Chapter 1

Introduction and Overview

During the past thirty years, dualities have been a cornerstone of progress in theoretical physics, and have motivated some of the most interesting and nontrivial new relations between physics and mathematics. Almost all such dualities have proven to have some basis in string theory. Prominent examples include the AdS/CFT correspondence [8] and mirror symmetry. More related to the present thesis is the Gromov-Witten/Donaldson-Thomas correspondence [9, 10, 11, 12], which related (via M-theory) the spaces of holomorphic maps into a Calabi-Yau threefold to the spaces of holomorphic curves on the threefold itself. In a somewhat different context, Chern-Simons theory with compact gauge group provided the first intrinsically three-dimensional interpretation of knot polynomials [13]; different descriptions of the theory then related the finite-type Vassiliev invariants with the Kontsevich integral [14].

Along these lines, this thesis is about developing new connections: between physical theories, between mathematical theories, and most importantly between physics and mathematics. We will begin by studying so-called refined Bogomol'nyi-Prasad-Sommerfield (BPS) invariants of Calabi-Yau threefolds and their wall-crossing behavior. We show that they can be described and calculated in many ways, including via melting crystal models, and we will conjecture that they are related to the motivic Donaldson-Thomas invariants of Kontsevich and Soibelman [15]. We will then turn to Chern-Simons theory with *noncompact* gauge group, a theory intrinsically different from the compact Chern-Simons theory that computes Jones polynomials. We use a multitude of approaches to understand the relation between the compact and noncompact theories, to calculate exact partition functions and new knot invariants, and to relate noncompact Chern-Simons theory to "quantum" hyperbolic geometry.

Refined BPS invariants

The BPS invariants of a Calabi-Yau threefold X can be thought of in several different ways. Physically, they describe the states of 1/2-BPS D-branes in type II string theory that is compactified on a product of X and four-dimensional Minkowski space, $X \times \mathbb{R}^{3,1}$. Equivalently, the same BPS invariants describe the bound states of supersymmetric point-like black holes in the low-energy supergravity theory on $\mathbb{R}^{3,1}$ [16]. Or, in a mathematical setting, BPS invariants describe objects in the derived category of coherent sheaves on X [17].

The actual Hilbert space of BPS states $\mathcal{H}_{BPS}$ in any of these descriptions depends on *stability conditions*, which in turn depend (say, in a type IIA duality frame) on the values of the Kähler moduli or "size parameters" of X [16, 17, 18]. In terms of D-branes, the stability conditions ensure roughly that a brane wraps a cycle of minimal volume, and that it cannot decay into a sum of noninteracting branes. Since $\mathcal{H}_{BPS}$ is a discrete object, it must be locally constant as a function of moduli. However, it can jump at special values of moduli where the stability conditions change. This happens at real codimension-1 wall in moduli space, and is a phenomenon known as wall crossing.

To study the properties of the space $\mathcal{H}_{BPS}$, it is useful to construct well-behaved supersymmetric indices that count its states. Often, such indices are sufficient for applications like approximating the entropy of black holes in string theory [19]. The simplest option for constructing an index is to observe that the Hilbert space $\mathcal{H}_{BPS}$ is graded by charge — in a string theory picture, this is the charge of the D-branes that make up various states. Then one can define an *unrefined index* $\Omega(\gamma)$ to be the signed count of charge-γ states in $\mathcal{H}_{BPS}$. For special values of Kähler moduli, the generating function of these unrefined indices is just the partition function of the well-known Gromov-Witten or Donaldson-Thomas invariants [20]. Indeed, as mentioned above, this is the context in which BPS invariants first became important in mathematics.

The *refined* BPS indices that play a main role in this thesis are defined by summing states in the Hilbert space $\mathcal{H}_{BPS}$ with an extra weight $(-y)^{2J_3}$ that keeps track of their spin content. In terms of four-dimensional supergravity, this spin is just the physical $\widetilde{SO(3)} \simeq SU(2)$ spin of massive point-like particles. The resulting index $\Omega^{ref}(\gamma; y)$ retains much more of the information in $\mathcal{H}_{BPS}$ and reduces to the unrefined $\Omega(\gamma)$ when $y \to 1$. Alternatively, in situations where $\mathcal{H}_{BPS}$ has a description as the cohomology of a classical D-brane moduli space $\mathcal{M}$ (*cf.* [21]), the refined index is associated to the Poincaré polynomial of $\mathcal{M}$, while the unrefined index is its Euler characteristic.

