Gromov-Witten Invariants: Crepant Resolutions and Simple Flops

Thesis by

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To the memory of my mom

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This thesis is dedicated to my family. They are behind me all the way...

Abstract

Let S be any smooth toric surface. We establish a ring isomorphism between the equivariant extended Chen-Ruan cohomology of the n-fold symmetric product stack $[\mathrm{Sym}^n(S)]$ of S and the equivariant extremal quantum cohomology of the Hilbert scheme $\mathrm{Hilb}^n(S)$ of n points in S. This proves a generalization of Ruan's Cohomological Crepant Resolution Conjecture for the case of $\mathrm{Sym}^n(S)$.

We determine the operators of small quantum multiplication by divisor classes on the orbifold quantum cohomology of $[\mathrm{Sym}^n(\mathcal{A}_r)]$, where $\mathcal{A}_r$ is the minimal resolution of the cyclic quotient singularity $\mathbb{C}^2/\mathbb{Z}_{r+1}$. Under the assumption of the nonderogatory conjecture, these operators completely determine the quantum ring structure, which gives an affirmative answer to Bryan-Graber's Crepant Resolution Conjecture on $[\mathrm{Sym}^n(\mathcal{A}_r)]$ and $\mathrm{Hilb}^n(\mathcal{A}_r)$. More strikingly, this allows us to complete a tetrahedron of equivalences relating the Gromov-Witten theories of $[\mathrm{Sym}^n(\mathcal{A}_r)]/\mathrm{Hilb}^n(\mathcal{A}_r)$ and the relative Gromov-Witten/Donaldson-Thomas theories of $\mathcal{A}_r \times \mathbb{P}^1$.

Finally, we prove a closed formula for an excess integral over the moduli space of degree d stable maps from unmarked curves of genus one to the projective space $\mathbb{P}^r$ for positive integers r and d. The result generalizes the multiple cover formula for $\mathbb{P}^1$ and reveals that any simple $\mathbb{P}^r$ flop of smooth projective varieties preserves the theory of extremal Gromov-Witten invariants of arbitrary genus. It also provides examples for which Ruan's Minimal Model Conjecture holds.

Contents

Chapter 1

Introduction

1.1 Main Results

1.1.1 The Crepant Resolution Conjecture

A principle in physics states that *string theory on an orbifold is equivalent to string theory on any crepant resolution of singularities.* Over the years, it has been put into various mathematical frameworks. Among them, we are particularly interested in the formulations pioneered by Ruan in the context of Gromov-Witten theory (see, e.g., [R, BG, CoIT, CoR]).

Let n be a positive integer and $\mathbb{T} = (\mathbb{C}^\times)^2$ a torus. The $\mathbb{T}$-equivariant cohomology of a point is simply $\mathbb{Q}[t_1, t_2]$. Given a smooth toric surface S, which comes with a $\mathbb{T}$-action, the symmetric group $\mathfrak{S}_n$ acts on the n-fold Cartesian product S^n by permuting coordinates. Thus, we obtain a quotient scheme $\mathrm{Sym}^n(S) := S^n/\mathfrak{S}_n$, the n-fold symmetric product of S, and a quotient stack $[\mathrm{Sym}^n(S)]$, the n-fold symmetric product stack of S. The stack $[\mathrm{Sym}^n(S)]$ is a smooth orbifold, whose coarse moduli space is none other than the symmetric product $\mathrm{Sym}^n(S)$. Further, it has a unique crepant resolution given by the Hilbert scheme $\mathrm{Hilb}^n(S)$ of n points in S.

The string theories of $[\mathrm{Sym}^n(S)]$ and $\mathrm{Hilb}^n(S)$ are expected to be equivalent. In order to make this precise in mathematics, Ruan [R] proposes the Cohomological Crepant Resolution Conjecture (CCRC), which asserts that *the Chen-Ruan cohomology ring of* $[\mathrm{Sym}^n(S)]$ *is isomorphic to the so-called quantum corrected cohomology ring of the crepant resolution* $\mathrm{Hilb}^n(S)$.

The first goal of this thesis, which covers Chapters 2 and 3, is to justify CCRC for symmetric products of smooth toric surfaces. In fact, we obtain a slightly stronger result, which says that there is a SYM-HILB correspondence between the degree zero Gromov-Witten theory of $[\mathrm{Sym}^n(S)]$ and the extremal Gromov-Witten theory of $\mathrm{Hilb}^n(S)$:

Theorem 1.1.1. *After making an appropriate change of variables and extending scalars to a suitable field F, there is a ring isomorphism*

$$L : H^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(S)]; F) \to H^*_{\mathbb{T}}(\mathrm{Hilb}^n(S); F),$$

which preserves gradings and is also an isometry with respect to Poincaré pairings. Here the left side is the equivariant extended Chen-Ruan cohomology whose structure constants are defined by three-point extended invariants of degree zero while the right side is the equivariant extremal quantum cohomology whose structure constants are defined by three-point extremal invariants.

The above result holds, in particular, for surfaces such as $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, Hirzeburch surfaces, and the total spaces of line bundles over $\mathbb{P}^1$.

The quantum corrected cohomology of the Hilbert scheme $\mathrm{Hilb}^n(S)$, in the sense of [R], is defined as the extremal quantum cohomology with the unique extremal quantum parameter being specialized to -1. The following result, being an immediate consequence of Theorem 1.1.1, justifies CCRC for $[\mathrm{Sym}^n(S)]$ and $\mathrm{Hilb}^n(S)$.

Corollary 1.1.2. *The equivariant Chen-Ruan cohomology of* $[\mathrm{Sym}^n(S)]$ *is isomorphic to the equivariant quantum corrected cohomology of* $\mathrm{Hilb}^n(S)$.

We may encode three-point extended $[\mathrm{Sym}^n(S)]$-invariants of degree zero in a three-point function (cf. Section 3.4), which depends only on the equivariant parameters t_1, t_2 and the quantum parameter u corresponding to the twisted divisor. Theorem 1.1.1 allows us to reconstruct the cup product of the Hilbert scheme from the Gromov-Witten invariants of the symmetric product stack as in the following statement, which is not covered by Ruan Conjecture, however.

Corollary 1.1.3. *The equivariant ordinary cohomology of* $\mathrm{Hilb}^n(S)$ *is isomorphic to a certain correction to the equivariant extended Chen-Ruan cohomology of* $[\mathrm{Sym}^n(S)]$*. Precisely, the cup product of* $\mathrm{Hilb}^n(S)$ *can be recovered from the three-point functions of* $[\mathrm{Sym}^n(S)]$ *by taking the limit $u \to i\infty$, where i is a square root of -1.*

If we have closed-form formulas for these symmetric product orbifold invariants, the cup product of the Hilbert scheme can be written down explicitly.

We also study a little bit about the relative Gromov-Witten theory of $S \times \mathbb{P}^1$ as it seems quite close to the orbifold theory and may yield an alternative way to compute the cup product of the Hilbert scheme of points. Indeed, the degree $(0, n)$ relative theory turns out to be equivalent to the extended Chen-Ruan cohomology of $[\mathrm{Sym}^n(S)]$, and thus it is also equivalent to the extremal quantum cohomology of $\mathrm{Hilb}^n(S)$ (see Section 5.1.1).

Given a positive integer r, we let $\mathcal{A}_r$ be the minimal resolution of the type A_r surface singularity. The second goal of the thesis, covering Chapters 4 and 5, is to compare the equivariant orbifold Gromov-Witten theories of the symmetric products of $\mathcal{A}_r$ with the equivariant Gromov-Witten theories of the crepant resolutions in the spirit of Bryan-Graber's Crepant Resolution Conjecture [BG]. The correspondence we obtain is stronger than Theorem 1.1.1 for the case of $\mathcal{A}_r$.

To formulate explicitly the correspondence, we consider the three-point functions

$$\langle\langle \alpha_1, \alpha_2, \alpha_3 \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} \in \mathbb{Q}(t_1, t_2)[[u, s_1, \ldots, s_r]],$$

which encode three-point extended Gromov-Witten invariants of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ (see (4.3.1)). These generating functions add a multiplicative structure to the equivariant Chen-Ruan cohomology $H^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)]; \mathbb{Q})$. The multiplication so obtained is called the small orbifold quantum product.

The $\mathbb{T}$-equivariant quantum cohomology of $\mathrm{Hilb}^n(\mathcal{A}_r)$ has been explored by Maulik and Oblomkov in [MO1], so we need only deal with the quantum ring of the orbifold $[\mathrm{Sym}^n(\mathcal{A}_r)]$. We fully cover two-point extended Gromov-Witten invariants of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ and find that the calculation of these invariants is tantamount to the question of counting certain branched covers of rational curves. Our discovery can be summarized in the following statement.

Theorem 1.1.4. *Two-point extended equivariant Gromov-Witten invariants of* $[\mathrm{Sym}^n(\mathcal{A}_r)]$ *are expressible in terms of equivariant orbifold Poincaré pairings and one-part double Hurwitz numbers.*

One-part double Hurwitz numbers, as shown by Goulden, Jackson, and Vakil [GJV], admit explicit closed formulas (cf. (4.8.4)), and therefore Theorem 1.1.4 provides a complete solution to the divisor operators, i.e., the operators of quantum multiplication by divisor classes. These operators correspond naturally to the divisor operators on the Hilbert scheme $\mathrm{Hilb}^n(\mathcal{A}_r)$:

Theorem 1.1.5. *Let L be as in Theorem 1.1.1. For any Chen-Ruan cohomology classes α_1, α_2 and divisor D, we have the following identity for three-point functions:*

$$\langle\langle \alpha_1, D, \alpha_2 \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \langle L(\alpha_1), L(D), L(\alpha_2) \rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}.$$

Here $\langle -,-,- \rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}$ *are the three-point functions of* $\mathrm{Hilb}^n(\mathcal{A}_r)$ *in variables* $t_1, t_2, q, s_1, \ldots, s_r$ *(see* (5.2.1)*), and we make the substitution* $q = -e^{iu}$*, where* $i^2 = -1$*.*

In addition to the relation to the Hilbert schemes, the orbifold theory is in connection with the relative Gromov-Witten theory of threefolds.

Theorem 1.1.6. *Given cohomology-weighted partitions* $\lambda_1(\vec{\eta}_1)$*,* $\lambda_2(\vec{\eta}_2)$ *of* n *and* $\alpha = 1(1)^n$*,* (2) *or* D_k*,* $k = 1, \ldots, r$ *(see Sections 3.2.1 and 5.1 for these classes), we have*

$$\langle\langle \lambda_1(\vec{\eta}_1), \alpha, \lambda_2(\vec{\eta}_2) \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1), \alpha, \lambda_2(\vec{\eta}_2)},$$

where the right hand side is a shifted partition function (cf. (5.1.1)*).*

1.1.2 Tetrahedron of equivalences

The above theorems form a triangle of equivalences. We can include the Donaldson-Thomas theory to make up a tetrahedron. In fact, Theorems 1.1.5 and 1.1.6, in conjunction with the results of [M, MO1, MO2], establish the following equivalences for divisor operators.

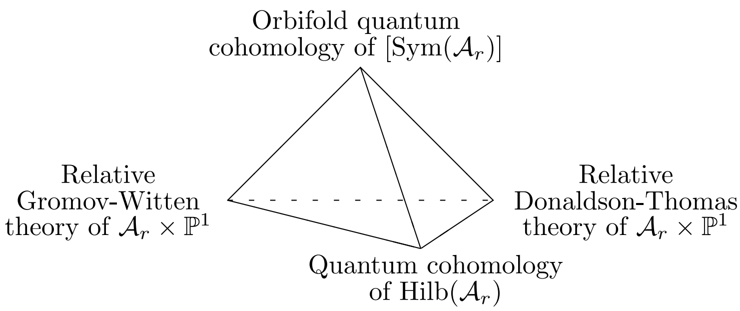

Figure 1.1. A tetrahedron of equivalences.

Before the study of the Gromov-Witten theory of $[\mathrm{Sym}^n(\mathcal{A}_r)]$, the case of the affine plane $\mathbb{C}^2$ was the only known example for the above tetrahedron to hold for all operators (cf. [BG, BP, OP1, OP2]). If the nonderogatory conjecture (Conjecture 5.3.1) of Maulik and Oblomkov is assumed, these four theories will be equivalent in our case of $\mathcal{A}_r$ as well. The base triangle of "equivalences" is the work of Maulik and Oblomkov. And the triangle facing the rightmost corner is worked out in this thesis:

Proposition 1.1.7. *Let* L *be as in Theorem 1.1.1 and* $\lambda_1(\vec{\eta}_1)$*,* $\lambda_2(\vec{\eta}_2)$*,* $\lambda_3(\vec{\eta}_3)$ *any cohomology-*

weighted partitions of n. Assuming the nonderogatory conjecture, the identities

$$\begin{aligned}\langle\langle\lambda_1(\vec{\eta}_1),\lambda_2(\vec{\eta}_2),\lambda_3(\vec{\eta}_3)\rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} &= \langle L(\lambda_1(\vec{\eta}_1)),L(\lambda_2(\vec{\eta}_2)),L(\lambda_3(\vec{\eta}_3))\rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}\\ &= \mathrm{GW}(\mathcal{A}_r\times\mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\lambda_2(\vec{\eta}_2),\lambda_3(\vec{\eta}_3)}\end{aligned}$$

hold under the substitution $q=-e^{iu}$.

Once the WDVV equations are used, we can make a more general statement on $[\mathrm{Sym}^n(\mathcal{A}_r)]$ and $\mathrm{Hilb}^n(\mathcal{A}_r)$.

Proposition 1.1.8. *Let $q=-e^{iu}$. Assuming the nonderogatory conjecture, the map L, in Theorem 1.1.1, equates the extended multipoint functions of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ to the multipoint functions of $\mathrm{Hilb}^n(\mathcal{A}_r)$. Moreover, these functions are rational functions in $t_1,t_2,q,s_1,\ldots,s_r$. (Multipoint functions are those with at least three insertions).*

This answers positively the Crepant Resolution Conjecture, proposed by Bryan and Graber, on the symmetric product case. We will see that Propositions 1.1.7 and 1.1.8 are valid in the case of $n=2,r=1$ even without presuming the nonderogatory conjecture (see Section 5.3.1).

1.1.3 Minimal Model Conjecture

The Hilbert scheme of points in a smooth surface S gives the unique crepant resolution of the symmetric product of S. In general, a singular variety may admit no crepant resolutions or more than one crepant resolution. The Minimal Model Conjecture, proposed by Ruan, is closely related to the Crepant Resolution Conjecture. It asserts that *if a variety X admits two different crepant resolutions Y_1 and Y_2, the Gromov-Witten theories of Y_1 and Y_2 are equivalent.*

A simple ordinary $\mathbb{P}^n$ flop is a K-equivalence with exceptional locus $Z=\mathbb{P}^n$. In [LLW], Lee, Lin, and Wang justify the genus zero Minimal Model Conjecture for the simple flop case. We will consider extremal Gromov Witten theories for simple flops in all genera and obtain the following result.

Theorem 1.1.9. *Let $f: X \dashrightarrow X'$ be a simple ordinary $\mathbb{P}^n$ flop. There is a correspondence which identifies extremal Gromov-Witten theories of X and X' for any genus.*

1.2 Setting for Chapters 2–5

1.2.1 Smooth toric surfaces

Throughout Chapters 2–5, we let S be a smooth toric surface, which is acted on by the torus $\mathbb{T} = (\mathbb{C}^\times)^2$.

The surface S is determined by a fan Σ that is a finite collection of strongly convex rational polyhedral cones σ contained in $N = \mathrm{Hom}(M, \mathbb{Z})$, where $M \cong \mathbb{Z}^2$. That is, S is obtained by gluing together affine toric varieties S_σ and S_τ along $S_{\sigma\cap\tau}$ for $\sigma, \tau \in \Sigma$. Here, for example, S_σ has coordinate ring $\mathbb{C}[\sigma^\vee \cap M]$, which is the $\mathbb{C}$-algebra with generators χ^m for $m \in \sigma^\vee \cap M$ and multiplication $\chi^m\chi^{m'} = \chi^{m+m'}$. Note that $\sigma^\vee \cap M$ is by definition the set of elements $m \in M$ satisfying $v(m) \geq 0$ for all $v \in \sigma$.

In addition, S has finitely many $\mathbb{T}$-invariant subvarieties, and so it has a finite number of $\mathbb{T}$-fixed points

$$x_1, \ldots, x_s.$$

(We do not study smooth toric surfaces without $\mathbb{T}$-fixed points as they are not interesting in equivariant theory.)

For each i, x_i is contained in

$$U_i := S_{\sigma_i}$$

for some $\sigma_i \in \Sigma$. As S is smooth and U_i possesses a unique $\mathbb{T}$-fixed point x_i, we see that U_i must be isomorphic to the affine plane with x_i corresponding to the origin. However, S is not necessarily the union $\bigcup_{i=1}^s U_i$.

We denote by

$$L_i \text{ and } R_i$$

the tangent weights at x_i.

1.2.2 Notation and convention

The following notations will be used in Chapters 2–5 without further comment. Some other notations will be introduced along the way.

1. To avoid doubling indices, we identify

$$A^i(X) = H^{2i}(X;\mathbb{Q}),\ A_i(X) = H_{2i}(X;\mathbb{Q}),\ \text{and } A_i(X;\mathbb{Z}) = H_{2i}(X;\mathbb{Z}),$$

just to name a few, for any complex variety X to appear in this article (note that we drop $\mathbb{Q}$ but not $\mathbb{Z}$). They will be referred to as cohomology or homology groups rather than Chow groups.

2. An orbifold $\mathcal{X}$ is a smooth Deligne-Mumford stack of finite type over $\mathbb{C}$. Denote by $c : \mathcal{X} \to X$ the canonical map to the coarse moduli space.

3. For every positive integer s, μ_s is the cyclic subgroup of $\mathbb{C}^\times$ of order s. For any finite group G, $\mathcal{B}G$ is the classifying stack of G, i.e., [Spec $\mathbb{C}/G$].

4. (a) $\mathbb{T} = (\mathbb{C}^\times)^2$ is always a two-dimensional torus, and t_1, t_2 are the generators of the $\mathbb{T}$-equivariant cohomology $A^*_{\mathbb{T}}(\text{point})$ of a point, that is, $A^*_{\mathbb{T}}(\text{point}) = \mathbb{Q}[t_1, t_2]$.

   (b) $V_{\mathfrak{m}} = V \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1, t_2)$ for each $\mathbb{Q}[t_1, t_2]$-module V.

5. Given any object $\mathcal{O}$, $\mathcal{O}^n$ means that $\mathcal{O}$ repeats itself n times.

6. For $i = 1, 2$, ϵ_i is a function on the set of nonnegative integers such that

$$\epsilon_i(m) = \begin{cases} 0 & \text{if } m < i; \\ 1 & \text{if } m \geq i. \end{cases}$$

7. Let σ be a partition of a nonnegative integer.

   (a) $\ell(\sigma)$ is the length of σ.

   (b) Unless otherwise stated, σ is presumed to be written as

$$\sigma = (\sigma_1, \ldots, \sigma_{\ell(\sigma)}) \text{ with } \sigma_1 \geq \cdots \geq \sigma_{\ell(\sigma)}.$$

   To make a emphasis, if σ_k is another partition, it is simply $(\sigma_{k1}, \ldots, \sigma_{k\ell(\sigma_k)})$.

   (c) Let $\vec{\alpha} := (\alpha_1, \ldots, \alpha_{\ell(\sigma)})$ be an $\ell(\sigma)$-tuple of cohomology classes associated to σ so that we may form a cohomology-weighted partition $\sigma(\vec{\alpha}) := \sigma_1(\alpha_1) \cdots \sigma_{\ell(\sigma)}(\alpha_{\ell(\sigma)})$. The group $\text{Aut}(\sigma(\vec{\alpha}))$ is defined to be the group of permutations on $\{1, 2, \ldots, \ell(\sigma)\}$ fixing

$$(\ (\sigma_1, \alpha_1), \ldots, (\sigma_{\ell(\sigma)}, \alpha_{\ell(\sigma)})\ ).$$

   Let $\text{Aut}(\sigma)$ be the group $\text{Aut}(\sigma(\vec{\alpha}))$ when all entries of $\vec{\alpha}$ are identical. Its order is simply $\prod_{i=1}^n m_i!$ if $\sigma = (1^{m_1}, \ldots, n^{m_n})$.

(d) $|\sigma| = n$ if $\sigma_1 + \cdots + \sigma_{\ell(\sigma)} = n$, and $o(\sigma) = \mathrm{lcm}(|\sigma_1|, \ldots, |\sigma_{\ell(\sigma)}|)$ is the order of any permutation of cycle type σ.

(e) $(2) := (1^{n-2}, 2)$ and $1 := (1^n)$ are partitions of length $n-1$ and length n respectively.

Chapter 2

Hilbert scheme of points

The Hilbert scheme of n points in S, denote by $\mathrm{Hilb}^n(S)$ or $S^{[n]}$, parametrizes zero-dimensional closed subscheme Z of S satisfying

$$\dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = n.$$

The Hilbert-Chow morphism

$$\rho : \mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S)$$

is defined by sending $[Z]$ to $\sum_{p \in S} \ell(\mathcal{O}_{Z,p})[p]$, where the length $\ell(\mathcal{O}_{Z,p})$ is simply the multiplicity of p in Z. The map ρ is not only a resolution of singularities, but it is also crepant, i.e., $K_{\mathrm{Hilb}^n(S)} = \rho^* K_{\mathrm{Sym}^n(S)}$. We remark that ρ actually gives a unique crepant resolution. (This is, however, no longer true for higher-dimensional varieties).

The action of $\mathbb{T}$ on S lifts to $\mathrm{Hilb}^n(S)$. Each element of the fixed locus $\mathrm{Hilb}^n(S)^{\mathbb{T}}$ has support in $(S^n)^{\mathbb{T}} = \{x_1, \ldots, x_s\}$, and $\mathrm{Hilb}^n(S)^{\mathbb{T}}$ is isolated.

2.1 Fixed-point basis and Nakajima basis

Fixed-point basis. As seen in [ES], there is a one-to-one correspondence between partitions of n and $\mathbb{T}$-fixed point of $\mathrm{Hilb}^n(\mathbb{C}^2)$. Indeed, for every partition λ, the corresponding $\mathbb{T}$-fixed points $\lambda(x_i) \in \mathrm{Hilb}^{|\lambda|}(U_i)$ is defined as follows: let $\mathbb{C}[u,v]$ be the coordinate ring for $U_i \cong \mathbb{C}^2$, $\lambda(x_i)$ is then the subscheme of U_i with ideal $\mathcal{I}_{\lambda(x_i)}$ being

$$(u^{\lambda_1}, vu^{\lambda_2}, \ldots, v^{\ell(\lambda)-1}u^{\lambda_{\ell(\lambda)}}, v^{\ell(\lambda)}).$$

The point $\lambda(x_i)$ is supported at x_i and is mapped to $|\lambda| \cdot [x_i] \in \mathrm{Sym}^{|\lambda|}(U_i)$ by the Hilbert-Chow morphism.

Note that each $\mathbb{T}$-fixed point of $\mathrm{Hilb}^n(S)$ is the sum

$$\lambda_1(x_1) + \cdots + \lambda_s(x_s)$$

for some partitions λ_i's satisfying $\sum_{i=1}^s |\lambda_i| = n$ (by "the sum" we mean the disjoint union of $\lambda_i(x_i)$'s). For $\widetilde{\lambda} := (\lambda_1, \ldots, \lambda_s)$, write

$$I_{\widetilde{\lambda}} = [\lambda_1(x_1) + \cdots + \lambda_s(x_s)],$$

which we call $\mathbb{T}$-fixed point classes of $\mathrm{Hilb}^n(S)$, and which form a basis for the localized cohomology $A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S))_{\mathfrak{m}}$.

Nakajima basis. Another important basis for $A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S))$ is the Nakajima basis, which we now describe. For further details, see [Gro, N1, N2, V, LQW].

Given a partition λ of n and an $\ell(\lambda)$-tuple $\vec{\eta} := (\eta_1, \ldots, \eta_{\ell(\lambda)})$ with entries in $A^*_{\mathbb{T}}(S)$. Let $|0\rangle = 1 \in A^0_{\mathbb{T}}(S^{[0]})$, we define

$$\mathfrak{a}_\lambda(\vec{\eta}) = \frac{1}{|\mathrm{Aut}(\lambda(\vec{\eta}))|} \prod_{i=1}^{\ell(\lambda)} \frac{1}{\lambda_i} \, \mathfrak{p}_{-\lambda_i}(\eta_i)|0\rangle,$$

where $\mathfrak{p}_{-\lambda_i}(\eta_i) : A^*_{\mathbb{T}}(S^{[k]}) \to A^{*+\lambda_i-1+\deg(\gamma)}_{\mathbb{T}}(S^{[k+\lambda_i]})$ are Heisenberg creation operators. (Sometimes we denote the class $\mathfrak{a}_\lambda(\vec{\eta})$ by $\mathfrak{a}_{\lambda_1}(\eta_1) \cdots \mathfrak{a}_{\lambda_{\ell(\lambda)}}(\eta_{\ell(\lambda)})$).

Choose a basis $\mathfrak{B}$ for $A^*_{\mathbb{T}}(S)$. The classes $\mathfrak{a}_\lambda(\vec{\eta})$'s, running through all partitions λ of n and all $\eta_i \in \mathfrak{B}$, give a basis for $A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S))$. They are referred to as the Nakajima basis associated to $\mathfrak{B}$.

We may also work with the Nakajima basis associated to the $\mathbb{T}$-fixed point classes

$$[x_1], \ldots, [x_s].$$

For partitions $\lambda_1, \ldots, \lambda_s$ of $n_1, \ldots, n_s$ respectively, we define $\ell(\widetilde{\lambda}) = \sum_{i=1}^s \ell(\lambda_i)$ and

$$\mathfrak{a}_{\widetilde{\lambda}} := \mathfrak{a}_{\lambda_{11}}([x_1]) \cdots \mathfrak{a}_{\lambda_{1\ell(\lambda_1)}}([x_1]) \cdots \mathfrak{a}_{\lambda_{s1}}([x_s]) \cdots \mathfrak{a}_{\lambda_{s\ell(\lambda_s)}}([x_s]).$$

The Chow degree of $\mathfrak{a}_{\widetilde{\lambda}}$ is

$$\sum_{i=1}^{s}(|\lambda_i| - \ell(\lambda_i)) + 2\ell(\widetilde{\lambda}) = (n - \ell(\widetilde{\lambda})) + 2\ell(\widetilde{\lambda}) = n + \ell(\widetilde{\lambda}).$$

In the case of U_i, we have exactly one $\mathbb{T}$-fixed point x_i. Hence,

$$\mathfrak{a}_{\widetilde{\lambda_i}},$$

denoting the classes $\mathfrak{a}_{\lambda_{i1}}([x_i]) \cdots \mathfrak{a}_{\lambda_{i\ell(\lambda_i)}}([x_i])|0\rangle$ (with $|\lambda_i| = n_i$) on $\mathrm{Hilb}^{n_i}(U_i)$, form a basis for $A^*_{\mathbb{T}}(\mathrm{Hilb}^{n_i}(U_i))_{\mathfrak{m}}$. The equivariant Poincaré pairing $\langle \bullet | \bullet \rangle$ of $A^*_{\mathbb{T}}(\mathrm{Hilb}^{n_i}(U_i))_{\mathfrak{m}}$ is determined by the formula

$$\langle \mathfrak{a}_{\widetilde{\lambda_i}} | \mathfrak{a}_{\widetilde{\mu_i}} \rangle = \delta_{\lambda_i,\mu_i} (-1)^{|\lambda_i| - \ell(\lambda_i)} (L_i R_i)^{\ell(\lambda_i)} \frac{1}{\mathfrak{z}_{\lambda_i}}, \ |\lambda_i| = |\mu_i| = n_i. \qquad (2.1.1)$$

2.2 Comparison to symmetric functions

Let $p_i(z) = \sum_{k=1}^{\infty} z_k^i$ be the i^{th} power sum. Given a partition μ, write

$$p_\mu(z) = \frac{1}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{\ell(\mu)} \frac{1}{\mu_i} p_{\mu_i}(z).$$

This family of symmetric functions forms a basis for the ring R_{Sym} of symmetric functions over $\mathbb{Q}(t_1, t_2)$. Let $\alpha_i = R_i/L_i$. We denote the integral Jack symmetric functions corresponding to α_i and the partitions μ by

$$J_\mu^{\alpha_i}(z),$$

which actually provide an orthogonal basis for R_{Sym} (cf. [S]).

The relationship between $\mathbb{T}$-fixed point basis and Nakajima basis is indeed the relationship between Jack symmetric functions and the power sums. More precisely, the Nakajima basis element $\mathfrak{a}_{(\lambda_1,\ldots,\lambda_s)}$ is identified with

$$\otimes_{i=1}^{s} L_i^{\ell(\lambda_i)} p_{\lambda_i}(z(i)),$$

while the $\mathbb{T}$-fixed point class $I_{(\mu_1,\ldots,\mu_s)}$ is identified with

$$\otimes_{i=1}^{s} L_i^{|\mu_i|} J_{\mu_i}^{\alpha_i}(z(i)).$$

For more details, see [N1], [V] or [LQW].

For $i = 1, \ldots, s$, let λ_i be a partition of n_i. As the fixed-point classes $[\mu_i(x_i)]$'s (with $|\mu_i| = n_i$) span $A^*_{\mathbb{T}}(\mathrm{Hilb}^{n_i}(U_i))_{\mathfrak{m}}$, we can write $\mathfrak{a}_{\widetilde{\lambda_i}} = \sum_{|\mu_i|=n_i} c_{\lambda_i,\mu_i}[\mu_i(x_i)]$ for some $c_{\lambda_i,\mu_i} \in \mathbb{Q}(t_1, t_2)$. By the above identifications,

$$\mathfrak{a}_{(\lambda_1,\ldots,\lambda_s)} = \sum_{|\mu_i|=n_i;i=1,\ldots,s} c_{\lambda_1,\mu_1} \cdots c_{\lambda_s,\mu_s} I_{(\mu_1,\ldots,\mu_s)}. \tag{2.2.1}$$

2.3 Three-point functions

Extremal Gromov-Witten invariants. The kernel of the morphism

$$\rho_* : A_1(\mathrm{Hilb}^n(S); \mathbb{Z}) \to A_1(\mathrm{Sym}^n(S); \mathbb{Z})$$

is one-dimensional and is generated by an effective rational curve class β_n that is dual to $-a_2(1)\mathfrak{a}_1(1)^{n-2}$. For every positive integer k, we let

$$\overline{M}_{0,k}(\mathrm{Hilb}^n(S), d)$$

be the moduli space parametrizing stable maps from genus zero, k-pointed, nodal curves to $\mathrm{Hilb}^n(S)$ of degree $d\beta_n$.

Let $e_i : \overline{M}_{0,k}(\mathrm{Hilb}^n(S), d) \to \mathrm{Hilb}^n(S)$ be the evaluation map at the i^{th} marked point. Although $\mathrm{Hilb}^n(S)$ is not necessarily compact, the k-point, $\mathbb{T}$-equivariant, extremal Gromov-Witten invariant

$$\langle \alpha_1, \ldots, \alpha_k \rangle_d^{\mathrm{Hilb}^n(S)} := \int_{[\overline{M}_{0,k}(\mathrm{Hilb}^n(S),d)]^{\mathrm{vir}}_{\mathbb{T}}} e_1^*(\alpha_1) \cdots e_k^*(\alpha_k) \tag{2.3.1}$$

is well-defined when all α_i's are $\mathbb{T}$-fixed point classes because the space of stable maps meeting these classes is compact. Since $\mathbb{T}$-fixed point basis spans the cohomology $A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S))_{\mathfrak{m}}$, the invariant $\langle \alpha_1, \ldots, \alpha_k \rangle_d^{\mathrm{Hilb}^n(S)}$ with insertions being any classes on $\mathrm{Hilb}^n(S)$ can be defined by writing each α_i in terms of fixed-point classes and by linearity. Another interpretation of (2.3.1) is to treat the integral as a sum of residue integrals over $\mathbb{T}$-fixed connected components of $\overline{M}_{0,k}(\mathrm{Hilb}^n(S), d)$ via virtual localization formula. The invariants in both treatments take values in $\mathbb{Q}(t_1, t_2)$.

