction

Inflationary cosmology [69, 70, 71] has come to play a central role in our modern understanding of the universe. Long appreciated as a solution to the horizon and flatness problems, the success of inflation-like perturbations (adiabatic, Gaussian, approximately scale-invariant) at explaining a multitude of observations has led most cosmologists to believe that some implementation of inflation is likely to be responsible for determining the initial conditions of our observable universe.

Nevertheless, our understanding of the fundamental workings of inflation lags behind our progress in observational cosmology. Although there are many models, we do not have a single standout candidate for a specific particle-physics realization of the inflaton and its dynamics. The fact that the scale of inflation is likely to be near the Planck scale opens the door to a number of unanticipated physical phenomena. Less often emphasized is our tenuous grip on the deep question of whether inflation actually delivers on its promise: providing a dynamical mechanism that turns a wide variety of plausible initial states into the apparently finely tuned conditions characteristic of our observable universe.

The point of inflation is to make the conditions of the hot, dense, smooth Big Bang seem natural. One can take the attitude that the initial conditions of the universe are simply to be accepted, rather than explained — we only have one universe, and should learn to deal with it, rather than seek explanations for the particular state in which we find it. In that case, there would never be any reason to contemplate inflation. The reason why inflation seems compelling is because we are more ambitious: we would like to understand why the universe seems to be one way, rather than some other way. By its own standards, the inflationary paradigm bears the burden of establishing that inflation is itself natural (or at least more natural than the alternatives).

It has been recognized for some time that there is tension between this goal and the underlying structure of classical mechanics (or quantum mechanics, for that matter). A key feature of classical mechanics is conservation of information: the time-evolution map from states at one time to states at some later time is invertible and volume-preserving, so that the earlier states can be unambiguously recovered from the later states. This property is encapsulated by Liouville's theorem, which states that a distribution function in the space of states remains constant along trajectories; roughly speaking, a certain number of states at one time always evolves into precisely the same number of states at any other time. In quantum mechanics, an analogous property is guaranteed by unitarity of the time-evolution operator; most of our analysis here will be purely classical, but we will refer to the conservation of the number of states as "unitarity" for convenience.

The conflict with the philosophy of inflation is clear. Inflation attempts to account for the apparent fine-tuning of our early universe by offering a mechanism by which a relatively natural early condition will robustly evolve into an apparently finely-tuned later condition. But if that evolution is unitary, it is impossible for any mechanism to evolve a large number of states into a small number, so the number of initial conditions corresponding to inflation must be correspondingly small, calling into question their status as "relatively natural." This point has been emphasized by Penrose [72], and has been subsequently discussed elsewhere [73, 74, 75, 76, 77, 78, 79, 80, 81]. As long as it operates within the framework of unitary evolution, the best inflation can do is to move the set of initial conditions that creates a smooth, flat universe at late times from one part of phase space to another part; it cannot increase the size of that set.

As a logical possibility, the true evolution of the universe may be non-unitary. Indeed, discussions of cosmology often proceed as if this were the case, as we discuss below. The justification for this perspective is that a comoving patch of space is smaller at earlier times, and therefore can accommodate fewer modes of quantum fields. But there is nothing in quantum field theory, or anything we know about gravity, to indicate that evolution is fundamentally non-unitary. The simplest resolution is to imagine that there are a large number of states that are not described by quantum fields in a smooth background (e.g., with Planckian spacetime curvature or the quantum-mechanical version thereof). Even if we don't have a straightforward description of the complete set of such states, the underlying principle of unitarity is sufficient to imply that they must exist.

It seems clear that inflation does *something*. If nothing else, the conditions required to begin inflation (a patch of space dominated by potential energy over a region larger than the corresponding Hubble length [82]) are algorithmically simple; they are easy to specify, in contrast with a wildly inhomogeneous early universe with conditions delicately tuned so that the inhomogeneities would smooth out as it evolved forward in time. It may very well be that, while proto-inflationary initial conditions are an extremely small subset of the space of all possible initial conditions, they are nevertheless what is naturally produced in some theory of quantum cosmology or multiverse dynamics. We argue that this is the best way to understand the role of inflation, rather than as a solution to the horizon and flatness problems.

This chapter has two goals. First, we use the canonical measure on the space of solutions to Einstein's equations developed by Gibbons, Hawking, and Stewart [73] to quantify the amount of fine-tuning involved in the flatness and homogeneity of the universe. Second, we attempt to clarify what is "nice" about the initial conditions required for inflation, in contrast with those of the conventional Big Bang cosmology. We do this by studying how classical trajectories leave the domain of validity of classical physics by entering a regime where quantum effects are necessarily important. A history of the universe, extrapolated into the past, will ultimately reach a point of Planckian curvatures where we should put a cutoff on our ability to describe it classically. But there is more than one kind of cutoff, depending on which quantity first reaches the Planck scale: the Hubble parameter, the background energy density, or the size of perturbations. Inflation acts to divert trajectories (evolving toward the past) so that they hit the perturbation cutoff before the Hubble cutoff; therefore, all inflationary trajectories starting at the Hubble cutoff (with sub-Planckian perturbations) lead to smooth universes at late times.

Along the way we encounter a surprise: the flatness problem doesn't exist. Considering the measure on purely Robertson-Walker cosmologies (without perturbations) as a function of spatial curvature, there is a divergence at zero curvature. In other words, curved RW cosmologies are a set of measure zero. This divergence has been noticed previously [73, 80], but was characterized as a feature that arose at large scale factors, rather than small curvatures. We argue for the most straightforward interpretation of the result: the flatness problem does not exist, as almost all solutions are spatially flat. Our intuition to the contrary is due to choosing an ill-defined measure on the space of initial conditions.

This divergence has no physical relevance, as the real world is not described by a perfectly Robertson-Walker metric. Nevertheless, it serves as a cautionary example for the importance of considering the space of initial conditions in a mathematically rigorous way, rather than relying on our intuition. We therefore perform a similar analysis for the case of perturbed universes, to verify that there is not any hidden divergence at perfect homogeneity. We find that there is not; any individual perturbation can be written as an oscillator with a time-dependent mass, and the measure is flat in the usual space of coordinate and momentum. The homogeneity of the universe represents a true fine tuning; there is no reason for the universe to be smooth.

The lesson of our investigation is that the state of the universe does appear unnatural from the point of view of the canonical measure on the space of trajectories, and that no choice of unitary evolution can alleviate that fine-tuning, whether it be inflation or any other mechanism. Inflation can alter the set of initial conditions that leads to a universe like ours, but it cannot make it any larger. Inflation does not remove the need for a theory of initial conditions, it simply brings that need into sharper focus.

6.2 The Evolution of Our Comoving Patch

Ever since Isaac Newton, the paradigm for fundamental physics has been information-conserving dynamical laws applied to initial data. A consequence of information conservation is reversibility: the state of the system at any one time is sufficient to recover its initial state, or indeed any state in the past or future.

Both quantum mechanics and classical mechanics feature this kind of unitary evolution.[1]

[1] The collapse of the wave function in quantum mechanics is an apparent exception. We will not address

In the Hamiltonian formulation of classical mechanics, a state is an element of phase space, specified by coordinates $q^a(t)$ and momenta $p_a(t)$. Time evolution is governed by Hamilton's equations,

$$\dot{q}^a = \frac{\partial \mathcal{H}}{\partial p_a}\,, \quad \dot{p}_a = -\frac{\partial \mathcal{H}}{\partial q^a}\,, \tag{6.1}$$

where $\mathcal{H}$ is the Hamiltonian. In quantum mechanics, a state is given by a wave function $|\psi(t)\rangle$ which defines a ray in Hilbert space. Time evolution is governed by the Schrödinger equation,

$$\hat{\mathcal{H}}|\Psi\rangle = i\partial_t|\Psi\rangle, \tag{6.2}$$

where $\hat{\mathcal{H}}$ is the Hamiltonian operator, or equivalently by the von Neumann equation,

$$\partial_t\hat{\rho} = -i[\hat{\mathcal{H}}, \hat{\rho}]\,, \tag{6.3}$$

where $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$ is the density operator. In either formalism, knowledge of the state at any one moment of time is sufficient (given the Hamiltonian) to determine the state at all other times. Even though we don't yet know the complete laws of fundamental physics, the most conservative assumption we could make would be to preserve the concept of unitarity. Even without knowing the Hamiltonian or the space of states, we will see that the principle of unitarity alone offers important insights into cosmological fine-tuning problems.

Although the assumption of unitary evolution seems like a mild one, there are challenges to applying the idea directly to an expanding universe. We can only observe a finite part of the universe, and the physical size of that part changes with time. The former feature

this phenomenon, implicitly assuming something like the many-worlds interpretation, in which wave function collapse is only apparent and the true evolution is perfectly unitary.

implies that the region we observe is not a truly closed system, and the latter implies that the set of field modes within this region is not fixed. Both aspects could be taken to imply that, even if the underlying laws of fundamental physics are perfectly unitary, it would nevertheless be inappropriate to apply the principle of unitarity to the the part of the universe we can observe.

In this chapter we will take the stance that it is nevertheless sensible to proceed under the assumption that the degrees of freedom describing our observable universe evolve according to unitary dynamical laws, even if that assumption is an approximation. In this section we offer the justification for this assumption. In particular we discuss two separate parts to this claim: that the observable universe evolves autonomously (as a closed system), and that this autonomous evolution is governed by unitary laws.

6.2.1 Autonomy

We live in an expanding universe that is approximately homogeneous and isotropic on large scales. We can therefore consider our universe as a perturbation of an exactly homogenous and isotropic (Robertson-Walker) background spacetime. Defining a particular map from the background to our physical spacetime involves a choice of gauge. Nothing that we are going to do depends on how that gauge is chosen, as long as it is defined consistently throughout the history of the universe. Henceforth we assume that we've chosen a gauge.

The map from the RW background spacetime to our universe provides two crucial elements: a foliation into time slices, and a congruence of comoving geodesic worldlines. The time slicing allows us to think of the universe as a fixed set of degrees of freedom evolving through time, obeying Hamilton's equations. At each moment in time there exists an exact value of the (background) Hubble parameter and all other cosmological parameters.

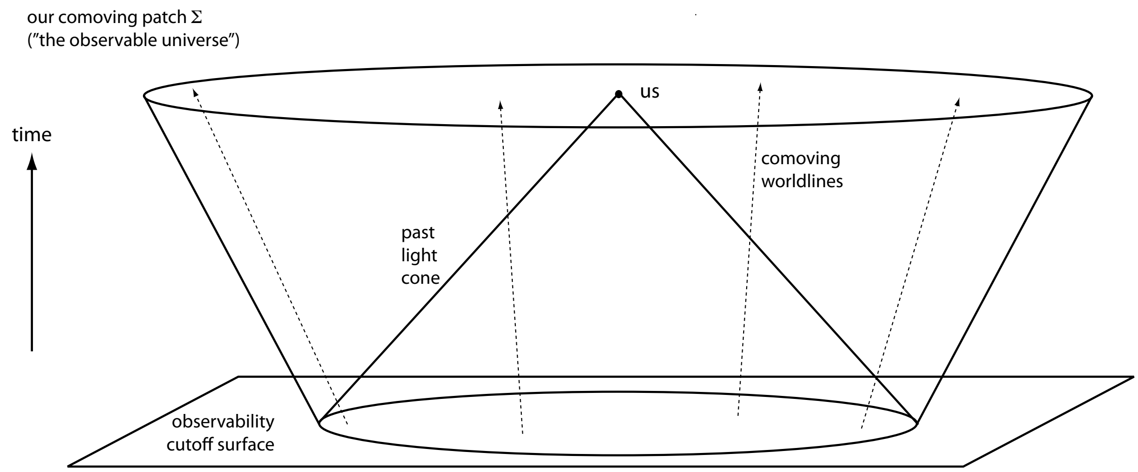

Figure 6.1: The physical system corresponding to our observable universe. Our comoving patch is defined by the interior of the intersection of our past light cone with a cutoff surface, for example the surface of last scattering. This illustration is not geometrically faithful, as the expansion is not linear in time. Despite the change in physical size, we assume that the space of states is of equal size at every moment.

The notion of comoving worldlines, orthogonal to spacelike hypersurfaces of constant Hubble parameter, allows us to define what we mean by our comoving patch. If there is a Big Bang singularity in our past, there is a corresponding particle horizon, defined by the intersection of our past light cone with the singularity. However, independent of the precise nature of the Big Bang, there is an effective limit to our ability to observe the past; in practice this is provided by the surface of last scattering, although in principle observations of gravitational waves or other particles could extend the surface backwards. The precise details of where we draw the surface aren't important to our arguments. What matters is that there exists a well-defined region of three-space interior to the intersection of our past light cone with the observability surface past which we can't see. Our comoving patch, Σ, is simply the physical system defined by the extension of that region forward in time via comoving worldlines, as shown in Figure 6.1.

Our assumption is that this comoving patch can be considered as a set of degrees of freedom evolving autonomously through time, free of influence from the rest of the universe. This is clearly an approximation, as an observer stationed close to the boundary of our patch would see particles pass both into and out of that region; our comoving patch isn't truly a closed system. However, the fact that the observable universe is homogenous implies that the net effect of that flux of particles is very small. In particular, we generally don't believe that what happens inside our observable universe depends in any significant way on what happens outside.

Note that we are not necessarily assuming that our observable universe is in a pure quantum state, free of entanglement with external degrees of freedom; such entanglements don't affect the local dynamics of the internal degrees of freedom, and therefore are completely compatible with the von Neumann equation (6.3). We are, however, assuming that the appropriate Hamiltonian is local in space. Holography implies that this is not likely to be strictly true, but it seems like an effective approximation for the universe we observe.

6.2.2 Unitarity

Autonomy implies that we can consider our comoving patch as a fixed set of degrees of freedom, evolving through time. Our other crucial assumption is that this evolution is unitary (reversible). Even if the underlying fundamental laws of physics are unitary, it is not completely obvious that the effective evolution of our comoving patch evolves this way. Indeed, this issue is at the heart of the disagreement between those who have emphasized the amount of fine-tuning required by inflationary initial conditions [72, 76, 78, 79] and those who have argued that they are natural [77, 81].

The issue revolves around the time-dependent nature of the cutoff on modes of a quantum field in an expanding universe. Since we are working in a comoving patch, there is a natural infrared cutoff given by the size of the patch, a length scale of order $\lambda_{\rm IR} \sim aH_0^{-1}$, where a is the scale factor (normalized to unity today) and H_0 is the current Hubble parameter. But there is also an ultraviolet cutoff at the Planck length, $\lambda_{\rm UV} \sim L_P = 1/\sqrt{8\pi G}$. Clearly the total number of modes that fit in between these two cutoffs increases with time as the universe expands. It is therefore tempting to conclude that the space of states is getting larger.

We can't definitively address this question in the absence of a theory of quantum gravity, but for purposes of this chapter we will assume that the space of states is *not* getting larger — which would violate the assumption of unitarity — but the nature of the states is changing. In particular, the subset of states that can usefully be described in terms of quantum fields on a smooth spacetime background is changing, but those are only a (very small) minority of all possible states.

