THEORETICAL INVESTIGATION

OF

DETACHED SHOCK WAVES

Thesis by

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ABSTRACT

The problems associated with the detached shock wave are considered from the analytical standpoint in this report. For considering the general case for the detached shock wave, the non-stationary isentropic differential equation is derived. In general the stationary detached shock wave is curved and thus, the flow back of the shock is rotational. The effect of rotational flow upon the velocity and pressure distribution over a circular cylinder is analyzed for a parabolic velocity distribution in the undisturbed region.

The basic equations for both normal and oblique shock waves are presented and the significance of these equations to the problem of detached shock is discussed. The conditions for the shock wave to be detached are presented and the mathematical formulation of the Tricomi type of differential equation for the detached shock wave is given.

The first approximation to the location of the detached shock wave is derived and the analytical results are correlated with the experimental data for spheres obtained from the supersonic wind tunnel and the ballistic range. The agreement was found to be satisfactory.

The existence and uniqueness of a potential

solution for an infinite wedge with normal detached shock wave moving at constant velocity is presented. It is shown that even for an infinite wedge with normal detached shock wave the potential solution dees not exist.

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I. INTRODUCTION AND SUMMARY

With the introduction of rocket and gas turbine jet power plants, the speeds of missiles and airplanes have reached and exceeded the speed of sound. With the attainment of supersonic speeds, such problems as high temperature due to high Mach number, flutter at supersonic speeds, and detached shock wave problems in the transonic range of Mach number are becoming serious. If the transition is rapid through the transonic Mach number range which can be considered as being in the range of Mach number of .95 to about 1.4, the problems of detached shock wave are not serious. At the higher subsonic Mach numbers the shock waves are formed usually on the body, (Cf. Ref. 1). At a Mach number of one the shock wave is formed ahead of the body and is detached for all bodies. The detached shock waves, which are formed only for the supersonic flow, will be considered in this thesis. Part of the flow behind such a detached shock wave is always subsonic because the shock wave is normal to the axis of symmetry and at some distance away from the normal shock position the flow behind the shock will be equal to Mach number of 1.0 and for points on the shock wave beyond this location the flow is supersonic. The detached shock wave is always curved and thus, the Mach number back of the shock wave varies from the lowest value back of the normal shock wave to the free stream Mach number at some finite distance from the body where the shock wave has become a Mach wave due to the interaction of the shock wave and the expansion wave from the surface. The flow in this case will be divided into two regions, the main flow being supersonic and the subsonic region is bounded by the shock wave and the surface of the body as shown in Fig. 1. Thus, the problem is slightly different from the lower transonic flow problem in which the free stream flow is subsonic and there is a supersonic region imbedded in the subsonic flow as discussed in Ref. 2.

The mathematical solution to the problem is difficult because of the mixed character of the differential equation. For the subsonic region the differential equation is of the elliptic type and in the supersonic region the differential equation is of the hyperbolic type. One other physical feature of the detached shock wave which makes the exact analytical solution difficult to obtain is that the flow is not irrotational after the detached shock wave because the entropy is not constant for all streamlines when the curvature of the shock wave is large, as for example, the flow at supersonic speed for blunt body. At lower supersonic Mach numbers where the detached shock is not curved appreciably, the vorticity of the flow after the shock wave will not be large. Even without the effect of vorticity, the solution to the mixed flow problem is difficult because of the mixed type of differential equation. There is at present no general mathematical method for solving such a mixed flow problem. Tricomi in Ref. 3 has considered the problem and has proven the uniqueness and existence of the solution under certain cnditions. Frankl in Ref. 4 has shown that for a particular mixed flow problem the solution would be unique, but he did not show how to obtain such a solution.

If we consider the supersonic flow over a wedge, as shown in Figs. 1a and 1b, the shock wave will be detached for free stream Mach number of 1.0 to a critical value of M_{cr_1}, which depends upon the wedge angle, at which Mach number the flow would be attached to the surface and part of the flow will be subsonic as shown by the chart in Ref. 5. For a finite wedge the detached shock wave will be curved and even for the attached shock wave for the critical Mach number, M_{cr_1}, the shock is curved as shown in Fig. 1b. The attached shock wave will become straight in the region from the nose to the intersection of the expansion waves with the shock for the finite wedge when the Mach number after the shock wave is equal to one, and we shall call this the second critical Mach number, M_{cr_2}. For this Mach number the pressure distribution over the wedge will become constant and for higher Mach numbers the pressure will also remain constant. At Mach numbers lower than this critical value, Mcr_2, the pressure distribution over the wedge surface will not be constant because the flow has to speed up from a subsonic Mach number at the nose to Mach number of one at the shoulder, where the wedge intersects the parallel side. At this point the flow will become supersonic by expanding around the corner as a Prandtl-Meyer flow, which is discussed in Ref. 6.

The flow behavior for cone with detached shock waves is very similar to that for a two-dimensional wedge. Some preliminary experimental data have indicated that the transition from the detached to attached condition is rather gradual without any large discontinuity as indicated in Ref. 7.

All the earlier references for analyzing the supersonic flow over bodies of revolution, as given in Refs. 8, 9, 10, and 11 have considered the disturbances to be small so that the shock waves were always attached to the body. As soon as the supersonic Mach number becomes low for a given cone angle and the shock wave was detached, the above results would no longer apply. For supersonic flow over two-dimensional bodies with sharp leading edges, the flow was also analyzed from the standpoint of small disturbance as given in Refs. 12, 13, and 14. Again, as soon as the shock wave became detached the results would no longer apply because the method did not give the solution for the detached shock conditions.

Recently there has been some work done to analyze the supersonic flow with detached shock waves. In Ref. 15 a numerical method was used to solve the subsonic region for simple bodies. In obtaining these solutions, assumptions were made with regards to the sonic line and the vorticity. Guderley discussed the nature of flow condition with detached shock wave in Ref. 16 and he obtained some mathematical relationships for the problem by neglecting vorticity and using an approximation to the shock polar.

The detached shock wave is in general a function of the free stream Mach number, wedge or cone angle at the nose, and angle of attack. If a double wedge airfoil or other sharp nosed airfoil is in a supersonic flow with attached shock wave, the shock wave will become detached as soon as the angle of attack reaches a definite value for a given supersonic Mach number. If the deflection is high enough for moveable control surfaces such as aileron, rudder, or elevator at supersonic speeds, a detached shock wave will form and the flow will become of the mixed type again, subsonic flow imbedded in the supersonic flow. For missiles and airplanes, it is very important to be able to predict the aerodynamic characteristics with detached shock wave for bodies, lifting surfaces and moveable control surfaces, in order to be able to determine the aerodynamic loads for structural design purposes and to predict the performance and dynamic stability. This problem is becoming more serious because the speeds of the latest research and military airplanes are approaching and exceeding the speed of sound.

The problem of detached shock wave was first noticed by the writer when 5"-HVAR rockets with 10% biconvex wings with pressure orifices were fired to obtain free flight aerodynamic data for supersonic and transonic Mach numbers. The maximum velocity attained by these vehicles was about 1.9 at launching and decreased to about .75 Mach number before striking the ground. The pressure data was telemetered during the flight and the vehicles were tracked with Radar to obtain the altitude and velocity during the flight. At the supersonic speeds where the shock wave was attached to the airfoil the agreement between the theory and flight data for the variation of the pressure coefficient with Mach number was fairly good, but for the transonic range of Mach number, the pressure variation with Mach number was exactly opposite to that predicted by the linear supersonic theory, as shown in Ref. 17.

The present analytical research work was undertaken to obtain some information about the behavior and characteristics of detached shock waves. The ultimate objective of the research project is to derive a theory for predicting the complete aerodynamic characteristics of bodies and lifting surfaces with detached shock wave. This objective is extremely difficult to obtain immediately due to the mathematical difficulties encountered in solving mixed flow problems and also to the lack of adequate experimental data on detached shock waves.

The present Thesis gives some of the results that were obtained for the problems of detached shock waves. In Section II the general differential equation for non-stationary adiabatic flow is developed as well as the non-stationary differential equation for vorticity in a compressible fluid. The general non-stationary equation was developed in detail because of later application of the equation for analyzing the detached shock wave for an infinite wedge in Section VIII.

Section III is devoted to investigate the effect of vorticity upon the velocity and pressure distribution over a body. The flow behind a detached shock that is curved is rotational and also subsonic for a definite region bounded by the shock wave and the body. To obtain some idea about the effect of vorticity for subsonic flow, parabolic velocity distribution at infinite distance ahead of the body was assumed for an incompressible flow. The effects of rotational flow upon the velocity and pressure distribution for the circular cylinder with and without circulation were investigated.

In Section IV the conditions for a propagating normal shock wave are discussed in detail. For a given condition ahead of the normal shock and velocity of propagation, the conditions behind the shock were determined in a convenient form for investigating the detached shock wave. The results presented in this Section were applied for determining the first approximation to the detached shock wave distance from the body in Section VII.

Section V is devoted to the investigation of the stationary oblique shock wave. Various authors have discussed the oblique shock wave theory, but in this Section the shock polar for the oblique shock wave is shown to be of fundamental importance in analyzing the flow back of the detached shock wave. By the use of the shock polar the velocity magnitude and direction at the detached shock wave can be determined. Also, the Mach number at which the shock would attach to the pointed body can be determined from the use of shock polar. For sharp-nosed, finite two-dimensional bodies, the Mach number at which the velocity and pressure becomes constant over the nose can be determined from the shock polar. Shock polar for oblique shock wave plays an important part in formulating the boundary condition for the mixed flow problem or for the region behind the detached shock wave. The results discussed in this Section are used in Section VI for the investigation of the condition for the shock wave to become detached from the body.

In Section VI, the results presented in Sections IV and V are applied for the investigation of the physical conditions, shape and types of flow for detached shock wave for finite and infinite wedges. A detailed discussion of the detached shock wave in the physical and hodograph planes for a two-dimensional finite wedge is presented in this Section. The significance of the critical Mach numbers is given in terms of the shock polar. The differential equation, which is the type first considered by Tricomi in Ref. 3, for the flow behind the detached shock wave is discussed.

In Section VII a theory for the first approximation to the distance between the nose of a symmetrical body at zero angle of attack is presented. The assumption is made that the curvature of the detached shock is small and thus, the flow behind the shock is assumed to be irrotational.

The theory is applied to determine the first approximation to the shock distance for circular cylinder and sphere with and without the correction for the compressibility effects. The correlation of the predicted and actual detached shock distance for the sphere is discussed.

In Section VIII the detached shock wave for an infinite wedge is investigated by considering the non-stationary differential equation of motion for compressible fluid. The flow is assumed to be isentropic and adiabatic back of the normal detached shock wave. It is also assumed that the normal shock wave propagates away from the nose of the wedge at constant velocity. With these assumptions the general non-stationary differential equation is transformed into non-dimensional stationary differential equation. From the analysis of the existence and uniqueness of a potential solution for an infinite wedge with normal detached shock wave moving at constant speed, it is shown that no potential flow exists back of a detached shock wave.

II. NON-STATIONARY ISENTROPIC DIFFERENTIAL EQUATION

A. Fundamental Equations and Assumptions

The hydrodynamical equation of general two-dimensional motion in terms of the Cartesian coordinate system is given as discussed in Refs. 18 and 19 by the vector equation,

$$\frac{D\bar{q}}{Dt} = -\frac{1}{\rho}\nabla P, \tag{2.1a}$$

where the differential operator in the expanded form is

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \bar{q}\cdot\nabla(\ ) = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y}.$$

The body forces, viscosity, and heat condition are neglected in the consideration of the fluid motion to simplify the analysis. The Eq. (2.1a) can be expanded and be written as

$$\frac{\partial\bar{q}}{\partial t} + \bar{q}\cdot\nabla\bar{q} = -\frac{1}{\rho}\nabla P, \tag{2.1b}$$

or in terms of the vorticity, $\bar{\Omega} = \nabla\times\bar{q}$, we have

$$\frac{\partial\bar{q}}{\partial t} - \bar{q}\times\bar{\Omega} = -\frac{1}{\rho}\nabla P - \nabla\frac{q^2}{2}. \tag{2.1c}$$

From the condition of conservation of the mass, we have the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\bar{q}) = 0 \tag{2.2a}$$

or in the expanded form this equation becomes

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \bar{q} + \bar{q} \cdot \nabla \rho = 0. \tag{2.2b}$$

Thus, this can be written in another convenient form

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \bar{q} = 0, \tag{2.2c}$$

which is used often in the analysis of the fluid flow. For the steady flow the $\frac{\partial}{\partial t}$ terms disappears; and thus, the equation becomes

$$\rho \nabla \cdot \bar{q} + \bar{q} \cdot \nabla \rho = 0.$$

To obtain the relationship between the pressure and density, we shall assume the flow to be adiabatic, then we have

$$\frac{p}{\rho^{\gamma}} = \text{constant}, \tag{2.3}$$

where

$\gamma = \frac{C_p}{C_v}$, ratio of the specific heats. The flow process takes place rapidly so there can be no heat conduction, and thus, the adiabatic assumption is good as indicated by the experimental data. We shall also assume the fluid to be a perfect gas so the equation of state relating the pressure, density, and temperature takes the form

$$\frac{p}{\rho} = RT \tag{2.4}$$

where R is the gas constant. The velocity of sound in the fluid medium is given by

$$a^2 = \frac{dp}{d\rho} = \frac{\gamma p}{\rho} \tag{2.5}$$

for an adiabatic flow.

B. Non-Stationary Vorticity Differential Equation

To obtain the differential equation for the vorticity, which is necessary in the analysis of flow with vorticity, operate on the equation of motion, Eq. (2.1c), with $\nabla\times$ to obtain

$$\frac{\partial(\nabla\times\bar{q})}{\partial t} - \nabla\times(\bar{q}\times\bar{\Omega}) = -\nabla\times\left(\frac{1}{\rho}\nabla p\right)$$

or

$$\frac{\partial\bar{\Omega}}{\partial t} - \nabla\times(\bar{q}\times\bar{\Omega}) = \frac{1}{\rho^2}\nabla\rho\times\nabla p$$

If the pressure is a function of the density, the general vorticity equation for the compressible fluid becomes

$$\frac{D\bar{\Omega}}{Dt} = \bar{\Omega}\cdot\nabla\bar{q} - \bar{\Omega}\,\nabla\cdot\bar{q}. \tag{2.6}$$

Crocco in Ref. 20 obtained the differential equation for the vorticity for a steady compressible flow and Vazsonyi in Ref. 21 obtained the non-stationary differential equation for the vorticity in a slightly different form than

Eq. (2.6).

The Eq. (2.6) gives the rate of change of vorticity of a fluid particle as it moves about in the region. For the incompressible fluid the second term on the right hand side of the equation does not exist because the divergence of the velocity is equal to zero. This equation also shows that if the fluid particle does not have vorticity at a particular time, the vorticity will always remain zero unless some discontinuity such as curved shock wave and coefficient of viscosity not equal to zero will produce rotational flow after the shock. Once the fluid particle has vorticity, it will continue to be a rotational flow.

C. Velocity Potential for Irrotational Flow

If the flow is irrotational, $\nabla \times \bar{q} = 0$, it is possible to introduce the velocity potential which is given by

$$\varphi(b) = \int_a^b \bar{q} \cdot d\bar{r} \qquad (2.7)$$

The gradient of this velocity potential gives the velocity

$$\bar{q} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\underline{i} + \frac{\partial\varphi}{\partial y}\underline{j}\,, \qquad (2.8a)$$

where

$$u = \frac{\partial\varphi}{\partial x}$$
$$v = \frac{\partial\varphi}{\partial y}. \qquad (2.8b)$$

The equation of motion, Eq. (2.1b), becomes for an irrotational motion in terms of the velocity potential

$$\nabla\frac{\partial\varphi}{\partial t} + \nabla\frac{q^2}{2} = -\frac{1}{\rho}\nabla P.$$

This can be integrated in terms of the space derivatives to obtain

$$\frac{\partial\varphi}{\partial t} + \frac{q^2}{2} + \int\frac{dP}{\rho} = F(t), \qquad (2.9a)$$

where $F(t)$ is an arbitrary function of time. And for an adiabatic flow, the integral can be expressed as

$$\int\frac{dP}{\rho} = \frac{\gamma}{\gamma-1}\frac{P}{\rho}, \qquad (2.10)$$

which is the expression for the enthalpy of the gas

$$I = c_v T + \frac{P}{\rho} = \frac{\gamma}{\gamma-1}\frac{P}{\rho}. \qquad (2.11)$$

Thus, Eq. (2.9a) becomes

$$\frac{\partial\varphi}{\partial t} + \frac{q^2}{2} + \frac{\gamma}{\gamma-1}\frac{P}{\rho} = F(t). \qquad (2.9b)$$

D. Non-Stationary Differential Equation for Irrotational and Isentropic Compressible Fluid.

To obtain the general non-stationary differential equation in terms of the velocity, multiply the equation of motion, Eq. (2.1b), by $\bar{q}$.

$$\bar{q}\cdot\frac{\partial\bar{q}}{\partial t}+\bar{q}\cdot[\bar{q}\cdot\nabla\bar{q}]=-\frac{1}{\rho}\bar{q}\cdot\nabla p=-\frac{a^2}{\rho}\bar{q}\cdot\nabla\rho$$

since $a^2=\frac{dp}{d\rho}$.

Substituting the expression for $\frac{\bar{q}\cdot\nabla\rho}{\rho}$ from this equation into the continuity equation (2.2b), we obtain

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t}+\nabla\cdot\bar{q}-\frac{\bar{q}}{a^2}\cdot\frac{\partial\bar{q}}{\partial t}-\frac{\bar{q}}{a^2}\cdot[\bar{q}\cdot\nabla\bar{q}]=0$$

or

$$\frac{a^2}{\rho}\frac{\partial\rho}{\partial t}+a^2\nabla\cdot\bar{q}-\bar{q}\cdot\frac{\partial\bar{q}}{\partial t}-\bar{q}\cdot[\bar{q}\cdot\nabla\bar{q}]=0 \qquad (2.12)$$

By taking the $\frac{\partial}{\partial t}$ of Eq. (2.9b) we obtain the expression

$$\frac{\partial^2\varphi}{\partial t^2}+\frac{\partial}{\partial t}\left(\frac{q^2}{2}\right)+\frac{a^2}{\rho}\frac{\partial\rho}{\partial t}=\frac{dF}{dt}=0 \qquad (2.13)$$

for $F(t)=$ constant, which is valid if the velocity potential is taken to be independent of time, $\frac{\partial\varphi}{\partial t}=0$, in the undisturbed region. Substitute the expression for $\frac{a^2}{\rho}\frac{\partial\rho}{\partial t}$ from this equation into Eq. (2.12) to obtain the non-stationary differential equation for the velocity,

$$\frac{\partial^2\varphi}{\partial t^2}+2\bar{q}\cdot\frac{\partial\bar{q}}{\partial t}+\bar{q}\cdot[\bar{q}\cdot\nabla\bar{q}]=a^2\nabla\cdot\bar{q}, \qquad (2.14a)$$

where a is determined by Eq. (2.5).

For an isentropic irrotational motion, the differential equation for the velocity potential can be obtained by substituting $\bar{q} = \nabla\varphi$ into Eq. (2.14a) to obtain

$$\frac{\partial^2\varphi}{\partial t^2} + 2\nabla\varphi\cdot\nabla\frac{\partial\varphi}{\partial t} + \nabla\varphi\cdot\nabla\left(\frac{\nabla\varphi\cdot\nabla\varphi}{2}\right) = a^2\nabla\cdot\nabla\varphi. \qquad (2.15a)$$

One can see that due to non-stationary motion there are two extra terms appearing in Eqs. (2.14a) and (2.15a), which are the variation of the velocity and the second derivative of the velocity potential with respect to time. Since it is necessary to consider these equations in detail for the solution of the detached shock wave for an infinite wedge, it will be convenient to write Eqs. (2.14a) and (2.15a) in terms of the two-dimensional velocity components instead of using the vector form. The equations become

$$\frac{\partial^2\varphi}{\partial t^2} + 2\left(u\frac{\partial u}{\partial t} + v\frac{\partial v}{\partial t}\right) + u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} + uv\frac{\partial u}{\partial y} + v^2\frac{\partial v}{\partial y} = a^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (2.14b)$$

and

$$\frac{\partial^2\varphi}{\partial t^2} + 2\left(\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x\partial t} + \frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y\partial t}\right) + \left(\frac{\partial\varphi}{\partial x}\right)^2\frac{\partial^2\varphi}{\partial x^2} + 2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}$$

$$+ \left(\frac{\partial\varphi}{\partial y}\right)^2\frac{\partial^2\varphi}{\partial y^2} = a^2\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right). \qquad (2.15b)$$

The right hand terms can be transferred to the left side to obtain.

$$\frac{\partial^2\varphi}{\partial t^2} + 2\left(u\frac{\partial u}{\partial t} + v\frac{\partial v}{\partial t}\right) + (u^2 - a^2)\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} + uv\frac{\partial u}{\partial y} + (v^2 - a^2)\frac{\partial v}{\partial y} = 0 \qquad (2.14c)$$

and

$$\frac{\partial^2\varphi}{\partial t^2}+2\left(\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x\partial t}+\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y\partial t}\right)+\left[\left(\frac{\partial\varphi}{\partial x}\right)^2-a^2\right]\frac{\partial^2\varphi}{\partial x^2}+2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}+\left[\left(\frac{\partial\varphi}{\partial y}\right)^2-a^2\right]\frac{\partial^2\varphi}{\partial y^2}=0. \quad (2.15c)$$

The general solution for the velocity potential is extremely difficult ot obtain because the equation is non-linear and also to the fact that there is another independent variable, time, being introduced. The solution of the steady state condition, which does not have the partial derivative with respect to time, is very difficult and only special cases have been solved analytically as discussed in Ref. 1.

If the velocity potential is taken to be independent of time for the undisturbed region, the partial derivative of the velocity potential with respect to time will be equal to zero, $\frac{\partial\varphi}{\partial t}=0$, for the undisturbed region and the value of $F(t)$ for Eq. (2.9b) will be equal to a constant. The Eq. (2.9b) for this condition becomes

$$\frac{\partial\varphi}{\partial t}+\frac{q^2}{2}+\frac{\gamma}{\gamma-1}\frac{p}{\rho}=F(t)=A=\text{constant}. \quad (2.9c)$$

At infinite distance ahead of the body in the undisturbed region, $\frac{\partial\varphi}{\partial t}=0$, the velocity, pressure and density will be constant, q_1, p_1, and ρ_1. The constant of Eq. (2.9c) in terms of these quantities is given by

$$\frac{q_1^2}{2}+\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}=A. \quad (2.16)$$

The velocity of sound for the stagnation condition in the undisturbed region can be determined by using the energy equation for the compressible fluid and is given by

$$a_0^2 = (\gamma-1)\left[\frac{q_1^2}{2} + \frac{a_1^2}{\gamma-1}\right] = a_1^2\left[1 + \frac{\gamma-1}{2}M_1^2\right], \tag{2.17}$$

where $a_1^2 = \frac{\gamma P_1}{\rho_1}$

and $M_1 = \frac{q_1}{a_1}$ is the Mach number of the undisturbed flow. From Eqs. (2.16) and (2.17) we see that the constant can be expressed in terms of the stagnation velocity of sound for the undisturbed region as follows

$$A = \frac{a_0^2}{\gamma-1} . \tag{2.18}$$

The local velocity of sound for non-stationary flow can be determined from Eqs. (2.9c) and (2.18), and the result is

$$a^2 = (\gamma-1)\left[\frac{a_0^2}{\gamma-1} - \frac{q^2}{2} - \frac{\partial\varphi}{\partial t}\right] \tag{2.19a}$$

or

$$a^2 = a_0^2\left[1 - \left(\frac{\gamma-1}{2}\right)\frac{q^2}{a_0^2} - \frac{(\gamma-1)}{a_0^2}\frac{\partial\varphi}{\partial t}\right], \tag{2.19b}$$

where $a^2 = \frac{\gamma P}{\rho}$.

