Topics in Supersymmetry Breaking and Gauge/Gravity Dualities

Thesis by

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Abstract

The thesis covers two topics in string theory and quantum field theory. First, we realize metastable vacua in various supersymmetric gauge theories. Specifically, we consider the Coulomb branch of any $\mathcal{N} = 2$ supersymmetric gauge theory, and perturb it by a superpotential and engineer a metastable vacuum at a point. We also study its relation to Kähler normal coordinates and Fayet-Iliopoulos terms. Having studied the metastable construction, we apply this to general gauge mediation. We show how to compute the current correlators when the hidden sector is strongly coupled in specific examples.

Next, we consider gauge/gravity dualities. We apply dualities to the investigation of various strongly coupled field theories. In one example, we construct M-theory supergravity solutions with the nonrelativistic Schrödinger symmetry starting from the warped AdS_5 metric with $\mathcal{N} = 1$ supersymmetry. We impose that the lightlike direction is compact by making it a nontrivial $U(1)$ bundle over the compact space. In another example, we show that, in a gravity theory with a Chern-Simons coupling, the Reissner-Nordström black hole in anti-de Sitter space is unstable depending on the value of the Chern-Simons coupling. The analysis suggests that the final configuration is likely to be a spatially modulated phase.

Contents

Introduction and Outline of Thesis

Supersymmetry extends the usual Poincaré symmetry by adjoining symmetries that relate bosons and fermions. The Coleman-Mandula theorem [1] states that supersymmetry is essentially the only possible extension of the Poincaré symmetry. Supersymmetry exhibits many virtues such as being a possible solution to the hierarchy problem and to the gauge coupling unification, and it has captivated high energy theorists for decades. In addition to being a viable model to describe nature above the TeV scale, supersymmetric gauge theories have been analyzed in place of more realistic systems to study physical phenomena such as confinement or chiral symmetry breaking, since supersymmetry places additional constraints over the system and much more information can be inferred compared to non-supersymmetric theories. The low energy effective theory of $\mathcal{N}=2$ supersymmetric gauge theory with $SU(2)$ gauge was found by [2, 3] and subsequent works extended this to more general $\mathcal{N}=2$ supersymmetric gauge theories. See [4] for a review. For a class of $\mathcal{N}=1$ supersymmetric gauge theories, Seiberg duality helps us examine some of the properties of the theories even when the coupling is strong [5, 6].

However, supersymmetry has to be spontaneously broken to describe nature, and it is natural to think that supersymmetry is broken at the TeV scale to suppress the radiative corrections to the Higgs boson mass. Although it is generally difficult to construct a dynamically broken supersymmetric model, it was pointed out in [7] that what we need is only a metastable vacuum, as long as the tunneling rate to supersymmetric vacua can be parametrically suppressed. That is, if the lifetime of a metastable vacuum is longer than the age of the universe, there is no problem considering it as stable in all practical purposes.

In part I of the thesis, we examine closely a class of supersymmetric gauge theories where metastable vacua exist. We will find a general method to construct such theories and investigate their properties. The models with metastable vacua can be used as a hidden sector for a realistic supersymmetric model to describe particle physics. We discuss how to obtain information of the supersymmetry breaking effect to the visible sector in several examples.

In part II of the thesis, we turn to another interesting topic: gauge/gravity dualities. The original and most concrete version is the correspondence between the $AdS_5 \times S^5$ type IIB supergravity and $\mathcal{N}=4$ super Yang-Mills theory. The initial observation was that there are two different descriptions of the low energy dynamics of D3-branes [8]. A more detailed prescription for the matching of the two theories was made in [9, 10]. In [10], the Kaluza-Klein spectrum of the supergravity theory is matched to the BPS operators in the $\mathcal{N} = 4$ Yang-Mills theory. Since then, numerous checks have been made to support the correspondence between the two theories. Also, many additional examples are found that have gauge/gravity dual pictures. For a comprehensive review of works on gauge/gravity dualities, see [11].

In addition to understanding gauge/gravity dualities and searching for the evidence in support of them, given the overwhelming evidence in favor of the dualities, another useful direction to pursue is to use gravity theories to study some strongly coupled gauge theories. Being strong/weak dualities, strongly coupled gauge theories correspond to classical solutions in gravity theories. Even though we do not have full information of the gauge theories, a substantial part of the classical gravity solutions can be determined. In this case, gravity solutions give us the properties of the universality class of strongly coupled gauge theories. We consider two applications of gauge/gravity dualities. First, we will study field theories with nonrelativistic scale invariance, called Schrödinger symmetry by constructing a supergravity solution in M-theory that has the required symmetry properties. Then, we will consider gauge theories with chiral anomalies for some global $U(1)$ symmetry, whose corresponding gravity theories have a Chern-Simons coupling. We will show that, in some cases, an inhomogeneous phase is favored over a uniform phase and investigate precise conditions for the occurrence of such a phase.

Part I

Metastable Vacua in Supersymmetric Gauge Theories

Chapter 1

Introduction

Presently, all physical phenomena up to the TeV scale are well described by the standard model. Despite its beautiful success, however, it is conceivable that a new theory has to exist beyond the TeV scale. One reason is that we need a Higgs boson to break electroweak symmetry, and for the radiative corrections to the mass of the Higgs boson not to give an unacceptable large contribution, the cutoff scale should be of the TeV scale. This is called the naturalness problem, and one way to address this problem is to consider supersymmetric gauge theories [12–14]. Usually, the mass of a boson receives a quadratic divergence. In contrast, the fermion mass obtains only a logarithmic divergence due to the fact that an additional chiral symmetry appears in the absence of the mass term. In the presence of supersymmetry, the mass of a boson is related with that of its fermionic partner. As a consequence, a quadratic divergence is absent from the Higgs mass under renormalization. In addition to addressing the naturalness problem, supersymmetric gauge theories provide the lightest supersymmetric particle (LSP) as a natural candidate for dark matter.

Of course, supersymmetry should be broken below the TeV scale to describe our non-supersymmetric world. We can introduce supersymmetry breaking terms by hand and in this case, they should break supersymmetry softly in the sense that quadratic divergences are absent [15–17]. But a better way will be to break supersymmetry dynamically: without an artificial introduction of soft terms, they are generated in the low energy effective theory dynamically. However, it is not easy to realize dynamical supersymmetry breaking. For example, $\mathcal{N}=1$ supersymmetric gauge theories with massive and vector-like matter fields have supersymmetric vacua due to the nonzero Witten index [18]. Therefore, the models that realize dynamical supersymmetry breaking vacua are rather elaborate.

However, it was pointed out by [7] that we do not actually need a complete supersymmetry breaking. Even though supersymmetry is restored at some moduli of the theory, it may be possible that there exist metastable vacua whose decay rate to the supersymmetric state is extremely low and practically zero. The metastable vacua have appeared previously in models of supersymmetry breaking [19–21], and in a superstring setup [22]. The novelty in the discovery of [7] is that metastable vacua occur even in very simple supersymmetric gauge theories. They found out that $SU(N_c)$

supersymmetric gauge theories with massive N_f hypermultiplets have metastable vacua for $N_c+1 \leq N_f < 3N_C/2$. Following their idea and techniques, various phenomenological models have been proposed [23–31]. Moreover, one can construct supersymmetry breaking models in the context of string theories as a low energy theories on D-branes [32–45]. We will later consider another way to realize metastable vacua by studying the Coulomb branch of $\mathcal{N} = 2$ supersymmetric gauge theory perturbed by a small superpotential [46, 47].

Realizing supersymmetry breaking is not enough for construction of a realistic model. Tree level supersymmetry breaking is not attractive due to the constraint given by the supertrace [48]: quite generally, the constraint predicts an existence of a supersymmetric partner lighter than its ordinary partner particle. To circumvent the problem, we may assume that there exists a hidden sector where supersymmetry is broken, but there is no direct tree level coupling between the hidden sector and the observable sector, which contains observable particles and their superpartners. The observable sector is usually a variation of the minimal supersymmetric standard model (MSSM).

There are several ways to realize the idea of the mediation between the two sectors. One is via gravity; since it couples to the stress-energy tensor, it virtually couples to every field. The higher dimensional operators are suppressed by powers of the Planck scale. One of such higher dimensional operators give supersymmetric mass terms, called the μ-term to the Higgs particles, so the μ-term is of the order of the supersymmetry breaking scale, which is of order 100GeV. This is one of the nice features of this approach. However, in this approach, there is typically a significant flavor symmetry violation. Note that the only place where the flavor symmetry is violated in the standard model is in the Yukawa couplings. The problem is that the mechanism that breaks the flavor symmetry at some higher scale may also introduce additional terms that break the flavor symmetry in supergravity models and there is no obvious reason to prohibit that. Another problem for the flavor symmetry is that it is not plausible that the ultraviolet theory has any kind of global symmetry. This comes from the study of the string theory, which does not seem to possess exact global symmetries.

Another mediation mechanism is the gauge mediation. In this scenario, some of the hidden-sector fields are gauged by the gauge group of the observable sector. As an example, in a simple model of gauge mediation, there is a chiral superfield $\mathcal{M}$ in the hidden sector that obtains a nonzero F-term expectation value by some dynamical supersymmetry breaking.

$$\mathcal{M} = M + \theta^2 F \, , \qquad F \neq 0 \, . \tag{1.1}$$

The hidden sector superfield $\mathcal{M}$ couples to the messenger chiral superfields $\tilde{\Phi}$ and Φ that are charged under the gauge group of the observable sector via the interaction

$$\mathcal{L}_{int} = \int d^2\theta \mathcal{M} \tilde{\Phi} \Phi \, . \tag{1.2}$$

In the case when the messengers $\tilde{\Phi}$ and Φ are weakly coupled in the hidden sector, the nonzero F component of $\mathcal{M}$ splits the mass spectrum of $\tilde{\Phi}$ and Φ and it can be computed explicitly. This splitting does not affect the mass spectrum of observable particles at tree level. However, since they are charged under the gauge group of the observable sector, loop corrections to the masses of observable particles contain Feynman diagrams with messengers, which means there are loop corrections to the mass spectrum.

A similar consideration can be made in the case when there are no apparent messenger fields and the hidden sector is inherently strongly coupled. In such general settings, a precise definition of gauge mediation is proposed by [49], which states that a model is called gauge mediated if the model decouples into the observable sector and the hidden sector when the gauge couplings of the observable sector vanish. An attractive feature of this scenario is that the mediation automatically respects the flavor symmetry since gauge symmetry is flavor blind. Also, the mechanism is highly predictive in the sense that just a few parameters that determine the soft supersymmetry breaking determine all parameters in the observable sector while, for example, the MSSM has 100 or more adjustable parameters. However, unlike the gravity mediated models, there is no apparent way to produce the μ-term in this mechanism since the μ-term is supersymmetric. This problem can be handled in various explicit models, but we do not discuss this issue in the thesis. The definition encompasses most models with the supersymmetry breaking sector and the messengers, and models with direct mediation. Many of the physically observable effects mediated to the observable sector can be encoded by the current-current correlators of the global symmetry of the hidden sector. This allows us to extract the soft supersymmetry breaking terms even in the strongly coupled hidden sector and we will see some of such examples [50].

The organization of part I is as follows. In chapter 2, we will review the basics of supersymmetric gauge theories for later discussions. In chapter 3, we will show that, for a generic choice of a point on the Coulomb branch of any $\mathcal{N} = 2$ supersymmetric gauge theory, it is possible to find a superpotential perturbation which generates a metastable vacuum at the point. We will also study its relation to Kähler normal coordinates and Fayet-Iliopoulos terms. In chapter 4, we consider general gauge mediation and compute current correlators when the hidden sector is strongly coupled.

Chapter 2

Supersymmetric Gauge Theories

In this chapter, we present basic elements of supersymmetric gauge theories as a preparation for later discussions. First, we introduce the superfield notation and construct $\mathcal{N} = 1$ supersymmetric gauge theories using superfields. Then we explain $\mathcal{N} = 2$ supersymmetric gauge theories in terms of $\mathcal{N} = 1$ superfields and then introduce $\mathcal{N} = 2$ superspace formalism for vector multiplets for abelian gauge theories to discuss Fayet-Iliopoulos terms.

2.1 $\mathcal{N} = 1$ Superfields

In the four-dimensional flat spacetime, a nice way to realize supersymmetry is to introduce superspace coordinates. These consist of the usual space-time coordinates x^{μ}, and additional fermionic coordinates θ^{α} and $\bar{\theta}_{\dot{\alpha}}$. The convention follows that of [51]. The supersymmetry transformation acts on functions of the superspace by the operators

$$
\begin{aligned}
Q_{\alpha} &= \frac{\partial}{\partial\theta^{\alpha}} - i\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_{\mu} \ , \\
\bar{Q}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu} \ ,
\end{aligned}
\tag{2.1}
$$

which satisfy the anticommutation relation $\{Q_{\alpha}, \bar{Q}_{\dot{\alpha}}\} = 2i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}$. It is convenient to introduce the supercovariant derivatives

$$
\begin{aligned}
D_{\alpha} &= \frac{\partial}{\partial\theta^{\alpha}} + i\sigma^{\mu}\alpha\dot{\alpha}\bar{\theta}^{\dot{\alpha}}\partial_{\mu} \ , \\
\bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^{\alpha}\sigma^{\mu}\alpha\dot{\alpha}\partial_{\mu} \ ,
\end{aligned}
\tag{2.2}
$$

which anticommute with Q_{α} and $\bar{Q}_{\dot{\alpha}}$ and have their own anticommutation relation $\{D_{\alpha}, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}$.

A superfield is a function on the superspace with coordinates $(x^{\mu}, \theta^{\alpha}, \bar{\theta}_{\dot{\alpha}})$. Since fermionic coordinates anticommute, a series expansion with respect to these coordinates always terminates at some finite order. Note that a sum or a product of two superfields is also a superfield. A general superfield, however, contains too many degrees of freedom to describe a physical system. So we need to impose some constraints on the superfield. One way is to impose the condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{2.3}$$

to a superfield Φ. If we introduce coordinates $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, this constraint is equivalent to saying that Φ is a function of only y^μ and θ^α and not of $\bar{\theta}_{\dot{\alpha}}$. That is, the superfield Φ has the expansion

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^\alpha\theta_\alpha F(y)\ . \tag{2.4}$$

In terms of x, θ and $\bar{\theta}$, this can be written as

$$\Phi(x,\theta,\bar{\theta}) = A(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu A - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2 A + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta\theta F(x)\ . \tag{2.5}$$

We call a superfield Φ that satisfies (2.3) a chiral superfield. Note that a product of two chiral superfields is also a chiral superfield.

There is another kind of constrained superfield that will be used later. It is called a vector superfield and a vector superfield V satisfies the relation $V = V^\dagger$. It is used to describe a gauge field. For an abelian gauge theory, V transforms as

$$V \to V + \Lambda + \Lambda^\dagger \tag{2.6}$$

under the supersymmetric version of the gauge transformation. Here Λ is a chiral superfield. We can choose a gauge, called the Wess-Zumino gauge, where V takes the form

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D\ . \tag{2.7}$$

Supersymmetry is broken by the choice of the gauge, but still there is an ordinary gauge symmetry $A_\mu \to A_\mu + \partial_\mu f$ for some function $f(x)$. The supersymmetric version of the field strength in the abelian theory is

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V\ . \tag{2.8}$$

W_α satisfies $\bar{D}_{\dot{\alpha}}W_\beta = 0$. That is, W_α is a chiral superfield. In components, it is written as

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha\ , \tag{2.9}$$

where the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

For non-abelian gauge symmetry, the transformation law (2.6) is replaced with

$$e^{-2V} \to e^{-i\Lambda^\dagger} e^{-2V} e^{i\Lambda} \ , \tag{2.10}$$

where V and Λ are Lie-algebra valued vector and chiral superfields, respectively. The field strength is given by

$$W_\alpha = \frac{1}{8}\bar{D}^2 e^{2V} D_\alpha e^{-2V} \ . \tag{2.11}$$

2.2 $\mathcal{N} = 1$ Lagrangians in Superfield Formalism

Using the chiral and vector superfields introduced in the previous section, it is possible to write down a Lagrangian that has $\mathcal{N} = 1$ supersymmetry manifestly. By performing dimensional analysis or direct calculation, it can be readily checked that the θ^2 component of a chiral superfield, or the $\theta^2\bar{\theta}^2$ component of a vector superfield yields a total derivative under a supersymmetric transformation. Therefore, for a chiral superfield Φ, a general $\mathcal{N} = 1$ supersymmetric Lagrangian has the form

$$\mathcal{L} = \int d^4\theta K(\Phi, \Phi^\dagger) + \int d^2\theta \mathcal{W}(\Phi) + h.c. \ , \tag{2.12}$$

where $h.c.$ denotes the hermitian conjugate of its previous term. $K(\Phi, \Phi^\dagger)$ is called the Kähler potential and is a real function of Φ and $\Phi^\dagger$.

For a vector superfield V for an abelian symmetry, there is a corresponding chiral superfield W_α, which is a field strength, as defined in (2.8). Using this, the kinetic term for the gauge field is given by

$$\mathcal{L} = \frac{1}{4g^2}\int d^2\theta W^\alpha W_\alpha + h.c. \ . \tag{2.13}$$

For a non-abelian theory, we only need an additional trace. In general, there is a θ-term in the Lagrangian of the form $\theta\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, and to account for this, it is better to use a complex coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$. In this case, the Lagrangian for a non-abelian theory is

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\mathrm{Tr}\int d^2\theta \ \tau W^\alpha W_\alpha + h.c. \ . \tag{2.14}$$

2.2.1 *R*-symmetry

In supersymmetric gauge theories, there is sometimes a global symmetry called an R-symmetry. The supercharges are in a representation of the R-symmetry. In $\mathcal{N} = 1$ supersymmetric gauge theories, the supercharges are in $U(1)$ representation: Q^α has charge $+1$ and $Q_{\dot{\alpha}}$ -1. This in turn implies that the fermionic superspace coordinate θ (resp. $\bar{\theta}$) has charge -1 (resp. $+1$) under the R-symmetry. The superfields in the theory transform appropriately if there is an R-symmetry. Note that the superpotential $\mathcal{W}(\Phi)$ has R-charge +2 as can be checked in (2.12).

2.3 $\mathcal{N} = 2$ Lagrangian in $\mathcal{N} = 1$ Superfield Formalism

We have seen that $\mathcal{N} = 1$ supersymmetry has two kinds of superfields: one is a chiral superfield and the other is a vector superfield. In this section, a Lagrangian that has $\mathcal{N} = 2$ symmetry will be described in terms of these $\mathcal{N} = 1$ superfields. Note that in this case, the R-symmetry is $SU(2) \times U(1)$: in addition to the $U(1)$ R-symmetry that is manifest in $\mathcal{N} = 1$ notation, there is an additional $SU(2)$ R-symmetry under which a pair of (Dirac) supercharges transform as a doublet. Let Φ be a chiral superfield and V a vector superfield. Suppose both are in the adjoint representation of the gauge symmetry. Then, combining them, we can construct an $\mathcal{N} = 2$ vector multiplet (Φ, V). The $\mathcal{N} = 2$ Lagrangian can be written as

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\left(\int d^2\theta \frac{\partial^2 \mathcal{F}}{\partial\Phi^a\partial\Phi^b} W^{a\alpha}W^b_\alpha + 2\int d^2\theta d^2\bar{\theta}(\Phi^\dagger e^{2gV})^a \frac{\partial\mathcal{F}(\Phi)}{\partial\Phi^a}\right) , \qquad (2.15)$$

where a, b are indices for the adjoint representation of the gauge symmetry. $\mathcal{F}(\Phi)$ is called a $\mathcal{N} = 2$ prepotential and is a holomorphic function of Φ. Note that $\mathcal{F}(\Phi)$ determines the Lagrangian completely. $\mathcal{N} = 2$ symmetry constrains the system very tightly so that this is the most general form of the Lagrangian with only one $\mathcal{N} = 2$ vector multiplet, and the only effect of the renormalization is to change the prepotential $\mathcal{F}(\Phi)$.

However, there is another $\mathcal{N} = 2$ multiplet, called a hypermultiplet, and this consists of a chiral superfield Q and an anti-chiral superfield $\tilde{Q}^\dagger$ in some representation of the gauge group. The $\mathcal{N} = 2$ Lagrangian for hypermultiplets $(Q_i, \tilde{Q}^i)$, where i denotes flavor symmetry index, can be written as

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\left(Q^{i\dagger}e^{-2V}Q_i + \tilde{Q}^i e^{2V}\tilde{Q}_i^\dagger\right) + \int d^2\theta\left(\sqrt{2}\tilde{Q}_i\Phi Q_i + m_i^j\tilde{Q}^i Q_j\right) + h.c. . \qquad (2.16)$$

Note that, unlike vector multiplets, a mass term is allowed without breaking $\mathcal{N} = 2$ supersymmetry.

2.4 Supersymmetry-Preserving Vacua of $\mathcal{N} = 2$ Abelian Theory with Fayet-Iliopoulos Terms

An abelian version of (2.15) of the $\mathcal{N} = 2$ Lagrangian of vector multiplets A_i is written as

$$\mathcal{L} = \frac{1}{2}\mathrm{Im}\left[\int d^4\theta\,\mathcal{F}_i(A_k)\bar{A}^i + \frac{1}{2}\int d^2\theta\,\mathcal{F}_{ij}(A_k)W^i_\alpha W^{\alpha\,j}\right] , \qquad (2.17)$$

where the gauge group is $U(1)^n$, $i = 1, \cdots, n$ and $\mathcal{F}_{i_1 i_2 \ldots} = \partial_{i_1}\partial_{i_2}\ldots\mathcal{F}(a_i)$. It is not possible to add an ordinary superpotential and still have $\mathcal{N} = 2$ supersymmetry. However, we can consider adding

$\mathcal{N}=2$ Fayet-Iliopoulos (FI) terms to the Lagrangian and, in special cases, they can be written as superpotentials. But it is more convenient to discuss the FI-terms in terms of the $\mathcal{N}=2$ superspace language to see explicitly that the theory with FI terms preserves the full $\mathcal{N}=2$ supersymmetry in an appropriate sense.

2.4.1 $\mathcal{N}=2$ Superspace and Superfields

We shall work in the $\mathcal{N}=2$ superspace conventions of [52] with two anticommuting coordinates θ^α and $\tilde{\theta}^\alpha$. The standard realization of $\mathcal{N}=2$ supersymmetry on this space is through the operators

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \ , \qquad \tilde{Q}_\alpha = \frac{\partial}{\partial\tilde{\theta}^\alpha} - i(\sigma^\mu\bar{\tilde{\theta}})_\alpha\partial_\mu \ , \tag{2.18}$$

and their conjugates. A generic chiral $\mathcal{N}=2$ superfield $\mathcal{A}$ can be constructed from two chiral $\mathcal{N}=1$ superfields, Φ and G, along with a chiral $\mathcal{N}=1$ spinor superfield W_α as

$$\mathcal{A}(\tilde{y},\theta,\tilde{\theta}) = \Phi(\tilde{y},\theta) + i\sqrt{2}\tilde{\theta}W(\tilde{y},\theta) + \tilde{\theta}^2 G(\tilde{y},\theta) \ , \tag{2.19}$$

where $\tilde{y}^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} + i\tilde{\theta}\sigma^\mu\bar{\tilde{\theta}}$. The $\mathcal{N}=1$ superfields admit further component expansions

$$\begin{aligned} \Phi(\tilde{y},\theta) &= \phi(\tilde{y}) + \sqrt{2}\theta\psi(\tilde{y}) + \theta^2 F(\tilde{y}) \ , \\ W_\alpha(\tilde{y},\theta) &= -i\lambda_\alpha(\tilde{y}) + \theta_\gamma\left(\delta_\alpha^\gamma D(\tilde{y}) - \frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu\theta\right)_\alpha^\gamma F_{\mu\nu}(\tilde{y})\right) - i\theta^2\xi_\alpha(\tilde{y}) \ , \\ G(\tilde{y},\theta) &= \tilde{F}(\tilde{y}) + \sqrt{2}\theta\eta(\tilde{y}) + \theta^2 C(\tilde{y}) \ . \end{aligned} \tag{2.20}$$

Note that W_α does not satisfy any constraints so it is not quite the superfield with which we are used to constructing $\mathcal{N}=1$-invariant actions. In particular, $F_{\mu\nu}$ does not satisfy the Bianchi identity and ξ_α is not proportional to $\left(\sigma^\mu\partial_\mu\bar{\lambda}\right)_\alpha$.

Let us now consider a chiral superfield $\mathcal{A}_D$ satisfying the additional reducing constraint

$$\left(D^{a\,\alpha}D^b_\alpha\right)\mathcal{A}_D = \left(\overline{D}^a_{\dot{\alpha}}\overline{D}^{b\,\dot{\alpha}}\right)\mathcal{A}_D^\dagger \ . \tag{2.21}$$

This again admits an expansion of the sort (2.19),

$$\mathcal{A}_D(\tilde{y},\theta,\tilde{\theta}) = \Phi_D(\tilde{y},\theta) + i\sqrt{2}\tilde{\theta}W_D(\tilde{y},\theta) + \tilde{\theta}^2 G_D(\tilde{y},\theta) \ . \tag{2.22}$$

The corresponding $\mathcal{N}=1$ expansion, though, becomes

$$
\begin{aligned}
\Phi_D(\tilde{y},\theta) &= \phi_D(\tilde{y}) + \sqrt{2}\theta\psi_D(\tilde{y}) + \theta^2 F_D(\tilde{y}) \,, \\
W_{\alpha\, D}(\tilde{y},\theta) &= -i\lambda_{\alpha\, D}(\tilde{y}) + \theta_\gamma \left(\delta_\alpha^\gamma D_D(\tilde{y}) - \frac{i}{2} \left(\sigma^\mu \bar{\sigma}^\nu \theta\right)_\alpha^\gamma F_{\mu\nu\, D}(\tilde{y}) \right) + \theta^2 \sigma^\mu_{\alpha\dot{\beta}} \partial_\mu \bar{\lambda}^{\dot{\beta}}_D(\tilde{y}) \,, \\
G_D(\tilde{y},\theta) &= \bar{F}_D(\tilde{y}) + i\sqrt{2}\left(\theta\sigma^\mu\partial_\mu\bar{\psi}_D(\tilde{y})\right) - \theta^2 \partial^2 \bar{\phi}_D(\tilde{y}) \,,
\end{aligned}
\tag{2.23}
$$

with

$$
\partial_{[\mu} F_{\nu\rho]\, D} = 0 \,. \tag{2.24}
$$

Note that the reduced $\mathcal{N}=2$ superfield differs from the unconstrained one in several ways. In addition to $F_{\mu\nu\, D}$ satisfying the Bianchi identity, the θ^2 component of $W_{\alpha\, D}$ as well as the θ and θ^2 components of G_D are no longer independent but instead are given by total derivatives of other component fields. Finally, the bottom component of G_D is an auxiliary field, which is not independent but instead set to the complex conjugate of the top component of Φ_D, denoted F_D.

2.4.2 FI Terms and Nonlinear Realization of $\mathcal{N}=2$ Supersymmetry

To describe $\mathcal{N}=2$ Lagrangian in terms of $\mathcal{N}=2$ superfields, we introduce two types of $\mathcal{N}=2$ chiral superfields:[1] a generic $\mathcal{N}=2$ chiral superfield $\mathcal{A}$ and a reduced $\mathcal{N}=2$ chiral superfield $\mathcal{A}_D$ which satisfies the constraint (2.21). Using $\mathcal{N}=2$ chiral superfields $\mathcal{A}_i$, the Lagrangian (2.17) is written as, up to an overall factor,

$$
\mathcal{L} = \frac{1}{2}\mathrm{Im}\left[\int d^2\theta\, d^2\tilde{\theta}\left(\mathcal{F}(\mathcal{A}_i) - \mathcal{A}^i \mathcal{A}_{D\, i}\right)\right] \,. \tag{2.25}
$$

To write FI terms, we introduce vectors of auxiliary components for both $\mathcal{A}$ and $\mathcal{A}_D$,

$$
Y = \begin{pmatrix} i\left(F - \tilde{F}\right) \\ F + \tilde{F} \\ \sqrt{2}D \end{pmatrix} \,, \qquad Y_D = \begin{pmatrix} i\left(F_D - \bar{F}_D\right) \\ F_D + \bar{F}_D \\ \sqrt{2}D_D \end{pmatrix} \,, \tag{2.26}
$$

where D and D_D are the usual $\theta\tilde{\theta}$ coefficients of $\mathcal{A}$ and $\mathcal{A}_D$, respectively. The addition of the FI terms to the original $\mathcal{N}=2$ action leads to

$$
S = \frac{1}{2}\mathrm{Im}\left[\int d^4x\, d^2\theta\, d^2\tilde{\theta}\left(\mathcal{F}(\mathcal{A}_i) - \mathcal{A}^i\mathcal{A}_{D\, i}\right)\right] + \frac{1}{2}\mathrm{Re}\int d^4x\, \left(E_i Y^i + M^i Y_{D\, i}\right) \,. \tag{2.27}
$$

E_i and M^i are electric/magnetic charge vectors, respectively. We will first consider the case where the third components in Y and Y_D do not contribute to the action. That is, we consider the case

[1] We use the notation of Antoniadis et al. [53]. See [54] for an $SU(2)$-covariant approach that is equivalent.

where the third component of each E_i and M^i vanishes. Introducing n complex numbers e_i and m^i, the vectors E_i and M^i is expressed as

$$E_i = \begin{pmatrix} \mathrm{Im}\, e_i \\ \mathrm{Re}\, e_i \\ 0 \end{pmatrix} , \qquad M^i = \begin{pmatrix} \mathrm{Im}\, m^i \\ \mathrm{Re}\, m^i \\ 0 \end{pmatrix} . \tag{2.28}$$

In terms of the $\mathcal{N}=1$ superfield language, the addition of such FI terms is equivalent to the addition of the superpotential

$$W = e_i a^i + m^i a_{Di} . \tag{2.29}$$

To recover the $\mathcal{N}=1$ version of the action, we note that integrating out $\mathcal{A}_D$ imposes the reducing constraint on $\mathcal{A}$, up to a subtlety involving m^j that we will address later.

The first term of (2.27) is manifestly invariant under the full $\mathcal{N}=2$ supersymmetry. Indeed, in the absence of the second term, it is straightforward to write (2.27) in components and demonstrate that integrating out $\mathcal{A}_D$ when $E_i = M^j = 0$ simply causes $\mathcal{A}$ to become a reduced $\mathcal{N}=2$ chiral superfield. The situation of vanishing E_i and M^j is also one in which the action (2.27) clearly preserves $\mathcal{N}=2$ supersymmetry because it is simply the top component of an $\mathcal{N}=2$ chiral superfield, which transforms into a total derivative.

Written in component form, the FI terms in (2.27) are given by

$$\frac{1}{2}\mathrm{Re} \int d^4x \left(E_i Y^i + M^i Y_{D\,i}\right) = \frac{1}{2} \int d^4x \left\{ \mathrm{Re}\left(e_i F^i + \bar{e}_i \tilde{F}^i\right) + \mathrm{Re}\,\left(2mF_D\right) \right\} . \tag{2.30}$$

Because F_D is the θ^2 component of a reduced $\mathcal{N}=2$ superfield, it is easy to see that it transforms into a total derivative under the action of all $\mathcal{N}=2$ generators (2.18). This is not so for F and $\tilde{F}$, though, as there are two problematic non-derivative transformations

$$\epsilon \tilde{Q} F = \sqrt{2}\epsilon\xi , \qquad \epsilon Q \tilde{F} = \sqrt{2}\epsilon\eta . \tag{2.31}$$

If $\mathcal{A}$ were reduced, the ξ_α would be proportional to $(\sigma^\mu \partial_\mu \bar{\lambda})_\alpha$ while $\eta_\alpha \sim (\sigma^\mu \partial_\mu \bar{\psi})_\alpha$. In this case, the RHS of (2.31) would consist of total derivatives and invariance of (2.30) would be assured. As it stands, however, (2.30) is not preserved by either Q_α or $\tilde{Q}_\alpha$.

We can try to improve the situation by suitably adjusting the transformation law of $\mathcal{A}_D$. In particular, because $\mathcal{A}_D$ appears in (2.27) only via a term which is linear in both $\mathcal{A}$ and $\mathcal{A}_D$, we can try to absorb the terms on the RHS of (2.31) by suitably shifting the transformation laws of component fields of $\mathcal{A}_D$. Indeed, expanding in components we see that ξ^i and η^i appear in the $\mathcal{A}\mathcal{A}_D$ term as

$$\int d^2\theta\, d^2\tilde{\theta}\, \left(-\mathcal{A}^i \mathcal{A}_{D\,i}\right) = \ldots + \frac{1}{2}\mathrm{Im}\,\left(\lambda_{D\,i}\xi^i + \psi_{D\,i}\eta^i\right) + \ldots \tag{2.32}$$

This means that the full action (2.27) can be made invariant under the full $\mathcal{N}=2$ supersymmetry if we modify the transformation laws of λ_D and ψ_D under $\tilde{Q}$ and Q from

$$\left(\epsilon\tilde{Q}\right)\lambda_{\alpha\,D}=\sqrt{2}\epsilon_\alpha\bar{F}_D\ ,\qquad (\epsilon Q)\,\psi_{\alpha\,D}=\sqrt{2}\epsilon_\alpha F_D\ , \tag{2.33}$$

to

$$\left(\epsilon\tilde{Q}\right)\lambda_{\alpha\,D}=\sqrt{2}\epsilon_\alpha\left(\bar{F}_D-ie\right)\ ,\qquad (\epsilon Q)\,\psi_{\alpha\,D}=\sqrt{2}\left(F_D-i\bar{e}\right)\ . \tag{2.34}$$

Because of the inhomogeneous terms now present in these transformation laws, the realization is no longer linear. We can linearize one of the supercharges, though, by shifting F_D appropriately. In particular, if we take $F_D\to F_D+i\bar{e}$ then the Q transformation of $\psi_{\alpha\,D}$ becomes linear. This can be understood by noting that such a shift also effectively removes the e_iF^i term from the action. On the other hand, taking $F_D\to F_D-i\bar{e}$ renders the $\tilde{Q}$ transformation of $\lambda_{\alpha\,D}$ linear. In this case, the shift of F_D effectively removes the $\bar{e}_i\tilde{F}^i$ term from the action. Note that, while we have a choice to linearly realize either supersymmetry, it is impossible to simultaneously do so for the full $\mathcal{N}=2$ supersymmetry algebra. This implies that at most $\mathcal{N}=1$ supersymmetry can be realized in vacua of the model (2.27). That such a nonlinear realization of a subset of supercharges is possible despite the general arguments of [55] was first pointed out in [56].

2.4.3 Conditions for Supersymmetry-Preserving Vacua

To study the conditions for a given vacuum to preserve some fraction of the $\mathcal{N}=2$ supersymmetry, it is sufficient to look at the transformation laws of the fermions in (2.27). Grouping the supercharges $Q,\tilde{Q}$ and fermions ψ^j,λ^j into $SU(2)_R$ doublets $\mathcal{Q}_I=(Q,\tilde{Q})^T$ and $\Psi^j_I=(\psi^j,\lambda^j)^T$, we can write these simply as

$$\epsilon^{\alpha\,K}\mathcal{Q}_{\alpha\,K}\Psi^j_{\beta\,I}\sim\epsilon_{IJ}\left(\mathcal{M}^{(j)}\right)^J_{\ K}\epsilon^K_\beta\ , \tag{2.35}$$

where[2]

$$\left(\mathcal{M}^{(j)}\right)^J_{\ K}=\begin{pmatrix}0 & \tilde{F}^j\\ F^j & 0\end{pmatrix}\ . \tag{2.36}$$

Consequently, we see that a vacuum preserves the Q ($\tilde{Q}$) supercharges when the expectation values of the F^j ($\tilde{F}^j$) vanish for all j. To compute these expectation values, we start by integrating out

[2]In general, $\mathcal{M}^{(j)}$ will take the form $\begin{pmatrix}D^j & \tilde{F}^j\\ F^j & -D^j\end{pmatrix}$ but we have set $D^j=0$ because we only consider adding electric and magnetic F-terms to the theory.

$\mathcal{A}_D$. The relevant part in the Lagrangian (2.27) is

$$\begin{aligned} S &= \ldots - \frac{1}{2}\mathrm{Im}\int d^4x\, d^2\theta\, d^2\tilde{\theta}\, \mathcal{A}^i \mathcal{A}_{D\,i} + \frac{1}{2}\mathrm{Re}\int d^4x\, \left(E_i Y^i + M^i Y_{D\,i}\right) \\ &= \ldots + \frac{1}{2}\mathrm{Re}\int d^4x\, \left[iF_{D\,j}\left(\tilde{F}^j - \bar{F}^j\right) + \bar{e}_j\left(\tilde{F}^j + \bar{F}^j\right) + 2m^j F_{D\,j}\right] . \end{aligned} \tag{2.37}$$

It is easy to see that the result is simply to set

$$\tilde{F}^j = \bar{F}^j + 2im^j \, . \tag{2.38}$$

The expectation value of F^j is then obtained by studying the F-term potential

$$\frac{1}{2}\mathrm{Im}\left(\tilde{F}^j \tau_{jk} F^k\right) + \frac{1}{2}\mathrm{Re}\left(\bar{e}_j(\tilde{F}^j + \bar{F}^j)\right) , \tag{2.39}$$

and concluding that

$$\begin{aligned} \bar{F}^j &= -\left(\mathrm{Im}\,\tau^{-1}\right)^{jk}\left(e_k + m^\ell \tau_{\ell k}\right) , \\ \tilde{F}^j &= -\left(\mathrm{Im}\,\tau^{-1}\right)^{jk}\left(e_k + m^\ell \bar{\tau}_{\ell k}\right) . \end{aligned} \tag{2.40}$$

Consequently, we see that vacua for which $(e + \tau m)_j = 0$ preserve the Q supercharges while vacua for which $(e + \bar{\tau} m) = 0$ preserve the $\tilde{Q}$ supercharges.

2.4.4 Inclusion of D-terms

To this point, we have only considered the addition of F-terms to the theory. But one can also consider the addition of D terms to the story. That is the third components of E_i and M^i in (2.28) need not vanish. This situation has been discussed by [57] in the context of IIB constructions.

In general, FI-terms are characterized by the $2n$ 3-vectors $\vec{E}_j$ and $\vec{Y}^j$ of (2.28), which transform under $SU(2)_R$. With our choice of basis, non-vanishing D-terms correspond to $\vec{E}_j$ and/or $\vec{Y}_j$ having nonzero third components.[3] Let us now suppose, for a moment, that a supersymmetric vacuum exists. Using an $SU(2)_R$ rotation, we can change the original supercharges Q_1 and Q_2 into another set of supercharges Q'_1 and Q'_2 such that the Q'_1 annihilate the vacuum. Generalizing (2.36) along with (2.40), the transformation matrix $\mathcal{M}$ is now given by

$$\left(\mathcal{M}^{(j)}\right)^J_{\ K} = -\left(\mathrm{Im}\,\tau^{-1}\right)^{jk}\begin{pmatrix} \xi_k + \xi^l_D \bar{\tau}_{lk} & e_k + m^l \bar{\tau}_{lk} \\ \bar{e}_k + \bar{m}^l \bar{\tau}_{lk} & -\xi_k - \xi^l_D \bar{\tau}_{lk} \end{pmatrix} , \tag{2.41}$$

where ξ_k and ξ^l_D are real and generically nonzero. Since we assume that Q'_1 annihilates the vacuum,

[3] In general, we will have nonzero D-terms for any choice of basis if $\vec{E}_i$ and $\vec{Y}^j$ are not all coplanar.

the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ should be annihilated by $\left(\mathcal{M}^{(j)}\right)^{J}_{\ K}$ for all j. To that end, we want

$$\xi_k + \xi_D^l \bar{\tau}_{lk} = 0 \qquad \text{and} \qquad e_k + m^l \tau_{lk} = 0 \qquad \text{for all } k \ . \tag{2.42}$$

The first condition cannot be satisfied, though, unless $\xi_k = \xi_D^k = 0$ for all k since $\mathrm{Im}\,\tau_{lk}$ is positive definite. Therefore, in $\mathcal{N} = 2$ supersymmetric language, a necessary condition to have a supersymmetric vacuum is that the $2n$ vectors $\vec{E}_j$ and $\vec{M}^j$ lie on a common plane. For $SU(2)$ gauge theory, this is always possible since there are only two vectors, but for higher gauge groups, generic FI terms necessarily break all of the supersymmetry.

