Chapter 1

Introduction

This thesis is comprised primarily of work from three independent papers, written in collaboration with Sean Carroll, Tim Dulaney, and Heywood Tam. The original motivation for the projects undertaken came from revisiting the standard assumption of spatial isotropy during inflation. Each project relates to the spontaneous breaking of Lorentz symmetry—in early Universe cosmology or in the context of effective field theory, in general. Here I motivate and introduce the three projects, presented in chapters 2, 3, and 4. At the end of this chapter I provide some more technical background that helps to contextualize the subsequent chapters.

1.1 The Big Picture

I like the way that physics tries to answer big questions. For example particle physicists answer "What are we made of?" by searching for the elementary constituents of matter and for mathematical structure within which these constituents and their interactions can be understood. Cosmologists approach "Where are we?" and "How did we come to be?" by using modern physics theory, astrophysical observations, logic, and intuition to construct a plausible and consistent picture of the Universe and its evolution.[1] The work in this thesis grew from thinking about what might have occurred very early on in our Universe's history.

[1] Recently, even the situation of our Universe within a hypothetical larger set of universes has been a topic of research.

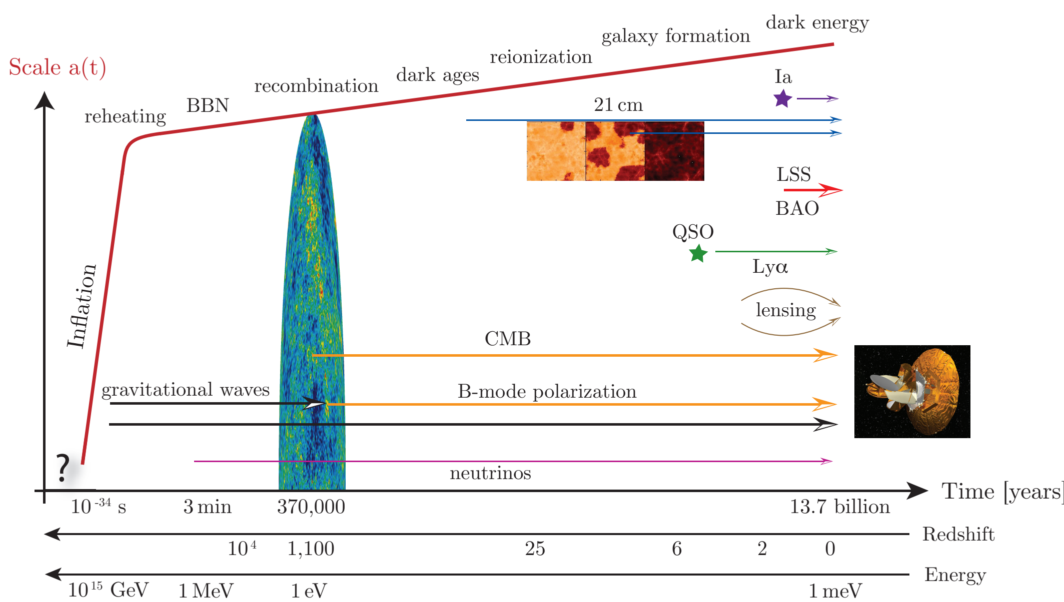

Figure 1.1: Image taken with permission from [4]. History of the Universe, including key events in the history of our Universe and some of the observable objects/features that have or could in principle provide(d) us with information about its evolution. *Acronyms*: BBN (Big Bang Nucleosynthesis), LSS (Large-Scale Structure), BAO (Baryon Acoustic Oscillations), QSO (Quasi-Stellar Objects; Quasars), Lyα (Lyman-alpha), CMB (Cosmic Microwave Background), Ia (Type Ia supernovae), 21 cm (hydrogen 21 cm-transition).

We already have a remarkably consistent and detailed picture and history of our Universe. A cartoon of this history is provided in Fig 1.1. In particular, it has been established that the Universe has been expanding (at various rates, indicated by the slope of the solid red line in Fig 1.1) for the entire traceable history of our Universe—about 14 billion years. In particular, we are quite confident that the Universe began[2] in a very hot, dense state. The name for this beginning is "the Big Bang". We don't really know how the Big Bang occurred, or what happened in the primordial era just after the Big Bang.

Returning to the cartoon history of our Universe, the vertical axis in Fig. 1.1 is the scale factor, which characterizes the expansion of the Universe. The horizontal axis is time—or equivalently decreasing temperature[3] or decreasing redshift of light in the Universe; the Universe cools and the wavelength of light gets stretched (*i.e.*, light is redshifted[4]) as it expands. The labeled arrows indicate signals from the past that we might observe at earth today. For example, observations of light from distant type Ia supernovae have provided an important measure of the local expansion rate of the Universe.

For the purposes of motivating the work in this thesis, the important thing to notice in Fig. 1.1 is that one of the longest arrows comes from the Cosmic Microwave Background (CMB). CMB radiation is light that last scattered off of a plasma of photons, electrons, and protons when the Universe was at a temperature of about 1 eV; at this temperature almost all free electrons and protons combined into neutral hydrogen. This point in history is known as *recombination* (electrons and protons recombined into neutral hydrogen), and the place from which the CMB photons reach-

[2] "Began" might be a controversial word to use here. It could be that the Universe collapsed into a very hot, dense, state and then began expanding again, or the hot dense state may have been born from a parent universe.

[3] Temperature (T) is related to energy (E) by

$$T = E/k_B, \qquad k_B = 8.62 \times 10^{-5}\,\frac{\text{eV}}{\text{K}}, \tag{1.1}$$

where k_B is Boltzmann's constant, K is Kelvin and eV is electron-Volts.

[4] In the visual spectrum, red light has longer wavelength than green light, which has longer wavelength than blue light. Thus the name "redshifting" for light with longer wavelength and "blueshifting" for light with shorter wavelength.

ing us today came is known as the *surface of last scattering*.[5,6] Before recombination the Universe was effectively opaque—filled with a plasma of electrons, protons, and photons, all scattering off of each other frequently. When the collision rates slowed down enough[7] so that electrons and protons could combine into neutral hydrogen ("recombination"), the Universe became effectively transparent; light could free-stream without colliding much with other particles. So the farthest back we can effectively see is the distance that light has traveled from the time of recombination.

Arrows that reach farther back than the CMB arrow in Fig. 1.1 are from primordial gravitational waves or neutrinos; gravitons and neutrinos decoupled from the primordial plasma earlier than photons and so have been free-streaming for longer. We think that gravitational wave and neutrino backgrounds analogous to the CMB exist and we hope to eventually observe them (or "observe" their nonexistence), but we haven't yet because our instruments are not sensitive enough. Thus the CMB is currently our best window to the very early Universe.[8] The Wilkinson Microwave Anisotropy Probe (WMAP) is pictured at the end of the CMB arrow in Fig. 1.1. That is because WMAP has served as our eyes looking on the CMB window. As will be discussed in subsequent sections, WMAP has taken very high resolution pictures of the CMB that have moved us into an age of precision (early) cosmology. The Planck satellite is currently taking even higher resolution pictures of the CMB and should advance early Universe cosmology even further.

The spectrum of CMB photons is thermal, and very nearly uniform across the

[5]Though the CMB radiation last scattered throughout 3-dimensional space at roughly the same time, from our vantage point on earth we see a bubble of radiation—a 2-dimensional surface. The photons at that surface formed another surface back when they last scattered long ago.