For a steady flow condition, the local velocity of sound is given by the first two terms of Eq. (2.19b) and the variation of the velocity potential with respect to time does not exist. In terms of the velocity potential the Eq. (2.19b) becomes

$$a^2 = a_0^2\left\{1 - \frac{(\gamma-1)}{2a_0^2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] - \frac{(\gamma-1)}{a_0^2}\frac{\partial\varphi}{\partial t}\right\}. \qquad (2.19c)$$

Thus, the value for a can be solved by using this equation and by substituting Eq. (2.19c) into Eq. (2.15b); the velocity potential appears explicitly as the only dependent variable.

In deriving the non-stationary partial differential equation for the velocity potential, Eq. (2.15b), it was assumed that there was no disturbance which would cause rotational flow. The differential equation will hold for region back of discontinuity, such as normal shock wave, if the vorticity is equal to zero. With vorticity the solution to the fluid flow must be obtained by solving the vorticity equation, Eq. (2.6). The general non-stationary differential equation for an isentropic irrotational flow was derived for the velocity potential since the equation was necessary for the analysis of the detached shock wave flow for an infinite wedge in Section VIII.

III. FLOW WITH VORTICITY

When the shock wave is not attached to the body at supersonic speeds, the detached shock wave intersects at right angles the axis of symmetry for a symmetrical body at zero angle of attack. In the general case the detached shock is always curved and at some point the shock wave is normal to the free stream flow direction. The detached shock wave gradually curves and becomes equal to the Mach angle at some finite distance from the finite body. With such a curved detached shock wave, the Mach number is subsonic back of the normal part of the wave and the Mach number after the shock wave becomes equal to the free stream Mach number where the shock wave angle is equal to the Mach angle. The subsonic region of flow for a wedge with a detached shock wave is shown in Fig. 1a. From this figure it is seen that the subsonic region for a two-dimensional wedge is bounded by the detached shock, wedge surface to the corner and the sonic line that goes from the corner to the shock wave. Within this region the flow is subsonic, but it is also rotational because of the curved shock wave which causes the entropy to vary for the streamlines. The Mach number after the normal part of the shock wave is always subsonic and becomes more subsonic as the free stream becomes more supersonic. The relationship between the Mach number after the normal shock and the free stream Mach number will be discussed in Section IV.

In order to obtain more information regarding the effects of subsonic rotational flow after the detached shock, an analysis was made for an incompressible flow with parabolic distribution of velocity at infinity. The vorticity distribution in this case will be a linear function of y in the undisturbed region of the flow. Tsien in Ref. 22 had studied the effect of linear velocity distribution, which corresponds to constant vorticity, upon the aerodynamic characteristics of Joukowsky Airfoil. Kuo in Ref. 23 had considered the same problem and used another method to derive the Blasius' equations for flow with constant vorticity.

The problem of flow with parabolic velocity distribution is also encountered in a wind tunnel if the velocity is not constant in the test section. The effects of this type of flow upon the loads on objects in the duct where the velocity distribution is not constant at the station where the object is placed can be estimated by applying the analysis that will be presented in this Section.

A. Fundamental Equations and Assumptions

It will be assumed that the flow is incompressible, constant, steady, non-viscous, and two-dimensional. The general dynamical equation of motion was shown in Section II to be

$$\frac{\partial \bar{q}}{\partial t} - \bar{q} \times \nabla \times \bar{q} = -\frac{1}{\rho} \nabla p - \nabla \frac{q^2}{2}. \qquad (2.1c)$$

For an incompressible two-dimensional flow the equation becomes

$$\frac{\partial \bar{q}}{\partial t} - \bar{q} \times \bar{\Omega} = -\nabla \left(\frac{p}{\rho} + \frac{q^2}{2} \right) \qquad (3.1)$$

where

$$\bar{\Omega} = \nabla \times \bar{q} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \underline{k}. \qquad (3.2)$$

Now for the steady flow condition the Eq. (3.1) reduces to

$$\bar{q} \times \bar{\Omega} = \nabla \left(\frac{p}{\rho} + \frac{q^2}{2} \right) \qquad (3.3)$$

since $\frac{\partial}{\partial t} = 0$. It can be seen from this equation that if the vorticity is equal to zero, $\bar{\Omega} = 0$, the right hand term must be equal to a constant,

$$\frac{p}{\rho} + \frac{q^2}{2} = \text{constant} \qquad (3.4)$$

which is the Bernoulli's equation for the incompressible fluid. To show that the vorticity for a steady flow must be permanent, operate with $\nabla \times$ on Eq. (2.1c) to obtain

$$\frac{\partial \bar{\Omega}}{\partial t} + \bar{q} \cdot \nabla \bar{\Omega} = \bar{\Omega} \cdot \nabla \bar{q} - \bar{\Omega} \nabla \cdot \bar{q}$$

which becomes

$$\frac{D\bar{\Omega}}{Dt} = \bar{\Omega}\cdot\nabla\bar{q} - \bar{\Omega}\,\nabla\cdot\bar{q} \tag{3.5a}$$

This equation gives the rate of change of vorticity, $\bar{\Omega}$, as the particle moves about. For a two-dimensional incompressible motion, the vorticity equation becomes

$$\frac{D\bar{\Omega}}{Dt} = 0 \,, \tag{3.5b}$$

since the velocity does not vary in the direction of the vorticity, and for a steady motion, the equation reduces to

$$\bar{q}\cdot\nabla\bar{\Omega} = 0, \tag{3.5c}$$

which shows that the vorticity of the particle does not change as the particle moves. If the flow is irrotational, $\bar{\Omega} = \nabla\times\bar{q} = 0$, the flow will always remain irrotational unless vorticity is produced by a discontinuity such as a detached shock wave or any other similar disturbance which produces non-uniformity in the energy for the streamlines.

These considerations have shown for a two-dimensional incompressible motion that the vorticity is constant for each streamline. Consider the case where the velocity distribution at infinite distance ahead of the body is parabolic,

$$u = U_0\left(1 + k\frac{y^2}{c^2}\right) \tag{3.6}$$

$$v = 0 \,,$$

where k is a non-dimensional constant, C is a representative length of the body, and U_0 is the undisturbed flow velocity along the x-axis. The vorticity at $x=-\infty$ can be determined from Eq. (3.6) and is given by

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=-2\frac{U_0 k}{C^2}y, \tag{3.7}$$

which is a linear function of y.

The equation of continuity for steady incompressible flow is

$$\nabla\cdot\bar{q}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{3.8}$$

A stream function, ψ, which satisfies this equation is introduced and is defined by

$$u=\frac{\partial\psi}{\partial y}\quad,\quad v=-\frac{\partial\psi}{\partial x}. \tag{3.9}$$

Substitute this into Eq. (3.2) to obtain the vorticity equation in terms of the stream function,

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\Omega. \tag{3.10a}$$

Due to the fact that the vorticity is associated with the particle and maintains the strength, each streamline at infinite distance away from the body will have a definite vorticity and will maintain the strength throughout the field. Hence, the vorticity equation can be considered as

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=f(\psi) \tag{3.10b}$$

and the problem reduces to solving this equation with the proper boundary conditions. The general solution for this equation is very difficult to obtain. We shall assume that the velocity curvature factor, k, is small and also that the disturbance due to the body is small such that the deflection of the streamlines is small. If the velocity curvature factor, k, is equal to zero, which would correspond to a uniform velocity distribution at $x=-\infty$, the Eq. (3.10a) for the stream function reduces to the Laplace equation.

For the undisturbed region the stream function can be determined by using Eqs. (3.6) and (3.9) and is

$$\psi_0=U_0\left\{y+\frac{ky^3}{3c^2}\right\}. \tag{3.11}$$

The vorticity equation for the undisturbed region is

$$\frac{\partial^2\psi_0}{\partial x^2}+\frac{\partial^2\psi_0}{\partial y^2}=\frac{2U_0k}{c^2}y \tag{3.12a}$$

and from Eq. (3.11) y can be expressed in terms of the undisturbed stream function, ψ_0, so that Eq. (3.12a) becomes

$$\frac{\partial^2\psi_0}{\partial x^2}+\frac{\partial^2\psi_0}{\partial y^2}=f(\psi_0). \tag{3.12b}$$

To simplify the solution of the problem, we shall introduce the stream function ψ_1 which is due to the presence of the body and is defined as

$$\psi = \psi_0 + \psi_1 \tag{3.13}$$

But substituting this equation into Eq. (3.10b), the general vorticity equation becomes

$$\nabla^2(\psi_0 + \psi_1) = f(\psi_0 + \psi_1) \tag{3.14a}$$

and by substituting the relationship given by Eq. (3.12b), the equation for ψ_1 becomes

$$\nabla^2\psi_1 = f(\psi_0 + \psi_1) - f(\psi_0). \tag{3.14b}$$

The right hand side of this equation can be approximated by the use of derivative for small disturbance due to the presence of the body so that Eq. (3.14b) becomes

$$\nabla^2\psi_1 = \frac{df(\psi_0)}{d\psi_0}\psi_1 . \tag{3.14c}$$

It is seen by comparing Eqs. (3.12a) and (3.12b) that

$$f(\psi_0) = \frac{2U_0 k}{c^2} y \tag{3.15}$$

for a parabolic distribution of velocity at infinite distance ahead of the body. By using Eqs. (3.11) and (3.15) the derivative of the $f(\psi_0)$ with respect to ψ_0 is

$$\frac{df}{d\psi_0} = \frac{2k}{c^2} \frac{1}{\left[1 + k\left(\frac{y^2}{c^2}\right)\right]} . \tag{3.16a}$$

If the value of k is small such that

$$k\frac{y^2}{c^2} << 1.0$$

the right hand side of Eq. (3.16a) can be expanded to obtain

$$\frac{df}{d\psi_0} = \frac{2k}{c^2}\left[1 - k\frac{y^2}{c^2} + k^2\frac{y^4}{c^4} - \cdots\cdots\right]. \tag{3.16b}$$

The Eq. (3.14c) becomes after substituting Eq. (3.16b)

$$\nabla^2\psi_1 = \frac{2k}{c^2}\left[1 - k\frac{y^2}{c^2} + k^2\frac{y^4}{c^4} - \cdots\right]\psi_1 \tag{3.17}$$

Since it was assumed that k small and also the disturbance stream function ψ_1 was small, the product of these terms will be second order so will be neglected for the first approximation. Thus, Eq. (3.17) for the first approximation becomes

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} = 0. \tag{3.18}$$

which is a Laplace equation.* Hence, the solution of this equation combined with the undisturbed stream function will give the first approximation to the solution of the basic Eq. (3.10b). In this Section the first approximation solution for the vorticity flow over circular cylinder will be

* The second approximation for the disturbance stream function is obtained by considering Eq. (3.17) with the first term on the right hand side and the solution is expressed in terms of the modified Bessel functions.

presented.

B. Circular Cylinder in Shear Flow

If the circular cylinder is located at the origin as indicated in Fig. 2, and the radius is taken to be $c/2$, the undisturbed velocity u at any point in the fluid is

$$u = \frac{\partial \psi_0}{\partial y} = U_0 \left[1 + k \frac{y^2}{c^2} \right]. \tag{3.19}$$

And on the surface of the cylinder the value of y in terms of the angle is

$$y = \frac{c}{2} \sin \theta .$$

Thus, the undisturbed velocity on the surface becomes in terms of the polar coordinate

$$u = U_0 \left[1 + \frac{k}{4} \sin^2 \theta \right]. \tag{3.20}$$

At the surface the normal component of the resultant fluid velocity must be equal to zero; hence, the normal component of the velocity due to the undisturbed velocity must be equal and opposite to the normal disturbance velocity due to ψ_1. The normal velocity at the surfaces produced by ψ_0 is

$$V_r = U \cos \theta = U_0 \left[\cos\theta + \frac{k}{4} \sin^2\theta \cos\theta \right], \tag{3.21a}$$

which becomes by using trigonometric relations

$$V_r = U_0\left[\left(1+\frac{k}{16}\right)\cos\theta - \frac{k}{16}\cos 3\theta\right]. \qquad (3.21b)$$

The normal velocity component due to the disturbance stream function is

$$V_r = \left(\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\right)_{r=\frac{c}{2}} \qquad (3.22)$$

and this must be equal and opposite to the value given by Eq. (3.21b) to satisfy the boundary condition at the surface,

$$\left(\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\right)_{r=\frac{c}{2}} = -U_0\left[\left(1+\frac{k}{16}\right)\cos\theta - \frac{k}{16}\cos 3\theta\right]. \qquad (3.23)$$

The solution of the Laplace equation for the disturbance stream function which must vanish at infinite distance away from the body is

$$\psi_1 = U_0\left[a_0\log r + b_0\theta + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)\frac{1}{r^n}\right., \qquad (3.24)$$

where a_n's and b_n's are undetermined coefficients. To determine these coefficients, substitute this equation into Eq. (3.23) to obtain

$$\left\{\frac{1}{r}U_0\left[b_0 + \sum_{n=1}^{\infty}(-na_n\sin n\theta + nb_n\cos n\theta)\frac{1}{r^n}\right\}_{r=\frac{c}{2}} = -U_0\left[\left(1+\frac{k}{16}\right)\cos\theta - \frac{k}{16}\cos 3\theta\right]. \qquad (3.25)$$

Since there is no sine term on the right hand side of this equation, the coefficients, a_n's, must vanish. The coefficients that do not vanish are

$$b_1 = -\left(\frac{c}{2}\right)^2\left(1+\frac{k}{16}\right) \tag{3.26}$$

$$b_3 = \left(\frac{c}{2}\right)^4 \frac{k}{48} \; .$$

Hence, the resultant stream function is

$$\psi = \psi_0 + \psi_1 = U_0\left[r\sin\theta + \frac{kr^2}{3c^2}\sin^3\theta\right] + U_0\left[-\left(\frac{c}{2}\right)^2\left(1+\frac{k}{16}\right)\frac{\sin\theta}{r} + \left(\frac{c}{2}\right)^4\frac{k}{48}\,\frac{\sin 3\theta}{r^3}\right], \tag{3.27a}$$

or

$$\psi = U_0\left\{\left[r - \frac{c^2}{4}\left(1+\frac{k}{16}\right)\frac{1}{r}\right]\sin\theta + \frac{k}{3}\left[\frac{r^3}{c^2}\sin^3\theta + \frac{c^4}{256}\,\frac{\sin 3\theta}{r^3}\right]\right\}, \tag{3.27b}$$

which is similar to the expression obtained by Tsien in Ref. 22 for constant vorticity distribution. At the surface of the cylinder which corresponds to $r=\frac{c}{2}$ and substituting this into Eq. (3.27b), the result is

$$\psi = 0 \, ,$$

which satisfies the boundary condition for the circular cylinder. The tangential velocity at any point can be calculated by means of Eq. (3.27b) and is equal to

$$v_\theta = -\frac{\partial\psi}{\partial r} = -U_0\left\{\left[1+\frac{c^2}{4}\left(1+\frac{k}{16}\right)\frac{1}{r^2}\right]\sin\theta + \frac{k}{3}\left[\frac{3r^2}{c^2}\sin^3\theta - \frac{3c^4}{256}\,\frac{\sin 3\theta}{r^4}\right]\right\}. \tag{3.28}$$

The normal velocity on the surface vanishes since

$$v_r = \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)_{r=\frac{c}{2}} = 0$$

and by substituting $r=\frac{c}{2}$ into Eq. (3.28), the velocity on the surface of the cylinder is given by

$$(V_\theta)_{r=\frac{c}{2}} = -U_0\left\{\left(2+\frac{k}{16}\right)\sin\theta + \frac{k}{4}\left(\sin^3\theta - \frac{\sin 3\theta}{4}\right)\right\}. \qquad (3.29)$$

At the stagnation point the velocity is zero, which corresponds to $(V_\theta)_{r=\frac{c}{2}}=0$, and this occurs at $\theta=0$ and π for all values of k . These locations are the same as for the flow without vorticity and this should be expected since the parabolic velocity distribution at $x=-\infty$ is symmetrical about the x-axis. For uniform flow at infinite distance ahead of the cylinder, the velocity on the surface of the cylinder is given by

$$(V_\theta)_{r=\frac{c}{2}} = -2U_0\sin\theta, \qquad (3.30)$$

which can be obtained by placing $k=0$ in Eq. (3.29). This means that the solution for the velocity on the surface given by Eq. (3.29) will approach in the limit as k approaches zero, which corresponds to the flow becoming irrotational, the correct value for the uniform flow at infinity.

It is interesting to determime the location for the maximum velocity on the cylinder for flow with linear vorticity distribution. To obtain the location and the magnitude of the maximum velocity, differentiate the Eq.

(3.29) with respect to θ and set it equal to zero and solve the equation for θ as

$$\frac{dV_\theta}{d\theta} = -U_0\left\{(2+\frac{k}{16})\cos\theta + \frac{k}{4}(3\sin^2\theta\cos\theta - \frac{3\cos 3\theta}{4})\right\} = 0 . \quad (3.31)$$

By utilizing the trigonometric relationships, the equation reduces to

$$(2+\frac{22}{16}k)\cos\theta - \frac{3k}{2}\cos^3\theta = 0 .$$

Thus, the real solution of this equation is

$$\cos\theta = 0 . \quad (3.32)$$

This means that the maximum velocity locations are for $\theta = \pm\frac{\pi}{2}$ which corresponds to the intersections of the cylinder with the y-axis and is the same location as for uniform flow without vorticity. The magnitude for the maximum velocity on the surface of the cylinder is obtained by substituting $\theta = \pm \pi/2$ into Eq. (3.29) and the result is

$$(V_\theta)_{r=c/2} = -U_0(2+\frac{3}{8}k) . \quad (3.33)$$

Therefore, the maximum velocity on the cylinder is a function of k which depends on the vorticity of the undisturbed flow. If we let $k=0$ in this equation, which would correspond to uniform flow, the maximum velocity becomes equal to twice the free stream velocity.

In considering the detached shock wave, it is important to determine the effect of vorticity upon the velocity distribution along the negative x-axis. Only the radial velocity exists along the x-axis since it is the axis of symmetry. The radial velocity at any point is given by

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{U_0}{r}\left\{\left[r - \frac{c^2}{4}\left(1+\frac{k}{16}\right)\frac{1}{r}\right]\cos\theta + \frac{k}{3}\left[\frac{r^3}{c^2}3\sin^2\theta\cos\theta + \frac{c^4}{256}\,\frac{3\cos 3\theta}{r^3}\right]\right\}. \tag{3.34}$$

Along the negative x-axis, $\theta = \pi$, and therefore, the equation becomes

$$V_{r_{\theta=\pi}} = U_0\left\{-\left[1 - \frac{c^2}{4}\left(1+\frac{k}{16}\right)\frac{1}{r^2}\right] - k\left(\frac{c^4}{256 r^4}\right)\right\}. \tag{3.35}$$

If we let $k = 0$ in this equation, the result is

$$V_{r_{\theta=\pi}} = -U_0\left(1 - \frac{c^2}{4r^2}\right), \tag{3.36}$$

which is the correct expression for the flow without vorticity. For a given ratio of $\frac{V_{r_{\theta=\pi}}}{U_0}$, the location on the negative x-axis of this velocity ratio will be farther away from the origin for the rotational flow than for the uniform velocity as can be seen by comparing Eqs. (3.35) and (3.36). In other words, the effect of vorticity has made the location for a given decrease in free stream velocity farther away than for the case of uniform flow.

C. Circular Cylinder with Circulation in Flow with Vorticity

Consider the case of circular cylinder with circulation in an incompressible flow with parabolic velocity distribution at $x=-\infty$ as indicated in Fig. 2.

The stream function for a two-dimensional vortex motion is

$$\psi_2 = \frac{\Gamma}{2\pi} \log r \;, \tag{3.37}$$

where Γ is the strength of the circulation and will be considered positive for the clockwise direction. Since the circulation does not affect the normal velocity at the surface of the cylinder, the resultant first approximation stream function will be the sum of Eqs. (3.27b) and (3.37),

$$\psi = \psi_0 + \psi_1 + \psi_2 = U_0\left\{\left[r - \frac{c^2}{4}\left(1+\frac{k}{16}\right)\frac{1}{r}\right]\sin\theta + \frac{k}{3}\left[\frac{r^3}{c^2}\sin^3\theta + \frac{c^4}{256}\frac{\sin 3\theta}{r^3}\right]\right\} + \frac{\Gamma}{2\pi}\log r. \tag{3.38}$$

This equation will satisfy the boundary condition of no flow across the surface of the cylinder as previously. The introductiom of the circulation is to introduce another term to the stream function.

The tangential velocity is obtained by differentiating Eq. (3.38) with respect to r , and is

$$V_\theta = -\frac{\partial \psi}{\partial r} = -U_0 \left\{ \left[1 + \frac{c^2}{4}\left(1 + \frac{k}{16}\right)\frac{1}{r^2} \right] \sin\theta + \frac{k}{3}\left[\frac{3r^2}{c^2}\sin^3\theta \right.\right. \tag{3.39}$$

$$\left.\left. - \frac{3c^4}{256}\,\frac{\sin 3\theta}{r^4} \right] \right\} - \frac{\Gamma}{2\pi r} .$$

On the surface of the cylinder, the velocity is obtained by substituting $r = \frac{c}{2}$ in this equation, and the result is

$$(V_\theta)_{r=\frac{c}{2}} = -U_0 \left\{ \left(2 + \frac{k}{16}\right) \sin\theta + \frac{k}{4}\left[\sin^3\theta - \frac{\sin 3\theta}{4} \right] \right\} - \frac{\Gamma}{\pi c} . \tag{3.40}$$

The stagnation points on the surface can be determined by placing Eq. (3.40) equal to zero and solving for θ and the equation for determining θ is

$$\left(2 - \frac{k}{8}\right) \sin\theta + \frac{k}{2}\sin^3\theta + \frac{\Gamma}{\pi c U_0} = 0 . \tag{3.41}$$

The solutions to Eq. (3.41) are:

$$\sin\theta_s = \tfrac{1}{3}(A + B - b)$$

$$\sin\theta_s = \tfrac{1}{3}(\omega^2 A + \omega B - b) \tag{3.42}$$

$$\sin\theta_s = \tfrac{1}{3}(\omega A + \omega^2 B - b) ,$$

where

$$A^3 = \frac{1}{2}\left\{ \frac{-54\Gamma}{\pi c U_0 k} + \left[\left(\frac{54\Gamma}{\pi c U_0 k}\right)^2 + 108\left(\frac{4}{k} - \frac{1}{4}\right)^3 \right]^{1/2} \right\}$$

$$B^3 = \frac{1}{2}\left\{ \frac{-54\Gamma}{\pi c U_0 k} - \left[\left(\frac{54\Gamma}{\pi c U_0 k}\right)^2 + 108\left(\frac{4}{k} - \frac{1}{4}\right)^3 \right]^{1/2} \right\}$$

$$\omega = \frac{-1 + i\sqrt{3}}{2}$$

Hence, it can be seen that the location of the stagnation point depends upon the vorticity k , circulation Γ and the velocity U_0 .

The pressure distribution on the surface of the cylinder in shear flow can be determined by applying the Bernoulli's equation. For each streamline the value of the constant given by

$$\frac{P}{\rho} + \frac{q^2}{2} = \text{constant}$$

must hold for incompressible flow and the value of constant will be different for each streamline depending upon the vorticity. If there is no vorticity, the constant will be the same throughout the field. For the streamline along the x-axis, the equation for pressure on the surface of the cylinder is given by

$$P_0 + \frac{1}{2}\rho_0 U_0^2 = P + \frac{1}{2}\rho_0 V_\theta^2 \qquad (3.43)$$

and the pressure coefficient is

$$C_p = \frac{P - P_0}{\frac{1}{2}\rho_0 U_0^2} = 1 - \frac{V_\theta^2}{U_0^2} \qquad (3.44)$$

Substitute Eq. (3.40) into Eq. (3.44) to obtain the pressure coefficient on the surface as

$$C_p = 1 - \left[U_0 \left\{ (2 + \frac{k}{16}) \sin\theta + \frac{k}{4}\left[\sin^3\theta - \frac{\sin 3\theta}{4}\right]\right\} + \frac{\Gamma}{\pi c} \right]^2 \frac{1}{U_0^2} \qquad (3.45)$$

This equation shows that the effect of vorticity, k, for the same circulation strength is to make the pressure coefficient more negative than for the flow without vorticity.