Chapter 3

Metastable Vacua in Supersymmetric Gauge Theories

In this chapter, we consider the Coulomb branch of $\mathcal{N}=2$ supersymmetric gauge theory, perturbed by a small superpotential. Let us first recall the Seiberg-Witten theory, which is the low energy effective field theory of a $\mathcal{N}=2$ supersymmetric gauge theory possibly with hypermultiplets. For $U(N)$ gauge group, the D-term contributes to the potential of the form

$$V_D \sim \mathrm{Tr}[\Phi, \Phi^\dagger] \,. \tag{3.1}$$

Hence vacuum states are characterized by the condition and Φ and $\Phi^\dagger$ are simultaneously diagonalizable. For a vacuum state, let a_i, $i=1,\ldots,N$ be the diagonal values of Φ in an appropriate basis. Not all the a_i cannot, however, be distinct since there is the Weyl symmetry that permutes a_i, which means that physical quantities are a gauge invariant combination, such as $u_r = \sum_i a_i^r$, where $r=1,\ldots,N$. For $SU(N)$ gauge group, the condition $u_1 = 0$ is imposed. For each distinct set of u_r, we have a distinct gauge inequivalent vacuum. The set of the gauge inequivalent vacua is called the moduli space. The low energy effective Lagrangian is shown in (2.17), which we write here again for convenience.

$$\mathcal{L} = \frac{1}{2}\mathrm{Im}\left[\int d^4\theta\, \mathcal{F}_i(A_k)\bar{A}^i + \frac{1}{2}\int d^2\theta\, \mathcal{F}_{ij}(A_k) W^i_\alpha W^{\alpha\, j}\right] \,, \tag{3.2}$$

where $\mathcal{F}_{i_1 i_2 \ldots} = \partial_{i_1}\partial_{i_2}\ldots\mathcal{F}(a_i)$. The Lagrangian is completely determined by a holomorphic prepotential $\mathcal{F}$ which depends on the moduli. Due to the holomorphy of the prepotential, its quantum correction is severely restricted, and it is possible to obtain an expression of the prepotential [2,3,58–60].

Viewed in $\mathcal{N}=1$ language, the scalar field Φ is the lowest component of a chiral superfield. If we add a superpotential for this chiral superfield, it will break $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$. However, as long as the superpotential is very small, we may ignore its effect on the metric, or on the Kähler potential, to the leading order. The simplest example of this kind will be the pure

$\mathcal{N}=2$ $SU(2)$ gauge theory with the addition of a small superpotential $W=\lambda\mathrm{Tr}\Phi^2$ with small λ. However, it was already shown in [7] that there are no metastable vacua in this setup. Hence we need to consider more complicated superpotentials. Fortunately, we will see that there is a systematic way to find a superpotential that generates metastable vacua. This follows from the fact that the sectional curvature of the Coulomb branch moduli space is positive semi-definite.

In the next section, we show that a metastable vacuum can occur at a generic point on the Coulomb branch with an appropriate choice of a superpotential. After developing the general framework, we discuss the case of the $SU(2)$ theory without matter multiplet [3] in detail to show how the mechanism works. In this case, the potential can be drawn as a three-dimensional graph, where we can see how a metastable vacuum is generated explicitly. The case of a general $SU(N)$ gauge group discussed in a similar fashion, but the calculation is more involved. This is considered in appendix B.

3.1 General Consideration

In this section, we show how to construct metastable vacua in the Coulomb branch of an arbitrary $\mathcal{N}=2$ supersymmetric gauge theory with gauge group G, possibly with hypermultiplets, by introducing a small superpotential. The key property of $\mathcal{N}=2$ gauge theory is that the metric for the moduli space is (the rigid limit of) special Kähler. The effective Lagrangian at the Coulomb branch is generically $\mathcal{N}=2$ $U(1)^{\mathrm{rank}\ G}$ supersymmetric gauge theory and is described by

$$\mathcal{L}_{eff}=\mathrm{Im}\frac{1}{4\pi}\left[\int d^4\theta\partial_i\mathcal{F}(A)\bar{A}^i+\frac{1}{2}\int d^2\theta\partial_i\partial_j\mathcal{F}(A)W^i_\alpha W^{\alpha j}\right] , \tag{3.3}$$

where $i,j=1,\ldots,\mathrm{rank}\ G$. We sometimes denote $\mathcal{F}_{ij}$ by τ_{ij}, which is the period matrix of the Seiberg-Witten curve. The metric on the moduli space $\mathcal{M}$ is given by

$$g_{i\bar{j}}=\mathrm{Im}\tau_{ij}=\mathrm{Im}\frac{\partial^2\mathcal{F}(a)}{\partial a^i\partial a^j} . \tag{3.4}$$

Later in this section, we will show that this relation implies that any sectional curvature of the curvature operator R is positive semi-definite. That is, for any given holomorphic vector field $w\in T\mathcal{M}$,

$$\langle w,R(v,v)w\rangle\geq 0 , \qquad \text{for all } v\in T\mathcal{M}_p \quad \text{and all } p\in\mathcal{M} .$$

We call such curvature operator semi-positive.[1] The curvature is called positive if the equality holds only when $v=w=0$. In our case, the tensor $\langle w,R(\cdot,\cdot)w\rangle$ is strictly positive definite at almost every point on the moduli space.

[1] That the Ricci curvature of the Coulomb branch is positive semi-definite was noted in [61]. Here we are making a stronger statement that the sectional curvatures are positive semi-definite.

For a generic point in the moduli space where the curvature is positive, we can show that a suitable superpotential exists that generates a metastable vacuum at the point. Of course, the superpotential has to be small so that it does not affect the Kähler potential significantly. Suppose we parameterize the moduli space using some coordinate system x^i $(i = 1, 2, \ldots, \mathrm{rank}\, G)$ near a point p. We may introduce the Kähler normal coordinates z^i [62, 63] as

$$z^i = x'^i + \frac{1}{2}\Gamma^i_{jk} x'^j x'^k + \frac{1}{6} g^{i\bar{m}} \partial_l (g_{n\bar{m}} \Gamma^n_{jk}) x'^j x'^k x'^l + O(x^4), \tag{3.5}$$

where connections are evaluated at p and $x' = x - x(p)$. The expansion is terminated at the cubic order since higher order terms are not relevant for our purpose in this section. Then the metric in the z coordinate system is

$$g_{i\bar{j}}(z, \bar{z}) = \tilde{g}_{i\bar{j}} + \tilde{R}_{i\bar{j}k\bar{l}} z^k \bar{z}^{\bar{l}} + O(z^3) \ ,$$

where $\tilde{}$ means evaluation at p. The inverse metric is given by

$$g^{i\bar{j}}(z, \bar{z}) = \tilde{g}^{i\bar{j}} + \tilde{R}^{i\bar{j}}{}_{k\bar{l}} z^k \bar{z}^{\bar{l}} + O(z^3) \ . \tag{3.6}$$

Let us consider a superpotential

$$W = k_i z^i \ . \tag{3.7}$$

Note that there are global coordinates for the moduli space. For example, $u_r = \mathrm{tr}(\phi^r)$ are global coordinates in $\mathcal{N} = 2$ $SU(N)$ gauge theory, and we can write down W in terms of u_r by coordinate transformation. The corresponding superpotential is then expressed by replacing u_r with $\mathrm{tr}(\phi^r)$, $r = 2, \ldots, N$. Note that we might have used exact Kähler normal coordinates, which means including all orders in (3.5). In this case, the superpotential would not be defined globally in the moduli space, and the corresponding singular points in the moduli space would change the following analysis. This case will be considered more carefully in the next sections.

Suppose k_i is so small that corrections to the Kähler potential is negligible. Then the leading potential is given by

$$V = g^{i\bar{j}} k_i \bar{k}_{\bar{j}} + k_i \bar{k}_{\bar{j}} \tilde{R}^{i\bar{j}}{}_{k\bar{l}} z^k \bar{z}^{\bar{l}} + O(z^3) \ . \tag{3.8}$$

If $\tilde{R}$ is positive, the potential indeed gives a metastable vacuum at p. If $\tilde{R}$ is semi-positive, there could be some flat directions. However, if k_i is not along the null direction of $\tilde{R}$, and the tensor $k_i \bar{k}_{\bar{j}} \tilde{R}^{ij}{}_{k\bar{l}}$ has positive-definite eigenvalues, we get a metastable vacuum. Generically, these conditions can be satisfied. For example, in the semi-classical region of the $\mathcal{N} = 2$ $SU(N)$ gauge theory without hypermultiplets, which we study in appendix A, we can make metastable vacua at any point. In the examples studied in appendix B, there arise flat directions in k_i because we choose a highly symmetric point, which is not sufficiently generic. Even in these cases, we can find a superpotential to generate a metastable vacuum by choosing k_i appropriately.

Now, let us prove the assertion that the curvature R is semi-positive. Since we are interested in the local behavior, we can use a^i $(i = 1, \ldots, \text{rank } G)$ as coordinates in which the metric is given by (3.4). In $\mathcal{N} = 2$ $SU(N)$ supersymmetric gauge theory, these a^i are the periods of a meromorphic one-form describing the Coulomb branch. An important fact is that each τ_{ij} is holomorphic. In components, we want to show

$$w^j \bar{w}^{\bar{m}} g_{\bar{m}i} R^i_{\ jk\bar{l}} v^k \bar{v}^{\bar{l}} \geq 0 \,, \qquad \text{for all} \quad v, w \,. \tag{3.9}$$

Since τ_{ij} is holomorphic,

$$\begin{aligned} R^i_{\ jk\bar{l}} &= -\partial_{\bar{l}} \left(g^{\bar{q}i} \partial_k g_{j\bar{q}} \right) \\ &= -(\partial_{\bar{l}} g^{\bar{q}i}) \partial_k g_{j\bar{q}} \\ &= g^{\bar{q}p} g^{i\bar{n}} \partial_{\bar{l}} g_{p\bar{n}} \partial_k g_{j\bar{q}} \,. \end{aligned} \tag{3.10}$$

Plugging this into the LHS of (3.9),

$$\begin{aligned} w^j \bar{w}^{\bar{m}} g_{\bar{m}i} g^{\bar{q}p} g^{i\bar{n}} \partial_{\bar{l}} g_{p\bar{n}} \partial_k g_{j\bar{q}} v^k \bar{v}^{\bar{l}} &= w^j \bar{w}^{\bar{n}} g^{\bar{q}p} \partial_{\bar{l}} g_{p\bar{n}} \partial_k g_{j\bar{q}} v^k \bar{v}^{\bar{l}} \\ &= g^{\bar{q}p} (w^j v^k \partial_k g_{j\bar{q}}) (\bar{w}^{\bar{n}} \bar{v}^{\bar{l}} \partial_{\bar{l}} g_{\bar{n}p}) \geq 0 \,, \end{aligned} \tag{3.11}$$

since $g^{\bar{q}p}$ is positive definite. Therefore, (3.9) is satisfied. For a given holomorphic vector field w, $P_{k\bar{q}} = w^j \partial_k g_{j\bar{q}}$ is holomorphic, so its determinant is 0 only on a complex codimension one subspace of the moduli space unless it is a constant. Thus, generically $P_{k\bar{q}} v^k$ is nonzero for nonzero v, which implies (3.11) is strictly positive for any nonzero v. We found that the curvature is semi-positive and that the tensor $w^j \bar{w}^{\bar{m}} g_{\bar{m}i} R^i_{\ jk\bar{l}}$ is strictly positive definite at almost every point on the moduli space.

The superpotential $W = k_i z^i$ can be expressed in terms of global coordinates of the moduli space, such as $u_r = \text{tr}(\phi^r)$ for $SU(N)$, by coordinate transformation near the metastable vacuum. Generally terms quadratic and cubic order in u_r's are needed (higher order terms are not relevant for the metastability), and the superpotential would contain multiple-trace operators. On the other hand, gauge theories realized in string theory often have superpotentials consisting of single-trace terms only [64–66].[2] To see when the superpotential can be chosen as a sum of single-trace terms, let us consider $\mathcal{N} = 2$ $SU(N)$ gauge theory. For $SU(2)$, the situation is easy since any multiple-trace operator can be expressed in terms of a single-trace operator. This is not the case when the gauge group is $SU(3)$. However, in this case, we can show that the superpotential $W = k_i z^i$ can be deformed in such a way that W turns into a single-trace operator without destabilizing the

[2] For discussion of theories with multiple-trace superpotentials, see [67].

metastable point given by the bosonic potential $V = g^{i\bar{j}}\partial_i W \bar{\partial}_{\bar{j}} \bar{W}$. To see this, let $u = \mathrm{tr}\phi^2$ and $v = \mathrm{tr}\phi^3$ be the two coordinates for $SU(3)$ and let $u' = u - u_0$ and $v' = v - v_0$ be the coordinates centered at (u_0, v_0). We can express $u_i = \mathrm{tr}\phi^i$ $(i = 0, 2, 3, \ldots, 9)$ as polynomials of u' and v'. They are all independent generically. To construct the superpotential that generates a metastable vacuum at $u' = v' = 0$, we can ignore terms that are quartic and higher order in u' and v'. Hence u_i span a 9-dimensional subspace of the 10-dimensional cubic polynomial space. But the missing polynomial can be set to vanish by using deformation analogous to the one used in appendix B.1, which does not disturb the metastability. For higher N, we have not been able to find out whether it is possible to construct a single-trace superpotential that can generate a metastable vacuum at a generic point in the moduli space. But at the origin of the moduli space, for any $SU(N)$, the single-trace superpotential

$$W = \lambda \left(\frac{1}{N} u_N + \frac{(N-1)^2}{6N^3} \frac{1}{\Lambda^{2N}} u_{3N} \right)$$

for small coupling constant λ produces a metastable vacuum, where Λ is the scale of the gauge theory, as we show in appendix B.

3.1.1 $SU(2)$ Seiberg-Witten Theory

We can apply our mechanism to produce metastable vacua when the Seiberg-Witten theory is strongly coupled. Let us demonstrate this at the origin in pure $\mathcal{N} = 2$ $SU(2)$ gauge theory [3]. We first construct an appropriate superpotential using the Kähler normal coordinate near the origin of the moduli space. Since all expressions for the periods and metric are given in terms of the hypergeometric functions explicitly, we can easily determine the effective potential produced by superpotential perturbation. Let $u = \mathrm{tr}\phi^2$ be the modulus of the theory. The elliptic curve that describes the moduli space of the $SU(2)$ Seiberg-Witten theory is

$$y^2 = (x^2 - u)^2 - \Lambda^4 \, .$$

The periods of the theory are given by

$$\begin{aligned} \frac{\partial a}{\partial u} &= \frac{\sqrt{2}}{2} (e_2 - e_1)^{\frac{-1}{2}} (e_4 - e_3)^{\frac{-1}{2}} F\left(\frac{1}{2}, \frac{1}{2}, 1, z\right) \, , \\ \frac{\partial a_D}{\partial u} &= \frac{\sqrt{2}}{2} \left[(e_1 - e_2)(e_4 - e_3)\right]^{\frac{-1}{2}} F\left(\frac{1}{2}, \frac{1}{2}, 1, 1 - z\right) \, , \end{aligned} \tag{3.12}$$

where

$$z = \frac{(e_1 - e_4)(e_3 - e_2)}{(e_2 - e_1)(e_4 - e_3)} \, ,$$

and

$$e_1 = -\sqrt{u - \Lambda^2}, \quad e_2 = \sqrt{u - \Lambda^2}, \quad e_3 = \sqrt{u + \Lambda^2}, \quad e_4 = -\sqrt{u + \Lambda^2} \, .$$

The periods determine the metric in the a coordinate by

$$\tau = \frac{\partial a_D/\partial u}{\partial a/\partial u} \ .$$

We are going to use the metric in the u coordinate. This can be expanded near the origin:

$$g_{u\bar{u}} = \mathrm{Im}\tau \left|\frac{da}{du}\right|^2 = r(1 + su^2 + \bar{s}\bar{u}^2 - tu\bar{u}) + O(u^3) \ , \tag{3.13}$$

where $r = 0.174\Lambda^{-2}$, $s = 0.125\Lambda^{-4}$, and $t = 0.0522\Lambda^{-4}$.

We can use the Kähler normal coordinate z given by (3.5) to choose a superpotential

$$W = mz, \qquad z = u + \frac{1}{3}su^3 \ ,$$

for small, real coupling constant m. The corresponding effective potential is

$$V = \frac{m^2}{g_{z\bar{z}}} = \frac{m^2}{g_{u\bar{u}}} \left|1 + su^2\right|^2 \ .$$

The graphs for the potential are drawn in Figure 3.1 in two different scales. Although the metastable vacuum is visible when magnified near the origin, it can hardly be seen at the scale of the graph on the right. The potential is almost flat near the origin, and the metastable vacuum is generated by a tiny dip! Interestingly, this is not due to some small parameters of the theory. Actually, other than the scale Λ, there are no additional parameters that we can put in the theory if we consider a metastable vacuum at the origin. The near-flatness of the potential around the origin is generated without fine-tuning.

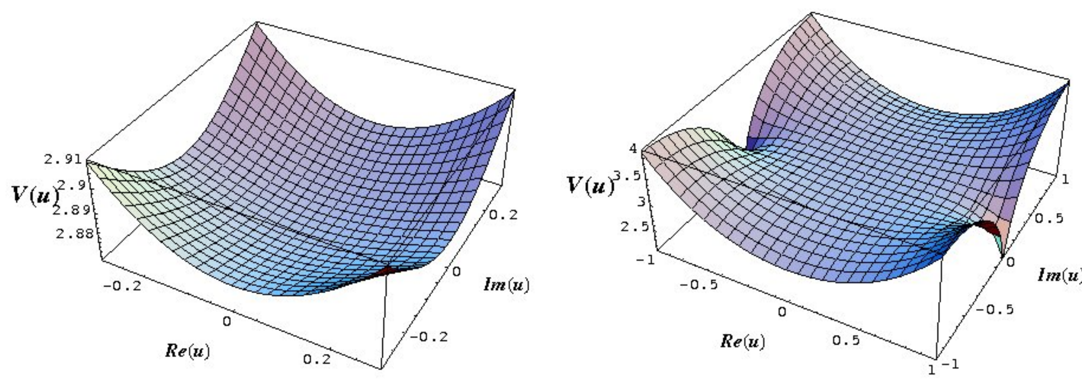

Figure 3.1: The appearance of a metastable vacuum at the origin. Note difference of scales.

We can consider a more general superpotential

$$W = mz + \frac{1}{2}\alpha z^2 + \frac{1}{3}\beta z^3 \ . \tag{3.14}$$

We have to set $\alpha = 0$ to have a local minimum at $z = 0$. Using (3.6), the effective potential $V = g^{z\bar{z}} \left|\partial W/\partial z\right|^2$ becomes

$$V = m^2 R z\bar{z} + m g^{u\bar{u}} \beta zz + m g^{u\bar{u}} \bar{\beta} \bar{z}\bar{z} + \text{constant} + O(z^3) \ ,$$

where $R = R^{z\bar{z}}{}_{z\bar{z}} = R^{u\bar{u}}{}_{u\bar{u}} = -g^{u\bar{u}}\partial_{\bar{u}}\partial_u \log g_{u\bar{u}} = 0.150\Lambda^{-2}$. In this case, it is straightforward to read the range to have a local minimum at $z = 0$. We need

$$mR \pm 2g^{u\bar{u}}\beta > 0.$$

Hence $\left|\frac{\beta}{m}\right| < \frac{g_{u\bar{u}}R}{2} = 0.0261\Lambda^{-4}$. In the u coordinate system, 3.14 becomes

$$W = m\left[u + \frac{1}{3}\left(s + \frac{\beta}{m}\right)u^3\right] + O(u^4) \ .$$

So, we want $s + \beta/m$ to lie between $(0.125 - 0.0261)\Lambda^{-4}$ and $(0.125 + 0.0261)\Lambda^{-4}$. We can confirm numerically that, precisely in this range, do we have a metastable vacuum at the origin.

We can consider also a superpotential that makes a metastable vacuum at some point other than the origin. This is possible for any points because the curvature is positive everywhere except at the two singular points, where it diverges. Also, for $SU(2)$ case, any polynomial of u can be expressed as single-trace form.

Now that we have found a metastable vacuum, we want to check its longevity. Notice that the $SU(2)$ Seiberg-Witten theory has only one dimensionful parameter Λ. In Figure 3.1, we have set it to be 1. If we change this, the coordinate u in the graph scales. Therefore, by sending the scale Λ to some limit, we may have a long-lived metastable vacuum at the origin. To see this, consider slices of the potential around the origin. Cutting through the real and imaginary axes, the potential looks like Figure 3.2:

We see that the characteristic feature of the graph is that it gets really flattened near the origin, and the local minimum at the origin and the peak of the graph are almost of the same height. But the distance between the origin, where the metastable vacuum is located, and the supersymmetric vacua can be arbitrarily large by setting Λ large. In such a case, we use the triangular approximation [68] instead of the thin-wall approximation [69]. The tunneling rate is proportional to e^{-S} where

$$S \sim \frac{(\Delta u/\Lambda)^4}{V_+} \ , \tag{3.15}$$

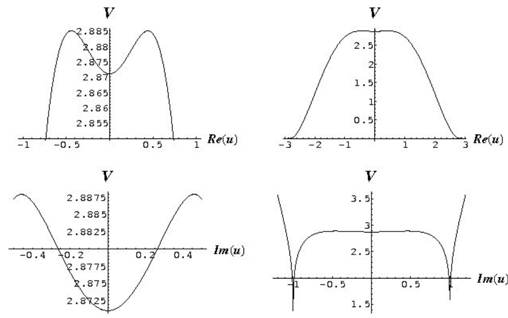

Figure 3.2: The real and imaginary slices of the potential through the origin, shown in two different scales.

where Δu is the distance between the peak and the origin and, and V_+ is the difference of the potentials between at the peak and at the origin. We insert Λ to make the u field of dimension 1. The distance Δu is proportional to Λ^2. V_+ is proportional to the mass parameter m in (3.14). Therefore, we can make the bounce action arbitrarily large: we choose m and Λ such that $m/\Lambda \ll 1$. This limit agrees with our assumption that we have added a small $\mathcal{N}=2$-to-$\mathcal{N}=1$ supersymmetry breaking term.

Since the superpotential $W = m\left(u + \frac{1}{3}su^3\right)$ has a cubic interaction, it introduces supersymmetric vacua when $u = u_0 = \pm\sqrt{-1/s}$. We have to consider the tunneling rate to decay into those vacua. However, the distance from 0 to u_0 is also set by the scale Λ. Therefore, for sufficiently large Λ, the decay process is arbitrarily suppressed.

3.1.2 Decay Rate of Metastable Vacua

In the previous subsection, we considered the decay rate of the metastable vacuum at the origin of the moduli space of the $SU(2)$ Seiberg-Witten theory. Extending the idea, let us estimate the decay rate of metastable vacua constructed using the curvature for a general $\mathcal{N}=2$ theory. We do not have an explicit expression for the effective potential. However, we can make a general argument that metastable vacua can be arbitrarily long-lived by choosing parameters appropriately. Note that whenever there appears a massless monopole or dyon in the moduli space, the metric diverges. In such a case, the effective potential vanishes and we get a supersymmetric vacuum at that point. The set of supersymmetric vacua is a subvariety of the moduli space. Additionally, the superpotentials introduce more supersymmetric vacua. Therefore, it is difficult to compute the exact tunneling rate. But we can estimate its dependence on the scale Λ and the typical scale of k_i. We consider the most efficient path to go from the metastable vacuum to a supersymmetric one. We expect that the shapes of such one-dimensional slices enable us to use the triangular approximation [68], just as in

$SU(2)$ case. Equation (3.15) in this case becomes

$$S \sim \frac{(\Delta Z)^4}{V_+} \ . \tag{3.16}$$

Here ΔZ is the distance between the metastable and supersymmetric vacua in z coordinates, scaled by some power of Λ to have a mass dimension 1, and V_+ is the difference of the effective potential at the metastable vacuum and that at the supersymmetric vacuum. Since the metric of the moduli space is determined by one dimensionful parameter Λ, $(\Delta Z)^4$ is proportional to Λ^4. If the coordinates x in (3.5) has mass dimension n, the typical value of the potential goes like $k_i^2\Lambda^{2n-2}$(each x_i might have different dimensions, e.g., $u_r = \mathrm{tr}\phi^r$ for $SU(N)$ case, but they can be made to have the same dimension by multiplying Λ appropriately). Then the bounce action S scales like Λ^{6-2n}/k^2. As long as this quantity is large enough, metastable vacua are long-lived.

3.2 Kähler Normal Coordinates and Fayet-Iliopoulos Terms

The construction described in the previous section makes use of the truncated Kähler normal coordinates as shown in (3.5). That is, given a point u_0 in the moduli space $\mathcal{M}$, we obtain globally defined superpotentials that are metastable at u_0 generically. Such a truncation may not seem to impact the physics to a great degree. Nevertheless, we will demonstrate that if one deforms the theory by a superpotential built from *exact* Kähler normal coordinates then the supersymmetry-breaking vacuum at u_0 becomes instead a supersymmetry-preserving one.

In general, Kähler normal coordinates z^i can be written in terms of special coordinates a^i along the moduli space as [62, 63]

$$z^i = \Delta a^i + g^{i\bar{j}}(a_0)\sum_{n=2}^{\infty}\frac{1}{n!}\partial_{i_3}\dots\partial_{i_n}\Gamma_{\bar{j}i_1i_2}(a_0)\Delta a^{i_1}\Delta a^{i_2}\dots\Delta a^{i_n} \ , \tag{3.17}$$

where

$$\Delta a^i \equiv a^i - a_0^i. \tag{3.18}$$

In contrast to the previous section, z^i keep all higher terms of Δa^i. In special coordinates, the connections $\Gamma_{\bar{j}i_1i_2}$ take a particularly simple form

$$\Gamma_{\bar{j}i_1i_2} = \frac{1}{2i}\mathcal{F}_{ji_1i_2} = \frac{1}{2i}\partial_{i_2}\tau_{ji_1} = \frac{1}{2i}\partial_{i_1}\partial_{i_2}a_{D\,j}, \tag{3.19}$$

where $a_{D\,i} = \partial_i\mathcal{F}$. This allows us to recognize the infinite series in (3.17) as a Taylor expansion of $a_{D\,j}$ about the point a_0^i. In fact, we can easily sum the series and write the exact Kähler normal coordinates z^i as

$$
\begin{aligned}
z^i &= \Delta a^i + \left(\frac{1}{\tau_0 - \bar{\tau}_0}\right)^{ij} \sum_{n=2}^{\infty} \frac{1}{n!} \frac{\partial^n a_{D\,j}(a_0)}{\partial a^{i_1} \cdots \partial a^{i_n}} \Delta a^{i_1} \Delta a^{i_2} \ldots \Delta a^{i_n} \\
&= \Delta a^i + \left(\frac{1}{\tau_0 - \bar{\tau}_0}\right)^{ij} \left(a_{D\,j}(a) - a_{D\,j}(a_0) - \tau_{0\,jk} \Delta a^k\right) \\
&= \left(\frac{1}{\tau_0 - \bar{\tau}_0}\right)^{ij} \left(a_{D\,j}(a) - \bar{\tau}_{0\,jk} a^k\right) + \text{const},
\end{aligned}
\tag{3.20}
$$

where $\tau_{0\,ij} = \tau_{ij}(a_0)$. This means that, up to irrelevant constant terms that we shall hereafter drop, the superpotential (3.7) is a specific linear combination of electric and magnetic FI terms

$$
W = e_i a^i + m^i a_{D\,i}, \tag{3.21}
$$

where

$$
e_i = -k_j \left(\frac{1}{\tau_0 - \bar{\tau}_0}\right)^{jk} \bar{\tau}_{0\,ki}, \qquad m^i = k_j \left(\frac{1}{\tau_0 - \bar{\tau}_0}\right)^{ji} . \tag{3.22}
$$

In particular the FI parameters satisfy

$$
e_i + m^j \bar{\tau}_{0\,ij} = 0 . \tag{3.23}
$$

We are therefore able to identify the theory with superpotential (3.7) as the classic model of partial supersymmetry breaking first introduced by Antoniadis et al. [53].[3] In the next section, we will study more closely how supersymmetry is realized at a_0^i.

While our focus is on deformed Seiberg-Witten theory, the structure considered makes a natural appearance in flux compactifications of type II superstring theory [71–73]. In fact, it appears as a geometric engineering limit of such a compactification, where the coefficients (e_i, m^i) of FI terms $W = \sum_i e_i a^i + m^i a_{Di}$ are identified with the amounts of fluxes. The scalar potential constructed from the superpotential W is invariant under the monodromy transformation of (a^i, a_{Di}) provided the fluxes (e_i, m^i) are also transformed appropriately. Thus, the potential is single-valued if we consider it as a function on the space of fluxes as well as on the Calabi-Yau moduli space. By contrast, in the field theory limit, the fluxes are frozen and become non-dynamical parameters, and the potential is multivalued in the Coulomb branch moduli space. This is caused since the field theory limit defined at a generic point in the Coulomb branch breaks down at massless dyon points because of the appearance of extra light particles at these points. It is exactly around each of these singular points where W is multivalued in the field theory limit.

[3] For local supersymmtric theories, see [70].

3.3 SUSY or non-SUSY at a_0^i

As we have reviewed in section 2.4, the condition (3.23) at a_0^i implies that the vacuum at a_0^i is a supersymmetric one which preserves the non-manifest $\tilde{Q}$ supercharges. When the superpotential is truncated as in the previous subsections, however, we break invariance under $\tilde{Q}$ at the level of the action and the a_0^i then become supersymmetry-breaking vacua.

At first glance this might seem strange because the higher order terms of (3.17) that we neglect when truncating do not affect the value of the scalar potential at τ_0, which is given by

$$V = \bar{k}_i \left(\mathrm{Im}\,\tau_0^{-1}\right)^{ij} k_j = \overline{(e_i + m^j \tau_{0\,ki})} \left(\mathrm{Im}\,\tau_0^{-1}\right)^{ij} \left(e_j + m^\ell \tau_{0\,\ell j}\right) \ . \tag{3.24}$$

Because this quantity is manifestly positive,[4] our intuition suggests that a_0^i should be a supersymmetry-breaking vacuum.

It is important to note, however, that having positive energy (3.24) is not sufficient for a vacuum to be supersymmetry breaking because we are, in principle, free to shift our definition of energy by a constant amount. It is the specific quantity that appears in the supersymmetry algebra which matters and to determine this may require a bit more work. In the truncated theory, the situation is actually pretty simple because there are vacua at the singular points in moduli space which preserve the manifest $\mathcal{N}=1$ supersymmetry. Setting the energy of these vacua to zero fixes any ambiguity and leaves us with the result (3.24).

The theory with full superpotential (3.7), on the other hand, exhibits no such vacua. The reason for this is that the superpotential is singular at the degeneration points. In fact, the full superpotential is actually multivalued on the moduli space with branch points where the supersymmetric vacua of the truncated theory would otherwise be. This change in the global structure of the theory suggests that we have to reexamine our definition of energy. To do so, let us start with the $\mathcal{N}=2$ formulation (2.27). In the conventional approach, where the Q_α supercharges are linearly realized, we shift $F_{D\,j} \to F_{D\,j} + i\bar{e}_j$ in the action (2.27) which effectively removes the $\tilde{F}^j$ from the second term of (2.39). In this case, the scalar potential is easily seen to be

$$V = \overline{(e_i + m^j \tau_{ki})} \left(\mathrm{Im}\,\tau^{-1}\right)^{ij} \left(e_j + m^\ell \tau_{\ell j}\right) \ , \tag{3.25}$$

in accordance with our result for the energy (3.24) of the a_0^i vacuum above. That this quantity fails to vanish at a_0^i simply means that the $\mathcal{N}=1$ supersymmetry generated by the Q_α is broken there.

On the other hand, to linearly realize the $\tilde{Q}_\alpha$ supercharges, we saw before that it is necessary to instead shift $F_{D\,j} \to F_{D\,j} - i\bar{e}_j$ in (2.27). This effectively removes the $\bar{F}^j$ from the second term of

[4] The combination of $e_i + m^k \bar{\tau}_{0\,ki} = 0$ and $\mathrm{Im}\,\tau_0 > 0$ imply that $e_i + m^k \tau_{0\,ki} \neq 0$.

(2.39), leading to the scalar potential

$$\begin{aligned}\tilde{V} &= \overline{(e_i + m^k \bar{\tau}_{ki})} \left(\mathrm{Im}\, \tau^{-1}\right)^{ij} \left(e_j + m^\ell \bar{\tau}_{\ell j}\right) \\ &= V + 4\mathrm{Im}\, \left(\bar{e}_i m^i\right) \ .\end{aligned} \tag{3.26}$$

In other words, if we choose to linearly realize the $\mathcal{N} = 1$ supersymmetry preserved by the vacuum at a_0^i, the definition of energy (3.26) appropriate for that choice differs from (3.25) by a constant shift.[5] As expected, this suitably defined energy vanishes at a_0^i.

3.4 Critical Points, Stability, and Non-supersymmetric Vacua

It should now be clear that the theory obtained by adding a superpotential (3.7) constructed from exact Kähler normal coordinates is significantly different from that obtained by truncating the series (3.17). This also suggests that the vacuum structure away from a_0^i may be fundamentally different as well.

This opens up a new problem, though, namely to understand the full vacuum structure of theories of the form (3.2) in the presence of superpotentials

$$W = e_i a^i + m^i a_{D\,i} \ , \tag{3.27}$$

for generic choices of e_i and m^j. In this section, we will take some initial steps along these lines. More specifically, we classify non-supersymmetric critical points, study the conditions for stabilizing them, and demonstrate that, in the simple example of a rank two gauge group, one can engineer stable vacua which break the full $\mathcal{N} = 2$ supersymmetry in part of the perturbative regime by choosing the e_i and m^j appropriately.

3.4.1 Stability Conditions and Supersymmetric Vacua

The principal object that controls the vacuum structure is the scalar potential constructed from (3.27). To start, let us write it in a covariant manner with respect to the Kähler metric $g_{i\bar{j}}$ of the Coulomb branch

$$V = \left(\overline{\nabla}_{\bar{i}} \overline{W}\right) g^{\bar{i}j} \left(\nabla_j W\right) \ . \tag{3.28}$$

Critical points of this potential satisfy

$$\nabla_k V = \left(\overline{\nabla}_{\bar{i}} \overline{W}\right) g^{\bar{i}j} \left(\nabla_k \nabla_j W\right) = 0 \ , \tag{3.29}$$

[5]From the analysis of section 2.4.3, we also see that it is the vanishing of $\tilde{V}$ that is required for preservation the corresponding $\mathcal{N} = 1$ supersymmetry.

while stability is determined by studying the second partials

$$\begin{aligned}\overline{\nabla}_{\bar\ell}\nabla_k V &= \left(\overline{\nabla}_{\bar\ell}\overline{\nabla}_{\bar\imath}\overline{W}\right) g^{\bar\imath j}\left(\nabla_k\nabla_j W\right) + \left(\overline{\nabla}_{\bar\imath}\overline{W}\right) g^{\bar\imath j} R^m{}_{j\bar\ell k}\left(\nabla_m W\right) \ , \\ \nabla_\ell\nabla_k V &= \left(\overline{\nabla}_{\bar\imath}\overline{W}\right) g^{\bar\imath j}\left(\nabla_\ell\nabla_k\nabla_j W\right) \ .\end{aligned} \tag{3.30}$$

From this, we see that the easiest way to find critical points is to impose either $\nabla_i W = 0$ or $\nabla_k\nabla_j W = 0$. For the former, it immediately follows from (3.30) that the resulting critical points are stable. For the latter, the same is also true at generic points provided $\nabla_\ell\nabla_k\nabla_j W$ also vanishes because the $R^m{}_{j\bar\ell k}$ term of (3.30) is positive (semi-)definite.

These two types of vacua are in fact nothing other than the supersymmetric ones we have studied thus far. To see this, we simply evaluate $\nabla_i W$ and $\nabla_j\nabla_k W$ in special coordinates, for which $g_{i\bar\jmath} = \operatorname{Im}\tau_{ij}$. Because the only nonvanishing Christoffel connections are

$$\Gamma^i_{jk} = -g_{k\bar\ell}\partial_j g^{i\bar\ell} \tag{3.31}$$

and their conjugates, this is particularly simple and results in

$$\begin{aligned}\nabla_i W &= e_i + \tau_{ij} m^j \ , \\ \nabla_i\nabla_j W &= -\frac{1}{2i}\mathcal{F}_{ijn}\left(\operatorname{Im}\tau\right)^{nk}\left(e_k + \bar\tau_{ks} m^s\right) \ .\end{aligned} \tag{3.32}$$

The supersymmetric vacuum that preserves Q is simply the $\nabla_i W = 0$ case while the supersymmetric vacuum that preserves $\tilde{Q}$ corresponds to $\nabla_i\nabla_j W = 0$. Note that there is no issue with stability of the latter because $\nabla_k\nabla_i\nabla_j W = 0$ when $(e + \bar\tau m)_j = 0$.[6]

3.4.2 Non-supersymmetric Vacua

While it is comforting to see the supersymmetric vacua and their stability emerging naturally from this framework, it is at the same time disappointing that the simplest ways to realize critical points of the potential fail to yield anything new.

In principle, the mechanism by which new critical points of the potential can be found is quite simple. We need a mixture of sorts of the supersymmetric and hidden supersymmetric cases where $g^{\bar\imath j}\overline{\nabla}_{\bar\imath}\overline{W}$ and $\nabla_k\nabla_j W$ are both nonzero but, in a suitable basis, have complementary components vanishing so that the contraction in (3.29) is zero. Unfortunately, there is no apparent reduction in complexity of (3.30) in this case so it is difficult to spell out simple conditions for a vacuum constructed in such a manner to be stable.

To describe the idea more precisely, let us drop the covariant notation of (3.29) and (3.30) and

[6] Even though this second order analysis only guarantees stability when the positive semi-definite term involving $R^m{}_{j\bar\ell k}$ does not have any flat directions, we know from the fact that the $(e + \bar\tau m)_j = 0$ vacua preserve an $\mathcal{N} = 1$ supersymmetry that the higher order analysis required when this condition fails must lead to stability.

instead reexpress the various derivatives of V in special coordinates as

$$\partial_q V = -\frac{1}{2i} F^f \mathcal{F}_{qfe} \tilde{F}^e \ , \tag{3.33}$$

and

$$\begin{aligned} \partial_p \partial_q V &= -\frac{1}{2i} F^f \left(\mathcal{F}_{pqfe} - \frac{1}{2i} \left[\mathcal{F}_{pfm} \left(\mathrm{Im}\, \tau^{-1} \right)^{mn} \mathcal{F}_{qne} + (p \leftrightarrow q) \right] \right) \tilde{F}^e \ , \\ \bar{\partial}_{\bar{p}} \partial_q V &= \frac{1}{4} \left[\bar{\tilde{F}}^a \overline{\mathcal{F}}_{pam} \left(\mathrm{Im}\, \tau^{-1} \right)^{mn} \mathcal{F}_{qnb} \tilde{F}^b + F^a \mathcal{F}_{qam} \left(\mathrm{Im}\, \tau^{-1} \right)^{mn} \overline{\mathcal{F}}_{pnb} \bar{F}^b \right] \ , \end{aligned} \tag{3.34}$$

where F^a and $\tilde{F}^b$ are the auxiliary field expectation values of (2.40).