Extremal quantum product. We will explore the following three-point function of $\mathrm{Hilb}^n(S)$:

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle^{\mathrm{Hilb}^n(S)}(q) := \sum_{d=0}^{\infty} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_d^{\mathrm{Hilb}^n(S)} q^d.$$

Let $\{\epsilon\}$ be a basis for $A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S))_{\mathfrak{m}}$ and $\{\epsilon^\vee\}$ its dual basis. Define the extremal quantum product $\cup_{\mathrm{crep}}$ for $A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S))_{\mathfrak{m}}$ as follows: Given any classes $a, b \in A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S))_{\mathfrak{m}}$,

$$a \cup_{\mathrm{crep}} b := \sum_{\epsilon} \langle a, b, \epsilon \rangle^{\mathrm{Hilb}^n(S)}(q)\ \epsilon^\vee.$$

Equivalently, $a \cup_{\mathrm{crep}} b$ is defined to be the unique element satisfying

$$\langle a \cup_{\mathrm{crep}} b | c \rangle = \langle a, b, c \rangle^{\mathrm{Hilb}^n(S)}(q), \ \forall c \in A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S))_{\mathfrak{m}}.$$

By extending scalars, the vector space $A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S)) \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1, t_2)((q))$ endowed with the multiplication $\cup_{\mathrm{crep}}$ is referred to as the extremal quantum cohomology ring of $\mathrm{Hilb}^n(S)$.

2.4 The product formula

For each $\mathbb{T}$-fixed connected component Γ, we denote by $\gamma : \Gamma \to \overline{M}_{0,3}(\mathrm{Hilb}^n(S), d)$ the natural inclusion and by N_Γ^{vir} the virtual normal bundle to Γ.

In this section, we are going to express our three-point invariant in $\mathfrak{a}_{\widetilde{\lambda}}, \mathfrak{a}_{\widetilde{\mu}}, \mathfrak{a}_{\widetilde{\nu}}$ in terms of Gromov-Witten invariants of Hilbert schemes of points in the affine plane. First of all, let us see a vanishing statement.

Proposition 2.4.1. $\langle \mathfrak{a}_{\widetilde{\lambda}}, \mathfrak{a}_{\widetilde{\mu}}, \mathfrak{a}_{\widetilde{\nu}} \rangle_d^{\mathrm{Hilb}^n(S)}$ *does not vanish only if*

$$|\lambda_i| = |\mu_i| = |\nu_i| \text{ for each } i = 1, \ldots, s. \tag{2.4.1}$$

Proof. Suppose (2.4.1) fails, we would like to show

$$\gamma^*(e_1^*(\mathfrak{a}_{\widetilde{\lambda}}) \cdot e_2^*(\mathfrak{a}_{\widetilde{\mu}}) \cdot e_3^*(\mathfrak{a}_{\widetilde{\nu}})) = 0$$

for every connected component Γ. By (2.2.1), we merely have to verify

$$\gamma^*(e_1^*(I_{\widetilde{\sigma}}) \cdot e_2^*(I_{\widetilde{\tau}}) \cdot e_3^*(I_{\widetilde{\theta}})) = 0 \tag{2.4.2}$$

for $|\sigma_i| = |\lambda_i|, |\tau_i| = |\mu_i|, |\theta_i| = |\nu_i|$, $\forall i = 1, 2, 3$. We note that the images of all $\mathbb{T}$-fixed stable maps in Γ go to the same point after composition with the Hilbert-Chow morphism. That is, $\rho \circ e_i(\Gamma)$'s are the same for each i, which means that at least one of $e_1^{-1}(\sigma_1(x_1) + \cdots + \sigma_s(x_s)), e_2^{-1}(\tau_1(x_1) + \cdots + \tau_s(x_s)), e_3^{-1}(\theta_1(x_1) + \cdots + \theta_s(x_s))$ does not meet Γ. Thus, (2.4.2) follows. □

It remains to study the three-point invariants

$$\left\langle \mathfrak{a}_{\widetilde{\lambda}}, \mathfrak{a}_{\widetilde{\mu}}, \mathfrak{a}_{\widetilde{\nu}} \right\rangle_d^{\mathrm{Hilb}^n(S)}$$

under condition (2.4.1): $n_i := |\lambda_i| = |\mu_i| = |\nu_i|$ for each $i = 1, \ldots, s$ and $\sum_{i=1}^s n_i = n$. We fix such n_i and partitions λ_i, μ_i, ν_i of n_i throughout the remainder of this section. Let

$$U = \mathrm{Hilb}^{n_1}(U_1) \times \cdots \times \mathrm{Hilb}^{n_s}(U_s),\ P = \rho^{-1}(n_1[x_1] + \cdots + n_s[x_s]).$$

In fact, $P \cong \rho_1^{-1}(n_1[x_1]) \times \cdots \times \rho_s^{-1}(n_s[x_s]) \subseteq U$, where $\rho_i : \mathrm{Hilb}^{n_i}(S) \to \mathrm{Sym}^{n_i}(S)$ is the Hilbert-Chow morphism, $\forall i$. (In case $n_i = 0$, $\rho_i^{-1}(n_i[x_i])$ will be missing from the product).

Let $N = \{i = 1, \ldots, s : n_i \geq 1\}$. As each $\rho_i^{-1}(n_i[x_i])$ is irreducible and has complex dimension $n_i - 1$ for $i \in N$, P is then irreducible and has dimension $n - |N|$.

Let $\xi = \mu_1(x_1) + \cdots + \mu_s(x_s) \in S^{[n]}$ and $\xi_{\widetilde{\mu}} = \mu_1(x_1) \times \cdots \times \mu_s(x_s) \in U$. We have $T_{\xi_{\widetilde{\mu}}} U = T_\xi S^{[n]}$. Indeed,

$$\begin{aligned} T_{\xi_{\widetilde{\mu}}} U &= \bigoplus_{i \in N} \mathrm{Hom}_{\mathcal{O}_{U_i}}(I_{\mu_i(x_i)}, \mathcal{O}_{\mu_i(x_i)}) = \bigoplus_{i \in N} \mathrm{Hom}_{\mathcal{O}_{S,x_i}}(I_{\xi,x_i}, \mathcal{O}_{\xi,x_i}) \\ &= \mathrm{Hom}_{\mathcal{O}_S}(I_\xi, \mathcal{O}_\xi) = T_\xi S^{[n]}. \end{aligned}$$

Denote by $\iota_P, \jmath_P$ the inclusion of P into $\mathrm{Hilb}^n(S)$ and U respectively. We have a simple lemma.

Lemma 2.4.2. $\iota_P^*(\mathfrak{a}_{\widetilde{\lambda}}) = \jmath_P^*(\mathfrak{a}_{\widetilde{\lambda_1}} \otimes \cdots \otimes \mathfrak{a}_{\widetilde{\lambda_s}})$.

Proof. By (2.2.1), it suffices to show that

$$\iota_P^* I_{(\mu_1, \ldots, \mu_s)} = \jmath_P^*([\mu_1(x_1)] \otimes \cdots \otimes [\mu_s(x_s)]). \tag{2.4.3}$$

Let ξ and $\xi_{\widetilde{\mu}}$ be the points as in the discussion preceding the lemma. We can see that the left side of (2.4.3) is given by

$$\sum_{\eta \in P^{\mathbb{T}}} i_{\eta *}\Big(\frac{(\iota_P \circ i_\eta)^* I_{(\mu_1,\ldots,\mu_s)}}{e_{\mathbb{T}}(N_{\eta/P})}\Big) = i_{\xi *}\Big(\frac{e_{\mathbb{T}}(T_\xi S^{[n]})}{e_{\mathbb{T}}(N_{\xi/P})}\Big),$$

where $i_\eta : \{\eta\} \to P$ is the natural inclusion. Similarly, the right side of (2.4.3) coincides with $i_{\xi_{\widetilde{\mu}} *}(e_{\mathbb{T}}(T_{\xi_{\widetilde{\mu}}} U)/e_{\mathbb{T}}(N_{\xi_{\widetilde{\mu}}/P}))$. Thus, the equality (2.4.3) follows from $T_{\xi_{\widetilde{\mu}}} U = T_\xi S^{[n]}$. □

To determine the three-point invariant $\langle \mathfrak{a}_{\widetilde{\lambda}}, \mathfrak{a}_{\widetilde{\mu}}, \mathfrak{a}_{\widetilde{\nu}} \rangle_d^{\mathrm{Hilb}^n(S)}$, we only need to consider those connected components of $\overline{M}_{0,3}(\mathrm{Hilb}^n(S), d)^{\mathbb{T}}$ whose images under the map $\rho \circ e_i$ are the point

$$n_1[x_1] + \cdots + n_s[x_s], \ \forall i = 1, 2, 3.$$

Observe that any $\mathbb{T}$-fixed stable map f, representing an element in these components, factors through P. Hence, we may calculate $\langle \mathfrak{a}_{\widetilde{\lambda}}, \mathfrak{a}_{\widetilde{\mu}}, \mathfrak{a}_{\widetilde{\nu}} \rangle_d$ over connected components lying in

$$\coprod_{d_1+\cdots+d_s=d} \overline{M}_{0,3}(P, (d_1, \ldots, d_s))^{\mathbb{T}}.$$

Write $\Gamma_{d_1,\ldots,d_s}$ for Γ whenever Γ is contained in the moduli space $\overline{M}_{0,3}(P, (d_1, \ldots, d_s))$. On the other hand, we also note that $\Gamma_{d_1,\ldots,d_s}$'s form a complete set of $\mathbb{T}$-fixed connected components of $\overline{M}_{0,3}(U, (d_1, \ldots, d_s))$. The following diagram summarizes their relationships:

$$\begin{array}{ccccc} & & \Gamma_{d_1,\ldots,d_s} & & \\ & \swarrow^{\gamma_U} & \downarrow{\scriptstyle \gamma_P} & \searrow^{\gamma} & \\ \overline{M}_{0,3}(U, (d_1, \ldots, d_s)) & \hookleftarrow & \overline{M}_{0,3}(P, (d_1, \ldots, d_s)) & \hookrightarrow & \overline{M}_{0,3}(\mathrm{Hilb}^n(S), d_1 + \cdots + d_s) \end{array}$$

Here γ_U, γ_P are the natural inclusion γ with the target replaced with $\overline{M}_{0,3}(U, (d_1, \ldots, d_s))$ and $\overline{M}_{0,3}(P, (d_1, \ldots, d_s))$ respectively.

Proposition 2.4.3. *The extremal invariants of* $\mathrm{Hilb}(S)$ *can be expressed in terms of the invariants of* $\mathrm{Hilb}(U_i)$*'s. Precisely,*

$$\langle \mathfrak{a}_{\widetilde{\lambda}}, \mathfrak{a}_{\widetilde{\mu}}, \mathfrak{a}_{\widetilde{\nu}} \rangle_d^{\mathrm{Hilb}^n(S)} = \sum_{d_1+\cdots+d_s=d} \prod_{i=1}^{s} \Big\langle \mathfrak{a}_{\widetilde{\lambda_i}}, \mathfrak{a}_{\widetilde{\mu_i}}, \mathfrak{a}_{\widetilde{\nu_i}} \Big\rangle_{d_i}^{\mathrm{Hilb}^{n_i}(U_i)}.$$

Proof. First of all, we would like to show that

$$\left\langle \mathfrak{a}_{\widetilde{\lambda}}, \mathfrak{a}_{\widetilde{\mu}}, \mathfrak{a}_{\widetilde{\nu}} \right\rangle_d^{\mathrm{Hilb}^n(S)} = \sum_{d_1+\cdots+d_s=d} \langle \otimes_{i=1}^s \mathfrak{a}_{\widetilde{\lambda_i}}, \otimes_{i=1}^s \mathfrak{a}_{\widetilde{\mu_i}}, \otimes_{i=1}^s \mathfrak{a}_{\widetilde{\nu_i}} \rangle^U_{(d_1,\dots,d_s)}. \tag{2.4.4}$$

Denote by $\bar{e}_i : \overline{M}_{0,3}(P,(d_1,\dots,d_s)) \to P$ the evaluation map at the i^{th} marked point. With notation as in the above discussion, the invariant $\left\langle \mathfrak{a}_{\widetilde{\lambda}}, \mathfrak{a}_{\widetilde{\mu}}, \mathfrak{a}_{\widetilde{\nu}} \right\rangle_d^{\mathrm{Hilb}^n(S)}$ is

$$\sum_{d_1+\cdots+d_s=d} \sum_{\Gamma_{d_1,\dots,d_s}} \int_{\Gamma_{d_1,\dots,d_s}} \frac{\gamma_P^*(\bar{e}_1^*\iota_P^*(\mathfrak{a}_{\widetilde{\lambda}}) \cdot \bar{e}_2^*\iota_P^*(\mathfrak{a}_{\widetilde{\mu}}) \cdot \bar{e}_3^*\iota_P^*(\mathfrak{a}_{\widetilde{\nu}}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\Gamma_{d_1,\dots,d_s}})}.$$

On the other hand, we find that $\langle \otimes_{i=1}^s \mathfrak{a}_{\widetilde{\lambda_i}}, \otimes_{i=1}^s \mathfrak{a}_{\widetilde{\mu_i}}, \otimes_{i=1}^s \mathfrak{a}_{\widetilde{\nu_i}} \rangle^U_{(d_1,\dots,d_s)}$ is

$$\sum_{\Gamma_{d_1,\dots,d_s}} \int_{\Gamma_{d_1,\dots,d_s}} \frac{\gamma_P^*(\bar{e}_1^*\jmath_P^*(\otimes_{i=1}^s \mathfrak{a}_{\widetilde{\lambda_i}}) \cdot \bar{e}_2^*\jmath_P^*(\otimes_{i=1}^s \mathfrak{a}_{\widetilde{\mu_i}}) \cdot \bar{e}_3^*\jmath_P^*(\otimes_{i=1}^s \mathfrak{a}_{\widetilde{\nu_i}}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\Gamma_{d_1,\dots,d_s},U})},$$

where $N^{\mathrm{vir}}_{\Gamma_{d_1,\dots,d_s},U}$ is the virtual normal bundle to $\Gamma_{d_1,\dots,d_s}$ in $\overline{M}_{0,3}(U,(d_1,\dots,d_s))$. By Lemma 2.4.2, it is given by

$$\sum_{\Gamma_{d_1,\dots,d_s}} \int_{\Gamma_{d_1,\dots,d_s}} \frac{\gamma_P^*(\bar{e}_1^*\iota_P^*(\mathfrak{a}_{\widetilde{\lambda}}) \cdot \bar{e}_2^*\iota_P^*(\mathfrak{a}_{\widetilde{\mu}}) \cdot \bar{e}_3^*\iota_P^*(\mathfrak{a}_{\widetilde{\nu}}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\Gamma_{d_1,\dots,d_s},U})}.$$

Hence (2.4.4) is a consequence of the equality

$$\frac{1}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\Gamma_{d_1,\dots,d_s}})} = \frac{1}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\Gamma_{d_1,\dots,d_s},U})} \text{ for each } \Gamma_{d_1,\dots,d_s}. \tag{2.4.5}$$

Now we prove (2.4.5). Given any $\mathbb{T}$-fixed stable map $[f : (\Sigma, p_1, p_2, p_3) \to P]$ in the $\mathbb{T}$-fixed connected component $\Gamma_{d_1,\dots,d_s}$. For

$$X = S^{[n]} \quad \text{or} \quad X = U,$$

in order to verify (2.4.5), we need only examine the infinitesimal deformations of f with the source curves fixed. In fact, it suffices to check two things:

$$\frac{e_{\mathbb{T}}(H^0(\Sigma_v, f^*TX)^{\mathrm{mov}})}{e_{\mathbb{T}}(H^1(\Sigma_v, f^*TX)^{\mathrm{mov}})} \text{ and } \frac{e_{\mathbb{T}}(H^0(\Sigma_e, f^*TX)^{\mathrm{mov}})}{e_{\mathbb{T}}(H^1(\Sigma_e, f^*TX)^{\mathrm{mov}})}$$

are independent of X for every connected contracted component Σ_v and noncontracted irreducible component Σ_e. The first independence holds due to the fact that Σ_v is of genus 0 and

$T_{f(\Sigma_v)}S^{[n]} = T_{f(\Sigma_v)}U$. Thus, it remains to justify the second independence. Since $\Sigma_e \cong \mathbb{P}^1$, f^*TX is a direct sum of line bundles over Σ_e, i.e., $f^*TX = \bigoplus_{i=1}^{2n} \mathcal{O}_{\Sigma_e}(\ell_i^X)$ for some integers ℓ_i^X's, and we also have

$$\frac{e_{\mathbb{T}}(H^0(\Sigma_e, f^*TX)^{\mathrm{mov}})}{e_{\mathbb{T}}(H^1(\Sigma_e, f^*TX)^{\mathrm{mov}})} = \prod_{i=1}^{2n} \frac{e_{\mathbb{T}}(H^0(\Sigma_e, \mathcal{O}_{\Sigma_e}(\ell_i^X))^{\mathrm{mov}})}{e_{\mathbb{T}}(H^1(\Sigma_e, \mathcal{O}_{\Sigma_e}(\ell_i^X))^{\mathrm{mov}})}. \tag{2.4.6}$$

Note that the $\mathbb{T}$-action on $f(\Sigma_e)$ (independent of X) induces a natural action on Σ_e and hence actions on $\mathcal{O}_{\Sigma_e}(\ell_i^X)$'s. Suppose that $\mathbb{T}$ acts on $\mathcal{O}_{\Sigma_e}(1)|_{p_0}$ with weight w, then the $\mathbb{T}$-weight of $\mathcal{O}_{\Sigma_e}(\ell)|_{p_0}$ is ℓw. This means that $T_{f(p_0)}X$ comes with weights $\ell_1^X w, \ldots, \ell_{2n}^X w$. But $T_{f(p_0)}S^{[n]} = T_{f(p_0)}U$, so the numbers $\ell_1^{S^{[n]}}, \ldots, \ell_{2n}^{S^{[n]}}$ are exactly $\ell_1^U, \ldots, \ell_{2n}^U$ after a suitable reordering. By (2.4.6), this shows the independence and ends the proof of (2.4.5).

Further, the equality

$$\langle \otimes_{i=1}^s \mathfrak{a}_{\widetilde{\lambda_i}}, \otimes_{i=1}^s \mathfrak{a}_{\widetilde{\mu_i}}, \otimes_{i=1}^s \mathfrak{a}_{\widetilde{\nu_i}} \rangle^U_{(d_1,\ldots,d_s)} = \prod_{i=1}^s \left\langle \mathfrak{a}_{\widetilde{\lambda_i}}, \mathfrak{a}_{\widetilde{\mu_i}}, \mathfrak{a}_{\widetilde{\nu_i}} \right\rangle^{\mathrm{Hilb}^{n_i}(U_i)}_{d_i}$$

holds. This is clear for $(d_1, \ldots, d_s) = (0, \ldots, 0)$; in general, the result follows from the Product formula [Be] in equivariant context. Combining it with (2.4.4), we obtain the proposition. □

Proposition 2.4.4. *The three-point function* $\left\langle \mathfrak{a}_{\widetilde{\lambda}}, \mathfrak{a}_{\widetilde{\mu}}, \mathfrak{a}_{\widetilde{\nu}} \right\rangle^{\mathrm{Hilb}^n(S)}(q)$ *is an element of* $\mathbb{Q}(t_1, t_2, q)$ *and is given by*

$$\prod_{i=1}^s \left\langle \mathfrak{a}_{\widetilde{\lambda_i}}, \mathfrak{a}_{\widetilde{\mu_i}}, \mathfrak{a}_{\widetilde{\nu_i}} \right\rangle^{\mathrm{Hilb}^{n_i}(U_i)}(q).$$

Proof. Each term $\langle \mathfrak{a}_{\widetilde{\lambda_i}}, \mathfrak{a}_{\widetilde{\mu_i}}, \mathfrak{a}_{\widetilde{\nu_i}} \rangle^{\mathrm{Hilb}^{n_i}(U_i)}(q)$ is a rational function in t_1, t_2, q (cf. [OP1]). By Propositions 2.4.1 and 2.4.3,

$$\begin{aligned}
\left\langle \mathfrak{a}_{\widetilde{\lambda}}, \mathfrak{a}_{\widetilde{\mu}}, \mathfrak{a}_{\widetilde{\nu}} \right\rangle^{\mathrm{Hilb}^n(S)}(q) &= \sum_{d=0}^{\infty} \Big(\sum_{d_1+\ldots+d_s=d} \prod_{i=1}^s \left\langle \mathfrak{a}_{\widetilde{\lambda_i}}, \mathfrak{a}_{\widetilde{\mu_i}}, \mathfrak{a}_{\widetilde{\nu_i}} \right\rangle^{\mathrm{Hilb}^{n_i}(U_i)}_{d_i} \Big) q^d \\
&= \prod_{i=1}^s \sum_{d_i=0}^{\infty} \left\langle \mathfrak{a}_{\widetilde{\lambda_i}}, \mathfrak{a}_{\widetilde{\mu_i}}, \mathfrak{a}_{\widetilde{\nu_i}} \right\rangle^{\mathrm{Hilb}^{n_i}(U_i)}_{d_i} q^{d_i} \\
&= \prod_{i=1}^s \left\langle \mathfrak{a}_{\widetilde{\lambda_i}}, \mathfrak{a}_{\widetilde{\mu_i}}, \mathfrak{a}_{\widetilde{\nu_i}} \right\rangle^{\mathrm{Hilb}^{n_i}(U_i)}(q),
\end{aligned}$$

as desired. □

Together with (2.1.1), we have the following result on Poincaré pairings.

Proposition 2.4.5. *The equivariant Poincaré pairing on* $\mathrm{Hilb}^n(S)$ *can be written in terms of* *the pairings on the Hilbert schemes of points in the affine plane. More precisely,*

$$\langle \mathfrak{a}_{\widetilde{\mu}} | \mathfrak{a}_{\widetilde{\nu}} \rangle = \prod_{i=1}^{s} \delta_{\mu_i,\nu_i} (-1)^{|\mu_k|-\ell(\mu_k)} (L_i R_i)^{\ell(\mu_i)} \frac{1}{\mathfrak{z}_{\mu_i}}.$$

In other words, $\mathfrak{a}_{\widetilde{\mu}}$*'s give an orthogonal basis for* $A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S))_{\mathfrak{m}}$. □

Chapter 3

Symmetric Product Stack

Let X be a smooth complex variety. Given any finite set N, let $\mathfrak{S}_N$ be the symmetric group on N and

$$X^N = \{(x_i)_{i\in N} : x_i\text{'s are elements of } X\},$$

a set of $|N|$-tuples of elements of X. We denote by $\mathfrak{S}_n$ the group $\mathfrak{S}_{\{1,\ldots,n\}}$ and by X^n the set $X^{\{1,\ldots,n\}}$.

The symmetric group $\mathfrak{S}_n$ acts on the n-fold product X^n by permutation of coordinates. The quotient scheme $\mathrm{Sym}^n(X) := X^n/\mathfrak{S}_n$ is referred to as the n-fold symmetric product of X and is the coarse moduli space of the quotient stack $[\mathrm{Sym}^n(X)]$ defined as follows:

- An object over U is a pair $(p : P \to U, f : P \to X^n)$ where p is a principal $\mathfrak{S}_n$-bundle, and f is a $\mathfrak{S}_n$-equivariant morphism.

- Suppose that $(p' : P' \to U', f' : P' \to X^n)$ is another object, a morphism from (p', f') to (p, f) is a Cartesian diagram

$$\begin{array}{ccc} P' & \xrightarrow{\alpha} & P \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ U' & \xrightarrow{\beta} & U \end{array}$$

  such that $f' = \alpha \circ f$.

Note that the stack $[\mathrm{Sym}^n(X)]$ is an orbifold with atlas $X^n \to [X^n/\mathfrak{S}_n]$.

3.1 Inertia stack

There is a natural stack associated to the symmetric product, i.e., the inertia stack

$$I[\mathrm{Sym}^n(X)] := \coprod_{s\in\mathbb{N}} HomRep(\mathcal{B}\mu_s, [\mathrm{Sym}^n(X)]),$$

where $HomRep(\mathcal{B}\mu_s, [\mathrm{Sym}^n(X)])$ is the stack of representable morphisms from the classifying stack $\mathcal{B}\mu_s$ to $[\mathrm{Sym}^n(X)]$. Moreover, $I[\mathrm{Sym}^n(X)]$ is isomorphic to the disjoint union of orbifolds

$$\coprod_{[g]\in C} [X^n_g/C(g)], \tag{3.1.1}$$

where C is the set of conjugacy classes, X^n_g is the g-fixed locus of X^n, and $C(g)$ is the centralizer of g. The component $[X^n/\mathfrak{S}_n]$ is called the untwisted sector while all other components are called twisted sectors. As there is a one-to-one correspondence between the conjugacy classes of $\mathfrak{S}_n$ and the partitions of n, these sectors can be labeled with the partitions of n. If $[g]$ is the conjugacy class corresponds to the partition λ and $\overline{C(g)} := C(g)/\langle g\rangle$, we may write

$$X(\lambda) := X^n_g/C(g), \text{ and } \overline{X(\lambda)} := X^n_g/\overline{C(g)} \text{ (see below).}$$

The Chen-Ruan cohomology

$$A^*_{\mathrm{orb}}([\mathrm{Sym}^n(X)])$$

is by definition the cohomology $A^*(I[\mathrm{Sym}^n(X)])$ of the inertia stack ([ChR1]). By (3.1.1), it is

$$\bigoplus_{[g]\in C} A^*(X^n_g/C(g)) = \bigoplus_{[g]\in C} A^*(X^n_g)^{C(g)}.$$

(For any orbifold $\mathcal{Y}$ with coarse moduli space Y, we identify $A^*(\mathcal{Y}) = A^*(Y)$ by the pushforward $c_* : A^*(\mathcal{Y}) \to A^*(Y)$ defined by $c_*([\mathcal{V}]) = \frac{1}{s}[c(\mathcal{V})]$, where $\mathcal{V}$ is a closed integral substack and s is the order of the stabilizer of a generic geometric point of $\mathcal{V}$).

The age (or the degree shifting number) of the sector $[X(\lambda)]$ is given by

$$\mathrm{age}(\lambda) := n - \ell(\lambda).$$

Additionally, the Chen-Ruan cohomology is graded by ages. If $\alpha \in A^i(X(\lambda))$, the orbifold

(Chow) degree of α is defined to be $i+\mathrm{age}(\lambda)$. In other words,

$$A^*_{\mathrm{orb}}([\mathrm{Sym}^n(X)]) = \bigoplus_{|\lambda|=n} A^{*-\mathrm{age}(\lambda)}(X(\lambda)).$$

We may rigidify the inertia stack to remove the actions of μ_s's. Each $\mathcal{B}\mu_s$ acts on the stack $HomRep(\mathcal{B}\mu_s, [\mathrm{Sym}^n(X)])$, and the quotient by this action is a stack of gerbes banded by μ_s to $[\mathrm{Sym}^n(X)]$. The stack

$$\overline{I}[\mathrm{Sym}^n(X)] := \coprod_{s\in\mathbb{N}} HomRep(\mathcal{B}\mu_s, [\mathrm{Sym}^n(X)])/\mathcal{B}\mu_s,$$

is called the rigidified inertia stack of $[\mathrm{Sym}^n(X)]$. One of the reasons why we mention this is that the rigidified stack is where the evaluation maps land (see (3.3.1)).

For more details on the rigidification procedure, consult [ACV, AGV1, AGV2]. In fact, the procedure amounts to removing the action of the permutation g from (3.1.1) (note that g acts trivially on X^n_g). Concretely, the stack $\overline{I}[\mathrm{Sym}^n(X)]$ is the disjoint union

$$\coprod_{[g]\in C} [X^n_g/\overline{C(g)}] \text{ (or } \coprod_{|\lambda|=n} [\overline{X(\lambda)}]).$$

However, its coarse moduli space $\coprod_{[g]\in C} X^n_g/C(g)$ is identical to that of the inertia stack.

3.2 Bases

3.2.1 A description

Assume that X admits a $\mathbb{T}$-action. We can see easily that there are induced $\mathbb{T}$-actions on the spaces $X^n_g/C(g)$ ($\forall g \in \mathfrak{S}_n$) and $I[\mathrm{Sym}^n(X)]$. So we may put the above cohomologies into an equivariant context by considering $\mathbb{T}$-equivariant cohomologies.

Now we would like to understand the module structure of the equivariant Chen-Ruan cohomology $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])$; particularly, we need a precise description of bases.

Given a partition λ of n, we would like to give a basis for the cohomology $A^*_{\mathbb{T}}(X^n_g)^{C(g)}$, where $g \in \mathfrak{S}_n$ has cycle type λ. The permutation g has a cycle decomposition, i.e., a product of disjoint cycles (including 1-cycles),

$$g = g_1 \cdots g_{\ell(\lambda)},$$

with g_i being a λ_i-cycle. For each i, let N_i be the minimal subset of $\{1,\ldots,n\}$ such that $g_i \in \mathfrak{S}_{N_i}$. Thus $|N_i| = \lambda_i$ and $\coprod_{i=1}^{\ell(\lambda)} N_i = \{1,\ldots,n\}$. It is clear that

$$X_g^n = \prod_{i=1}^{\ell(\lambda)} X_{g_i}^{N_i}, \text{ and } X_{g_i}^{N_i} \cong X.$$

To the partition λ, we associate an $\ell(\lambda)$-tuple $\vec{\eta} = (\eta_1 \ldots \eta_{\ell(\lambda)})$ with entries in $A_{\mathbb{T}}^*(X)$. Let us put

$$g(\vec{\eta}) = (|\mathrm{Aut}(\lambda(\vec{\eta}))| \prod_{i=1}^{\ell(\lambda)} \lambda_i)^{-1} \sum_{h \in C(g)} \bigotimes_{i=1}^{\ell(\lambda)} g_i^h(\eta_i) \in A_{\mathbb{T}}^*(X_g^n)^{C(g)}. \tag{3.2.1}$$

This requires some explanations:

- $g_i^h := h^{-1} g_i h$.
- Let N be a subset of $\{1,\ldots,n\}$. For each $|N|$-cycle $\alpha \in \mathfrak{S}_N$ and η a class on X, let $\alpha(\eta)$ be the pullback of η by the obvious isomorphism $X_\alpha^N \cong X$.
- Two classes $\bigotimes_{i=1}^{\ell(\lambda)} g_i^{h_1}(\eta_i)$ and $\bigotimes_{i=1}^{\ell(\lambda)} g_i^{h_2}(\eta_i)$ on X_g^n coincide for some $h_1, h_2 \in C(g)$, and a straightforward verification shows that each term $\bigotimes_{i=1}^{\ell(\lambda)} g_i^h(\eta_i)$ repeats precisely $|\mathrm{Aut}(\lambda(\vec{\eta}))| \prod_{i=1}^{\ell(\lambda)} \lambda_i$ times. Hence, $(|\mathrm{Aut}(\lambda(\vec{\eta}))| \prod_{i=1}^{\ell(\lambda)} \lambda_i)^{-1}$ is a normalization factor to ensure that no repetition occurs in (3.2.1).
- If $g = k_1 \cdots k_{\ell(\lambda)}$ is another cycle decomposition with λ_i-cycles k_i's, then there exists $h \in C(g)$ such that
$$\bigotimes_{i=1}^{\ell(\lambda)} k_i(\eta_i) = \bigotimes_{i=1}^{\ell(\lambda)} g_i^h(\eta_i).$$

  Thus, the expression (3.2.1) is independent of the cycle decomposition.

Let $\mathfrak{B}$ be a basis for $A_{\mathbb{T}}^*(X)$. The classes $g(\vec{\eta})$'s, with η_i's elements of $\mathfrak{B}$, form a basis for $A_{\mathbb{T}}^*(X_g^n)^{C(g)}$.