IV. CONDITIONS FOR PROPAGATING NORMAL SHOCK WAVE

For detached shock wave at supersonic speeds, the shock will be normal to the axis of symmetry for a symmetrical body and for general case the shock will be normal to the free stream direction at some point on the detached shock wave. When the shock wave is normal to the free stream velocity, the velocity aft of the shock will be subsonic. The relationships for the normal shock wave which are useful for the analysis of the detached shock wave will be presented in this Section. In Refs. 5, 24, and 25, a discussion of the normal shock wave is given. For the first approximation of the distance ahead of the body for the detached shock wave on the axis of symmetry, the condition for the normal shock wave propagating into fluid at rest is used and the detailed derivation of this approximation is presented in Section VII.

A. General Propagating Normal Shock Wave

The general case of normal shock wave will be discussed in which the shock wave is assumed to be moving at constant velocity and the velocities in front and back of the shock wave are finite. It will be assumed that the flow is adiabatic and steady and that the viscosity and heat conduction effects are neglected.

With these assumptions, consider the case where the flow is in a duct as shown in Fig. 3, with conditions ahead of the shock designated by subscript $(\ \ )_1$, and for conditions after the shock wave designated by subscript $(\ \ )_2$ and the normal shock wave is propagating at a constant velocity λ_t. To simplify the analysis, introduce the relative motion of the fluid with respect to the moving shock wave which is given by

$$V_i = u_i - \lambda_t , \tag{4.1}$$

where $i=1,2$ refers to conditions before and after the shock wave.

From the conservation of the mass, the mass flow per unit time across the shock wave must be constant. The continuity equation is

$$\rho_1 V_1 = \rho_2 V_2 = m , \tag{4.2}$$

where m is the mass flow through the shock wave for unit time. The conservation of the momentum across the shock is given by

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \tag{4.3}$$

which involves only the relative velocities with respect to the shock. The third equation is the condition for the conservation of energy and is

$$\tfrac{1}{2} V_1^2 + i_1 = \tfrac{1}{2} V_2^2 + i_2 , \tag{4.4}$$

where i is the enthalpy of the gas

$$i = c_v T + \frac{p}{\rho} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}.$$

These three equations, Eqs. (4.2), (4.3), and (4.4) are sufficient to determine the three unknown quantities u_2, p_2, and ρ_2. It is important to notice that for a general normal shock wave the velocities relative to the shock wave determine the conditions after the shock. Since only relative velocities are involved in these basic equations, the shock conditions are invariant under the translation with constant velocity.

The change in the relative velocity across a normal shock can be found by using the three basic equations to obtain

$$V_1 - V_2 = (V_1 - V_2)\left(\frac{\gamma+1}{2\gamma}\frac{a_*^2}{V_1 V_2} + \frac{\gamma-1}{2\gamma}\right), \qquad (4.5)$$

where a_* is the velocity of sound for $M = \frac{V}{a_*} = 1.0$. The solutions to this equation are

$$V_1 = V_2$$

and

$$V_1 V_2 = a_*^2. \qquad (4.6)$$

The first solution is a trivial case and the second one is used to obtain the relationship between the velocites in front and back of the normal shock wave. The Mach number before and after the shock is given by

$$M_2^2 = \frac{1 + \frac{\gamma-1}{2} M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}}, \tag{4.7}$$

where $M_1 = \frac{V_1}{a_1}$ and $M_2 = \frac{V_2}{a_2}$. The Mach numbers are based upon the flow velocities relative to the shock wave and the equation is identical for the case where the shock wave is stationary, $L_t = 0$. In terms of the Mach number of the relative velocity with respect to the shock, the change in velocity across the shock is given by

$$\left(\frac{V_2 - V_1}{V_1}\right) = \left(\frac{2}{\gamma+1}\right)\left(\frac{1 - M_1^2}{M_1^2}\right). \tag{4.8}$$

For supersonic Mach numbers, $M_1 > 1.0$, this equation shows that the velocity across the shock always decreases.

The increase in pressure across a shock is given by

$$\left(\frac{P_2 - P_1}{P_1}\right) = \frac{2\gamma}{\gamma+1}(M_1^2 - 1). \tag{4.9}$$

This equation shows that the pressure increases across the shock and thus, the wave in a supersonic flow must be a compression shock. In terms of the Mach number ahead of the shock, the increase in density across a normal shock is given by

$$\left(\frac{\rho_2 - \rho_1}{\rho_1}\right) = \frac{M_1^2 - 1}{\left(1 + \frac{\gamma-1}{2} M_1^2\right)} \tag{4.10}$$

B. Propagating Normal Shock Wave in a Fluid at Rest

The equations derived in the previous part were based upon the fluid being in motion both ahead and after the normal shock and also the shock wave was propagating at constant velocity L_t. For the ease where the fluid ahead of the shock is at rest and the shock wave is moving through the fluid, the velocity u_1 is equal to zero, but the velocities L_t and u_2 are not equal to zero. It was shown that as long as the state P_1, ρ_1, and T_1 ahead of the shock and the velocity of the propagation of the normal shock wave are known, the state after the shock is uniquely determined. Hence, the solution to the problem is obtained by substituting $u_1=0$ into the equations derived for the general normal shock wave condition. Assume that the shock wave is propagating from left to right as shown in Fig. 3. The equation for V_1 becomes

$$V_1 = u_1 - L_t = -L_t \tag{4.11a}$$

and

$$M_1 = \left(\frac{L_t}{a_1}\right), \tag{4.11b}$$

where $a_1^2 = \frac{\gamma P_1}{\rho_1}$, the velocity of sound for the undisturbed flow ahead of the shock wave.

Due to the normal shock wave propagating at a constant velocity into the fluid at rest, the velocity of the fluid after the shock also moves at constant velocity and in the same direction as the shock wave. The increment of velocity imparted by the normal shock to the fluid is obtained by using Eq. (4.8) and the result is

$$u_2 = \left(\frac{2}{\gamma+1}\right)\left(\frac{M_1^2 - \mathcal{L}_t}{M_1^2}\right)\mathcal{L}_t \tag{4.12}$$

or

$$\frac{u_2}{\mathcal{L}_t} = \frac{2}{\gamma+1}\left(\frac{M_1^2 - 1}{M_1^2}\right).$$

Thus, knowing $\mathcal{L}_t$ and a_1, the increment of velocity imparted by the shock to the fluid can be calculated by this equation. When the propagation velocity is equal to the speed of sound for the undisturbed flow, the increment of velocity imparted to the fluid is zero and this corresponds to the propagation of sound. For stronger shock waves, which would correspond to a definite value for u_2, the propagation velocity will be supersonic. Explosion waves and other similar disturbances must travel at speed greater than the speed of sound for the undisturbed flow.

The increase in the pressure across the shock wave can be determined by Eq. (4.9) and the result is

$$\left(\frac{P_2 - P_1}{P_1}\right) = \frac{2\gamma}{\gamma+1}\left[\left(\frac{\mathcal{L}_t}{a_1}\right)^2 - 1\right] \tag{4.13}$$

and the pressure ratio is

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\left(\frac{c_t}{a_1}\right)^2 - \left(\frac{\gamma-1}{\gamma+1}\right).$$

The increase in the density across the shock is obtained by using Eq. (4.10) and is expressed by

$$\left(\frac{\rho_2-\rho_1}{\rho_1}\right) = \frac{\left(\frac{c_t}{a_1}\right)^2 - 1}{\left[1+\frac{\gamma-1}{2}\left(\frac{c_t}{a_1}\right)^2\right]} \qquad (4.14a)$$

and the density ratio is

$$\frac{\rho_2}{\rho_1} = \left(\frac{\gamma+1}{2}\right)\frac{\left(\frac{c_t}{a_1}\right)^2}{\left[1+\frac{\gamma-1}{2}\left(\frac{c_t}{a_1}\right)^2\right]}.$$

These basic equations, Eqs. (4.12), (4.13), and (4.14), for propagating normal shock wave into an undisturbed fluid region indicate that if the state ahead of the shock wave and one of the parameters for the state of the fluid after the shock are known, the propagation velocity can be determined. The propagation velocity is given by

$$c_t = \left(\frac{\gamma+1}{4}\right)u_2 + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 u_2^2 + a_1^2} \qquad (4.22a)$$

$$c_t = a_1\sqrt{1+\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2-p_1}{p_1}\right)} \qquad (4.22b)$$

$$c_t = a_1\sqrt{\frac{1+\frac{\rho_2-\rho_1}{\rho_1}}{1-\left(\frac{\gamma-1}{2}\right)\left(\frac{\rho_2-\rho_1}{\rho_1}\right)}} \qquad (4.22c)$$

For determining the normal shock wave location ahead of a given body, the increment of velocity, Eq. (4.12), produced by the moving shock wave will be utilized in Section VII.

For supersonic Mach numbers close to one, the detached shock wave is nearly normal with very little curvature. Thus, the normal shock wave results presented in this Section can be utilized. At higher supersonic Mach numbers the detached shock waves are curved so the normal shock wave results cannot be applied except where the shock is actually normal to the free stream velocity. The curved shock wave will be considered in the next Section from the standpoint of the shock polar.

V. BASIC EQUATIONS AND SIGNIFICANCE OF SHOCK POLAR

In the investigation of the flow conditions immediately behind the detached shock wave, the shock polar, which is disucssed in Refs. 5, 24, and 25, is very important and thus, will be considered in detail in this Section. The condition for the shock to become detached as the Mach number is lowered for a given sharp nosed body is clearly given by the shock polar. Also some idea about the subsonic region behind the detached shock wave can be gained by using the shock polar. In the physical plane the shape and location of the detached shock are very difficult to determing mathematically because of the mixed flow nature of the boundary condition, but in the hodograph plane the shock wave corresponds to the shock polar. This brief discussion illustrates the importance of the shock polar in the study of the detached shock wave.

A. Fundamental Equations for the Oblique Shock Wave

It will be assumed in the present analysis for the oblique shock wave that the shock is stationary, the flow is steady, the viscosity and heat conduction effects are neglected, and the process is adiabatic. The oblique shock wave can be analyzed by considering it bo be moving at a constant speed and the flow before and after the shock having a finite constant velocity, similar to the general analysis of the normal shock wave in Section IV. The flow will be assumed to be two-dimensional in the derivation of the shock polar. Even for three dimensional shock wave, the shock polar will apply to determine the state variables back of the shock in terms of known state variables ahead of the shock, because a local element of the three-dimensional shock is approximately a two-dimensional shock.

With these assumptions, a supersonic flow over a two-dimensional corner as shown in Fig. 4 with the origin placed at the corner will be considered. At the corner there must be an oblique compression shock to change the flow direction so that the flow after it crosses the oblique shock will be parallel to the surface. The change in the velocity direction is caused by the oblique shock. It will be assumed that the angle of the corner is such that the oblique shock is attached and straight.

To obtain the basic equations for the oblique shock, consider an element of the shock as shown in Fig. 6. By considering the conservation of mass across the shock, the continuity equation is

$$\rho_1 u_1 \sin\theta_w = \rho_2 (u_2 \sin\theta_w - v_2 \cos\theta_w) = m \, , \qquad (5.1)$$

where the subscript 1 refers to the uniform condition ahead of the shock and subscript 2 refers to the condition immediately behind the shock and θ_w is the oblique shock wave angle. The velocities u's and v's are the component of the velocities in the x and y directions.

The conservation of the momentum in the direction normal to the shock wave is given by

$$p_1 + \rho_1 u_1^2 \sin^2\theta_w = p_2 + \rho_2 (u_2 \sin\theta_w - v_2 \cos\theta_w)^2 \tag{5.2a}$$

and the conservation of momentum parallel to the shock is given by

$$\rho_1 u_1^2 \sin\theta_w \cos\theta_w = \rho_2 (u_2 \sin\theta_w - v_2 \cos\theta_w)(u_2 \cos\theta_w + v_2 \sin\theta_w) . \tag{5.2b}$$

For an adiabatic process the conservation of energy across the shock wave, according to Eq. (4.4) is given by

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{1}{2}(u_2^2 + v_2^2) + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} = \left(\frac{\gamma+1}{\gamma-1}\right)\frac{a_*^2}{2} = constant. \tag{5.3}$$

The boundary condition on the surface requires that the velocity after the shock be parallel to the surface and thus, the relation between the flow deflection angle θ and the components of the velocity after the shock is given by

$$\tan\theta = \frac{v_2}{u_2} . \tag{5.4a}$$

Eqs. (5.1) and (5.2b) give the condition that the velocity parallel to the shock wave is not altered by the compression shock since there is no force acting in the tangential direction and hence, the tangential velocity relation is given by

$$V_T = u_1 \cos\theta_w = u_2 \cos\theta_w + v_2 \sin\theta_w \tag{5.5}$$

This means that only the normal velocity to the shock wave is altered from supersonic to subsonic speed. Thus, the flow across an oblique shock wave is always deflected towards the shock. The velocity after the oblique shock wave may be supersonic or subsonic depending upon the magnitude of the initial velocity and the deflection angle. The shock wave angle can be expressed as

$$\tan\theta_w = \frac{u_1 - u_2}{v_2} \tag{5.6}$$

Eqs. (5.1), (5.2a), (5.2b), (5.3), and (5.4a) are the five equations necessary to solve for the five unknowns for the oblique shock wave, P_2, ρ_2, u_2, v_2, and θ_w for given P_1, ρ_1, u_1, and θ. To obtain the shock polar, it is necessary to eliminate θ_w, P_2, and ρ_2 from these basic equations and obtain the equation relating the velocity as function of u_1 and u_2

$$v_2^2 = (u_1 - u_2)^2 \frac{(u_1 u_2 - a_*^2)}{\left(\frac{2}{\gamma+1} u_1^2 - u_1 u_2 + a_*^2\right)} . \tag{5.7a}$$

This equation gives the velocity component v_2 in terms of

u_1, u_2, and a_*^2. For a given flow problem u_1 and a_* are known and it is desired to solve for u_2 and v_2 that will satisfy this equation. The equation can also be written as

$$v_2 = \pm(u_1 - u_2)\sqrt{\frac{u_2 - \frac{a_*^2}{u_1}}{\frac{2}{\gamma+1}u_1 + \frac{a_*^2}{u_1} - u_2}} \qquad (5.7b)$$

B. Physical Significance of Shock Polar

Eq. (5.7a) is cubic in terms of u_2 and is symmetrical about the u-axis. The shock polar is obtained by plotting Eq. (5.7a) in the hodograph plane which has u and v as the coordinate axes. For each value of the free stream velocity u_1 with constant critical velocity a_*, there will correspond in the hodograph plane a particular shock polar curve as indicated in Fig. 5. The only part of the shock polar that has any physical significance is plotted in Fig. 5. With the use of the shock polar it is possible to determine the conditions after the oblique shock for a given free stream velocity and the deflection angle. A convenient shock polar diagram for analyzing the flow after oblique shock is given in Ref. 5.

The shock polar is divided by the critical sonic velocity a_* into two regions as indicated by Fig. 5. For region within the circle with radius $q = a_*$, the flow after the oblique shock wave is completely subsonic. For region outside the circle the flow after the oblique shock is supersonic. When the free stream velocity is equal to the critical velocity a_* , the shock polar degenerates into a point at Q of Fig. 5. For all other supersonic speeds the shock polar will exist. The shock angle with respect to the free stream is given by the angle PTM of Fig. 5 as shown by Eq. (5.6). As the flow deflection angles become smaller and the flow is supersonic after the shock, the shock angle θ_w approaches the Mach angle which corresponds to a weak disturbance.

For a given flow deflection of the supersonic velocity u_1 , there are two possible oblique shock waves which corresponds to two points of intersection with the shock polar, N and T , as indicated in Fig. 5. Thus, there will be two shock wave angles which will produce the same flow deflection. At point T the flow after the shock is supersonic and the disturbance is not large. At point N the flow is subsonic after the shock and the shock wave angle is close to being normal to the free stream. The disturbance corresponding to point N is large and is approaching the normal shock condition. There are two values for u_2 when there is no flow deflection, points P and R of Fig. 5. At the point P , $u_1 = u_2$, and there is no change in velocity which corresponds to a Mach wave. At the point R the velocity is subsonic and the shock wave is normal to the free stream velocity. As the flow deflection angle increases, there is a maximum deflection angle for a given free stream velocity and this point corresponds to a tangent to the shock polar as indicated by L in Fig. 5. The point of tangency to the shock polar will always be in the subsonic region. For all shock conditions given by points on the arc LP, the shock wave is weak since the flow is close to Mach number of one from L to S and supersonic for the region S to P. The shock condition given by the arc RL of the shock polar corresponds to the strong shock since the velocity after the shock is subsonic and thus, the disturbance due to the shock is large.

For a two-dimensional body with attached shock wave, the shock corresponds to the weak shock. When the flow deflection angle for a finite wedge exceeds the maximum value for a given free stream supersonic velocity, the shock wave will be detached and curved. the detached shock wave is represented by both strong and weak shocks, and in the hodograph plane the shock wave corresponds to the shock polar, RSP of Fig. 5. For an infinite wedge with the wedge angle greater than the maximum deflection angle for a given free stream velocity, the detached shock wave will not be stationary as will be discussed in Section VIII. The curved detached shock wave may be considered to be composed of small elements of oblique shock waves. The flow behind each point of the detached shock may be determined from the shock polar and thus, the flow immediately behind the detached shock can be completely determined even though the region is not uniform. This is the fundamental importance of the shock polar for the study of the detached shock wave problem.

VI. DETACHMENT CONDITIONS FOR SHOCK WAVE

In the two previous sections, the basic equations for the normal and oblique shock waves were discussed from the physical standpoint. These results together with some of the mathematical knowledge about the mixed flow problem presented in Refs. 3, 4, and 16 will be used in the present section to discuss the condition, shape, and types of flow for detached shock wave for wedges and cones. The flow at infinity will be assumed to be steady and uniform and that the effects of viscosity and heat conduction are negligible.

A. Mathematical Problem for Detached Shock Wave

With the detached shock wave as shown in Figs. 6a and 6b for physical and hodograph planes, the boundary conditions of the flow is reduced to the mixed sub- and supersonic flows. The region ABCO is subsonic imbedded in the supersonic flow for the rest of the region. If the supersonic free stream Mach number is close to unity, the detached shock wave will have small curvature and the vorticity back of the shock will also be small. For this condition, if the vorticity of the flow is negligible, the velocity potential and stream function can be used to analyze the flow back of the detached shock wave. With these assumptions the following differential equation for the stream function was obtained by Frankl in Ref. 4.

$$\frac{\partial^2 \psi}{\partial \sigma^2} + K \frac{\partial^2 \psi}{\partial \theta^2} = 0, \qquad (6.1)$$

where σ is a function of the velocity and is given by

$$\sigma = \int_{\tau}^{(2\beta+1)^{-1}} \frac{(1-\tau)^{\beta}}{2\tau} d\tau$$

$$\beta = \frac{1}{\gamma - 1}$$

$$\tau = \frac{q^2}{q_m^2}$$

$$K = \frac{1-(2\beta+1)\tau}{(1-\tau)^{2\beta+1}}$$

and θ is the inclination of the velocity. Thus, when the velocity is supersonic the value of K is negative and when the velocity is subsonic the value of K is positive. Hence, it can be seen that the differential equation is hyperbolic type for supersonic region and elliptic type for subsonic region. The mathematical problem for determining the flow over a wedge with detached shock wave of small curvature reduces to solving Eq. (6.1) in the region AOCDB of the hodograph plane as indicated by Fig. 6b with the proper boundary condition. The Eq. (6.1) is the type that was first considered by Tricomi (Cf. Ref. 13) in which he proved the uniqueness and existence of the solution. In the region for which the solution to Eq. (6.1) is desired, the segment AB of the boundary is part of the shock polar, OA is the segment of the u-axis, OC is the side of the wedge, CD and BD are segments of the characteristics, B and C are points on the sonic circle.

Frankl in Ref. 4 has shown that for a wedge angle less than 54 degrees the Eq. (6.1) has a unique solution, but did not show the method for obtaining this unique solution. Guderley in Ref. 16 has treated the same problem and by using the approximate shock polar was able to show that the transition of the shock wave from attached to detached condition was continuous. This reference also does not give the solution for the two-dimensional edge with detached shock wave.

Since the mathematical solution to Eq. (6.1) is difficult to obtain, a discussion of the physical condition for the shock wave to become detached and the condition back of the detached shock wave will be presented in this section.

B. Detached Shock Wave for Finite Wedge

For a two-dimensional wedge as shown in Fig. 6a, the shock wave becomes detached from the nose for a particular free stream supersonic velocity when the angle of the wedge exceeds the maximum deflection angle. When the shock becomes detached, a stagnation is formed at the nose of the wedge. Up to the maximum critical flow deflection there was no stagnation point at the nose because the oblique shock ~~was~~ was strong enough to deflect the free stream velocity to coincide with the surface of the wedge.

From the shock polar it is seen that the pressure on the wedge surface will be constant if the Mach number M_2 back of the oblique shock wave is supersonic or equal to unity. The shock wave is attached to the nose and is straight up to the point on the shock wave where the Prandtl-Meyer expansion waves from the corner, (Cf. Fig. 7a), intersects the shock wave. The shock wave will be curved in the region where the expansion waves from the corner interact with the shock wave.

When the Mach number after the oblique shock is equal to unity or the free stream Mach number $M_1 = M_{cr_2}$, the shock wave from the nose will be straight and the angle of the wave is given by the shock polar. The shock wave will become curved when the expansion wave from the corner intersects the shock wave. Thus, for this case there is a triangular region OCB (Cf. Fig. 7b) in which the Mach number is constant and equal to 1.0. The pressure on the wedge surface must also be constant. In the hodograph plane of Fig. 6b, the region OCB of the physical plane corresponds to a point B.

For the wedge angle, point O in Fig. 7c, which is between the maximum critical flow deflection and the deflection for which $M_2 = 1.0$, the Mach number after the oblique shock is subsonic. The flow on the surface must speed up from the subsonic speed at the nose to Mach number of 1.0 at the corner. Thus, the pressure on the surface will not be constant but will become less positive as it approaches the corner. The shock wave will be attached to the surface as shown in Fig. 7d but will be curved even before the expansion waves from the corner intersect the shock wave because of the decreasing pressure on the wedge surface. The shock wave angle at the nose is given by the shock polar. In the hodograph plane of Fig. 7c, the region OCB of the physical plane does not correspond to a point as in the previous case, but corresponds to the region BOCD with CD and BD being a segment of the chracteristics. The differential Eq. (6.1) must be solved for this region.

The maximum deflection point, which corresponds to point L of Fig. 5 and to free stream Mach number, $M_1 = M_{cr_1}$, the shock wave is attached but it is again curved before the intersection of the expansion wave from the corner with the shock wave. The velocity is subsonic at the nose and must speed up to Mach number of 1.0 at the corner; hence, the pressure on the surface will not be constant. To determine this pressure distribution over the wedge, it is necessary to solve the Eq. (6.1) in the hodograph region $B O' C' D'$ of Fig. 7c. The subsonic region in the hodograph plane as shown by Fig. 7c is larger than for the wedge angle less than the maximum critical flow deflection.

For all wedge angles larger than the maximum deflection angle, the shock wave becomes detached (Cf. Fig. 6a) and in the hodograph plane the mixed flow region is represented by AOCDB as shown in Fig. 6b. At the nose for these cases there is a stangation point. The detached shock wave will be normal at the axis of symmetry and will curve and become equal to the Mach wave at a finite distance from the wedge. The shock wave is curved even before the expansion wave from the corner intersects the shock wave. The flow condition through this detached shock wave is given by the shock polar, ABE of Fig. 6b. The strong shock wave is represented by the arc of the shock polar from the maximum deflection angle point to the intersection of the shock polar with the u-axis. The segments in the hodograph plane CD and BD are characteristics which start from the sonic circle. To obtain the distribution of the pressure on the wedge surface, it is necessary to solve the Eq. (6.1) in the region AOCDB with the proper boundary condition. This equation is for an irrotational flow so the curvature of the detached shock wave should not be large for the solution of the equation to give reasonable agreement with experimental data. The rotational condition of the flow aft of the shock wave should be considered in obtaining the exact solution to the flow region AOCDB. When the supersonic free stream Mach number is close to unity, the detached shock wave will become more normal and the curvature will be small. For this condition the flow back of the shock wave will have small vorticity which may be neglected and consequently, the solution to Eq. (6.1) should give close agreement with experimental data.

