THE PARTITION OF AMPLITUDES FOR AN SV-WAVE BY ZOEPPRITZ'S METHOD

by

Enrique F. Silgado

A Thesis Submitted as Partial Fulfillment of the Requirements for the Degree of Master of Science in Geophysics

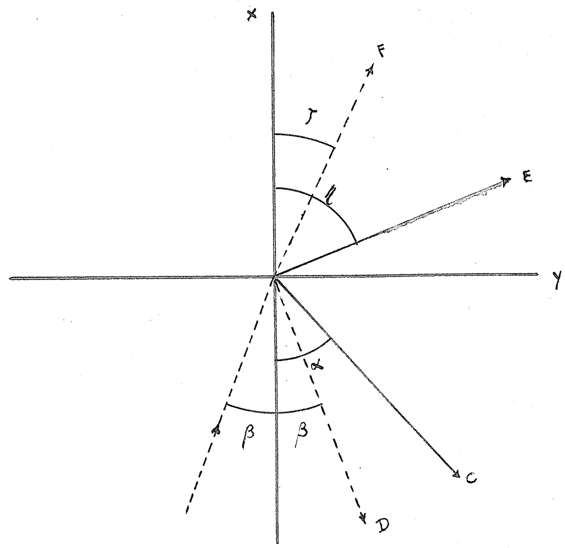
California Institute of Technology

Pasadena, California

January, 1944

The Partition of Amplitudes for an SV-Wave by Zoeppritz's Method

In the figure shown, we have an incident shear wave and four derived rays



Where:

β the angle of incidence of a shear wave of velocity $\mathcal{V}_1$

α is the angle of incidence of the generating compressional wave of velocity V_1

η the angle of refraction of the generating compressional wave of velocity V_2

$\jmath$ the angle of refraction of the generated shear wave of velocity $\mathcal{V}_2$

Our problem is to find the amplitude ratios under specific boundary conditions. First, let us deduce the relations between the different amplitudes in the usual manner (Macelwane 1.).

We are dealing with plane waves in isotropic media and the boundary surface between both media is a plane. The energy of the incident wave is partitioned among the generated waves.

The displacement ϕ of the particle at x,y at the time t is given by

$$\phi = A\varepsilon^{ip\left(t - \frac{y\sin\alpha + x\cos\alpha}{V}\right)}$$

where $p = \frac{2\pi}{T}$ and T is the period.

The symbol ψ is used for the amplitude of the shear wave. The displacements are given by the formulas:

a) For the incident compressional wave

$$\phi_1 = A\varepsilon^{ip\left(t - \frac{y\sin\alpha + x\cos\alpha}{V_1}\right)}$$

b) For the incident shear wave

$$\psi_1 = B\varepsilon^{ip\left(t - \frac{y\sin\beta + x\cos\beta}{\vartheta_1}\right)}$$

c) For the reflected compressional wave

$$\phi_1' = C\varepsilon^{ip\left(t - \frac{y\sin\alpha - x\cos\alpha}{V_1}\right)}$$

d) For the reflected shear wave

$$\psi_1' = D\varepsilon^{ip\left(t - \frac{y\sin\beta - x\cos\beta}{\vartheta_1}\right)}$$

e) For the refracted compressional wave

$$\phi_2 = E\,\varepsilon^{ip\left(t - \frac{y \sin\eta + x\cos\eta}{V_2}\right)}$$

f) For the refracted shear wave

$$\psi_2 = F\,\varepsilon^{ip\left(t - \frac{y \sin\gamma + x\cos\gamma}{\vartheta_2}\right)}$$

The components of the displacement of the shear wave are given by the following equations:

Normal	Tangential
$U_b = -\psi_1 \sin\beta$	$V_b = \psi_1 \cos\beta$
$U_c = -\phi_1' \cos\alpha$	$V_c = \phi_1' \sin\alpha$
$U_d = \psi_1' \sin\beta$	$V_d = \psi_1' \cos\beta$
$U_e = \phi_2 \cos\eta$	$V_e = \phi_2 \sin\eta$
$U_f = \psi_2 \sin\gamma$	$V_f = -\psi_2 \cos\gamma$

Boundary conditions:

a) Equality of the sums of the normal displacements

$$\sum u_1 = \sum u_2$$

b) Equality of the tangential displacements

$$\sum v_1 = \sum v_2$$

c) Equality of the sums of the normal stresses

$$\sum (X_x)_1 = \sum (X_x)_2$$

$$\sum \left(\lambda\theta + 2\mu \frac{\partial u}{\partial x}\right)_1 = \sum \left(\lambda\theta + 2\mu \frac{\partial u}{\partial x}\right)_2$$

d) Equality of the sums of the tangential stresses across the interface

$$\sum (X_y)_1 = \sum (X_y)_2$$

$$\sum \left[\mu\left(\frac{\partial v}{\partial x}\right) + \frac{\partial u}{\partial y}\right]_1 = \sum \left[\mu\left(\frac{\partial V}{\partial x}\right) + \frac{\partial u}{\partial y}\right]_2$$

