PERMUTATION DECOMPOSITIONS OF (0, 1)-MATRICES AND DECOMPOSITION TRANSVERSALS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California 91109

1971

(Submitted August 24, 1970)

ACKNOWLEDGMENTS

I wish to thank Professor H. J. Ryser for introducing me to the subject of Combinatorial Mathematics, for the many hours he has listened to my ideas, and particularly for his help in the preparation of this thesis. I also wish to thank my wife Helen for her patience during the time I have been a graduate student.

For their financial support of this research I would like to thank the U. S. Army Research Office (Durham) and also the Mathematics Departments of Syracuse University and California Institute of Technology for teaching assistantships.

ABSTRACT

The central problem of this thesis is the study of sums of disjoint partial permutation matrices ("permutation decompositions"). This problem has as its origin the result of G. Birkoff that an order n (0,1)-matrix having k 1's in every row and column can be written as a sum of k permutation matrices (partial permutation matrices of "size" and order n).

The thesis divides into two main parts. In the first part (Chapters II, III) we first deal with the existence of permutation decompositions of a given (0,1)-matrix where each of the summands has a specified size and secondly, with some applications consisting of reformulating certain identification problems of Combinatorics in terms of permutation decompositions. The general existence problem remains unsolved. For more than two distinct sizes in the proposed permutation decomposition of a (0,1)-matrix A, a more subtle invarient than numbers of 1's in submatrices of A is required.

The second part of this thesis is concerned with "transversals" of permutation decompositions. The specific goal is to make some contribution toward resolving the conjecture of H. J. Ryser that every odd order latin square has a "transversal". Chapter IV is preliminary, and deals with "generalized traces" of 3-dimensional (0,1)-matrices. A more fruitful approach is considered in Chapter V. There the conjecture of Ryser is generalized and the apparently central concept of a "square" n-tuple of positive integers is introduced. Such square "lists"

are characterized in terms of tournament score vectors. A weaker structure than a latin square, that of a "pair configuration", is also introduced and for such structures the concept of a square list is more intimately connected with the existence of a "transversal". The generalized conjecture is proven only in special cases.

TABLE OF CONTENTS

I. INTRODUCTION AND GENERAL TERMINOLOGY

An n-tuple of positive integers will be called a list (of length n). If in a particular case 0 entries are allowed, then such a list will be called a non-negative list. A permutation matrix of size p is a (0,1)-matrix containing p 1's, no two of which are in the same line (row or column). A matrix sum

$$A = P_1 + P_2 + \dots + P_\ell \qquad (1.1)$$

where A is a (0,1)-matrix and the P_i are permutation matrices will be called a permutation decomposition of A. The integer array

$$L = 1 \cdot P_1 + 2 \cdot P_2 + \dots + \ell \cdot P_\ell \qquad (1.2)$$

will be referred to as a partial latin square. No implicit assumption that such a partial latin square can be completed to a latin square is to be read into this definition [1]. Let p_i be the size of P_i in (1.1) or (1.2). The permutation decomposition (1.1) and the partial latin square (1.2) will both be said to have list $(p_1, p_2, \dots, p_n)$. Such a list will also be called feasible for the (0,1)-matrix A when A has such a decomposition (1.1). In the second chapter we will consider the problem of determining all feasible lists for a given (0,1)-matrix.

Certain natural questions which can be asked of a given (0,1)-matrix A, e.g., if it is permutable into circulant form, can be reformulated equivalently by asking whether A has a permutation decomposition of a particular form. Some such considerations will be found in Chapter III. The concept of a "generalized trace" of a 3-dimensional

(0, 1)-matrix will be introduced in Chapter IV. Although this concept is related to problems on the existence of "transversals" of a decomposition (1.1), it does not appear to be the most fruitful approach to such questions. A conjecture on the existence of a "transversal" for each of the permutation decompositions in certain (large) classes will generalize the conjecture of Ryser [15, p. 72] that every latin square of odd order has a transversal. Material concerning this generalization forms the substance of Chapter V.

II. EXISTENCE OF DECOMPOSITIONS WITH SPECIFIED LIST

As will quickly become apparent, the problem of determining all feasible lists of a given (0, 1)-matrix (in terms of some other invarients of the matrix) increases in complexity with the number of distinct integers appearing in the list $(p_1, p_2, \ldots, p_\ell)$. The simplest case is that of an order n (0, 1)-matrix and list $(n, n, \ldots, n)$. It is resolved by the following well-known theorem.

<u>THEOREM 2.1</u> (G. Birkoff)

An order n (0, 1)-matrix is a sum of ℓ size n permutation matrices iff every line of A has sum ℓ.

The condition that every line sum of A is = ℓ is clearly necessary and the sufficiency of this condition can be proved in several ways, e.g., [13, p. 57]. Since $\frac{1}{\ell}A$ is doubly stochastic it follows that Theorem 2.1 is a special case of Birkoff's Theorem for double stochastic matrices, namely that they form the convex hull of the permutation matrices of that order. Alternatively, Theorem 2.1 can easily be derived from (and is essentially equivalent to):

<u>THEOREM 2.2</u> (Kőnig-Egerváry)

The maximal number of 1's, no 2 in a line, in a (0, 1)-matrix, is equal to the minimum number of lines containing all the 1's.

In turn, this is equivalent to the "Max-Flow Min-Cut" Theorem of Ford and Fulkerson and to the "labelling process" of the Theory of

Flows in Networks [3]. Because of these and other essentially equivalent formulations, theorems of the general nature of Theorems 2.1, 2.2, 2.3, etc. can be proven in many settings. With the necessity of choosing one we will prove Theorems 2.4, 2.7 by using the "Max-Flow Min-Cut" Theorem which will be started after certain notation is introduced. The advantages of using this theorem are in the brevity of proofs and in the "standard form" that such proofs take. The theorems to follow constitute a selection and by no means exhaust the use of the Max-Flow Min-Cut Theorem in "list-type" problems.

The following notation and terminology is that of [3]. An (undirected) graph is a finite set of elements (nodes) together with a collection of 2-subsets of these nodes (arcs between the nodes). Nodes x, y are adjacent if the 2-subset $\{x, y\}$ is an arc of the graph. In an undirected graph such an arc will be denoted either (x, y) or (y, x) whereas in a directed graph exactly one of (x, y), (y, x) holds for every arc $\{x, y\}$. We introduce a function (capacity function) whose domain is the set of all arcs and whose range is the non-negative real numbers. Its value on the arc (x, y) will be denoted $c(x, y)$. A directed graph having two distinguished nodes, s("source") and t ("sink") and having a capacity function c defined on all arcs will henceforth be termed a network. A flow of value v from s to t in a network $\mathcal{N}$ is a non-negative real valued function f, defined on all arcs of $\mathcal{N}$ (a typical value being denoted $f(x, y)$) satisfying:

$$\sum \{f(x,y) - f(y,x)\} = \begin{cases} v, & x = s \\ 0, & x \neq s \text{ or } t \\ -v, & x = t \end{cases} \tag{2.1}$$

$$f(x,y) \leq c(x,y) \tag{2.2}$$

The sum in (2.1) is over all y for which there is an arc (x, y) or (y, x) with the convention that if (x, y) isn't an arc of $\mathcal{N}$, then $f(x, y) = 0$. Inequality (2.2) is to hold for all arcs $(x, y) \in \mathcal{N}$. If g is a real-valued function defined on the arcs of $\mathcal{N}$ and if X, Y are collections of nodes of $\mathcal{N}$, then by (X, Y) we will mean the totality of all (directed) arcs (x, y) with $x \in X$ and $y \in Y$. Also $g(X, Y)$ denotes $\sum g(x, y)$ the sum being over (X, Y). A <u>cut</u> in $\mathcal{N}$ is a collection of arcs $(X, \overline{X})$ with $s \in X$, $t \in \overline{X}$, where $\overline{X}$ is the set of all nodes of $\mathcal{N}$ not in X. The <u>cut capacity</u> of $(X, \overline{X})$ is $c(X, \overline{X})$. We can now state the Max-Flow Min-Cut Theorem of Ford and Fulkerson [3, p. 11]:

<u>THEOREM 2.3</u>

For any network $\mathcal{N}$ the maximum value ("maximum flow") of v in (2.1) is equal to $\min c(X, \overline{X})$ ("minimum cut capacity"). Here the maximum is computed over all real valued f consistent with (2.2) and the minimum is computed over all cuts.

As a sample application of Theorem 2.3 we consider a "class decomposition theorem". Let $R = (r_1, r_2, \ldots, r_m)$, $R' = (r_1', r_2', \ldots, r_m')$ be m-tuples of non-negative integers with $r_i' \leq r_i$, $i = 1, 2, \ldots, m$ which we will denote more briefly by $R' \leq R$. Similarly, let $S = (s_1, s_2, \ldots, s_n)$,

$S' = (s_1', s_2', \ldots, s_n')$ be n-tuples of non-negative integers with $S' \leq S$. By $\mathfrak{A}(R,S)$ we shall mean the class of all $m \times n$ (0, 1)-matrices having row (column) sums R(S) and where

$$\sum_{i=1}^{m} r_i = \sum_{i=1}^{n} s_j$$

holds. Let A' be an $e \times f$ submatrix of an $m \times n$ matrix A formed by the intersection of rows $\{i_1, i_2, \ldots, i_e\}$ and columns $\{j_1, j_2, \ldots, j_f\}$. By $\sigma(A')$ $(\tau(A'))$ we will mean these sets of row (column) indices of A'. Finally $N_1(A)$ denotes the number of 1's in the matrix A.