As the wedge angle approaches 180 degrees, the region of subsonic flow in the hodograph plane becomes larger for a given supersonic flow. In the physical plane the subsonic region will be larger and thus, the shock will be located at greater distance away from the nose and the region with nearly normal shock wave will also be larger. For these large wedge angles, the shock will be detached to a higher supersonic Mach number as indicated by experimental data for spheres given in Refs. 26 and 27.

C. Detached Shock Wave for Infinite Wedge

The shock polar applies to flow over an infinite wedge. The difference in the flow back of an attached shock wave for a finite and infinite wedge is that for an infinite wedge the velocity and pressure on the wedge surface is constant even though the Mach number after the shock is subsonic. This difference in the flow aft of the attached shock wave is caused by the lack of corner to influence the shock wave for the infinite wedge. If the Mach number after the shock wave is less than 1.0, the velocity after the shock will be uniform and constant. The flow back of the shock is represented by the singular point in the hodograph plane. Thus, there is only uniform supersonic flow ahead of the shock and uniform subsonic flow after the oblique shock. The oblique shock wave will be straight and will not be curved as for the finite wedge.

The problem of determining the subsonic flow after the attached oblique shock wave for an infinite wedge reduces to the shock polar and it is not necessary to solve the Tricomi problem Eq. (6.1) as for the finite wedge. If the shock wave is attached, the pressure and velocity on the surface of the infinite wedge can be determined from the shock polar without any difficulty. Hence, the Tricomi problem for the mixed flow condition simplifies as the wedge becomes infinite.

For the maximum critical deflection condition, the oblique shock wave will be attached to the nose for an infinite wedge with uniform subsonic flow after the shock wave. For the finite wedge at this same condition the shock wave will be attached to the nose and will be curved as indicated in Fig. 7d. Thus, the flows over the infinite and finite wedges with the same wedge angle and free stream supersonic Mach number at the maximum critical deflection condition is not the same.

When the deflection angle exceeds the maximum critical value, the shock wave for the infinite wedge must become detached. Since the infinite wedge has no finite length, the detached shock wave will not be fixed at any particular point but will move away from the nose. For the case of finite wedge, the detached shock wave was located at some finite distance away from the nose, the distance depending upon the wedge angle, length, and the free stream supersonic Mach number. The detached shock waves for the infinite and finite wedges with the same wedge angle are entirely different. A more detailed analysis of the detached shock wave for an infinite wedge is given in Section VIII. For the finite wedge the detached shock wave can be analyzed by solving Eq. (6.1), which is for the stationary condition. For the infinite wedge the detached shock wave must be analyzed from the general non-stationary differential equation, Eq. (2.15c) discussed in Section II. A particular type of solution for the infinite wedge with detached shock wave is presented in Section VIII.

D. Detached Shock Wave for Three-Dimensional Cone

For a finite cone the shock polar can again be applied to determine the flow after the oblique shock wave as for the two-dimensional problem. The principal difference between the two-dimensional wedge problem and the axial symmetrical cone problem is that for the cone the flow must be analyzed from the standpoint of conical flow. In Ref. 28 the Taylor and Maccall solutions for cones at different Mach numbers were calculated and tabulated. The analytical results indicated that the oblique shock wave would be attached even for cone angles greater than the maximum critical deflection angles. This means that for conical body it is possible to have an attached strong shock wave, discussed in Section V, to exist. In the two-dimensional problem the strong shock wave was present only when the shock was detached from the nose of the body. The experimental data, presented in Ref. 7 for a 75 degree cone, illustrates the existence of the strong shock wave attached to the nose. The agreement for the free stream Mach number at which the shock is attached to the nose between the experimental result and the calculated data of Ref. 28 was very good.

The difference in the shape of the attached shock waves for infinite cones with same cone angle and free stream supersonic Mach number is similar to that for finite and infinite wedges. For the infinite cone due to lack of influence of the corner, the attached oblique shock wave will be straight even for subsonic Mach number after the shock. The detached shock wave will be stationary as shown in Ref. 29 for a finite cone, but for an infinite cone the detached shock will not be stationary and will behave similarly to that for an infinite wedge.

VII. FIRST APPROXIMATION FOR THE LOCATION OF DETACHED SHOCK WAVE

In this section an approximate theory for obtaining the first approximation to the distance between the nose of symmetrical body at zero angle and the detached shock wave will be presented. It will be assumed that the detachment conditions for the shock wave discussed in Section VI are fulfilled. Either the body is blunt so the shock wave is always detached or else the supersonic Mach number is low enough to cause the shock to be detached even for a body with sharp nose. The shape of the detached shock wave will not be predicted by the first approximate theory for determining the distance between the detached shock wave and the nose of a symmetrical body. To obtain the shape of the detached shock wave, it is necessary to solve the Tricomi problem, Eq. (6.1), in the mixed flow region and the rotationality of the flow must also be considered to obtain the correct shape for the detached shock wave.

The flow back of the detached shock wave will be assumed to be irrotational, adiabatic and steady. The viscosity and heat conduction effects will be neglected. For supersonic Mach number close to unity, the assumption of irrotational flow back of the shock wave is reasonable. To simplify the derivation for the detachment distance of the shock wave, it will be assumed that the flow at infinity is at rest and the body is moving through this fluid at constant supersonic speed.

Since the derivation of the first approximate theory for the detached shock wave distance was obtained, a NACA report, Ref. 30, has been published on the same subject. In this reference the shock wave distance is determined on the assumption of no entropy change across the normal shock wave. The free stream Mach number was taken to be slightly greater than 1.0; thus, the result does not hold except for Mach number close to unity as will be shown later. A comparison of the results obtained by the present theory and that of Ref. 30 for a sphere is presented in Fig. 13 together with the actual experimental data presented in Refs. 26 and 27. For these references the representative detached shock wave pictures obtained in the supersonic wind tunnel and the ballistic range are presented in Figs. 11 and 12. In the present approximate theory there is no restriction regarding the change in entropy through the shock wave and the exact normal shock wave result is used to obtain the change in the velocity through the shock.

A. First Approximation Theory for Locating the Detached Shock Wave

For these assumptions the flow back of the detached shock wave can be described by velocity potential and stream function. It was shown in Section IV that the velocity back of the normal shock wave was always subsonic and that the velocity became more subsonic as the free stream supersonic Mach number increased as shown by Eq. (4.7).

It will be first assumed that the flow back of the shock is incompressible and next a correction will be applied to take care for the compressibility effects. Further refinements to the analysis will not be made on account of the assumptions utilized to obtain the results. A solution to the Tricomi problem considering the effects of rotational flow should give the exact detachment distance.

When a finite body is moving in a fluid at rest at constant supersonic speed $\mathcal{L}_t$, the detached shock wave will be stationary with respect to the body. On the axis of symmetry, (Cf. Fig. 8), the velocity induced by the normal shock wave must be equal to the induced velocity at that point due to the body for the detached shock wave to be stationary with respect to the body. If the induced velocity due to the normal shock on the axis of symmetry is greater than the velocity induced by the body, the detached shock wave will move towards the body to a position where the two velocities are equal. If the induced velocity on the axis of symmetry due to the detached shock wave is less than that due to the body, the shock wave will move away from the body. At all points on the detached shock wave the induced velocities due to the shock wave and the body must be equal in order to have stationary detached shock wave with respect to the body. The first approximation theory for the detached shock wave distance was determined on this basis of equal induced velocities on the axis of symmetry.

In Section IV the velocity induced by a normal shock wave moving in a fluid at rest at constant supersonic speed L_t was shown to be

$$u_2 = \left(\frac{2}{\gamma+1}\right)\left(\frac{M_1^2-1}{M_1^2}\right)L_t \qquad (4.12)$$

or

$$\frac{u_2}{a_1} = \left(\frac{2}{\gamma+1}\right)\left(\frac{M_1^2-1}{M_1}\right) ,$$

where

$$M_1 = \frac{L_t}{a_1}$$

and $a_1^2 = \frac{\gamma P_1}{\rho_1}$ is the velocity of sound for the fluid at rest ahead of the shock wave. It can be seen from this equation that for $M_1 = 1.0$ which corresponds to the velocity of propagation of sound, there is no induced velocity by the normal shock. As the Mach number becomes more supersonic, the induced velocity u_2 becomes larger. The shock wave is fixed relative to the finite body and thus, the shock wave velocity U_t must be equal to that of the body. The state of the fluid ahead of the normal shock wave is known so the velocity of sound a_1 and the induced velocity given by Eq. (4.12) can be determined as a function of the Mach number $M_1 = \frac{U_t}{a_1}$. With the body moving in the negative x-direction as shown in Fig. 38, the induced velocity will have negative sign. The Eq. (4.12) is valid for a normal shock wave traveling at supersonic Mach numbers and there is no limitation that M_1 must be close to unity as in Ref. 30. For supersonic Mach numbers close to unity, the use of normal shock wave to calculate the detached shock wave should be a good approximation.

B. Detached Shock Wave Distance for Circular Cylinder

The shock wave will be detached, as shown in Fig. 38, for a circular cylinder of radius a moving at constant supersonic speed U_t in the negative x-direction in a fluid that is at rest. If we assume for the first approximation that the flow back of the detached shock is incompressible and irrotational, the flow back of the shock can be approximated by the complex potential (Cf. Refs. 18 and 19),

$$F(z) = \varphi + i\psi = \frac{U_t a^2}{z} \qquad (7.1a)$$

where φ is the velocity potential and ψ is the stream function.

In terms of the polar coordinates the complex potential becomes

$$F(Z) = \frac{4a^2}{r} e^{-i\theta}. \tag{7.1b}$$

The radial and tangential velocities at any point in the fluid are

$$\begin{aligned} V_r &= -\frac{4a^2}{r^2}\cos\theta \\ V_\theta &= -\frac{4a^2}{r^2}\sin\theta . \end{aligned} \tag{7.2}$$

On the negative x-axis the value of θ is π and the resultant velocity is

$$V_r = \frac{4a^2}{r^2}, \tag{7.3}$$

which is in the direction of the motion of the cylinder.

At the normal shock wave on the negative x-axis, the velocity induced by the shock wave must be equal to the velocity induced by the body as given by Eq. (7.3). Hence, the distance of the normal detached shock wave ahead of the body on the axis of symmetry can be obtained from Eq. (7.3) and Eq. (4.12) as

$$\frac{4a^2}{r^2} = \left(\frac{2}{\gamma+1}\right)\left(\frac{M_1^2-1}{M_1}\right) a_1 \tag{7.4}$$

The location of the detached shock wave on the negative x-axis is obtained by solving this equation and the result is

$$\left(\frac{r}{a}\right)_i = \sqrt{\left(\frac{\gamma+1}{2}\right)\left(\frac{M_1^2}{M_1^2-1}\right)}. \tag{7.5}$$

In terms of the distance l from the nose of the cylinder to the detached shock wave and the diameter d of the cylinder, the equation for the detached shock distance becomes

$$\left(\frac{l}{d}\right)_i = \frac{1}{2}\left\{\sqrt{\left(\frac{\gamma+1}{2}\right)\left(\frac{M_1^2}{M_1^2-1}\right)} - 1\right\} \tag{7.6}$$

and this is plotted in Fig. 8. For the velocity of the cylinder equal to the sound velocity, $M_1 = \frac{U}{a_1} = 1.0$ the distance $\left(\frac{l}{d}\right)_i$ becomes infinite and for $M_1 \to \infty$ the shock distance approaches the limit

$$\left(\frac{l}{d}\right)_i = \frac{1}{2}\left\{\sqrt{\left(\frac{\gamma+1}{2}\right)} - 1\right\}. \tag{7.7}$$

The Prandtl-Glauert correction (Cf. Ref. 1) for the compressibility effects can be used to determine the velocity distribution for the compressible fluid. The Mach number M_2 after the normal shock wave will be used to determine the correction for compressibility. For this assumption either the lateral dimension or the perturbation velocity can be corrected by $\frac{1}{\sqrt{1-M_2^2}}$ as discussed in Ref. 1. If we take the correction to be for the perturbed velocity, the increment in velocity for compressible flow along the negative x-axis is obtained from Eq. (7.3) and the result is

$$V_{r_c} = \frac{4a^2}{r^2}\frac{1}{\sqrt{1-M_2^2}} . \tag{7.8}$$

With this for the increment in velocity on the negative x-axis for the cylinder moving in a compressible fluid, the location for the normal detached shock wave can be determined in the same manner as for the incompressible fluid case. The result is obtained by equating Eqs. (4.12) and (7.8),

$$\frac{4a^2}{r^2\sqrt{1-M_2^2}} = \left(\frac{2}{\gamma+1}\right)\left(\frac{M_1^2-1}{M_1}\right)a_1 \tag{7.9}$$

and

$$\left(\frac{r}{a}\right)_c = (1-M_2^2)^{-1/4}\sqrt{\left(\frac{\gamma+1}{2}\right)\left(\frac{M_1^2}{M_1^2-1}\right)} \tag{7.10a}$$

or

$$\left(\frac{r}{a}\right)_c = (1-M_2^2)^{-1/4}\left(\frac{r}{a}\right)_i \tag{7.10b}$$

and

$$\left(\frac{\ell}{d}\right)_c = \frac{1}{2}\left\{(1-M_2^2)^{-1/4}\left(\frac{r}{a}\right)_i - 1\right\}. \tag{7.11}$$

The Mach number M_2 after the normal shock was shown in Section IV to be

$$M_2^2 = \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}} \tag{4.7}$$

in terms of the Mach number M_1 ahead of the shock. In Fig. 9 the Eq. (7.11) is plotted together with the result for the incompressible fluid to show the effect of compressibility upon the detached shock wave distance, and the result of Ref. 30 is plotted for comparison.

C. Detached Shock Wave Distance for Two-Dimensional Source

The detached shock distance for a two-dimensional body represented by a two-dimensional source can be considered by the method applied for the circular cylinder. For the body moving at constant velocity u_t in the negative x-direction, the induced velocity on the negative x-axis is given by

$$V_r = \frac{u_t h}{2\pi r} \tag{7.12}$$

where h is the thickness of the plate at $x = +\infty$. The shock distance for this blunt two-dimensional body for the assumption of incompressible flow back of the shock is obtained from Eqs. (4.12) and (7.12) and the result is

$$\left(\frac{r}{h}\right)_i = \left(\frac{\gamma+1}{4\pi}\right)\left(\frac{M_1^2}{M_1^2-1}\right) \tag{7.13}$$

and the distance from the nose of the body is

$$\left(\frac{l}{h}\right)_i = \frac{1}{2\pi}\left\{\left(\frac{\gamma+1}{2}\right)\left(\frac{M_1^2}{M_1^2-1}\right)-1\right\} \tag{7.14a}$$

For a Mach number of $M_1 = \frac{u_t}{a_1} = 1.0$, the normal detached shock wave becomes infinite and for $M_1 \to \infty$ the distance approaches asymptatically to

$$\left(\frac{\ell}{h}\right)_i = \left(\frac{\gamma-1}{4\pi}\right).$$

For the Prandtl-Glauert correction for the compressibility effects in the region back of the shock, the normal shock wave distance can be determined from Eqs. (4.12) and (7.13) as

$$\frac{L_f h}{2\pi r\sqrt{1-M_2^2}} = \left(\frac{2}{\gamma+1}\right)\left(\frac{M_1^2-1}{M_1}\right)a_1 \tag{7.14b}$$

or

$$\left(\frac{r}{h}\right)_c = (1-M_2^2)^{-1/2}\left(\frac{r}{h}\right)_i \tag{7.14c}$$

and

$$\left(\frac{\ell}{d}\right)_c = \left\{(1-M_2^2)^{-1/2}\left(\frac{r}{h}\right)_i - \frac{1}{2\pi}\right\} \tag{7.15}$$

D. Detached Shock Wave Distance for Sphere

1. Zero Order Correction for the Compressibility Effects to the Detached Shock Distance for Sphere

For irrotational, steady, and incompressible assumption for the flow back of the detached shock wave, the velocity potential for the sphere moving at a uniform velocity in the negative x-direction in terms of spherical coordinates (Cf. Ref. 19) is given by

$$\varphi_0 = \frac{U_f R^3}{2\omega^2}\cos\varpi, \tag{7.16}$$

where R is the radius of the sphere as indicated in Fig. 10. The radial velocity V_{ω_o} in the direction of ω is

$$V_{\omega_o} = \frac{\partial \varphi}{\partial \omega} = -\frac{L_1 R^3}{\omega^3} \cos \varpi. \qquad (7.17a)$$

Along the negative x-axis, which corresponds to $\varpi = \pi$, the radial velocity is given by

$$V_{\omega_o} = \frac{L_1 R^3}{\omega^3}, \qquad (7.17b)$$

which is in the direction of the motion of the sphere.

On the negative x-axis of symmetry the induced velocity due to the detached normal shock must be equal to the induced velocity due to the sphere for the first approximation for locating the detached shock distance as discussed in the first part of this Section. Hence, the shock distance can be determined by using Eqs. (4.19) and (7.17b) and the result is

$$\frac{L_1 R^3}{\omega^3} = \left(\frac{2}{\gamma+1}\right)\left(\frac{M_1^2-1}{M_1}\right) a, \qquad (7.18a)$$

or

$$\left(\frac{\omega}{R}\right)_o = \left\{\left(\frac{\gamma+1}{2}\right)\left(\frac{M_1^2}{M_1^2-1}\right)\right\}^{1/3} \qquad (7.18b)$$

and the normal shock distance ℓ from the nose of the sphere in terms of the diameter D is given by

$$\left(\frac{\ell}{D}\right) = \frac{1}{2}\left\{\left[\left(\frac{\gamma+1}{2}\right)\left(\frac{M_1^2}{M_1^2-1}\right)\right]^{1/3} - 1\right\}. \qquad (7.19)$$

When the velocity of sphere is equal to the sound velocity of the undisturbed fluid, $M_1 = 1.0$, the distance $(\ell/D)_0$ becomes infinite and when the velocity of the sphere approaches infinity, $M_1 \to \infty$, the shock distance approaches the asymptote

$$\left(\frac{\ell}{D}\right)_0 = \frac{1}{2}\left\{\left(\frac{\gamma+1}{2}\right)^{1/3} - 1\right\}. \tag{7.20}$$

Comparison of Eqs. (7.17) and (7.19) shows that the normal shock wave distance in terms of the diameter is very similar for circular cylinder and sphere. The Eq. (7.19) is plotted in Fig. 13 together with the result of Ref. 3 and the experimental data obtained in a supersonic wind tunnel, Ref. 26, and from the firing range, Ref. 27.

2. First Order Correction for the Compressibility Effects to the Detached Shock Distance for Sphere

The correction for the compressibility effects will be made by using the solution obtained by the expansion of the velocity potential φ in powers of the free stream Mach number which is the Rayleigh-Janzen method (Cf. Refs. 31 and 32) as

$$\varphi' = \varphi_0' + \varphi_1' M^2 + \varphi_2' M^4 + \cdots\cdots, \tag{7.21}$$

where

$$\varphi' = \frac{\varphi}{U}.$$

In Ref. 33 Kaplan used the method of Rayleigh-Janzen and Tamada in Ref. 34 used the method of Poggi, Ref. 35, to obtain the second order correction to the velocity potential for the irrotational flow of a compressible fluid past a sphere. The zero order correction to the velocity potential would correspond to the potential for the incompressible fluid which is given by Eq. (7.16). The detached shock wave distance for the sphere will be determined for the first and second order corrections to the velocity potential.

The first order correction φ_1' to the velocity potential for sphere of unit radius as obtained in Refs. 33 and 34 is given by

$$\varphi_1' = \left(\tfrac{1}{3}\omega^{-2} - \tfrac{1}{5}\omega^{-5} + \tfrac{1}{24}\omega^{-8}\right) P_1(\cos\vartheta) \tag{7.22}$$

$$+ \left(-\tfrac{3}{10}\omega^{-2} + \tfrac{27}{55}\omega^{-4} - \tfrac{3}{10}\omega^{-5} + \tfrac{3}{176}\omega^{-8}\right) P_3(\cos\vartheta),$$

where $P_n(\cos\vartheta)$'s are Legendre Polynomials (Cf. Refs. 36 and 37) and

$$P_1(\cos\vartheta) = \cos\vartheta$$

$$P_3(\cos\vartheta) = \tfrac{1}{8}(5\cos 3\vartheta + 3\cos\vartheta).$$

On the negative x-axis $\vartheta = \pi$, the value of these polynomials are

$$P_1(\cos\vartheta) = -1$$

$$P_3(\cos\vartheta) = -1.$$

The first order correction to the velocity at any point on the negative x-axis for a particular Mach number is obtained from Eq. (7.22) and is

$$V_{\omega_1} = UM^2\left(\frac{1}{15}\omega^{-3} + \frac{108}{55}\omega^{-5} - \frac{5}{2}\omega^{-6} + \frac{31}{66}\omega^{-9}\right). \qquad (7.23)$$

Thus, the resultant velocity on the negative x-axis for the sphere moving at constant velocity L_t in the direction of the negative x-axis for a sphere of unit radius is given by the sum of Eqs. (7.17b) and (7.23) as

$$V_\omega = L_t\omega^{-3} + L_t M_2^2\left(\frac{1}{15}\omega^{-3} + \frac{108}{55}\omega^{-5} - \frac{5}{2}\omega^{-6} + \frac{31}{66}\omega^{-9}\right), \qquad (7.24)$$

where M_2 is the Mach number after the normal shock wave. The corresponding shock distance for the first order correction is obtained by using Eqs. (4.19) and (7.24) and the result is

$$\left(1 + \frac{M_2^2}{15}\right)\omega^{-3} + M_2^2\left(\frac{108}{55}\omega^{-5} - \frac{5}{2}\omega^{-6} + \frac{31}{66}\omega^{-9}\right) = \left(\frac{2}{\gamma+1}\right)\left(\frac{M_1^2 - 1}{M_1^2}\right). \qquad (7.25)$$

From the normal shock wave condition discussed in Section IV, the relation between the Mach number before and after the shock wave is given by Eq. (4.12). Thus, knowing the Mach number $M_1 = \frac{L_t}{a_1}$, the Mach number M_2 can be determined which is used in Eq. (7.25) to obtain the effect of compressibility upon the shock wave distance. The normal detached shock distance corresponding to the first order correction to the velocity potential for supersonic Mach number M_1 is obtained by solving Eq. (7.25) for ω. A numerical graphical method was used to solve for ω, and the results are plotted in Fig. 15 together with the experimental data.