Substituting in (2) the values given by (1), we have for the normal displacements

$$U_b = - B \, \varepsilon^{ip\left(t - \frac{y \sin\beta + x \cos\beta}{v_1}\right)} \sin\beta$$

$$U_c = - C \, \varepsilon^{ip\left(t - \frac{y \sin\alpha - x \cos\alpha}{V_1}\right)} \cos\alpha$$

$$U_d = D\,\varepsilon^{ip\left(t - \frac{y \sin\beta - x\cos\beta}{\vartheta_1}\right)} \sin\beta$$

$$U_e = E\,\varepsilon^{ip\left(t - \frac{y \sin\eta + x\cos\eta}{V_2}\right)} \cos\eta$$

$$U_f = F\,\varepsilon^{ip\left(t - \frac{y \sin\gamma + x\cos\gamma}{\vartheta_2}\right)} \sin\gamma$$

The condition is at the interface X = 0, and taking into account the Snell's law:

$$\frac{\sin\alpha}{V_1} = \frac{\sin\beta}{\vartheta_1} = \frac{\sin\eta}{V_2} = \frac{\sin\gamma}{\vartheta_2}$$

and applying $\sum u_1 = \sum u_2$, we obtain

$$-B\sin\beta - C\cos\alpha - D\sin\beta = E\cos\eta + F\sin\gamma$$

Applying the boundary condition $\sum v_1 = \sum v_2$ for tangential displacements we get

$$B\cos\beta + C\sin\alpha + D\cos\beta = E\sin\eta - F\cos\gamma$$

For the equality of the sums of the normal stresses

$$\sum\left(\lambda\theta + 2\mu\frac{\partial u}{\partial x}\right)_1 = \sum\left(\lambda\theta + 2\mu\frac{\partial u}{\partial x}\right)_2$$

where $$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

using $$V_1 = \sqrt{\frac{\lambda + 2\mu}{\rho_1}}, \qquad \vartheta_1 = \sqrt{\frac{\mu}{\rho_1}}$$

$$V_2 = \sqrt{\frac{\lambda + 2\mu}{\rho_2}}, \qquad \vartheta_2 = \sqrt{\frac{\mu}{\rho_2}}$$

we get

$$B \sin 2\beta - C \frac{V_1}{\vartheta_1} \cos 2\beta + D \sin 2\beta = - E \frac{\rho_2 V_2}{\rho_1 \vartheta_1} \cos 2\gamma - F \frac{\rho_2 \vartheta_2}{\rho_1 \vartheta_1} \sin 2\gamma$$

For the boundary condition of the equality of tangential stresses

$$\sum \left[\mu \left(\frac{\partial v}{\partial x} \right) + \frac{\partial u}{\partial y} \right]_1 = \sum \left[\mu \left(\frac{\partial v}{\partial x} \right) + \frac{\partial u}{\partial y} \right]_2$$

we get

$$-B \cos 2\beta + C \frac{\vartheta_1}{V_1} \sin 2\alpha + D \cos 2\beta = E \frac{\rho_2 \vartheta_2^2}{\rho_1 \vartheta_1 V_2} \sin 2\eta + F \frac{\rho_2}{\rho_1} \frac{\vartheta_2}{\vartheta_1} \cos 2\gamma$$

Hence, the Zoeppritz's equations for an SV-Wave are:

$$\begin{array}{llllll}
-\sin\beta & -\frac{C}{B}\cos\alpha & +\frac{D}{B}\sin\beta & -\frac{E}{B}\cos\eta & -\frac{F}{B}\sin\gamma & = 0 \\
\cos\beta & -\frac{C}{B}\sin\alpha & +\frac{D}{B}\cos\beta & -\frac{E}{B}\sin\eta & +\frac{F}{B}\cos\gamma & = 0 \\
\sin 2\beta & -\frac{C}{B}\cos 2\beta & +\frac{D}{B}\sin 2\beta & +\frac{E}{B}\frac{\rho_2 V_2}{\rho_1 \vartheta_1}\cos 2\gamma & +\frac{F}{B}\frac{\rho_2 \vartheta_2}{\rho_1 \vartheta_1}\sin 2\gamma & = 0 \\
-\cos 2\beta & +\frac{C\vartheta_1}{BV_1}\sin 2\alpha & +\frac{D}{B}\cos 2\beta & +\frac{E}{B}\frac{\rho_2 \vartheta_2^2}{\rho_1 \vartheta_1 V_2}\sin 2\eta & -\frac{F}{B}\frac{\rho_2 \vartheta_2}{\rho_1 \vartheta_1}\cos 2\gamma & = 0
\end{array}$$