THEOREM 2.4

Let $A \in \mathfrak{A}(R,S)$ and let $\mathfrak{A}(R',S')$ be a class with $R' \leq R$ and $S' \leq S$. Then A can be written

$$A = B + R$$

with $B \in \mathfrak{A}(R',S')$ and R a (0, 1)-matrix iff for every submatrix A' of A we have

$$N_1(A') \geq \sum_{i \in \sigma(A')} r_i' - \sum_{j \notin \tau(A')} s_j'$$

Proof

The implication to the right is simply a matter of counting 1's. For convenience in the following diagram we take the submatrix A' as occupying the upper left corner of A:

$$\underset{A}{\overset{\longleftarrow f \longrightarrow}{e\updownarrow\left[\begin{array}{c|c} A' & A_1 \\ \hline A_2 & A_3 \end{array}\right]}} = \underset{B}{\overset{\longleftarrow f \longrightarrow}{e\updownarrow\left[\begin{array}{c|c} B' & B_1 \\ \hline B_2 & B_3 \end{array}\right]}} + \underset{R}{\overset{\longleftarrow f \longrightarrow}{e\updownarrow\left[\begin{array}{c|c} R' & R_1 \\ \hline R_2 & R_3 \end{array}\right]}}$$

Then,

$$\begin{aligned} N_1(A') &\geq N_1(B') \\ &\geq N_1(B') - N_1(B_3) \\ &= [N_1(B') + N_1(B_1)] - [N_1(B_1) + N_1(B_3)] \\ &= \sum_{i \in \sigma(A')} r'_i - \sum_{j \notin \tau(A')} s'_i \end{aligned}$$

as wanted.

For the converse we use the Max-Flow Min-Cut Theorem. We first set up a network η. The nodes of η are two distinguished nodes s, t together with a node (x_i) for each row of A and a node (y_j) for each column of A. All arcs are of one of the three forms: (s, x_i), $i = 1, 2, \ldots, m$; (x_i, y_j) (iff $a_{ij} = 1$); (y_j, t), $j = 1, 2, \ldots, n$ and the corresponding capacities are:

$$\begin{aligned} c(s, x_i) &= r'_i, & i &= 1, 2, \ldots, m \\ c(x_i, y_j) &= 1 & &(\text{iff } a_{ij} = 1) \\ c(y_j, t) &= s'_j, & j &= 1, 2, \ldots, n \quad . \end{aligned}$$

From the Integrity Theorem [3, p. 19] of the Theory of Flows in Networks, η has a flow of value τ where

$$\tau = \sum_{i=1}^{m} r_i' = \sum_{j=1}^{n} s_j'$$

iff A contains a (0, 1)-matrix $B \in \mathfrak{A}(R', S')$. By the Max-Flow Min-Cut Theorem (Theorem 2.3) η has such a flow iff

$$\tau \leq c(X, \overline{X}) \qquad (2.3)$$

for all cuts $(X, \overline{X})$. In the cut $(X, \overline{X})$ let the nodes $x_i \in X$ and the nodes $y_j \in \overline{X}$ define a submatrix A' in A. Then (2.3) can be rewritten as:

$$\sum_{i=1}^{m} r_i' \leq N_1(A') + \sum_{i \notin \sigma(A')} r_i' + \sum_{j \notin \tau(A')} s_j'$$

or,

$$N_1(A') \geq \sum_{i \in \sigma(A')} r_i' - \sum_{j \notin \tau(A')} s_j'$$

which completes the proof of the theorem.

A generalization of Theorem 2.1 due to Ryser [12, p. 551] is:

THEOREM 2.5

An $m \times n$ $(m \leq n)$ (0, 1)-matrix A has feasible list (m, m, . . . , m) (ℓ terms) iff $A \in \mathfrak{A}(R, S)$ with $R = (\ell, \ell, \ldots, \ell)$ and $S \leq (\ell, \ell, \ldots, \ell)$.

Proofs of this result are also possible using the Max-Flow Min-Cut Theorem or one of its equivalences.

In order to consider permutation decompositions whose summands are of a size strictly less than the minimal dimension (of the summands) we will need a lemma (Lemma 2.6). This in turn will lead to the best result of an "existence type" which has been obtained by this author (Theorem 2.7).

LEMMA 2.6

A $m \times n$ (0,1)-matrix A has feasible list $(p, p, \ldots, p)$ (ℓ terms) iff it contains ℓp 1's and has every line sum $\leq \ell$.

Proof

Several proofs of this lemma are now available in the literature. Two methods of proof are given in [2, p. 18]. The following proof, somewhat of an improvement over one of the proofs of the present author, is an unpublished result due to Richard A. Brualdi.

That the stated conditions are necessary is obvious so we turn to a proof of sufficiency. Without loss of generality we assume $m \leq n$ and adjoin a $(n-m) \times m$ 0 submatrix to A, so forming an order n (0,1)-matrix A'. $\frac{1}{\ell} A'$ has all of its line sums ≤ 1 and by the proof of The Theorem of Mendelsohn and Dulmage [7, p. 253] it follows that the order n matrix $\frac{1}{\ell} A$ can be imbedded in a doubly stochastic matrix B of order 2n-p:

$$B = \left(\begin{array}{c|c} \frac{1}{\ell} A' & B_1 \\ \hline B_2 & 0 \end{array}\right)$$

Birkoff's Theorem for doubly stochastic matrices implies that B contains 2n-p positive entries, no two in a line. Since B_1, B_2 are

$n \times (n-p)$ and $(n-p) \times n$ matrices, respectively,

$$(2n-p) - (n-p) - (n-p) = p$$

of these positive entries are in $\frac{1}{\ell} A'$, i. e., A has a partial permutation matrix P of size p as a (0, 1)-summand. Furthermore, if $\frac{1}{\ell} A'$ has a line sum = 1 then one of the 2n-p positive elements was in that line of $\frac{1}{\ell} A'$ since the corresponding line of B_1 or B_2 is a line of 0's. Consequently A - P contains $(\ell - 1)p$ 1's and has every line sum $\leq \ell - 1$. The above procedure can then be repeated until we obtain the desired decomposition after ℓ steps.

THEOREM 2.7

Let $A = (a_{ij})$ be a $m \times n$ (0, 1)-matrix of term rank p and let p' be a positive integer $\leq p$.

$$\underbrace{(p', p', \ldots, p'}_{\ell_1 = \ell \text{ terms}}, \underbrace{1, 1, \ldots, 1)}_{\ell_2 \text{ terms}} \tag{2.4}$$

is a feasible list for A for all $\ell \leq \bar{\ell}$ where

$$\bar{\ell} = \min\left[\frac{N_1(A')}{e+f+p'-m-n}\right] ,$$

the minimum being calculated over all $e \times f$ submatrices A' of A for which $e + f > m + n - p'$ and $[\alpha]$ denotes the greatest integer $\leq \alpha$.

Proof

Using A we set up a network $\mathcal{N}$. Then using Lemma 2.6 we apply Theorem 2.3 to get the desired result. The procedure is nearly identical with the proof of Theorem 2.4. The nodes of $\mathcal{N}$ consist of the two distinguished nodes s, t together with a node (x_i) for each row of A and a node (y_i) for each column of A. All (directed) arcs of $\mathcal{N}$ and their corresponding capacities are enumerated by:

$$\begin{array}{lll} \text{i)} & c(s, x_i) = \ell, & i = 1, 2, \ldots, m \\ \text{ii)} & c(x_i, y_j) = 1 & \text{iff } a_{ij} = 1 \\ \text{iii)} & c(y_j, t) = \ell, & j = 1, 2, \ldots, n \quad . \end{array}$$

From the Integrity Theorem [3, p. 19] of the Theory of Flows in Networks it follows that $\mathcal{N}$ has a flow of value $\ell p'$ iff A has an $m \times n$ submatrix B with $N_1(B) = \ell p'$ and having all line sums $\leq \ell$. Further, by Lemma 2.6, this is equivalent to A having feasible list (2.4). By Theorem 2.3 $\mathcal{N}$ has a flow $\ell p'$ iff

$$\ell p' \leq c(X, \overline{X}) \tag{2.5}$$

for all cuts $(X, \overline{X})$. In the cut $(X, \overline{X})$ let X contain e nodes x_i, $\overline{X}$ contain f nodes y_i and let the corresponding $e \times f$ submatrix of A be denoted A'. Then (2.5) can be rewritten as:

$$\ell p' \leq N_1(A') + \ell(m-e) + \ell(n-f)$$

or,

$$N_1(A') \geq \ell(e+f+p'-m-n)$$

from which the theorem follows.

Theorem 2.7 can be stated in the equivalent form that: $(p_1, p_2, \ldots, p_{\ell_1+\ell_2})$ is a feasible list for A iff for all submatrices A' of A we have

$$N_1(A') \geq \sum_{i=\Delta g+1}^{\infty} p_i^* \tag{2.6}$$

(with equality in the case A' = A) where $\{p_1^*, p_2^*, \dots\}$ is the sequence conjugate to the sequence $\{p_1, p_2, \dots, p_{\ell_1+\ell_2}\}$ and $\Delta g = g(A) - g(A') = (m+n) - (e+f)$, i.e., g(B) is the sum of the dimensions of B, for a name the "girth" of B. A stronger form of Theorem 2.7 (necessary and sufficient conditions for feasibility of a list containing <u>any</u> two positive integers) was obtained by Folkman and Fulkerson [2, p. 16] and takes the form (2.6).

<u>THEOREM 2.8</u>

Let $p_1 = p_2 = \dots = p_{\ell_1} = p$, $p_{\ell_1+1} = p_{\ell_1+2} = \dots = p_{\ell_1+\ell_2} = q$ be a sequence of positive integers and $\{p_i^*\}$ its conjugate sequence. Then $(p_1, p_2, \dots, p_{\ell_1}, p_{\ell_1+1}, \dots, p_{\ell_1+\ell_2})$ is a feasible list for a (0,1)-matrix A iff for every submatrix A' of A we have

$$N_1(A') \geq \sum_{i=\Delta g+1}^{\infty} p_i^*$$

(with equality in the case A' = A).