3. Second Order Correction for the Compressibility Effects to the Detached Shock Distance for Sphere

The second order correction φ_2' (Cf. Refs. 33 and 34) to the velocity potential is given for a sphere of unit radius by

$$\varphi_2' = (\gamma-1)\Big\{\Big(\frac{599}{16800}\omega^{-2} - \frac{3}{70}\omega^{-5} + \frac{1}{24}\omega^{-8} - \frac{13}{560}\omega^{-11} + \frac{13}{2800}\omega^{-14}\Big)P_1(\cos\vartheta)$$

$$+\Big(\frac{83133}{366520}\omega^{-4} - \frac{1}{10}\omega^{-5} + \frac{15}{176}\omega^{-8} - \frac{57}{1960}\omega^{-11} + \frac{23}{6800}\omega^{-14}\Big)P_3(\cos\vartheta)$$

$$+\Big(\frac{1}{7}\omega^{-5} - \frac{50517}{276640}\omega^{-6} + \frac{3}{52}\omega^{-8} - \frac{9}{1120}\omega^{-11} + \frac{5}{8512}\omega^{-14}\Big)P_5(\cos\vartheta)$$

$$+\Big(\frac{1325953}{4065600}\omega^{-2} - \frac{49}{150}\omega^{-5} - \frac{243}{1925}\omega^{-7} + \frac{1049}{3080}\omega^{-10} - \frac{3049}{18480}\omega^{-11} + \frac{7789}{369600}\omega^{-14}\Big)P_3(\cos\vartheta)$$

$$+\Big(\frac{-53}{150}\omega^{-2} + \frac{70620063}{95295200}\omega^{-4} - \frac{6}{150}\omega^{-5} - \frac{156}{275}\omega^{-7} + \frac{28}{55}\omega^{-8} + \frac{1137}{7150}\omega^{-10} - \frac{12631}{646800}\omega^{-11} + \frac{929}{56100}\omega^{-14}\Big)P_3(\cos\vartheta)$$

$$+\Big(\frac{5}{42}\omega^{-2} - \frac{6}{11}\omega^{-4} + \frac{2}{3}\omega^{-5} + \frac{135921}{640640}\omega^{-6} - \frac{60}{77}\omega^{-7} + \frac{253}{728}\omega^{-8} + \frac{87}{1540}\omega^{-10} - \frac{997}{14480}\omega^{-11} + \frac{85}{29568}\omega^{-14}\Big)P_5(\cos\vartheta)$$

(7.26)

where

$$P_5(\cos\vartheta) = \frac{1}{128}(63\cos 5\vartheta + 35\cos 3\vartheta + 30\cos\vartheta)$$

and on the negative x-axis $\vartheta = \pi$, the value of this Legendre Polynomial is

$$P_5(\cos\vartheta) = -1.$$

The second order correction to the velocity at any point on the negative x-axis for a particular Mach number back of the normal shock is obtained from Eq. (7.26) by differentiating with respect to the radius ω and the result is

$$V_{\omega_2} = UM^4\left\{\alpha_3\omega^{-3} + \alpha_5\omega^{-5} + \alpha_6\omega^{-6} + \alpha_7\omega^{-7} + \alpha_8\omega^{-8} + \alpha_9\omega^{-9} + \alpha_{11}\omega^{-11} + \alpha_{12}\omega^{-12} + \alpha_{15}\omega^{-15}\right\}, \quad (7.27)$$

where α's are constants and are given in the Appendix I. Thus, the resultant induced velocity on the negative x-axis for a sphere of unit radius moving at a constant velocity $U = L_t$ in the negative x-direction is given by the sum of these perturbed velocities, Eqs. (7.17b), (7.23), and (7.29) as

$$V_\omega = V_{\omega_0} + V_{\omega_1} + V_{\omega_2} = L_t\omega^{-3} + L_t M_2^2\left(\frac{1}{15}\omega^{-3} + \frac{108}{55}\omega^{-5} - \frac{5}{2}\omega^{-6} + \frac{31}{66}\omega^{-9}\right) \quad (7.28)$$

$$+ L_t M^4\left(\alpha_3\omega^{-3} + \alpha_5\omega^{-5} + \alpha_6\omega^{-6} + \alpha_7\omega^{-7} + \alpha_8\omega^{-8} + \alpha_9\omega^{-9} + \alpha_{11}\omega^{-11} + \alpha_{12}\omega^{-12} + \alpha_{15}\omega^{-15}\right),$$

where M_2 is the Mach number after the normal detached shock wave.

As discussed in the first part of this Section, the detached shock distance can be determined for the second order correction for the compressibility effects by equating the induced velocity given by Eq. (7.28) to the induced velocity due to the normal shock wave given by Eq. (4.19), and the result is

$$\omega^{-3}+M_1^2\left(\frac{1}{15}\omega^{-3}+\frac{108}{55}\omega^{-5}-\frac{5}{2}\omega^{-6}+\frac{31}{66}\omega^{-9}\right)+M_1^4\left(\alpha_3\omega^{-3}+\alpha_5\omega^{-5}+\alpha_6\omega^{-6}\right. \tag{7.29}$$

$$\left.+\alpha_7\omega^{-7}+\alpha_8\omega^{-8}+\alpha_9\omega^{-9}+\alpha_{11}\omega^{-11}+\alpha_{12}\omega^{-12}+\alpha_{15}\omega^{-15}\right)=\left(\frac{2}{\gamma+1}\right)\left(\frac{M_1^2-1}{M_1^2}\right).$$

The detached shock distance is determined by solving this equation for ω in terms of $M_1=\frac{U}{a_1}$, the Mach number for the sphere. A numberical-graphical method was used to solve for ω. The detached shock distance for the second order correction is plotted in Fig. 12 together with other results as function of Mach number of the sphere moving in a compressible fluid at rest.

4. Correlation of Calculated Detached Shock Wave Distance wi~~t~~h Experimental Data for Sphere

In Fig. 13 a comparison of the variation of the detached shock wave distance as function of Mach number according to the approximate theories is shown. For comparison the experimental results obtained in the wind tunnel, Ref. 26, and the ballistic range, Ref. 27, are also presented.

The detached shock wave distance as calculated by the approximate theory developed in this Section for a given free stream Mach number is less than the experimental results. The variation of the detached shock distance with Mach number is similar to the experimental results. At lower supersonic Mach numbers the first and second order corrections for the compressibility effects move the detached shock wave distance farther away from the body, but at higher supersonic Mach numbers the correction in shock distance for compressibility effects is small. The result of Ref. 30, which is also plotted in Fig. 13, shows that it is tangent to the curve given by Eq. (7.19) and gives negative detached shock distance for Mach numbers greater than 1.6.

The difference in the detached shock wave distance between the approximate theories and the experimental data is due to the neglection of vorticity, curvature of the shock, viscosity, more exact compressibility correction, interaction of subsonic and supersonic regions, and deceleration, which is present for the ballistic range data. It was shown in Section III that for a rotational flow corresponding to parabolic velocity distribution at infinite distance ahead of a body the location for a given velocity decrease was farther ahead of the body than for the case of flow without vorticity. Thus, by considering the rotationality of the flow back of the detached shock wave the shock distance will be farther away from the body and be closer to the experimental data.

At the detached shock wave only the condition on the axis of symmetry, where the shock wave is normal, was considered to determine the shock distance. The proper boundary conditions for the subsonic and supersonic regions should be considered as discussed in Section VI to obtain the correct shape and the location of the detached shock wave. The normal detached shock wave distance determined by the approximate theory presented in this Section is only an approximation to the actual distance.

VIII. DETACHED SHOCK WAVE FOR INFINITE WEDGE

In Section VII a first approximation theory for locating the detached shock wave distance from the nose of the body was discussed and the approximate theory applied to few symmetrical bodies with blunt nose. In this section the detached shock wave for an infinite wedge will be investigated by considering the non-stationary differential equation of motion for compressible fluid.

For an infinite wedge which has no length dimension and only angle, the possible types of detached shock waves for supersonic flow from the dimensional analysis standpoint are the normal and curved shock waves. A normal detached shock wave which is fixed in space is not possible because the wedge has no length dimension. The normal shock must move at constant speed or accelerate away from the wedge. If the normal detached shock moves with acceleration, the entropy back of the shock will not be constant; and hence, a potential flow does not exist back of this type of shock wave. The flow back of a normal detached shock wave which moves at constant speed would be irrotational.

If there exists a definite relationship between the radius of curvature of the detached shock wave and the distance from the wedge to the shock wave, a curved detached shock wave may be possible for an infinite wedge. The flow back of the curved shock wave will be rotational so that the potential flow cannot exist back of the shock.

Therefore, the only possible type of detached shock wave for a supersonic free stream flow for an infinite wedge with possible potential flow back of the shock is a normal detached shock wave moving at constant speed away from the wedge. In this section this type of detached shock wave will be analyzed for the infinite wedge.

A. Fundamental Equations and Assumptions

It will be assumed that the flow is isentropic and adiabatic, and the effects of viscosity and heat conduction will be assumed to be negligible. The detached shock will be assumed to be normal and moving at constant speed, and thus, the flow after the shock will be irrotational.

For these assumptions, the non-stationary differential equation is given by

$$\frac{\partial^2\varphi}{\partial t^2} + 2\left(\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x\partial t} + \frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y\partial t}\right) + \left(\frac{\partial\varphi}{\partial x}\right)^2\frac{\partial^2\varphi}{\partial x^2} + 2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y} + \left(\frac{\partial\varphi}{\partial y}\right)^2\frac{\partial^2\varphi}{\partial y^2} = a^2\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) \qquad (2.15b)$$

and

$$a^2 = a_0^2\left\{1 - \frac{(\gamma-1)}{2a_0^2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] - \frac{(\gamma-1)}{a_0^2}\frac{\partial\varphi}{\partial t}\right\}, \qquad (2.19c)$$

which were discussed in Section II. To simplify the Eq.

(2.15b) for the detached shock flow analysis for an infinite wedge as shown in Fig. 14a, the non-dimensional space parameters are introduced. The x and y distances are divided by the distance $L(t)$, which is the distance from the nose of the wedge to the normal detached shock wave, in order to obtain the non-dimensional space parameters,

$$\begin{aligned} \xi &= \frac{x}{L(t)} \\ \eta &= \frac{y}{L(t)} \\ \tau &= t \end{aligned} \tag{8.1}$$

It is also assumed that the normal shock wave is propagating in the negative x-direction at a constant velocity L_t ; hence, the distance of the detached shock wave from the wedge is only a function of time. The physical plane in which the wedge exists will be referred as z-plane, Fig. 16a, and the transformed non-dimensional space plane as ζ-plane, Fig. 14b. The z and ζ in terms of the components can be written as

$$z = x + iy \tag{8.2}$$

$$\zeta = \xi + i\eta . \tag{8.3}$$

In the z-plane the velocity potential is given in the general form as

$$\varphi = \varphi(x, y, t) \tag{8.4}$$

and in the ζ-plane the potential is given by

$$\varphi = \varphi(\xi, \eta, \tau). \tag{8.5}$$

To determine the differential equation in terms of the non-dimensional space parameters, it is necessary to use the relationships connecting the partial derivatives shich are obtained from Eqs. (8.1), (8.4), and (8.5),

$$\begin{aligned}
\frac{\partial(\;)}{\partial x} &= \frac{\partial(\;)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial(\;)}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial(\;)}{\partial \tau}\frac{\partial \tau}{\partial x} = \frac{1}{L}\frac{\partial(\;)}{\partial \xi} \\
\frac{\partial(\;)}{\partial y} &= \frac{\partial(\;)}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial(\;)}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial(\;)}{\partial \tau}\frac{\partial \tau}{\partial y} = \frac{1}{L}\frac{\partial(\;)}{\partial \eta} \\
\frac{\partial(\;)}{\partial t} &= \frac{\partial(\;)}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial(\;)}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial(\;)}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{L_t}{L}\left(\xi\frac{\partial(\;)}{\partial \xi} + \eta\frac{\partial(\;)}{\partial \eta}\right) + \frac{\partial(\;)}{\partial \tau},
\end{aligned} \tag{8.6}$$

where $L_t = \frac{\partial L}{\partial t}$ is the velocity of the detached shock wave. Thus, the partial derivatives of the velocity potential in terms of the non-dimensional space parameters are obtained by using the relationship given by Eq. (8.6) and the results are

$$\begin{aligned}
\frac{\partial \varphi}{\partial x} &= \frac{1}{L}\frac{\partial \varphi}{\partial \xi} & \qquad \frac{\partial \varphi}{\partial y} &= \frac{1}{L}\frac{\partial \varphi}{\partial \eta} \\
\frac{\partial^2 \varphi}{\partial x^2} &= \frac{1}{L^2}\frac{\partial^2 \varphi}{\partial \xi^2} & \qquad \frac{\partial^2 \varphi}{\partial y^2} &= \frac{1}{L^2}\frac{\partial^2 \varphi}{\partial \eta^2}
\end{aligned} \tag{8.8}$$

and

$$\begin{aligned}
\frac{\partial^2 \varphi}{\partial x \partial t} &= -\frac{L_t}{L}\left(\frac{\xi}{L}\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\eta}{L}\frac{\partial^2 \varphi}{\partial \xi \partial \eta}\right) - \frac{L_t}{L^2}\frac{\partial \varphi}{\partial \xi} + \frac{1}{L}\frac{\partial^2 \varphi}{\partial \xi \partial \tau} \\
\frac{\partial^2 \varphi}{\partial y \partial t} &= -\frac{L_t}{L}\left(\frac{\xi}{L}\frac{\partial^2 \varphi}{\partial \xi \partial \eta} + \frac{\eta}{L}\frac{\partial^2 \varphi}{\partial \eta^2}\right) - \frac{L_t}{L^2}\frac{\partial \varphi}{\partial \eta} + \frac{1}{L}\frac{\partial^2 \varphi}{\partial \eta \partial \tau}
\end{aligned} \tag{8.9}$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial \tau^2} - \frac{2L_t}{L}\left(\xi\frac{\partial^2 \varphi}{\partial \xi \partial \tau} + \eta\frac{\partial^2 \varphi}{\partial \eta \partial \tau}\right) + \frac{L_t^2}{L^2}\left(\xi^2\frac{\partial^2 \varphi}{\partial \xi^2} + \eta^2\frac{\partial^2 \varphi}{\partial \eta^2}\right) + \frac{2L_t^2}{L^2}\xi\eta\frac{\partial^2 \varphi}{\partial \xi \partial \eta} + 2\frac{L_t^2}{L^2}\left(\xi\frac{\partial \varphi}{\partial \xi} + \eta\frac{\partial \varphi}{\partial \eta}\right).$$

By substituting Eqs. (8.8) and (8.9) into Eq. (2.15b), the differential equation for the velocity potential in terms of the non-dimensional space parameters is obtained as

$$\frac{\partial^2\varphi}{\partial\tau^2}-\frac{2L_t}{L}\left(\xi\frac{\partial^2\varphi}{\partial\xi\partial\tau}+\eta\frac{\partial^2\varphi}{\partial\eta\partial\tau}\right)+\frac{L_t^2}{L^2}\left(\xi^2\frac{\partial^2\varphi}{\partial\xi^2}+\eta^2\frac{\partial^2\varphi}{\partial\eta^2}\right)-\frac{2L_t}{L^3}\left(\xi\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi^2}+\eta\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial\eta^2}\right)$$

$$+\frac{2L_t^2}{L^2}\xi\eta\frac{\partial^2\varphi}{\partial\xi\partial\eta}+\frac{2L_t^2}{L^2}\left(\xi\frac{\partial\varphi}{\partial\xi}+\eta\frac{\partial\varphi}{\partial\eta}\right)-\frac{2L_t}{L^3}\left(\eta\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi\partial\eta}+\xi\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial\eta\partial\xi}\right)-\frac{2L_t}{L^3}\left[\left(\frac{\partial\varphi}{\partial\xi}\right)^2+\left(\frac{\partial\varphi}{\partial\eta}\right)^2\right] \tag{8.10}$$

$$+\frac{2}{L^2}\left(\frac{\partial\varphi}{\partial\xi}\frac{\partial^2\varphi}{\partial\xi\partial\tau}+\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial\eta\partial\tau}\right)+\frac{1}{L^4}\left[\left(\frac{\partial\varphi}{\partial\xi}\right)^2\frac{\partial^2\varphi}{\partial\xi^2}+2\frac{\partial\varphi}{\partial\xi}\frac{\partial\varphi}{\partial\eta}\frac{\partial^2\varphi}{\partial\xi\partial\eta}+\left(\frac{\partial\varphi}{\partial\eta}\right)^2\frac{\partial^2\varphi}{\partial\eta^2}\right]=\frac{a^2}{L^2}\left(\frac{\partial^2\varphi}{\partial\xi^2}+\frac{\partial^2\varphi}{\partial\eta^2}\right).$$

and

$$a^2=a_0^2\left\{1-\frac{(\gamma-1)}{a_0^2}\frac{1}{L^2}\left[\left(\frac{\partial\varphi}{\partial\xi}\right)^2+\left(\frac{\partial\varphi}{\partial\eta}\right)^2\right]-\frac{(\gamma-1)}{a_0^2}\left[-\frac{L_t}{L}\left(\xi\frac{\partial\varphi}{\partial\xi}+\eta\frac{\partial\varphi}{\partial\eta}\right)+\frac{\partial\varphi}{\partial\tau}\right]\right\}. \tag{8.11}$$

B. Non-dimensional Differential Equations for Infinite Wedge.

For a normal detached shock wave moving at constant speed into a supersonic free stream flow, the velocity back of the detached shock can be determined from the normal shock Eqs. (4.11b) and (4.12). It is seen from these equations that, as long as the supersonic free stream flow is steady and the propagating normal shock velocity is constant, the flow back of the shock wave will also be constant. Thus, the boundary condition for a normal detached shock wave moving at constant speed away from an infinite wedge is that the velocity at the shock wave will be constant.

At the surface of the infinite wedge, the normal velocity must be equal to zero in order to satisfy the condition of no flow across the solid boundary. This normal velocity must be zero for all values of time or for all locations of the normal detached shock wave.

To satisfy these boundary conditions for an infinite wedge with a normal detached shock wave moving at constant velocity, and also to satisfy the non-stationary differential Eq. (8.10), a possible solution for the velocity potential is given by

$$\varphi = a_0 L(t)\, \Phi(\xi,\eta), \tag{8.12}$$

where a_0 is the speed of sound for the stafnation condition back of the detached shock wave and is evaluated by using Eq. (2.9c). $L(t)$ is the detached shock distance from the wedge, and $\Phi(\xi,\eta)$ is the velocity potential which is only function of the non-dimensional space parameters.

The partial derivatives of the velocity potential in terms of the non-dimensional potential, Eq. (8.12), are given by

$$\begin{aligned} \frac{\partial\varphi}{\partial\xi} &= a_0 L \frac{\partial\Phi}{\partial\xi} & \qquad \frac{\partial\varphi}{\partial\eta} &= a_0 L \frac{\partial\Phi}{\partial\eta} \\ \frac{\partial^2\varphi}{\partial\xi^2} &= a_0 L \frac{\partial^2\Phi}{\partial\xi^2} & \qquad \frac{\partial^2\varphi}{\partial\eta^2} &= a_0 L \frac{\partial^2\Phi}{\partial\eta^2} \end{aligned} \tag{8.13}$$

and

$$\frac{\partial \varphi}{\partial \tau} = a_0 L_t \bar{\Phi} \qquad \frac{\partial^2 \varphi}{\partial \eta \partial \tau} = a_0 L_t \frac{\partial \bar{\Phi}}{\partial \eta} \tag{8.14}$$

$$\frac{\partial^2 \varphi}{\partial \xi \partial \tau} = a_0 L_t \frac{\partial \bar{\Phi}}{\partial \xi}$$

$$\frac{\partial^2 \varphi}{\partial \tau^2} = 0 \; ,$$

since the detached shock was assumed to be moving at constant velocity. The differential equation for the velocity potential, which is only function of non-dimensional space parameters, is obtained by substituting Eqs. (8.13) and (8.14) into Eq. (3.10) and the result is

$$\left[\frac{L_t^2}{a_0^2}\xi^2 - 2\frac{L_t}{a_0}\xi\frac{\partial \bar{\Phi}}{\partial \xi} + \left(\frac{\partial \bar{\Phi}}{\partial \xi}\right)^2\right]\frac{\partial^2 \bar{\Phi}}{\partial \xi^2} + \left[\frac{L_t^2}{a_0^2}\eta^2 - 2\frac{L_t}{a_0}\eta\frac{\partial \bar{\Phi}}{\partial \eta} + \left(\frac{\partial \bar{\Phi}}{\partial \eta}\right)^2\right]\frac{\partial^2 \bar{\Phi}}{\partial \eta^2} + \left[2\frac{L_t^2}{a_0^2}\xi\eta - 2\frac{L_t}{a_0}\eta\frac{\partial \bar{\Phi}}{\partial \xi} - 2\frac{L_t}{a_0}\xi\frac{\partial \bar{\Phi}}{\partial \eta}\right.$$

$$\left. + 2\frac{\partial \bar{\Phi}}{\partial \xi}\frac{\partial \bar{\Phi}}{\partial \eta}\right]\frac{\partial^2 \bar{\Phi}}{\partial \xi \partial \eta} = \left\{1 - \frac{(\gamma-1)}{2}\left[\left(\frac{\partial \bar{\Phi}}{\partial \xi}\right)^2 + \left(\frac{\partial \bar{\Phi}}{\partial \eta}\right)^2\right] - (\gamma-1)\frac{L_t}{a_0}\left[-\left(\xi\frac{\partial \bar{\Phi}}{\partial \xi} + \eta\frac{\partial \bar{\Phi}}{\partial \eta}\right) + \bar{\Phi}\right]\right\}\left(\frac{\partial^2 \bar{\Phi}}{\partial \xi^2} + \frac{\partial^2 \bar{\Phi}}{\partial \eta^2}\right) \tag{8.15a}$$

and

$$a^2 = a_0^2\left\{1 - \frac{(\gamma-1)}{2}\left[\left(\frac{\partial \bar{\Phi}}{\partial \xi}\right)^2 + \left(\frac{\partial \bar{\Phi}}{\partial \eta}\right)^2\right] - (\gamma-1)\frac{L_t}{a_0}\left[-\left(\xi\frac{\partial \bar{\Phi}}{\partial \xi} + \eta\frac{\partial \bar{\Phi}}{\partial \eta}\right) + \bar{\Phi}\right]\right\} \tag{8.16}$$

If we introduce the parameter $\lambda = L_t/a_0$, which is the ratio of the velocity of the detached shock wave to the velocity of sound for the stagnation condition, evaluated by using Eq. (2.9c), after the shock, the differential equation for $\bar{\Phi}$ becomes

$$\left[\lambda^2\xi^2 - 2\lambda\xi\frac{\partial \bar{\Phi}}{\partial \xi} + \left(\frac{\partial \bar{\Phi}}{\partial \xi}\right)^2\right]\frac{\partial^2 \bar{\Phi}}{\partial \xi^2} + \left[\lambda^2\eta^2 - 2\lambda\eta\frac{\partial \bar{\Phi}}{\partial \eta} + \left(\frac{\partial \bar{\Phi}}{\partial \eta}\right)^2\right]\frac{\partial^2 \bar{\Phi}}{\partial \eta^2} + \left[2\lambda^2\xi\eta - 2\lambda\eta\frac{\partial \bar{\Phi}}{\partial \xi}\right.$$

$$\left. - 2\lambda\xi\frac{\partial \bar{\Phi}}{\partial \eta} + 2\frac{\partial \bar{\Phi}}{\partial \xi}\frac{\partial \bar{\Phi}}{\partial \eta}\right]\frac{\partial^2 \bar{\Phi}}{\partial \xi \partial \eta} = \left\{1 - \frac{(\gamma-1)}{2}\left[\left(\frac{\partial \bar{\Phi}}{\partial \xi}\right)^2 + \left(\frac{\partial \bar{\Phi}}{\partial \eta}\right)^2\right] - (\gamma-1)\lambda\left[-\left(\xi\frac{\partial \bar{\Phi}}{\partial \xi} + \eta\frac{\partial \bar{\Phi}}{\partial \eta}\right) + \bar{\Phi}\right]\right\} \tag{8.15b}$$

$$\left(\frac{\partial^2 \bar{\Phi}}{\partial \xi^2} + \frac{\partial^2 \bar{\Phi}}{\partial \eta^2}\right).$$

Hence, the problem for the infinite wedge with the normal detached shock wave moving at constant speed reduces to solving this particular differential equation with proper boundary conditions. The first boundary condition is that at the normal detached shock wave the velocity potential must be continuous and the velocity after the shock as shown in Section IV is also constant and is a function of the condition ahead of the shock and the propagating velocity of the shock wave. The second boundary condition is that the normal velocity at the wedge surface must be equal to zero.

C. Existence and Uniqueness of the Velocity Potential for an Infinite Wedge with Normal Detached Shock Wave

The existence and the uniqueness of a potential solution for an infinite wedge can be determined by investigating the differential equation for the non-dimensional velocity potential, Eq. (8.15b), and the boundary conditions.

Immediately back of the normal shock wave moving at constant velocity into a uniform supersonic flow, the general energy equation, discussed in Section II, is

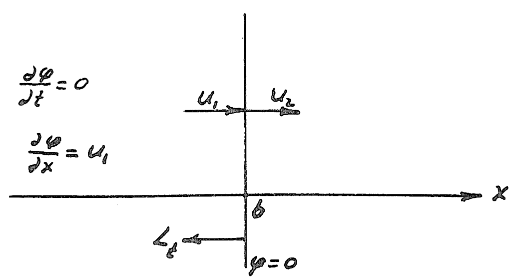
$$\frac{\partial \varphi}{\partial t} + \frac{q^2}{2} + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \left(\frac{\partial \varphi}{\partial t}\right)_2 + \frac{q_2^2}{2} + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} = F(t) = \frac{a_0^2}{\gamma-1} \tag{2.9c}$$

where $(\ )_2$ refers to the conditions immediately after the normal shock and a_0 is the velocity of sound for the stagnation condition back of the shock including the non-stationary effects. In this equation the values of g_2 , P_2 , and ρ_2 are known for given free stream condition and velocity of propagation of the normal shock.

