Part II

Chern-Simons Theory with Complex Gauge Group

Chapter 5

Perturbation theory around a nontrivial flat connection

We now change focus to consider Chern-Simons theory with noncompact, complex gauge group, and in particular the various approaches to computing its perturbative partition functions that were outlined in the Introduction.

We begin here by reviewing some basic features of Chern-Simons theory with complex gauge group and its perturbative expansion, following mainly [39] and [40]. The notations we introduce will be used throughout Part II. We then pursue the most traditional approach to Chern-Simons theory, based on the evaluation of Feynman diagrams. This analysis will show that the perturbative coefficients in the Chern-Simons partition functions have a very special structure, motivating the definition of "arithmetic TQFT." We define a "geometric" flat $SL(2,\mathbb{C})$ Chern-Simons connection that is related to the hyperbolic structure on a hyperbolic three-manifold M, and conjecture that in the background of this connection $SL(2,\mathbb{C})$ Chern-Simons theory *is* such an arithmetic TQFT. In the last section of the chapter, we substantiate this claim with explicit computations for the knots $\mathbf{4_1}$ and $\mathbf{5_2}$. The discussions of Feynman diagrams and arithmeticity follow [3].

Starting in Section 5.2, some understanding of hyperbolic geometry will be helpful. We will introduce several basic concepts as needed, but also refer the reader to Section 8.1 for a more thorough treatment of hyperbolic three-manifolds. Similarly, the computations in Section 5.4 will require some familiarity with the Volume Conjecture, which is described much more fully in Sections 6.2 and 7.4.

5.1 Basics

Let us denote a compact gauge group by G, its noncompact complexification by $G_{\mathbb{C}}$, and the respective Lie algebras of these groups as $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$. We can assume that G, $G_{\mathbb{C}}$ are reductive. As noted in the Introduction, the Chern-Simons action for a complex gauge field $\mathcal{A}$ can be written as a sum of two (classically) topological terms, one for $\mathcal{A}$ and one for $\bar{\mathcal{A}}$:

$$\begin{aligned} S \;=\; & \frac{t}{8\pi}\int_M \mathrm{Tr}\Big(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big) \\ & + \frac{\bar{t}}{8\pi}\int_M \mathrm{Tr}\Big(\bar{\mathcal{A}}\wedge d\bar{\mathcal{A}}+\frac{2}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\Big)\,. \end{aligned} \tag{5.1.1}$$

The field $\mathcal{A}$ here is a locally-defined $\mathfrak{g}_{\mathbb{C}}$-valued one-form on the euclidean three-manifold M. The two coupling constant t and $\bar{t}$ can be written as

$$t = k+\sigma\,, \qquad t = k-\sigma\,, \tag{5.1.2}$$

and different physical unitarity structures force the "level" k to be an integer and σ to be either real or imaginary [39]. For example, in the case $G_{\mathbb{C}} = SL(2,\mathbb{C})$, the above action can be recast as the action for euclidean gravity with negative cosmological constant by writing $\mathcal{A}$ as a vielbein and a spin connection, $\mathcal{A} = -(w+ie)$ [96], and under the resulting unitarity structure the coupling σ must be real. Unlike in the case of compact gauge group, the level k does *not* undergo a shift in the quantum theory [44].

It is explained in [39] that introducing a noncompact gauge group is a perfectly acceptable option in Chern-Simons theory. In Yang-Mills theories, a noncompact gauge group would lead to a kinetic term that is not positive definite, and hence to unbounded energy (or an ill-defined path integral). In Chern-Simons theory with complex gauge group the kinetic term *is* indefinite, but this is no problem: the Hamiltonian of the theory vanishes due to topological invariance, so the "energy" is always exactly zero.

The classical solutions, or extrema of the action (5.1.1), are flat connections, *i.e.* connections that obey

$$\mathcal{A}+\mathcal{A}\wedge\mathcal{A}=0\,, \qquad \bar{\mathcal{A}}+\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}=0\,. \tag{5.1.3}$$

Flat $G_{\mathbb{C}}$-connections on a three-manifold M are completely determined by their holonomies, *i.e.* by a homomorphism

$$\rho\,:\;\pi_1(M)\to G_{\mathbb{C}}\,, \tag{5.1.4}$$

up to conjugacy (*i.e.* up to gauge transformations). Thus the moduli space of classical solutions can be written as

$$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}; M) = \text{Hom}(\pi_1(M),\, G_{\mathbb{C}})/\!\!/G_{\mathbb{C}}\,. \tag{5.1.5}$$

As discussed in the introduction, one can then consider perturbation theory around a given flat connection $\mathcal{A}^{(\rho)} \in \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}; M)$ corresponding to the homomorphism ρ. Since the classical action is a sum of terms for $\mathcal{A}$ and $\bar{\mathcal{A}}$, the *perturbative* expansion of the partition function around $\mathcal{A}^{(\rho)}$ will factorize as (*cf.* [40, 41])

$$Z^{(\rho)}(M) = Z^{(\rho)}(M;t)\, Z^{(\rho)}(M;\bar{t})\,. \tag{5.1.6}$$

One may *hope* that the full non-perturbative path integral obeys a relation of the form

$$Z(M) = \int \mathcal{D}\mathcal{A}\, e^{iS} = \sum_{\rho} Z^{(\rho)}(M;t)\, Z^{(\rho)}(M;\bar{t})\,, \tag{5.1.7}$$

summing over all flat connections. Here, however, we merely focus on the perturbative pieces $Z^{(\rho)}(M;t)$.

By standard methods of quantum field theory — essentially a stationary phase approximation to the path integral — each component $Z^{(\rho)}(M;t)$ can be expanded in inverse powers of t. Explicitly, let us define Planck's constant

$$\boxed{\hbar = \frac{2\pi i}{t}} \tag{5.1.8}$$

and expand

$$Z^{(\rho)}(M;\hbar) \;=\; \exp\left(\frac{1}{\hbar}S_0^{(\rho)} - \frac{1}{2}\delta^{(\rho)}\log\hbar \,+\, \sum_{n=0}^{\infty} S_{n+1}^{(\rho)}\hbar^n\right)\,. \tag{5.1.9}$$

The coefficients $S_n^{(\rho)}$ (and $\delta^{(\rho)}$) completely characterize $Z^{(\rho)}(M;t)$ to all orders in perturbation theory.

5.2 Coefficients and Feynman diagrams

Let us examine each term in the expansion (5.1.9) more carefully. As already mentioned in the introduction, the leading term $S_0^{(\rho)}$ is the value of the classical holomorphic Chern-Simons functional

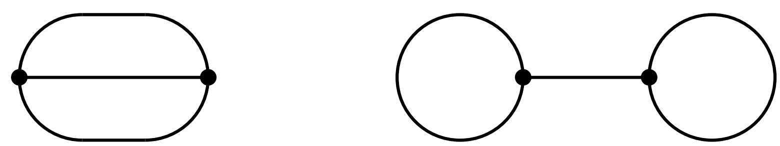
$$\frac{1}{4}\int_M \text{Tr}\Big(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big) \tag{5.2.1}$$

evaluated on a flat gauge connection $\mathcal{A}^{(\rho)}$ that corresponds to a homomorphism ρ. It is also easy to understand the integer coefficient $\delta^{(\rho)}$ in (5.1.9). The homomorphism ρ defines a flat $G_{\mathbb{C}}$ bundle over M, which we denote as E_ρ. Letting $H^i(M; E_\rho)$ be the i-th cohomology group of M with coefficients in the flat bundle E_ρ, the coefficient $\delta^{(\rho)}$ is given by

$$\delta^{(\rho)} = h^1 - h^0 , \tag{5.2.2}$$

where $h^i := \dim H^i(M; E_\rho)$. Both this term and $S_1^{(\rho)}$ come from the "one-loop" contribution to the path integral (1.0.2); $S_1^{(\rho)}$ can be expressed in terms of the Ray-Singer torsion [97] of M with respect to the flat bundle E_ρ (*cf.* [13, 44, 98]),

$$S_1^{(\rho)} = \frac{1}{2} \log \left(\frac{T(M; E_\rho)}{2} \right) . \tag{5.2.3}$$



a) ***b)***

Figure 5.1: Two kinds of 2-loop Feynman diagrams that contribute to $S_2^{(\rho)}$.

The geometric interpretation of the higher-order terms $S_n^{(\rho)}$ with $n > 1$ is more interesting, yet less obvious. To understand it better, we note that the saddle-point approximation to the path integral (1.0.2) gives an expression for $S_n^{(\rho)}$ as a sum of Feynman diagrams with n loops. For example, the relevant diagrams for $n = 2$ are shown schematically in Figure 5.1. Since the Chern-Simons action (5.1.1) is cubic, all vacuum diagrams (that is, Feynman diagrams with no external lines) are closed trivalent graphs; lines in such graphs have no open end-points. Therefore, a Feynman diagram with n loops ($n > 1$) has $3(n-1)$ line segments with end-points meeting at $2(n-1)$ trivalent vertices. Each such diagram contributes an integral of the form (*cf.* [42])

$$\int_{M^{2n-2}} L^{3n-3} , \tag{5.2.4}$$

where M^{2n-2} denotes a product of $2n-2$ copies of M and L^{3n-3} denotes a wedge product of a 2-form $L(x,y) \in \Omega^2(M_x \times M_y; \mathfrak{g}_{\mathbb{C}} \otimes \mathfrak{g}_{\mathbb{C}})$. The 2-form $L(x,y)$, called the "propagator," is a solution to the first-order PDE,

$$d_{\mathcal{A}^{(\rho)}} L(x,y) \;=\; \delta^3(x,y)\,, \tag{5.2.5}$$

where $\delta^3(x,y)$ is a δ-function 3-form supported on the diagonal in $M_x \times M_y$ and $d_{\mathcal{A}^{(\rho)}}$ is the exterior derivative twisted by a flat connection $\mathcal{A}^{(\rho)}$ on the $G_{\mathbb{C}}$ bundle E_ρ.

For example, suppose that M is a geodesically complete hyperbolic 3-manifold of finite volume. As we pointed out in the Introduction, such 3-manifolds provide some of the most interesting examples for Chern-Simons theory with complex gauge group. Every such M can be represented as a quotient

$$M \;=\; \mathbb{H}^3/\Gamma \tag{5.2.6}$$

of the hyperbolic space $\mathbb{H}^3$ by a discrete, torsion-free subgroup $\Gamma \subset PSL(2,\mathbb{C})$, which is a holonomy representation of the fundamental group $\pi_1(M)$ into $\mathrm{Isom}^+(\mathbb{H}^3) = PSL(2,\mathbb{C})$. Furthermore, one can always choose a spin structure on M such that this holonomy representation lifts to a map from $\pi_1(M)$ to $SL(2,\mathbb{C})$. In what follows, we call this representation "geometric" and denote the corresponding flat $SL(2,\mathbb{C})$ connection as $\mathcal{A}^{(\mathrm{geom})}$.

In order to explicitly describe the flat connection $\mathcal{A}^{(\mathrm{geom})}$, recall that $\mathbb{H}^3$ can be defined as the upper half-space with the standard hyperbolic metric

$$ds^2 \;=\; \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2)\,, \qquad x_3 > 0\,. \tag{5.2.7}$$

The components of the vielbein and spin connection corresponding to this metric can be written as

$$\begin{aligned} e^1 &= \frac{dx_1}{x_3} & \qquad w^1 &= \frac{dx_2}{x_3} \\ e^2 &= \frac{dx_2}{x_3} & \qquad w^2 &= -\frac{dx_1}{x_3} \\ e^3 &= \frac{dx_3}{x_3} & \qquad w^3 &= 0\,. \end{aligned}$$

These satisfy $de^a + \epsilon^{abc} w_b \wedge e_c = 0$. It is easy to check that the corresponding $(P)SL(2,\mathbb{C})$ connection

$$\mathcal{A}^{(\mathrm{geom})} \;=\; -(w + ie) \;=\; \frac{1}{2x_3}\begin{pmatrix} dx_3 & 2dx_1 - 2idx_2 \\ 0 & -dx_3 \end{pmatrix} \tag{5.2.8}$$

is flat, *i.e.* obeys (1.0.3) on page 6. In Chern-Simons theory with gauge group $G_{\mathbb{C}} = (P)SL(2,\mathbb{C})$, this gives an explicit expression for the flat gauge connection that corresponds to the hyperbolic structure on M. In a theory with a larger gauge group, one can also define a geometric connection $\mathcal{A}^{(\mathrm{geom})}$ by embedding (5.2.8) into a larger matrix. We note, however, that $\mathcal{A}^{(\mathrm{geom})}$ constructed in this way is not unique and depends on the choice of embedding. Nevertheless, we will continue to use the notation $\mathcal{A}^{(\mathrm{geom})}$ even in the higher-rank case whenever the choice of embedding introduces no confusion.

The action of Γ on $\mathbb{H}^3$ can be conveniently expressed by identifying a point $(x_1, x_2, x_3) \in \mathbb{H}^3$ with a quaternion $q = x_1 + x_2 i + x_3 j$ and defining

$$\gamma : q \mapsto (aq+b)/(cq+d)\ , \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C})\,. \tag{5.2.9}$$

Explicitly, setting $z = x_1 + i x_2$, we find

$$\gamma(z + x_3 j) \ = \ z' + x_3' j\ , \tag{5.2.10}$$

where

$$z' \ = \ \frac{(az+b)(\bar{c}\bar{z}+\bar{d}) + a\bar{c}x_3^2}{|cz+d|^2 + |c|^2 x_3^2}\ , \qquad x_3' \ = \ \frac{x_3}{|cz+d|^2+|c|^2x_3^2}\ . \tag{5.2.11}$$

Let $L_0(x,y)$ be the propagator for $\mathbb{H}^3$, *i.e.* a solution to equation (5.2.5) on $\mathbb{H}^3 \times \mathbb{H}^3$ with the non-trivial flat connection $\mathcal{A}^{(\mathrm{geom})}$. Then, for a hyperbolic quotient space (5.2.6), the propagator $L(x,y)$ can simply be obtained by summing over images:

$$L(x,y) \ = \ \sum_{\gamma\in\Gamma} L_0(x, \gamma y)\,. \tag{5.2.12}$$

5.3 Arithmeticity

Now, we would like to consider what kind of values the perturbative invariants $S_n^{(\rho)}$ can take. For $n \geq 1$, the $S_n^{(\rho)}$'s are given by sums over Feynman diagrams, each of which contributes an integral of the form (5.2.4). *A priori* the value of every such integral can be an arbitrary complex number (complex because we are studying Chern-Simons theory with complex gauge group) that depends on the 3-manifold M, the gauge group $G_{\mathbb{C}}$, and the classical solution ρ. However, for a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ and for the flat connection $\mathcal{A}^{(\mathrm{geom})}$ associated with the hyperbolic structure on M, we find that the $S_n^{(\mathrm{geom})}$'s are significantly more restricted.

Most basically, one might expect that the values of $S_n^{(\mathrm{geom})}$'s are *periods* [99]

$$S_n^{(\mathrm{geom})} \in \mathcal{P}. \tag{5.3.1}$$

Here $\mathcal{P}$ is the set of all periods, satisfying

$$\mathbb{Q} \subset \overline{\mathbb{Q}} \subset \mathcal{P} \subset \mathbb{C}. \tag{5.3.2}$$

By definition, a period is a complex number whose real and imaginary parts are (absolutely convergent) integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ defined by polynomial inequalities with rational coefficients [99]. Examples of periods are powers of π, special values of L-functions, and logarithmic Mahler measures of polynomials with integer coefficients. Thus, periods can be transcendental numbers, but they form a countable set, $\mathcal{P}$. Moreover, $\mathcal{P}$ is an algebra; a sum or a product of two periods is also a period.

Although the formulation of the perturbative invariants $S_{n\geq 1}^{(\mathrm{geom})}$ in terms of Feynman diagrams naturally leads to integrals of the form (5.2.4), which have the set of periods, $\mathcal{P}$, as their natural home, here we make a stronger claim and conjecture that for $n > 1$ the $S_n^{(\mathrm{geom})}$'s are algebraic numbers, *i.e.* they take values in $\overline{\mathbb{Q}}$. As we indicated in (5.3.2), the field $\overline{\mathbb{Q}}$ is contained in $\mathcal{P}$, but leads to a much stronger condition on the arithmetic nature of the perturbative invariants $S_n^{(\mathrm{geom})}$. In order to formulate a more precise conjecture, we introduce the following definition:

Definition: A perturbative quantum field theory is called *arithmetic* if, for all $n > 1$, the perturbative coefficients $S_n^{(\rho)}$ take values in some algebraic number field $\mathbb{K}$,

$$S_n^{(\rho)} \;\in\; \mathbb{K}\,, \tag{5.3.3}$$

and $S_1^{(\rho)} \in \mathbb{Q} \cdot \log \mathbb{K}\,.$

Therefore, to a manifold M and a classical solution ρ an arithmetic topological quantum field theory (arithmetic TQFT for short) associates an algebraic number field $\mathbb{K}$,

$$(M, \rho) \rightsquigarrow \mathbb{K}\,. \tag{5.3.4}$$

This is very reminiscent of arithmetic topology, a program proposed in the sixties by D. Mumford, B. Mazur, and Yu. Manin, based on striking analogies between number theory and low-dimensional topology. For instance, in arithmetic topology, 3-manifolds correspond to algebraic number fields and knots correspond to primes.

Usually, in perturbative quantum field theory the normalization of the expansion parameter is a matter of choice. Thus, in the notations of the present paper, a rescaling of the coupling constant $\hbar \to \lambda\hbar$ by a numerical factor λ is equivalent to a redefnition $S_n^{(\rho)} \to \lambda^{1-n} S_n^{(\rho)}$. While this transformation does not affect the physics of the perturbative expansion, it certainly is important for the arithmetic aspects discussed here. In particular, the above definition of arithmetic QFT is preserved by such a transformation only if $\lambda \in \mathbb{Q}$. In a theory with no canonical scale of $\hbar$, it is natural to choose it in such a way that makes the arithmetic nature of the perturbative coefficients $S_n^{(\rho)}$ as simple as possible. However, in some cases (which include Chern-Simons gauge theory), the coupling constant must obey certain quantization conditions, which, therefore, can lead to a "preferred" normalization of $\hbar$ up to irrelevant $\mathbb{Q}$-valued factors.

We emphasize that our definition of arithmetic QFT is perturbative. In particular, it depends on the choice of the classical solution ρ. In the present context of Chern-Simons gauge theory with complex gauge group $G_{\mathbb{C}}$, there is a natural choice of ρ when M is a geodesically complete hyperbolic 3-manifold, namely the geometric representation that corresponds to $\mathcal{A}^{(\text{geom})}$. In this case, we conjecture:

Arithmeticity conjecture: *As in* (5.2.6)*, let M be a geodesically complete hyperbolic 3-manifold of finite volume, and let $\rho = \text{geom}$ be the corresponding discrete faithful representation of $\pi_1(M)$ into $PSL(2,\mathbb{C})$. Then the perturbative Chern-Simons theory with complex gauge group $G_{\mathbb{C}} = PSL(2,\mathbb{C})$ (or its double cover, $SL(2,\mathbb{C})$) in the background of a non-trivial flat connection $\mathcal{A}^{(\text{geom})}$ is arithmetic on M.*

In fact, we can be a little bit more specific. In all the examples that we studied, we find that, for M as in (5.2.6) and for all values of $n > 1$, the perturbative invariants $S_n^{(\text{geom})}$ take values in the trace field of Γ,

$$S_n^{(\text{geom})} \in \mathbb{Q}(\text{tr}\Gamma)\,, \tag{5.3.5}$$

where, by definition, $\mathbb{Q}(\text{tr}\Gamma)$ is the minimal extension of $\mathbb{Q}$ containing $\text{tr}\gamma$ for all $\gamma \in \Gamma$. We conjecture that this is the case in general, namely that the $SL(2,\mathbb{C})$ Chern-Simons theory on a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is arithmetic with $\mathbb{K} = \mathbb{Q}(\text{tr}\Gamma)$. This should be contrasted with the case of a compact gauge group, where one usually develops perturbation theory in the background of a trivial flat connection, and the perturbative invariants $S_n^{(\rho)}$

turn out to be rational numbers.

Of course, in this conjecture it is important that the representation ρ is fixed, *e.g.* by the hyperbolic geometry of M as in the case at hand. As we shall see below, in many cases the representation ρ admits continuous deformations or, put differently, comes in a family. For geometric representations, this does not contradict the famous rigidity of hyperbolic structures because the deformations correspond to incomplete hyperbolic structures on M. In a sense, the second part of this section is devoted to studying such deformations. As we shall see, the perturbative $G_{\mathbb{C}}$ invariant $Z^{(\rho)}(M;\hbar)$ is a function of the deformation parameters, which on the *geometric branch*[1] can be interpreted as shape parameters of the associated hyperbolic structure.

In general, one might expect the perturbative coefficients $S_{n>1}^{(\rho)}$ to be rational functions of these shape parameters. Note that, if true, this statement would imply the above conjecture, since at the point corresponding to the complete hyperbolic structure the shape parameters take values in the trace field $\mathbb{Q}(\mathrm{tr}\Gamma)$. This indeed appears to be the case, at least for several simple examples of hyperbolic 3-manifolds that we have studied.

For the arithmeticity conjecture to hold, it is important that $\hbar$ is defined (up to $\mathbb{Q}$-valued factors) as in (5.1.8) on page 83, so that the leading term $S_0^{(\rho)}$ is a rational multiple of the classical Chern-Simons functional. This normalization is natural for a number of reasons. For example, it makes the arithmetic nature of the perturbative coefficients $S_n^{(\rho)}$ as clear as possible. Namely, according to the above arithmeticity conjecture, in this normalization $S_1^{(\mathrm{geom})}$ is a period, whereas $S_{n>1}^{(\mathrm{geom})}$ take values in $\overline{\mathbb{Q}}$. However, although we are not going to use it here, we note that another natural normalization of $\hbar$ could be obtained by a redefinition $\hbar \to \lambda\hbar$ with $\lambda \in (2\pi i)^{-1} \cdot \mathbb{Q}$. As we shall see below, this normalization is especially natural from the viewpoint of the analytic continuation approach. In this normalization, the arithmeticity conjecture says that all $S_{n>1}^{(\mathrm{geom})}$ are expected to be periods. More specifically, it says that $S_n^{(\rho)} \in (2\pi i)^{n-1} \cdot \overline{\mathbb{Q}}$, suggesting that the n-loop perturbative invariants $S_n^{(\mathrm{geom})}$ are periods of (framed) mixed Tate motives $\mathbb{Q}(n-1)$. In this form, the arithmeticity of perturbative Chern-Simons invariants discussed here is very similar to the motivic interpretation of Feynman integrals in [100].

Finally, we note that, for some applications, it may be convenient to normalize the path integral (1.0.2) by dividing the right-hand side by $Z(\mathbf{S}^3;\hbar)$. (Since $\pi_1(\mathbf{S}^3)$ is trivial, we have

[1] *i.e.* on the branch containing the discrete faithful representation.

$Z(\mathbf{S}^3;\hbar) = Z^{(0)}(\mathbf{S^3};\hbar)$.) This normalization does not affect the arithmetic nature of the perturbative coefficients $S_n^{(\mathrm{geom})}$ because, for $M = \mathbf{S}^3$, all the $S_n^{(0)}$'s are rational numbers. Specifically,

$$Z(\mathbf{S}^3;\hbar) \;=\; \left(\frac{\hbar}{i\pi}\right)^{r/2}\left(\frac{\mathrm{Vol}\Lambda_{\mathrm{wt}}}{\mathrm{Vol}\Lambda_{\mathrm{rt}}}\right)^{1/2}\prod_{\alpha\in\Lambda_{\mathrm{rt}}^+} 2\sinh\left(\hbar(\alpha\cdot\varrho)\right), \tag{5.3.6}$$

where the product is over positive roots $\alpha \in \Lambda_{\mathrm{rt}}^+$, r is the rank of the gauge group, and ϱ is half the sum of the positive roots, familiar from the Weyl character formula. Therefore, in the above conjecture and in eq. (5.3.5) we can use the perturbative invariants of M with either normalization.

The arithmeticity conjecture discussed here is a part of a richer structure: the quantum $G_{\mathbb{C}}$ invariants are only the special case $x = 0$ of a collection of functions indexed by rational numbers x which each have asymptotic expansions in $\hbar$ satisfying the arithmeticity conjecture and which have a certain kind of modularity behavior under the action of $SL(2,\mathbb{Z})$ on $\mathbb{Q}$ [101]. A better understanding of this phenomenon and its interpretation is a subject of ongoing research.

5.4 Examples

In this section, we explicitly calculate the perturbative $SL(2,\mathbb{C})$ invariants $S_n^{(\mathrm{geom})}$ for the simplest hyperbolic knot complements in S^3, *i.e.* for $M = S^3\backslash K$. It was conjectured in [40] that such invariants can be extracted from the asymptotic expansion of knot invariants computed by Chern-Simons theory with compact gauge group $G = SU(2)$ in a double-scaling limit

$$k\to\infty\,,\qquad \hbar = \frac{i\pi}{N}\to 0\,,\qquad \frac{N}{k+2}\equiv 1\,. \tag{5.4.1}$$

Here, k is the (unnormalized) level of the $SU(2)$ theory, and N is the dimension of an $SU(2)$ representation associated to a Wilson loop in S^3 supported on the given knot K. (This will be explained in much greater detail in Sections 6.2 and 7.4.) The relevant $SU(2)$ knot invariant is

$$\mathcal{J}(K;N) = \frac{J_N\big(K;\,q\big)}{J_N\big(\mathrm{unknot};\,q\big)}\,,\qquad q = e^{2\hbar} = e^{\frac{2\pi i}{N}}\,, \tag{5.4.2}$$

where $J_N(K;q)$ is the "N-colored Jones polynomial" of K, normalized such that

$$J_N(\mathrm{unknot};q) = \frac{q^{N/2}-q^{-N/2}}{q^{1/2}-q^{-1/2}} \tag{5.4.3}$$

is the "quantum dimension" of the N-dimensional representation of $SU(2)$.

According to the Generalized Volume Conjecture [40], the invariant $\mathcal{J}(K;N)$ should have the asymptotics

$$\mathcal{J}(K;N) \;\sim\; \frac{Z^{(\mathrm{geom})}(M;\, i\pi/N)}{Z(\mathbf{S}^1\times\mathbf{D}^2;\, i\pi/N)} \;\sim\; N^{3/2}\,\exp\left(\sum_{n=0}^{\infty} s_n \left(\frac{2\pi i}{N}\right)^{n-1}\right) \tag{5.4.4}$$

as $N\to\infty$, where

$$\begin{aligned} S_n^{(\mathrm{geom})} &= s_n\cdot 2^{n-1} \qquad (n\neq 1) \\ S_1^{(\mathrm{geom})} &= s_1 + \frac{1}{2}\log 2 \qquad (n=1) \end{aligned} \tag{5.4.5}$$

are the perturbative $SL(2,\mathbb{C})$ invariants of $M=\mathbf{S}^3\setminus K$. (Here, we view the solid torus, $\mathbf{S}^1\times\mathbf{D}^2$, as the complement of the unknot in the 3-sphere.) Specifically, one has

$$s_0 \;=\; i\big(\mathrm{Vol}(M) \,+\, i\,\mathrm{CS}(M)\big) \tag{5.4.6}$$

(Volume Conjecture) and s_1 is the Ray-Singer torsion of M twisted by a flat connection, *cf.* eq. (5.2.3). The Arithmeticity Conjecture of Section 5.3 predicts that

$$s_1 \;\in\; \mathbb{Q}\cdot\log\mathbb{K}\,, \qquad\qquad s_n \;\in\; \mathbb{K} \qquad (n\geq 2), \tag{5.4.7}$$

where $\mathbb{K}$ is the trace field of the knot. We will present numerical computations supporting this conjecture for the two simplest hyperbolic knots $\mathbf{4_1}$ and $\mathbf{5_2}$, following our work in [3].

The formulas for $\mathcal{J}(K;N)$ in both cases are known explicitly, see *e.g.* [102]. One has

$$\mathcal{J}(\mathbf{4_1};N) = \sum_{m=0}^{N-1}(q)_m(q^{-1})_m\,, \tag{5.4.8}$$

$$\mathcal{J}(\mathbf{5_2};N) = \sum_{m=0}^{N-1}\sum_{k=0}^{m} q^{-(m+1)k}(q)_m^2/(q^{-1})_k\,, \tag{5.4.9}$$

where $(q)_m=(1-q)\cdots(1-q^m)$ is the q-Pochhammer symbol as in Section 8.3. The first few values of these invariants are

N	$\mathcal{J}(\mathbf{4_1};N)$	$\mathcal{J}(\mathbf{5_2};N)$
1	1	1
2	5	7
3	13	$18-5q$
4	27	$40-23q$
5	$44-4q^2-4q^3$	$46-55q-31q^2+q^3$
6	89	$120-187q$
7	$100-14q^2-25q^3-25q^4-14q^5$	$-154q-88q^2+47q^3+58q^4+77q^5$
8	$187-45q-45q^3$	$-84-407q-150q^2+96q^3$

Using formula (5.4.8) for N of the order of 5000 and the numerical interpolation method explained in [103] and [104], the values of s_n for $0 \le n \le 27$ were computed to very high precision, resulting in

$$s_0 = \frac{1}{2\pi^2 i}\,D\big(e^{\pi i/3}\big), \qquad s_1 = -\frac{1}{4}\,\log 3\,, \tag{5.4.10}$$

the first in accordance with the volume conjecture and the second in accordance with the first statement of (5.4.7), since $\mathbb{K} = \mathbb{Q}\big(\sqrt{-3}\big)$ in this case. Moreover the numbers

$$s'_n = s_n \cdot \big(6\sqrt{-3}\big)^{n-1} \qquad (n \ge 2) \tag{5.4.11}$$

are very close to rational numbers with relatively small and highly factored denominators:

n	2	3	4	5	6	7	8	9
s'_n	$\frac{11}{12}$	2	$\frac{1081}{90}$	98	$\frac{110011}{105}$	$\frac{207892}{15}$	$\frac{32729683}{150}$	$\frac{139418294}{35}$

n	10	...	27
s'_n	$\frac{860118209659}{10395}$	...	$\frac{240605060980290369529478710291172763261781986098552}{814172781296875}$

confirming the second prediction in (5.4.7). Here $D(z)$ is the Bloch-Wigner dilogarithm (8.1.3), *cf.* Section 8.1. (That s_n is a rational multiple of $(\sqrt{-3})^{n-1}$, and not merely an element of $\mathbb{Q}(\sqrt{-3})$ is a consequence of (5.4.4) and the fact that $\mathcal{J}(\mathbf{4_1};N)$ is real.)