The proof of this theorem requires considerably more machinery than that developed for Theorem 2.7 and in particular in [2] the "Circulation Theorem" from Network Flow Theory is used. Very roughly, the basic idea is that for a given matrix A and integer values p, q, ℓ_1, ℓ_2, a circulating network is set up, the 1's of the matrix corresponding to certain nodes of the circuit. By imposing unit capacity on certain arcs the "on-off" nature of the integral flow (of a maximal circulation) through a node, as determined by the Circulation Theorem, splits the (0, 1)-matrix A into a matrix sum B + C such that Lemma 2.6 is applicable to B and also to C with q replacing p. From the nature of the proof it is clearly plausible that Theorem 2.8 (i.e., necessary and sufficient conditions for <u>two</u> distinct integers in the feasible list) is the best possible result of its kind that can be obtained from Network Flow Theory or its equivalences.

It was also proven in [2] that if the non-negative list $P = (p_1, p_2, \ldots, p_\ell)$ majorizes the non-negative list $Q = (q_1, q_2, \ldots, q_\ell)$ where P is feasible for a (0, 1)-matrix A, then so also is Q feasible for A, and that a maximal feasible list $(p_1, p_2, \ldots, p_\ell)$ ("maximal" with respect to the partial order defined by majorization) has exactly k positive components where k = maximal line sum of A. This summarizes all known results of an encouraging nature on the feasible list problem. It is easy to find results of an opposite nature. For example, the necessary and sufficient conditions of Theorems 2.7, 2.8 take the form:

$$g > n \rightarrow m(g) = \min_{A' \text{ of girth } g} N_1(A')$$

must be sufficiently large. Does a knowledge of all values of m(g) offer a means of determining which lists are feasible for a matrix A? The answer is "no" and we provide a counterexample. The following two (0, 1)-matrices,

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

have identical values of m(g) namely:

g	≤5	6	7	8	9	10
m(g)	0	1	2	4	6	9

yet (5, 3, 1) is feasible for A but not for B.

III. APPLICATIONS USING THE EXISTENCE OF DECOMPOSITIONS OF SPECIAL FORMS

The object of this chapter is to show that some natural questions concerning (0,1)-matrices can be equivalently reformulated in terms of permutation decompositions. No attempt is made to be exhaustive but rather, simply to give some examples. We will first consider permuted forms of circulants.

THEOREM 3.1

A (0,1)-matrix A of order n can be permuted (i.e., by permutations of rows and columns) into circulant form iff

i) there are "n-cycle" permutation matrices P^*, Q^* ("n-cyclic" in the sense that they correspond to cyclic permutations on n letters) such that $P^*AQ^* = A$,

or equivalently, iff

ii) there is a permutation decomposition of A, $A = P_1 + P_2 + \dots + P_\ell$, with each P_i of size n and n-cycle permutation matrices P^*, Q^* such that $P^*P_iQ^* = P_i$, $i = 1, 2, \dots, \ell$.

Proof

We first prove i). Let P, Q be size n permutation matrices such that PAQ is a circulant:

$$PAQ = \sum_{i=1}^{\ell} C^{d_i} \tag{3.1}$$

where C is the (row) permutation matrix corresponding to the cycle $(2, 3, \ldots, n, 1)$ and the d_i are non-negative integers. Define

$$\begin{aligned} P^* &= P^{-1}CP \\ Q^* &= QC^{-1}Q^{-1} \end{aligned} \tag{3.2}$$

so

$$\begin{aligned} P^*AQ^* &= P^{-1}(CPAQC^{-1})Q^{-1} \\ &= P^{-1} \sum_{i=1}^{\ell} C^{d_i} Q^{-1} \\ &= A \ . \end{aligned}$$

Conversely, given n-cycle permutation matrices, P^*, Q^* with $P^*AQ^* = A$ we can write these n-cycles in the forms:

$$\begin{aligned} P^* &= P^{-1}CP \\ Q^* &= QC^{-1}Q^{-1} \end{aligned}$$

then

$$P^{-1}CPAQC^{-1}Q^{-1} = A$$

so

$$C(PAQ) = (PAQ)C \ .$$

Since C is nonderogatory and C, PAQ commute then PAQ is a polynomial in C [5, p. 78], i.e., PAQ is a circulant.

We use the above equivalence to prove ii). If A can be permuted into circulant form (3.1) then defining P^*, Q^* as in (3.2) we have

$$P^* A Q^* = A$$

$$= P^{-1} \sum_{i=1}^{\ell} C^{d_i} Q^{-1} \quad .$$

Defining $P_i = P^{-1} C^{d_i} Q^{-1}$, $i = 1, 2, \ldots, \ell$, we have a permutation decomposition of A and

$$P^* P_i Q^* = P^{-1} C P P^{-1} C^{d_i} Q^{-1} Q C^{-1} Q^{-1}$$

$$= P^{-1} C^{d_i} Q^{-1}$$

$$= P_i \quad .$$

The converse implication is trivial.

Let $P_1 (P_2)$ be an m-cycle (n-cycle) permutation matrix corresponding to a row permutation $\pi_1 (\pi_2)$. For any i, $0 \le i < mn$ there is a unique representation

$$i = i_1 n + i_2 \ , \quad 0 \le i_1 < m \ , \quad 0 \le i_2 < n \tag{3.3}$$

and the Kronecker product $P_1 \times P_2$, $[(P_1)_{ij} P_2]$, is a permutation matrix of order and size mn corresponding to a row permutation π defined by

$$\pi(i) = \pi_1(i_1) n + \pi_2(i_2) \tag{3.4}$$

(For notational convenience we are indexing the rows of an order N matrix $0, 1, \ldots, N-1$). If $\pi^k(i) = \pi^\ell(i)$ for some k, ℓ with $0 \le k, \ell < mn$ then for (3.4) we have

$$\pi_1^k(i_1) n + \pi_2^k(i_2) = \pi_1^\ell(i_1) n + \pi_2^\ell(i_2)$$

from which, using (3. 3) we have

$$\pi_j^k(i_j) = \pi_j^{\ell}(i_j), \qquad j = 1, 2$$

which in turn imply $k \equiv \ell \pmod m$ and $k \equiv \ell \pmod n$. If $(m, n) = 1$ then $k \equiv \ell \pmod{mn}$, i. e., $k = \ell$ so the powers $\pi^k(i)$ are distinct but otherwise they are not. We conclude: $P_1 \times P_2$ is an n-cycle matrix iff $(m, n) = 1$.

Using this fact together with Theorem 3. 1 we have, for example:

COROLLARY 3. 2

Let A, B be (0, 1)-matrices of orders m, n, respectively. If A, B are both permuted forms of circulants and $(m, n) = 1$ then $A \times B$ is a permuted form of a circulant.

Proof

Let $A = P_1 + P_2 + \ldots + P_k$, $B = Q_1 + Q_2 + \ldots + Q_\ell$ be permutation decompositions with $P^{(1)}P_iQ^{(1)} = P_i$, $i = 1, 2, \ldots, k$ and $P^{(2)}Q_jQ^{(2)} = Q_j$, $j = 1, 2, \ldots, \ell$ where $P^{(1)}$, $Q^{(1)}$ are m-cycle permutation matrices and $P^{(2)}$, $Q^{(2)}$ are n-cycle permutation matrices. Then

$$A \times B = \sum_{i=1}^{k} \sum_{j=1}^{\ell} P_i \times Q_j$$

is a permutation decomposition. We define the cycle permutation matrices $P = P^{(1)} \times P^{(2)}$, $Q = Q^{(1)} \times Q^{(2)}$. Then

$$\begin{aligned} P(P_i \times Q_j)Q &= (P^{(1)} \times P^{(2)})(P_i \times Q_j)(Q^{(1)} \times Q^{(2)}) \\ &= P^{(1)}P_iQ^{(1)} \times P^{(2)}Q_jQ^{(2)} \\ &= P_i \times Q_j \end{aligned}$$

So $A \times B$ is a permuted circulant by condition i) of Theorem 3.1.

The condition $(m, n) = 1$ is necessary in the Corollary. An example is given by $A \times A$ where A is the circulant

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

There are four distinct circulants, up to permutations of rows and columns, in the class of order 9 (0,1)-circulants having four 1's per line. These four equivalence classes can be represented by the four circulants:

I	$c^0 + c^1 + c^2 + c^3$
II	$c^0 + c^1 + c^2 + c^4$
III	$c^0 + c^1 + c^3 + c^4$
IV	$c^0 + c^1 + c^3 + c^6$

none of which is a permuted form of $A \times A$. This follows, since a row of $A \times A$ has inner product = 2 with four rows and = 1 with the other four rows. The 4×9 submatrix determined by the former four rows has one column sum = 0. Types II, III circulants have the same set of inner products as $A \times A$ but the corresponding 4×9 submatrices have all column sums positive.

It is of interest to note the close similarity of the statement of Theorem 3.1 (i)) and a Theorem of E. T. Parker which we now state in a form slightly more general than it appears in [9, p. 351].

THEOREM 3.3

Let A be a real non-singular matrix of order n. Suppose there are size n permutation matrices P, Q such that

$$PAQ = A \quad .$$

Then there are size n permutation matrices P', Q' such that

$$PA'P^{-1} = A'$$

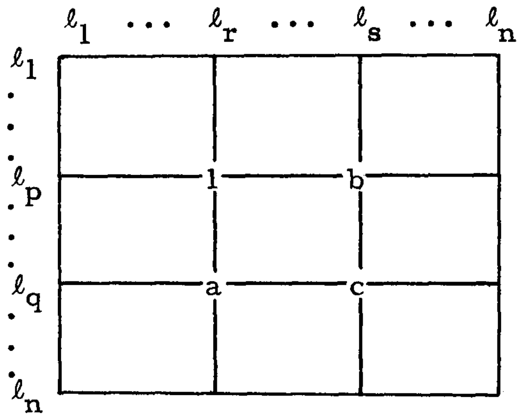
where

$$P'AQ' = A' \quad .$$

Another problem which can be phrased in terms of permutation decompositions is that of necessary and sufficient conditions that a loop be a group. Let $\mathcal{L}$ be a loop of order n. Label its elements 1, 2, . . . , n, the identity (e) receiving the label 1. The multiplication table (Cayley table) of $\mathcal{L}$ can then be represented by a normalized latin square L of order n based on the integers 1, 2, . . . , n. L is "normalized" in the sense that the first row and column contain 1, 2, . . . , n in their natural order corresponding to e being a 2-sided identity.