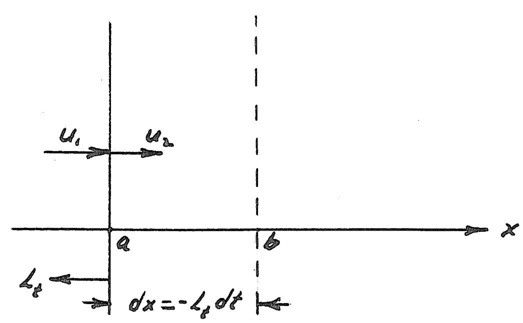
To evaluate the arbritrary function $F(t)$ of the energy equation at the shock wave moving at constant speed L_t , it is necessary to know the partial derivative of the velocity potential with respect to time. If the velocity potential, φ , is continuous throughout the flow field with discontinuous first and higher order derivatives, the partial derivative of the velocity potential with respect to time at the shock wave is determined by the following analysis. Ahead of the shock wave define

$$\frac{\partial \varphi}{\partial t} = 0$$

and at time $t = t_0$ the normal shock wave is located at point $x = b$ and the boundary conditions are



At $t = t_0 + dt$ the normal shock wave has moved to point $x = a$



and for a continuous velocity potential the increment of velocity potential at point $x = a$ is

$$d\varphi_a = -u_1 L_t\, dt \tag{8.16}$$

and the increment of velocity potential at the point b, which is fixed, is

$$d\varphi_b = d\varphi_a + u_2 L_t\, dt, \tag{8.17a}$$

By substituting Eq. (8.16) into this equation, one obtains

$$d\varphi_b = -(u_1 - u_2) L_t\, dt. \tag{8.17b}$$

Hence, the partial derivative of the velocity potential with respect to time at the moving normal shock wave is

$$\left(\frac{\partial\varphi}{\partial t}\right)_2 = \left(\frac{\partial\varphi_b}{\partial t}\right) = -(u_1 - u_2) L_t \tag{8.18}$$

The constant for the non-stationary energy equation can be evaluated immediately back of the normal shock by substituting Eq. (8.18) into Eq. (2.9c) to obtain

$$-(u_1 - u_2) L_t + \frac{a_{o_2}^2}{\gamma - 1} = \frac{a_o^2}{\gamma - 1}, \tag{8.19}$$

where

$$\frac{a_{o_2}^2}{\gamma - 1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}. \tag{8.20}$$

For a normal shock wave moving at constant velocity, the energy equation in terms of the relative velocities with respect to the shock is

$$\frac{V_1^2}{2}+\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}=\frac{V_2^2}{2}+\frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}= \text{constant}, \tag{8.21a}$$

where

$$V_1 = u_1 + L_f$$

$$V_2 = u_2 + L_f \tag{8.22}$$

Substitute Eq. (8.22) into Eq. (8.21a) to obtain

$$\frac{u_1^2}{2}+u_1 L_f+\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}=\frac{u_2^2}{2}+u_2 L_f+\frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}\,. \tag{8.21b}$$

The stagnation velocity of sound for fluid ahead of the shock wave is

$$\frac{a_{0_1}^2}{2}=\frac{u_1^2}{2}+\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} \tag{8.23}$$

Eliminate the pressure and density from Eq. (8.21b) by using Eqs. (8.20) and (8.23) to obtain

$$u_1 L_f+\frac{a_{0_1}^2}{\gamma-1}=u_2 L_f+\frac{a_{0_2}^2}{\gamma-1}$$

Hence, we have

$$\frac{a_{0_1}^2}{\gamma-1}-\frac{a_{0_2}^2}{\gamma-1}=-(u_1-u_2)L_f\,. \tag{8.24}$$

By substituting Eq. (8.24) into Eq. (8.19) the following result is obtained

$$\frac{a_{0_1}^2}{\gamma-1} = \frac{u_1^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} = \frac{a_0^2}{\gamma-1} \tag{8.25}$$

which shows that the constant for the non-stationary energy equation, Eq. (2.9c), for both sides of the normal shock wave moving at constant velocity is the same,

$$\left(\frac{\partial\varphi}{\partial t}\right)_2 + \frac{u_2^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2} = \frac{u_1^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} = \frac{a_0^2}{\gamma-1}\,. \tag{8.26}$$

For an infinite wedge with normal detached shock wave moving at constant velocity, the partial derivative of the velocity potential with respect to time in terms of non-dimensional velocity potential and space parameters can be determined from Eq. (8.6) and is

$$\frac{\partial\varphi}{\partial t} = -a_0 L_t\left(\xi\frac{\partial\Phi}{\partial\xi} + \eta\frac{\partial\Phi}{\partial\eta}\right) + a_0 L_t \Phi. \tag{8.27}$$

Immediately after the shock wave, the boundary conditions are

$$x = -L(t) \qquad \xi = -1.0$$

$$u = u_2 \qquad \left(\frac{\partial\Phi}{\partial\eta}\right)_2 = \left(\frac{\partial^2\Phi}{\partial\eta^2}\right)_2 = \left(\frac{\partial^2\Phi}{\partial\xi\partial\eta}\right)\cdots = 0$$

$$v = 0 \qquad \left(\frac{\partial\Phi}{\partial\xi}\right)_2 = \frac{u_2}{a_0}.$$

For these boundary conditions the partial derivative of the velocity potential with respect to time becomes

$$\left(\frac{\partial\varphi}{\partial t}\right)_2 = u_2 L_t + a_0 L_t \Phi_2 \tag{8.28}$$

Thus, the value of the non-dimensional velocity potential, $\bar{\Phi}_2$, at the shock wave is determined by equating Eqs. (8.18) and (8.28) and is

$$\bar{\Phi}_2 = -\frac{u_1}{a_0} \tag{8.29}$$

This is the value for $\bar{\Phi}$ at the shock wave that must be used in order to satisfy the non-stationary energy Eq. (2.9c). On the surface of the wedge, to satisfy the condition of no flow across the boundary the partial derivative of the non-dimensional velocity potential with respect to the normal to the surface must be equal to zero,

$$\frac{\partial \bar{\Phi}}{\partial n} = 0. \tag{8.30}$$

The existence and uniqueness of the potential solution for an infinite wedge with normal shock wave moving at constant velocity can be determined by considering the non-stationary differential equation, Eq. (8.15b), and the proper boundary conditions. At the shock wave the differential equation, Eq. (8.15b), becomes

$$\left[\lambda^2 + 2\lambda\left(\frac{\partial \bar{\Phi}}{\partial \xi}\right)_2 + \left(\frac{\partial \bar{\Phi}}{\partial \xi}\right)_2^2\right] \frac{\partial^2 \bar{\Phi}}{\partial \xi^2} = \frac{a_2^2}{a_0^2} \frac{\partial^2 \bar{\Phi}}{\partial \xi^2}. \tag{8.31a}$$

and the velocity of sound given by Eq. (8.16) becomes

$$\frac{a_2^2}{a_0^2} = 1 - \frac{(\gamma-1)}{2}\left(\frac{\partial \bar{\Phi}}{\partial \xi}\right)_2 - (\gamma-1)\lambda\left[\left(\frac{\partial \bar{\Phi}}{\partial \xi}\right)_2 + \bar{\Phi}_2\right]. \tag{8.32}$$

Since $\left(\frac{\partial \bar{\phi}}{\partial \xi}\right)_2 = \frac{u_2}{a_0}$, the Eq. (8.31a) reduces to

$$\left(\lambda^2 + 2\lambda \frac{u_2}{a_0} + \frac{u_2^2}{a_0^2}\right) \frac{\partial^2 \bar{\phi}}{\partial \xi^2} = \frac{a_2^2}{a_0^2} \frac{\partial^2 \bar{\phi}}{\partial \xi^2} . \tag{8.31b}$$

The values of $\lambda = \frac{L_f}{a_0}$ which will satisfy this equation* is determined from the equation

$$\left(\lambda^2 + 2\lambda \frac{u_2}{a_0} + \frac{u_2^2}{a_0^2}\right) = \frac{a_2^2}{a_0^2} \tag{8.32}$$

and the solutions are

$$\left(\lambda + \frac{u_2}{a_0}\right) = \left(\frac{L_f}{a_0} + \frac{u_2}{a_0}\right) = \pm \frac{a_2}{a_0} \tag{8.33a}$$

The positive sign applies for the shock wave moving towards the negative x-axis at constant velocity and the negative sign applies for the shock wave moving towards the positive x-axis. In terms of the relative velocity of the fluid with respect to the normal shock, the Eq. (8.33a) becomes

$$V_2 = L_f + u_2 = \pm a_2 . \tag{8.33b}$$

Thus, the Mach number of the relative velocity of the fluid with respect to the shock is equal to unity,

$$M_2' = \frac{V_2}{a_2} = 1.0 , \tag{8.34}$$

whether the normal shock is propagating towards the positive or negative x-axis. This shows that the boundary

* $\frac{\partial^2 \bar{\phi}}{\partial \xi^2} = 0$ leads to a uniform flow with no wedge

condition at the shock wave and the differential Eq. (8.15b) is satisfied only if $M_2' = 1.0$, which corresponds to a sonic disturbance.

The conclusion of this analysis for the existence and uniqueness of a potential solution for an infinite wedge with normal shock wave moving at constant speed is that no potential flow exists back of a detached shock wave. For all finite bodies there will be no potential flow back of the detached shock wave since the shock must be curved, and consequently, the flow back of the shock will be rotational. The only possible type of detached shock wave with irrotational flow back of the shock is a normal shock wave. A possible solution for an infinite wedge is a normal shock wave moving at constant speed, but the analysis indicates that the flow will not be potential back of the normal shock wave. Therefore, no potential flow can exist back of detached shock wave. The results of the existence and uniqueness of the potential solution for an infinite wedge will also apply to infinite cone where the shock wave will be normal and cannot be stationary.

APPENDIX I

Coefficients for the Second Order Correction to the Velocity as given by Eq. (7.27) are:

$$\alpha_3 = .2122314$$

$$\alpha_5 = .8187377$$

$$\alpha_6 = -.3333333$$

$$\alpha_7 = .8347244$$

$$\alpha_8 = -10.3090910$$

$$\alpha_9 = 10.1682984$$

$$\alpha_{11} = 2.8150032$$

$$\alpha_{12} = -4.8219218$$

$$\alpha_{15} = .6153508$$

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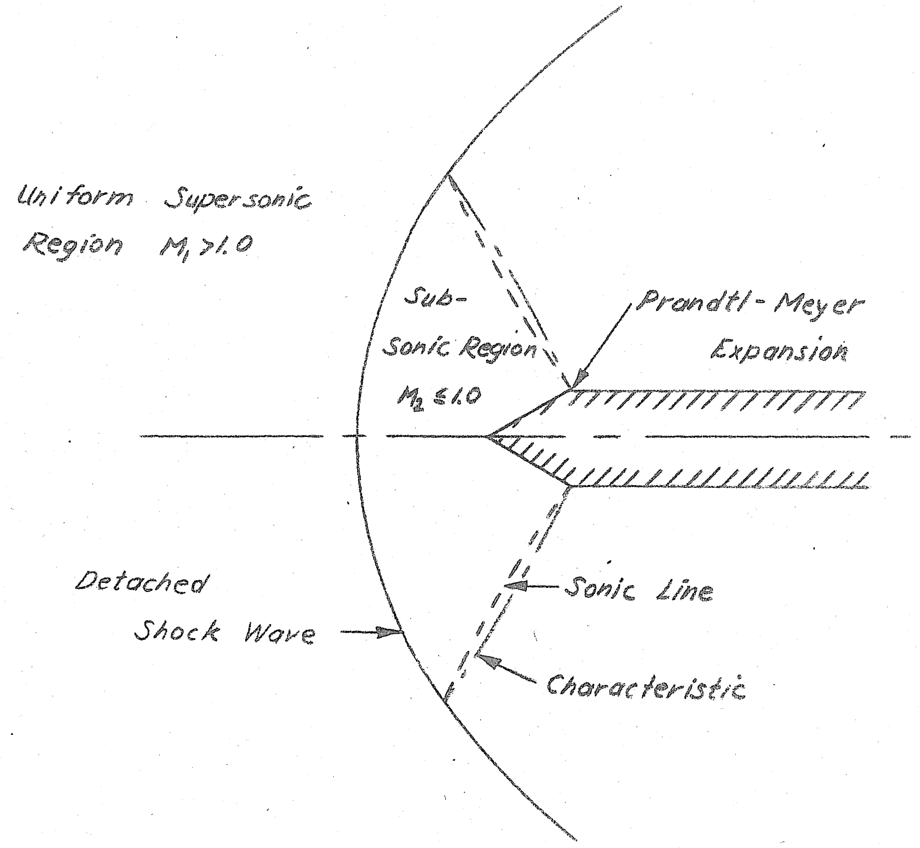