The problem of the present investigation is to study the special case in which

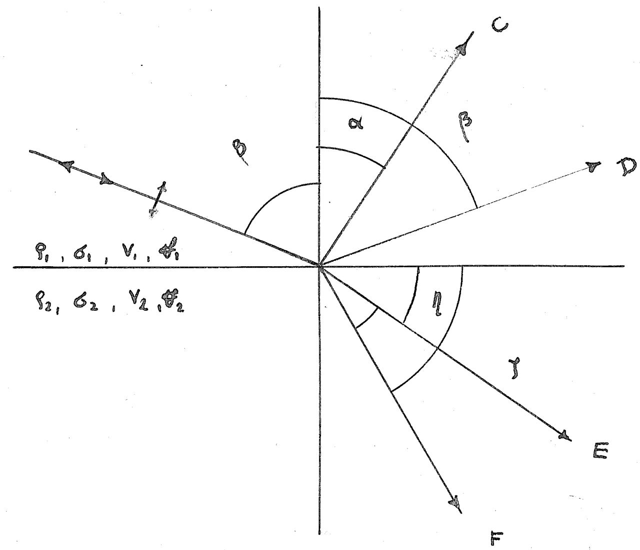
$$\frac{V_2}{V_1} = 1.10$$

$$\frac{\vartheta_2}{\vartheta_1} = 1.09$$

$$\frac{\rho_2}{\rho_1} = 1.09$$

$$\sigma_1 = 0.25$$

$$\sigma_2 = 0.26$$



Especially, we want to find the behavior of the relative amplitudes ratios $\frac{C}{B}$, $\frac{D}{B}$, $\frac{E}{B}$ and $\frac{F}{B}$ of a SV-Wave incident in medium I and transmitted from medium I to medium II. As we assumed $\frac{\sin \eta}{\sin \alpha} = \frac{V_2}{V_1} = 1.10$, it is clear that sin η becomes greater than the unity for $\alpha = 90^{\circ}$. Physically it would mean that the refracted ray will cease to exist beyond 90°, when the incident ray of the shear wave is $31^{\circ}\ 45'$. That leads to values imaginaries of the angle η. Also for values of α for $\gamma = 90^{\circ}$.

Consequently the complex amplitudes are in the form $(p + iq)$. Calculated values of $\frac{C}{B}$, $\frac{D}{B}$, $\frac{E}{B}$ and $\frac{F}{B}$ are given in the following tables. They were plotted as a function of the angle of incidence of the SV-Wave β.

TABLES

1) For $\beta = 0$

Results:

$\alpha = 0$

$\eta = 0$

$\gamma = 0$

The Zoeppritz equations take the form:

$$
\begin{array}{lllll}
 & -\frac{C}{B} & -\frac{E}{B} & & = 0 \\
1 & +\frac{D}{B} & & +\frac{F}{B} & = 0 \\
 & -1.73\frac{C}{B} & +2.07\frac{E}{B} & & = 0 \\
-1 & +\frac{D}{B} & & -1.19\frac{F}{B} & = 0
\end{array}
$$

$\frac{C}{B} = 0$

$\frac{D}{B} = -0.09$

$\frac{E}{B} = 0$

$\frac{F}{B} = -0.91$

2) For $\alpha = 20^{\circ}$

Results:

$\beta = 11^{\circ}\ 23'$

$\gamma = 12^{\circ}\ 24'$

$\eta = 22^{\circ}\ 5'$

$$-0.197 - 0.939\frac{C}{B} + 0.197\frac{D}{B} - 0.926\frac{E}{B} - 0.215\frac{F}{B} = 0$$

$$0.980 + 0.342\frac{C}{B} + 0.980\frac{D}{B} - 0.376\frac{E}{B} + 0.976\frac{F}{B} = 0$$

$$0.386 - 1.595\frac{C}{B} + 0.386\frac{D}{B} + 1.877\frac{E}{B} + 0.501\frac{F}{B} = 0$$

$$-0.922 + 0.366\frac{C}{B} + 0.922\frac{D}{B} + 0.282\frac{E}{B} - 1.079\frac{F}{B} = 0$$

$$\frac{C}{B} = -0.036$$

$$\frac{D}{B} = -0.070$$

$$\frac{E}{B} = 0.021$$

$$\frac{F}{B} = -0.916$$

3) For $\alpha = 30$

Results:

$\beta = 16^\circ\ 46'$

$\gamma = 18^\circ\ 19'$

$\eta = 33^\circ\ 22'$

$$-0.288 \quad -0.866\frac{C}{B} \quad +0.288\frac{D}{B} \quad -0.835\frac{E}{B} \quad -0.314\frac{F}{B} \quad = 0$$

$$0.957 \quad +0.5\ \frac{C}{B} \quad +0.957\frac{D}{B} \quad -0.55\ \frac{E}{B} \quad +0.949\frac{F}{B} \quad = 0$$

$$0.552 \quad -1.44\ \frac{C}{B} \quad +0.552\frac{D}{B} \quad +1.66\ \frac{E}{B} \quad +0.71\ \frac{F}{B} \quad = 0$$

$$-0.833 \quad +0.494\frac{C}{B} \quad +0.833\frac{D}{B} \quad +0.615\frac{E}{B} \quad -0.954\frac{F}{B} \quad = 0$$

$$\frac{C}{B} = -0.047$$

$$\frac{D}{B} = -0.046$$

$$\frac{E}{B} = 0.033$$

$$\frac{F}{B} = -0.917$$

4) For $\alpha = 45$

Results:

$\beta = 24^\circ\ 5'$

$\eta = 51^\circ\ 3'$

$\zeta = 26^\circ$

$$-0.408 \quad -0.707\frac{C}{B} \quad +0.408\frac{D}{B} \quad -0.629\frac{E}{B} \quad -0.438\frac{F}{B} \quad = 0$$

$$0.913 \quad +0.707\frac{C}{B} \quad +0.913\frac{D}{B} \quad -0.778\frac{E}{B} \quad +0.899\frac{F}{B} \quad = 0$$

$$0.745 \quad -1.58\ \frac{C}{B} \quad +0.745\frac{D}{B} \quad +1.274\frac{E}{B} \quad +0.938\frac{F}{B} \quad = 0$$

$$-0.667 \quad +0.570\frac{C}{B} \quad +0.667\frac{D}{B} \quad +0.655\frac{E}{B} \quad -0.733\frac{F}{B} \quad = 0$$

$$\frac{C}{B} = -0.050$$

$$\frac{D}{B} = -0.015$$

$$\frac{E}{B} = 0.042$$

$$\frac{F}{B} = -0.924$$

5) For $\alpha = 55^{\circ}$

Results:

$\beta = 28^{\circ}\ 13'$

$\gamma = 30^{\circ}\ 59'$

$\eta = 64^{\circ}\ 17'$

$$-0.473 - 0.573\frac{C}{B} + 0.473\frac{D}{B} - 0.434\frac{E}{B} - 0.515\frac{F}{B} = 0$$

$$0.881 + 0.819\frac{C}{B} + 0.881\frac{D}{B} - 0.901\frac{E}{B} + 0.857\frac{F}{B} = 0$$

$$0.833 - 0.955\frac{C}{B} + 0.833\frac{D}{B} + 0.971\frac{E}{B} + 1.05\frac{F}{B} = 0$$

$$-0.552 + 0.535\frac{C}{B} + 0.552\frac{D}{B} + 0.524\frac{E}{B} - 0.558\frac{F}{B} = 0$$

$$\frac{C}{B} = -0.042$$

$$\frac{D}{B} = 0.022$$

$$\frac{E}{B} = 0.085$$

$$\frac{F}{B} = -0.924$$

6) For $\eta = 90^{\circ}$

Results:

$\alpha = 65^{\circ}\ 23'$

$\beta = 31^{\circ}\ 45'$

$\gamma = 34^{\circ}\ 51'$

$$-0.526 - 0.417\frac{C}{B} + 0.526\frac{D}{B} - 0.571\frac{F}{B} = 0$$

$$0.850 + 0.909\frac{C}{B} + 0.850\frac{D}{B} - \frac{E}{B} + 0.821\frac{F}{B} = 0$$

$$0.895 - 0.772\frac{C}{B} + 0.895\frac{D}{B} + 0.720\frac{E}{B} + 1.116\frac{F}{B} = 0$$

$$-0.446 + 0.431\frac{C}{B} + 0.446\frac{D}{B} - 0.413\frac{F}{B} = 0$$

$\frac{C}{B} = 0.081$

$\frac{D}{B} = 0.057$

$\frac{E}{B} = 0.211$

$\frac{F}{B} = -0.927$

7) For $\alpha = 90^{\circ}$

Results:

$\beta = 35^{\circ}\ 20'$

$\gamma = 38^{\circ}\ 55'$

$\sin \eta > 1$

-0.57		$+\ 0.57\frac{D}{B}$	$-0.451\frac{E}{B}$	$-0.62\frac{F}{B}$	$= 0$
0.81	$+\ \frac{C}{B}$	$+\ 0.81\frac{D}{B}$	$-1.10\ \frac{E}{B}$	$+\ 0.78\frac{F}{B}$	$= 0$
0.92	$-0.55\frac{C}{B}$	$+\ 0.92\frac{D}{B}$	$+\ 0.45\ \frac{E}{B}$	$+\ 1.14\frac{F}{B}$	$= 0$
-0.32		$+\ 0.32\frac{D}{B}$	$+\ 0.661\frac{E}{B}$	$-0.26\frac{F}{B}$	$= 0$