In general, rather than looking for stable vacua at fixed e_i and m^j, we will find it easier to reverse our thinking and approach the problem in a manner analogous to our previous construction in section 3.1. That is, we instead specify a point u_0 along the Coulomb branch at which we would like to engineer a stable critical point and develop an algorithm for obtaining values e_i and m^j that do the job, if such exist.

To aid in this task, let us first use (3.33) and (3.34) to study the general structure of supersymmetry-breaking vacua. The first thing to note is that the vectors F^a and $\tilde{F}^a$ at such a vacuum can never be parallel. The reason for this is that a critical point for which they are parallel satisfies $\bar{F}^{\bar{p}} \bar{\partial}_{\bar{p}} \partial_q V F^q = 0$ from (3.33) and (3.34) and

$$\begin{pmatrix} e^{i\phi} F^p & e^{-i\phi} \bar{F}^{\bar{p}} \end{pmatrix} \begin{pmatrix} \partial_p \bar{\partial}_{\bar{q}} V & \partial_p \partial_q V \\ \bar{\partial}_{\bar{p}} \bar{\partial}_{\bar{q}} V & \partial_q \bar{\partial}_{\bar{p}} V \end{pmatrix} \begin{pmatrix} e^{-i\phi} \bar{F}^{\bar{q}} \\ e^{i\phi} F^q \end{pmatrix} = 2\mathrm{Re}\left(e^{2i\phi} F^p \partial_p \partial_q V F^q \right) \ , \tag{3.35}$$

where ϕ is a real phase. There is always a choice of ϕ for which this is negative so we see that such a critical point can never be stable. Incidentally, this means that for real e_i and m^j, achieving a metastable supersymmetry-breaking vacuum is impossible since $F^a = \tilde{F}^a$ in this case. Since neither $e_i + \tau_{ij} m^j = 0$ nor $e_i + \bar{\tau}_{ij} m^j = 0$ is attainable either, the only possible minimum occurs when the metric is singular. That is, when we have a dyon condensation point and the dyon charge is proportional to (e_i, m^j), the effective potential vanishes at that point and we have a supersymmetric vacuum there.

Let us now consider a coordinate transformation matrix $Q^i{}_{i'}$ under which $\mathcal{F}_{qfe}$ transforms as

$$\mathcal{F}_{qfe} \to \mathcal{F}'_{q'f'e'} = \mathcal{F}_{qfe} Q^q{}_{q'} Q^f{}_{f'} Q^e{}_{e'} \ . \tag{3.36}$$

Because F^a and $\tilde{F}^a$ are not parallel, we can always perform a coordinate transformation $Q^i{}_{i'}$ so that the only non-vanishing component of F^a ($\tilde{F}^a$) is the first (second) one. In this basis, $\mathcal{F}_{q12} = 0$ for all q.

It should now clear how to engineer a critical point at u_0 that can potentially be stabilized. Given $\mathcal{F}$, we use a coordinate transformation (3.36) so that $\mathcal{F}_{q12} = 0$ for all q. Such a coordinate transformation should generically exist because we have $(N-1)^2$ degrees of freedom in Q to satisfy only $N-1$ conditions. With such a Q, we then choose values of F^a and $\tilde{F}^a$ as

$$F = Q \begin{pmatrix} \zeta \\ 0 \\ 0 \\ \vdots \end{pmatrix} , \qquad \tilde{F} = Q \begin{pmatrix} 0 \\ \xi \\ 0 \\ \vdots \end{pmatrix} . \tag{3.37}$$

Once such a choice is made, we can generically solve (3.37) for the corresponding values of e_i and m^j because this is a system of $2(N-1)$ linear equations in $2(N-1)$ variables.

Now that we have constructed critical points, we must turn to the question of their stability. Given that we can actually engineer families of critical points parametrized by ζ and ξ, one might hope that there is enough freedom left over to achieve stability as well. Studying this issue is very complicated in practice, though, so to demonstrate the principle in action we focus on the most basic example we can find. It is clear that vacua of this sort cannot be generated when the gauge group has rank 1, so we turn instead to the rank 2 case of $SU(3)$ Seiberg-Witten theory.

3.4.3 An $SU(3)$ Example

In what follows, we shall work exclusively in the perturbative regime $a_i \gg \Lambda$, where the Seiberg-Witten prepotential $\mathcal{F}$ appearing in (3.2) takes the approximate form [74–76]

$$\mathcal{F}(a_i) = \frac{i}{4\pi} \sum_{i<j}^{3} (a_i - a_j)^2 \ln \left[\frac{(a_i - a_j)^2}{\Lambda^2} \right] . \tag{3.38}$$

We will henceforth set $\Lambda = 1$ and use the coordinate basis

$$x = a_2 - a_1 , \qquad y = a_3 - a_2 . \tag{3.39}$$

In terms of these, the prepotential is given by a simple expression

$$\mathcal{F} = \frac{i}{4\pi} \left(x^2 \ln x^2 + y^2 \ln y^2 + (x+y)^2 \ln (x+y)^2 \right) . \tag{3.40}$$

The various derivatives we shall need when studying (3.33) and (3.34) are now easily evaluated. We start with the period matrix

$$\tau = \frac{i}{2\pi}\begin{pmatrix} 6+\ln x^2+\ln(x+y)^2 & 3+\ln(x+y)^2 \\ 3+\ln(x+y)^2 & 6+\ln y^2+\ln(x+y)^2 \end{pmatrix}, \tag{3.41}$$

and proceed to its derivatives

$$\partial_x \tau_{ij} = \mathcal{F}_{xij} = \frac{i}{\pi(x+y)}\begin{pmatrix} 2+\frac{y}{x} & 1 \\ 1 & 1 \end{pmatrix}, \qquad \partial_y \tau_{ij} = \mathcal{F}_{yij} = \frac{i}{\pi(x+y)}\begin{pmatrix} 1 & 1 \\ 1 & 2+\frac{x}{y} \end{pmatrix}, \tag{3.42}$$

and second derivatives

$$\begin{aligned} \mathcal{F}_{xxij} &= \frac{1}{i\pi(x+y)^2}\begin{pmatrix} 2+2\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2 & 1 \\ 1 & 1 \end{pmatrix}, \\ \mathcal{F}_{xyij} &= \frac{1}{i\pi(x+y)^2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ \mathcal{F}_{yyij} &= \frac{1}{i\pi(x+y)^2}\begin{pmatrix} 1 & 1 \\ 1 & 2+2\left(\frac{x}{y}\right)+\left(\frac{x}{y}\right)^2 \end{pmatrix}. \end{aligned} \tag{3.43}$$

In this simple example, we can set $\mathcal{F}_{q12} = 0$ using a transformation of the form (3.36) with Q given by

$$Q = \begin{pmatrix} -x & x+y+\sqrt{(x+y)^2-xy} \\ x+y+\sqrt{(x+y)^2-xy} & -y \end{pmatrix}. \tag{3.44}$$

To find a choice of e_i, m^j for which the potential has a critical point at (x_0, y_0), we turn then to the equations

$$F = Q\begin{pmatrix} \zeta \\ 0 \end{pmatrix}, \qquad \tilde{F} = Q\begin{pmatrix} 0 \\ \xi \end{pmatrix}, \tag{3.45}$$

where here ζ and ξ are nonzero constants that we are free to choose, Q is as in (3.44), and $F, \tilde{F}$ are given in terms of $e_i, m^j, \tau_{k\ell}$ as in (2.40). Given our result (3.44) for Q, (3.45) is equivalent to the requirement

$$F = -\zeta\begin{pmatrix} -x \\ x+y+\sqrt{(x+y)^2-xy} \end{pmatrix}, \quad \tilde{F} = -\xi\begin{pmatrix} x+y+\sqrt{(x+y)^2-xy} \\ -y \end{pmatrix}. \tag{3.46}$$

As mentioned before, we generically expect that it is possible to choose e_i, m^j for any nonzero choice of ζ and ξ such that (3.46) is satisfied at a fixed point (x_0, y_0). From this point onward, we will assume that the situation is indeed generic and take the existence of such a solution for granted.

3.4.3.1 Stability

Engineering a critical point is one matter, but achieving stability is the real challenge. However, as we will now demonstrate through a simple scaling argument, it is possible to take advantage of the freedom to adjust ζ and ξ to choose FI terms that engineer *stable* supersymmetry-breaking vacua in part of the perturbative regime.

In particular, let us consider the regime $y \gg x \gg 1$. We will now show that if we choose ζ and ξ to be of order 1, the critical point constructed by solving (3.45) is always locally stable. Expanding the Hessian

$$H = \begin{pmatrix} \partial_p \bar{\partial}_{\bar{q}} V & \partial_p \partial_q V \\ \bar{\partial}_{\bar{p}} \bar{\partial}_{\bar{q}} V & \partial_q \bar{\partial}_{\bar{p}} V \end{pmatrix} \tag{3.47}$$

at the critical point, it is straightforward to check whether the eigenvalues $\lambda_1 \ldots \lambda_4$ of H are all positive. In the limit mentioned above, H scales near infinite y as follows:

$$H = \begin{pmatrix} h_{11}y^2 & h_{12}y & h_{13}y & h_{14} \\ h_{21}y & h_{22} & h_{14} & \frac{h_{24}}{y} \\ h_{31}y & h_{41} & h_{11}y^2 & h_{21}y \\ h_{41} & \frac{h_{42}}{y} & h_{12}y & h_{22} \end{pmatrix} , \tag{3.48}$$

where the h_{ij} depend logarithmically on y (and on x, ξ, ζ). To leading order in y, the four eigenvalues are

$$h_{11}y^2 \quad \text{and} \quad \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{11}} , \tag{3.49}$$

with multiplicity two for each. Since the matrix $\partial_p \bar{\partial}_{\bar{q}} V$ is positive definite from (3.30), the critical point is locally stable.

To illustrate potential subtleties that can arise when studying stability, let us also consider a second regime, namely $x \sim y \gg 1$. If we use r to denote the scale of x and y, the quantities appearing in (3.34) behave at large r as

$$\begin{aligned} \tau_{ij} &\sim \ln r , \\ \mathcal{F}_{ijk} &\sim r^{-1} , \\ \mathcal{F}_{ijk\ell} &\sim r^{-2} , \\ F &\sim \zeta r , \\ \tilde{F} &\sim \xi r . \end{aligned} \tag{3.50}$$

This means that $\partial_p \partial_q V \sim \zeta\xi$ while $\bar{\partial}_{\bar{p}} \partial_q V \sim \zeta^2 (\ln r)^{-1} + \xi^2 (\ln r)^{-1}$ at large r. If we take ζ and ξ to be of order 1 in this case, then the $\partial_p \partial_q V$ terms dominate and the Hessian necessarily has at least one negative eigenvalue.

Given the above scalings, though, one might naively think that stability can be achieved by taking ζ to be very large, say $\zeta \sim r$ for example, and ξ to be small, as in $\xi \sim r^{-1}$, because this ensures that the dominant contribution to the Hessian comes from $\bar{\partial}_{\bar{p}}\partial_q V$. This looks good for stability but unfortunately $\bar{\partial}_{\bar{p}}\partial_q V$ has an obvious flat direction in this case proportional to $\tilde{F}^q$ because

$$F^a \mathcal{F}_{qam} \tilde{F}^q = 0 \; . \tag{3.51}$$

The corresponding zero eigenvalue is generically lifted by the next-leading contribution to the Hessian, which comes from the off-diagonal term $\partial_p \partial_q V$. This means that the leading correction to this zero eigenvalue is, in fact, negative and our critical point is actually unstable.

3.5 Connection with Flux Compactifications

Until now, we have mainly focused on the field theory perspective of Seiberg-Witten theories deformed by electric and magnetic FI terms. Here we will briefly discuss the geometric realization of the vacua that we have studied so far in the context of string theory compactifications in the presence of NS and RR fluxes. In a series of papers [77–80], Seiberg-Witten theories were geometrically engineered in Type IIA and IIB string theories compactified on Calabi-Yau manifolds in a rigid limit of special geometry. For example, in type IIB, $SU(N)$ Seiberg-Witten theory was realized on a geometry constructed as a $K3$ fibration over a $\mathbf{P}^1$ base. Near the singular locus of $K3$ over $\mathbf{P}^1$, the Calabi-Yau manifold becomes

$$z + \frac{\Lambda^{2N}}{z} + 2W_{A_{N-1}}(x_1, u_i) + 2x_2^2 + 2x_3^2 = 0,$$

where $W_{A_{N-1}}(x_k, u_i)$ corresponds to the characteristic polynomial of the Seiberg-Witten theory.

Non-vanishing NS and RR fluxes, $H = H_{RR} + \tau_{\rm st} H_{NS}$, generate a superpotential [81] and lift the vacuum degeneracy in the Calabi-Yau manifold [73],

$$\begin{aligned} W_{GVW} &= \int H \wedge \Omega = \int_{B_i} H \int_{A_i} \Omega - \int_{A_i} H \int_{B_i} \Omega \\ &\equiv e_i a^i + m^i a_{D\,i} \; , \end{aligned}$$

where $(A_i, B_i) = \delta_{ij}$ comprise a symplectic basis of three-cycles. Since the integrals of the holomorphic 3-form, Ω, are naturally identified with the periods of Seiberg-Witten theory while turning on generic fluxes yields a set of complex valued (e_i, m^j), we can realize the model treated here by adding fluxes appropriately.

Supersymmetry-breaking in Calabi-Yau compactifications of this sort have also appeared in connection with brane/antibrane systems in [72] and more recently in [57,82–87]. In particular, a notion of geometric transition involving gauge/gravity duality was generalized to the non-supersymmetric setting, allowing configurations of branes and antibranes to be studied using the same sort of abelian gauge theory with FI terms considered here. Because the vacua studied here have natural realizations on the flux side in this context, it would be interesting to follow the geometric transition in reverse and study them from this perspective.

The flux realization of the model with FI terms also gives us a clear picture of how the potential behaves near a singular point in moduli space. At such a point, a massless dyon with charges (n_i^e, n_i^m) emerges and the corresponding cycle $\gamma = n_i^e A_i + n_i^m B_i$ shrinks. When we turn on generic FI-terms, the scalar potential diverges there for the simple reason that non-zero fluxes penetrate the cycle

$$\int_\gamma H = n_i^e e_i - n_i^m m_i \neq 0 \ ,$$

and render infinite the energy cost associated with closing it up.

Chapter 4

Current Correlators for General Gauge Mediation

So far, we considered a general way to engineer metastable vacua by deforming $\mathcal{N} = 2$ supersymmetric gauge theories by a small superpotential. One way to construct a realistic model is to use a hidden sector to break supersymmetry and mediate its supersymmetry breaking effect to the visible sector by the effect of gauge coupling. Gauge mediation has been an attractive mechanism for mediating the supersymmetry breaking effects. Meade, Seiberg, and Shih [49] presented a very general definition of the gauge mediation mechanism that includes most models with the supersymmetry breaking sector and the messengers, and models with the direct mediation. They also computed several physical quantities such as gaugino and sfermion masses on a general ground. Many of the physically observable effects mediated to the observable sector, which is sometimes taken to be a variant of the Minimal Supersymmetric Standard Model (MSSM), can be encoded by the current-current correlators of the global symmetry of the hidden sector. This allows us to extract the soft supersymmetry breaking terms even in the strongly coupled hidden sector.[1]

In this chapter, we provide methods to compute the zero-momentum current correlators in some classes of strongly coupled gauge theories. The way we calculate the current correlators is to weakly couple the system to some "spectator" gauge theory. Sometimes the coupled system is solvable, in which case, by taking the decoupling limit, we are able to obtain useful information about the current correlators.

We will review the idea of general gauge mediation of [49] briefly and then understand that the correlators of the global currents essentially characterize a wide range of gauge mediation models. Then we go on to the basic scheme of our computation of current correlators. To illustrate the technique, we provide a number of computable examples.

[1] For an earlier work for gauge mediation with strongly coupled hidden sector, see [88].

4.1 Review of General Gauge Mediation

In this section, we briefly review the definition of the gauge mediation and the determination of the effect of the hidden sector via current correlations in [49]. Following their definition, a model has the gauge mediation mechanism if the theory decouples into the observable sector and a separate hidden sector that breaks supersymmetry (SUSY) in the limit the observable sector gauge couplings α_i all vanish. In this setup, we may be able to compute various quantities in perturbation theory in the gauge coupling α_i, but the hidden sector may be strongly coupled. The information of the hidden sector, however, can be parametrized by the current correlation functions in the hidden sector. In the supersymmetric gauge theory, a global current superfield $\mathcal{J}^{\mathcal{A}}$ has the component form

$$\mathcal{J}^{\mathcal{A}} = J^A + i\theta j^A - i\bar{\theta}\bar{j}^A - \theta\sigma^\mu\bar{\theta}j^A_\mu + \frac{1}{2}\theta^2\bar{\theta}\bar{\sigma}^\mu\partial_\mu j^A - \frac{1}{2}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{j}^A - \frac{1}{4}\theta^2\bar{\theta}^2\Box J^A \; , \tag{4.1}$$

and satisfies the current conservation conditions

$$\bar{D}^2\mathcal{J}^A = D^2\mathcal{J}^A = 0 \; , \tag{4.2}$$

with j_μ satisfying $\partial^\mu j^A_\mu = 0$. Here A is an index for the adjoint representation of the global symmetry group. The current-current correlators have the following general forms:

$$\begin{aligned}
\langle J^A(x)J^B(0)\rangle &= \delta^{AB}\frac{1}{x^4}C_0(x^2M^2) \; , \\
\langle j^A_\alpha(x)\bar{j}^B_{\dot{\alpha}}(0)\rangle &= -i\delta^{AB}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\left(\frac{1}{x^4}C_{1/2}(x^2M^2)\right) \; , \\
\langle j^A_\mu(x)j^B_\nu(0)\rangle &= \delta^{AB}(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\left(\frac{1}{x^4}C_1(x^2M^2)\right) \; , \\
\langle j^A_\alpha(x)j^B_\beta(0)\rangle &= \delta^{AB}\epsilon_{\alpha\beta}\frac{1}{x^5}B_{1/2}(x^2M^2) \; ,
\end{aligned} \tag{4.3}$$

where M is the characteristic mass scale of the theory. $B_{1/2}$ may be complex but all C_a are real. There could also be nonzero one-point function $\langle J(x)\rangle$, but it vanishes for nonabelian currents. When supersymmetry is not broken spontaneously, we have the relations

$$C_0 = C_{1/2} = C_1 \; , \qquad \text{and} \qquad B_{1/2} = 0 \; . \tag{4.4}$$

Since supersymmetry is restored in UV, whether SUSY is spontaneously broken or not,

$$\lim_{x\to 0} C_0(x^2M^2) = \lim_{x\to 0} C_{1/2}(x^2M^2) = \lim_{x\to 0} C_1(x^2M^2) \; , \qquad \text{and} \qquad \lim_{x\to 0} B_{1/2}(x^2M^2) = 0 \; . \tag{4.5}$$

We now gauge the global current by coupling it to a gauge field. The part of the original Lagrangian for the gauge field is given by

$$\begin{aligned}\mathcal{L} &= \frac{1}{8\pi}\mathrm{Im}\left(\tau\mathrm{Tr}\int d^2\theta^2 W^\alpha W_\alpha\right) + \dots \\ &= \frac{1}{2g^2}D^A D^A - \frac{i}{g^2}\lambda^A\sigma^\mu D_\mu\bar{\lambda}^A - \frac{1}{4g^2}F^A_{\mu\nu}F^{A\mu\nu} + \dots ,\end{aligned} \tag{4.6}$$

where $\mathrm{Im}\tau = \frac{4\pi}{g^2}$ and we use normalization such that $\mathrm{Tr}T^A T^B = \delta^{AB}$. After integrating out the hidden sector, its effect is determined by the current correlation functions. Here we will only consider one gauge group and not a product of gauge groups, such as in the MSSM, for simplicity. Ignoring higher derivative terms, the change of the effective Lagrangian is

$$\delta\mathcal{L}_{eff} = \frac{1}{2}\tilde{C}_0(0)D^A D^A - \tilde{C}_{1/2}(0)i\lambda^A\sigma^\mu\partial_\mu\bar{\lambda}^A - \frac{1}{4}\tilde{C}_1(0)F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{2}(M\tilde{B}_{1/2}(0)\lambda^A\lambda^A + c.c.) + \dots . \tag{4.7}$$

Here $\tilde{C}_a$ and $\tilde{B}$ are Fourier transforms of C_a and B, respectively:

$$\begin{aligned}\tilde{C}_a\left(\frac{p^2}{M^2};\frac{M}{\Lambda}\right) &= \int d^4x e^{ipx}\frac{1}{x^4}C_a(x^2M^2) , \\ M\tilde{B}_{1/2}\left(\frac{p^2}{M^2}\right) &= \int d^4x e^{ipx}\frac{1}{x^5}B_{1/2}(x^2M^2) .\end{aligned} \tag{4.8}$$

The gaugino and sfermion masses are also determined by the current correlation functions. The gaugino mass at tree level can be read off from the change of the Lagrangian (4.7):

$$M_\lambda = g^2 M\tilde{B}_{1/2}(0) . \tag{4.9}$$

When the one point function $\langle J\rangle$ is zero, the sfermion mass occurs at one loop. In this case, we have to know the current correlation functions at the momentum of order M, the typical scale of the hidden sector. The sfermion mass is then

$$m^2_{\tilde{f}} = g^4 c_{2f} A , \tag{4.10}$$

where c_{2f} is the quadratic Casimir of the representation of f under the gauge group and

$$A = -\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}\left(3\tilde{C}_1\left(\frac{p^2}{M^2}\right) - 4\tilde{C}_{1/2}\left(\frac{p^2}{M^2}\right) + \tilde{C}_0\left(\frac{p^2}{M^2}\right)\right) . \tag{4.11}$$

4.2 Basic Idea

In this section, we present the basic idea to compute the current correlators in non-supersymmetric vacua, which encode the gaugino and sfermion masses. Schematically, the full theory can be written in the following form:

$$\mathcal{L} = \mathcal{L}_{hid} + \mathcal{L}_{int} + \mathcal{L}_{MSSM} \ , \tag{4.12}$$

where $\mathcal{L}_{hid}$ is the supersymmetry breaking hidden sector, $\mathcal{L}_{MSSM}$ is the visible MSSM sector, and $\mathcal{L}_{int}$ is the interaction between the two sectors, which transmits the supersymmetry breaking effects to the MSSM. For simplicity, hereafter, we assume the observable sector is the MSSM. When we integrate out $\mathcal{L}_{hid} + \mathcal{L}_{int}$, we produce the soft terms that break supersymmetry in the MSSM Lagrangian.

$$\mathcal{L} \rightarrow \mathcal{L}_{eff} = \mathcal{L}_{MSSM} + \delta\mathcal{L}_{soft} \ . \tag{4.13}$$

This can be done explicitly if the hidden sector is weakly coupled, but it is hard to do so for the strongly coupled case in general.

However, we can circumvent the difficulties in certain cases. As we have reviewed in the previous section, many of the parameters of the soft terms can be determined by calculating the current correlation functions. To achieve this, we will replace the MSSM with another theory such that we have control over the combined theory. That is, we are going to consider the following Lagrangian:

$$\mathcal{L}_{g'} = \mathcal{L}_{hid} + \mathcal{L}'_{int} + \mathcal{L}'_{spec} \ . \tag{4.14}$$

Here, if the gauge coupling constant g' for the spectator Lagrangian goes to zero, the spectator fields decouple. Supposing this new theory is solvable, we integrate out the hidden sector and obtain

$$\mathcal{L}_{g',eff} = \mathcal{L}'_{spec} + \delta\mathcal{L}_{soft} \ . \tag{4.15}$$

Now, by taking g' small and extracting nontrivial terms, we get the desired soft terms for the original theory.

In the following sections, we will be more specific and gauge the flavor symmetry of the hidden sector. Suppose the hidden sector is a gauge theory with gauge group G_1 and global symmetry group G_2, and each field lives in the representation (A_i, B_i) under $G_1 \times G_2$. Now by weakly gauging G_2, we get the theory with the product gauge group $G_1 \times G_2$:

$$\mathcal{L} = \mathcal{L}_{G_1} + \mathcal{L}_{int} + \mathcal{L}_{G_2} \ . \tag{4.16}$$

Let the dynamical scales associated with gauge groups G_1 and G_2 be Λ_1 and Λ_2, respectively, and g' gauge coupling for the gauge group G_2. We set the scale $\Lambda_1 \gg \Lambda_2$, so that we can treat the gauge interaction for G_2 to be very weak at the scale we probe. If we can integrate out the whole Lagrangian (4.16) in this limit, we are able to extract information about the current correlation functions of the flavor symmetry of the hidden sector. Note that this probe Lagrangian should not be confused with the Lagrangian for the real visible sector, although the structure is very similar. Here, this is just a probe to extract the information of the hidden sector. In the next section, we provide certain classes of examples in which we can follow this procedure.

Before we go on, let us briefly mention that we do not really have to gauge the full global symmetry. It is actually enough to gauge any subgroup H of the global symmetry group G, because the global currents are only sensitive to group theory factors. To see this, suppose the global current for G is in a representation R. Let the generators of G be $\{T^a\}$ and H $\{t^\alpha\}$. The global current correlators can be written as

$$\langle J^a(x) J^b(0) \rangle = f(x) \delta^{ab} \; , \tag{4.17}$$

where we have omitted Lorentz indices. This is the only possible form by symmetry. The current correlators come with the group theory factor

$$\langle J^a(x) J^b(0) \rangle_G = \tilde{f}(x) D_G(R) \delta^{ab} \; , \tag{4.18}$$

where $D_G(R)$ is the Dynkin index of the group G in the representation R and we have omitted the Lorentz indices. Now, let us decompose the representation in terms of the representations of the subgroup H by $R = \oplus_i r_i$ where r_i is a representation of H. Then, the correlators we get by gauging the subgroup can be written as

$$\langle J^\alpha(x) J^\beta(0) \rangle_H = \tilde{f}(x) \sum_i D_H(r_i) \delta^{\alpha\beta}. \tag{4.19}$$

Therefore, we can multiply the currents obtained by gauging the subgroup by the group theory factors to get the desired current correlators.

$$\langle J^a(x) J^b(0) \rangle_G = \frac{D_G(R)}{\sum_i D_H(r_i)} \langle J^a(x) J^b(0) \rangle_H . \tag{4.20}$$

4.3 Examples

4.3.1 $\mathcal{L} = 2$ SQCD with a Superpotential

Here we will consider $\mathcal{N} = 2$ $SU(N_c)$ Seiberg-Witten theory with N_f hypermultiplets [2, 3] with an appropriate superpotential as the hidden sector. This theory has $U(N_f)$ global symmetry. We will weakly gauge the $SU(N_f)$ part of the global symmetry. So the hidden sector is given by the

Lagrangian

$$\begin{aligned}\mathcal{L} =&\frac{1}{8\pi}\mathrm{Im}\left[\tau\left(\mathrm{Tr}\int d^2\theta W^\alpha W_\alpha + 2\int d^2\theta d^2\bar{\theta}\Phi^\dagger e^{-2V}\Phi\right)\right] + \int d^2\theta W(\Phi) + c.c. \\ &+ \int d^2\theta d^2\bar{\theta}\left(Q_a^\dagger e^{-2V} Q_a + \tilde{Q}_a e^{2V} \tilde{Q}_a{}^\dagger\right) + \int d^2\theta\left(\sqrt{2}\tilde{Q}_a\Phi Q_a + M\tilde{Q}_a Q_a\right) + c.c\ ,\end{aligned} \tag{4.21}$$

where $a = 1, \ldots, N_f$ and $W(\Phi)$ is a superpotential for the adjoint chiral superfield Φ. Note that M in the mass term $M\tilde{Q}_a Q_a$ is proportional to the $N_f \times N_f$ identity matrix, which is of the most general form to preserve $SU(N_f)$ global symmetry. We will consider the Coulomb branch of the hidden sector, whose special coordinates are denoted by a^i where $i = 1, \ldots, N_c$ with $\sum a^i = 0$. With a suitable choice of the superpotential $W(\Phi)$, the hidden sector can be in a metastable SUSY breaking state. For example, we can use a Kähler normal coordinate truncated at some finite order as a superpotential to get a metastable SUSY breaking state at a generic point in the Coulomb branch as we see in Chapter 3. Note that a^i may have nonzero F-component, which we denote by F^i. Our purpose is to compute the current-current correlators of the $SU(N_f)$ global current below the typical scale of a^i.

To achieve this, we gauge the flavor symmetry of the hypermultiplets. Since our purpose is to calculate the current-current correlators, we need not gauge the flavor symmetry using the MSSM. To facilitate computations, it is better to gauge the flavor symmetry by another $\mathcal{N} = 2$ $SU(N_f)$ SW gauge theory. So the total Lagrangian can be written as

$$\begin{aligned}\mathcal{L}' =&\frac{1}{8\pi}\mathrm{Im}\left[\tau\left(\mathrm{Tr}\int d^2\theta W^\alpha W_\alpha + 2\int d^2\theta d^2\bar{\theta}\Phi^\dagger e^{-2V}\Phi\right)\right] + \int d^2\theta W(\Phi) + c.c. \\ &+ \int d^2\theta d^2\bar{\theta}\left(Q^\dagger e^{-2V-2V'} Q + \tilde{Q} e^{2V+2V'} \tilde{Q}^\dagger\right) \\ &+ \int d^2\theta\left(\sqrt{2}\tilde{Q}_a\Phi Q_a + M\tilde{Q}_a Q_a + \sqrt{2}\tilde{Q}_a\Phi'_{ab}Q_b\right) + c.c. \\ &+ \frac{1}{8\pi}\mathrm{Im}\left[\tau'\left(\mathrm{Tr}\int d^2\theta W'^\alpha W'_\alpha + 2\int d^2\theta d^2\bar{\theta}\Phi'^\dagger e^{-2V'}\Phi'\right)\right]\ ,\end{aligned} \tag{4.22}$$

where primes denote similar multiplets in the $SU(N_f)$ gauge theory and the trace in the last line is over the flavor $SU(N_f)$ indices. Also, in the second line, the first term may be explicitly expressed as

$$Q_a^{\dagger i}\left(e^{-2V}\right)_i^{\ j}\left(e^{-2V'}\right)^a_{\ b} Q_j^b\ , \tag{4.23}$$

where $i, j = 1, \ldots, N_c$ and $a, b = 1, \ldots, N_f$. The gauge coupling of the spectator gauge theory is assumed to be very weak, so $\tau' = 4\pi i/g'^2$ with g' very small.

Treating the theory as $\mathcal{N} = 2$ $SU(N_c) \times SU(N_f)$ SW gauge theory, the low energy effective theory in the Coulomb branch is given by

$$\begin{aligned}\mathcal{L}'_{eff} =& \frac{1}{8\pi}\mathrm{Im}\left[\int d^2\theta \left(\mathcal{F}_{ij}W^{\alpha i}W^j_\alpha + 2\mathcal{F}_{ia}W^{\alpha i}W'^a_\alpha + \mathcal{F}_{ab}W'^{\alpha a}W'^b_\alpha\right)\right] \\ &+ \frac{1}{4\pi}\mathrm{Im}\left[\int d^2\theta d^2\bar{\theta}\left(\mathcal{F}_i\bar{a}^i + \mathcal{F}_a\bar{m}^a\right)\right] ,\end{aligned} \tag{4.24}$$

where a^i and m^a are eigenvalues of Φ and Φ', respectively, and subscripts under $\mathcal{F}$ denote differentiations. Note that the prepotential $\mathcal{F}$ depends both on a^i and m^a. Usually, it is hard to compute $\mathcal{F}$ for product gauge groups. However, using the fact that the spectator gauge theory is weakly coupled, we may obtain necessary information out of $\mathcal{F}_0$ of the original $\mathcal{N}=2$ $SU(N_c)$ SW theory. To see this, note that the low energy effective theory of (4.21) is

$$\mathcal{L}_{eff} = \frac{1}{8\pi}\mathrm{Im}\left[\int d^2\theta\mathcal{F}_{0ij}W^{i\alpha}W^j_\alpha + 2\int d^2\theta d^2\bar{\theta}\mathcal{F}_{0i}\bar{a}^i\right] , \tag{4.25}$$

where $\mathcal{F}_0(a)$ is the prepotential for $\mathcal{N}=2$ $SU(N_c)$ SW theory. Let us consider the case where g' goes to zero. In such a limit, m^a is treated as constant. Note that the dynamics of a^i in (4.24) and (4.25) agree when $\mathcal{F}_i = \mathcal{F}_{0i}$ and $m^a \sim 0$. The condition $m^a \sim 0$ is necessary since we set all nonabelian mass parameters to vanish in (4.25). It really means that the typical scale of m^a is much smaller than that of a^i. That is, of the moduli space of the Coulomb branch of the product gauge group, in the region where $|a| \gg |m|$, we may interchange $\mathcal{F}_i$ and $\mathcal{F}_{0i}$ freely. Therefore, if our interest is to consider the low energy effective theory whose typical scale of m is much lower than that of a, we may use the prepotential of the original $\mathcal{N}=2$ $SU(N_c)$ SW theory to compute quantities.

In the Seiberg-Witten curve language, one can obtain the prepotential in this limit by taking appropriate mass deformation of the curve, and regarding it as the moduli of the gauge theory. To see this, let us compare (4.21) and (4.22). In the limit where the second gauge group $SU(N_f)$ decouples, the only difference is the term $\sqrt{2}\tilde{Q}_a\Phi'_{ab}Q_b$ in the superpotential. The adjoint scalar component of Φ'_{ab} acts as a mass term for the hypermultiplets. Therefore, by identifying the massive deformation parameter $\tilde{m^a}$ to be $\sqrt{2}m^a$, and computing the period integrals, we can obtain the prepotential $\mathcal{F}$ and its derivatives in the limit of $|m| \ll |a|$.

4.3.1.1 Calculation of $\tilde{B}_{1/2}(0)$

From (4.24), we can read off the coefficients of D'^2, $\lambda'\sigma^\mu\partial_\mu\bar{\lambda}'$, $F'_{\mu\nu}F'^{\mu\nu}$ and $\lambda'\lambda'$ of the spectator gauge theory. Let us first consider the coefficient of $\lambda'\lambda'$. To show what is going on explicitly, let us write down the Lagrangian in terms of the component fields ignoring terms involving derivatives:

$$
\begin{aligned}
\mathcal{L}'_{eff} = \ldots + & g_{IJ}F^I\bar{F}^J + \frac{1}{2\pi}g_{IJ}D^I D^J + \frac{1}{8\sqrt{2}\pi}\mathcal{F}_{IJK}D^I\psi^J\lambda^K + \frac{1}{8\sqrt{2}\pi}\bar{\mathcal{F}}_{IJK}D^I\bar{\psi}^j\bar{\lambda}^K \\
& + \frac{i}{16\pi}\mathcal{F}_{IJK}\bar{F}^I\psi^J\psi^K - \frac{i}{16\pi}\mathcal{F}_{IJK}F^I\lambda^J\lambda^K + c.c. \\
& + \frac{i}{32\pi}\mathcal{F}_{IJKL}(\psi^I\psi^J)(\lambda^K\lambda^L) + W_I F^I - \frac{1}{2}W_{IJ}\psi^I\psi^J + c.c.
\end{aligned}
\tag{4.26}
$$

Here $I = (i, a)$ and the prepotential $\mathcal{F}$ has the superfields $A^I = (a^i, m^a)$ as its arguments. Note that $W_a = 0$, and $g_{IJ} = \frac{1}{4\pi}\mathrm{Im}\mathcal{F}_{IJ}$ is the metric for the product group gauge theory. After integrating out the auxiliary fields F^I, we can read off the coefficients of $\lambda^I\lambda^J$, which are given by

$$
\mathcal{L}'_{eff} = \ldots + \frac{i}{16\pi}\mathcal{F}_{IJK}g^{KL}\bar{W}_L\lambda^I\lambda^J + \ldots\ . \tag{4.27}
$$

If the hidden sector is very weakly interacting with the spectator gauge theory, $g^{KL}\bar{W}_L$ should nearly be the same as $(g_0^{ij}\bar{W}_j, 0)$ where g_0^{ij} is the inverse metric of the original $\mathcal{N} = 2$ $SU(N_c)$ gauge theory. To see this, note that among the components of the metric g_{IJ}, g_{ab} is very large compared with g_{ij} and g_{ia} in the limit $g' \to 0$. From the expression

$$
\delta^K_J = g_{IJ}g^{JK} = \begin{pmatrix} g_{ij} & g_{ib} \\ g_{aj} & g_{ab} \end{pmatrix} \begin{pmatrix} g^{jk} & g^{jc} \\ g^{bk} & g^{bc} \end{pmatrix}\ , \tag{4.28}
$$

we see that g^{ij} is much larger than g^{ib} and g^{ab} and this is more so as $g' \to 0$. Also g^{ij} is the inverse of g_{ij}, which is the same as g_{0ij}, all in the limit $g' \to 0$. Therefore, $g^{aL}\bar{W}_L$ can be ignored compared to $g^{ij}\bar{W}_j$, and

$$
g^{IJ}\bar{W}_J = (g_0^{ij}\bar{W}_j, 0) =: (-F^i, 0)\ . \tag{4.29}
$$

Next, we need to integrate out λ^i fields in (4.27). When we do this, we get

$$
\mathcal{L} = \ldots - \frac{i}{16\pi}\left(F^i\mathcal{F}_{iab} - (\mathcal{F}_{ijm}F^m)^{-1}\mathcal{F}_{aik}\mathcal{F}_{bjl}F^kF^l\right)\lambda^a\lambda^b + \ldots\ , \tag{4.30}
$$

where $(\mathcal{F}_{ijm}F^m)^{-1}$ is the inverse of the matrix $X_{ij} = (\mathcal{F}_{ijm}F^m)$ and we use the relation (4.29) in our limit. There are two terms in the coefficient of $\lambda^a\lambda^b$. The first term is the usual one, but the second is from integrating out the gaugino λ^i in the hidden sector. However, when we gauge the $SU(N_f)$ flavor current, the second term can be neglected. The point is that, in the second term, $\mathcal{F}$ is differentiated only once by m^a. Note that $\mathcal{F}_0$ in the original $\mathcal{N} = 2$ $SU(N_c)$ SW theory depends smoothly on the symmetric polynomials of m^a's, such as $\sum_{a\leq b} m^am^b, \sum_{a\leq b\leq c} m^am^bm^c, \ldots$. Hence, when differentiated once, $\mathcal{F}_0$ necessarily contains at least one factor of m^a and so goes to zero when $|m|/|a| \to 0$. On the other hand, $\mathcal{F}_0$ itself or its second derivatives by m^a need not vanish when m^a

is small. Since $\mathcal{F}_{0i} = \mathcal{F}_i$ in our limit, in the region of the moduli space where $|a| \gg |m|$, the first term dominates and we can safely ignore the second term. Therefore, when the hidden sector fields are integrated out, we may say that the relevant gaugino mass term is

$$\mathcal{L} = \ldots - \frac{i}{16\pi} F^k \mathcal{F}_{kab} \lambda^a \lambda^b + \ldots \ . \tag{4.31}$$

Note that, if we gauged the $U(N_f)$ flavor current, the prepotential would depend on $\sum_a m^a$ and we would not be able to ignore the second term in (4.30).