FIG. 1a: DETACHED SHOCK WAVE FOR TWO-DIMENSIONAL WEDGE $M_1 > 1.0$

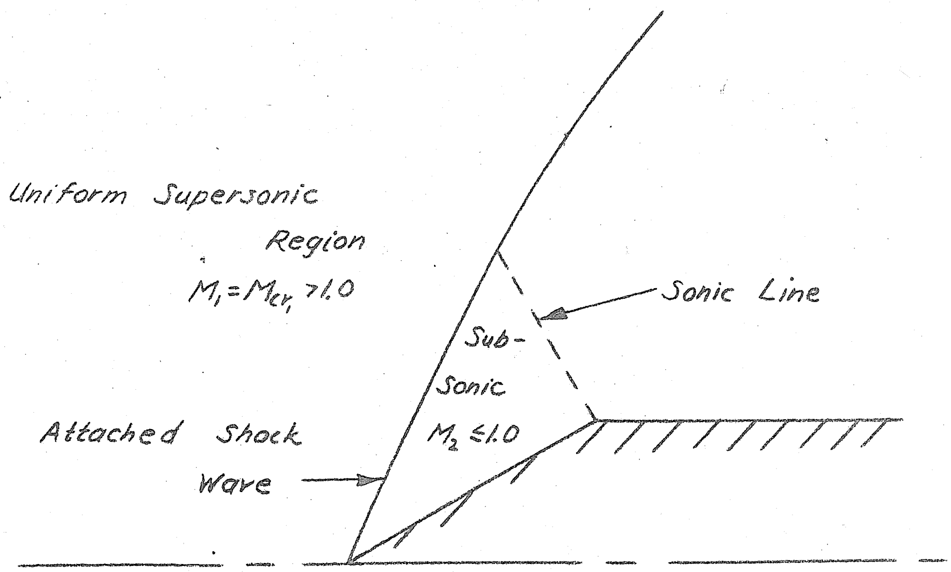


FIG. 1b: ATTACHED SHOCK WAVE FOR TWO-DIMENSIONAL WEDGE
$M_1 = M_{cr_1}$

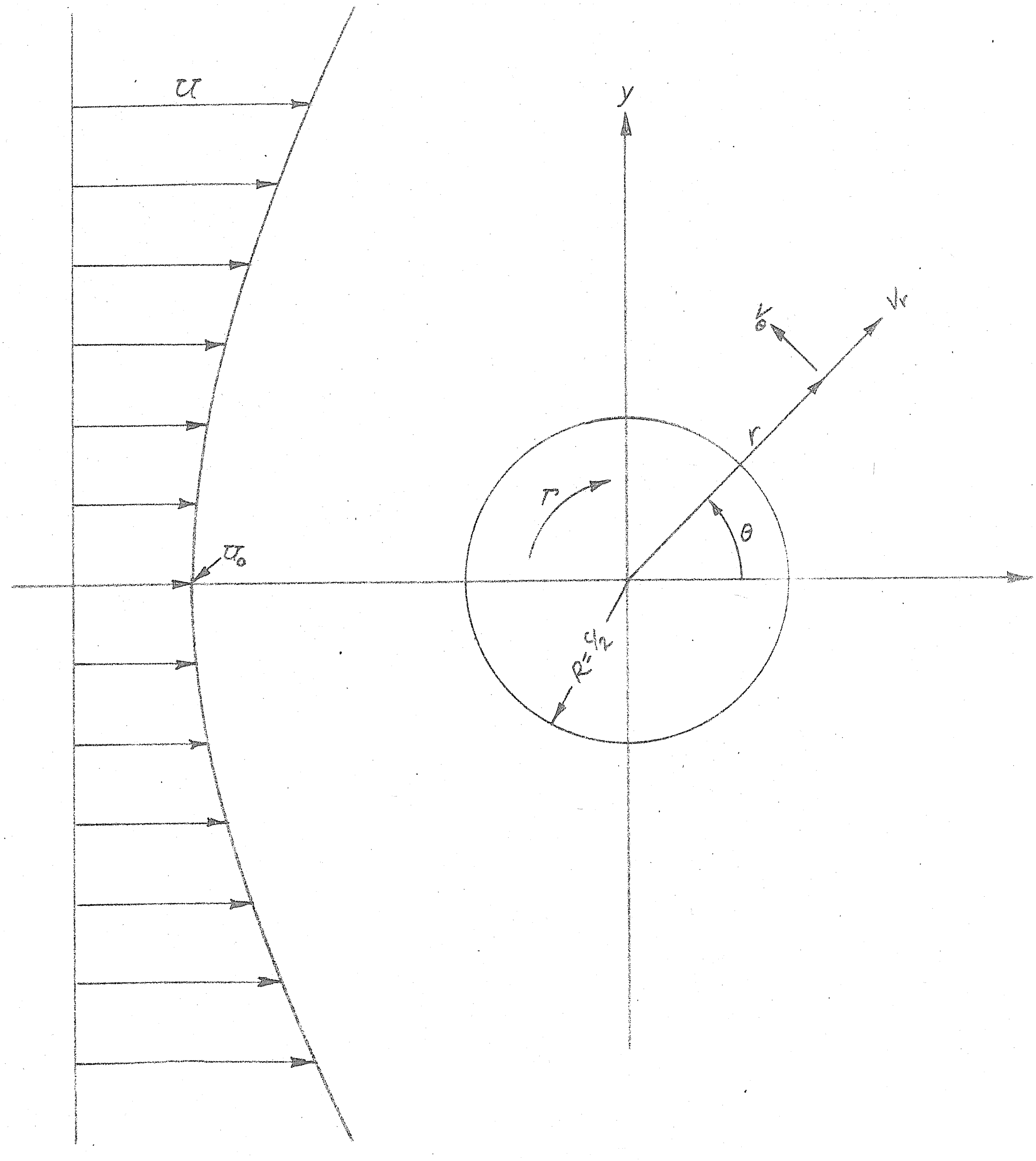


FIG. 2: CIRCULAR CYLINDER IN SHEAR FLOW

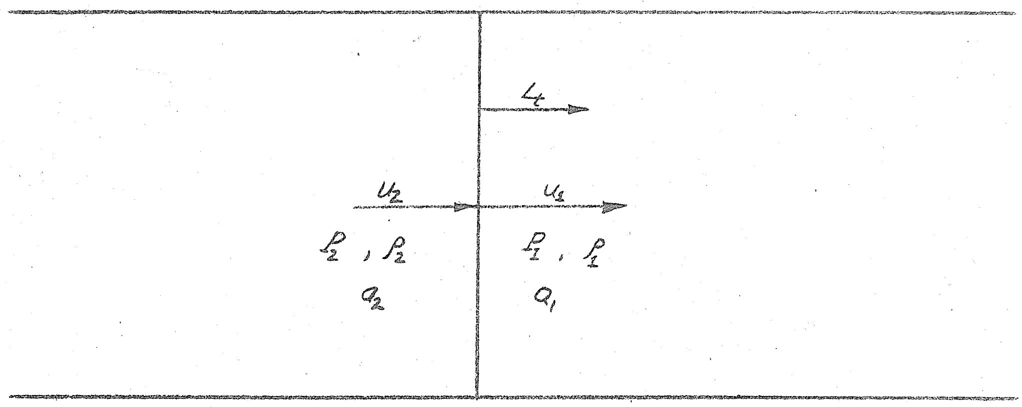


FIG. 3: PROPAGATING NORMAL SHOCK WAVE

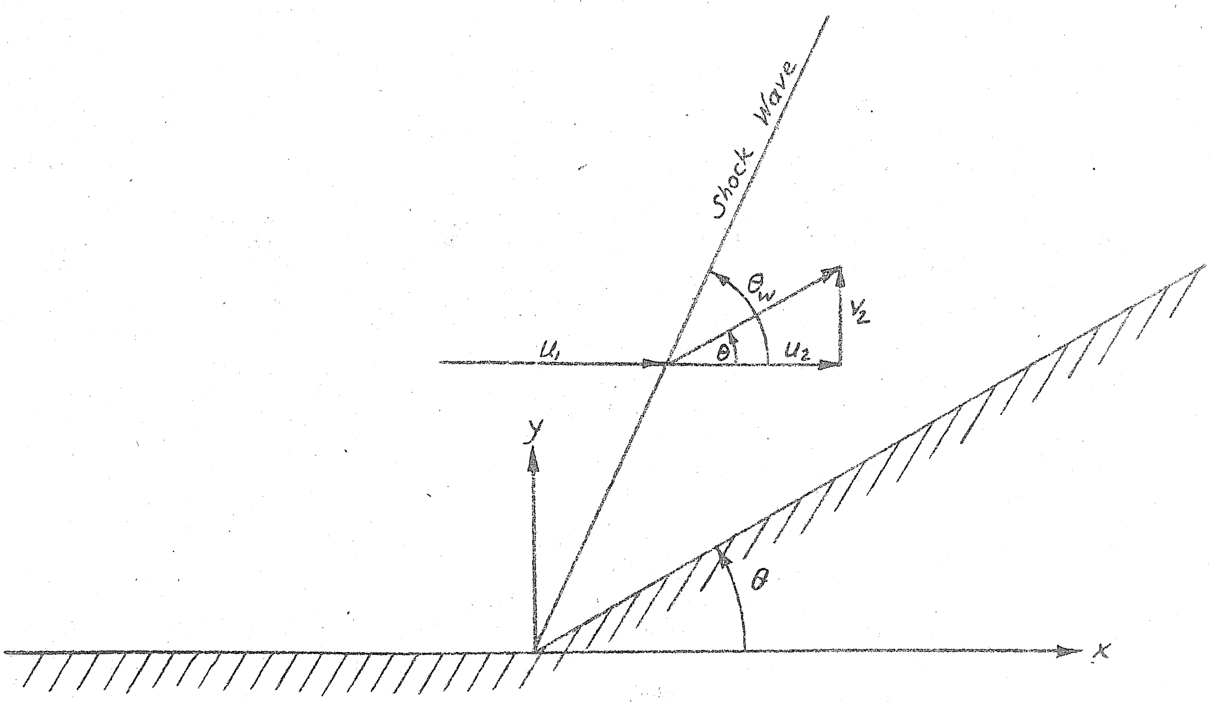


FIG. 4: STATIONARY OBLIQUE SHOCK WAVE

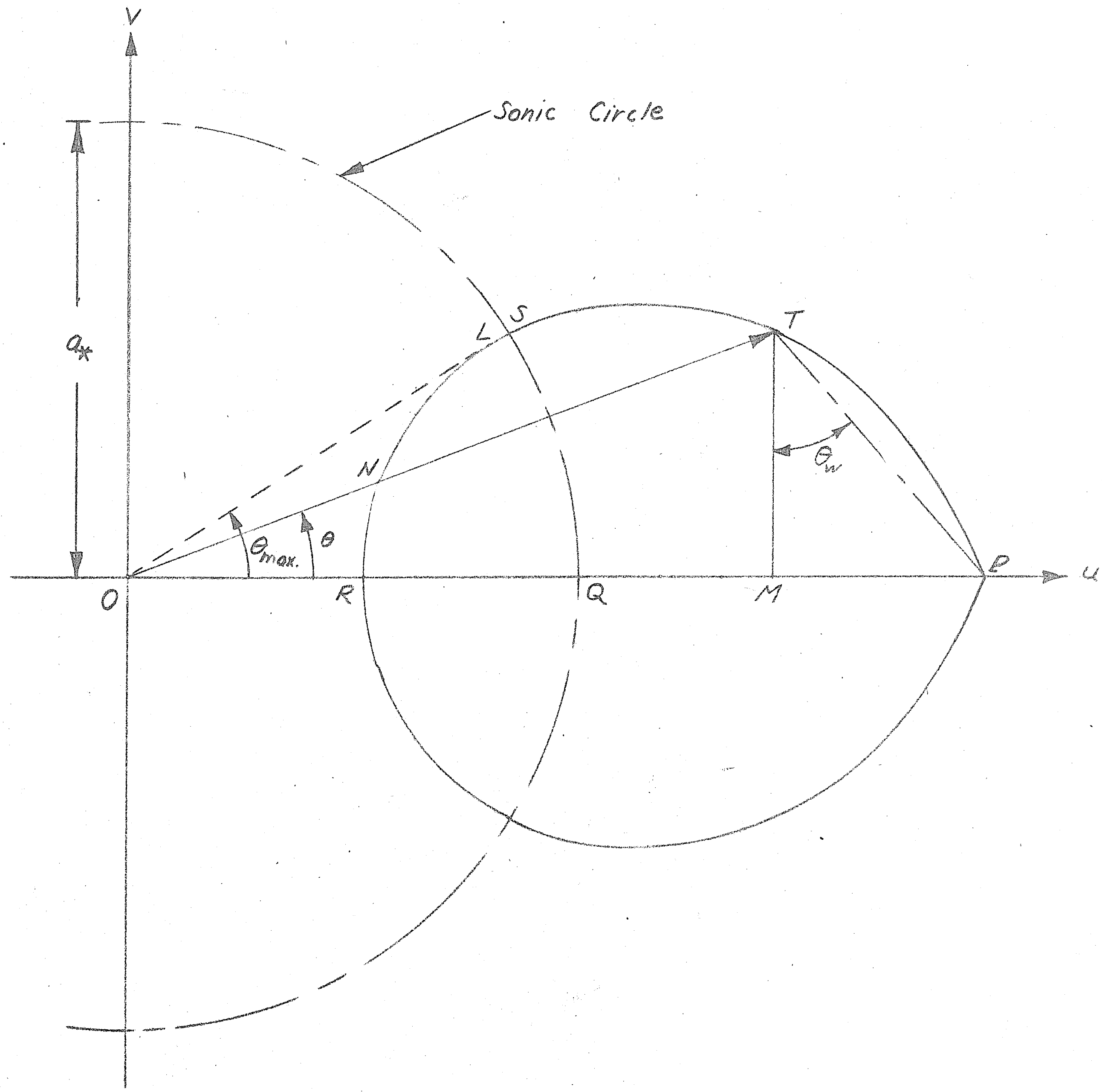
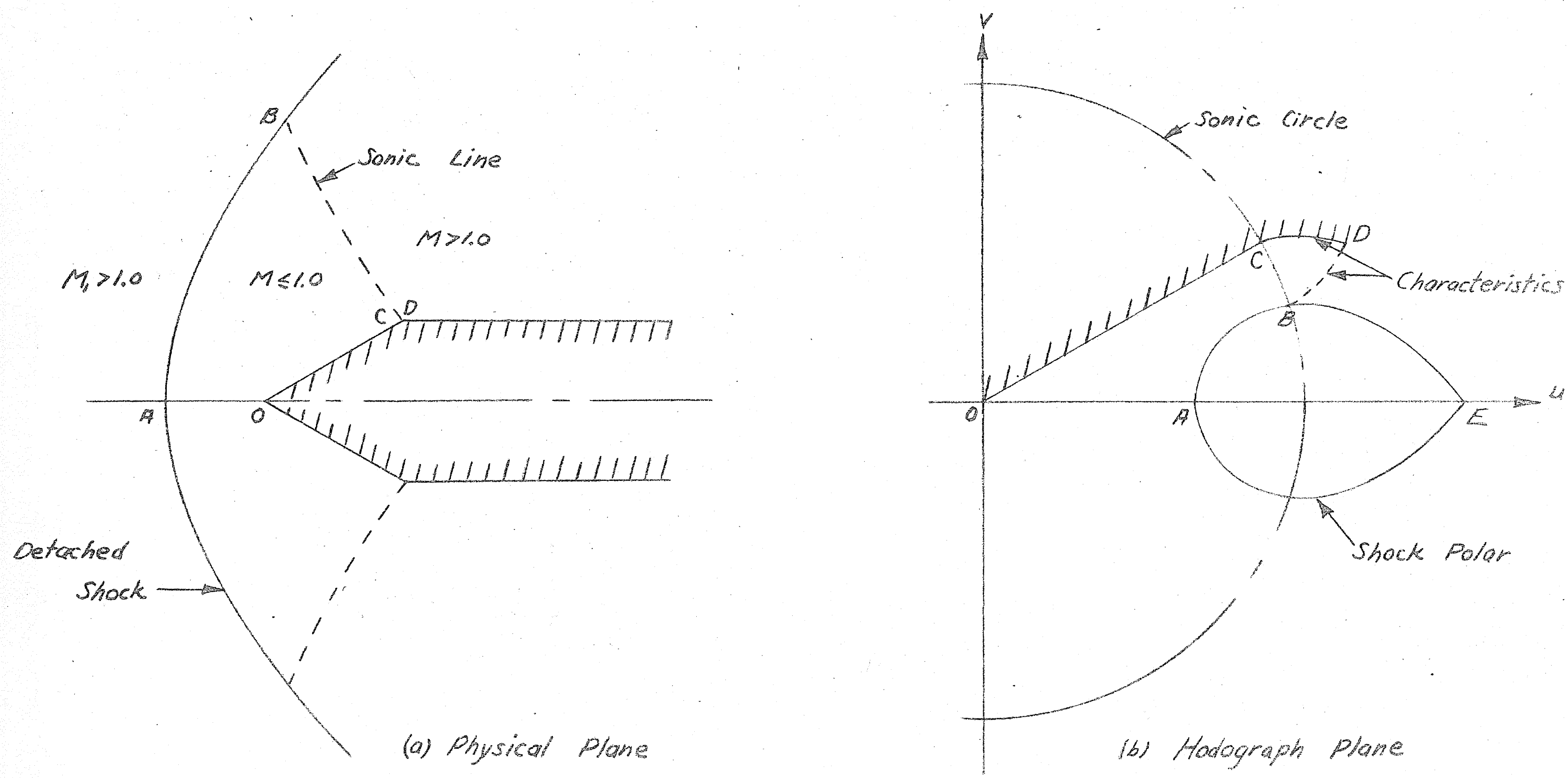


FIG. 5: SHOCK POLAR FOR AIR (γ = 1.405)



(a) Physical Plane

(b) Hodograph Plane

FIG. 6: DETACHED SHOCK WAVE FOR A FINITE TWO-DIMENSIONAL WEDGE IN THE PHYSICAL AND HODOGRAPH PLANES

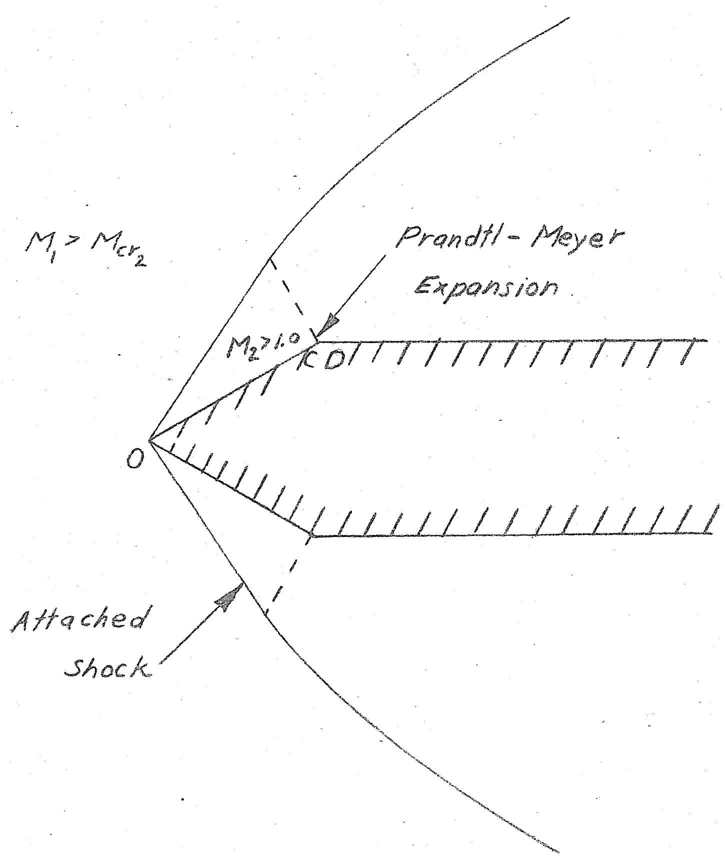


FIG. 7a: ATTACHED SHOCK WAVE FOR TWO-DIMENSIONAL WEDGE FOR $M_1 > M_{cr_2}$

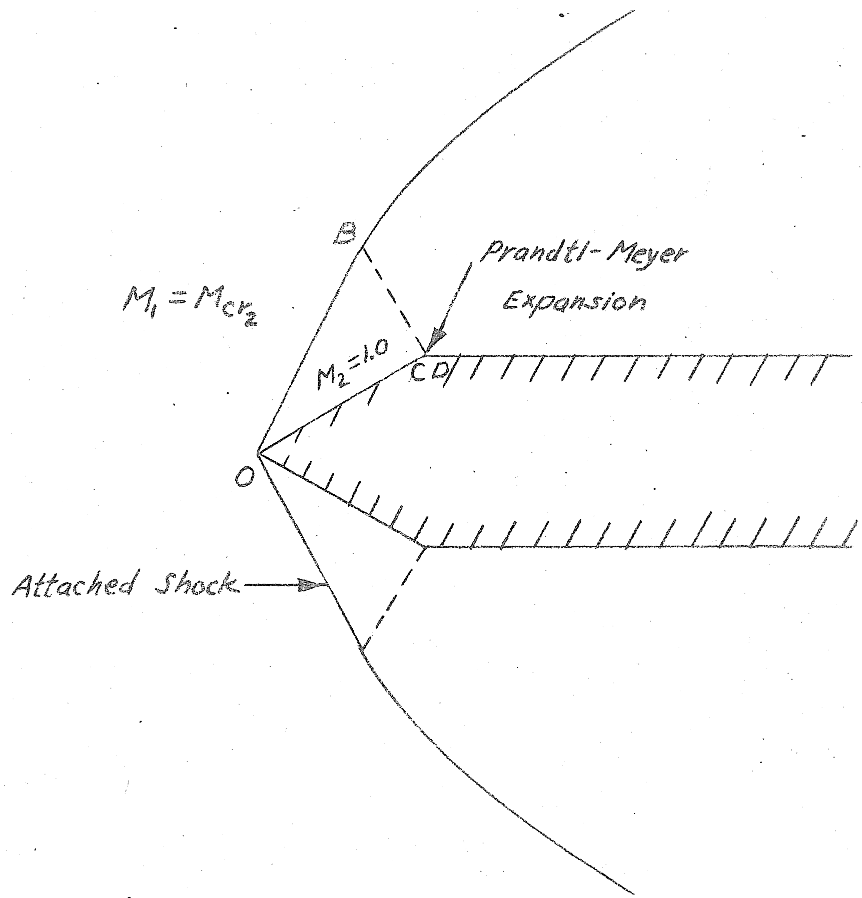


FIG. 7b: ATTACHED SHOCK WAVE FOR TWO-DIMENSIONAL WEDGE FOR $M_1 = M_{cr_2}$.

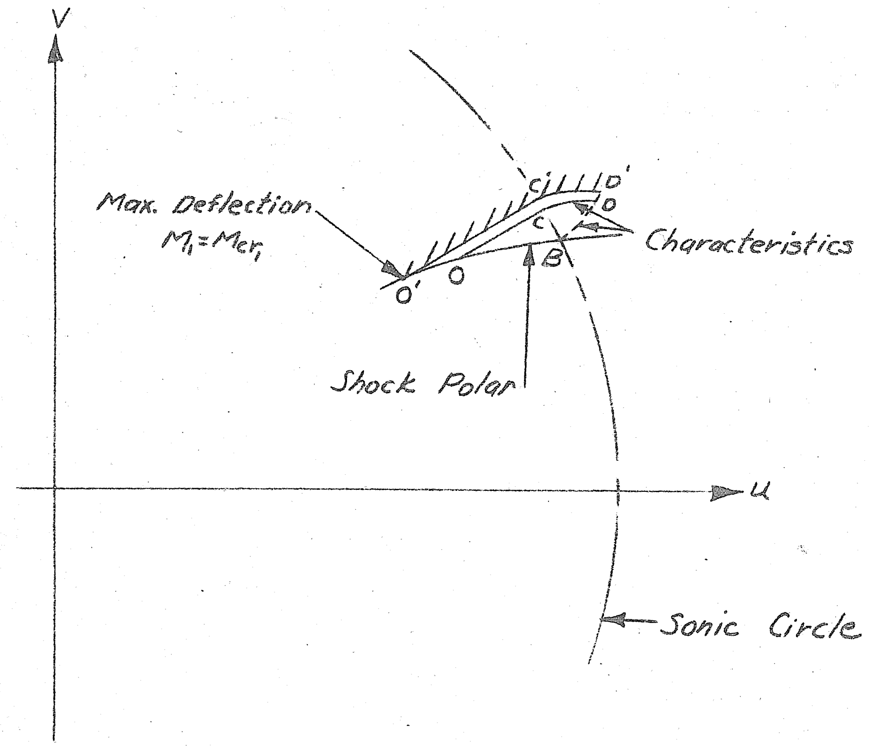


FIG. 7c: ATTACHED SHOCK WAVE FOR TWO-DIMENSIONAL WEDGE IN HODOGRAPH PLANE FOR $M_{cr_1} < M_1 < M_{cr_2}$ AND $M_1 = M_{cr_2}$

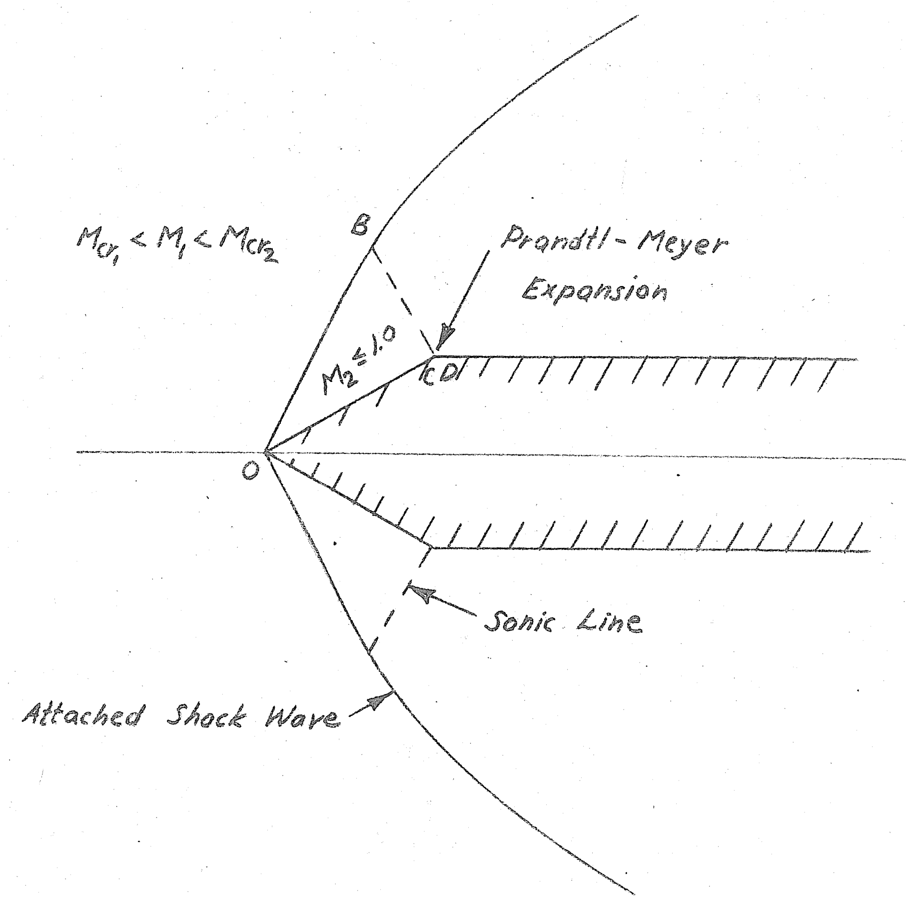


FIG. 7d: ATTACHED SHOCK WAVE FOR TWO-DIMENSIONAL WEDGE IN PHYSICAL PLANE FOR $M_{cr_1} < M_2 < M_{cr_2}$

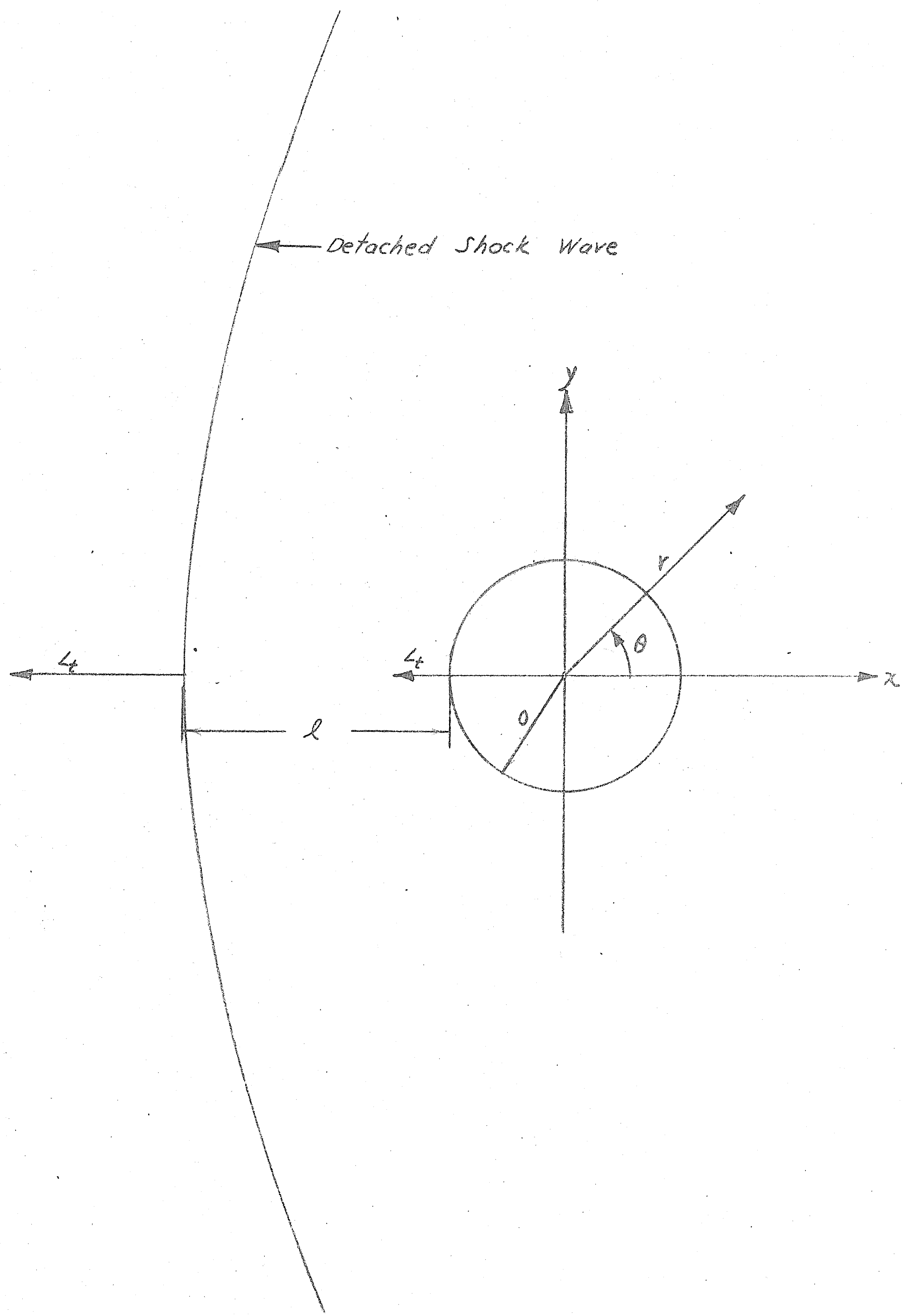


FIG. 8: DETACHED SHOCK WAVE FOR CIRCULAR CYLINDER

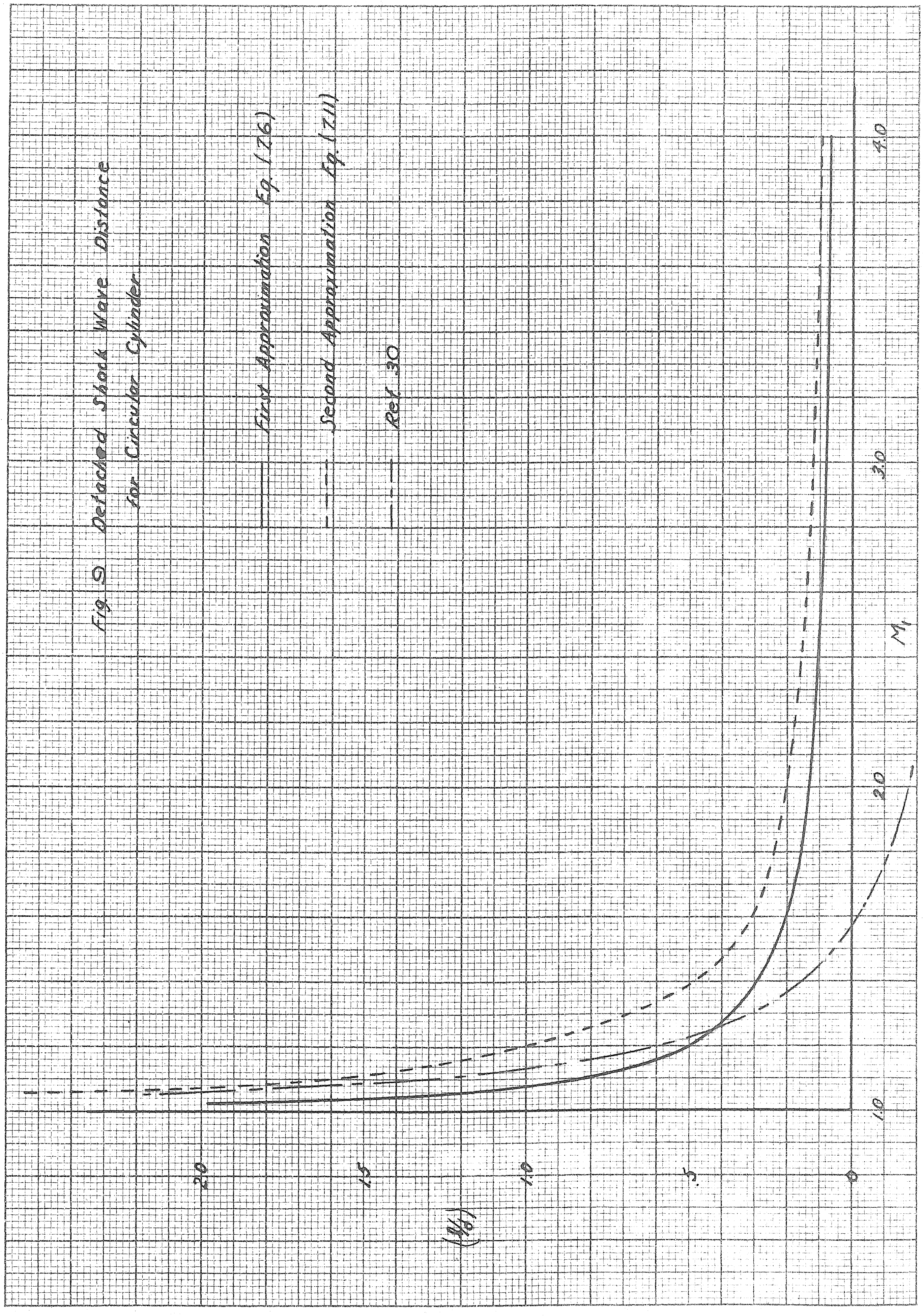


Fig. 9 Detached Shock Wave Distance
for Circular Cylinder
First Approximation Eq. (7.6)
Second Approximation Eq. (7.11)
Ref. 30
M_1
1.0
2.0
3.0
4.0
0
.5
1.0
1.5
2.0
(Δ/D)