Actually, in this case one can prove the correctness of the expansion rigorously: the two formulas in (5.4.10) were proved in [105] and the rationality of the numbers s'_n defined by (5.4.11) in [106, 107] and [108], and can therefore check that the numerically determined values are the true ones (see also [101] for a generalization of this analysis). In the case of $\mathbf{5_2}$, such an analysis has not been done, and the numerical interpolation method is therefore needed. If one tries to do this directly using eq. (5.4.9), the process is very time-consuming because, unlike the figure-8 case, there are now $\mathrm{O}(N^2)$ terms. To get around this difficulty, we use the formula

$$\sum_{k=0}^{m} \frac{q^{(m+1)k}}{(q)_k} = (q)_m \sum_{k=0}^{m} \frac{q^{k^2}}{(q)_k^2}\,, \tag{5.4.12}$$

which is proved by observing that both sides vanish for $m=-1$ and satisfy the recursion $t_m = (1-q^m)t_{m-1} + q^{m^2}/(q)_m$. This proof gives a way to successively compute each t_m in $\mathrm{O}(1)$ steps (compute $y=q^m$ as q times the previous y, $(q)_m$ as $1-y$ times its previous value, and then t_m by the recursion) and hence to compute the whole sum in (5.4.9) in only $\mathrm{O}(N)$ steps. The interpolation method can therefore be carried out to just as high precision as in the figure-eight ($\mathbf{4_1}$) case.

The results are as follows. The first coefficient is given to high precision by

$$s_0 = -\frac{3}{2\pi}\Big(\mathrm{Li}_2(\alpha) + \frac{1}{2}\log(\alpha)\log(1-\alpha)\Big) + \frac{\pi}{3}\,, \tag{5.4.13}$$

in accordance with the prediction (5.4.6), where $\alpha = 0.87743\cdots - 0.74486\ldots i$ is the root of

$$\alpha^3 - \alpha^2 + 1 = 0 \tag{5.4.14}$$

with negative imaginary part. The next four values are (again numerically to very high precision)

$$s_1 = \frac{1}{4}\log\frac{1+3\alpha}{23}\,, \qquad s_2 = \frac{198\alpha^2 + 1452\alpha - 1999}{24\cdot 23^2},$$
$$s_3 = \frac{465\alpha^2 - 465\alpha + 54}{2\cdot 23^3}\,, \qquad s_4 = \frac{-2103302\alpha^2 + 55115\alpha + 5481271}{240\cdot 23^5}\,,$$

in accordance with the arithmeticity conjecture since $\mathbb{K} = \mathbb{Q}(\alpha)$ in this case.

These coefficients are already quite complicated, and the next values even more so. We can simplify them by making the rescaling

$$s'_n = s_n\,\lambda^{n-1} \tag{5.4.15}$$

($i.e.$, by expanding in powers of $2\pi i/\lambda N$ instead of $2\pi i/N$), where

$$\lambda = \alpha^5(3\alpha-2)^3 = \alpha^{-1}(\alpha^2-3)^3\,. \qquad (5.4.16)$$

(This number is a generator of $\mathfrak{p}^3$, where $\mathfrak{p} = (3\alpha - 2) = (\alpha^2 - 3)$ is the unique ramified prime ideal of $\mathbb{K}$, of norm 23.) We then find

$$\begin{aligned}
s'_2 &= -\tfrac{1}{24}\bigl(12\,\alpha^2 - 19\,\alpha + 86\bigr),\\
s'_3 &= -\tfrac{3}{2}\bigl(2\,\alpha^2 + 5\,\alpha - 4\bigr),\\
s'_4 &= \tfrac{1}{240}\bigl(494\,\alpha^2 + 12431\,\alpha + 1926\bigr),\\
s'_5 &= -\tfrac{1}{8}\bigl(577\,\alpha^2 - 842\,\alpha + 1497\bigr),\\
s'_6 &= \tfrac{1}{10080}\bigl(176530333\,\alpha^2 - 80229954\,\alpha - 18058879\bigr),\\
s'_7 &= -\tfrac{1}{240}\bigl(99281740\,\alpha^2 + 40494555\,\alpha + 63284429\bigr),\\
s'_8 &= -\tfrac{1}{403200}\bigl(3270153377244\,\alpha^2 - 4926985303821\,\alpha - 8792961648103\bigr),\\
s'_9 &= \tfrac{1}{13440}\bigl(9875382391800\,\alpha^2 - 939631794912\,\alpha - 7973863388897\bigr),\\
s'_{10} &= -\tfrac{1}{15966720}\bigl(188477928956464660\,\alpha^2 + 213430022592301436\,\alpha + 61086306651454303\bigr),\\
s'_{11} &= -\tfrac{1}{1209600}\bigl(517421716298434577\,\alpha^2 - 286061854126193276\,\alpha - 701171308042539352\bigr),
\end{aligned}$$

with much simpler coefficients than before, and with each denominator dividing $(n+2)!$. These highly nontrivial numbers give a strong experimental confirmation of the conjecture.

We observe that in both of the examples treated here the first statement of the conjecture (5.4.7) can be strengthened to

$$\exp(4s_1) \;\in\; \mathbb{K}\,. \qquad (5.4.17)$$

It would be interesting to know if the same statement holds for all hyperbolic knot complements (or even all hyperbolic 3-manifolds). Another comment in this vein is that (5.4.6) also has an arithmetic content: one knows that the right-hand side of this equation is in the image under the extended regulator map of an element in the Bloch group (or, equivalently, the third algebraic K-group) of the number field $\mathbb{K}$.

Chapter 6

Geometric quantization

In Chapter 5, we introduced Chern-Simons perturbation theory and described the traditional approach to computing perturbative partition functions via Feynman diagrams. Now we want to consider another approach, based on the quantization of moduli space spaces of flat connections. Combined with the existence of a perturbative expansions (5.1.9), it will yield a powerful method for calculating perturbative invariants (Section 6.3). We will also find that quantization of Chern-Simons theory with complex gauge group $G_{\mathbb{C}}$ is closely related to quantization for compact group G, justifying a third approach to computing $G_{\mathbb{C}}$ invariants: "analytic continuation" (Section 6.2). Here, we mainly use analytic continuation to find the operators $\hat{A}_i$ that annihilate $G_{\mathbb{C}}$ partition functions, as explained in the introduction. In Chapter 9, we will revisit analytic continuation, employing it more directly to find classical integral expressions for partition functions.

Most of the results in this section follow [3]. We also include a short discussion of "brane quantization" for Chern-Simons theory with complex gauge group from [3], as it is closely related to geometric quantization.

6.1 Quantization of $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$

As a TQFT, Chern-Simons gauge theory (with any gauge group) associates a Hilbert space $\mathcal{H}_\Sigma$ to a closed Riemann surface Σ and a vector in $\mathcal{H}_\Sigma$ to every 3-manifold M with boundary Σ. We denote this vector as $|M\rangle \in \mathcal{H}_\Sigma$. If there are two such manifolds, M_+ and M_-, glued along a common boundary Σ (with matching orientation), then the quantum invariant $Z(M)$ that Chern-Simons theory associates to the closed 3-manifold $M = M_+ \cup_\Sigma M_-$ is given by the inner product of two vectors $|M_+\rangle$ and $|M_-\rangle$ in $\mathcal{H}_\Sigma$

$$Z(M) \;=\; \langle M_+ | M_- \rangle\,. \tag{6.1.1}$$

Therefore, in what follows, our goal will be to understand Chern-Simons gauge theory on manifolds with boundary, from which invariants of closed manifolds without boundary can be obtained via (6.1.1).

Since the Chern-Simons action (5.1.1) is first order in derivatives, the Hilbert space $\mathcal{H}_\Sigma$ is obtained by quantizing the classical phase space, which is the space of classical solutions on the 3-manifold $\mathbb{R} \times \Sigma$. Such classical solutions are given precisely by the flat connections on the Riemann surface Σ. Therefore, in a theory with complex gauge group $G_\mathbb{C}$, the classical phase space is the moduli space of flat $G_\mathbb{C}$ connections on Σ, modulo gauge equivalence,

$$\mathcal{M}_{\text{flat}}(G_\mathbb{C}, \Sigma) \;=\; \text{Hom}(\pi_1(\Sigma), G_\mathbb{C})/\text{conj}. \tag{6.1.2}$$

As a classical phase space, $\mathcal{M}_{\text{flat}}(G_\mathbb{C}, \Sigma)$ comes equipped with a symplectic structure ω, which can also be deduced from the classical Chern-Simons action (5.1.1). Since we are interested only in the "holomorphic" sector of the theory, we shall look only at the kinetic term for the field $\mathcal{A}$ (and not $\bar{\mathcal{A}}$); it leads to a holomorphic symplectic 2-form on $\mathcal{M}_{\text{flat}}(G_\mathbb{C}, \Sigma)$:

$$\omega \;=\; \frac{i}{4\hbar} \int_\Sigma \text{Tr}\; \delta\mathcal{A} \wedge \delta\mathcal{A}\,. \tag{6.1.3}$$

We note that this symplectic structure does not depend on the complex structure of Σ, in accord with the topological nature of the theory. Then, in Chern-Simons theory with complex gauge group $G_\mathbb{C}$, the Hilbert space $\mathcal{H}_\Sigma$ is obtained by quantizing the moduli space of flat $G_\mathbb{C}$ connections on Σ with symplectic structure (6.1.3):

$$\text{quantization of } (\mathcal{M}_{\text{flat}}(G_\mathbb{C}, \Sigma), \omega) \quad \rightsquigarrow \quad \mathcal{H}_\Sigma\,. \tag{6.1.4}$$

Now, let us consider a closed 3-manifold M with boundary $\Sigma = \partial M$, and its associated state $|M\rangle \in \mathcal{H}_\Sigma$. In a (semi-)classical theory, quantum states correspond to Lagrangian submanifolds of the classical phase space. Recall that, by definition, a Lagrangian submanifold $\mathcal{L}$ is a middle-dimensional submanifold such that the restriction of ω to $\mathcal{L}$ vanishes,

$$\omega|_\mathcal{L} \;=\; 0\,. \tag{6.1.5}$$

For the problem at hand, the phase space is $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ and the Lagrangian submanifold $\mathcal{L}$ associated to a 3-manifold M with boundary $\Sigma = \partial M$ consists of flat connections on Σ that can be extended to classical solutions on all of M [40]. Since the space of classical solutions on M is the moduli space of flat $G_{\mathbb{C}}$ connections on M,

$$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M) = \text{Hom}(\pi_1(M), G_{\mathbb{C}})/\text{conj.}, \tag{6.1.6}$$

it follows that

$$\mathcal{L} \ = \ \iota\big(\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M)\big) \tag{6.1.7}$$

is the image of $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M)$ under the map

$$\iota : \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M) \to \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma) \tag{6.1.8}$$

induced by the natural inclusion $\pi_1(\Sigma) \to \pi_1(M)$. One can show that $\mathcal{L} \subset \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ is indeed Lagrangian with respect to the symplectic structure (6.1.3).

Much of what we described so far is very general and has an obvious analogue in Chern-Simons theory with arbitrary gauge group. However, quantization of Chern-Simons theory with complex gauge group has a number of good properties. In this case the classical phase space $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ is an algebraic variety; it admits a complete hyper-Kähler metric [109], and the Lagrangian submanifold $\mathcal{L}$ is a holomorphic subvariety of $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$. The hyper-Kähler structure on $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ can be obtained by interpreting it as the moduli space $\mathcal{M}_H(G, \Sigma)$ of solutions to Hitchin's equations on Σ. Note that this requires a choice of complex structure on Σ, whereas $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma) \cong \mathcal{M}_H(G, \Sigma)$ as a complex symplectic manifold does not. Existence of a hyper-Kähler structure on $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ considerably simplifies the quantization problem in any of the existing frameworks, such as geometric quantization [45], deformation quantization [46, 47], or "brane quantization" [48].

The hyper-Kähler moduli space $\mathcal{M}_H(G, \Sigma)$ has three complex structures that we denote as I, J, and $K = IJ$, and three corresponding Kähler forms, ω_I, ω_J, and ω_K (*cf.* [110]). In the complex structure usually denoted by J, $\mathcal{M}_H(G, \Sigma)$ can be identified with $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ as a complex symplectic manifold with the holomorphic symplectic form (6.1.3),

$$\omega \ = \ \frac{1}{\hbar}(\omega_K + i\omega_I)\,. \tag{6.1.9}$$

Moreover, in this complex structure, $\mathcal{L}$ is an algebraic subvariety of $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$. To be more precise, it is a (finite) union of algebraic subvarieties, each of which is defined by polynomial equations $A_i = 0$. In the quantum theory, these equations are replaced by corresponding operators $\widehat{A}_i$ acting on $\mathcal{H}_\Sigma$ that annihilate the state $|M\rangle$.

Now, let us consider in more detail the simple but important case when Σ is of genus 1, that is $\Sigma = T^2$. In this case, $\pi_1(\Sigma) \cong \mathbb{Z} \times \mathbb{Z}$ is abelian, and

$$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, T^2) \;=\; (\mathbb{T}_{\mathbb{C}} \times \mathbb{T}_{\mathbb{C}})/\mathcal{W} \,, \tag{6.1.10}$$

where $\mathbb{T}_{\mathbb{C}}$ is the maximal torus of $G_{\mathbb{C}}$ and $\mathcal{W}$ is the Weyl group. We parametrize each copy of $\mathbb{T}_{\mathbb{C}}$ by complex variables $l = (l_1, \ldots, l_r)$ and $m = (m_1, \ldots, m_r)$, respectively. Here, r is the rank of the gauge group $G_{\mathbb{C}}$. The values of l and m are eigenvalues of the holonomies of the flat $G_{\mathbb{C}}$ connection over the two basic 1-cycles of $\Sigma = T^2$. They are defined up to Weyl transformations, which act diagonally on $\mathbb{T}_{\mathbb{C}} \times \mathbb{T}_{\mathbb{C}}$.

The moduli space of flat $G_{\mathbb{C}}$ connections on a 3-manifold M with a single toral boundary, $\partial M = T^2$, defines a complex Lagrangian submanifold

$$\mathcal{L} \subset (\mathbb{T}_{\mathbb{C}} \times \mathbb{T}_{\mathbb{C}})/\mathcal{W} \,. \tag{6.1.11}$$

(More precisely, this Lagrangian submanifold comprises the top-dimensional (stable) components of the moduli space of flat $G_{\mathbb{C}}$ connections on Σ.) In particular, a generic irreducible component of $\mathcal{L}$ is defined by r polynomial equations

$$A_i(l, m) \;=\; 0 \,, \qquad i = 1, \ldots, r \,, \tag{6.1.12}$$

which must be invariant under the action of the Weyl group $\mathcal{W}$ (which simultaneously acts on the eigenvalues $l_1, \ldots, l_r$ and $m_1, \ldots, m_r$).

In the quantum theory, the equations (6.1.12) are replaced by the operator equations,

$$\widehat{A}_i(\hat{l}, \hat{m}) \; Z(M) \;=\; 0 \,, \qquad i = 1, \ldots, r \,. \tag{6.1.13}$$

For $\Sigma = T^2$ the complex symplectic structure (6.1.3) takes a very simple form

$$\omega \;=\; \frac{i}{\hbar} \sum_{i=1}^{r} du_i \wedge dv_i \,, \tag{6.1.14}$$

where we introduce new variables u and v (defined modulo elements of the cocharacter lattice $\Lambda_{\text{cochar}} \;=\; \text{Hom}(U(1), \mathbb{T})$), such that $l = -e^v$ and $m = e^u$. In the quantum theory, u and v are replaced by operators $\hat{u}$ and $\hat{v}$ that obey the canonical commutation relations

$$[\hat{u}_i, \hat{v}_j] \;=\; -\hbar \, \delta_{ij} \,. \tag{6.1.15}$$

As one usually does in quantum mechanics, we can introduce a complete set of states $|u\rangle$ on which $\hat{u}$ acts by multiplication, $\hat{u}_i|u\rangle = u_i|u\rangle$. Similarly, we let $|v\rangle$ be a complete basis, such that $\hat{v}_i|v\rangle = v_i|v\rangle$. Then, we can define the wave function associated to a 3-manifold M either in the u-space or v-space representation, respectively, as $\langle u|M\rangle$ or $\langle v|M\rangle$. We will mostly work with the former and let $Z(M;u) := \langle u|M\rangle$.

We note that a generic value of u does not uniquely specify a flat $G_{\mathbb{C}}$ connection on M or, equivalently, a unique point on the representation variety (6.1.7). Indeed, for a generic value of u, equations (6.1.12) may have several solutions that we label by a discrete parameter α. Therefore, in the u-space representation, flat $G_{\mathbb{C}}$ connections on M (previously labeled by the homomorphism $\rho \in \mathcal{L}$) are now labeled by a set of continuous parameters $u = (u_1, \ldots, u_r)$ and a discrete parameter α:

$$\rho \quad \longleftrightarrow \quad (u, \alpha)\,. \tag{6.1.16}$$

The perturbative $G_{\mathbb{C}}$ invariant $Z^{(\rho)}(M;\hbar)$ can then be written in this notation as $Z^{(\alpha)}(M;\hbar,u) = \langle u,\alpha|M\rangle$. Similarly, the coefficients $S_n^{(\rho)}$ in the $\hbar$-expansion can be written as $S_n^{(\alpha)}(u)$.

To summarize, in the approach based on quantization of $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}},\Sigma)$ the calculation of $Z^{(\rho)}(M;\hbar)$ reduces to two main steps: *i*) the construction of quantum operators $\widehat{A}_i(\hat{l},\hat{m})$, and *ii*) the solution of Schrödinger-like equations (6.1.13). Below we explain how to implement each of these steps.

6.2 Analytic continuation and the Volume Conjecture

For a generic 3-manifold M with boundary $\Sigma = T^2$, constructing the quantum operators $\widehat{A}_i(\hat{l},\hat{m})$ may be a difficult task. However, when M is the complement of a knot K in the 3-sphere,

$$M = \mathbf{S}^3 \setminus K\,, \tag{6.2.1}$$

there is a simple way to find the $\widehat{A}_i$'s. Indeed, as anticipated in the introduction, these operators also annihilate the polynomial knot invariants $P_{G,R}(K;q)$, which are defined in terms of Chern-Simons theory with compact gauge group G,

$$\widehat{A}_i(\hat{l},\hat{m})\; P_{G,R}(K;q) = 0\,. \tag{6.2.2}$$

The operator $\hat{l}_i$ acts on the set of polynomial invariants $\{P_{G,R}(K;q)\}$ by shifting the highest weight $\lambda = (\lambda_1, \ldots, \lambda_r)$ of the representation R by the i-th basis elements of the weight lattice Λ_{wt}, while the operator $\hat{m}_j$ acts simply as multiplication by $q^{\lambda_j/2}$. Let us briefly explain how this comes about.

In general, the moduli space $\mathcal{M}_{\mathrm{flat}}(G_{\mathbb{C}}, \Sigma)$ is a complexification of $\mathcal{M}_{\mathrm{flat}}(G, \Sigma)$. The latter is the classical phase space in Chern-Simons theory with compact gauge group G and can be obtained from $\mathcal{M}_{\mathrm{flat}}(G_{\mathbb{C}}, \Sigma)$ by requiring all the holonomies to be "real," *i.e.* in G. Similarly, restricting to real holonomies in the definition of $\mathcal{L}$ produces a Lagrangian submanifold in $\mathcal{M}_{\mathrm{flat}}(G, \Sigma)$ that corresponds to a quantum state $|M\rangle$ in Chern-Simons theory with compact gauge group G. In the present example of knot complements, restricting to such "real" holonomies means replacing $\mathbb{T}_{\mathbb{C}}$ by $\mathbb{T}$ in (6.1.10) and taking purely imaginary values of u_i and v_i in equations (6.1.12). Apart from this, the quantization problem is essentially the same for gauge groups G and $G_{\mathbb{C}}$. In particular, the symplectic structure (6.1.14) has the same form (with imaginary u_i and v_i in the theory with gauge group G) and the quantum operators $\widehat{A}_i(\hat{l}, \hat{m})$ annihilate both $P_{G,R}(K;q)$ and $Z^{(\alpha)}(M;\hbar,u)$ computed, respectively, by Chern-Simons theories with gauge groups G and $G_{\mathbb{C}}$.

In order to understand the precise relation between the parameters in these theories, let us consider a Wilson loop operator, $W_R(K)$, supported on $K \subset \mathbf{S}^3$ in Chern-Simons theory with compact gauge group G. It is labeled by a representation $R = R_\lambda$ of the gauge group G, which we assume to be an irreducible representation with highest weight $\lambda \in \mathfrak{t}^\vee$. The path integral in Chern-Simons theory on $\mathbf{S}^3$ with a Wilson loop operator $W_R(K)$ computes the polynomial knot invariant $P_{G,R}(K;q)$, with $q = e^{2\hbar}$. Using (6.1.1), we can represent this path integral as $\langle R|M\rangle$, where $|R\rangle$ is the result of the path integral on a solid torus containing a Wilson loop $W_R(K)$, and $|M\rangle$ is the path integral on its complement, M.

In the semi-classical limit, the state $|R\rangle$ corresponds to a Lagrangian submanifold of $\mathcal{M}_{\mathrm{flat}}(G, T^2) = (\mathbb{T} \times \mathbb{T})/\mathcal{W}$ defined by the fixed value of the holonomy $m = e^u$ on a small loop around the knot. The relation between $m = e^u$, which is an element of the maximal torus $\mathbb{T}$ of G, and the representation R_λ is given by the invariant quadratic form $-\mathrm{Tr}$ (restricted to $\mathfrak{t}$). Specifically,

$$m \;=\; \exp \hbar(\lambda^* + \rho^*) \in \mathbb{T}\,, \tag{6.2.3}$$

where λ^* is the unique element of $\mathfrak{t}$ such that $\lambda^*(x) = -\mathrm{Tr}\,\lambda x$ for all $x \in \mathfrak{t}$, and ρ^* is the analogous dual of the Weyl vector (half the sum of positive weights of G). For example, for the N-colored Jones polynomial, corresponding to the N-dimensional representation of $SU(2)$, (6.2.3) looks like $m = \exp \hbar N$. In "analytic continuation," we analytically continue this relation to $m \in \mathbb{T}_{\mathbb{C}}$. A perturbative partition function arises as

$$\hbar \to 0\,, \qquad \lambda^* \to \infty \qquad \text{for } \; u = \hbar(\lambda^* + \rho^*) \text{ fixed}\,. \tag{6.2.4}$$

A more detailed explanation of the limit (6.2.3), will appear in Chapter 7, where we discuss Wilson loops for complex as well as compact groups.

For a given value of $m = e^u$, equations (6.1.12) have a finite set of solutions l_α, labeled by α. Only for one particular value of α is the perturbative $G_{\mathbb{C}}$ invariant $Z^{(\alpha)}(M;\hbar,u)$ related to the asymptotic behavior of $P_{G,R}(K;q)$. This is the value of α which maximizes $\mathrm{Re}\big(S_0^{(\alpha)}(M;u)\big)$. For this α, we have:

$$\frac{Z^{(\alpha)}(M;\hbar,u)}{Z(\mathbf{S}^3;\hbar)} \;=\; \text{asymptotic expansion of } P_{G,R}(K;q)\,. \tag{6.2.5}$$

For hyperbolic knots and u sufficiently close to 0, this "maximal" value of α is always $\alpha = \mathrm{geom}$.

It should nevertheless be noted that the analytic continuation described here is not as "analytic" as it sounds. In particular, the limit (6.2.4) is very subtle and requires much care. As explained in [40], in taking this limit it is important that values of $q = e^{2\hbar}$ avoid roots of unity. If one takes the limit with $\hbar^{-1} \in \frac{1}{i\pi}\mathbb{Z}$, which corresponds to the allowed values of the coupling constant in Chern-Simons theory with compact gauge group G, then one can never see the exponential asymptotics (5.1.9) with $\mathrm{Im}(S_0^{(\rho)}) > 0$. The exponential growth characteristic to Chern-Simons theory with complex gauge group emerges only in the limit with $\hbar = u/(\lambda^* + \rho^*)$ and u generic.[1]

6.3 Hierarchy of differential equations

The system of Schrödinger-like equations (6.1.13) determines the perturbative $G_{\mathbb{C}}$ invariant $Z^{(\alpha)}(M;\hbar,u)$ up to multiplication by an overall function of $\hbar$, which can be fixed by suitable boundary conditions.

[1]The subtle behavior of the compact Chern-Simons partition function as the level $k = i\pi/\hbar$ and the representation λ^* are continued away from integers has recently been analyzed carefully in [41]. In particular, the transition from polynomial to exponential growth is explained there.

In order to see in detail how the perturbative coefficients $S_n^{(\alpha)}(u)$ may be calculated and to avoid cluttering, let us assume that the rank $r = 1$. (A generalization to arbitrary values of r is straightforward.) In this case, $A(l, m)$ is the so-called A-polynomial of M, originally introduced in [111], and the system (6.1.13) consists of a single equation

$$\widehat{A}(\hat{l}, \hat{m})\, Z^{(\alpha)}(M; \hbar, u) \;=\; 0\,. \tag{6.3.1}$$

In the u-space representation the operator $\hat{m} = \exp(\hat{u})$ acts on functions of u simply via multiplication by e^u, whereas $\hat{l} = \exp(\hat{v} + i\pi) = \exp(\hbar \frac{d}{du})$ acts as a "shift operator":

$$\hat{m} f(u) \;=\; e^u f(u)\,, \qquad \hat{l} f(u) \;=\; f(u+\hbar)\,. \tag{6.3.2}$$

In particular, the operators $\hat{l}$ and $\hat{m}$ obey the relation

$$\hat{l}\hat{m} \;=\; q^{\frac{1}{2}} \hat{m}\hat{l}\,, \tag{6.3.3}$$

which follows directly from the commutation relation (6.1.15) for $\hat{u}$ and $\hat{v}$, with

$$q \;=\; e^{2\hbar}\,. \tag{6.3.4}$$

We would like to recast eq. (6.3.1) as an infinite hierarchy of differential equations that can be solved recursively for the perturbative coefficients $S_n^{(\alpha)}(u)$. Just like its classical limit $A(l, m)$, the operator $\widehat{A}(\hat{l}, \hat{m})$ is a polynomial in $\hat{l}$. Therefore, pushing all operators $\hat{l}$ to the right, we can write it as

$$\widehat{A}(\hat{l}, \hat{m}) \;=\; \sum_{j=0}^{d} a_j(\hat{m}, \hbar)\, \hat{l}^j \tag{6.3.5}$$

for some functions $a_j(m, \hbar)$ and some integer d. Using (6.3.2), we can write eq. (6.3.1) as

$$\sum_{j=0}^{d} a_j(m, \hbar) Z^{(\alpha)}(M; \hbar, u + j\hbar) \;=\; 0\,. \tag{6.3.6}$$

Then, substituting the general form (5.1.9) of $Z^{(\alpha)}(M; \hbar, u)$, we obtain the equation

$$\sum_{j=0}^{d} a_j(m, \hbar) \exp\left[\frac{1}{\hbar} S_0^{(\alpha)}(u + j\hbar) - \frac{1}{2}\delta^{(\alpha)} \log\hbar + \sum_{n=0}^{\infty} \hbar^n S_{n+1}^{(\alpha)}(u + j\hbar)\right] \;=\; 0\,. \tag{6.3.7}$$

Since $\delta^{(\alpha)}$ is independent of u, we can just factor out the $-\frac{1}{2}\delta^{(\alpha)} \log\hbar$ term and remove it from the exponent. Now we expand everything in $\hbar$. Let

$$a_j(m, \hbar) \;=\; \sum_{p=0}^{\infty} a_{j,p}(m) \hbar^p \tag{6.3.8}$$

and

$$\sum_{n=-1}^{\infty} \hbar^n S_{n+1}(u+j\hbar) \;=\; \sum_{r=-1}^{\infty}\sum_{m=-1}^{r} \frac{j^{r-m}}{(r-m)!}\hbar^r S_{m+1}^{(r-m)}(u)\,, \tag{6.3.9}$$

suppressing the index α to simplify notation. We can substitute (6.3.8) and (6.3.9) into (6.3.7) and divide the entire expression by $\exp(\sum_n \hbar^n S_{n+1}(u))$. The hierarchy of equations then follows by expanding the exponential in the resulting expression as a series in $\hbar$ and requiring that the coefficient of every term in this series vanishes (see also [112]). The first four equations are shown in Table 6.1.

$$\sum_{j=0}^{d} e^{jS_0'} a_{j,0} = 0$$

$$\sum_{j=0}^{d} e^{jS_0'}\Big(a_{j,1} + a_{j,0}\big(\tfrac{1}{2}j^2 S_0'' + jS_1'\big)\Big) = 0$$

$$\sum_{j=0}^{d} e^{jS_0'}\Big(a_{j,2} + a_{j,1}\big(\tfrac{1}{2}j^2 S_0'' + jS_1'\big) + a_{j,0}\Big(\frac{1}{2}\big(\frac{1}{2}j^2 S_0'' + jS_1'\big)^2 + \frac{j^3}{6}S_0''' + \frac{j^2}{2}S_1'' + jS_2'\Big)\Big) = 0$$

$$\begin{aligned}\sum_{j=0}^{d} e^{jS_0'}\Big(&a_{j,3} + a_{j,2}\big(\tfrac{1}{2}j^2 S_0'' + jS_1'\big) + a_{j,1}\Big(\frac{1}{2}\big(\frac{1}{2}j^2 S_0'' + jS_1'\big)^2 + \frac{j^3}{6}S_0''' + \frac{j^2}{2}S_1'' + jS_2'\Big) \\ &+ a_{j,0}\Big(\frac{1}{6}\big(\frac{1}{2}j^2 S_0'' + jS_1'\big)^3 + \big(\frac{1}{2}j^2 S_0'' + jS_1'\big)\big(\frac{j^3}{6}S_0''' + \frac{j^2}{2}S_1'' + jS_2'\big) \\ &+ \frac{j^4}{24}S_0^{(4)} + \frac{j^3}{6}S_1''' + \frac{j^2}{2}S_2'' + jS_3'\Big)\Big) = 0\end{aligned}$$

$$\cdots$$

Table 6.1: Hierarchy of differential equations derived from $\widehat{A}(\hat{l},\hat{m})\; Z^{(\alpha)}(M;\hbar,u) = 0$.

The equations in Table 6.1 can be solved recursively for the $S_n(u)$'s, since each S_n first appears in the $(n+1)^{\text{st}}$ equation, differentiated only once. Indeed, after S_0 is obtained, the remaining equations feature the $S_{n\geq 1}$ linearly the first time they occur, and so determine these coefficients uniquely up to an additive constant of integration.

The first equation, however, is somewhat special. Since $a_{j,0}(m)$ is precisely the coefficient of l^j in the classical A-polynomial $A(l,m)$, we can rewrite this equation as

$$A(e^{S_0'(u)}, e^u) = 0\,. \tag{6.3.10}$$

This is exactly the classical constraint $A(e^{v+i\pi}, e^u) = 0$ that defines the complex Lagrangian submanifold $\mathcal{L}$, with $S_0'(u) = v + i\pi$. Therefore, we can integrate along a branch ($l_\alpha =$

$e^{v_\alpha + i\pi}, m = e^u$) of $\mathcal{L}$ to get the value of the classical Chern-Simons action (5.2.1) [40]

$$S_0^{(\alpha)}(u) = \text{const} + \int^u \theta|_{\mathcal{L}} \,, \tag{6.3.11}$$

where $\theta|_{\mathcal{L}}$ denotes a restriction to $\mathcal{L}$ of the Liouville 1-form on $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$,

$$\theta = v du + i\pi du \,. \tag{6.3.12}$$

The expression (6.3.11) is precisely the semi-classical approximation to the wave function $Z^{(\alpha)}(M; \hbar, u)$ supported on the Lagrangian submanifold $\mathcal{L}$, obtained in the WKB quantization of the classical phase space $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$. By definition, the Liouville form θ (associated with a symplectic structure ω) obeys $d\theta = i\hbar\omega$, and it is easy to check that this is indeed the case for the forms ω and θ on $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ given by eqs. (6.1.14) and (6.3.12), respectively.