Next, we will look for a theory at the scale of $|a|$ which gives the low energy effective theory described above. That is, we hide all signals of the hidden sector into the gauge coupling of the spectator gauge theory and consider $\mathcal{N} = 2$ $SU(N_f)$ SW theory

$$\mathcal{L} = \frac{1}{8\pi} \mathrm{Im} \left[\tau_2 \left(\mathrm{Tr} \int d^2\theta W'^{\alpha} W'_{\alpha} + 2 \int d^2\theta d^2\bar{\theta} \Phi'^{\dagger} e^{-2V'} \Phi' \right) \right] , \tag{4.32}$$

where $\tau_2 = 4\pi i/g'^2 + \theta^2 F_\tau$ is the gauge coupling of the probe Lagrangian at the scale of $|a|$.

This is a theory at the scale of $|a|$. We will go to the low energy effective theory in the Coulomb branch where the superfield Φ' has eigenvalues m^a, $a = 1, \ldots, N_f$(so $\sum m^a = 0$). Although $|a| \gg |m|$, the coupling g' is so small that one loop correction is enough when we consider the dynamics at the scale of $|m|$. The prepotential at one loop is given by

$$\mathcal{F}(m) = \frac{\tau_2}{2} \sum_a \left(m^a - \frac{\sum_b m^b}{N_c} \right)^2 + \frac{i}{4\pi} \sum_{a<b} (m^a - m^b)^2 \log \frac{(m^a - m^b)^2}{\Lambda^2} , \tag{4.33}$$

and the low energy effective theory is

$$\mathcal{L} = \frac{1}{8\pi} \mathrm{Im} \left[\int d^2\theta \mathcal{F}_{ab} W'^{\alpha a} W'^{b}_{\alpha} + 2 \int d^2\theta d^2\bar{\theta} \mathcal{F}_a \bar{m}^a \right] . \tag{4.34}$$

Note that the θ^2 component of $\mathcal{F}$ is not corrected by the one-loop effect. So the gaugino mass term is given by

$$\mathcal{L} = \ldots - \frac{1}{8\pi} F_\tau \lambda^a \lambda^a + \ldots \ . \tag{4.35}$$

Note that we use coordinates for the second gauge group such that $\sum_{a=1}^{N_f} \lambda^a = 0$. Comparing this with (4.31), we obtain

$$F_\tau \left(\delta_{ab} - \frac{1}{N_c} \right) = \frac{i}{2} F^k \mathcal{F}_{kab} , \tag{4.36}$$

at $m^a = 0$. Note that, since $\mathcal{F}_k$ depends on the masses m^a by the combination $m^a - \sum_b m^b/Nc$, taking derivatives with respect to m_a and m_b, F_{kab} in the right-hand side has the same index structure as that in the left-hand side. From (4.7),

$$M\tilde{B}_{1/2}(0)\left(\delta_{ab}-\frac{1}{N_c}\right)=\frac{i}{8\pi}F^k\mathcal{F}_{0kab}\ . \tag{4.37}$$

It is instructive to actually calculate the gaugino mass using this formula in the semiclassical regime. That is, we check the expressions in the case where the expectation value of the chiral superfield Φ of the hidden sector is much larger than the scale of the hidden sector gauge theory. Additionally, we assume that the hypermultiplets Q and $\tilde{Q}$ are massless: i.e., $M=0$ in (4.22). The chiral superfield Φ_{ij} has nonzero F-term $F_{\Phi ij}$ where i and j are the gauge group indices for the hidden sector. Note that we are in the Coulomb branch of the hidden sector. We use gauge transformation such that $\langle\Phi_{ij}\rangle$ is diagonal with diagonal elements $a_i(\sum a_i=0)$. Let F_i be the corresponding F-term of a_i. We will calculate the gaugino mass in this setup. We start with the Lagrangian (4.22) and go to the low energy effective theory at the scale of $|m|$, where $\langle\Phi'\rangle$ has eigenvalues of m_a with the constraint $\sum m_a=0$. The a_i dependent part of the prepotential $\mathcal{F}$, given by [89] with the constraint $\sum m_a=0$ built in, is

$$\begin{aligned}\mathcal{F}=&\frac{2\pi}{g^2}\sum_i\left(a_i-\frac{\sum_j a_j}{N_c}\right)^2\\&+\frac{i}{4\pi}\sum_{i<j}(a_i-a_j)^2\log\frac{(a_i-a_j)^2}{\Lambda^2}\\&-\frac{i}{8\pi}\sum_{i,b}\left(a_i-m_b-\frac{\sum a}{N_c}+\frac{\sum m}{N_f}\right)^2\log\frac{\left(a_i-m_b-\frac{\sum a}{N_c}+\frac{\sum m}{N_f}\right)^2}{\Lambda''^2}+\dots\ ,\end{aligned} \tag{4.38}$$

where $\tau=4\pi i/g^2$, $\tau'=4\pi i/g'^2$. Λ is the scale for the first gauge group and Λ'' is some scale which does not change the answer that follows. Then

$$\mathcal{F}_{kab}=-\frac{i}{2\pi}\sum_i\frac{1}{a_i-m_b}\left(\delta_{ik}-\frac{1}{N_c}\right)\left(\delta_{ab}-\frac{1}{N_f}\right)\ , \tag{4.39}$$

after imposing tracelessness conditions. Since the scale of the spectator gauge theory does not enter into the expression, we may set $\mathcal{F}_{0kab}=\mathcal{F}_{kab}$. In the limit $m_b\to 0$, we have, from (4.37),

$$M\tilde{B}_{1/2}(0)=\frac{1}{16\pi^2}\sum_k\frac{F_k}{a_k}\ , \tag{4.40}$$

which gives the usual one-loop gaugino mass through (4.9)

$$M_\lambda=\frac{g^2}{16\pi^2}\sum_k\frac{F_k}{a_k}\ , \tag{4.41}$$

if we identify $\sqrt{2}a_k$ with masses of the messengers(The F-term of $\sqrt{2}a_k$ is $\sqrt{2}F_k$).

4.3.1.2 Calculation of $\tilde{C}_a(0)$ and Sfermion Masses

Using the same technique, we may compute $\tilde{C}_a(p^2/M^2)$ at zero momentum. Note that the effect of integrating out the hidden sector fields is to change the gauge coupling τ' to τ_2 as shown in (4.32). Also, in this case, all $\tilde{C}_a(0)$ are the same. Using (4.7),

$$\tilde{C}_a(0) = \frac{1}{4\pi}\mathrm{Im}\left(\tau_2(a) - \tau'\right) \ . \tag{4.42}$$

When we go to the low energy effective theory, $\tau_2(a)$ gets renormalized. But since g' is very small, it is enough to consider only the one-loop effect. So $\tau_2(a)$ gets only additive renormalization of the form $\log m$, which is independent of a. Hence we have

$$\frac{\partial\tau_2(a)}{\partial a^k}\left(\delta_{ab} - \frac{1}{N}\right) = \mathcal{F}_{kab} \ . \tag{4.43}$$

Integrating this equation, we are able to obtain $\tau_2(a)$, which then may be fed into (4.42) to get $\tilde{C}_a(0)$. The additive constant is determined by noting that $\tilde{C}_a(0)$ goes to zero in the limit where $a \to \infty$.

Since we are required to calculate $\tilde{C}_a(p^2/M^2)$ when p^2/M^2 is of order 1, the information we have just obtained is not enough to calculate the sfermion masses of the MSSM. Alternatively, we can introduce a matter multiplet charged with respect to $SU(N_f)$ and evaluate its low energy effective action terms.

4.3.1.3 Generalization to Other Gauge Groups

Let us briefly comment on how the expressions (4.37) and (4.43) change for other non-abelian groups. We start from (4.32). Let T_A be a basis of the adjoint representation of the flavor symmetry group and H_a be a basis of the Cartan subalgebra. The prepotential as a function of the $\mathcal{N} = 2$ adjoint chiral superfield Φ' at classical level is given by

$$\mathcal{F}(\Phi') = \frac{1}{2}\frac{\tau_2}{c_{adj}}\mathrm{Tr}_{adj}(\Phi'^2) \ , \tag{4.44}$$

where c_{adj} is the Dynkin index for the adjoint representation:

$$\mathrm{Tr}_{\mathrm{adj}}(T_A T_B) = c_{adj}\delta_{AB} \ . \tag{4.45}$$

In the Coulomb branch, all massive modes are integrated out and Φ' is diagonalizable due to D-term constraint. Hence $\Phi' = \sum_a m^a H_a$ and the prepotential above becomes

$$\mathcal{F}(a) = \frac{1}{2}\frac{\tau_2}{c_{adj}}\mathrm{Tr}_{adj}(H_a H_b) m^a m^b = \frac{\tau_2}{c_{adj}} m^a m^b \sum_\alpha \alpha_a \alpha_b \ , \tag{4.46}$$

where the summation is over all positive roots of the flavor group. Therefore $\lambda\lambda$ term in (4.34) becomes

$$\mathcal{L} = \ldots - \frac{1}{4\pi}\frac{F_\tau}{c_{adj}} \sum_\alpha \alpha_a \alpha_b \lambda^a \lambda^b \ldots \ . \tag{4.47}$$

We now compare this with the $\lambda\lambda$ term in the low energy theory of the product gauge group (4.31). Therefore

$$F_\tau \sum_\alpha \alpha_a \alpha_b = \frac{i c_{adj}}{4} F^k \mathcal{F}_{kab} \ . \tag{4.48}$$

Since the θ^2 component of τ does not receive corrections at one loop, this F_τ can be used to calculate $M\tilde{B}_{1/2}(0)$. That is, the part of $\lambda\lambda$ consisting of the Cartan subalgebra part in (4.7) is

$$\delta\mathcal{L}_{eff} = \ldots - M\tilde{B}_{1/2}(0)\frac{1}{c_{adj}} \sum_\alpha \alpha_a \alpha_b \lambda^a \lambda^b \ldots \ . \tag{4.49}$$

Therefore the relation corresponding to (4.37) is

$$M\tilde{B}_{1/2}(0) \sum_\alpha \alpha_a \alpha_b = \frac{i}{16\pi} c_{adj} F^k \mathcal{F}_{0kab} \ . \tag{4.50}$$

Similarly, corresponding to (4.43), we have

$$\frac{\partial \tau_a(a)}{\partial a^k} \sum_\alpha \alpha_a \alpha_b = \frac{1}{8\pi} c_{adj} \mathcal{F}_{0kab} \ . \tag{4.51}$$

4.3.1.4 Hypermultiplet Condensation

It may be of interest to check whether the hypermultiplet bilinear $\tilde{Q}_a Q_b$ develops a nonzero expectation value. The Lagrangian for the product $SU(N_c) \times SU(N_f)$ gauge theory (4.22) has some terms containing the F component of Φ':

$$\mathcal{L}' = \ldots + \frac{1}{g'^2}\mathrm{Tr}\left(\bar{F}_{\Phi'} F_{\Phi'}\right) + \tilde{Q}_a F_{\Phi' ab} Q_b + c.c. + \ldots \ . \tag{4.52}$$

Therefore we can calculate the expectation values of the bilinear $\tilde{Q}_a Q_b$ by differentiating the partition function with respect to $F_{\Phi'}$. More precisely, we will calculate the traceless part of $\langle \tilde{Q}_a Q_b \rangle$ since the source $F_{\Phi'}$ is traceless. To get the effective Lagrangian for the spectator gauge theory, we start with (4.26) and set all vevs for the fermions to zero. Then we integrate out the F and D terms for the hidden sector. The equation of motion for D^i sets $D^i = 0$. The equation of motion for F^i is

$$\bar{F}^{\bar{j}} = -g^{i\bar{j}} g_{i\bar{a}} \bar{F}^{\bar{a}} - g^{i\bar{j}} W_i \; , \tag{4.53}$$

Plugging the result into the Lagrangian, we get

$$\mathcal{L}'_{eff} = -g^{j\bar{i}} (g_{a\bar{i}} F^a + \bar{W}_{\bar{i}})(g_{j\bar{b}} \bar{F}^{\bar{b}} + W_j) + g_{a\bar{b}} F^a \bar{F}^{\bar{b}} \; . \tag{4.54}$$

Now we can read off the linear term in F^a:

$$\begin{aligned} \mathcal{L}'_{eff} &= \ldots - g^{j\bar{i}} g_{a\bar{i}} W_j F^a + \ldots \\ &= \ldots - g_0^{j\bar{i}} g_{0a\bar{i}} W_j F^a + \ldots \; , \end{aligned} \tag{4.55}$$

where the second line follows in the limit $g' \to 0$. But as we argued below (4.31), $g_{i\bar{a}}$ vanishes as m^a goes to 0 since we differentiate the prepotential $\mathcal{F}$ only once with respect to the second gauge indices to get $g_{i\bar{a}}$. Therefore, the linear term vanishes and the hypermultiplet bilinear can have at best an expectation value of the form

$$\langle \tilde{Q}_a(0) Q_b(0) \rangle = h(a) \delta_{ab} \; , \tag{4.56}$$

for some function $h(a)$. This does not break $U(N_f)$ symmetry. Also, Q or $\tilde{Q}$ cannot have nonzero expectation values perturbatively in the superpotential W. The reason is that we have a $U(1)$ subgroup of $SU(2)$ R-symmetry of $\mathcal{N} = 2$ theory even after including a superpotential if we assign charge $+2$ to the superpotential. Since Q and $\tilde{Q}$ have charge $+1$, it cannot be expressed as a series in W. Therefore, although there could be hypermultiplet condensation, $U(N_f)$ symmetry is still preserved. Note that, if we gauged $U(N_f)$ symmetry instead, the bilinear would be diagonal with ath diagonal element $g^{i\bar{j}} g_{i\bar{a}} W_j$. In this case, this would not vanish when $m^a \to 0$. However, by symmetry, $g_{i\bar{a}}$ would be the same for all a for a fixed i. Hence the vev of the bilinear would be proportional to the identity matrix, and $U(N_f)$ symmetry would still exist. Of course, the answer does not depend on which global symmetry we gauge. Therefore $h(a)$ in (4.56) is determined and we have, for any index c,

$$\langle \tilde{Q}_a(0) Q_b(0) \rangle = g_0^{j\bar{i}} g_{0c\bar{i}} W_j \delta_{ab} \; , \text{for the metric } g_0 \text{ of } U(N_f) \; . \tag{4.57}$$

Note that we actually calculate $\langle \tilde{Q}_a(0) Q_b(0) \rangle$ at the scale of $|m|$. But $\langle \tilde{Q}_a(0) Q_b(0) \rangle$ both at the scale of $|m|$ and at the scale of $|a|$ have the same form since when we go from the scale $|a|$ to $|m|$, we receive only perturbative effects, and this does not change $\langle \tilde{Q}_a(0) Q_b(0) \rangle$.

Having derived the formula for the quark condensate (4.57), let us verify it in the semiclassical

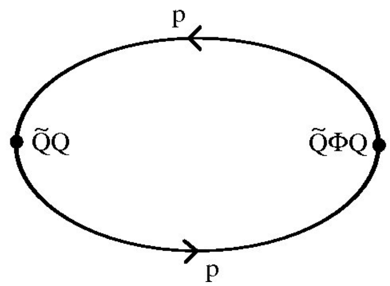

Figure 4.1: One loop diagram contributing hypermultiplet condensation.

regime. The leading contribution of $\sqrt{2}\tilde{Q}^i_a F^j_{\Phi i} Q^a_j$ to $\langle \tilde{Q}^i_a Q^b_i \rangle$ for small F_Φ, shown in Figure 4.1, is

$$\begin{aligned} \langle \tilde{Q}^i_a Q^b_i \rangle &= \sum_i \int_0^{\Lambda_0} \frac{d^4p}{(2\pi)^4} \frac{\bar{F}_i \delta^b_a}{(p^2 + (\sqrt{2}a_i)^2)^2} \\ &= -\frac{1}{16\pi^2} \delta^b_a \sum_i \bar{F}_i \log(\sqrt{2}a_i)^2 \ . \end{aligned} \tag{4.58}$$

Note that the interaction is insensitive to the cutoff Λ_0. Let us compare this with (4.57). The relevant part of the prepotential is similar to (4.38):

$$\begin{aligned} \mathcal{F} =& \frac{2\pi}{g^2} \sum_i (a_i)^2 \\ &+ \frac{i}{4\pi} \sum_{i<j} (a_i - a_j)^2 \log \frac{(a_i - a_j)^2}{\Lambda^2} - \frac{i}{8\pi} \sum_{i,a} (a_i - m_a)^2 \log \frac{(a_i - m_a)^2}{\Lambda''^2} + \dots \ . \end{aligned} \tag{4.59}$$

Note we do not impose the constraint $\sum m_a = 0$ since (4.57) is valid when gauging $U(N_f)$ symmetry. The metric component g_{ia} at weak coupling is given by

$$g_{a\bar{i}} = \frac{3}{16\pi^2} + \frac{1}{16\pi^2} \log \frac{(a_i - m_a)^2}{\Lambda''^2} \ . \tag{4.60}$$

Let us go to the limit $m^a \to 0$. Then $g_0^{j\bar{i}} W_j = -\bar{F}^{\bar{i}}$ and using (4.57),

$$\langle \tilde{Q}^k_a Q^b_k \rangle = -\bar{F}^{\bar{k}} g_{0c\bar{k}} \delta^b_a = -\frac{1}{16\pi^2} \delta^b_a \sum_k \bar{F}_k \log(\sqrt{2}a_k)^2 \ . \tag{4.61}$$

The result agrees with (4.58).

4.3.2 Geometrically Realized Models

As another controllable model, we study geometrically induced supersymmetry breaking configuration in Type IIB string theory on A_2-fibered geometry. This has been studied in [87] somewhat in a different context. Consider A_2 fibred geometry [66] defined by

$$x^2 + y^2 + z\,(z - m_1(t - a_1))\,(z + m_2(t - a_2)) = 0.$$

There are three singular points, $t = a_{1,2}$ and $a_3 = (m_1 a_1 + m_2 a_2)/(m_1 + m_2)$. Wrapping N_c anti-$D5$ and N_f $D5$ branes on two $\mathbf{S}^2$s that resolve the singularities at $t = a_1$ and a_2 respectively, we can construct a supersymmetry breaking configuration. We do not wrap any brane at $t = a_3$, because this can decay into a lower energy configuration. Our present setup does not have an unstable mode as has been discussed in [90]. Therefore we can take field theory limit. Here we claim that as long as the size of $\mathbf{S}^2$ at $t = a_2$ is much bigger than that at $t = a_1$, there is a field theory description for this brane/anti-brane system. According to the conjecture proposed in [43, 82, 87, 91], there is a glueball description with respect to the hidden sector gauge group corresponding to a partial geometric transition. Thus it is reasonable to claim that the low energy field theory description is an interacting product $U(1) \times U(N_f)$ gauge theory. The kinetic term is

$$\int d^4\theta K(S_1)\Phi_2\Phi_2^\dagger + \mathrm{Im}\left[\int d^4\theta \bar{S}_1 \frac{\partial \mathcal{F}_{0,0}(S_1)}{\partial S_1}\right] + \int d^2\theta \left[\tau\left(S_1\right) W^\alpha W_\alpha\right] + c.c. + \dots \;, \tag{4.62}$$

where $\dots$ includes higher derivative terms and $U(1)$ gauge kinetic terms. $\mathcal{F}_{0,0}(S_1)$ is the prepotential for the geometry after the transition,

$$2\pi i \mathcal{F}_{0,0} = \frac{S_1^2}{2}\left[\log\left(\frac{S_1}{m_1\Lambda_0^2}\right) - \frac{3}{2}\right].$$

where Λ_0 is a cutoff scale of this description. The superpotential terms are

$$W = \alpha S_1 + N_c \frac{\partial \mathcal{F}_{0,0}}{\partial S_1} + \tilde{W}_2(\Phi_2, S_1)\;. \tag{4.63}$$

In application to phenomenology we will identify a subgroup of $SU(N_f)$ as the standard model gauge group. So the adjoint field Φ_2 for $SU(N_f)$ gauge group should be integrated out by taking $m_2 \to \infty$. In the limit, the superpotential $\tilde{W}_2$ becomes a relatively simple function,

$$\tilde{W}_2 = \int_{\Lambda_0}^{a_2}\left[\frac{m_1}{2}(t - a_1) + \frac{1}{2}\sqrt{m_1^2(t - a_1)^2 - \frac{4S_1}{m_1}}\right] dt.$$

Our goal in this section is to compute the function $\tau(S_1)$ and extract C_is and $B_{1/2}$ from it. In the open string description we can say that this S_1 dependence is generated by the bifundamental matter. On the other hand, after the transition in closed string point of view, it is generated by closed string modes. To extract the interacting part, we use the glueball description for $U(N_f)$ gauge group as well and assume that the glueball fields and $U(1) \subset U(N_f)$ gauge supermultiplet w_α

are background fields. Following [92, 93], we use glueball description for evaluating the interacting part even though the $SU(N_f)$ theory is weakly coupled and is not confined. Turning on these backgrounds modifies the geometry slightly. At the leading order of the modification, we read off the kinetic term for the gauge group. The low energy description is given by

$$\mathcal{L} = \mathrm{Im}\left(\int d^4\theta \bar{S}_i \frac{\partial \mathcal{F}_0}{\partial S_i} + \int d^2\theta \frac{1}{2}\frac{\partial \mathcal{F}_0}{\partial S_i \partial S_j} w_i w_j\right) + \int d^2\theta W(S_i) + c.c. \,, \tag{4.64}$$

where $W(S_i)$ is Gukov-Vafa-Witten superpotential [81] generated by the flux. Solving the equation of motion for F^1, we obtain the potential

$$V = \frac{1}{g_{11}}\left|g_{12}\bar{F}^2 + \partial_1 W\right|^2 - g_{22}F^2\bar{F}^2 - F^2\partial_2 W - \bar{F}^2\bar{\partial}_2\bar{W}, \tag{4.65}$$

where we ignored $U(1)$ gauge fields. The metric is defined by $\mathrm{Im}\,\partial_i\partial_j\mathcal{F}_0$. Since we are interested in coefficients of correlation functions $B_{1/2}$ and C_i, which are related to linear terms in S_2 and F^2, we can put these to be zero when we evaluate the minimum of the potential,

$$V(S_2 = 0, F^2 = 0) = \frac{1}{g_{11}}\left|\partial_1 W\right|^2.$$

To find the minimum it is useful to expand the prepotential $\mathcal{F}_0$ for A_2 geometry as

$$\mathcal{F}_0 = \sum_{b=0}^{\infty} {S_2}^b \mathcal{F}_{0,b}(S_1),$$

where we ignored S_1 independent part, which can be combined with the classical action of $SU(N_f)$ and construct the one-loop running coupling constant. In the matrix model computation, $\mathcal{F}_{0,b}$ are contributions of diagrams with b boundaries, which are perturbatively calculable order by order. With this expansion, the superpotential and metric become

$$\begin{aligned} W(S_2 = 0) &= \alpha_1 S_1 + N_c \frac{\partial \mathcal{F}_{0,0}(S_1)}{\partial S_1} + N_f \mathcal{F}_{0,1}(S_1), \\ g_{11}(S_2 = 0) &= \mathrm{Im} \frac{\partial^2 \mathcal{F}_{0,0}}{\partial S_1 \partial S_1}. \end{aligned}$$

In our setup, the disk and annulus amplitudes are exactly known [94–96],

$$
\begin{aligned}
2\pi i\mathcal{F}_{0,1} &= S_1\Bigg(\log\frac{\Delta+\sqrt{\Delta^2-\frac{4S_1}{m_1}}}{2\Lambda_0}+\frac{\Delta}{\Delta+\sqrt{\Delta^2-\frac{4S_1}{m_1}}}-\frac{1}{2}\Bigg)\\
&\simeq S_1\log\frac{\Delta}{\Lambda_0}-\frac{S_1^2}{2m_1\Delta^2}+\ldots,\\
2\pi i\mathcal{F}_{0,2} &= \frac{1}{2}\log\left(\Delta+\sqrt{\Delta^2-\frac{4S_1}{m_1}}\right)-\frac{1}{2}\log\left(2\sqrt{\Delta^2-\frac{4S_1}{m_1}}\right)\\
&\simeq \frac{S_1}{2m_1\Delta^2}+\ldots,
\end{aligned}
$$

where $\Delta=a_1-a_2$. With these expressions, we see that $W(S_2=0)$ reproduces the superpotential in (4.63) in the limit $m_2\to\infty$ and $S_2\to 0$. At the leading order, the minimum of the potential is given by

$$
\langle S_1\rangle^{|N_c|}=(m_1\Lambda_0)^{|N_c|}\left(\frac{\bar\Delta}{\bar\Lambda_0}\right)^{N_f}e^{2\pi i\bar\alpha_1}. \tag{4.66}
$$

Note that $N_f>0>N_c$. Since there is an exponential suppression factor, vev of S_1 exists in physical region, which we regard as a dynamical scale of the theory on the anti-D5 branes.

Expanding the potential (4.65) around the minimum we can read off coefficients of linear terms in S_2 which yield the gaugino mass term for $SU(N_f)$ part,

$$
\begin{aligned}
\frac{2\pi i}{g_{YM}^2}m_\lambda &= \frac{2\pi i F_1}{16\pi^2}\left[-|N_c|\frac{\partial^2\mathcal{F}_{0,1}}{\partial S_1\partial S_1}+2N_f\frac{\partial\mathcal{F}_{0,2}}{\partial S_1}\right]\Bigg|_{\langle S_1\rangle}-\frac{|F_1|^2}{32\pi^2 i\Lambda^4}\frac{\partial^2\mathcal{F}_{0,1}}{\partial S_1\partial S_1}\Bigg|_{\langle S_1\rangle}\\
&\simeq \frac{1}{16\pi^2}\left[\frac{|N_c|+N_f}{m_1\Delta^2}F_1+\frac{|F_1|^2}{2i\,m_1\Delta^2\Lambda^4}\right],
\end{aligned} \tag{4.67}
$$

where we supplied a dimensionful parameter Λ. In the field theory limit, we take the string scale to be infinity, keeping the scale Λ finite which should be identified with the scale of S_1 in (4.66) in our model. The α_1, which is the size of $\mathbf{P}^1$, also has to scale appropriately [78].[2] The vev of F^1 is cut-off independent and a finite quantity in the limit,

$$
F^1=-g_{11}^{-1}\partial_1 W\big|_0\simeq\beta\Lambda^4,
$$

where the β is defined by $2i\,\mathrm{Im}\bar\alpha_1\sim\beta\log\Lambda_0$,[3] which encodes geometric data of the $\mathbf{P}^1$. On the

[2]In the geometric engineering one focuses on the leading effect of the small parameter Λ/M_{st}. Geometric quantities scale with the small parameter, for example the potential scales $V\sim(\Lambda/M_{st})^4$. Thus the original string scale in the potential cancels and it becomes field theory scale vacuum energy $V\sim\mathcal{O}(\Lambda^4)$.

[3]Note that without loss of generality we can take the phase of Δ/Λ_0 to be real. With this normalization, we defined the β.

other hand, another correlation functions can be read off from the linear term in F^2.

$$
\begin{aligned}
2\pi i C_i(0) &= \frac{-2\pi i}{16\pi^2}\mathrm{Re}\left[\mathrm{Im}\left(\frac{\partial\mathcal{F}_{0,1}}{\partial S_1}\right)\Lambda^{-4}\bar{F}^1 + |N_c|\frac{\partial\mathcal{F}_{0,1}}{\partial S_1} - 2N_f\mathcal{F}_{0,2}\right]\Bigg|_{\langle S_1\rangle} \\
&\simeq \frac{1}{16\pi^2}\left[\mathrm{Im}\left(\frac{\langle S_1\rangle}{m_1\Delta^2}\right)\Lambda^{-4}\mathrm{Re}F^1 + (|N_c| + N_f)\mathrm{Re}\left(\frac{\langle S_1\rangle}{m_1\Delta^2}\right)\right]. \qquad (4.68)
\end{aligned}
$$

Finally let us comment on the diagrammatical computation of the gaugino mass. Although our present geometric configuration does not include an unstable mode, we do not know explicitly the UV Lagrangian for the brane/anti-brane system. Thus it is not easy to compute the correlation functions studied above from matrix model computations directly. However, the flop of the $\mathbf{S}^2$ wrapping the anti-brane is a smooth process because its physical volume can never be zero [87, 91]. The new geometry yields the brane/brane configuration. The world volume theory on the branes is quiver gauge theory with a superpotential,

$$
W_{SUSY} = \frac{m_1}{2}\mathrm{tr}(\Phi_1 - a_1)^2 + \frac{m_2}{2}\mathrm{tr}(\Phi_2 - a_2)^2 + Q_{12}\Phi_2 Q_{21} + Q_{21}\Phi_1 Q_{12}.
$$

Using this explicit Lagrangian and technology developed in [92, 93, 97, 98], we can compute the non-perturbative effect from perturbative Feynman diagram computations. In fact, explicit formulae for $\mathcal{F}_{0,0}$, $\mathcal{F}_{0,1}$ and $\mathcal{F}_{0,2}$ have been perturbatively computed by this method.

Part II

Gauge/Gravity Dualities and Their Applications

Chapter 5

Introduction

We have seen in Part I of the thesis that supersymmetric gauge theories play a vital role in constructing a realistic model for a theory above the TeV scale. But there are many known cases where supersymmetric gauge theories are related to gravity theories. In the second half of the thesis, we will discuss the understanding and applications of gauge/gravity dualities. The most well-studied example is the correspondence between the $AdS_5 \times S^5$ supergravity and $\mathcal{N} = 4$ super Yang-Mill theory [8–10]. The correspondence can be thought of as an equivalence between the two descriptions describing the low energy dynamics of N multiple parallel D3-branes in flat space in type IIB string theory. The low energy limit can be equivalently thought of as keeping the energy scale fixed while sending the string length $l_s = \sqrt{\alpha'}$ to 0. On the one hand, the system is described by open strings on the D3-branes and the closed strings in the ten-dimensional bulk. In the low energy description, the D3-branes is described by $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory and it decouples from the free bulk dynamics described by closed strings. The gauge coupling constant becomes $g_{YM}^2 = g_s$ where g_s is the string coupling constant. On the other hand, we may view the D3-branes as a source of the energy-momentum tensor and consider its effect on the metric and other fields in supergravity. The metric is then given by

$$\begin{aligned} ds^2 &= f(r)^{-\frac{1}{2}} dx_\mu dx^\mu + f(r)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2) , \\ f(r) &= 1 + \frac{R^4}{r^4} , \qquad R^4 = 4\pi g_s {\alpha'}^2 N , \end{aligned} \tag{5.1}$$

where $dx_\mu dx^\mu$ is the four-dimensional Minkowski metric and $d\Omega_5^2$ is the metric for the unit five sphere. Sending α' to 0 and keeping the energy fixed means keeping $U = \frac{r}{R^2}$ fixed where $R^4 = 4\pi g_s N {\alpha'}^2$. In that limit, the metric becomes

$$ds^2 = R^2 \left(\frac{dU^2}{U^2} + U^2 dx_\mu dx^\mu \right) + R^2 d\Omega_5^2 . \tag{5.2}$$

That is, the near horizon region of the geometry becomes $AdS_5 \times S^5$. In the limit, the near horizon dynamics decouples from the free bulk physics. Combining the two, it is natural to identify $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory and the supergravity theory in $AdS_5 \times S^5$. Of course, this argument is not rigorous since we do not treat string theory non-perturbatively. Moreover, the supergravity description is valid when $R \gg l_s$ or $g_s N \gg 1$, while the $\mathcal{N} = 4$ super-Yang-Mills theory is perturbatively described when $g_{YM}^2 N = g_s N \ll 1$ in the large N limit. Despite the difficulty, there is overwhelming evidence that the correspondence is correct: for example, operators with some amount of supersymmetry does not receive quantum corrections when the coupling constant g_s changes, so it is possible to compare these operators in two different descriptions [10, 99].

Moreover it is believed that the essence of the correspondence does not depend on supersymmetry [100], so it makes sense to discuss non-supersymmetric versions of the correspondence also. It is a strong/weak duality, which makes it difficult to prove while beneficial to use. For example, we can learn about a strong-coupling behavior of a field theory by studying classical solutions of its gravity dual. Also, a gravity theory is lacking a UV definition, and the dual field theory may provide a way to define the gravity theory rigorously.

If we assume gauge/gravity dualities, we can study the strong-coupling behavior of some field theories by its classical dual gravity solutions. Even though we do not know an exact pair, sometimes a classical gravity geometry is determined by symmetries of the corresponding field theory to a great degree. Hence we can extract much information for a field theory if we assume the existence of a gravity dual.

Along this line, we will consider a field theory with Schrödinger symmetry, which is a nonrelativistic version of scale symmetry [101]. The Schrödinger symmetry is an extension of the nonrelativistic Galilean symmetry [102, 103]. Just as in the relativistic scale symmetry, there is a dilatation operator, by which time and space scale differently. However, unlike its relativistic counterpart, there is only one special conformal operator. One of the most interesting physical examples with such symmetry is a set of fermions in an optical lattice with the magnetic field. The magnetic field induces Feshbach resonances and the strength of attraction is tunable arbitrarily [104–107]. The interaction of the fermion gas arises mainly through the binary s-wave collisions with the scattering length a. When the attraction between the fermions is very weak, the fermions favor to form Cooper pairs, forming a Bardeen-Cooper-Schrieffer (BCS) state. On the other hand, if the attraction is sufficiently strong, two fermions form a bound state and the system is effectively described by the Bose-Einstein condensation (BEC). In both the BCS and BEC limits, the system can be described as a weakly interacting system with the interaction parameter ak_F, where k_F is the Fermi momentum. As we change the magnetic field strength, there is an intermediate regime, called the unitarity limit, where the scattering length a becomes infinite. In this regime, we expect to see a nonrelativistic version of scale symmetry, i.e., Schrödinger symmetry. Perturbation theory does not work well here since ak_F

diverge, but the gauge/gravity correspondence may provide a useful technique to study the problem.

Another application of gauge/gravity dualities is to study phase transitions of field theories. Finite temperature states in a field theory correspond to black hole solutions in the dual gravity theory [108]. A non-zero charge density solution in the field theory can be realized by turning on the chemical potential in the grand canonical ensemble. This corresponds to a charged black hole solution in the gravity theory. An instability of a black hole signals a phase transition in the corresponding field theory. The instability may occur due to charged or neutral scalar fields as discussed in [109–111]. But a supergravity theory typically has Chern-Simons terms, and it may cause an instability [112], which we will verify in Chapter 7.

Let us consider a five-dimensional gravitational system with the Maxwell field and its Chern-Simons term. By the gauge/gravity duality, the system is dual to a four-dimensional gauge theory. The effect of the Chern-Simons term can be analyzed as follows [10]. Suppose the geometry of the gravity solution is of the form $M \times X$ where M is a five-dimensional space that asymptotes to AdS_5 and X is some compact five-dimensional space. The Chern-Simons term appears in the gravity action as

$$S_{CS} = k \int_M A \wedge dA \wedge dA \ , \tag{5.3}$$

where k is some constant. Since we are going to study the Maxwell field, we confine ourselves to the consideration of abelian gauge group only, even though the extension to the non-abelian gauge group is straightforward. Note that the space M, being asymptotically AdS_5, has a boundary ∂M, which means that the action (5.3) is not gauge invariant: under the gauge variation $\delta A = d\Lambda$ for a zero-form Λ,

$$\delta S_{CS} = k \int_{\partial M} \Lambda \wedge dA \wedge dA \ . \tag{5.4}$$

Note that the change δS_{CS} of the Chern-Simons action depends only on the values of the gauge field on the boundary. From the view point of the boundary field theory, there is a $U(1)$ global symmetry corresponding to the $U(1)$ gauge symmetry of the gauge field A in the bulk. The corresponding global current J is coupled to an external source field A on the boundary. The change of the action (5.4) can be thought of as the change of the action of the boundary theory. The global current J couples to the external source field A in the form $\int A_\mu J^\mu d^4x$. Under the change of the source field $\delta A_\mu = \partial_\mu \Lambda$, this term changes by $-\int \Lambda \partial_\mu J^\mu d^4x$ after partial integration. Therefore, we obtain the relation $\partial_\mu J^\mu \sim kF \wedge F$, which tells us that the effect of the chiral anomaly of the global $U(1)$ symmetry is proportional to the coefficient of the Chern-Simons action k.

Given a gravity action with Chern-Simons term, we may consider a charged black hole solution. There is the Reissner-Nordström black hole in a gravity theory without Chern-Simons term. The additional terms in the equations of motion due to Chern-Simons term vanish in that background. Therefore, we may think that the Reissner-Nordström black hole is still a valid solution in the presence of Chern-Simons terms. However, we will show later that the fluctuation analysis shows that there are unstable modes, depending on the Chern-Simons coupling and temperature. Such an instability is interesting since it exhibits a spatially modulated phase. In condensed matter physics, a spatially modulated phase, called the Fulde-Ferrell-Larkin-Ovchinnikov phase, occurs when two kinds of fermions with different Fermi surfaces condense with non-vanishing total momentum [113, 114]. [115] studied a similar effect in QCD. Also, in finite density QCD, the chiral density wave studied in [116, 117] exhibits such a spatially modulated phase. In addition, the Brazovskii model [118] generates a spatially modulated phase, and it has been applied to a variety of physical problems [119]. In this model, a non-standard dispersion relation is postulated so that the fluctuation spectrum has a minimum at non-zero momentum. Gravity theories with the Chern-Simons term may provide dual descriptions for these systems.

The organization of part II is as follows. In chapter 6, we construct M-theory supergravity solutions with the nonrelativistic Schrödinger symmetry starting from the warped AdS_5 metric with $\mathcal{N} = 1$ supersymmetry. We impose that the lightlike direction is compact by making it a non-trivial $U(1)$ bundle over the compact space. In chapter 7, we show that, in a gravity theory with a Chern-Simons coupling, the Reissner-Nordström black hole in anti-de Sitter space is unstable depending on the value of the Chern-Simons coupling. The analysis suggests that the final configuration is likely to be a spatially modulated phase.

Chapter 6

Supersymmetric Nonrelativistic Geometries in M-theory

In this chapter, we are going to consider an example of gauge/gravity dualities applied to the study of a strongly coupled field theory. We construct M-theory supergravity solutions with Schrödinger symmetry starting from the warped AdS_5 metric with $\mathcal{N}=1$ supersymmetry.