Suppose that $\hat{g}$ is another permutation of cycle type λ, i.e., $\hat{g} = g^\alpha$ for some $\alpha \in \mathfrak{S}_n$. The classes $g(\vec{\eta})$ and $\hat{g}(\vec{\eta})$ are identical in $A_{\mathbb{T},\mathrm{orb}}^*([\mathrm{Sym}^n(X)])$ though they are related by ring isomorphism $\alpha^* : A_{\mathbb{T}}^*(X_g^n)^{C(g)} \to A_{\mathbb{T}}^*(X_{\hat{g}}^n)^{C(\hat{g})}$ induced by α. In fact, α^* sends $g(\vec{\eta})$ to $\hat{g}(\vec{\eta})$ and is independent of the choice of α due to the fact that $\vartheta^* : A_{\mathbb{T}}^*(X_g^n)^{C(g)} \to A_{\mathbb{T}}^*(X_g^n)^{C(g)}$ is the identity for every $\vartheta \in C(g)$.

We use the cohomology-weighted partition

$$\lambda_1(\eta_1) \cdots \lambda_{\ell(\lambda)}(\eta_{\ell(\lambda)}) \quad \text{or simply} \quad \lambda(\vec{\eta})$$

to denote the class $g(\vec{\eta})$ (and hence $\hat{g}(\vec{\eta})$).

Now the classes $\lambda(\vec{\eta})$'s, running over all partitions λ of n and all $\eta_i \in \mathfrak{B}$, serve as a basis for the Chen-Ruan cohomology $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])$. For classes $\lambda(\vec{\eta}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])$ and $\rho(\vec{\xi}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(X)])$, keep in mind that the class

$$\lambda_1(\eta_1)\cdots\lambda_{\ell(\lambda)}(\eta_{\ell(\lambda)})\rho_1(\xi_1)\cdots\rho_{\ell(\rho)}(\xi_{\ell(\rho)}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^{n+m}(X)])$$

is denoted by

$$\lambda(\vec{\eta})\rho(\vec{\xi}).$$

We use the shorthand

$$(2)$$

for the divisor class $2(1)1(1)^{n-2}$. Also, we define the age of $\lambda(\vec{\eta})$, denoted by

$$\mathrm{age}(\lambda(\vec{\eta})),$$

to be the age of the sector $[X(\lambda)]$, i.e., $n-\ell(\lambda)$.

3.2.2 Fixed-point classes

We can work with $\lambda(\vec{\eta})$'s with η_k's in the localized cohomology $A^*_{\mathbb{T}}(X)_{\mathfrak{m}}$ to give a basis for $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])_{\mathfrak{m}}$.

Assume that X has exactly p $\mathbb{T}$-fixed points $z_1,\ldots,z_p$. For partitions $\sigma_1,\ldots,\sigma_p$, we denote the class

$$\sigma_{11}([z_1])\cdots\sigma_{1\ell(\sigma_1)}([z_1])\cdots\sigma_{p1}([z_p])\cdots\sigma_{p\ell(\sigma_p)}([z_p])$$

by

$$\widetilde{\sigma} := (\sigma_1,\ldots,\sigma_p).$$

The classes $\widetilde{\sigma}$'s form a basis for $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])_{\mathfrak{m}}$. Note also that each $\widetilde{\sigma}$ corresponds to a $\mathbb{T}$-fixed point, which we denote by

$$[\widetilde{\sigma}],$$

in the sector indexed by the partition $(\sigma_{11},\ldots,\sigma_{1\ell(\sigma_1)},\ldots,\sigma_{p1},\ldots,\sigma_{p\ell(\sigma_p)})$. So we refer to $\widetilde{\sigma}$'s as $\mathbb{T}$-fixed point classes.

Moreover, given $\widetilde{\delta} \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])_{\mathfrak{m}}$ and $\widetilde{\sigma} \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(X)])_{\mathfrak{m}}$ $(m \leq n)$, we say that

$$\widetilde{\delta} \supset \widetilde{\sigma}$$

if σ_k is a subpartition of δ_k, $\forall k = 1, \dots, p$; in this case, we let

$$\widetilde{\delta} - \widetilde{\sigma} := (\delta_1 - \sigma_1, \dots, \delta_p - \sigma_p) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^{n-m}(X)])_{\mathfrak{m}}.$$

(e.g., the difference $(1,1,2,2,3) - (1,2,3)$ of two partitions is the partition $(1,2)$.)

$\mathbb{T}$-weights. Given any fixed-point class $\widetilde{\sigma}$, let

$$t(\widetilde{\sigma}) = e_{\mathbb{T}}(T_{[\widetilde{\sigma}]}\bar{I}[\mathrm{Sym}^m(X)]).$$

A simple analysis shows that

$$t(\widetilde{\sigma}) = \prod_{k=1}^{p} e_{\mathbb{T}}(T_{z_k}X)^{\ell(\sigma_k)}. \tag{3.2.2}$$

Thus, for each $\widetilde{\delta} \supset \widetilde{\sigma}$, $t(\widetilde{\delta}) = t(\widetilde{\sigma})t(\widetilde{\delta} - \widetilde{\sigma})$.

Coefficients with respect to fixed-point basis. For $\theta(\vec{\xi}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(X)])_{\mathfrak{m}}$, we have

$$\theta(\vec{\xi}) = \sum_{\widetilde{\sigma}} \frac{\langle \theta(\vec{\xi}) | \widetilde{\sigma} \rangle}{\langle \widetilde{\sigma} | \widetilde{\sigma} \rangle} \widetilde{\sigma},$$

where $\langle \bullet | \bullet \rangle$ are $\mathbb{T}$-equivariant orbifold pairings on $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(X)])_{\mathfrak{m}}$. Now let

$$\alpha_{\theta(\vec{\xi})}(\widetilde{\sigma}) := \frac{\langle \theta(\vec{\xi}) | \widetilde{\sigma} \rangle}{\langle \widetilde{\sigma} | \widetilde{\sigma} \rangle}$$

be the components of $\theta(\vec{\xi})$ relative to $\widetilde{\sigma}$'s. We have two properties by direct verification:

(1) Suppose $\lambda(\vec{\eta})$, $\rho(\vec{\varepsilon}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])_{\mathfrak{m}}$ have explicit forms

$$\prod_{i=1}^{n} \prod_{j=1}^{m_i} i(\eta_{ij}) \text{ and } \prod_{i=1}^{n} \prod_{j=1}^{\ell_i} i(\varepsilon_{ij})$$

respectively, we have

$$\langle \lambda(\vec{\eta}) | \rho(\vec{\varepsilon}) \rangle = \begin{cases} 0 & \text{if } m_i \neq \ell_i \text{ for some } i; \\ \prod_{i=1}^{n} \langle \prod_{j=1}^{m_i} i(\eta_{ij}) \cdot \prod_{j=1}^{m_i} i(\varepsilon_{ij}) \rangle & \text{if } m_i = \ell_i \text{ for each } i. \end{cases}$$

(2) Given $\eta_1, \ldots, \eta_n \in A^*_{\mathbb{T}}(X)_{\mathfrak{m}}$ and $\mathbb{T}$-fixed points $y_1, \ldots, y_n$ of X. For $m \leq n$, the coefficient $\alpha_{i(\eta_1)\cdots i(\eta_n)}(i([y_1]) \cdots i([y_n]))$ equals

$$\sum \alpha_{i(\xi_1)\cdots i(\xi_m)}(i([y_1]) \cdots i([y_m]))\alpha_{i(\xi_{m+1})\cdots i(\xi_n)}(i([y_{m+1}]) \cdots i([y_n])),$$

where the sum is over all possible $i(\xi_1) \cdots i(\xi_m)$ and $i(\xi_{m+1}) \cdots i(\xi_n)$ such that

$$i(\xi_1) \cdots i(\xi_n) = i(\eta_1) \cdots i(\eta_n).$$

We may combine (1) with (2) to get a general statement, which presents an algorithm to calculate the coefficient $\alpha_{\lambda(\vec{\eta})}(\widetilde{\delta})$.

Proposition 3.2.1. *Given* $\lambda(\vec{\eta}), \widetilde{\delta} \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(X)])_{\mathfrak{m}}$ *and* $\widetilde{\sigma} \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(X)])_{\mathfrak{m}}$ *with* $\widetilde{\delta} \supset \widetilde{\sigma}$,

$$\alpha_{\lambda(\vec{\eta})}(\widetilde{\delta}) = \sum_{P} \alpha_{\theta(\vec{\xi})}(\widetilde{\sigma})\alpha_{\mu(\vec{\gamma})}(\widetilde{\delta} - \widetilde{\sigma}), \tag{3.2.3}$$

where the index P *under the summation symbol means that the sum is taken over all possible* $\theta(\vec{\xi}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^m(X)])_{\mathfrak{m}}$ *and* $\mu(\vec{\gamma}) \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^{n-m}(X)])_{\mathfrak{m}}$ *satisfying* $\lambda(\vec{\eta}) = \theta(\vec{\xi})\mu(\vec{\gamma})$.

In the proposition, $\widetilde{\delta}$ is separated into two parts $\widetilde{\sigma}$ and $\widetilde{\delta} - \widetilde{\sigma}$. In general, we can break it as many parts as possible. The form (3.2.3) is, however, convenient for later use.

3.3 Extended Gromov-Witten theory of orbifolds

To make our exposition as self-contained as possible, we review some relevant background on orbifold Gromov-Witten theory. We take the algebro-geometric approach in the sense of Abramovich, Graber and Vistoli's works [AGV1, AGV2]. The reader may also want to consult the original work [ChR2] of Chen and Ruan in symplectic category.

In what follows, we utilize the isomorphism

$$A_1(\mathrm{Sym}^n(X); \mathbb{Z}) \cong A_1(X^n; \mathbb{Z})^{\mathfrak{S}_n} \cong A_1(X; \mathbb{Z}).$$

3.3.1 The space of twisted stable maps

For any curve class $\beta \in A_1(X;\mathbb{Z})$, the moduli space

$$\overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)^1$$

parametrizes genus zero, k-pointed, twisted stable map (or orbifold stable map in [ChR2])

$$f:(\mathcal{C},\mathcal{P}_1,\dots,\mathcal{P}_k)\to[\mathrm{Sym}^n(X)]$$

with the following conditions:

- $(\mathcal{C},\mathcal{P}_1,\dots,\mathcal{P}_k)$ is an twisted nodal k-pointed curve. The marking $\mathcal{P}_i$ is an étale gerbe banded by μ_{r_i}, where r_i is the order of the stabilizer of the twisted point. Moreover, over a node, $\mathcal{C}$ has a chart isomorphic to Spec $\mathbb{C}[u,v]/(uv)/\mu_s$ where μ_s acts on Spec $\mathbb{C}[u,v]$ by $\xi\cdot(u,v)=(\xi u,\xi^{-1}v)$, and the canonical map $c:\mathcal{C}\to C$ is given by $x=u^s, y=v^s$ in this chart.
- f is a representable morphism and gives rise to a genus zero, k-pointed, stable map $f_c:(C,c(\mathcal{P}_1),\dots,c(\mathcal{P}_k))\to\mathrm{Sym}^n(X)$ of degree β by passing to coarse moduli spaces. Note that the canonical map $c:\mathcal{C}\to C$ is an isomorphism away from the nodes and marked gerbes and that whenever we say that f is of degree β, we actually mean f_c is.

There are evaluation maps on the moduli space $\overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)$, which take values in the rigidified inertia stack. At the level of Spec($\mathbb{C}$)-points, the i^{th} evaluation map

$$\mathrm{ev}_i:\overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)\to\overline{I}[\mathrm{Sym}^n(X)] \tag{3.3.1}$$

is defined by $[f:(\mathcal{C},\mathcal{P}_1,\dots,\mathcal{P}_k)\to[\mathrm{Sym}^n(X)]]\longmapsto[f|_{\mathcal{P}_i}:\mathcal{P}_i\to[\mathrm{Sym}^n(X)]]$.

The moduli space $\overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)$ can be decomposed into open and closed substacks:

$$\overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)=\coprod_{\sigma_1,\dots,\sigma_k}\overline{M}([\mathrm{Sym}^n(X)],\sigma_1,\dots,\sigma_k;\beta).$$

Here $\overline{M}([\mathrm{Sym}^n(X)],\sigma_1,\dots,\sigma_k;\beta)=\mathrm{ev}_1^{-1}([\overline{X(\sigma_1)}])\cap\cdots\cap\mathrm{ev}_k^{-1}([\overline{X(\sigma_k)}])$, which can be empty for monodromy reason (e.g., the component $\overline{M}([\mathrm{Sym}^3(X)],(2),(2),(2);\beta)$ is empty), and the union

[1] [AGV1] and [AGV2] adopt the notation $\mathcal{K}$ instead of $\overline{M}$. Also, we just describe the Spec($\mathbb{C}$)-points of the moduli stack in this article. This is enough because our main purpose is the calculation of Gromov-Witten invariants.

is taken over all partitions $\sigma_1, \ldots, \sigma_k$ of n. Keep in mind that the substack carries a virtual class $[\overline{M}([\mathrm{Sym}^n(X)], \sigma_1, \ldots, \sigma_k; \beta)]^{\mathrm{vir}}$ of dimension

$$-K_{[\mathrm{Sym}^n(X)]} \cdot \beta + n \cdot \dim(X) + k - 3 - \sum_{i=1}^{k} \mathrm{age}(\sigma_i).$$

The twisted map f that represents an element of $\overline{M}([\mathrm{Sym}^n(X)], \sigma_1, \ldots, \sigma_k; \beta)$ amounts to the following commutative diagram

$$\begin{array}{ccc} P_{\mathcal{C}} & \xrightarrow{f'} & X^n \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{C} & \xrightarrow{f} & [\mathrm{Sym}^n(X)] \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ C & \xrightarrow{f_c} & \mathrm{Sym}^n(X) \end{array} \tag{3.3.2}$$

where π is the natural map, $P_{\mathcal{C}} := \mathcal{C} \times_{[\mathrm{Sym}^n(X)]} X^n$ is a scheme by representability of f, and f' is $\mathfrak{S}_n$-equivariant. Away from the marked points and nodes, $P_{\mathcal{C}}$ is a principal $\mathfrak{S}_n$-bundle of C. It is branched over the markings with ramification types $\sigma_1, \ldots, \sigma_k$.

Additionally, there is such a diagram

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\tilde{f}} & X \\ {\scriptstyle p}\downarrow & & \\ (C, c(\mathcal{P}_1), \ldots, c(\mathcal{P}_k)) & & \end{array} \tag{3.3.3}$$

associated to f that $p : \tilde{C} \to C$ is an admissible cover branched over $c(\mathcal{P}_1), \ldots, c(\mathcal{P}_k)$ with monodromy given by $\sigma_1, \ldots, \sigma_k$, and $\tilde{f} : \tilde{C} \to X$ is a degree β morphism such that if $\Sigma \subset C$ is a rational curve possessing less than 3 special points, then there is a component of $p^{-1}(\Sigma)$ which is not $\tilde{f}$-contracted. In fact, (3.3.3) is induced by the diagram (3.3.2) by taking f' mod $\mathfrak{S}_{n-1}$ and composing with the n^{th} projection.

The diagram (3.3.3) will be particularly helpful later in the descriptions of $\mathbb{T}$-fixed loci for the space of twisted stable maps to $[\mathrm{Sym}^n(\mathcal{A}_r)]$. The reader should look closely at the above notation. We will use (3.3.2) and (3.3.3) and the symbols there mostly without further comment.

3.3.2 Gromov-Witten invariants

For any cohomology classes $\alpha_i \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(S)])$ $(i = 1, \ldots, k)$, the k-point equivariant Gromov-Witten invariant is defined by

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{\beta}^{[\mathrm{Sym}^n(S)]} := \int_{[\overline{M}([\mathrm{Sym}^n(S)],\beta)]^{\mathrm{vir}}_{\mathbb{T}}} \mathrm{ev}_1^*(\alpha_1) \cdots \mathrm{ev}_k^*(\alpha_k), \tag{3.3.4}$$

where the symbol $[\ \ ]^{\mathrm{vir}}_{\mathbb{T}}$ stands for the $\mathbb{T}$-equivariant virtual class. The underlying moduli space is not necessarily compact, but the above definition makes sense because of a similar treatment given in Section 2.3. Moreover, it is convenient to express the integral in (3.3.4) as a sum of integrals against the virtual fundamental classes of the components $\overline{M}([\mathrm{Sym}^n(S)], \sigma_1, \ldots, \sigma_k; \beta)$'s.

In the context of orbifolds, it is in reality more natural to study the Gromov-Witten theory in twisted degrees, i.e., in curve classes of

$$A_{\mathrm{orb},1}([\mathrm{Sym}^n(S)]; \mathbb{Z}) = A_0([\overline{S((2))}]; \mathbb{Z}) \oplus A_1([\mathrm{Sym}^n(S)]); \mathbb{Z}).$$

This makes a lot of sense because the direct sum matches $A_1(\mathrm{Hilb}^n(S); \mathbb{Z})$ (cf. Section 5.2.1).

Let us identify $A_0([\overline{S((2))}]; \mathbb{Z})$ with $\mathbb{Z}$. To define the k-point extended Gromov-Witten invariant $\langle \alpha_1, \ldots, \alpha_k \rangle_{(a,\beta)}^{[\mathrm{Sym}^n(S)]}$ of twisted degree $(a, \beta) \in \mathbb{Z} \oplus A_1(S; \mathbb{Z})$ with $a \geq 0$, we include additional a unordered markings in the twisted stable map of degree β above such that these markings go to the age one sector under the corresponding evaluation maps. To make this precise, we present a formula:

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{(a,\beta)}^{[\mathrm{Sym}^n(S)]} = \frac{1}{a!} \langle \alpha_1, \ldots, \alpha_k, (2)^a \rangle_{\beta}^{[\mathrm{Sym}^n(S)]}. \tag{3.3.5}$$

Note that in the expression, the last a insertions are all (2) and the invariant is defined to be zero in case $a < 0$. For later convenience of explanation, we refer to the markings associated to $\alpha_1, \ldots, \alpha_k$ as distinguished marked points and to the other a markings as simple marked points. Also the markings corresponding to the twisted sectors are called twisted and are otherwise called untwisted.

The expression (3.3.5) is almost identical to the nonextended version except for the appearance of the factor $\frac{1}{a!}$ due to the fact that we do not order simple markings. Additionally, we say that $\langle \alpha_1, \ldots, \alpha_k \rangle_{(a,\beta)}^{[\mathrm{Sym}^n(S)]}$ is of nonzero (resp. zero) degree if it is a Gromov-Witten invariant (up to a multiple) of nonzero (resp. zero) degree and that $\langle \alpha_1, \ldots, \alpha_k \rangle_{(a,\beta)}^{[\mathrm{Sym}^n(S)]}$ is multipoint if $k \geq 3$.

Like ordinary Gromov-Witten theory, if $\beta \neq 0$ or $k \geq 3$, we have a forgetful morphism

$$ft_{k+1} : \overline{M}([\mathrm{Sym}^n(S)], \sigma_1, \ldots, \sigma_k, 1; \beta) \to \overline{M}([\mathrm{Sym}^n(S)], \sigma_1, \ldots, \sigma_k; \beta)$$

defined by forgetting the last untwisted marked points. The (untwisted) divisor equation holds as well in the orbifold case. Unfortunately, we are not allowed to forget twisted markings in general.

3.4 Extended Chen-Ruan product and the product formula

In the remainder of this chapter, we will be primarily interested in three-point extended degree zero invariants. Let us simplify our notation a little. For partitions $\sigma_1, \ldots, \sigma_m$ of n, let

$$\overline{M}([\mathrm{Sym}^n(S)], \sigma_1, \ldots, \sigma_m; a) = \bigcap_{i=1}^{m} \mathrm{ev}_i^{-1}([\overline{S(\sigma_i)}]) \cap \bigcap_{j=1}^{a} \mathrm{ev}_{m+j}^{-1}([\overline{S((2))}])$$

be an open and closed substack of the moduli space $\overline{M}_{m+a}([\mathrm{Sym}^n(S)], 0)$. Given any $\alpha_i \in A^*_{\mathbb{T}}([\mathrm{Sym}^n(S)])$ for $i = 1, \ldots, m$, we put

$$\langle \alpha_1, \ldots, \alpha_m \rangle_a^{[\mathrm{Sym}^n(S)]} := \langle \alpha_1, \ldots, \alpha_m \rangle_{(a,0)}^{[\mathrm{Sym}^n(S)]}.$$

We encode the invariants in a three-point function:

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle^{[\mathrm{Sym}^n(S)]}(u) = \sum_a \langle \alpha_1, \alpha_2, \alpha_3 \rangle_a^{[\mathrm{Sym}^n(S)]} u^a.$$

Now let $\{\gamma\}$ be a basis for $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(S)])$ and $\{\gamma^\vee\}$ its dual basis. Define the extended Chen-Ruan product $\cup_{\mathrm{orb}}$ for $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(S)])$ in this way:

$$\alpha_1 \cup_{\mathrm{orb}} \alpha_2 = \sum_\gamma \langle \alpha_1, \alpha_2, \gamma \rangle^{[\mathrm{Sym}^n(S)]}(u)\, \gamma^\vee.$$

We call the vector space $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(S)]) \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1,t_2)((u))$ with the multiplication $\cup_{\mathrm{orb}}$ the extended Chen-Ruan cohomology ring of $[\mathrm{Sym}^n(S)]$. (Note that the associativity of $\cup_{\mathrm{orb}}$ follows from the WDVV equation.)

We use the notation

$$\vec{g} = (g_1, \ldots, g_s) \tag{3.4.1}$$

to represent an s-tuple of partitions or an s-tuple of nonnegative integers. In the case of integers, we say that

$$|\vec{g}| = \ell$$

if the entries of $\vec{g}$ add up to ℓ.

Fixed loci. Given s-tuples $\sigma_1, \ldots, \sigma_m$ of partitions and an s-tuple $\vec{a}$ of nonnegative integers with $|\vec{a}| = d$. Let

$$\mathcal{M}(\vec{\sigma}_1, \ldots, \vec{\sigma}_m; \vec{a})$$

be the union of $\mathbb{T}$-fixed components of $\overline{M}([\mathrm{Sym}^n(S)], \sigma'_1, \ldots, \sigma'_m; d)$ (here σ'_i admits a decomposition $(\sigma_{i1}, \ldots, \sigma_{is})$) with the following configuration:

Let $[f : \mathcal{C} \to [\mathrm{Sym}^n(S)]]$ be any element. As discussed earlier, it comes naturally with diagram (3.3.2). In addition, the twisted stable map f has the following properties:

- The associated admissible cover $\tilde{C}$ of C is ramified with monodromy $\sigma_1, \ldots, \sigma_m, (2)^d$ and has components $\tilde{C}_k$, $k = 1, \ldots, s$. Each $\tilde{C}_k$, if nonempty, is contracted by $\tilde{f}$ to x_k. ($\tilde{C}_k$ is possibly empty or disconnected. Empty sets are included just for the simplicity of notation.)

- The cover $\tilde{C}_k \to C$ is ramified with monodromy $\sigma_{1k}, \ldots, \sigma_{mk}, (2)^{a_k}, 1^{d-a_k}$ for every k. □

Suppose that $|\sigma_{ik}| = n_k$ for some n_k, $\forall i = 1, \ldots, m$, we have a natural morphism

$$\phi : \mathcal{M}(\vec{\sigma}_1, \ldots, \vec{\sigma}_m; \vec{a}) \to \prod_{k=1}^{s} \overline{M}([\mathrm{Sym}^{n_k} U_k], \sigma_{1k}, \ldots, \sigma_{mk}; a_k)^{\mathbb{T}}$$

defined as follows: Let $[f : \mathcal{C} \to [\mathrm{Sym}^n(S)]]$ be an element of $\mathcal{M}(\vec{\sigma}_1, \ldots, \vec{\sigma}_m; \vec{a})$. For each k, we stabilize the target of the covering $\tilde{C}_k \to C$ and the domain accordingly (by forgetting those simple markings of C over which the points of $\tilde{C}_k$ are unramified). The output is the following setting

$$\begin{array}{ccc} \tilde{C}_k^{\mathrm{st}_k} & \longrightarrow & \{x_i\} \\ \big\downarrow & & \\ C^{\mathrm{st}_k} & & \end{array} \tag{3.4.2}$$

with the vertical map being an admissible covering. It gives rise to a $\mathbb{T}$-fixed twisted stable map $f_k : \mathcal{C}_k \to [\mathrm{Sym}^{n_k}(U_k)]$, which represents a $\mathbb{T}$-fixed point of $\overline{M}([\mathrm{Sym}^{n_k} U_k], \sigma_{1k}, \ldots, \sigma_{mk}; a_k)$. We then take $\phi([f]) := ([f_1], \ldots, [f_s])$.

Now we focus on $m = 3$, in which case the morphism ϕ is surjective, and both its source and target have dimension d. Given any $\mathbb{T}$-fixed connected component $F(\vec{a}) \subseteq \mathcal{M}(\vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3; \vec{a})$ and denote by π_k the k^{th} projection, we let

$$F_k(\vec{a}) = \pi_k \circ \phi(F(\vec{a})).$$

The collection $\prod_{k=1}^{s} F_k(\vec{a})$'s form a complete set of $\mathbb{T}$-fixed connected components of the product space $\prod_{k=1}^{s} \overline{M}([\mathrm{Sym}^{n_k} U_k], \vec{\sigma}_{1k}, \vec{\sigma}_{2k}, \vec{\sigma}_{3k}; a_k)$.

Determination. We want to investigate the invariant $\langle \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu} \rangle_d^{[\mathrm{Sym}^n(S)]}$. It is clearly zero if the condition

$$|\lambda_k| = |\mu_k| = |\nu_k| = n_k \text{ for each } k = 1, \ldots, s \tag{3.4.3}$$

fails. In general, we have the following product formula.

Proposition 3.4.1. *Given any $\mathbb{T}$-fixed point classes $\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}$,*

$$\langle \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu} \rangle_d^{[\mathrm{Sym}^n(S)]} = \sum_{a_1 + \cdots + a_s = d} \prod_{k=1}^{s} \langle \widetilde{\lambda_k}, \widetilde{\mu_k}, \widetilde{\nu_k} \rangle_{a_k}^{[\mathrm{Sym}^{n_k}(U_k)]}.$$

Proof. The statement is obvious when (3.4.3) does not hold. So let us assume (3.4.3). The only fixed loci that can make contribution to the three-point extended invariant

$$\langle \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu} \rangle_d^{[\mathrm{Sym}^n(S)]} \tag{3.4.4}$$

are $\mathcal{M}(\vec{\lambda}, \vec{\mu}, \vec{\nu}; \vec{a})$'s with $|\vec{a}| = d$. Precisely, (3.4.4) is given by

$$\frac{1}{d!} \sum_{|\vec{a}|=d} \sum_{F(\vec{a})} \int_{F(\vec{a})} \frac{\iota_{F(\vec{a})}^* (\mathrm{ev}_1^*(\widetilde{\lambda}) \cdot \mathrm{ev}_2^*(\widetilde{\mu}) \cdot \mathrm{ev}_3^*(\widetilde{\nu}))}{e_{\mathbb{T}}(N_{F(\vec{a})}^{\mathrm{vir}})},$$

where $F(\vec{a}) \subset \mathcal{M}(\vec{\lambda}, \vec{\mu}, \vec{\nu}; \vec{a})$ runs over all $\mathbb{T}$-fixed connected components. Given any $\mathbb{T}$-fixed component $F(\vec{a}) \subset \mathcal{M}(\vec{\lambda}, \vec{\mu}, \vec{\nu}; \vec{a})$ and $[f] \in F(\vec{a})$, we have

$$e_{\mathbb{T}}(H^i(\mathcal{C}, f^* T[\mathrm{Sym}^n(S)])) = \phi^* \bigotimes_{k=1}^{s} e_{\mathbb{T}}(H^i(\mathcal{C}_k, f_k^* T[\mathrm{Sym}^{n_k} U_k]))$$

for $i = 0, 1$, and so

$$e_{\mathbb{T}}(N^{\mathrm{vir}}_{F(\vec{a})}) = \phi^* \bigotimes_{k=1}^{s} e_{\mathbb{T}}(N^{\mathrm{vir}}_{F_k(\vec{a})}).$$

Moreover, by (3.2.2), we check immediately that

$$\iota^*_{F(\vec{a})}(\mathrm{ev}_1^*(\widetilde{\lambda}) \cdot \mathrm{ev}_2^*(\widetilde{\mu}) \cdot \mathrm{ev}_3^*(\widetilde{\nu})) = \prod_{k=1}^{s} \iota^*_{F_k(\vec{a})}(\mathrm{ev}_1^*(\widetilde{\lambda_k}) \cdot \mathrm{ev}_2^*(\widetilde{\mu_k}) \cdot \mathrm{ev}_3^*(\widetilde{\nu_k})).$$

Hence the contribution of $\mathcal{M}(\vec{\lambda}, \vec{\mu}, \vec{\nu}; \vec{a})$ to (3.4.4) equals

$$\frac{1}{a_1! \cdots a_s!} \sum_{F(\vec{a})} \prod_{k=1}^{s} \int_{F_k(\vec{a})} \frac{\iota^*_{F_k(\vec{a})}(\mathrm{ev}_1^*(\widetilde{\lambda_k}) \cdot \mathrm{ev}_2^*(\widetilde{\mu_k}) \cdot \mathrm{ev}_3^*(\widetilde{\nu_k}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{F_k(\vec{a})})},$$

where the prefactor accounts for the distribution of simple marked points. The sum is nothing but

$$\prod_{k=1}^{s} \frac{1}{a_k!} \sum_{F_k(\vec{a})} \int_{F_k(\vec{a})} \frac{\iota^*_{F_k(\vec{a})}(\mathrm{ev}_1^*(\widetilde{\lambda_k}) \cdot \mathrm{ev}_2^*(\widetilde{\mu_k}) \cdot \mathrm{ev}_3^*(\widetilde{\nu_k}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{F_k(\vec{a})})}.$$

Since $F_k(\vec{a})$'s run through all connected components of $\overline{M}([\mathrm{Sym}^{n_k} U_k], \vec{\lambda}_{1k}, \vec{\lambda}_{2k}, \vec{\lambda}_{3k}; a_k)^{\mathbb{T}}$, (3.4.4) and $\sum_{|\vec{a}|=d} \prod_{k=1}^{s} \langle \widetilde{\lambda_k}, \widetilde{\mu_k}, \widetilde{\nu_k} \rangle_{a_k}^{[\mathrm{Sym}^{n_k}(U_k)]}$ coincide. □

Put another way, we have the following.

Corollary 3.4.2. *For any $\mathbb{T}$-fixed point classes $\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}$,*

$$\langle \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu} \rangle^{[\mathrm{Sym}^n(S)]}(u) = \prod_{k=1}^{s} \langle \widetilde{\lambda_k}, \widetilde{\mu_k}, \widetilde{\nu_k} \rangle^{[\mathrm{Sym}^{n_k}(U_k)]}(u).$$

Moreover, any extended three-point function is a rational function in t_1, t_2, e^{iu}, where $i^2 = -1$.

Proof. The first statement is immediate from Proposition 3.4.1. The second statement is due to the fact that each $\langle \widetilde{\lambda_k}, \widetilde{\mu_k}, \widetilde{\nu_k} \rangle^{[\mathrm{Sym}^{n_k}(U_k)]}(u)$ is an element of $\mathbb{Q}(t_1, t_2, e^{iu})$. □

The orbifold Poincaré pairing $\langle \bullet | \bullet \rangle$ on $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^{n_k}(U_k)])_{\mathfrak{m}}$ is determined by

$$\langle \widetilde{\lambda_k} | \widetilde{\mu_k} \rangle = \delta_{\lambda_k, \mu_k} (L_k R_k)^{\ell(\lambda_k)} \frac{1}{\mathfrak{z}_{\lambda_k}}, \ |\lambda_k| = |\mu_k| = n_k. \tag{3.4.5}$$

The argument of Proposition 3.4.1 may be applied to show that the orbifold pairing on $[\mathrm{Sym}^n(S)]$ is expressible in terms of those on $[\mathrm{Sym}^{n_k}(U_k)]$'s.

Proposition 3.4.3. *The equivariant orbifold Poincaré pairing on* $[\mathrm{Sym}^n(S)]$ *is determined by the formula:*

$$\langle \widetilde{\lambda} | \widetilde{\mu} \rangle = \prod_{k=1}^{s} \delta_{\lambda_k, \mu_k} (L_k R_k)^{\ell(\lambda_k)} \frac{1}{\mathfrak{z}_{\lambda_k}}.$$

Thus, $\widetilde{\lambda}$*'s provide an orthogonal basis for* $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(S)])_{\mathfrak{m}}$.