Refined indices were first introduced in the special case of Gromov-Witten/Donaldson-Thomas theory. They allowed topological strings (*i.e.* Gromov-Witten theory) to compute equivariant instanton sums in four-dimensional gauge theory [22, 23]. For toric Calabi-Yau's, ordinary Gromov-Witten/Donaldson-Thomas generating functions could be calculated using the topological vertex of [24, 25], and a refined vertex was constructed to compute the corresponding refined generating functions [26]. Moreover, using large-N duality [27] and the relation between topological strings and compact Chern-Simons theory [28], it was realized that the refined partition functions should be related to homological invariants of knots (which categorify Jones, etc., polynomials) [29, 30].

All these previous applications of refined BPS invariants were restricted exclusively to the Gromov-Witten/Donaldson-Thomas chamber of Calabi-Yau moduli space (*i.e.* the special choice of moduli for which $\Omega^{ref}(\gamma)$ are refined Donaldson-Thomas invariants). In this thesis, we want to move beyond Gromov-Witten/Donaldson-Thomas theory and analyze refined BPS invariants in all chambers of Kähler moduli space, focusing in particular on their wall-crossing behavior.

In Chapter 2, we define refined indices more carefully, and generalize the wall-crossing formulas derived by Denef and Moore [20] from the unrefined to the refined case. There are some important differences between unrefined and refined invariants, such as a dependence of $\Omega^{ref}(\gamma)$ on complex structure (or hypermultiplet) moduli as well as the potential existence of new walls in Kähler moduli space where $\Omega^{ref}(\gamma)$ could jump. We give a (non)example of the latter in Section 2.4. In Chapter 3, we apply refined wall crossing to the resolved conifold geometry $\mathcal{O}(-1,-1) \to \mathbb{P}^1$, and derive a picture of refined generating functions in an infinite set of chambers, analogous to an unrefined description presented by [31]. Moreover, we relate the generating function in each chamber to a statistical melting crystal model of refined "pyramid partitions" with varying boundary conditions, which generalize the refined topological vertex. These models will suggested a new combinatorial interpretation of wall crossing in [1], which has since been extended beyond the conifold [32, 33].

In Chapter 4, we arrive at our main mathematical conjecture: that refined invariants are equivalent to the motivic Donaldson-Thomas invariants of Kontsevich and Soibelman [15]. Kontsevich and Soibelman defined a version of Donaldson-Thomas invariants for Calabi-Yau categories that depend on a stability condition (just like physical BPS invariants), and obey a very general wall-crossing formula. It has previously been argued [34] that the "classical" or unrefined versions of motivic Donaldson-Thomas invariants are equivalent to physical unrefined BPS invariants. Motivic invariants, however, naturally depend on an extra parameter '$\mathbb{L}$' or 'q' (the motive of the affine line), which we argue is to be identified with the refined spin variable y. We substantiate our claim both theoretically, by matching refined and motivic wall-crossing formulas, and with direct examples from $SU(2)$ Seiberg-Witten theory.

The ultimate goal of the present program would be to study not the refined indices $\Omega^{ref}(\gamma)$ but the entire Hilbert space $\mathcal{H}_{BPS}$, its full dependence on *all* moduli, and its homological properties. We have by now come quite close to doing this, and hope it will be the subject of future interesting work.

Chern-Simons theory

In the second part of this thesis, we shift our focus to three-dimensional Chern-Simons gauge theory with complex, noncompact gauge group. Chern-Simons theory is a preeminent example of a topological quantum field theory (TQFT). By now, Chern-Simons theory with *compact* gauge group G is a mature subject with a history going back to the 1980's (see *e.g.* [35, 14] for excellent reviews), and has a wide range of applications, ranging from invariants of knots and 3-manifolds [13] on one hand, to condensed matter physics [36, 37] and to string theory [38] on the other.