We first recall what Schrödiner symmetry is [120]. Let us start with the Galilean algebra in $(1{+}d)$ dimensions consisting of the particle number operator N, the Hamiltonian H, spatial momenta P_i, rotations M_{ij} and Galilean boosts K_i. The last symmetry acts on the spacetime as

$$t \to t \;, \qquad x_i \to x_i - v_i t \;, \quad \text{for some constant vector } v_i \;. \tag{6.1}$$

An interesting feature of this algebra is that it is a subalgebra of the Poincaré algebra in $1+(d+1)$ dimensions. That can be most easily shown by introducing the light-cone coordinates $x^{\pm} = x^0 \pm x^{d+1}$. Then we consider a subalgebra of the Poincaré algebra that commutes with the lightcone momentum $\tilde{P}_-$, where tilde denotes elements in the Poincaré algebra. Then the following identification can be made:

$$N = -\tilde{P}_- \;, \quad H = -\tilde{P}_+ \;, \quad P_i = \tilde{P}_i \;, \quad M_{ij} = \tilde{M}_{ij} \;, \quad K_i = \tilde{M}_{-i} \;. \tag{6.2}$$

Just as the Poincaré algebra can be extended to include scale symmetry generator $\tilde{D}$, we can add a dilatation generator D to the Galilean algebra by the identification $D = \tilde{D} + 2(z-1)\tilde{M}_{-+}$ for some number z, called the dynamical exponent. The commutation relation of D with other generators are

$$\begin{aligned} &[D, P_i] = -iP_i \;, \qquad [D, H] = -izH \;, \\ &[D, K_i] = i(z-1) \;, \qquad [D, N] = i(z-2)N \;. \end{aligned} \tag{6.3}$$

Note that, when $z=2$, the particle number operator N commutes with all other generators. In that case, we can extend the algebra further by adding a special conformal transformation generator C

that satisfies

$$[C, P_i] = iK_i \ , \qquad [D, C] = 2iC \ , \qquad [H, C] = iD \ . \tag{6.4}$$

The special conformal generator C can be identified with $-\tilde{K}_-$ in the Poincaré algebra. The final algebra that contains D and C in addition to the Galilean algebra is the Schrödinger algebra.

Note that the Schrödinger algebra can be thought of as a subalgebra of the Poincaré algebra in one higher dimension, such that its elements commute with the lightcone momentum $\tilde{P}_-$. $\tilde{P}_-$ is identified with the particle number generator N, which takes discrete values. Therefore, it is natural that the direction associated with $\tilde{P}_-$ in the gravity dual is compact. We impose this condition by making it a non-trivial $U(1)$ bundle over the compact space.

One motivation to consider M-theory supergravity to find a solution with Schrödinger symmetry with non-trivial $U(1)$ bundle is that the mass-deformed limit of three-dimensional $\mathcal{N} = 8$ maximally supersymmetric gauge theory has a concrete description in M-theory supergravity [121, 122]. Let us first see what developments have been made in the understanding of the mass-deformed theory in the field theoretic Lagrangian description.

The Lagrangian description of the maximally supersymmetric gauge theory in three dimensions was found by Bagger and Lambert [123–125] (see also [126]), by developing the idea of [127]. However, it was difficult to increase the rank of the gauge group. This is in some sense related to the fact that the maximally supersymmetric M2-brane solution does not have an adjustable parameter. Later, Aharony et al. [128] constructed $\mathcal{N} = 6$ $U(N) \times U(N)$ Chern-Simons-matter theory (ABJM theory) that describes multiple M2-branes on the orbifold $\mathbb{C}^4/\mathbb{Z}_k$, where k becomes the level of the Chern-Simons action in the field theory. This orbifold provides us with an adjustable parameter, which enables us to treat weakly coupled field theories in some limit. A mass-deformed version of ABJM theory was considered in [129] and its vacuum structure was identified in [130]. Especially, in the most symmetric vacuum, the system has $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ symmetry. The mass term breaks the relativistic scaling symmetry. However, there is a nonrelativistic limit of this theory that has the Schrödinger symmetry [131, 132].

Turning our attention to the gravity side, we can turn on an anti-self-dual four-form flux for multiple M2-branes in flat space, which corresponds to adding a fermionic mass term to the field theory. The four-form flux polarizes M2-branes into M5-branes [121, 133] and the discrete set of vacua of the theory has a one-to-one correspondence with the partition of N, the number of M2-branes [122]. For multiple M2-branes on the orbifold $\mathbb{C}^4/\mathbb{Z}_k$, we do not have a clear answer yet, but expect that a similar kind of solutions with desirable properties may be found.

Note that the Chern-Simons-matter theory is a good model to study the nonrelativistic limit since gauge fields are not propagating. Therefore, it is natural to seek for a supergravity solution that corresponds to the nonrelativistic limit of the mass-deformed ABJM theory. Assuming the classical analysis of the vacuum structure of the field theory is still applicable to the supergravity limit, the solutions will have $SU(2)\times SU(2)\times U(1)\times \mathbb{Z}_2$ global symmetry and several additional $U(1)$ symmetries corresponding to the nonrelativistic particle number symmetry, depending on which fields to retain in the nonrelativistic limit [131]. In the most supersymmetric case, it has 14 supercharges. Although we were not able to find a supergravity solution with the same number of supersymmetries, we will present a class of supersymmetric solutions with the Schrödinger symmetry in two space dimensions in M-theory, and then consider a specific case with the same global bosonic symmetry of the nonrelativistic limit of the mass-deformed ABJM theory.

The Schrödinger symmetry considered in [134, 135][1] breaks the AdS symmetry explicitly due to the term $-\frac{dx^{+2}}{r^4}$ in the metric, where x^+ is one of the two lightlike coordinates. Soon after, the geometry was embedded in string theory [120, 138, 139]. The supergravity solutions with the Schrödinger symmetry does not have supersymmetry mainly due to the term $-\frac{dx^{+2}}{r^4}$ in the metric and the lightlike three form flux H_3 that supports it. Supersymmetry can be recovered if the coefficient of $\frac{dx^{+2}}{r^4}$ depends on the compact space [140]. However, in their case, the coefficient is necessarily negative in some region of the compact space and the stability of the spacetime is not guaranteed. This problem was remedied and supersymmetric solutions were obtained with negative coefficient of $\frac{dx^{+2}}{r^4}$ by turning on some lightlike fluxes, which can be related either to a Killing vector that leaves some Killing spinors invariant [141], or to the properties of the Calabi-Yau structure [142]. Also, it is possible to explicitly break the AdS symmetry by adding a term dx^+C to the metric where C is a one-form that does not depend on the worldvolume coordinates [142, 143]. There are also proposals where the breaking occurs due to the fact that the lightlike direction is compact without explicitly adding a term to the AdS metric [144, 145].

In the following, we will explore supergravity solutions having the Schrödinger symmetry in M-theory. As we mentioned above, we make the compact lightlike direction a non-trivial $U(1)$ bundle over the compact space. We begin with the $\mathcal{N}=1$ warped AdS_5 solutions in M-theory given in [146], and modify the geometry to obtain the Schrödinger symmetry. Initially the AdS_5 solution has eight supercharges and they reduce to two after the modification in general. However, there is a special case when there remain six supercharges, which is the same number as in the DLCQ of AdS. After general remarks, we specialize to a specific example with $SU(2)\times SU(2)\times U(1)$ isometry. We consider the Kaluza-Klein spectrum of the theory, and show that the non-trivial $U(1)$ bundle structure of the lightlike compact direction sets an upper bound for the nonrelativistic particle number for given quantum numbers of the compact space. The initial motivation to consider a Schrödinger invariant geometry with $SU(2)\times SU(2)\times U(1)$ was to find a candidate theory for the dual of the nonrelativistic mass-deformed ABJM theory. In line with this, we also provide a non-supersymmetric solution with the same global symmetry briefly at the end.

[1] See [136] for an earlier discussion, whose relation is explained in [137].

6.1 General Consideration

In this section, we will deform the supergravity solutions given in [146] in such a way that the resulting solutions have the Schrödinger symmetry.

6.1.1 Warped AdS_5 solutions in M-theory

Before dealing with nonrelativistic solutions, let us describe the general $\mathcal{N} = 1$ supersymmetric solutions of the supergravity limit of M-theory consisting of a warped product of AdS_5 and a six-dimensional space considered in [146]. We summarize our supergravity notation in appendix C. The metric is of the form

$$ds^2 = e^{2\lambda} \left[ds^2_{AdS_5} + ds^2_{M_6}\right] \ , \tag{6.5}$$

and the four-form flux lies along the compact six dimensions. The overall coefficient e^{λ} is a warping factor that depends on M_6. The authors of [146] obtained the most general condition for $\mathcal{N} = 1$ supersymmetry, and then specialized to a special case where the six-dimensional manifold M_6 is a complex manifold with a Hermitian metric. In this case, the supersymmetry condition becomes significantly simplified and they can obtain many explicit solutions. Let us describe the manifold M_6 first. The metric of M_6 is given by

$$ds^2_{M_6} = e^{-6\lambda(y)} \left[\hat{g}_{ij}(x,y)dx^i dx^j + \sec^2\zeta(y)dy^2\right] + \frac{1}{9}\cos^2\zeta(y)(d\psi + \hat{P})^2 \ . \tag{6.6}$$

There is a four-dimensional Kähler manifold M_4, whose metric is $\hat{g}_{ij}dx^i dx^j$. The complex structure of the metric is independent of y and ψ. $\frac{\partial}{\partial\psi}$ is a Killing vector of M_6 and the y dependence of the metric warps the spacetime. $\hat{P}$ is the canonical Ricci-form connection defined by the Kähler metric $\hat{g}$. That is, the Ricci form $R = d\hat{P}$. $\hat{P}$ is independent of y and ψ. ζ is a function of y which is implicitly defined by

$$2y = e^{3\lambda}\sin\zeta \ . \tag{6.7}$$

We fix the AdS_5 radius to be 1. The four-form field strength is given by

$$\begin{aligned} F_4^{(0)} &= -(\partial_y e^{-6\lambda})\hat{V}_4 + \frac{1}{3}dy \wedge (d\psi + \hat{P}) \wedge \hat{L} \\ \hat{L} &= \frac{1}{3}\cos^2\zeta \hat{*}_4 d\hat{P} - 4e^{-6\lambda}\hat{J} \ , \end{aligned} \tag{6.8}$$

where $\hat{V}_4$ is the volume form and $\hat{J}$ is the Kähler form of M_4. In addition to these, we have two more constraints:

$$\begin{aligned} \partial_y \hat{J} &= -\frac{2}{3}y d\hat{P} \ , \\ \partial_y \log\sqrt{\hat{g}} &= -3y^{-1}\tan^2\zeta - 2\partial_y \log\cos\zeta \ . \end{aligned} \tag{6.9}$$

Given these conditions, the Bianchi identity and the equations of motion for $F_4^{(0)}$, and the Einstein equations are all satisfied.

6.1.2 Deformation to Solutions with Schrödinger Symmetry

Let us first write the AdS_5 metric in a form that will be suitable for later analysis:

$$ds^2_{AdS_5} = \frac{-2dx^+dx^- + dx_1^2 + dx_2^2 + dr^2}{r^2} \ . \tag{6.10}$$

The DLCQ of AdS_5 makes the x^- direction compact. The modification we do here is to make x^- a coordinate for a $U(1)$ bundle over the compact space. In the case when the $U(1)$ bundle is non-trivial, the lightlike direction is necessarily compact and breaks AdS_5 symmetry down to the Schrödinger symmetry.[2] Let us call the geometry Sch_5.

Note that making the lightlike direction compact makes it subtle to deal with the system in the supergravity approximation. The situation gets better if we add large momenta along the compact lightlike direction [120]. This will involve making a black hole solution that asymptotes to the geometry that we give below. We will not consider such a finite temperature/finite density solution here, but we note that the compact lightlike direction changes the causal structure of the spacetime drastically. In particular, any two points in the geometry can be joined by a timelike or lightlike curve: Suppose we want to connect some point $P = (x^+, x^-, x^i, r)$ to $Q = (0, 0, 0, 0)$ using a timelike curve when $x^+ < 0$. Due to the periodic identification, we can equally start at $P = (x^+, x^- - N\Delta x^-, x^i, r)$ for some large N where Δx^- is the period of the x^- direction. For large enough N, there is indeed a timelike curve connecting the points P and Q. This is a property that is expected for the dual theory of a nonrelativistic system.

Note that we can also add a term proportional to $\frac{dx^{+2}}{r^4}$, which does not break the Schrödinger symmetry. The coefficient depends on the compact space. Such a possibility was explored previously in [140]. Specifically, we consider the following metric:

$$\begin{aligned} ds^2 &= e^{2\lambda}\left[ds^2_{Sch_5} + ds^2_{M_6}\right] \ , \\ ds^2_{Sch_5} &= -f(y)\frac{dx^{+2}}{r^4} + \frac{-2dx^+(dx^- + A) + dx_1^2 + dx_2^2 + dr^2}{r^2} \ , \\ ds^2_{M_6} &= e^{-6\lambda(y)}\left[ds^2_{M_4} + \sec^2\zeta(y)dy^2\right] + \frac{1}{9}\cos^2\zeta(y)(d\psi + \hat{P})^2 \ , \\ ds^2_{M_4} &= \hat{g}_{ij}(x, y)dx^i dx^j \ . \end{aligned} \tag{6.11}$$

A is a gauge field on M_4 and $f(y)$ is some function that depends only on y. We need to determine these two quantities. To support this geometry, we turn on the four-form field strength along the

[2] There was a paper [147] that also considers modification of the warped AdS_5 solutions of [146]. They added dx^+C component to the metric, where C is a globally defined one-form on the compact space, which means the $U(1)$ bundle corresponding to the the x^- direction is trivial.

lightlike direction:

$$F_4 = F_4^{(0)} + \frac{1}{r^3}s(y)dx^+ \wedge dr \wedge dA - \frac{1}{2r^2}s'(y)dx^+ \wedge dy \wedge dA\ . \tag{6.12}$$

We demand that A depends only on x^i, and not on y: otherwise, the second term includes a part proportional to $dx^+ \wedge dr \wedge dy \wedge \partial_y(dA)$, and then it is impossible to satisfy the equations of motion for F_4. $F_4^{(0)}$ is the original four-form field strength of the warped AdS_5 solution, and $s(y)$ is some function to be determined. By construction, $dF_4 = 0$. Just as in the original warped AdS_5 solution, we also require $F_4 \wedge F_4 = 0$. This requires

$$\hat{L} \wedge dA = 0\ . \tag{6.13}$$

Let us consider the equations of motion for F_4 first. The dual seven-form F_7 is given by

$$\begin{aligned} F_7 = *_{11}F_4 =& F_7^{(0)} + e^{6\lambda}\frac{1}{r^5}dx^+ \wedge dx_1 \wedge dx_2 \wedge dr \wedge A \wedge \left[2\lambda'(y)dy \wedge (d\psi + \hat{P}) + \hat{*}_4\hat{L}\right] \\ &+ \frac{s(y)}{3r^4}dx^+ \wedge dx_1 \wedge dx_2 \wedge dy \wedge (d\psi + \hat{P}) \wedge \hat{*}_4 dA \\ &+ \frac{s'(y)}{6r^5}e^{6\lambda}\cos^2\zeta dx^+ \wedge dx_1 \wedge dx_2 \wedge dr \wedge (d\psi + \hat{P}) \wedge \hat{*}_4 dA\ , \end{aligned} \tag{6.14}$$

where $F_7^{(0)}$ is the seven-form field strength of the corresponding warped AdS_5 solution. Since we only consider the case when $F_4 \wedge F_4 = 0$, the equation of motion of F_4 is satisfied when $dF_7 = 0$. This is satisfied provided

$$\begin{aligned} dA &= \pm\hat{*}_4 dA\ , \\ d\hat{P} \wedge \hat{*}_4 dA &= 0\ , \\ dA \wedge \hat{*}_4\hat{L} &= 0\ , \end{aligned} \tag{6.15}$$

as well as

$$\pm 12e^{6\lambda}\lambda' + 8s(y) + \partial_y(s'(y)e^{6\lambda}\cos^2\zeta) = 0\ . \tag{6.16}$$

The last equation is satisfied when

$$\begin{aligned} s(y) &= -2y \qquad \text{if } dA \text{ is self-dual}\ , \\ s(y) &= 2y \qquad \text{if } dA \text{ is anti-self-dual}\ . \end{aligned} \tag{6.17}$$

due to the relation (6.7). In the cases we are interested, y takes values between two roots of $\cos\zeta = 0$. Since (6.16) is a second order differential equation and the coefficient of $s''(y)$ vanishes when $\cos\zeta = 0$, the other solution necessarily blows up when $\cos\zeta = 0$. Therefore, $s(y) = 2y$ is the regular solution we want. The third equation implies $dA \wedge \hat{J} = 0$. We will see presently that the

Einstein equations are also satisfied by choosing the coefficient $f(y)$ of $\frac{dx^{+2}}{r^4}$ appropriately. However, it is possible that the coefficient can take both positive and negative values over the compact space and, in the example that we consider in the next section, indeed this is the case. This is analogous to the situation considered in [140], where the coefficient of $\frac{1}{r^4}dx^{+2}$ is a harmonic function, which implies that it is necessarily negative in some region of the compact space. They show that there is an instability of a field with sufficiently large particle number due to the unboundedness of the Hamiltonian H(the conjugate momentum to x^+). Supersymmetry cannot guarantee H is positive since there is no dynamical supercharge. We expect a similar instability in our geometry unless $f(y)$ vanishes. However, as we will see in section 6.1.3, when $f(y) = 0$, there are two dynamical supercharges and the Hamiltonian H is bounded by the condition $\{Q, Q^\dagger\} = H$ for dynamical supercharges Q and $Q^\dagger$.

To sum up, if there is a harmonic (anti)self-dual two-form dA that satisfies

$$d\hat{P} \wedge dA = 0 \ , \qquad \hat{J} \wedge dA = 0 \ , \tag{6.18}$$

then we can construct a supergravity solution with the Schrödinger symmetry as described above.[3] Note that A is a one-form on M_4 and does not depend on y. Since $\partial_y \hat{J} = -\frac{2}{3} y d\hat{P}$ and $\hat{P}$ is independent of y, if (6.18) is satisfied at one y, it is automatically satisfied for all y.

One case where a solution is easily found is when the manifold M_4 is Kähler-Einstein and y and ψ give a $\mathbb{C}\mathrm{P}^1$ bundle over M_4. The isometry of $\mathbb{C}\mathrm{P}^1$ is broken to $U(1)$ by the warping factor that depends on y. In this case, $d\hat{P}$, the Ricci form, is proportional to $\hat{J}$. Since $d\hat{P}$ is y-independent, $\hat{J}$ factorizes into a y-dependent function and a y-independent form. Hence, given a harmonic (anti)self-dual two-form dA with $\hat{J} \wedge dA = 0$, we can construct a Schrödinger solution. To do that, the dimension of the second cohomology class has to be greater than 1, which means we cannot construct our solution on $\mathbb{C}\mathrm{P}^3$. However, there are cases when the dimension of the second cohomology class is greater than 1, and we will consider such an example where the manifold M_4 is $S^2 \times S^2$.

Given the above requirement, the Einstein equations are satisfied by choosing a suitable $f(y)$.

[3] dA represents a non-trivial element of the second cohomology class $H^2(M_4)$. For this to be a non-trivial element of $H^2(M_6)$, we need to assume a global structure of the six-dimensional complex manifold M_6. In the examples of [146], M_6 is taken to be a $\mathbb{C}\mathrm{P}^1$ bundle over the Kähler base M_4. Then the Gysin sequence $0 \to H^2(M_4) \to H^2(M_6)$ implies dA is also a non-trivial element of $H^2(M_6)$ as long as the orientability condition is satisfied.

Let us first introduce the following vielbeins:

$$
\begin{aligned}
E^0 &= e^{\lambda}\left(\frac{1+f(y)}{2}\frac{1}{r^2}dx^+ + dx^- + A\right) \,, \\
E^1 &= e^{\lambda}\frac{1}{r}dx_1 \,, \qquad E^2 = e^{\lambda}\frac{1}{r}dx_2 \,, \\
E^3 &= e^{\lambda}\left(\frac{1-f(y)}{2}\frac{1}{r^2}dx^+ - (dx^- + A)\right) \,, \\
E^4 &= e^{\lambda}\frac{dr}{r} \,, \\
E^y &= e^{-2\lambda}\sec\zeta dy \,, \qquad E^{\psi} = \frac{1}{3}e^{\lambda}\cos\zeta(d\psi + \hat{P}) \,, \\
E^i &= e^{-2\lambda}\hat{e}^i \,, \qquad i = 1,2,3,4 \,,
\end{aligned}
\tag{6.19}
$$

where $\hat{e}^i$ are vielbeins for the metric $ds^2_{M_4}$ in (6.11). Knowing that the original warped AdS_5 solution satisfies the Einstein equations of motion, all we need to check is the change of the component $G_{03} = \kappa_{11}^2 T_{03}$ of the Einstein equation. This will be satisfied if

$$
-f(y) + yf'(y) - \frac{1}{12}e^{6\lambda}\cos^2\zeta f''(y) = 0 \,. \tag{6.20}
$$

There are two linearly independent solutions and one obvious solution is $f(y) = \beta y$ for an arbitrary constant β. In the case when y and ψ combine to give topologically a two-sphere S^2, $\cos\zeta = 0$ at the two poles of the sphere, and we take the solution $f(y) = \beta y$ as the smooth solution. The other solution diverges when $\cos\zeta = 0$.

6.1.3 Supersymmetry

The Killing spinor equation is given by

$$
\delta\Psi_A = D_A\epsilon = \nabla_A\epsilon + \frac{1}{12}\left(\Gamma_A\mathbf{F}^{(4)} - 3\mathbf{F}^{(4)}_A\right)\epsilon = \partial_A\epsilon + \frac{1}{4}\omega_{ABC}\Gamma^{BC} + \frac{1}{12}\left(\Gamma_A\mathbf{F}^{(4)} - 3\mathbf{F}^{(4)}_A\right)\epsilon \,, \tag{6.21}
$$

where ϵ is a Killing spinor and

$$
\begin{aligned}
\mathbf{F}^{(4)} &= \frac{1}{4!}F_{ABCD}\Gamma^{ABCD} \,, \\
\mathbf{F}^{(4)}_A &= \frac{1}{2}\left[\Gamma_A, \mathbf{F}^{(4)}\right] \,.
\end{aligned}
\tag{6.22}
$$

We use $A, B, \ldots$ for vielbein indices and $M, N, \ldots$ for coordinate indices of eleven dimensions. Our strategy is to divide the operator D_A into two: one is independent of β and A, while the other is not. Then, given a Killing spinor ϵ of the corresponding AdS solution, we impose the condition that ϵ is annihilated by β, A-dependent part. Let us denote by $\Delta\partial_A$ the change of the derivative ∂_A due to the presence of β and A, and similarly denote by $\Delta\omega_A$ the change of the connection $\omega_{ABC}\Gamma^{BC}$.

If we define a matrix ${\Lambda^A}_M$ by $E^A = {\Lambda^A}_M dx^M$, ${(\Lambda^T)^{-1}}_A{}^M \partial_M = \partial_A$. Then it is easy to see that the only components that depend on β are ${(\Lambda^T)^{-1}}_0{}^-$ and ${(\Lambda^T)^{-1}}_3{}^-$, and those that depend on A are ${(\Lambda^T)^{-1}}_i{}^-$. Therefore, we keep Killing spinors of the AdS solution when it is independent of x^-. We will see later that the Killing spinors consistent with the compactification of x^- are all independent of x^-. Hence it does not give a new condition.

Next, let us consider the change of the connection $\Delta\omega_A$. They are given by

$$\begin{aligned}
&\Delta\omega_1 = \Delta\omega_2 = \Delta\omega_4 = \Delta\omega_y = \Delta\omega_\psi = 0 \ , \\
&\Delta\omega_0 = \Delta\omega_3 = \beta e^{2\lambda}\left(-\sin\zeta\Gamma^4 + \cos\zeta\Gamma^y\right)\Gamma^+ + e^{5\lambda}\mathbf{F}^{(2)} \ , \\
&\Delta\omega_i = -e^{5\lambda}F_{ij}\Gamma^j\Gamma^+ \ .
\end{aligned} \tag{6.23}$$

Here $\Gamma^+ = \Gamma^0 + \Gamma^3$ and $\mathbf{F}^{(2)}$ is a product of gamma matrices $\frac{1}{2}F_{ij}\Gamma^{ij}$ where $F = dA$ and $\frac{1}{2}F_{ij}\hat{e}^i\hat{e}^j = F$. The change in the four-form field strength is

$$\Delta\mathbf{F}^{(4)} = e^{5\lambda}\left(-\sin\zeta\Gamma^4 + \cos\zeta\Gamma^y\right)\Gamma^+\mathbf{F}^{(2)} \ . \tag{6.24}$$

The condition that the differential operators D_0 and D_3 still annihilate a Killing spinor ϵ of the AdS solution imposes

$$\beta\,\Gamma^+\epsilon = 0; \ , \qquad \mathbf{F}^{(2)}\epsilon = 0 \ . \tag{6.25}$$

The second equation is satisfied if, for example, the manifold M_4 is Kähler-Einstein and the two-form field strength F is a $(1,1)$-form on M_4. To see this, let us decompose gamma matrices and spinors into AdS_5 and M_6 parts (note that we are looking for a Killing spinor of the original AdS_5 geometry which survives after we change the metric to the Sch_5 geometry). First, we decompose the eleven-dimensional gamma matrices as

$$\begin{aligned}
\Gamma^a &= \gamma^a \otimes \tau_7 \ , \\
\Gamma^m &= 1 \otimes \tau^m \ ,
\end{aligned} \tag{6.26}$$

where $a = 0, \ldots, 4$ and $m = 1, \ldots, 6$ are orthonormal indices for AdS_5 and M_6, respectively, and $\tau_7 = \tau_1 \ldots \tau_6$. They satisfy

$$\begin{aligned}
\{\gamma^a, \gamma^b\} &= -2\eta^{ab} \ , \\
\{\tau^m, \tau^n\} &= 2\delta^{mn} \ ,
\end{aligned} \tag{6.27}$$

where $\eta^{ab} = \mathrm{diag}(-1,1,1,1,1)$. Note $\tau_7^2 = -1$.

The Killing spinor ϵ is decomposed as $\psi(x) \otimes e^{\frac{\lambda}{2}}\xi(y)$ for $x \in AdS_5$ and $y \in M_6$. ψ satisfies the

Killing spinor equation for AdS_5:

$$\partial_a\psi - \frac{1}{4}\omega_{abc}\gamma^{bc}\psi = \frac{1}{2}i\gamma_a\psi \ . \tag{6.28}$$

There are two types of Killing spinors of AdS_5. They are given as (see, for example, [148])

$$\psi^+ = r^{-\frac{1}{2}}\psi_0^+ \ , \qquad \psi^- = (r^{\frac{1}{2}} + ir^{-\frac{1}{2}}x^\mu\gamma_\mu)\psi_0^- \ , \tag{6.29}$$

where $-i\gamma^r\psi_0^\pm = \pm\psi_0^\pm$. ψ^+ generates a Poincaré supersymmetry and ψ^- a superconformal one. In our case, the lightlike direction x^- is compactified. The coordinate ψ^+ depends only on r, so ψ^+ survives compactification. The coordinate ψ^- is position dependent, and to be periodic in x^-, it should not have x^- dependence. This is the same as requiring that $\gamma^+\psi_0^- = 0$. Hence half of the superconformal supersymmetries survive compactification.

The Killing spinor equation $D_a\epsilon = 0$ implies that ξ has to satisfy

$$\left(\tau^m\nabla_m\lambda + \frac{1}{6}e^{-3\lambda}\mathbf{F}_0^{(4)} - i\tau_7\right)\xi = 0 \ , \tag{6.30}$$

where $\mathbf{F}_0^{(4)}$ is a gamma matrix expression using τ^m constructed from the four-form field strength (6.8). Let us multiply the above equation by $\mathbf{F}^{(2)}$, where now $\mathbf{F}^{(2)}$ is made up of τ^m matrices:

$$\mathbf{F}^{(2)}\left(\tau^m\nabla_m\lambda + \frac{1}{6}e^{-3\lambda}\mathbf{F}_0^{(4)} - i\tau_7\right)\xi = 0 \ . \tag{6.31}$$

From (6.15), we obtain $\hat{L}\wedge F = \hat{L}\wedge \hat{*}_4 F = 0$. This implies $\{\mathbf{F}^{(2)}, \hat{\mathbf{L}}\} = 0$ since $\{\tau^{mn}, \tau^{pq}\} = 2\gamma^{mnpq} - 4\delta^{pq}_{mn}$. Also, since we assume M_4 is Kähler-Einstein, $[\mathbf{F}^{(2)}, \hat{\mathbf{L}}]$ is proportional to $F_{ij}\hat{J}^j{}_k\Gamma^{ik}$, which vanishes if F is a (1,1)-form. Now, we can simplify the expression (6.31) in the form $\mathbf{Q}\mathbf{F}^{(2)}\xi = 0$ where $\mathbf{Q}$ is some linear combination of gamma matrices. By examining the explicit expression, we see that $\mathbf{Q}$ has determinant $(1-4y^2(\lambda')^2)^4$, which does not vanish. Therefore we conclude $\mathbf{F}^{(2)}\xi = 0$. The remaining constraints come from examining $D_i\xi = 0$ for $i = \theta_1, \phi_1, \theta_2, \phi_2$. In the case $\mathbf{F}^{(2)}\xi = 0$, they impose an additional condition

$$\Gamma^+\left(1 + \sin\zeta\Gamma^4 - \cos\zeta\Gamma^y\right)\epsilon = 0 \ . \tag{6.32}$$

This is satisfied if $\Gamma^+\epsilon = 0$.

In conclusion, a Killing spinor of the AdS solution survive if it satisfies $\Gamma^+\epsilon = 0$. Therefore, at each point, a Killing spinor has to lie in some four-dimensional space. This does not necessarily mean that there are four Killing spinors, since higher order integrability condition may not be satisfied. In fact, a superconformal supercharge cannot satisfy $\Gamma^+\epsilon = 0$. To see this, note that a superconformal supercharge is represented in the Poincaré coordinates as in the second expression in (6.29). $\Gamma^+\epsilon = 0$ translates into $\gamma^+\psi^- = 0$, which is written as

$$\gamma^+ \left[r^{\frac{1}{2}} + i r^{-\frac{1}{2}} (x^i\gamma^i - x^+\gamma^- - x^-\gamma^+) \right] \psi_0^- = 0 \; . \tag{6.33}$$

At $x = 0$, this implies $\gamma^+\psi_0^- = 0$. Now, we move γ^+ to the right. Then, since $\{\gamma^+, \gamma^-\} = 2$, we end up getting $\psi_0^- = 0$, which means the only solution to this equation is the trivial one. Hence no superconformal supersymmetries survive, which means there remain only two Poincaré supercharges that are annihilated by γ^+.

When $\beta = 0$, there can be more supercharges since the first equation of (6.25) is trivial and all we require is (6.32) as well as $\mathbf{F}^{(2)}\epsilon = 0$. We have already considered the case when $\Gamma^+\epsilon = 0$. Another possibility is that ϵ is annihilated by the second factor. Under the decomposition, this can be rewritten as

$$(1 \pm i \sin\zeta\tau_7 - \cos\zeta\tau_y)\,\xi = 0 \; , \tag{6.34}$$

depending on $-i\gamma^r\psi^\pm = \pm\psi^\pm$.

Now, we will prove that (6.30) implies (6.34) with plus sign in the second term if M_4 is Kähler-Einstein and F is anti-self-dual. $\mathbf{F}^{(2)}\mathbf{F}^{(2)}\epsilon = 0$ implies $\tau^{3456}\epsilon = -\epsilon$ if F is anti-self-dual. The indices for M_6 are such that $\{y, \psi, \theta_1, \phi_1, \theta_2, \phi_2\} \leftrightarrow \{1, 2, 3, 4, 5, 6\}$. For a Kähler-Einstein manifold M_4, $\hat{L}$ is given by [146]

$$\hat{L} = \left(\frac{\cos^2\zeta(1 + 6y\lambda')}{e^{6\lambda} - 4y^2} - 4e^{-6\lambda} \right) \hat{J} \; . \tag{6.35}$$

Define $\hat{\mathbf{J}} = \frac{1}{2} e^{6\lambda} \hat{J}_{ij}\tau^{ij}$ where $\hat{J} = \frac{1}{2}\hat{J}_{ij}\hat{e}^i\hat{e}^j$, and define $\hat{\mathbf{L}}$ similarly. Since $\hat{J}$ is self-dual,

$$\hat{\mathbf{J}}\hat{\mathbf{J}} = \frac{1}{2}\{\hat{\mathbf{J}}, \hat{\mathbf{J}}\} = e^{12\lambda}(\tau^{3456} - 1) \; , \tag{6.36}$$

We can rewrite (6.30) as

$$\left(e^{3\lambda}\lambda' \cos\zeta\tau_1 + e^{3\lambda}\lambda'\tau_{3456} + \frac{1}{6}\tau_{12}\hat{\mathbf{L}} - i\tau_7 \right) \xi = 0 \; . \tag{6.37}$$

By multiplying by $\tau_7 \left(e^{3\lambda}\lambda' \cos\zeta\tau_1 - e^{3\lambda}\lambda'\tau_{3456} + \frac{1}{6}\tau_{12}\hat{\mathbf{L}} + i\tau_7 \right)$ to the left, up to an overall factor, we obtain (6.34) with plus sign in the second term if we use (6.35), (6.36) and $\tau^{3456}\epsilon = -\epsilon$. That implies that the corresponding Killing spinor in the AdS_5 part is a Poincaré supercharge. Therefore, when $\beta = 0$, we have four Poincaré supercharges.

However, there should be additional supercharges that we might have overlooked when we analyze (6.32). Indeed, if we keep all four Poincaré supercharges of the AdS solution, there are two kinematical supercharges and two dynamical ones.[4] In this case, the commutator of the special

[4] For a related discussion about Schrödinger superalgebra, see [149–152].

conformal generator C and a dynamical supercharge Q produces a superconformal supercharge S: $[C,Q] \sim S$. Therefore, there has to be a way to obtain a superconformal supersymmetry. To see how it comes about, let us look at the expression for a superconformal supersymmetry in AdS_5 space

$$\psi^- = (r^{\frac{1}{2}} + ir^{-\frac{1}{2}} x^\mu \gamma_\mu)\psi_0^- = \left[r^{\frac{1}{2}} + ir^{-\frac{1}{2}}(x^i\gamma^i - x^+\gamma^- - x^-\gamma^+)\right]\psi_0^- \ , \tag{6.38}$$

with $-i\gamma^r\psi_0^- = -\psi_0^-$. Since x^- is compactified, we demand $\gamma^+\psi_0^- = 0$. If we set $\epsilon = \psi^- \otimes e^{\frac{\lambda}{2}}\xi(y)$ with ψ^- as just given, (6.32) becomes

$$(1 + i\sin\zeta\tau_7 - \cos\zeta\tau_y)\,\xi = 0 \ , \tag{6.39}$$

which we have already verified. Therefore, two superconformal supercharges that are constructed from ψ_0^- with $\gamma^+\psi_0^- = 0$ survive.

In this section, we have shown that, if M_4 is Kähler-Einstein and $F = dA$ is a harmonic anti-self-dual two-form of type (1,1) on M_4, it preserves two Poincaré supercharges when $\beta \neq 0$. This corresponds to the kinematical supercharges. If $\beta = 0$, we additionally have two dynamical supercharges and two superconformal supercharges, adding up to six in total. The number of surviving supercharges are the same as those of the discrete light cone quantization (DLCQ) of the AdS solution. Note that the presence of the dynamical supercharges guarantees that the Hamiltonian H(the conjugate momentum to the x^+ coordinate) is positive definite: $\{Q, Q^\dagger\} = H$ for dynamical supercharges Q and $Q^\dagger$.

6.2 Specific Example

Here we present a specific example of the above analysis. We consider the case when the four-dimensional manifold M_4 is $S^2 \times S^2$ and y and ψ describes a $\mathbb{CP}^1$ bundle, but warped by the y coordinate. The symmetry of the six-dimensional compact space is $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ where the $U(1)$ is related to $\frac{\partial}{\partial\psi}$ and $\mathbb{Z}_2$ exchanges the two spheres. Such a solution may be interesting since this is the symmetry of the nonrelativistic limit of ABJM theory [131,132]. Let us first consider the warped AdS_5 solution.

6.2.1 Warped AdS_5 solution before modification

This solution appeared in [146] as a specific example. The base manifold M_4 is $S^2 \times S^2$ of the same radius, and is a Kähler-Einstein manifold. The six dimensional manifold M_6 has $SU(2) \times SU(2) \times$

$U(1)$ symmetry and also a $\mathbb{Z}_2$ symmetry that switches the two S^2s. The metric is given by

$$
\begin{aligned}
ds^2_{11} &= e^{2\lambda(y)}\left[ds^2_{AdS_5} + ds^2_{M_6}\right] \,, \\
ds^2_{AdS_5} &= \frac{-2dx^+dx^- + dx_1^2 + dx_2^2 + dr^2}{r^2} \,, \\
ds^2_{M_6} &= \frac{1}{3}e^{-6\lambda}(1-y^2)(d\theta_1^2 + \sin\theta_1^2 d\phi_1^2 + d\theta_2^2 + \sin\theta_2^2 d\phi_2^2) \\
&\quad + e^{-6\lambda}\sec^2\zeta dy^2 + \frac{1}{9}\cos^2\zeta(d\psi + \hat{P})^2 \,,
\end{aligned}
\tag{6.40}
$$

where

$$
\begin{aligned}
\hat{P} &= A_1 + A_2 \,, \\
A_1 &= -\cos\theta_1 d\phi_1 \,, \qquad A_2 = -\cos\theta_2 d\phi_2 \,, \\
e^{6\lambda} &= \frac{2(1-y^2)^2}{2+cy+2y^2} \,, \\
\cos^2\zeta &= \frac{-3y^4 - 2cy^3 - 6y^2 + 1}{(1-y^2)^2} \,.
\end{aligned}
\tag{6.41}
$$

The four-form field strength is given by

$$
F_4 = p_1(y)\omega_1 \wedge \omega_2 + p_2(y) dy \wedge (d\psi + \hat{P}) \wedge (\omega_1 + \omega_2) \,, \tag{6.42}
$$

where $\omega_i = dA_i$ and

$$
p_1(y) = \frac{4y^3 + 3cy^2 + 12y + c}{18(y^2-1)} \,, \qquad p_2(y) = \frac{y^4 - 6y^2 - 2cy - 3}{9(y^2-1)^2} \,. \tag{6.43}
$$

The coordinates θ_1 and ϕ_1 parametrize one S^2, and θ_2 and ϕ_2 the other S^2. The period of ψ is 2π to have a smooth geometry. The coordinates y and ψ combine to give a S^2 fibration over $S^2 \times S^2$. However, due to the y dependence here and there, only $U(1)$ symmetry survives. Also c is constant, $0 \le c < 4$ and y runs between the two roots of the equation $\cos^2\zeta = 0$. Since $\cos^2\zeta > 0$ for $y = 0$, one root is positive and the other negative. It preserves 8 supercharges.

6.2.2 Transformation to Schrödinger Solution

Now we modify the geometry (6.40) according to section 6.1.2. We make x^- a non-trivial $U(1)$ bundle over $S^2 \times S^2$ with gauge field $A = n(A_1 - A_2)$, where n is some integer. The metric is given by

$$
\begin{aligned}
ds^2_{11} &= e^{2\lambda(y)}\left[ds^2_{Sch_5} + ds^2_{M_6}\right] \ , \\
ds^2_{Sch_5} &= -\beta y \frac{dx^{+2}}{r^4} + \frac{-2dx^+(dx^- + A) + dx_1^2 + dx_2^2 + dr^2}{r^2} \ , \\
ds^2_{M_6} &= \frac{1}{3} e^{-6\lambda}(1-y^2)(d\theta_1^2 + \sin\theta_1^2 d\phi_1^2 + d\theta_2^2 + \sin\theta_2^2 d\phi_2^2) \\
&\quad + e^{-6\lambda}\sec^2\zeta dy^2 + \frac{1}{9}\cos^2\zeta(d\psi + \hat{P})^2 \ ,
\end{aligned}
\tag{6.44}
$$

Note that dA is anti-self-dual, $dA \wedge d\hat{P} = 0$ and A does not depend on y. The four-form flux is modified as follows:

$$
\begin{aligned}
F_4 &= p_1(y)\omega_1 \wedge \omega_2 + p_2(y) dy \wedge (d\psi + \hat{P}) \wedge (\omega_1 + \omega_2) \\
&\quad + 2ny\frac{1}{r^3} dx^+ \wedge dr \wedge (\omega_1 - \omega_2) - n\frac{1}{r^2} dx^+ \wedge dy \wedge (\omega_1 - \omega_2) \ ,
\end{aligned}
\tag{6.45}
$$

Note that the solution exists for each $c \in [0, 4)$ and each integer n. Given the general analysis in the previous section, the equations of motion for the four-form field and the metric are guaranteed to be satisfied. Note that $-\beta y$, the coefficient of $\frac{1}{r^4} dx^{+2}$, takes both positive and negative values over the compact space. As mentioned in section 6.1.2, this signals an instability due to the unboundedness of the Hamiltonian [140] unless we set $\beta = 0$.