The semi-classical expression (6.3.11) gives the value of the classical Chern-Simons functional (5.1.1) evaluated on a flat gauge connection $\mathcal{A}^{(\rho)}$, labeled by a homomorphism ρ. As we explained in (6.1.16), the dependence on ρ is encoded in the dependence on a continuous holonomy parameter u, as well as a discrete parameter α that labels different solutions $v_\alpha(u)$ to $A(e^{v+i\pi}, e^u) = 0$, at a fixed value of u. In other words, α labels different branches of the Riemann surface $A(l, m) = 0$, regarded as a cover of the complex plane $\mathbb{C}$ parametrized by $m = e^u$,

$$A(l_\alpha, m) = 0 \,. \tag{6.3.13}$$

Since $A(l, m)$ is a polynomial in both l and m, the set of values of α is finite (in fact, its cardinality is equal to the degree of $A(l, m)$ in l). Note, however, that for a given choice of α there are infinitely many ways to lift a solution $l_\alpha(u)$ to $v_\alpha(u)$; namely, one can add to $v_\alpha(u)$ any integer multiple of $2\pi i$. This ambiguity implies that the integral (6.3.11) is defined only up to integer multiples of $2\pi i u$,

$$S_0^{(\alpha)}(u) - \text{const} = \int^u \log l_\alpha(u') \, du' = \int^u v_\alpha(u') \, du' \; + \; i\pi u \qquad (\text{mod } 2\pi i u) \,. \tag{6.3.14}$$

In practice, this ambiguity can always be fixed by imposing suitable boundary conditions on $S_0^{(\alpha)}(u)$, and it never affects the higher-order terms $S_n^{(\alpha)}(u)$. Therefore, since our main goal is to solve the *quantum* theory (to all orders in perturbation theory) we shall not worry about this ambiguity in the classical term. As we illustrate later (see Section 6.6), it will always be easy to fix this ambiguity in concrete examples.

Before we proceed, let us remark that if M is a hyperbolic 3-manifold with a single torus boundary $\Sigma = \partial M$ and $\mathcal{A}^{(\text{geom})}$ is the "geometric" flat $SL(2, \mathbb{C})$ connection associated with a hyperbolic metric on M (not necessarily geodesically complete), then the integral (6.3.14) is essentially the complexified volume function, $i(\mathrm{Vol}(M;u) + i\mathrm{CS}(M;u))$, which combines the (real) hyperbolic volume and Chern-Simons invariants[2] of M. Specifically, the relation is [40, 114]:

$$S_0^{(\mathrm{geom})}(u) = \frac{i}{2}\Big[\mathrm{Vol}(M;u) + i\mathrm{CS}(M;u)\Big] + v_{\mathrm{geom}}(u)\,\mathrm{Re}(u) \,+\, i\pi u\,, \tag{6.3.15}$$

modulo the integration constant and multiples of $2\pi i u$.

6.4 Classical and quantum symmetries

A vast supply of 3-manifolds with a single toral boundary can be obtained by considering knot complements as in (6.2.1); our main examples throughout this thesis are of this type. As discussed above, the Lagrangian subvariety $\mathcal{L} \in \mathcal{M}_{\mathrm{flat}}(G_{\mathbb{C}}, \Sigma)$ for any knot complement M is defined by polynomial equations (6.1.12). Such an $\mathcal{L}$ contains multiple branches, indexed by α, corresponding to the different solutions to $\{A_i(l,m) = 0\}$ for fixed m. In this section, we describe relationships among these branches and the corresponding perturbative invariants $Z^{(\alpha)}(M;\hbar,u)$ by using the symmetries of Chern-Simons theory with complex gauge group $G_{\mathbb{C}}$.

Before we begin, it is useful to summarize what we already know about the branches of $\mathcal{L}$. As mentioned in the previous discussion, there always exists a geometric branch — or in the case of rank $r > 1$ several geometric branches — when M is a hyperbolic knot complement. Like the geometric branch, most other branches of $\mathcal{L}$ correspond to genuinely nonabelian representations $\rho : \pi_1(\Sigma) \to G_{\mathbb{C}}$. However, for any knot complement M there also exists an "abelian" component of $\mathcal{L}$, described by the equations

$$l_1 = \ldots = l_r = 1\,. \tag{6.4.1}$$

Indeed, since $H_1(M)$ is the abelianization of $\pi_1(M)$, the representation variety (6.1.6) always

[2]Here u describes the cusp of a hyperbolic metric on M. For example, imaginary u parametrizes a conical singularity. See, *e.g.* [113, 114, 115] and our discussion in Section 8.1 for more detailed descriptions of $\mathrm{Vol}(M;u)$ and $\mathrm{CS}(M;u)$. In part of the literature (*e.g.* in [114]), the parameters (u,v) are related to those used here by $2u_{\mathrm{here}} = u_{\mathrm{there}}$ and $2(v_{\mathrm{here}} + i\pi) = v_{\mathrm{there}}$. We include a shift of $i\pi$ in our definition of v so that the complete hyperbolic structure arises at $(u,v) = (0,0)$.

has a component corresponding to abelian representations that factor through $H_1(M) \cong \mathbb{Z}$,

$$\pi_1(M) \to H_1(M) \to G_{\mathbb{C}}\,. \tag{6.4.2}$$

The corresponding flat connection, $\mathcal{A}^{(\text{abel})}$, is characterized by the trivial holonomy around a 1-cycle of $\Sigma = T^2$ which is trivial in homology $H_1(M)$; choosing it to be the 1-cycle whose holonomy was denoted by $l = (l_1, \ldots, l_r)$ we obtain (6.4.1). Note that, under projection to the u-space, the abelian component of $\mathcal{L}$ corresponds to a single branch that we denote by $\alpha = \text{abel}$.

The first relevant symmetry of Chern-Simons theory with complex gauge group $G_{\mathbb{C}}$ is conjugation. We observe that for every flat connection $\mathcal{A}^{(\rho)}$ on M, with $\rho = (u, \alpha)$, there is a conjugate flat connection $\mathcal{A}^{(\bar{\rho})} := \overline{\mathcal{A}^{(\rho)}}$ corresponding to a homomorphism $\bar{\rho} = (\bar{u}, \bar{\alpha})$. We use $\bar{\alpha}$ to denote the branch of $\mathcal{L}$ "conjugate" to branch α; the fact that branches of $\mathcal{L}$ come in conjugate pairs is reflected in the fact that eqs. (6.1.12) have real (in fact, integer) coefficients. The perturbative expansions around $\mathcal{A}^{\rho}$ and $\mathcal{A}^{\bar{\rho}}$ are very simply related. Namely, by directly conjugating the perturbative path integral and noting that the Chern-Simons action has real coefficients, we find[3] $\overline{Z^{\alpha}(M;\hbar,u)} = Z^{(\bar{\alpha})}(M;\bar{\hbar},\bar{u})$. The latter partition function is actually in the antiholomorphic sector of the Chern-Simons theory, but we can just rename $(\bar{\hbar}, \bar{u}) \mapsto (\hbar, u)$ (and use analyticity) to obtain a perturbative partition function for the conjugate branch in the holomorphic sector,

$$Z^{(\bar{\alpha})}(M;\hbar,u) = \overline{Z^{(\alpha)}}(M;\hbar,u)\,. \tag{6.4.3}$$

Here, for any function $f(z)$ we define $\bar{f}(z) := \overline{f(\bar{z})}$. In particular, if f is analytic, $f(z) = \sum f_n z^n$, then $\bar{f}$ denotes a similar function with conjugate coefficients, $\bar{f}(z) = \sum \bar{f}_n z^n$.

In the case $r = 1$, the symmetry (6.4.3) implies that branches of the classical $SL(2,\mathbb{C})$ A-polynomial come in conjugate pairs v_α and $v_{\bar{\alpha}}(u) = \overline{v_\alpha}(u)$. Again, these pairs arise algebraically because the A-polynomial has integer coefficients. (See *e.g.* [111, 116] for a detailed discussion of properties of $A(l,m)$.) Some branches, like the abelian branch, may be self-conjugate. For the abelian branch, this is consistent with $S_0^{(abel)} = 0$. The geometric

[3] More explicitly, letting $I_{CS}(\hbar, \mathcal{A}) = -\frac{1}{4\hbar}\int_M \text{Tr}\big(\mathcal{A}\wedge\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big)$, we have

$$\overline{Z^{\alpha}(M;\hbar,u)} = \left(\int_{(u,\alpha)} \mathcal{D}\mathcal{A}\, e^{I_{CS}(\hbar,\mathcal{A})}\right)^* = \int_{(\bar{u},\bar{\alpha})} \mathcal{D}\bar{\mathcal{A}}\, e^{I_{CS}(\bar{\hbar},\bar{\mathcal{A}})} = Z^{(\bar{\alpha})}(M;\bar{\hbar},\bar{u})\,.$$

branch, on the other hand, has a distinct conjugate because $\mathrm{Vol}(M;0) > 0$; from (6.3.15) we see that its leading perturbative coefficient obeys

$$S_0^{(\mathrm{conj})}(u) = \frac{i}{2}\Big[-\mathrm{Vol}(M;\bar{u}) + i\mathrm{CS}(M;\bar{u})\Big] + \overline{v_{\mathrm{geom}}}(u)\mathrm{Re}(u) - i\pi u\,. \tag{6.4.4}$$

In general, we have

$$S_0^{(\bar{\alpha})}(u) = \overline{S_0^{(\alpha)}}(u) \qquad (\mathrm{mod}\ 2\pi u)\,. \tag{6.4.5}$$

Now, let us consider symmetries that originate from geometry, *i.e.* symmetries that involve involutions of M,

$$\tau: \quad M \to M\,. \tag{6.4.6}$$

Every such involution restricts to a self-map of $\Sigma = \partial M$,

$$\tau|_{\Sigma}: \quad \Sigma \to \Sigma\,, \tag{6.4.7}$$

which, in turn, induces an endomorphism on homology, $H_i(\Sigma)$. Specifically, let us consider an orientation-preserving involution τ which induces an endomorphism $(-1,-1)$ on $H_1(\Sigma) \cong \mathbb{Z}\times\mathbb{Z}$. This involution is a homeomorphism of M; it changes our definition of the holonomies,

$$m_i \to \frac{1}{m_i} \qquad \text{and} \qquad l_i \to \frac{1}{l_i}\,, \tag{6.4.8}$$

leaving the symplectic form (6.1.14) invariant. Therefore, it preserves both the symplectic phase space $\mathcal{M}_{\mathrm{flat}}(G_{\mathbb{C}},\Sigma)$ and the Lagrangian submanifold $\mathcal{L}$ (possibly permuting some of its branches).

In the basic case of rank $r=1$, the symmetry (6.4.8) corresponds to the simple, well-known relation $A(l^{-1},m^{-1}) = A(l,m)$, up to overall powers of l and m. Similarly, at the quantum level, $\widehat{A}(\hat{l}^{-1},\hat{m}^{-1}) = \widehat{A}(\hat{l},\hat{m})$ when $\widehat{A}(\hat{l},\hat{m})$ is properly normalized. Branches of the A-polynomial are individually preserved, implying that the perturbative partition functions (and the coefficients $S_n^{(\alpha)}$) are all *even*:

$$Z^{(\alpha)}(M;\hbar,-u) = Z^{(\alpha)}(M;\hbar,u)\,, \tag{6.4.9}$$

modulo factors of $e^{2\pi u/\hbar}$ that are related to the ambiguity in $S_0^{(\alpha)}(u)$. Note that in the $r=1$ case one can also think of the symmetry (6.4.8) as the Weyl reflection. Since, by definition, holonomies that differ by an element of the Weyl group define the same point in the moduli space (6.1.10), it is clear that both $\mathcal{M}_{\mathrm{flat}}(G_{\mathbb{C}},\Sigma)$ and $\mathcal{L}$ are manifestly invariant under this symmetry. (For $r > 1$, Weyl transformations on the variables l and m lead to new, independent relations among the branches of $\mathcal{L}$.)

Finally, let us consider a more interesting "parity" symmetry, an orientation-reversing involution

$$P: \quad M \to \overline{M} \tag{6.4.10}$$

$$P|_\Sigma: \quad \Sigma \to \overline{\Sigma} \tag{6.4.11}$$

that induces a map $(1, -1)$ on $H_1(\Sigma) \cong \mathbb{Z}\times\mathbb{Z}$. This operation by itself cannot be a symmetry of the theory because it does not preserve the symplectic form (6.1.14). We can try, however, to combine it with the transformation $\hbar \to -\hbar$ to get a symmetry of the symplectic phase space $(\mathcal{M}_{\text{flat}}(G_\mathbb{C}, \Sigma), \omega)$. We are still not done because this combined operation changes the orientation of both Σ and M, and unless $\overline{M} \cong M$ the state $|M\rangle$ assigned to M will be mapped to a different state $|\overline{M}\rangle$. But if M is an amphicheiral[4] manifold, then both $m_i \to \frac{1}{m_i}$ and $l_i \to \frac{1}{l_i}$ (independently) become symmetries of the theory, once combined with $\hbar \to -\hbar$. This now implies that solutions come in signed pairs, $v_\alpha(u)$ and $v_{\hat{\alpha}}(u) = -v_\alpha(u)$, such that the corresponding perturbative $G_\mathbb{C}$ invariants satisfy

$$Z^{(\hat{\alpha})}(M; \hbar, u) = Z^{(\alpha)}(M; -\hbar, u)\,. \tag{6.4.12}$$

For the perturbative coefficients, this leads to the relations

$$S_0^{(\hat{\alpha})}(u) = -S_0^{(\alpha)}(u) \qquad (\text{mod } 2\pi u)\,, \tag{6.4.13}$$

$$S_n^{(\hat{\alpha})}(u) = (-1)^{n+1} S_n^{(\alpha)}(u) \qquad n \geq 1\,. \tag{6.4.14}$$

Assuming that the $\sim \pm\frac{i}{2}\text{Vol}(M; u)$ behavior of the geometric and conjugate branches is unique, their signed and conjugate pairs must coincide for amphicheiral 3-manifolds. $S_0^{(\text{geom})}$, then, is an even analytic function of u with strictly real series coefficients; at $u \in i\mathbb{R}$, the Chern-Simons invariant $\text{CS}(M; u)$ will vanish.

6.5 Brane quantization

Now, let us briefly describe how a new approach to quantization based on D-branes in the topological A-model [48] can be applied to the problem of quantizing the moduli space

[4] A manifold is called chiral or amphicheiral according to whether the orientation cannot or can be reversed by a self-map.

of flat connections, $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$. This discussion is not crucial for the rest of the chapter — although it can be useful for obtaining a better understanding of Chern-Simons theory with complex gauge group — and the reader not interested in this approach may skip directly to Section 6.6.

In the approach of [48], the problem of quantizing a symplectic manifold N with symplectic structure ω is solved by complexifying (N, ω) into a new space (Y, Ω) and studying the A-model of Y with symplectic structure $\omega_Y = \text{Im}\Omega$. Here, Y is a complexification of N, *i.e.* a complex manifold with complex structure $\mathcal{I}$ and an antiholomorphic involution

$$\tau : Y \to Y\,, \tag{6.5.1}$$

such that N is contained in the fixed point set of τ and $\tau^*\mathcal{I} = -\mathcal{I}$. The 2-form Ω on Y is holomorphic in complex structure $\mathcal{I}$ and obeys

$$\tau^*\Omega = \overline{\Omega} \tag{6.5.2}$$

and

$$\Omega|_N = \omega\,. \tag{6.5.3}$$

In addition, one needs to pick a unitary line bundle $\mathfrak{L} \to Y$ (extending the "prequantum line bundle" $\mathfrak{L} \to N$) with a connection of curvature $\text{Re}\Omega$. This choice needs to be consistent with the action of the involution τ, meaning that $\tau : Y \to Y$ lifts to an action on $\mathfrak{L}$, such that $\tau|_N = \text{id}$. To summarize, in brane quantization the starting point involves the choice of Y, Ω, $\mathfrak{L}$, and τ.

In our problem, the space $N = \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ that we wish to quantize is already a complex manifold. Indeed, as we noted earlier, it comes equipped with the complex structure J (that does not depend on the complex structure on Σ). Therefore, its complexification[5] is $Y = N \times \overline{N}$ with the complex structure on $\overline{N}$ being prescribed by $-J$ and the complex structure on Y being $\mathcal{I} = (J, -J)$. The tangent bundle $TY = TN \oplus T\overline{N}$ is identified with the complexified tangent bundle of N, which has the usual decomposition $\mathbb{C}TN = T^{1,0}N \oplus T^{0,1}N$. Then, the "real slice" N is embedded in Y as the diagonal

$$N \ni x \mapsto (x, x) \in N \times \overline{N}\,. \tag{6.5.4}$$

[5]Note that, since in our problem we start with a hyper-Kähler manifold $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$, its complexification Y admits many complex structures. In fact, Y has holonomy group $Sp(n) \times Sp(n)$, where n is the quaternionic dimension of $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$.

In particular, N is the fixed point set of the antiholomorphic involution $\tau : Y \to Y$ which acts on $(x,y) \in Y$ as $\tau : (x,y) \mapsto (y,x)$.

Our next goal is to describe the holomorphic 2-form Ω that obeys (6.5.2) and (6.5.3) with[6]

$$\omega = \frac{t}{2\pi i}(\omega_K + i\omega_I) - \frac{\bar{t}}{2\pi i}(\omega_K - i\omega_I) \tag{6.5.5}$$

Note, that $(\omega_K + i\omega_I)$ is holomorphic on $N = \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ and $(\omega_K - i\omega_I)$ is holomorphic on $\overline{N}$. Moreover, if we take $\bar{t}$ to be a complex conjugate of t, the antiholomorphic involution τ maps $t(\omega_K + i\omega_I)$ to $\bar{t}(\omega_K - i\omega_I)$, so that $\tau^*\omega = \omega$. Therefore, we can simply take

$$\Omega = \frac{t}{2\pi i}(\omega_K^{(1)} + i\omega_I^{(1)}) - \frac{\bar{t}}{2\pi i}(\omega_K^{(2)} - i\omega_I^{(2)}) \tag{6.5.6}$$

where the superscript $i = 1,2$ refers to the first (resp. second) factor in $Y = N \times \overline{N}$. It is easy to verify that the 2-form Ω defined in this way indeed obeys $(\text{Im}\Omega)^{-1}\text{Re}\Omega = \mathcal{I}$. Moreover, one can also check that if $\bar{t}$ is a complex conjugate of t then the restriction of $\omega_Y = \text{Im}\Omega$ to the diagonal (6.5.4) vanishes, so that the "real slice" $N \subset Y$, as expected, is a Lagrangian submanifold in (Y, ω_Y).

Now, the quantization problem can be realized in the A-model of Y with symplectic structure $\omega_Y = \text{Im}\Omega$. In particular, the Hilbert space $\mathcal{H}_\Sigma$ is obtained as the space of $(\mathcal{B}_{cc}, \mathcal{B}')$ strings,

$$\mathcal{H}_\Sigma = \text{ space of } (\mathcal{B}_{cc}, \mathcal{B}') \text{ strings} \tag{6.5.7}$$

where $\mathcal{B}_{cc}$ and $\mathcal{B}'$ are A-branes on Y (with respect to the symplectic structure ω_Y). The brane $\mathcal{B}'$ is the ordinary Lagrangian brane supported on the "real slice" $N \subset Y$. The other A-brane, $\mathcal{B}_{cc}$, is the so-called canonical coisotropic brane supported on all of Y. It carries a Chan-Paton line bundle of curvature $F = \text{Re}\Omega$. Note that for $[F]$ to be an integral cohomology class we need $\text{Re}(t) \in \mathbb{Z}$. Since in the present case the involution τ fixes the "real slice" pointwise, it defines a hermitian inner product on $\mathcal{H}_\Sigma$ which is positive definite.

[6]Notice that while elsewhere we consider only the "holomorphic" sector of the theory (which is sufficient in the perturbative approach), here we write the complete symplectic form on $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ that follows from the classical Chern-Simons action (5.1.1), including the contributions of both fields $\mathcal{A}$ and $\bar{\mathcal{A}}$.

6.6 Examples

To conclude this chapter, we give several examples of explicit computations using the hierarchy of differential equations described in Section 6.3. We focus on the gauge group $SL(2,\mathbb{C})$. We will begin with the trefoil, whose higher-order perturbative invariants on the non-abelian branch of flat connections actually vanish. This behavior was also noticed by direct analytic continuation of the colored Jones polynomial in [117]. We then consider figure-eight knot $\mathbf{4_1}$, the simplest hyperbolic knot, which has new, interesting, non-trivial perturbative invariants to all orders. It is quite possible to apply the hierarchy of differential equations to other knots as well (for example, explicit operators $\hat{A}$ for twist knots appear in [118]), but computations start to become somewhat inefficient. In Chapter 9, we will employ a different method (the state integral model) to find perturbative invariants for knots like $\mathbf{4_1}$ and $\mathbf{5_2}$ and to verify that they satisfy $\hat{A} \cdot Z = 0$.

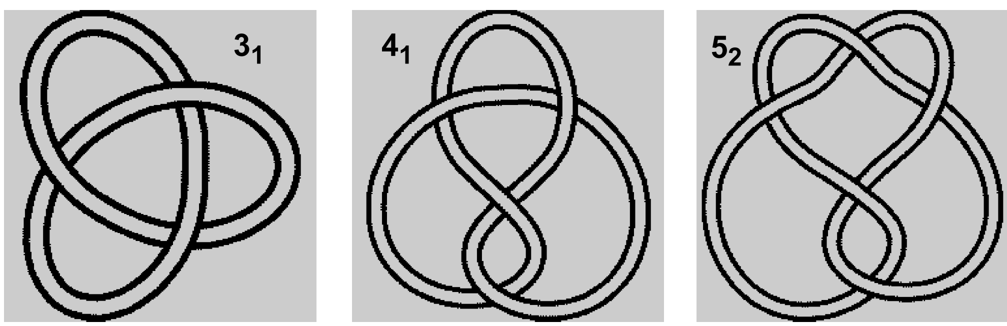


Figure 6.1: Recurrent examples: the trefoil knot $\mathbf{3_1}$, figure-eight knot $\mathbf{4_1}$, and "three-twist" knot $\mathbf{5_2}$, courtesy of `KnotPlot`.

6.6.1 Trefoil

The classical $SL(2,\mathbb{C})$ A-polynomial for the trefoil knot is

$$A(l,m) = (l-1)(l+m^6)\,. \tag{6.6.1}$$

It is a special case of the A-polynomial for (p,q) torus knots, which takes the form $(l-1)(lm^{pq}+1)$. This A-polynomial has two branches of solutions, an

$$\text{abelian branch}: \qquad l_{abel} = 1\,, \tag{6.6.2}$$

and a

$$\text{non-abelian branch :} \qquad l_{na} = -m^6 \,. \tag{6.6.3}$$

Since the trefoil is a chiral knot, the A-polynomial is not invariant under the transformation $m \to m^{-1}$, which takes the right-handed trefoil to the left-handed trefoil. (It was explained in Section 6.4 how the quantum partition function must change under this mirror action.)

As explained earlier, *cf.* (6.2.2), a simple way to find the operator $\widehat{A}(\hat{l}, \hat{m})$ is to note that it also annihilates the polynomial invariants of the knot K computed by Chern-Simons theory with compact gauge group G. In the present case, $G = SU(2)$ and the equation (6.2.2) takes the form of a recursion relation on the set of colored Jones polynomials, $\{J_n(K;q)\}$. Specifically, using the fact that $\hat{l}$ acts by shifting the value of the highest weight of the representation and writing $\widehat{A}(\hat{l}, \hat{m}) = \sum_{j=0}^{d} a_j(\hat{m}, q)\hat{l}^j$ as in (6.3.5), we obtain

$$\sum_{j=0}^{d} a_j(q^{n/2}, q) J_{n+j}(K;q) = 0 \,. \tag{6.6.4}$$

It is easy to verify that the colored Jones polynomials of the trefoil knot indeed satisfy such a recursion relation, with the coefficients (see also[7] [119, 120]):

$$a_0(\hat{m}, q) = q^2 m^6 (q^3 m^4 - 1) \,, \tag{6.6.5a}$$

$$a_1(\hat{m}, q) = \sqrt{q}(-q^5 m^{10} + q^4 m^6 + q^3 m^4 - 1) \,, \tag{6.6.5b}$$

$$a_2(\hat{m}, q) = 1 - q m^4 \,. \tag{6.6.5c}$$

Note that in the classical limit $q = e^{2\hbar} \to 1$, we have

$$\widehat{A}(\hat{l}, \hat{m}) \underset{\hbar \to 0}{\longrightarrow} (1 - m^4) A(l, m) \,. \tag{6.6.6}$$

We could divide all the a_j's by the extra factor $(1 - m^4)$ in(6.6.6) (or a q-deformation thereof) to obtain a more direct correspondence between $\widehat{A}(\hat{l}, \hat{m})$ and $A(l, m)$, but this does not affect any of the following calculations.

To find the exact perturbative invariant $Z^{(\alpha)}(M; \hbar, u)$ for each α, we reduce $\hat{m}$ to a classical variable m (since in the u-space representation the operator $\hat{m}$ just acts via ordinary

[7] Note that [119, 120] look at asymptotics of the colored Jones polynomial normalized by its value at the unknot, while the $SL(2, \mathbb{C})$ Chern-Simons partition function should agree with the unnormalized colored Jones polynomial. Hence, we must divide the expressions for a_j given there by $(m^2 q^{j/2} - q^{-j/2})$ to account for the difference, introducing a few factors of $q^{1/2}$ in our formulas.

multiplication), expand each of the above a_j's as $a_j(m,q) = \sum_{p=0}^{\infty} a_{j,p}(m)\hbar^p$, and substitute the $a_{j,p}$ into the hierarchy of differential equations derived in Section 6.1 and displayed in Table 6.1. The equations are then solved recursively on each branch α of the A-polynomial to determine the coefficients $S_n^{(\alpha)}(u)$.

Non-abelian branch

For the non-abelian branch, which one would naively expect to be the more non-trivial one, all higher-order perturbative invariants actually vanish. It is easy to check directly that

$$Z^{(na)}(\mathbf{3_1};\hbar;u) = \exp\left(\frac{3u^2 + i\pi u}{\hbar}\right) \tag{6.6.7}$$

is an exact solution to $\hat{A}(\hat{l},\hat{m}) \cdot Z = 0$. This implies that

$$S_0^{(na)}(u) = 3u^2 + i\pi u \qquad (\mathrm{mod}\ 2\pi i u)\,, \tag{6.6.8}$$

and agrees with the fact that the volume of the (non-hyperbolic) trefoil knot complement is zero.

Moreover, all higher-order invariants $S_n^{(na)}(u)$ must be constants. In [117, 121], it is independently argued (via direct analytic continuation of the colored Jones polynomial) that this is indeed the correct behavior, and that $S_1^{(na)}$ is a nonzero constant, while all other $S_n^{(na)}$ vanish.

Abelian branch

The abelian branch, surprisingly, is non-trivial. It senses the fact that the operator $\hat{A}$ does contain more information than the classical A-polynomial. Setting $S_0(u) = 0$ to single out the abelian branch, we find that

$$\begin{aligned}
S_1(u) &= \log \frac{m\left(m^2-1\right)}{m^4-m^2+1}\,, \\
S_2(u) &= \frac{2m^4}{\left(m^4-m^2+1\right)^2}\,, \\
S_3(u) &= -\frac{2\left(m^{12}-4m^8+m^4\right)}{\left(m^4-m^2+1\right)^4}\,,
\end{aligned}$$

$$S_4(u) = \frac{4m^4\left(m^{16}+2m^{14}-23m^{12}-4m^{10}+60m^8-4m^6-23m^4+2m^2+1\right)}{3\left(m^4-m^2+1\right)^6},$$

$$S_5(u) = -\frac{2m^4}{3\left(m^4-m^2+1\right)^8}\left(m^{24}+8m^{22}-84m^{20}-144m^{18}+868m^{16}+200m^{14}-1832m^{12}\right.$$
$$\left.+200m^{10}+868m^8-144m^6-84m^4+8m^2+1\right),$$

$$\ldots.$$

Since the abelian branch is self-conjugate (in the language of Section 6.4), the powers of m appearing here are all balanced; for $u \in i\mathbb{R}$ or $u \in \mathbb{R}$ the coefficients are real. Unfortunately, since this *is* an abelian branch, all the perturbative invariants at $u = 0$ are uninteresting rational numbers.

6.6.2 Figure-eight knot

The classical A-polynomial of the figure-eight knot is

$$A(l,m) = (l-1)(m^4-(1-m^2-2m^4-m^6+m^8)l+m^4l^2)\,. \tag{6.6.9}$$

Observe that $A(l^{-1},m^{-1}) \sim A(l,m)$ and that $A(l^{-1},m) \sim A(l,m)$, the latter reflecting the fact that the figure-eight knot is amphicheiral. The zero locus of this A-polynomial has three branches: the abelian branch $l_{\text{abel}} = 1$ (or $v_{\text{abel}} = -i\pi$) from the first factor, and two other branches from the second, given explicitly by

$$l_{\text{geom,conj}}(m) = \frac{1-m^2-2m^4-m^6+m^8}{2m^4} \pm \frac{1-m^4}{2m^2}\Delta(m) \tag{6.6.10}$$

or

$$v_{\text{geom,conj}}(u) = \log(-l_{\text{geom,conj}}(e^u))\,, \tag{6.6.11}$$

where we have defined

$$\Delta(m) = i\sqrt{-m^{-4}+2m^{-2}+1+2m^2-m^4}\,. \tag{6.6.12}$$

Note that $\Delta(m)$ is an analytic function of $m = e^u$ around $u = 0$, as are $l(m)$ and $v(u)$. In particular, $\Delta(1) = \sqrt{-3}$ is the generator of the trace field $\mathbb{K} = \mathbb{Q}(\text{tr}\Gamma)$ for the figure-eight knot.

Since there are only two non-abelian branches, they must (as indicated in Section 6.4) be the geometric and conjugate ones. By looking at the behavior of these two branches in the neighborhood of the point $(l, m) = (-1, 1)$ that corresponds to the complete hyperbolic structure, where the geometric branch must have maximal volume, and using (6.3.15), one can check that the identification in (6.6.10) is the correct one. (It is also possible to check this directly once the hyperbolic structure of the figure-eight knot complement is understood, *cf.* Section 8.1.1.)