Let M be the normalized latin square obtained by permuting the rows and columns of L to bring a particular different 1 to the (1, 1) position. The condition M = L for all n-1 possible M is a necessary and sufficient condition that $\mathcal{L}$ be a group. The essence of the proof of this statement is to be found in [16, p. 4]. We will now give a precise and detailed proof.

For any $\ell_p, \ell_q, \ell_r, \ell_s \in \mathcal{L}$ consider the following structure in L



(3.5)

Upon arbitrary permutation of rows and columns of L the same entries 1, a, b still uniquely determine the same entry c. Thus the condition M = L for all n-1 possible M is equivalent to Zassenhaus' rectangle rule, namely that every subconfiguration

$$\begin{bmatrix} 1 & b \\ a & * \end{bmatrix}$$

have the same integer in its (2, 2) entry.

If $\mathcal{L}$ is a group, this condition is easily seen to be satisfied. Referring to (3.5) we have $\ell_p \ell_r = e$ so that $\ell_r^{-1} = \ell_p$ and

$$\ell_q \ell_s = (\ell_q \ell_r)(\ell_r^{-1} \ell_s) = (\ell_q \ell_r)(\ell_p \ell_s)$$

i.e., the integer c is uniquely determined by a, b so that M = L.

Conversely, if the rectangle rule holds, then again from (3.5) we have $\ell_p \ell_r = e$ and $(\ell_q \ell_r)(\ell_p \ell_s) = \ell_q \ell_s$ so that $\mathcal{L}$ satisfies

$$cd = e \rightarrow (x'd)(cy') = x'y' \quad \text{for all} \quad x', y' \in \mathcal{L} \tag{3.6}$$

In particular, since e is a 2-sided identity, from cd = e we have

$$(x'd)c = x' \quad \text{for all} \quad x' \in \mathcal{L} \tag{3.7}$$

$$d(cx') = x' \quad \text{for all} \quad x' \in \mathcal{L} \tag{3.8}$$

Also using (3.7) we have dc = (ed)c = e so that for each element $\ell \in \mathcal{L}$ there is a (unique) 2-sided inverse $\ell^{-1} \in \mathcal{L}$, i.e., an element such that $\ell\ell^{-1} = \ell^{-1}\ell = e$. Consequently, (3.6) can be rewritten

$$(x'\ell^{-1})(\ell y') = x'y' \quad \text{for all} \quad \ell, x', y' \in \mathcal{L} \tag{3.9}$$

Finally we show $\mathcal{L}$ is associative and hence a group. Let $x, y, z \in \mathcal{L}$ then from (3.8)

$$(xy)z = (xy)(y^{-1}(yz))$$

which is of the form (3.9) so

$$(xy)z = x(yz)$$

completing the proof.

IV. GENERALIZED TRACES OF 3-DIMENSIONAL (0, 1)-MATRICES

Consider an arbitrary permutation decomposition

$$A = P_1 + P_2 + \dots + P_\ell \tag{4.1}$$

of a $m \times n$ (0, 1)-matrix A. From (4.1) we can define a 3-dimensional (0, 1)-matrix $B = (b_{ijk})$ by

$$b_{ijk} = \begin{cases} 1 & \text{if } (P_k)_{ij} = 1, \; i = 1, \dots, m; \; j = 1, \dots, n; \; k = 1, \dots, \ell \\ 0 & \text{otherwise} \end{cases}$$

Henceforth we will use expressions like "k-dimensional hyperplane" to denote the k-dimensional analogue of the 2-dimensional concept of a "line". So, if in the above definition we take the third subscript as a "vertical" coordinate, then B is formed by stacking the P_i's one above another and hence every "vertical" 2-dimensional (hyper)plane of B contains at most ℓ 1's. We define a <u>generalized diagonal</u> of a $m \times n \times \ell$ (0, 1)-matrix as $\nu = \min(m, n, \ell)$ positions within B, no two in the same (2-dimensional) plane. A <u>generalized trace</u> of B is the sum of the elements on a generalized diagonal. Note, if B has a generalized trace = t then there is a $m \times n$ permutation matrix P of size = min(m, n) such that P, P_{i_j}, $j = 1, 2, \dots, t$, have a 1 in the same matrix position where $P_{i_1}, P_{i_2}, \dots, P_{i_\ell}$ is some relabeling of the terms of (4.1). Such a P is one possible definition of a "transversal" (of length t) of an arbitrary permutation decomposition. It is closely related but not identical with the concept as it will be introduced in Chapter V.

Let B be a fixed 3-dimensional (0, 1)-matrix not necessarily constructed from some permutation decomposition (4. 1). We are interested in the generalized traces of B. First we summarize the situation in 2-dimensions where the analogous problem has been completely resolved by Mesner [6].

Let A be a m × n (0, 1)-matrix. The <u>trace sequence</u> $\sigma_A = \{\sigma_1, \sigma_2, \ldots, \sigma_u\}$ of A where

$$\tilde{\sigma}(A) = \sigma_1 < \sigma_2 < \ldots < \sigma_u = \bar{\sigma}(A) \quad (4.2)$$

lists all distinct generalized traces of A. The maximum and minimum values are determinable by the König-Egerváry Theorem, namely $\bar{\sigma}(A)$ = term rank of A = minimal number of lines containing all the 1's of A and $\tilde{\sigma}(A) = \min(m,n) - \bar{\sigma}(\bar{A}) = \min(m,n)$ - term rank of $\bar{A}$ = min(m, n) - minimal number of lines containing all the 1's of $\bar{A}$. Here $\bar{A}$ denotes the (0, 1)-complement of A. Intermediate values are mainly accounted for by Mesner's Theorem 1:

<u>THEOREM 4. 1</u>

Let A be a m × n (0, 1)-matrix such that neither A nor $\bar{A}$ is a rearranged direct sum of J matrices (matrices all of whose entries are = 1). Then for any integer σ satisfying $\tilde{\sigma} \le \sigma \le \bar{\sigma}$, A has generalized trace σ.

The basic tool in Mesner's proof of Theorem 4. 1 was

LEMMA 4.2

Let A be an order n (0,1)-matrix with trace n. Then A has no generalized trace = n-1 iff it is a simultaneous row and column permutation of a direct sum of square J matrices.

A proof that a simultaneous permutation of a direct sum of square J matrices can have no generalized trace n-1 is straightforward and Mesner's proof of the converse implication will be considered later. The case not handled by Theorem 4.1, i.e., when A is a rearranged direct sum of J matrices is lengthy but not difficult, and available in [6, p. 91].

For a general (0,1)-matrix B of size $m \times n \times \ell$ both the problem of (i) $\overline{\sigma}(B)$ (clearly $\tilde{\sigma}(B) = \min(m, n, \ell) - \overline{\sigma}(\overline{B})$) and (ii) intermediate values, are much more complicated. We will make only very modest contributions to their solution. Of particular interest is $\overline{\sigma}(B_L)$ where B_L is the (0,1)-cube ("permutation cube") constructed from the permutation decomposition (1.1) associated with an order n latin square L of the form (1.2). It has been conjectured by Ryser that $\overline{\sigma}(B_L) = n$ when L is of odd order n [15, p. 72] and that always $\overline{\sigma}(B_L) \geq n-1$.

For d-dimensional (0,1)-matrices A with $d > 2$ we no longer have a König-Egerváry Theorem, a theorem which was solely responsible for the useful equivalence of maximal generalized trace in the 2-dimensional case. We can define "term ranks" and "covering numbers" of d-dimensional matrices, $d \geq 2$, relative to hyperplanes of all dimensions k with $1 \leq k \leq d-1$ but we will want to restrict ourselves here to the case $k = d-1$; i.e., the term rank (ρ) of A will be the maximum number of 1's of A such that no two are in the same (d-1)-dimensional hyperplane and the covering number (c) of A is the minimal number of (d-1)-dimensional hyperplanes which contain all the 1's of A. The König-Egerváry Theorem claims $c = \rho$ in 2 dimensions, the crux of the argument being in showing $c \le \rho$ since $c \ge \rho$ is a triviality for any dimension d. We conjecture the following:

CONJECTURE 4.3

> Let the term rank (ρ) and covering number (c) of a d-dimensional, $d \ge 2$, (0,1)-matrix be defined in terms of (d-1)-dimensional hyperplanes. Then $c \le (d-1)\rho$.

Little progress has been made with this conjecture. The weaker inequality $c \le d\rho$ is a triviality since we can cover all 1's in the matrix by the d hyperplanes through each of ρ 1's, no two in a (d-1)-dimensional hyperplane. For $\rho = 1$ the cases $d = 3$, $d = 4$ have been verified exhaustively as has the case $d = 3$, $\rho = 2$.

The conjectured upper bound is always attainable at least in the case where the existence of a finite projective plane of order $n = d-1$ is known. This follows since the existence of a projective plane of order $n \ge 3$ is equivalent to the existence of n-1 orthogonal latin squares of order n which are in turn equivalent to the existence of an $n^2 x(n+1)$ array of integers $\in \{1, 2, \ldots, n\}$ such that each $n^2 \times 2$ subarray consists of all the 2-samples of $\{1, 2, \ldots, n\}$ [13, p. 82]. Viewing the array as the coordinates of n^2 1's in a (n+1)-dimensional, order n (0,1)-matrix A, we have $c = n = d-1$ since A has the same number of 1's in each (d-1)-dimensional hyperplane. Also from the "$\lambda = 1$" property of a projective plane it follows that each pair of d-tuples agree in exactly one coordinate, so in particular $\rho = 1$. For general ρ we take the direct sum B of ρ copies of A so that the term rank and covering number are each multiplied by a factor ρ, i.e., $c = (d-1)\rho$. The n = 2 case is handled similarly since the array

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

has the desired properties.