We will specifically be interested in a version of Chern-Simons gauge theory with complex gauge group $G_{\mathbb{C}}$. Although at first it may appear merely as a variation on the subject, the physics of this theory is qualitatively different from that of Chern-Simons theory with compact gauge group. For example, one important difference is that to a compact Riemann surface Σ Chern-Simons theory with compact gauge group associates a finite-dimensional Hilbert space $\mathcal{H}_\Sigma$, whereas in a theory with non-compact (and, in particular, complex) gauge group the Hilbert space is infinite-dimensional. Due to this and other important differences that will be explained in further detail below, Chern-Simons gauge theory with complex gauge group remains a rather mysterious subject. The first steps toward understanding this theory were made in [39] and, more recently, in [40, 41].

As in a theory with a compact gauge group, the classical action of Chern-Simons gauge theory with complex gauge group $G_{\mathbb{C}}$ is purely topological — that is, independent of the metric on the underlying 3-manifold M. However, since the gauge field $\mathcal{A}$ (a $\mathfrak{g}_{\mathbb{C}}$-valued 1-form on M) is now complex, one can write two topological terms in the action, involving $\mathcal{A}$ and $\bar{\mathcal{A}}$:

$$\begin{aligned} S \;&=\; \frac{t}{8\pi}\int_M \mathrm{Tr}\Big(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\,\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big) \\ &\quad+\frac{\bar{t}}{8\pi}\int_M \mathrm{Tr}\Big(\bar{\mathcal{A}}\wedge d\bar{\mathcal{A}}+\frac{2}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\Big)\,. \end{aligned} \tag{1.0.1}$$

Although in general the complex coefficients ("coupling constants") t and $\bar{t}$ need not be complex conjugate to each other, they are not entirely arbitrary. Thus, if we write $t = k+\sigma$ and $\bar{t} = k-\sigma$, then consistency of the quantum theory requires the "level" k to be an integer, $k \in \mathbb{Z}$, whereas unitarity requires σ to be either real, $\sigma \in \mathbb{R}$, or purely imaginary, $\sigma \in i\mathbb{R}$ [39].

Given a 3-manifold M (possibly with boundary), Chern-Simons theory associates to M a "quantum $G_{\mathbb{C}}$ invariant" that we denote as $Z(M)$. Physically, $Z(M)$ is the partition function of the Chern-Simons gauge theory on M, defined as a Feynman path integral

$$Z(M) \;=\; \int \mathcal{D}\mathcal{A}\; e^{iS} \tag{1.0.2}$$

with the classical action (1.0.1). Since the action (1.0.1) is independent of the choice of metric on M, one might expect that the quantum $G_{\mathbb{C}}$ invariant $Z(M)$ is a topological invariant of M. This is essentially correct even though independence of metric is less obvious in the quantum theory, and $Z(M)$ does turn out to be an interesting invariant. How then does one compute $Z(M)$?

One approach is to use the topological invariance of the theory. In Chern-Simons theory with compact gauge group G, the partition function $Z(M)$ can be efficiently computed by cutting M into simple "pieces," on which the path integral (1.0.2) is easy to evaluate. Then, via "gluing rules," the answers for individual pieces are assembled together to produce

$Z(M)$. In practice, there may exist many different ways to decompose M into basic building blocks, resulting in different ways of computing $Z(M)$.

Although a similar set of gluing rules should exist in a theory with complex gauge group $G_{\mathbb{C}}$, they are expected to be more involved than in the compact case. The underlying reason for this was already mentioned: in Chern-Simons theory with complex gauge group the Hilbert space is infinite dimensional (as opposed to a finite-dimensional Hilbert space in the case of compact gauge group G). One consequence of this fact is that finite sums which appear in gluing rules for Chern-Simons theory with compact group G turn into integrals over continuous parameters in a theory with non-compact gauge group. This is one of the difficulties one needs to face in computing $Z(M)$ non-perturbatively, *i.e.* as a closed-form function of complex parameters t and $\bar{t}$.

A somewhat more modest goal is to compute $Z(M)$ perturbatively, by expanding the integral (1.0.2) in inverse powers of t and $\bar{t}$ around a saddle point (a classical solution). In Chern-Simons theory, classical solutions are flat gauge connections, that is gauge connections $\mathcal{A}$ which obey

$$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \; = \; 0 \,, \tag{1.0.3}$$

and similarly for $\bar{\mathcal{A}}$. A flat connection on M is determined by its holonomies, that is by a homomorphism

$$\rho : \pi_1(M) \to G_{\mathbb{C}} \,, \tag{1.0.4}$$

modulo gauge transformations, which act via conjugation by elements in $G_{\mathbb{C}}$.