3.5 Ruan conjecture

3.5.1 SYM-HILB correspondence

In this section, we give a SYM-HILB correspondence that relates the Gromov-Witten theories of Hilbert scheme of points and symmetric products.

Let $q = -e^{iu}$, where $i^2 = -1$, and $F = \mathbb{Q}(i, t_1, t_2, q)$. Define

$$L : A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(S)]) \to A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S))$$

by

$$L(\widetilde{\lambda}) = (-i)^{\mathrm{age}(\widetilde{\lambda})} \mathfrak{a}_{\widetilde{\lambda}}.$$

As we have a bijection between the bases on both sides, L extends to a $\mathbb{Q}(i, t_1, t_2)((u))$-linear isomorphism. However, extended three-point functions of $[\mathrm{Sym}^n(S)]$ and three-point functions of $\mathrm{Hilb}^n(S)$ are elements of F. So we may view L as an F-linear isomorphism.

Note that $\widetilde{\lambda}$ has orbifold Chow degree

$$2\ell(\widetilde{\lambda}) + \mathrm{age}(\widetilde{\lambda}) = n + \ell(\widetilde{\lambda}),$$

which matches the Chow degree of $\mathfrak{a}_{\widetilde{\lambda}}$. Further, L is an isometry:

Proposition 3.5.1. *For every fixed-point class* $\widetilde{\lambda}$*,*

$$\langle \widetilde{\lambda} | \widetilde{\lambda} \rangle = \langle L(\widetilde{\lambda}) | L(\widetilde{\lambda}) \rangle.$$

Proof. Propositions 3.4.3 and 2.4.5 say that $\langle \widetilde{\lambda} | \widetilde{\lambda} \rangle = (-1)^{\mathrm{age}(\widetilde{\lambda})} \langle \mathfrak{a}_{\widetilde{\lambda}} | \mathfrak{a}_{\widetilde{\lambda}} \rangle$. □

Theorem 3.5.2. *The map*

$$L : A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(S)]) \otimes_{\mathbb{Q}[t_1,t_2]} F \to A^*_{\mathbb{T}}(\mathrm{Hilb}^n(S)) \otimes_{\mathbb{Q}[t_1,t_2]} F$$

is an F-algebra isomorphism. In particular, equivariant Ruan conjecture holds for $\mathrm{Sym}^n(S)$.

Proof. The proof relies on the affine plane case. Indeed, Okounkov-Pandharipande and Bryan-Graber determine the structures of $\mathbb{T}$-equivariant quantum cohomology rings of $\mathrm{Hilb}^n(\mathbb{C}^2)$ and $[\mathrm{Sym}^n(\mathbb{C}^2)]$ respectively. Also, these rings are related by the correspondence

$$L_{\mathbb{C}^2} : A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathbb{C}^2)]) \otimes_{\mathbb{Q}[t_1,t_2]} F \to A^*_{\mathbb{T}}(\mathrm{Hilb}^n(\mathbb{C}^2)) \otimes_{\mathbb{Q}[t_1,t_2]} F,$$

which sends $\mu_1([0]) \cdots \mu_{\ell(\mu)}([0])$ to $(-i)^{\mathrm{age}(\mu)} \mathfrak{a}_{\mu_1}([0]) \cdots \mathfrak{a}_{\mu_{\ell(\mu)}}([0])$. Bryan and Graber show that $L_{\mathbb{C}^2}$ preserves three-point functions. That is, for any Chen-Ruan classes $\delta_1, \delta_2, \delta_3$ on the orbifold $[\mathrm{Sym}^n(\mathbb{C}^2)]$, we have

$$\langle \delta_1, \delta_2, \delta_3 \rangle^{[\mathrm{Sym}^n(\mathbb{C}^2)]}(u) = \langle L_{\mathbb{C}^2}(\delta_1), L_{\mathbb{C}^2}(\delta_2), L_{\mathbb{C}^2}(\delta_3) \rangle^{\mathrm{Hilb}^n(\mathbb{C}^2)}(q). \tag{3.5.1}$$

As $U_k \cong \mathbb{C}^2$, we denote the associated correspondence by

$$L_{U_k} : A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(U_k)]) \otimes_{\mathbb{Q}[t_1,t_2]} F \to A^*_{\mathbb{T}}(\mathrm{Hilb}^n(U_k)) \otimes_{\mathbb{Q}[t_1,t_2]} F.$$

Given any equivariant Chen-Ruan classes $\alpha_1, \alpha_2, \alpha_3$ on $[\mathrm{Sym}^n(S)]$, let us show the identity

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle^{[\mathrm{Sym}^n(S)]}(u) = \langle L(\alpha_1), L(\alpha_2), L(\alpha_3) \rangle^{\mathrm{Hilb}^n(S)}(q).$$

It is, however, enough to establish that for all $\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}$ satisfying (3.4.3),

$$\langle \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu} \rangle^{[\mathrm{Sym}^n(S)]}(u) = \langle L(\widetilde{\lambda}), L(\widetilde{\mu}), L(\widetilde{\nu}) \rangle^{\mathrm{Hilb}^n(S)}(q). \tag{3.5.2}$$

In fact, by (3.5.1) and Corollary 3.4.2, the invariant $\langle \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu} \rangle^{[\mathrm{Sym}^n(S)]}(u)$ is given by

$$\prod_{k=1}^{s} \langle \widetilde{\lambda_k}, \widetilde{\mu_k}, \widetilde{\nu_k} \rangle^{[\mathrm{Sym}^{n_k}(U_k)]}(u) = \prod_{k=1}^{s} \langle L_{U_k}(\widetilde{\lambda_k}), L_{U_k}(\widetilde{\mu_k}), L_{U_k}(\widetilde{\nu_k}) \rangle^{\mathrm{Hilb}^{n_k}(U_k)}(q).$$

Clearly, $\mathrm{age}(\widetilde{\sigma}) = \sum_{k=1}^{s} \mathrm{age}(\sigma_k)$ for any fixed-point class $\widetilde{\sigma}$. By applying Proposition 2.4.4, we obtain (3.5.2).

In addition, according to Proposition 3.5.1, L preserves Poincaré pairings, and so we have an equality:

$$\langle L(\alpha_1 \cup_{\mathrm{orb}} \alpha_2) | L(\alpha_3) \rangle = \langle L(\alpha_1) \cup_{\mathrm{crep}} L(\alpha_2) | L(\alpha_3) \rangle.$$

This yields $L(\alpha_1 \cup_{\mathrm{orb}} \alpha_2) = L(\alpha_1) \cup_{\mathrm{crep}} L(\alpha_2)$ and proves the isomorphism of algebras. □

3.5.2 The cup product structure on the Hilbert scheme

An upshot of Theorem 3.5.2 is that three-point degree zero invariants of Hilbert schemes is expressible in terms of degree zero invariants of symmetric product stacks.

Corollary 3.5.3. *Given cohomology-weighted partitions $\lambda_1(\vec{\eta_1}), \lambda_2(\vec{\eta_2}), \lambda_3(\vec{\eta_3})$, the degree zero invariant $\left\langle \mathfrak{a}_{\lambda_1(\vec{\eta_1})}, \mathfrak{a}_{\lambda_2(\vec{\eta_2})}, \mathfrak{a}_{\lambda_3(\vec{\eta_3})} \right\rangle_0^{\mathrm{Hilb}^n(S)}$ is given by*

$$i^{\sum_{k=1}^3 \mathrm{age}(\lambda_k)} \lim_{u \to +i\infty} \langle \lambda_1(\vec{\eta_1}), \lambda_2(\vec{\eta_2}), \lambda_3(\vec{\eta_3}) \rangle^{[\mathrm{Sym}^n(S)]}(u). \tag{3.5.3}$$

Proof. It follows immediately from Theorem 3.5.2 by taking $q \to 0$. □

This is the precise statement of Corollary 1.1.3. Since the ordinary cup product of $\mathrm{Hilb}^n(S)$ is defined by three-point degree zero invariants, the corollary says that the three-point extended $[\mathrm{Sym}^n(S)]$-invariants of degree zero completely determine the cup product of $\mathrm{Hilb}^n(S)$. This may shed new light on the explicit calculation of the ordinary cohomology ring of the Hilbert scheme of points.

Chapter 4

Orbifold quantum cohomology of the symmetric product of $\mathcal{A}_r$

In Chapters 4 and 5,[1] we basically restrict our attention to the case where S is the minimal resolution of type A_r surface singularity.

4.1 Resolutions of cyclic quotient surface singularities

We fix a positive integer r once and for all. Let the cyclic group μ_{r+1} act on $\mathbb{C}^2$ by the diagonal matrices

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

where $\zeta \in \mu_{r+1}$. The quotient $\mathbb{C}^2/\mu_{r+1}$ is a surface singularity. We denote by

$$\pi : \mathcal{A}_r \to \mathbb{C}^2/\mu_{r+1}$$

its minimal resolution. It is actually well-known that π can be obtained via a sequence of $\lfloor \frac{r+1}{2} \rfloor$ blow-ups at the unique singularity. The exceptional locus $\mathrm{Ex}(\pi)$ of π is a chain of (-2)-curves,

$$\bigcup_{i=1}^{r} E_i,$$

[1] Some results in these two chapters have also appeared in [CG] but the materials presented here are due solely to myself.

with E_{i-1} and E_i intersect transversally. The intersection numbers of the exceptional curves are given by

$$E_i \cdot E_j = \begin{cases} -2 & \text{if } i = j; \\ 1 & \text{if } |i-j| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the intersection matrix is negative definite (as expected from the general theory of complex surfaces). Additionally, $E_1, \ldots, E_r$ give a basis for $A_1(\mathcal{A}_r; \mathbb{Z})$. We also have two noncompact curves E_0 and E_{r+1} attached to E_1 and E_r respectively. The curve E_0 (resp. E_{r+1}) can be arranged to map to the μ_{r+1}-orbit of x-axis (resp. y-axis).

The natural action of $\mathbb{T}$ on $\mathbb{C}^2$ comes with tangent weights t_1 and t_2 at the origin. It commutes with the μ_{r+1}-action, so we have an induced $\mathbb{T}$-action on the quotient $\mathbb{C}^2/\mu_{r+1}$ and thus on the resolved surface $\mathcal{A}_r$. We fix these actions of $\mathbb{T}$ throughout Chapters 4 and 5.

The $\mathbb{T}$-invariant curves on $\mathcal{A}_r$ are $E_1, \ldots, E_r$, and there are $r+1$ $\mathbb{T}$-fixed points $x_1, \ldots, x_{r+1}$, which are the nodes of the chain $\cup_{i=0}^{r+1} E_i$ of curves. We may assume that

$$\{x_i\} = E_{i-1} \cap E_i.$$

Let L_i and R_i be respectively the weights of the $\mathbb{T}$-action on the tangent spaces to E_{i-1} and E_i at x_i. We have

$$L_1 = (r+1)t_1, \; R_{r+1} = (r+1)t_2,$$

and the following equalities

$$L_i + R_i = t_1 + t_2, \; R_i = -L_{i+1},$$

for each $i = 1, \ldots r$.

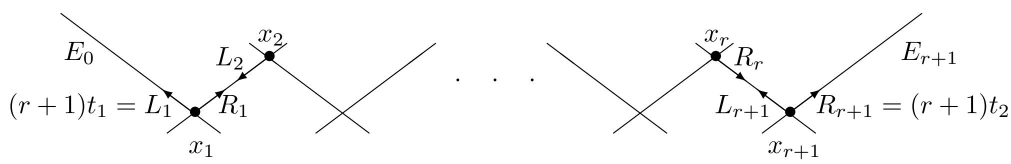

Figure 4.1. The middle chain is the exceptional locus $\mathrm{Ex}(\pi)$. The labeled vectors stand for the tangent weights at the fixed points.

The above information will be sufficient for our calculation of Gromov-Witten invariants.

One can also compute explicitly to obtain

$$(L_i, R_i) = ((r-i+2)t_1 + (1-i)t_2,\ (-r+i-1)t_1 + it_2).$$

Obviously, $L_k R_k \equiv -(r+1)^2 t_1^2 \mod (t_1+t_2)$ for $k = 1, \ldots, r+1$. It is convenient to take

$$\tau = -(r+1)^2 t_1^2.$$

In this manner,

$$t(\widetilde{\delta}) \equiv \tau^{\ell(\widetilde{\delta})} \mod (t_1+t_2). \tag{4.1.1}$$

Here $\ell(\widetilde{\delta})$ is the sum $\sum_{k=1}^{r+1} \ell(\delta_k)$.

4.2 Connected invariants

Let

$$\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)$$

be the component of $\overline{M}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)$ parametrizing connected covers (i.e., each cover $\tilde{C}$ associated to $[f : \mathcal{C} \to [\mathrm{Sym}^n(\mathcal{A}_r)]] \in \overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)$ is connected).

We define k-point connected Gromov-Witten invariant as the contribution of the component $\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)$ to the extended Gromov-Witten invariant. That is,

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{\beta}^{[\mathrm{Sym}^n(\mathcal{A}_r)],\mathrm{conn}} = \int_{[\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)],\beta)]^{\mathrm{vir}}_{\mathbb{T}}} \mathrm{ev}_1^*(\alpha_1) \cdots \mathrm{ev}_k^*(\alpha_k).$$

Note that $\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(\mathcal{A}_r)], \beta)$ is compact whenever $\beta \neq 0$, in which case the corresponding connected invariant is an element of $\mathbb{Q}[t_1, t_2]$.

Similarly, the connected invariant has an extended version. We define k-point extended connected invariant by

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{(a,\beta)}^{[\mathrm{Sym}^n(\mathcal{A}_r)],\mathrm{conn}} = \frac{1}{a!} \langle \alpha_1, \ldots, \alpha_k, (2)^a \rangle_{\beta}^{[\mathrm{Sym}^n(\mathcal{A}_r)],\mathrm{conn}}.$$

4.3 Orbifold quantum product and divisor operators

We may view curve classes $E_1, \dots, E_r$ as a basis for $A_1(\mathrm{Sym}^n(\mathcal{A}_r); \mathbb{Z})$. Let $\{\omega_1, \dots, \omega_r\}$ be the dual basis of $\{E_1, \dots, E_r\}$ with respect to the Poincaré pairing. For any classes $\alpha_1, \dots, \alpha_k \in A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)])$, we define the extended k-point function of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ by

$$\langle\langle \alpha_1, \dots, \alpha_k \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \sum_{a=0}^{\infty} \sum_{\beta \in A_1(\mathcal{A}_r;\mathbb{Z})} \langle \alpha_1, \dots, \alpha_k \rangle_{(a,\beta)}^{[\mathrm{Sym}^n(\mathcal{A}_r)]} u^a s_1^{\beta\cdot\omega_1} \cdots s_r^{\beta\cdot\omega_r}, \tag{4.3.1}$$

and denote by

$$\langle \alpha_1, \dots, \alpha_k \rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}$$

the usual k-point function $\langle\langle \alpha_1, \dots, \alpha_k \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}|_{u=0}$.

Now let $\{\gamma\}$ be a basis for the Chen-Ruan cohomology $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)])$ and $\{\gamma^\vee\}$ its dual basis. Define the small orbifold quantum product on $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)])$ in this way:

$$\alpha_1 *_{\mathrm{orb}} \alpha_2 = \sum_{\gamma} \langle\langle \alpha_1, \alpha_2, \gamma \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} \gamma^\vee.$$

By extending scalars, we work with

$$QA^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)]),$$

which is defined as the vector space $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)]) \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1,t_2)((u, s_1, \dots, s_r))$ endowed with quantum multiplication $*_{\mathrm{orb}}$.

The extended k-point functions were first studied by Bryan and Graber [BG] in the case of $[\mathrm{Sym}^n(\mathbb{C}^2)]$ so as to link the Gromov-Witten theory of $[\mathrm{Sym}^n(\mathbb{C}^2)]$ to that of $\mathrm{Hilb}^n(\mathbb{C}^2)$. When it comes to the whole group of multipoint functions, it is clear that the extended and the usual versions share the same information. However, extended three-point functions are a wider group than the usual three-point functions, and the quantum product defined above retains more information than the usual small quantum product.

We are going to study the operators

$$D *_{\mathrm{orb}} -$$

on the (small) quantum cohomology of the orbifold $[\mathrm{Sym}^n(\mathcal{A}_r)]$ for divisor classes D. We refer to them as divisor operators. We let

$$D_k = 1(1)^{n-1}1(\omega_k),\ k = 1, \ldots, r.$$

These classes, along with (2), form a basis for divisors on $[\mathrm{Sym}^n(\mathcal{A}_r)]$. Thus, the divisor operators are determined by

$$(2) *_{\mathrm{orb}} -,\ D_1 *_{\mathrm{orb}} -, \ldots, D_r *_{\mathrm{orb}} -,$$

which are governed by two-point extended invariants to be calculated in this chapter.

4.4 Fixed loci

Fix a nonnegative integer a throughout the rest of this section. We shorten our notation by declaring

$$\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \sigma_1, \ldots, \sigma_k; (a, \beta)) = \overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \sigma_1, \ldots, \sigma_k, (2)^a; \beta).$$

Also, as in (3.4.1), we use

$$\vec{g} = (g_1, \ldots, g_{r+1})$$

to denote an $(r+1)$-tuple, whose entries are either all partitions or all nonnegative integers. Moreover, given a partition σ_0 and a multi-partition $\vec{\sigma}$, we put

$$\hat{\sigma} := (\sigma_0, \vec{\sigma}) = (\sigma_0, \ldots, \sigma_{r+1}),$$

which we also realize as a partition of $\sum_{k=0}^{r+1} |\sigma_k|$.

Let us now describe the fixed loci that will play an important role in our virtual localization calculation.

Given nonnegative integers i, j, s with $1 \leq i \leq j \leq r$ and $s \leq a$. We consider effective curve classes

$$\mathcal{E}_{ij} = E_i + \cdots + E_j.$$

(Note that $\mathcal{E}_{ii} = E_i$). For each $b_0^L \in \{0, \ldots, s\}$ and $u_0^L \in \{0, \ldots, a-s\}$, put $b_0^R = s - b_0^L$ and

$u_0^R = a - s - u_0^L$. We let

$$\{\overline{M}_0^{b_0^L,\sigma_0,u_0^L}(1)\} \ \ (\text{resp. } \{\overline{M}_0^{b_0^L,\sigma_0,u_0^L}(2)\}) \tag{4.4.1}$$

be the set consisting of all $\mathbb{T}$-fixed connected components of the moduli space

$$\overline{M}^{\circ}([\mathrm{Sym}^{|\lambda_0|}(\mathcal{A}_r)], \lambda_0, \rho_0, (2)^s, 1^{a-s};\ d\mathcal{E}_{ij})$$

such that each point $[f_0 : \mathcal{C} \to [\mathrm{Sym}^{|\lambda_0|}(\mathcal{A}_r)]] \in \overline{M}_0^{b_0^L,\sigma_0,u_0^L}(1)$ (resp. $\overline{M}_0^{b_0^L,\sigma_0,u_0^L}(2)$) has the following properties:

(i) f_0 has its source curve decomposed as

$$\mathcal{C} = \mathcal{C}_{L0} \cup \mathcal{D}_0 \cup \mathcal{C}_{R0}.$$

Here $\mathcal{C}_{k0}$'s are disjoint f_0-contracted components, $\mathcal{D}_0$ is a chain of noncontracted components with $f_{0*}([\mathcal{D}_0]) = d\mathcal{E}_{ij}$, and $\mathcal{C}_{k0} \cap \mathcal{D}_0 = \{\mathcal{P}_k\}$ is a twisted point, $k = L, R$.

Let D_0, C, P_k be coarse moduli spaces of $\mathcal{D}_0, \mathcal{C}, \mathcal{P}_k$ respectively ($k = L, R$) and $\tilde{C}_0$ the admissible cover associated to $\mathcal{C}$.

(ii) $\tilde{C}_0 := \tilde{C}_{L0} \cup \tilde{D}_0 \cup \tilde{C}_{R0}$ is connected with admissible covers $\tilde{D}_0 \to D_0$ and $\tilde{C}_{k0} \to C_{k0}$ ($k = L, R$). Moreover,

- each irreducible component of the cover $\tilde{D}_0 \to D_0$ is totally branched over two points (either nodes or markings) and branched nowhere else.
- the covering $\tilde{C}_{L0} \to C_{L0}$ is branched with monodromy

$$\lambda_0, (2)^{b_0^L}, 1^{u_0^L}, \sigma_0 \ \ (\text{resp. } \lambda_0, \rho_0, (2)^{b_0^L}, 1^{u_0^L}, \sigma_0),$$

around markings and P_L while the covering $\tilde{C}_{R0} \to C_{R0}$ is branched with monodromy

$$\rho_0, (2)^{b_0^R}, 1^{u_0^R}, \sigma_0 \ \ (\text{resp. } (2)^{b_0^R}, 1^{u_0^R}, \sigma_0),$$

around markings and P_R.

(iii) In the cover $\tilde{D}_0$, there exists a unique chain ε formed by rational curves not contracted by $\tilde{f}_0$. Additionally,

- ε possesses $j-i+1$ irreducible components which are mapped to $E_i, \ldots, E_j$ with the same degree d under the map $\tilde{f}_0$.
- the contracted components attached to the two ends of ε collapse to x_i and x_{j+1} respectively. □

Now we turn our attention to the fixed locus on the moduli space

$$\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda, \wp, (a, d\mathcal{E}_{ij})).$$

We fix $\vec{b}^L$ and $\vec{b}^R$, tuples of nonnegative integers, with $|\vec{b}^L| = u_0^L$ and $|\vec{b}^R| = u_0^R$. We define

$$\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})[i,j,s] = \{\overline{M}^{b_0^L,\sigma_0,u_0^L}(1)\}$$

to be the set of $\mathbb{T}$-fixed loci of $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda, \wp, (a, d\mathcal{E}_{ij}))$ (so $\Lambda = \hat{\lambda}$ and $\wp = \hat{\rho}$ as partitions) such that any element $[f : \mathcal{C} \to [\mathrm{Sym}^n(\mathcal{A}_r)]] \in \overline{M}^{b_0^L,\sigma_0,u_0^L}(1)$ of these fixed loci satisfies the following properties:

(a) The domain curve $\mathcal{C}$ of f decomposes into three pieces

$$\mathcal{C} = \mathcal{C}_L \cup \mathcal{D} \cup \mathcal{C}_R, \tag{4.4.2}$$

where $\mathcal{C}_k$'s are disjoint f-contracted components; $\mathcal{D}$ is a chain of noncontracted components, which maps to $[\mathrm{Sym}^n(\mathcal{A}_r)]$ with degree $d\mathcal{E}_{ij}$; and the intersection $\mathcal{C}_k \cap \mathcal{D} := \{\mathcal{Q}_k\}$ is a twisted point, $k = L, R$.

As in (3.3.3), there is an associated morphism $\tilde{f} : \tilde{C} \to \mathcal{A}_r$. Let D, C, C_k, Q_k be coarse moduli spaces of $\mathcal{D}, \mathcal{C}, \mathcal{C}_k, \mathcal{Q}_k$ respectively ($k = L, R$).

(b) $\mathcal{C}_L$ carries $b_0^L + u_0^L + 1$ marked points, and $\mathcal{C}_R$ carries the other $b_0^R + u_0^R + 1 = a - b_0^L - u_0^L + 1$ marked points.

(c) The covering $\tilde{C} \to C$ has components

$$\tilde{C}_k := \tilde{C}_{Lk} \cup \tilde{D}_k \cup \tilde{C}_{Rk}, \quad k = 0, \ldots, r+1. \tag{4.4.3}$$

For $k \neq 0$, $\tilde{C}_k$, if nonempty, is contracted to x_k in $\mathcal{A}_r$. (Note that, as in Chapter 3, $\tilde{C}_k$ is possibly empty or disconnected for $k \neq 0$, and we include empty sets just for the simplicity of notation).

(d) For $k = 0, \ldots, r+1$,

- the covering $\coprod_{k=0}^{r+1} \tilde{C}_{Lk} \to C_L$ (resp. $\coprod_{k=0}^{r+1} \tilde{C}_{Rk} \to C_R$) is ramified with monodromy

$$\hat{\lambda}, (2)^{b_0^L + u_0^L}, \hat{\sigma} \quad (\text{resp. } \hat{\rho}, (2)^{b_0^R + u_0^R}, \hat{\sigma}),$$

  around markings and Q_L (resp. Q_R);

- each irreducible component of the cover $\tilde{D}_k \to D$ is totally branched over two points and branched nowhere else;

- each $\tilde{C}_{Lk} \to C_L$ (resp. $\tilde{C}_{Rk} \to C_R$) is a covering ramified with monodromy

$$\lambda_k, (2)^{b_k^L}, 1^{b_0^L + u_0^L - b_k^L}, \sigma_k \quad (\text{resp. } \rho_k, (2)^{b_k^R}, 1^{b_0^R + u_0^R - b_k^R}, \sigma_k),$$

  around markings and Q_L (resp. Q_R).

(e) The diagram of maps

$$\begin{array}{ccc} \tilde{C}_0 & \xrightarrow{\tilde{f}|_{\tilde{C}_0}} & \mathcal{A}_r \\ \downarrow & & \\ C & & \end{array} \tag{4.4.4}$$

corresponds to $[f_0] \in \overline{M}_0^{b_0^L, \sigma_0, u_0^L}(1)$ above. □

Note that $\mathcal{F}^{\vec{\sigma}}_{\lambda_0, \sigma_0, \rho_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})[i, j, s]$ does not exist for certain parameters. If it does, it is indexed by $\overline{M}_0^{b_0^L, \sigma_0, u_0^L}(1)$'s. Each fixed locus $\overline{M}^{b_0^L, \sigma_0, u_0^L}(1)$ is, however, a union of $\mathbb{T}$-fixed connected components in general.

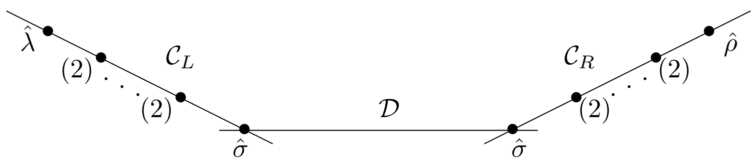

Figure 4.2. This is the configuration of a typical domain curve $\mathcal{C}$ for $\overline{M}^{b_0^L, \sigma_0, u_0^L}(1)$. Each straight line represents a chain of curves. All markings and $\mathcal{Q}_k$'s are labeled with their monodromy and there are $b_0^k + u_0^k$ copies of (2) on $\mathcal{C}_k$, $k = L, R$. In case $b_0^k + u_0^k = 0$, $\mathcal{C}_k$ is simply a twisted point. Details on the covering $\tilde{C}$ associated to $\mathcal{C}$ are included in the above properties.

Define

$$\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho},\vec{b}^L \mid \vec{b}^R)[i,j,s] := \{\overline{M}^{b_0^L,\sigma_0,u_0^L}(2)\}$$

in an analogous manner. The differences occur in properties (b), (d), and (e). Precisely, (b) the curve $\mathcal{C}_L$ carries $b_0^L + u_0^L + 2$ marked points while the curve $\mathcal{C}_R$ carries the other $b_0^R + u_0^R$ marked points; (d) the covering $\coprod_{k=0}^{r+1} \tilde{C}_{Lk} \to C_L$ (resp. $\coprod_{k=0}^{r+1} \tilde{C}_{Rk} \to C_R$) is ramified with monodromy $\hat{\lambda}, \hat{\rho}, (2)^{b_0^L+u_0^L}, \hat{\sigma}$ (resp. $(2)^{b_0^R+u_0^R}, \hat{\sigma}$) around markings and Q_L (resp. Q_R), and the monodromy associated to the cover $\tilde{C}_{Lk} \to C_L$(resp. $\tilde{C}_{Rk} \to C_R$) is now $\lambda_k, \rho_k, (2)^{b_k^L}, 1^{b_0^L+u_0^L-b_k^L}, \sigma_k$ (resp. $(2)^{b_k^R}, 1^{b_0^R+u_0^R-b_k^R}, \sigma_k$); (e) the diagram (4.4.4) corresponds to $[f_0] \in \overline{M}_0^{b_0^L,\sigma_0,u_0^L}(2)$.

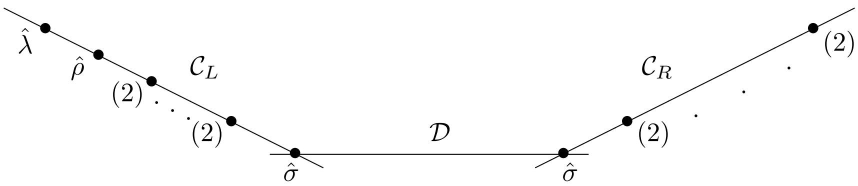

Figure 4.3. This is the configuration of a typical domain curve $\mathcal{C}$ for $\overline{M}^{b_0^L,\sigma_0,u_0^L}(2)$. There are $b_0^k + u_0^k$ copies of (2) on $\mathcal{C}_k$, $k = L, R$. $\mathcal{C}_L$ is always a twisted curve. $\mathcal{C}_R$ is of dimension $\epsilon_2(b_0^R + u_0^R)$; in particular, it is a twisted point when $b_0^R + u_0^R \leq 1$.

4.5 Valuations

Given moduli space $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda, \wp, (a, d\mathcal{E}_{ij}))$ as above. For each $\mathbb{T}$-fixed connected component F, the virtual normal bundle to F is denoted by

$$N_F^{\mathrm{vir}}.$$

Let $[f : \mathcal{C} \to [\mathrm{Sym}^n(\mathcal{A}_r)] \in F$ and $\coprod_v \mathcal{C}_v$ the union of one-dimensional, contracted, connected components of $\mathcal{C}$. We have a natural morphism

$$\phi_F : F \to F^c := \prod_v \overline{M}_{0,val(v)},$$

defined by $\phi_F([f]) = ([c(\mathcal{C}_v)])_v$. That is, all noncontracted components, zero-dimensional contracted components, stack structures at special points and the map f are forgotten. Also, $val(v)$ denotes the number of special points on $\mathcal{C}_v$.

Let

$$\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s]$$

be the union $\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda},\vec{b}^L \mid \vec{b}^R,\vec{\rho})[i,j,s] \cup \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho},\vec{b}^L \mid \vec{b}^R)[i,j,s]$.

The indices $b_0^L,\sigma_0,u_0^L,(k)$ $(k=L,R)$ from $\overline{M}^{b_0^L,\sigma_0,u_0^L}(k)$ and $\overline{M}_0^{b_0^L,\sigma_0,u_0^L}(k)$ are going to be suppressed. We simply write $\overline{M}$, $\overline{M}_0$. For each $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s]$, we let

$$\overline{M}_{\mathbb{T}}$$

be the collection of all $\mathbb{T}$-fixed connected components of $\overline{M}$.

There are other $\mathbb{T}$-fixed loci on the moduli space $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)],\Lambda,\wp,(a,d\mathcal{E}_{ij}))$. The reason why $\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s]$'s are singled out will be discussed later. It will turn out that these fixed loci are enough for our study of two-point extended invariants as a consequence of (t_1+t_2)-valuation below.

Proposition 4.5.1. *If*

$$\overline{M} \in \bigcup \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s] \quad \textit{and } F \in \overline{M}_{\mathbb{T}}, \tag{4.5.1}$$

where the union ranges over all possible parameters, the inverse Euler class $\frac{1}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})}$ *has valuation* 1 *with respect to* (t_1+t_2). *Otherwise, it has valuation at least* 2.

Proof. Consider any $\mathbb{T}$-fixed connected component F of $\overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)],\Lambda,\wp;(a,\beta))$. We let $f:\mathcal{C}\to[\mathrm{Sym}^n(\mathcal{A}_r)]$ be a twisted stable map representing a point of F. As discussed earlier, there are a morphism $\tilde{f}:\tilde{C}\to\mathcal{A}_r$ and an ordinary stable map $f_c:C\to\mathrm{Sym}^n(\mathcal{A}_r)$ associated to f. Recall that $\tau=-(r+1)^2t_1^2$. To establish the assertion, we need to analyze the contribution of following situations (cf. [GP]) to the inverse Euler class $\frac{1}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})}$.