[6]Think about the sky on a cloudy day. The light reaching our eyes from the sky last scattered off of water molecules forming the clouds in the atmosphere. We can't see past the clouds because even though much of the light we're seeing made its way from the sun through a jagged path within the clouds and out the other side, the light's characteristics changed substantially during all of the collisions it had within the cloud, and what we see is light with those characteristics—not the characteristics of direct sunlight. The CMB light is analogous to light that last scattered off of the clouds.

[7]Due to expansion the plasma became less dense, hence less collisions per time.

[8]Figuring out what happened in the very early Universe is not only interesting as an answer to the "Where did we come from?" question, but physics at very high energy scales played an important role at this time, so we could also learn more about high energy physics by looking this far back in cosmic history.

entire sky. That CMB radiation is thermal is a profound fact; it seems to imply that the region from which CMB photons came at recombination must have been in thermal equilibrium; in particular the photons must have been in causal contact.[9] The thermal spectrum and near uniformity of the CMB, along with its nearly scale-invariant spectrum[10] of deviations from uniformity have led many to believe that the Universe must have undergone a period of rapid expansion during the first moments after the Big Bang. Without such a period of rapid expansion or some other nonstandard sequence of events, the uniformity of the CMB appears to be an extraordinary accident. If we trace back the evolution of the Universe assuming that just the kind of matter we observe today to dominate the energy density of the Universe determined the dynamics of our Universe's expansion, then the expansion of the Universe would have always been decelerating and CMB photons separated by about a degree on the sky or more couldn't have been in causal contact before or after the time of last scattering. So photons across the entire sky could not have reached thermal equilibrium, which means the uniform, thermal spectrum across the entire sky would be an extremely unlikely coincidence. This is known as the Horizon Problem. A period of accelerating expansion of the Universe could have allowed the CMB photons on our sky to reach thermal equilibrium before the time of last scattering and, in this sense, solves the Horizon Problem [5]. Such a period of accelerating expansion in the early Universe is known as *inflation*.

I, like many (including the authors of Fig. 1.1), find inflation to be compelling. As I'll touch on later in this introduction, not only does inflation solve the Horizon Problem, but it also provides an explanation of the small, nearly scale-invariant energy density anisotropies that seeded structure formation in the Universe and can account for the pattern of small temperature variation across the CMB sky. But until we develop technology good enough to measure, for example, primordial gravitational waves or neutrinos, the best evidence we have that inflation did or did not occur is our measurement of the CMB. It's *indirect* evidence.

[9] Photons come into equilibrium by interacting with each other. If they were never in causal contact, then they couldn't have interacted and thus couldn't have reached equilibrium.

[10] I'll discuss what's meant by "scale-invariant spectrum" in §1.9.2.

Cosmologists must often make do with indirect evidence due to the very nature of cosmology. Unlike physicists who are trying to uncover the laws of nature in our neighborhood, cosmologists cannot design and repeat experiments that in effect recreate events that we expect to occur. For example, particle physicists have the luxury of building big machines that smash particles together and then measuring what comes out in order to test whether Higgs bosons exist. Cosmologists do not have the luxury of recreating the Big Bang and then measuring what happens subsequently. On the other hand, in some sense both particle physicists and cosmologists run into the same basic problem; it's technologically impossible (whether in principle or just given practical constraints) to test certain theories directly. We must then get creative and clever; we must come up with theories that subsume experimentally verified theories and find ways to test such new theories indirectly, given our technological capabilities. Speculation is inevitably part of the creative process by which advancements in such cases are made.

Indeed, a problem that remains even if we're right that inflation did occur is *how* it occurred. By what mechanism did the Universe inflate?[11] It's productive to speculate about the multitude of theoretical mechanisms of inflation and then try to figure out ways to find astrophysical signatures (*e.g.*, signatures on the CMB) that support or rule out such mechanisms. It was out of this kind of creative process—speculating about the primordial Universe and how we might see its features through the window of the CMB—that this thesis emerged.

1.2 How This Thesis Emerged

The motivation for the work in this thesis came from the possibility that rotational symmetry (*i.e.*, isotropy) was broken during inflation. There are several reasons why this possibility had not been seriously considered until recently:

- We observe isotropy to be a very nearly exact local symmetry today. (Local

[11] Remember: I mentioned above that ordinary matter (the kind of matter that we know is fueling the expansion of the Universe today) cannot give rise to inflation.

isotropy leads to conservation of angular momentum, for example, and allows for the classification of particles by their spin.)

- The CMB, at least at a glance, appears to be very nearly statistically isotropic. (More on this in §1.4.)

- Under straightforward assumptions about the nature of the fields involved during an inflationary epoch, statistical isotropy of the CMB and of the Universe on large scales is a consequence of inflation. (More on this in §1.6.)

But we should keep an open mind. Slightly anisotropic inflation is an interesting possibility. A generic signature of slight anisotropy during inflation on the CMB was postulated and studied in [6]. The work in this thesis emerged after thinking about particular models of inflation that could yield an anisotropic inflationary scenario leading to the generic signature put forth in [6].

1.3 Synopsis

Chapters 2 and 3 address stability issues in a popular class of models that give rise to the breaking of Lorentz Symmetry: *æther* models. Here, "æther" refers to a dynamical fixed-norm vector field. Spatial rotations being a subgroup of the Lorentz group, in particular æther models can give rise to the breaking of rotational invariance. The project out of which chapter 2 emerged transformed into one very different from the project we originally set out to do. From a study of the evolution of æther fields in an expanding Universe, we were eventually led to study more generally the effective field theory of spontaneously broken Lorentz symmetry in flat space. In the course of the original project, we found obstructions to the smooth evolution of initial data, and later realized that this was a symptom of much more general problems in these theories. Chapter 3 is a study of the one æther theory that we found to be well behaved. Chapter 2 is also interesting from a perspective independent of cosmology; it brings together three of the most powerful concepts in modern theoretical physics: gauge symmetries, spontaneous symmetry breaking, and effective field theory. In a

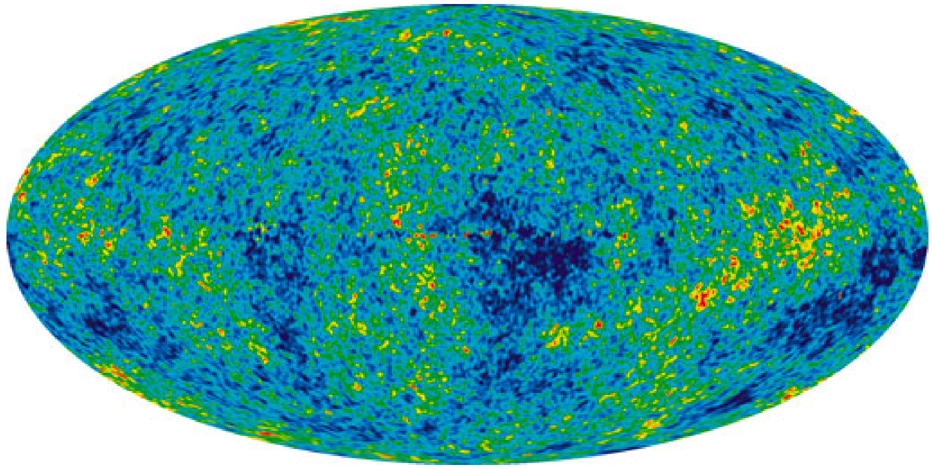

Figure 1.2: WMAP 7-year full-sky Mollweide projection map of the cosmic microwave background. WMAP measures temperature differences across the sky. The colors represent temperatures according to a linear scale, ranging from -200 μK to $+200$ μK. The root mean square variation is on the order of tens of μK. From independent measurements, we know that the average temperature of the CMB is 2.725 K. That means the temperature across the entire sky varies, roughly speaking, by only about one part in 100,000. Credit: WMAP Science Team.

few words, usually we talk about *internal* symmetries being spontaneously broken; but what happens in theories in which *space-time* symmetry is spontaneously broken?