$\frac{C}{B} = -0.152\ -0.0501$ $\qquad \left|\frac{C}{B}\right| = 0.152$

$\frac{D}{B} = \ 0.052\ -0.0501$ $\qquad \left|\frac{D}{B}\right| = 0.072$

$\frac{E}{B} = -0.025\ -0.0831$ $\qquad \left|\frac{E}{B}\right| = 0.086$

$\frac{F}{B} = -0.933\ -0.0301$ $\qquad \left|\frac{F}{B}\right| = 0.933$

8) For $\gamma = 90^\circ$

Results:

$\sin\beta = 0.91$ $\beta = 65^\circ\ 30'$

$\sin\alpha = 1.59$ α = imaginary

$\sin\eta = 1.75$ η = imaginary

$$-0.91 - 1.241\frac{C}{B} + 0.91\frac{D}{B} - 1.431\frac{E}{B} - \frac{F}{B} = 0$$

$$0.41 - 1.59\frac{C}{B} + 0.41\frac{D}{B} - 1.75\frac{E}{B} = 0$$

$$0.73 + 1.14\frac{C}{B} + 0.73\frac{D}{B} - 2.07\frac{E}{B} = 0$$

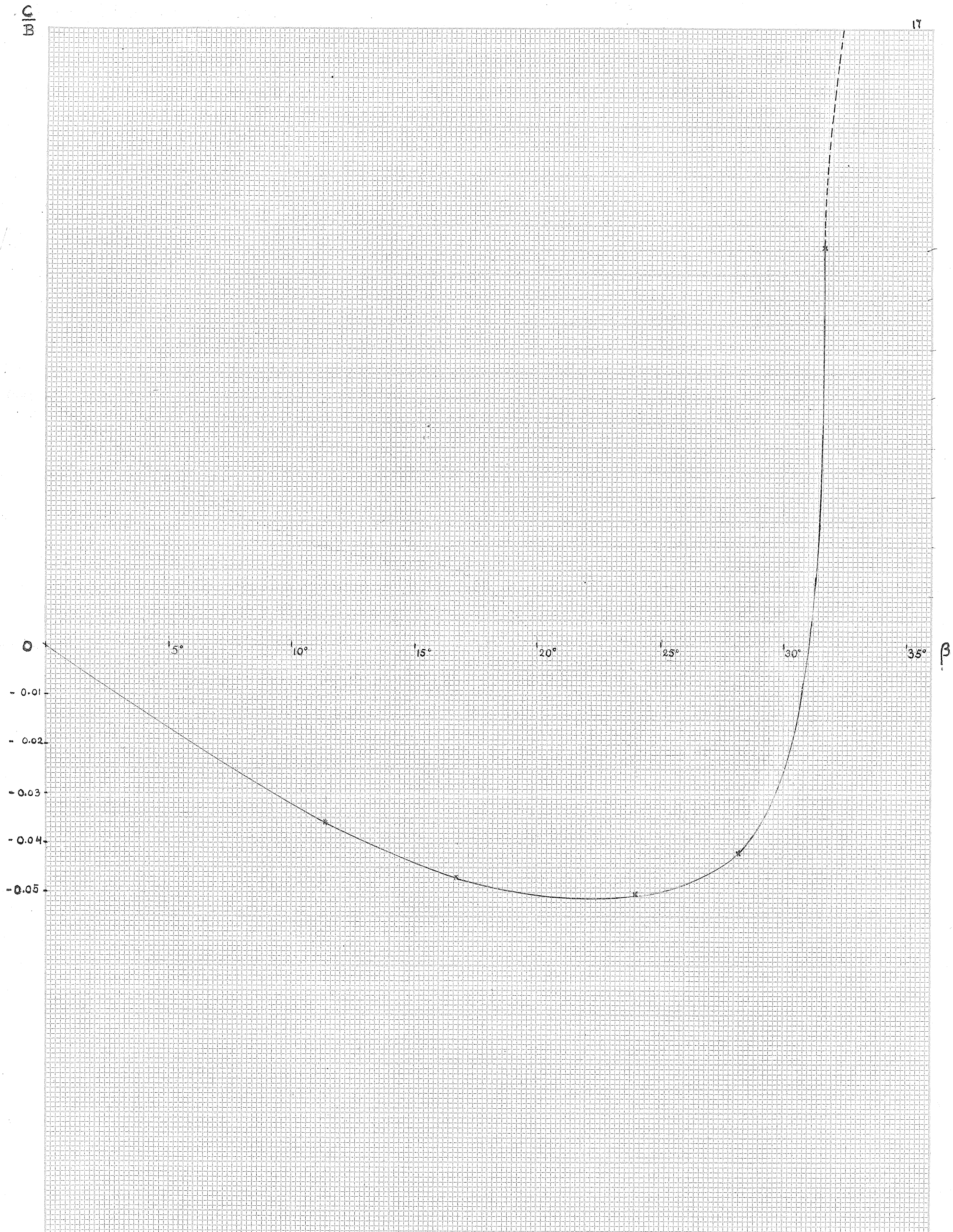
$$0.66 + 2.241\frac{C}{B} - 0.66\frac{D}{B} + 3.351\frac{E}{B} + 1.19\frac{F}{B} = 0$$