Note that dA is an anti-self-dual two-form of type (1,1) in M_4. Hence, according to the argument in section 6.1.3, there are two kinematical supercharges when $\beta \neq 0$, and six supercharges when $\beta = 0$. The six supercharges consist of two kinematical, two dynamical, and two superconformal supercharges. Especially, when $\beta = 0$, the Hamiltonian will be bounded below due to the presence of the dynamical supercharges.

6.2.3 Solution with Plane Wave Boundary

In the previous sections, we use the Poincaré coordinate system for (deformed)AdS_5. In general, the AdS_{n+2} metric in Poincaré coordinates is given by

$$
ds^2 = \frac{-2dx^+ dx^- + d\vec{x}^2 + dr^2}{r^2} \ , \tag{6.46}
$$

where $\vec{x} = (x_1, \ldots, x_{n-1})$. The boundary is $\mathbb{R}^{1,n}$. There is another coordinate system in which the boundary approaches the plane wave metric [144, 145, 153]. It is given by

$$
ds^2 = \frac{-2dx'^+ dx'^- - \vec{x}'^2 dx'^{+2} + d\vec{x}'^2 + dr^2}{r^2} - dx'^{+2} \ . \tag{6.47}
$$

The relation between the two coordinate systems is

$$
\begin{aligned}
x^{+} &= \tan x'^{+} \ , \\
r &= r' \sec x'^{+} \ , \\
\vec{x} &= \vec{x}' \sec x'^{+} \ , \\
x^{-} &= x'^{-} + \frac{1}{2}(r'^{2} + \vec{x}'^{2}) \tan x'^{+} \ .
\end{aligned}
\tag{6.48}
$$

Note that $\frac{\partial}{\partial x^{-}}$ and $\frac{\partial}{\partial x'^{-}}$ generate the same flow in different coordinates: both are related to the number operator of the Schrödinger algebra. This also suggests that not much will change even if the x'^{-} direction is a line bundle over the compact space. That is, instead of (6.44), we may consider the metric

$$
\begin{aligned}
ds_{11}^2 &= e^{2\lambda(y)} \left[ds_{Sch_5}^2 + ds_{M_6}^2 \right] \ , \\
ds_{Sch_5}^2 &= -\beta y \frac{dx^{+2}}{r^4} + \frac{-2dx^{+}(dx^{-} + A) - (x_1^2 + x_2^2)dx^{+2} + dx_1^2 + dx_2^2 + dr^2}{r^2} - dx^{+2} \\
&\quad + e^{-6\lambda} \sec^2 \zeta dy^2 + \frac{1}{9} \cos^2 \zeta (d\psi + \hat{P})^2 \ .
\end{aligned}
\tag{6.49}
$$

This and (6.44) are related by an obvious coordinate transformation. The first term $-\beta y \frac{dx^{+2}}{r^4}$ may look troublesome at first, but actually $-\frac{dx^{+2}}{r^4}$ itself is invariant under (6.48). This form of the metric may be useful since the time direction in this coordinate system is associated to the harmonic oscillator potential

$$
H_{osc} = H + C \tag{6.50}
$$

of the Schrödinger algebra. Here H generates the time translation in the Poincaré coordinates and C is the special conformal generator.

6.3 Kaluza-Klein Mass Spectrum

The fact that the lightlike compact direction is a non-trivial bundle over the compact space has an interesting consequence on the spectrum of the Kaluza-Klein states. We will show below that the nonrelativistic particle number is bounded above by the quantum numbers of the compact space. It seems at first a bit strange that there is such a bound. However, we can view the system from the compact space point of view and consider the Kaluza-Klein particles charged under the momentum conjugate to the x^{-} coordinate. Due to the non-trivial gauge field A, we can think that the Kaluza-Klein particles are in a magnetic monopole background field. Then it is well known [154] that the quantum numbers of the compact space of a wave function describing a Kaluza-Klein particle is bounded below by the "electric" charge of the particle, which in this case means the $U(1)$ charge along the x^- direction. The eigenstates are expressed as monopole harmonics. Below, we will follow the classical analysis, but in a way that can be more easily applicable to our situation.

Let us first consider the three sphere S^3 as a preparation. The metric is given by

$$ds^2_{S^3} = (d\psi - \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2 \ , \tag{6.51}$$

where $0 \leq \psi \leq 4\pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The manifest symmetry is $SU(2) \times U(1)$ of $SO(4)$. The Killing vectors are

$$\begin{aligned} L_1 &= \sin\phi \frac{\partial}{\partial\theta} + \cos\phi \left[\cot\theta \frac{\partial}{\partial\phi} + \csc\theta \frac{\partial}{\partial\psi}\right] \ , \\ L_2 &= \cos\phi \frac{\partial}{\partial\theta} - \sin\phi \left[\cot\theta \frac{\partial}{\partial\phi} + \csc\theta \frac{\partial}{\partial\psi}\right] \ , \\ L_3 &= \frac{\partial}{\partial\phi} \ , \\ L_\psi &= \frac{\partial}{\partial\psi} \ . \end{aligned} \tag{6.52}$$

They satisfy $[L_i, L_j] = \sum_k \epsilon_{ijk} L_k$ and $[L_i, L_\psi] = 0$, which comprise $SU(2) \times U(1)$ Lie algebra. We will construct a wave function $\Phi(\psi, \theta, \phi)$ carrying definite quantum numbers of $SU(2)$ and $U(1)$. First, let us demand

$$L_\psi \Phi = -i m_\psi \Phi \ . \tag{6.53}$$

Since ψ has period 4π, $m_\psi \in \frac{\mathbb{Z}}{2}$. For $SU(2)$ part, the analysis is very similar to the standard angular momentum analysis in quantum mechanics. For the l representation of $SU(2)$, let us consider the highest state $(l, m) = (l, l)$. It will be annihilated by $L_+ = L_1 + iL_2$. It is easy to see that

$$L_+ e^{-im\phi} f(\theta) = 0 \quad \text{for} \quad f(\theta) = \left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)^{m_\psi} \sin^l\theta \ . \tag{6.54}$$

The wave function is then given by $\Phi(\psi, \theta, \phi) = e^{-im_\psi\psi} e^{-im\phi} f(\theta)$. Since we want a wave function not to diverge at $\theta = 0$ or 2π, $|m_\psi| \leq l$. By applying the lowering operator $L_- = L_1 - iL_2$ repeatedly, we obtain a wave function with definite quantum numbers (m_ψ, l, m):

$$\Phi_{m_\psi, l, m} = e^{-im_\psi\psi} e^{-im\phi} (1-u^2)^{-\frac{l}{2}} \left(\frac{1-u}{1+u}\right)^{-\frac{m_\psi}{2}} \frac{d^{l-m}}{du^{l-m}} \left(\frac{1-u}{1+u}\right)^{m_\psi} (1-u^2)^l \ , \tag{6.55}$$

where $u = \cos\theta$. Since $\Phi_{m_\psi, l, -l-k}$ has to vanish for any $k = 1, 2, \ldots$, $l \pm m_\psi$ has to be integral and positive. In particular, l can be half-integral since m_ψ can. The Laplacian of S^3 is written as

$$\Delta = L_1^2 + L_2^2 + L_3^2 \ , \tag{6.56}$$

which means the eigenvalues of the Laplacian Δ is $l(1+1)$ with $l \in \frac{\mathbb{Z}}{2}$. That is, $\frac{1}{4}L(L+2)$ with $L \in \mathbb{Z}$.

In sum, for a given quantum number (l, m) of $SU(2)$, the possible m_ψ range from $-l$ to l with spacing 1. Of course, the fact that the possible values of m_ψ are finite for a given pair of (l, m) is obvious since S^3 has actually $SO(4)$ symmetry and for a given value of the quadratic Casimir, there are finite number of states. However, the analysis we have done shows that the finiteness can be derived by using $SU(2) \times U(1)$ symmetry alone as well as the existence of a well-defined wave function. For example, we would arrive at the same conclusion even though the coefficient of $(d\psi - \cos\theta d\phi)^2$ in (6.51) were different from 1.

Let us turn to the case we are interested in. The metric is given in (6.44). There are two sets of $SU(2)$ Killing vectors $L^{(1)}_{1,2,3}$ and $L^{(2)}_{1,2,3}$ satisfying $[L^{(i)}_a, L^{(j)}_b] = \sum_c \epsilon_{abc}\delta^{ij} L^{(i)}_c$. Explicitly,

$$
\begin{aligned}
L_1^{(1)} &= \sin\phi_1 \frac{\partial}{\partial\theta_1} + \cos\phi_1 \left[\cot\theta_1 \frac{\partial}{\partial\phi_1} + \csc\theta_1 \left(\frac{\partial}{\partial\psi} + n\frac{\partial}{\partial x^-}\right)\right] , \\
L_2^{(1)} &= \cos\phi_1 \frac{\partial}{\partial\theta_1} - \sin\phi_1 \left[\cot\theta_1 \frac{\partial}{\partial\phi_1} + \csc\theta_1 \left(\frac{\partial}{\partial\psi} + n\frac{\partial}{\partial x^-}\right)\right] , \\
L_3^{(1)} &= \frac{\partial}{\partial\phi_1} , \\
L_1^{(2)} &= \sin\phi_2 \frac{\partial}{\partial\theta_2} + \cos\phi_2 \left[\cot\theta_2 \frac{\partial}{\partial\phi_2} + \csc\theta_2 \left(\frac{\partial}{\partial\psi} - n\frac{\partial}{\partial x^-}\right)\right] , \\
L_2^{(2)} &= \cos\phi_2 \frac{\partial}{\partial\theta_2} - \sin\phi_2 \left[\cot\theta_2 \frac{\partial}{\partial\phi_2} + \csc\theta_2 \left(\frac{\partial}{\partial\psi} - n\frac{\partial}{\partial x^-}\right)\right] , \\
L_3^{(2)} &= \frac{\partial}{\partial\phi_2} .
\end{aligned}
\tag{6.57}
$$

For each S^2, the only change from the analysis of S^3 is that $\frac{\partial}{\partial\psi}$ is replaced by $\frac{\partial}{\partial\psi} \pm n\frac{\partial}{\partial x^-}$. Denoting the quantum numbers for $U(1)_\psi$ and $U(1)_{x^-}$ by m_ψ and N, respectively, then we have the following constraints for given quantum numbers $(l_1, m_1; l_2, m_2)$ of $SU(2) \times SU(2)$:

$$
\begin{aligned}
-l_1 \le m_\psi + nN \le l_1 , \\
-l_2 \le m_\psi - nN \le l_2 .
\end{aligned}
\tag{6.58}
$$

In particular, N has to satisfy $|nN| \le l_1 + l_2$.

To see some implication of this result, let us consider the massive Klein-Gordon equation in eleven dimensions:

$$
\frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}g^{MN}\partial_N\Phi) - m^2\Phi = 0 . \tag{6.59}
$$

Due to the warping factor, the Laplacian becomes a little complicated. The result can be written as

$$
e^{-2\lambda}\left[-2r^2\frac{\partial^2\Phi}{\partial x^+\partial x^-} + r^2\frac{\partial^2\Phi}{\partial x_1^2} + r^2\frac{\partial^2\Phi}{\partial x_2^2} + r^5\frac{\partial}{\partial r}\left(r^{-3}\frac{\partial\Phi}{\partial r}\right) - M^2\Phi\right] = 0 . \tag{6.60}
$$

We put all y dependence except the overall factor into a function M^2, which is given by

$$\begin{aligned} -M^2 = \beta y \frac{\partial^2 \Phi}{\partial x^{-2}} - e^{2\lambda} m^2 \Phi + \frac{e^{6\lambda}}{(1-y^2)^2} \frac{\partial}{\partial y} \left[(1-y^2)^2 \cos^2 \zeta \frac{\partial \Phi}{\partial y} \right] \\ + 9 \sec^2 \zeta \frac{\partial^2 \Phi}{\partial \psi^2} + \frac{3e^{6\lambda}}{1-y^2} \left[(\Delta_1 + \Delta_2)\Phi - 2\frac{\partial^2 \Phi}{\partial \psi^2} - 2n^2 \frac{\partial^2 \Phi}{\partial x^{-2}} \right] . \end{aligned} \tag{6.61}$$

Δ_1 and Δ_2 are the Casimir operators of the two $SU(2)$ isometry groups, which are given by $\Delta_i = (L_1^{(i)})^2 + (L_2^{(i)})^2 + (L_3^{(i)})^2$ using (6.57). For a wave function with definite quantum numbers of $SU(2) \times SU(2) \times U(1)_\psi$ and definite particle number, this equation becomes an ordinary second order differential equation in y. Note that the last term in (6.61) looks problematic since, by increasing the momenta along the ψ and x^- directions, this part can be negative and large in absolute value. However, this cannot happen since the quantum numbers m_ψ and N are bounded. That is, from (6.58), we have

$$l_1(l_1+1) + l_2(l_2+1) \geq l_1^2 + l_2^2 \geq (m_\psi + nN)^2 + (m_\psi - nN)^2 = 2m_\psi^2 + 2n^2 N^2 . \tag{6.62}$$

It implies that the operator

$$\mathcal{O} = \Delta_1 + \Delta_2 - 2\frac{\partial^2}{\partial \psi^2} - 2n^2 \frac{\partial^2}{\partial x^{-2}} \tag{6.63}$$

cannot have positive eigenvalues. Therefore, the last three terms in (6.61)(multiplied by $e^{-2\lambda}$) gives positive contribution to the mass parameter M^2. That is, when β vanishes, the Kaluza-Klein mode does not suffer an instability due to the violation of the Breitenlohner-Freedman bound.

If we solve (6.61) and get the spectrum of the mass parameter M, the scaling dimensions and the correlation functions can be computed [134, 135]. Let $\nu = \sqrt{M^2 + 4}$. The scaling dimension Δ of the corresponding operator in the field theory is given by $\Delta = 2 + \nu$ and the two point correlation function of two such operators is given by

$$\langle \mathcal{O}_1(x,t) \mathcal{O}_2(0,0) \rangle \sim \delta_{\Delta_1 \Delta_2} \theta(t) \frac{1}{t^{\Delta_1}} e^{-\frac{iNx^2}{2t}} , \tag{6.64}$$

where Δ_i are the scaling dimensions of $\mathcal{O}_i$. $\Delta = 2 - \nu$ is possible if $0 < \nu < 1$ [134, 155].

6.4 Solution with No Supersymmetry

In the absence of supersymmetry, there may be many solutions with the symmetries we want. The solution given here can be thought of as a deformation of the non-supersymmetric $AdS_5 \times \mathbb{C}\mathrm{P}^3$ solution in [156]. As such, the solution here does not preserve any supersymmetry. We simply state the solution since it is straightforward to check that the solution satisfies the equations of motion. We take the lightlike direction x^- to be a non-trivial $U(1)$ bundle over the compact direction. That is, an invariant combination is $dx^- + nA$ where A is a gauge potential given below. For each integer n there is a solution. The metric is given by

$$\begin{aligned}
ds^2 =& -10n^2\frac{dx^{+2}}{r^4} + \frac{-2dx^+(dx^- + nA) + dx_1^2 + dx_2^2 + dr^2}{r^2} + \frac{1}{2}ds^2_{N_6} \ ,\\
ds^2_{N_6} =& \frac{d\alpha^2}{f(\alpha)} + f(\alpha)\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(d\chi + \cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2)^2\\
&+ \cos^2\frac{\alpha}{2}(d\theta_1^2 + \sin^2\theta_1^2 d\phi_1^2) + \sin^2\frac{\alpha}{2}(d\theta_2^2 + \sin^2\theta_2^2 d\phi_2^2) \ ,\\
A =& \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 + \cos\alpha(d\chi + \cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2) \ ,\\
f(\alpha) =& 1 - \frac{k}{\sin^4\alpha} \ .
\end{aligned} \tag{6.65}$$

The parameter k is some constant and $\theta_i \in [0,\pi]$, $\phi_i \in [0,2\pi]$ and $\chi \in [0,4\pi]$. The four-form field strength F_4 is given by

$$F_4 = \frac{\sqrt{2}}{16}\omega_2 \wedge \omega_2 + \frac{n}{\sqrt{2}}\frac{1}{r^3}dx^+ \wedge dr \wedge \omega_2 + 12n\frac{1}{r^5}dx^+ \wedge dx_1 \wedge dx_2 \wedge dr \ , \tag{6.66}$$

where $\omega_2 = dA$ is proportional to the Kähler form. N_6 is a six-dimensional compact manifold. It is a variant of $\mathbb{C}\mathrm{P}^3$: When $k = 0$, N_6 becomes $\mathbb{C}\mathrm{P}^3$. k is fixed once we require the manifold N_6 to be smooth. If $k = 0$, N_6 is smooth since it is $\mathbb{C}\mathrm{P}^3$, in which case the global symmetry is $SU(4)$. To get reduced symmetry, we want to take non-zero k. For non-zero k, since $f(\alpha)$ is supposed to be positive, α runs between the two roots of $\sin^4\alpha = k$. Calling the roots $\pm\alpha_0$, near α_0, the metric becomes

$$ds^2 = \tan\alpha_0\left[du^2 + \cos^2\alpha_0 u^2(d\chi + \cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2)^2\right] + \dots \ , \tag{6.67}$$

where $\alpha = \alpha_0 + u^2$. Since χ has period 4π, $\cos\alpha_0 = \frac{1}{2}$ to have a smooth geometry. Therefore, $k = \frac{9}{16}$ and $\alpha \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$. In this case, the surviving global symmetry is $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$. Note that this construction can be easily generalized to the case when the compact six-dimensional manifold is Kähler-Einstein.

Chapter 7

Gravity Dual of Spatially Modulated Phase

In this chapter, we show that the five-dimensional Maxwell theory with a Chern-Simons coupling in the Reissner-Nordström black hole geometry has tachyonic modes. This instability has an interesting property that it happens only at non-vanishing momenta, suggesting a spatially modulated phase transition in the holographically dual field theory.

In three dimensions, the Maxwell theory becomes massive when the Chern-Simons term is included [157, 158]. In higher dimensions, the Chern-Simons term starts with a higher power in gauge fields, but it can contribute to quadratic fluctuations if there is a non-zero background gauge field. We will show that the Maxwell theory in five dimensions with the Chern-Simons term becomes tachyonic if we turn on a constant electric field. In contrast, a background magnetic field does not cause instability, but it makes the gauge field massive as in three dimensions.[1]

Chern-Simons terms abound in supergravity theories, and charged black hole solutions in these theories provide an interesting laboratory in which to study the instability and its implications since these solutions carry background electric fields. The near-horizon geometry of the five-dimensional extremal Reissner-Nordström black hole in AdS_5 is $AdS_2 \times \mathbb{R}^3$ with the gauge field strength proportional to the volume form of AdS_2. The background electric field causes mixing of the gauge field with the metric at the quadratic order, and we will take it into account in our stability analysis. We find a critical value α_{crit} of the Chern-Simons coupling α above which the near-horizon geometry becomes unstable for some range of momenta k in $\mathbb{R}^3$. Interestingly, the range excludes $k = 0$, *i.e.*, the instability happens only at non-zero spatial momentum.

The Reissner-Nordström black hole solution in AdS_5 gives a holographic description of a thermodynamic state in the dual conformal field theory at finite temperature T and chemical potential

[1]To our knowledge, [159] is the first paper to point out that the Chern-Simons term in five dimensions induces instability. They considered a system consisting of two non-Abelian gauge fields coupled to an adjoint scalar field with a tachyonic mass as a holographic model of QCD and reduced it to four dimensions before studying its spectrum. Though their setup and analysis are different from ours, the dispersion relation we derive in section 7.1 is related to theirs. We will point this out at an appropriate place in section 7.1.

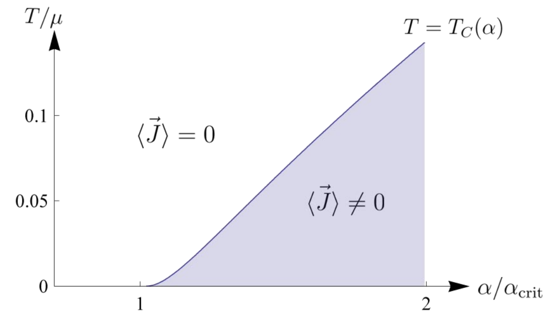

Figure 7.1: Critical temperature as a function of the Chern-Simons coupling α. The shaded region indicates a phase with a non-zero expectation value of the conserved current $\vec{J}$ which is helical and position dependent.

μ.[2] We find that, for $\alpha > \alpha_{\rm crit}$, there is a critical temperature $T_{\rm c}(\alpha)$ below which the black hole solution becomes unstable, as shown in Figure 7.1. The instability happens at a range of momenta, which becomes wider as T is lowered but never includes $k = 0$, as shown in Figure 7.2. We find an interesting subtlety in the zero temperature limit; the unstable range is wider than the range expected from the analysis near the horizon of the extremal black hole. It turns out that the near-horizon analysis gives a sufficient but not necessary condition since there are unstable modes in the full Reissner-Nordström solution which do not reduce to normalizable modes in $AdS_2 \times \mathbb{R}^3$ in the near-horizon limit.

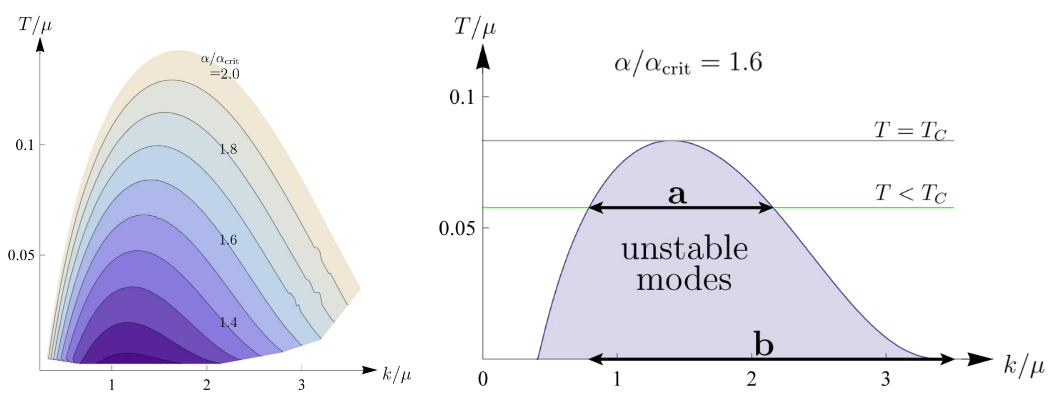

Figure 7.2: The left figure indicates unstable regions for various values of the Chern-Simons coupling α. The right figure is for a particular choice of the Chern-Simons coupling $\alpha = 1.6\alpha_{\rm crit}$. The critical temperature T_C is the maximum temperature with unstable modes. The figure indicates the unstable range **a** for some temperature $T < T_C$. The range **b** is derived from the near-horizon analysis at $T = 0$. Note that the actual range of unstable momenta is wider.

In the dual field theory in $(3 + 1)$ dimensions, the instability of the Reissner-Nordström solution can be interpreted as a signal of a novel phase transition at finite chemical potential where the charge

[2] [160, 161] studied the thermodynamic properties of the Reissner-Nordström AdS black hole. Its relation to Fermi liquid is discussed in [162].

current $\vec{J}(x)$ dual to the gauge field develops a position dependent expectation value of the form,

$$\langle \vec{J}(x) \rangle = \mathrm{Re}\left(\vec{u}e^{ikx}\right), \tag{7.1}$$

with non-zero momentum k. The constant vector $\vec{u}$ is circularly polarized as

$$\vec{k} \times \vec{u} = \pm i|k|\vec{u}, \tag{7.2}$$

where the sign is correlated to the sign of the Chern-Simons couping as we will explain later. The vacuum expectation value (7.1) is helical and breaks translational and rotational symmetries in three spatial dimensions, while preserving a certain combination of the two. The configuration reminds us of the cholesteric phase of liquid crystals.

We use the Maxwell theory with the Chern-Simons term coupled to the gravity in AdS_5 as a phenomenological model of quantum critical phenomena in the spirit of [163, 164]. To have an explicit description of the field content and interactions of the dual field theory, we need to identify a specific superstring construction where the instability takes place. We examined the simplest case of the three-charge black hole in the type IIB superstring theory on $AdS_5 \times S^5$ and found that the Chern-Simons coupling of the low energy gravity theory barely satisfies the stability bound. More specifically, when the three charges are the same, the effective Chern-Simons coupling α is only 0.4% less than the critical value α_{crit} for the instability. There is a limit of an extreme ratio of charges, where an effective α coincides with α_{crit} and the black hole becomes marginally stable.

This seems to indicate that if we survey a wider class of examples, we may be able to find a theory with a Chern-Simons coupling large enough to cause an instability. Generally speaking, the Chern-Simons coupling for a gauge field in AdS_5 is proportional to the chiral anomaly of the corresponding current in the dual conformal field theory [10]. In particular, for the type IIB superstring theory on AdS_5 times a toric Sasaki-Einstein manifold, the Chern-Simons coupling is determined by the toric data, or equivalently by the combinatorial data of the quiver diagram for the dual gauge theory [165]. It would be interesting to find an explicit example where the Chern-Simons coupling exceeds the stability bound. Or, one may try to prove that such theories are all in the Swampland [61, 166].

We should also point out that another type of instability of rotating charged black holes was suggested in [167, 168]. While the Chern-Simons term seems to play a role there, we have found no obvious connection to the instability discussed here. Effects of the bulk Chern-Simons terms on hydrodynamics of the dual field theories have been studied in [169–172]. In [173, 174], dispersion relations of hydrodynamic waves in the Reissner-Nordström geometry with the Chern-Simons term are discussed. Since the authors of these papers relied on power series expansions around $k = 0$, they did not observe the instability we found since the range of instability is away from $k = 0$ as shown in Figure 7.2.

This chapter is organized as follows. In section 7.1, we show that the five-dimensional Maxwell theory with the Chern-Simons term is unstable in the presence of a constant electric field. The metric is treated as non-dynamical in this analysis. In section 7.2, we turn on the metric fluctuation and study the stability of the near-horizon geometry of the extremal Reissner-Nordström black hole in AdS_5. In section 7.3, we generalize the analysis of section 7.2 to the full Reissner-Nordström solution. We solve the linearized equations around the black hole geometry and identify the critical temperature $T_{\rm crit}$ of the phase transition. We examine the onset of the phase transition and interpret the result from the point of view of the dual field theory. In section 7.4, we show that the three-charge black hole in the type IIB superstring theory on $AdS_5 \times S^5$ is barely stable against the type of instability we discussed.

7.1 Maxwell Theory with Chern-Simons Term

It is well known that the three-dimensional Maxwell theory with the Chern-Simons term is massive [157, 158]. The equation of motion for the 2-form field strength F is given by

$$d^*F + \alpha F = 0, \tag{7.3}$$

where α is the Chern-Simons coupling constant. Applying d^* to this equation and using the Bianchi identity $dF = 0$, one finds

$$\Box F = d^*d^*F = -\alpha d^*F = \alpha^2 F.$$

Thus, the Chern-Simons term in three dimensions induces the mass $|\alpha|$ of the gauge field.

Surprisingly, we find that the Chern-Simons term in five dimensions can turn the Maxwell theory tachyonic. In this section, we will demonstrate this by treating gravity as non-dynamical. Coupling to gravity will be studied in the following sections. Consider the following Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}\sqrt{-g}F_{IJ}F^{IJ} + \frac{\alpha}{3!}\epsilon^{IJKLM}A_I F_{JK} F_{LM}, \tag{7.4}$$

with the equation of motion,

$$\partial_J(\sqrt{-g}F^{JI}) + \frac{\alpha}{2}\epsilon^{IJKLM}F_{JK}F_{LM} = 0. \tag{7.5}$$

We use the almost positive convention for the metric g_{IJ} $(I, J = 0, ..., 4)$. Choose a background solution $F^{(0)}$ and linearize (7.5) around it by substituting $F = F^{(0)} + f$ in (7.5). The linearized equation for f is given by

$$\partial_J(\sqrt{-g}f^{JI}) + \alpha\epsilon^{IJKLM}F^{(0)}_{JK}f_{LM} = 0. \tag{7.6}$$

If $F^{(0)}$ is magnetic, this equation is similar to (7.3); the fluctuation f_{IJ} is massive and the configuration is stable. If $F^{(0)}$ is electric, on the other hand, (7.6) has tachyonic modes as we now explain.

Suppose the five-dimensional space is flat $\mathbb{R}^{1,4}$, regard it as the product $\mathbb{R}^{1,1} \times \mathbb{R}^3$, and use coordinates $(x^{\mu=0,1}, y^{i=2,3,4})$. Let us turn on a constant electric field in the x^1 direction,

$$\begin{aligned} F^{(0)}_{\mu\nu} &= E\epsilon_{\mu\nu}, \\ F^{(0)}_{\mu i} &= 0, \quad F^{(0)}_{ij} = 0. \end{aligned} \tag{7.7}$$

The equation of motion (7.6) is then,

$$\begin{aligned} &\partial^\mu f_{\mu\nu} + \partial^i f_{i\nu} = 0, \\ &\partial^\mu f_{\mu i} + \partial^j f_{ji} - 2\alpha E \epsilon_{ijk} f_{jk} = 0. \end{aligned} \tag{7.8}$$

Our ϵ-symbol convention is such that $\epsilon_{01} = 1$ and $\epsilon_{234} = 1$. By multiplying $\epsilon_{ijk}\partial_j$ to the second equation, we obtain

$$\left(\partial^\mu \partial_\mu + \partial^j \partial_j\right) f_i - 4\alpha E \epsilon_{ijk} \partial_j f_k = 0, \tag{7.9}$$

where

$$f_i = \frac{1}{2}\epsilon_{ijk} f_{jk}.$$

To derive (7.9), we used the Bianchi identities,

$$\partial_i f_{\mu j} - \partial_j f_{\mu i} = \partial_\mu f_{ij}, \quad \epsilon_{ijk}\partial_i f_{jk} = 2\partial^i f_i = 0.$$

In the momentum basis $e^{ip_\mu x^\mu + ik_i y^i}$, the operator $\epsilon_{ijk}\partial_j$ has eigenvalues $\pm k$ and 0, where $k = |\vec{k}|$. However, the eigenvalue 0 corresponds to $f_i \sim k_i$, which is prohibited by the Bianchi identity $k^i f_i = 0$. Thus, the linearized equation (7.9) gives the dispersion relation,[3]

$$\begin{aligned} (p_0)^2 - (p_1)^2 &= k^2 \pm 4\alpha E k \\ &= (k \pm 2\alpha E)^2 - 4\alpha^2 E^2. \end{aligned} \tag{7.10}$$

We find tachyonic modes in $\mathbb{R}^{1,1}$ in the range of $0 < k < 4|\alpha E|$.

[3] At this point, we should note that there is a similarity of this dispersion relation to eq. (17) of [159] if we set $m_\rho = m_{a_1}$ in the paper and interpret m_ρ^2 as being equal to $(p_1)^2$.

It is instructive to compare this with the case when we turn on a constant magnetic field,

$$F^{(0)}_{34} = -F^{(0)}_{43} = B,$$
$$F^{(0)}_{IJ} = 0 \quad \text{(otherwise)}. \tag{7.11}$$

By repeating the previous analysis, we find the dispersion relation,

$$(p_0)^2 - (p_1)^2 - (k_2)^2 = \left(\sqrt{(k_3)^2 + (k_4)^2 + 4\alpha^2 B^2} + 2|\alpha B|\right)^2.$$

In particular, when $k_3 = k_4 = 0$, the equation gives $p_0^2 - p_1^2 - k_2^2 = (4\alpha B)^2$, reproducing the topologically massive gauge field in three dimensions.

In the following sections, we will examine stability of the extremal Reissner-Nordström black hole in AdS_5. If the boundary theory is on $\mathbb{R}^{1,3}$, the near-horizon geometry of an extremal black hole takes the form $AdS_2 \times \mathbb{R}^3$ with an electric field proportional to the volume form of AdS_2. In such a configuration, the effective mass squared in AdS_2 is again given by the right-hand side of (7.10). The configuration is unstable if $-4\alpha^2 E^2$ violates the Breitenlohner-Freedman bound m^2_{BF} in AdS_2, namely,

$$4\alpha^2 E^2 > |m^2_{BF}| = \frac{1}{4r_2^2}, \tag{7.12}$$

where r_2 is the curvature radius of AdS_2. If this inequality is satisfied, the instability happens for non-zero momenta in the range,

$$2|\alpha E|\left(1 - \sqrt{1 - \frac{1}{16\alpha^2 E^2 r_2^2}}\right) < k < 2|\alpha E|\left(1 + \sqrt{1 - \frac{1}{16\alpha^2 E^2 r_2^2}}\right). \tag{7.13}$$

It is interesting to note that the zero momentum $k = 0$ is excluded from the instability range. Thus, the condensate of the gauge field happens for non-zero momentum in the $\mathbb{R}^3$ direction of the near-horizon geometry.

As we shall see in the next section, the value of α for the minimal gauged supergravity is such that $4\alpha^2 E^2$ exceeds the stability bound as in (7.12). This, however, does not mean that extremal charged black holes in the minimal gauged supergravity are unstable since we must take into account the coupling of the Maxwell field to other degrees of freedom in the supergravity theory. We will perform this analysis in the next section.

7.2 Coupling to Gravity

The background electric field causes mixing of the gauge field with the metric at the quadratic order, and it modifies the stability condition. In this section, we will study stability of the near-horizon geometry of the extremal Reissner-Nordström solution in AdS_5. It is a solution to the Maxwell theory with the Chern-Simons term coupled to the Einstein gravity with negative cosmological constant,

$$16\pi G_5 \mathcal{L} = \sqrt{-g}\left(R + \frac{12}{\ell^2} - \frac{1}{4}\ell^2 F_{IJ}F^{IJ}\right) + \frac{\alpha}{3!}\ell^3 \epsilon^{IJKLM} A_I F_{JK} F_{LM}. \tag{7.14}$$

The curvature radius r_5 of the AdS_5 solution in this theory is equal to ℓ. In the following, we will work in the unit of $\ell = 1$. This is also the Lagrangian density of the minimal gauged supergravity in five dimensions [175]. In this case, supersymmetry determines the Chern-Simons coupling α as

$$\alpha = \frac{1}{2\sqrt{3}}. \tag{7.15}$$

In this and next sections, we will treat (7.14) as a phenomenological Lagrangian with α as its parameter.

7.2.1 $AdS_2 \times \mathbb{R}^3$

Let us first consider the extremal black hole solution which is asymptotic to AdS_5 in the Poincaré coordinates. It describes the dual conformal field theory on $\mathbb{R}^{1,3}$ with non-zero chemical potential and at zero temperature. The near-horizon geometry of the extremal black hole is $AdS_2 \times \mathbb{R}^3$ with the metric

$$ds^2 = \frac{-(dx^0)^2 + (dx^1)^2}{12(x^1)^2} + d\vec{y}^2, \quad \vec{y} = (y^2, y^3, y^4). \tag{7.16}$$

Note that the curvature radius r_2 of AdS_2 is $1/\sqrt{12}$; the curvature is stronger near the horizon. The electric field strength near the horizon is proportional to the volume form of AdS_2 and is given by

$$F^{(0)}_{01} = \frac{E}{12(x^1)^2}, \quad E = \pm 2\sqrt{6}. \tag{7.17}$$

For the minimal gauged supergravity with α given by (7.15),

$$4\alpha^2 E^2 = 8 > |m^2_{BF}| = \frac{1}{4r_2^2} = 3. \tag{7.18}$$

Thus, if gravity is treated as non-dynamical, the gauge field fluctuation near the horizon violates the Breitenlohner-Freedman bound for this value of α.

We decompose the metric g_{IJ} into the background $g^{(0)}_{IJ}$ and the fluctuation h_{IJ} as $g_{IJ} = g^{(0)}_{IJ} + h_{IJ}$. The indices are raised/lowered by using the background metric. Notice that $g^{IJ} = g^{IJ(0)} - h^{IJ} + O(h^2)$ so that $g^{IJ}g_{JK} = \delta^I{}_K$. In the presence of the background electric field $F^{(0)}_{\mu\nu}$, the unstable gauge field components $f_{\mu i}, f_{ij} \neq 0$ mix with the off-diagonal elements $h_\mu^{\ i}$ of the metric perturbation through the gauge kinetic term,

$$F_{IJ}F^{IJ} = 4F^{(0)\mu\nu}h_\mu{}^i f_{\nu i} + \dots . \tag{7.19}$$

Thus, in the stability analysis, we have to take into account the mixing. One can think of $h_\mu{}^i$ as the Kaluza-Klein gauge field upon reduction on $\mathbb{R}^3$. Since we are considering a sector with non-zero momentum $\vec{k}$ along $\mathbb{R}^3$, the Kaluza-Klein gauge field on AdS_2 has mass $\vec{k}^2$.

To examine the stability of the black hole solution, we can apply the standard linear perturbation theory. In the present situation, however, there is a simpler way as we describe here. Suppose that the momentum $\vec{k}$ on $\mathbb{R}^3$ is in the y^2 direction. To derive the effective action for the Kaluza-Klein gauge field h^i_μ in AdS_2, it is convenient to reduce the Einstein action in (7.14) along the $y^{3,4}$ directions first. This gives rise to two gauge fields $(h_\mu{}^i, h_2{}^i)$ $(i = 3,4)$ on $AdS_2 \times \mathbb{R}_{y^2}$, with the effective Lagrangian

$$\begin{aligned}&\sqrt{-g^{(0)}_{5d}}\,(R+12)\\ &\to \sqrt{-g^{(0)}_{3d}}\left[-\sum_{i=3,4}\left(\frac{1}{4}K^i_{\mu\nu}K^{i\mu\nu} + \frac{1}{2}K^i_{\mu 2}K^{i\mu 2}\right) + (\text{terms not involving } h_\mu{}^i, h_2{}^i)\right],\end{aligned} \tag{7.20}$$

where the gauge field strengths are

$$K^i_{\mu\nu} = \partial_\mu h_\nu{}^i - \partial_\nu h_\mu{}^i,\ K^i_{\mu 2} = \partial_\mu h_2{}^i - \partial_2 h_\mu{}^i \quad (\mu,\nu = 0,1;\ i = 3,4).$$

Upon further reduction in the y^2 direction with momentum k, the effective Lagrangian density for the Kaluza-Klein gauge field is

$$\mathcal{L}_{eff} = -\sqrt{-g^{(0)}_{2d}}\sum_{i=3,4}\left[\frac{1}{4}K^i_{\mu\nu}K^{i\mu\nu} + \frac{1}{2}\left|\partial_\mu h_2{}^i - ikh_\mu{}^i\right|^2\right]. \tag{7.21}$$

We see that the off-diagonal elements $h_\mu{}^i$ $(i = 3,4)$ of the metric fluctuation give rise to two massive gauge fields of mass $|k|$ on AdS_2 with $h_2{}^i$ serving as the requisite Stückelberg fields.