1. *Infinitesimal deformations and obstructions of f with $\mathcal{C}$ held fixed:*

   (a) Any contracted component contributes zero (t_1+t_2)-valuation. Let $\mathcal{C}'\subset\mathcal{C}$ be a contracted component and pick any connected component Z of the cover associated to $\mathcal{C}'$. We see that Z contributes

   $$\frac{e_{\mathbb{T}}(H^1(Z,\tilde{f}^*T\mathcal{A}_r))}{e_{\mathbb{T}}(H^0(Z,\tilde{f}^*T\mathcal{A}_r))} \tag{4.5.2}$$

   and is collapsed by $\tilde{f}$ to x_k for some k. So the numerator is, by Mumford's relation, congruent modulo t_1+t_2 to

   $$\Lambda^\vee(L_k)\Lambda^\vee(R_k)\equiv\tau^g,$$

where $g = \mathrm{rank}(H^0(Z,\omega_Z))$ and $\Lambda^\vee(t) = \sum_{i=0}^{g} c_i(H^0(Z,\omega_Z)^\vee)t^{g-i}$. The denominator of (4.5.2) is $e_{\mathbb{T}}(T_{x_k}\mathcal{A}_r)$. Thus, the contribution of Z is simply

$$\tau^{g-1} \mod (t_1+t_2).$$

In other words, the contribution of $\mathcal{C}'$, being the product of the contributions of such Z's, is not divisible by t_1+t_2.

(b) The nodes joining contracted curves to noncontracted curves have zero (t_1+t_2)-valuation because each of them gives some positive power of τ modulo (t_1+t_2).

(c) Noncontracted curves: Suppose $\mathcal{D}$ is a noncontracted component with $\tilde{D}$ its associated (possibly disconnected) covering. Its contribution is

$$\frac{e_{\mathbb{T}}(H^1(\mathcal{D}, f^*T[\mathrm{Sym}^n(\mathcal{A}_r)]))^{\mathrm{mov}}}{e_{\mathbb{T}}(H^0(\mathcal{D}, f^*T[\mathrm{Sym}^n(\mathcal{A}_r)]))^{\mathrm{mov}}} = \frac{e_{\mathbb{T}}(H^1(\tilde{D}, \tilde{f}^*T\mathcal{A}_r))^{\mathrm{mov}}}{e_{\mathbb{T}}(H^0(\tilde{D}, \tilde{f}^*T\mathcal{A}_r))^{\mathrm{mov}}}.$$

Here $(\ )^{\mathrm{mov}}$ stands for the moving part. It is clear from (a) that each $\tilde{f}$-contracted component of $\tilde{D}$ has zero (t_1+t_2)-valuation. However, any irreducible component Σ of $\tilde{D}$ that is not $\tilde{f}$-contracted contributes

$$\frac{t_1+t_2}{\tau} \mod (t_1+t_2)^2. \tag{4.5.3}$$

This can be seen as follows.

Assume that $\tilde{f}$ maps Σ to $E := \tilde{f}(\Sigma)$ with degree $\ell > 0$. Let $S_1 = \{0,\ldots,2\ell-2\} - \{\ell-1\}$ and $S_2 = \{0,\ldots,2\ell\} - \{\ell\}$.

The moving part of $e_{\mathbb{T}}(H^1(\Sigma, \tilde{f}^*T\mathcal{A}_r)))$ arises from

$$H^1(\Sigma, \tilde{f}^*N_{E/\mathcal{A}_r}) = H^0(\Sigma, \omega_\Sigma \otimes \tilde{f}^*N_{E/\mathcal{A}_r}^\vee)^\vee.$$

The curve E having self-intersection -2 implies $N_{E/\mathcal{A}_r} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, and so the invertible sheaf $\omega_\Sigma \otimes \tilde{f}^*N_{E/\mathcal{A}_r}^\vee$ has degree $2\ell-2$. Hence, the moving part is

$$(t_1+t_2)\prod_{k\in S_1} \frac{k(\frac{\ell-1}{\ell}(r+1)t_1) + (2\ell-2-k)(\frac{1-\ell}{\ell}(r+1)t_1)}{2\ell-2} \mod (t_1+t_2)^2$$

(which is simply (t_1+t_2) for $\ell=1$). We further simplify it to get

$$(t_1+t_2)\tau^{\ell-1}\prod_{k=1}^{\ell-1}(\frac{\ell-k}{\ell})^2 \mod (t_1+t_2)^2. \tag{4.5.4}$$

On the other hand, $e_{\mathbb{T}}(H^0(\Sigma,\tilde{f}^*T\mathcal{A}_r))^{\mathrm{mov}}$ equals $e_{\mathbb{T}}(H^0(\Sigma,\tilde{f}^*TE))^{\mathrm{mov}}$, that is congruent modulo (t_1+t_2) to

$$\prod_{k\in S_2}\frac{k(-(r+1)t_1)+(2\ell-k)((r+1)t_1)}{2\ell}\equiv\tau^{\ell}\prod_{k=1}^{\ell-1}(\frac{\ell-k}{\ell})^2. \tag{4.5.5}$$

Dividing (4.5.4) by (4.5.5) gives (4.5.3).

2. *Infinitesimal automorphisms of $\mathcal{C}$:*

We need only investigate the nonspecial points p, which lie on noncontracted curves Σ and are mapped to fixed points. In fact, each p gives the weight of the tangent space to Σ at p. This has zero (t_1+t_2)-valuation.

3. *Infinitesimal deformations of $\mathcal{C}$:*

Given any node $\mathcal{P}$ joining two curves $\mathcal{V}_1$ and $\mathcal{V}_2$. Let P, V_1, V_2 be coarse moduli spaces of $\mathcal{P}, \mathcal{V}_1, \mathcal{V}_2$ respectively and $\mathrm{Stab}(\mathcal{P})$ the stabilizer of $\mathcal{P}$. In each of the following, we study the contribution arising from smoothing the node $\mathcal{P}$.

(a) $\mathcal{V}_1$ and $\mathcal{V}_2$ are noncontracted: We may assume that the restriction of f_c to V_k is a d_k-sheeted covering

$$f_c|_{V_k}: V_k\to\Sigma_k:=f_c(V_k)\cong\mathbb{P}^1$$

for some $d_k>0$, $k=1,2$. The node-smoothing contribution is

$$|\mathrm{Stab}(\mathcal{P})|\;(\frac{w_1}{d_1}+\frac{w_2}{d_2})^{-1}, \tag{4.5.6}$$

where w_k is the tangent weight of the rational curve Σ_k at the fixed point $f_c(P)$. Thus, (4.5.6) is proportional to $(t_1+t_2)^{-1}$ only if $d_1=d_2$ and w_1+w_2 is a multiple of t_1+t_2.

(b) $\mathcal{V}_1$ is noncontracted but $\mathcal{V}_2$ is contracted: Let w be the tangent weight of V_1 at the node P and $\mathcal{L}$ the tautological line bundle formed by the cotangent space $T^*_P V_2$ (see

Chapter 6). Denote by ψ the first Chern class of $\mathcal{L}$. The node smoothing contributes

$$\frac{|\mathrm{Stab}(\mathcal{P})|}{w-\psi}. \tag{4.5.7}$$

So, neither (t_1+t_2) nor $(t_1+t_2)^{-1}$ is generated in this case.

Thus, only 1(c) and 3(a) may produce any power of (t_1+t_2). We conclude that F gives positive (t_1+t_2)-valuation because the number of noncontracted curves is more than the number of nodes joining them.

Suppose F is a $\mathbb{T}$-fixed component described in (4.5.1), in which case we have a unique chain of noncontracted rational components for the cover associated to $\mathcal{C}$. The discussion in 3(a) shows that each node in the chain gives (t_1+t_2)-valuation -1. In total, the node smoothing contributes $i-j$ in valuation. On the other hand, the chain has $j-i+1$ irreducible components. By the result of 1(c), $\frac{1}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})}$ has valuation 1, which establishes the first assertion.

Assume that F is not as in (4.5.1). If the associated cover has at least two disjoint chains of noncontracted rational curves, a (t_1+t_2)-valuation at least 2 is obtained because each chain gives valuation at least 1. Otherwise, the cover has a unique chain but property (e) (and hence (iii)) in Section 4.4 is not fulfilled for each i, j, s. In this case, we have the same consequence by the discussion in 3(a) and the result of 1(c). This shows the second assertion. □

4.6 Counting branched covers

Later, we will count certain coverings of (chains of) rational curves. Let us now review some related notions and fix notation.

For partitions $\eta_1, \ldots, \eta_s$ of n, the Hurwitz number

$$H(\eta_1, \ldots, \eta_s)$$

is the weighted number of possibly disconnected covers $\pi : X \to (\mathbb{P}^1, p_1, \ldots, p_s)$ such that π are branched over $p_1, \ldots, p_s$ with ramification profiles $\eta_1, \ldots, \eta_s$ and unbranched away from $p_1, \ldots, p_s$. (Each cover is counted with weight 1 over the size of its automorphism group).

The Hurwitz number $H(\eta_1, \ldots, \eta_s)$ is essentially a combinatorial object. It can be described combinatorially by

$$\frac{1}{n!}|\mathcal{H}(\eta_1, \ldots, \eta_s)|.$$

Here $\mathcal{H}(\eta_1, \ldots, \eta_s)$ is the set consisting of $(g_1, \ldots, g_s) \in \prod_{i=1}^s \mathfrak{S}_n$ satisfying (i) for each $i = 1, \ldots, s$, g_i has cycle type η_i; (ii) $g_1 \cdots g_s = 1$.

Let us introduce some other Hurwitz-type numbers. Let

$$\mathcal{H}_\sigma(\eta_1, \ldots, \eta_s \mid \tau_1, \ldots, \tau_t)$$

be the subset of $\mathcal{H}(\eta_1, \ldots, \eta_s, \tau_1, \ldots, \tau_t)$ such that each element $(g_1, \ldots, g_s, h_1, \ldots, h_t)$ has an additional property that $g_1 \cdots g_s$ has cycle type σ (and so $h_1 \cdots h_t$ has the same cycle type as well). Put

$$H_\sigma(\eta_1, \ldots, \eta_s \mid \tau_1, \ldots, \tau_t) := \frac{|\mathcal{H}_\sigma(\eta_1, \ldots, \eta_s \mid \tau_1, \ldots, \tau_t)|}{n!}$$

(in case σ is a vacuous partition, we set $H_\sigma(\eta_1, \ldots, \eta_s \mid \tau_1, \ldots, \tau_t) = 1$).

We readily find the following relations.

Lemma 4.6.1. *The number $H_\sigma(\eta_1, \ldots, \eta_s \mid \tau_1, \ldots, \tau_t)$ is exactly the product*

$$|C(\sigma)| \; H(\eta_1, \ldots, \eta_s, \sigma) \; H(\sigma, \tau_1, \ldots, \tau_t).$$

Moreover, we have

$$H(\eta_1, \ldots, \eta_s, \tau_1, \ldots, \tau_t) = \sum_{|\sigma|=n} H_\sigma(\eta_1, \ldots, \eta_s \mid \tau_1, \ldots, \tau_t).$$

4.7 Localization contributions

4.7.1 Reduction

From now on, fix cohomology-weighted partitions $\mu_1(\vec{\eta}_1)$ and $\mu_2(\vec{\eta}_2)$ of n with $\eta_{k\ell}$'s 1 or divisors on $\mathcal{A}_r$. We concentrate on the two-point extended invariant

$$\langle \mu_1(\vec{\eta}_1), \mu_2(\vec{\eta}_2) \rangle_{(a,\beta)}^{[\mathrm{Sym}^n(\mathcal{A}_r)]} \tag{4.7.1}$$

of twisted degree (a, β), $\beta \neq 0$. We will leave out the superscript $[\mathrm{Sym}^n(\mathcal{A}_r)]$ when there is no likelihood of confusion.

Let us write

$$\mu_i(\vec{\eta}_1) = \kappa_i(\vec{\eta}_{i1}) \theta_i(\vec{\eta}_{i2})$$

where the entries of $\vec{\eta}_{i1}$'s are all 1 and the entries of $\vec{\eta}_{i2}$'s are all divisors, $i = 1, 2$. We may assume that

$$\ell(\kappa_1) \le \ell(\kappa_2).$$

Use the identity $1 = \sum_{k=1}^{r+1} \frac{1}{L_k R_k}[x_k]$, we see readily that (4.7.1) is a $\mathbb{Q}(t_1, t_2)$-linear combination of the invariants of the form

$$\left\langle \kappa_{11}([x_{m_1}]) \cdots \kappa_{1\ell(\kappa_1)}([x_{m_{\ell(\kappa_1)}}])\theta_1(\vec{\eta}_{12}), \mu_2(\vec{\eta}_2) \right\rangle_{(a,\beta)}. \tag{4.7.2}$$

Additionally, (4.7.2) is an element of $\mathbb{Q}[t_1, t_2]$ as the first insertion has compact support. Also, the sum of the degrees of the insertions is at most 1 larger than the virtual dimension. Precisely, the difference is

$$\ell(\kappa_1) - \ell(\kappa_2) + 1.$$

Thus, the invariant (4.7.2) is a linear polynomial if $\ell(\kappa_1) = \ell(\kappa_2)$; otherwise, it is a rational number.

Assume that β is not a multiple of $\mathcal{E}_{ij}$ for any i, j. Clearly, the fixed loci (4.5.1) make no contribution. By Proposition 4.5.1, the invariant (4.7.2) is zero by divisibility of $(t_1 + t_2)^2$ (each of the two insertions is a linear combination of fixed-point classes with coefficients being 0 or having nonnegative $(t_1 + t_2)$-valuation; for details, consult the discussion preceding Lemma 4.7.3). It follows that (4.7.1) is zero as well. So we can now set our mind on the invariant

$$\langle \mu_1(\vec{\eta}_1), \mu_2(\vec{\eta}_2) \rangle_{(a, d\mathcal{E}_{ij})}, \quad d, i, j > 0. \tag{4.7.3}$$

We fix positive integers i, j, d with $i \le j$ from here on. Let $\beta = d\mathcal{E}_{ij}$. By virtual localization, (4.7.2) can be expressed as a sum of residue integrals over $\mathbb{T}$-fixed loci. By Proposition 4.5.1, the invariant (4.7.2) is $\alpha(t_1 + t_2)$ for some rational number α, and it suffices to evaluate (4.7.2) over all $\mathbb{T}$-fixed loci lying in the union $\coprod^{i,j} \mathcal{F}^{\vec{\sigma}}_{\lambda_0, \rho_0, \sigma_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{\rho}; \vec{b}^L, \vec{b}^R)[i, j, s]$, where $\coprod^{i,j}$ means that only i, j are fixed and the other parameters can vary.

4.7.2 The setup for localization

Given $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0, \rho_0, \sigma_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{\rho}; \vec{b}^L, \vec{b}^R)[i, j, s]$ and $F \in \overline{M}_{\mathbb{T}}$, we let

$$\iota_F : F \to \overline{M}([\mathrm{Sym}^n(\mathcal{A}_r)], \Lambda, \wp, (a, d\mathcal{E}_{ij}))$$

be the natural inclusion (as partitions, $\Lambda = \hat{\lambda}$, and $\wp = \hat{\rho}$).

Let $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})[i,j,s]$ (resp. $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda}, \vec{\rho}, \vec{b}^L \mid \vec{b}^R)[i,j,s]$). As mentioned earlier, there are natural morphisms

$$\phi_F : F \to \overline{M}_{0,b_0^L+u_0^L+2} \times \overline{M}_{0,b_0^R+u_0^R+2} \text{ (resp. } \overline{M}_{0,b_0^L+u_0^L+3} \times \overline{M}_{0,b_0^R+u_0^R+1})$$

for $F \in \overline{M}_{\mathbb{T}}$ and

$$\phi_{\overline{M}_0} : \overline{M}_0 \to \overline{M}_{0,b_0^L+u_0^L+2} \times \overline{M}_{0,b_0^R+u_0^R+2} \text{ (resp. } \overline{M}_{0,b_0^L+u_0^L+3} \times \overline{M}_{0,b_0^R+u_0^R+1}).$$

Obviously, $F^c = \overline{M}_0^c$. We intend to calculate our Gromov-Witten invariants by localization, which will be reduced to integrals over F^c's. So it is necessary to understand the degree $\deg(\phi_F)$ of the morphism ϕ_F.

For $F \in \overline{M}_{\mathbb{T}}$ with $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})[i,j,s]$, we let $[f : \mathcal{C}_L \cup \mathcal{D} \cup \mathcal{C}_R \to [\mathrm{Sym}^n(\mathcal{A}_r)]] \in F$ (in the notation of Section 4.4) be a typical element. We obtain

$$\deg(\phi_F) = m_1 \cdot m_2.$$

Here

- $m_1 = c_0(o(\hat{\sigma})^{-1} \prod_{k=1}^{r+1} |C(\sigma_k)|)^{\varepsilon(F)}$ is a factor arising from the nodes, which are glued over the rigidified inertia stack. Here c_0 is an overall factor coming from nodes of the cover $\tilde{C}_0 \to C$ (we do not have to give a careful description here as c_0 will be cancelled by an identical term in $\deg(\phi_{\overline{M}_0})$), and $\varepsilon(F) = \epsilon_1(b_0^L + u_0^L) + \epsilon_1(b_0^R + u_0^R) + j - i$ (the terms $\epsilon_1(b_0^L + u_0^L)$ and $\epsilon_1(b_0^R + u_0^R)$ record the dimensions of C_L and C_R respectively).

- m_2 is given by

$$d^{j-i+1} m_0 \prod_{k=1}^{r+1} H(\lambda_k, (2)^{b_k^L}, 1^{b_0^L+u_0^L-b_k^L}, \sigma_k)\ H(\sigma_k, \sigma_k)^{j-i+1}\ H(\sigma_k, (2)^{b_k^R}, 1^{b_0^R+u_0^R-b_k^R}, \rho_k),$$

  where d^{j-i+1} is an automorphism factor that takes care of the restriction $f|_{\mathcal{D}}$ forgotten by ϕ_F, m_0 is the contribution of $\tilde{C}_0$, and the other terms account for the overall contribution of the disconnected curve $\coprod_{k=1}^{r+1} \tilde{C}_k$.

Also, the degree of $\phi_{\overline{M}_0}$ can be calculated in a similar fashion. That is,

$$\deg(\phi_{\overline{M}_0}) = c_0(\frac{1}{o(\sigma_0)})^{\varepsilon(F)} d^{j-i+1} m_0.$$

By Lemma 4.6.1, we may write $\deg(\phi_F)$ as

$$\deg(\phi_{\overline{M}_0})(\frac{o(\sigma_0)}{o(\hat{\sigma})})^{\varepsilon(F)} \prod_{k=1}^{r+1} H_{\sigma_k}(\lambda_k, (2)^{b_k^L}, 1^{b_0^L+u_0^L-b_k^L} \mid (2)^{b_k^R}, 1^{b_0^R+u_0^R-b_k^R}, \rho_k). \tag{4.7.4}$$

Similarly, for $F \in \overline{M}_{\mathbb{T}}$ with $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda}, \vec{\rho}, \vec{b}^L \mid \vec{b}^R)[i,j,s]$, $\deg(\phi_F)$ is given by

$$\deg(\phi_{\overline{M}_0})(\frac{o(\sigma_0)}{o(\hat{\sigma})})^{\varepsilon(F)} \prod_{k=1}^{r+1} H_{\sigma_k}(\lambda_k, \rho_k, (2)^{b_k^L}, 1^{b_0^L+u_0^L-b_k^L} \mid (2)^{b_k^R}, 1^{b_0^R+u_0^R-b_k^R}). \tag{4.7.5}$$

Now $\varepsilon(F)$ is set to be $1 + \epsilon_2(b_0^R + u_0^R) + j - i$.

Remark. The term $(\frac{o(\sigma_0)}{o(\hat{\sigma})})^{\varepsilon(F)}$ will cancel with a similar term in $\frac{1}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})}$ (see Lemma 4.7.1 below). Moreover, forgetting the indices involving the partition 1 does not change the value of the Hurwitz-type numbers. We did not do this in the above formulas so as to keep track of the ramification profiles corresponding to the simple marked points.

4.7.3 Virtual normal bundles

Let us determine $\frac{1}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})}$ modulo $(t_1+t_2)^2$ for each connected component F described in (4.5.1). The following outcome should be within our expectation. Recall again that $\tau = -(r+1)^2 t_1^2$.

Lemma 4.7.1. *Given $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda}, \vec{\rho}; \vec{b}^L, \vec{b}^R)[i,j,s]$ and any connected component $F \in \overline{M}_{\mathbb{T}}$, we have the congruence equation*

$$\frac{1}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})} \equiv (\frac{o(\hat{\sigma})}{o(\sigma_0)})^{\varepsilon(F)} \frac{\tau^{\frac{1}{2}(a-s-\ell(\vec{\lambda})-\ell(\vec{\rho}))}}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})} \quad \mathrm{mod}\ (t_1+t_2)^2.$$

Here $\varepsilon(F)$'s are as in (4.7.4)*,* (4.7.5) *respectively.*

Proof. We just investigate the case where $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})$ and $F \in \overline{M}_{\mathbb{T}}$, the other case being similar.

Let $p = \sum_{k=0}^{r+1} b_k^L$ and $q = \sum_{k=0}^{r+1} b_k^R$, and so $p+q = a$. Assume that $p, q > 0$. Pick any point $[f] \in F$. Again, we follow the notation of Section 4.4. The contribution of the contracted component $\mathcal{C}_L$ is

$$\frac{e_{\mathbb{T}}(H^1(\mathcal{C}_L, f^*[\mathrm{Sym}^n(\mathcal{A}_r)]))}{e_{\mathbb{T}}(H^0(\mathcal{C}_L, f^*[\mathrm{Sym}^n(\mathcal{A}_r)]))} \equiv \tau^{\sum_k (g_k-1)} \mod (t_1+t_2).$$

Here g_k's are the genera of connected components of the covering associated to $\mathcal{C}_L$. We find, by Riemann-Hurwitz formula, that $\sum_k (g_k - 1) = \frac{1}{2}(p - \ell(\hat{\lambda}) - \ell(\hat{\sigma}))$. Hence $\mathcal{C}_L$ contributes

$$\tau^{\frac{1}{2}(p-\ell(\hat{\lambda})-\ell(\hat{\sigma}))} \mod (t_1+t_2).$$

Similarly, $\mathcal{C}_R$ contributes

$$\tau^{\frac{1}{2}(q-\ell(\hat{\rho})-\ell(\hat{\sigma}))} \mod (t_1+t_2).$$

And the contribution of nodes joining contracted components to $\mathcal{D}$ is

$$\tau^{2\ell(\hat{\sigma})} \mod (t_1+t_2).$$

These three contributions, taken together, yield

$$\tau^{\frac{1}{2}(a-\ell(\hat{\lambda})-\ell(\hat{\rho})+2\ell(\hat{\sigma}))} \mod (t_1+t_2).$$

One can check that the same formula holds when $p = 0$ or $q = 0$.

As for the cover $\tilde{C}_{L0} \cup \tilde{D}_0 \cup \tilde{C}_{R0}$, by a similar argument, the combined contribution of $\tilde{C}_{L0}, \tilde{C}_{R0}$, and nodes joining $\tilde{C}_{L0}, \tilde{C}_{R0}$ to $\tilde{D}_0$ is given by

$$\tau^{\frac{1}{2}(s-\ell(\lambda_0)-\ell(\rho_0)+2\ell(\sigma_0))} \mod (t_1+t_2).$$

Further, the covers $\tilde{D}_1, \ldots, \tilde{D}_{r+1}$ (including the nodes inside) contribute

$$\frac{1}{\tau^{\ell(\vec{\sigma})}} \mod (t_1+t_2).$$

We now study the infinitesimal deformations of $\mathcal{C}$. Let $k = L, R$. When $\mathcal{C}_k$ is a curve, smoothing the node $\mathcal{P}_k$ joining $\mathcal{C}_k$ to $\mathcal{D}$ contributes

$$\frac{o(\hat{\sigma})}{w_k - \psi_k},$$

where w_k is the $\mathbb{T}$-weight of the tangent space to $c(\mathcal{D})$ at the point $c(\mathcal{P}_k)$, and ψ_k is the class associated to $T^*_{c(\mathcal{P}_k)}\mathcal{C}_k$ (cf. (4.5.7)). By property (e) in Section 4.4, $\tilde{f} : \tilde{C}_0 \to \mathcal{A}_r$ corresponds to the point $[f_0 : \mathcal{C}_{L0} \cup \mathcal{D}_0 \cup \mathcal{C}_{R0} \to [\mathrm{Sym}^{|\lambda_0|}(\mathcal{A}_r)]] \in \overline{M}_0$, so

$$\frac{o(\sigma_0)}{w_k - \psi_k}$$

is the factor smoothing nodes joining $\mathcal{C}_{k0}$ and $\mathcal{D}_0$ and is $o(\sigma_0)/o(\hat{\sigma})$ times the preceding factor. Similarly, the overall contributions of node smoothing inside $\mathcal{D}$ and node smoothing inside $\mathcal{D}_0$ differ by a factor $(o(\hat{\sigma})/o(\sigma_0))^{j-i}$. Hence, deformations of $\mathcal{C}$ contribute $(o(\hat{\sigma})/o(\sigma_0))^{\varepsilon(F)}$ times those of $\mathcal{C}_{L0} \cup \mathcal{D}_0 \cup \mathcal{C}_{R0}$, and the term

$$(\frac{o(\hat{\sigma})}{o(\sigma_0)})^{\varepsilon(F)} \frac{1}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})}$$

is the combined contribution of the deformations of $\mathcal{C}$ and the unique noncontracted connected component $\tilde{C}_0$ of the associated cover $\tilde{C}$.

Putting all these together, we get

$$\begin{aligned}
\frac{1}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})} &\equiv (\frac{o(\hat{\sigma})}{o(\sigma_0)})^{\varepsilon(F)} \frac{1}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})} \cdot \frac{\tau^{\frac{1}{2}(a-\ell(\hat{\lambda})-\ell(\hat{\rho})+2\ell(\hat{\sigma}))}}{\tau^{\frac{1}{2}(s-\ell(\lambda_0)-\ell(\rho_0)+2\ell(\sigma_0))}} \cdot \frac{1}{\tau^{\ell(\vec{\sigma})}} \\
&\equiv (\frac{o(\hat{\sigma})}{o(\sigma_0)})^{\varepsilon(F)} \frac{\tau^{\frac{1}{2}(a-s-\ell(\vec{\lambda})-\ell(\vec{\rho}))}}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})} \mod (t_1+t_2)^2,
\end{aligned}$$

as desired. □

4.7.4 Vanishing and relation to connected invariants

Let us look closely at the invariant (4.7.2) with $\beta = d\mathcal{E}_{ij}$. We will go back to (4.7.3) in the end.

For every nonnegative integer s, let

$$I(s)$$

be the contribution of the $\mathbb{T}$-fixed loci $\coprod^{i,j,s} \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho};\vec{b}^L,\vec{b}^R)[i,j,s]$ (all but i,j,s vary) to the invariant (4.7.2) with $\beta = d\mathcal{E}_{ij}$. We claim the following.

Proposition 4.7.2. *For any $s < a$,*

$$I(s) \equiv 0 \mod (t_1+t_2)^2.$$

Now fix a nonnegative integer $s < a$ as well. For simplicity, we drop the index $[i, j, s]$ from the notation of fixed loci.

We would like to deduce Proposition 4.7.2 by replacing the first two insertions with $\mathbb{T}$-fixed point classes. Fix $\mathbb{T}$-fixed point classes $\vec{A}$, $\vec{B}$. Define

$$\mathcal{I} := \sum_{\overline{M}} \sum_{F \in \overline{M}_{\mathbb{T}}} \int_F \frac{\iota_F^*(\mathrm{ev}_1^*(\vec{A})\mathrm{ev}_2^*(\vec{B}))}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})}, \tag{4.7.6}$$

where $\overline{M}$ is taken over all possible $\mathbb{T}$-fixed loci in $\coprod_{\sigma_0, b_0^L, u_0^L, \vec{\sigma}, \vec{b}^L, \vec{b}^R} \mathcal{F}^{\vec{\sigma}}_{\lambda_0, \rho_0, \sigma_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{\rho}; \vec{b}^L, \vec{b}^R)$. The coefficient

$$\frac{\langle \kappa_{11}([x_{m_1}]) \cdots \kappa_{1\ell(\kappa_1)}([x_{m_{\ell(\kappa_1)}}])\theta_1(\vec{\eta}_{12}) | \vec{A}\rangle}{\langle \vec{A} | \vec{A} \rangle} \cdot \frac{\langle \mu_2(\vec{\eta}_2) | \vec{B} \rangle}{\langle \vec{B} | \vec{B} \rangle},$$

is either zero or has nonnegative valuation with respect to $t_1 + t_2$, so Proposition 4.7.2 follows from the following lemma.

Lemma 4.7.3.

$$\mathcal{I} \equiv 0 \mod (t_1 + t_2)^2.$$

Proof of Lemma 4.7.3

Lemma 4.7.3 is clear if the condition

$$\lambda_k \subset A_k \text{ and } \rho_k \subset B_k, \quad \forall k = 1, \ldots, r+1 \tag{4.7.7}$$

does not hold, in which case $\mathcal{I}$ is identically zero. Now we assume (4.7.7), and the idea of the proof in this case is to relate $\mathcal{I}$ to certain connected invariants. We put

$$\bar{\lambda}_k = A_k - \lambda_k, \bar{\rho}_k = B_k - \rho_k.$$

That is, we may write $\vec{A} = ((\lambda_1, \bar{\lambda}_1), \ldots, (\lambda_{r+1}, \bar{\lambda}_{r+1}))$ and $\vec{B} = ((\rho_1, \bar{\rho}_1), \ldots, (\rho_{r+1}, \bar{\rho}_{r+1}))$. Let

$$\bar{A} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{r+1}) \text{ and } \bar{B} = (\bar{\rho}_1, \ldots, \bar{\rho}_{r+1})$$

be $\mathbb{T}$-fixed point classes.

First of all, it is good to have some observations on hand.

Lemma 4.7.4. *For every partition σ_0 and $(r+1)$-tuples $\vec{b}^L, \vec{b}^R$, $\vec{\sigma}$,*

$$J_1(\sigma_0; b_0^L, u_0^L) := \sum_{\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0, \sigma_0, \rho_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})} \deg(\phi_{\overline{M}_0}) \int_{\overline{M}_0^c} \frac{\iota^*_{\overline{M}_0}(\mathrm{ev}_1^*(\bar{A})\mathrm{ev}_2^*(\bar{B}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})}$$

is

$$\sum_{\overline{M} \in \mathcal{F}^{\vec{\theta}}_{\lambda_0, \sigma_0, \rho_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{c}^L \mid \vec{c}^R, \vec{\rho})} \deg(\phi_{\overline{M}_0}) \int_{\overline{M}_0^c} \frac{\iota^*_{\overline{M}_0}(\mathrm{ev}_1^*(\bar{A})\mathrm{ev}_2^*(\bar{B}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})},$$

and

$$J_2(\sigma_0; b_0^L, u_0^L) := \sum_{\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0, \rho_0, \sigma_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{\rho}, \vec{b}^L \mid \vec{b}^R)} \deg(\phi_{\overline{M}_0}) \int_{\overline{M}_0^c} \frac{\iota^*_{\overline{M}_0}(\mathrm{ev}_1^*(\bar{A})\mathrm{ev}_2^*(\bar{B}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})}$$

is

$$\sum_{\overline{M} \in \mathcal{F}^{\vec{\theta}}_{\lambda_0, \rho_0, \sigma_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{\rho}, \vec{c}^L \mid \vec{c}^R)} \deg(\phi_{\overline{M}_0}) \int_{\overline{M}_0^c} \frac{\iota^*_{\overline{M}_0}(\mathrm{ev}_1^*(\bar{A})\mathrm{ev}_2^*(\bar{B}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})},$$

for any $\vec{c}^L$, $\vec{c}^R$ and $\vec{\theta}$ satisfying $|\theta_k| = |\sigma_k|$ for each $k = 1, \dots, r+1$. Here the collections of $\mathbb{T}$-fixed loci under the summation symbols are all nonempty.