Chapter 4 is a study of a model that can give rise to anisotropy during inflation through a mechanism very different than that of æther theories. The model, first set forth in the context of anisotropic inflation in [7], is built on standard single field inflation, but includes a nonstandard coupling of the inflaton field to a $U(1)$ gauge field. We study the stability of the model and also (more importantly) the spectra of cosmological perturbations in the theory.

In the remainder of this chapter I shall review more carefully some more technical background needed to contextualize chapters 2, 3, and 4—especially chapter 4.

1.4 CMB Temperature Correlations

As mentioned in §1.1, the Cosmic Microwave Background (CMB) is light that has been more-or-less free-streaming toward us since the time at which the Universe had cooled (through expansion) to a temperature at which hydrogen ions and electrons

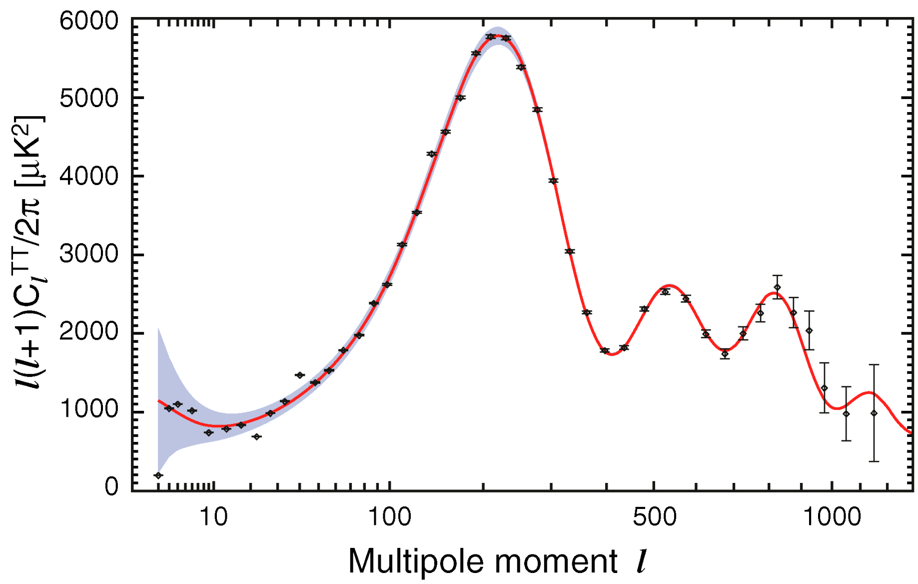

Figure 1.3: Spectrum of CMB multipole coefficients from 7-year WMAP data. (The C_ls plotted here are multiplied by $\bar{T}^2$ as compared to the definition in (1.6), *i.e.*, these are the temperature difference multipole moments, not the *fractional* temperature difference multipole moments.) Credit: WMAP Science Team.

recombined to form neutral hydrogen—about 370,000 years (about 10^{-5} times the age of the Universe) after the Big Bang. The CMB radiation has a very nearly uniform temperature across the sky, but there are small variations. See Fig. 1.2. A systematic way to look for patterns of those variations is to compute correlations between various points on the map of temperature differences that we've measured across the entire sky.[12]

It's convenient to decompose the fractional temperature difference as a function of position on the sky, $\hat{e}$ (where $\hat{e} \cdot \hat{e} = 1$), into spherical harmonics:

$$\frac{T(\hat{e}) - \bar{T}}{\bar{T}} = \frac{\Delta T(\hat{e})}{\bar{T}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_l^m(\hat{e}), \tag{1.2}$$

where $\bar{T}$ is the average temperature of the CMB, and the spherical harmonic functions

[12] I primarily consulted [8] and [9] while writing the following two sections.

are normalized such that

$$\int d\hat{e}\, Y_{l'}^{m'*}(\hat{e})\, Y_l^m(\hat{e}) = \frac{4\pi}{2l+1}\delta_{ll'}\delta_{mm'}. \tag{1.3}$$

In principle, the temperature difference depends not only on a direction in the sky, but also on the vantage point from which the measurement is made (in our case, the Earth). In other words, the a_{lm}s are implicitly functions of vantage point, $\vec{x}$. On the other hand, based on the Copernican principle we'd expect a given correlation function averaged across directions on the sky to be approximately equal to the same function averaged over different vantage points, but for a fixed direction. An averaging over vantage points is known as a *cosmic mean*. And the difference between a local measurement and the cosmic mean is known as *cosmic variance*. We can make generic predictions from inflation only for cosmic means. Thus, since our measurements are necessarily local, we're always limited by cosmic variance.

We expect the distribution of temperature perturbations to be random, but with a certain distribution. For example, if the distribution is Gaussian, then the pattern of fluctuations on the sky should be completely characterized by the two-point function,

$$\left\langle \frac{\Delta T(\hat{e}_1)}{\bar{T}} \frac{\Delta T(\hat{e}_2)}{\bar{T}} \right\rangle, \tag{1.4}$$

where $\langle \ldots \rangle$ denotes the cosmic mean. If the distribution is non-Gaussian, then higher order correlation functions are needed to completely characterize the pattern of fluctuations.

The covariance of the two-point temperature correlation is defined as follows,

$$\begin{aligned} C_{ll',mm'} &\equiv \langle {a_{l'm'}}^* a_{lm} \rangle \\ &= \left(\frac{2l+1}{4\pi}\right)\left(\frac{2l'+1}{4\pi}\right) \int d\hat{e}d\hat{e}' Y_l^{m*}(\hat{e}) Y_{l'}^{m'}(\hat{e}') \left\langle \frac{\Delta T(\hat{e})}{\bar{T}} \frac{\Delta T(\hat{e}')}{\bar{T}} \right\rangle. \end{aligned} \tag{1.5}$$

In practice, we cannot measure the cosmic mean; the "covariance" we measure doesn't have $\langle \ldots \rangle$. However, for a statistically isotropic and homogeneous Universe,

$$C_{ll',mm'} = \delta_{mm'}\delta_{ll'}C_l \tag{1.6}$$

where the C_l are known as multipole moments. If the temperature variation is governed by a Gaussian distribution, then an observed C_l^{obs} is the average over $2l+1$ independent a_{lm}s, squared, and it can be shown that the cosmic variance for a given C_l^{obs} is

$$\left\langle \left(\frac{C_l^{obs} - C_l}{C_l} \right)^2 \right\rangle = \frac{2}{2l+1}. \tag{1.7}$$

The dipole moment, C_1, for example, has a variance of 67%. This fact, and complications having to do with Earth's motion around the sun make any predictions we might have for the cosmic mean dipole moment practically incomparable with experiment.