The quantum A-polynomial $\hat{A}(\hat{l}, \hat{m}) = \sum_{j=0}^{3} a_j(\hat{m}, q)\hat{l}^j$ for the figure-eight knot has coefficients

$$a_0(\hat{m}, q) = \frac{q\hat{m}^2}{(1+q\hat{m}^2)(-1+q\hat{m}^4)}\,, \tag{6.6.13a}$$

$$a_1(\hat{m}, q) = \frac{1+(q^2-2q)\hat{m}^2-(q^3-q^2+q)\hat{m}^4-(2q^3-q^2)\hat{m}^6+q^4\hat{m}^8}{q^{1/2}\hat{m}^2(1+q^2\hat{m}^2-q\hat{m}^4-q^3\hat{m}^6)}\,, \tag{6.6.13b}$$

$$a_2(\hat{m}, q) = -\frac{1-(2q^2-q)\hat{m}^2-(q^5-q^4+q^3)\hat{m}^4+(q^7-2q^6)\hat{m}^6+q^8\hat{m}^8}{q\hat{m}^2(1+q\hat{m}^2-q^5\hat{m}^4-q^6\hat{m}^6)}\,, \tag{6.6.13c}$$

$$a_3(\hat{m}, q) = -\frac{q^4\hat{m}^2}{q^{1/2}(1+q^2\hat{m}^2)(-1+q^5\hat{m}^4)}\,. \tag{6.6.13d}$$

In the classical limit $q = e^{2\hbar} \to 1$, we have

$$\widehat{A}(\hat{l}, \hat{m}) \underset{\hbar\to 0}{\longrightarrow} \frac{A(l, m)}{m^2(m^2-1)(m^2+1)^2}\,. \tag{6.6.14}$$

Again, we could multiply all the a_j's by the denominator of (6.6.14) (or a q-deformation thereof) to obtain a more direct correspondence between $\widehat{A}(\hat{l}, \hat{m})$ and $A(l, m)$.

Geometric branch

For the geometric branch, the solution to the first equation in Table 6.1 is chosen to be

$$S_0^{(\text{geom})}(u) = \frac{i}{2}\text{Vol}(\mathbf{4_1}; 0) + \int_0^u du\, v_{\text{geom}}(u) + i\pi u\,, \tag{6.6.15}$$

with $v_{\text{geom}}(u)$ as in (6.6.11). The integration constant $i(\text{Vol}(0) + i\text{CS}(0))$ is not important for determining the remaining coefficients (since only derivatives of S_0 appear in the equations), but we have fixed it by requiring that $S_0^{(\text{geom})}(0) = \frac{i}{2}(\text{Vol}(\mathbf{4_1}; 0) + i\text{CS}(\mathbf{4_1}; 0)) = \frac{i}{2}\text{Vol}(\mathbf{4_1}; 0) = (1.01494\ldots)i$, as expected for the classical action of Chern-Simons theory.

Substituting the above $S_0^{(\text{geom})}(u)$ into the hierarchy of equations, the rest are readily solved[8] for the subleading coefficients. The first eight functions $S_n^{(\text{geom})}(u)$ appear below:

[8] It is computationally advantageous to express everything in terms of $m = e^u$ and $m\frac{d}{dm} = \frac{d}{du}$, *etc.*, when implementing this on a computer.

$$S_1(u) = -\frac{1}{2}\log\left(\frac{-i\Delta(m)}{2}\right),$$

$$S_2(u) = \frac{-1}{12\Delta(m)^3 m^6}\left(1 - m^2 - 2m^4 + 15m^6 - 2m^8 - m^{10} + m^{12}\right),$$

$$S_3(u) = \frac{2}{\Delta(m)^6 m^6}\left(1 - m^2 - 2m^4 + 5m^6 - 2m^8 - m^{10} + m^{12}\right),$$

$$S_4(u) = \frac{1}{90\Delta(m)^9 m^{16}}\left(1 - 4m^2 - 128m^4 + 36m^6\right.$$
$$+1074m^8 - 5630m^{10} + 5782m^{12} + 7484m^{14} - 18311m^{16} + 7484m^{18}$$
$$\left.+5782m^{20} - 5630m^{22} + 1074m^{24} + 36m^{26} - 128m^{28} - 4m^{30} + m^{32}\right),$$

$$S_5(u) = \frac{2}{3\Delta(m)^{12} m^{18}}\left(1 + 5m^2 - 35m^4 + 240m^6 - 282m^8 - 978m^{10}\right.$$
$$+3914m^{12} - 3496m^{14} - 4205m^{16} + 9819m^{18} - 4205m^{20} - 3496m^{22}$$
$$\left.+3914m^{24} - 978m^{26} - 282m^{28} + 240m^{30} - 35m^{32} + 5m^{34} + m^{36}\right),$$

$$S_6(u) = \frac{-1}{945\Delta(m)^{15} m^{28}}\left(1 + 2m^2 + 169m^4 + 4834m^6\right.$$
$$-24460m^8 + 241472m^{10} - 65355m^{12} - 3040056m^{14} + 13729993m^{16}$$
$$-15693080m^{18} - 36091774m^{20} + 129092600m^{22} - 103336363m^{24}$$
$$-119715716m^{26} + 270785565m^{28} - 119715716m^{30} - 103336363m^{32}$$
$$+129092600m^{34} - 36091774m^{36} - 15693080m^{38} + 13729993m^{40} - 3040056m^{42}$$
$$\left.-65355m^{44} + 241472m^{46} - 24460m^{48} + 4834m^{50} + 169m^{52} + 2m^{54} + m^{56}\right),$$

$$S_7(u) = \frac{4}{45\Delta(m)^{18} m^{30}}\left(1 + 47m^2 - 176m^4 + 3373m^6 + 9683m^8\right.$$
$$-116636m^{10} + 562249m^{12} - 515145m^{14} - 3761442m^{16} + 14939871m^{18}$$
$$-15523117m^{20} - 29061458m^{22} + 96455335m^{24} - 71522261m^{26}$$
$$-80929522m^{28} + 179074315m^{30} - 80929522m^{32} - 71522261m^{34}$$
$$+96455335m^{36} - 29061458m^{38} - 15523117m^{40} + 14939871m^{42}$$
$$-3761442m^{44} - 515145m^{46} + 562249m^{48} - 116636m^{50}$$
$$\left.+9683m^{52} + 3373m^{54} - 176m^{56} + 47m^{58} + m^{60}\right),$$

$$S_8(u) = \frac{1}{9450\Delta(m)^{21} m^{40}}\left(1 + 44m^2 - 686m^4\right.$$
$$-25756m^6 + 25339m^8 - 2848194m^{10} - 28212360m^{12}$$

$$+216407820m^{14} - 1122018175m^{16} - 266877530m^{18}$$
$$+19134044852m^{20} - 76571532502m^{22} + 75899475728m^{24}$$
$$+324454438828m^{26} - 1206206901182m^{28} + 1153211096310m^{30}$$
$$+1903970421177m^{32} - 5957756639958m^{34} + 4180507070492m^{36}$$
$$+4649717451712m^{38} - 10132372721949m^{40} + 4649717451712m^{42}$$
$$+4180507070492m^{44} - 5957756639958m^{46} + 1903970421177m^{48}$$
$$+1153211096310m^{50} - 1206206901182m^{52} + 324454438828m^{54}$$
$$+75899475728m^{56} - 76571532502m^{58} + 19134044852m^{60}$$
$$-266877530m^{62} - 1122018175m^{64} + 216407820m^{66} - 28212360m^{68}$$
$$-2848194m^{70} + 25339m^{72} - 25756m^{74} - 686m^{76} + 44m^{78} + m^{80}\big) .$$

Table 6.3: Perturbative invariants $S_n^{(\mathrm{geom})}(u)$ up to eight loops.

According to (5.2.3), the coefficient $S_1^{(\mathrm{geom})}(u)$ in the perturbative Chern-Simons partition function should be related to the Reidemeister-Ray-Singer torsion of M twisted by $\mathcal{A}^{(\mathrm{geom})}$, which has been independently computed. Our function matches[9] that appearing in *e.g.* [98], up to a shift by $-\log\pi$. The constants of integration for the remaining coefficients have been fixed by comparison to the asymptotics of the colored Jones polynomial, using (5.4.5) and the results of Section 5.4. Note that at $u = 0$, the arithmetic invariants of Section 5.4 at the complete hyperbolic structure should be reproduced.

Conjugate branch

For the "conjugate" branch, the solution for $S_0(u)$ is now chosen to be

$$S_0^{(\mathrm{conj})}(u) = -\frac{i}{2}\mathrm{Vol}(\mathbf{4_1};0) + \int_0^u du\, v_{\mathrm{conj}}(u) + i\pi u \qquad (\mathrm{mod}\ 2\pi u), \tag{6.6.16}$$

so that $S_0^{(\mathrm{conj})}(u) = -S_0^{(\mathrm{geom})}(u)$. As for the geometric branch, this is then substituted into remainder of the hierarchy of equations. Calculating the subleading coefficients, the

[9]To compare with [98], note that $k_{\mathrm{there}} = k_{\mathrm{here}} = i\pi/\hbar$, and $u_{\mathrm{there}} = 2u_{\mathrm{here}}$. The shift by $-\log(\pi)$ is directly related to a jump in the asymptotics of the colored Jones polynomial at $u = 0$.

constants of integration can all be fixed so that

$$S_n^{(\mathrm{conj})}(u) = (-1)^{n+1} S_{n+1}^{(\mathrm{geom})}(u), \tag{6.6.17}$$

This is precisely what one expects for an amphicheiral knot when a conjugate pair of branches coincides with a "signed" pair, as discussed in Section 6.4.

Abelian branch

For completeness, we can also mention perturbation theory around an abelian flat connection $\mathcal{A}^{(\mathrm{abel})}$ on M, although it has no obvious counterpart in the state integral model.

For an abelian flat connection $\mathcal{A}^{(\mathrm{abel})}$, the classical Chern-Simons action (5.2.1) vanishes. This is exactly what one finds from (6.3.14):

$$S_0^{(\mathrm{abel})}(u) = \int_0^u du\, v_{\mathrm{abel}}(u) + i\pi u = 0 \qquad (\mathrm{mod}\ 2\pi i u)\,, \tag{6.6.18}$$

fixing the constant of integration so that $S_0^{(\mathrm{abel})}(0) = 0$. From the hierarchy of differential equations, the first few subleading coefficients are

$$\begin{aligned}
S_1^{(\mathrm{abel})}(u) &= \log \frac{m(m^2-1)}{1-3m^2+m^4}, \\
S_2^{(\mathrm{abel})}(u) &= 0 \\
S_3^{(\mathrm{abel})}(u) &= \frac{4(m^2-1)^2}{(1-3m^2+m^4)^3}\big(1-7m^2+16m^4-7m^6+m^8\big), \\
S_4^{(\mathrm{abel})}(u) &= 0 \\
S_5^{(\mathrm{abel})}(u) &= \frac{4(m^2-1)^2}{3(1-3m^2+m^4)^6}\big(41-656m^2+4427m^4-16334m^6+35417m^8-46266m^{10} \\
&\quad +35417m^{12}-16334m^{14}+4427m^{16}-656m^{18}+41m^{20}\big) \\
S_6^{(\mathrm{abel})}(u) &= 0 \ \ \ldots.
\end{aligned}$$

Table 6.4: Perturbative invariants $S_n^{(\mathrm{abel})}(u)$ up to six loops.

In the language of Section 6.4, the abelian branch must be its own "signed pair," guaranteeing that all even $S_{2k}^{(\mathrm{abel})}(u)$ vanish.

Chapter 7

Wilson loops for complex gauge groups

The goal of this chapter is to explain the relationship between Chern-Simons theory with Wilson loops and Chern-Simons theory on knot complements, and to use this relationship to elucidate the limit taken in "analytic continuation" of Chern-Simons theory with compact gauge group G to Chern-Simons theory with complex gauge group $G_{\mathbb{C}}$.

For compact gauge groups, the relation between Wilson loops and knot complements have been understood for a long time [13, 122]. Part of our discussion here reviews and explains these results in greater detail, as they provide the groundwork for the complex case. The main idea of the approach in (*e.g.*) [122] is to rewrite a Wilson loop as a new path integral for quantum mechanics of particles moving along the loop. This has two advantages: it allows a generalization to complex gauge group (taking the trace of a holonomy in a given representation does *not*); and it most directly connects boundary conditions and representations.

Unfortunately, the generalization to complex gauge group requires a certain number of mathematical preliminaries. Thus, we spend the first part of the chapter reviewing principal series representations of complex groups and their realizations via the orbit method, following [123, 124, 125]. In the second part of the chapter, we incorporate these results into Wilson loops and Chern-Simons theory.

7.1 Some representation theory

Finite and infinite-dimensional representations

The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is generated as a complex vector space by the elements E, F, H satisfying

$$[E,F] = H\,, \quad [H,E] = 2E\,, \quad [H,F] = -2F\,. \tag{7.1.1}$$

In the defining representation, these elements correspond to the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{7.1.2}$$

One exponentiates (with no i's) to obtain the group $SL(2,\mathbb{C})$. The compact subgroup $SU(2)$ of $SL(2,\mathbb{C})$ has a Lie algebra $\mathfrak{su}(2)$ generated as a real vector space over anti-Hermitian generators (*a.k.a.* Pauli matrices):

$$\mathfrak{su}(2) = \langle iH, E-F, i(E+F)\rangle_{\mathbb{R}}\,. \tag{7.1.3}$$

All representations of $\mathfrak{su}(2)$ can be extended to be *holomorphic* representations of $\mathfrak{sl}(2,\mathbb{C})$. They act most generally as

$$\begin{aligned} \rho_\lambda(E) &= -\lambda z + z^2\partial_z\,, \\ \rho_\lambda(F) &= -\partial_z\,, \\ \rho_\lambda(H) &= -\lambda + 2z\partial_z \end{aligned} \tag{7.1.4}$$

on holomorphic functions $f(z)$ of a single complex variable. Here, ρ is a complex-linear homomorphism on the generators of the Lie algebra. The quadratic Casimir of this representation is

$$C_2(\lambda) = \rho_\lambda\left(\frac{1}{2}H^2 + EF + FE\right) = \frac{1}{2}\lambda(\lambda+2)\,. \tag{7.1.5}$$

While representation (7.1.4) is a perfectly good representation of $\mathfrak{sl}(2,\mathbb{C})$ or its subalgebras for any $\lambda \in \mathbb{C}$, it is not always integrable to representations of $SL(2,\mathbb{C})$ or its subgroups. If ρ_λ *is* integrable, then it integrates generically to

$$\rho_\lambda\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z) = (-bz+d)^\lambda f\left(\frac{az-c}{-bz+d}\right)\,. \tag{7.1.6}$$

For $SU(2)$ and $SL(2,\mathbb{C})$, it is sufficient to check that this is well-defined on the maximal tori generated by H, and in particular that $\rho(I) = \rho_\lambda(e^{2\pi i H}) := e^{2\pi i \rho_\lambda(H)} = 1$. This forces λ to be an integer. Indeed, when $\lambda \in \mathbb{Z}_{\geq 0}$, ρ_λ can be restricted to a finite-dimensional representation acting on polynomials in z of degree $\leq \lambda$. The dimension of this representation is $N = \lambda + 1$.

In the case of $SU(2)$, these finite-dimensional representations are unitary. For $SL(2,\mathbb{C})$ however, they are not. In order to find unitary representations of $SL(2,\mathbb{C})$, one must break its complex structure. That is, consider $\mathfrak{sl}(2,\mathbb{C})$ to be a *real* Lie algebra[1] generated over the basis $\{E,\, F,\, H,\, \tilde{E} := iE,\, \tilde{F} := iF,\, \tilde{H} := iH\}$. We seek representations ρ with $\rho(\tilde{T}) \neq i\rho(T)$ for $T = E, F, H$.

It is convenient to complexify the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ (a second time), and to extend ρ complex-linearly under this second complexification. Working over $\mathbb{C}$, we then redefine generators

$$\begin{aligned} 2E_L &= E - i\tilde{E}, \qquad & 2E_R &= E + i\tilde{E}\,, \\ 2F_L &= F - i\tilde{F}, \qquad & 2F_R &= F + i\tilde{F}\,, \\ 2H_L &= H - i\tilde{H}, \qquad & 2H_R &= H + i\tilde{H}\,. \end{aligned} \tag{7.1.7}$$

Each (E_L, F_L, H_L) and (E_R, F_R, H_R) generate independent (commuting) $\mathfrak{sl}(2,\mathbb{C})$ subalgebras, and we obtain an isomorphism $\mathfrak{sl}(2,\mathbb{C})_\mathbb{C} \simeq \mathfrak{sl}(2,\mathbb{C})_L \times \mathfrak{sl}(2,\mathbb{C})_R$. Now, generic nonholomorphic representations of $\mathfrak{sl}(2,\mathbb{C})$ should be given as $\rho_{\lambda_L,\lambda_R} = \rho_{\lambda_L} \times \rho_{\lambda_R}$, using (7.1.4) for each of the two factors $\mathfrak{sl}(2,\mathbb{C})_{L,R}$. The representation space consists of nonholomorphic functions $f(z,\bar{z})$, with ρ_{λ_L} acting on z and ρ_{λ_R} acting on $\bar{z}$. The corresponding quadratic Casimir is

$$C_2(\lambda_L, \lambda_R) = \lambda_L(\lambda_L + 2) + \lambda_R(\lambda_R + 2)\,. \tag{7.1.8}$$

When naively integrating the representation $\rho_{\lambda_L,\lambda_R}$ described above, one obtains

$$\rho_{\lambda_L,\lambda_R}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z,\bar{z}) = (-bz+d)^{\lambda_L}\,\overline{(-bz+d)}^{\lambda_R} f\left(\frac{az-c}{-bz+d}, \frac{\bar{a}\bar{z}-\bar{c}}{-\bar{b}\bar{z}+\bar{d}}\right). \tag{7.1.9}$$

This expression is well-defined only when $\boxed{\lambda_L - \lambda_R \in \mathbb{Z}}$. This condition can be seen by writing the multiplicative factor above as

$$|-bz+d|^{\lambda_L+\lambda_R}\left(\frac{-bz+d}{|-bz+d|}\right)^{\lambda_L-\lambda_R}.$$

[1] In this description, we have $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{spin}(3,1)$.

Then, for $\rho_\lambda(e^{2\pi\tilde{H}}) = 1$ the exponent of the phase must be an integer. It is clear from (7.1.9) that when $\lambda_R = 0$ we just obtain the old holomorphic representations, with $\lambda = \lambda_L$.

For a pair $\lambda_L, \lambda_R \in \mathbb{C}$ satisfying $\lambda_L - \lambda_R \in \mathbb{Z}$, the expression (7.1.9) defines a *generalized principal series representation* of $SL(2,\mathbb{C})$. This infinite-dimensional representation is more commonly labelled as $\mathcal{P}^{\Bbbk,w}$ (*cf.* [123]), with

$$w = -(\lambda_L + 1) - (\lambda_R + 1) \in \mathbb{C}, \qquad \Bbbk = -(\lambda_L - \lambda_R) \in \mathbb{Z}\,. \tag{7.1.10}$$

For w imaginary and arbitrary $\Bbbk$, the principal series representations are unitary under the standard inner product on $L^2(\mathbb{C})$. For real $0 < w < 2$ and $\Bbbk = 0$, the representations are unitary under a non-standard inner product, and are called the complementary series. In terms of $\Bbbk$ and w, the quadratic Casimir is

$$\frac{1}{4}(w^2 + \Bbbk^2) - 1\,. \tag{7.1.11}$$

Representations related by $(w, \Bbbk) \leftrightarrow (-w, -\Bbbk)$ are isomorphic, and this is the only equivalence in the series. The unitary principal series and complementary series are the only unitary representations of $SL(2,\mathbb{C})$. They are irreducible unless $w \in \mathbb{Z}$, $|k| < |w|$, and $k \equiv w \bmod 2$ [123].

Induction

Given a group G, a subgroup $H \subset G$, and a representation σ of H, one can form an *induced representation* $\mathrm{ind}_H^G(\sigma)$. This acts on the space of functions on G valued in the vector space of the representation σ that satisfy

$$f(xh) = \sigma(h^{-1})f(x) \tag{7.1.12}$$

for $h \in H$, $x \in G$. This space, formally $C_c^\infty(H\backslash G;\sigma)$, can equivalently be thought of as sections of a bundle on the coset space G/H. The action of G on $f \in C_c^\infty(H\backslash G;\sigma)$ is by the left-regular representation

$$\mathrm{ind}_H^G(\sigma)(g) \cdot f(x) := f(g^{-1}x)\,. \tag{7.1.13}$$

For a general semi-simple reductive group, *all* so-called admissible representations are generated by the operation of induction from simpler representations of parabolic subgroups, and then by taking quotients to obtain irreducible representations. In the case of $SL(2,\mathbb{C})$, the story is very simple. The only nontrivial parabolic subgroup is the Borel subgroup B of upper-triangular matrices. It has a decomposition $B = MAN$ corresponding to the decomposition $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}$ with

$$\begin{aligned} \mathfrak{a} &= \langle H\rangle, & A &= \{\exp tH\} \quad (t \in \mathbb{R})\,, \\ \mathfrak{m} &= \langle iH\rangle, & M &= \{\exp tiH\} \simeq U(1)\,, \\ \mathfrak{n} &= \langle E, iE\rangle, & N &= \{\exp(tE + t'iE)\}\,. \end{aligned} \tag{7.1.14}$$

Then, letting $\nu \in (\mathfrak{a}^*)_{\mathbb{C}}$ and $\kappa \in \mathfrak{m}^*$ be (real) linear functionals defined by

$$\nu(H) = -iw\,, \quad \kappa(iH) = k\!\!\!\!\!\!\;\,;\qquad \nu(iH) = \kappa(H) = 0\,, \tag{7.1.15}$$

and letting ρ be the half-sum of positive restricted roots satisfying

$$\rho(H) = 2\,, \qquad \rho(iH) = 0\,, \tag{7.1.16}$$

the principal series representation $\mathcal{P}^{k\!\!\!\!\,, w}$ is induced from the representation $\exp(i\nu + \rho) \otimes \exp i\kappa \otimes id$ of $B = MAN$:

$$\mathbb{P}^{k\!\!\!\!\,, w} = \mathrm{ind}_B^{G_{\mathbb{C}}}(e^{i\nu + i\kappa + \rho})\,. \tag{7.1.17}$$

Observe that the defining data here is a pair of linear functionals κ and ν, corresponding to the parameters $k\!\!\!\!\,$ and w, that act on the maximal torus $T_{\mathbb{C}} \simeq GL(1)$ of the group $G_{\mathbb{C}}$. The functional acting on the compact part of $T_{\mathbb{C}}$ is quantized. This is true in general for maximal parabolic subgroups: principal series representations are induced from representations of the maximal torus that act trivially on the off-diagonal part of the parabolic subgroup. The representation $T_{\mathbb{C}}$ (here $e^{i\nu+i\kappa+\rho}$) is called the quasi-character of the induced representation.

7.2 Representations as coadjoint orbits

The mathematical machinery that connects representations with conjugacy classes (and for us Wilson loops with boundary conditions) is the Borel-Weil-Bott theorem [126] and generalizations thereof to noncompact groups (developed by Kirillov and others). The basic idea is that irreducible representations a group $G_{(\mathbb{C})}$ can be obtained by the geometric quantization of its coadjoint orbits. Excellent reviews of the topic can be found in [124] or

[125] (see also [127] and [48] in the physics literature). Here, we will give a brief overview of the general procedure, stating necessary results, and then specializing to the example of $SU(2)$ and $SL(2,\mathbb{C})$.

The adjoint action of G on $\mathfrak{g}$ can be written as $g : X \to gXg^{-1}$. Likewise, there is a coadjoint action of G on the dual space $\mathfrak{g}^*$. Since $\mathfrak{g}^*$ and $\mathfrak{g}$ can be identified for reductive groups by a non-degenerate trace form, we can write weights $\lambda \in \mathfrak{g}^*$ as matrices; then the coadjoint action is $g : X \to g\lambda g^{-1}$. The coadjoint *orbit*

$$\Omega_\lambda = \{g\lambda g^{-1}\}_{g\in G} \tag{7.2.1}$$

through a point λ has the geometry of the coset G/H_λ (or sometimes a quotient thereof), where H_λ is the stabilizer of λ:

$$H_\lambda = \{h \in G \mid h\lambda h^{-1} = \lambda\}\,. \tag{7.2.2}$$

In the case of compact or complex G, H_λ is conjugate to the maximal torus of G for generic λ. The left-regular G-action on G/H_λ is equivalent to the coadjoint action on Ω_λ, and is the action used for representations.

The coadjoint orbit $\Omega_\lambda \simeq H_\lambda$ has a natural G-invariant symplectic structure. The symplectic form can be written as

$$\omega = -\mathrm{Tr}(\lambda\, g^{-1}dg \wedge g^{-1}dg) \tag{7.2.3}$$

for $g \in G/H_\lambda$. To geometrically quantize Ω_λ one must form a line bundle $\mathcal{L} \to \Omega_\lambda$ with curvature ω, choose a *polarization* that effectively cuts out half the degrees of freedom (half the coordinates) on Ω_λ, and construct a Hilbert space V as the space of square-integrable polarization-invariant sections of $\mathcal{L}$. Considering λ again to be a linear functional $\lambda \in \mathfrak{g}^*$, the line bundle $\mathcal{L}$ can is obtained as a quotient of $\mathbb{C} \times G$ by the representation $e^{i\lambda}$ of $H_\lambda \in G$ acting on $\mathbb{C}$; in other words, it is essentially the space $C^\infty(G, e^{i\lambda})$ of the induced representation $\mathrm{ind}_{H_\lambda}^G(e^{i\lambda})$ described in Section 7.1. The line bundle exists if and only if the representation $e^{i\lambda}$ is integrable.

In the case of compact group G, where $H_\lambda \simeq \mathbb{T}$ is a maximal torus, one can induce a complex structure on G/H_λ via the equivalence

$$G/H_\lambda \simeq G_{\mathbb{C}}/B\,, \tag{7.2.4}$$

where B is the upper-triangular Borel subgroup of $G_{\mathbb{C}}$ containing H_λ. Then, to obtain the unitary finite-dimensional representations, one must choose a *holomorphic* polarization. Specifically, if $-\lambda$ is the dominant weight of a sought representation R, then the space $H^0_{\bar\partial}(G/H_\lambda; e^{i\lambda})$ of holomorphic sections of the bundle $\mathcal{L}$ is finite dimensional and G acts on it in the left-regular representation to furnish R.

In the case of complex group $G_{\mathbb{C}}$, principal series representations are obtained by considering *real* polarizations on the line bundle over $G_{\mathbb{C}}/H_\lambda$, where λ is now a non-complex-linear element of the dual $\mathfrak{g}^*_{\mathbb{C}}$. The set of all sections of this line bundle is the representation space of a representation induced from H_λ by $e^{i\lambda}$. To get the desired representation induced from the *Borel* subgroup $B = H_\lambda N$ by $e^{i\lambda} \otimes id$, the polarization is chosen precisely such that sections are independent of coordinates in N. (This process can also be extended to describe induction from generic non-maximal parabolic subgroups, *e.g.* by choosing λ such that its isotropy group H_λ is non-minimal.)

In both compact and complex cases, the representations of the group $G_{(\mathbb{C})}$ end up being described by choices of coadjoint orbits. Since a coadjoint orbit is defined by an element $\lambda \in \mathfrak{g}^*_{(\mathbb{C})} \simeq \mathfrak{g}_{(\mathbb{C})}$ *up to conjugacy*, this establishes an equivalence between representations and conjugacy classes of $\mathfrak{g}_{(\mathbb{C})}$.

Example: $SU(2)$

In the simplest case of $SU(2)$, let us write a matrix $g \in SU(2)$ as

$$g = \begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix}, \qquad w,\, z \in \mathbb{C}. \tag{7.2.5}$$

with $|w|^2 + |z|^2 = 1$. The right action of a diagonal matrix $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$ in the maximal torus $H = \mathbb{T}$ acts by sending $(w, z) \mapsto (e^{i\theta} w, e^{i\theta} z)$. From this, we immediately see that $SU(2)$ has the geometry of S^3, and the H action is just rotation in the S^1 fiber of the Hopf bundle $S^1 \to S^3 \to S^2$. Therefore, a generic coadjoint orbit looks like

$$\Omega \simeq SU(2)/\mathbb{T} = S^2 \simeq \mathbb{P}^1\,. \tag{7.2.6}$$

We already know what all *holomorphic* bundles on $\mathbb{P}^1$ look like: they are tensor powers of the canonical line bundle, $\mathcal{O}(\tilde\lambda)$ for $-\tilde\lambda \in \mathbb{Z}$. These have curvature

$$\omega = -\mathrm{Tr}(\lambda g^{-1} dg \wedge g^{-1} dg)\,, \tag{7.2.7}$$

where λ is the matrix

$$\lambda = \frac{\tilde{\lambda}}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7.2.8}$$

or equivalently the corresponding element of $(\mathfrak{g}^*)_{\mathbb{C}}$ acting as $\mathrm{Tr}[\lambda, \cdot]$. Moreover, for $\tilde{\lambda} < 0$, the bundle $\mathcal{O}(-\tilde{\lambda})$ has an $(|\tilde{\lambda}| + 1)$-dimensional space of square-integrable holomorphic sections. A basis of this space is given by the polynomials $\{1, z, ..., z^{-\tilde{\lambda}}\}$ in local coordinates on $SU(2)/H$ where $w \neq 0$; this is precisely the $(|\tilde{\lambda}| + 1)$-dimensional representation of $SU(2)$. It is easy to see that the generators of the Lie algebra $\mathfrak{su}(2)$ act on local functions $f(z)$ exactly as in (7.1.4).

By using the symplectic form (7.2.6), and working in projective coordinates $(w : z)$ on $\mathbb{P}^1$, it is also possible to show that the variables $\bar{w}$ and $\bar{z}$ become identified after quantization with the conjugate momenta

$$\tilde{\lambda}\bar{w} \sim -\partial_w, \quad \tilde{\lambda}\bar{z} \sim -\partial_z. \tag{7.2.9}$$

Example: $SL(2, \mathbb{C})$

Let us write an element of $SL(2, \mathbb{C})$ in complex coordinates as

$$g = \begin{pmatrix} w & -x \\ z & y \end{pmatrix}, \quad wy + zx = 1, \qquad z, w, x, y \in \mathbb{C}\,. \tag{7.2.10}$$

We can also put $w = b+ic$, $y = b-ic$, $z = d+ie$, $x = d-ie$ (four new complex coordinates); then $SL(2, \mathbb{C}) = \{b^2 + c^2 + d^2 + e^2 = 1\} \simeq T^*S^3 \simeq T^*(SU(2))$.