Let us dualize the Kaluza-Klein field strength $K^i_{\mu\nu}$ on AdS_2 and write it as a function K_i times the volume form,

$$K^i_{01} = \frac{K_i}{12(x^1)^2}.$$

The equations of motion for $f_i = \frac{1}{2}\epsilon_{ijk}f_{jk}$ and K_j are derived from the Lagrangian density, which is (7.4) plus (7.21) with the coupling (7.19). They can be organized into the form,

$$\begin{aligned} &\left(\Box_{AdS_2} + \partial^j \partial_j\right) f_i - 4\alpha E \epsilon_{ijk} \partial_j f_k + E \epsilon_{ijk} \partial_j K_k = 0, \\ &E\, \Box_{AdS_2} f_i + \left(\Box_{AdS_2} + \partial^j \partial_j\right) \epsilon_{ijk} \partial_j K_k = 0. \end{aligned} \tag{7.22}$$

The effective mass m of these fields in AdS_2 can then be computed by solving

$$\det \begin{pmatrix} m^2 - k^2 - 4\alpha E k & E \\ E m^2 & m^2 - k^2 \end{pmatrix} = 0, \tag{7.23}$$

where $k = \pm|k|$. We find

$$m^2 = \frac{1}{2}\left(2k^2 + E^2 + 4\alpha E k \pm \sqrt{E^4 + 8\alpha E^3 k + 4(1+4\alpha^2)E^2 k^2}\right). \tag{7.24}$$

Minimizing m^2 with respect to k and choosing the minus sign in (7.24), we obtain the lowest value of m^2 as

$$m^2_{\min} = \frac{E^2\left(-64\alpha^6 - 24\alpha^4 + 6\alpha^2 - \left(16\alpha^4 + 4\alpha^2 + 1\right)^{3/2} + 1\right)}{2\left(4\alpha^2+1\right)^2}.$$

Substituting $E = 2\sqrt{6}$ for the near-horizon geometry, we find numerically that the lowest value of m^2 violates the Breitenlohner-Freedman bound if

$$|\alpha| > \alpha_{\rm crit} = 0.2896\ldots. \tag{7.25}$$

The value of α for the minimal gauged supergravity is

$$\alpha = \frac{1}{2\sqrt{3}} = 0.2887\ldots.$$

Thus, the supergravity theory is stable against the fluctuation of the gauge field, but barely so (with a margin less than 0.4%).

7.2.2 $AdS_2 \times S^3$

For completeness, let us consider the case when the boundary theory is on $\mathbb{R} \times S^3$. The near-horizon geometry is $AdS_2 \times S^3$. Let us denote the curvature radii of AdS_2 and S^3 by r_2 and r_3, respectively. They are related to the electric field strength E and the cosmological constant Λ, which is -6 in the

$AdS_2 \times \mathbb{R}^3$ limit, by

$$\Lambda = -\frac{1}{2r_2^2} + \frac{2}{r_3^2},$$
$$E^2 = \frac{2}{r_2^2} + \frac{4}{r_3^2}. \tag{7.26}$$

Note that, in the limit $r_3 \to \infty$, where S^3 becomes $\mathbb{R}^3$, this reproduces $E = \pm 2\sqrt{6}$ in our unit.

As in the previous case, we consider fluctuations of the metric $g_{IJ} = g_{IJ}^{(0)} + h_{IJ}$ and the gauge field $F_{IJ} = F_{IJ}^{(0)} + f_{IJ}$ from their classical values indicated by $^{(0)}$. We expand the Einstein equation,

$$R_{IJ} - \frac{1}{2} g_{IJ} R = \frac{1}{2}\left(F_{IK}F_J{}^K - \frac{1}{4} g_{IJ} F_{KL} F^{KL}\right), \tag{7.27}$$

and the Maxwell equation modified by the Chern-Simons term,

$$\sqrt{-g}\nabla_J F^{JI} + \frac{\alpha}{2}\epsilon^{IJKLM} F_{JK} F_{LM} = 0, \tag{7.28}$$

to the linear order in h_{IJ} and f_{IJ}.

The linearized equations for f_{ij} and K_i, where K_i is defined such that $K_i(\text{vol AdS}_2)_{\mu\nu} = 2\nabla_{[\mu} h_{\nu]i}$, can be written as

$$(\Box_{AdS_2} + \Delta_{S^3})\, f - 4\alpha E d^* f + E dK = 0,$$
$$E \Box_{AdS_2} f + \left(\Box_{AdS_2} + \Delta_{S^3} + \frac{4}{r_3^2}\right) dK = 0. \tag{7.29}$$

These equations are similar to (7.22), except for the last term $\frac{4}{r_3^2}$ in the second equation. Here * means the Hodge dual on S^3. Since d^* is hermitian when acting on the space of two-forms on S^3, decompose f into its eigenstate. Its eigenvalue is known to be $k = \pm(n+2)/r_3$, where $n = 0, 1, 2, \ldots$. Since $\Delta_{S^3} = -(d^*)^2$ when acting on f satisfying the Bianchi identify $df = 0$, we can set $\Delta_{S^3} = -k^2$.

The mass m on AdS_2 then satisfies the determinant equation,

$$\det \begin{pmatrix} m^2 - k^2 - 4\alpha E k & E \\ E m^2 & m^2 - k^2 + 4/r_3^2 \end{pmatrix} = 0. \tag{7.30}$$

This can be solved to obtain

$$m^2 = \frac{1}{2}\left[2k^2 + E^2 + 4\alpha E k - \frac{4}{r_3^2} \pm \sqrt{E^4 + 8\alpha E^3 k + \frac{16}{r_3^4} + \frac{32\alpha k}{r_3^2} + 4E^2\left(k^2 + 4\alpha^2 k^2 - \frac{2}{r_3^2}\right)}\right]. \tag{7.31}$$

In the limit of $r_3 \to \infty$, this reduces to the previous result (7.24).

We have numerically checked that, for a wide range of Λ and E, the Breitenlohner-Freedman bound is not violated in the minimal gauged supergravity, where $\alpha = \frac{1}{2\sqrt{3}}$. It is interesting to note that, in the limit of $\Lambda \to 0$ but with non-zero E, the lowest m^2 in (7.31) saturates the Breitenlohner-Freedman bound [176], which is

$$-\frac{1}{4r_2^2} = -\frac{1}{12}(E^2 - 2\Lambda) = -\frac{E^2}{12}. \tag{7.32}$$

7.3 Phase Transition and Critical Temperature

In the last section, we studied the instability of the near-horizon region of the extremal Reissner-Nordström solution. This gives a sufficient condition for the solution to be unstable. However, as we will see in this section, the condition turns out to be not necessary. To clarify the nature of the phase transition and identify the critical temperature, we study linear perturbation to the full Reissner-Nordström black hole in AdS_5.

7.3.1 Geometry and Equations

The Reissner-Nordström black hole has the metric

$$ds^2 = -H(r)dt^2 + \frac{1}{H(r)}dr^2 + r^2 d\vec{y}^2 \,, \qquad \vec{y} = (y^2, y^3, y^4) \,. \tag{7.33}$$

Note $\sqrt{-g^{(0)}} = r^3$. The gauge field strength is given by

$$F^{(0)} = \frac{Q}{r^3} dt \wedge dr \,. \tag{7.34}$$

The function $H(r)$ is given by

$$H(r) = r^2 \left[1 - \left(1 + \frac{\mu^2}{3r_+^2} \right) \left(\frac{r_+}{r} \right)^4 + \frac{\mu^2}{3r_+^2} \left(\frac{r_+}{r} \right)^6 \right] \,, \tag{7.35}$$

where $Q = -2\mu r_+^2$.

The equation of motion coming from the variation of the gauge field a_i is

$$\partial_\mu(\sqrt{-g^{(0)}} f^{\mu i}) + \partial_j(\sqrt{-g^{(0)}} f^{ji}) - 2\alpha \frac{Q}{r^3} \epsilon_{ijk} f_{jk} - \partial_\rho \left(\sqrt{-g^{(0)}} \frac{Q}{r^3} \epsilon^{\mu\rho} h_\mu^{\ i} \right) = 0 \,. \tag{7.36}$$

In the black hole background (7.33), it becomes

$$-\frac{r}{H(r)} \partial_t f_{ti} + \partial_r (r H(r) f_{ri}) + \frac{1}{r} \partial_j f_{ji} - 2\alpha \frac{Q}{r^3} \epsilon_{ijk} f_{jk} + Q K^i = 0 \,, \tag{7.37}$$

where $K^i = \partial_t h_r{}^i - \partial_r h_t{}^i$. By operating $\epsilon_{ijk}\partial_j$ on this equation, we obtain

$$-\frac{r}{H(r)}\partial_t^2 f_i + \partial_r(H(r)r\partial_r f_i) + \frac{1}{r}\Delta_{\mathbb{R}^3} f_i - 4\alpha\frac{Q}{r^3}\epsilon_{ijk}\partial_j f_k + Q\epsilon_{ijk}\partial_j K^k = 0\ , \tag{7.38}$$

where $f_i = \frac{1}{2}\epsilon_{ijk} f_{jk}$ and $\Delta_{\mathbb{R}^3} = \partial_{y^2}^2 + \partial_{y^3}^2 + \partial_{y^4}^2$.

To obtain the equation of motion that comes from the variation of the off-diagonal metric elements, let us use the Kaluza-Klein reduction in the presence of momentum $\vec{k}$ along the y^2 direction. The effective Lagrangian has the form

$$\mathcal{L}_{eff} = -r^3\sqrt{-g_{2d}^{(0)}}\sum_{i=3,4}\left[\frac{1}{4}r^2 K^i_{\mu\nu}K^{i\mu\nu} + \frac{1}{2}\left|\partial_\mu h_2{}^i - ikh_\mu{}^i\right|^2\right]\ . \tag{7.39}$$

The r^3 factor comes from the volume form on the $\mathbb{R}^3$ directions with coordinates $\vec{y}$. The equation of motion coming from the variation with respect to the metric is given by

$$\partial_\nu(r^5 K^{\nu\mu i}) + r^3(-ik)(\partial^\mu h_2{}^i - ikh^{\mu i}) - Q\epsilon^{\mu\rho} f_{\rho i} = 0\ . \tag{7.40}$$

Acting on the operator $\epsilon_{\alpha\mu}\partial_\beta g^{\alpha\beta(0)}\frac{1}{r^3}$, we can eliminate the term containing $\partial^\mu h_2{}^i$. Using $\epsilon_{\alpha\mu}\partial_\nu + \epsilon_{\mu\nu}\partial_\alpha + \epsilon_{\nu\alpha}\partial_\mu = 0$ in two dimensions, we obtain

$$-\frac{1}{2}\epsilon_{\mu\nu}\partial_\beta g^{\alpha\beta(0)}\frac{1}{r^3}\partial_\alpha r^5 K^{\nu\mu i} + k^2 K^i - Q\partial_\mu(g^{\mu\nu(0)}\frac{1}{r^3}f_{\nu i}) = 0\ . \tag{7.41}$$

Further operating $\epsilon_{ijk}\partial_j$ on the equation,

$$\partial_\beta[g^{\alpha\beta(0)}\frac{1}{r^3}\partial_\alpha(r^5\epsilon_{ijk}\partial_j K^k)] + \Delta^2_{\mathbb{R}^3}\epsilon_{ijk}\partial_j K^k + Q\partial_\mu(g^{\mu\nu(0)}\frac{1}{r^3}\partial_\nu f_i) = 0\ . \tag{7.42}$$

More explicitly,

$$\left(-\frac{1}{r^3 H(r)}\partial_t^2 + \partial_r H(r)\frac{1}{r^3}\partial_r\right)\left(r^5\epsilon_{ijk}\partial_j K^k + Qf_i\right) + \Delta^2_{\mathbb{R}^3}\epsilon_{ijk}\partial_j K^k = 0\ . \tag{7.43}$$

We have two sets of equations of motion (7.38) and (7.43). To simplify them, let us perform the following rescaling,

$$r \to \frac{r_+}{u}\ , \qquad t \to \frac{t}{r_+}\ , \qquad \vec{y} \to \frac{\vec{x}}{r_+}\ , \tag{7.44}$$

and make the change of variables,

$$\begin{aligned} &f_i(r) \to \phi(r), \\ &\epsilon_{ijk}\partial_j K^k \to \frac{1}{\sqrt{3}r_+^2}u^3\psi(r), \end{aligned} \tag{7.45}$$

and set $q = \frac{\mu}{\sqrt{3}r_+}$. The temperature T is

$$T = \frac{r_+}{2\pi}\left(2 - \frac{\mu^2}{3r_+^2}\right) . \tag{7.46}$$

With the rescaled variables, the Reissner-Nordström black hole is

$$ds^2 = \frac{1}{u^2}\left(-\tilde{H}(u)dt^2 + \frac{1}{\tilde{H}(u)}du^2\right) + \frac{1}{u^2}d\vec{x}^2 , \tag{7.47}$$

where

$$\tilde{H}(u) = 1 - (1+q^2)u^4 + q^2u^6 . \tag{7.48}$$

In these coordinates, the AdS_5 boundary is at $u = 0$ and the black hole horizon is located at $u = 1$.

Suppose that the fields ϕ and ψ have time dependence $e^{-i\omega t}$. Then the equations of motion for the fields ϕ and ψ give the following set of ordinary differential equations,

$$\begin{aligned} &\frac{\omega^2}{\tilde{H}(u)}\phi + u\partial_u\left(\tilde{H}(u)u^{-1}\partial_u\phi\right) - k^2\phi + 8\sqrt{3}\alpha qku^2\phi - 2qu^2\psi = 0 , \\ &\frac{\omega^2}{\tilde{H}(u)}\left(\psi - 6qu^2\phi\right) + u^{-1}\partial_u\left[\tilde{H}(u)u^3\partial_u\left(u^{-2}(\psi - 6qu^2\phi)\right)\right] - k^2\psi = 0 . \end{aligned} \tag{7.49}$$

Introducing a new function $\xi = \psi - 6qu^2\phi$, the equations can be written as

$$\begin{aligned} &\frac{\omega^2}{\tilde{H}(u)}\phi + u\partial_u(\tilde{H}(u)u^{-1}\partial_u\phi) - (k^2 + 8\sqrt{3}\alpha qku^2 + 12q^2u^4)\phi - 2qu^2\xi = 0 , \\ &\frac{\omega^2}{\tilde{H}(u)}\xi + u\partial_u(\tilde{H}(u)u^{-1}\partial_u\xi) - 6qk^2u^2\phi - (k^2 - 8u^2 - 9u^2q^2 + 12q^2u^4)\xi = 0 . \end{aligned} \tag{7.50}$$

Interestingly, the two equations can be diagonalized by a u-independent matrix. That is, for some linear combinations ϕ_1 and ϕ_2 of ϕ and ξ, we have

$$\frac{\omega^2}{\tilde{H}(u)}\phi_i(u) + u\partial_u(\tilde{H}(u)u^{-1}\partial_u\phi_i(u)) - \kappa_i(u)\phi_i(u) = 0 , \tag{7.51}$$

where $i = 1, 2$ and

$$\begin{aligned} \kappa_i(u) =& k^2 - 4\sqrt{3}\alpha kqu^2 \\ & - 2u^2\left(2 + q^2(2-6u^2) \mp \sqrt{4 + 4q^4 - 8\sqrt{3}\alpha kq - 8\sqrt{3}kq^3\alpha + q^2(8 + k^2(3+12\alpha^2))}\right) , \end{aligned} \tag{7.52}$$

where $\kappa_1(\kappa_2)$ chooses the minus (plus) sign on the right-hand side. Our numerical analysis shows that only $\phi_2(u)$ can be an unstable mode. It is related to the fact that, in the extremal limit, κ_2 corresponds to the smaller mass-squared in (7.24) in the near-horizon limit.

7.3.2 Numerical Analysis

To solve the equations of motion (7.51) numerically, we impose the in-going boundary condition near the horizon $u = 1$, and then evolve the solution to $u = 0$, the AdS_5 boundary. The asymptotic behavior of ϕ_i near $u = 0$ is either $\phi_i \sim u^2$ or constant. The former is normalizable and the latter is non-normalizable. To find normalizable modes in the full Reissner-Nordström solution, we scan the initial conditions and see when the fields vanish at $u = 0$.

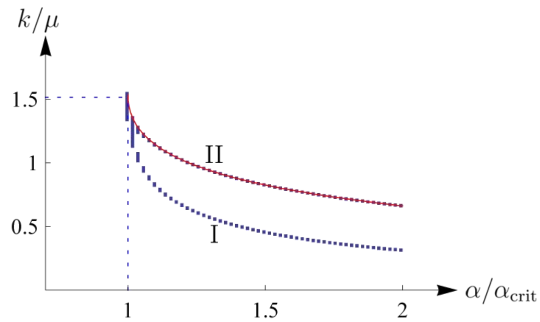

Figure 7.3: For a given value of the Chern-Simons coupling α, there is a discrete set of momenta k for which static solutions exist. The curves I and II indicate two of such momenta for each α. The red curve is the lower end of the momentum range that violates the Breitenlohner-Freedman bound near the horizon. Note that the red curve coincides with the curve II. However, there is another curve I with a lower momentum. This means that the near-horizon analysis gives a sufficient but not necessary condition for the instability. Both curves end at the same critical value of α.

First, let us consider the zero temperature limit ($q = \sqrt{2}$) and search for static solutions ($\omega = 0$), which signal the onset of an instability. The behavior of the fields near $u = 1$ can be found from (7.51) as

$$\phi_i = (1-u)^{-\frac{1}{2}+\sqrt{\frac{\kappa_i(1)+3}{12}}}(1+\ldots)\,, \tag{7.53}$$

where terms in $\ldots$ vanish at $u = 1$. In the actual numerical calculation in this section, we include several subleading terms to improve accuracy. For a given Chern-Simons coupling α, static modes appear at discrete values of momentum k. The lowest two modes are plotted in Figure 7.3. As mentioned before, only the second field ϕ_2 has normalizable static solutions.

The two curves in Figure 7.3 are denoted as I and II. Both curves terminate at $\alpha/\alpha_{\text{crit}} = 1$ and $k/\mu = 1.52\ldots$. The critical value of the Chern-Simons coupling $\alpha_{\text{crit}} = 0.2896\ldots$ is the one we found from the stability analysis of the near-horizon geometry in the previous section. The curves are supposed to extend over $k/\mu = 1.52\ldots$ and come back to the right in a bell-shaped curves. The upper branches of the curves represent the upper bounds of unstable modes. However, we have not been able to plot them due to inaccuracy of our numerical computation.

We also found out a static solution at zero momentum. However, for this solution, the curl of the off-diagonal metric component $\epsilon_{ijk}\partial_j K^k$ is constant on $\mathbb{R}^3$. This means that K^i is linear in $\mathbb{R}^3$, and the solution is not normalizable.

We note that the curve II fits with the red curve which is at the lower end of the momentum range that violates the Breitenlohner-Freedman bound in the near-horizon $AdS_2 \times \mathbb{R}^3$ geometry. As we saw in the previous section, the near-horizon geometry is unstable in this momentum range, thus the full Reissner-Nordström solution should also be unstable. In fact the momentum range that violates the Breitenlohner-Freedman bound is specified by $\kappa_2(1) < -3$, where $\phi_2(u)$ oscillates infinitely many times as they approach the horizon as can be seen from (7.53). On general ground, we expect an instability to occur in this range [177].

Interestingly, the instability condition $\kappa_2(1) < -3$ of the near-horizon geometry is not necessary for the instability of the full solution. This is because there is yet another curve I, located outside of this momentum range. What happens is that the curve I corresponds to a normalizable perturbation to the full Reissner-Nordström geometry, but the corresponding mode becomes non-normalizable in the near-horizon limit. Our numerical analysis shows that the critical Chern-Simons coupling $\alpha_{\rm crit}$ for the curve I is the same as that for the curve II, even though the value of $\alpha_{\rm crit}$ was derived from the near-horizon analysis.

To see that these static solutions indeed signal instability, let us turn on ω with positive imaginary part in (7.51). We impose the in-going boundary condition, which is

$$\phi_i = e^{-\frac{|\omega|}{12(1-u)}} (1-u)^{\frac{7}{36}|\omega|} (1+\ldots) \ , \tag{7.54}$$

in the zero temperature limit and

$$\phi_i = (1-u)^{\frac{|\omega|}{4-2q^2}} (1+\ldots) \ , \tag{7.55}$$

at a positive temperature. Figure 7.4 shows the negative frequency squared as a function of the momentum k at zero and finite temperature. It shows that the upper and lower curves in Figure 7.3 are boundaries of unstable modes.

The occurrence of instability by the Chern-Simons coupling is summarized concisely in Figure 7.1 and Figure 7.2. For each Chern-Simon coupling α, Figure 7.2 shows an unstable region in the momentum-temperature plane. This is related to the curve I in Figure 7.3. The range of unstable momenta never includes $k = 0$. The highest temperature with unstable modes is denoted as $T_C(\alpha)$. Figure 7.1 shows this critical temperature as a function of α. Below the critical temperature $T_C(\alpha)$, we expect an instability and the charge current gets a position dependent expectation value of the form of (7.1).

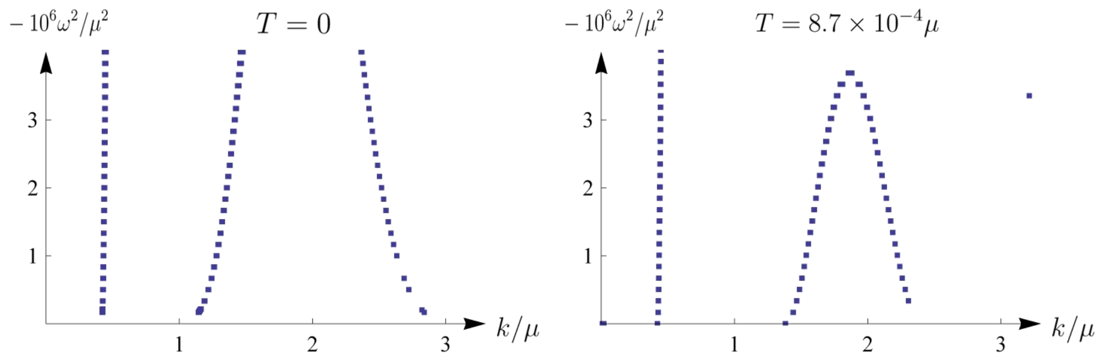

Figure 7.4: Left: Negative frequency squared as a function of momentum k at zero temperature when $\alpha = 1.6\alpha_{\text{crit}}$. Only positive $-\omega^2$ is plotted. The curves starting around 1 and 3 join to represent a tachyonic dispersion relation for the unstable mode predicted by the near-horizon analysis. The curve starting below 1 is also expected to be connected with another line in the higher momentum region to form a larger bell-shaped curve, but the large momentum part is difficult to analyze numerically. The zero momentum static solution does not extend to an unstable mode. Right: Negative frequency squared as a function of momentum at temperature $T = 8.7 \times 10^{-4}\mu$.

7.3.3 Spontaneous Current Generation

The vacuum expectation value of the current $\vec{J}$ in the dual field theory can be evaluated by extracting the asymptotic behavior of the corresponding gauge field toward the boundary of AdS_5. In the absence of the Chern-Simons term, it is well known that $\langle \vec{J} \rangle$ is given by the normalizable part of $\sqrt{-g^{(0)}} f^{ri}$ evaluated at $r \to \infty$. The normalizable mode of f_{ri} decays, but that effect is compensated by the scaling behaving of the metric so that we find a finite limiting value in the low temperature phase. The Chern-Simons term gives rise to an additional term of the form $\alpha\mu\epsilon_{ijk}f_{jk}$. However, it vanishes at the boundary and does not contribute to the expectation value. Thus, the vacuum expectation value of the current in the low temperature phase is given by $\sqrt{-g^{(0)}} f^{ri}$ evaluated at the boundary of AdS_5. It takes the form,

$$\langle \vec{J}(x) \rangle = \mathrm{Re}\left(\vec{u}e^{ikx}\right), \tag{7.56}$$

where the polarization vector $\vec{u}$ obeys

$$\vec{k} \times \vec{u} = \pm i|k|\vec{u}. \tag{7.57}$$

From the analysis in the previous section, it is clear that we should choose the plus (minus) sign when αE is positive (negative). Namely, the sign of the Chern-Simons coupling determines whether the circular polarization of the current expectation is clockwise or counterclockwise. This configuration breaks translational and rotational symmetries, but a certain combination of the two is preserved. The polarization of the current is helical and reminds us of the cholesteric phase of liquid crystals.

Since the gauge field mixes with the metric fluctuation $h_{\mu i}$ in the bulk, the corresponding component T_{0i} of the energy-momentum tensor has a non-zero expectation value at the boundary. This is expected since the non-zero current in the spatial direction means that there is a momentum density.

7.3.4 Spontaneous Breaking of Internal Symmetry

So far, we have considered the case when the gauge group in the bulk is $U(1)$. Since the $U(1)$ current commutes with itself, its expectation value does not break the $U(1)$ global symmetry on the boundary.

To realize spontaneous breaking of an internal symmetry, one possibility would be to choose the gauge group to be non-abelian. The Chern-Simons term can be written in five dimensions if there is a symmetric tensor d_{abc} in the Lie algebra, such as in $SU(n)$ with $n \geq 3$. Suppose we turn on an electric field strength in a direction T^a in the Lie algebra. According to [178, 179], the gauge kinetic term can generate instability in directions in the Lie algebra that do not commute with T^a. This breaks the symmetry homogeneously. On the other hand, the Chern-Simons term can cause a spatially modulated instability in directions where $d_{abc} \neq 0$ with T^a. The competition of the two effects would be decided by the relative strength of the gauge coupling and the Chern-Simons coupling. It would be interesting to study such an effect in a more explicit manner to identify the gravity dual of a spatially modulated phase with spontaneous breaking of an internal symmetry.

7.4 Three-Charge Black Holes in Type IIB Theory

The consistent truncation of the type IIB theory on $AdS_5 \times S^5$ to the $U(1)^3$ gauged supergravity in five dimensions was given in [180]. The bosonic action contains three gauge fields for $U(1)^3$ and three scalar fields X_1, X_2, X_3 subject to the constraint $X_1 X_2 X_3 = 1$, in addition to the metric. This low energy theory admits the three-charge black hole solutions of [181]. Here we will examine the stability of the near-horizon region of the three-charge black holes in the extremal limit.

7.4.1 Case with Equal Charges

Let us consider the case when the three charges are identical, which implies the scalar fields are constant $X_1 = X_2 = X_3 = 1$. In this case, both the Lagrangian and the black hole configuration are symmetric under exchange of the three gauge fields F_1, F_2, F_3. It is convenient to take their linear combinations as

$$\begin{aligned} F &= \frac{1}{\sqrt{3}}\left(F_1 + F_2 + F_3\right) \ , \\ F_+ &= \frac{1}{\sqrt{3}}\left(F_1 + \omega F_2 + \omega^2 F_3\right) \ , \\ F_- &= \frac{1}{\sqrt{3}}\left(F_1 + \omega^2 F_2 + \omega F_3\right) . \end{aligned} \tag{7.58}$$

They are eigenstates of the $\mathbb{Z}_3$ permutation with eigenvalues 1 and $\omega^{\pm 1}$, where $\omega = e^{2\pi i/3}$. In the black hole geometry, the $\mathbb{Z}_3$ invariant gauge field F has an electric component with $E = 2\sqrt{6}$, and $F_\pm = 0$. Similarly, fluctuations of the scalar fields from $X_1 = X_2 = X_3 = 1$ can be organized into eigenstates with eigenvalues $\omega^{\pm 1}$ under the $\mathbb{Z}_3$ permutation.

The $\mathbb{Z}_3$ invariant sector is the minimal gauged supergravity with $\alpha = 1/2\sqrt{3}$. To the quadratic order, the metric and the $\mathbb{Z}_3$ invariant gauge field do not mix with other fields. Thus, the stability analysis with respect to them is exactly the same as the one we performed in the previous section. The three-charge black hole is barely stable in this sector, being within 0.4 % of the stability bound.

Since the gauge fields $F_\pm$ have zero expectation value on the black hole geometry, the $\mathbb{R}^3$ components of these gauge fields do not couple with other degrees of freedom in the quadratic order. It is convenient to write them as

$$F_\pm = \frac{1}{\sqrt{2}}(f^{(1)} \pm i f^{(2)}).$$

With the standard normalization of their kinetic terms, the Chern-Simons term takes the form,

$$\frac{1}{8\sqrt{3}}\epsilon^{IJKLM} F_{IJ}\left(a_K^{(1)} f_{LM}^{(1)} + a_K^{(2)} f_{LM}^{(2)}\right), \tag{7.59}$$

where $a_I^{(i)}$ are the vector potentials for $f_{IJ}^{(i)}$ ($i = 1.2$). To the quadratic order, we can take the $\mathbb{Z}_3$ invariant F_{IJ} to be its background value $F_{IJ}^{(0)}$.

Since these gauge fields do not couple to other fields in the quadratic order, their linearized equations of motion are

$$\partial_J(\sqrt{-g} f^{(i)JI}) + \frac{1}{4\sqrt{3}}\epsilon^{IJKML} F_{JK}^{(0)} f_{LM}^{(i)} = 0, \quad (a = 1, 2). \tag{7.60}$$

Comparing this with (7.6), we find $\alpha = \pm 1/4\sqrt{3}$. Since $E = \pm 2\sqrt{6}$ as in the previous example, the mass squared is given by

$$-4\alpha^2 E^2 = -2. \tag{7.61}$$

It is greater than the Breitenlohner-Freedman bound ($m_{BF}^2 = -3$ in our unit), and quadratic fluctuations in these gauge fields are stable.

7.4.2 Case with Non-Equal Charges

Next, let us consider the case when the three charges are different. The five-dimensional Lagrangian is derived in [180],

$$
\begin{aligned}
&16\pi G_5 \mathcal{L} \\
&= \sqrt{-g}\left(R - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 + 4\sum_a X_a^{-1} - \frac{1}{4}\sum_a X_a^{-2}(F^a)^2\right) + \frac{1}{4}\epsilon^{IJKLM}F^1_{IJ}F^2_{KL}A^3_M \ .
\end{aligned}
\tag{7.62}
$$

X_a are functions of the two scalars ϕ_1 and ϕ_2 subject to the constraint $X_1X_2X_3 = 1$. The Lagrangian admits AdS_5 black holes parametrized by three charges q_1, q_2 and q_3. The metric is given by

$$
\begin{aligned}
&ds^2 = -(H_1H_2H_3)^{-\frac{2}{3}}h(r)dt^2 + (H_1H_2H_3)^{\frac{1}{3}}\left(\frac{dr^2}{h(r)} + r^2 d\Omega_3^2\right) \ , \\
&H_a(r) = 1 + \frac{q_a}{r^2} \ , \qquad q_a = \mu\sinh^2\beta_a \ , \qquad a = 1,2,3 \ , \\
&X_a = H_a^{-1}(H_1H_2H_3)^{\frac{1}{3}} \ , \\
&h(r) = 1 - \frac{\mu}{r^2} + r^2 H_1H_2H_3 \ , \\
&A_a = (1 - H_a^{-1})\coth\beta_a dt \ .
\end{aligned}
\tag{7.63}
$$

This is the metric whose foliating transverse space is S^3. If it is $\mathbb{R}^3$ instead, the S^3 metric $d\Omega_3^2$ is replaced by the flat metric and $h(r)$ and A^a are replaced with

$$
\begin{aligned}
h(r) &= -\frac{\mu}{r^2} + r^2 H_1H_2H_3 \ , \\
A_a &= \frac{1 - H_a^{-1}}{\sinh\beta_a}dt \ .
\end{aligned}
\tag{7.64}
$$

Given the charges q_a, it may be possible to choose μ such that the black hole becomes extremal. That is, the inner and the outer horizons coincide. For the extremal case, the near-horizon geometry is $AdS_2 \times S^3$ or $AdS_2 \times \mathbb{R}^3$: if the horizon occurs at $r = r_0$, for small $\rho = r - r_0$, $h(r) = \frac{1}{2}h''(r_0)\rho^2$. Hence the geometry becomes

$$
ds^2 = \frac{1}{a_1}\left[-\rho^2 dt^2 + \frac{d\rho^2}{\rho^2}\right] + \frac{1}{a_2}d\Omega_3^2 \ , \tag{7.65}
$$

where $a_1 = \frac{1}{2}(H_1H_2H_3)^{-\frac{1}{3}}h''$, and $a_2 = (H_1H_2H_3)^{-\frac{1}{3}}r_0^{-2}$ for S^3 and $a_2^{-1}d\Omega_3^2$ is replaced with the flat metric for $\mathbb{R}^3$. H_a and h'' are implicitly evaluated at $r = r_0$.

We want to analyze the linear fluctuations near the horizon in the extremal limit. Let $F_a = F_a^{(0)} + f_a = F_a^{(0)} + da_a$ and $g_{IJ} = g^{(0)} + h_{IJ}$. If we focus on the fluctuations of the a_i and $h_{\mu i}$

fields only, we find that the linear fluctuations of the scalar fields ϕ_1 and ϕ_2 do not couple to them. Therefore, we may use the background value of the scalar fields. In this case, we can derive the equations of motion as in the previous case, and the result is

$$
\begin{aligned}
&(\Delta_2+\Delta_3)f^1 - X_1^2 E_3 d^* f^2 - X_1^2 E_2 d^* f^3 + E_1 dK = 0 \ , \\
&- X_2^2 E_3 d^* f^1 + (\Delta_2+\Delta_3)f^2 - X_2^2 E_1 d^* f^3 + E_2 dK = 0 \ , \\
&- X_3^2 E_2 d^* f^1 - X_3^2 E_1 d^* f^2 + (\Delta_2+\Delta_3)f^3 + E_3 dK = 0 \ , \\
&\frac{E_1}{X_1^2}\Delta_2{}^* f^1 + \frac{E_2}{X_2^2}\Delta_2{}^* f^2 + \frac{E_3}{X_3^2}\Delta_2{}^* f^3 + (\Delta_2+\Delta_3+4a_2)dK = 0 \ .
\end{aligned} \tag{7.66}
$$

E_a are the electric fields such that $dA_a = E_a(H_1H_2H_3)^{-\frac{1}{6}}dt\wedge dr$. The above four equations give a mass matrix equation

$$
\det\begin{pmatrix}
m^2-k^2 & -E_3X_1^2k & -E_2X_1^2k & E_1 \\
-E_3X_2^2k & m^2-k^2 & -E_1X_2^2k & E_2 \\
-E_2X_3^2k & -E_1X_3^2k & m^2-k^2 & E_3 \\
\frac{E_1}{X_1^2}m^2 & \frac{E_2}{X_2^2}m^2 & \frac{E_3}{X_3^2}m^2 & m^2-k^2+4a_2
\end{pmatrix} = 0 \ . \tag{7.67}
$$

Solving this equation for m^2, we obtain the mass spectrum.

When two of the three charges are the same, we can analyze the mass spectrum analytically for the $AdS_2\times\mathbb{R}^3$ geometry. In this case, only the ratio of the charges matter. Let the charge assignments be $(q_a) = (1,q,q)$. Demanding $f(r_0) = f'(r_0) = 0$ at some $r = r_0$, we obtain the relation $q = x(2x+1)$ and $\mu = 4x(1+x)^3$ where $x = r_0^2$. Let us parametrize the extremal solutions in terms of x. Then the various functions at the horizon are given by

$$H_1 = x^{-1}(1+x) \ , \qquad H_2 = H_3 = 2(1+x) \ , \qquad H_1H_2H_3 = 4x^{-1}(1+x)^3 \ , \tag{7.68}$$

$$X_1 = 2^{\frac{2}{3}}x^{\frac{2}{3}} \ , \qquad X_2 = X_3 = 2^{-\frac{1}{3}}x^{-\frac{1}{3}} \ , \tag{7.69}$$

$$E_1 = 2^{\frac{7}{3}}x^{\frac{5}{6}} \ , \qquad E_2 = E_3 = 2^{\frac{1}{3}}x^{-\frac{2}{3}}(1+2x)^{\frac{1}{2}} \ , \tag{7.70}$$

$$a_1 = 2^{\frac{4}{3}}x^{-\frac{2}{3}}(1+4x) \ , \qquad \Lambda = -2^{\frac{1}{3}}x^{-\frac{2}{3}}(1+4x) \ . \tag{7.71}$$

When the two charges are the same, there is a $\mathbb{Z}_2$ symmetry exchanging the two charges. Since the gravity is insensitive to this exchange, only the combination f^2+f^3 couples to the metric component and f^2-f^3 decouples. The decoupled mode is analyzed by considering the mass matrix in (7.67) with the eigenvector $(0,1,-1,0)$ for some m^2. Due to the fact that $E_2 = E_3$ and $X_2 = X_3$, the only condition that we need to satisfy is

$$m^2 - k^2 + E_1X_2^2k = 0 \ . \tag{7.72}$$

Therefore m^2 has the minimum value when $k = \frac{E_1 X_2^2}{2}$, in which case $m^2 = -\frac{E_1^2 X_2^4}{4} = -2^{\frac{4}{3}} x^{\frac{1}{3}}$. The Breitenlohner-Freedman bound is $-\frac{a_1}{4} = -2^{-\frac{2}{3}} x^{-\frac{2}{3}}(1+4x)$. Their ratio is $\frac{4x}{1+4x}$, which is always lower than 1. That is, the mass squared is always above the bound.

Of course, it is possible that there are other modes that go below the bound. But this turns out not to be the case. To see this, let us evaluate the determinant (7.67) when the mass-squared m^2 takes the value $-\frac{1}{4}a_1$, which is the Breitenlohner-Freedman bound. Then this is a function of k and x. We can check that this function is always positive, meaning that the roots of the determinant equation, which are the possible values of the mass-squared, are all greater than the Breitenlohner-Freedman bound.

When all three charges are different, we have not been able to solve the equations analytically, so we resorted to a numerical method. Given three charges, we first adjust the parameter μ in (7.63) so that it gives an extremal black hole. Then we evaluate the metric and the functions at the horizon and solve the mass matrix equation (7.67) for m^2. In both $AdS_2 \times S^3$ or $AdS_2 \times \mathbb{R}^3$, however, no unstable modes are found for a large range of the three charges. The bound is always barely satisfied.

In this section, we studied stability of the three-charge black hole in the near-horizon limit. As we saw in the previous section, the near-horizon analysis gives a sufficient but not necessary condition for the instability at $T = 0$. However, the critical value of the Chern-Simons coupling is given correctly from the near-horizon analysis. Thus, we expect that our conclusion in this section would not be modified even if we perform the analysis in the full black hole geometry.

Chapter 8

Conclusions

In this thesis, we first discussed metastable vacua in various supersymmetric gauge theories and examined the consequence when they play a role in the construction of realistic particle physics models. Next, we applied the gauge/gravity duality technique to study strongly coupled field theories.

In Part I, we showed that there is a general method to construct a small superpotential of the scalar field in the vector multiplet in the $\mathcal{N}=2$ supersymmetric gauge theories such that it produces a metastable vacuum generically at any point in the moduli space in the Coulomb branch. The superpotential is intimately related to Kähler normal coordinates: the superpotential is a linear combination of truncated Kähler normal coordinates to some finite order. We also observe that when all orders of Kähler normal coordinates are included in the construction of the superpotential, the superpotential actually becomes a linear combination of electric and magnetic Fayet-Iliopoulos terms. Hence a previously metastable point preserves $\mathcal{N}=1$ supersymmetry if we include all higher order terms of Kähler normal coordinates. The occurrence of supersymmetry is related to the fact that Kähler normal coordinates are not globally defined in the moduli space.