The observed multipole moments (normalized by $l(l+1)/2\pi$) as measured by WMAP are plotted in Fig. 1.3. From the detailed shape of the spectrum of multipole moments, cosmologists have been able to extract some of the most precise values for cosmological parameters to date—such as the Hubble rate, the age of the Universe, the curvature of space, the percent energy density in the Universe from dark matter, baryons, and dark energy, etc. One can read how such parameters are extracted in texts such as [9] or [8]. In the next section we'll get a feel for how the spectrum of energy density fluctuations before the surface of last scattering is ultimately related to the CMB spectrum.

1.5 From Primordial Perturbations to CMB Temperature Correlations

Inflation not only explains how our current horizon volume might have been in causal contact in our past, but it can also provide an explanation for the pattern of temperature fluctuations that we observe in the CMB. The key is that energy density fluctuations generated during inflation give rise to temperature fluctuations on the

CMB today because of effects such as, for instance, what's known as the Sachs-Wolfe effect: the relative redshifting of photons that emerge from regions with higher energy density compared to their neighbors. To trace the evolution of photons from the primordial era—the end of inflation for our purposes—given just the energy density perturbation spectrum at that time to the surface of last scattering, and then through time and space until they're reaching us today is a somewhat complicated task, but suffice it to say that given a spectrum of energy density perturbations at the end of inflation a pattern of temperature variations on the CMB sky can be predicted given established nuclear physics, thermodynamics, scattering theory, and general relativity. The remarkable thing is that a particular form of the energy density spectrum is *predicted* at the end of inflation, and this form of the energy density spectrum indeed leads to a CMB temperature power spectrum that matches well with the one we measure!

More specifically, the fractional deviation from the mean temperature of the CMB in a particular direction, $\hat{e}$, on the sky is given by

$$\frac{T(\hat{e}) - \bar{T}}{\bar{T}} = \frac{\Delta T(\hat{e})}{\bar{T}} = \int d\vec{k} \sum_{l} \left(\frac{2l+1}{4\pi}\right) (-i)^l \delta_\varepsilon(\vec{k}) P_l(\hat{k} \cdot \hat{e}) \Theta_l(k), \tag{1.8}$$

where $\delta_\varepsilon(\vec{k})$ is the Fourier transform of the primordial energy density perturbation, $(\varepsilon(\vec{x}) - \varepsilon_0)/\varepsilon_0$, P_l is a Legendre polynomial, $k \equiv \sqrt{\vec{k} \cdot \vec{k}}$, and Θ_l is a function assumed to be governed by statistically isotropic physics that characterizes the evolution of photons from the primordial era until today. For example, the part of Θ_l due to the Sachs-Wolfe effect is proportional to the spherical Bessel function, $j_l(kr_L)$, where r_L is the radial coordinate of the surface of last scattering. It turns out that for small l the Sachs-Wolfe effect is the dominant effect. For larger l, the important parts of Θ_l are more complicated and, for example, account for the dynamics of the photon-nucleon-electron plasma before recombination.

The covariance given this form of $\Delta T/\bar{T}$ can be seen to be

$$C_{ll',mm'} = \frac{(2l'+1)(2l+1)}{(4\pi)^2}\int d\vec{k}d\vec{k}'\sum_{l_1}\sum_{l_2}\frac{(2l_1+1)(2l_2+1)}{(4\pi)^2}(-i)^{l_1+l_2}\Theta_{l_1}(k)\Theta_{l_2}(k')$$
$$\times\langle\delta_\varepsilon(\vec{k})\delta_\varepsilon(\vec{k}')\rangle\int d\hat{e}d\hat{e}'Y_l^{m*}(\hat{e})Y_{l'}^{m'}(\hat{e}')P_{l_1}(\hat{k}\cdot\hat{e})P_{l_2}(\hat{k}'\cdot\hat{e}')$$
$$= \int d\vec{k}d\vec{k}'(-i)^{l+l'}\Theta_l(k)\Theta_{l'}(k')\langle\delta_\varepsilon(\vec{k})\delta_\varepsilon(\vec{k}')\rangle Y_l^{m*}(\hat{k})Y_{l'}^{m'}(\hat{k}'), \quad (1.9)$$

where we've used the identities,

$$P_l(\hat{e}_1\cdot\hat{e}_2) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_l^m(\hat{e}_1)Y_l^{m*}(\hat{e}_2), \quad (1.10)$$

and (1.3) in the last line. Again, here $\langle\ldots\rangle$ denotes the cosmic mean. In concert with the ergodic theorem,[13] inflation gives us a prediction for $\langle\delta_\varepsilon(\vec{k})\delta_\varepsilon(\vec{k}')\rangle$. In inflation, we calculate $\langle\delta_\varepsilon(\vec{x})\delta_\varepsilon(\vec{y})\rangle$ interpreting $\langle\ldots\rangle$ as a quantum average—an average over histories. Then the ergodic theorem says that averaging over histories should give the same result as a cosmic average—an average over vantage points.

Now if $\langle\delta_\varepsilon(\vec{x})\delta_\varepsilon(\vec{y})\rangle$ is translationally invariant it must depend only on $\vec{x}-\vec{y}$. In that case

$$\langle\delta_\varepsilon(\vec{k})\delta_\varepsilon(\vec{k}')\rangle = \int d\vec{x}\int d\vec{y}f(\vec{x}-\vec{y})e^{i(\vec{x}\cdot\vec{k}+\vec{y}\cdot\vec{k}')}$$
$$= \int d\vec{x}_+e^{i\vec{x}_+\cdot(\vec{k}+\vec{k}')}\int d\vec{x}_-f(\vec{x}_-)e^{i\vec{x}_-\cdot(\vec{k}-\vec{k}')}$$
$$= (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')\int d\vec{x}_-f(\vec{x}_-)e^{i\vec{x}_-\cdot(\vec{k}-\vec{k}')}$$
$$\equiv (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}')P(\vec{k}). \quad (1.11)$$

Here we've defined the power spectrum, $P(\vec{k})$. Plugging (1.11) back into (1.9) we find

$$C_{ll',mm'} = \int d\vec{k}(-i)^{l-l'}\Theta_l(k)\Theta_{l'}(k)(2\pi)^3P(\vec{k})Y_l^{m*}(\hat{k})Y_{l'}^{m'}(\hat{k}), \quad (1.12)$$

[13]See, *e.g.*, Appendix D in [9].

where we've used the fact that $Y_l^m(-\hat{e}) = (-1)^l Y_l^m(\hat{e})$. If the power spectrum is rotationally invariant, then $P(\vec{k}) = P(k)$ and we recover (1.6):

$$\begin{aligned} C_{ll',mm'} &= \int dk (-i)^{l-l'} \Theta_l(k) \Theta_{l'}(k) (2\pi)^3 P(k) \int d\hat{k} Y_l^{m*}(\hat{k}) Y_{l'}^{m'}(\hat{k}) \\ &= \delta_{ll'} \delta_{mm'} \left(\frac{4\pi}{2l+1} \int dk \left(\Theta_l(k)\right)^2 (2\pi)^3 P(k) \right). \end{aligned} \tag{1.13}$$

If the power spectrum is not rotationally invariant, then the covariance matrix does not simplify to the above diagonal form. The form of the covariance given a primordial power spectrum that slightly deviates from isotropic due to a preferred direction, $\hat{n}$, during the primordial era,

$$P(\vec{k}) = P_0(k) \left(1 + g(k)(\hat{n} \cdot \hat{k})^2 + \ldots \right) \tag{1.14}$$

was worked out in [6]. The absence of odd powers of $\hat{n} \cdot \hat{k}$ follows from the identity $\langle \delta_\varepsilon(\vec{k}) \delta_\varepsilon(\vec{q}) \rangle = \langle \delta_\varepsilon(\vec{q}) \delta_\varepsilon(\vec{k}) \rangle$.[14] It is also assumed that the effect of the preferred direction must be small (else we'd be able to see a signature by eye on the CMB), thus terms of higher order in the "small" vector $\hat{n}$ should be negligible. In other words, they worked out the effect of a small primordial power quadrupole on the CMB covariance. They found that in addition to diagonal elements ($m = m', l = l'$) there are in general nonzero off-diagonal elements when $l = l' \pm 2$ and/or $m' \pm 2$ and/or $m' \pm 1$, depending on the direction of $\hat{n}$.