There is a single Cartan subgroup (or maximal torus) $H = \mathbb{T}_{\mathbb{C}}$, and almost all of $SL(2, \mathbb{C})$ is conjugate to it. In the coadjoint orbit $SL(2, \mathbb{C})/\mathbb{T}_{\mathbb{C}}$, there is a rescaling symmetry $(w, z, x, y) \mapsto (aw, az, a^{-1}x, a^{-1}y)$ for $a \in \mathbb{C}^*$. A slightly nontrivial argument shows that $SL(2, \mathbb{C})/T_{\mathbb{C}}$ is isomorphic to $\{b^2 + c^2 + f^2 = 1\} \in \mathbb{C}^3$ (we have essentially set $f^2 = zx$ and used the scaling to set $x = 1$ or $z = 1$, or both, except at a point). Therefore, a genetic coadjoint orbit looks like

$$\Omega \simeq SL(2, \mathbb{C})/\mathbb{T}_C \simeq T^*S^2. \tag{7.2.11}$$

Unlike the case of $SU(2)$, this is a noncompact manifold, and will lead to infinite-dimensional Hilbert spaces (representations) upon geometric quantization.

In the case of $SU(2)$, we implicitly extended the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ to an isomorphism $(\mathfrak{g}^*)_{\mathbb{C}} \simeq \mathfrak{g}$ by complex linearity. We need do the same for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, viewed as a *real* Lie algebra in order to access principal series representations. To simplify notation, let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ (rather than using $\mathfrak{g}_{\mathbb{C}}$). Then, as in Section 7.1, the complexification of this real Lie algebra satisfies $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g} \times \mathfrak{g}$. An appropriate complexification of the nondegenerate trace form ReTr is given by $\langle\, , \,\rangle : \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g} \simeq (\mathfrak{g} \times \mathfrak{g}) \times \mathfrak{g} \to \mathbb{C}$, such that

$$\langle (X_L, X_R), Y\rangle = \frac{1}{2}\mathrm{Tr}(X_L Y) + \frac{1}{2}\mathrm{Tr}(X_R \bar{Y}\,, \tag{7.2.12}$$

where $\bar{Y}$ denotes the usual complex conjugation in the ("broken") complex structure on $\mathfrak{sl}(2,\mathbb{C})$. This leads to an identification of linear functionals and matrices

$$\nu \in (\mathfrak{a}^*)^{\mathbb{C}} \;\leftrightarrow\; \nu = -\frac{iw}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \kappa \in (\mathfrak{m}^*)^{\mathbb{C}} \;\leftrightarrow\; \kappa = -\frac{i\not{k}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2.13}$$

The pair $\nu \oplus \kappa \in (\mathfrak{m}^* \oplus \mathfrak{a}^*)^{\mathbb{C}} \subset (\mathfrak{g}^*)^{\mathbb{C}}$ is identified with

$$\nu \oplus \kappa \;\;\leftrightarrow\;\; (X_L, X_R) = (\nu + \kappa, \nu - \kappa) \;\; \in \mathfrak{sl}(2,\mathbb{C})^{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})\,. \tag{7.2.14}$$

The natural symplectic form resulting from the above isomorphism is

$$\omega_{\nu,\kappa} = \frac{1}{2}\mathrm{Tr}((\nu+\kappa)g^{-1}dg \wedge g^{-1}dg) + \frac{1}{2}\mathrm{Tr}((\nu-\kappa)\overline{g^{-1}dg \wedge g^{-1}dg}). \tag{7.2.15}$$

Due to the compact cycle S^2 in the coadjoint orbit $\Omega \simeq T^*S^2$, this form can is the curvature of a line bundle $\mathcal{L}$ if and only if $k \in \mathbb{Z}$. Moreover, when the line bundle $\mathcal{L}$ exists, it will be unique (since T^*S^2 is simply-connected). Using a polarization in which sections of $\mathcal{L}$ depend only on a single complex projective coordinate $(w : z)$, we see that the sections induced from $e^{i\nu + i\kappa + \rho}$ will transform as

$$f(aw, az, \bar{a}\bar{w}, \bar{a}\bar{z}) = a^{-\frac{1}{2}(w+\not{k})-1}\bar{a}^{-\frac{1}{2}(w-\not{k})-1} f(w, z, \bar{w}, \bar{z}), \quad a \in \mathbb{C}^*. \tag{7.2.16}$$

This, of course, is precisely the right transformation for the principal series representation $\mathcal{P}^{\not{k},w}$.

The conjugate momenta to the complex projective coordinates $(w : z)$ and $(\bar{w} : \bar{z})$ are

$$x \sim \frac{-2}{w+\not{k}}\partial_z, \quad y \sim \frac{-2}{w+\not{k}}\partial_w, \quad \bar{x} \sim \frac{-2}{w-\not{k}}\bar{\partial}_z, \quad \bar{y} \sim \frac{-2}{w-\not{k}}\bar{\partial}_w\,. \tag{7.2.17}$$

On sections of $\mathcal{L}$, one has that

$$z\partial_z + w\partial_w = -\frac{1}{2}(w+\not{k}) - 1, \qquad \bar{z}\bar{\partial}_z + \bar{w}\bar{\partial}_w = -\frac{1}{2}(w-\not{k}) - 1. \tag{7.2.18}$$

7.3 Quantum mechanics for Wilson loops

For compact gauge groups, Wilson loops in a representation R supported on a curve C are typically written as

$$W_R(C;\mathcal{A}) = \mathrm{Tr}_R\,\mathrm{Hol}_C\mathcal{A} = \mathrm{Tr}_R\left[P\oint_C e^{\mathcal{A}}\right]. \tag{7.3.1}$$

Unfortunately, "Tr_R" does not entirely make sense for infinite-dimensional representations of complex gauge groups. It may be possible to replace this trace with a distributional character from the mathematical theory of reductive Lie groups in order to define infinite-dimensional Wilson loops. However, a better approach was suggested during the early development of Chern-Simons theory with compact gauge group: Wilson loops as in (7.3.1) can alternatively be written as path integrals in a quantum mechanical theory of particles moving along the loop [122, 128, 28]. Specifically, one has

$$W_R(K;\mathcal{A}) = \int_{LG/LH}\mathcal{D}g\,e^{iS[g]}, \qquad S[g] = \int_C \mathrm{Tr}(\lambda g^{-1}(d+\mathcal{A})g)\,, \tag{7.3.2}$$

For compact groups, λ is the highest weight of the finite-dimensional representation R, identified with an element of $\mathfrak{g}$ via Tr. In this section, we will explain why (7.3.2) makes sense for compact groups (a thorough discussion seems absent from the literature). Then we will show that a version of this formula works perfectly for noncompact groups when λ is identified with the quasi-character of an induced representation.

In the path integral (7.3.2), we let $0 \geq t < 2\pi$ be the time coordinate along the loop C. The path integral goes over maps $g : C \to G$, in other words over the loop group LG. There is, however, a gauge symmetry: any right-translation $g \mapsto gh$, with h belonging to the isotropy group H_λ of λ for all t, leaves the integrand e^{iS} invariant. (This holds for compact or noncompact G.) Thus one really integrates over the space LG/LH.

Let $\mathcal{A}_t dt$ be the component of $\mathcal{A}$ along the loop C. The Lagrangian of the action in (7.3.2) is $L = \mathrm{Tr}(\lambda g^{-1}(\partial_t g - A_t g))$. If we somewhat naively take g_{ij} to be our coordinates, we find that canonical momenta are

$$p_{ij} = \frac{\partial L}{\partial(\partial_t g_{ij})} = (\lambda g^{-1})_{ji}.$$

We can then construct a (local) symplectic form for this system as

$$\omega = dp_{ij}\wedge dg_{ij} = \mathrm{Tr}(\lambda dg^{-1}\wedge dg) = \mathrm{Tr}((-\lambda)g^{-1}dg\wedge g^{-1}dg). \tag{7.3.3}$$

Finally, to find the (classical) Hamiltonian, we calculate

$$\mathcal{H} = p_{ij}\partial_t g_{ij} - L = -\mathrm{Tr}(\lambda g^{-1} A_t g). \tag{7.3.4}$$

In order to justify (7.3.2), we will proceed to quantize the path integral in a Hamiltonian formalism, and rewrite the partition function as a trace over an appropriate Hilbert space. To this end, consider the system at some fixed time t. The classical phase space of the system is just G/H_λ, since all elements of presumed canonical momenta λg^{-1} are just algebraic functions of the elements of g. (Put another way, this is a first-order Lagrangian, so both momenta and coordinates are contained in g.) But then the symplectic form (7.3.3) on G/H_λ is nothing but the canonical symplectic form (7.2.3) on the coadjoint orbit of weight $-\lambda \in \mathfrak{g}^*$, at least if G is compact. Therefore, we immediately conclude that the Hilbert space of this quantum mechanical system is the space of the representation with highest weight λ.

Somewhat less trivially, we claim that the quantization of the Hamiltonian $i\mathcal{H} = -i\mathrm{Tr}(\lambda g^{-1}\mathcal{A}_t g)$ is precisely the matrix $\mathcal{A}_t$ acting on the Hilbert space in the representation with highest weight λ. In other words, conjugation by a quantum operator g effectively converts matrices into the right coadjoint orbit representation! Then we can simply "cut" the Wilson loop in the path integral at time $t = 0$ and use to Hamiltonian formalism to see that

$$\int \mathcal{D}g\, e^{iS[g]} = \mathrm{Tr}_{\text{Hilbert space}}\, T \,\exp\left(\int_0^{2\pi} i\mathcal{H}(t)\, dt\right), \tag{7.3.5}$$

becomes precisely the holonomy of the gauge connection $\mathcal{A}$ (where T is a time ordering that because the Wilson loop path ordering).

For $G = SU(2)$, the claim that the Hamiltonian is just the matrix $\mathcal{A}$ acting on the Hilbert space can be proven explicitly by writing $\mathcal{A}_t = aH + bE + cF$ and simply expressing $-i\mathrm{Tr}(\lambda g^{-1} \cdot g)$ in terms of quantum operators for each generator H, E, F. For example, with λ as in (7.2.8) and operators as in (7.2.9) we find

$$\begin{aligned} -i\mathrm{Tr}(\lambda g^{-1} E g) &= -\frac{\tilde{\lambda}}{2}\mathrm{Tr}\left[\begin{pmatrix} \bar{w} & \bar{z} \\ z & -w \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix}\right] \\ &= -\tilde{\lambda} z \bar{w} \\ &= -z\partial_w \\ &\simeq -\tilde{\lambda} z + z^2 \partial_z\,, \end{aligned}$$

and similarly $-i\text{Tr}(\lambda g^{-1}Hg) = -\tilde{\lambda} + 2z\partial_z$ and $-i\text{Tr}(\lambda g^{-1}Fg) = -\partial_z$.

Wilson loops for complex groups

In the case of $G_{\mathbb{C}} = SL(2,\mathbb{C})$, we propose that a Wilson loop in representation $\mathcal{P}^{\not{k},w}$ be defined by the path integral (7.3.2), but with an action

$$S = -\frac{1}{2}\int_C \text{Tr}[(\nu+\kappa)g^{-1}(d+\mathcal{A})g + (\nu-\kappa)\overline{g^{-1}(d+\mathcal{A})g}], \tag{7.3.6}$$

This modified action will lead exactly to the symplectic form (7.2.15) relevant for constructing principal series representations from coadjoint orbits.

To show that the resulting Hamiltonian $i\mathcal{H} = \frac{i}{2}\text{Tr}[(\nu+\kappa)g^{-1}\mathcal{A}g + (\nu-\kappa)\bar{g}^{-1}\bar{\mathcal{A}}\bar{g}]$ is again just $\mathcal{A}$ in the $\mathcal{P}^{\not{k},w}$ representation, we can use the operator expressions (7.2.17) and explicitly derive the operators for basis elements $\mathcal{A} = H, \tilde{H}, E, \tilde{E}, F, \tilde{F}$ just as in the $SU(2)$ case.

For other complex gauge groups, this procedure can be easily generalized. The action (7.3.6) will still involve a sum of two terms, but ν and κ need to be replaced by the relevant linear functionals that induce the desired principal series representation.

7.4 Wilson loops *vs.* boundary conditions

As a culmination of the above theory and proposal for defining Wilson loops with infinite-dimensional representations, we can finally derive the relation between a partition function on a knot complement and a partition function with a knot in Chern-Simons theory with complex gauge group. We obtain a map between boundary conditions and representations, and use it to explain the limits used in the “analytic continuation” of Chapter 6.

As a starting point, for arbitrary gauge group G and a Euclidean 3-manifold M, consider the action

$$I_{CS}[\mathcal{A}] = \frac{1}{4\pi}\int_M \text{Tr}\big(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big)\,. \tag{7.4.1}$$

If we put the theory on a manifold of topology $\mathbb{R}\times\Sigma$ (or $S^1\times\Sigma$), where Σ is some 2-surface, we can split the gauge field into a component A^t along “time” $\mathbb{R}$, and a “spatial” component

$\mathcal{A}^\perp$. With suitable boundary conditions on $\mathcal{A}^t$ at ∂M, we can integrate by parts and rewrite

$$I_{CS}[\mathcal{A}] = \frac{1}{4\pi}\int_M \mathrm{Tr}(\mathcal{A}^\perp \wedge d\mathcal{A}^\perp) + \frac{1}{2\pi}\int_M F^\perp \wedge \mathcal{A}^t\,, \tag{7.4.2}$$

where $F^\perp = d\mathcal{A}^\perp + \mathcal{A}^\perp \wedge \mathcal{A}^\perp$ is the curvature of $A^\perp$. Then the field A^t becomes nondynamical, and we can integrate it out to obtain a condition

$$F^\perp = 0 \tag{7.4.3}$$

which the new path integral in $\mathcal{A}^\perp$ must obey. This is the origin of the reasoning that we restrict to moduli spaces of flat connections in geometric quantization: the classical phase space of the theory consists of flat connections $\mathcal{A}^\perp$ on Σ (at fixed t), modulo gauge transformations.

We want to specialize the theory to compact and complex gauge groups, and to show how condition (7.4.3) is altered by Wilson loops. The compact case was considered in (*e.g.*) [122], but we review it because it will be relevant for analytic continuation.

Representations and boundary conditions for compact groups

Chern-Simons theory with compact gauge group (say $SU(N)$) has an action $S = kI_{CS}$. Gauge-invariance under large gauge transformations on M force the constant k to be an integer. By fixing orientation on M, one usually takes k to be a positive integer – the level of the theory.

Now put the theory on $S^1 \times \Sigma$ and add a Wilson loop in representation λ parallel to the S^1. Let t be the coordinate on S^1 as before, and let (x_1, x_2) be local coordinates on Σ, so that the Wilson loop goes through $(x_1, x_2) = 0$. From Section 7.3, we know that we can write the path integral of the theory with the Wilson loop insertion as $\int \mathcal{D}\mathcal{A}\,\mathcal{D}g\, e^{iS}$, where the full effective action is

$$\begin{aligned} S &= kI_CS + \int_C \mathrm{Tr}(\lambda g^{-1}(d+\mathcal{A})g) \\ &= kI_{CS} + \int_M \delta(x) dx_1 \wedge dx_2 \wedge \mathrm{Tr}(\lambda g^{-1}(d+\mathcal{A}^t)g)\,. \end{aligned} \tag{7.4.4}$$

Therefore, we can again integrate out $\mathcal{A}^t$. However, instead of complete flatness in the perpendicular directions, we now obtain the constraint[2]

$$\frac{k+h^\vee}{2\pi} F^\perp + g(\lambda+\rho)g^{-1}\delta(x)dx_1 \wedge dx_2 = 0. \tag{7.4.5}$$

[2]When performing this integration in the full quantum theory for compact gauge groups, two

In other words, $\mathcal{A}^\perp$ must be flat everywhere on Σ *except* at the point $x = 0$. At fixed time t_0, we can take a small disc $D \subset \Sigma$ around $x = 0$ and integrate $F^\perp$ on it to find

$$\frac{k + h^\vee}{2\pi} \int_D F^\perp = -g(t_0)(\lambda + \rho) g^{-1}(t_0) = \frac{k}{2\pi} \log \mathrm{Hol}_{\partial D}(\mathcal{A}^\perp).$$

For $SU(2)$ (for example), this argument shows that the holonomy of $\mathcal{A}^\perp$ linking the Wilson loop is always conjugate to

$$\mathrm{Hol}_{\partial D}(\mathcal{A}^\perp) \overset{conj}{\sim} \begin{pmatrix} e^{\frac{i\pi}{k+2}N} & 0 \\ 0 & e^{-\frac{i\pi}{k+2}N} \end{pmatrix}, \tag{7.4.6}$$

where $N = \tilde{\lambda} + 1 \sim \lambda + \rho$ is the dimension of a given representation.

Representations and boundary conditions for complex groups

Let us consider $SL(2, \mathbb{C})$ theory. The Chern-Simons action is (5.1.1) is $\frac{t}{2} I_{CS}[\mathcal{A}] + \frac{\bar{t}}{2} I_{CS}[\bar{\mathcal{A}}]$. The appropriate Wilson loop action is given by (7.3.6). Treating $\mathcal{A}$ and $\bar{\mathcal{A}}$ as independent fields, we can separately integrate out both $\mathcal{A}^t$ and $\bar{\mathcal{A}}^t$, yielding *two* constraints:

$$\frac{t}{4\pi} F^\perp - \frac{1}{2} g(\nu + \kappa) g^{-1}\, \delta(x) dx_1 \wedge dx_2 = 0, \tag{7.4.7a}$$
$$\frac{\bar{t}}{4\pi} \bar{F}^\perp - \frac{1}{2} \bar{g}(\nu - \kappa) \bar{g}^{-1}\, \delta(x) dx_1 \wedge dx_2 = 0. \tag{7.4.7b}$$

These only makes sense if F and $\bar{F}$ obey conjugate equations.

We can also look at restrictions imposed by unitarity. As explained in Section 5.1, there are two possible unitarity structures for Chern-Simons theory with complex gauge group. In both structures, constraints (7.4.7) become compatible if unitary principal series representations ($w \in i\mathbb{R}$, $k \in \mathbb{Z}$) are used for the Wilson loop. In the Chern-Simons unitarity structure relevant for Euclidean quantum gravity, with $t, \bar{t} \in \mathbb{R}$, representations with $w \in \mathbb{R}$ are also allowed. Therefore, it may be possible to have Wilson loops with complementary series representations as well.[3]

---

matching shifts of "coupling constants" happen: the Chern-Simons level k is shifted to $k + h^\vee$ (where $h^\vee$ is the dual Coxeter number of G), and the weight λ is shifted to $\lambda + \rho$ (where ρ is half the sum of positive roots). Neither of these happen in the complex case [48].

[3]This may explain a short remark by Witten in [39] that the unitarity structure relevant for Euclidean quantum gravity is related to complementary series representations.

The holonomy of $\mathcal{A}$ is given generically by

$$\mathrm{Hol}_{\partial D}(\mathcal{A}^{\perp}) \overset{conj}{\sim} \begin{pmatrix} e^{-\frac{i\pi}{t}(w+k)} & 0 \\ 0 & e^{\frac{i\pi}{t}(w+k)} \end{pmatrix}. \tag{7.4.8}$$

Again, for more general complex gauge group this expression will simply be generalized to contain various eigenvalues of the linear functionals or quasi-characters that characterize principal series representations.

TQFT

The above arguments relate representations and conjugacy classes of the holonomy of $\mathcal{A}$. Another way of phrasing the results is that the Chern-Simons path integral on a solid torus $D^2 \times S^1$ with a knot in its center is always an exact delta-function that forces the holonomy of $\mathcal{A}$ around the (contractible) cycle on the boundary of this torus to be determined by the representation on the knot. By cutting solid neighborhoods of knots out of three-manifolds, this then shows that a partition function with a Wilson loop is equivalent to a partition function on the loop's complement with fixed boundary conditions. A somewhat longer discussion of this relation appears in the review [6].

Analytic continuation

Let us finally use these results to motivate the limit used in the analytic continuation of Chern-Simons theory with compact group G to Chern-Simons theory with complex group $G_{\mathbb{C}}$.

In Chapter 6, we considered $SL(2,\mathbb{C})$ (say) Chern-Simons theory on a knot complement, with a boundary condition that the holomony of $\mathcal{A}$ on a small loop linking the knot was

$$\mathrm{Hol}_{bdy}(\mathcal{A}) \overset{conj}{\sim} \begin{pmatrix} e^{u} & 0 \\ 0 & e^{-u} \end{pmatrix}. \tag{7.4.9}$$

Comparing this to the holonomy (7.4.6) of $SU(2)$ theory, we see that

$$u \equiv \frac{i\pi N}{k+2} = N\hbar\,. \tag{7.4.10}$$

Thus, one must "analytically continue" at this fixed value of $N/(k+2)$. For general groups

G and $G_{\mathbb{C}}$, the relation will be

$$u \equiv \frac{i\pi(\lambda+\rho)}{k+h^{\vee}} = (\lambda+\rho)\hbar\,. \tag{7.4.11}$$

From the relation (7.4.8), we also finally see that u is related to a principal series $SL(2,\mathbb{C})$ representation $\mathcal{P}^{\not{k},w}$ via

$$u = (-)\frac{i\pi(w+\not{k})}{t}\,. \tag{7.4.12}$$

Chapter 8

The state integral model

In this chapter, we introduce a "state integral" model for $Z^{(\rho)}(M;\hbar)$ in the simplest case of $G_{\mathbb{C}} = SL(2,\mathbb{C})$. Our construction will rely heavily on the work of Hikami [52, 53], where a new invariant of hyperbolic 3-manifolds was introduced using ideal triangulations. The resulting invariant is very close to the state integral model we are looking for. However, in order to make it into a useful tool for computing $Z^{(\rho)}(M;\hbar)$ we will need to understand Hikami's construction better and make a number of important modifications. In particular, as we explain below, Hikami's invariant is written as a certain integral along a path in the complex plane (or, more generally, over a hypersurface in complex space) which was ill-defined[1] in the original work [52, 53]. Another issue that we need to address is how to incorporate in Hikami's construction a choice of the homomorphisms (5.1.4) (page 82),

$$\rho : \pi_1(M) \to SL(2,\mathbb{C})\,. \tag{8.0.1}$$

(The original construction assumes very special choices of ρ that we called "geometric" in Section 5.2.) It turns out that these two questions are not unrelated and can be addressed simultaneously, so that Hikami's invariant can be extended to a state sum model for $Z^{(\rho)}(M;\hbar)$ with an arbitrary ρ.

To properly describe the state integral model, we will need a more thorough review of the properties of hyperbolic manifolds and hyperbolic triangulations, presented in Section

[1] The choice mentioned in [52, 53] is to integrate over the real axis (resp. real subspace) of the complex parameter space. While this choice is in some sense natural, it encounters some very bad singularities and a closer look shows that it cannot be correct.

8.1. The state integral model that we give here depends on ideal hyperbolic triangulations.[2] We will also take a small digression in Section 8.3 to define and discuss the main properties of the *quantum dilogarithm*. This function is central to the state integral model, and has also appeared previously in the description of motivic BPS invariants, back in Chapter 4.

Throughout this chapter, we work in the u-space representation for $SL(2,\mathbb{C})$ partition functions. In particular, we use the identification (6.1.16) and denote the perturbative $SL(2,\mathbb{C})$ invariant as $Z^{(\alpha)}(M;\hbar,u)$. The discussion here follows [3].

8.1 Hyperbolic geometry

The construction of a state integral model described in the rest of this chapter applies to orientable hyperbolic 3-manifolds of finite volume (possibly with boundary) and uses ideal triangulations in a crucial way. Therefore, we begin this section by reviewing some relevant facts from hyperbolic geometry (more details can be found in [113, 129, 130]).

Recall that hyperbolic 3-space $\mathbb{H}^3$ can be represented as the upper half-space $\{(x_1,x_2,x_3)|\, x_3>0\}$ with metric (5.2.7) of constant curvature -1. The boundary $\partial\mathbb{H}^3$, topologically an $\mathbf{S}^2$, consists of the plane $x_3=0$ together with the point at infinity. The group of isometries of $\mathbb{H}^3$ is $PSL(2,\mathbb{C})$, which acts on the boundary via the usual Möbius transformations. In this picture, geodesic surfaces are spheres of any radius which intersect $\partial\mathbb{H}^3$ orthogonally.

An ideal tetrahedron Δ in $\mathbb{H}^3$ has by definition all its faces along geodesic surfaces, and all its vertices in $\partial\mathbb{H}^3$ — such vertices are called *ideal points*. After Möbius transformations, one can fix three of the vertices at $(0,0,0)$, $(1,0,0)$, and infinity. The coordinate of the fourth vertex $(x_1,x_2,0)$, with $x_2\geq 0$, defines a complex number $z=x_1+ix_2$ called the *shape parameter* (sometimes also called *edge parameter*). At various edges, the faces of the tetrahedron Δ form dihedral angles $\arg z_j$ $(j=1,2,3)$ as indicated in Figure 8.1, with

$$z_1=z\,,\quad z_2=1-\frac{1}{z}\,,\quad z_3=\frac{1}{1-z}\,. \tag{8.1.1}$$

The ideal tetrahedron is noncompact, but has finite volume given by

$$\mathrm{Vol}(\Delta_z)\;=\;D(z)\,, \tag{8.1.2}$$

[2]It is nevertheless fairly clear that it should work for general three-manifolds, in part because any three-manifold with boundary *has* an ideal topological triangulation even when its hyperbolic volume is zero. A description of the state sum model for torus knots will appear in [5].

where $D(z)$ is the Bloch-Wigner dilogarithm function, related to the usual dilogarithm Li_2 (see Section 8.3) by

$$D(z) = \mathrm{Im}\big(\mathrm{Li}_2(z)\big) + \arg(1-z)\,\log|z|\,. \tag{8.1.3}$$

Note that any of the z_j can be taken to be the shape parameter of Δ, and that $D(z_j) = \mathrm{Vol}(\Delta_z)$ for each j. We will allow shape parameters to be any complex numbers in $\mathbb{C}-\{0,1\}$, noting that for $z \in \mathbb{R}$ an ideal tetrahedron is degenerate and that for $\mathrm{Im}\,z < 0$ it technically has negative volume due to its orientation.

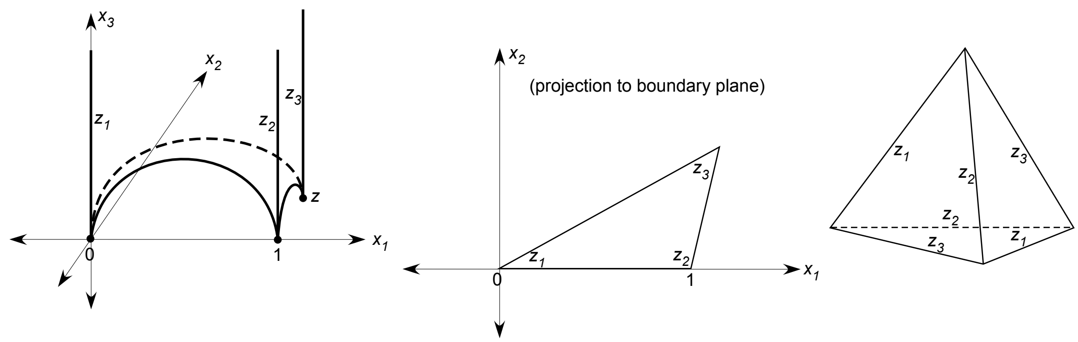


Figure 8.1: An ideal tetrahedron in $\mathbb{H}^3$.

A hyperbolic structure on a 3-manifold is a metric that is locally isometric to $\mathbb{H}^3$. A 3-manifold is called hyperbolic if it admits a hyperbolic structure that is geodesically complete and has finite volume. Most 3-manifolds are hyperbolic, including the vast majority of knot and link complements in $\mathbf{S}^3$. Specifically, a knot complement is hyperbolic as long as the knot is not a torus or satellite knot [131]. Every closed 3-manifold can be obtained via Dehn surgery on a knot in $\mathbf{S}^3$, and for hyperbolic knots all but finitely many such surgeries yield hyperbolic manifolds [113].

By the Mostow rigidity theorem [132, 133], the complete hyperbolic structure on a hyperbolic manifold is unique. Therefore, geometric invariants like the hyperbolic volume are actually topological invariants. For the large class of hyperbolic knot complements in $\mathbf{S}^3$, the unique complete hyperbolic structure has a parabolic holonomy with unit eigenvalues around the knot. In $SL(2,\mathbb{C})$ Chern-Simons theory, this structure corresponds to the "geometric" flat connection $\mathcal{A}^{(\mathrm{geom})}$ with $u = 0$. As discussed in Section 5.2, hyperbolic manifolds with complete hyperbolic structures can also be described as quotients $\mathbb{H}^3/\Gamma$.

Given a hyperbolic knot complement, one can deform the hyperbolic metric in such a way that the holonomy u is not zero. Such deformed metrics are unique in a neighborhood of $u = 0$, but they are not geodesically complete. For a discrete set of values of u, one can add in the "missing" geodesic, and the deformed metrics coincide with the unique complete hyperbolic structures on closed 3-manifolds obtained via appropriate Dehn surgeries on the knot in $\mathbf{S}^3$. For other values of u, the knot complement can be completed by adding either a circle S^1 or a single point, but the resulting hyperbolic metric will be singular. For example, if $u \in i\mathbb{R}$ one adds a circle and the resulting metric has a conical singularity. These descriptions can easily be extended to link complements (*i.e.* multiple cusps), using multiple parameters u_k, one for each link component.

Any orientable hyperbolic manifold M is homeomorphic to the interior of a compact 3-manifold $\bar{M}$ with boundary consisting of finitely many tori. (The manifold M itself can also be thought of as the union of $\bar{M}$ with neighborhoods of the cusps, each of the latter being homeomorphic to $T^2 \times [0, \infty)$.) All hyperbolic manifolds therefore arise as knot or link complements in closed 3-manifolds. Moreover, every hyperbolic manifold has an ideal triangulation, *i.e* a finite decomposition into (possibly degenerate[3]) ideal tetrahedra; see *e.g.* [113, 134].

To reconstruct a hyperbolic 3-manifold M from its ideal triangulation $\{\Delta_i\}_{i=1}^N$, faces of tetrahedra are glued together in pairs. One must remember, however, that vertices of tetrahedra are not part of M, and that the combined boundaries of their neighborhoods in M are not spheres, but tori. (Thus, some intuition from simplicial triangulations no longer holds.) There always exists a triangulation of M whose edges can all be oriented in such a way that the boundary of every face (shared between two tetrahedra) has two edges oriented in the same direction (clockwise or counterclockwise) and one opposite. Then the vertices of each tetrahedron can be canonically labeled 0, 1, 2, 3 according to the number of edges entering the vertex, so that the tetrahedron can be identified in a unique way with one of the two numbered tetrahedra shown in Figure 8.3 of the next subsection. This at the same time orients the tetrahedron. The orientation of a given tetrahedron Δ_i may not agree with that of M; one defines $\epsilon_i = 1$ if the orientations agree and $\epsilon_i = -1$ otherwise. The edges of each tetrahedron can then be given shape parameters $(z_1^{(i)}, z_2^{(i)}, z_3^{(i)})$, running counterclockwise around each vertex (viewed from outside the tetrahedron) if $\epsilon_i = 1$ and clockwise if $\epsilon_i = -1$.