The above-described $n^2 x (n+1)$ array can also be interpreted as a (0,1)-matrix, namely the incidence matrix of the affine plane of order n. For example we view A above as:

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right)$$

$$\underbrace{\text{"1" "2"}}_{\substack{\text{1st} \\ \text{coordinate}}} \quad \cdots \quad \underbrace{\text{"1" "2"}}_{\substack{d^{th} \\ \text{coordinate}}}$$

and in this way we can obtain a (0,1), 2-dimensional, and slightly more general formulation of Conjecture 3.1. Let A be an arbitrary $m \times n$ (0,1)-matrix and let d = d(A) ("dimension" of A) be the minimum number of parts into which the columns of A must be partitioned such that in each row there is at most one 1 in each part. Let $\rho = \rho(A)$ be the maximum number of rows of A such that every pair has inner product = 0. Finally, let $\epsilon = \epsilon_1(A)$ denote the 1-width of A (the minimum number of columns of A which contain at least one 1 in every row). Then the reformulation is:

CONJECTURE 4.4

A (0,1)-matrix having $d > 1$ and no row of 0's satisfies $\epsilon \leq (d-1)\rho$.

A small contribution toward resolving this conjecture is given by the following theorem but a corresponding result for an arbitrary number of rows of the incidence matrix of a finite projective plane remains unproven.

THEOREM 4.5

Let A be an $m \times n$ (0,1)-matrix having row sums = $r > 1$, column sums = s and such that every pair of rows has a positive inner product. Then $\epsilon \leq (d-1)\rho$.

Proof

If $s = 1$ then $m = 1$ and if $m = 1$ then $\epsilon = 1$. Similarly $s = m$ implies $\epsilon = 1$. In all these cases the conclusion is immediate so there is no loss in assuming $s < m$, $s > 1$ and $m > 1$. Also, since every pair of rows has a positive inner product then $\rho(A) = 1$ and $\epsilon_1(A) \leq r$ so we can further restrict ourselves to the case where $d(A)-1 < r$, i.e., where $d(A) \leq r$. The maximum number of columns of A such that no two have a positive inner product is

$$\leq \left\lceil \frac{m}{s} \right\rceil$$

and therefore from the definition of d(A),

$$d(A) \geq \frac{n}{\lceil m/s \rceil} \quad .$$

But mr = ns so n = (m/s)r and

$$d(A) \geq \frac{n}{\lceil m/s \rceil} \geq r \quad .$$

The remaining case has d(A) = r where $s \mid m$ and every row contains exactly one 1 within each part of a partition of the columns of A determined by d(A). Clearly $\epsilon_1(A) = m/s$ so defining an average inner product $(\bar{\lambda})$ among distinct pairs of rows of A we have the following inequalities:

$$\bar{\lambda} = \frac{ns(s-1)}{m(m-1)} = \frac{r(s-1)}{m-1} \geq 1$$

$$d(A) = r \geq \frac{m-1}{d-1}$$

$$\frac{m-1}{s-1} - \epsilon_1(A) = \frac{m-1}{s-1} - \frac{m}{s} = \frac{m-s}{s(s-1)} > 0$$

$$\epsilon_1(A) < \frac{m-1}{s-1} \leq d(A)$$

or,

$$\epsilon_1(A) \leq d(A)-1$$

in all cases, which completes the proof.

We now turn to the problem of intermediate values in the trace sequence of a 3-dimensional (0, 1)-matrix B. There does not appear to be any simple alternative description of those B which omit an integer between $\tilde{\sigma}(B)$, $\bar{\sigma}(B)$ in (4. 2). A few mild constraints can be obtained on such "gaps" and one such result will be considered in Theorem 4. 7. First we will consider a procedure for obtaining some (0, 1)-cubes B having trace n, no generalized trace n-1 from the corresponding 2-dimensional analogues described in Lemma 4. 2.

Let A be an order n (0, 1)-matrix having trace n but no generalized trace = n-1. Consider an arbitrary partition $P = \{p_1, p_2, \ldots, p_r\}$ of the columns of A into (non-empty) parts p_i. Form an order n (0, 1)-cube B according to the following rule:

> For each $i = 1, 2, \ldots, n$ if the i^{th} column of A is in P_j then the i^{th} horizontal plane of B has the k^{th} column of A as its k^{th} column for all $k \in P_j$ and all other entries are set = 0.

Now consider the patterns of the positive elements of two of the line sum matrices of B. Clearly the pattern formed by the vertical line sums is that of A. The other pattern we will want to consider is formed by the horizontal line sums which are in the direction of the columns of A. Since A has trace n, B has trace n and the elements of the diagonal of A cause the second pattern of B to be that of a simultaneous permutation of the rows and columns of a direct sum of square J matrices. From the construction it follows that these two patterns completely determine B in the sense that a position in B contains a 1 iff it projects onto a non-zero element in both patterns. But then B has no generalized trace = n-1. For suppose it did and consider the projected images of the n elements of a generalized diagonal of sum = n-1 in the two patterns. Since both patterns are patterns of simultaneous permutations of direct sums of square J matrices it follows that the image of the 0 element of the generalized diagonal is a non-zero element in both patterns. Since the patterns determine B this 0 is a 1--a contradiction showing there is no generalized trace = n-1.

Although this construction produces a large class of cubes it does not exhaust the class $\mathcal{B}_n^{(3)}$ of order n cubes having trace n but no generalized trace = n-1. So far it has proven impossible to find a characterization of those $B \in \mathcal{B}_n^{(3)}$ which are constructable from a 2-dimensional $A \in \mathcal{B}_n^{(2)}$, and which has a generalization characterizing all cubes in $\mathcal{B}_n^{(3)}$. We will now consider Mesner's proof of Lemma 4.2 since an attempt at an analogous proof in 3 dimensions sheds considerable light on the difficulties of characterizing $\mathcal{B}_n^{(3)}$.

Proof of Lemma 4.2

As previously noted, the sufficiency of the conditions is straightforward so we will restrict ourselves to showing that if A has trace n and no generalized trace = n-1 then A is a simultaneous permutation of a direct sum of square J matrices. First note that A can have no principal order 2 submatrix of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{4.3}$$

or its transpose, since otherwise A has a generalized trace = n-1.

Thus,

$$a_{ij} = 1 \rightarrow a_{ji} = 1 \quad .$$

Similarly there can be no principal order 3 submatrix of the form

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{4.4}$$

since again A would have a generalized trace = n-1. Thus,

$$a_{ij} = 1 \quad \text{and} \quad a_{jk} = 1 \rightarrow a_{ij} = 1 \quad .$$

Defining an equivalence relation "$\sim$" on the n integers $1, 2, \ldots, n$ by

$$i \sim j \quad \text{iff} \quad a_{ij} = 1$$

then we have $i \sim i$, $i \sim j \rightarrow j \sim i$, $i \sim j$ and $j \sim k \rightarrow i \sim k$ so the equivalence classes defined by $\sim$ correspond to (simultaneously permuted) summands of square J matrices and the lemma follows.

Note that the truth of Lemma 4.2 and the fact that equivalence relations define equivalence classes are essentially equivalent statements. It is also of interest that although there are higher order ($n \geq 4$) principal subconfigurations ("forbidden" configurations) which are excluded from the (0,1)-matrix A if it is to be $\mathcal{D}_n^{(2)}$, such subconfigurations themselves contain a subconfiguration (4.3) (or its transpose) or a subconfiguration (4.4). It is in these aspects that the same approach fails for $\mathcal{D}_n^{(3)}$. We can introduce a triadic relation:

$$(i, j, k) \quad \text{iff} \quad b_{ijk} = 1$$

so that if $B \in \mathcal{B}_n^{(3)}$ we have (i,i,i), $(i,j,j) \to (j,i,i)$, (i,j,k) and $(j,k,j) \to (k,i,i)$, etc. and in general if in any principal order n' subconfiguration of A n'-1 1's can be found, no two in a plane, then another implication holds. However, not only do we lack an analogue of equivalence classes, but also it can be shown that even restricting $B \in \mathcal{B}_n^{(3)}$ to those which have at most one 1 per plane is not sufficient to eliminate the existence of forbidden configurations of any order k, $1 < k \leq n$ which contains no forbidden subconfiguration of a smaller order.

As a final topic in this chapter we will consider a previously-mentioned mild constraint on the 3-dimensional trace sequence.

LEMMA 4.6

If B is a $m \times n \times \ell$ (0,1)-matrix with full trace $\nu = \min(m, n, \ell)$ but with no generalized trace $= \nu-1$ or $\nu-2$ then $B = J$.

Proof

First consider the principal order ν submatrix B' of $B = (b_{ijk})$. B' has trace ν and no generalized trace $= \nu-1$ or $\nu-2$. We claim $b_{ijk} = 1$ for $(i,j,k) \in B'$. For $i=j=k$ this is part of the hypotheses. It is also clear when there are exactly two distinct integers among i, j, k since for example, one of the two nonplanar positions (i,i,j), (j,j,i) is to be found in every plane numbered i or j and therefore B', and hence B, has generalized trace $\nu-1$ ($\nu-2$) if one (both) of these positions contains a 0. Finally, if i, j, k are three distinct integers, the principal order 3 subcube containing b_{ijk} also contains b_{jij} and b_{kki} which are both = 1 by the previous argument. But $b_{jij} = 1$ and $b_{kki} = 1 \to$

$b_{ijk} = 1$ if a generalized trace $= \nu-1$ is not to occur. Hence $B' = J$.