As mentioned earlier, the work presented in subsequent chapters of this thesis ultimately grew from thinking about models that could give rise to such a slightly statistically anisotropic spectrum. Indeed, a model of inflation that successfully reproduces a spectrum of the slightly anisotropic form in [6] is presented in chapter 4. Our main work was to calculate $g(k)$ in the model. Chapters 2 and 3 came from thinking about another class of models that could, on the face, lead to a small amount of anisotropy during inflation: æther models. I give a very brief introduction to æther models in §1.7 below.

[14] From the identity and the definition of the power spectrum, $P(\vec{k})$, it follows that $P(\vec{k}) = P(-\vec{k})$.

But before moving on to æther models, I'll provide evidence that it is actually rather difficult to construct a consistent model with enough[15] anisotropic inflation.

1.6 The Cosmic No-Hair Theorem

Under some reasonable conditions, it can be shown that a large class of inflationary scenarios tend to wash out anisotropy. More precisely, Bob Wald proved the following theorem, known now as the *cosmic no-hair theorem* [10]:

If a space-time

- is initially expanding,
- can be foliated by homogeneous hypersurfaces,[16]
- evolves according to Einstein's equations with a positive cosmological constant,

$$G^\mu_\nu = -\Lambda\delta^\mu_\nu + 8\pi G T^\mu_\nu, \quad \Lambda > 0, \tag{1.15}$$

- contains matter with stress-energy, T^μ_ν, that satisfies the dominant and strong energy conditions,

$$T_{\mu\nu}t^\mu t^\nu \geq 0, \qquad T_{\mu\nu}t^\nu T^{\mu\lambda}t_\lambda \leq 0, \qquad \text{and} \quad T_{\mu\nu}t^\mu t^\nu \geq \frac{1}{2}T^\lambda_\lambda t^\sigma t_\sigma, \tag{1.16}$$

  for all timelike t_μ (*i.e.* for all t^μ such that $t_\mu t^\mu < 0$),

then

the space-time evolves exponentially (on a timescale of $\sqrt{3/\Lambda}$) toward one with de Sitter geometry. De Sitter space can be parametrized as follows,

$$ds^2 = -dt^2 + e^{2Ht}d\vec{x}\cdot d\vec{x} \qquad \text{where } H \text{ is constant,} \tag{1.17}$$

[15] The point will be that most models with enough inflation to solve the Horizon problem predict that any initial anisotropy will be completely wiped out early on during the inflationary era.

[16] All such space-times, which are homogeneous but perhaps anisotropic, fall into a Bianchi classification [11]. There's a slight caveat here: All Bianchi models *except* Bianchi type IX fall under the purview of the cosmic no-hair theorem.

and in particular is isotropic, flat, and has no distinguishing feature (no *hair*) other than the rate of expansion, H. In other words, any energy density besides that of the cosmological constant becomes totally negligible on a timescale set by $\sqrt{3/\Lambda}$.

In the course of his proof, Wald shows in particular that the shear, $\sigma_{\mu\nu}$, which characterizes the anisotropy of the space-time, satisfies the following equation,

$$\sigma_{\mu\nu}\sigma^{\mu\nu} \leq \frac{2\Lambda}{\sinh^2(t/\sqrt{3/\Lambda})}, \tag{1.18}$$

where t is proper time.

Now in order to solve the horizon problem, there must have been at least sixty e-folds of inflation.[17] That is, the scale factor must have increased by a factor of e^{60} during an initial phase when the scale factor was accelerating. If the matter that drives inflation acts like a cosmological constant during inflation, then the Hubble parameter during inflation is approximately $\sqrt{\Lambda/3}$, and $t/\sqrt{3/\Lambda}$ is about equal to the number of e-foldings. Within just about 5 e-foldings of inflation, the denominator on the right-hand side of (1.18) is 1000s, and after 60 e-foldings, it's about 10^{51}; anisotropy becomes very small (in units of $\sqrt{\Lambda} \sim H$) within just a few e-foldings of inflation, and it's minuscule after 60 e-folds.

This means that in order for anisotropy to persist even in small amounts during inflation, at least one of the premises in the cosmic no-hair theorem must *not* apply.

1.7 Æther

An obvious way to avoid Wald's theorem is to source a small anisotropy with matter that *does not* satisfy the dominant or strong energy conditions. This is how Einstein-æther theories (æther theories, for short), popularized by Jacobson and Mattingly in [12], can avoid the no-hair theorem. Usually "æther theory" refers to a theory

[17] See, *e.g.*, [9].

with normal Einstein gravity, plus a dynamical fixed-norm *timelike* vector field that breaks boost invariance of the vacuum and effectively leads to a universal preferred rest frame. Einstein gravity plus a cosmological constant and a dynamical fixed-norm *spacelike* vector field that breaks rotational invariance was considered as a toy model of anisotropic inflation in [6]. It was in this context that I first became interested in æther theories. But æther theories are independently interesting as effective field theories of the spontaneous breaking of Lorentz invariance. They are effective models that include preferred frame effects while leaving diffeomorphism invariance intact. For more on reasons that theorists are interested in æther theories, see, *e.g.*, [13].

The toy æther model in [6] was later shown to be classically unstable [14]. Chapter 2 revisits the stability of æther theories more generally.

1.8 Hairy Inflation

Another way to avoid the cosmic no-hair theorem is to couple matter that could source anisotropy to the matter field that sources inflation (the inflaton field). That's the idea of the model we study in chapter 4, which was originally named "hairy inflation" by the authors of [7]. The model is built on top of standard single field inflation, but unlike in standard single field inflation, there's a coupling between the inflaton field and a $U(1)$ gauge field that retards the dissipation of the energy density in the $U(1)$ gauge field enough to allow for a small persistent anisotropy during inflation. In chapter 4 we give a pedagogical explanation of model. The main work of chapter 4 was in calculating the spectrum of cosmological perturbations in the anisotropic background of the model. Since the model is built on top of standard single field inflation and since the results we found for cosmological perturbations in the model ought to be compared to the results from more "standard" models, below we finish this introductory chapter with a brief review of standard single-field slow-roll inflation.