$\frac{C}{B} = -0.076 + 0.081i$ $\left|\frac{C}{B}\right| = 0.111$

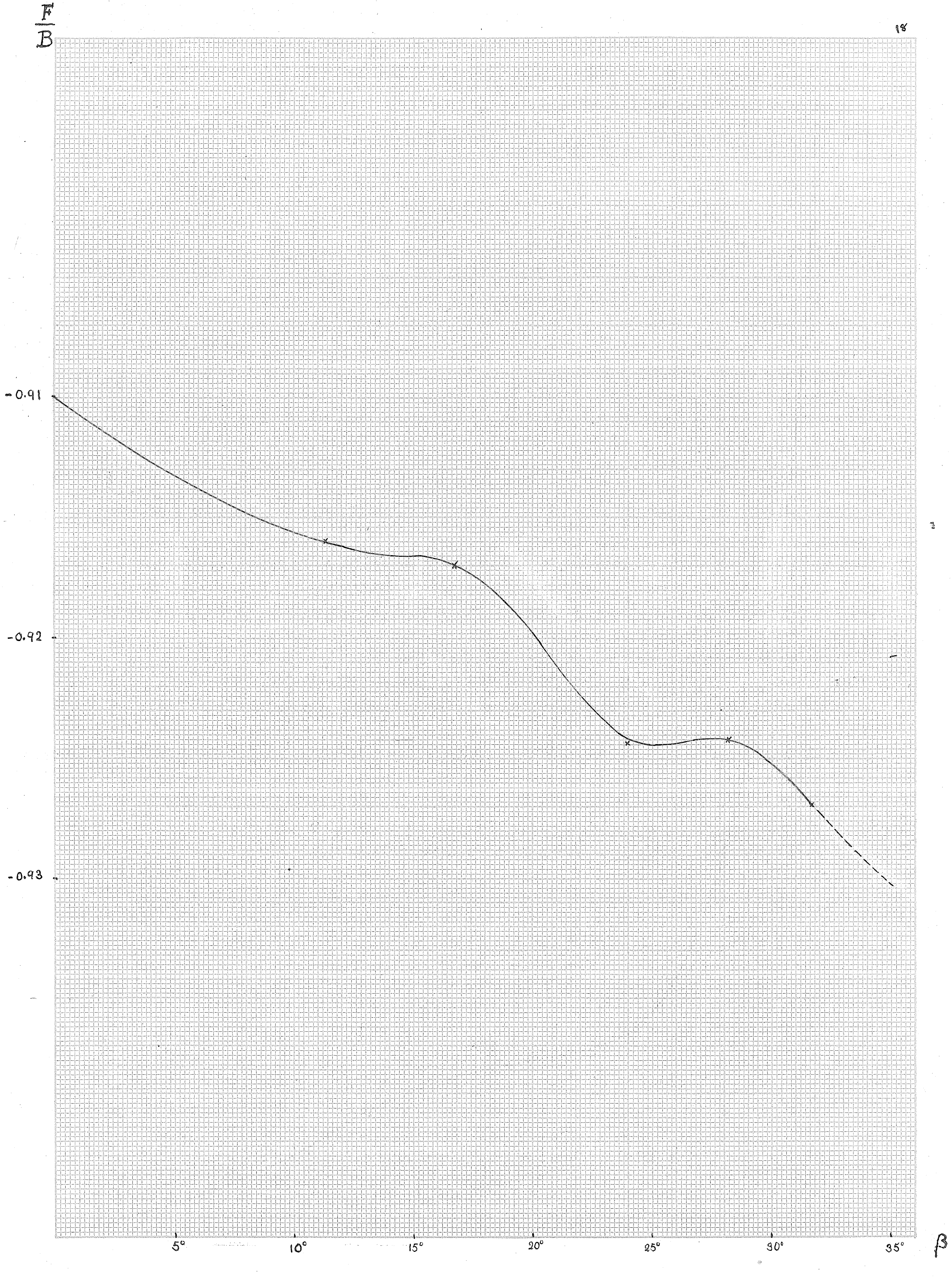
$\frac{D}{B} = -0.066 + 1.62\ i$ $\left|\frac{D}{B}\right| = 1.625$