Now consider a position $(i, j, k) \in B \setminus B'$. There are at most two of $i, j, k \in \{1, 2, \ldots, \nu\}$ and hence the deletion of all three planes through (i, j, k) will delete at most two of the ν 1's of the main diagonal of B. If only one 1 is deleted then we have constructed a generalized trace $= \nu-1$ if $b_{ijk} = 0$. If two are deleted and $b_{ijk} = 0$ then it is in a position (i, j, k) with i, j, k all distinct, two among $\{1, 2, \ldots, \nu\}$ and one not in $\{1, 2, \ldots, \nu\}$, e.g., (i, j, k) with $i, j \in \{1, 2, \ldots, \nu\}$, $i \neq j$ and $k \notin \{1, 2, \ldots, \nu\}$ so that $b_{iii} = 1$ and $b_{jjj} = 1$ have been deleted. If we replace these two of the ν 1's on the main diagonal of B with $b_{ijk} = 0$ and $b_{jii} = 1$ we have constructed a generalized trace $= \nu-1$. Consequently, in all cases $b_{ijk} = 1$.

This lemma can be applied to show that gaps in the 3-dimensional trace sequence, when they occur, can be of length at most 1.

THEOREM 4.7

If an $m \times n \times \ell$ (0,1)-matrix B has no generalized trace $= \sigma$ where $\tilde{\sigma} < \sigma < \bar{\sigma}$, then it has both $\sigma \pm 1$ as generalized traces.

Proof

Suppose B has a generalized trace = t but neither generalized trace = t+1 nor = t+2 where $t+2 < \bar{\sigma}$. Let u be the smallest integer $> t+2$ for which there is a generalized trace = u in B. The planes of B can be permuted to a matrix B_1 having t 1's in the initial positions on its diagonal. Deleting all planes through these 1's we obtain a matrix B_2 having trace 0 and no generalized trace = 1 or 2. From Lemma 4.6

(taking (0, 1)-complements) B_2 is a matrix of 0's so in particular B cannot contain any J subcube J_k of order $k > t$ (such a J_k must contain a 1 all of whose coordinates are $\geq k > t$, i.e., such a 1 would have to be in B_2). On the other hand, B can also be permuted to a matrix B_1' having a leading principal order u submatrix B_2' with trace u and no generalized trace = u-1 or u-2. Again using Lemma 4.6, $B_2' = J$ of order u, a contradiction since $u > t$.

V. ON THE CONJECTURE THAT EVERY LATIN SQUARE OF ODD ORDER HAS A TRANSVERSAL

It has been conjectured [15, p. 72] that in every latin square of odd order n, n different symbols can be found with no two in the same line. Or more briefly, the conjecture is that every latin square of odd order has a transversal. This chapter will attempt to determine the separate contributions of various substructures of an odd ordered latin square to the (conjectured) existence of a transversal. In particular, we will consider how the existence of a transversal depends upon: (1) all symbols filling a square array; (2) the array being of odd order; (3) each symbol occurring the same number of times; (4) any row and any column of the array having exactly one element in common, etc.

We first generalize our notion of a transversal. A transversal of a partial latin square

$$L = 1 \cdot P_1 + 2 \cdot P_2 + \dots + \ell \cdot P_\ell \qquad (5.1)$$

is a set of ℓ different symbols of L, no two in a line. These ℓ symbols define a permutation matrix P of size ℓ (the dimensions of P being those of L) such that P, P_i have a 1 in the same matrix position for each of the ℓ P_i's of the permutation decomposition

$$A = P_1 + P_2 + \dots + P_\ell \qquad (5.2)$$

corresponding to (5.1). There will be no ambiguity in calling P a transversal of the permutation decomposition (5.2).

Let $(p_1, p_2, \dots, p_\ell)$ be an arbitrary list and consider the set of all partial latin squares with this list. We ask, which lists guarantee transversals? The conjecture which follows is then a generalization of the conjecture of Ryser. We need to introduce one definition. A list $(p_1, p_2, \ldots, p_n)$ is <u>square</u> if for every k-subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$ we have

$$p_{i_1} + p_{i_2} + \ldots + p_{i_k} \geq k^2$$

for all $k = 1, 2, \ldots, n$. Equivalently $(p_1, p_2, \ldots, p_n)$ is a square list iff upon reordering and resubscripting such that

$$p_1 \leq p_2 \leq \ldots \leq p_n$$

we have

$$p_1 + p_2 + \ldots + p_k \geq k^2 \quad , \qquad k = 1, 2, \ldots, n \quad .$$

<u>CONJECTURE 5.1</u>

Every partial latin square $L = 1 \cdot P_1 + 2 \cdot P_2 + \ldots + \ell \cdot P_\ell$ with list $\mathcal{L} = (p_1, p_2, \ldots, p_\ell)$, $p_1 \leq p_2 \leq \ldots \leq p_\ell$, has a transversal iff

i) $\mathcal{L}$ is square

and

ii) if $p_1 + p_2 + \ldots + p_t = t^2$ for any $t \leq \ell$ with p_t even, then there are an odd number of components of $\mathcal{L}$ equal to p_t.

Some indications of the possible truth of this conjecture can be found in the following considerations. The conjecture does hold for the square list $(p_1, p_2, \ldots, p_\ell)$ with $p_i = 2i-1$. For consider any partial latin square L based on such a list. For $k = 1, 2, \ldots, \ell$ the lines occupied by k integers, $1, 2, \ldots, k$ of L, no two in a line, contain at most 2k of the 2k+1 integers = k+1. Then for $k = 1, 2, \ldots, \ell-1$ any set of k integers $1, 2, \ldots, k$, no two in a line, can be extended to an analogous set of k+1 integers $1, 2, \ldots, k+1$, so L has a transversal. Similarly, let

$$(p_1, \ldots, p_t, p_{t+1}, \ldots, p_\ell) \tag{5.3}$$

be a square list with $p_1 \leq p_2 \leq \ldots \leq p_\ell$ and $p_1 + p_2 + \ldots + p_t = t^2$. Further, suppose every partial latin square with list $(p_1, p_2, \ldots, p_t)$ has a transversal and a similar statement also holds for the (easily shown) square list $(p_{t+1}-2t, \ldots, p_\ell-2t)$. Then every partial latin square with list (5.3) has a transversal. This follows since any transversal of a sub-partial latin square with list $(p_1, p_2, \ldots, p_t)$ is contained within 2t lines of a partial latin square described by (5.3). Also if the conditions of Conjecture 5.1 fail on $(p_1, p_2, \ldots, p_t)$ they also fail on $(p_1, p_2, \ldots, p_t, \ldots, p_\ell)$ and it follows there is a partial latin square with list (5.3) and no transversal, if the same statement can be said of $(p_1, p_2, \ldots, p_t)$. Conjecture 5.1 has been checked for all $\ell \leq 4$ using

<u>LEMMA 5.2</u>

Let $(p_1, p_2, \ldots, p_n)$ be a square list with $p_1 \leq p_2 \leq \ldots \leq p_n$. Then there is a square list $(p_1', p_2', \ldots, p_n')$ with $p_1' \leq p_2' \leq \ldots \leq p_n'$, $p_i' \leq p_i$, $i = 1, 2, \ldots, n$ and $p_1' + p_2' + \ldots + p_n' = n^2$.

Proof

If $p_1 + p_2 + \ldots + p_n = n^2$ then we take $p_i' = p_i$, $i = 1, 2, \ldots, n$. Otherwise $p_1 + p_2 + \ldots + p_n > n^2$ and let t be the smallest integer for which $p_1 + p_2 + \ldots + p_k > k^2$ for all $k \geq t$. We claim

$$p_1 \leq p_2 \leq \ldots \leq p_{t-1} \leq p_t - 1 \leq p_{t+1} \leq \ldots \leq p_n \tag{5.4}$$

and the k^{th} partial sum of

$$p_1 + p_2 + \ldots + p_{t-1} + (p_t - 1) + p_{t+1} + \ldots + p_n \tag{5.5}$$

is at least k^2 for all $k = 1, 2, \ldots, n$. By the definition of t, either $t = 1$ in which case $p_1 \geq 2$ so (5.4) is satisfied and the k^{th} partial sum of (5.5) is at least k^2 for $k = 1, 2, \ldots, n$ or $t > 1$ and $p_1 + \ldots + p_{t-1} = (t-1)^2$. In the latter case using $p_1 + \ldots + p_k \geq k^2$ for $k = t-2$ (when $t > 2$) and for $k = t$ we conclude $p_{t-1} \leq 2t-3$ and $p_t \geq 2t-1$ so $p_t - p_{t-1} \geq 2$. Then (5.4) holds and the k^{th} partial sum ($k = 1, 2, \ldots, n$) of (5.5) is again $\geq k^2$. The procedure can be iterated and after each step the total sum is reduced by one. After $(p_1 + \ldots + p_n) - n^2$ steps we have the desired list.

A square list $(p_1, p_2, \ldots, p_n)$ with $p_1 + p_2 + \ldots + p_n = n^2$ will be called an exact square list.

We now turn to some further properties of square lists. A tournament score vector of length n is an n-tuple of non-negative integers $(q_1, q_2, \ldots, q_n)$ listing the number of wins q_i by the i^{th} player in a round-robin tournament between n competitors (allowing no draws). Such n-tuples have been characterized by Landau and Ryser [4;14] by the conditions

$$q_{i_1} + q_{i_2} + \ldots + q_{i_\ell} \geq \binom{\ell}{2} \tag{5.6}$$

for all ℓ-subsets $\{i_1, i_2, \ldots, i_\ell\}$ of $\{1, 2, \ldots, n\}$ and for all $\ell = 1, 2, \ldots, n$, with equality for $\ell = n$. If the q_i's are reordered such that $q_1' \leq q_2' \leq \ldots \leq q_n'$ then (5.6) is equivalent to

$$q_1' + q_2' + \ldots + q_\ell' \geq \binom{\ell}{2}, \qquad \ell = 1, 2, \ldots, n .$$

LEMMA 5.3

Let $s_1, s_2, \ldots, s_k$ be positive integers and let $r_1, r_2, \ldots, r_k$ be non-negative integers satisfying:

i) $r_1 \leq r_2 \leq \ldots \leq r_k$

ii) $r_1 + r_2 + \ldots + r_\ell \geq \binom{\ell}{2}, \qquad \ell = 1, 2, \ldots, k-1$

iii) $r_1 + r_2 + \ldots + r_k \leq \binom{k}{2}$.