1.9 Standard Slow-Roll Inflation

1.9.1 Background Equations

Assuming that over large distances (over cosmological scales) the Universe is homogeneous and isotropic,[18] the space-time metric in the Universe is well parametrized by,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2(\sin^2\theta d\phi^2 + d\theta^2)\right). \tag{1.19}$$

Here, K is equal to -1, 0, or $+1$, corresponding to whether the geometry of the Universe is open, flat, or closed (respectively). Matter that supports this geometry must also be homogeneous and isotropic. In that case its stress-energy tensor should take the form

$$T^\mu_\nu = \begin{pmatrix} -\rho(t) & & & \\ & p(t) & & \\ & & p(t) & \\ & & & p(t) \end{pmatrix}. \tag{1.20}$$

Einstein's field equations, $G^\mu_\nu = R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu = 8\pi G T^\mu_\nu$ yield the following two independent differential equations:[19]

$$3\frac{K}{a^2} + 3H^2 = 8\pi G\rho \tag{1.21}$$

and

$$-6\frac{\ddot{a}}{a} = -6(\dot{H} + H^2) = 8\pi G(\rho + 3p), \tag{1.22}$$

where $\dot{}$ denotes a derivative with respect to time and $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter.

[18] Indeed, from our vantage point the density of galaxies and other astrophysical objects on average over a variety of very large scales appears to be about the same in every direction. This observation along with the copernican principle (roughly speaking, that our neighborhood is a typical one) provides evidence that the assumption of homogeneity and isotropy of the Universe is a good one. The CMB provides even better evidence.

[19] The following two equations correspond to $(-G^t_t = -8\pi G T^t_t)$ and $\left(G^\mu_\mu - 2G^t_t = 8\pi G(T^\mu_\mu - 2T^t_t)\right)$, respectively.

Another equation that's useful, and related to the above two equations through the identity $\nabla_\mu G^\mu_\nu = 0$ is the continuity equation:

$$\dot{\rho} = -3H(\rho + p). \tag{1.23}$$

The horizon problem can be solved if the Universe underwent accelerated expansion before recombination. From (1.22) we can see that accelerated expansion requires matter that satisfies $\rho + 3p < 0$. A homogeneous, canonical scalar field, ϕ, which has

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{1.24}$$

can clearly satisfy the condition $\rho + 3p < 0$ if $\dot{\phi}^2 < V(\phi)$. The accelerated expansion is rapid if $\dot{H} \ll H^2$. Such rapid expansion occurs if $\dot{\phi}^2 \ll V(\phi)$. And expansion is nearly exponential ($a(t) \approx e^{Ht}$ where H is constant) if all derivatives of H are small. Slow-roll inflation is just the realization of this scenario—of nearly exponential expansion. The field ϕ is called an "inflaton" in this case. The slow-roll conditions relate derivatives of H to functions of the scalar field and its derivatives, thus giving the conditions that the inflaton field and its potential must satisfy in order for (rapid) inflation to occur. Let's quickly derive these relations. We'll set $K = 0$ for simplicity.

Define

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \qquad \text{and} \qquad \delta \equiv \frac{\ddot{H}}{2H\dot{H}}, \tag{1.25}$$

and note the identity,

$$\dot{\epsilon} = 2H\epsilon(\epsilon + \delta). \tag{1.26}$$

Rearranging (1.21) and (1.22) we get

$$-\frac{\dot{H}}{H^2} = \epsilon = 4\pi G\frac{\rho + p}{H^2} = 4\pi G\left(\frac{\dot{\phi}}{H}\right)^2. \tag{1.27}$$

Differentiating this equation we get

$$4\pi G\frac{d}{dt}\dot{\phi}^2 = 2H^3\epsilon\delta. \tag{1.28}$$

From the above two equations we can see why "slow-roll" is an appropriate name: the velocity and acceleration of the inflaton field, ϕ, must be small compared to the Hubble rate in order for inflation to occur.

We can also derive consistency relations for the form of the inflaton potential. The continuity equation (1.23) implies,

$$V'(\phi)\dot{\phi} = -3H\dot{\phi}^2 - \frac{1}{2}\frac{d}{dt}\dot{\phi}^2 = -\frac{H^3}{4\pi G}\epsilon(3+\delta), \tag{1.29}$$

and plugging (1.27) into (1.21) we get

$$V(\phi) = \frac{H^2}{8\pi G}(3-\epsilon). \tag{1.30}$$

Combining these two equations and (1.27) we see

$$\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = 4\epsilon^2\frac{H^2}{\dot{\phi}^2}\left(\frac{3+\delta}{3-\epsilon}\right)^2 = 16\pi G\,\epsilon\left(\frac{3+\delta}{3-\epsilon}\right)^2 \approx 16\pi G\,\epsilon. \tag{1.31}$$

Differentiating this equation we get

$$2\frac{V'(\phi)}{V(\phi)}\dot{\phi}\left(\frac{V''(\phi)}{V(\phi)} - \left(\frac{V'(\phi)}{V(\phi)}\right)^2\right) \approx 16\pi G\,\dot{\epsilon}, \tag{1.32}$$

and using (1.29) and (1.30) to sub in for $\frac{V'(\phi)}{V(\phi)}\dot{\phi}$ this leads to

$$\left(\frac{V''(\phi)}{V(\phi)}\right) \approx -8\pi G\,\frac{\dot{\epsilon}}{2H\epsilon} + \left(\frac{V'(\phi)}{V(\phi)}\right)^2 \approx 8\pi G(\epsilon-\delta). \tag{1.33}$$

Insisting that the magnitudes of ϵ and δ are much much less than one, equations

(1.31) and (1.33) lead to flatness conditions on the inflaton potential.

Inflation ends when the inflaton reaches the minimum of its potential. Once the inflaton nears the minimum of its potential, it begins oscillating about its minimum and decaying into other matter fields. This is called "reheating".

1.9.2 Perturbations from Single-Field Slow-Roll Inflation

So how does slow-roll inflation give rise to primordial density perturbations? Let's consider the evolution of the quantum-mechanical degrees of freedom in a slow-roll inflation model, with the dynamical inflaton field, ϕ, and the gravitational field, $g_{\mu\nu}$. The quantum-mechanical degrees of freedom are the small space-time dependent fluctuations of these fields about the homogeneous background values in a slow-roll inflation scenario as described above. So

$$\phi = \bar{\phi}(t) + \delta\phi(t, \vec{x}) \qquad \text{and} \qquad g_{\mu\nu} = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \vec{x}), \tag{1.34}$$

where the background fields are barred. Given a homogeneous background, it's standard to Fourier transpose the perturbations:

$$\delta f(t, \vec{x}) \equiv \int \frac{d\vec{k}}{(2\pi)^3} \delta f(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}}. \tag{1.35}$$

The perturbation δf is promoted to a quantum-mechanical operator, so

$$\delta f(t, \vec{k}) = \chi(k, t)\hat{a}_{\vec{k}} + \chi^*(k, t)\hat{a}^\dagger_{-\vec{k}},$$
$$\text{where} \qquad [\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = (2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}'), \quad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0 = [\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] \tag{1.36}$$

where here $\hat{}$ denotes a quantum operator, χ is an appropriately normalized mode function, $k = |\vec{k}|$ is the wavelength of the mode, and $k_{phys} = |\vec{k}|/a$ is the physical wavelength of the mode.