For a given M with cusps or conical singularities specified by holonomy parameters u_k,

[3]It is conjectured and widely believed that nondegenerate tetrahedra alone are always sufficient.

the shape parameters $z_j^{(i)}$ of the tetrahedra Δ_i in its triangulation are fixed by two sets of conditions. First, the product of the shape parameters $z_j^{(i)}$ at every edge in the triangulation must be equal to 1, in order for the hyperbolic structures of adjacent tetrahedra to match. More precisely, the sum of some chosen branches of $\log z_j^{(i)}$ (equal to the standard branch if one is near the complete structure) should equal $2\pi i$, so that the total dihedral angle at each edge is 2π. Second, one can compute holonomy eigenvalues around each torus boundary in M as a product of $z_j^{(i)}$'s by mapping out the neighborhood of each vertex in the triangulation in a so-called developing map, and following a procedure illustrated in, *e.g.*, [114]. There is one distinct vertex "inside" each boundary torus. One then requires that the eigenvalues of the holonomy around the kth boundary are equal to $e^{\pm u_k}$. These two conditions will be referred to, respectively, as *edge* and *cusp* relations.

Every hyperbolic 3-manifold has a well-defined class in the Bloch group [135]. This is a subgroup[4] of the quotient of the free $\mathbb{Z}$-module $\mathbb{Z}[\mathbb{C} - \{0, 1\}]$ by the relations

$$[x] - [y] + [\frac{y}{x}] - [\frac{1-x^{-1}}{1-y^{-1}}] + [\frac{1-x}{1-y}] = 0\,. \tag{8.1.4}$$

This five-term or pentagon relation accounts for the fact that a polyhedron with five ideal vertices can be decomposed into ideal tetrahedra in multiple ways. The five ideal tetrahedra in this polyhedron (each obtained by deleting an ideal vertex) can be given the five shape parameters x, y, $y/x, \ldots$ appearing above. The signs of the different terms correspond to orientations. Geometrically, an instance of the five-term relation can be visualized as the 2-3 Pachner move, illustrated in Figure 8.2.

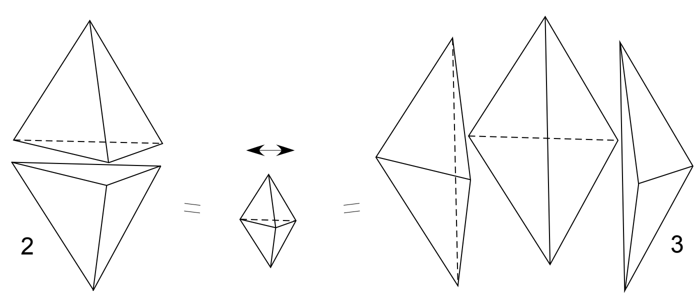


Figure 8.2: The 2-3 Pachner move.

The class $[M]$ of a hyperbolic 3-manifold M in the Bloch group can be computed by summing (with orientation) the shape parameters $[z]$ of any ideal triangulation, but it

[4]Namely, the kernel of the map $[z] \mapsto 2z \wedge (1-z) \in \mathbb{C}^* \wedge_{\mathbb{Z}} \mathbb{C}^*$ acting on this quotient module.

is *independent* of the triangulation. Thus, hyperbolic invariants of 3-manifolds may be obtained by functions on the Bloch group — *i.e.* functions compatible with (8.1.4). For example, the Bloch-Wigner function (8.1.3) satisfies

$$D(x) - D(y) + D\big(\frac{y}{x}\big) - D\big(\frac{1-x^{-1}}{1-y^{-1}}\big) + D\big(\frac{1-x}{1-y}\big) = 0\,, \tag{8.1.5}$$

and the hyperbolic volume of a manifold M triangulated by ideal tetrahedra $\{\Delta_i\}_{i=1}^N$ can be calculated as

$$\mathrm{Vol}(M) = \sum_{i=1}^{N} \epsilon_i D(z^{(i)})\,. \tag{8.1.6}$$

The symbols ϵ_i here could be removed if shape parameters were assigned to tetrahedra in a manner independent of orientation, noting that reversing the orientation of a tetrahedron corresponds to sending $z \mapsto 1/z$ and that $D(1/z) = -D(z)$. This is sometimes seen in the literature.

The complexified volume $i(\mathrm{Vol}(M) + i\mathrm{CS}(M))$ is trickier to evaluate. For a hyperbolic manifold with a spin structure, corresponding to full $SL(2,\mathbb{C})$ holonomies, this invariant is defined modulo $4\pi^2$. Here, we will outline a computation of the complexified volume modulo π^2, following [136]; for the complete invariant modulo $4\pi^2$, see [137]. To proceed, one must first make sure that the three shape parameters $z^{(i)} = z_1^{(i)}$, $z_2^{(i)}$, and $z_3^{(i)}$ are specifically assigned to edges $[v_1^{(i)}, v_2^{(i)}]$, $[v_1^{(i)}, v_3^{(i)}]$, and $[v_1^{(i)}, v_2^{(i)}]$ (respectively) in each tetrahedron Δ_i of an oriented triangulation of M, where $[v_a^{(i)}, v_b^{(i)}]$ denotes the edge going from numbered vertex $v_a^{(i)}$ to numbered vertex $v_b^{(i)}$. One also chooses logarithms $(w_1^{(i)}, w_2^{(i)}, w_3^{(i)})$ of the shape parameters such that

$$e^{w_1^{(i)}} = \pm z_1^{(i)}\,, \quad e^{w_2^{(i)}} = \pm z_2^{(i)}\,, \quad e^{w_3^{(i)}} = \pm z_3^{(i)}\,, \tag{8.1.7a}$$

$$w_1^{(i)} + w_2^{(i)} + w_3^{(i)} = 0 \quad \forall\, i\,, \tag{8.1.7b}$$

and defines integers $(q^{(i)}, r^{(i)})$ by

$$w_1^{(i)} = \mathrm{Log}(z^{(i)}) + \pi i q^{(i)}\,, \qquad w_2^{(i)} = -\mathrm{Log}(1 - z^{(i)}) + \pi i r^{(i)}\,, \tag{8.1.8}$$

where Log denotes the principal branch of the logarithm, with a cut from 0 to $-\infty$. For a consistent labeling of the triangulation, called a "combinatorial flattening," the sum of log-parameters $w_j^{(i)}$ around every edge must vanish, and the (signed[5]) sum of log-parameters

[5]Signs arise from tetrahedron orientations and the sense in which a path winds around edges; see [136], Def. 4.2.

along the two paths generating $\pi_1(T^2) = \mathbb{Z}^2$ for any boundary (cusp) T^2 must equal twice the logarithm of the $SL(2,\mathbb{C})$ holonomies around these paths.[6] The complexified volume is then given, modulo π^2, as

$$i(\mathrm{Vol}(M) + i\mathrm{CS}(M)) \;=\; \sum_{i=1}^{N} \epsilon_i \mathrm{L}(z^{(i)}; q^{(i)}, r^{(i)}) - \sum_{\text{cusps } k} (v_k \overline{u}_k + i\pi u_k)\,, \tag{8.1.9}$$

with

$$\mathrm{L}(z;q,r) = \mathrm{Li}_2(z) + \frac{1}{2}(\mathrm{Log}(z) + \pi i q)(\mathrm{Log}(1-z) + \pi i r) + \frac{\pi^2 qr}{2} - \frac{\pi^2}{6}\,. \tag{8.1.10}$$

The function $\mathrm{L}(z;q,r)$, a modified version of the Rogers dilogarithm, satisfies a five-term relation in an extended Bloch group that lifts (8.1.4) in a natural way to the space of log-parameters.[7]

8.1.1 Example: figure-eight knot complement

Let K be the figure-eight knot (shown in Figure 6.1) and let M be its complement in the 3-sphere. In order to compute the perturbative invariants $S_n^{(\rho)}$ for an arbitrary $\rho:\ \pi_1(M) \to SL(2,\mathbb{C})$ we first need to review the classical geometry of M in more detail and, in particular, to describe the moduli space of flat $SL(2,\mathbb{C})$ connections on M. As we already mentioned earlier, the knot group $\pi_1(M)$ is generated by two elements, a and b, such that $a^{-1}bab^{-1}a = ba^{-1}bab^{-1}$. The corresponding representation into $SL(2,\mathbb{C})$ is given by

$$\rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \rho(b) = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}, \tag{8.1.11}$$

where $\zeta = (-1+\sqrt{-3})/2$ is the cube root of unity, $\zeta^3 = 1$.

The complement of the figure-eight knot can be also represented as a quotient space $\mathbb{H}^3/\Gamma$ (5.2.6), where the holonomy group Γ is generated by the above two matrices. Specifically,

---

[6]Explicitly, in the notation of Section 6.1 and above, the sum of log-parameters along the two paths in the neighborhood of the kth cusp must equal $2u_k$ and $2v_k + 2\pi i$, respectively.

[7]The branch of Li_2 in (8.1.10) is taken to be the standard one, with a cut from 1 to $+\infty$. Note, however, that we could also take care of ambiguities arising from the choice of dilogarithm branch by rewriting (8.1.10) in terms of the function $\tilde{\mathrm{L}}(w) = \int_{-\infty}^{w} \frac{t\,dt}{1-e^{-t}}$. This is a well-defined holomorphic function $: \mathbb{C} \to \mathbb{C}/4\pi^2\mathbb{Z}$ because all the residues of $t/(1-e^t)$ are integer multiples of $2\pi i$, and it coincides with a branch of the function $\mathrm{Li}_2(e^w) + w\log(1-e^w)$.

we have

$$\Gamma \cong PSL(2, \mathcal{O}_{\mathbb{K}})\,, \tag{8.1.12}$$

where $\mathcal{O}_{\mathbb{K}}$ is the ring of integers in the imaginary quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$. The fundamental domain, $\mathcal{F}$, for Γ is described by a geodesic pyramid in $\mathbb{H}^3$ with one vertex at infinity and the other four vertices at the points:

$$\begin{aligned} v_1 &= j \\ v_2 &= \frac{1}{2} - \frac{\sqrt{3}}{6} i + \sqrt{\frac{2}{3}} j \\ v_3 &= \frac{1}{2} + \frac{\sqrt{3}}{6} i + \sqrt{\frac{2}{3}} j \\ v_4 &= \frac{1}{\sqrt{3}} i + \sqrt{\frac{2}{3}} j\,. \end{aligned} \tag{8.1.13}$$

Explicitly, we have

$$\mathcal{F} = \{z + x_3 j \in \mathbb{H}^3 \mid z \in \mathcal{F}_z,\ x_3^2 + |z|^2 \geq 1\}\,, \tag{8.1.14}$$

where $z = x_1 + i x_2$ and

$$\begin{aligned} \mathcal{F}_z = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z),\ \frac{1}{\sqrt{3}}\mathrm{Re}(z) \leq \mathrm{Im}(z),\ \mathrm{Im}(z) \leq \frac{1}{\sqrt{3}}(1 - \mathrm{Re}(z))\} \\ \cup \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq \frac{1}{2},\ -\frac{1}{\sqrt{3}}\mathrm{Re}(z) \leq \mathrm{Im}(z) \leq \frac{1}{\sqrt{3}}\mathrm{Re}(z)\} \end{aligned} \tag{8.1.15}$$

is the fundamental domain of a 2-torus with modular parameter $\tau = \zeta$. The region of large values of x_3 in $\mathcal{F}$ corresponds to the region near the cusp of the figure-eight knot complement M.

The standard triangulation of the figure eight knot complement comprises two ideal tetrahedra of opposite simplicial orientations, as in Figure 8.3, glued together in the only nontrivial consistent manner possible,

$$M = \Delta_z \cup \Delta_w\,. \tag{8.1.16}$$

Here, z and w are complex numbers, representing the shapes of the ideal tetrahedra; we take Δ_z to be positively oriented and Δ_w to be negatively oriented. As explained in Section 8.1, the shape parameters z and w must obey edge relations, which in the case of the figure-eight knot reduce to a single algebraic relation (see *e.g.* Chapter 4 of [113] and Section 15 of [136]):

$$(z-1)(w-1) = z^2 w^2\,. \tag{8.1.17}$$

The shape parameters z and w are related to the $SL(2,\mathbb{C})$ holonomy eigenvalue, l, along the longitude[8] of the knot in the following way

$$\frac{z^2}{z-1} = -l\,, \qquad \frac{w^2}{w-1} = -l^{-1}\,, \tag{8.1.18}$$

which automatically solves the edge condition (8.1.17). Similarly, the holonomy eigenvalue $m = e^u$ around the noncontractible meridian of the torus is given by

$$zw = m^2. \tag{8.1.19}$$

By eliminating z and w from (8.1.17), (8.1.18), and (8.1.19), one obtains the non-abelian irreducible component of the A-polynomial for the figure-eight knot from (6.6.9),

$$m^4 - (1 - m^2 - 2m^4 - m^6 + m^8)l + m^4 l^2 = 0\,. \tag{8.1.20}$$

Given the decomposition (8.1.16) of the 3-manifold M into two tetrahedra, we can find its volume by adding the (signed) volumes of Δ_z and Δ_w,

$$\mathrm{Vol}(M;u) = \mathrm{Vol}(\Delta_z) - \mathrm{Vol}(\Delta_w) = D(z) - D(w)\,, \tag{8.1.21}$$

where the volume of an ideal tetrahedron is given by (8.1.2). Similarly, following the prescription in Section 8.1, the complexified volume can be given by[9]

$$i(\mathrm{Vol}(M;u) + i\mathrm{CS}(M;u)) = L(z;0,0) - L(w;0,0) - v\bar{u} - i\pi u\,. \tag{8.1.22}$$

Notice that due to the edge relation (8.1.17) the total volumes (8.1.21), (8.1.22) are functions of one complex parameter, or, equivalently, a point on the zero locus of the A-polynomial, $A(l,m) = A(-e^v, e^u) = 0$.

From the relation (8.1.18) we find that the point $(l,m) = (-1,1)$ corresponding to the complete hyperbolic structure on M is characterized by the values of z and w which solve the equation

$$z^2 - z + 1 = 0\,. \tag{8.1.23}$$

In order to obtain tetrahedra of positive (signed) volume, we must choose z to be the root of this equation with a positive imaginary part, and w its inverse:

$$z = \frac{1+i\sqrt{3}}{2}\,, \qquad w = \frac{1-i\sqrt{3}}{2}\,. \tag{8.1.24}$$

---

[8]A 1-cycle on $\Sigma = T^2$ which is contractible in the knot complement M.

[9]This expression differs slightly from the one given in [136], because we use $2v + 2\pi i$ rather than $2v$ as the logarithm of the "longitudinal" holonomy, as mentioned in Footnote 6 of Section 8.1.

These value correspond to regular ideal tetrahedra, and maximize (respectively, minimize) the Bloch-Wigner dilogarithm function $D(z)$.

8.2 Hikami's invariant

We can now describe Hikami's geometric construction. Roughly speaking, to compute the invariant for a hyperbolic manifold M, one chooses an ideal triangulation of M, assigns an infinite-dimensional vector space V or V^* to each tetrahedron face, and assigns a matrix element in $V \otimes V \otimes V^* \otimes V^*$ to each tetrahedron. These matrix elements depend on a small parameter $\hbar$, and in the classical $\hbar \to 0$ limit they capture the hyperbolic structure of the tetrahedra. The invariant of M is obtained by taking inner products of matrix elements on every pair of identified faces (gluing the tetrahedra back together), subject to the cusp conditions described above in the classical limit.

To describe the process in greater detail, we begin with an orientable hyperbolic manifold that has an oriented ideal triangulation $\{\Delta_i\}_{i=1}^N$, and initially forget about the hyperbolic structures of these tetrahedra. As discussed in Section 8.1 and indicated in Figure 8.3, each tetrahedron comes with one of two possible orientations of its edges, which induces an ordering of its vertices $v_j^{(i)}$ (the subscript j here is not to be confused with the shape parameter subscript in (8.1.1)), an ordering of its faces, and orientations on each face. The latter can be indicated by inward or outward-pointing normal vectors. The faces (or their normal vectors) are labelled by $p_j^{(i)}$, in correspondence with opposing vertices. The normal face-vectors of adjacent tetrahedra match up head-to-tail (and actually define an oriented dual decomposition) when tetrahedra are glued to form M.

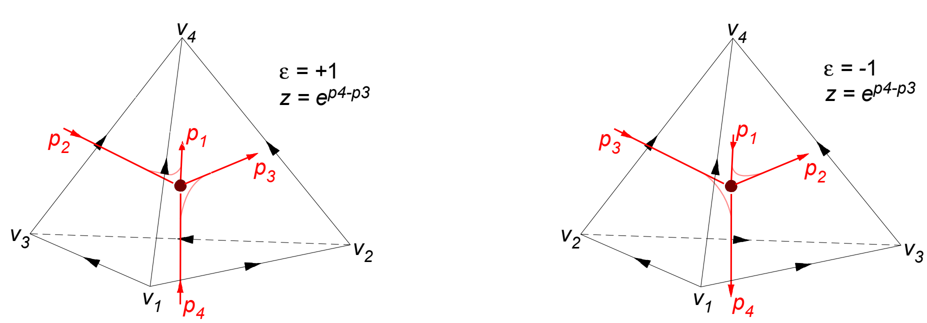


Figure 8.3: Oriented tetrahedra, to which matrix elements $\langle p_1, p_3|\mathbf{S}|p_2, p_4\rangle$ (left) and $\langle p_2, p_4|\mathbf{S}^{-1}|p_1, p_3\rangle$ (right) are assigned.

Given such an oriented triangulation, one associates a vector space V to each inward-oriented face, and the dual space V^* to each outward-oriented face. (Physicists should think of these spaces as "Hilbert" spaces obtained by quantizing the theory on a manifold with boundary.) The elements of V are represented by complex-valued functions in one variable, with adjoints given by conjugation and inner products given by integration. Abusing the notation, but following the very natural set of conventions of [52, 53], we denote these complex variables by $p_j^{(i)}$, which we used earlier to label the corresponding faces of the tetrahedra. As a result, to the boundary of every tetrahedron Δ_i one associates a vector space $V \otimes V \otimes V^* \otimes V^*$, represented by functions of its four face labels, $p_1^{(i)}$, $p_2^{(i)}$, $p_3^{(i)}$, and $p_4^{(i)}$.

To each tetrahedron one assigns a matrix element $\langle p_1, p_3|\mathbf{S}|p_2, p_4\rangle$ or $\langle p_2, p_4|\mathbf{S}^{-1}|p_1, p_3\rangle$, depending on orientation as indicated in Figure 8.3. Here, the matrix $\mathbf{S}$ acts on functions $f(p_1, p_2) \in V \otimes V$ as

$$\mathbf{S} \;=\; e^{\hat{q}_1\hat{p}_2/2\hbar}\,\Phi_\hbar(\hat{p}_1 + \hat{q}_2 - \hat{p}_2)\,, \tag{8.2.1}$$

where $\hat{p}_i f = p_i f$ and $\hat{q}_i f = 2\hbar\frac{\partial}{\partial p_i} f$. The function $\Phi_\hbar$ is the quantum dilogarithm, to be described in the next subsection. Assuming that $\int \frac{idq}{4\pi\hbar}|q\rangle\langle q| = \int dp\,|p\rangle\langle p| = 1$ and $\langle p|q\rangle = e^{\frac{pq}{2i\hbar}}$ (the exact normalizations are not important for the final invariant), one obtains via Fourier transform

$$\langle p_1, p_3|\mathbf{S}|p_2, p_4\rangle = \frac{\delta(p_1 + p_3 - p_2)}{\sqrt{-4\pi i\hbar}}\Phi_\hbar(p_4 - p_3 + i\pi + \hbar)\, e^{\frac{1}{2\hbar}\left(p_1(p_4 - p_3) + \frac{i\pi\hbar}{2} - \frac{\pi^2 - \hbar^2}{6}\right)}\,, \tag{8.2.2}$$

$$\langle p_2, p_4|\mathbf{S}^{-1}|p_1, p_3\rangle = \frac{\delta(p_1 + p_3 - p_2)}{\sqrt{-4\pi i\hbar}}\frac{1}{\Phi_\hbar(p_4 - p_3 - i\pi - \hbar)}e^{\frac{1}{2\hbar}\left(p_1(p_3 - p_4) - \frac{i\pi\hbar}{2} + \frac{\pi^2 - \hbar^2}{6}\right)}\,. \tag{8.2.3}$$

In the classical limit $\hbar \to 0$, the quantum dilogarithm has the asymptotic $\Phi_\hbar(p) \sim \frac{1}{2\hbar}\mathrm{Li}_2(-e^p)$. One therefore sees that the classical limits of the above matrix elements look very much like exponentials of $\frac{1}{2\hbar}$ times the complexified hyperbolic volumes of tetrahedra. For example, the asymptotic of (8.2.2) coincides with $\exp(L(z;\cdot,\cdot)/(2\hbar))$ if we identify $e^{p_4 - p_3}$ with z and e^{-2p_1} with $1/(1-z)$. For building a quantum invariant, however, only half of the variables p_j really "belong" to a single tetrahedron. Hikami's claim [52, 53] is that if we only identify a shape parameter

$$z^{(i)} = e^{p_4^{(i)} - p_3^{(i)}}\,, \tag{8.2.4}$$

for every tetrahedron Δ_i, the classical limit of the resulting quantum invariant will completely reproduce the hyperbolic structure and complexified hyperbolic volume on M.[10]

[10]Eqn. (8.2.4) is a little different from the relation appearing in [52, 53], because our convention for assigning shape parameters to edges based on orientation differs from that of [52, 53].

To finish calculating the invariant of M, one glues the tetrahedra back together and takes inner products in every pair V and V^* corresponding to identified faces. This amounts to multiplying together all the matrix elements (8.2.2) or (8.2.3), identifying the $p_j^{(i)}$ variables on identified faces (with matching head-to-tail normal vectors), and integrating over the $2N$ remaining p's. To account for possible toral boundaries of M, however, one must revert back to the hyperbolic structure. This allows one to write the holonomy eigenvalues $\{e^{u_k}\}$ as products of shape parameters $(z^{(i)}, 1-1/z^{(i)}, 1/(1-z^{(i)}))$, and, using (8.2.4), to turn every cusp condition into a linear relation of the form $\sum p\text{'s} = 2u_k$. These relations are then inserted as delta functions in the inner product integral, enforcing global boundary conditions. In the end, noting that each matrix element (8.2.2)-(8.2.3) also contains a delta function, one is left with $N - b_0(\Sigma)$ nontrivial integrals, where $b_0(\Sigma)$ is the number of connected components of $\Sigma = \partial M$. For example, specializing to hyperbolic 3-manifolds with a single torus boundary $\Sigma = T^2$, the integration variables can be relabeled so that Hikami's invariant takes the form

$$H(M;\hbar,u) = \frac{1}{(4\pi\hbar)^{N/2}} \int \prod_{i=1}^{N} \Phi_\hbar\big(\, g_i(\mathbf{p},2u) + \epsilon_i(i\pi+\hbar)\,\big)^{\epsilon_i}\, e^{f(\mathbf{p},2u,\hbar)/2\hbar}\, dp_1 \ldots dp_{N-1}\,. \tag{8.2.5}$$

The g_i are linear combinations of $(p_1,\ldots,p_{N-1},2u)$ with integer coefficients, and f is a quadratic polynomial, also with integer coefficients for all terms involving p_k's or u. In the classical limit, this integral can naively be evaluated in a saddle-point approximation, and Hikami's claim is that the saddle-point relations coincide precisely with the edge conditions for the triangulation on M. There is more to this story, however, as we will see in Section 8.4.

8.3 Quantum dilogarithm

Since quantum dilogarithms play a key role here, we take a little time to discuss some of their most important properties.

Somewhat confusingly, there are at least three distinct—though related—functions which have occurred in the literature under the name "quantum dilogarithm":

i) The function $\mathrm{Li}_2(x;q)$ is defined for $x,\, q \in \mathbb{C}$ with $|x|,\, |q| < 1$ by

$$\mathrm{Li}_2(x;q) \;=\; \sum_{n=1}^{\infty} \frac{x^n}{n\,(1-q^n)}\,; \tag{8.3.1}$$

its relation to the classical dilogarithm function $\mathrm{Li}_2(x) = \sum\limits_{n=1}^{\infty} \dfrac{x^n}{n^2}$ is that

$$\mathrm{Li}_2\bigl(x;\, e^{2\hbar}\bigr) \;\sim\; -\frac{1}{2\hbar}\,\mathrm{Li}_2(x) \qquad \text{as } \hbar \to 0\,. \tag{8.3.2}$$

ii) The function $(x;q)_\infty$ is defined for $|q| < 1$ and all $x \in \mathbb{C}$ by

$$(x;q)_\infty \;=\; \prod_{r=0}^{\infty} (1 - q^r x)\,, \tag{8.3.3}$$

and is related to $\mathrm{Li}_2(x;q)$ for $|x| < 1$ by

$$(x;q)_\infty \;=\; \exp\bigl(-\mathrm{Li}_2(x;q)\bigr)\,. \tag{8.3.4}$$

It is also related to the function $\mathbf{E}(x)$ of Chapter 4 as

$$\mathbf{E}(x) = \bigl(-\,q^{\frac{1}{2}} x;\, q\bigr)^{-1}\,. \tag{8.3.5}$$

Finally,

iii) the function $\Phi(z;\tau)$ is defined for $\mathrm{Re}(\tau) > 0$ and $2|\mathrm{Re}(z)| < 1 + \mathrm{Re}(\tau)$ by

$$\Phi(z;\tau) \;=\; \exp\biggl(\frac{1}{4}\int_{\mathbb{R}^{(+)}} \frac{e^{2xz}}{\sinh x\, \sinh \tau x}\,\frac{dx}{x}\biggr) \tag{8.3.6}$$

(here $\mathbb{R}^{(+)}$ denotes a path from $-\infty$ to ∞ along the real line but deformed to pass over the singularity at zero). It is related to $(x;q)_\infty$ by

$$\Phi(z;\tau) \;=\; \begin{cases} \dfrac{\bigl(-\,\mathbf{e}(z+\tau/2);\, \mathbf{e}(\tau)\bigr)_\infty}{\bigl(-\,\mathbf{e}((z-1/2)/\tau);\, \mathbf{e}(-1/\tau)\bigr)_\infty} & \text{if } \mathrm{Im}(\tau) > 0\,, \\[2ex] \dfrac{\bigl(-\,\mathbf{e}((z+1/2)/\tau);\, \mathbf{e}(1/\tau)\bigr)_\infty}{\bigl(-\,\mathbf{e}(z-\tau/2);\, \mathbf{e}(-\tau)\bigr)_\infty} & \text{if } \mathrm{Im}(\tau) < 0\,. \end{cases} \tag{8.3.7}$$

(Here and in future we use the abbreviation $\mathbf{e}(x) = e^{2\pi i x}$.)

It is the third of these functions, in the normalization

$$\Phi_\hbar(z) \;=\; \Phi\bigl(\frac{z}{2\pi i};\, \frac{\hbar}{i\pi}\bigr)\,, \tag{8.3.8}$$

which occurs in our "state integral" and which we will take as our basic "quantum dilogarithm," but all three functions play a role in the analysis, so we will describe the main properties of all three here. We give complete proofs, but only sketchily since none of this material is new. For further discussion and proofs, see, *e.g.*, [138, 139, 140, 141, 142], and [143] (subsection II.1.D).

1. The asymptotic formula (8.3.2) can be refined to the asymptotic expansion

$$\mathrm{Li}_2\big(x;e^{2\hbar}\big) \;\sim\; -\frac{1}{2\hbar}\,\mathrm{Li}_2(x)-\frac{1}{2}\log(1-x)-\frac{x}{1-x}\,\frac{\hbar}{6}+0\hbar^2+\frac{x+x^2}{(1-x)^3}\,\frac{\hbar^3}{90}+\cdots \tag{8.3.9}$$

as $\hbar\to 0$ with x fixed, in which the coefficient of $\hbar^{n-1}$ for $n\geq 2$ is the product of $-2^{n-1}B_n/n!$ (here B_n is the nth Bernoulli number) with the negative-index polylogarithm $\mathrm{Li}_{2-n}(x)\in\mathbb{Q}\big[\frac{1}{1-x}\big]$. More generally, one has the asymptotic formula

$$\mathrm{Li}_2\big(xe^{2\lambda\hbar};\,e^{2\hbar}\big)\;\sim\;-\sum_{n=0}^{\infty}\frac{2^{n-1}B_n(\lambda)}{n!}\,\mathrm{Li}_{2-n}(x)\,\hbar^{n-1} \tag{8.3.10}$$

as $\hbar\to 0$ with λ fixed, where $B_n(t)$ denotes the nth Bernoulli polynomial.[11] Both formulas are easy consequences of the Euler-Maclaurin summation formula. By combining (8.3.7), (8.3.4), and (8.3.10), one also obtains an asymptotic expansion

$$\Phi_\hbar(z+2\lambda\hbar)=\exp\left(\sum_{n=0}^{\infty}\frac{2^{n-1}B_n(1/2+\lambda)}{n!}\hbar^{n-1}\mathrm{Li}_{2-n}(-e^z)\right)\,. \tag{8.3.11}$$

(To derive this, note that in (8.3.7) $(-\mathbf{e}(z\pm 1/2)/\tau;\mathbf{e}(\pm 1/\tau))_\infty\sim 1$ to all orders in $\hbar$ as $\hbar\to 0$.)

2. The function $(x;q)_\infty$ and its reciprocal have the Taylor expansions (*cf.* (4.1.14))

$$(x;q)_\infty\;=\;\sum_{n=0}^{\infty}\frac{(-1)^n}{(q)_n}\,q^{\frac{n(n-1)}{2}}\,x^n\,,\qquad \frac{1}{(x;q)_\infty}\;=\;\sum_{n=0}^{\infty}\frac{1}{(q)_n}\,x^n \tag{8.3.12}$$

around $x=0$, where

$$(q)_n\;=\;\frac{(q;q)_\infty}{(q^{n+1};q)_\infty}\;=\;(1-q)(1-q^2)\cdots(1-q^n) \tag{8.3.13}$$

is the nth q-Pochhammer symbol. These, as well as formula (8.3.4), can be proved easily from the recursion formula $(x;q)_\infty=(1-x)(qx;q)_\infty$, which together with the initial value $(0;q)_\infty=1$ determines the power series $(x;q)_\infty$ uniquely. (Of course, (8.3.4) can also be proved directly by expanding each term in $\sum_r\log(1-q^rx)$ as a power series in x.) Another

[11] This is the unique polynomial satisfying $\int_x^{x+1}B_n(t)dt=x^n$, and is a monic polynomial of degree n with constant term B_n.