It is an interesting problem to see such a construction in string or M-theory setup. There are several works [182,183] that attempt to realize the perturbed $\mathcal{N}=2$ supersymmetric gauge theories by deforming M5-branes appropriately in M-theory. Especially, in [183], it was shown that the scalar potential that results from a small superpotential perturbation of $\mathcal{N}=2$ theory can be described by the off-shell M5-brane worldvolume action. Being non-supersymmetric, the M5 configuration is non-holomorphic, but it is still a harmonic embedding and they give a geometric explanation for the stability of the M5 configuration.

In addition to developing a method to generate metastable vacua, we apply it to the construction of realistic models. Following the definition of the general gauge mediation in [49], we computed correlation functions and gaugino masses in various models. According to the definition, we are interested in the properties of the system when the gauge couplings of the observable sector are small. In that case, for computational purposes, we may replace the observable sector with some ‘spectator’ gauge theory. Then, sometimes the system can be solved even in strongly coupled regime, and we obtain correlation functions in this way.

Part II of the thesis was devoted to the study of gauge/gravity dualities. We studied the applications of gauge/gravity dualities to possible condensed matter problems. First, we consider a field theory with nonrelativistic version of conformal symmetry, called the Schrödinger symmetry. In search of a gravity dual of the nonrelativistic limit of the mass-deformed version of ABJM, we found a supergravity solution in M-theory with $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ global symmetry. However, it turns out that the number of supercharges are at most 6, which is less than the required number 14 to be a proper dual of the nonrelativistic limit of the mass-deformed ABJM theory. Also, it was difficult to introduce a term proportional to $-\frac{dx^{+2}}{r^4}$ in the metric, which makes the x^+ direction timelike. The problem is mainly that it is difficult to keep the supersymmetry with this term. However, this term was considered in the paper [141, 184] preserving some supersymmetries. A more thorough analysis on M-theory solutions with Schrödinger symmetry is done in [185], where it was shown that the most general M-theory solution with $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ global symmetry and $\mathcal{N} = 2$ super-Schrödinger symmetry cannot have more than six supercharges. As discussed in [185], the impossibility may be due to the fact that the gravity geometry may be singular, which is the case for the level $k = 1$ of ABJM theory with mass-deformation, and it cannot be resolved by taking any limit. In that case, either nonrelativistic mass-deformed ABJM theory does not exist quantum mechanically, or the duality does not work in the nonrelativistic limit. Or it is possible that the ground state of the field theory may break some of the supersymmetries present in the theory. The last possibility is supported by [186], who computed the Witten index and pointed out that the classical analysis of nonrelativistic Chern-Simons-matter theories should receive substantial quantum corrections when $N/k > 1$, which includes the parameter region where the supergravity approximation is valid.

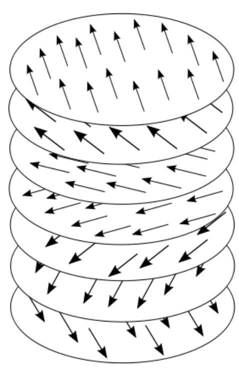

Figure 8.1: The configuration of the gauge fields in the spatially modulated phase

In the last chapter, we study five-dimensional gravity theories with Chern-Simons term, that are dual to four-dimensional gauge theories with $U(1)$ chiral anomaly. We show that if the Chern-Simons coupling α is above some critical value α_C, there exists unstable modes above some critical temperature $T_C(\alpha)$ that depends on α. The corresponding phase is spatially modulated; some of the gauge field components point along helical directions, as shown in Figure 8.1. It will be interesting to investigate the final configuration of the instability. In addition to the works in Chapter 7 related to the five-dimensional gravity theories with Chern-Simons term, there are recent studies in the same theory with the addition of the magnetic fields [187–189]. It would be interesting to include magnetic fields in our analysis and see whether a spatially modulated phase still exists, and in which condition it is favored over the uniform phase.

Appendix A

Semiclassical Consideration

Although it is very complicated to derive explicit geometric quantities for general points of the moduli space of $\mathcal{N}=2$ $SU(N)$ supersymmetric gauge theory, there are simple expressions available in the semiclassical region. This is the region where only perturbative corrections are enough. For simplicity, we consider the case without hypermultiplets. Since there is a possibility that the curvature is semipositive and the flat directions are not lifted for any choice of k_i in $W = k_i z^i$, it is useful to see that this actually does not happen in this regime. In the semiclassical approximation, we have to consider the region in which each A_i in (3.3) is different from each other to prevent enhanced gauge symmetry. The prepotential $\mathcal{F}(A)$ in (3.3) is given by, up to nonperturbative corrections [74]
,

$$\mathcal{F}(A) = \frac{N\tau_0}{2}\sum_i\left(A_i - \frac{\sum_j A_j}{N}\right)^2 + \frac{i}{4\pi}\sum_{i<j}(A_i - A_j)^2\log\frac{(A_i - A_j)^2}{\Lambda^2}\ .$$

Here A_i are the coordinates of a point p in the moduli space. Of course, they are not independent and subject to the constraint $\sum_i A_i = 0$. From this, it is straightforward to derive the various metric and their derivative components. In particular, the curvature does not vanish and is of order $O(g^4)$. Hence we see that the nonzero curvature is induced by the perturbative effects. The derivatives of the metric are given by

$$\partial_k g_{j\bar{q}} = \frac{1}{2\pi}\left(\delta_{j\bar{q}}\delta_{jk}\sum_m \frac{1}{A_j - A_m} - \frac{\delta_{j\bar{q}}}{A_j - A_k} - \frac{\delta_{jk} - \delta_{\bar{q}k}}{A_j - A_{\bar{q}}}\right)\ .$$

If we contract this with a vector w^j at p,

$$P_{k\bar{q}} = w^j\partial_k g_{j\bar{q}} = \frac{1}{2\pi}\sum_m\left(\frac{w^k - w^m}{A_k - A_m}\right)\delta_{k\bar{q}} - \frac{w^k - w^{\bar{q}}}{A_k - A_{\bar{q}}}\ , \tag{A.1}$$

where we implicitly omit terms whose denominators vanish. Note that this is precisely the expression that entered (3.11). $P_{k\bar{q}}$ in (A.1), treated as a matrix, is nonsingular at least at one value of w^j:

When $w^j = A_j$,

$$P_{k\bar{q}} = \frac{N}{2\pi}\left(\delta_{k\bar{q}} - \frac{1}{N}\right) \ ,$$

which is non-degenerate (note that the vector $(1, \ldots, 1)$ does not count). This implies $P_{k\bar{q}}$ is non-degenerate for generic choices of w^j. In (3.11), $g^{\bar{q}p}$ is positive definite. So the equality holds only when $v = 0$ for the above given w. Therefore, we can choose a superpotential to make a metastable vacuum at any point in the semiclassical regime.

Appendix B

Metastable Vacua at the Origin of the $SU(N)$ Moduli Space

In section 2, we showed how our mechanism applied to the simplest case when the gauge group is $SU(2)$. We can extend this to the more general $SU(N)$. For simplicity, we will consider the $SU(N)$ theory without hypermultiplet. Though it is hard to find an explicit form of the moduli space metric for the $SU(N)$ theory and compute its curvature, it turns out to be possible at the origin of the moduli space. This result in turn determines the normal coordinates and hence the superpotential which generates a metastable vacuum at the origin. Later, we consider a deformation of the superpotential so that it becomes a single-trace operator.

Let $u_r = \mathrm{tr}(\phi^r)$, $i = 1, \ldots, N$. These parameterize the moduli space. They become $u_r = \sum_i (a_i)^r$ at weak coupling where a_i are the expectation values of the eigenvalues of the chiral supermultiplet. It is more convenient to use the symmetric polynomials whose expressions at weak coupling are given by

$$s_r = (-1)^r \sum_{i_1 < \ldots < i_r} a_{i_1} \ldots a_{i_r}, \qquad r = 2, \ldots, N \, .$$

At strong coupling, these are defined by

$$r s_r + \sum_{\alpha=0}^{r} s_{r-\alpha} u_\alpha = 0, \qquad r = 1, 2, \ldots \, . \tag{B.1}$$

The moduli space are given by the elliptic curve [58, 59]:

$$y^2 = P(x)^2 - \Lambda^2 \, , \qquad \text{where} \qquad P(x) = \sum_{\alpha=0}^{N} s_\alpha x^{N-\alpha} \, . \tag{B.2}$$

At the origin of the moduli space, all $s_r = 0$ and $P(x) = x^N$. s_0 is defined to be 1 and $s_1 = 0$ for $SU(N)$ case.

We choose the basis cycles α_i and β_j such that their intersection form is $(\alpha_i, \beta_j) = \delta_{ij}$, $i, j =$

$1, \ldots, N-1$. Then

$$a_{Di} = \oint_{\alpha_i} \lambda \, , \qquad a_j = \oint_{\beta_j} \lambda \, , \tag{B.3}$$

where

$$\lambda = \sum_{\alpha=0}^{N-1} (N-\alpha) s_\alpha x^{N-\alpha} \frac{dx}{y} \, .$$

There is an overall constant in front of λ which can be determined by examining the classical limit. But it can be absorbed in the coefficients k_α in the superpotential $W = k_\alpha z^\alpha$. So the exact coefficient is not necessary. Since

$$\frac{\partial \lambda}{\partial s_\alpha} = -\frac{x^{N-\alpha}}{y} + d\left(\frac{x^{N+1-\alpha}}{y}\right) \, ,$$

the differentials of a_D and a are

$$\frac{\partial a_{Di}}{\partial s_\alpha} = -\oint_{\alpha_i} \frac{x^{N-\alpha}}{y}, \qquad \frac{\partial a_j}{\partial s_\alpha} = -\oint_{\beta_j} \frac{x^{N-\alpha}}{y} \, . \tag{B.4}$$

Since we are going to compute the connection and curvature at the origin, we also need expressions for multiple differentiation. Differentiating the above equation with respect to s_β,

$$\begin{aligned} \frac{\partial^2 \lambda}{\partial s_\alpha \partial s_\beta} &\simeq \frac{x^{N-\alpha}}{y^3} P(x) x^{N-\beta} dx \\ &= \sum_{\rho=0}^{N} s_\rho \frac{x^{3N-\alpha-\beta-\rho}}{y^3} dx \, , \end{aligned} \tag{B.5}$$

where $\simeq$ means equality up to exact pieces. Differentiating once more,

$$\frac{\partial^3 \lambda}{\partial s_\alpha \partial s_\beta \partial s_\gamma} \simeq \sum_{\rho=0}^{N} s_\rho \frac{-3x^{4N-\alpha-\beta-\gamma-\rho}}{y^5} P(x) dx + \frac{x^{3N-\alpha-\beta-\gamma}}{y^3} dx \, .$$

These are general expressions. Now we consider the values at the origin of the moduli space. Using the relations

$$\begin{aligned} d\left(\frac{x^{N-k}}{y}\right) &= (N-k)\frac{x^{N-k-1}}{y} dx - \frac{N x^{3N-k-1}}{y^3} dx \, , \\ d\left(\frac{x^{3N-k}}{y^3}\right) &= (3N-k)\frac{x^{3N-k-1}}{y^3} dx - \frac{3N x^{5N-k-1}}{y^5} dx \, , \end{aligned} \tag{B.6}$$

it follows that

$$\begin{aligned} \frac{\partial^2 \lambda}{\partial s_\alpha \partial s_\beta} &\simeq \frac{N-\alpha-\beta+1}{N} \frac{x^{N-\alpha-\beta}}{y} dx \, , \\ \frac{\partial^3 \lambda}{\partial s_\alpha \partial s_\beta \partial s_\gamma} &\simeq \frac{(\alpha+\beta+\gamma-2N-1)(N-\alpha-\beta-\gamma+1)}{N^2} \frac{x^{N-\alpha-\beta-\gamma}}{y} dx \, . \end{aligned} \tag{B.7}$$

When the moduli are set to the origin, the curve is given by $y^2 = x^{2N} - 1$. Here we set the scale Λ of the theory to 1. We place the branches on the unit circle as follows [190, 191]: The nth branch lies along the angle $\frac{2\pi}{N}(2n-2)$ to $\frac{2\pi}{N}(2n-1)$. The α_n cycle encloses the nth branch. The γ_n cycle runs between $n-1$ and nth branches (indices are modulo n). For example, when $N = 4$, the branches are distributed as in Figure B.1.

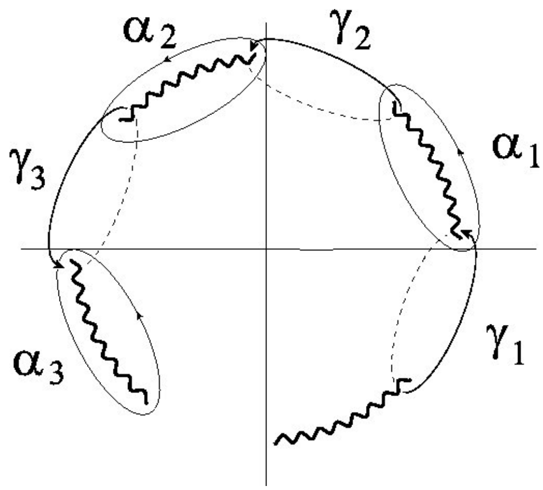

Figure B.1: The cycles α_i and γ_i, and the branches for the moduli $s_\alpha = 0$ when $N = 4$.

We choose the cycles β_n by

$$\beta_n = \sum_{i \le n} \gamma_i \ . \tag{B.8}$$

Then the intersection matrix for α_m and β_n are given by $(\alpha_m, \beta_n) = \delta_{mn}$. Since we are considering the moduli space at the origin, the periods have many relations among each other. These eventually determine all periods in terms of one function. Let us start with the period

$$\begin{aligned} \frac{\partial a_{Dm}}{\partial s_\alpha} &= -\oint_{\alpha_m} \frac{x^{N-\alpha}}{y} dx \\ &= -2 \int_{2\pi(m-1)/N}^{2\pi(m-\frac{1}{2})/N} \frac{e^{i\theta(N-\alpha)} e^{i\theta}}{\sqrt{e^{2iN\theta} - 1}} i d\theta \ . \end{aligned} \tag{B.9}$$

By changing integration variables, we get the recursion relation

$$\frac{\partial a_{Dm+1}}{\partial s_\alpha} = e^{\frac{2\pi i}{N}(N-\alpha+1)} \frac{\partial a_{Dm}}{\partial s_\alpha} \ .$$

That is,

$$\frac{\partial a_{Dm}}{\partial s_\alpha} = e^{\frac{2\pi i}{N}(N-\alpha+1)(m-1)} \frac{\partial a_{D1}}{\partial s_\alpha} \ ,$$

and the same relations hold for their differentiations with respect to s_β (resp. s_β and s_γ) by replacing α with $\alpha + \beta$ (resp. $\alpha + \beta + \gamma$). Also, an analogous result can be drawn for a by using the cycle β_n

in (B.8). Moreover, a_{D1} and a_1 are related by

$$\begin{aligned}
\frac{\partial a_1}{\partial s_\alpha} &= e^{-\frac{i\pi}{N}(N-\alpha+1)}\frac{\partial a_{D1}}{\partial s_\alpha} \ , \\
\frac{\partial^2 a_1}{\partial s_\alpha \partial s_\beta} &= e^{-i\frac{\pi}{N}(N-\alpha-\beta+1)}\frac{\partial^2 a_{D1}}{\partial s_\alpha \partial s_\beta} \ , \\
\frac{\partial^3 a_1}{\partial s_\alpha \partial s_\beta \partial s_\gamma} &= e^{-i\frac{\pi}{N}(N-\alpha-\beta-\gamma+1)}\frac{\partial^3 a_{D1}}{\partial s_\alpha \partial s_\beta \partial s_\gamma} \ ,
\end{aligned} \tag{B.10}$$

which can also be obtained by change of integration variables. So, let us define

$$h(\alpha) = 2\int_0^{\pi/N} \frac{e^{i\theta(N-\alpha+1)}}{\sqrt{1-e^{2Ni\theta}}}d\theta \ ,$$

so that

$$\frac{\partial a_{D1}}{\partial s_\alpha} = -h(\alpha) \ .$$

The rest are determined by the above relations.

The metric is given by

$$g_{\alpha\bar{\beta}} = \frac{1}{2i}\sum_j \left(\frac{\partial a_{Dj}}{\partial s_\alpha}\frac{\partial \bar{a}_j}{\partial \bar{s}_{\bar{\beta}}} - \frac{\partial a_j}{\partial s_\alpha}\frac{\partial \bar{a}_{Dj}}{\partial \bar{s}_{\bar{\beta}}} \right) \ .$$

By substitution, we reach

$$g_{\alpha\bar{\beta}} = A_{\alpha,\bar{\beta}}h(\alpha)\bar{h}(\bar{\beta}) \ ,$$

where

$$\begin{aligned}
A_{\alpha,\bar{\beta}} =& \frac{1}{2i}\sum_{i=1}^{N-1}\sum_{j=1}^{i} e^{\frac{2\pi i}{N}\left[(N-\alpha+1)(i-1)-(N-\bar{\beta}+1)(j-1)+\frac{1}{2}(N-\bar{\beta}+1)\right]} \\
&- \frac{1}{2i}\sum_{i=1}^{N-1}\sum_{j=1}^{i} e^{\frac{2\pi i}{N}\left[(N-\alpha+1)(j-1)-(N-\bar{\beta}+1)(i-1)-\frac{1}{2}(N-\alpha+1)\right]} \ .
\end{aligned} \tag{B.11}$$

The summation can be done straightforwardly. This is nonzero only when $\alpha = \bar{\beta}$ provided $\alpha, \bar{\beta} \leq N$. Evaluating when $\alpha = \bar{\beta}$, we get

$$A_{\alpha,\bar{\beta}} = \frac{N}{2\sin\frac{\pi(\bar{\beta}-1)}{N}}\delta_{\alpha,\bar{\beta}} \ .$$

When evaluating $\partial_\gamma g_{\alpha\bar{\beta}}$, we get a very similar expression but with $A_{\alpha+\gamma,\bar{\beta}}$ instead of $A_{\alpha,\bar{\beta}}$. Since $\alpha+\gamma$ can be $N+1$, in which case $A_{\alpha+\gamma,\bar{\beta}}$ is non-zero, it may cause a problem. But, fortunately, such terms do not contribute by (B.7) . $A_{\rho+\gamma+\alpha,\bar{\beta}}$ is nonzero when $\rho+\gamma+\alpha = \bar{\beta}+2N$ and we have to take this into account.

The results of the computation are summarized as follows:

$$
\begin{aligned}
g_{\alpha\bar\beta} =&\delta_{\alpha\bar\beta}\frac{N}{2\sin\frac{\pi(\bar\beta-1)}{N}}|h(\bar\beta)|^2\ ,\\
g^{\alpha\bar\beta} =&\delta_{\alpha\bar\beta}\frac{2\sin\frac{\pi(\bar\beta-1)}{N}}{N}|h(\bar\beta)|^{-2}\ ,\\
\partial_\gamma g_{\alpha\bar\beta} =&-\delta_{\gamma+\alpha,\bar\beta}\frac{N-\bar\beta+1}{2\sin\frac{\pi(\bar\beta-1)}{N}}|h(\bar\beta)|^2\ ,\\
\partial_\rho\partial_\gamma g_{\alpha\bar\beta} =&-\delta_{\rho+\gamma+\alpha,\bar\beta}\frac{(N-\bar\beta+1)(\bar\beta-2N-1)}{2N\sin\frac{\pi(\bar\beta-1)}{N}}|h(\bar\beta)|^2\\
&+\delta_{\rho+\gamma+\alpha,\bar\beta+2N}\frac{(\bar\beta-1)^2}{2N\sin\frac{\pi(\bar\beta-1)}{N}}|h(\bar\beta)|^2\ ,\\
\partial_{\bar\delta}\partial_\gamma g_{\beta\bar\rho} =&\delta_{\beta+\gamma,\bar\rho+\bar\delta}\frac{(N-\beta-\gamma+1)^2}{2N\sin\frac{\pi(\beta+\gamma-1)}{N}}|h(\beta+\gamma)|^2\ ,\\
\Gamma^\alpha_{\ \beta\gamma} =&g^{\alpha\bar\delta}\partial_\beta g_{\gamma\bar\delta}=-\delta_{\alpha,\beta+\gamma}\frac{N-\alpha+1}{N}\ ,\\
R^\alpha_{\ \beta\gamma\bar\delta} =&g^{\alpha\bar p}g^{q\bar p}\partial_{\bar\delta}g_{q\bar p}\partial_\gamma g_{\beta\bar\rho}-g^{\alpha\bar\rho}\partial_{\bar\delta}\partial_\gamma g_{\beta\bar\rho}\\
=&\begin{cases}-\delta_{\alpha+\bar\delta,\beta+\gamma}\frac{(N-\alpha-\bar\delta+1)^2}{N^2}\left|\frac{h(\alpha+\bar\delta)}{h(\alpha)}\right|^2\frac{\sin\frac{\pi(\alpha-1)}{N}}{\sin\frac{\pi(\alpha+\bar\delta-1)}{N}} & \text{for } \alpha+\bar\delta>N\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
\tag{B.12}
$$

Let us try $W=\lambda s_\alpha$ as our starting superpotential where λ is a small coupling constant. Due to the curvature formula, $R^\alpha_{\ \alpha\beta\bar\gamma}$ for fixed α is a diagonal matrix with some zeroes on the diagonal unless $\alpha=N$. Hence the only plausible case is $W=\lambda s_N$. In this case,

$$
R^N_{\ N\gamma\bar\delta}=\delta_{\gamma\bar\delta}\frac{(\bar\delta-1)^2}{N^2}\left|\frac{h(N+\bar\delta)}{h(N)}\right|^2\frac{\sin\frac{\pi}{N}}{\sin\frac{\pi(\bar\delta-1)}{N}}\ , \tag{B.13}
$$

which is manifestly positive-definite.

The correction we need to add to make a normal coordinate is given by (3.5). Using the following values

$$
\begin{aligned}
g_{N\bar N} &= \frac{N}{2\sin\frac{\pi}{N}}|h(N)|^2\ ,\\
\partial_\alpha g_{N-\alpha,\bar N} &= -\frac{1}{2\sin\frac{\pi}{N}}|h(N)|^2\ ,\\
\partial_\alpha\partial_\beta g_{N-\alpha-\beta,\bar N} &= \frac{1}{2\sin\frac{\pi}{N}}\frac{N+1}{N}|h(N)|^2\ ,\\
\partial_N\partial_N g_{N,\bar N} &= \frac{1}{2\sin\frac{\pi}{N}}\frac{(N-1)^2}{N}|h(N)|^2\ ,
\end{aligned}
\tag{B.14}
$$

we have

$$W = \lambda z^N = \lambda \left(s_N - \frac{1}{2N} \sum_{\alpha+\beta=N} s_\alpha s_\beta + \frac{N+1}{6N^2} \sum_{\alpha+\beta+\gamma=N} s_\alpha s_\beta s_\gamma + \frac{(N-1)^2}{6N^2} (s_N)^3 \right) . \tag{B.15}$$

In the case $N = 2$, we have $W = \lambda u + \frac{1}{24}\lambda u^3$, $u = -s_2$, which is the superpotential that we used to check the metastability for $SU(2)$.

B.1 Deformation to a Superpotential with Single-trace Terms

The superpotential (B.15) is not a sum of u_r where $u_r = \mathrm{tr}(\phi^r)$. Actually, s_α is given by the implicit relation (B.1) and there are quadratic and cubic terms in s in (B.15). For $N = 2$ and 3, the superpotentials are already of single-trace type because we have few independent coordinates $(s_2, \ldots, s_N)$. For $N = 2$, it is trivial. Let us consider $N = 3$. Here, all $\partial_\alpha g_{\beta,\bar{3}}$ vanish since $\alpha + \beta = 3$ cannot be satisfied both being greater than or equal to 2. Considering other terms also similarly, the only terms we get are s_3 and $(s_3)^3$. $s_3 = -u_3/3$ and $(s_3)^3 = -u_9/3$ up to cubic orders of u_2 and u_3. But, for large N, this does not work. So we have to consider a deformation.

We will first consider a general deformation and apply this to our case. Given a superpotential $W = k_\alpha z^\alpha$, consider a deformation of the form

$$W = k_\alpha z^\alpha + \frac{\alpha_{\alpha\beta}}{2} z^\alpha z^\beta + \frac{\beta_{\alpha\beta\gamma}}{3} z^\alpha z^\beta z^\gamma . \tag{B.16}$$

We may add quartic or higher degree terms in z. This will not change the local behavior of the leading potential near p, however. From the inverse metric (3.6), the leading effective potential is given by

$$\begin{aligned} V &= \left(g^{\alpha\bar{\delta}} + R^{\alpha\bar{\delta}}{}_{\rho\bar{\lambda}} z^\rho \bar{z}^{\bar{\lambda}} \right) \left(k_\alpha + \alpha_{\alpha\beta} z^\beta + \beta_{\alpha\beta\gamma} z^\beta z^\gamma \right) \left(\bar{k}_{\bar{\delta}} + \bar{\alpha}_{\bar{\delta}\bar{\beta}} \bar{z}^{\bar{\beta}} + \bar{\beta}_{\bar{\delta}\bar{\beta}\bar{\gamma}} \bar{z}^{\bar{\beta}} \bar{z}^{\bar{\gamma}} \right) \\ &= k_\alpha \bar{k}^\alpha + \alpha_{\alpha\beta} \bar{k}^\alpha z^\beta + \bar{\alpha}_{\bar{\alpha}\bar{\beta}} k^{\bar{\alpha}} \bar{z}^{\bar{\beta}} + \beta_{\alpha\beta\gamma} \bar{k}^\alpha z^\beta z^\gamma + \bar{\beta}_{\bar{\delta}\bar{\beta}\bar{\gamma}} k^{\bar{\delta}} \bar{z}^{\bar{\beta}} \bar{z}^{\bar{\gamma}} \\ &\quad + \left(k_\rho \bar{k}_{\bar{\lambda}} R^{\rho\bar{\lambda}}{}_{\beta\bar{\gamma}} + \alpha_{\alpha\beta} \bar{\alpha}^\alpha{}_{\bar{\gamma}} \right) z^\beta \bar{z}^{\bar{\gamma}} + O(z^3) , \end{aligned} \tag{B.17}$$

where $g^{\alpha\bar{\beta}}$ and $g_{\alpha\bar{\beta}}$ are used to raise and lower indices. All tensors are evaluated at the origin. If we demand a deformation leave the local minimum invariant, $\alpha_{\alpha\beta}$ should satisfy

$$\alpha_{\alpha\beta} \bar{k}^\beta = 0 . \tag{B.18}$$

Given such $\alpha_{\alpha\beta}$, (B.17) becomes

$$V = k_\alpha \bar{k}^\alpha + M_{\alpha\bar{\beta}} z^\alpha \bar{z}^{\bar{\beta}} + L_{\alpha\beta} z^\alpha z^\beta + \bar{L}_{\bar{\alpha}\bar{\beta}} \bar{z}^{\bar{\alpha}} \bar{z}^{\bar{\beta}} , \tag{B.19}$$

where $M_{\alpha\bar{\beta}} = k_\rho \bar{k}_{\bar{\delta}} R^{\rho\bar{\delta}}{}_{\alpha\bar{\beta}} + \alpha_{\gamma\alpha}\bar{\alpha}^{\gamma}{}_{\bar{\beta}}$ and $L_{\alpha\beta} = \bar{k}^\gamma \beta_{\gamma\alpha\beta}$. The second term is positive definite, so it tends to give a local minimum at p. But the last two terms develop tachyonic directions. So, roughly, when $\beta_{\gamma\alpha\beta}$ is smaller than the order of $k_\rho R^\rho{}_{\gamma\alpha\beta}$ schematically, we have a metastable minimum.

We now consider a specific deformation. Note that the last term of (B.15) can be converted into $-\frac{(N-1)^2}{6N^5}(u_N)^3$ to cubic order, and this is

$$-\frac{(N-1)^2}{6N^3} u_{3N} \ ,$$

to the same order.[1] So the last term is fine. That is, if we express u_{3N} in terms of $u_2, \ldots, u_N$, we have a term $\frac{1}{N^2}(u_N)^3$, but all other terms are of quartic and higher orders.

To deform the first three terms of (B.15), note first that, from (B.1),

$$\begin{aligned} u_N &= -N s_N - \sum_{\alpha=1}^{N-1} s_{N-\alpha} u_\alpha \\ &= -N s_N - \sum_{\alpha=1}^{N-1} s_{N-\alpha}(-\alpha s_\alpha - \sum_{\beta=1}^{\alpha-1} s_{\alpha-\beta} u_\beta) \\ &= -N s_N + \sum_{\alpha=1}^{N-1} \alpha s_{N-\alpha} s_\alpha - \sum_{\alpha=1}^{N-1}\sum_{\beta=1}^{\alpha-1} \beta s_{N-\alpha} s_{\alpha-\beta} s_\beta + O(s^4) \ . \end{aligned} \tag{B.20}$$

Therefore,

$$\frac{1}{N} u_N = -s_N + \frac{1}{2}\sum_{\alpha+\beta=N} s_\alpha s_\beta - \frac{1}{3}\sum_{\alpha+\beta+\gamma=N} s_\alpha s_\beta s_\gamma + O(s^4) \ . \tag{B.21}$$

We can invert (3.5) and get

$$s^\rho = z^\rho - \frac{1}{2} g^{\rho\bar{\alpha}} \partial_\beta g_{\gamma\bar{\alpha}} z^\beta z^\gamma + O(z^3) \ . \tag{B.22}$$

We will consider a superpotential

$$W = \lambda \left(\frac{1}{N} u_N + \frac{(N-1)^2}{6N^3} u_{3N} \right) \ , \tag{B.23}$$

for small coupling constant λ. Note that we have set the scale Λ of the theory to 1.

This is indeed a sum of single-trace operators. We will see that this superpotential produces a metastable vacuum at the origin. Note that $(s_N)^3 = -\frac{1}{N} u_{3N} + O(s^4)$. Using (B.15), (B.21), and

[1] Actually, the chiral ring is modified due to instantons as discussed in appendix A of [192]. We will discuss this effect later.

(B.22), we can express W in terms of z^α:

$$\begin{aligned}-\lambda^{-1}W =&z^N + \frac{1-N}{2N}\sum_{\alpha+\beta=N} z^\alpha z^\beta \\ &+\left(\frac{1}{3}-\frac{N+1}{6N^2}\right)\sum_{\alpha+\beta+\gamma=N} z^\alpha z^\beta z^\gamma - \sum_{\alpha+\delta=N} g^{\delta\bar\rho}\partial_\gamma g_{\beta\bar\rho} z^\alpha z^\beta z^\gamma + O(z^4)\ .\end{aligned} \tag{B.24}$$

Referring to (B.16), the deformation corresponds to

$$\begin{aligned}-\lambda^{-1}\alpha_{\alpha\beta} &= \frac{1-N}{N}\delta_{\alpha+\beta,N} \\ -\lambda^{-1}\beta_{\alpha\beta\gamma} &= \left(1-\frac{N+1}{2N^2}\right)\delta_{\alpha+\beta+\gamma,N} - 3\sum_{\delta,\bar\rho}\delta_{N-\delta,(\alpha} g^{\delta\bar\rho}\partial_\gamma g_{\beta)\bar\rho}\ ,\end{aligned} \tag{B.25}$$

where $(\cdots)$ in indices denotes symmetrization.

When we deform the superpotential W according to (B.16), the tree level potential is given by (B.17) . From this, we see that deformations given by $\alpha_{\alpha\beta}$ and $\beta_{\alpha\beta\gamma}$ such that

$$\begin{aligned}\alpha_{\alpha\beta}\bar k^\beta &= 0\ , \\ \beta_{\alpha\beta\gamma}\bar k^\gamma &= 0\end{aligned} \tag{B.26}$$

leave the metastable vacuum at the origin of the effective potential. Since the metric is diagonal at the origin, these amount to requiring $\alpha_{\alpha N} = \beta_{\alpha\beta N} = 0$. Note that $g^{\delta\bar\rho}$ vanish unless $\delta = \bar\rho$ and $\partial_\gamma g_{\beta\bar\rho}$ vanish unless $\gamma+\beta = \bar\rho$. Considering all combinations of indices, $\alpha_{\alpha N} = \beta_{\alpha\beta N} = 0$.

As noted before, we also have instanton corrections on the chiral ring. Quantum mechanically [192],

$$u_{3N} = \sum_{m=0}^{1}\begin{pmatrix}2m\\ m\end{pmatrix}\Lambda^{2Nm}\frac{1}{2\pi i}\oint_C z^{3N}\frac{P'(z)}{P(z)^{2m+1}}dz\ ,$$

where C is a large contour around $z=\infty$.

The $m=0$ term gives the classical relation, i.e., $u_{3N} = \frac{1}{N^2}(u_N)^3 + O(u^4)$. The $m=1$ term gives the instanton correction. This changes the coefficients $\alpha_{\alpha\beta}$ and $\beta_{\alpha\beta\gamma}$. However, the relations (B.26) are still satisfied since the additional contribution to $\alpha_{\alpha\beta}$ (resp. $\beta_{\alpha\beta\gamma}$) occurs only when $\alpha+\beta = N$ (resp. $\alpha+\beta+\gamma = N$). We conclude that the superpotential (B.23) gives a metastable vacuum at the origin.

B.2 Large N Behavior

The first and the second terms of (B.23) are both of order N^{-1} when expressed in terms of s_α. Hence we may consider a deformation that eliminates the second term. This turns out not to be possible.

Since λ is just an overall coefficient, we can set it to -1 in the following discussion. Note that u_{3N} is $-N(z^N)^3$ in z^α coordinates to cubic order. Hence (B.25) change to

$$\begin{aligned}
\alpha_{\alpha\beta} &= \frac{1-N}{N}\delta_{\alpha+\beta,N} \ , \\
\beta_{\alpha\beta\gamma} &= \left(1-\frac{N+1}{2N^2}\right)\delta_{\alpha+\beta+\gamma,N} - 3\delta_{N-\delta,(\alpha}g^{\delta\bar{\rho}}\partial_\gamma g_{\beta)\bar{\rho}} - \frac{(N-1)^2}{2N^2}\delta_{\alpha,N}\delta_{\beta,N}\delta_{\gamma,N} \ .
\end{aligned} \tag{B.27}$$

Since we have shown that $\alpha_{\alpha N} = \beta_{\alpha\beta N} = 0$ were it not for the additional term, we have

$$\beta_{\alpha\beta N} = -\frac{(N-1)^2}{2N^2}\delta_{\alpha,N}\delta_{\beta,N} \ .$$

Then $L_{\alpha\beta} = g^{N\bar{N}}\beta_{\alpha\beta N}$ in (B.19) are all zero except when $\alpha = \beta = N$ and

$$L_{NN} = -\frac{(N-1)^2}{N^3}\frac{\sin\frac{\pi}{N}}{|h(N)|^2} \ .$$

Since $h(N) \sim 1/N$, this scales like N^0 for large N. But $M_{\alpha\bar{\beta}}$ are given by, using (B.13),

$$\begin{aligned}
M_{\alpha\bar{\beta}} &= R^{N\bar{N}}{}_{\alpha\bar{\beta}} + \alpha_{\gamma\alpha}\bar{\alpha}^{\gamma}{}_{\bar{\beta}} = g^{N\bar{N}}R^{N}{}_{N\alpha\bar{\beta}} + g^{\gamma\bar{\delta}}\alpha_{\gamma\alpha}\bar{\alpha}_{\bar{\delta}\bar{\beta}} \\
&= \delta_{\alpha\bar{\beta}}\frac{2(\bar{\beta}-1)^2}{N^3}\frac{\left|h(N+\bar{\beta})\right|^2}{\left|h(N)\right|^4}\frac{(\sin\frac{\pi}{N})^2}{\sin\frac{\pi(\bar{\beta}-1)}{N}} \\
&\qquad + \left(\frac{1-N}{N}\right)^2\sum_{\gamma}\frac{2\sin\frac{\pi(\gamma-1)}{N}}{N}\left|h(\gamma)\right|^{-2}\delta_{\alpha+\gamma,N}\delta_{\bar{\beta}+\gamma,N} \ .
\end{aligned} \tag{B.28}$$

$M_{\alpha\bar{\beta}}$ are diagonal and the second term vanishes when $\alpha = \bar{\beta} = N$. Therefore,

$$M_{N\bar{N}} = \frac{2(N-1)^2}{N^3}\frac{\left|h(2N)\right|^2}{\left|h(N)\right|^4}\sin\frac{\pi}{N} \ .$$

Since $h(2N)$ scales like N^{-2}, $M_{N\bar{N}}$ scales like N^{-2}. Since L_{NN} introduces saddle point behavior at the origin along the N-th direction and $M_{N\bar{N}}$ is not large enough to lift it, the metastability could not be maintained in the large N limit if we did not include the second term of (B.23) . Actually, $L_{NN}/M_{N\bar{N}} > 1/2$ for all N(we have explicit formulae), so we cannot remove the second term of (B.23) for any N.

The components of (B.13) is of order N^{-2} when $\gamma = \delta$ is near N and of order N^{-1} when $\gamma = \delta$ is near $N/2$. So although the metastable vacuum at the origin persists for any finite N, the mechanism to make metastable vacua using the curvature becomes harder and harder to implement as N increases in the current setup.

Appendix C

Notation for the Eleven-dimensional Supergravity

We mostly follow the notation of [193]. In the supergravity approximation of M theory, the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2\kappa_{11}^2}\left[\int d^{11}x\sqrt{-g}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4\right] . \tag{C.1}$$

The quantity $|F_p|^2$ is defined by

$$|F_p|^2 = \frac{1}{p!} g^{M_1 N_1} \dots g^{M_p N_p} F_{M_1 \dots M_p} F_{N_1 \dots N_p} . \tag{C.2}$$

Indices $M, N, \dots$ run from 1 to 11 and denote coordinate indices. The metric is mostly positive. The vielbein indices are denoted by $A, B, \dots$. The equation of motion for A_3 is

$$\begin{aligned} dF_4 &= 0 , \\ d * F_4 + \frac{1}{2} F_4 \wedge F_4 &= 0 . \end{aligned} \tag{C.3}$$

The equation of motion for the metric g_{MN} is

$$G_{MN} = \kappa_{11}^2 T_{MN} , \tag{C.4}$$

where G_{MN} is the Einstein tensor, and

$$T_{MN} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{A_3}}{\delta g^{MN}} , \tag{C.5}$$

where S_{A_3} denotes the part of the action excluding the Ricci scalar term. Explicitly,

$$T_{MN} = -\frac{1}{4\kappa_{11}^2}\left(\frac{1}{4!} g_{MN} F^{M_1 \dots M_4} F_{M_1 \dots M_4} - \frac{2}{3!} F_{M M_1 M_2 M_3} F_N{}^{M_1 M_2 M_3}\right) . \tag{C.6}$$

In terms of the gamma matrices,

$$T_{MN} = -\frac{1}{4\kappa_{11}^2}\frac{1}{32}\mathrm{Tr}(\Gamma_M \mathbf{F}^{(4)} \Gamma_N \mathbf{F}^{(4)}) \ . \tag{C.7}$$

where $\mathbf{F}^{(4)} = \frac{1}{4!} F_{MNPQ} \Gamma^{MNPQ}$. The eleven gamma matrices satisfy the relation

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} \ , \tag{C.8}$$

where $\eta^{AB} = \mathrm{diag}(-1, 1, \ldots, 1)$.

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