Proof. The first identity follows as $\mathcal{F}^{\vec{\sigma}}_{\lambda_0, \sigma_0, \rho_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})$ and $\mathcal{F}^{\vec{\theta}}_{\lambda_0, \sigma_0, \rho_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{c}^L \mid \vec{c}^R, \vec{\rho})$ have the same number of elements and the same configuration for the unique noncontracted connected component of the associated cover (see the description in Section 4.4). The second identity holds for similar reasons. □

We apply Proposition 4.5.1 to the connected invariant

$$\langle \bar{A}, \bar{B}, (2)^s, 1^{a-s} \rangle^{\mathrm{conn}}_{d\mathcal{E}_{ij}} \tag{4.7.8}$$

and find that (4.7.8) is given by

$$\sum_{\sigma_0, b_0^L, u_0^L} (J_1(\sigma_0; b_0^L, u_0^L) + J_2(\sigma_0; b_0^L, u_0^L)) \mod (t_1 + t_2)^2.$$

As $a - s > 0$, (4.7.8) is zero. We have

$$\sum_{\sigma_0, b_0^L, u_0^L} (J_1(\sigma_0; b_0^L, u_0^L) + J_2(\sigma_0; b_0^L, u_0^L)) \equiv 0 \mod (t_1 + t_2)^2. \tag{4.7.9}$$

Here is an elementary but helpful combinatorial fact.

Lemma 4.7.5. *Given nonnegative integers k, p and $p_1, \ldots, p_k$ with $p_1 + \cdots + p_k = p$. For any nonnegative integer $m \leq p$,*

$$\binom{p}{p_1, \ldots, p_k} = \sum_{m_1, \ldots, m_k} \binom{m}{m_1, \ldots, m_k} \binom{p-m}{p_1 - m_1, \ldots, p_k - m_k}.$$

Note that $\binom{\ell}{\ell_1, \ldots, \ell_k}$ is declared to be 0 if ℓ is smaller than some of ℓ_i's or if some entries are negative integers. □

We continue the proof of Lemma 4.7.3. Let

$$\theta = \frac{1}{2}(a - s + \ell(\vec{\lambda}) + \ell(\vec{\rho})).$$

For every $(r+1)$-tuple $\vec{q}$ with $|\vec{q}| = a - s$, let

$$Q(\vec{q}) = \{(\vec{b}^L, \vec{b}^R) \mid b_k^L + b_k^R = q_k, \ \forall k = 1, \ldots, r+1\}.$$

Fix σ_0, b_0^L, u_0^L, we consider two cases:

(1) The contribution of $\mathcal{F}^{\vec{\sigma}}_{\lambda_0, \sigma_0, \rho_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})$'s to $\mathcal{I}$ with the constraint $(\vec{b}^L, \vec{b}^R) \in Q(\vec{q})$ is

$$\sum_{(\vec{b}^L, \vec{b}^R) \in Q(\vec{q})} \sum_{\vec{\sigma}} \sum_{\overline{M}} \sum_{F \in \overline{M}_{\mathbb{T}}} \int_F \frac{\iota_F^*(\mathrm{ev}_1^*(A)\mathrm{ev}_2^*(B))}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})} \mod (t_1 + t_2)^2, \tag{4.7.10}$$

where $\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0, \sigma_0, \rho_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})$ runs through all $\mathbb{T}$-fixed loci. By (4.1.1), for each $F \in \overline{M}_{\mathbb{T}}$,

$$\iota_F^*(\mathrm{ev}_1^*(A) \cdot \mathrm{ev}_2^*(B)) \equiv \tau^{\ell(\vec{\lambda}) + \ell(\vec{\rho})} \ \iota_{\overline{M}_0}^*(\mathrm{ev}_1^*(\bar{A}) \cdot \mathrm{ev}_2^*(\bar{B})) \mod (t_1 + t_2).$$

Applying the pushforward ϕ_{F*} and Lemma 4.7.1, (4.7.10) is given by

$$\tau^\theta \sum_{(\vec{b}^L, \vec{b}^R) \in Q(\vec{q})} \sum_{\vec{\sigma}} \sum_{\overline{M} \in \mathcal{F}^{\vec{\sigma}}_{\lambda_0, \sigma_0, \rho_0; b_0^L, u_0^L}(\vec{\lambda}, \vec{b}^L \mid \vec{b}^R, \vec{\rho})} \sum_{F \in \overline{M}_{\mathbb{T}}} \deg(\phi_F)$$
$$\cdot \left(\frac{o(\hat{\sigma})}{o(\sigma_0)}\right)^{\varepsilon(F)} \int_{\overline{M}_0^c} \frac{\iota_{\overline{M}_0}^*(\mathrm{ev}_1^*(\bar{A})\mathrm{ev}_2^*(\bar{B}))}{e_{\mathbb{T}}(N_{\overline{M}_0}^{\mathrm{vir}})} \mod (t_1 + t_2)^2.$$

By (4.7.4), (4.7.10) is congruent modulo $(t_1+t_2)^2$ to

$$\tau^\theta \sum_{(\vec{b}^L,\vec{b}^R)\in Q(\vec{q})} \binom{u_0^L}{b_1^L,\dots b_{r+1}^L}\binom{u_0^R}{b_1^R,\dots,b_{r+1}^R}$$
$$\cdot \sum_{\vec{\sigma}}\prod_{k=1}^{r+1} H_{\sigma_k}(\lambda_k,(2)^{b_k^L},1^{b_0^L+u_0^L-b_k^L}\mid(2)^{b_k^R},1^{b_0^R+u_0^R-b_k^R},\rho_k)$$
$$\cdot \sum_{\overline{M}\in\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b_0^L,u_0^L}(\vec{\lambda},\vec{b}^L\mid\vec{b}^R,\vec{\rho})} \deg(\phi_{\overline{M}_0})\int_{\overline{M}_0^c}\frac{\iota^*_{\overline{M}_0}(\mathrm{ev}_1^*(\bar{A})\mathrm{ev}_2^*(\bar{B}))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})},$$

where the product $\binom{u_0^L}{b_1^L,\dots b_{r+1}^L}\binom{u_0^R}{b_1^R,\dots,b_{r+1}^R}$ is the number of choices to distribute simple ramification points lying above simple markings.

By Lemmas 4.6.1, 4.7.4 and 4.7.5, (4.7.10) is simplified to

$$\binom{a-s}{q_1,\dots,q_{r+1}}\tau^\theta\prod_{k=1}^{r+1}H(\lambda_k,(2)^{q_k},1^{a-q_k},\rho_k)J_1(\sigma_0;b_0^L,u_0^L) \mod (t_1+t_2)^2.$$

(2) By a similar argument, the contribution of $\mathcal{F}^{\vec{\sigma}}_{\lambda_0,\rho_0,\sigma_0;b_0^L,u_0^L}(\vec{\lambda},\vec{\rho},\vec{b}^L\mid\vec{b}^R)$'s to $\mathcal{I}$ with the constraint $(\vec{b}^L,\vec{b}^R)\in Q(\vec{q})$ is

$$\binom{a-s}{q_1,\dots,q_{r+1}}\tau^\theta\prod_{k=1}^{r+1}H(\lambda_k,(2)^{q_k},1^{a-q_k},\rho_k)J_2(\sigma_0;b_0^L,u_0^L) \mod (t_1+t_2)^2.$$

In total, $\mathcal{I}$ is given by

$$H\cdot\sum_{\sigma_0;b_0^L,u_0^L}(J_1(\sigma_0;b_0^L,u_0^L)+J_2(\sigma_0;b_0^L,u_0^L)) \mod (t_1+t_2)^2, \tag{4.7.11}$$

where $H:=\sum_{|\vec{q}|=a-s}\binom{a-s}{q_1,\dots,q_{r+1}}\tau^\theta\prod_{k=1}^{r+1}H(\lambda_k,(2)^{q_k},1^{a-q_k},\rho_k)$. By (4.7.9),

$$\mathcal{I}\equiv 0 \mod (t_1+t_2)^2.$$

This shows Lemma 4.7.3 and ends the proof of Proposition 4.7.2. □

4.8 Combinatorial descriptions of two-point extended invariants

Now we return to the invariant (4.7.3). We will interpret it combinatorially in terms of orbifold Poincaré pairings and connected invariants. In general, we have the following fact on two-point extended invariants of nonzero degree.

Theorem 4.8.1. *Given partitions μ_1, μ_2 of n and $\ell(\mu_1)$-tuple $\vec{\eta}_1$, $\ell(\mu_2)$-tuple $\vec{\eta}_2$ with entries* 1 *or divisors on $\mathcal{A}_r$. For any curve class $\beta \neq 0$, the invariant*

$$\langle \mu_1(\vec{\eta}_1), \mu_2(\vec{\eta}_2) \rangle_{(a,\beta)} \tag{4.8.1}$$

is given by the sum

$$\sum \langle \theta(\vec{\xi}_1) | \theta(\vec{\xi}_2) \rangle \, \langle \nu_1(\vec{\gamma}_1), \nu_2(\vec{\gamma}_2) \rangle^{\text{conn}}_{(a,\beta)} \, . \tag{4.8.2}$$

Here the sum is taken over all possible cohomology-weighted partitions $\theta(\vec{\xi}_1)$, $\theta(\vec{\xi}_2)$, $\nu_1(\vec{\gamma}_1)$, $\nu_2(\vec{\gamma}_2)$ satisfying $\mu_1(\vec{\eta}_1) = \theta(\vec{\xi}_1)\nu_1(\vec{\gamma}_1)$ and $\mu_2(\vec{\eta}_2) = \theta(\vec{\xi}_2)\nu_2(\vec{\gamma}_2)$. (In particular, ν_1, ν_2 are subpartitions of μ_1, μ_2 respectively and $\mu_1 - \nu_1 = \theta = \mu_2 - \nu_2$).

Proof of Theorem 4.8.1

The statement is clear if β is not a multiple of $\mathcal{E}_{ij}$ for each i, j because both (4.8.1) and (4.8.2) vanish. Now fix $i, j, d > 0$ and let $\beta = d\mathcal{E}_{ij}$.

We learn by Proposition 4.7.2 that $I(a)$ is the only possible contribution to (4.7.2). In other words, only

$$\mathcal{F}_{\sigma_0,b}(\lambda_0, \rho_0; \vec{\sigma}) := \mathcal{F}_1 \cup \mathcal{F}_2,$$

ranging over all possible $\lambda_0, \rho_0, \sigma_0, b, \vec{\sigma}$, can possibly make a contribution. Here

$$\mathcal{F}_1 = \mathcal{F}^{\vec{\sigma}}_{\lambda_0,\sigma_0,\rho_0;b,0}(\vec{\sigma}, (0,\ldots,0) \mid (0,\ldots,0), \vec{\sigma})[i,j,a], \tag{4.8.3}$$

$$\mathcal{F}_2 = \mathcal{F}^{(1^n)}_{\lambda_0,\rho_0,\sigma_0;b,0}(\vec{\sigma}, \vec{\sigma}, (0,\ldots,0) \mid (0,\ldots,0))[i,j,a]. \tag{4.8.4}$$

(With notation of Section 4.4, the admissible cover $\tilde{C}$ corresponding to any of these fixed loci has all those simple ramification points that are branched over simple markings in the connected component $\tilde{C}_0$, and each $\tilde{C}_k$ ($k \neq 0$) is either empty or a chain of rational curves.)

As mentioned earlier, in order to evaluate the invariant (4.8.1), it is enough to perform localization calculations over $\mathcal{F}_{\sigma_0,b}(\lambda_0,\rho_0;\vec{\sigma})$'s because (4.8.1) is a linear combination of invariants of the form (4.7.2).

We have a lemma on the inverse Euler classes of virtual normal bundles.

Lemma 4.8.2. *Given $F \in \overline{M}_{\mathbb{T}}$ with $\overline{M} \in \mathcal{F}_1 \cup \mathcal{F}_2$, we have*

$$\frac{1}{e_{\mathbb{T}}(N_F^{\mathrm{vir}})} = (\frac{o(\hat{\sigma})}{o(\sigma_0)})^{\varepsilon_k(F)} \frac{1}{t(\widetilde{\sigma})\ e_{\mathbb{T}}(N_{\overline{M}_0}^{\mathrm{vir}})},$$

for $\overline{M} \in \mathcal{F}_k$, $k = 1, 2$. Here $\varepsilon_1(F) = \epsilon_1(b) + \epsilon_1(a-b) + j - i$ and $\varepsilon_2(F) = 1 + \epsilon_2(a-b) + j - i$.

Proof. All contracted connected components of the associated cover are necessarily of genus 0. The proof of Lemma 4.7.1 can be carried through. □

We let

$$\mathcal{I}(\nu_1,\nu_2) \text{ and } \mathcal{I}(\nu_1,\nu_2;\vec{\sigma})$$

be the contributions to (4.8.1) of $\coprod_{\sigma_0,b,\vec{\sigma}} \mathcal{F}_{\sigma_0,b}(\nu_1,\nu_2;\vec{\sigma})$ and $\coprod_{\sigma_0,b} \mathcal{F}_{\sigma_0,b}(\nu_1,\nu_2;\vec{\sigma})$ respectively.

Now we compute $\mathcal{I}(\nu_1,\nu_2;\vec{\sigma})$. In order for the contribution not to vanish, the partitions ν_1 and ν_2 must be subpartitions of μ_1 and μ_2 respectively. Let us assume $\nu_1 \subset \mu_1$, $\nu_2 \subset \mu_2$. The configurations (4.8.3) and (4.8.4) force $\mu_1 - \nu_1 = \mu_2 - \mu_2$. We set $\theta = \mu_1 - \nu_1$.

Lemma 4.8.3. *Take a fixed locus $\overline{M} \in \coprod_{b,\sigma_0} \mathcal{F}_{\sigma_0}^{\vec{\sigma}}(\nu_1,\nu_2,\vec{\sigma})$. For $k = 1, 2$ and each $F \in \overline{M}_{\mathbb{T}}$,*

$$\iota_F^* \mathrm{ev}_k^*(\mu_k(\vec{\eta}_k)) = t(\widetilde{\sigma}) \sum_{P_k} \alpha_{\theta(\vec{\xi}_k)}(\widetilde{\sigma})\ \iota_{\overline{M}_0}^* \mathrm{ev}_k^*(\nu_k(\vec{\gamma}_k)). \tag{4.8.5}$$

Here P_k means that we take the sum over all possible $\theta(\vec{\xi}_k)$, $\nu_k(\vec{\gamma}_k)$ satisfying the equality $\mu_k(\vec{\eta}_k) = \theta(\vec{\xi}_k)\nu_k(\vec{\gamma}_k)$.

Proof. The left side of (4.8.5) is $\sum_{\widetilde{\delta} \supset \widetilde{\sigma}} \alpha_{\mu_k(\vec{\eta}_k)}(\widetilde{\delta})\ t(\widetilde{\delta})$. By Proposition 3.2.1, it equals

$$\sum_{\widetilde{\delta} \supset \widetilde{\sigma}} \sum_{P_k} \alpha_{\theta(\vec{\xi}_k)}(\widetilde{\sigma}) \alpha_{\nu_1(\vec{\gamma}_k)}(\widetilde{\delta} - \widetilde{\sigma})\ t(\widetilde{\delta}) = t(\widetilde{\sigma}) \sum_{P_k} \alpha_{\theta(\vec{\xi}_k)}(\widetilde{\sigma}) \sum_{\widetilde{\epsilon}} \alpha_{\nu_1(\vec{\gamma}_k)}(\widetilde{\epsilon})\ t(\widetilde{\epsilon}),$$

which gives the right side of (4.8.5). □

It follows from Lemma 4.8.3 that for each $F \in \overline{M}_{\mathbb{T}}$, $\iota_F^*(\mathrm{ev}_1^*(\mu_1(\vec{\eta}_1)) \cdot \mathrm{ev}_2^*(\mu_2(\vec{\eta}_2)))$ coincides with

$$t(\widetilde{\sigma})^2 \sum_Q \alpha_{\theta(\vec{\xi}_1)}(\widetilde{\sigma})\alpha_{\theta(\vec{\xi}_2)}(\widetilde{\sigma})\iota^*_{\overline{M}_0}(\mathrm{ev}_1^*(\nu_1(\vec{\gamma}_1)) \cdot \mathrm{ev}_2^*(\nu_2(\vec{\gamma}_2))). \tag{4.8.6}$$

In the formula, the index Q means that the sum is over all possible $\theta(\vec{\xi}_1)$, $\theta(\vec{\xi}_2)$, $\nu_1(\vec{\gamma}_1))$ and $\nu_2(\vec{\gamma}_2))$ satisfying $\mu_1(\vec{\eta}_1) = \theta(\vec{\xi}_1)\nu_1(\vec{\gamma}_1)$ and $\mu_2(\vec{\eta}_2) = \theta(\vec{\xi}_2)\nu_2(\vec{\gamma}_2)$. Applying (4.8.6) and Lemma 4.8.2, the contribution $\mathcal{I}(\nu_1, \nu_2; \vec{\sigma})$ is

$$\frac{t(\widetilde{\sigma})}{a!} \sum_Q \alpha_{\theta(\vec{\xi}_1)}(\widetilde{\sigma})\alpha_{\theta(\vec{\xi}_2)}(\widetilde{\sigma}) \sum_{\sigma_0, b, \overline{M}_0} \mathbb{H}(\widetilde{\sigma}) \int_{\overline{M}_0} \frac{\iota^*_{\overline{M}_0}(\mathrm{ev}_1^*(\nu_1(\vec{\gamma}_1)) \cdot \mathrm{ev}_2^*(\nu_2(\vec{\gamma}_2))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\overline{M}_0})},$$

where $\mathbb{H}(\widetilde{\sigma}) := \prod_{k=1}^{r+1} H(\sigma_k, \sigma_k)$ is a product of Hurwitz numbers. Thus, $\mathcal{I}(\nu_1, \nu_2; \vec{\sigma})$ is simplified to

$$\mathbb{H}(\widetilde{\sigma})t(\widetilde{\sigma}) \sum_Q \alpha_{\theta(\vec{\xi}_1)}(\widetilde{\sigma})\alpha_{\theta(\vec{\xi}_2)}(\widetilde{\sigma}) \left\langle \nu_1(\vec{\gamma}_1), \nu_2(\vec{\gamma}_2) \right\rangle^{\mathrm{conn}}_{(a, d\mathcal{E}_{ij})}.$$

Adding up all possible $\mathcal{I}(\nu_1, \nu_2; \vec{\sigma})$'s, we obtain

$$\mathcal{I}(\nu_1, \nu_2) = \sum_Q \sum_{\widetilde{\sigma}} \mathbb{H}(\widetilde{\sigma})t(\widetilde{\sigma})\alpha_{\theta(\vec{\xi}_1)}(\widetilde{\sigma})\alpha_{\theta(\vec{\xi}_2)}(\widetilde{\sigma}) \left\langle \nu_1(\vec{\gamma}_1), \nu_2(\vec{\gamma}_2) \right\rangle^{\mathrm{conn}}_{(a, d\mathcal{E}_{ij})}.$$

Moreover,

$$\langle \theta(\vec{\xi}_1) | \theta(\vec{\xi}_2) \rangle = \sum_{\widetilde{\sigma}} \alpha_{\theta(\vec{\xi}_1)}(\widetilde{\sigma})\alpha_{\theta(\vec{\xi}_2)}(\widetilde{\sigma}) \langle \widetilde{\sigma} | \widetilde{\sigma} \rangle = \sum_{\widetilde{\sigma}} \alpha_{\theta(\vec{\xi}_1)}(\widetilde{\sigma})\alpha_{\theta(\vec{\xi}_2)}(\widetilde{\sigma})\mathbb{H}(\widetilde{\sigma})t(\widetilde{\sigma}).$$

This implies that

$$\mathcal{I}(\nu_1, \nu_2) = \sum_Q \langle \theta(\vec{\xi}_1) | \theta(\vec{\xi}_2) \rangle \left\langle \nu_1(\vec{\gamma}_1), \nu_2(\vec{\gamma}_2) \right\rangle^{\mathrm{conn}}_{(a, d\mathcal{E}_{ij})}.$$

Consequently, by taking into account of all $\mathcal{I}(\nu_1, \nu_2)$'s, we deduce that (4.8.1) equals

$$\sum \langle \theta(\vec{\xi}_1) | \theta(\vec{\xi}_2) \rangle \left\langle \nu_1(\vec{\gamma}_1), \nu_2(\vec{\gamma}_2) \right\rangle^{\mathrm{conn}}_{(a, d\mathcal{E}_{ij})},$$

where the sum is taken over all possible choices stated in the theorem. This finishes the proof. □

For partitions μ, ν of n, we denote the Hurwitz number $H(\mu, \nu, (2)^b)$ by $H^g_{\mu,\nu}$, where $g = \frac{1}{2}(b + 2 - \ell(\mu) - \ell(\nu))$ is determined by the Riemann-Hurwitz formula, and we refer to it as the double Hurwitz number. In general, it is not easy to obtain a closed formula for $H^g_{\mu,\nu}$. However, when $\nu = (n)$, we have the following fact due to Goulden, Jackson, and Vakil.

Proposition 4.8.4 ([GJV]). *Given any partition $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})$ of n. The so-called one-part double Hurwitz number $H^g_{\mu,(n)}$ is the coefficient of t^{2g} in the power series expansion of*

$$\frac{(2g+\ell(\mu)-1)!\; n^{2g+\ell(\mu)-2}}{|\mathrm{Aut}(\mu)|}\frac{t/2}{sinh(t/2)}\prod_{i=1}^{\ell(\mu)}\frac{sinh(\mu_i t/2)}{\mu_i t/2}.$$

Given nonnegative integers a_1, a_2, let g_{a_1}, g_{a_2}, $g(a)$ be integers satisfying

$$\begin{aligned} a_k &= 2g_{a_k} - 1 + \ell(\mu_k),\ \ k = 1, 2 \\ g(a) &= \frac{1}{2}(a - \ell(\mu_1) - \ell(\mu_2) + 2). \end{aligned}$$

For $k = 1, 2$, we put

$$[\mu_k(\vec{\gamma}_k)] = \frac{|\mathrm{Aut}(\mu_k)|}{|\mathrm{Aut}(\mu_k(\vec{\gamma}_k))|}.$$

Our connected invariants can be expressed in terms of one-part double Hurwitz numbers.

Theorem 4.8.5. *Assume that γ_{1k}, $\gamma_{2\ell}$'s are $E_1, \ldots, E_r$ or 1 and β is a nonzero effective curve class. If $\beta = d\mathcal{E}_{ij}$ for some d, i, j and all γ_{1k}, $\gamma_{2\ell}$'s are either E_i or E_j, the invariant*

$$\langle \mu_1(\vec{\gamma}_1), \mu_2(\vec{\gamma}_2)\rangle^{\mathrm{conn}}_{(a,\beta)} \tag{4.8.7}$$

is given by

$$\frac{(t_1+t_2)(-1)^{g(a)}(-1-\delta_{1,r})^{\ell(\mu_1)+\ell(\mu_2)}d^{a-1}[\mu_1(\vec{\gamma})][\mu_2(\vec{\gamma}_2)]}{n^{a-2}}\sum_{a_1+a_2=a}\frac{H^{g_{a_1}}_{\mu_1,(n)}H^{g_{a_2}}_{\mu_2,(n)}}{a_1!a_2!}, \tag{4.8.8}$$

where $\delta_{1,r}$ is the Kronecker delta (which is 1 if $r = 1$ and 0 otherwise). Otherwise, (4.8.7) *vanishes. Thus, by Proposition 4.8.4, there is an explicit closed formula for the invariant* (4.8.7).

Proof. Let $r > 1$. Each insertion of (4.8.7) is a linear combination of fixed-point classes with coefficients zero or having nonnegative $(t_1 + t_2)$-valuation. According to Lemma 4.5.1, (4.8.7) is divisible by $t_1 + t_2$.

As mentioned earlier, (4.8.7) is a polynomial in t_1, t_2. So if at least one of γ_{1k}, $\gamma_{2\ell}$'s is 1, the invariant must be zero because the sum of the degrees of the insertions is at most $\ell(\mu_1)+\ell(\mu_2)-1$, which is the virtual dimension.

Assume that all γ_{1k}, $\gamma_{2\ell}$'s are $E_1, \ldots, E_r$, in which case (4.8.7) is proportional to $(t_1 + t_2)$. By Lemma 4.5.1 again, the invariant is zero if β is not a multiple of $\mathcal{E}_{ij}$ for all i, j.

Now we assume further that $\beta = d\mathcal{E}_{ij}$ for some d, i, j. We may evaluate (4.8.7) modulo $(t_1 + t_2)^2$, so any $\mathbb{T}$-fixed locus that contributes a factor $(t_1 + t_2)^k$ for some $k \geq 2$ may be ruled out. That is, it is enough to investigate those $\mathbb{T}$-fixed loci defined in (4.4.1) ($s = a$), which we denote by F's. However, in order for the contributions of these loci to (4.8.7) not to vanish, the ramification points lying above the distinguished markings must map to x_i or x_{j+1}. As a result, (4.8.7) vanishes if one of γ_{1k}, $\gamma_{2\ell}$'s is E_k for some $k \neq i, j$. This completes the proof of the second assertion.

Now we show the first assertion. Let

$$P = -\frac{1}{L_i}[x_i],\ Q = -\frac{1}{R_{j+1}}[x_{j+1}].$$

It remains to evaluate (4.8.7) with γ_{1k}, $\gamma_{2\ell}$'s from E_i or E_j. We check that

$$E_i \sim P,\ E_i \sim Q$$

("$\sim$" means that the difference between the left side and the right side can be written in terms of classes $[x_{i+1}], \ldots, [x_j]$ and 1 as long as we are working modulo $(t_1 + t_2)$); similarly,

$$E_j \sim P,\ E_j \sim Q.$$

By the vanishing claims just verified, we can replace all γ_{1k}'s with P and all $\gamma_{2\ell}$'s with Q. The invariant (4.8.7) is not exactly the resulting invariant

$$\langle \mu_{11}(P) \cdots \mu_{1\ell(\mu_1)}(P), \mu_{21}(Q) \cdots \mu_{2\ell(\mu_2)}(Q) \rangle^{\mathrm{conn}}_{(a, d\mathcal{E}_{ij})}.$$

Instead, it is congruent modulo $(t_1 + t_2)^2$ to

$$J := [\mu_1(\vec{\gamma}_1)][\mu_2(\vec{\gamma}_2)] \langle \mu_{11}(P) \cdots \mu_{1\ell(\mu_1)}(P), \mu_{21}(Q) \cdots \mu_{2\ell(\mu_2)}(Q) \rangle^{\mathrm{conn}}_{(a, d\mathcal{E}_{ij})}.$$

We can thus execute localization calculations over those F's with one more constraint on the source curve $\mathcal{C}_0$: $\mathcal{C}_{L0}$ carries the marking corresponding to μ_1, and $\mathcal{C}_{R0}$ carries the marking corresponding to μ_2 because the ramification points associated to μ_1 (resp. μ_2) are mapped to

x_i (resp. x_{j+1}). This means that in (4.4.1), $\lambda_0 = \mu_1$, $\rho_0 = \mu_2$, and $\sigma_0 = (k)$.

To summarize, we only have to consider such $\mathbb{T}$-fixed loci, denoted by F_{a_1,a_2}, where the source curve $\mathcal{C}$ decomposes into three pieces $\mathcal{C}_{a_1} \cup \Sigma \cup \mathcal{C}_{a_2}$: $\mathcal{C}_{a_k}$ is a contracted component carrying a_k simple markings, and its unique distinguished marking corresponds to μ_k; the intersection $\mathcal{C}_{a_1} \cap \mathcal{C}_{a_2}$ is empty; the cover $\tilde{C}_{a_k}$ associated to $\mathcal{C}_{a_k}$ is of genus g_{a_k}; and Σ is a chain of noncontracted components, which connects $\mathcal{C}_{a_1}$ and $\mathcal{C}_{a_2}$, and the two twisted points of intersection have stack structures given by the monodromy (n). Note that $\mathcal{C}_{a_k}$'s are twisted points whenever they contain less than three special points and are otherwise twisted curves.

In this way, we reduce our calculation to the integral over

$$\overline{M}(\mathcal{B}\mathfrak{S}_n, \mu_1, (n);\ a_1) \times \overline{M}(\mathcal{B}\mathfrak{S}_n, \mu_2, (n); a_2),$$

followed by division by the product of the automorphism factor d^{j-i+1} and the distribution factor $a_1!a_2!$ of simple marked points.

Let

$$\epsilon_1 : \overline{M}(\mathcal{B}\mathfrak{S}_n, \mu_1, (n);\ a_1) \to \overline{M}_{0,a_1+2}$$

be the natural morphism mapping $\mathcal{C}_{a_1}$ to its coarse moduli space C_{a_1} (the node $\mathcal{C}_{a_1} \cap \Sigma$ is mapped to the marking Q_1) and $\mathcal{L}_1$ the tautological line bundle formed by the cotangent space $T^*_{Q_1} C_{a_1}$. Let $\psi_1 = c_1(\mathcal{L}_1)$. We define $\epsilon_2 : \overline{M}(\mathcal{B}\mathfrak{S}_n, \mu_2, (n);\ a_2) \to \overline{M}_{0,a_1+2}$ and ψ_2 in a similar way.

To proceed, let us summarize the contributions of virtual normal bundles. Set $\theta = (r+1)t_1$.

- Contracted components: For $k = 1, 2$, $\mathcal{C}_{a_k}$ contributes

$$(-1)^{g_{a_k}-1}\theta^{2g_{a_k}-2} \mod (t_1 + t_2).$$

- A chain of noncontracted components: The contribution of each node smoothing is just

$$(\frac{t_1 + t_2}{d})^{-1}.$$

All other node contributions are

$$L_k R_k, \quad k = i, \dots, j+1,$$

each of which equals $-\theta^2$ mod t_1+t_2. Furthermore, all noncontracted curves contribute

$$(\frac{t_1+t_2}{-\theta^2})^{j-i+1} \mod (t_1+t_2)^2.$$

Hence the total contribution equals

$$-\theta^2 d^{j-i}(t_1+t_2) \mod (t_1+t_2)^2.$$

- Smoothing nodes joining a contracted curve to a noncontracted curve: The contributions are given by

$$\frac{1}{\frac{1}{n}(\frac{nR_i}{d}-\epsilon_1^*\psi_1)}, \quad \frac{1}{\frac{1}{n}(\frac{nL_{j+1}}{d}-\epsilon_2^*\psi_2)}.$$

The contribution I_{a_1,a_2} of the fixed locus F_{a_1,a_2} to J is congruent modulo $(t_1+t_2)^2$ to

$$\begin{aligned} &-\theta^2 d^{j-i}(t_1+t_2)\ \frac{[\mu_1(\vec{\gamma}_1)][\mu_2(\vec{\gamma}_2)]}{d^{j-i+1}a_1!a_2!}\ \theta^{\ell(\mu_1)}(-\theta)^{\ell(\mu_2)}\cdot\frac{(-1)^{a_1}}{\theta^4} \\ \times &\prod_{k=1}^{2}(-1)^{g_{a_k}}\theta^{2g_{a_k}}\frac{1}{n^{a_k-1}}\ (\frac{d}{\theta})^{a_k}\int_{\overline{M}(\mathcal{B}\mathfrak{S}_n,\mu_k,(n);\ a_k)}\epsilon_k^*\psi_k^{a_k-1}. \end{aligned}$$

Note that each factor in the second line is replaced with 1 in case $a_k=0$. Simplifying the expression yields

$$\frac{(t_1+t_2)(-1)^{g(a)+\ell(\mu_1)+\ell(\mu_2)}d^{a-1}[\mu_1(\vec{\gamma}_1)][\mu_2(\vec{\gamma}_2)]}{n^{a-2}a_1!a_2!}\prod_{k=1}^{2}\int_{\overline{M}(\mathcal{B}\mathfrak{S}_n,\mu_k,(n);a_k)}\epsilon_k^*\psi_k^{a_k-1}.$$

For $a_k>0$,

$$\int_{\overline{M}(\mathcal{B}\mathfrak{S}_n,\mu_k,(n);\ a_k)}\epsilon_k^*\psi_k^{a_k-1}=\deg(\epsilon_k)\int_{\overline{M}_{0,a_k+2}}\psi_k^{a_k-1}=H^{g_{a_k}}_{\mu_k,(n)}.$$

We conclude that

$$I_{a_1,a_2}\equiv\frac{(t_1+t_2)(-1)^{g(a)+\ell(\mu_1)+\ell(\mu_2)}d^{a-1}[\mu_1(\vec{\gamma}_1)][\mu_2(\vec{\gamma}_2)]}{n^{a-2}}\cdot\frac{H^{g_{a_1}}_{\mu_1,(n)}H^{g_{a_2}}_{\mu_2,(n)}}{a_1!a_2!} \mod (t_1+t_2)^2.$$

Keeping in mind that (4.8.7) is a multiple of t_1+t_2, so we obtain (4.8.8).