The justification for this view comes from the assumed reversibility of the underlying laws. Consider the macrostate of our universe today — the set of all microstates compatible with the macroscopic configuration we observe. For any given amount of energy density, there are two solutions to the Friedmann equation, one with positive expansion rate and one with negative expansion rate (unless the expansion rate is precisely zero, when the solution is unique). So there is an equal number of microstates that are similar to our current configuration, except that the universe is contracting rather than expanding. As the universe contracts, each of those states must evolve into some unique future state; therefore, the number of states accessible to the universe for different values of the Hubble parameter (or different moments in time) is constant.

Most of the states available when the universe is smaller, however, are not described by quantum fields on a smooth background. This is reflected in the fact that spatial inhomogeneities would be generically expected to grow, rather than shrink, as the universe contracted. The effect of gravity on the state counting becomes significant, and in particular we would expect copious production of black holes. These would appear as white holes in the time-reversed expanding description. Therefore, the overwhelming majority of states at early times that could evolve into something like our current observable universe are not relatively smooth spacetimes with gently fluctuating quantum fields; they are expected to be wildly inhomogeneous, filled with white holes or at least Planck-scale curvatures.

We do not know enough about quantum gravity to explicitly enumerate these states, although some attempts to describe them have been made (see e.g., [83]). But the point is that we don't have to know how to describe them; the underlying assumption of unitarity implies that they are there, whether we can describe them or not. (Similarly, the Bekenstein-Hawking entropy formula is conventionally taken to imply a large number of states for macroscopic black holes, even if there is no general description for what those individual states are.)

This argument is not new, and it is often stated in terms of the entropy of our comoving patch. In the current universe, this entropy is dominated by black holes, and has a value of order $S_\Sigma(t_0) \sim 10^{104}$ [84]. If all the matter were part of a single black hole it would be as large as $S_\Sigma(\mathrm{BH}) \sim 10^{122}$. In the radiation-dominated era, when inhomogeneities were small and local gravitational effects were negligible, the entropy was of order $S_\Sigma(\mathrm{RD}) \sim 10^{88}$. If we assume that the entropy is the logarithm of the number of macroscopically indistinguishable microstates, and that every microstate within the current macrostate corresponds to a unique predecessor at earlier times, it is clear that the vast majority of states from which our present universe might have evolved don't look anything like the smooth radiation-dominated configuration we actually believe existed (since $\exp[10^{104}] \gg \exp[10^{88}]$).

This distinction between the number of states implied by the assumption of unitarity and the number of states that could reasonably be described by quantum fields on a smooth background is absolutely crucial for the question of how finely-tuned are the conditions necessary to begin inflation. If we were to start with a configuration of small size and very high density, and consider only those states described by field theory, we would dramatically undercount the total number of states. Unitarity could possibly be violated in an ultimate theory, but we will accept it for the remainder of this chapter.

6.3 The Canonical Measure

With these considerations in mind, we turn to a quantitative examination of the space of solutions in classical general relativity. Despite subtleties associated with coordinate invariance, GR can be cast as a conventional Hamiltonian system, with an infinite-dimensional phase space and a set of constraints. In classical mechanics the state of the system is described by a point γ in a phase space Γ, with canonical coordinates q^a and momenta p_a. The index a goes from 1 to n, so that phase space is $2n$-dimensional. Evolution according to Hamilton's equations (6.1) is generated by a Hamiltonian phase flow with tangent vector

$$V = \frac{\partial \mathcal{H}}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial \mathcal{H}}{\partial q^a}\frac{\partial}{\partial p_a}, \tag{6.4}$$

where $\mathcal{H}$ is the Hamiltonian.

Phase space is a symplectic manifold, which means that it naturally comes equipped with a symplectic form, which is a closed 2-form on Γ:

$$\omega = \sum_{a=1}^{n} \mathrm{d}p_a \wedge \mathrm{d}q^a\,, \qquad \mathrm{d}\omega = 0\,. \tag{6.5}$$

Here Γ is $2n$-dimensional, where n is the number of coordinates q^a. The existence of the symplectic form provides us with a unique measure on phase space,

$$\Omega = \frac{(-1)^{n(n-1)/2}}{n!}\omega^n \,. \tag{6.6}$$

This is the Liouville measure, a $2n$-form on Γ. It corresponds to the usual way of integrating distributions over regions of phase space,

$$\int f(\gamma)\Omega = \int f(q^a, p_a) d^n q d^n p \,. \tag{6.7}$$

The Liouville measure is conserved under Hamiltonian evolution. If we begin with a region $A \subset \Gamma$, and it evolves into a region A', Liouville's theorem states that

$$\int_A \Omega = \int_{A'} \Omega \,. \tag{6.8}$$

The infinitesimal version of this result is that the Lie derivative of Ω with respect to the vector field V vanishes,

$$\mathcal{L}_V \Omega = 0 \,. \tag{6.9}$$

These results can be traced back to the fact that the original symplectic form ω is also invariant under the flow:

$$\mathcal{L}_V \omega = 0 \,, \tag{6.10}$$

so any form constructed from powers of ω will be invariant.

In classical statistical mechanics, the Liouville measure can be used to assign weights to different distributions on phase space. That's not equivalent to assigning *probabilities* to different sets of states, which requires some additional assumption. However, since the Liouville measure is the only naturally-defined measure on phase space, it is natural to assume that it is proportional to the probability in the absence of further information; this is essentially Laplace's "Principle of Indifference." Indeed, in statistical mechanics we typically assume that microstates are distributed with equal probability with respect to the Liouville measure, consistent with known macroscopic constraints.

In cosmology, we don't typically imagine choosing a random state of the universe, subject to some constraints. When we consider questions of fine-tuning, however, we often consider what a randomly-chosen *history* of the universe would be like. In other words, we implicitly assume a measure on the space of solutions to Einstein's equations, with respect to which we can argue that a certain class of solutions (such as spatially flat universes, or universes that are approximately homogeneous on large scales) are unnaturally finely tuned, suggesting some deeper explanation than random chance. The assumption of some sort of measure is absolutely necessary for making sense of cosmological fine-tuning arguments; otherwise all we can say is that we live in the universe we see, and no further explanation is needed. (Note that this measure on the space of solutions to Einstein's equation is conceptually distinct from a measure on observers in a multiverse, which is sometimes used to calculate expectation values for cosmological parameters based on the anthropic principle.)

Gibbons, Hawking, and Stewart (GHS; [73]) showed how the Liouville measure on phase space could be used to define a unique measure on the space of solutions (see also [74, 75, 80]). In general relativity we impose the Hamiltonian constraint, so we can consider the $(2n-1)$-dimensional constraint hypersurface of fixed Hamiltonian,

$$C = \Gamma/\{\mathcal{H} = \mathcal{H}_*\}\,. \tag{6.11}$$

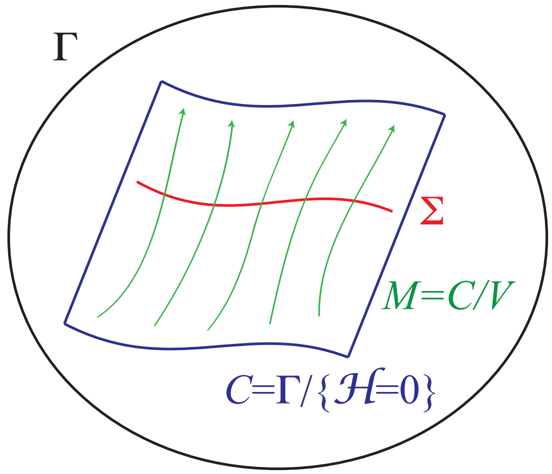

Figure 6.2: Γ is the phase space of the system. C is the constraint hypersurface of constant Hamiltonian (here chosen to be zero). M is the set of all classical trajectories in C, and Σ is a transverse surface through which each trajectory passes only once. Physical quantities should be independent of the choice of surface.

For Robertson-Walker cosmology, the Hamiltonian precisely vanishes for either open or closed universes, so we can take $\mathcal{H}_* = 0$. Then we consider the space of classical trajectories within this constraint hypersurface:

$$M = C/V \,, \tag{6.12}$$

where the quotient by the evolution vector field V means that two points are equivalent if they are connected by a classical trajectory. Note that this is well-defined, in the sense that points in C always stay within C, because the Hamiltonian is conserved. The construction is shown in Fig. 6.2.

As M is a submanifold of Γ, the measure is constructed by pulling back the symplectic form from Γ to M and raising it to the $(n-1)$th power. GHS constructed a useful explicit form by choosing the nth coordinate on phase space to be the time, $q^n = t$, so that the conjugate momentum becomes the Hamiltonian itself, $p_n = \mathcal{H}$. The symplectic form is then

$$\omega = \widetilde{\omega} + \mathrm{d}\mathcal{H} \wedge \mathrm{d}t \,, \tag{6.13}$$

where

$$\widetilde{\omega} = \sum_{a=1}^{n-1} \mathrm{d}p_a \wedge \mathrm{d}q^a \,. \tag{6.14}$$

The pullback of ω onto C then has precisely the same coordinate expression as (6.14), and we will simply refer to this pullback as $\widetilde{\omega}$ from now on. It is automatically transverse to the Hamiltonian flow ($\widetilde{\omega}(V) = 0$), and therefore defines a well-defined symplectic form on the space of trajectories M. The associated measure is

$$\widetilde{\Omega} = \frac{(-1)^{(n-1)(n-2)/2}}{(n-1)!} \widetilde{\omega}^{n-1} \,. \tag{6.15}$$

We will refer to this as the GHS measure; it is the unique measure on the space of trajectories that is positive, independent of arbitrary choices, and respects the appropriate symmetries [73].

To evaluate the measure we need to define coordinates on the space of trajectories. We can choose a hypersurface Σ in phase space that is transverse to the evolution trajectories, and use the coordinates on phase space restricted to that hypersurface. An important property of the GHS measure is that the integral over a region within a hypersurface is independent of which hypersurface we chose, so long as it intersects the same set of trajectories; if S_1 and S_2 are subsets, respectively, of two transverse hypersurfaces Σ_1 and Σ_2 in C, with the property that the set of trajectories passing through S_1 is the same as that passing through S_2, then

$$\int_{S_1} \widetilde{\Omega} = \int_{S_2} \widetilde{\Omega}\,. \tag{6.16}$$

The property that the measure on trajectories is local in phase space has a crucial implication for studies of cosmological fine-tuning. Imagine that we specify a certain set of trajectories by their macroscopic properties today; e.g., cosmological solutions that are approximately homogeneous, isotropic, and spatially flat, suitably specified in terms of canonical coordinates and momenta. It is immediately clear that the measure on this set is independent of the choice of Hamiltonian. Therefore, *no choice of Hamiltonian can make the current universe more or less finely tuned.* No new early-universe phenomena can change the measure on a set of universes specified at late times, because we can always evaluate the measure on a late-time hypersurface without reference to the behavior of the universe at any earlier time. At heart, this is a direct consequence of Liouville's theorem.

Gibbons and Turok interpreted the measure as the flux of a divergence-free "magnetic field," which is thereby converted into a surface integral [80]. The magnetic field is a one-form given by the Hodge dual (defined on the $(2n-1)$-dimensional constraint hypersurface C) of the two-form $\widetilde{\omega}$ raised to the $(n-1)$ power:

$$B \equiv *_C(\widetilde{\omega})^{n-1}\,. \tag{6.17}$$

In components, where $i \in \{1, \cdots 2n-1\}$ is a coordinate label on C,

$$B_i = \frac{1}{2^{n-1}(n-1)!}\epsilon_{ij_1j_2\cdots j_{2n-2}}(\widetilde{\omega})_{j_1j_2}(\widetilde{\omega})_{j_3j_4}\cdots(\widetilde{\omega})_{j_{2n-2}j_{2n-1}}\,. \tag{6.18}$$

The magnetic field B is divergenceless (from $\mathrm{d}\widetilde{\omega} = 0$), and parallel to the evolution vector V.

The integral of B projected into a hypersurface in C is independent of deformations of the hypersurface, and is equal to the integral of $\widetilde{\Omega}$. Hence, given some transverse hypersurface Σ in C representing M, the measure can be written in either of two forms,

$$\mu = \int_\Sigma \widetilde{\Omega} = \int_\Sigma B_i n^i \,, \tag{6.19}$$

where n^i is a unit vector in C orthogonal to Σ. With this formalism established, we can apply the measure to cosmological spacetimes.

6.4 Flatness

In this section, we evalute the measure on the space of solutions to Einstein's equations in minisuperspace (Robertson-Walker) cosmology with a scalar field. The metric is given by

$$ds^2 = -N^2 \mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{6.20}$$

where the spatial curvature parameter k can be normalized to -1, 0, or $+1$ (so that $a(t_0)$ is not normalized to unity). N is the lapse function, which acts as a Lagrange multiplier. The energy density of the scalar field ϕ is

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,. \tag{6.21}$$

The Lagrangian for this system is

$$L = -3N^{-1}a\dot{a}^2 + 3Nak + \frac{1}{2}N^{-1}a^3\dot{\phi}^2 - Na^3V(\phi)\,, \tag{6.22}$$

where we have chosen units where $8\pi G = 1$. The canonical coordinates can be taken to be the lapse function N, the scale factor a, and the scalar field ϕ. We can do a Legendre transformation to get the conjugate momenta,

$$p_N = 0\,, \quad p_a = -6N^{-1}a\dot{a}\,, \quad p_\phi = N^{-1}a^3\dot{\phi}\,. \tag{6.23}$$

The Hamiltonian is then given by

$$\mathcal{H} = N\left(-\frac{p_a^2}{12a} + \frac{p_\phi^2}{2a^3} + a^3V(\phi) - 3ak\right). \tag{6.24}$$

Varying with respect to N gives the Hamiltonian constraint, $\mathcal{H} = 0$, which is just the Friedmann equation,

$$H^2 = \frac{1}{3}\left(\rho_{\dot{\phi}} + \rho_V + \rho_k\right), \tag{6.25}$$

where we have defined

$$\rho_{\dot{\phi}} = \frac{1}{2}\dot{\phi}^2\,, \quad \rho_V = V(\phi)\,, \quad \rho_k = -3\frac{k}{a^2}\,. \tag{6.26}$$

Henceforth we will set $N = 1$, and we are left with a four-dimensional phase space,

$$\Gamma = \{\phi, p_\phi, a, p_a\}\,, \tag{6.27}$$

with the canonical measure

$$\omega = (dp_a \wedge da + dp_\phi \wedge d\phi)|_{\mathcal{H}=0}, \tag{6.28}$$

which is just the Liouville measure subject to the constraint that $\mathcal{H} = 0$. To enforce the constraint, we can use the Friedmann equation to eliminate a from the measure, which yields

$$a = \sqrt{\frac{3k}{V + \dot{\phi}^2/2 - 3H^2}}\,. \tag{6.29}$$

Upon substitution into (6.28), the measure simplifies to

$$\omega = \frac{1}{|k|}\left(\frac{3k}{V + \dot{\phi}^2/2 - 3H^2}\right)^{5/2}\left(\frac{1}{3}(V - \dot{\phi}^2 - 3H^2)d\dot{\phi}\wedge d\phi + (V' + 3H\dot{\phi})dH \wedge d\phi + \dot{\phi} dH \wedge d\dot{\phi}\right). \tag{6.30}$$