Given a gauge equivalence class of the flat connection $\mathcal{A}$, or, equivalently, a conjugacy class of the homomorphism ρ, one can define a "perturbative partition function" $Z^{(\rho)}(M)$ by expanding the integral (1.0.2) in inverse powers of t and $\bar{t}$. Since the classical action (1.0.1) is a sum of two terms, the perturbation theory for the fields $\mathcal{A}$ and $\bar{\mathcal{A}}$ is independent. As a result, to all orders in perturbation theory, the partition function $Z^{(\rho)}(M)$ factorizes into a product of "holomorphic" and "antiholomorphic" terms:

$$Z^{(\rho)}(M) \; = \; Z^{(\rho)}(M;t) Z^{(\rho)}(M;\bar{t}) \,. \tag{1.0.5}$$

This holomorphic factorization is only a property of the perturbative partition function. The exact, non-perturbative partition function $Z(M)$ depends in a non-trivial way on both

t and $\bar{t}$, and the best one can hope for is that it can be written in the form (*cf.* [40, 41])

$$Z(M) \;=\; \sum_{\rho} Z^{(\rho)}(M;t) Z^{(\rho)}(M;\bar{t})\,, \tag{1.0.6}$$

where the sum is over classical solutions (1.0.3) or, equivalently, conjugacy classes of homomorphisms (1.0.4).

In the greater part of this thesis, we study the perturbative partition function $Z^{(\rho)}(M)$. Due to the factorization (1.0.5), it suffices to consider only the holomorphic part $Z^{(\rho)}(M;t)$. Moreover, since the perturbative expansion is in the inverse powers of t, it is convenient to introduce a new expansion parameter $\hbar \;=\; 2\pi i/t$, which plays the role of Planck's constant (the semiclassical limit corresponds to $\hbar \to 0$). In general, the perturbative partition function $Z^{(\rho)}(M;\hbar)$ is an asymptotic power series in $\hbar$. To find its general form one applies the stationary phase approximation to the integral (1.0.2):

$$Z^{(\rho)}(M;\hbar) \;=\; \exp\left(\frac{1}{\hbar} S_0^{(\rho)} - \frac{1}{2}\delta^{(\rho)} \log \hbar \,+\, \sum_{n=0}^{\infty} S_{n+1}^{(\rho)} \hbar^n \right)\,. \tag{1.0.7}$$

This is the general form of the perturbative partition function in Chern-Simons gauge theory with any gauge group, compact or otherwise, computed with standard rules of perturbation theory [13, 42, 43, 44] that will be discussed in more detail below. Roughly speaking, $S_0^{(\rho)}$ is the value of the Chern-Simons functional evaluated on a flat gauge connection $\mathcal{A}^{(\rho)}$ associated with the homomorphism ρ, and each subleading coefficient $S_n^{(\rho)}$ is obtained by summing over Feynman diagrams with n loops.

In Chern-Simons theory with compact gauge group, perturbation theory is often developed in the background of a trivial (or reducible) flat connection $\mathcal{A}^{(\rho)}$. As a result, the perturbative coefficients $S_n^{(\rho)}$ have a fairly simple structure; they factorize into a product of topological invariants of M — the finite type (Vassiliev) invariants and variations thereof — and group-theory factors [14]. In particular, they are rational numbers. In contrast, Chern-Simons theory with complex gauge group naturally involves perturbation theory in the background of genuinely non-abelian (non-reducible) flat connections. Physically, this is a novelty that has not been properly addressed in previous literature. We shall see that the information about a non-abelian flat connection and the 3-manifold M is mixed within the $S_n^{(\rho)}(M)$ in a non-trivial way, and results in $S_n^{(\rho)}(M)$'s that are *not* finite type invariants and are typically not valued in $\mathbb{Q}$.