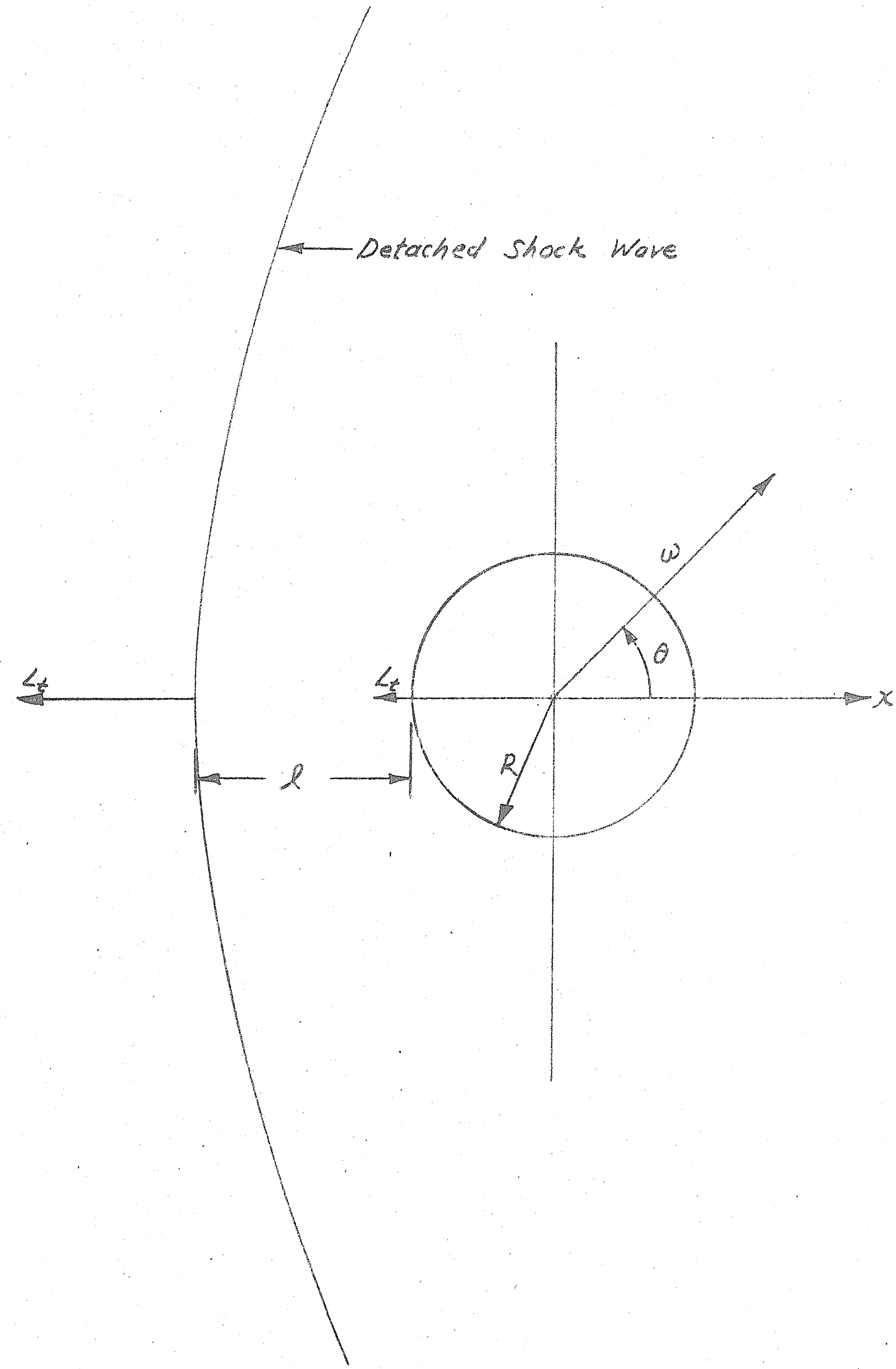
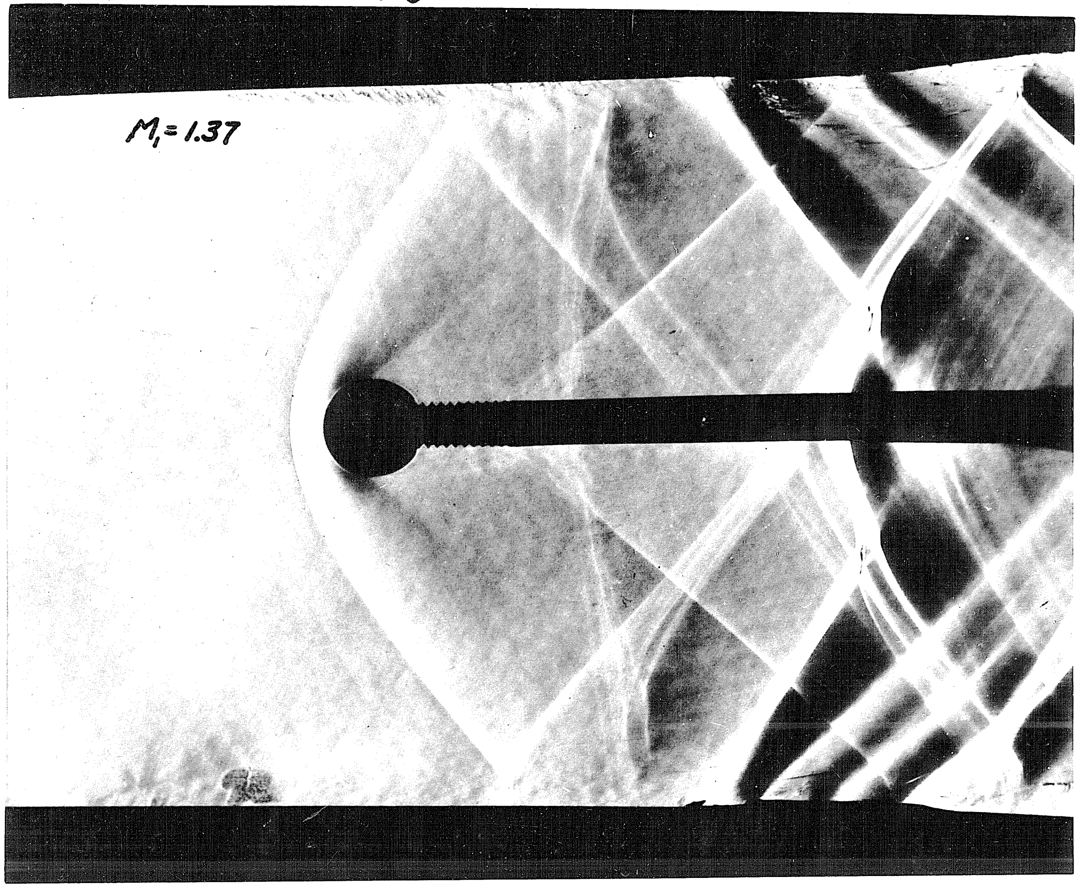


FIG. 10: DETACHED SHOCK WAVE FOR SPHERE



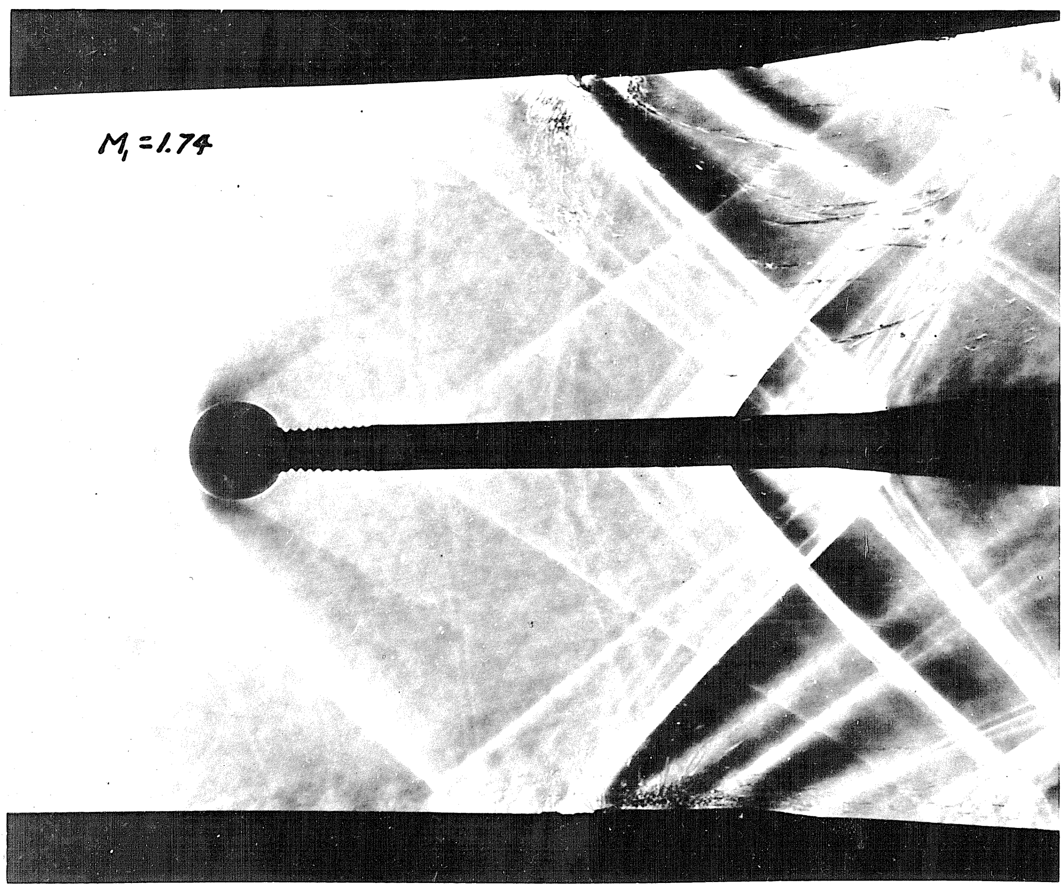


Fig. 11. Detached Shock Wave for .266 Inch Diameter Sphere From Supersonic Tunnel (Ref. 26)

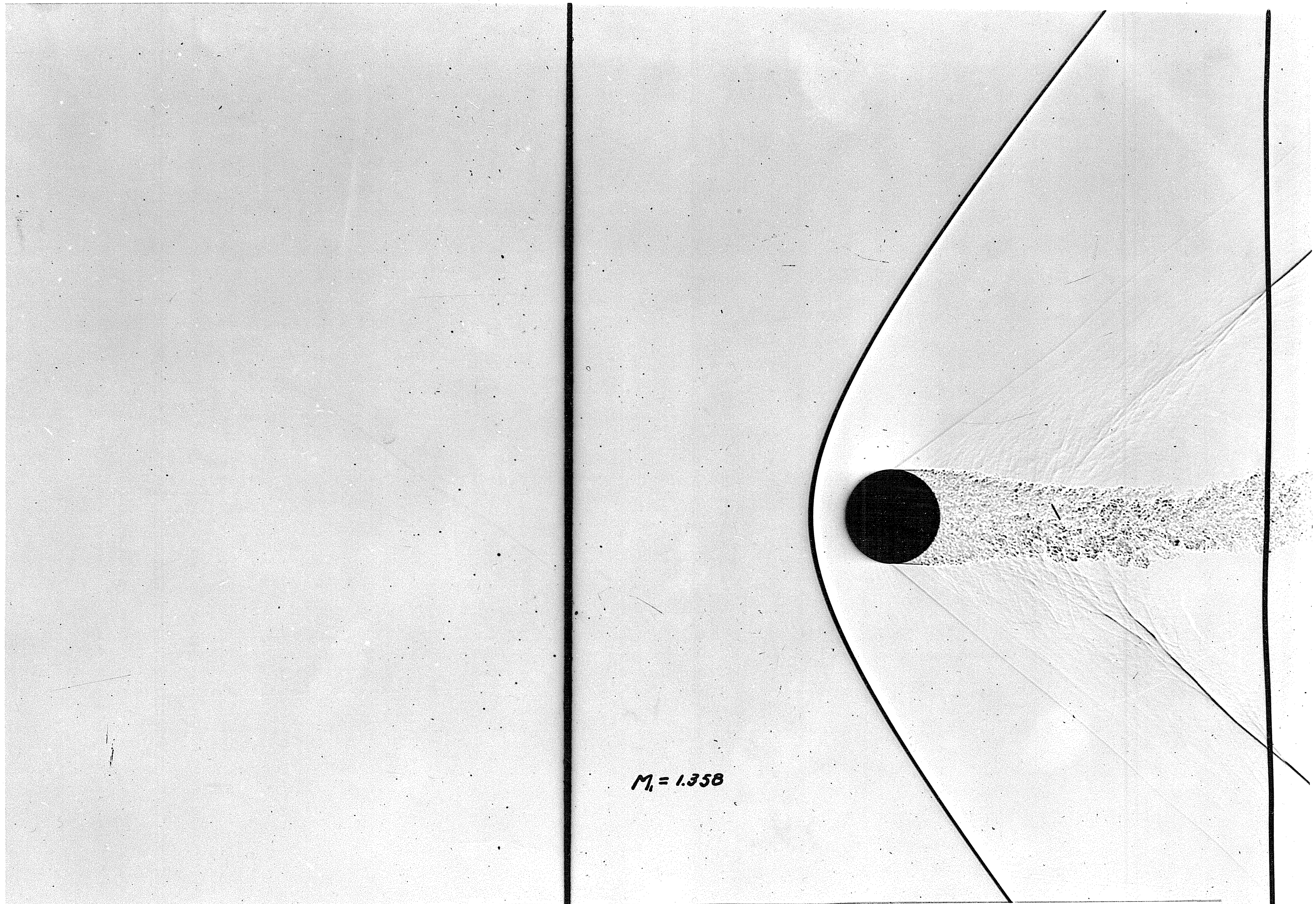


Fig. 12 Detached Shock Wave for 9/16 Inch Diameter Sphere From Aberdeen Ballistic Range (Ref. 27)

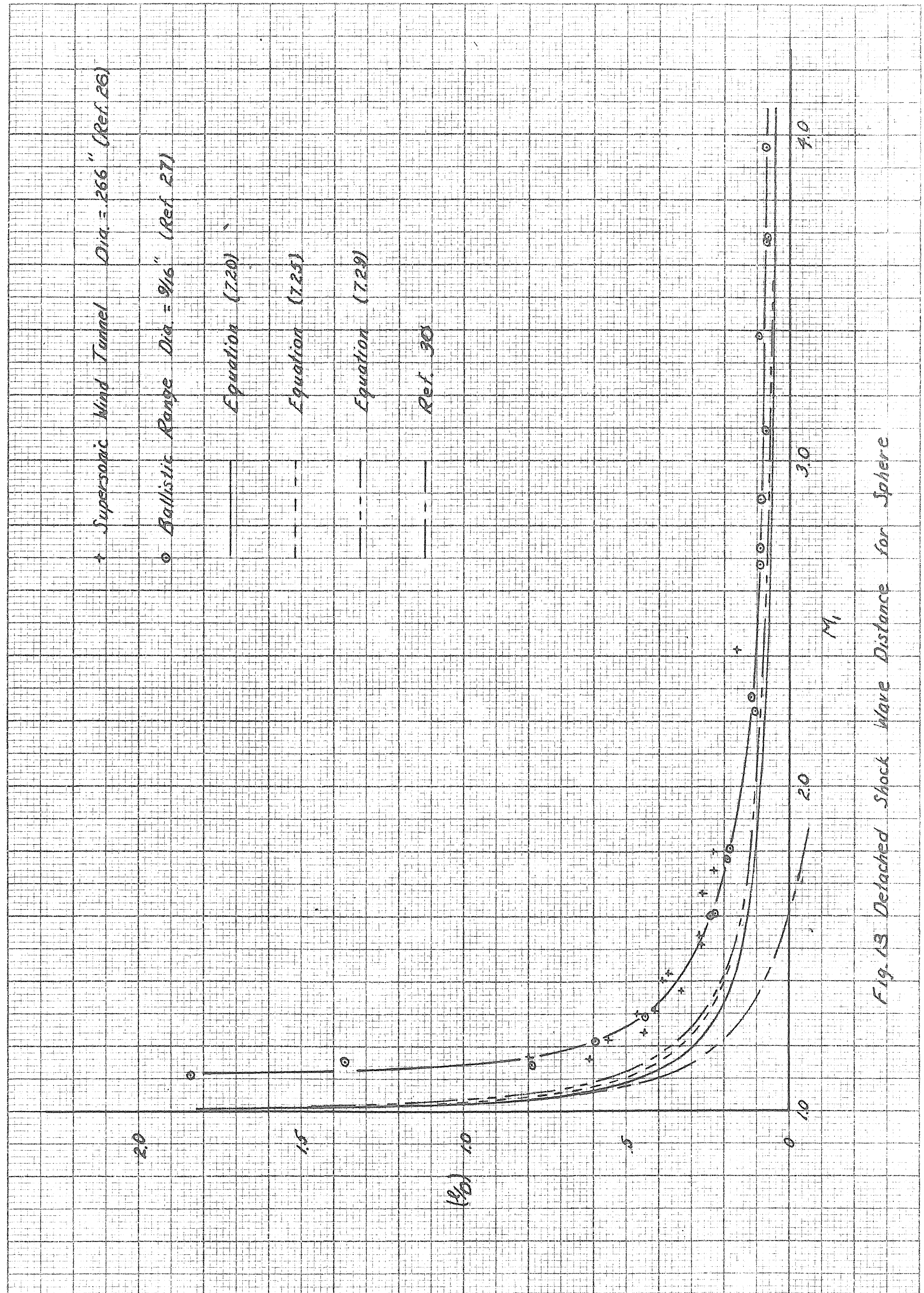


Fig. 13 Detached Shock Wave Distance for Sphere

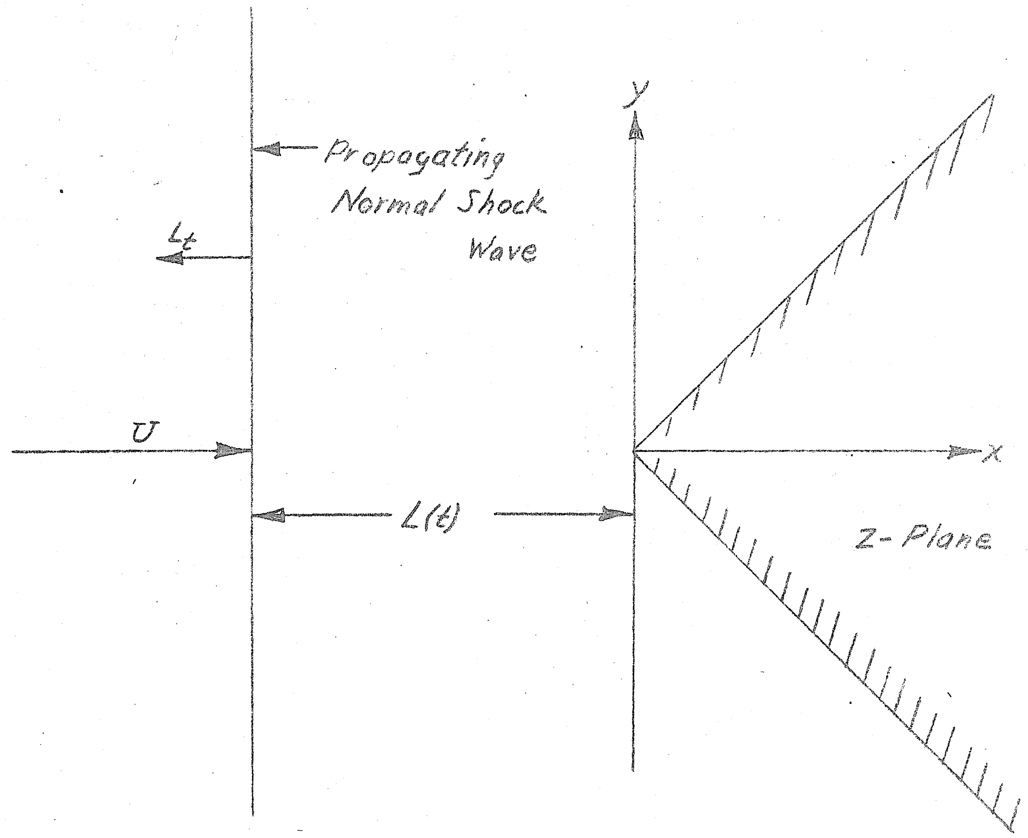


FIG. 14a: DETACHED SHOCK WAVE FOR INFINITE WEDGE

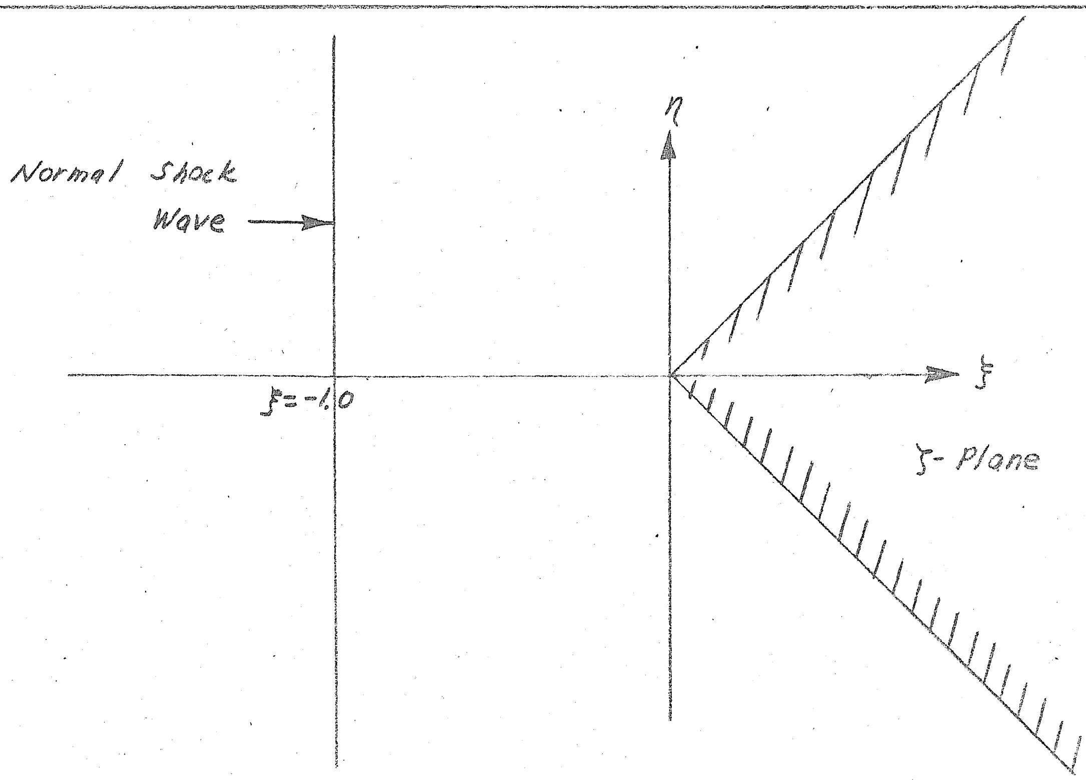


FIG. 14.b: NORMAL DETACHED SHOCK WAVE FOR INFINITE WEDGE IN THE NON-DIMENSIONAL ζ-PLANE