Energy density perturbations during inflation are thought to arise in the following way. The Universe is inflating due to an inflaton rolling down its (flat) potential and is in its quantum-mechanical ground state, so, *e.g.*, $\langle\delta\phi\rangle = 0$. But there are necessarily vacuum fluctuations with nonzero dispersion, $\langle\delta\phi\,\delta\phi\rangle \neq 0$. By the equivalence principle, when curvature is unimportant (for modes with physical wavelength much less than the Hubble radius), the normalization of quantum-mechanical modes is canonical.[20] As curvature becomes important for a given mode (*i.e.*, as the physical wavelength of a given mode becomes greater than the Hubble radius) the quantum-mechanical correlations are frozen into classical density perturbations that form the seeds of structure formation and lead to statistical temperature correlations on the CMB sky.

Canonically normalizing the modes takes a bit of work. It's done by expanding the action

$$S = \int \sqrt{-g}d^4x \left(\frac{R}{16\pi G} - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right) \tag{1.37}$$

to quadratic order in the perturbations $\delta\phi$ and $\delta g_{\mu\nu}$, eliminating non-dynamical degrees of freedom, and combining the dynamical fields into combinations so that the kinetic term in the action is canonically normalized. It's the field variables with canonically normalized kinetic terms that get canonically quantized.

It is convenient to use conformal time,

$$d\eta = a(t)dt. \tag{1.38}$$

In Newtonian gauge, the metric fluctuation may be decomposed as follows,

$$ds^2 = a(\eta)^2\left[-(1+2\Phi)d\eta^2 + (\delta_{ij}(\eta)(1-2\Psi) + \partial_i E_j + \partial_j E_i + 2E_{ij})dx^i dx^j\right], \tag{1.39}$$

[20]There are ambiguities having to do with renormalization that I'm glossing over here.

where E_j is transverse ($\delta^{ij}\partial_i E_j = 0$) and E_{ij} is symmetric, transverse and traceless.

After solving several constraint equations derived from the quadratic action and substituting those solutions back into the action, using the background equations of motion, and integrating by parts several times, the quadratic action can be expressed

$$S^{(2)} = \int d\eta \int \frac{d\vec{k}}{(2\pi)^3}\Bigg(\frac{1}{2}r'(\eta,-\vec{k})\,r'(\eta,\vec{k}) - \frac{1}{2}\left(\vec{k}\cdot\vec{k} - \frac{z''}{z}\right) r(\eta,-\vec{k})\,r(\eta,\vec{k}) + \frac{1}{2}\sum_{s=+,\times}\left[\tilde{h}'_s(\eta,-\vec{k})\,\tilde{h}'_s(\eta,\vec{k}) - \left(\vec{k}\cdot\vec{k} - \frac{a''}{a}\right)\tilde{h}_s(\eta,-\vec{k})\,\tilde{h}_s(\eta,\vec{k})\right]\Bigg), \tag{1.40}$$

where $'$ denotes derivatives with respect to conformal time,

$$z \equiv a\frac{\dot{\phi}}{H}, \tag{1.41}$$

and where

$$r(\eta,\vec{k}) \equiv a\left(\delta\phi(\eta,\vec{k}) + \frac{\dot{\phi}}{H}\Psi(\eta,\vec{k})\right), \tag{1.42}$$

and

$$\tilde{h}_+(\eta,\vec{k}) = \frac{a}{\sqrt{8\pi G}}\left(\frac{(e_1^i e_1^j - e_2^i e_2^j)}{\sqrt{2}}E_{ij}(\eta,\vec{k})\right), \tag{1.43}$$

$$\tilde{h}_\times(\eta,\vec{k}) = \frac{a}{\sqrt{8\pi G}}\left(\frac{(e_1^i e_2^j + e_2^i e_1^j)}{\sqrt{2}}E_{ij}(\eta,\vec{k})\right), \tag{1.44}$$

where $\vec{e}_1$ and $\vec{e}_2$ are two unit 3-vectors satisfying $\vec{e}_a \cdot \vec{e}_b = \delta_{ab}$ and $\vec{e}_a \cdot \vec{k} = 0$. The fields $2\sqrt{8\pi G}\tilde{h}_{+,\times}/a(\eta)$ are the two gravitational wave amplitudes. When $|\vec{k}| \ll aH$, the quantity $-r(\eta,\vec{k})H/a\dot{\phi}$ is equal to the so-called curvature perturbation, $\zeta(\vec{k},\eta)$.[21]

[21] There is a gauge where the spatial part of the metric perturbation takes the form $\delta g_{ij} = a^2 e^{2\zeta}[\exp\gamma]_{ij}$, $\gamma_{ii} = 0$, $\partial_i\gamma_{ij} = 0$. See *e.g.*, [15] pg. 4.

The equations of motion for r and $\tilde{h}_s$ are

$$r'' = -\left(\vec{k}\cdot\vec{k} - \frac{z''}{z}\right) r \qquad \text{and} \qquad \tilde{h}_s'' = -\left(\vec{k}\cdot\vec{k} - \frac{a''}{a}\right)\tilde{h}_s. \tag{1.45}$$

To quantize, we promote r and $\tilde{h}_s$ to operators as in (1.36). The mode functions χ_r and $\chi_{\tilde{h}_s}$ must solve the above equations of motion. First notice that

$$\frac{a''}{a} = \frac{d}{dt}a^2 H = a^2 H^2 (2-\epsilon) \tag{1.46}$$

and

$$\begin{aligned} \frac{z''}{z} = \frac{1}{\sqrt{\epsilon}}\frac{d}{dt}a^2\sqrt{\epsilon}H\left(1+\frac{\dot{\epsilon}}{2H\epsilon}\right) &= \frac{1}{\sqrt{\epsilon}}\frac{d}{dt}a^2\sqrt{\epsilon}H(1+\epsilon+\delta) \\ &= a^2H^2\left((2+\delta)(1+\epsilon+\delta)+(\dot{\epsilon}/H+\dot{\delta}/H)\right). \end{aligned} \tag{1.47}$$

During slow-roll inflation, $|\dot{\epsilon}/H|, |\dot{\delta}/H| \ll |\epsilon|, |\delta| \ll 1$ and so $H \approx$ constant. That means $a \approx -\frac{1}{H\eta}$ where $\eta \to -\infty$ in the past and so

$$\frac{a''}{a} \approx (2-\epsilon)/\eta^2, \qquad \frac{z''}{z} \approx (2+2\epsilon+3\delta)/\eta^2. \tag{1.48}$$

Using these expressions for a''/a and z''/z and treating ϵ and δ as constants, the solutions to (1.45) are Hermite polynomials. Setting ϵ and δ to zero, the solutions become even simpler. The solution to

$$f'' = -(k^2 - 2/\eta^2) f \tag{1.49}$$

is

$$f = c_1(k)\left(1-\frac{i}{k\eta}\right)e^{-ik\eta} + c_2(k)\left(1+\frac{i}{k\eta}\right)e^{ik\eta}. \tag{1.50}$$

Notice that in the long wavelength limit, when $k\eta \gg 1$,[22] the equations of motion for r and $\tilde{h}_s$ are just harmonic oscillator equations. Invoking the equivalence principle, we use this fact to normalize the mode functions. The mode function should satisfy,

$$\chi_k \partial_\eta \chi_k^* - \chi_k^* \partial_\eta \chi_k = i \qquad \text{(long wavelength limit)}. \tag{1.51}$$