A primary example of non-abelian representations in the complex case comes from considering hyperbolic 3-manifolds, which, in a sense, constitute the richest and the most interesting class of 3-manifolds. A hyperbolic structure on a 3-manifold M corresponds to a discrete faithful representation of the fundamental group $\pi_1(M)$ into $\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL(2,\mathbb{C})$, the group of orientation-preserving isometries of 3-dimensional hyperbolic space $\mathbb{H}^3$. Adding a choice of spin structure, this lifts to a representation $\rho : \pi_1(M) \to SL(2,\mathbb{C})$, which can then be composed with a morphism ϕ to any larger algebraic group $G_\mathbb{C}$ to obtain a representation $\rho : \pi_1(M) \to G_\mathbb{C}$. The flat connection associated to such a ρ is non-reducible (in fact, for a complete hyperbolic structure the holonomies of the connection are parabolic), and the corresponding perturbative coefficients $S_n^{(\rho)}$ are interesting new invariants of the hyperbolic 3-manifold M. See Table 6.3 on page 117 for the simplest example of this type.

A direct computation of the perturbative invariants $S_n^{(\rho)}$ via Feynman diagrams is straightforward in principle, but quickly becomes unwieldy as the number of loops n grows. Thus, it is useful to look for alternative methods of defining and computing these invariants. Altogether, different physical descriptions and quantizations of Chern-Simons theory lead us to the following four approaches:

1. **Feynman diagrams**, as already mentioned.

2. **Geometric quantization of** $\mathcal{M}_{\mathrm{flat}}(G_\mathbb{C};\Sigma)$, the moduli space of flat connections on the boundary Σ of a three-manifold M, which serves as the classical phase space of Chern-Simons theory.

3. **"Analytic continuation"** from Chern-Simons theory with compact gauge group G to its complexification $G_\mathbb{C}$.

4. **State sum model** obtained by decomposing M into tetrahedra, assigning a simple partition function to each tetrahedron, and integrating out boundary conditions as the tetrahedra are glued back together.

The first three have been previously employed to tackle Chern-Simons theory with complex gauge group, while the fourth is completely new. Used in conjunction, these methods lead to very powerful results, mathematically and physically.

We will begin by describing the "traditional" approach of Feynman diagrams in Chapter 5. They will lead us to define the concept of an Arithmetic TQFT, and conjecture that

Chern-Simons theory with complex gauge group belongs to this special class. For hyperbolic M, the arithmeticity of Chern-Simons theory will be very closely related to the arithmeticity of hyperbolic invariants.

In Chapter 6, we then consider the geometric quantization of Chern-Simons theory with complex gauge group on a three-manifold M with boundary Σ. One advantage of Chern-Simons theory with complex gauge group is that the classical phase space $\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$ is a hyper-Kähler manifold, a fact that considerably simplifies the quantization problem in any of the existing frameworks (such as geometric quantization [45], deformation quantization [46, 47], or "brane quantization" [48]). We will see that the partition functions $Z^{(\rho)}(M;\hbar)$ obey a system of Schrödinger-like equations $\hat{A}_i Z^{(\rho)}(M;\hbar) = 0$, which, together with appropriate boundary conditions, uniquely determine $Z^{(\rho)}(M;\hbar)$.

Combining geometric quantization with "analytic continuation" will lead to a very efficient computation of the operators $\hat{A}_i$, since it will turn out that they also act on and annihilate partition functions of Chern-Simons theory with compact gauge group. Mathematically, perhaps the most interesting consequence of combining "analytic continuation" with geometric quantization is (almost) a physical proof of the volume conjecture. We will discuss this in Chapter 6 as well. Note that "analytic continuation" involves a very subtle limit of the compact Chern-Simons invariants, and, in particular, does not contradict the fact that the complex Chern-Simons invariants turn out to be qualitatively different from compact ones.

Our main examples throughout this work involve knot or link complements in closed three-manifolds. (In particular, all hyperbolic manifolds are of this type.) In Chern-Simons theory with compact gauge group, however, knot invariants are typically associated with the expectation values of Wilson loops in closed manifolds. In Chapter 7, we define Wilson loops also that carry infinite-dimensional irreducible representations of complex-gauge group, and explain how their expectation values are equivalent to partition functions on knot/link complements. The discussion will also clarify the limiting process involved in "analytic continuation."

In Chapter 8, we finally proceed to the fourth approach: the state sum model for Chern-Simons theory with complex gauge group. This involves cutting a manifold M into tetrahedra, assigning to each tetrahedron a partition function — specifically, an element of a Hilbert space $\mathcal{H}$ associated to the tetrahedron boundary — and taking inner products in these boundary Hilbert spaces to glue the tetrahedra back together. Conceptually, such a cutting and gluing procedure should always be possible in TQFT; it was often employed to study Chern-Simons theory with compact gauge group, where boundary Hilbert spaces are finite-dimensional (*cf.*. [49, 50, 51]). In the complex case, boundary Hilbert spaces are infinite-dimensional, so that what one seeks is really a state *integral* model. We construct such a model for the case $G_{\mathbb{C}} = SL(2,\mathbb{C})$ and for M hyperbolic based on the work of K. Hikami [52, 53]. Extensions to completely general M and $G_{\mathbb{C}}$ should be possible, though they have not yet been fully developed.