The case $r=1$ is similar, and so we omit the proof. □

By applying the intersection matrix with respect to the curve classes $E_1,\ldots,E_r$, we arrive at the following statement which is a little shorter than Theorem 4.8.5.

Corollary 4.8.6. *Let γ_{1k}, $\gamma_{2\ell}$'s be 1 or divisors on $\mathcal{A}_r$ and β a nonzero effective curve class.*

If $\beta = d\mathcal{E}_{ij}$ *for some* d, i, j*, the connected invariant* $\langle \mu_1(\vec{\gamma}_1), \mu_2(\vec{\gamma}_2)\rangle^{\text{conn}}_{(a,\beta)}$ *is given by*

$$\frac{(t_1+t_2)(-1)^{g(a)} d^{a-1} [\mu_1(\vec{\gamma}_1)][\mu_2(\vec{\gamma}_2)] \prod_{k=1}^{\ell(\mu_1)} (\mathcal{E}_{ij}\cdot\gamma_{1k}) \prod_{k=1}^{\ell(\mu_2)} (\mathcal{E}_{ij}\cdot\delta_k)}{n^{a-2}} \sum_{a_1+a_2=a} \frac{H^{g_{a_1}}_{\mu_1,(n)} H^{g_{a_2}}_{\mu_2,(n)}}{a_1! a_2!}.$$

Otherwise, it is zero. □

Theorems 4.8.1 and 4.8.5 provide an effective method to compute two-point extended invariants of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ of nonzero degrees. With the equations in the following proposition, this also determines the divisor operators as a consequence of three-point extended invariants of zero degree being determined by the Gromov-Witten theory of $[\mathrm{Sym}^n(\mathbb{C}^2)]$.

Proposition 4.8.7. *Given any classes* $\alpha_1, \ldots, \alpha_k \in A^*_{\mathbb{T},\mathrm{orb}}[\mathrm{Sym}^n(\mathcal{A}_r)]$*. We have*

$$\langle\langle \alpha_1, \ldots, \alpha_k, (2)\rangle\rangle = \frac{d}{du}\langle\langle \alpha_1, \ldots, \alpha_k\rangle\rangle, \tag{4.8.9}$$

and for each $\ell = 1, \ldots, r$*,*

$$\langle\langle \alpha_1, \ldots, \alpha_k, D_\ell\rangle\rangle = \langle\langle \alpha_1, \ldots, \alpha_k, D_\ell\rangle\rangle|_{s_1,\ldots,s_r=0} + s_\ell \frac{d}{ds_\ell}\langle\langle \alpha_1, \ldots, \alpha_k\rangle\rangle. \tag{4.8.10}$$

Proof. By definition,

$$\langle \alpha_1, \ldots, \alpha_k, (2)\rangle_{(a,\beta)} = (a+1)\langle \alpha_1, \ldots, \alpha_k\rangle_{(a+1,\beta)},$$

and by the untwisted divisor equation ($\beta \neq 0$ or $k \geq 3$),

$$\langle \alpha_1, \ldots, \alpha_k, D_\ell\rangle_{(a,\beta)} = (\omega_\ell \cdot \beta)\langle \alpha_1, \ldots, \alpha_k\rangle_{(a,\beta)}.$$

These relations yield (4.8.9) and (4.8.10). (Note, however, that (4.8.10) is read as

$$\langle\langle \alpha_1, \ldots, \alpha_k, D_\ell\rangle\rangle = s_\ell \frac{d}{ds_\ell}\langle\langle \alpha_1, \ldots, \alpha_k\rangle\rangle$$

for $k \geq 3$.) □

The sine function $\sin(u)$ is a rational function of e^{iu}, where $i^2 = -1$. It is straightforward to verify that extended three-point functions involving (2) or D_ℓ are rational functions in t_1, t_2, $e^{iu}, s_1, \ldots, s_r$ by the above equations.

On the other hand, we treat (4.8.9) as a twisted divisor equation because it provides a means of pulling the twisted divisor (2) out. If we substitute $q = -e^{iu}$, we immediately obtain a relation on differential operators:

$$\frac{d}{du} = iq\frac{d}{dq}.$$

With this, (4.8.9) seems quite close to the usual divisor equation. They are still different, though.

Chapter 5

Comparison to other theories

5.1 Relative Gromov-Witten theory of threefolds

Fix k distinct points $p_1, \ldots, p_k$ of $\mathbb{P}^1$. Given a positive integer n and partitions $\lambda_1, \ldots, \lambda_k$ of n, let

$$\overline{M}^{\bullet}_g(S \times \mathbb{P}^1, (\beta, n); \lambda_1, \ldots, \lambda_k)$$

be the moduli space parametrizing genus g relative stable maps (cf. [L1, L2]) to $S \times \mathbb{P}^1$ relative to $S \times p_1, \ldots, S \times p_k$ with the following data:

- the domains are nodal curves of genus g and are allowed to be disconnected;
- the relative stable maps have degree $(\beta, n) \in A_1(S \times \mathbb{P}^1; \mathbb{Z})$ and have nonzero degree on any connected components;
- the maps are ramified over the divisor $S \times p_i$ with ramification type λ_i. The ramification points are taken to be marked and ordered.

Given any cohomology weighed partition $\lambda_i(\vec{\eta}_i)$, $i = 1, \ldots, k$, we have an evaluation map

$$\mathrm{ev}_{ij} : \overline{M}^{\bullet}_g(S \times \mathbb{P}^1, (\beta, n); \lambda_1, \ldots, \lambda_k) \to S$$

corresponding to the ramification point of type λ_{ij} over the divisor $S \times p_i$. The genus g relative invariant in the cohomology-weighed partitions $\lambda_1(\vec{\eta}_1), \ldots, \lambda_k(\vec{\eta}_k)$ is defined by

$$\langle \lambda_1(\vec{\eta}_1), \ldots, \lambda_k(\vec{\eta}_k) \rangle^{S \times \mathbb{P}^1}_{g,\beta} = \frac{1}{\prod_{i=1}^k |\mathrm{Aut}(\lambda_i(\vec{\eta}_i))|} \int_{[\overline{M}^{\bullet}_g(S \times \mathbb{P}^1, (\beta,n); \lambda_1, \ldots, \lambda_k)]^{\mathrm{vir}}_{\mathbb{T}}} \prod_{i=1}^{k} \prod_{j=1}^{l(\lambda_i)} \mathrm{ev}^*_{ij}(\eta_{ij}).$$

Let $B_1, \ldots, B_\ell$ be a basis for $A_1(S;\mathbb{Z})$ and $B_1^*, \ldots, B_\ell^*$ its dual basis. We define the partition function by

$$\mathrm{Z}'(S\times\mathbb{P}^1)_{\lambda_1(\vec{\eta_1}),\ldots,\lambda_k(\vec{\eta_k})} = \sum_{g,\beta}\langle\lambda_1(\vec{\eta_1}),\ldots,\lambda_k(\vec{\eta_k})\rangle_{g,\beta}^{S\times\mathbb{P}^1} u^{2g-2} s_1^{\beta\cdot B_1}\cdots s_\ell^{\beta\cdot B_\ell}.$$

However, we are more interested in the following shifted generating function

$$\mathrm{GW}(S\times\mathbb{P}^1)_{\lambda_1(\vec{\eta_1}),\ldots,\lambda_k(\vec{\eta_k})} = u^{2n-\sum_{i=1}^k \mathrm{age}(\lambda_i)}\mathrm{Z}'(S\times\mathbb{P}^1)_{\lambda_1(\vec{\eta_1}),\ldots,\lambda_k(\vec{\eta_k})}. \tag{5.1.1}$$

5.1.1 Degree $(0,n)$ case

In this section, we study the following (truncated) shifted partition function

$$\mathrm{GW}_u(S\times\mathbb{P}^1)_{\lambda_1(\vec{\eta_1}),\ldots,\lambda_m(\vec{\eta_m})} = \mathrm{GW}(S\times\mathbb{P}^1)_{\lambda_1(\vec{\eta_1}),\ldots,\lambda_k(\vec{\eta_k})}|_{s_1=0,\ldots,s_\ell=0}.$$

Like the treatment of the fixed-point basis for $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(S)])$, we denote the cohomology-weighted partition

$$\sigma_{11}([x_1])\cdots\sigma_{1\ell(\sigma_1)}([x_1])\cdots\sigma_{s1}([x_s])\cdots\sigma_{s\ell(\sigma_s)}([x_s])$$

associated to fixed-point classes by

$$\widetilde{\sigma}.$$

Evidently, the shifted partition function $\mathrm{GW}_u(S\times\mathbb{P}^1)_{\lambda_1(\vec{\eta_1}),\ldots,\lambda_m(\vec{\eta_m})}$ is determined by shifted partition functions $\mathrm{GW}_u(S\times\mathbb{P}^1)_{\widetilde{\sigma_1},\ldots,\widetilde{\sigma_m}}$'s.

Let i be the square root of -1 appearing in the SYM-HILB correspondence introduced in Section 3.5. We can translate the three-point shifted partition functions to the three-point functions of $[\mathrm{Sym}^n(S)]$ and $\mathrm{Hilb}^n(S)$ by the following equations.

Proposition 5.1.1. *For any cohomology-weighted partitions $\lambda_1(\vec{\eta_1}), \lambda_2(\vec{\eta_2}), \lambda_3(\vec{\eta_3})$,*

$$\begin{aligned}\mathrm{GW}_u(S\times\mathbb{P}^1)_{\lambda_1(\vec{\eta_1}),\lambda_2(\vec{\eta_2}),\lambda_3(\vec{\eta_3})} &= \langle\lambda_1(\vec{\eta_1}),\lambda_2(\vec{\eta_2}),\lambda_3(\vec{\eta_3})\rangle^{[\mathrm{Sym}^n(S)]}(u)\\ &= (-i)^{\sum_{k=1}^3\mathrm{age}(\lambda_k)}\langle\mathfrak{a}_{\lambda_1}(\vec{\eta_1}),\mathfrak{a}_{\lambda_2}(\vec{\eta_2}),\mathfrak{a}_{\lambda_3}(\vec{\eta_3})\rangle^{\mathrm{Hilb}^n(S)}(-e^{iu}).\end{aligned}$$

Proof. By Theorem 3.5.2, it suffices show that

$$\mathrm{GW}_u(S \times \mathbb{P}^1)_{\widetilde{\lambda},\widetilde{\mu},\widetilde{\nu}} = \langle \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu} \rangle^{[\mathrm{Sym}^n(S)]}(u). \tag{5.1.2}$$

Both sides are zero if the condition $|\lambda_k| = |\mu_k| = |\nu_k|$ for each $k = 1, \dots, s$ is violated. Now assume that the condition holds. Since the relative stable maps collapse along the S-direction to $x_1, \dots, x_s$, the shifted partition function $\mathrm{GW}_u(S\times\mathbb{P}^1)_{\widetilde{\lambda},\widetilde{\mu},\widetilde{\nu}}$ is simply $\prod_{k=1}^s \mathrm{GW}_u(S\times\mathbb{P}^1)_{\widetilde{\lambda_k},\widetilde{\mu_k},\widetilde{\nu_k}}$. Moreover, $\mathrm{GW}_u(S \times \mathbb{P}^1)_{\widetilde{\lambda_k},\widetilde{\mu_k},\widetilde{\nu_k}}$ and $\mathrm{GW}_u(U_k \times \mathbb{P}^1)_{\widetilde{\lambda_k},\widetilde{\mu_k},\widetilde{\nu_k}}$ obviously coincide. As the equality

$$\mathrm{GW}_u(U_k \times \mathbb{P}^1)_{\widetilde{\lambda_k},\widetilde{\mu_k},\widetilde{\nu_k}} = \langle \widetilde{\lambda_k}, \widetilde{\mu_k}, \widetilde{\nu_k} \rangle^{[\mathrm{Sym}^n(U_k)]}(u)$$

is guaranteed by results of [BP, BG] for the $\mathbb{C}^2$-case, the identity (5.1.2) follows immediately from Corollary 3.4.2. □

Remark. The functions $\mathrm{GW}_u(S \times \mathbb{P}^1)_{\lambda_1(\vec{\eta_1}),\lambda_2(\vec{\eta_2}),\lambda_3(\vec{\eta_3})}$ are elements of $\mathbb{Q}(t_1, t_2, e^{iu})$, and we may also recover the cup product of $\mathrm{Hilb}^n(S)$ from the limits

$$\lim_{u\to+i\infty} \mathrm{GW}_u(S \times \mathbb{P}^1)_{\lambda_1(\vec{\eta_1}),\lambda_2(\vec{\eta_2}),\lambda_3(\vec{\eta_3})}.$$

5.1.2 $\mathcal{A}_r$ case

Assume that the partition functions $\mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta_1}),\dots,\lambda_k(\vec{\eta_k})}$ are defined by using the basis $E_1, \dots, E_r$. Our results in the preceding chapter recover certain relative Gromov-Witten invariants by the following equalities.

Theorem 5.1.2. *For $\alpha = 1(1)^n$, (2) or D_k, $k = 1, \dots, r$,*

$$\langle\langle \lambda_1(\vec{\eta_1}), \alpha, \lambda_2(\vec{\eta_2}) \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta_1}),\alpha,\lambda_2(\vec{\eta_2})}. \tag{5.1.3}$$

Proof. When specialized to $s_1 = \cdots = s_r = 0$, the equality (5.1.3) follows from Proposition 5.1.1. In particular, (5.1.3) is valid for $\alpha = 1(1)^n$ without the constraint.

For $\alpha = (2)$ or D_k, the coefficients of $u^i s_1^{j_1} \dots s_r^{j_r}$, where $j_1 + \cdots + j_r > 0$, match up on both sides of (5.1.3) by a direct comparison of Proposition 4.4 in [M] with our results in Section 4.8. Hence, (5.1.3) follows as well in this case. □

5.2 Quantum cohomology of Hilbert schemes of points

5.2.1 Quantum cup product

Let $\rho_* : A_1(\mathrm{Hilb}^n(\mathcal{A}_r);\mathbb{Z}) \to A_1(\mathrm{Sym}^n(\mathcal{A}_r);\mathbb{Z})$ be the homomorphism induced by the Hilbert-Chow morphism $\rho : \mathrm{Hilb}^n(\mathcal{A}_r) \to \mathrm{Sym}^n(\mathcal{A}_r)$. There are isomorphisms

$$A_1(\mathrm{Hilb}^n(\mathcal{A}_r);\mathbb{Z}) \cong Ker(\rho_*) \oplus A_1(\mathrm{Sym}^n(\mathcal{A}_r);\mathbb{Z}) \cong Ker(\rho_*) \oplus A_1(\mathcal{A}_r;\mathbb{Z}).$$

Let ℓ be the class dual to the divisor $-\mathfrak{a}_1(1)^{n-2}a_2(1)$ on $\mathrm{Hilb}^n(\mathcal{A}_r)$. It is an effective rational curve class generating the kernel $Ker(\rho_*^{\mathrm{HC}})$. For any classes $\alpha_1, \ldots, \alpha_k$ on $\mathrm{Hilb}^n(\mathcal{A}_r)$, we consider the k-point function

$$\langle \alpha_1, \ldots, \alpha_k \rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)} = \sum_{d=0}^{\infty} \sum_{\beta \in A_1(\mathcal{A}_r;\mathbb{Z})} \langle \alpha_1, \ldots, \alpha_k \rangle_{(d\ell,\beta)}^{\mathrm{Hilb}^n(\mathcal{A}_r)} q^d s_1^{\beta\cdot\omega_1} \cdots s_r^{\beta\cdot\omega_r}. \tag{5.2.1}$$

Now given any basis $\{\delta\}$ for $A_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r))$ and $\{\delta^\vee\}$ its dual basis. Define the small quantum cup product $*_{\mathrm{crep}}$ on $A_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r))$ by the three-point functions as follows:

$$\alpha_1 *_{\mathrm{crep}} \alpha_2 = \sum_{\delta} \langle \alpha_1, \alpha_2, \delta \rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)} \delta^\vee.$$

Like the orbifold case, we define

$$QA_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r))$$

as the vector space $A_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r)) \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1,t_2)((q, s_1, \ldots, s_r))$ with the multiplication $*_{\mathrm{crep}}$.

5.2.2 SYM-HILB correspondence

In Section 4.8, we provide a combinatorial description of any divisor operator on the ring $A_{\mathbb{T},\mathrm{orb}}^*([\mathrm{Sym}^n(\mathcal{A}_r)])$. In [MO1], on the other hand, any divisor operator on $A_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r))$ is expressed in terms of the action of affine Lie algebra $\hat{\mathfrak{gl}}(r+1)$ on the basic representations. These two expressions are actually equivalent via the correspondence given in Section 3.5.

We make the substitution $q = -e^{iu}$ as in Section 3.5 and consider the map L there. Put

$$F = \mathbb{Q}(i, t_1, t_2)((u, s_1, \ldots, s_r)) \text{ and } K = \mathbb{Q}(t_1, t_2)((u, s_1, \ldots, s_r)).$$

The map L extends to a F-linear isomorphism

$$L : QA^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)]) \otimes_K F \to QA^*_{\mathbb{T}}(\mathrm{Hilb}^n(\mathcal{A}_r)) \otimes_K F.$$

Further, we have the following SYM-HILB correspondence.

Theorem 5.2.1. *The F-linear isomorphism L respects quantum multiplication by divisors:*

$$L(D *_{\mathrm{orb}} \alpha) = L(D) *_{\mathrm{crep}} L(\alpha) \tag{5.2.2}$$

for any class α and divisor D.

Proof. For cohomology-weighted partitions $\lambda_1(\vec{\eta}_1)$, $\lambda_2(\vec{\eta}_2)$ and $\alpha = (2)$ or D_k,

$$\begin{aligned} \langle\langle \lambda_1(\vec{\eta}_1), \alpha, \lambda_2(\vec{\eta}_2)\rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} &= \mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\alpha,\lambda_2(\vec{\eta}_2)} \\ &= \langle L(\lambda_1(\vec{\eta}_1)), L(\alpha), L(\lambda_2(\vec{\eta}_2))\rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}. \end{aligned}$$

Indeed, the first equality is Theorem 5.1.2 while the second equality is Proposition 6.6 in [MO1].

As L preserves Poincaré pairings, it follows from the above equalities that

$$\langle L(\lambda_1(\vec{\eta}_1) *_{\mathrm{orb}} \alpha) \mid L(\lambda_2(\vec{\eta}_2))\rangle = \langle L(\lambda_1(\vec{\eta}_1)) *_{\mathrm{crep}} L(\alpha) \mid L(\lambda_2(\vec{\eta}_2))\rangle.$$

This implies that L respects quantum multiplication by (2) and D_k's. The equality (5.2.2) now follows due to the fact that (2) and D_k's give a basis for divisor classes. □

5.3 The Crepant Resolution Conjecture

Before discussing the full version of Bryan-Graber Crepant Resolution Conjecture, let us study a simple example.

5.3.1 An example

We would like to give an explicit expression for the divisor operator $D_1 *_{\mathrm{orb}} -$ on the quantum ring $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^2(\mathcal{A}_1)])$. Let us substitute $q = -e^{iu}$ so that

$$\sin(\gamma u) = \frac{1}{2i}((-q)^\gamma - \frac{1}{(-q)^\gamma}).$$

Consider the following basis

$$\mathfrak{B} := \{1(E_1)1(E_1),\ 2(E_1),\ 1(1)1(E_1),\ 2(1),\ 1(1)1(1)\},$$

whose elements are ordered according to their orbifold degrees. The matrix representation of the operator $D_1 *_{\mathrm{orb}} -$ with respect to $\mathfrak{B}$ is given by

$$\begin{pmatrix} 2\theta(1-\frac{1}{1+sq}-\frac{1}{1+s/q}) & i\theta(\frac{1}{1+sq}-\frac{1}{1+s/q}) & -1 & 0 & 0 \\ -2i\theta(\frac{1}{1+sq}-\frac{1}{1+s/q}) & \theta(2-\frac{1}{1+sq}-\frac{1}{1+s/q}-\frac{2}{1-s}) & 0 & -1 & 0 \\ 2t_1t_2 & 0 & \frac{-\theta(1+s)}{1-s} & 0 & -\frac{1}{2} \\ 0 & 4t_1t_2 & 0 & 0 & 0 \\ 0 & 0 & 4t_1t_2 & 0 & 0 \end{pmatrix},$$

where $\theta := t_1 + t_2$ and $s := s_1$. This is also the matrix representation of the operator

$$L(D_1) *_{\mathrm{crep}} -$$

with respect to the ordered basis $L(\mathfrak{B})$(cf. [MO1]).

It is straightforward to check that $D_1 *_{\mathrm{orb}} -$ has distinct eigenvalues. In particular, we have a basis $\{v_1, \ldots, v_5\}$ of eigenvectors. By quantum multiplication by D_1 and the identity 1, we find

$$\begin{aligned} v_i *_{\mathrm{orb}} v_i &= a_i v_i, \quad \text{for some } a_i \neq 0; \\ v_i *_{\mathrm{orb}} v_j &= 0, \quad \forall i \neq j. \end{aligned}$$

So by replacing v_i with v_i/a_i, we may assume that $\{v_1, \ldots, v_5\}$ is an idempotent basis; in which case,

$$1 = \sum_{i=1}^{5} v_i. \tag{5.3.1}$$

Moreover, the Vandermonde matrix associated to the eigenvalues of $D_1 *_{\mathrm{orb}} -$ is invertible. In other words, by (5.3.1) the set

$$\{1, D_1, D_1^2, D_1^3, D_1^4\}$$

is a basis for the quantum cohomology $QA^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^2(\mathcal{A}_1)])$. Similarly, $L(D_1)$ generates the quantum ring $QA^*_{\mathbb{T}}(\mathrm{Hilb}^2(\mathcal{A}_1)) \otimes_K F$. We conclude that

$$L : QA^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^2(\mathcal{A}_1)]) \otimes_K F \to QA^*_{\mathbb{T}}(\mathrm{Hilb}^2(\mathcal{A}_1)) \otimes_K F$$

is indeed an F-algebra isomorphism. □

This simple example raises the question: Do divisor classes generate the whole quantum ring? In response to this, one may wish to examine the eigenvalues of divisor operators for bigger n. This, however, seems a difficult task to perform directly.

If one of the operators $(2) *_{\mathrm{orb}} -$, $D_k *_{\mathrm{orb}} -$'s turns out to have distinct eigenvalues, the ring structure will be determined, and L will be an F-algebra isomorphism. The hypothesis has yet to be entirely verified and may seem a little too good to be true. It is reasonable to expect something weaker (maybe certain combinations of these operators work).

5.3.2 Nonderogatory Conjecture

We name the following nonderogatory conjecture, but we claim no originality for the statement. The reader is urged to consult [MO1] for a partial evidence of the conjecture.

Conjecture 5.3.1 ([MO1])**.** *Let L be as in Section 5.2.2. The commuting family of the operators*

$$L((2)) *_{\mathrm{crep}} -,\ L(D_1) *_{\mathrm{crep}} -, \dots, L(D_r) *_{\mathrm{crep}} -$$

on the quantum cohomology of $\mathrm{Hilb}^n(\mathcal{A}_r)$ *is nonderogatory. That is, its joint eigenspaces are one-dimensional.*

Let us briefly explain some consequences of the nonderogatory conjecture on our quantum cohomology rings. We set

$$R = \mathbb{Q}(i, t_1, t_2, q, s_1, \dots, s_r) \text{ and } q = -e^{iu}.$$

Since the quantum ring $A^*_{\mathbb{T}}(\mathrm{Hilb}^n(\mathcal{A}_r)) \otimes_{\mathbb{Q}[t_1,t_2]} R$ is semisimple, it admits a basis, say $\{v_1, \dots, v_m\}$, of idempotent eigenvectors summing to the identity 1. Note that the basis elements are also the simultaneous eigenvectors for $L((2)) *_{\mathrm{crep}} -$, $L(D_1) *_{\mathrm{crep}} -, \dots, L(D_r) *_{\mathrm{crep}} -$.

Suppose that $e_{0k}, e_{1k}, \dots, e_{rk}$ are respectively the eigenvalues of the operators

$$L((2)) *_{\mathrm{crep}} -, L(D_1) *_{\mathrm{crep}} -, \dots, L(D_r) *_{\mathrm{crep}} -$$

corresponding to the eigenvector v_k. The nonderogatory property ensures that we can find numbers $a_0, a_1, \dots, a_r$ such that

$$\sum_{j=0}^{r} a_j e_{j1}, \dots, \sum_{j=0}^{r} a_j e_{jm}$$

is a sequence of distinct elements. Therefore, the Vandermonde argument given earlier shows that the element $a_0 \cdot L((2)) + \sum_{j=1}^{r} a_j \cdot L(D_j)$ generates $A_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r)) \otimes_{\mathbb{Q}[t_1,t_2]} R$. This implies that $a_0 \cdot (2) + \sum_{j=1}^{r} a_j \cdot D_j$ generates the quantum cohomology of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ over R as well. We thus obtain the following "corollary".[1]

"Corollary" 5.3.2. *The divisor classes* (2) *and* $D_1, \dots, D_r$ *generate the quantum cohomology ring* $QA_{\mathbb{T},\mathrm{orb}}^*([\mathrm{Sym}^n(\mathcal{A}_r)])$*, and any extended three-point function is a rational function in* t_1, t_2*,* $e^{iu}, s_1, \dots, s_r$*. Under the substitution* $q = -e^{iu}$*, the map*

$$L : QA_{\mathbb{T},\mathrm{orb}}^*([\mathrm{Sym}^n(\mathcal{A}_r)]) \otimes_K F \to QA_{\mathbb{T}}^*(\mathrm{Hilb}^n(\mathcal{A}_r)) \otimes_K F$$

gives an isomorphism of F*-algebras.* □

On the other hand, we can match the orbifold Gromov-Witten theory with the relative Gromov-Witten theory.

"Corollary" 5.3.3. *The equality*

$$\langle\langle \lambda_1(\vec{\eta}_1), \lambda_2(\vec{\eta}_2), \lambda_3(\vec{\eta}_3) \rangle\rangle = \mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1), \lambda_2(\vec{\eta}_2), \lambda_3(\vec{\eta}_3)}$$

holds for any cohomology-weighted partitions $\lambda_1(\vec{\eta}_1)$*,* $\lambda_2(\vec{\eta}_2)$*,* $\lambda_3(\vec{\eta}_3)$ *of* n*.* □

5.3.3 Multipoint functions

Once the nonderogatory conjecture holds, all extended three-point functions are known by "Corollary" 5.3.2. In this situation, we are actually able to generalize "Corollary" 5.3.2 to cover multipoint invariants. This can be done by proceeding in an analogous manner to Okounkov and Pandharipande's determination of multipoint invariants of $\mathrm{Hilb}^n(\mathbb{C}^2)$ (cf. [OP1]).

[1] Whenever we put a double quotation mark " ", we emphasize that the statement or word inside comes with the hypothesis of the nonderogatory conjecture.

Let $\mathcal{B}$ be a basis for the Chen-Ruan cohomology $A^*_{\mathbb{T},\mathrm{orb}}([\mathrm{Sym}^n(\mathcal{A}_r)])$. We recall the WDVV equation from [AGV2], but we write it in terms of extended functions to better suit our needs. For the time being, we drop the superscript $[\mathrm{Sym}^n(\mathcal{A}_r)]$.

Proposition 5.3.4 ([AGV2]). *Given Chen-Ruan classes $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \ldots, \beta_k$. Let S be the set $\{1, \ldots, k\}$, we have*

$$\begin{aligned} & \sum_{S_1 \coprod S_2 = S} \sum_{\gamma \in \mathcal{B}} \langle\langle \alpha_1, \alpha_2, \beta_{S_1}, \gamma \rangle\rangle \langle\langle \gamma^\vee, \beta_{S_2}, \alpha_3, \alpha_4 \rangle\rangle \\ = & \sum_{S_1 \coprod S_2 = S} \sum_{\gamma \in \mathcal{B}} \langle\langle \alpha_1, \alpha_3, \beta_{S_1}, \gamma \rangle\rangle \langle\langle \gamma^\vee, \beta_{S_2}, \alpha_2, \alpha_4 \rangle\rangle. \end{aligned}$$

Here, for instance, $\langle\langle \alpha_1, \alpha_2, \beta_{S_1}, \gamma \rangle\rangle := \langle\langle \alpha_1, \alpha_2, \beta_{i_1}, \ldots, \beta_{i_\ell}, \gamma \rangle\rangle$ if $S_1 = \{i_1, \ldots, i_\ell\}$.

"Proposition" 5.3.5. *All extended multipoint functions of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ can be determined from extended three-point functions and are rational functions in t_1, t_2, $e^{iu}, s_1, \ldots, s_r$.*

Proof. We may see this by induction. Suppose that any extended m-point function with $m \leq k$ is known and is a rational function in t_1, t_2, $e^{iu}, s_1, \ldots, s_r$. To determine extended $(k+1)$-point function, it suffices to study

$$N := \langle\langle \alpha_0, \alpha_1, \ldots, \alpha_k \rangle\rangle$$

for $\alpha_0 = (2)^\ell *_{\mathrm{orb}} D_1^{m_1} *_{\mathrm{orb}} \cdots *_{\mathrm{orb}} D_r^{m_r}$, where ℓ, $m_1, \ldots, m_r$ are nonnegative integers. We may assume that $\ell + m_1 + \cdots + m_r \geq 2$ in light of Proposition 4.8.7 and the fundamental class axiom. Let us write $\alpha_0 = D *_{\mathrm{orb}} \delta$ for some $D = (2)$ or D_j. Clearly,

$$N = \sum_{\gamma \in \mathcal{B}} \langle\langle D, \delta, \gamma \rangle\rangle \langle\langle \gamma^\vee, \alpha_1, \ldots, \alpha_k \rangle\rangle.$$

Let $S = \{1, \ldots, k-2\}$. By the WDVV equation,

$$\begin{aligned} & \sum_{\gamma \in \mathcal{B}} \langle\langle D, \delta, \gamma \rangle\rangle \langle\langle \gamma^\vee, \alpha_S, \alpha_{k-1}, \alpha_k \rangle\rangle + \sum_{\gamma \in \mathcal{B}} \langle\langle D, \delta, \alpha_S, \gamma \rangle\rangle \langle\langle \gamma^\vee, \alpha_{k-1}, \alpha_k \rangle\rangle \\ = & \sum_{\gamma \in \mathcal{B}} \langle\langle D, \alpha_{k-1}, \gamma \rangle\rangle \langle\langle \gamma^\vee, \alpha_S, \delta, \alpha_k \rangle\rangle + \sum_{\gamma \in \mathcal{B}} \langle\langle D, \alpha_{k-1}, \alpha_S, \gamma \rangle\rangle \langle\langle \gamma^\vee, \delta, \alpha_k \rangle\rangle \\ & + (\text{terms with extended } m\text{-point functions, } 3 \leq m \leq k). \end{aligned}$$

This says that N is determined by lower-point functions and extended $(k+1)$-point functions with a δ-insertion. By replacing $D *_{\mathrm{orb}} \delta$ with δ if necessary and continuing the above procedure, we conclude that N can be calculated from lower-point functions and is a rational function in

$t_1, t_2, e^{iu}, s_1, \ldots, s_r$. By induction, our claim is thus justified. □

"Corollary" 5.3.6 (The Crepant Resolution Conjecture)**.** *Let* $q = -e^{iu}$ *and* $k \geq 3$. *For any Chen-Ruan classes* $\alpha_1, \ldots, \alpha_k$ *on* $[\mathrm{Sym}^n(\mathcal{A}_r)]$, *we have*

$$\langle\langle \alpha_1, \ldots, \alpha_k \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \langle L(\alpha_1), \ldots, L(\alpha_k) \rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}.$$

In particular, $\langle \alpha_1, \ldots, \alpha_k \rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \langle L(\alpha_1), \ldots, L(\alpha_k) \rangle^{\mathrm{Hilb}^n(\mathcal{A}_r)}|_{q=-1}$.