The corresponding magnetic field is

$$B_i \equiv \left(B_\phi, B_{\dot{\phi}}, B_H\right) = \frac{1}{|k|}\left(\frac{-3k}{3H^2 - V - \dot{\phi}^2/2}\right)^{5/2}\left(-\dot{\phi}, V' + 3H\dot{\phi}, -\frac{1}{3}(V - \dot{\phi}^2 - 3H^2)\right)\,. \tag{6.31}$$

We are now left with a three-dimensional reduced phase space, and a two-dimensional space of trajectories. The measure is defined by choosing some transverse surface Σ, and integrating the B-field dotted into an orthogonal one-form n_i.

$$\mu = \int_\Sigma B_i n^i\,. \tag{6.32}$$

One possible choice of the transverse surface Σ is to fix the Hubble parameter,

$$\Sigma : \{H = H_*\}\,. \tag{6.33}$$

The measure evaluated on a surface of constant H is then

$$\mu = \int_{H=H_*} B_H d\dot{\phi}\, d\phi \tag{6.34}$$

$$= \int_{H=H_*} \frac{1}{|k|} \left(\frac{-3k}{3H_*^2 - V - \dot{\phi}^2/2} \right)^{5/2} (V - \dot{\phi}^2 - 3H_*^2) d\phi d\dot{\phi}\,. \tag{6.35}$$

It is convenient to rewrite this by changing variables from $(\phi, \dot{\phi})$ to $(\rho_{\{\dot{\phi}}, \rho_k)$, and using the Friedmann equation (6.25). For simplicity, we will look at the potential $V(\phi) = m^2\phi^2/2$, although our results don't depend on this choice. The measure then becomes

$$\mu = \frac{3^{3/2}}{2m|k|} \int_{H=H_*} \left(\frac{-k}{\rho_k} \right)^{5/2} \frac{3\rho_{\dot{\phi}} + \rho_k}{\rho_{\dot{\phi}}{}^{1/2} (3H_*^2 - \rho_{\dot{\phi}} - \rho_k)^{1/2}} d\rho_{\dot{\phi}} d\rho_k \tag{6.36}$$

It is clear that the integrals over both $\rho_{\dot{\phi}}$ and ρ_k diverge. The divergence with respect to $\rho_{\dot{\phi}}$ occurs at large values, and is easily regulated by limiting our attention to densities smaller than some fixed number. With respect to curvature, however, there is a divergence as

$$\rho_k \to 0\,, \tag{6.37}$$

where the integrand goes as $\rho_k^{5/2}$. We might imagine regularizing this divergence by removing a region of size ϵ around $\rho_k = 0$, and letting $\epsilon \to 0$. We would find that all of the measure is dominated by nearly flat universes, in the following sense: Let $\mu(a, b)$ be the measure obtained by integrating over all values of $\rho_{\dot{\phi}}$ less than the cutoff, and values of ρ_k with $a < \rho_k < b$. Then we have

$$\lim_{\epsilon \to 0} \frac{\mu(a, b)}{\mu(\epsilon, a)} = 0 \tag{6.38}$$

for any $b > a > 0$. (An analogous conclusion holds for negative curvatures.) In other words, solutions with $\rho_k \neq 0$ are a set of measure zero.

There is a straightforward interpretation of this result: the flatness problem does not exist. If we were to somehow imagine randomly choosing a Robertson-Walker universe, it would be spatially flat with probability one. We feel that this interpretation is the most sensible one, even though it runs counter to the conventional presentation of the flatness problem.[2] The usual statement of the flatness problem notes that even a very small deviation from flatness at early times grows into an appreciable amount of curvature at late times. While this is true, it only becomes a "problem" when we presume a measure — in particular, some approximately-flat measure over values of the curvature parameter on some initial-condition surface in the early universe. The lesson of the GHS measure is that this reasonable-seeming intuition is wrong; the correct measure is very far from flat, and is strongly concentrated on precisely flat universes.

Notice that the hypersurface $H = H_*$ intersects all trajectories exactly once if $k \leq 0$. However, our conclusion remains valid even for closed universes, since the divergence $\rho_k^{-5/2}$ is present in all three components of (6.31). In principle, one can imagine deforming the $H = H_*$ surface to one that intersects all trajectories exactly once, and the divergence still remains. Alternatively, we could have chosen to eliminate p_a or p_ϕ instead of a in the measure. In this case, since $\frac{d}{dt}(\dot{\phi}a^3) = V'a^3$, the transverse surface $\dot{\phi}a^3 =$ constant would intersect all trajectories once, as long as the potential V for ϕ is monotonic. However, the physical meaning of this transverse surface is less intuitive, so we use instead the $H = H_*$ surface in our analysis.

[2]This divergence was noted in the original GHS paper [73], where it was attributed to "universes with very large scale factors" due to a different choice of variables. This seems to be beside the point, as any open universe will eventually have a large scale factor. It is also discussed by Gibbons and Turok [80], who correctly attribute it to nearly-flat universes. However, they advocate discarding all such universes as physically indistinguishable, and concentrating on the non-flat universes. To us, this seems to be throwing away almost all the solutions, and keeping a set of measure zero.

We should be clear about the implications of this result. The real world is not perfectly Robertson-Walker. If there are super-Hubble-radius perturbations (which are not suppressed, according to the analysis in the next section), in any one patch the measured value of the curvature parameter will deviate from unity. However, we draw the lesson that it is worthwhile doing a careful analysis of cosmological fine-tuning using a well-defined measure on the space of histories, as the results can differ substantially from a naive analysis.

6.5 Homogeneity

We now generalize our previous analysis of minisuperspace cosmology by including scalar perturbations to examine the horizon problem. Although the horizon problem is usually formulated in terms of the absence of causal contact between widely separated points in the early universe, for our purposes we can think of it as the statement that perturbation modes with large wavelengths have small amplitudes. While the set of all perturbations defines a large-dimensional phase space, we can keep things simple by looking at a single mode at a time. We will find that, in contrast with the surprising result of the last section, the measure on perturbations is just what we would expect.

To calculate the measure for scalar perturbations, we need to first compute the corresponding action. For the cases pertinent to our discussion (background domination by a perfect fluid or a scalar field), the calculation of the action has already been done in [85], which we will follow closely below. After obtaining the action, we can isolate the dynamical variables and construct the symplectic two-form on phase space, which can then be used to compute the measure on the set of solutions to Einstein's equations.

6.5.1 Action for a perfect fluid background

We will first calculate the measure for the solutions of scalar perturbations to Einstein's equations for a flat FRW background filled with a perfect fluid. The metric for this setting is

$$ds^2 = a^2(\eta)\left[-(1+2\phi)d\eta^2 + 2B_{,i}d\eta dx^2 + ((1-2\psi)\delta_{ij} + 2E_{,ij})dx^i dx^j\right], \tag{6.39}$$

where ϕ, ψ, $E_{,ij}$, and $B_{,i}$ are scalar perturbations to the metric. In this section, we will be using the conformal time η in addition to t. Derivatives with respect to η are denoted by the superscript $'$.

Up to second order in the perturbations, the gravitational part of the action is

$$\begin{aligned}\delta^{(2)}S_{gr} \;=\; & \frac{1}{2}\int d^4x a^2(-6\psi'^2 - 12\bar{H}(\phi+\psi)\psi' - 9\bar{H}^2(\phi+\psi)^2 \\ & -2\psi_{,i}(2\phi_{,i} - \psi_{,i}) - 4\bar{H}(\phi+\psi)(B-E')_{,ii} + 4\bar{H}\psi' E_{,ii} \\ & -4\psi'(B-E')_{,ii} - 4\bar{H}\psi_{,i}B_{,i} + 6\bar{H}^2(\phi+\psi)E_{,ii} \\ & -4\bar{H}E_{,ii}(B-E')_{,jj} + 4\bar{H}E_{,ii}B_{,jj} + 3\bar{H}^2E_{,ii}^2 + 3\bar{H}^2B_{,i}B_{,i}) \\ & +\text{total derivatives,}\end{aligned} \tag{6.40}$$

where $\bar{H} = a'/a$ (while $H = \dot{a}/a$). The dynamical quantity for hydrodynamical matter is $\xi^\alpha(x^\beta)$, the deviation of test particles from their trajectory in the unperturbed FRW universe. From this we can compute the matter part of the action,

$$\begin{aligned}\delta^{(2)}S_m \;=\; & \int d^4x[\frac{1}{2}\rho\phi^2 + p(\frac{3}{2}\psi^2 - 3\phi\psi + \phi E_{,ii} - \psi E_{,ii} + \frac{1}{2}E_{,ii}E_{,jj} \\ & -E_{,ij}E_{,ij} + \frac{1}{2}B_{,i}B_{,i}) + (\rho+p)(\frac{1}{2}\xi^{i\prime}\xi^{i\prime} + B_{,i}\xi^{i\prime} + \phi\xi^i_{,i}) \\ & -\frac{1}{2}c_s^2(\rho+p)(3\psi - E_{,ii} - \xi^i_{,i})^2]a^4 + \text{total derivatives,}\end{aligned} \tag{6.41}$$

where c_s is the adiabatic speed of sound in the fluid; $\beta \equiv \bar{H}^2 - \bar{H}'$; and ρ and p are the unperturbed energy density and pressure of the fluid. Combining (6.41) with (6.40), we obtain the total action quadratic in the scalar perturbations,

$$\begin{aligned}
\delta^{(2)}S &= \delta^{(2)}S_{gr} + \delta^{(2)}S_m \\
&= \frac{1}{2}\int d^4x(a^2(-6[\psi'^2 + 2\bar{H}\phi\psi' + (\bar{H}^2 - \frac{\beta}{3c_s^2})\phi^2] \\
&\quad -4(\psi' + \bar{H}\phi)(B - E')_{,ii} - 2\psi_{,i}(2\phi_{,i} - \psi_{,i}) \\
&\quad +2\beta(\xi^{i'} + B_{,i})(\xi^{i'} + B_{,i}) - 2\beta c_s^2(3\psi - E_{,ii} - \xi^i_{,i} + \frac{1}{c_s^2\phi^2})^2) \\
&\quad +\text{total derivatives.}
\end{aligned} \tag{6.42}$$

We now introduce the Mukhanov-Sasaki variable v:

$$v = \frac{1}{\sqrt{2}}(\phi_v - 2z\psi), \tag{6.43}$$

where $z \equiv a\beta^{1/2}/\bar{H}c_s$ and $\phi_v = -2a(\psi' + \bar{H}\phi)/(c_s\beta^{1/2})$ is the velocity potential of the fluid. Using constraints obtained by varying (6.42) with respect to ϕ, ψ, and $E_{,ii}$, the action takes on the simple form

$$\delta^{(2)}S = \frac{1}{2}\int d^4x\left(v'^2 - c_s^2 v_{,i}v_{,i} + \frac{z''}{z}v^2 + \text{total derivatives}\right). \tag{6.44}$$

This is the just the action for a scalar field with a time-varying mass. The fact that the we can express the action in terms of the Mukhanov-Sasaki variable v alone implies that there is only one dynamical degree of freedom present. The momentum p_v conjugate to v

is simply v', and the Hamiltonian is given by

$$\mathcal{H} = \frac{p_v^2}{2} - \left(c_s k^2 + \frac{z''}{z} \right) v^2. \tag{6.45}$$

6.5.2 Action during inflation

We now repeat the calculation for the case where the background is filled with a canonical scalar field $\mathcal{S}$ instead of a perfect fluid. The gravitational part of the action remains the same. The scalar-field contribution to the action is

$$S_{\mathcal{S}} = d^4x\sqrt{-g}\left(\frac{1}{2}\mathcal{S}_{;\alpha}\mathcal{S}^{;\alpha} - V(\mathcal{S})\right). \tag{6.46}$$

Expanding all quantities to second order in the perturbations, we have

$$\begin{aligned}
\delta^{(2)}S &= \delta^{(2)}S_{gr} + \delta^{(2)}S_{\mathcal{S}} \\
&= \frac{1}{2}\int a^2[-6\psi'^2 - 12\bar{H}\phi\psi' - 2\psi_{,i}(2\phi_{,i} - \psi_{,i}) - 2(\bar{H}' + 2\bar{H}^2)\phi^2 \\
&\quad +(\delta\mathcal{S}'^2 - \delta\mathcal{S}_{,i}\delta\mathcal{S}_{,i} - V_{,\mathcal{S}\mathcal{S}}a^2\delta\mathcal{S}^2) + 2(\bar{\mathcal{S}}'(\phi + 3\psi)'\delta\mathcal{S} - 2V_{,\mathcal{S}}a^2\phi\delta\mathcal{S}) \\
&\quad +4(B - E')_{,ii}(\bar{\mathcal{S}}\delta\mathcal{S}/2 - \psi' - \bar{H}\phi)] + \text{total derivatives}.
\end{aligned} \tag{6.47}$$

As before, we introduce a gauge-invariant quantity analogous to the Mukhanov-Sasaki variable,

$$v = a(\delta\mathcal{S} + (\bar{\mathcal{S}}'/\bar{H})\psi). \tag{6.48}$$

In terms of v, the action (6.47) simplifies to

$$\delta^{(2)}S = \frac{1}{2}\int\left(v'^2 - v_{,i}v_{,i} + \frac{z''}{z}v^2 + \text{total derivatives}\right), \tag{6.49}$$

where $z = a\bar{\mathcal{S}}'/\bar{H}$. Similar to the perfect fluid case, the action is just that for a scalar field with a time-varying mass, only that we now have $c_s^2 = 1$.

6.5.3 Computation of the measure

Given the actions (6.42) and (6.47), we can straightforwardly compute the invariant measure on phase space. One caveat is that now the Hamiltonian is time-dependent, so the carrier manifold of the Hamiltonian has an odd number of dimensions. We can retain the symplecticity of a time-dependent Hamiltonian system (which requires an even number of dimensions) by promoting time to be an addition canonical coordinate $q^{n+1} = t$. The conjugate momentum is then the negative value of the Hamiltonian, $p_{n+1} = -\mathcal{H}$. We can then construct an extended Hamiltonian $\mathcal{H}_+ = \mathcal{H}(p, q, t) + p_{n+1}$, which is explicitly time-independent, and from which we can derive the original Hamiltonian's equations ($\dot{q}^i = \partial\mathcal{H}_+/\partial p_i$ and $\dot{p}_i = -\partial\mathcal{H}_+/\partial q^i$), plus two additional trivial equations $\dot{t} = 1$ and $\dot{\mathcal{H}} = \partial\mathcal{H}/\partial t$.