Chapter 9 is then devoted to examples of computations in the state integral model. Schematically, the state integral model expresses $Z^{(\rho)}(M;\hbar)$ as a multi-dimensional integral of a product of *quantum dilogarithm* functions, on which classical saddle-point methods can be used to extract the invariants $S_n^{(\rho)}(M)$. We consider in detail the complements of the figure-eight knot $\mathbf{4_1}$ and the knot $\mathbf{5_2}$, computing $S_n^{(\rho)}(M)$ to high order. We also compare the integrals of the state integral model to similar expressions obtained by direct analytic continuation of compact G-invariants in some special cases, showing that the latter can also be used to find $S_n^{(\rho)}(M)$'s.

Future directions and the quantum dilogarithm

There are many directions in which to continue the studies of refined BPS invariants and complex Chern-Simons theory that have begun here. They are both still relatively unexplored fields. In the case of BPS invariants, it would be very exciting to find a proof of refined = motivic directly in physics, extending the proof of the classical Kontsevich-Soibelman formula in gauge theory given by [34]. Some progress along these lines was recently made in [54]. There is also much yet to be understood about the wall-crossing (or "locus-crossing") behavior of refined invariants in hypermultiplet moduli space, and about the existence of potential new walls in the vector multiplet moduli space. Ultimately, one would like to describe the full "categorical" dependence of the Hilbert space $\mathcal{H}_{BPS}$ itself on moduli.

In the case of Chern-Simons theory with complex gauge group, an immediate goal (and a subject of current research) is to generalize the state integral model to arbitrary manifolds and gauge groups. It is also not fully understood how to obtain the Schrödinger-like operators $\hat{A}$ directly in geometric quantization. Perhaps more importantly, all the results that appear in this work describe Chern-Simons theory and knot invariants in the perturbative regime, and it would be exciting to move beyond this and understand Chern-Simons theory with complex gauge group nonperturbatively. The first steps in this direction were made in [41]. It is quite possible that a fully developed state integral model will be able to complete the program.

A final related direction concerns an intriguing connection between BPS invariants and the Chern-Simons state integral model. At the most rudimentary level, one find the same special function — a quantum dilogarithm — appearing in both. The simplest definition for the quantum dilogarithm is via the infinite product

$$\mathbf{E}_q(x) = \prod_{r=1}^{\infty} \left(1 + q^{r-1/2}x\right)^{-1}\,, \qquad |q| < 1\,. \tag{1.0.8}$$

It obeys a remarkable "pentagon" identity: if operators x_1 and x_2 are such that $x_1x_2 = qx_2x_2$, then

$$\mathbf{E}_q(x_1)\,\mathbf{E}_q(x_2) = \mathbf{E}_q(x_2)\,\mathbf{E}_q(q^{-\frac{1}{2}}x_1x_2)\,\mathbf{E}_q(x_1)\,. \tag{1.0.9}$$

The central role of the quantum dilogarithm and (1.0.9) throughout this thesis (see especially Chapters 2, 4, 8, and 9) has motivated its presence in the title.

Elsewhere in physics, the quantum dilogarithm appears as the generating function of a gas of free (charged) bosons. In a more specialized context, it also features as an ingredient in open topological string partition functions. In mathematics, the quantum dilogarithm is ubiquitous in representation theory of quantum groups and (noncompact) affine Lie algebras.

In the context of both BPS invariants and Chern-Simons theory, the quantum dilogarithm function really signals the presence of an entire structural apparatus involving quantizations of complex tori and cluster transformations [55] acting on triangulated surfaces. These triangulated surfaces were recently given physical meaning in terms of BPS wall crossing by [56]. In Chern-Simons theory, the quantization of triangulated surfaces is the quantization of boundary moduli spaces. It would be truly interesting to connect these two pictures via a physical duality — this will hopefully be the subject of future work.