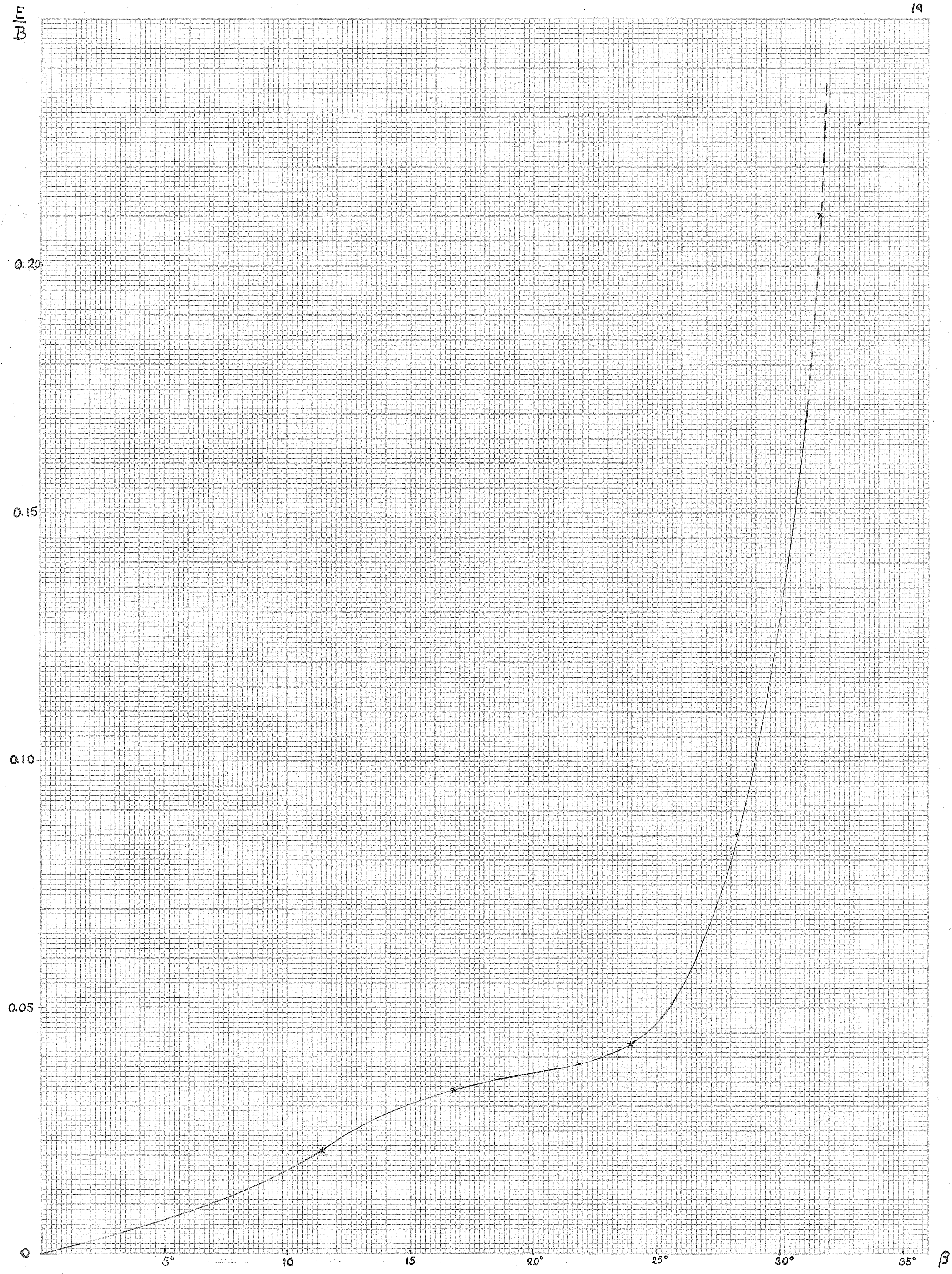
$\frac{E}{B} = 0.288 + 0.307i$ $\left|\frac{E}{B}\right| = 0.420$