With t promoted to a coordinate, the time-dependent Hamiltonian system also comes equipped naturally with a closed symplectic two-form, now with an additional term:

$$\omega = \sum_{a=1}^{n} dp_a \wedge dq^a - d\mathcal{H} \wedge dt. \tag{6.50}$$

The invariance of the form of Hamilton's equations ensures that the Lie derivative of ω with respect to the vector field generated by $\mathcal{H}_+$ vanishes. The top exterior power of ω is then guaranteed to be conserved under the extended Hamiltonian flow, and can thus play the role of the Liouville measure for the augmented system. The GHS measure can then be obtained by pulling-back the Liouville measure onto a hypersurface intersecting the trajectories and satisfying the constraint $\mathcal{H}_+ = 0$.

In our case, the original system, with coordinate v and conjugate momentum p_v, is augmented to one with two coordinates v and η, and their conjugate momenta p_v and $-\mathcal{H}$. The extended Hamiltonian, $\mathcal{H}_+ = p_v^2/2 - (c_s^2 k^2 - z''(t)/z(t))v^2 - \mathcal{H}$, is explicitly time-independent (identically zero), and its conservation is analogous to the Friedmann equation constraint in the analysis of the flatness problem. Using (6.50), the GHS measure ω_{GHS} for the perturbation is

$$\begin{aligned}
\omega_{GHS} &= dp_v \wedge dv - (d\mathcal{H} \wedge d\eta)|_{\mathcal{H}=p_v^2/2-(c_s^2k^2-z''(t)/z(t))v^2} \\
&= dp_v \wedge dv - d\left(\frac{p_v^2}{2} - \left(c_s^2 k^2 + \frac{z''}{z}\right) v^2\right) \wedge d\eta \\
&= dp_v \wedge dv - p_v(dp_v \wedge d\eta) + 2v\left(c_s^2 k^2 + \frac{z''}{z}\right) dv \wedge d\eta\,. \qquad (6.51)
\end{aligned}$$

One convenient hypersurface in which we can evalute the flux of trajectories is $\eta =$ constant. As $d\eta = dt/a$ is always positive, this surface intersects all trajectories exactly once. The flux of trajectories crossing this surface is unity, the coefficient of the first term in (6.51). This implies that all values for v and p_v are equally likely. There is nothing in the measure that would explain the small observed values of perturbations at early times. Hence, the observed homogeneity of our universe does imply considerable fine-tuning; unlike the flatness problem, the horizon problem is real.

6.6 Tracing Perturbations Backwards

Having established that the nearly homogeneous nature of our present universe represents a true fine-tuning problem, we turn to the relationship of inflation to this problem. Before delving into a general discussion, in this section we address a specific calculation: the evolution of perturbations backwards from the present day to the early universe. This will help us understand the difference between universes with and without inflation; in particular, how trajectories intersect cutoff surfaces defined at the Planck scale.

For the purpose of illustration, we will consider a highly-simplified picture, in which we compare the evolution of the energy density of perturbations in two scenarios — one where the universe is matter-dominated throughout its history and another where the universe undergoes a period of inflation prior to matter domination. Our analysis in this section draws on results from [86] and [87].

6.6.1 Relation to Planck-scale cutoffs

All of our discussion has been in the context of classical general relativity. We know that such a description can't be valid in all regimes; in particular, in cosmology, physical quantities will inevitably reach the Planck scale at some early time. This can be accounted for by imposing appropriate cutoffs, denoting boundaries of the phase space past which classical gravity no longer applies. In a smooth background, either the Hubble parameter or the energy density could reach the Planck scale; in principle we could also consider spatial curvature, but according to the Friedmann equation it will always be sub-Planckian if the Hubble parameter and the (positive) energy density are sub-Planckian. If we restrict ourselves to flat universes, a single cutoff when $H = m_{Pl}$ suffices. For perturbations, the classical equations fail when the gauge-invariant energy density $\widetilde{\delta\rho}$ becomes larger than m_{Pl}^4.

We therefore have two separate cutoff surfaces, for the Hubble parameter and for the perturbations. A universe that looks like ours at late times will, when evolved backwards in time, intersect one or the other of these surfaces. An important feature of inflation is that it changes which surface is relevant. Without inflation, trajectories typically hit the

Hubble parameter cutoff long before they hit the perturbation cutoff; with inflation, they typically hit the perturbation cutoff first.

To see this explicitly, we derive equations to evolve scalar perturbations in the current universe backwards in time. Although the canonical variables in the measure are the Mukhanov-Sasaki variable v and its conjugate p_v, we will focus instead on the gauge-invariant energy density $\widetilde{\delta\rho}$, since its physical meaning is much clearer. In terms of v and p_v, $\widetilde{\delta\rho}$ for a perfect-fluid background can be expressed as

$$\widetilde{\delta\rho} = \frac{1}{D}\left[\left(f_1 g_1 + f_1 g_2' + f_1 g_2 h_2 - f_2 g_1' - f_2 g_2 h_1\right) v + \left(-f_1 g_2 + f_2 g_1\right) p_v\right], \tag{6.52}$$

where

$$\begin{aligned}
f_1 &= -\frac{2}{a^2}(k^2 + 3\bar{H}^2), && (6.53)\\
f_2 &= -\frac{6}{a^2}\bar{H}, && (6.54)\\
g_1 &= -\frac{2a}{\sqrt{2}}\left(\frac{\bar{H}}{\beta^{1/2}c_s} + \frac{\beta^{1/2}}{\bar{H}c_s}\right), && (6.55)\\
g_2 &= -\frac{2a}{\sqrt{2}\beta^{1/2}c_s}, && (6.56)\\
h_1 &= -(c_s^2 k^2 + 2\bar{H}' + (1+3c_s^2)\bar{H}^2), && (6.57)\\
h_2 &= -3(1+c_s^2)\bar{H}. && (6.58)
\end{aligned}$$

For simplicity we assume that the perfect fluid is matter rather than radiation, although qualitatively similar results would be obtained for a more detailed calculation. In this case (6.52) simplifies to

$$\widetilde{\delta\rho} = \frac{1458\sqrt{3}}{\eta^8}(2v - \eta p_v). \tag{6.59}$$

Likewise, we can relate $\widetilde{\delta\rho}$ during inflation, which can be calculated using (6.70) and

(6.89), to v and p_v:

$$\widetilde{\delta\rho} = \left[\left(f_1 - \frac{f_2 h_1}{h_2}\right)\frac{\bar{\phi}'}{2k^2 a}\frac{z'}{z} + \frac{f_2}{h_2}\right] v - \left[\left(f_1 - \frac{f_2 h_1}{h_2}\right)\frac{\bar{\phi}'}{2k^2 a}\right] p_v, \tag{6.60}$$

where f_1, f_2 are as defined above, and

$$h_1 = a\left(\frac{2\bar{H}}{\bar{\phi}'} + \frac{\bar{\phi}'}{\bar{H}}\right) \tag{6.61}$$

$$h_2 = \frac{2a}{\bar{\phi}'}. \tag{6.62}$$

6.6.2 Evolution of perturbations in a matter-dominated universe

For the case in which the universe is matter-dominated throughout its history, the scale factor and Hubble parameter are given by

$$a = \left(\frac{t}{t_0}\right)^{2/3} = \left(\frac{\eta}{3t_0}\right)^2 \tag{6.63}$$

$$\bar{H} = \frac{2}{\eta}, \tag{6.64}$$

where t_0 is the age of the universe.

The evolution of perturbations is most easily described by considering the gauge-invariant form of Einstein's equations. We first define the following gauge-invariant quantities:

$$\Phi = \phi - \frac{1}{a}[a(B - E')]', \tag{6.65}$$

$$\Psi = \psi + \frac{a'}{a}(B - E'), \tag{6.66}$$

$$\widetilde{\delta\rho} = \delta\rho - \rho'(B - E'), \tag{6.67}$$

$$\widetilde{\delta p} = \delta p - p'(B - E'). \tag{6.68}$$

$$\tag{6.69}$$

In terms of these variables, Einstein's equations become

$$-k^2\Psi - 3\bar{H}(\Psi' + \bar{H}\Psi) = \frac{1}{2}a^2\widetilde{\delta\rho} \tag{6.70}$$

$$\Psi'' + 3\bar{H}\Psi' + (2\bar{H}' + \bar{H}^2)\Psi = \frac{1}{2}a^2\widetilde{\delta p}, \tag{6.71}$$

where we have used the fact $\Phi = \Psi$ (due to the absence of anisotropic stress) to simplify the equations.

Combining these two equations and considering only adiabatic perturbations, we have

$$\Psi'' + 3(1 + c_s^2)\bar{H}\Psi' + c_s^2k^2\Psi + (2\bar{H}' + (1 + 3c_s^2)\bar{H}^2)\Psi = 0. \tag{6.72}$$

For non-relativistic matter ($c_s = 0$), this equation simplifies to

$$\Psi'' + \frac{6}{\eta}\Psi' = 0, \tag{6.73}$$

which admits the solution

$$\Psi = a_1 + \frac{a_2}{\eta^5} \quad = \quad b_1 + b_2 H^{5/3} \tag{6.74}$$

where a_1, a_2, b_1, and b_2 are constants of integration.

Using (6.70), we can now solve for the gauge-invariant energy density of the perturbation $\widetilde{\delta\rho}$,

$$\begin{aligned} \widetilde{\delta\rho} &= -\frac{2}{a^2}\left(\frac{k^2}{a^2}\Psi + 3\bar{H}\dot{\Psi} + 3\bar{H}^2\Psi\right) \\ &= -2b_1k^2H^{4/3} - 2b_2k^2H^3 - 6b_1H^2 + 9b_2H^{11/3}. \end{aligned} \tag{6.75}$$

Of the two modes present, we are interested in the growing mode (terms with coefficient b_1):

$$\begin{aligned} \widetilde{\delta\rho_g} &= -2b_1(k^2H^{4/3} + 3H^2) \\ &= \frac{3H_0^2}{k^2 + 3H_0^2}\left(\frac{\widetilde{\delta\rho}(H_0)}{\rho}\right)\left(k^2\left(\frac{H}{H_0}\right)^{4/3} + 3H^2\right), \end{aligned} \tag{6.76}$$

where H_0 is the current Hubble parameter.

6.6.3 Evolution of perturbations in a matter-dominated universe preceded by inflation

For perturbations during inflation, we choose a model of inflation in which the inflaton is a canonical scalar field $\mathcal{S}$ in a potential that has the form of an exponential, so that all relevant quantities can be calculated analytically. In particular, the potential for the $\mathcal{S}$ is

$$V(\mathcal{S}) = ge^{-\lambda\mathcal{S}}, \tag{6.77}$$

where g and λ are constants.

With this potential, the background scalar field $\bar{\mathcal{S}}$ obeys

$$\bar{\mathcal{S}} = \frac{1}{\lambda}\ln\left(\frac{8\pi Gg\epsilon^2 t^2}{3-\epsilon}\right), \tag{6.78}$$

where the slow-roll parameter $\epsilon = -\dot{H}/H^2 = \lambda^2/2$, and

$$\dot{\bar{\mathcal{S}}} = \frac{2}{\lambda t} = \frac{2\epsilon H}{\lambda}. \tag{6.79}$$

This potential leads to power-law inflation, with $a \propto t^{1/\epsilon}$, and $H = 1/\epsilon t$.

The parameter z as defined above is given by

$$z = \frac{a\dot{\bar{\mathcal{S}}}}{H} = \frac{a\bar{\mathcal{S}}'}{\bar{H}} = \frac{2\epsilon t^{1/\epsilon}}{\lambda} \propto a, \tag{6.80}$$

Perturbations during inflation are, again, most simply described by the gauge-invariant form of Einstein's equations. We introduce the following gauge-invariant variables

$$u = \frac{2\Psi}{(\rho+p)^{1/2}} = \frac{2\Psi}{\dot{\bar{\mathcal{S}}}}, \tag{6.81}$$

$$\theta = \frac{1}{z} = \sqrt{\frac{1}{3}\frac{1}{a}}\left(1+\frac{p}{\rho}\right)^{-1/2}, \tag{6.82}$$

where ρ and p are the energy density and pressure of the background.

Using these gauge-invariant variables, two of Einstein's equations simplify to

$$\nabla^2 u = z\left(\frac{v}{z}\right)', \tag{6.83}$$

$$v = \theta\left(\frac{u}{\theta}\right)' \tag{6.84}$$

where z and v have been defined in (6.49) and (6.48). Note that we have switched back to the conformal time η. For the exponential potential, as t and a run from zero to infinity, $\eta = -\epsilon/(1-\epsilon)t^{-(1-\epsilon)/\epsilon}$ increases from negative infinity to zero.

Combining the two equations yields, in Fourier space,

$$u'' + \left(k^2 - \frac{\theta''}{\theta}\right) u = 0. \tag{6.85}$$

Defining $U = u/\theta$, (6.85) becomes

$$U'' + 2\frac{\theta'}{\theta}U' + k^2 U = 0, \tag{6.86}$$

which has the solution

$$U = (-k\eta)^{-\frac{1+\epsilon}{2(1-\epsilon)}}\left[c_3 J_{\frac{1+\epsilon}{2(1-\epsilon)}}(-k\eta) + c_4 J_{\frac{1+\epsilon}{2(1-\epsilon)}}(-k\eta)\right], \tag{6.87}$$

where c_3 and c_4 are integration constants, and $J_\alpha(x)$ is the Bessel function of the first kind of order α.

Applying the initial condition that $\Psi \to \eta^{\frac{-1}{1-\epsilon}} e^{-ik\eta}$ at very large $-k\eta$ (which can be obtained from solving Einstein's equations in the WKB approximation), $c_4 = ic_3$, implying that

$$U = d_3(-k\eta)^{-\frac{1+\epsilon}{2(1-\epsilon)}}\mathrm{Han}_{\frac{1+\epsilon}{2(1-\epsilon)}}(-k\eta), \tag{6.88}$$

where $\mathrm{Han}_\alpha(x)$ is the Hankel function of order α, and d_3 is a constant of integration. The scalar perturbation Ψ then becomes

$$\Psi = \frac{\dot{\mathcal{S}}}{2}\theta U = d_4\dot{\mathcal{S}}\frac{1}{a}(-k\eta)^{-\frac{1+\epsilon}{2(1-\epsilon)}}\mathrm{Han}_{\frac{1+\epsilon}{2(1-\epsilon)}}(-k\eta), \tag{6.89}$$

where d_4 is a constant and $\dot{\mathcal{S}}$ is given in (6.79). The gauge-invariant energy density of the perturbation $\widetilde{\delta\rho_i}$ can now be calculated by substituting Ψ and Ψ' into (6.70). We omit the explicit expression here, as it is very lengthy, and not particularly illuminating.