Then

$$\sum_{i=1}^{k} r_i s_i \leq \sum_{\substack{i,j=1 \\ i<j}}^{k} s_i s_j$$

with strict inequality when strict inequality holds in iii) (the right-hand sum is to be taken = 0 when k = 1).

Proof

Define $r_i' = r_i$, $i = 1, 2, \ldots, k-1$, $r_k' = r_k + \binom{k}{2} - (r_1 + \ldots + r_{k-1})$. Then $(r_1', r_2', \ldots, r_k')$ is a tournament score vector by the Theorem of Landau and Ryser [4, 14] so there is an order k $(0,1)$-matrix $A = (a_{ij})$ satisfying:

$$a_{ii} = 0 \quad , \quad i = 1,2,\dots,k$$

$$a_{ij} = 1 \text{ iff } a_{ji} = 1 \;, \quad \text{for all } i \neq j$$

$$\sum_{j=1}^{n} a_{ij} = r_i' \;, \quad i = 1,2,\dots,k \;.$$

From these properties we have

$$\sum_{\substack{i,j=1\\ i<j}}^{k} s_i s_j = \sum_{i,j=1}^{k} s_i a_{ij} s_j$$

$$= \sum_{i=1}^{k} \left(\sum_{j=1}^{k} a_{ij} s_j \right) s_i$$

$$\geq \sum_{i=1}^{k} r_i' s_i$$

and moreover

$$\sum_{i=1}^{k} r_i' s_i \geq \sum_{i=1}^{k} r_i s_i$$

with strict inequality if $r_k' \neq r_k$.

Using this lemma we can "strengthen" the Landau-Ryser characterization of a tournament score vector.

LEMMA 5.4

If $T = (q_1, q_2, \dots, q_n)$, a n-tuple of integers, satisfies

i) $q_i - q_{i+1} \leq 1 \;, \quad i = 1,2,\dots,n-1$

ii) $q_1 + q_2 + \dots + q_\ell \geq \binom{\ell}{2} \;, \quad \ell = 1,2,\dots,n$

with equality for $\ell = n$ then T is a tournament score vector.

<u>Proof</u>

Suppose not, so that recording the components of T we have

$$q_1' \leq q_2' \leq \dots \leq q_n'$$

and there is a smallest integer k such that $q_1' + q_2' + \dots + q_k' < \binom{k}{2}$. Let a permutation $(j_1, j_2, \dots, j_n)$ of $(1, 2, \dots, n)$ be defined by:

$$q_{j_i} = q_i', \quad i = 1, 2, \dots, n$$

and let

$$s = \max_{i \leq i \leq k} \{j_i\} .$$

The values $j_1, j_2, \dots, j_k$ partition the interval $[0, s]$ into k parts of lengths $s_1, s_2, \dots, s_k$ $(s = s_1 + s_2 + \dots + s_k)$ such that the closed interval of length s_i has its right-hand endpoint at j_i. Then by i),

$$\sum_{i=1}^{s} q_i \leq \sum_{i=1}^{k} \left\{ \binom{q_i' + s_i}{2} - \binom{q_i'}{2} \right\}$$

$$= \sum_{i=1}^{k} \binom{s_i}{2} + \sum_{i=1}^{k} s_i q_i' .$$

Lemma 5.3 can be applied to the second sum to give

$$\sum_{i=1}^{s} q_i < \sum_{i=1}^{k} \binom{s_i}{2} + \sum_{\substack{i,j=1 \\ i<j}}^{k} s_i s_j$$

$$= \binom{s_1 + s_2 + \dots + s_k}{2} = \binom{s}{2}$$

which contradicts ii) and so proves the lemma.

Lemma 5.4 gives the following determination of exact square lists.

THEOREM 5.5

There is a 1—1 correspondence between exact square lists $(p_1, p_2, \ldots, p_n)$, $p_1 \le p_2 \le \ldots \le p_n$ and tournament score vectors $(q_1, q_2, \ldots, q_n)$ satisfying $q_i - q_{i+1} \le 1$, $i = 1, 2, \ldots, n-1$ given by

$$p_i \leftrightarrow q_i \qquad \text{where} \qquad q_i = p_i - i \tag{5.7}$$

Proof

If $(q_1, q_2, \ldots, q_n)$ is a tournament score vector with $q_i - q_{i+1} \le 1$, $i = 1, 2, \ldots, n-1$, then letting $p_i = q_i + i$ we have $p_1 \le p_2 \le \ldots \le p_n$ and

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} q_i + \sum_{i=1}^{k} i \ge \frac{k(k-1)}{2} + \frac{k(k+1)}{2} = k^2, \qquad k = 1, 2, \ldots, n$$

with equality for $k = n$. Conversely, if $(p_1, p_2, \ldots, p_n)$ is an exact square list with $p_1 \le p_2 \le \ldots \le p_n$ then with $q_i = p_i - i$, $(q_1, q_2, \ldots, q_n)$ satisfies the hypotheses of Lemma 5.4 and so is a tournament score vector.

Some immediate consequences of Theorem 5.5 are the following.

1) If $(p_1, p_2, \ldots, p_n)$ is a square list with $p_1 \le p_2 \le \ldots \le p_n$ then $p_{i_1} + p_{i_2} + \ldots + p_{i_k} \ge \binom{k}{2} + i_1 + i_2 + \ldots + i_k$ $(i_1, i_2, \ldots, i_k$ being pairwise distinct).

2) The set of all exact square lists of length n partitions into disjoint

classes with each class determined by a different tournament score vector of length n. The members of the class determined by $Q = (q_1, q_2, \ldots, q_n)$ with

$$q_1 \le q_2 \le \ldots \le q_n \tag{5.8}$$

are in 1—1 correspondence (namely the correspondence (5.7)) with the reorderings $q_1', q_2', \ldots, q_n'$ of Q with the property that $q_i' - q_{i+1}' \le 1$ for $i = 1, 2, \ldots, n-1$. Therefore in the class corresponding to Q we have

$$\prod_j \prod_{i=1}^{\nu_j - 1} \binom{t_i + t_{i+1}}{t_i} \tag{5.9}$$

exact square lists where the first product is over the number of occurrences of

$$q_{k+1} - q_k > 1 \tag{5.10}$$

in (5.8) (a term is taken = 1 if $\nu_j = 1$) and the terms of the innermost product are determined by

$$q_{\mu+1} = \ldots = q_{\mu+t_1} < q_{\mu+t_1+1} = \ldots = q_{\mu+t_1+t_2} < q_{\mu+t_1+t_2+1}$$

$$= \ldots \ldots = q_{\mu+t_1+t_2+\ldots+t_{\nu_j}} \tag{5.11}$$

where (5.11) is a maximal length subsequence of (5.8) in which (5.10) doesn't occur. Formula (5.9) can be established by induction on ν_j. Let q(n) be the number of tournament score vectors $(q_1, q_2, \ldots, q_n)$ with $q_1 \le q_2 \le \ldots \le q_n$ and p(n) be the number of exact square lists $(p_1, p_2, \ldots, p_n)$ with $p_1 \le p_2 \le \ldots \le p_n$. For small n we have the following table. The values of q(n) agree with those in [8, p. 68]. The values of p(n) have been calculated using formula (5.9) above.

n	q(n)	p(n)
1	1	1
2	1	2
3	2	5
4	4	16
5	9	59
6	22	247
7	59	1,111
8	167	5,302
9	490	26,376
10	1,480	135,670

From these data it is plausible that $\frac{(\sqrt{3})^n q(n)}{p(n)} \sim 3$. Note it has been conjectured that $\frac{q(n+1)}{q(n)} \sim 4$, see, e.g., [8, p. 67] so that if both conjectures are correct, then $\frac{p(n+1)}{p(n)} \sim 4\sqrt{3}$.

3) From the correspondence (5.7) it is reasonable to suspect that the problem of determining an explicit formula for p(n) is at least no simpler than the difficult and unsolved problem [10;11] of determining an explicit formula for q(n). The number ($p_1(n)$) of exact square lists of length n having some proper square sublist can be determined from the values p(k), k < n:

$$p_1(n) = \sum_{k=1}^{n-1} (-1)^{k+1} \sum \prod_{i=1}^{k+1} p(\alpha_i - \alpha_{i-1})$$

where the inner sum is over all sequences $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ of length k satisfying

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k < \alpha_n = n \quad .$$

This is immediate from the sieve formula by noting $p_0(n) = p(n) - p_1(n)$ is the number of length n square lists having no proper square sublist and by making use of the form of an exact square list having a proper square sublist as noted in the comments following Conjecture 5.1.

We now turn to a consideration of partial latin squares and begin by introducing some definitions. A graph has been defined in Chapter II. An undirected graph each of whose nodes has been assigned a positive integer such that adjacent nodes have distinct integers assigned will be called a labeled graph. A maximal complete subgraph of such a labeled graph is a complete subgraph on distinctly labeled nodes and maximal in the sense that it isn't properly contained in any other such subgraph. A latin graph is a labeled graph G for which there is a partial latin square L and a 1—1 correspondence between nodes labeled i and partial latin square elements "i" such that nodes labeled i, j are adjacent in G iff the corresponding "i", "j" are in a line in L. The derived graph G' of a labeled graph G is a (undirected, unlabeled) graph having as nodes the maximal complete subgraphs of G and an arc between such nodes iff the corresponding maximal complete subgraphs in G have a common node.