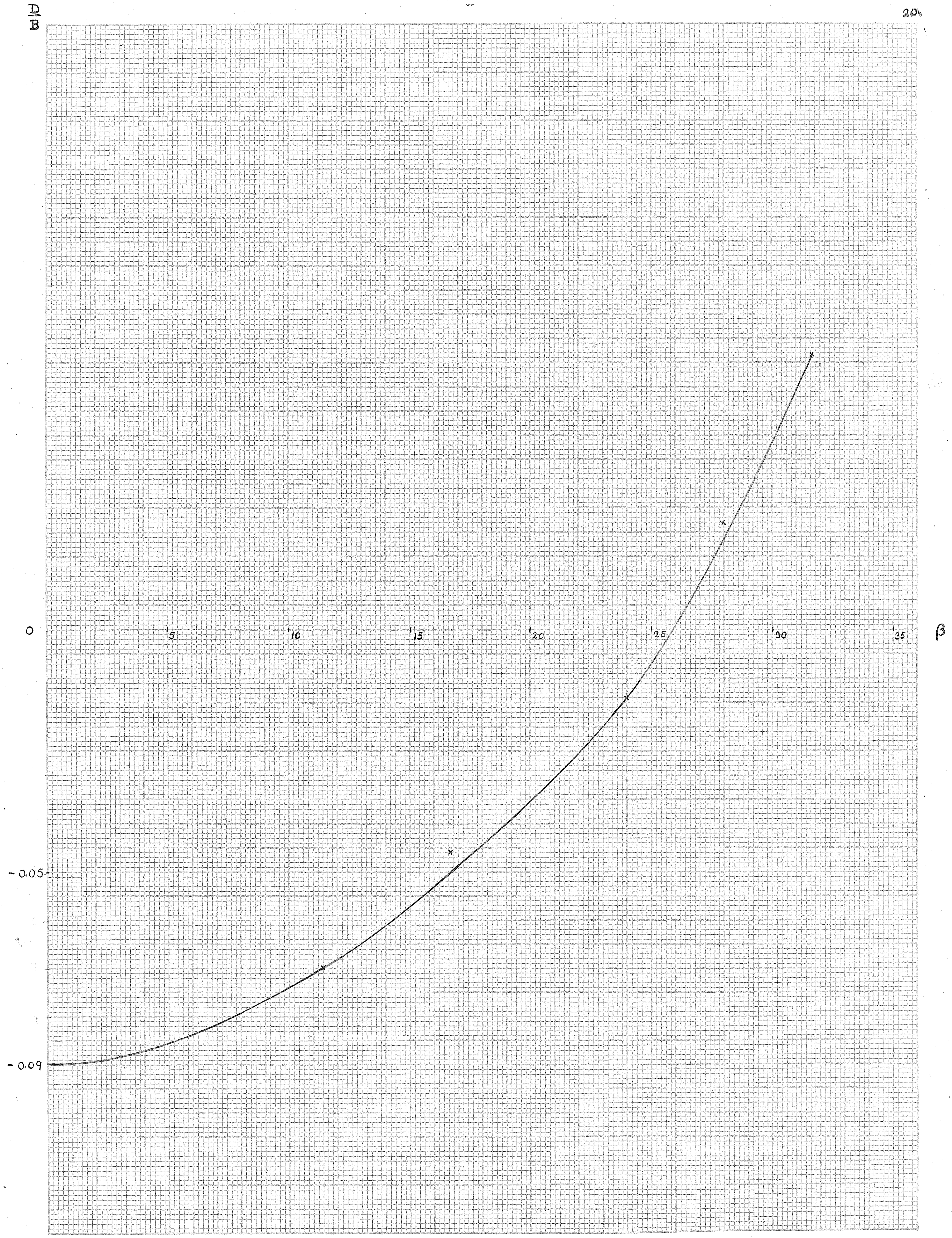
$\frac{F}{B} = -0.431 - 1.16\ i$ $\left|\frac{F}{B}\right| = 1.12$



C/B
0
5°
10°
15°
20°
25°
30°
35°
β
- 0.01
- 0.02
- 0.03
- 0.04
- 0.05


F/B
-0.91
-0.92
-0.93
5°
10°
15°
20°
25°
30°
35°
β


E/B
0.20
0.15
0.10
0.05
0
5°
10
15
20°
25°
30°
35°
β



D/B
0
5
10
15
20
25
30
35
β
- 0.05
- 0.09

REFERENCES

ZOEPPRITZ, KARL - "Uber Erdbebenwellen, VIIB: Uber Reflexion und Durchgang seismicher Wellen durch Unstetigkeitsflachen". Nachrichten der Koniglichen Gesellshaft der Wissenschaften zu Gottingen, Mathematisch-physikalische Klasse, 1919 pp. 57-84.

KNOTT, C.G. - "On the Reflection and Refraction of Elastic Waves, with Seismological Applications." Philosophical Magazine, Fifth Series, Vol. 48, 1899, pp. 64-67.

BERLAGE, H.P., JR. - "Naherungsformeln zur Berechnung der Amplituden der elastichen Wellen, die keim Durchgang einer gegebenen Welle durch eine Unstetigkeistsflache entstehen." Gerlands Beitrage zur Geophysik, Band 26, 1930, pp. 131-140.

JEFFREYS, HAROLD - "Reflexion and Refraction of Elastic Waves." Monthly Notices of the Royal Astronomical Society, Geophysical Supplement, Vol. 1, 1926, pp. 321-334.

WIECHERT, E. and ZOEPPRITZ, KARL - "Uber Erdbebenwellen, I und II." Nachrichten der Koniglichen Gesellschaft der Wissenschaften zu Gottingen, Mathematisch-physikalishe Klasse, 1907, pp. 415-549.

SLICHTER, L.B., and GABRIEL, V.G. - "Studies in Reflected Seismic Waves." Gerlands Beitrage zur Geophysik, Band 38, 1933, pp. 228-256.

MACELWANE, J.B. - Geodynamics (Part I of Theoretical Seismology, page 169).

ACKNOWLEDGMENTS

I am indebted to Dr. Beno Gutenberg of California Institute of Technology, not only for his suggestion of the problem, but also for his valuable direction and advice.