To describe the evolution of perturbations for a matter-dominated universe preceded by inflation, we can match solutions obtained in this section with those found in Section 6.6.2. In particular, at the transition, a, H, and $\widetilde{\delta\rho}$ have to be continuous.

6.6.4 Results

Our results are shown in Fig. 3, which extrapolates a set of trajectories backwards from the present day to the early universe, both with and without inflation. For the universe that is entirely matter-dominated, the trajectories reach the $H = m_{Pl}$ cutoff before intersecting the $\widetilde{\delta\rho} = m_{Pl}^4$ cutoff, while the opposite is true for the universe that inflated prior to the matter-domination era. It is important to keep in mind that, although the trajectories intersect the Placnkian cutoff surface very differently in these two scenarios, the number of trajectories contained in each band are identical by Liouville's theorem. Inflation merely diverts the trajectories, rather than increasing the number of states that evolve into our current universe.

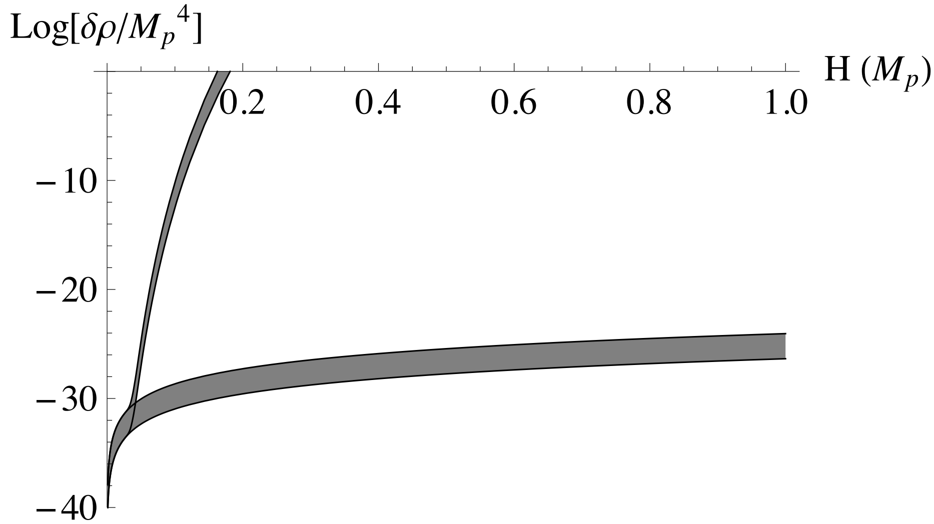

Figure 6.3: Log plot of the energy density of perturbation versus the Hubble parameter in a universe that is matter-dominated entirely (lower curve), and a universe that first undergoes inflation and then becomes matter-dominated (upper curve). The wavenumber shown here is $10^{-55}m_{Pl}$. Energy densities in the range of 10^{-122} to $10^{-121}m_{Pl}$ today ($H_0 = 10^{-60}m_{Pl}$) are plotted. For the upper bands, inflation ends at $H_i = 10^{-3}m_{Pl}$. The slow-roll parameter, ϵ, is chosen to be 0.1. This corresponds to $\lambda = 2.24$, and results in approximately 70 e-folds.

This result will seem more familiar if we turn it around: to obtain a universe with small perturbations at late times, in a purely matter-dominated cosmology we would have to start with extremely small perturbations when the Hubble parameter is near the Planck scale. With inflation, in contrast, we can start with the Hubble parameter at the Planck scale and *any* sub-Planckian value of the perturbations. (In our classical analysis, any such perturbations will be inflated to incredibly small values; in the real world, we expect that the observed perturbations are due to quantum fluctuations.)

6.7 What is Inflation Good For?

We have used the invariant measure on cosmological solutions to Einstein's equation to quantitatively investigate the amount of fine-tuning required to explain the initial conditions for our universe. Interestingly, we find that a careful analysis makes the flatness problem disappear; in the context of purely Robertson-Walker cosmologies, the measure diverges on flat universes. In the case of deviations from homogeneity, however, we recover something closer to the conventional result; in appropriate variables, the measure on the phase space of any particular mode of perturbation is flat, so that a generic universe would be expected to be highly inhomogeneous.

Now let's turn to the implications of this analysis for inflation. As we have discussed, the assumptions of unitarity and autonomy when applied to our comoving patch imply that any set of states at late times necessarily corresponds to an equal number of states at early times, as implied by Liouville's theorem. The situation is illustrated in Fig. 6.4. The diagram portrays the space of states for our comoving patch, foliated into slices corresponding to states with specific values of the Hubble parameter. As long as we restrict our attention to approximately Robertson-Walker universes, this is a valid description. For realistic values of the cosmological parameters, the Hubble parameter evolves monotonically in time, so that time evolution moves states through the foliation without doubling back. Furthermore, within this approximation trajectories that start with the same Hubble parameter at an initial time with have equal Hubble parameters at all times, since they share identical background cosmologies.

Liouville's theorem then implies that a given number of states on one slice through phase space will evolve into an equal number of states at any other time. In the figure we illustrate this schematically for two different choices of Hamiltonian: both have exactly the same field content and general form of the action, but in one there is a scalar field potential that allows for inflation, while in the other the corresponding potential does not support inflation. The canonical variables and the invariant measure on phase space will be the same for these two models, so they can be directly compared. It is clear that, assuming unitary evolution for

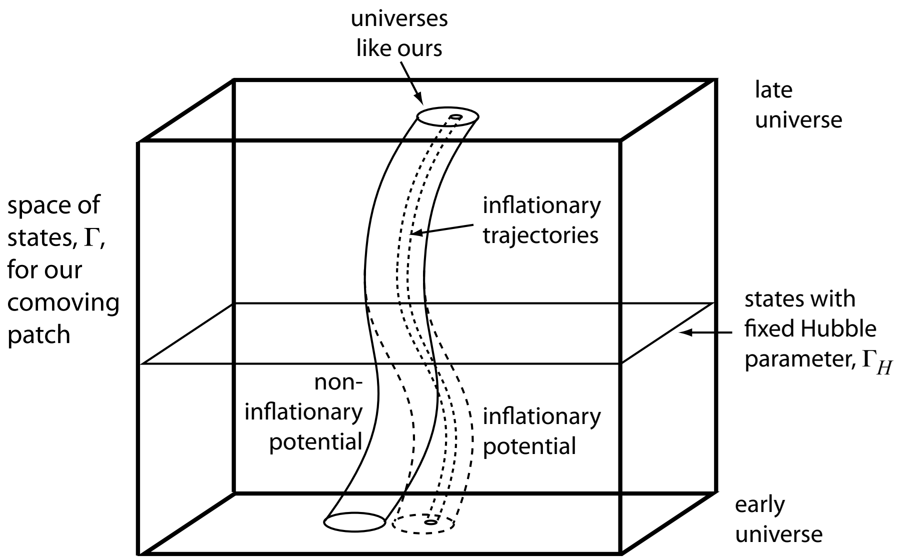

Figure 6.4: Γ is the space of states for the system defined by our comoving patch. For universes that are approximately Robertson-Walker, it can be foliated into subspaces of states with particular values of the Hubble parameter. Liouville's theorem implies that the choice of Hamiltonian does not affect the volume of a region of phase space as it evolves through time; this is illustrated schematically in the case of two theories with the same number of degrees of freedom, but different scalar-field potentials. Even with a potential that allows for inflation, the fraction of universes that actually inflate is very small. (This depiction is not to scale.)

our comoving patch, the different choice of scalar potential can only deflect the trajectories in some overall way. It cannot serve to focus or spread the trajectories, which would violate Liouville's theorem. Therefore, whether or not a theory allows for inflation has no impact on the total fraction of initial conditions that lead to a universe that looks like ours at late times.

Moreover, even with a Hamiltonian that permits inflation, the vast majority of cosmological solutions do not pass through an inflationary phase [74, 76, 80]. This is easily seen by imagining collapsing universes; it is extremely unlikely that a thermal plasma of fields will "anti-reheat" into a coherent inflaton field that then rolls slowly up its potential. Even if we restrict to the minisuperspace approximation, most trajectories roll quickly down the potential or start at the bottom of the potential, rather than the 60 or more e-folds of slow-roll phase that is required.

It is sometimes claimed that inflation is an "attractor" (see e.g., [88]), which would seem to be at odds with this picture. It is a basic feature of Hamiltonian mechanics that there are no attractors for closed systems; attractors only occur for systems with dissipation. Inflation appears to be an attractor if we only consider the behavior of the scalar inflaton field, without including gravity; the scalar sector by itself is dissipative due to Hubble friction. If the entire phase space is considered, it follows immediately that there are no attractors.

6.7.1 The universe is not chosen randomly

This basic argument has been appreciated for some time; indeed, its essential features were outlined by Penrose [72] even before inflation was invented. Nevertheless, it has failed to make an important impact on most discussions of inflationary cosmology. Attitudes toward this line of inquiry fall roughly into three camps: a small camp who believe that the implications of Liouville's theorem represent a significant challenge to inflation's purported ability to address fine-tuning problems [76, 78, 79, 80]; an even smaller camp who explicitly argue that the allowed space of initial conditions is much smaller than the space of later conditions, in apparent conflict with the principles of unitary evolution [77, 81]; and a very large camp who choose to ignore the issue or keep their opinions to themselves.

We would like to stake out a judicious middle ground. On the one hand, we believe that unitary evolution is to be respected, even at early times when the vast majority of states are not described by quantum field theory on smooth spacetime backgrounds. Therefore, inflation does not increase the fraction of states that evolve into reasonable universes; it merely alters their trajectories. On the other hand, the way in which the trajectories are altered by inflation is extremely suggestive. Even though the *number* of states that undergo inflation is much smaller than the number that do not, even when we restrict attention to trajectories that evolve into universes like the one we see, the *character* of those states is very different. We believe that the benefit of inflation is not that it makes universes like ours more numerous in the space of all possible universes, but that it provides a more reasonable target for a true theory of initial conditions, from quantum cosmology or elsewhere. (This is a possible reading of [77, 81], although those authors seem to exclude non-smooth initial conditions *a priori*, rather than relying on some well-defined theory of initial conditions.)

We have mentioned that, in the space of all trajectories that pass through states similar to our universe today, ones that include a period of inflation are a very small fraction. But it should be noted that something similar (although not quantitatively as strong) could be said about ordinary Big Bang cosmologies. Given the coarse-grained features of our universe today — the spatial geometry, distribution of matter and radiation, and so on — the overwhelming majority of microstates with those features did *not* arise from much smoother earlier states. This holds true even if the coarse-grained description includes our specific observations along our past light cone, such as the temperature anisotropies of the cosmic microwave background. Given only this information and no additional assumptions about the microstate, far more trajectories describe universes in which the apparent homogeneity of early times arises as a conspiracy of accidental cancelations between different effects, rather than an actually smooth early state. At the level of cosmological perturbations, this arises from the fact that we simply discard the decaying solution of every single mode; keeping them would admit universes that were more inhomogeneous in the past, and smoothed out to our present state. More generally, this can be seen by considering universes like ours that are collapsing rather than expanding; we would generically expect inhomogeneities to grow during the collapse. Most universes that look like ours today are simply time-reversed versions of such solutions, describing long series of thermodynamically unlikely coincidences.

However, we can admit that such universes seem bizarre to us. If we picked a trajectory for the universe randomly according to the canonical measure, most would never look like our present universe. Of those that did, only a very small minority would start smooth and evolve in what we think of as the conventional matter. However, those that do start smooth have a certain advantage over the others: we can easily say which ones they are, simply by referring to their macroscopic features at early times. (Namely, "they start smooth.") In contrast, the majority of initial states that grow into our universe today show no signs of being ready to do so at early times; there is no way to know which ones they are. The fact that they will ultimately smooth out is hidden in extremely subtle correlations between a multitude of degrees of freedom.[3] It seems much easier to imagine that an ultimate theory of initial conditions will produce states that are simple to describe rather than ones that feature an enormous number of mysterious and inaccessible correlations. In other words, it's true that a randomly-chosen universe like ours will begin in a wildly inhomogeneous state; but there's good reason to think that our universe was not chosen at random.

6.7.2 Inflation as an easy target

Given that we need some theory of initial conditions to explain why our universe was not chosen at random, the question becomes whether inflation provides any help to this unknown

[3] For a more familiar example, consider a glass of water with an ice cube that melts over the course of an hour. At the end of the melting process, if we reverse the momentum of every molecule in the glass, we will describe an initial condition that evolves into an ice cube. But there's no way of knowing that, just from the macroscopically available information; it looks just like a regular glass of water.

theory. We would like to suggest that it does, in two familiar ways: the required initial state does not need to be as big, or as smooth, as in conventional Big Bang cosmology.

First, inflation allows the initial patch of spacetime with a Planck-scale Hubble parameter to be physically small, while conventional cosmology does not. If we extrapoloate a matter- and radiation-dominated universe from today backwards in time, a comoving patch of size H_0^{-1} today corresponds to a physical size $\sim 10^{-26} H_0 \sim 10^{34} m_{Pl}^{-1}$ when $H = m_{Pl}$. In contrast, with inflation, the same patch needs to be no larger than the Planck length when $H = m_{Pl}$, as emphasized by Kofman, Linde, and Mukhanov [77, 81]. If our purported theory of initial conditions, whether quantum cosmology or baby-universe nucleation or some other scheme, has an easier time making small patches of space than large ones, inflation would be an enormous help.

The other advantage is in the degree of smoothness required. At the end of the previous section we calculated that a perfect-fluid universe with Planckian Hubble parameter would have to be extremely homogeneous to be compatible with the current universe, while an analogous inflationary patch could accommodate any amount of sub-Planckian perturbations. While the actual number of trajectories may be smaller in the case of inflation, there is a sense in which the requirements seem more natural. Within the set of initial conditions that experience sufficient inflation, all such states give us reasonable universes at late times; in a more conventional Big Bang cosmology, the perturbations require an additional substantial fine tuning. Again, we have a relatively plausible target for a future theory of initial conditions: as long as inflation occurs, and the perturbations are not initially super-Planckian, we will get a reasonable universe.

These features of inflation are certainly not novel; it is well-known that inflation allows for the creation of a universe such as our own out of a small and relatively small bubble of false vacuum energy. We are nevertheless presenting the point in such detail because we believe that the usual sales pitch for inflation is misleading; inflation does offer important advantages over conventional Friedmann cosmologies, but not necessarily the ones that are often advertised. In particular, inflation does not by itself make our current universe more likely; the number of trajectories that end up looking like our present universe is unaffected by the possibility of inflation, and even when it is allowed only a tiny minority of solutions feature it. Rather, inflation provides a specific kind of "nice" set-up for a true theory of initial conditions — one that is yet to be definitively developed.

Appendix A

Solutions to the Linearized Equations of Motion

We start by finding the solution to the equations of motion, linearized about a timelike, fixed-norm background, A_μ. Then, showing less details, we find the solutions to the equations of motion linearized about a spacelike background. Finally, we put the solutions in both cases into the compact form of (A.26)–(A.28). Our results agree with the solutions for Goldstone modes found in [44].