In the approximation where $\epsilon = \delta = 0$, it's clear that the correctly normalized mode functions are

$$\chi_{r\,k} = \chi_{\tilde{h}_s\,k} = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) e^{-ik\eta}. \tag{1.52}$$

We can now calculate the two-point function for r and for $\tilde{h}$. In general for a field f with mode expansion $f(\eta, \vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3}\left(\chi_k(\eta) e^{i\vec{k}\cdot\vec{x}} \hat{a}_{\vec{k}} + \chi_k^*(\eta) e^{-i\vec{k}\cdot\vec{x}} \hat{a}_{\vec{k}}^\dagger\right)$, it's not hard to show that

$$\begin{aligned} \langle f(\eta, \vec{x}) f(\eta, \vec{y})\rangle &= \int \frac{d\vec{k}}{(2\pi)^3} \int \frac{d\vec{q}}{(2\pi)^3} \chi_k(\eta)\chi_q^*(\eta) e^{i(\vec{k}\cdot\vec{x} - \vec{q}\cdot\vec{y})}[a_{\vec{k}}, a_{\vec{q}}^\dagger] \\ &= \int \frac{d\vec{k}}{(2\pi)^3} \chi_k(\eta)\chi_k^*(\eta) e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \qquad (1.53) \\ &\equiv \int \frac{d\vec{k}}{(2\pi)^3} P_f(k;\eta) e^{i\vec{k}\cdot(\vec{x}-\vec{y})}, \qquad (1.54) \end{aligned}$$

where $P_f(k;\eta)$ is the power spectrum at time η. It's also not hard to show for the Fourier-transformed functions that

$$\begin{aligned} \langle f(\eta, \vec{k}) f(\eta, \vec{q})\rangle &= \chi_k(\eta)\chi_q^*(\eta)[a_{\vec{k}}, a_{-\vec{q}}^\dagger] \\ &= \chi_k(\eta)\chi_k^*(\eta)(2\pi)^3 \delta^{(3)}(\vec{k} + \vec{q}) \qquad (1.55) \\ &= P_f(k;\eta)(2\pi)^3 \delta^{(3)}(\vec{k} + \vec{q}). \qquad (1.56) \end{aligned}$$

As mentioned above, the $\delta^{(3)}(\vec{k} + \vec{q})$ dependence can be independently derived from

[22] Note that $\frac{z''}{z} \approx \frac{a''}{a} \approx (aH)^2$ during slow-roll.

the fact that the position space two-point function is invariant under translations.

Let's think for a moment about the short wavelength limit (when $k \ll aH$). The equations then take the form

$$f'' = \frac{\gamma''}{\gamma} f, \tag{1.57}$$

the solution to which is

$$f = c_1\gamma + c_2\gamma \int \frac{d\eta}{\gamma^2}. \tag{1.58}$$

For $\gamma = z$ or a during slow-roll inflation, $\int \frac{d\eta}{\gamma^2} \sim a^{-3}$. Thus the exact solution for χ_{rk}/z and for $\chi_{\tilde{h}_s k}/a$ when $k \ll aH$ is a constant plus a decaying part. In other words $r(\eta, \vec{k})/z$ and $\tilde{h}_s(\eta, \vec{k})/a$ are conserved outside the Hubble horizon. In particular, for modes that cross the Hubble horizon well before the end of inflation (when ϵ and δ are much much less than one), the solution (1.52) should be a very good approximation just after Horizon crossing ($-\frac{1}{\eta} \approx aH > k$). Then we know that much after horizon crossing the amplitudes of r/z and $\tilde{h}/a$ should be conserved. That means

$$P_{r/z}(k; \eta > \eta_{\times,k}) \approx \frac{1}{z^2}\frac{1}{2k}\left(1 - \frac{i}{k\eta}\right)\left(1 + \frac{i}{k\eta}\right) \approx \frac{1}{z^2}\frac{1}{2k}\left(\frac{1}{k\eta}\right)^2 \approx \text{constant}, \tag{1.59}$$

where $\eta_{\times,k}$ is η at horizon crossing for wavelength k. Recalling that $z = a\dot{\phi}/H$ we can see that

$$P_{r/z}(k; \eta \gg \eta_{\times,k}) \approx \frac{H^2}{2k^3}\left(\frac{H}{\dot{\phi}}\right)^2\bigg|_{\text{Horzion crossing}} = \frac{H^2}{2k^3}\left(\frac{\epsilon}{4\pi G}\right)^{-1}\bigg|_{\text{Horzion crossing}}. \tag{1.60}$$

Similarly,

$$P_{\tilde{h}/a}(k; \eta \gg \eta_{\times,k}) \approx \frac{H^2}{2k^3}\bigg|_{\text{Horzion crossing}}. \tag{1.61}$$

We see that $P(k) \propto k^{-3}$ for both r and $\tilde{h}$. This means that the position space two-point correlation function,

$$P(\vec{x}-\vec{y}) = \int \frac{d\vec{k}}{(2\pi)^3} P(k) e^{i\vec{k}\cdot(\vec{x}-\vec{y})}, \tag{1.62}$$

is invariant under scale transformations, $\vec{k} \to \lambda^{-1}\vec{k}$, $\vec{x} \to \lambda\vec{x}$. For this reason, a power spectrum proportional to k^{-3} is called *scale invariant*. Actually, if we'd used the more precise expressions (1.48) for a''/a and z''/z and used the corresponding more precise Hermite polynomials as our mode functions, we would have found a very slightly scale-noninvariant power spectrum (with the deviation from scale invariance controlled by ϵ and δ). Measurements of the CMB (and the distribution of structures in the Universe) do indeed indicate that the primordial power spectrum is nearly scale invariant. And measurements are now getting sensitive enough to probe very slight deviations from scale invariance. So, given that the slow-roll parameters are related to the shape of the inflaton potential, in a sense we're on the brink of being able to probe the form of the inflaton potential.

I mentioned earlier that the two gravitational wave amplitudes are

$$h_s = 2\sqrt{8\pi G}\tilde{h}_s/a(\eta), \qquad \text{where} \qquad s = + \text{ or } \times, \tag{1.63}$$

and that the curvature perturbation ζ is equal to $-r(\eta,\vec{k})H/a\dot{\phi}$ outside the horizon. Thus for modes outside the horizon the ratio of power in gravitational waves and the curvature power is

$$\frac{P_{h_+} + P_{h_\times}}{P_\zeta} \approx 2(2\sqrt{8\pi G})^2 \left(\frac{\epsilon}{4\pi G}\right) = 16\epsilon, \tag{1.64}$$

where ϵ is evaluated near horizon crossing. This is known as the tensor-to-scalar ratio. It turns out that the gravitational wave power spectrum and the curvature perturbation power spectrum are expected to be conserved outside the horizon in very generic circumstances, even after inflation ends and the evolution of the background space-time changes substantially [16],[17]. I mentioned above that deviations of the power spectrum from scale invariance are controlled by the slow-roll parameters. Comparing these deviations to the size of the tensor-to-scalar ratio is an important cross-check of slow-roll models. There are other such cross-checks that can be made, such as comparing the sizes of non-Gaussianities to the tensor-to-scalar ratio and to deviations from scale invariance. The theoretical predictions for ratios of such observable quantities (which should be *numbers*, independent of slow-roll parameters) are known as consistency conditions. We will derive a consistency condition for hairy inflation models in chapter 4.