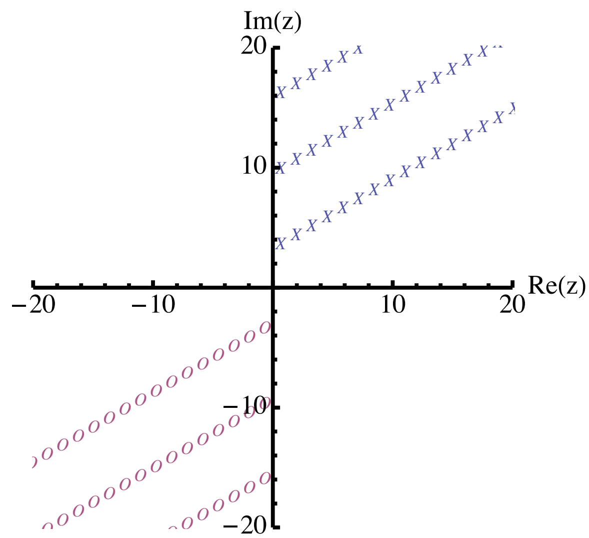


Figure 8.4: The complex z-plane, showing poles (X's) and zeroes (O's) of $\Phi_\hbar(z)$ at $\hbar = \frac{3}{4}e^{i\pi/3}$.

famous result, easily deduced from (8.3.12) using the identity $\sum_{m-n=k} \frac{q^{mn}}{(q)_m(q)_n} = \frac{1}{(q)_\infty}$ for all $k \in \mathbb{Z}$, is the Jacobi triple product formula

$$(q;q)_\infty\,(x;q)_\infty\,(qx^{-1};q)_\infty \;=\; \sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{k(k-1)}{2}}\,x^k\,, \qquad (8.3.14)$$

relating the function $(x;q)_\infty$ to the classical Jacobi theta function.

3. The function $\Phi(z;\tau)$ defined (initially for $\mathrm{Re}(\tau)>0$ and $|\mathrm{Re}(z)|<\frac{1}{2}+\frac{1}{2}\mathrm{Re}(\tau)$) by (8.3.6) has several functional equations. Denote by $I(z;\tau)$ the integral appearing in this formula. Choosing for $\mathbb{R}^{(+)}$ the path $(-\infty,-\varepsilon]\cup\varepsilon\exp([i\pi,0])\cup[\varepsilon,\infty)$ and letting $\varepsilon\to 0$, we find

$$I(z;\tau) \;=\; \frac{2\pi i}{\tau}\left(\frac{1+\tau^2}{12}-z^2\right)\,+\,2\int_0^\infty\left(\frac{\sinh 2xz}{\sinh x\,\sinh\tau x}\;-\;\frac{2z}{\tau x}\right)\frac{dx}{x}\,. \qquad (8.3.15)$$

Since the second term is an even function of z, this gives

$$\Phi(z;\tau)\;\Phi(-z;\tau)\;=\;\mathbf{e}\Big(\frac{\tau^2-12z^2+1}{24\tau}\Big)\,. \qquad (8.3.16)$$

From (8.3.15) we also get

$$I(z+1/2;\tau)\;-\;I(z-1/2;\tau)\;=\;-\frac{4\pi i z}{\tau}\,+\,4\int_0^\infty\left(\frac{\cosh 2xz}{\sinh\tau x}\;-\;\frac{1}{\tau x}\right)\frac{dx}{x}\,. \qquad (8.3.17)$$

The integral equals $-\log(2\cos(\pi z/\tau))$ (proof left as an exercise). Dividing by 4 and exponentiating we get the first of the two functional equations

$$\frac{\Phi(z-1/2;\tau)}{\Phi(z+1/2;\tau)}\;=\;1\,+\,\mathbf{e}\big(z/\tau\big)\,,\qquad \frac{\Phi(z-\tau/2;\tau)}{\Phi(z+\tau/2;\tau)}\;=\;1\,+\,\mathbf{e}\big(z\big)\,, \qquad (8.3.18)$$

and the second can be proved in the same way or deduced from the first using the obvious symmetry property

$$\Phi(z;\tau) \;=\; \Phi(z/\tau;1/\tau)\,, \tag{8.3.19}$$

of the function Φ. (Replace x by x/τ in (8.3.6).)

4. The functional equations (8.3.18) show that $\Phi(z;\tau)$, which in its initial domain of definition clearly has no zeros or poles, extends (for fixed τ with $\mathrm{Re}(\tau)>0$) to a meromorphic function of z with simple poles at $z\in\Xi(\tau)$ and simple zeros at $z\in-\Xi(\tau)$, where

$$\Xi(\tau) \;=\; \bigl(\mathbb{Z}_{\ge 0}+\tfrac12\bigr)\,\tau \;+\; \bigl(\mathbb{Z}_{\ge 0}+\tfrac12\bigr) \;\subset\; \mathbb{C}\,. \tag{8.3.20}$$

In terms of the normalization (8.3.8), this says that $\Phi_\hbar(z)$ has simple poles at $z\in\tilde{\Xi}(\hbar)$ and simple zeroes at $z\in-\tilde{\Xi}(\hbar)$, where

$$\tilde{\Xi}(\hbar) = (2\mathbb{Z}_{\ge 0}+1)i\pi + (2\mathbb{Z}_{\ge 0}+1)\hbar\,. \tag{8.3.21}$$

This is illustrated in Figure 8.4. Equation (8.3.7) expressing Φ in terms of the function $(x;q)_\infty$ also follows, because the quotient of its left- and right-hand sides is a doubly periodic function of z with no zeros or poles and hence constant, and the constant can only be ±1 (and can then be checked to be $+1$ in several ways, e.g., by evaluating numerically at one point) because the right-hand side of (8.3.7) satisfies the same functional equation (8.3.16) as $\Phi(z;\tau)$ by virtue of the Jacobi triple product formula (8.3.14) and the well-known modular transformation properties of the Jacobi theta function.

5. From (8.3.15) we also find the Taylor expansion of $I(z;\tau)$ at $z=0$,

$$I(z;\tau) \;=\; 4\sum_{k=0}^{\infty} C_k(\tau)\,z^k\,,$$

with coefficients $C_k(\tau)=\tau^{-k}C_k(1/\tau)$ given by

$$\begin{aligned}
C_0(\tau) \;&=\; \frac{\pi i}{24}\bigl(\tau+\tau^{-1}\bigr)\,, \qquad C_1(\tau) \;=\; \int_0^\infty\left(\frac{1}{\sinh x\,\sinh\tau x}-\frac{1}{\tau x^2}\right)dx\,,\\
C_2(\tau) \;&=\; -\frac{\pi i}{2\tau}\,, \qquad C_k(\tau) \;=\; 0 \quad \text{for } k\ge 4 \text{ even},\\
C_k(\tau) \;&=\; \frac{2^{k-1}}{k!}\int_0^\infty \frac{x^{k-1}\,dx}{\sinh x\,\sinh\tau x} \quad \text{for } k\ge 3 \text{ odd.}
\end{aligned}$$

By expanding $1/\sinh(x)$ and $1/\sinh(\tau x)$ as power series in e^{-x} and $e^{-\tau x}$ we can evaluate the last of these expressions to get

$$C_k(\tau) \;=\; \frac{2^{k+1}}{k!}\sum_{m,n>0,\ \text{odd}}\int_0^\infty e^{-mx-n\tau x}\,x^{k-1}\,dx \;=\; \frac{2}{k}\sum_{s\in\Xi(\tau)} s^{-k} \quad (k\ge 3 \text{ odd})$$

with $\Xi(\tau)$ as in (8.3.20). Dividing by 4 and exponentiating gives the Weierstrass product expansion

$$\Phi(z;\tau) \;=\; \exp\biggl(\frac{\pi i}{24}\bigl(\tau+\tau^{-1}\bigr) \;+\; C_1(\tau)\,z \;-\; \frac{\pi i z^2}{2\tau}\biggr) \prod_{s\in\Xi(\tau)} \biggl(\frac{s+z}{s-z}\,e^{-2z/s}\biggr) \tag{8.3.22}$$

of $\Phi(z;\tau)$. From this expansion, one finds that $\Phi(z;\tau)$ extends meromorphically to $\mathbb{C}\times\bigl(\mathbb{C}\smallsetminus(-\infty,0]\bigr)$ with simple poles and simple zeros for $z\in\Xi(\tau)$ and $z\in-\Xi(\tau)$ and no other zeros or poles. (This analytic continuation can also be deduced by rotating the path of integration in (8.3.6), e.g. by replacing $\int_{\mathbb{R}(+)}$ by $\int_{\mathbb{R}(+)/\sqrt{\tau}}$ for z sufficiently small.)

6. The quantum dilogarithm is related via the Jacobi triple product formula to the Jacobi theta function, which is a Jacobi form, i.e., it has transformation properties not only with respect to the lattice translations $z\mapsto z+1$ and $z\mapsto z+\tau$ but also with respect to the modular transformations $\tau\mapsto\tau+1$ and $\tau\mapsto -1/\tau$. The function $\Phi(z;\tau)$ has the lattice transformation properties (8.3.18) and maps to its inverse under $(z,\tau)\mapsto(z/\tau,\,-1/\tau)$, but it does not transform in a simple way with respect to $\tau\mapsto\tau+1$. Nevertheless, it has an interesting modularity property of a different kind (cocycle property) which is worth mentioning here even though no use of it will be made in the remainder of this thesis. Write (8.3.7) as

$$\Phi(z;\tau) \;=\; \frac{S(z;\tau)}{S(z/\tau;-1/\tau)}\,, \qquad S(z;\tau) \;=\; \begin{cases} \prod_{n\,>\,0\ \text{odd}}\bigl(1+q^{n/2}\mathbf{e}(z)\bigr) & \text{if } \operatorname{Im}(\tau)>0,\\ \prod_{n\,<\,0\ \text{odd}}\bigl(1+q^{n/2}\mathbf{e}(z)\bigr)^{-1} & \text{if } \operatorname{Im}(\tau)<0, \end{cases}$$

where $q=\mathbf{e}(\tau)$. The function $S(z;\tau)$ has the transformation properties

$$S(z;\tau)=S(z+1;\tau)=(1+q^{1/2}\mathbf{e}(z))\,S(z+\tau;\tau)=S(z+\tfrac12;\tau+1)=S(z;\tau+2)$$

amd from these we deduce by a short calculation the two three-term functional equations

$$\Phi(z;\tau) \;=\; \Phi\bigl(z\pm\frac12,\tau+1\bigr)\,\Phi\bigl(\frac{z\mp\tau/2}{\tau+1},\,\frac{\tau}{\tau+1}\bigr) \tag{8.3.23}$$

of Φ. This is highly reminiscent of the fact (cf. [144]) that the holomorphic function

$$\psi(\tau) \;=\; f(\tau)-\tau^{-2s}f(-1/\tau)\,, \qquad f(\tau) \;=\; \begin{cases} \sum_{n>0} a_n q^n & \text{if } \operatorname{Im}(\tau)>0,\\ -\sum_{n<0} a_n q^n & \text{if } \operatorname{Im}(\tau)<0 \end{cases}$$

associated to a Maass cusp form $u(\tau)$ on $SL(2,\mathbb{Z})$ with spectral parameter s, where a_n are the normalized coefficients in the Fourier-Bessel expansion of u, satisfies the Lewis functional equation $\psi(\tau) = \psi(\tau+1) + (\tau+1)^{-2s}\psi\big(\frac{\tau}{\tau+1}\big)$ and extends holomorphically from its initial domain of definition $\mathbb{C} \smallsetminus \mathbb{R}$ to $\mathbb{C} \smallsetminus (-\infty, 0]$.

7. Finally, the quantum dilogarithm functions satisfy various five-term relations, of which the classical five-term functional equation of $\mathrm{Li}_2(x)$ is a limiting case, when the arguments are non-commuting variables. The simplest and oldest is the identity

$$(Y;q)_\infty \, (X;q)_\infty \;=\; (X;q)_\infty \, (-YX;q)_\infty \, (Y;q)_\infty \tag{8.3.24}$$

for operators X and Y satisfying $XY = qYX$ (*cf.* Eqn. (4.1.21) for $\mathbf{E}(x)$). From this one deduces the "quantum pentagon relation"

$$\Phi_\hbar(\hat{p}) \, \Phi_\hbar(\hat{q}) \;=\; \Phi_\hbar(\hat{q}) \, \Phi_\hbar(\hat{p}+\hat{q}) \, \Phi_\hbar(\hat{p}) \tag{8.3.25}$$

for operators $\hat{p}$ and $\hat{q}$ satisfying $[\hat{q}, \hat{p}] = 2\hbar$. Letting $\mathbf{S}_{ij}$ be a copy, acting on the i^{th} and j^{th} factors of $V \otimes V \otimes V$, of the $\mathbf{S}$-matrix introduced in (8.2.1), we deduce from (8.3.25) the operator identity

$$\mathbf{S}_{23}\mathbf{S}_{12} \;=\; \mathbf{S}_{12}\mathbf{S}_{13}\mathbf{S}_{23} \,. \tag{8.3.26}$$

It is this very special property which guarantees that the gluing procedure used in the definition of (8.2.5) is invariant under 2-3 Pachner moves on the underlying triangulations and produces a true hyperbolic invariant [52, 53, 145]. This identity is also related to the interesting fact that the fifth power of the operator which maps a nicely behaved function to the Fourier transform of its product with $\Phi_\hbar(z)$ (suitably normalized) is a multiple of the identity [142].

8.4 A state integral model for $Z^{(\rho)}(M;\hbar)$

Now, let us return to the analysis of the integral (8.2.5) and compare it with the perturbative $SL(2,\mathbb{C})$ invariant $Z^{(\alpha)}(M;\hbar,u)$. Both invariants compute quantum (*i.e.* $\hbar$-deformed) topological invariants of hyperbolic 3-manifolds and, thus, are expected to be closely related. However, in order to establish a precise relation, we need to face two problems mentioned in the beginning of this section:

i) the integration contour is not specified in (8.2.5), and

ii) the integral (8.2.5) does not depend on the choice of the classical solution α.

These two problems are related, and can be addressed by studying the integral (8.2.5) in various saddle-point approximations. Using the leading term in (8.3.11), we can approximate it to leading order as

$$H(M;\hbar,u) \underset{\hbar\to 0}{\sim} \int e^{\frac{1}{\hbar}V(p_1,\ldots,p_{N-1},u)}\,dp_1\ldots dp_{N-1}\,, \tag{8.4.1}$$

with the "potential"

$$V(\mathbf{p},u) \;=\; \frac{1}{2}\sum_{i=1}^{N}\epsilon_i \mathrm{Li}_2(-\exp(g_i(\mathbf{p},2u)+i\pi\epsilon_i))+\frac{1}{2}f(\mathbf{p},2u,\hbar=0)\,. \tag{8.4.2}$$

As explained below (8.3.11), the branches of Li_2 must be chosen appropriately to coincide with the half-lines of poles and zeroes of the quantum dilogarithms in (8.2.5). The leading contribution to $H(M;\hbar,u)$ will then come from the highest-lying critical point through which a given contour can be deformed.

It was the observation of [53] that the potential V always has one critical point that reproduces the classical Chern-Simons action on the "geometric" branch, that is a critical point $\mathbf{p}^{(\mathrm{geom})}(u)$ such that

$$l_{\mathrm{geom}}(u) = \exp\left[\tfrac{d}{du}V(\mathbf{p}^{(\mathrm{geom})}(u),u)\right] \tag{8.4.3}$$

and

$$S_0^{(\mathrm{geom})}(u) = V(\mathbf{p}^{(\mathrm{geom})}(u),u)\,. \tag{8.4.4}$$

This identification follows from the fact that both $S_0^{(\mathrm{geom})}(u)$ and matrix elements (8.2.2)-(8.2.3) in the limit $\hbar\to 0$ are related to the complexified volume function $i(\mathrm{Vol}(M;u)+i\mathrm{CS}(M;u))$. We want to argue presently that in fact *every* critical point of V corresponds to a classical solution in Chern-Simons theory (that is, to a branch of $A(l,m)=0$) in this manner, with similar relations

$$l_\alpha(u) = e^{\frac{d}{du}V(\mathbf{p}^{(\alpha)}(u),u)} \tag{8.4.5}$$

and

$$S_0^{(\alpha)}(u) = V(\mathbf{p}^{(\alpha)}(u),u) \tag{8.4.6}$$

for some α. In particular, $l_\alpha(u)$ as given by (8.4.5) and $m=e^u$ obey (6.3.13).

To analyze generic critical points of V, observe that the critical point equations take the form

$$2\frac{\partial}{\partial p_j}V(\mathbf{p},u) = -\sum_{i=1}^{N}\epsilon_i G_{ji}\log(1+\exp(g_i(\mathbf{p},2u)+i\pi\epsilon_i))+\frac{\partial}{\partial p_j}f(\mathbf{p},2u,0)=0 \qquad \forall\, j\,, \tag{8.4.7}$$

where $G_{ji} = \frac{\partial}{\partial p_j} g_i(\mathbf{p}, 2u)$ are some constants and the functions $\frac{\partial}{\partial p_j} f(\mathbf{p}, 2u, 0)$ are linear. Again, the cuts of the logarithm must match the singularities of the quantum dilogarithm. Exponentiating (8.4.7), we obtain another set of conditions

$$r_j(\mathbf{x}, m) = 1 \quad \forall\, j\,, \tag{8.4.8}$$

where

$$r_j(\mathbf{x}, m) = \exp\Big(2\frac{\partial}{\partial p_j} V(\mathbf{p}, u)\Big) = \prod_i (1 - e^{g_i})^{-\epsilon_i G_{ji}} \exp\Big(\frac{\partial}{\partial p_j} f\Big) \tag{8.4.9}$$

are all *rational* functions of the variables $x_j = e^{p_j}$ and $m = e^u$. Note that an entire family of points $\{\mathbf{p} + 2\pi i \mathbf{n} \,|\, \mathbf{n} \in \mathbb{Z}^{N-1}\}$ maps to a single $\mathbf{x}$, and that all branch cut ambiguities disappear in the simpler equations (8.4.8). Depending on $\arg(\hbar)$ and the precise form of f, solutions to (8.4.8) either "lift" uniquely to critical points of the potential V, or they lift to a family of critical points at which V differs only by integer multiples of $2\pi i u$.

Now, the system (8.4.8) is algebraic, so its set of solutions defines a complex affine variety

$$\mathcal{R} = \{(x_1, \ldots, x_{N-1}, m) \in \mathbb{C}^N \,|\, r_j(x_1, \ldots, x_{N-1}, m) = 0 \quad \forall j\}\,, \tag{8.4.10}$$

which is closely related to the representation variety $\mathcal{L}$ given by $A(l, m) = 0$. Both generically have complex dimension one. Noting that $s(x_1, \ldots, x_{N-1}, m) = \exp\big(\frac{\partial}{\partial u} V\big)$ is also a rational function, we can define a rational map $\phi : \mathbb{C}^N \to \mathbb{C}^2$ by

$$\phi(x_1, \ldots, x_{N-1}, m) = (s(x_1, \ldots, x_{N-1}, m), m)\,. \tag{8.4.11}$$

The claim in [53] that one critical point of V always corresponds to the geometric branch of $\mathcal{L}$ means that $\overline{\phi(\mathcal{R})}$ (taking an algebraic closure) always intersects $\mathcal{L}$ nontrivially, along a subvariety of dimension 1. Thus, some irreducible component of $\overline{\phi(\mathcal{R})}$, coming from an irreducible component of $\mathcal{R}$, must coincide with the entire irreducible component of $\mathcal{L}$ containing the geometric branch. Every solution $\mathbf{x} = \mathbf{x}(m)$ in this component of $\mathcal{R}$ corresponds to a branch of the A-polynomial. Moreover, if such a solution $\mathbf{x}^{(\alpha)}$ (corresponding to branch α) can be lifted to a real critical point $\mathbf{p}^{(\alpha)}(u)$ of V, then one must have relations (8.4.5) and (8.4.6).

This simple algebraic analysis shows that *some* solutions of (8.4.8) will cover an entire irreducible component of the curve $\mathcal{L}$ defined by $A(l, m) = 0$. We cannot push the general argument further without knowing more about the reducibility of $\mathcal{R}$. However, we can look at some actual examples. Computing $V(\mathbf{p},u)$ for thirteen hyperbolic manifolds with a single torus boundary,[12] we found in every case that solutions of (8.4.8) completely covered all non-abelian branches $\alpha \neq \text{abel}$; in other words, $\overline{\phi(\mathcal{R})} = \mathcal{L}'$, with $\mathcal{L}' = \{A(l,m)/(l-1) = 0\}$. For six of these manifolds, we found unique critical points $\mathbf{p}(u)$ corresponding to every non-abelian branch of $\mathcal{L}$ at $\arg(\hbar) = i\pi$. Motivated by these examples, it is natural to state the following conjecture:

Conjecture 2: *Every critical point of V corresponds to some branch α, and all $\alpha \neq$ abel are obtained in this way. Moreover, for every critical point $\mathbf{p}^{(\alpha)}(u)$ (corresponding to some branch α) we have* (8.4.5)-(8.4.6) *and to all orders in perturbation theory:*

$$Z^{(\alpha)}(M;\hbar,u) = \sqrt{2}\int_{C_\alpha}\prod_{i=1}^{N}\Phi_\hbar\big(\,g_i(\mathbf{p},2u)+\epsilon_i(i\pi+\hbar)\,\big)^{\epsilon_i}\,e^{\frac{1}{2\hbar}f(\mathbf{p},2u,\hbar)-u}\prod_{j=1}^{N-1}\frac{dp_j}{\sqrt{4\pi\hbar}}\,, \quad (8.4.12)$$

where C_α is an arbitrary contour with fixed endpoints which passes through $\mathbf{p}^{(\alpha)}(u)$ and no other critical point.

A slightly more conservative version of this conjecture might state that only those α that belong to the same irreducible component of $\mathcal{L}$ as the geometric branch, $\alpha = \text{geom}$, are covered by critical points of V. Indeed, the "abelian" branch with $l_{\text{abel}} = 1$ is not covered by the critical points of V and it belongs to the separate component $(l-1)$ of the curve $A(l,m) = 0$. It would be interesting to study the relation between critical points of V and irreducible components of $\mathcal{L}$ further, in particular by looking at examples with reducible A-polynomials aside from the universal $(l-1)$ factor.

The right-hand side of (8.4.12) is the proposed state integral model for the *exact* perturbative partition function of $SL(2,\mathbb{C})$ Chern-Simons theory on a hyperbolic 3-manifold M with a single torus boundary $\Sigma = T^2$. (A generalization to 3-manifolds with an arbitrary number of boundary components is straightforward.) This state integral model is a modified version of Hikami's invariant (8.2.5). Just like its predecessor, eq. (8.4.12) is based on an ideal triangulation $\{\Delta_i\}_{i=1}^N$ of a hyperbolic 3-manifold M and inherits topological

[12]Namely, the complements of hyperbolic knots $\mathbf{4_1}(\mathbf{k2_1})$, $\mathbf{5_2}(\mathbf{k3_2})$, $\mathbf{12n_{242}}$ ($\mathbf{3_1}$, (-2,3,7)-pretzel knot), $\mathbf{6_1}(\mathbf{k4_1})$, $\mathbf{6_3}(\mathbf{k6_{43}})$, $\mathbf{7_2}(\mathbf{k4_2})$, $\mathbf{7_3}(\mathbf{k5_{20}})$, $\mathbf{7_4}(\mathbf{k6_{28}})$, $\mathbf{10_{132}}(\mathbf{K5_9})$, $\mathbf{10_{139}}(\mathbf{K5_{22}})$, and $\mathbf{11n_{38}}(\mathbf{K5_{13}})$, as well as the one-punctured torus bundles L^2R and LR^3 over $\mathbf{S}^1$ (also knot complements, but in a manifold other than $\mathbf{S}^3$).

invariance from the pentagon identity (8.3.25) of the quantum dilogarithm.

However, in writing (8.4.12) we made two important modifications to Hikami's invariant (8.2.5). First, we introduced contours C_α running across the sadle points $\mathbf{p}_\alpha$, which now encode the choice of a classical solution in Chern-Simons theory. Second, in (8.4.12) we introduced an extra factor of $\sqrt{8\pi\hbar}e^{-u}$, which is needed to reproduce the correct asymptotic behavior of $Z^{(\alpha)}(M;\hbar,u)$. To understand this correction factor, we must look at the higher-order terms in the expansion of $Z^{(\alpha)}(M;\hbar,u)$. By using (8.3.11), one can continue the saddle-point approximations described above to arbitrary order in $\hbar$. The result has the expected form (5.1.9),

$$Z^{(\alpha)}(M;\hbar,u) = \exp\left(\frac{1}{\hbar}S_0^{(\alpha)}(u) - \frac{1}{2}\delta^{(\alpha)}\log\hbar + \sum_{n=0}^{\infty} S_{n+1}^{(\alpha)}(u)\hbar^n\right), \tag{8.4.13}$$

with the correct leading term $S_0^{(\alpha)}(u)$ that we already analyzed above, *cf.* eq. (8.4.6).

Let us examine the next-leading logarithmic term. Its coefficient $\delta^{(\alpha)}$ receives contributions from two places: from the prefactor $(4\pi\hbar)^{-(N-1)/2}$ in (8.4.12), and from the standard Gaussian determinant. The former depends on the total number of tetrahedra, N, in the triangulation of M and therefore must be cancelled (at least partially) since the total integral (8.4.12) is a topological invariant and cannot depend on N. This is indeed what happens. For example, in a saddle-point approximation around a nondegenerate critical point $\mathbf{p}^{(\alpha)}(u)$, the contribution of the Gaussian determinant goes like $\sim \hbar^{(N-1)/2}$ and exactly cancels the contribution of the prefactor $\sim \hbar^{-(N-1)/2}$. An example of such critical point is the critical point $\mathbf{p}^{(\text{geom})}(u)$ corresponding to the geometric branch. Therefore, the asymptotic expansion of the integral (8.4.12) around the critical point $\mathbf{p}^{(\text{geom})}(u)$ has the form (8.4.13) with

$$\delta^{(\text{geom})} = 0\,, \tag{8.4.14}$$

which is the expected result.[13] Indeed, as explained *e.g.* in [40, 98], the rigidity of the flat connection $\mathcal{A}^{(\text{geom})}$ associated with a hyperbolic structure on M implies $h^0 = h^1 = 0$, so that (5.2.2) gives $\delta^{(\text{geom})} = 0$.

[13] Recall that throughout this work $Z^{(\rho)}(M;\hbar)$ (resp. $Z^{(\alpha)}(M;\hbar,u)$) stands for the *unnormalized* perturbative $G_{\mathbb{C}}$ invariant. A normalized version, obtained by dividing by $Z(\mathbf{S}^3)$, has an asymptotic expansion of the same form (5.1.9) (resp. (8.4.13)) with the value of $\delta^{(\alpha)}$ shifted by $\dim(G)$ for every α. This is easy to see from (5.3.6).

Using (8.3.11) , one can also calculate the higher-order perturbative coefficients $S_n^{(\alpha)}(u)$, with $n \geq 1$. In the following section, we carry out this analysis to high order for the figure-8 knot complement and find perfect agreement with the results obtained by methods of Section 6.6. (Other interesting examples and further checks will appear elsewhere [5].)

Note that the $S_n^{(\alpha)}(u)$'s do not depend on the details of the contours C_α. The only part of (8.4.12) which actually depends on the details of C_α is exponentially suppressed and is not part of the perturbative series (8.4.13). Finally, we also note that we use only those critical points of the full integrand (8.4.12) which correspond to critical points of V in the limit $\hbar \to 0$. For any fixed $\hbar > 0$, the actual integrand has many other critical points which become trapped in the half-line singularities of quantum dilogarithms as $\hbar \to 0$, so the integrals over them do not have a well-behaved limit.

We conclude this section by observing that Conjecture 2 implies Conjecture 1. From (8.3.11), we can write an asymptotic double series expansion (for very small p):

$$\begin{aligned}\Phi_\hbar(p_0+p) &= \exp\left(\sum_{n=0}^{\infty} B_n\big(\frac{1}{2}+\frac{p}{2\hbar}\big)\,\mathrm{Li}_{2-n}\big(-e^{p_0}\big)\,\frac{(2\hbar)^{n-1}}{n!}\right)\\ &= \exp\left(\sum_{k=-1}^{\infty}\sum_{j=0}^{\infty}\frac{B_{k+1}(1/2)\,2^k}{(k+1)!j!}\mathrm{Li}_{1-j-k}(-e^{p_0})\,\hbar^k p^j\right). \end{aligned} \tag{8.4.15}$$

Using this formula and taking into account the shifts by $\pm(i\pi+\hbar)$, we expand every quantum dilogarithm appearing in the integrand of (8.4.12) around a critical point $\mathbf{p}^{(\alpha)}$. At each order in $\hbar$, the state integral model then reduces to an integral of a polynomial in $\mathbf{p}$ with a Gaussian weight. Due to the fact that Li_k is a rational function for $k \leq 0$, the coefficients of these polynomials are all rational functions of the variables $\mathbf{x}^{(\alpha)} = \exp(\mathbf{p}^{(\alpha)})$ and m. Therefore, the resulting coefficients $S_n^{(\alpha)}(m)$, for $n > 1$, will also be rational functions of $\mathbf{x}^{(\alpha)}$ and m. At $m = 1$, the solutions $\mathbf{x}^{(\alpha)}(m)$ to the rational equations (8.4.8) all belong to some algebraic number field $\mathbb{K} \subset \overline{\mathbb{Q}}$, leading immediately to Conjecture 1. In particular, for the geometric branch $\alpha = \mathrm{geom}$, the field $\mathbb{K}$ is nothing but the trace field $\mathbb{Q}(\mathrm{tr}\,\Gamma)$.

Chapter 9

Saddle points and new invariants

In this chapter, we begin by performing explicit computations with the state integral model of Chapter 8, using the algorithm outlined very briefly at the end of Section 8.4. We use the hyperbolic figure-eight knot $\mathbf{4_1}$ and the knot $\mathbf{5_2}$ (or rather their complements in the three-sphere) as our main examples. In both cases, we how the equations of geometric quantization $\hat{A} \cdot Z = 0$ can be easily verified directly within the state integral model. Of course, for $\mathbf{4_1}$ our coefficients $S_n(u)$ will agree with those obtained in Section 6.6 directly from geometric quantization. Unlike geometric quantization, however, the state integral model fixes (almost) all potentially undetermined constants in the invariants $S_n(u)$; one can check that for $\mathbf{4_1}$ and $\mathbf{5_2}$, these invariants reduce to the arithmetic invariants found in Section 5.4 at $u = 0$.

The knot $\mathbf{5_2}$ is not amphicheiral, and its perturbative invariants display several novel features. In particular, the $\mathbf{5_2}$ knot complement has three branches of non-abelian flat connections, two of which are conjugate to each other (but are not a signed pair), and one which is a real self-conjugate branch. Unlike the case of self-conjugate abelian branches, the non-abelian real branch here has interesting arithmetic properties. Moreover, the conjugate branches have a non-vanishing Chern-Simons invariant.

In Section 9.3, we will then use functional equations for the quantum dilogarithm to obtain direct (though unrigorous) analytic continuations of the colored Jones polynomials of knots $\mathbf{4_1}$ and $\mathbf{5_2}$, as well as the non-hyperbolic trefoil $\mathbf{3_1}$. The resulting analytic continuations become integrals of products of quantum dilogarithms just like the partition functions of the state integral model. In some cases, the integrals are identical. Saddle-point approximations can be applied in this case as well, leading to the same perturbative $G_{\mathbb{C}}$ invariants.

The figure-eight example appeared first in our work [3]. The $\mathbf{5_2}$ example, verification of $\hat{A} \cdot Z = 0$, and direct analytic continuation will appear in [5]. The integral for the trefoil obtained by analytic continuation suggests the existence of a non-hyperbolic state integral model.