The equations of motion for a timelike (+) or spacelike (−) vector field are (3.16),

$$Q_\mu \equiv \left(\eta_{\mu\nu} \pm \frac{A_\mu A_\nu}{m^2}\right)(\beta_1 \partial_\rho \partial^\rho A^\nu + (\beta_* - \beta_1)\partial^\nu \partial_\rho A^\rho + \beta_4 G^\nu) = 0, \tag{A.1}$$

where G^ν is defined in (3.14) and $A^\mu Q_\mu = 0$ identically.

Timelike background. Consider perturbations about an arbitrary, constant (in space and time) timelike background $A_\mu = \bar{A}_\mu$ that satisfies the constraint: $\bar{A}_\mu \bar{A}^\mu = -m^2$. Define perturbations by $A_\mu = \bar{A}_\mu + \delta A_\mu$. Then, to first order in these perturbations, $\bar{A}^\mu Q_\mu = 0$ identically, and $\eta^{\mu\nu}\bar{A}_\mu \delta A_\nu = 0$ by the constraint. We can define a basis set of four Lorentz

4-vectors n^α, with components

$$n^0_\mu = \bar{A}_\mu/m \ , \qquad n^i_\mu \ ; \qquad i \in \{1,2,3\} \, , \tag{A.2}$$

such that

$$\eta^{\mu\nu} n^\alpha_\mu n^\beta_\nu = \eta^{\alpha\beta}. \tag{A.3}$$

The independent perturbations are $\delta a^\alpha \equiv \eta^{\mu\nu} n^\alpha_\mu \delta A_\nu$ for $\alpha = 1,2,3$. (δa^0 is zero at first order in perturbations due to the constraint.) It is then clear that there are three independent equations of motion at first order in pertubations (assuming the constraint) for the three independent perturbations,

$$\delta Q^i \equiv n^i_\nu \left(\beta_1 \partial_\rho \partial^\rho \delta A^\nu + (\beta_* - \beta_1) \partial^\nu \partial_\rho \delta A^\rho + \beta_4 n^0_\mu n^0_\rho \partial^\mu \partial^\rho \delta A^\nu \right) = 0, \tag{A.4}$$

where $i \in \{1,2,3\}$. We look for plane wave solutions for the δA:

$$\delta A_\mu = \int d^4k \, q_\mu(k) e^{i k_\nu x^\nu}. \tag{A.5}$$

Since $\eta^{\mu\nu} n^0_\mu \delta A_\nu = 0$, at first order,

$$q_\mu = c_j n^j_\mu \qquad \text{where} \qquad j \in \{1,2,3\}. \tag{A.6}$$

The equations of motion become the algebraic equations:

$$0 = \left(\beta_1 k_\rho k^\rho n^i_\nu n^{j\nu} + (\beta_* - \beta_1) n^i_\nu k^\nu n^j_\mu k^\mu + \beta_4 n^0_\mu n^0_\rho k^\mu k^\rho n^i_\nu n^{j\nu}\right) c_j \tag{A.7}$$

$$= \left(\beta_1 k_\rho k^\rho \delta^{ij} + (\beta_* - \beta_1) n^i_\nu k^\nu n^j_\mu k^\mu + \beta_4 n^0_\mu n^0_\rho k^\mu k^\rho \delta^{ij}\right) c_j \tag{A.8}$$

$$\equiv M^{ij} c_j. \tag{A.9}$$

The three independent solutions to these equations are given by setting an eigenvalue of the matrix M to zero and setting c_i to the corresponding eigenvector. Setting an eigenvalue of M equal to zero gives a dispersion relation,

$$\beta_1 k_\rho k^\rho + \beta_4 (n^0_\mu k^\mu)^2 = 0, \tag{A.10}$$

with two linearly independent eigenvectors,

$$(e_2)_i = \epsilon_{2ij} n^j_\mu k^\mu \qquad ; \qquad (e_3)_i = \epsilon_{3ij} n^j_\mu k^\mu. \tag{A.11}$$

The second eigenvalue of M gives the dispersion relation,

$$\beta_* k_\rho k^\rho + (\beta_* - \beta_1 + \beta_4)(n^0_\mu k^\mu)^2 = 0, \tag{A.12}$$

with corresponding eigenvector,

$$c_i = n^i_\mu k^\mu. \tag{A.13}$$

Spacelike background. The first-order linearized equations of motion about a spacelike background are:

$$\delta Q^a \equiv n^a_\nu \left(\beta_1 \partial_\rho \partial^\rho \delta A^\nu + (\beta_* - \beta_1)\partial^\nu \partial_\rho \delta A^\rho + \beta_4 n^3_\mu n^3_\rho \partial^\mu \partial^\rho \delta A^\nu\right) = 0 \tag{A.14}$$

where $a \in \{0,1,2\}$ and where, similarly to the timelike case, we have defined the set of four Lorentz 4-vectors, n^α_μ, to be

$$n^3_\mu = \bar{A}_\mu/m \qquad \text{and} \qquad n^a_\mu;\ a \in \{0,1,2\} \tag{A.15}$$

such that

$$\eta^{\mu\nu} n^\alpha_\mu n^\beta_\nu = \eta^{\alpha\beta}. \tag{A.16}$$

The independent perturbations are $\delta a^\alpha \equiv \eta^{\mu\nu} n^\alpha_\mu \delta A_\nu$ for $\alpha = 0,1,2$. (δa^3 is zero at first order in perturbations due to the constraint.)

Again we look for plane wave solutions of the form in (A.5). But now, since $\eta^{\mu\nu} n^3_\mu \delta A_\nu = 0$, at first order,

$$q_\mu = c_a n^a_\mu \qquad \text{where} \qquad a \in \{0,1,2\}. \tag{A.17}$$

The equations of motion become the algebraic equations:

$$= \Big(\beta_1 k_\rho k^\rho n^a_\nu n^{b\nu} + (\beta_* - \beta_1) n^a_\nu k^\nu n^b_\mu k^\mu + \beta_4 n^3_\mu n^3_\rho k^\mu k^\rho n^a_\nu n^{b\nu}\Big) c_b \tag{A.18}$$

$$= \Big(\beta_1 k_\rho k^\rho \eta^{ab} + (\beta_* - \beta_1) n^a_\nu k^\nu n^b_\mu k^\mu + \beta_4 n^3_\mu n^3_\rho k^\mu k^\rho \eta^{ab}\Big) c_b \tag{A.19}$$

$$\equiv M^{ab} c_b. \qquad a,b \in \{0,1,2\} \tag{A.20}$$

Two independent solutions correspond to the dispersion relation ($a \in \{0,1,2\}$)

$$\beta_1 k_\rho k^\rho + \beta_4 (n^3_\mu k^\mu)^2 = 0\,, \tag{A.21}$$

with corresponding eigenmodes

$$(e_1)_a = \epsilon_{a1b3} n^b_\mu k^\mu \qquad ; \qquad (e_2)_a = \epsilon_{ab23} n^b_\mu k^\mu. \tag{A.22}$$

The third solution corresponds to the dispersion relation

$$\beta_* k_\rho k^\rho - (\beta_* - \beta_1 - \beta_4)(n^3_\mu k^\mu)^2 = 0\,, \tag{A.23}$$

with corresponding eigenmode

$$c_a = \eta_{ab} n^b_\mu k^\mu. \tag{A.24}$$

General expression. We can express the solutions in the timelike and spacelike cases in a compact form by using the orthonormality of the n^α_μ, (A.3), along with (A.2), (A.15), and the fact that,[1]

$$\epsilon_{\alpha\beta\rho\sigma} n^\alpha_\mu n^\beta_\nu = \epsilon_{\mu\nu\alpha\beta} n^\alpha_\rho n^\beta_\sigma. \tag{A.25}$$

Then plugging (A.6) and (A.17) into (A.5) yields the solutions,

$$\delta A_\mu = \int d^4k\, q_\mu(k) e^{ik_\nu x^\nu} \tag{A.26}$$

[1]This follows from the invariance of the Levi-Civita tensor,

$$\epsilon_{\alpha\beta\gamma\delta} n^\alpha_\mu n^\beta_\nu n^\gamma_\rho n^\delta_\sigma = \epsilon_{\mu\nu\rho\sigma}$$

plus orthonormality, (A.3).

where either,

$$q_\mu(k) = i\alpha^\nu k^\rho \frac{\bar{A}^\sigma}{m}\epsilon_{\mu\nu\rho\sigma} \;\; \text{and} \;\; \beta_1 k_\rho k^\rho + \beta_4 \left(\frac{\bar{A}_\mu k^\mu}{m}\right)^2 = 0 \;\; \text{and} \;\; \alpha^\nu \bar{A}_\nu = 0, \tag{A.27}$$

where α^ν are real-valued constants or,

$$q_\mu = i\alpha\left(\eta_{\mu\nu} \pm \frac{\bar{A}_\mu \bar{A}_\nu}{m^2}\right)k^\nu \qquad \text{and} \qquad \beta_* k_\rho k^\rho \pm (\beta_* - \beta_1 \pm \beta_4)\left(\frac{\bar{A}_\mu k^\mu}{m}\right)^2 = 0, \tag{A.28}$$

where α is a real-valued constant. The reality of the α's follows from the condition, $q_\mu(k) = q^*_\mu(-k)$, that holds if and only if δA_μ in (A.5) is real. In (A.28), the "+" sign corresponds to the timelike background and the "−" sign to a spacelike background.

Appendix B

Additional Properties of the Goldstone Modes

B.1 Polarizations of Goldstone Modes

We enumerate here the Goldstone modes that arise when a symmetric two-index tensor acquires various forms of vacuum expectation values. Linearity implies that the Goldstone mode corresponding to a general vev is a superposition of these modes.

B.1.1 Time-time

Let's first consider the case where only the 00 component of $H_{\mu\nu}$ does not vanish. In that case, the three boost generators are broken, and we therefore have three Goldstone modes.

$$H_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to h_{\mu\nu} = \begin{pmatrix} 0 & -\beta_1 & -\beta_2 & -\beta_3 \\ -\beta_1 & 0 & 0 & 0 \\ -\beta_2 & 0 & 0 & 0 \\ -\beta_3 & 0 & 0 & 0 \end{pmatrix}. \tag{B.1}$$

Obviously, this choice of the vacuum expectation value preserves rotational invariance. Hence, none of the θ modes is excited.

B.1.2 Time-space

Now consider the case where one of the $0i$ components is nonzero. This breaks all three boosts, but only two of the three rotation generators. There are thus five Goldstone modes.

$$H_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow h_{\mu\nu} = \begin{pmatrix} -2\beta_1 & 0 & \theta_3 & -\theta_2 \\ 0 & -2\beta_1 & -\beta_2 & -\beta_3 \\ \theta_3 & -\beta_2 & 0 & 0 \\ -\theta_2 & -\beta_3 & 0 & 0 \end{pmatrix}. \tag{B.2}$$

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow h_{\mu\nu} = \begin{pmatrix} -2\beta_2 & -\theta_3 & 0 & \theta_1 \\ -\theta_3 & 0 & -\beta_1 & 0 \\ 0 & -\beta_1 & -2\beta_2 & -\beta_3 \\ \theta_1 & 0 & -\beta_3 & 0 \end{pmatrix}. \tag{B.3}$$

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow h_{\mu\nu} = \begin{pmatrix} -2\beta_3 & \theta_2 & -\theta_1 & 0 \\ \theta_2 & 0 & 0 & -\beta_1 \\ -\theta_1 & 0 & 0 & -\beta_2 \\ 0 & -\beta_1 & -\beta_2 & -2\beta_3 \end{pmatrix}. \tag{B.4}$$

B.1.3 Diagonal space-space

Now consider the case where one of the diagonal spatial elements does not vanish. This breaks one of the three boosts, and two of the rotations.

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow h_{\mu\nu} = \begin{pmatrix} 0 & -\beta_1 & 0 & 0 \\ -\beta_1 & 0 & \theta_3 & -\theta_2 \\ 0 & \theta_3 & 0 & 0 \\ 0 & -\theta_2 & 0 & 0 \end{pmatrix}. \tag{B.5}$$

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow h_{\mu\nu} = \begin{pmatrix} 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & -\theta_3 & 0 \\ -\beta_2 & -\theta_3 & 0 & \theta_1 \\ 0 & 0 & \theta_1 & 0 \end{pmatrix}. \tag{B.6}$$

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -\beta_3 \\ 0 & 0 & 0 & \theta_2 \\ 0 & 0 & 0 & -\theta_1 \\ -\beta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix}. \tag{B.7}$$

B.1.4 Off-diagonal space-space

Finally, we consider the case in which one of the off-diagonal spatial components is non-zero. This breaks two boosts and all rotations.

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow h_{\mu\nu} = \begin{pmatrix} 0 & -\beta_2 & -\beta_1 & 0 \\ -\beta_2 & -2\theta_3 & 0 & \theta_1 \\ -\beta_1 & 0 & 2\theta_3 & -\theta_2 \\ 0 & \theta_1 & -\theta_2 & 0 \end{pmatrix}. \tag{B.8}$$

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \rightarrow h_{\mu\nu} = \begin{pmatrix} 0 & -\beta_3 & 0 & -\beta_1 \\ -\beta_3 & 2\theta_2 & -\theta_1 & 0 \\ 0 & -\theta_1 & 0 & \theta_3 \\ -\beta_1 & 0 & \theta_3 & -2\theta_2 \end{pmatrix}. \tag{B.9}$$

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow h_{\mu\nu} = \begin{pmatrix} 0 & 0 & -\beta_3 & -\beta_2 \\ 0 & 0 & \theta_2 & -\theta_3 \\ -\beta_3 & \theta_2 & -2\theta_1 & 0 \\ -\beta_2 & -\theta_3 & 0 & 2\theta_1 \end{pmatrix}. \tag{B.10}$$

Notice that not all ten modes are independent. We can, for example, perform a rotation to diagonalize the three modes in B.1.4, so that they become a linear combination of the modes in B.1.3.

B.2 Proof That Gravitons Can Be Goldstone Bosons

We present here a proof that when all six generators are broken, two linear combinations of the resulting six Goldstone bosons have properties that agree with those of the graviton at lowest order.[1] The propagating Goldstone modes obey the dispersion relation $k^\mu k_\mu = 0$, the transverse conditions $k^\mu h_{\mu\nu} = 0$, and the four cardinal gauge conditions.

First consider the most general vacuum expectation value

$$H_{\mu\nu} = \begin{pmatrix} d & e & f & g \\ e & a & h & i \\ f & h & b & j \\ g & i & j & c \end{pmatrix}, \tag{B.11}$$

where the ten constants a, b, c, d, e, f, g, h, i, j are presumably determined by the potential V in (5.12). This choice of the vev might seem unnecessarily complicated (as it can be simplified by boosts and rotations). However, as will be shown below, Eq. (B.11) will simplify our analysis later on.