Proof. We suppress the indices $[\mathrm{Sym}^n(\mathcal{A}_r)]$ and $\mathrm{Hilb}^n(\mathcal{A}_r)$. The proof of "Proposition" 5.3.5 works as well for multipoint functions on $\mathrm{Hilb}^n(\mathcal{A}_r)$. What makes things nice is that we get exactly the same set of WDVV equations on both $[\mathrm{Sym}^n(\mathcal{A}_r)]$ and $\mathrm{Hilb}^n(\mathcal{A}_r)$ sides via L provided that we have the equalities:

$$\langle\langle \alpha_1, \alpha_2, \alpha_3, D \rangle\rangle = \langle L(\alpha_1), L(\alpha_2), L(\alpha_3), L(D) \rangle$$

for $D = (2)$ and D_j $(j = 1, \ldots, r)$. But these are clear by divisor equations and "Corollary" 5.3.2. Thus by a recursive argument, we conclude that L preserves (extended) multipoint functions, and the first claim follows. The second claim is now clear. □

5.3.4 Closing remarks

All "results" discussed above are honestly true for the case $n = 2$ and $r = 1$ since the divisor operator $D_1 *_{\mathrm{orb}} -$ has distinct eigenvalues and determines the orbifold quantum product.

Also, in the definition of the map L, we may choose $-i$ instead of i, in which setting the correct change of variables is $q = -e^{-iu}$. As a matter of fact, the transformation

$$q \longmapsto \frac{1}{q}$$

takes $\langle\langle \lambda_1(\vec{\eta}_1), \ldots, \lambda_k(\vec{\eta}_k) \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}$ to $(-1)^{\sum_{j=1}^k \mathrm{age}(\lambda_j)} \langle\langle \lambda_1(\vec{\eta}_1), \ldots, \lambda_k(\vec{\eta}_k) \rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}$. To illustrate this, just look at the matrix in Section 5.3.1. There we observe that terms involving q and $\frac{1}{q}$ agree up to a sign.

The calculation of $[\mathrm{Sym}^n(\mathcal{A}_r)]$-invariants in Section 5.1.1 gives an indication that these invariants might be closer, geometrically and combinatorially, to the relative invariants of $\mathcal{A}_r \times \mathbb{P}^1$ than the invariants of $\mathrm{Hilb}^n(\mathcal{A}_r)$. In reality, it is the form the relative invariants take that motivates our calculation. We do know that $\mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\ldots,\lambda_k(\vec{\eta}_k)}$ can be "reduced" to the three-point case by the degeneration formula (cf. [M]). It is, however, unclear if the WDVV equation "behaves" in a similar way to the degeneration formula. At the moment, we expect that the equality

$$\langle\langle \lambda_1(\vec{\eta}_1), \ldots, \lambda_k(\vec{\eta}_k)\rangle\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]} = \mathrm{GW}(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\ldots,\lambda_k(\vec{\eta}_k)}$$

should also be true. Particularly, the usual k-point function $\langle \lambda_1(\vec{\eta}_1), \ldots, \lambda_k(\vec{\eta}_k)\rangle^{[\mathrm{Sym}^n(\mathcal{A}_r)]}$ should be the coefficient of $u^{\sum_{i=1}^k \mathrm{age}(\lambda_i)-2n}$ in $\mathrm{Z}'(\mathcal{A}_r \times \mathbb{P}^1)_{\lambda_1(\vec{\eta}_1),\ldots,\lambda_k(\vec{\eta}_k)}$.

On the other hand, for $n \geq 2$, the smooth schemes $\mathrm{Hilb}^n(\mathcal{A}_r)$ and $\mathrm{Hilb}^n([\mathbb{C}^2/\mu_{r+1}])$ are two different crepant resolutions of the symmetric product $\mathrm{Sym}^n(\mathbb{C}^2/\mu_{r+1})$. Thus, it is quite possible that *the genus zero Gromov-Witten theories of the schemes* $\mathrm{Hilb}^n(\mathcal{A}_r)$ *and* $\mathrm{Hilb}^n([\mathbb{C}^2/\mu_{r+1}])$ *are equivalent for positive integers* n *and* r (note that the statement is obviously true for the case where $n = 1$ as the schemes $\mathrm{Hilb}^1(\mathcal{A}_r)$ and $\mathrm{Hilb}^1([\mathbb{C}^2/\mu_{r+1}])$ coincide). We will discuss a little about the Quantum Minimal Model Conjecture, which predicts that there exists an equivalence between the Gromov-Witten theories of two crepant resolutions, in the next chapter.

Chapter 6

Ordinary flops

6.1 Preliminaries

Let X be a nonsingular projective variety and $\overline{M}_{g,n}(X, \beta)$ the moduli space parametrizing genus g, n-pointed stable map $f : (C, p_1, \ldots, p_n) \to X$ of degree β.

We recall some basic notions before proceeding. For $i = 1, \ldots, n$, denote the i^{th} evaluation map by

$$e_i : \overline{M}_{g,n}(X, \beta) \to X.$$

There is also a map

$$ft : \overline{M}_{g,n+1}(X, \beta) \to \overline{M}_{g,n}(X, \beta)$$

defined by forgetting the last marked point and contracting any resulting unstable components. For $i = 1, \ldots, n$, the map ft carries a tautological section

$$s_i : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n+1}(X, \beta)$$

defined by

$$s_i([f : (C, p_1, \ldots, p_n) \to X]) = [f' : (C \cup \mathbb{P}^1, p_1, \ldots, p_{i-1}, [1:0], p_{i+1}, \ldots, p_n, [0:1]) \to X],$$

where $[1 : 0], [0 : 1] \in \mathbb{P}^1$, and $f' := f$ on C but $f'(\mathbb{P}^1) := f(p_i)$. Let ω_{ft} be the relative dualizing sheaf of ft. We have tautological line bundle

$$\mathbb{L}_i := s_i^* \omega_{ft},$$

whose fiber over $[f : (C, p_1, \ldots, p_n) \to X]$ is the cotangent line $T^*_{p_i}C$ at the i^{th} marked point, and we let

$$\psi_i := c_1(\mathbb{L}_i),$$

the first Chern class of $\mathbb{L}_i$.

The virtual dimension of $\overline{M}_{g,n}(X, \beta)$ is given by

$$\int_\beta c_1(T_X) + (1-g)(\dim X - 3) + n.$$

For $\alpha_1, \ldots, \alpha_n \in H^*(X)$, we have the n-point Gromov-Witten invariant

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n,\beta} := \int_{[\overline{M}_{g,n}(X,\beta)]^{\text{vir}}} e_1^*(\alpha_1) \cup \ldots \cup e_n^*(\alpha_n).$$

6.2 The main formula

The main goal of this chapter is to show the following formula.

Theorem 6.2.1. *With notation as above,*

$$\int_{[\overline{M}_{1,0}(\mathbb{P}^r,d)]^{\text{vir}}} e(R^1 ft_* e_1^*(\mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)})) = \frac{(-1)^{(r+1)d}(r+1)}{24d}.$$

When we set $r = 1$, we obtain the following equality:

$$\int_{[\overline{M}_{1,0}(\mathbb{P}^1,d)]^{\text{vir}}} e(R^1 ft_* e_1^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = \frac{1}{12d},$$

which was first computed in physics (cf. [BCOV]) and proved later in mathematics (cf. Proposition 5.2 in [GP]). Also, Theorem 6.2.1 has been independently discovered and proved by Iwao, Lee, Lin, and Wang [ILLW]. We remark that our proof, dating back to 2007, is totally different from theirs and is much simpler.

Proof of Theorem 6.2.1

For each $i = 0, \ldots, r$, let $\mathbb{T}_i = (\mathbb{C}^\times)^{r+1}$ act on $\mathbb{C}^{r+1}$ by

$$(s_0, \ldots, s_r) \cdot (x_0, \ldots, x_r) = ((s_i^{-1}s_0) \cdot x_0, \ldots, (s_i^{-1}s_r) \cdot x_r).$$

Now let $q_i = [0, \ldots, 0, 1, 0, \ldots, 0] \in \mathbb{P}^r$ (1 is the i^{th} component). Denote by

$$\pi_i : (\mathbb{P}^\infty)^{r+1} \to \mathbb{P}^\infty$$

the projection onto the i^{th} factor. Let

$$\begin{aligned} H &= c_1(\mathcal{O}_{\mathbb{P}(\oplus_{i=0}^r \pi_i^* \mathcal{O}_{\mathbb{P}^\infty}(1))}(1)), \\ t_i &= c_1(\pi_i^* \mathcal{O}_{\mathbb{P}^\infty}(1)). \end{aligned}$$

The first $\mathbb{T}_i$-equivariant Chern class of $\mathcal{O}_{\mathbb{P}^r}(-1)$ can be determined as follows:

$$c_1^{\mathbb{T}_i}(\mathcal{O}_{\mathbb{P}^r}(-1)) = t_i - H.$$

This means particularly that $\mathbb{T}_i$ acts on $\mathcal{O}_{\mathbb{P}^r}(-1)|_{q_i}$ trivially.

We denote $\mathbb{T}_i$'s by $\mathbb{T}$ if there is no danger of confusion. In order to establish Theorem 6.2.1, it suffices to show that

$$\int_{[\overline{M}_{1,0}(\mathbb{P}^r,d)]^{\text{vir}}_{\mathbb{T}}} \prod_{i=0}^{r} e_{\mathbb{T}_i}(R^1 ft_* e_1^*(\mathcal{O}_{\mathbb{P}^r}(-1)) = \frac{(-1)^{(r+1)d}(r+1)}{24d}. \tag{6.2.1}$$

The rest of the section is devoted to proving (6.2.1). By the virtual localization formula, the left side of (6.2.1) equals

$$\sum_{\Gamma} \frac{1}{|\mathbb{A}_\Gamma|} \int_{\overline{M}_\Gamma} \frac{\prod_{i=0}^r i_\Gamma^*(e_{\mathbb{T}_i}(R^1 ft_* e_1^*(\mathcal{O}_{\mathbb{P}^r}(-1))}{e_{\mathbb{T}}(N_\Gamma^{\text{vir}})}, \tag{6.2.2}$$

where

(i) Each Γ is a labeled graph, which is actually a connected tree. Its vertices v are labeled with $\mathbb{T}$-fixed loci of $\mathbb{P}^r$ and genera of the corresponding contracted components, and its edges e are labeled with the degrees d_e of their images. Moreover, $\overline{M}_\Gamma$ is the product of moduli spaces of pointed curves associated to Γ.

(ii) $i_\Gamma : \overline{M}_\Gamma \to \overline{M}_{1,0}(\mathbb{P}^r, d)$ is a finite morphism with image, realized as $\overline{M}_\Gamma / \mathbb{A}_\Gamma$, a connected component of $\overline{M}_{1,0}(\mathbb{P}^r, d)^{\mathbb{T}}$. Here $\mathbb{A}_\Gamma$ is a finite group of automorphisms acting on $\overline{M}_\Gamma$ and fits into the exact sequence of groups

$$1 \to \prod_{e} \mathbb{Z}/d_e \to \mathbb{A}_\Gamma \to \mathrm{Aut}(\Gamma) \to 1.$$

(iii) $i_\Gamma(\overline{M}_\Gamma)$'s form a complete set of connected components of $\overline{M}_{1,0}(\mathbb{P}^r, d)^{\mathbb{T}}$.

(iv) $e_{\mathbb{T}}(N_\Gamma^{\mathrm{vir}})$ is the $\mathbb{T}$-equivariant Euler class of the virtual normal bundle to $\overline{M}_\Gamma$.

The following lemma helps us get rid of superfluous fixed loci.

Lemma 6.2.2. *If a labeled graph has more than one edge, then it makes no contribution to* (6.2.2).

Proof. Suppose that such a graph Γ has more than one edge. Let $[f : C \to \mathbb{P}^r] \in \overline{M}_\Gamma$, and consider the normalization sequence

$$0 \to \mathcal{O}_C \to v_* \mathcal{O}_{\tilde{C}} \to \oplus_{j=1}^{\delta} \mathcal{O}_{n_j} \to 0.$$

Here $v : \tilde{C} \to C$ is the canonical normalization map and $n_1, \ldots, n_\delta$ are nodes of C.

Tensoring the sequence by $f^*\mathcal{O}_{\mathbb{P}^r}(-1)$ and taking cohomology, we get

$$\begin{aligned} 0 \quad & \to H^0(C, f^*\mathcal{O}_{\mathbb{P}^r}(-1)) \to H^0(\tilde{C}, v^* f^*\mathcal{O}_{\mathbb{P}^r}(-1)) \xrightarrow{\phi_f} \oplus_{j=1}^{\delta} \mathcal{O}_{\mathbb{P}^r}(-1)|_{f(n_j)} \\ & \xrightarrow{\theta_f} H^1(C, f^*\mathcal{O}_{\mathbb{P}^r}(-1)) \to H^1(\tilde{C}, v^* f^*\mathcal{O}_{\mathbb{P}^r}(-1)) \to 0. \end{aligned}$$

We claim that θ_f is not trivial. To see this, we examine the map ϕ_f. Let $C_1, \ldots, C_m$ be $(f \circ v)$-contracted components of $\tilde{C}$ and $C_{m+1}, \ldots, C_n$ the noncontracted components. As $H^0(C_i, (f \circ v)^*\mathcal{O}_{\mathbb{P}^r}(-1)) = 0$ for $i \geq m+1$, we have

$$H^0(\tilde{C}, v^* f^*\mathcal{O}_{\mathbb{P}^r}(-1)) = \bigoplus_{i=1}^{m} H^0(C_i, (f \circ v)^*\mathcal{O}_{\mathbb{P}^r}(-1)) \cong \mathbb{C}^m.$$

Moreover, a simple analysis shows that a connected contracted curve has nodes at least one less than the number of its irreducible components. As Γ is not a one-edge graph, we deduce that

$$\delta > m.$$

Therefore, ϕ_f cannot be surjective, and so θ_f is not a zero map.

By the claim, there exists a node n_k such that the $\mathbb{T}_i$-weight of $\mathcal{O}_{\mathbb{P}^r}(-1)|_{f(n_k)}$ divides $e_{\mathbb{T}_i}(H^1(C, f^*\mathcal{O}_{\mathbb{P}^r}(-1))$ for each i. Because $f(n_k) = q_s$ for some s, and $\mathbb{T}_s$ acts trivially on $\mathcal{O}_{\mathbb{P}^r}(-1)|_{q_s}$, it follows that

$$e_{\mathbb{T}_s}(H^1(C, f^*\mathcal{O}_{\mathbb{P}^r}(-1)) = 0.$$

That is, Γ does not contribute to (6.2.2). □

For $a, b = 0, \ldots, r$, let $\Gamma^d_{a,b}$ be the one-edge graph with the unique edge labeled with degree $d > 0$ and two endpoints labeled with points q_a, q_b and genera $g_a = 1$, $g_b = 0$:

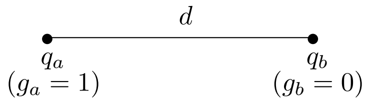

Figure 6.1. $\Gamma^d_{a,b}$.

This is, in fact, a graph in which the source curve C consists of a noncontracted component $C_e \cong \mathbb{P}^1$ and a genus one contracted component C_a labeled with q_a (the vertex labeled with q_b corresponds to a single point). By the above lemma, these are all graphs that can possibly make a contribution.

To proceed, it is helpful to make use of the identification

$$H^1(C, f^*\mathcal{O}_{\mathbb{P}^r}(-1)) \cong H^1(C_a, f^*\mathcal{O}_{\mathbb{P}^r}(-1)) \oplus H^1(\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^r}(-1)),$$

which, together with the following lemma, enables us to calculate $e_{\mathbb{T}_i}(H^1(C, f^*\mathcal{O}_{\mathbb{P}^r}(-1)))$.

Lemma 6.2.3. *Let* $\lambda = c_1(H^0(C_a, \omega_{C_a}))$. *We have the equalities*

$$e_{\mathbb{T}_i}(H^1(C_a, f^*\mathcal{O}_{\mathbb{P}^r}(-1))) = -\lambda + t_i - t_a,$$

and

$$e_{\mathbb{T}_i}(H^1(\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^r}(-1)) = (-1)^{d-1} \prod_{s=1}^{d-1} (\frac{st_a + (d-s)t_b}{d} - t_i)$$

for $i = 0, \ldots, r$.

Proof. By Serre's duality,

$$H^1(C_a, f^*\mathcal{O}_{\mathbb{P}^r}(-1)) = H^0(C_a, \omega_{C_a})^\vee \otimes \mathcal{O}_{\mathbb{P}^r}(-1)|_{q_a}.$$

This shows the first equality.

Now set $p_0 = [1:0],\ p_1 = [0:1] \in \mathbb{P}^1$. We know that $\mathbb{T}_i$ acts on $(f^*\mathcal{O}_{\mathbb{P}^r}(1) \otimes \omega_{\mathbb{P}^1})|_{p_0}$ and $(f^*\mathcal{O}_{\mathbb{P}^r}(1) \otimes \omega_{\mathbb{P}^1})|_{p_1}$ with weights

$$\begin{aligned} w_0 : &= t_a - t_i + \frac{t_b - t_a}{d} = \frac{d-1}{d} t_a + \frac{1}{d} t_b - t_i, \\ w_1 : &= t_b - t_i + \frac{t_a - t_b}{d} = \frac{1}{d} t_a + \frac{d-1}{d} t_b - t_i, \end{aligned}$$

respectively, and so

$$\begin{aligned} e_{\mathbb{T}_i}(H^0(\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^r}(1) \otimes \omega_{\mathbb{P}^1}) &= \prod_{s=0}^{d-2} \frac{sw_0 + (d-2-s)w_1}{d-2} \\ &= \prod_{s=0}^{d-2} (\frac{(s+1)t_a + [d-(1+s)]t_b}{d} - t_i) \\ &= \prod_{s=1}^{d-1} (\frac{st_a + (d-s)t_b}{d} - t_i). \end{aligned}$$

Thus,

$$\begin{aligned} e_{\mathbb{T}_i}(H^1(\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^r}(-1)) &= e_{\mathbb{T}_i}(H^0(\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^r}(1) \otimes \omega_{\mathbb{P}^1})^\vee) \\ &= (-1)^{d-1} \prod_{s=1}^{d-1} (\frac{st_a + (d-s)t_b}{d} - t_i), \end{aligned}$$

which proves the second equality. □

Now we study the inverse equivariant Euler class $1/e_{\mathbb{T}}(N^{\text{vir}}_{\Gamma^d_{a,b}})$ for all a, b and d:

(a) *Infinitesimal automorphisms of C*: At the point where C_e and C_a intersect, the tangent space to C_e contributes 1, while at the point labeled with q_b, the tangent space contributes

$$\frac{t_b - t_a}{d}.$$

(b) *Infinitesimal deformations of C*: It comes from the node joining C_a and C_e, and so it is given by

$$(\frac{t_a - t_b}{d} - \psi)^{-1},$$

where ψ is the first Chern class of the tautological line bundle formed by the cotangent space of C_a at the node.

(c) *Infinitesimal deformations of the stable map with the domain held fixed:*

- *Vertices*: The vertex labeled with q_a contributes

$$\frac{e_{\mathbb{T}}(H^1(C_a, f^*T\mathbb{P}^r))}{e_{\mathbb{T}}(H^0(C_a, f^*T\mathbb{P}^r))}.$$

As studied earlier, it is expressible in terms of the Hodge class and t_i's:

$$\prod_{j\neq a}(1+\frac{-\lambda}{t_a-t_j}).$$

On the other hand, the vertex labeled with q_b contributes

$$\frac{1}{e_{\mathbb{T}}(T_{q_b}\mathbb{P}^r)}=\prod_{j\neq b}\frac{1}{t_b-t_j}.$$

- *Flags*: The contribution is none other than the product $e_{\mathbb{T}}(T_{q_a}\mathbb{P}^r)\cdot e_{\mathbb{T}}(T_{q_b}\mathbb{P}^r)$, which is

$$\prod_{j\neq a}(t_a-t_j)\cdot\prod_{j\neq b}(t_b-t_j).$$

- *The unique edge*: We need to look at the moving part of $e_{\mathbb{T}}(H^0(C_e, f^*T\mathbb{P}^r))$, which can be determined by the Euler sequence on $\mathbb{P}^r$. Since a detailed discussion has been provided in [GP], we just write down the expression:

$$\frac{(-1)^d d^{2d}}{(d!)^2((t_a-t_b)^{2d}}\prod_{j\neq a,b}\frac{1}{\prod_{s=0}^{d}\frac{st_a+(d-s)t_b}{d}-t_j}.$$

Now we are ready to deduce (6.2.1). By Lemma 6.2.3, we have

$$\begin{aligned}\prod_{i=0}^{r} i^*_{\Gamma^d_{a,b}} e_{\mathbb{T}_i}(R^1 ft_* e_1^*(\mathcal{O}_{\mathbb{P}^r}(-1))) &= -\lambda\prod_{i\neq a}(-\lambda+t_i-t_a)\\ &\times\ (-1)^{d-1}[(d-1)!]^2\frac{(t_a-t_b)^{2d-2}}{d^{2d-2}}\\ &\times\ (-1)^{(d-1)(r-1)}\prod_{i\neq a,b}\prod_{s=1}^{d-1}(\frac{st_a+(d-s)t_b}{d}-t_i),\end{aligned}$$

for all a, b, and d.

By our expression of $1/e_{\mathbb{T}}(N^{\mathrm{vir}}_{\Gamma^d_{a,b}})$, we find that the contribution of each $\Gamma^d_{a,b}$ to (6.2.1) is given by

$$\begin{aligned}
&\int_{\overline{M}_{\Gamma^d_{a,b}}} \frac{\prod_{i=0}^r i^*_{\Gamma^d_{a,b}} e_{\mathbb{T}_i}(R^1 ft_* e_1^*(\mathcal{O}_{\mathbb{P}^r}(-1)))}{e_{\mathbb{T}}(N^{\mathrm{vir}}_{\Gamma^d_{a,b}})} \\
= \;& (-1)^{(r+1)(d+1)} \frac{t_b - t_a}{d} \frac{1}{\prod_{j\neq a,b}(t_a - t_j)(t_b - t_j)} \\
& \times \int_{\overline{M}_{1,1}} \lambda \prod_{i\neq a}(-\lambda + t_i - t_a)[\frac{d}{t_a - t_b}\sum_{k\geq 0}(\frac{d}{t_a-t_b}\psi)^k]\prod_{j\neq a}(t_a - t_j - \lambda) \\
= \;& (-1)^{(r+1)(d+1)} \frac{t_b - t_a}{d} \frac{1}{\prod_{j\neq a,b}(t_a - t_j)(t_b - t_j)} \cdot (-1)^r \prod_{j\neq a}(t_a - t_j)^2 \frac{d}{t_a - t_b}\int_{\overline{M}_{1,1}} \lambda \\
= \;& \frac{(-1)^{(r+1)d}}{24} \prod_{j\neq a,b} \frac{t_a - t_j}{t_b - t_j}.
\end{aligned}$$

As we are dealing with one-edge graph, we have $\mathbb{A}_{\Gamma^d_{a,b}} \cong \mathbb{Z}/(d)$ for all a, b, and d. In other words, $|\mathbb{A}_{\Gamma^d_{a,b}}| = d$. Using the well-known fact that

$$\int_{\overline{M}_{1,1}} \lambda = \frac{1}{24},$$

we obtain

$$\begin{aligned}
&\int_{[\overline{M}_{1,0}(\mathbb{P}^r,d)]^{\mathrm{vir}}_{\mathbb{T}}} \prod_{i=0}^r e_{\mathbb{T}_i}(R^1 ft_* e_1^*(\mathcal{O}_{\mathbb{P}^r}(-1))) \\
= \;& \sum_{a,b=0,a\neq b}^r \frac{1}{|\mathbb{A}_{\Gamma^d_{a,b}}|}\int_{\overline{M}_{\Gamma^d_{a,b}}} \frac{\prod_{i=0}^r i^*_{\Gamma^d_{a,b}} e_{\mathbb{T}_i}(R^1 ft_* e_1^*(\mathcal{O}_{\mathbb{P}^r}(-1)))}{e(N^{\mathrm{vir}}_{\Gamma^d_{a,b}})} \\
= \;& \frac{(-1)^{(r+1)d}}{24d} \sum_{a,b=0,a\neq b}^r \prod_{j\neq a,b} \frac{t_a - t_j}{t_b - t_j} \\
= \;& \frac{(-1)^{(r+1)d}(r+1)}{24d},
\end{aligned}$$

where the last equality is a consequence of the following beautiful identity.

Lemma 6.2.4. *Given $r+1$ variables $x_0, \ldots, x_r$,*

$$\sum_{k=1}^r \prod_{j\neq 0,k} \frac{x_0 - x_j}{x_k - x_j} = 1.$$

Proof. Write

$$R(x) := \sum_{k=1}^r \prod_{j\neq 0,k} \frac{x_0 - x_j}{x_k - x_j} = \frac{P(x)}{q_1 \cdots q_{r-1}},$$

where $x = (x_0, \ldots, x_r)$, $q_i = \prod_{j\geq i+1}(x_i - x_j)$ for $i = 1, \ldots, r-1$, and $P(x) \in \mathbb{Q}[x_0, \ldots, x_r]$.

First of all,

$$(x_1 - x_2)|P(x).$$

To see this, consider $P(x)$ as a polynomial with coefficients in $\mathbb{Q}[x_0, x_2, \ldots, x_r]$. Then

$$\begin{aligned} P(x) &= (x_0 - x_2)\ldots(x_0 - x_r)(x_2 - x_3)\ldots(x_2 - x_r)Q_1 \\ &\quad -(x_0 - x_1)(x_0 - x_3)\ldots(x_0 - x_r)(x_1 - x_3)\ldots(x_1 - x_r)Q_1 \\ &\quad +(x_1 - x_2)S, \end{aligned}$$

where $Q_1 = q_3 \ldots q_{r-1}$ and $S \in \mathbb{Q}[x_0, \ldots, x_r]$. Obviously, $P(x_0, x_2, x_2, x_3, \ldots, x_r) = 0$. Thus, $(x_1 - x_2)|P(x)$.

By a similar argument, all other factors of q_i divides $P(x)$ for each i, and the product $q_1 \ldots q_{r-1}$ divides $P(x)$ because q_i's are relatively prime. Hence we deduce that

$$P(x) = h \cdot q_1 \ldots q_{r-1}$$

for some $h \in Q[x_0, \ldots, x_r]$. Note that for each i, the degree of x_i in $P(x)$ is at most $r-1$ but the degree of x_i in the product $q_1 \ldots q_{r-1}$ is exactly $r-1$. This means that h is free of variables $x_1, \ldots, x_r$ and is thus a polynomial in at most one variable x_0.

Let us write $h = h(x_0)$, it remains to show that

$$h(x_0) = 1.$$

We need only check the cases $x_0 = n \in \mathbb{Z}$ and $x_i = i + x_0$ for $i \geq 1$:

$$\begin{aligned} h(n) &= \sum_{k=1}^{r} \frac{(-1)^{r-1} r!/k}{(k-1)\cdots 1 \cdot (-1) \cdots (k-r)} = \sum_{k=1}^{r} (-1)^{k-1} \frac{r!}{k!(r-k)!} \\ &= -(1-1)^r + 1 = 1. \end{aligned}$$

Therefore, the lemma is proved. □

6.3 Minimal Model Conjecture

The motivation behind Theorem 6.2.1 is the Quantum Minimal Model Conjecture, which is stated as follows.

Conjecture 6.3.1 ([R]). *K-equivalent projective varieties have isomorphic quantum cohomologies.*

In this section, we merely focus on a special kind of K-equivalences, which we now discuss.

Let X be a smooth complex projective variety and $X \to \bar{X}$ a flopping contraction, with the exceptional locus

$$Z \cong \mathbb{P}^r,$$

and the normal bundle

$$N_{Z/X} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}.$$

The corresponding $\mathbb{P}^r$ flop $f : X \dashrightarrow X'$ exists and is referred to as simple ordinary $\mathbb{P}^r$ flop.

The simple flop may be constructed in this way: Blowing up X along Z to get a morphism $\phi : Y \to X$. The exceptional divisor E of ϕ is

$$\mathbb{P}_Z(N_{Z/X}) = Z \times_{\mathbb{C}} Z',$$

where $Z' = \mathbb{P}^r$. It can be shown that there is a morphism $\phi' : Y \to X'$ obtained by blowing down E onto Z', and the normal bundle to Z' is given by

$$N_{Z'/X'} \cong \mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)}.$$

More details can be found in [LLW]. The varieties X and X' are K-equivalent since $\phi^* K_X = \phi'^* K_{X'}$.

The invariance of the genus zero Gromov-Witten theory under simple ordinary flops has been demonstrated in [LLW]. Theorem 6.2.1 suggests a similar result concerning genus one Gromov-Witten theory attached to the extremal ray. In the recent paper [ILLW], a more general setting is considered, namely ancestors and descendents are both included.

For later convenience of explanation, we introduce the extremal Gromov-Witten potential of X, which is defined by

$$\langle \alpha_1, \ldots, \alpha_n \rangle^X = \sum_{g=0}^{\infty} \sum_{d=0}^{\infty} \langle \alpha_1, \ldots, \alpha_n \rangle_{g,d\ell}\, q^d,$$

where ℓ is the line class in $Z \cong \mathbb{P}^r$. We define similarly the extremal Gromov-Witten potential $\langle -, \ldots, - \rangle^{X'}$ of X' but we replace the quantum parameter q with q^{-1}.

We consider the correspondence

$$\mathcal{F} = \phi'_* \phi^*$$

and set

$$\mathcal{F}(q) = q^{-1}.$$

Corollary 6.3.2. *The extremal Gromov Witten potentials are preserved by simple $\mathbb{P}^r$ flops. More precisely, we have the following equality:*

$$\mathcal{F}\left\langle \alpha_1, \ldots, \alpha_n \right\rangle^X = \left\langle \mathcal{F}\alpha_1, \ldots, \mathcal{F}\alpha_n \right\rangle^{X'}, \tag{6.3.1}$$

where X and X' are as above.

Proof. First of all, we assume that X and X' are not three-dimensional. By dimension reasoning, the positive degree, genus one, extremal Gromov-Witten theory is completely determined by zero-point Gromov-Witten invariants, and the genus g theory is trivial for $g \geq 2$. Moreover, we have

$$\overline{M}_{1,0}(W, d\ell) = \overline{M}_{1,0}(\mathbb{P}^r, d)$$

for $W = X, X'$ and the following equality:

$$\int_{[\overline{M}_{1,0}(W,d\ell)]^{\mathrm{vir}}} 1 = \int_{[\overline{M}_{1,0}(\mathbb{P}^r,d)]^{\mathrm{vir}}} e(R^1 ft_* e_1^*(\mathcal{O}_{\mathbb{P}^r}(-1)^{\oplus(r+1)})).$$

On the other hand, in [LLW] and [ILLW], the degree zero theory and the genus zero theory of X and X' have been determined, and the relations

$$\mathcal{F}(\ell^k) = (-1)^{r-k}\ell^k, \; k \geq 1$$

have also been established. Therefore, our statement follows from Theorem 6.2.1 and a direct comparison of both sides of (6.3.1).

Suppose now that X and X' are of dimension 3. The above argument for genus ≤ 1 also works but we also need to understand higher genus invariants, in which case, the main formula for the equality (6.3.1) to hold is simply

$$\int_{[\overline{M}_{g,0}(\mathbb{P}^1,d)]^{\mathrm{vir}}} e(R^1 ft_* e_1^*(\mathcal{O}_{\mathbb{P}^r}(-1) \oplus \mathcal{O}_{\mathbb{P}^r}(-1))) = \frac{d^{2g-3}|\chi(M_g)|}{(2g-3)!}, \tag{6.3.2}$$

where $\chi(M_g)$ is the orbifold Euler characteristic of M_g for $g \geq 2$. However, (6.3.2) has already been shown in [FP]. The proof is now complete. $\square$

The Quantum Minimal Model Conjecture is not independent of the Crepant Resolution Conjecture, which has been studied in the previous chapters. Indeed, if the crepant resolutions of a singular variety exist, they can be related to one another by K-equivalences. Thus, the Minimal Model Conjecture also tests the validity of the Crepant Resolution Conjecture.

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