9.1 Figure-eight knot $\mathbf{4_1}$

The state integral model (8.4.12) for the figure-eight knot complement gives:

$$Z^{(\alpha)}(M;\hbar,u) = \frac{1}{\sqrt{2\pi\hbar}} \int_{C_\alpha} dp \frac{\Phi_\hbar(p+i\pi+\hbar)}{\Phi_\hbar(-p-2u-i\pi-\hbar)} e^{-\frac{2}{\hbar}u(u+p)-u}\,. \tag{9.1.1}$$

There are two tetrahedra ($N=2$) in the standard triangulation of M, and so two quantum dilogarithms in the integral. There is a single integration variable p, and we can identify $g_1(p,u) = p,\ g_2(p,u) = -p-2u$, and $f(p,2u,\hbar) = -4u(u+p)$.

It will be convenient here to actually change variables $p \mapsto p-u-i\pi-\hbar$, removing the $(i\pi+\hbar)$ terms in the quantum dilogarithms, and obtaining the somewhat more symmetric expression

$$Z^{(\alpha)}(M;\hbar,u) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{2\pi i u}{\hbar}+u} \int_{C_\alpha} dp \frac{\Phi_\hbar(p-u)}{\Phi_\hbar(-p-u)} e^{-\frac{2pu}{\hbar}} \tag{9.1.2}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{2\pi i u}{\hbar}+u} \int_{C_\alpha} dp\, e^{\Upsilon(\hbar,p,u)}\,. \tag{9.1.3}$$

We define $e^{\Upsilon(\hbar,p,u)} = \frac{\Phi_\hbar(p-u)}{\Phi_\hbar(-p-u)} e^{-\frac{2pu}{\hbar}}$. Figures 9.1 and 9.2 show plots of $|e^{\Upsilon(\hbar,p,u)}|$ and $\log|e^{\Upsilon(\hbar,p,u)}| = \mathrm{Re}\,\Upsilon(\hbar,p,u)$ at $\hbar = i/3$ and two values of u. The half-lines of poles and zeroes of the two quantum dilogarithms combine into similar singularities for $\Upsilon(\hbar,p,u)$, as is depicted in Figure 9.3; note the splitting of these poles and zeroes by an amount $2u$.

After our change of variables, the "potential" function $V(p,u)$ as in (8.4.2) is now seen to be

$$V(p,u) = \frac{1}{2}\left[\mathrm{Li}_2(-e^{p-u}) - \mathrm{Li}_2(-e^{-p-u}) - 4pu + 4\pi i u\right]\,. \tag{9.1.4}$$

Instead of looking directly at $\frac{\partial}{\partial p}V = 0$ to find its critical points, we consider the simpler equation

$$r(x,m) = e^{2\frac{\partial}{\partial p}V} = \frac{x}{m^2(m+x)(1+mx)} = 1\,, \tag{9.1.5}$$

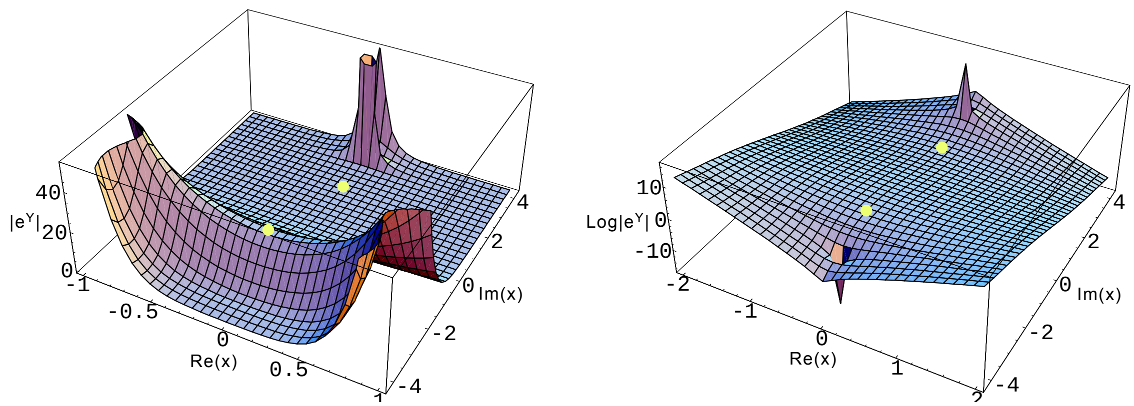


Figure 9.1: Plots of $|e^{\Upsilon(\hbar,p,u)}|$ and its logarithm at $u = 0$ and $\hbar = \frac{i}{3}$.

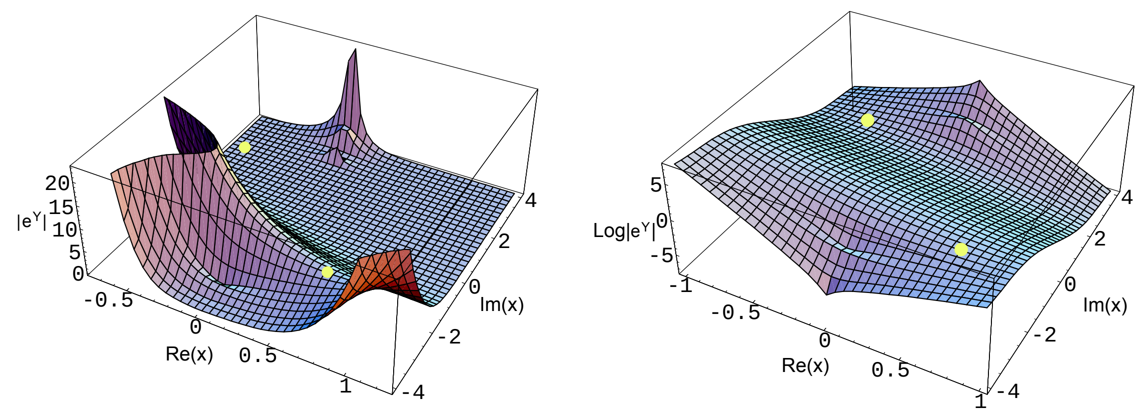


Figure 9.2: Plots of $|e^{\Upsilon(\hbar,p,u)}|$ and its logarithm at $u = \frac{1}{2}i$ and $\hbar = \frac{i}{3}$.

in terms of $x = e^p$ and $m = e^u$. This clearly has two branches of solutions, which both lift to true critical points of V, given by

$$p^{(\mathrm{geom,conj})}(u) = \log\left[\frac{1 - m^2 - m^4 \mp m^2\Delta(m)}{2m^3}\right], \tag{9.1.6}$$

with $\Delta(m)$ defined as in (6.6.12):

$$\Delta(m) = i\sqrt{-m^{-4} + 2m^{-2} + 1 + 2m^2 - m^4}\,. \tag{9.1.7}$$

The lift is unique if (say) $\hbar \in i\,\mathbb{R}_{>0}$. We claim that these correspond to the geometric and conjugate branches of the A-polynomial described in Section 6.6, which can be verified by calculating[1] $s(x,m) = \exp\left(\frac{\partial}{\partial u}V(p,u)\right) = \left[\frac{m+x}{x^3+mx^4}\right]^{1/2}$, and checking indeed that $s(x^{(\mathrm{geom,conj})}(m), m) = l_{\mathrm{geom,conj}}(m)$.

[1]Here $s(x,m)$ is the square root of a rational function rather than a rational function itself due to our redefined variables. Using (9.1.1) directly, we would have gotten a pure rational expression.

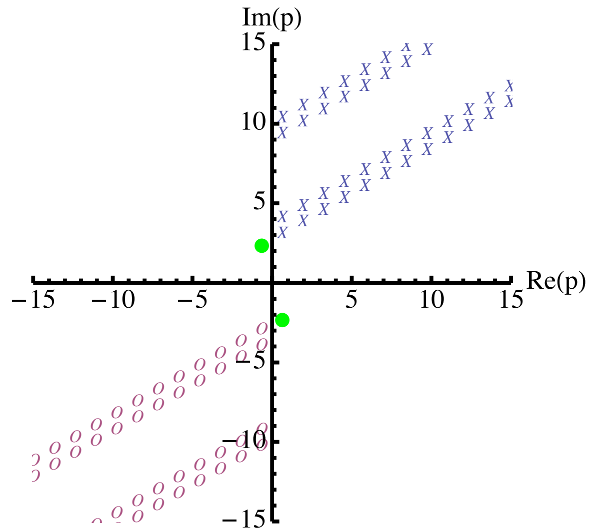


Figure 9.3: Poles, zeroes, and critical points of $e^{\Upsilon(\hbar,p,u)}$ for $u=\frac{1}{2}i$ and $\hbar=\frac{3}{4}e^{i\pi/6}$.

The two critical points of $\Lambda_\hbar$ which correspond to the critical points of V (as $\hbar\to 0$) are indicated in Figures 9.1 and 9.2. As mentioned at the end of Section 8.4, at any fixed $\hbar\neq 0$ there exist many other critical points of $\Lambda_\hbar$ that can be seen between consecutive pairs of poles and zeroes in Figures 9.1 and 9.2; however, these other critical points become trapped in half-line singularities as $\hbar\to 0$, and their saddle-point approximations are not well-defined.

We now calculate the perturbative invariants $S_n^{(\mathrm{geom})}(u)$ and $S_n^{(\mathrm{conj})}(u)$ by doing a full saddle-point approximation of the integral (9.1.2) on (dummy) contours passing through the two critical points. We begin by formally expanding $\Upsilon(\hbar,p,u)=\log\Lambda_\hbar(p,u)$ as a series in both $\hbar$ and p around some fixed point p_0:

$$\Upsilon(\hbar,p_0+p,u)=\sum_{j=0}^{\infty}\sum_{k=-1}^{\infty}\Upsilon_{j,k}(p_0,u)\,p^j\hbar^k\,. \tag{9.1.8}$$

Our potential $V(p,u)$ is identified with $\Upsilon_{0,-1}(p,u)+2\pi iu$, and critical points of V are defined by $\Upsilon_{1,-1}(p,u)=\frac{\partial}{\partial p}V(p,u)=0$. Let us also define

$$b(p,u):=-2\Upsilon_{2,-1}(p,u)=-\frac{\partial^2}{\partial p^2}V(p,u)\,. \tag{9.1.9}$$

Then at a critical point $p_0 = p^{(\alpha)}(u)$, the integral (9.1.2) becomes[2]

$$Z^{(\alpha)}(M;\hbar,u) = \frac{e^{u+\frac{1}{\hbar}V^{(\alpha)}(u)}}{\sqrt{2\pi\hbar}}\int_{C_\alpha} dp\, e^{-\frac{b^{(\alpha)}(u)}{2\hbar}p^2}\exp\left[\frac{1}{\hbar}\sum_{j=3}^{\infty}\Upsilon^{(\alpha)}_{j,-1}(u)\,p^j + \sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\Upsilon^{(\alpha)}_{j,k}(u)\,p^j\hbar^k\right], \tag{9.1.10}$$

where $V^{(\alpha)}(u) = V(p^{(\alpha)}(u),u)$, $b^{(\alpha)}(u) = b(p^{(\alpha)}(u),u)$, and $\Upsilon^{(\alpha)}_{j,k}(u) = \Upsilon_{j,k}(p^{(\alpha)}(u),u)$ are implicitly functions of u alone.

We can expand the exponential in (9.1.10), integrate each term using

$$\int dp\, e^{-\frac{b}{2\hbar}p^2}p^n = \begin{cases} (n-1)!!\left(\frac{\hbar}{b}\right)^{n/2}\sqrt{\frac{2\pi\hbar}{b}} & n \text{ even} \\ 0 & n \text{ odd}\end{cases}, \tag{9.1.11}$$

and re-exponentiate the answer to get a final result. The integrals in (9.1.11) are accurate up to corrections of order $\mathcal{O}(e^{-\text{const}/\hbar})$, which depend on a specific choice of contour and are ignored. Following this process, we obtain

$$Z^{(\alpha)}(M;\hbar,u) = \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2\pi\hbar}{b^{(\alpha)}}}e^{u+\frac{1}{\hbar}V^{(\alpha)}(u)}e^{S_2^{(\alpha)}\hbar+S_3^{(\alpha)}\hbar^2+\ldots} \tag{9.1.12}$$

$$= \exp\left[\frac{1}{\hbar}V^{(\alpha)}(u) - \frac{1}{2}\log b^{(\alpha)} + u + S_2^{(\alpha)}\hbar + S_3^{(\alpha)}\hbar^2 + \ldots\right], \tag{9.1.13}$$

where the coefficients S_n can be straightforwardly computed in terms of b and the Υ's. For example, $S_2^{(\alpha)} = \frac{15}{2(b^{(\alpha)})^3}\Upsilon^{(\alpha)}_{3,-1} + \frac{3}{(b^{(\alpha)})^2}\Upsilon^{(\alpha)}_{4,-1} + \Upsilon^{(\alpha)}_{0,1}$ and $S_3^{(\alpha)} = \frac{3465}{8(b^{(\alpha)})^6}(\Upsilon^{(\alpha)}_{3,-1})^4 +$ (sixteen other terms). In addition, we clearly have

$$S_0^{(\alpha)}(u) = V^{(\alpha)}(u), \tag{9.1.14}$$

$$\delta^{(\alpha)} = 0, \tag{9.1.15}$$

$$S_1^{(\alpha)}(u) = -\frac{1}{2}\log\frac{b^{(\alpha)}}{m^2}. \tag{9.1.16}$$

To actually evaluate the coefficients $\Upsilon_{i,j}(p,u)$, we refer back to the expansion (8.4.15) of the quantum dilogarithm in Section 8.4. We find

$$\Upsilon_{j,k}(p,u) = \frac{B_{k+1}(1/2)\,2^k}{(k+1)!j!}\left[\text{Li}_{1-j-k}(-e^{p-u}) - (-1)^j\text{Li}_{1-j-k}(-e^{-p-u})\right] \tag{9.1.17}$$

when $j \geq 2$ or $k \geq 0$, and that all $\Upsilon_{j,2k}$ vanish.

Therefore: to calculate the expansion coefficients $S_n^{(\alpha)}(u)$ around a given critical point, we substitute $p^{(\alpha)}(u)$ from (9.1.6) into equations (9.1.4) and (9.1.17) to obtain $V^{(\alpha)}$, $b^{(\alpha)}$,

[2] This expression assumes that $\Upsilon_{j,k} = 0$ when k is even, a fact that shall be explained momentarily.

and $\Upsilon^{(\alpha)}_{j,k}$; then we substitute these functions into expressions for the S_n and simplify. At the geometric critical point $p^{(\mathrm{geom})}(u)$, we obtain

$$V^{(\mathrm{geom})}(u) = \frac{1}{2}\left[\mathrm{Li}_2(-e^{p^{(\mathrm{geom})}(u)-u}) - \mathrm{Li}_2(-e^{-p^{(\mathrm{geom})}(u)-u}) - 4p^{(\mathrm{geom})}(u)\,u + 4\pi i u\right], \tag{9.1.18}$$

$$b^{(\mathrm{geom})}(u) = \frac{im^2}{2}\Delta(m)\,, \tag{9.1.19}$$

and it is easy to check with a little algebra that all the expansion coefficients $S_n^{(\mathrm{geom})}(u)$ reproduce *exactly*[3] what we found in Table 6.3 of Section 6.6 by quantizing the moduli space of flat connections. (This has been verified to eight-loop order.) Moreover, the present state integral model completely fixed all the constants of the $S_n^{(\mathrm{geom})}(u)$, which had to be fixed in Table 6.3 by comparison to analytic continuation of the Jones polynomial.

Similarly, at the conjugate critical point $p^{(\mathrm{conj})}(u)$, we have

$$V^{(\mathrm{conj})}(u) = -V^{(\mathrm{geom})}(u)\,, \qquad b^{(\mathrm{conj})}(u) = -\frac{im^2}{2}\Delta(m)\,, \tag{9.1.20}$$

and more generally $S_n^{(\mathrm{conj})} = (-1)^{n-1}S_n^{(\mathrm{geom})}$. It is not hard to actually prove this relation between the geomtric and conjugate critical points to all orders by inspecting the symmetries of $e^{\Upsilon(\hbar,p,u)}$. Thus, we find complete agreement with the results of Section 6.6.

9.1.1 Checking $\hat{A}\cdot Z = 0$

An alternative and more convenient way to check that the quantum A-polynomial annihilates the perturbative partition functions of the state integral model (more convenient than computing $S_n(u)$'s independently in both approaches) is to apply the operator $\hat{A}(\hat{l},\hat{m})$ directly to the state integral.

Let us write, as usual,

$$\hat{A}(\hat{l},\hat{m}) = \sum_j a_j(\hat{m},q)\hat{l}^j\,. \tag{9.1.21}$$

By virtue of the functional relation

$$\Phi_\hbar(p-\hbar) = (1+e^p)\Phi_\hbar(p+\hbar)\,, \tag{9.1.22}$$

[3]There appears to be a small "correction" of $\frac{1}{2}\log(-1) = \log(\pm i)$ in $S_1^{(\mathrm{geom})}$, comparing (9.1.16) with the value in Table 6.3. This merely multiplies the partition function by i and can be attributed to the orientation of the stationary-phase contour passing through the geometric critical point. We also allow the usual modulo $2\pi i u$ ambiguity in matching S_0.

the operator $\hat{l}$ has a very simple action on quantum dilogarithms. In the case of the figure-eight knot, shifts of u and relabeling of the integration variable p can be combined to show that

$$Z(u+j\hbar) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{2\pi i u}{\hbar}} \int dp\, e^{\frac{(u+j\hbar)(2ip+(2j-1)\hbar)}{\hbar}} \frac{\Phi_\hbar(p-u)}{\Phi_\hbar(-p-u)} \big(-e^{p-u-(2j-1)\hbar};q\big)_j\,, \quad (9.1.23)$$

where $(x;q)_j$ denotes the *finite* q-Pochhammer symbol $(x;q)_j = \prod_{r=0}^{j-1}(1-q^r x)$. Then we have

$$\hat{A}\cdot Z(u) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{2\pi i u}{\hbar}} \int dp\, \frac{\Phi_\hbar(p-u)}{\Phi_\hbar(-p-u)} \sum_{j=0}^{3} a_j(e^u,q)\big(-e^{p-u-(2j-1)\hbar};q\big)_j e^{\frac{(u+j\hbar)(2ip+(2j-1)\hbar)}{\hbar}}\,. \quad (9.1.24)$$

The classical saddle point of this integral is unchanged from the original case (9.1.2), and the very same saddle point methods used above to find the perturbative coefficients of $Z(u)$ can be applied to the integral here to show that it vanishes perturbatively to all orders. This is actually somewhat easier than computing $Z(u)$ itself, because the terms S_0 and S_1, appearing multiplicatively in front of (9.1.12) can be completely ignored, and there are no branch cut ambiguities to worry about.

It may be possible that the integral (9.1.24) can be shown to vanish identically, without using any perturbative expansions (although, as written, the integrand is certainly not zero). It would be interesting to explore this further.

9.2 Three-twist knot $\mathbf{5_2}$

The complement knot $\mathbf{5_2}$ can be divided into three hyperbolic tetrahedra. They can be chosen to all have negative orientation; at the complete hyperbolic structure, they all have equal (negative) volumes. See, *e.g.*, [137] for a full description of the hyperbolic structure. The resulting state sum model can be written as a two-dimensional integral

$$Z(u) = \frac{1}{2\sqrt{2\pi\hbar}}\, e^{-u} \int dp_x\, dp_y\, \frac{e^{\frac{1}{2\hbar}\left[(p_x+4u+i\pi+\hbar)(p_y+2u+i\pi+\hbar)-\frac{3}{2}i\pi\hbar+\frac{\pi^2-\hbar^2}{2}\right]}}{\Phi_\hbar(p_y)\Phi_\hbar(p_x+2u)\Phi_\hbar(p_x)}\,. \quad (9.2.1)$$

As in the case of the figure-eight knot, each quantum dilogarithm only involves a single integration variable, making the asymptotic expansions of the quantum dilogarithms very simple.

The A-polynomial for the knot $\mathbf{5_2}$ is

$$A(l,m) = (l-1)(m^{14}l^3+(-m^{14}+2m^{12}+2m^{10}-m^6+m^4)l^2+(m^{10}-m^8+2m^4+2m^2-1)l+1)\,. \tag{9.2.2}$$

There are three non-abelian branches, two conjugate to each other and one real. Since this knot is chiral, there is no symmetry $A(l,m^{-1}) \sim A(l,m)$.

Since the A-polynomial has three non-abelian branches, one expects to find three saddle points in the integrand of (9.2.1). In terms of the "downstairs" variables $x = e^{p_x}$ and $y = e^{p_y}$, the saddle points are given by the equations

$$m^4x(y+1) = -1\,, \qquad m^2(x+1)y\left(m^2x+1\right) = -1\,. \tag{9.2.3}$$

It is easy to solve for $y = \frac{m^4(-x)-1}{m^4x}$, but the subsequent equation for x is (as expected) a third-degree irreducible polynomial:

$$m^6x^3 + m^6x^2 + m^4x^2 + m^4x + m^2x^2 + x + 1 = 0\,. \tag{9.2.4}$$

The solutions to this polynomial have the same arithmetic structure as the solutions to the A-polynomial. It is most convenient to leave an algebraic dependence on x in the perturbative invariants $S_n^{(\alpha)}(u)$, reducing them as much as possible with the equation (9.2.4). The resulting general expressions for $S_n^{(\alpha)}(u)$ can then be specialized to various branches α by substituting in the three actual solutions of (9.2.4).

The leading coefficient $S_0^{(\alpha)}(u)$ is given by

$$S_0^{(\alpha)}(u) = \frac{1}{2}\left[-\mathrm{Li}_2(-e^{p_x}) - \mathrm{Li}_2(-e^{p_x+2u}) - \mathrm{Li}_2(-e^{p_y}) + (p_y+2u)(p_x+4u) + i\pi(p_x+p_y+6u) - \frac{\pi^2}{2}\right]\,, \tag{9.2.5}$$

and specializes as expected to Vol$+i$CS, its conjugate, and a nonzero real quantity (for real or imaginary u) when an appropriate lift of the solution to is used.

The other coefficients quickly become more complicated, but the present saddle-point methods can still calculate them easily up to about sixth order. For example, one has

$$S_1 = -\frac{1}{2}\log\left[-\frac{3m^6x^2+2m^6x+2m^4x+m^4+2m^2x+1}{2m^2}\right]\,, \tag{9.2.6}$$

and

$$S_2^{(\alpha)}(m) = \frac{1}{12\left(m^{16}-6m^{14}+11m^{12}-12m^{10}-11m^8-12m^6+11m^4-6m^2+1\right)^2} \tag{9.2.7}$$
$$\times \big(-2m^{34}x^2-2m^{34}x+16m^{32}x^2+12m^{32}x-3m^{32}-106m^{30}x^2-68m^{30}x$$
$$+30m^{30}+254m^{28}x^2+112m^{28}x-148m^{28}-182m^{26}x^2-78m^{26}x+384m^{26}$$
$$-370m^{24}x^2-148m^{24}x-641m^{24}+392m^{22}x^2-280m^{22}x+410m^{22}$$
$$-1654m^{20}x^2-336m^{20}x+250m^{20}+392m^{18}x^2-1922m^{18}x-1116m^{18}$$
$$-370m^{16}x^2-336m^{16}x-529m^{16}-182m^{14}x^2-280m^{14}x-1116m^{14}$$
$$+254m^{12}x^2-148m^{12}x+250m^{12}-106m^{10}x^2-78m^{10}x+410m^{10}$$
$$+16m^8x^2+112m^8x-641m^8-2m^6x^2-68m^6x+384m^6+12m^4x-148m^4$$
$$-2m^2x+30m^2-3\big)\,.$$

The same approach of applying the quantum A-polynomial directly to the state integral can be used to show that these expressions are fully compatible with geometric quantization. The quantum A-polynomial for $\mathbf{5_2}$ appears in [118].

9.3 Direct analytic continuation

Let us finally describe the approach of direct analytic continuation from the colored Jones polynomial. This works in a few special cases where the colored Jones polynomial has a closed form expression as a sum of products, or more precisely a sum of finite q-Pochhammer symbols. The trick is to use the functional identity (9.1.22) for the quantum dilogarithm to write each q-Pochhammer symbol as a ratio of quantum dilogarithms, and to approximate the sum by an integral as $\hbar \to 0$ (this was also done in [102]). Then perturbative invariants $S_n^{(\alpha)}(u)$ can then be derived via our now standard methods of saddle-point approximation.

The analytic continuations involved here have *not* been made rigorous, but they certainly seem to work for practical computations. Moreover, unlike the current formulation of the state integral model, they work just as well for non-hyperbolic knots as for hyperbolic ones.

Figure-eight 4_1

Normalized (as explained in Section 5.4) to agree with the Chern-Simons partition function, the $SU(2)$ colored Jones polynomial for the figure-eight knot is [146]

$$J_N(q) = -\sqrt{\frac{2i\hbar}{\pi}}\sinh u \sum_{j=0}^{N-1} q^{Nj} \prod_{k=1}^{j} (1-q^{-N}q^{-k})(1-q^{-N}q^{k}) \tag{9.3.1}$$

$$= -\sqrt{\frac{2\hbar}{\pi}}\sinh u \sum_{j=0}^{N-1} q^{Nj} \frac{\Phi_\hbar(-2u+i\pi+\hbar)}{\Phi_\hbar(-2u-i\pi-\hbar)} \frac{\Phi_\hbar(-2i-i\pi-\hbar-2j\hbar)}{\Phi_\hbar(-2u+i\pi+\hbar+2j\hbar)} \tag{9.3.2}$$

$$= -\sqrt{\frac{2\hbar}{\pi}} e^u (1-e^{\frac{-2\pi i u}{\hbar}}) \sum_{j=0}^{N-1} q^{Nj} \frac{\Phi_\hbar(-2i-i\pi-\hbar-2j\hbar)}{\Phi_\hbar(-2u+i\pi+\hbar+2j\hbar)}\,. \tag{9.3.3}$$

As usual, $q = e^{2\hbar}$, and in analytic continuation $u = \hbar N$ (*cf.* (6.2.4)). Equating $-i\pi-\hbar+2j\hbar$ with a new integration variable p, and letting $dp = 2\hbar$, this sum is approximated by an integral in the limit $\hbar \to 0$. Specifically,

$$J_N(q) \to \frac{1}{\sqrt{2\pi i\hbar}} \big(e^{\frac{i\pi u}{\hbar}} - e^{-\frac{i\pi u}{\hbar}}\big) \int_{-i\pi+\hbar-2u}^{i\pi-i\hbar} dp\, e^{-\frac{up}{\hbar}} \frac{\Phi_\hbar(p-2u)}{\Phi_\hbar(-p-2u)}\,. \tag{9.3.4}$$

This expression looks very similar to the state integral model (9.1.2), though it has not been possible yet to prove their equivalence. It has several nice properties, including a more manifest symmetry under $u \to -u$ and a preferred choice of contour. It turns out that (9.3.4) has exactly two saddle points, and doing abstract saddle point expansions around them yields exactly the same perturbative invariants $S_n^{(\mathrm{geom,conj})}(u)$ obtained from the state integral model. The preferred contour indicated here crosses the geometric saddle — perhaps this is not a surprise, since it is the saddle that should govern the leading asymptotics of the colored Jones.

Note that the two nonperturbative terms in the prefactor $\big(e^{\frac{i\pi u}{\hbar}} - e^{-i\frac{\pi u}{\hbar}}\big)$ are related by a difference of $2\pi i u/\hbar$ in the exponent, which has always been an ambiguity in the partition function. Also note that when $u\hbar = N$ is an integer (*i.e.* the dimension of an $SU(2)$ representation), the prefactor $\big(e^{\frac{i\pi u}{\hbar}} - e^{-i\frac{\pi u}{\hbar}}\big)$ vanishes. This is a general feature of the subtle analytic continuation of compact Chern-Simons partition functions, and (partly) explains why exponential growth only appears when $N \notin \mathbb{Z}$ (*cf.* [41]).

Three-twist $\mathbf{5_2}$

There is a sum-of-products expression for the colored Jones polynomial of the knot $\mathbf{5_2}$ at the complete hyperbolic structure $u = 0$. It is given by [102]

$$J_N(q) = -\sqrt{\frac{2i\hbar}{\pi}} \sum_{0\le k\le l<N} \frac{(q;q)_l^2}{(q^{-1};q^{-1})_k} q^{-k(l+1)} \,. \tag{9.3.5}$$

In the analytic continuation limit $\hbar \to 0$ (and $u = N\hbar$ fixed), letting the two summation indices be identified with integration variables p_x and p_y, this formally becomes exactly the same expression as the state integral model (9.2.1) at $u = 0$.

Trefoil $\mathbf{3_1}$

The analogous expression for the trefoil can be evaluated exactly, in accordance with fact that higher-order invariants vanish on the non-abelian branch (*cf.* Section 6.6). The colored Jones polynomial is

$$J_N(q) = -\sqrt{\frac{2i\hbar}{\pi}} \sinh u q^{1-N} \sum_{j=0}^{N-1} q^{-jN} \prod_{k=1}^{j} (1 - q^{k-N}) \tag{9.3.6}$$

$$\to \sqrt{\frac{2i\hbar}{\pi}} (e^u - e^{-u}) \Phi_\hbar(-2u + \hbar - i\pi) \int_0^{2\pi i + 2u} \frac{e^{-\frac{1}{2\hbar}(p(2u+2\pi i) - 2u + 2\hbar)}}{\Phi_\hbar(p - 2u - i\pi + \hbar)} \,. \tag{9.3.7}$$

After a formal shift of countour, this is just a Fourier transform of the quantum dilogarithm. The Fourier transform of $\Phi_\hbar(x)^{\pm 1}$, described in [140], is just another quantum dilogarithm $\Phi_\hbar(x)^{\pm 1}$. In fact, the result of the integral in(9.3.7) is a quantum dilogarithm that almost exactly cancels the prefactor $\Phi_\hbar(-2u + \hbar - i\pi)$, resulting in an expression that involves no quantum dilogarithms at all. We find

$$J_N(q) \sim e^{\frac{3u^2 + i\pi u}{\hbar}} \,, \tag{9.3.8}$$

up to some additional constants, just as in the geometric quantization analysis of Section 6.6.

The form of the expression (9.3.7) suggests that the $SL(2,\mathbb{C})$ Chern-Simons partition function for the trefoil knot complement might have a state integral model despite the fact that the trefoil is not a hyperbolic knot. Indeed, it is known that the trefoil knot complement does have *topological* ideal triangulation consisting of two tetrahedra, the same as the number of quantum dilogarithms appearing in (9.3.7). A topological ideal triangulation of a three-manifold is similar to a geometric one, with all vertices of the triangulation being located on the manifold's boundary. For a torus knot such as the trefoil, the "problem" with imposing a hyperbolic geometric structure on this triangulation is that gluing conditions require all tetrahedra to be flat when $u = 0$. However, this should be no problem for Chern-Simons theory: an $SL(2, \mathbb{C})$ structure does not care whether tetrahedra are "flat." These ideas are explained further in [5].