This vacuum expectation value gives the following Goldstone excitations:

[1]During the preparation of this manuscript, we became aware of the recent work by Kostelecky and Potting [65], in which they gave a proof that a version of this Lorentz-violating theory of gravity is identical to linearized gravity in the cardinal gauge.

$$h_{00} = -2e\beta_1 - 2f\beta_2 - 2g\beta_3 \tag{B.12}$$

$$h_{01} = -(a+d)\beta_1 - h\beta_2 - i\beta_3 + g\theta_2 - f\theta_3 \tag{B.13}$$

$$h_{02} = -h\beta_1 - (b+d)\beta_2 - j\beta_3 - g\theta_1 + e\theta_3 \tag{B.14}$$

$$h_{03} = -i\beta_1 - j\beta_2 - (c+d)\beta_3 + f\theta_1 - e\theta_2 \tag{B.15}$$

$$h_{11} = -2e\beta_1 + 2i\theta_2 - 2h\theta_3 \tag{B.16}$$

$$h_{22} = -2f\beta_2 - 2j\theta_1 + 2h\theta_3 \tag{B.17}$$

$$h_{33} = -2g\beta_3 + 2j\theta_1 - 2i\theta_2 \tag{B.18}$$

$$h_{12} = -f\beta_1 - e\beta_2 - i\theta_1 + j\theta_2 + (a-b)\theta_3 \tag{B.19}$$

$$h_{13} = -g\beta_1 - e\beta_3 + h\theta_1 + (c-a)\theta_2 - j\theta_3 \tag{B.20}$$

$$h_{23} = -g\beta_2 - f\beta_3 + (b-c)\theta_1 - h\theta_2 + i\theta_3. \tag{B.21}$$

We would now like to demonstrate that it is possible for the Goldstone modes resulting from a completely general vev to have a polarization tensor that agrees with that of a graviton (in GR) propagating in the z direction in some gauge. In general relativity, we have the freedom to add to any solution of the linearized Einstein's equations the pure gauge mode $k_{(\mu|}\xi_{|\nu)}$. Therefore, the familiar $+$ and $\times$ polarizations in the transverse-traceless gauge,

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\alpha x^\alpha}, \tag{B.22}$$

are not the most general form that the graviton in general relativity can take.

For a graviton propagating in the z direction, we have $k^\mu = (\omega, 0, 0, \omega)$. If we set $\xi_\mu = \frac{1}{\omega}(-p_{00}, -p_{01}, -p_{02}, -p_{03})$, the polarization $p^{(gauge)}_{\mu\nu}$ of the most general gauge mode $h^{(gauge)}_{\mu\nu} = p^{(gauge)}_{\mu\nu} e^{ik_\alpha x^\alpha}$ can be written as

$$p^{(gauge)}_{\mu\nu} = \begin{pmatrix} p_{00} & p_{01} & p_{02} & (p_{03} - p_{00})/2 \\ p_{01} & 0 & 0 & -p_{01} \\ p_{02} & 0 & 0 & -p_{02} \\ (p_{03} - p_{00})/2 & -p_{01} & -p_{02} & -p_{03} \end{pmatrix}, \tag{B.23}$$

where p_{00}, p_{01}, p_{02}, and p_{03} are constants. Thus, the most general form that the graviton can assume in GR is the sum of (B.22) and (B.23)[2]:

$$h^{(general)}_{\mu\nu} = \begin{pmatrix} p_{00} & p_{01} & p_{02} & -p_{00} \\ p_{01} & h_+ & h_\times & -p_{01} \\ p_{02} & h_\times & -h_+ & -p_{02} \\ -p_{00} & -p_{01} & -p_{02} & p_{00} \end{pmatrix} e^{ik_\alpha x^\alpha}. \tag{B.24}$$

Note that because the Goldstone modes are all traceless, we have also set $p_{00} = -p_{03}$ above. We now want to see if the polarizations of the Goldstone bosons resulting from the most general vev (B.11) can be matched onto (B.24).

[2]Here, we are restricting ourselves to graviton solutions of the form $e^{ik_\alpha x^\alpha}$. If we relax this assumption, it is conceivable that there are other possible functional forms. This is analogous to electromagnetism in the axial gauge, in which $A_\mu \propto z e^{ik_\alpha x^\alpha}$ is needed to describe a plane-wave photon in the z direction. Thus, the field becomes unbounded at spatial infinity, and it is questionable whether our effective theory is valid.

To match (B.11) onto (B.24), we have to satisfy the following conditions:

$$
\begin{aligned}
h_{00} &= -h_{03} \\
h_{01} &= -h_{31} \\
h_{02} &= -h_{32} \\
h_{00} &= h_{33}.
\end{aligned}
\qquad \text{(B.25)}
$$

These four conditions leave in the six Goldstone modes two degrees of freedom, exactly the right number to describe the graviton, which has two polarizations.

At this point, it is convenient to define new fields by linearly combining the Goldstone modes:

$$
\begin{aligned}
M_1 &= -(h_{00}+h_{33}) \\
&= (2e+i)\beta_1+(2f+j)\beta_2+(2g+c+d)\beta_3 \\
&\quad -f\theta_1+e\theta_2 \qquad \text{(B.26)} \\
M_2 &= -(h_{01}+h_{31}) \\
&= (a+d+g)\beta_1+h\beta_2+(i+e)\beta_3 \\
&\quad -h\theta_1-(g+c-a)\theta_2+(f+j)\theta_3 \qquad \text{(B.27)} \\
M_3 &= -(h_{02}+h_{32}) \\
&= h\beta_1+(b+d+g)\beta_2+(j+f)\beta_3 \\
&\quad +(g+c-b)\theta_1+h\theta_2-(e+i)\theta_3 \qquad \text{(B.28)} \\
M_4 &= -h_{00}+h_{33} \\
&= 2e\beta_1+2f\beta_2+2j\theta_1-2i\theta_2 \qquad \text{(B.29)} \\
M_5 &= h_{11}\equiv h_+ \\
&= -2e\beta_1+2i\theta_2-2h\theta_3 \qquad \text{(B.30)} \\
M_6 &= h_{12}\equiv h_\times \\
&= -f\beta_1-e\beta_2-i\theta_1+j\theta_2+(a-b)\theta_3. \qquad \text{(B.31)}
\end{aligned}
$$

In this new basis, the physical degrees of freedom are made very transparent: M_5 and M_6 are the usual $+$ and $\times$ gravitons. The four conditions (B.25) now become $M_1 = M_2 = M_3 = M_4 = 0$.

These six linear equations relating the two bases can be written as a matrix equation

$$\mathbf{A}\vec{\zeta} = \vec{M}, \tag{B.32}$$

where $\vec{\zeta} = (\beta_1, \beta_2, \beta_3, \theta_1, \theta_2, \theta_3)$ and $\vec{M} = (M_1, M_2, M_3, M_4, M_5, M_6)$ are the Goldstone modes in the original basis and new basis, respectively. This gives immediately the constraint $\det(\mathbf{A}) \neq 0$, since otherwise the matrix $\mathbf{A}$ is singular and the new basis spanned by $\vec{M}$ is incomplete.

To express $h_{\mu\nu}$ in the new basis spanned by $\vec{M}$, we first invert Eq. (B.32) to solve for $\vec{\zeta} = \mathbf{A}^{-1}\vec{M}$, which can then be substituted into Eqs. (B.12) – (B.21).

B.2.1 The two transverse linear combinations of the six Goldstone modes

We now proceed to show that two linear combinations of the Goldstone modes (M_5 and M_6) obey the dispersion relation $k^\mu k_\mu = 0$ and are transverse to the momentum ($k^\mu h_{\mu\nu} = 0$).

Setting all $M_i = 0$ except for M_5, all the conditions in (B.25) would be satisfied, and we have

$$h^{(5)}_{\mu\nu} = \begin{pmatrix} c^5_{00} & c^5_{01} & c^5_{02} & -c^5_{00} \\ c^5_{01} & 1 & 0 & -c^5_{01} \\ c^5_{02} & 0 & -1 & -c^5_{02} \\ -c^5_{00} & -c^5_{01} & -c^5_{02} & c^5_{00} \end{pmatrix} M_5, \tag{B.33}$$

which has exactly the form of (B.24) if $h_\times = 0$. M_5 therefore corresponds to the + polarization of the graviton. The constants c^5_{ij} are computed straightforwardly using Eqs. (B.12) – (B.21).

Similarly, if we turn off all the M_i's except M_6, all the conditions (B.25) are satisfied, and the polarization tensor of the Goldstone mode M_6 becomes

$$h^{(6)}_{\mu\nu} = \begin{pmatrix} c^6_{00} & c^6_{01} & c^6_{02} & -c^6_{00} \\ c^6_{01} & 0 & 1 & -c^6_{01} \\ c^6_{02} & 1 & 0 & -c^6_{02} \\ -c^6_{00} & -c^6_{01} & -c^6_{02} & c^6_{00} \end{pmatrix} M_6, \tag{B.34}$$

which agrees with (B.24) if $h_+ = 0$, and therefore represents the $\times$ polarization. As before, the constants c^6_{ij} are computed using Eqs.(B.12) – (B.21). Note that because M_5 and M_6 are nonzero, it is in general impossible to set all c^5_{ij} and $c^6_{ij} = 0$. That is, no choice of $H_{\mu\nu}$ corresponds to the transverse-traceless gauge conventionally used to describe the graviton.

Because the kinetic terms in the Lagrangian of our theory are those in the Einstein-Hilbert action, the equations of motion of these Goldstone modes (valid for all six modes $M_{1\to 6}$) to leading order are simply given by the linearized Einstein equation in vacuum

$$\partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma{}_\nu - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\sigma} = 0. \tag{B.35}$$

Substituting the + mode, Eq. (B.33), into Eq. (B.35) and setting the 4-momentum to

$k^\mu = (\omega, 0, 0, k)$ gives

$$2G_{00} = 0 \tag{B.36}$$

$$2G_{01} = c^5_{01} k(\omega - k) = 0 \tag{B.37}$$

$$2G_{02} = c^5_{02} k(\omega - k) = 0 \tag{B.38}$$

$$2G_{03} = 0 \tag{B.39}$$

$$2G_{11} = (\omega^2 - k^2) - (\omega - k)^2 c^5_{00} = 0 \tag{B.40}$$

$$2G_{12} = 0 \tag{B.41}$$

$$2G_{13} = c^5_{12} \omega(k - \omega) = 0 \tag{B.42}$$

$$2G_{22} = -(\omega^2 - k^2) - (\omega - k)^2 c^5_{00} = 0 \tag{B.43}$$

$$2G_{23} = c^5_{23} \omega(k - \omega) = 0 \tag{B.44}$$

$$2G_{33} = 0. \tag{B.45}$$

In general, c^5_{ij} do not vanish and Eqs. (B.36) – (B.45) imply that $\omega = k$. That is, $h^{(5)}_{\mu\nu}$ propagates along the z direction at the speed of light, as expected.

If instead we substitute the $\times$ mode (Eq. (B.34)) into Eq. (B.35) and again set the 4-momentum $k^\mu = (\omega, 0, 0, k)$, we obtain the same equations, except that now

$$2G_{11} = -(\omega - k)^2 c^6_{00} = 0 \tag{B.46}$$

$$2G_{12} = (\omega^2 - k^2) = 0 \tag{B.47}$$

$$2G_{22} = -(\omega - k)^2 c^6_{00} = 0, \tag{B.48}$$

and $c^5_{ij} \rightarrow c^6_{ij}$ in (B.36) – (B.45). Clearly, the solution is still $\omega = k$. Thus, $h^6_{\mu\nu}$ also propagates along z at the speed of light.

Finally, the fact that these modes are transverse can be shown by direct computation:

$$\begin{aligned} k^\mu h_{\mu\nu}^{(general)} &= k^\mu (h_{\mu\nu}^{TT} + p_{\mu\nu}^{gauge} e^{ik_\alpha x^\alpha}) \\ &= \frac{1}{2} k^\mu (k_\mu \xi_\nu + k_\nu \xi_\mu) \\ &= \frac{1}{2} (k^2 \xi_\nu + k_\nu k^\mu \xi_\mu) \\ &= 0, \end{aligned} \tag{B.49}$$

since the graviton obeys $k^2 = 0$ and the gauge modes are traceless (i.e., $k_\mu \xi^\mu = 0$).

In summary, we have shown that there are two special linear combinations (M_5 and M_6) of the six Goldstone modes that have a polarization tensor identical to that of a graviton in general relativity; obey the normal dispersion relation $k^2 = 0$; and are transverse to the momentum k^μ.

B.2.2 The remaining four linear combinations

In this section, we demonstrate that the remaining four linear combinations do not propagate upon imposing the equations of motion. The four remaining modes (M_1 to M_4) are given respectively by

$$h_{\mu\nu}^{(1)} = \begin{pmatrix} c_{00}^1 & c_{01}^1 & c_{02}^1 & c_{03}^1 \\ c_{01}^1 & 0 & 0 & -c_{01}^1 \\ c_{02}^1 & 0 & 0 & -c_{02}^1 \\ c_{03}^1 & -c_{01}^1 & -c_{02}^1 & c_{00}^1 \end{pmatrix} M_1 \tag{B.50}$$

$$h_{\mu\nu}^{(2)} = \begin{pmatrix} c_{00}^2 & c_{01}^2 & c_{02}^2 & -c_{00}^2 \\ c_{01}^2 & 0 & 0 & c_{13}^2 \\ c_{02}^2 & 0 & 0 & -c_{02}^2 \\ -c_{00}^2 & c_{13}^2 & -c_{02}^2 & c_{00}^2 \end{pmatrix} M_2 \tag{B.51}$$

$$h_{\mu\nu}^{(3)} = \begin{pmatrix} c_{00}^3 & c_{01}^3 & c_{02}^3 & -c_{00}^3 \\ c_{01}^3 & 0 & 0 & -c_{01}^3 \\ c_{02}^3 & 0 & 0 & c_{23}^3 \\ -c_{00}^3 & -c_{01}^3 & c_{23}^3 & c_{00}^3 \end{pmatrix} M_3 \tag{B.52}$$

$$h_{\mu\nu}^{(4)} = \begin{pmatrix} c_{00}^4 & c_{01}^4 & c_{02}^4 & c_{03}^4 \\ c_{01}^4 & 0 & 0 & -c_{13}^4 \\ c_{02}^4 & 0 & 1 & c_{23}^4 \\ c_{03}^4 & c_{13}^4 & c_{23}^4 & c_{00}^4 - c_{33}^4 \end{pmatrix} M_4, \tag{B.53}$$

where $c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4$ are constants determined by Eqs. (B.12) – (B.21).