THEOREM 5.6

A labeled graph is latin iff the following three conditions hold:

i) every node is in at most two maximal complete subgraphs

ii) any two nodes are in at most one maximal complete subgraph

iii) the derived graph is bipartite.

Proof

If a labeled graph G is latin, then the necessity of the conditions is clear from the structure of any partial latin square L_G corresponding to G. Conversely, let a labeled graph G satisfy the three conditions; we will construct a corresponding partial latin square L_G by a simple induction on the number ($|G|$) of nodes of G. The result is clear for $|G| = 1$. Now assume $|G| > 1$ and let G_1 be a subgraph of G obtained by deleting a node (N) and all associated arcs. We claim G_1 also satisfies the three conditions of the theorem. This follows since the deleted arcs take one of three forms: (a) all possible arcs between N and all nodes of at most two maximal complete subgraphs of G_1; (b) a single arc between N and a node of a maximal complete subgraph of G_1 together possibly with all possible arcs between N and all nodes of a distinct maximal complete subgraph of G_1; (c) at most two arcs, one between each of N and a node in distinct maximal complete subgraphs of G_1. If a node (M) of G_1 were in three maximal complete subgraphs of G_1 then there exists three nodes of G_1, one in each of these maximal complete subgraphs, with no two adjacent. From the relationship between G_1, G these nodes are not adjacent in G and M is consequently in three maximal complete subgraphs of G--a contradiction showing i) holds for G_1. A similar argument proves ii) for G_1.

iii) follows since the derived graph G_1' is a subgraph of G', the derived graph of G.

We now claim L_G can be constructed by appropriately entering the label of N into a partial latin square L_{G_1}, which exists by the induction hypothesis. There are three possibilities. If N is an isolated node of G then L_G can be taken in the form of a direct sum of L_{G_1} and the label of N (note there is no restriction on the dimensions of L_G). If N is in a single maximal complete subgraph of G, consisting of at least two nodes then its label can be entered somewhere in the line corresponding to this maximal complete subgraph in G_1. The final case is where N is in two maximal complete subgraphs of G. By condition iii) one of these lines is a row and the other is a column, so their intersection determines a unique position in G. This position contains no label since by i) N is the unique node in both maximal complete subgraphs and isn't a node of G_1. Moreover the label of N appears in neither line in L_{G_1} since these lines are determined (condition ii)) by the maximal complete subgraphs on distinctly labeled nodes in G. Thus the label of N can be entered to form L_G, so completing the proof.

Note that a transversal of a permutation decomposition $A = P_1 + P_2 + \ldots + P_\ell$ with list $(p_1, p_2, \ldots, p_\ell)$ is equivalent in a corresponding latin graph G, having p_i nodes labeled "i" for $i = 1, 2, \ldots, \ell$, to the existence of ℓ distinctly labeled nodes, no two in the same maximal complete subgraph.

Some idea of the dependency of the existence of a transversal in a (odd order) latin square can be obtained by weakening the conditions found in Theorem 5.2. For example, let us call a labeled graph G a weak latin graph if every node of G is adjacent to at most two j-labeled nodes for all j. By a "transversal" of such a graph we will mean exactly the notion described in the previous paragraph. Then all weak latin graphs with list $(p_1, p_2, \ldots, p_\ell)$, $p_1 \leq p_2 \leq \ldots \leq p_\ell$ have transversals iff $p_i \geq 2i-1$ for $i = 1, 2, \ldots, \ell$. The proof that such a list guarantees the existence of a transversal is essentially the agrument given following Conjecture 5.1 and will not be repeated here. Conversely, suppose $p_i \geq 2i-1$ for $i = 1, 2, \ldots, k-1$ but $p_k \not\geq 2k-1$, i.e., $2k-3 \leq p_k \leq 2k-2$. Then it is sufficient to exhibit a graph G with list $(p_1', p_2', \ldots, p_k')$ where $p_1' = p_2' = \ldots = p_k' = 2k-2$ and having no transversal, since a graph described by smaller components can be found as a subgraph. We construct G as k-1 copies of the graph G' on 2k nodes labeled 1, 1, 2, 2, ..., k, k with every pair of distinctly labeled nodes of G' adjacent.

Singling out condition i) of Conjecture 5.1, we would like to weaken the form of partial latin squares to a structure for which the existence of "transversals" for all structures in the class described by a list $\mathcal{L} = (p_1, p_2, \ldots, p_n)$ is completely determined by whether or not $\mathcal{L}$ is square. It is believed this is accomplished with the following definition.

Let $(p_1, p_2, \ldots, p_n)$ be a non-negative list and $\{x_1, x_2, \ldots, x_m\}$ be an ordered m-set of objects. From the latter we form 2m-1 elements y_i of the two types: x_i, $x_i \cup x_{i+1}$. Let $S_1, S_2, \ldots, S_n$ be sets with S_i containing p_i of the elements y_j, such that each x_k occurs at most once among the p_i y_j's. Such a collection of sets and elements will be called a pair configuration. We will say that such a configuration has a <u>pair system of distinct representatives</u> (also "transversal") if there is a mapping $f: \{S_i\} \rightarrow \{y_i\}$ such that no two $f(S_i)$ contain a common x_j.

Generalizing what we mean by an incidence matrix, we can completely describe the pair configuration by a $(0, 1, 1\!-\!\!-\!1)$-array of the form:

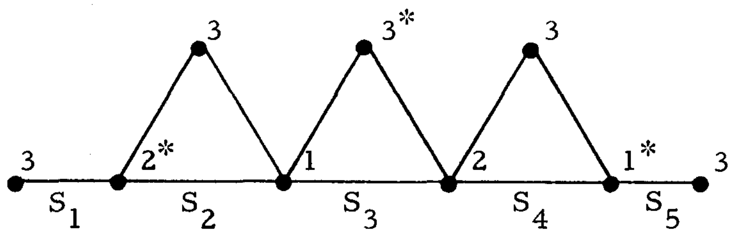
$$\begin{array}{c} \\ S_1 \\ S_2 \\ \vdots \\ S_n \end{array} \begin{array}{c} x_1 x_2 \cdots x_m \\ \left[\begin{array}{c} \\ \\ (0,1,1\!-\!\!-\!1) \\ \\ \\ \end{array} \right] \end{array} \begin{array}{c} \\ \rightarrow p_1 \\ \rightarrow p_2 \\ \vdots \\ \rightarrow p_n \end{array}$$

where a "pair", i.e., "1—1" representing the element $x_i \cup x_{i+1}$ is entered in columns i and i+1. A pair system of distinct representatives in this array corresponds to n non-zero entries, one out of each row, no two sharing a common column.

An example is perhaps in order; e.g., the partial latin square

	S_4	S_2			
	3				
		3			
S_1		2*	3		
S_3	2	1		3*	
S_5	1*				3

the latin graph



and the pair array

	S_1	S_2	S_3	S_4	S_5	
"1"	0	1—	—1	1—	—1*	→ 2
"2"	1—	—1*	1—	—1	0	→ 2
"3"	1	1	1*	1	1	→ 5

all describe the same combinatorial situation. In each case the unique transversal is indicated by asterisks.

It is readily seen from examples that a necessary and sufficient condition for a pair array to have a transversal, if such exists in a form similar to the analogous Philip Hall Theorem for (0, 1)-arrays, must involve ideas of "overlapped" elements. Here two elements are "overlapped" by a third element if the third element shares a different column with each of them. Moreover, in terms of overlapped elements, the intuitive concept of a "connected" pair array can be rigorously defined. Necessary and sufficient conditions on the list $(p_1, p_2, \ldots, p_n)$

in order that: (i) for all arrays with this list every t entries in distinct columns contain a pair of elements with an overlapping element and (ii) the class of connected pair arrays with this list be non-empty, are easily obtained but will not be proven here. In both cases the conditions take the form of linear inequalities in the p_i's. Further structure within the class of all pair configurations with list $(p_1, p_2, \ldots, p_n)$ was previously alluded to, and is conjectured with:

<u>CONJECTURE 5.7</u>

Every pair configuration with list $\mathcal{L} = (p_1, p_2, \ldots, p_n)$ has a pair system of distinct representatives (transversal) iff $\mathcal{L}$ is a square list.

Possibly the sufficiency part of this conjecture can be proven by induction on n although no such proof has yet been found. If in applying such an induction argument there is a proper exact sublist of $\mathcal{L}$ then it follows that up to relabeling we have $p_1 \le p_2 \le \ldots \le p_t < p_{t+1} \le p_{t+2} \le \ldots \le p_n$ for some integer t, $1 \le t < n$, with $p_1 + p_2 + \ldots + p_t = t^2$. By the induction hypothesis the subarray on the first t rows of the pair array has a transversal and the elements of such a transversal occupy at most 2t columns. For $i > t$, row i has $\ge p_i - 2t$ elements entirely outside these columns. $(p_{t+1} - 2t, p_{r+2} - 2t, \ldots, p_n - 2t)$ is square and the induction hypothesis can again be applied. The two partial transversals then together give a transversal of the entire configuration.

Another technique is given by the use of a "left-hand element",

i. e., a non-zero element of a pair array which occupies a column furthest to the left. Such an element can, without loss of generality, be taken as a 1 and hence the remaining (n-1)-rowed pair array not using this column must have row sums $\geq p_i - 1$. Such a condition is very restrictive and in particular this argument by itself proves that every pair with list $(p_1, p_2, \ldots, p_n)$, $p_i = n$, $i = 1, 2, \ldots, n$ has a transversal.

Using the above ideas and some constructions, Conjecture 5.7 has been verified for all $n \leq 5$ and also for most lists with $n = 6$.

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