Again, using the linearized Einstein's equations, the mode M_1 (B.50) has the following equations of motion:

$$2G_{00} = -(c^1_{00}\omega^2 + 2kc^1_{03}\omega + c^1_{00}k^2) = 0 \tag{B.54}$$

$$2G_{01} = c^1_{01}k(\omega - k) = 0 \tag{B.55}$$

$$2G_{02} = c^2_{02}k(\omega - k) = 0 \tag{B.56}$$

$$2G_{03} = 0 \tag{B.57}$$

$$2G_{11} = c^1_{00}\omega^2 + 2kc^1_{03}\omega + c^1_{00}k^2 = 0 \tag{B.58}$$

$$2G_{12} = 0 \tag{B.59}$$

$$2G_{13} = c^1_{01}\omega(k - \omega) = 0 \tag{B.60}$$

$$2G_{22} = c^1_{00}\omega^2 + 2kc^1_{03}\omega + c^1_{00}k^2 = 0 \tag{B.61}$$

$$2G_{23} = c^1_{02}\omega(k - \omega) = 0 \tag{B.62}$$

$$2G_{33} = 0. \tag{B.63}$$

In general, the constants c^1_{ij} do not vanish and the only way to satisfy all these conditions is to set $\omega = k = 0$. This mode therefore does not propagate. It is straightforward to repeat the analysis for the other three modes, and it can be shown that their equations of motion lead to $\omega = k = 0$.

This analysis is thus in agreement with that by Kostelecky and Potting [65]: in this Lorentz-violating theory, only two linear combinations of the six Goldstone modes propagate and obey the dispersion relation $k_\mu k^\mu = 0$ and the transverse condition $k_\mu \epsilon^{\mu\nu} = 0$. Also, because of the form (5.13) of the Goldstone modes, the cardinal gauge conditions are all satisfied. The four remaining linear combinations do not propagate. Thus, at lowest order, the theory contains two propagating modes with properties identical to the graviton in linearized general relativity.

B.3 Proof of the Necessity of Breaking All Six Generators to Get Goldstone Gravitons

We now discuss a systematic way of determining the number of Goldstone modes that result for a given vev. We construct a 10×6 matrix $\mathbf{N}$ where each row corresponds to one of the ten components of $h_{\mu\nu}$, and each column corresponds to one of the six generators of the Lorentz group (θ_i and β_i, $i \in 1, 2, 3$).

$$\mathbf{N} = \begin{pmatrix} -2H_{01} & -2H_{02} & -2H_{03} & 0 & 0 & 0 \\ -(H_{00}+h_{11}) & -H_{12} & -H_{13} & 0 & H_{03} & -H_{02} \\ -H_{12} & -(H_{00}+H_{22}) & -H_{23} & -H_{03} & 0 & H_{01} \\ -H_{13} & -H_{23} & -(H_{00}+H_{33}) & H_{02} & -H_{01} & 0 \\ -2H_{01} & 0 & 0 & 0 & 2H_{13} & -2H_{12} \\ -H_{02} & -H_{01} & 0 & -H_{13} & H_{23} & H_{11}-H_{22} \\ -H_{03} & 0 & -2H_{01} & H_{12} & H_{33}-H_{11} & -H_{23} \\ 0 & -2H_{02} & 0 & -2H_{23} & 0 & 2H_{12} \\ 0 & -H_{03} & -H_{02} & H_{22}-H_{33} & -H_{12} & H_{13} \\ 0 & 0 & -2H_{03} & 2H_{23} & -2H_{13} & 0 \end{pmatrix}. \qquad \text{(B.64)}$$

The entries $\mathbf{N}$ are the coefficients of the θ_i and β_i in the ten components of $h_{\mu\nu}$. The rank of this matrix is the number of Goldstone modes. The possible ranks of this matrix are three, five, and six. This is different in the vector case, in which the rank of the corresponding 4×6 matrix is always three, consistent with the fact that there are always three Goldstone modes.

We found in Appendix B that a necessary and sufficient condition for the theory to contain two linear combinations of the Goldstone modes is

$$\det(\mathbf{A}) \neq 0, \qquad \text{(B.65)}$$

which is equivalent to $\text{Rank}(\mathbf{A}) = 6$. Since the rows of $\mathbf{A}$ are just linear combinations of those of $\mathbf{N}$, the rank of the former is necessarily less than or equal to the latter. Thus, for vevs that do not break all six generators, the number of Goldstone modes < 6, implying that

$$\text{Rank}(\mathbf{N}) < 6 \tag{B.66}$$

$$\Rightarrow \quad \text{Rank}(\mathbf{A}) < 6 \tag{B.67}$$

$$\Leftrightarrow \quad \det(\mathbf{A}) = 0, \tag{B.68}$$

violating the condition (B.65). This implies the lack of two linear combinations of the Goldstone modes that behave like the graviton in general relativity. However, as was discussed, it is still possible that the theory contains massless excitations that behave like the graviton; they are just not Goldstone in origin.

Bibliography

[1] S. M. Carroll and H. Tam, Phys. Rev. D **78**, 044047 (2008) [arXiv:hep-ph/0802.0521].

[2] S. M. Carroll, T. R. Dulaney, M. I. Gresham, and H. Tam, Phys. Rev. D **79**, 065011 (2009) [arXiv:hep-th/0812.1049].

[3] S. M. Carroll, T. R. Dulaney, M. I. Gresham, and H. Tam, Phys. Rev. D **79**, 065012 (2009) [arXiv:hep-th/0812.1050].

[4] S. M. Carroll, H. Tam and I. K. Wehus, Phys. Rev. D **80**, 025020 (2009) [arXiv:hep-th/0904.4680].

[5] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. B **429**, 263 (1998) [arXiv:hep-ph/9803315].

[6] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. B **436**, 257 (1998) [arXiv:hep-ph/9804398].

[7] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [arXiv:hep-ph/9905221].

[8] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [arXiv:hep-th/9906064].

[9] T. G. Rizzo, JHEP **0509**, 036 (2005) [arXiv:hep-ph/0506056].

[10] S. M. Carroll, G. B. Field, and R. Jackiw, Phys. Rev. D **41**, 1231 (1990).

[11] S. R. Coleman and S. L. Glashow, Phys. Lett. B **405**, 249 (1997) [arXiv:hep-ph/9703240].

[12] D. Colladay and V. A. Kostelecky, Phys. Rev. D **58**, 116002 (1998) [arXiv:hep-ph/9809521].

[13] J. Soda and S. Kanno, [arXiv:gr-qc/0612069].

[14] V. A. Kostelecky and S. Samuel, Phys. Rev. D **40**, 1886 (1989).

[15] T. Jacobson and D. Mattingly, Phys. Rev. D **64**, 024028 (2001) [arXiv:gr-qc/0007031].

[16] T. Jacobson and D. Mattingly, Phys. Rev. D **70**, 024003 (2004) [arXiv:gr-qc/0402005].

[17] S. M. Carroll and E. A. Lim, Phys. Rev. D **70**, 123525 (2004) [arXiv:hep-th/0407149].

[18] E. A. Lim, Phys. Rev. D **71**, 063504 (2005) [arXiv:astro-ph/0407437].

[19] C. Eling and T. Jacobson, Phys. Rev. D **69**, 064005 (2004) [arXiv:gr-qc/0310044].

[20] V. A. Kostelecky, Phys. Rev. D **69**, 105009 (2004) [arXiv:hep-th/0312310].

[21] R. Bluhm, S. H. Fung, and V. A. Kostelecky, [arXiv:hep-th/0712.4119].

[22] M. V. Libanov and V. A. Rubakov, JHEP **0508**, 001 (2005) [arXiv:hep-th/0505231].

[23] H. C. Cheng, M. A. Luty, S. Mukohyama, and J. Thaler, JHEP **0605**, 076 (2006) [arXiv:hep-th/0603010].

[24] L. Ackerman, S. M. Carroll, and M. B. Wise, Phys. Rev. D **75**, 083502 (2007) [arXiv:astro-ph/0701357].

[25] T. R. Dulaney, M. I. Gresham, and M. B. Wise, [arXiv:astro-ph/0801.2950].

[26] C. M. Will and K. J. Nordtvedt, Astrophys. J. **177**, 757 (1972).

[27] M. Gasperini, Class. Quant. Grav. **4**, 485 (1987).

[28] C. Eling, T. Jacobson, and D. Mattingly, [arXiv:gr-qc/0410001].

[29] J. B. Jimenez and A. L. Maroto, JCAP **0902**, 025 (2009) [arXiv:astro-ph/0811.0784].

[30] V. A. Kostelecky and R. Lehnert, Phys. Rev. D **63**, 065008 (2001) [arXiv:hep-th/0012060].

[31] J. W. Elliott, G. D. Moore, and H. Stoica, JHEP **0508**, 066 (2005) [arXiv:hep-ph/0505211].

[32] D. Mattingly, Living Rev. Rel. **8**, 5 (2005) [arXiv:gr-qc/0502097].

[33] C. M. Will, Living Rev. Rel. **9**, 3 (2005) [arXiv:gr-qc/0510072].

[34] T. Jacobson, PoS **QG-PH**, 020 (2007) [arXiv:gr-qc/0801.1547].

[35] N. Arkani-Hamed, H. C. Cheng, M. A. Luty, and S. Mukohyama, JHEP **0405**, 074 (2004) [arXiv:hep-th/0312099].

[36] N. Arkani-Hamed, H. C. Cheng, M. A. Luty, S. Mukohyama, and T. Wiseman, JHEP **0701**, 036 (2007) [arXiv:hep-ph/0507120].

[37] S. M. Carroll, M. Hoffman, and M. Trodden, Phys. Rev. D **68**, 023509 (2003) [arXiv:astro-ph/0301273].

[38] S. L. Dubovsky and S. M. Sibiryakov, Phys. Lett. B **638**, 509 (2006) [arXiv:hep-th/0603158].

[39] C. Eling, B. Z. Foster, T. Jacobson, and A. C. Wall, Phys. Rev. D **75**, 101502 (2007) [arXiv:hep-th/0702124].

[40] V. A. Kostelecky, [arXiv:hep-ph/0104227].

[41] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, and R. Rattazzi, JHEP **0610**, 014 (2006) [arXiv:hep-th/0602178].

[42] R. Bluhm, N. L. Gagne, R. Potting, and A. Vrublevskis, Phys. Rev. D **77**, 125007 (2008) [Erratum: Phys. Rev. D **79**, 029902 (2009)] [arXiv:hep-th/0802.4071].

[43] J. L. Chkareuli, C. D. Froggatt, and H. B. Nielsen, Nucl. Phys. B **821**, 65 (2009) [arXiv:hep-th/0610186].

[44] B. M. Gripaios, JHEP **0410**, 069 (2004) [arXiv:hep-th/0408127].

[45] M. A. Clayton, [arXiv:gr-qc/0104103].

[46] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University (1992), http://www.loc.gov/catdir/description/prin031/92011585.html.

[47] Y. Nambu, Prog. Theor. Phys. Suppl. **190** (1968).

[48] B. Himmetoglu, C. R. Contaldi, and M. Peloso, Phys. Rev. Lett. **102**, 111301 (2009) [arXiv:astro-ph/0809.2779].

[49] B. Himmetoglu, C. R. Contaldi, and M. Peloso, Phys. Rev. D **79**, 063517 (2009) [arXiv:astro-ph/0812.1231].

[50] M. D. Seifert, Phys. Rev. D **76**, 064002 (2007) [arXiv:gr-qc/0703060].

[51] T. R. Dulaney and M. I. Gresham, [arXiv:gr-qc/0805.1078].

[52] T. S. Koivisto and D. F. Mota, JCAP **0808**, 021 (2008) [arXiv:astro-ph/0805.4229].

[53] S. Kanno and J. Soda, Phys. Rev. D **74**, 063505 (2006) [arXiv:hep-th/0604192].

[54] B. Li, D. Fonseca Mota and J. D. Barrow, Phys. Rev. D **77**, 024032 (2008) [arXiv:astro-ph/0709.4581].

[55] Y. Nambu, Phys. Rev. Lett. **4** 380 (1960).

[56] J. Goldstone, Nuovo Cim. **19**, 154 (1961).

[57] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[58] J. D. Bjorken, Annals Phys. **24**, 174 (1963).

[59] J. Bjorken, [arXiv:hep-th/0111196].

[60] P. R. Phillips, Phys. Rev. **146**, 966 (1966).

[61] H. C. Ohanian, Phys. Rev. **184**, 1305 (1969).

[62] P. Kraus and E. T. Tomboulis, Phys. Rev. D **66**, 045015 (2002) [arXiv:hep-th/0203221].

[63] Z. Berezhiani and O. V. Kancheli, [arXiv:hep-th/0808.3181].

[64] B. McElrath, [arXiv:gr-qc/0812.2696].

[65] V. A. Kostelecky and R. Potting, Phys. Rev. D **79**, 065018 (2009) [arXiv:gr-qc/0901.0662].

[66] R. Bluhm and V. A. Kostelecky, Phys. Rev. D **71**, 065008 (2005) [arXiv:hep-th/0412320].

[67] R. Bluhm, Int. J. Mod. Phys. D **16**, 2357 (2008) [arXiv:hep-th/0607127].

[68] G. D. Moore and A. E. Nelson, JHEP **0109**, 023 (2001) [arXiv:hep-ph/0106220].

[69] A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[70] A. D. Linde, Phys. Lett. B **108**, 389 (1982).

[71] A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[72] R. Penrose, in S. W. Hawking and W. Israel. *General Relativity: An Einstein Centenary Survey,* Cambridge University Press (1979), p. 581638.

[73] G. W. Gibbons, S. W. Hawking, and J. M. Stewart, Nucl. Phys. B **281**, 736 (1987).

[74] S. W. Hawking and D. N. Page, Nucl. Phys. B **298**, 789 (1988).

[75] D. H. Coule, Class. Quant. Grav. **12**, 455 (1995) [arXiv:gr-qc/9408026].

[76] S. Hollands and R. M. Wald, Gen. Rel. Grav. **34**, 2043 (2002) [arXiv:gr-qc/0205058].

[77] L. Kofman, A. Linde, and V. F. Mukhanov, JHEP **0210**, 057 (2002) [arXiv:hep-th/0206088].

[78] S. Hollands and R. M. Wald, [arXiv:hep-th/0210001].

[79] S. M. Carroll and J. Chen, [arXiv:hep-th/0410270].

[80] G. W. Gibbons and N. Turok, [arXiv:hep-th/0609095].

[81] A. Linde, [arXiv:hep-th/0705.0164].

[82] T. Vachaspati and M. Trodden, Phys. Rev. D **61**, 023502 (1999) [arXiv:gr-qc/9811037].

[83] S. D. Mathur, J. Phys. Conf. Ser. **140**, 012009 (2008) [arXiv:hep-th/0803.3727].

[84] C. A. Egan and C. H. Lineweaver, Astrophys. J. **710**, 1825 (2010) [arXiv:astro-ph/0909.3983].

[85] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rept. **215**, 203 (1992).

[86] V. F. Mukhanov, *Physical Foundations of Cosmology*, Cambridge University Press (2005).

[87] S. Weinberg, *Cosmology*, Oxford University Press (2008).

[88] M. Trodden and S. M. Carroll, [arXiv:astro-ph/0401547].